Homework 5

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General Linear 1st Order Systems

1. (Differential calculus for matrices.) Suppose $A(t)$ and $B(t)$ are differentiable. Prove (3.77) and (3.78).

   Proof: We begin by proving (3.77). Let A be an $n \times m$ matrix and B be a $m \times q$ matrix. By definition of matrix multiplication, we have

$$[AB]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

   Since derivatives of matrices are defined as term-by-term differentiation, we have that

$$\begin{aligned}
[\dot{AB}]_{ij} &= \frac{d}{dt} \sum_{k=1}^{m} a_{ik}(t) b_{kj}(t) \\
&= \sum_{k=1}^{m} \frac{d}{dt} a_{ik}(t) b_{kj}(t) \\
&= \sum_{k=1}^{m} (\dot{a}_{ik}(t) b_{kj}(t) + a_{ik}(t) \dot{b}_{kj}(t)) \\
&= [\dot{A}B]_{ij} + [A\dot{B}]_{ij} \\
\implies \frac{d}{dt} AB &= \dot{A}B + A\dot{B}.
\end{aligned}$$

   Now, to show (3.78), notice that

$$AA^{-1} = \mathbb{I}$$

   and so, let $B = A^{-1}$. Then notice

$$\begin{aligned}
\frac{d}{dt} AA^{-1} &= \frac{d}{dt} AB \\
&= \frac{d}{dt} \mathbb{I} \\
&= \mathbf{0}.
\end{aligned}$$

   Thus, by the above proof, we have

$$\begin{aligned}
\frac{d}{dt} AB &= \dot{A}B + A\dot{B} \\
&= 0 \\
\implies A\dot{B} &= -\dot{A}B \\
\implies \dot{B} &= -A^{-1}\dot{A}B \\
\implies \dot{A^{-1}} &= -A^{-1}\dot{A}A^{-1}
\end{aligned}$$

   as was desired.

2. Modify Problem 3.27 to solve the inhomogeneous initial value problem

$$\dot{x} = A(t)x(t) + f(t), \qquad x(0) = x_0$$

where

$$A(t) = \begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix}, \qquad f(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

Soln. Let us begin by solving the homogeneous problem. Note that the IVP corresponds to the following linear system:

$$\begin{aligned} \dot{x}_1 &= tx_1 \\ \dot{x}_2 &= x_1 + tx_2 \end{aligned}$$

To build the principle matrix, first consider the initial condition $x_0 = \delta_1 = (1,0)^T$. Then solving the first equation, we have

$$\begin{aligned} \frac{dx_1}{dt} &= tx_1 \\ \int_{x_1(t_0)}^{x_1(t)} \frac{1}{x_1} dx_1 &= \int_{t_0}^{t} s ds \\ \implies \ln|x_1(t)| &= \frac{t^2}{2} - \frac{t_0^2}{2} + C \\ \implies x_1(t) &= e^{1/2(t^2 - t_0^2)} \qquad (C = 0 \text{ by I.C.}) \end{aligned}$$

and so

$$\frac{dx_2}{dt} = e^{1/2(t^2-t_0^2)} + tx_1(t)$$

solving the homogeneous equation gives us the same form as $x_1(t)$, but by the initial condition δ_1, we have that the homogeneous solution is identically zero. For the particular solution, use the ansatz $x_{2,p} = A(t-t_0)e^{1/2(t^2-t_0^2)}$ so that $\dot{x}_{2,p} = Ae^{1/2(t^2-t_0^2)} + tx_{2,p}$ hence $A = 1$ and

$$x_2 = (t-t_0)e^{1/2(t^2-t_0^2)}$$

Thus,

$$\phi_1 = \begin{pmatrix} e^{1/2(t^2-t_0^2)} \\ (t-t_0)e^{1/2(t^2-t_0^2)} \end{pmatrix}.$$

Now, we consider the initial condition $x_0 = \delta_2 = (0,1)^T$. From our work above, we have that $x_1(t) = 0$ and $x_2(t) = e^{1/2(t^2-t_0^2)}$ so that

$$\phi_2 = \begin{pmatrix} 0 \\ e^{1/2(t^2-t_0^2)} \end{pmatrix}.$$

Thus, our principle matrix is given

$$\begin{aligned} \Pi(t,t_0) &= [\phi_1, \phi_2] \\ &= \begin{pmatrix} e^{1/2(t^2-t_0^2)} & 0 \\ (t-t_0)e^{1/2(t^2-t_0^2)} & e^{1/2(t^2-t_0^2)} \end{pmatrix} \end{aligned}$$

so that our solution is given by

$$x(t) = \begin{pmatrix} e^{1/2(t^2-t_0^2)} & 0 \\ (t-t_0)e^{1/2(t^2-t_0^2)} & e^{1/2(t^2-t_0^2)} \end{pmatrix} x_0$$

$$x(t) = \Pi(t,t_0)x_0 + \int_{t_0}^{t} \Pi(t,s)f(s)ds$$

via variation of parameters. Thus, from our work in problem 1, we have

$$x(t) = \begin{pmatrix} e^{1/2(t^2-t_0^2)} & 0 \\ (t-t_0)e^{1/2(t^2-t_0^2)} & e^{1/2(t^2-t_0^2)} \end{pmatrix} x_0 + \int_{t_0}^{t} \begin{pmatrix} e^{1/2(t^2-s^2)} & 0 \\ (t-s)e^{1/2(t^2-s^2)} & e^{1/2(t^2-s^2)} \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds$$

Notice

$$\int_{t_0}^{t} \begin{pmatrix} e^{1/2(t^2-s^2)} & 0 \\ (t-s)e^{1/2(t^2-s^2)} & e^{1/2(t^2-s^2)} \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds = \int_{t_0}^{t} \begin{pmatrix} 0 \\ se^{1/2(t^2-s^2)} \end{pmatrix} ds$$

and that

$$\int_{t_0}^{t} se^{1/2(t^2-s^2)}ds = e^{1/2t^2} \int_{t_0}^{t} se^{-1/2s^2}ds$$

let $u = 1/2s^2$ so that $du = sds$ and so the above integral becomes

$$\begin{aligned} e^{1/2t^2} \int_{t_0}^{t} se^{-1/2s^2}ds &= e^{1/2t^2} \int_{1/2t_0^2}^{1/2t^2} e^{-u}du \\ &= -e^{1/2t^2} \left[e^{-u}\right] \Big|_{1/2t_0^2}^{1/2t^2} \\ &= -e^{1/2t^2} \left[e^{-1/2t^2} - e^{-1/2t_0^2}\right] \\ &= e^{1/2(t^2-t_0^2)} - 1 \end{aligned}$$

so that our general solution is given by

$$x(t) = \begin{pmatrix} e^{1/2(t^2-t_0^2)} & 0 \\ (t-t_0)e^{1/2(t^2-t_0^2)} & e^{1/2(t^2-t_0^2)} \end{pmatrix} x_0 + \begin{pmatrix} 0 \\ e^{1/2(t^2-t_0^2)} - 1 \end{pmatrix}$$

Reduction of Order

3. Use reduction of order to find the general solution of the following equations:

   (i) $t\ddot{x} - 2(t+1)\dot{x} + (t+2)x = 0$, $\phi_1(t) = e^t$.

   Soln. Let us begin by rewriting the differential equation as a first order system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1-\frac{2}{t} & 2+\frac{2}{t} \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

   Since e^t is a solution to the original equation, we have $\phi_1 = (e^t, e^t)^T$ is a solution to the system. Now, transforming the coordinates to $\dot{y} = X^{-1}(AX - \dot{X})y$, we have

$$X = \begin{pmatrix} e^t & 0 \\ e^t & 1 \end{pmatrix}, \quad \dot{X} = \begin{pmatrix} e^t & 0 \\ e^t & 0 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} e^{-t} & 0 \\ -1 & 1 \end{pmatrix}$$

so that the transformed system becomes

$$\begin{aligned}
\dot{y} &= \begin{pmatrix} e^{-t} & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ -t-\frac{2}{t} & 2+\frac{2}{t} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ e^t & 1 \end{pmatrix} - \begin{pmatrix} e^t & 0 \\ e^t & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} e^{-t} & 0 \\ -1 & 1 \end{pmatrix} \left[\begin{pmatrix} e^t & 1 \\ e^t & 2+\frac{2}{t} \end{pmatrix} - \begin{pmatrix} e^t & 0 \\ e^t & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} e^{-t} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 2+\frac{2}{t} \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{-t} \\ 0 & 1+\frac{2}{t} \end{pmatrix}
\end{aligned}$$

So that we now have the linear system

$$\begin{aligned}
\dot{y}_1 &= e^{-t} y_2 \\
\dot{y}_2 &= (1+\tfrac{2}{t}) y_2
\end{aligned}$$

from the second equation, via separation of variables, we find $y_2 = t^2 e^t$ and so $\dot{y}_1 = t^2, \implies y_1 = \frac{1}{3}t^3$. Converting back to x:

$$\begin{aligned}
X \begin{pmatrix} \frac{1}{3}t^3 \\ t^2 e^t \end{pmatrix} &= \begin{pmatrix} e^t & 0 \\ e^t & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3}t^3 \\ t^2 e^t \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3}t^3 e^t \\ \frac{1}{3}t^3 e^t + t^2 e^t \end{pmatrix}
\end{aligned}$$

so that our other solution is $x_2(t) = \frac{1}{3}t^3 e^t$ and the general solution is

$$x(t) = C_1 e^t + C_2 t^3 e^t$$

(ii) $t^2\ddot{x} - 3t\dot{x} + 4x = 0$, $\phi_1(t) = t^2$.

Soln. As above, we write the differential equation as a first order system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{4}{t^2} & \frac{3}{t} \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

and since $x_1(t) = t^2$ is a solution to the differential equation, $\phi_1(t) = (t^2, 2t)^T$ is a solution to our system. Thus, transforming the coordinates, we have

$$\begin{aligned}
\dot{y} &= X^{-1}(AX - \dot{X}) \\
&= \begin{pmatrix} \frac{1}{t^2} & 0 \\ -\frac{2}{t} & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ -\frac{4}{t^2} & \frac{3}{t} \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 2t & 1 \end{pmatrix} - \begin{pmatrix} 2t & 0 \\ 2 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{1}{t^2} & 0 \\ -\frac{2}{t} & 1 \end{pmatrix} \left[\begin{pmatrix} 2t & 1 \\ 2 & \frac{3}{t} \end{pmatrix} - \begin{pmatrix} 2t & 0 \\ 2 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{1}{t^2} & 0 \\ -\frac{2}{t} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & \frac{3}{t} \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{t^2} \\ 0 & \frac{1}{t} \end{pmatrix}.
\end{aligned}$$

So now, we have the system

$$\begin{aligned}
\dot{y}_1 &= \tfrac{1}{t^2} y_2 \\
\dot{y}_2 &= \tfrac{1}{t} y_2.
\end{aligned}$$

By separation of variables, from the second equation, we have $y_2 = t$ and so $\dot{y}_1 = \frac{1}{t} \implies y_1 = \ln|t|$. Transforming back to our original coordinates, we have

$$\begin{pmatrix} t^2 & 0 \\ 2t & 1 \end{pmatrix} \begin{pmatrix} \ln|t| \\ t \end{pmatrix} = \begin{pmatrix} t^2 \ln|t| \\ 2t\ln|t| + t \end{pmatrix}$$

so that $x_2(t) = t^2 \ln|t|$ and our general solution is

$$x(t) = C_1 t^2 + C_2 t^2 \ln|t|.$$

4. Verify that the second-order equation
$$\ddot{x} + (1-t^2)x = 0$$
can be factorized as
$$\left(\frac{d}{dt} - t\right)\left(\frac{d}{dt} + t\right)x = 0$$
(note that the order is important.) Use this to find the solution.

Soln. Verifying the factorization, notice
$$\begin{aligned}
\left(\frac{d}{dt} - t\right)\left(\frac{d}{dt} + t\right)x &= \left(\frac{d}{dt} - t\right)\left(\frac{dx}{dt} + tx\right) \\
&= \frac{d}{dt}\left(\frac{dx}{dt} + tx\right) - t\left(\frac{dx}{dt} + tx\right) \\
&= \frac{d^2x}{dt^2} + x + t\frac{dx}{dt} - t\frac{dx}{dt} - t^2x \\
&= \frac{d^2x}{dt^2} + x - t^2x \\
&= \ddot{x} + (1-t^2)x.
\end{aligned}$$
Now, notice that if $\left(\frac{d}{dt} + t\right)x = 0$, the differential equation is satisfied. Thus, solving this equation, we have
$$\begin{aligned}
&\frac{dx}{dt} + tx = 0 \\
\implies &\frac{dx}{dt} = -tx \\
\implies &\int \frac{dx}{x} = -\int tdt \\
\implies &\ln|x| = -\frac{t^2}{2} + C \\
\implies &x(t) = Ce^{-t^2/2}
\end{aligned}$$
so that $x_1 = e^{-t^2/2}$ is a solution to the differential equation. Now, we write the differential equation as a linear system:
$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t^2-1 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$
Thus, $\phi_1 = (e^{-t^2/2}, -te^{-t^2/2})^T$ is a solution to the above system. Now, apply the change of coordinates $\dot{y} = X^{-1}(AX - \dot{X})y$ where
$$X = \begin{pmatrix} e^{-t^2/2} & 0 \\ -te^{-t^2/2} & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ t^2-1 & 0 \end{pmatrix}, \quad \dot{X} = \begin{pmatrix} -te^{-t^2/2} & 0 \\ (t^2-1)e^{-t^2/2} & 0 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} e^{t^2/2} & 0 \\ t & 1 \end{pmatrix}$$
so that
$$\begin{aligned}
\dot{y} &= \begin{pmatrix} e^{t^2/2} & 0 \\ t & 1 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ t^2-1 & 0 \end{pmatrix}\begin{pmatrix} e^{-t^2/2} & 0 \\ -te^{-t^2/2} & 1 \end{pmatrix} - \begin{pmatrix} -te^{-t^2/2} & 0 \\ (t^2-1)e^{-t^2/2} & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} e^{t^2/2} & 0 \\ t & 1 \end{pmatrix}\left[\begin{pmatrix} -te^{-t^2/2} & 1 \\ (t^2-1)e^{-t^2/2} & 0 \end{pmatrix} - \begin{pmatrix} -te^{-t^2/2} & 0 \\ (t^2-1)e^{-t^2/2} & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} e^{t^2/2} & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{t^2/2} \\ 0 & t \end{pmatrix}
\end{aligned}$$

so that we have the following system:

$$\begin{aligned}\dot{y}_1 &= e^{t^2/2} y_2 \\ \dot{y}_2 &= t y_2\end{aligned}$$

and from the second equation, notice $y_2 = e^{t^2/2}$ and so

$$y_1 = \int_{t_0}^{t} e^{s^2}\, ds.$$

Converting back to our original coordinates we find

$$\begin{aligned} x &= \begin{pmatrix} e^{-t^2/2} & 0 \\ -te^{-t^2/2} & 1 \end{pmatrix} \begin{pmatrix} \int_{t_0}^{t} e^{s^2} ds \\ e^{t^2/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-t^2/2} \int_{t_0}^{t} e^{s^2} ds \\ -te^{-t^2/2} \int_{t_0}^{t} e^{s^2} ds + e^{t^2/2} \end{pmatrix} \end{aligned}$$

so that $x_2 = e^{-t^2/2} \int_{t_0}^{t} e^{s^2} ds$ is a solution to the system. Thus, the general solution to the differential equation is

$$x(t) = C_1 e^{-t^2/2} + C_2 e^{-t^2/2} \int_{t_0}^{t} e^{s^2} ds.$$

Floquet Theory

5. Compute the monodromy matrix where $A(t)$ is of period 1 and given by

$$A(t) = \begin{cases} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, & 0 \leq t < \frac{1}{2}, \\ \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, & \frac{1}{2} \leq t < 1, \end{cases} \qquad \alpha \in \mathbb{C}.$$

Note that since $A(t)$ is not continuous you have to match solutions at every discontinuity such that the solutions are continuous.

For which values of α remain all solutions bounded? Show that the bound found in problem 3.31 is optimal by considering $A(t/T)$ as $T \to 0$.

(Note that we could approximate $A(t)$ by continuous matrices and obtain the same qualitative result with an arbitrary small error.)

Soln. For $0 \leq t < \frac{1}{2}$, we have

$$A(t) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

is constant for these values of t, we have $\Pi(t, t_0) = e^{(t-t_0)A}$, so that

$$\Pi(t, t_0) = \begin{pmatrix} e^{(t-t_0)\alpha} & (t-t_0)e^{(t-t_0)\alpha} \\ 0 & e^{(t-t_0)\alpha} \end{pmatrix} \qquad t \in [0, \tfrac{1}{2}).$$

For $\frac{1}{2} \leq t < 1$, since

$$\begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}^T = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

and that $\exp(A^T) = (\exp(A))^T$, we have

$$\Pi(t, t_0) = \begin{pmatrix} e^{(t-t_0)\alpha} & 0 \\ (t-t_0)e^{(t-t_0)\alpha} & e^{(t-t_0)\alpha} \end{pmatrix}, \qquad t \in [\tfrac{1}{2}, 1).$$

Now, since the monodromy matrix is defined as

$$m(t_0) = \Pi(t_0 + T, t_0) = \Pi(1, 0)$$

we use the fact that

$$\Pi(t, t_1) = \Pi(t, t_0)\Pi(t_0, t_1)$$

to find

$$\begin{aligned} \Pi(1, 0) &= \Pi(1, \tfrac{1}{2})\Pi(\tfrac{1}{2}, 0) \\ &= \begin{pmatrix} e^{(1-\frac{1}{2})\alpha} & 0 \\ (1-\frac{1}{2})e^{(1-\frac{1}{2})\alpha} & e^{(1-\frac{1}{2})\alpha} \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\alpha} & \frac{1}{2}e^{\frac{1}{2}\alpha} \\ 0 & e^{\frac{1}{2}\alpha} \end{pmatrix} \\ &= \begin{pmatrix} e^{\alpha} & \frac{1}{2}e^{\alpha} \\ \frac{1}{2}e^{\alpha} & \frac{5}{4}e^{\alpha} \end{pmatrix} \end{aligned}$$

thus

$$m(t_0) = \begin{pmatrix} e^{\alpha} & \frac{1}{2}e^{\alpha} \\ \frac{1}{2}e^{\alpha} & \frac{5}{4}e^{\alpha} \end{pmatrix}.$$

Now, to find the values of $\alpha \in \mathbb{C}$ for which solutions remain bounded, we find the Floquet multipliers, that is the eigenvalues of the monodromy matrix:

$$\begin{aligned}
\begin{vmatrix} e^{\alpha} - \lambda & \frac{1}{2}e^{\alpha} \\ \\ \frac{1}{2}e^{\alpha} & \frac{5}{4}e^{\alpha} - \lambda \end{vmatrix} &= (e^{\alpha} - \lambda)(\tfrac{5}{4}e^{\alpha} - \lambda) - \tfrac{1}{4}e^{2\alpha} \\
&= \lambda^2 - \lambda e^{\alpha} - \tfrac{5}{4}\lambda e^{\alpha} + \tfrac{5}{4}e^{2\alpha} - \tfrac{1}{4}e^{2\alpha} \\
&= \lambda^2 - \tfrac{9}{4}\lambda e^{\alpha} + e^{2\alpha} \\
&= 0 \\
\implies \lambda_{\pm} &= \frac{\frac{9}{4}e^{\alpha} \pm \sqrt{\frac{81}{16}e^{2\alpha} - 4e^{2\alpha}}}{2} \\
&= \frac{\frac{9}{4}e^{\alpha} \pm \sqrt{\frac{17}{16}e^{2\alpha}}}{2} \\
&= e^{\alpha}\left(\frac{9}{8} \pm \frac{\sqrt{17}}{8}\right).
\end{aligned}$$

Recall that if $|\lambda| \leq 1$, we have that the system is stable, hence bounded. Now, let $\alpha = a + ib$ for $a, b \in \mathbb{R}$ and let $x = \frac{9}{8} + \frac{\sqrt{17}}{8}$ and $y = \frac{9}{8} - \frac{\sqrt{17}}{8}$. Notice that $x, y > 0$ since $9 > \sqrt{17}$. Looking case by case, notice

$$\begin{aligned}
|\lambda_+| &= |e^{a+ib}x| \leq 1 \\
\implies e^{a} &\leq \frac{1}{x} \\
\implies a &\leq -\ln(x) && (\ln(\cdot) \text{ monotone}) \\
\mathrm{Re}(\alpha) &\leq -\ln(x)
\end{aligned}$$

similarly, for λ_-, we have

$$\begin{aligned}
|\lambda_-| &= |e^{a+ib}y| \leq 1 \\
\implies e^{a} &\leq \frac{1}{y} \\
\implies a &\leq -\ln(x) && (\ln(\cdot) \text{ monotone}) \\
\mathrm{Re}(\alpha) &\leq -\ln(x).
\end{aligned}$$

Thus, since $x > y$, and $\ln(\cdot)$ is monotone, we have that

$$\mathrm{Re}(\alpha) \leq -\ln\left(\frac{9}{8} + \frac{\sqrt{17}}{8}\right)$$

will guarantee stability. To show the bound in problem 3.31 is optimal, consider $A(t/T)$:

$$A(t/T) = \begin{cases} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, & 0 \leq \frac{t}{T} < \frac{1}{2}, \\ \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, & \frac{1}{2} \leq \frac{t}{T} < 1 \end{cases}, \quad \alpha \in \mathbb{C}$$

which we may rewrite as

$$A(t/T) = \begin{cases} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, & 0 \leq t < \frac{T}{2} \\ \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, & \frac{T}{2} \leq t < T \end{cases}, \quad \alpha \in \mathbb{C}$$

which has principle matrix solutions

$$\Pi(\tfrac{T}{2},0) = \begin{pmatrix} e^{\frac{T}{2}\alpha} & \frac{T}{2}e^{\frac{T}{2}\alpha} \\ 0 & e^{\frac{T}{2}\alpha} \end{pmatrix}$$

and

$$\Pi(T,\tfrac{T}{2}) = \begin{pmatrix} e^{\frac{T}{2}\alpha} & 0 \\ \frac{T}{2}e^{\frac{T}{2}\alpha} & e^{\frac{T}{2}\alpha} \end{pmatrix}$$

and notice that for any t, $\operatorname{eig}\left(\frac{A+A^*}{2}\right) = \operatorname{Re}(\alpha)$. Thus, by the bound in problem 3.31, we have

$$\|\Pi(t,t_0)\| \le e^{(t-t_0)\operatorname{Re}(\alpha)}$$

to show a bound for the operator norm is optimal, usually you find a norm 1 vector and take their product and find the norm of that product. For this, I have no idea how to do that without a specific norm, and I have no idea how to use $A(t/T)$.

6. Numerically solve the Mathieu equation (3.148). Re-create the stability diagram in figure 3.6 on page 96.

$$\ddot{x} = -\omega^2(1 + \varepsilon\cos(t))x.$$

To begin, we rewrite this differential equation as a system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2(1+\varepsilon\cos(t))x & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

to numerically solve this, we implemented RK4 in MATLAB for fixed values of ω, ε and solved for the initial conditions $\delta_1 = (1, 0)^T$, $\delta_2 = (0, 1)^T$ on the interval $[0, 2\pi]$. At the value 2π, we take the associated solutions for x and $\dot{x}$ corresponding to δ_1 and δ_2 and build the monodromy matrix to find the value of Δ ($\Delta = \mathrm{tr}(m(t_0))/2$). By varying ω and ε and computing Δ, we find the following stability diagram:

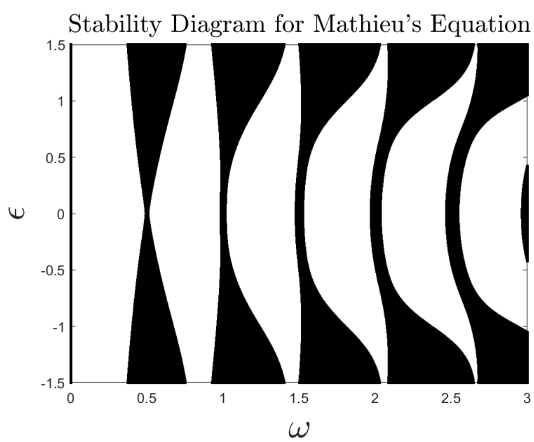

Figure 1: Stability region for Mathieu's equation. Dark region corresponds to $|\Delta| \geq 1$, light regions correspond to $|\Delta| < 1$, as in the textbook.