# Analysis Final

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1. (a) Consider a sequence $\{x_n\}$ on the interval [0,1]. If every *convergent* subsequence of $\{x_n\}$ has the same limit $x_0$. Prove that $\lim x_n = x_0$.

Proof: Let $\{x_n\}$ be a sequence on $[0, 1]$ and suppose that every convergent subsequence of $\{x_n\}$ has the same limit $x_0$. Suppose by way of contradiction that $\{x_n\}$ does not converge to $x_0$. Then for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\{x_{n_k}\}$ does not converge to $x_0$.
Fix $\epsilon > 0$. Then for some $n_k > N_1 \in \mathbb{N}$, since $\{x_{n_k}\}$ does not converge to $x_0$,

$$|x_{n_k} - x_0| \geq \epsilon$$

But since $\{x_{n_k}\}$ is a sequence on $[0, 1]$, a closed, bounded interval, Bolzano-Weierstrass guarantees that there exists a convergent subsequence $\{x_{n_{k_m}}\}$ of $\{x_{n_k}\}$. But by the above inequality, we have that for some $n_{k_m} > N_2 \in \mathbb{N}$,

$$|x_{n_{k_m}} - x_0| \geq \epsilon$$

That is, $\{x_{n_{k_m}}\}$ does not converge to $x_0$. But $\{x_{n_{k_m}}\}$ is a convergent subsequence of $\{x_n\}$, contradicting our hypothesis that every convergent subsequence of $\{x_n\}$ converges to $x_0$.

(b) Let $f_n(x) = x^n(1 - x^n)$. Show that $\{f_n\}$ is *not* a convergent sequence in $C[0, 1]$ with metric $d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)|$.

To begin, we will find the maximum value of $f_n$. Using the first derivative test, we find

$$f_n' = nx^{n-1}(1 - x^n) - nx^{2n-1} = 0$$

and we find $f_n$ attains a maximum value at

$$x_{max} = \left(\frac{1}{2}\right)^{1/n}$$

Plugging this value of $x$ into $f_n(x)$, we get

$$f_n(x_{max}) = \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

That is, for every $n$, the maximum value of $f_n$ on $[0, 1]$ is $\frac{1}{4}$. Now we will show that $f_n$ does not converge on $C[0, 1]$. We will do so by showing that $\{f_n\}$ is not Cauchy. Let $\epsilon = \frac{1}{32}$ and let $m, n > N \in \mathbb{N}$ and consider

$$\sup_{x\in[0,1]} |f_n(x) - f_m(x)|$$

and notice that

$$|f_n(x) - f_m(x)| \leq \sup_{x\in[0,1]} |f_n(x) - f_m(x)|$$

Now, choose $m = 2n$. Then

$$|f_n(x) - f_m(x)| = |f_n(x) - f_{2n}(x)| = |x^n(1 - x^n) - x^{2n}(1 - x^{2n})|$$

Let's check the point $\left(\frac{1}{2}\right)^{1/n}$:

$$\left| f_n\left(\frac{1}{2^{1/n}}\right) - f_{2n}\left(\frac{1}{2^{1/n}}\right) \right| = \left| \frac{1}{4} - \frac{3}{16} \right|$$

$$= \left| \frac{1}{16} \right| > \frac{1}{32}$$

So for any choice of $m, n > N$, choosing $m = 2n$, we get that $|f_n(x_{max}) - f_m(x_{max})| > \frac{1}{32}$ So $\{f_n\}$ is not Cauchy, and therefore does not converge.

2. (a) Let $(X, d)$ be a metric space. Show that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$, $\forall x, y \in X$, defines a metric on $X$, and that every subset $E \subset X$ is bounded with respect to the metric $\delta$.

Proof: To show $\delta(x, y)$ is a metric, we must show that symmetry, non-negativity, and the triangle inequality holds. To begin, we will show symmetry holds. Since $d(x, y)$ is a metric, notice that

$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)} = \delta(y, x)$$

So symmetry holds. Now we will show non-negativity holds. We have that $d(x, y) \geq 0$, so $1 + d(x, y) \geq 1$ and so $\delta(x, y) \geq 0$. Now we must show $\delta(x, y) = 0$ if and only if $x = y$. To begin, let $x = y$. Then

$$\delta(x, y) = \delta(x, x) = \frac{d(x, x)}{1 + d(x, x)} = \frac{0}{1} = 0$$

Now suppose $\delta(x, y) = 0$. Then $\frac{d(x,y)}{1+d(x,y)} = 0$ and so we must have $d(x, y) = 0$. And since $d(x, y)$ is a metric, we have $x = y$. So non-negativity holds.
Finally, we will show that $\delta(x, y)$ satisfies the triangle inequality.
To begin, let $x, y, z \in X$ and consider $\delta(x, z) = \frac{d(x,z)}{1+d(x,z)}$ and $\delta(z, y) = \frac{d(z,y)}{1+d(z,y)}$. Notice that since $d(x, y) \geq 0$ for any $x, y \in X$, we have that

$$\delta(x, z) = \frac{d(x, z)}{1 + d(x, z)} \geq \frac{d(x, z)}{1 + d(x, z) + d(z, y)}$$

similarly,

$$\delta(z, y) = \frac{d(z, y)}{1 + d(z, y)} \geq \frac{d(z, y)}{1 + d(z, y) + d(x, z)}$$

That is,

$$\frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \leq \delta(x, z) + \delta(z, y)$$

Now notice if we divide the numerator and denominator on the left hand side of the above equation by $d(x, z) + d(z, y)$, we get

$$\frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} = \frac{1}{\frac{1}{d(x,z)+d(z,y)} + 1}$$

And since $d(x, y)$ is a metric,

$$d(x, y) \leq d(x, z) + d(z, y)$$

$$\frac{1}{d(x,y)} \geq \frac{1}{d(x,z)+d(z,y)}$$

$$\frac{1}{d(x,y)} + 1 \geq \frac{1}{d(x,z)+d(z,y)} + 1$$

$$\frac{1}{\frac{1}{d(x,y)}+1} \leq \frac{1}{\frac{1}{d(x,z)+d(z,y)}+1}$$

That is,

$$\frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

Finally, we have

$$\delta(x,y) \leq \delta(x,z) + \delta(z,y)$$

And so the triangle inequality holds. Thus, $\delta(x,y)$ defines a metric.

Quickly note that since $d(x,y) \geq 0$ for all $x,y \in X$, $\delta(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq d(x,y)$

To see that any subset $E \subset X$ is bounded with respect to $\delta$, consider the following cases:

Case 1: $E \subset X$ is bounded with respect to $d$. That is, for any $x,y \in E$, $\sup_{x,y\in E} d(x,y) < M$ for some $M \in \mathbb{R}$. Then since $\delta(x,y) \leq \sup_{x,y\in E} \delta(x,y) \leq \sup_{x,y\in E} d(x,y)$

$$\delta(x,y) \leq \sup_{x,y\in E} \delta(x,y) \leq \sup_{x,y\in E} d(x,y) = M$$

That is, $E$ is bounded with respect to $\delta$.

Case 2: $E \subset X$ is unbounded with respect to $d$. That is, $\sup_{x,y\in E} d(x,y) = +\infty$. We wish to show that $E$ is bounded with respect to $\delta$. Notice the following:

$$\sup_{x,y\in E} \delta(x,y) = \sup_{x,y\in E} \frac{d(x,y)}{1+d(x,y)}$$

$$= \sup_{x,y\in E} \frac{1}{\frac{1}{d(x,y)}+1}$$

and since $d(x,y) \to \infty$, we have that $\delta(x,y) \to \frac{1}{0+1} = 1$.

So if $E$ is unbounded with respect to $d$, then $\sup_{x,y\in E} \delta(x,y) = 1$, meaning that $E$ is bounded with respect to $\delta$.

(b) Prove that $S = \{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} \subset \mathbb{R}$ is compact with the usual metric $d(x,y) = |x-y|$, $x,y \in \mathbb{R}$.

Proof: We will use the Heine-Borel Theorem to show $S$ is compact. That is, we must show that $S$ is closed and bounded. We will begin by showing that $S$ is bounded. Notice that $S = \{\frac{n}{n+1}\}_{n\in\mathbb{N}} \cup \{1\}$ and that $n < n+1$, so $\frac{n}{n+1} < 1$. Additionally, $1 \leq 1$. That is, 1 is an upper bound for $S$. I claim that $\frac{1}{2}$ is a lower bound for $S$. To show this, I will show that $\{\frac{n}{n+1}\}$ is increasing. Consider $\frac{n+1}{n+2} - \frac{n}{n+1}$ for any $n \in \mathbb{N}$:

$$\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)}$$

$$= \frac{1}{(n+2)(n+1)} \geq 0$$

That is, $\{\frac{n}{n+1}\}$ is an increasing sequence for all $n \in \mathbb{N}$. So $\frac{1}{2}$ is a lower bound for $\{\frac{n}{n+1}\}$, and is therefore a lower bound for $S$. That is, for any element $s \in S$, $\frac{1}{2} \leq s \leq 1$. Or, in terms of the given metric, $\sup_{x,y\in S} d(x,y) = |1 - 1/2| = 1/2$. So $S$ is bounded.

Now we must show that $S$ is closed. Consider $\mathbb{R} \setminus S$:

$$\mathbb{R} \setminus S = \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{3}{4}\right) \cup \ldots \cup (1, \infty)$$

Notice that $\mathbb{R} \setminus S$ is a countable union of open sets, and since an arbitrary union of open sets is open, we have that $\mathbb{R} \setminus S$ is open. Then $S$ is closed.

So we have that $S$ is closed and bounded, so by the Heine-Borel Theorem, we have that $S$ is compact.

3. (a) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin\left(\frac{\cos x}{2}\right)$ is a contraction mapping. (Hint: Use MVT for derivatives.)

   Proof: Recall that $\cos(x)$ is continuous on $\mathbb{R}$, and thus $\frac{\cos(x)}{2}$ is also continuous on $\mathbb{R}$. Also recall that $\sin(x)$ is continuous on $\mathbb{R}$, so $f(x) = \sin\left(\frac{\cos(x)}{2}\right)$ is continuous on $\mathbb{R}$.

   Now let $x, y \in \mathbb{R}$. We have that $f(x)$ is continuous on $[x, y]$, and so by the mean value theorem, there exists some $c \in (x, y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

   Rearranging, we find

$$|f(x) - f(y)| = |f'(c)||x - y|$$

   Well, $f'(x) = -\frac{\sin(x)}{2} \sin\left(\frac{\cos(x)}{2}\right)$, and $|f'(x)| = |\frac{\sin(x)}{2} \sin\left(\frac{\cos(x)}{2}\right)| \leq \frac{1}{2}|\sin\left(\frac{\cos(x)}{2}\right)| \leq \frac{1}{2}$ That is, we have

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

   for some $x, y \in \mathbb{R}$. By definition, $f$ is a contraction map.

   (b) Suppose $\gamma : [0, 1] \to \mathbb{R}^3$ be continuous with $\gamma(0) = (0, 0, 0)$, $\gamma(1) = (1, 1, 1)$. Show that the curve $\gamma(t)$ intersects the plane $x + y + z = 2$ in $\mathbb{R}^3$.

   Proof: We have $\gamma : [0, 1] \to \mathbb{R}$ is continuous. Let $f : \mathbb{R}^3 \to \mathbb{R}$ $f(x, y, z) = x + y + z$. $f$ is continuous since $f$ is the sum of polynomials, which are continuous. Since $\gamma$ and $f$ are continuous, we have that $f \circ \gamma$ is also continuous. Now, notice that

$$(f \circ \gamma)(0) = f((0, 0, 0)) = 0 + 0 + 0 = 0$$

   and

$$(f \circ \gamma)(1) = f((1, 1, 1)) = 1 + 1 + 1 = 3$$

   Then by the intermediate value theorem, there exists some value $c \in [0, 1]$ such that $(f \circ \gamma)(c) = 2$. That is to say, for some $c$, $\gamma(c)$ will map to a point on the plane $x + y + z = 2$.

4. (a) Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that $0 < f(x) < 1$ for all $0 \leq x < 1$, and $f(1) = 1$. Suppose in addition that $\lim_{x \to 1^-} \frac{f(1) - f(x)}{1 - x} = \ell > 1$. Prove that there is some number $c$ with $0 < c < 1$ such that $f(c) = c$. (Hint: Apply IVT to an appropriate function).

   Proof: Let $f : [0, 1] \to \mathbb{R}$ be continuous with $0 < f(x) < 1$ for all $0 \leq x < 1$ and $f(1) = 1$, and in addition, $\lim_{x \to 1^-} \frac{f(1) - f(x)}{1 - x} = \ell > 1$.

   Let $\{x_n\}$ be a sequence in $[0, 1]$ converging to 1. Then since $f$ is continuous, and from the given limit, for some $n > N \in \mathbb{N}$ $\frac{f(1) - f(x_n)}{1 - x_n} \geq 1$. Rearranging, we have

$$1 - f(x_n) > 1 - x_n$$

$$x_n - f(x_n) > 0$$

Let $g(x) = x - f(x)$. Notice that $g$ is continuous since $f$ is continuous, and $x$ is a polynomial. Now, notice that $g(0) = 0 - f(0) = -f(0) < 0$. Then we have the following inequality:

$$g(0) < 0 < g(x_n)$$

And by the intermediate value theorem, we have that there exists some $c \in (0, x_n)$ such that

$$g(c) = 0$$

That is, $c - f(c) = 0$ or $f(c) = c$.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with the property: $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Prove that $f$ is uniformly continuous.

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$ and fix $\epsilon > 0$. Since $\lim_{x\to\infty} f(x) = 0$, there exists some $m_1 \in \mathbb{R}$, whenever $x > m_1$, $m_1 > 0$ $|f(x) - 0| < \epsilon/3$.

Similarly, there exists some $m_2 \in \mathbb{R}$, $m_2 < 0$ such that whenever $x < m_2$, $|f(x) - 0| < \epsilon/3$. Now, let $M = \max\{|m_1|, |m_2|\}$ and consider the interval $[-M, M]$. Since this is a closed, bounded interval, and $f$ continuous on $\mathbb{R}$, we have that $f$ is uniformly continuous on $[-M, M]$.

That is, for some $\delta > 0$, whenever $|x - y| < \delta$ for $x, y \in [-M, M]$, $|f(x) - f(y)| < \epsilon/3$. To show $f$ is uniformly continuous continuous on $\mathbb{R}$, let $|x - y| < \delta$ for some $\delta > 0$ and consider the following three cases:

Case 1: Both $|x| \geq M$ and $|y| \geq M$. Then $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon/3 + \epsilon/3 = 2\epsilon/3 < \epsilon$.

Case 2: Both $x, y \in [-M, M]$. Then from above, we have that $|f(x) - f(y)| < \epsilon/3 < \epsilon$

Case 3: One of $x, y \in [-M, M]$ and the other in $(-\infty, -M) \cup (M, \infty)$. Without loss of generality, assume $x \in [-M, M]$ and $y \in (M, \infty)$. Then

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(M) + f(M) - f(y)| \\
&\leq |f(x) - f(M)| + |f(M)| + |f(y)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

In any of these cases, we have that $f$ is uniformly continuous on $\mathbb{R}$.

5. Suppose a real valued function $f(x, y)$ is defined on an open set $U \in \mathbb{R}^2$. Assume that the first partial derivatives of $f(x, y)$ exist and are uniformly bounded on $U$.

(a) Prove that $f(x, y)$ is continuous on $U$.

Proof: Let $x_0 \in U$ and since $U$ is open, we have for some $r > 0$, $B_r(x_0) \subseteq U$. Now let $\{\mathbf{x}_n\}$ be a sequence in $B_r(x_0)$ such that $x_n$ converges to $x_0$. Let $\{\mathbf{h}_n\}$ be a sequence in $B_r(x_0)$ defined by $\mathbf{h}_n = \mathbf{x}_n - x_0$. Notice that $\mathbf{h}_n + x_0 = \mathbf{x}_n \in B_r(x_0)$. Then by the mean value proposition, we have that

$$f(x_0 + \mathbf{x}_n) - f(x_0) = h_n^1 \frac{\partial f}{\partial x}(z_1) + h_n^2 \frac{\partial f}{\partial y}(z_2)$$

for some $z_1, z_2 \in B_r(x_0)$. Then since the first partial derivatives of $f$ are uniformly bounded, we have that

$$\left|\frac{\partial f}{\partial x}\right| \leq M, \quad \left|\frac{\partial f}{\partial y}\right| \leq M$$

for some $M \in \mathbb{R}$ for all $(x, y) \in U$. Define $\mathbf{M} = [M, M]^T$. Then we have

$$|f(x_0 + \mathbf{h}_n) - f(x_0)| \leq \langle \mathbf{M}, \mathbf{h}_n \rangle$$

Now, as $n \to \infty$, we have $|\mathbf{h}_n| \to 0$ so as $n \to \infty$,

$$|f(x_0 + \mathbf{h}_n) - f(x_0)| \to 0$$

Then $f(x_0 + \mathbf{h}_n) \to f(x_0)$ as $n \to \infty$. So $f$ is continuous at $x_0$. Since $x_0 \in U$ was arbitrary, we have that $f$ is continuous on $U$.

(b) Let $f(x, y) = \frac{xy^2}{x^2+y^4}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show explicitly that the first partial derivatives of $f(x, y)$ exist but are *not* bounded in an open neighborhood of $(0, 0)$. Then show that $f(x, y)$ is not continuous at $(0, 0)$.

We wish to show that the first partial derivatives of $f$ exist. To do so, we must show that $\lim_{t\to 0} \frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}$ and $\lim_{t\to 0} \frac{f(x_0,y_0+t)-f(x_0,y_0)}{t}$ exist for some point $(x_0, y_0) \in \mathbb{R}^2$. To begin, we will show $\frac{\partial f}{\partial x}$ exists. To do so, consider

$$\lim_{t\to 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t} = \lim_{t\to 0} \frac{\frac{(x_0+t)y_0^2}{(x_0+t)^2+y^4} - \frac{x_0 y_0^2}{x_0^2+y_0^4}}{t}$$

$$= \lim_{t\to 0} \frac{(x_0y_0^2 + ty_0^2)(x_0^2 + y_0^4) - x_0y_0^2((x_0 + t)^2 + y_0^4)}{t(x_0^2 + y_0^4)((x_0 + t)^2 + y_0^4)}$$

$$= \lim_{t\to 0} \frac{x_0^3y_0^2 + tx_0^2y_0^2 + ty_0^6 + x_0y_0^6 - x_0^3y_0^2 - 2tx_0^2y_0^2 - x_0y_0^6}{t(x_0^2 + y_0^4)((x_0 + t)^2 + y_0^4)}$$

$$= \lim_{t\to 0} \frac{y_0^6 - x_0^2y_0^2}{(x_0^2 + y_0^4)((x_0 + t)^2 + y_0^4)}$$

$$= \frac{y_0^6 - x_0^2y_0^2}{(x_0^2 + y_0^4)^2}$$

so $\frac{\partial f}{\partial x}$ exists. It remains to be seen that $\frac{\partial f}{\partial x}|_{(x,y)=(0,0)}$ exists. Using the limit definition of the derivative,

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t\to 0} \frac{\frac{t(0)^2}{t^2+0^4} - 0}{t}$$

$$= \lim_{t\to 0} \frac{0}{t}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = 0$$

Now we will show $\frac{\partial f}{\partial y}$ exists. Using the limit definition, we have

$$\frac{\partial f}{\partial y} = \lim_{t\to 0} \frac{f(x_0, y_0 + t) - f(x_0, y_0)}{t}$$

$$= \lim_{t\to 0} \frac{\frac{x_0(y_0+t)^2}{x_0^2+(y_0+t)^4} - \frac{x_0y_0^2}{x_0^2+y_0^4}}{t}$$

$$= \lim_{t\to 0} \frac{\frac{x_0y_0^2+2ty_0x_0+x_0t^2}{x_0^2+(y_0+t)^4} - \frac{x_0y_0^2}{x_0^2+y_0^4}}{t}$$

$$= \lim_{t\to 0} \frac{(x_0y_0^2+2tx_0y_0+x_0t^2)(x_0^2+y_0^4) - x_0y_0^2(x_0^2+(y_0+t)^4)}{t(x_0^2+y_0^4)(x_0^2+(y_0+t)^4)}$$

$$= \lim_{t\to 0} \frac{x_0^3y_0^2+2tx_0^3y_0+x_0^3t^2+x_0y_0^6+2tx_0y_0^5+t^2x_0y_0^4-x_0^3y_0^2-x_0y_0^2(y_0+t)^4}{t(x_0^2+y_0^4)(x_0^2+(y_0+t)^4)}$$

$$= \lim_{t\to 0} \frac{2tx_0^3y_0+t^2x_0^3-2tx_0y_0^5+t^2x_0y_0^4-6t^2x_0y_0^4-4t^3y_0^3x_0-t^4x_0y_0^2}{t(x_0^2+y_0^4)(x_0^2+(y_0+t)^4)}$$

$$= \lim_{t\to 0} \frac{2x_0^3y_0+tx_0^3-2x_0y_0^5+tx_0^4-6tx_0y_0^4-4t^2y_0^3x_0-t^3x_0y_0^2}{(x_0^2+y_0^4)(x_0^2+(y_0+t)^4)}$$

$$= \frac{2x_0^3y_0-2x_0y_0^5}{(x_0^2+y_0^4)^2}$$

Now it remains to be seen that $\frac{\partial f}{\partial y}$ exists at $(0,0)$.
Using the limit definition of partial derivatives,

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}$$

$$= \lim_{t\to 0}\frac{0(0)^2}{t(0^2+t^4)}$$

$$= \lim_{t\to 0} 0$$

$$= 0$$

So $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist for all $(x,y)\in\mathbb{R}^2$.

Now we must show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are unbounded in an open neighborhood around $(0,0)$. Let $B_r((0,0))$ be an open ball of radius $r>0$ around $(0,0)$ and notice that $\left(0,\frac{r}{2n}\right)\in B_r((0,0))$ for all $n\in\mathbb{N}$. Similarly, $\left(\frac{r}{2n},\frac{r}{2n}\right)\in B_r((0,0))$ for all $n\in\mathbb{N}$. Using the expressions we found for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and the sequences above, notice

$$\lim_{n\to\infty}\left|\frac{\partial f}{\partial x}\left(0,\frac{r}{2n}\right)\right| = \lim_{n\to\infty}\frac{\left(\frac{r}{2n}\right)^6-0}{\left(\frac{r}{2n}\right)^8}$$

$$= +\infty$$

and

$$\lim_{n\to\infty}\left|\frac{\partial f}{\partial y}\left(\frac{r}{2n},\frac{r}{2n}\right)\right| = 2$$

?? (I've had trouble finding a sequence that causes $\frac{\partial f}{\partial y}$ to diverge near 0 :( )
To see that $f(x,y)$ is discontinuous at $(0,0)$, it suffices to be shown that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Consider the limit along the path $y=x$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0}\frac{x^3}{x^2+x^4}$$

$$= \lim_{x\to 0}\frac{x}{1+x^2}$$

$$= 0$$

Now consider the path $y = \sqrt{x}$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2 + x^2}$$

$$= \lim_{x\to 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

So $\lim_{(x,y)\to(0,0)} f(x,y)$ along the path $y = x$ does not equal the limit along the path $y = \sqrt{x}$. That is, $f(x,y)$ is discontinuous at $(0,0)$.

**Extra Credit** Show that $T : C[0,\pi/2] \to C[0,\pi/2]$ defined by $T(f)(x) = \int_0^x f(t)\sin t dt$ is *not* a contraction map, yet it has a unique fixed point. Take $d(f,g) = \sup_{x\in[0,\pi/2]} |f(x) - g(x)|$ as the metric on $C[0,\pi/2]$. (Hint: Check $T^2$.)