Modern Algebra HW 10

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Section 23 Problems

2. Find $q(x)$ and $r(x)$ as described by the division algorithm so that $f(x) = g(x)q(x) + r(x)$ with $r(x) = 0$ or of degree less than degree of $g(x)$.

$f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2$ and $g(x) = 3x^2 + 2x - 3$ in $\mathbb{Z}_7[x]$.

To find $q(x)$ and $r(x)$, we will proceed by using long division. Notice

$$
\begin{array}{r}
5x^4 + 5x^2 - x \qquad\qquad\qquad\qquad\qquad\qquad \\
3x^2 + 2x - 3\overline{)x^6 + 3x^5 + 0x^4 + 0x^3 + 4x^2 - 3x + 2} \\
-(\underline{x^6 + 3x^5 - x^4}) \qquad\qquad\qquad\qquad\qquad\quad \\
x^4 + 0x^3 + 4x^2 - 3x + 2 \\
-(\underline{x^4 + 3x^3 - x^2}) \qquad\qquad\quad \\
-3x^3 + 5x^2 - 3x + 2 \\
-(\underline{-3x^3 - 2x^2 + 3x}) \qquad \\
-6x + 2
\end{array}
$$

So

$$x^6 + 3x^5 + 4x^2 - 3x + 2 = g(x)q(x) + r(x)$$

where $g(x) = 3x^2 + 2x - 3$, $q(x) = 5x^4 + 5x^2 - x$, and $r(x) = -6x + 2$.

12. Is $x^3 + 2x + 3$ an irreducible polynomial of $\mathbb{Z}_5[x]$? Why? Express it as a product of irreducible polynomials of $\mathbb{Z}_5[x]$.

Let $f(x) = x^3 + 2x + 3$ and let us begin by checking whether each element of $\mathbb{Z}_5$ gives us $f(x) = 0$:

$$
\begin{aligned}
f(0) &= 3 \\
f(1) &= 1 \\
f(2) &= 0 \\
f(3) &= 1 \\
f(4) &= 3
\end{aligned}
$$

From this, we can see that $(x-2)$ is a factor of $f(x)$. Let us find the quotient of $f(x)$ under division by $(x-2)$ by means of long division:

$$
\begin{array}{r}
x^2 + 2x + 1 \\
x-2\overline{)x^3 + 0x^2 + 2x + 3} \\
-\underline{(x^3 - 2x^2)} \\
2x^2 + 2x + 3 \\
-\underline{(2x^2 - 4x)} \\
x + 3 \\
-\underline{(x - 2)} \\
0
\end{array}
$$

So $f(x)$ factors to $(x-2)(x^2+2x+1)$. Now let $g(x) = x^2 + 2x + 1$. Notice that $x^2 + 2x + 1 = (x+1)^2$, so our factorization for $f(x)$ is:

$$f(x) = (x-2)(x+1)^2$$

16. Demonstrate that $x^3 + 3x^2 - 8$ is irreducible over $\mathbb{Q}$.

Notice that the coefficients for $x^3 + 3x^2 - 8$ are (in order of descending power): $1, 3, 0, -8$. Further notice that $1 \not\equiv 0 \mod 3$, $3 \equiv 0 \mod 3$, $0 \equiv 0 \mod 3$, and $-8 \not\equiv 0 \mod 3^2$. Then by the Eisenstein Criterion, we have that $x^3 + 3x^2 - 8$ is irreducible over $\mathbb{Q}$.

17. Demonstrate that $x^4 - 22x^2 + 1$ is irreducible over $\mathbb{Q}$.

Notice that the only zeros of this function in $\mathbb{Z}$ must be $x = \pm 1$. Testing these values for $f(x) = x^4 - 22x^2 + 1$, we find

$$
\begin{aligned}
f(1) &= 1 - 22 + 1 = -20 \\
f(-1) &= 1 - 22 + 1 = -20
\end{aligned}
$$

Then $f(x)$ is irreducible in $\mathbb{Z}$, and so by theorem 23.11, we have that $f(x)$ is irreducible over $\mathbb{Q}$.

28. Find all irreducible polynomials of degree 3 in $\mathbb{Z}_2[x]$.

Let us begin by listing all the degree 3 polynomials in $\mathbb{Z}_2[x]$:

$$
\begin{aligned}
&x^3 \\
&x^3 + 1 \\
&x^3 + x \\
&x^3 + x + 1 \\
&x^3 + x^2 \\
&x^3 + x^2 + 1 \\
&x^3 + x^2 + x \\
&x^3 + x^2 + x + 1
\end{aligned}
$$

Immediately, we can see that x^3, $x^3 + x$, $x^3 + x^2$, $x^3 + x^2 + x$ are reducible (factor out an x!). Upon further inspection, we can see $x^3 + 1$ and $x^3 + x^2 + x + 1$ have factors of $x - 1$ and are thus irreducible. So the irreducible polynomials of degree 3 in $\mathbb{Z}_2$ are

$$x^3 + x^2 + 1$$
$$x^3 + x + 1$$