Grad Problem #1

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Show that an object will take longer to go down than up with a general air resistive force.

From Newton's law, we have the differential equation

$$\frac{dv}{dt} = -g - f(v)$$

where the resistive force, $f(v)$, satisfies

$$\begin{aligned} -f(v) &< 0 \quad v > 0 \\ -f(v) &> 0 \quad v < 0 \end{aligned}$$

And since the magnitude of $f(v)$ must be the same for magnitudes of v being the same, we have $f(v)$ is odd, so $f(-v) = -f(v)$. Let T be the time it takes the object to reach the top of its trajectory. Then $v(T) = 0$. From the differential equation, we have (for the object going up)

$$\begin{aligned} \int_{v(0)}^{v(T)} \frac{dv}{g + f(v)} &= \int_0^T -dt \\ \int_0^{v_0} \frac{dv}{g + f(v)} &= T \end{aligned}$$

Now let us do the same analysis for the object moving downwards up to $t = 2T$:

$$\begin{aligned} \int_{v(T)}^{v(2T)} \frac{dv}{g + f(v)} &= \int_T^{2T} -dt \\ \int_{v(2T)}^{0} \frac{dv}{g + f(v)} &= T \end{aligned}$$

From this, we have

$$\int_0^{v(T)} \frac{dv}{g + f(v)} = \int_{v(2T)}^{0} \frac{dv}{g + f(v)}$$

Now, since the first integral corresponds to the object moving upwards, we have $f(v) > 0$, and in the second integral, we have $f(v) < 0$ since the object is moving downwards. Then we have

$$\frac{1}{g + f(v)} > \frac{1}{g + f(v)}$$

where the left fraction corresponds to the object falling downwards. This implies that $v(T) > -v(2T)$. Now we introduce the time variable $0 \leq \tau \leq T$. Notice

$$\begin{aligned} \int_{v(\tau)}^{v(T)} \frac{dv}{g + f(v)} &= \int_\tau^T -dt \\ &= -(T - \tau) \\ &= -(2T - \tau - T) \\ &= \int_T^{2T-\tau} -dt \\ &= \int_{v(T)}^{V(2t-\tau)} \frac{dv}{g + f(v)} \end{aligned}$$

So we have

$$\int_{v(\tau)}^{0} \frac{dv}{g+f(v)} = \int_{0}^{V(2t-\tau)} \frac{dv}{g+f(v)}$$

Similar to above, we have $-v(2T-\tau) < v(\tau)$. Integrating this inequality with respect to τ (by monotonicity of the integral), we have

$$\begin{gathered}
-\int_0^T v(2T-\tau)d\tau = \int_0^T v(\tau)d\tau \\
h(2T-T) - h(2T) < h(T) - h(0) \\
h(T) - h(2T) < h(T) \\
h(2T) > 0
\end{gathered}$$

Then the object is still above the ground at $t = 2T$, meaning it will take more time for the object to complete its trip down than going up!