Homework 7

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1. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

Soln. To find $H(x)$, let us inspect $\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx}$ since we want something of the form $\frac{d}{dx}[p(x)\frac{d\phi}{dx}]$. That is, we want

$$H(x)\frac{d^2\phi}{dx^2} + H(x)\alpha(x)\frac{d\phi}{dx} = \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right].$$

Expanding the right hand side of the above equation yields

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] = p'(x)\frac{d\phi}{dx} + p(x)\frac{d^2\phi}{dx^2}$$

which gives us

$$\begin{aligned} H(x) &= p(x) \\ p'(x) &= H(x)\alpha(x) \\ \implies H'(x) &= H(x)\alpha(x) \\ \implies \int \frac{dH}{H(x)}dx &= \int \alpha(x)dx \\ \implies H(x) &= e^{\int \alpha(x)dx} \end{aligned}$$

hence

$$\begin{aligned} p(x) &= e^{\int \alpha(x)dx} \\ \sigma(x) &= \beta(x)e^{\int \alpha(x)dx} \\ q(x) &= \gamma(x)e^{\int \alpha(x)dx} \end{aligned}$$

2. For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \text{with} \qquad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0,$$

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest, but no largest.

Soln. We seek solutions of the form $\phi = e^{mx}$ and obtain the relationship

$$m^2 + \lambda = 0$$

for $\lambda > 0$, we find

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

which yields, from the boundary conditions

$$\begin{aligned}
\frac{d\phi}{dx} &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x) \\
\implies \frac{d\phi}{dx}(0) &= \sqrt{\lambda}c_2 = 0 \\
\implies c_2 &= 0 \\
\implies \frac{d\phi}{dx}(L) &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}L) = 0 \\
\implies \sqrt{\lambda}L &= n\pi && (n \in \mathbb{Z}) \\
\implies \lambda_n &= \left(\frac{n\pi}{L}\right)^2 && (n \in \mathbb{N})
\end{aligned}$$

We note that the case $\lambda \le 0$ yields the trivial solution $\phi = 0$ (which is not an eigenfunction by definition) from the boundary conditions, so that $\lambda \le 0$ are not eigenvalues. From the above equation, we see that $\lambda_1 = \frac{\pi^2}{L^2}$ is the smallest eigenvalue and there is no largest eigenvalue since $\lambda_n \propto n^2$.

(b) The n^{th} eigenfunction has n zeros.

Soln. Consider the n^{th} eigenfunction $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$. Notice that

$$\cos\left(\frac{n\pi}{L}x\right) = 0 \implies \frac{n\pi}{L}x = \frac{\pi}{2} + k\pi, \qquad k \in \mathbb{Z}$$

and we wish to find the zeros that are within the interval $[0, L]$ immediately, we have $k \in \mathbb{N} \cup \{0\}$ and notice

$$\begin{aligned}
x &= \frac{L}{2n} + \frac{kL}{n} < L \\
\implies \frac{1}{2} &+ k < n
\end{aligned}$$

holds for $k = 0, 1, \dots, n-1$. Thus, the n^{th} eigenfunction has n zeros.

(c) The eigenfunctions are orthogonal.

Soln. Notice

$$\begin{aligned}
\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx &= \frac{1}{2}\int_0^L \left(\cos\left(\frac{\pi}{L}(n-m)x\right)+\cos\left(\frac{\pi}{L}(n+m)x\right)\right) \qquad (n \neq m) \\
&= \frac{1}{2}\left[\frac{1}{\pi/L(n-m)}\sin\left(\frac{\pi}{L}(n-m)x\right)+\frac{1}{\pi/L(n+m)}\sin\left(\frac{\pi}{L}(n+m)x\right)\right]\Bigg|_0^L \\
&= \frac{1}{2}\left[\frac{1}{\pi/L(n-m)}\sin(\pi(n-m))+\frac{1}{\pi/L(n+m)}\sin(\pi(n+m))\right] \\
&= 0
\end{aligned}$$

since $n-m \in \mathbb{Z}$ and $n+m \in \mathbb{Z}$.

(d) The solution can be expressed in terms of an eigenfunction expansion.

Soln. From part (a), we have that $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$ and by principle of superposition,

$$\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\right).$$

(e) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

*Soln.*For this problem, we have $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$, so from the Rayleigh quotient, we have

$$\begin{aligned}
\lambda &= \frac{\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \\
&= \frac{\phi(b)\frac{d\phi}{dx} - \phi(a)\frac{d\phi}{dx}(a) + \int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \\
&= \frac{\int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \geq 0. \qquad \text{(B.C.s)}
\end{aligned}$$

So we have that the eigenvalues of this problem are nonnegative, and notice that $\lambda = 0$ whenever $\phi =$ constant, but from the boundary conditions, we get $\phi \equiv 0$, so that $\lambda = 0$ is not an eigenvalue.

3. Redo Problem 2 for the Sturm-Liouville eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \text{with} \qquad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

(a) *Soln.* Consider the following cases:
Case 1: $\lambda > 0$. Then

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

and from the boundary conditions, we find

$$\begin{aligned}
\frac{d\phi}{dx} &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x) \\
\implies \frac{d\phi}{dx}(0) &= c_2\sqrt{\lambda} = 0 \\
\implies c_2 &= 0 \\
\implies \phi(L) &= c_1 \cos(\sqrt{\lambda}L) = 0 \\
\implies \sqrt{\lambda}L &= \frac{\pi}{2} + n\pi \qquad (n \in \mathbb{N} \cup \{0\}) \\
\implies \lambda_n &= \frac{\left(\frac{\pi}{2} + n\pi\right)^2}{L^2}.
\end{aligned}$$

And note that from the boundary conditions, the case $\lambda \le 0$ yields $\phi = 0$, so that $\lambda \le 0$ are not eigenvalues. Notice that the smallest eigenvalue is given by

$$\lambda_1 = \frac{\pi^2}{4L^2}$$

and has no largest since $\lambda_n \propto n^2$.

(b) *Soln.* Notice that, for the n^{th} eigenfunction, we have

$$\begin{aligned}
\cos\left(\frac{\frac{\pi}{2} + n\pi}{L}x\right) &= 0 \\
\implies \frac{\frac{\pi}{2} + n\pi}{L}x &= \frac{\pi}{2} + k\pi \\
\implies \left(\frac{1}{2} + n\right)x &= \frac{1}{2} + k \\
\implies x &= \frac{\frac{1}{2} + k}{\frac{1}{2} + n}
\end{aligned}$$

and notice that $x \in (0, L)$ whenever $k = 0, 1, \ldots, n-1$ so that $\phi_n(x)$ has n zeros on $(0, L)$.

(c) *Soln.* We consider

$$\begin{aligned}
\int_0^L \phi_n(x)\phi_m(x)dx &= \int_0^L \cos\left(\frac{\pi/2 + n\pi}{L}x\right)\cos\left(\frac{\pi/2 + m\pi}{L}x\right)dx \\
&= \frac{1}{2}\int_0^L \left[\cos\left(\frac{\pi}{L}(n+m+1)x\right) + \cos\left(\frac{\pi}{L}(n-m)x\right)\right]dx \\
&= \frac{1}{2}\left[\frac{1}{\pi/L(n+m+1)}\sin\left(\frac{\pi}{L}(n+m+1)x\right) + \frac{1}{\pi/L(n-m)}\sin\left(\frac{\pi}{L}(n-m)x\right)\right]\Bigg|_0^L \\
&= \frac{1}{2}\left[\frac{1}{\pi/L(n+m-1)}\sin(\pi(n+m+1)) + \frac{1}{\pi/L(n-m)}\sin(\pi(n-m))\right] \\
&= 0
\end{aligned}$$

since $n + m + 1 \in \mathbb{Z}$ and $n - m \in \mathbb{Z}$. Thus, the eigenfunctions are orthogonal.

(d) *Soln.* From part (a) we have

$$\phi_n(x) = \cos\left(\frac{\pi/2 + n\pi}{2} x\right)$$

is a solution to the differential equation for all $n \in \mathbb{N}$ and, by principle of super position, we have

$$\phi(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi/2 + n\pi}{2} x\right)$$

is also a solution.

(e) *Soln.* For this problem, we have $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$, so from the Rayleigh quotient, we have

$$\begin{aligned}
\lambda &= \frac{\phi \frac{d\phi}{dx}\Big|_a^b + \int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \\
&= \frac{\phi(b)\frac{d\phi}{dx} - \phi(a)\frac{d\phi}{dx}(a) + \int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \\
&= \frac{\int_a^b \left(\frac{d\phi}{dx}\right)^2 dx}{\int_a^b \phi^2 dx} \geq 0. && \text{(B.C.s)}
\end{aligned}$$

So we have that the eigenvalues of this problem are nonnegative, and notice that $\lambda = 0$ whenever $\phi =$ constant, but from the boundary conditions, that must mean $\phi \equiv 0$, so $\lambda = 0$ is not an eigenvalue.

4. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \qquad \text{with} \qquad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

Proof: Note that the above differential equation is a regular Sturm-Liouville equation with $p(x) = 1$, $\sigma(x) = 1$, $q(x) = -x^2$. Inspecting the Rayleigh coefficient, we find

$$\begin{aligned}
\lambda &= \frac{-p(x)\phi(x)\frac{d\phi}{dx}\Big|_0^L + \int_0^L \left[p(x)\left(\frac{d\phi}{dx}\right)^2 - q(x)\phi^2\right]dx}{\int_0^L \phi^2 \sigma dx} \\
&= \frac{-\phi(L)\frac{d\phi}{dx}(L) + \phi(0)\frac{d\phi}{dx} + \int_0^L \left[\left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2\right]dx}{\int_0^L \phi^2 dx} \\
&= \frac{\int_0^L \left[\left(\frac{d\phi}{dx}\right)^2 + (x\phi)^2\right]dx}{\int_0^L \phi^2 dx}
\end{aligned}$$

and since $\phi^2 \geq 0$, $\left(\frac{d\phi}{dx}\right)^2 \geq 0$, and $(x\phi)^2 \geq 0$, we have

$$\lambda \geq 0.$$

Now, I claim that $\lambda \neq 0$. To see this, notice, by setting $\lambda = 0$ in the Rayleigh quotient,

$$\begin{aligned}
&\frac{\int_0^L \left[\left(\frac{d\phi}{dx}\right)^2 + (x\phi)^2\right]dx}{\int_0^L \phi^2 dx} = 0 \\
&\implies \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx = -\int_0^L (x\phi)^2 dx
\end{aligned}$$

and since $\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx, \int_0^L (x\phi)^2 dx \geq 0$, by the above equality, it must be the case that $\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx = \int_0^L (x\phi)^2 dx = 0 \implies x\phi = 0 \implies \phi = 0$, which is not an eigenvector by definition. Hence, $\lambda = 0$ is not an eigenvalue of this problem.

5. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

since then $\int_a^b \{uL[v] - vL[u]\}dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) $\phi(a) = 0$ and $\phi(b) = 0$

Soln. For the remainder of the problem, let u, v satisfy the given boundary conditions. For this problem, notice

$$\begin{aligned} p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\ &= p(b)\left(0\cdot\frac{dv}{dx}(b) - 0\cdot\frac{du}{dx}(b)\right) - p(a)\left(0\cdot\frac{dv}{dx}(a) - 0\cdot\frac{du}{dx}(a)\right) \\ &= 0 \end{aligned}$$

so that these boundary conditions yield a self-adjoint problem.

(b) $\frac{d\phi}{dx}(a) = 0$ and $\phi(b) = 0$

Soln. Notice

$$\begin{aligned} p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\ &= p(b)\left(0\cdot\frac{dv}{dx}(b) - 0\cdot\frac{du}{dx}(b)\right) - p(a)\left(u(a)\cdot 0 - v(a)\cdot 0\right) \\ &= 0 \end{aligned}$$

so that the boundary conditions yield a self-adjoint problem.

(c) $\frac{d\phi}{dx}(a) - h\phi(a) = 0$ and $\frac{d\phi}{dx}(b) = 0$

Soln. Notice

$$\begin{aligned} p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\ &= p(b)\left(u(b)\cdot 0 - v(b)\cdot 0\right) - p(a)\left(u(a)\cdot hv(a) - v(a)\cdot hu(a)\right) \\ &= 0 - p(a)(hv(a)u(a) - hv(a)u(a)) \\ &= 0 \end{aligned}$$

so that these boundary conditions yield a self-adjoint problem.

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

Soln.

$$\begin{aligned} p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\ &= \left(u(a)p(b)\frac{dv}{dx}(b) - v(a)p(b)\frac{du}{dx}(b)\right) - \left(u(a)p(a)\frac{dv}{dx}(a) - v(a)p(a)\frac{du}{dx}(a)\right) \\ &= \left(u(a)p(a)\frac{dv}{dx}(a) - v(a)p(a)\frac{du}{dx}(a)\right) - \left(u(a)p(a)\frac{dv}{dx}(a) - v(a)p(a)\frac{du}{dx}(a)\right) \\ &= 0 \end{aligned}$$

so that these boundary conditions yield a self-adjoint problem.

(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ (self-adjoint only if $p(a) = p(b)$)

Soln.

$$\begin{aligned}
p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\
&= p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\
&= 0
\end{aligned}$$

so that these boundary conditions yield a self-adjoint problem.

(f) $\phi(b) = 0$ and (in the situation with $p(a) = 0$) $\phi(0)$ bounded and $\lim_{x\to a} p(x)\frac{d\phi}{dx} = 0$

Soln. We first consider the case $\phi(b) = 0$ and $p(a) = 0$. Notice

$$\begin{aligned}
p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - p(a)\left(u(a)\frac{dv}{dx}(a) - v(a)\frac{du}{dx}(a)\right) \\
&= p(b)\left(0\cdot\frac{dv}{dx}(b) - 0\cdot\frac{du}{dx}(b)\right) - 0\cdot\left(0\cdot\frac{dv}{dx}(a) - 0\cdot\frac{du}{dx}(a)\right) \\
&= 0
\end{aligned}$$

so that the problem is self adjoint.
Now consider the case $\phi(b) = 0$, $\phi(a)$ bounded and $\lim_{x\to a} p(x)\frac{d\phi}{dx} = 0$:

$$\begin{aligned}
p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b &= p(b)\left(u(b)\frac{dv}{dx}(b) - v(b)\frac{du}{dx}(b)\right) - \lim_{x\to a}\left(p(x)\left(u(x)\frac{dv}{dx}(x) - v(x)\frac{du}{dx}(x)\right)\right) \\
&= p(b)\left(0\cdot\frac{dv}{dx}(b) - 0\cdot\frac{du}{dx}(b)\right) - \lim_{x\to a}\left(p(x)\left(u(x)\frac{dv}{dx}(x) - v(x)\frac{du}{dx}(x)\right)\right) \\
&= -\lim_{x\to a} p(x)\left(u(x)\frac{dv}{dx}(x) - v(x)\frac{du}{dx}(x)\right) \\
&= -u(a)\lim_{x\to a} p(x)\frac{dv}{dx}(x) + v(a)\lim_{x\to a} p(x)\frac{du}{dx} && (\phi(a) \text{ bounded}) \\
&= 0
\end{aligned}$$

so that this situation yields a self-adjoint operator.