Math 4510
Topology

Problem Set 8

1. (#5 in 6.2) Prove Theorem 6.2.4: Let A_{τ_A} be a connected subspace of a space X_τ, and let B be a subset of X with $A \subseteq B \subseteq \mathrm{Cl}(A)$. Then B is also connected in the subspace topology.

   Proof: Let A_{τ_A} be a connected subspace of a space X_τ and B be a subset of X with $A \subseteq B \subseteq \mathrm{Cl}(A)$. By theorem 6.2.3, we also know that $\mathrm{Cl}(A)$ is connected.

   Assume by way of contradiction that B is disconnected. That is, there exist non-empty subsets of B, β_1, β_2 open in τ_B such that $\beta_1 \cup \beta_2 = B$ and $\beta_1 \cap \beta_2 = \emptyset$.

   Since β_1 and β_2 are τ_B-open, there exist τ-open sets U_1, U_2 such that

   $$\beta_1 = U_1 \cap B$$

   $$\beta_2 = U_2 \cap B$$

   And since $A \subseteq B$, we have that $A = A \cap B$. So

   $$A = A \cap [(U_1 \cap B) \cup (U_2 \cap B)]$$

   $$= (A \cap (U_1 \cap B)) \cup (A \cap (U_2 \cap B))$$
   $$= (A \cap B \cap U_1) \cup (A \cap B \cap U_2)$$
   $$= (A \cap U_1) \cup (A \cap U_2)$$
   $$= A \cap (U_1 \cup U_2)$$

   Let $V_1 = A \cap U_1$ and $V_2 = A \cap U_2$; which are open in τ_A since U_1 and U_2 are τ-open. So we have

   $$A = V_1 \cup V_2$$

   Now we will show that V_1, V_2 are non-empty. Recall that $\mathrm{Cl}(A) = A \cup A'$.

   Since $\beta_1 \neq \emptyset$ and $\beta_1 = U_1 \cap B$, we have that U_1 is nonempty. Let $x_0 \in B$. Then $x_0 \in \beta_1 \cup \beta_2$, and since $\beta_1 \cap \beta_2 = \emptyset$, x_0 is either exclusively in β_1 or β_2. Suppose without loss of generality that $x_0 \in \beta_1$. Then $x_0 \in U_1 \cap B$, and so $x_0 \in U_1$. Additionally, since $B \subseteq \mathrm{Cl}(A)$, $x_0 \in \mathrm{Cl}(A)$. Then every neighborhood N_{x_0} of x_0 intersects A nontrivially, so $x_0 \in A$. Then we have $x_0 \in A$ and $x_0 \in U_1$, so $x_0 \in A \cap U_1 = V_1$.

   So we have that $V_1 = A \cap U_1$ is non-empty. A similar argument holds for V_2.

   Recall that $\beta_1 \cap \beta_2 = \emptyset$ and that $\beta_1 = U_1 \cap B$ and $\beta_2 = U_2 \cap B$, so

   $$\beta_1 \cap \beta_2 = (U_1 \cap B) \cap (U_2 \cap B)$$

   $$= U_1 \cap B \cap U_2 \cap B$$

$$
\begin{aligned}
&= B \cap B \cap (U_1 \cap U_2) \\
&= B \cap (U_1 \cap U_2) \\
&= \emptyset
\end{aligned}
$$

and since $B \neq \emptyset$, we have that $U_1 \cap U_2 = \emptyset$.

Notice that

$$
\begin{aligned}
V_1 \cap V_2 &= (A \cap U_1) \cap (A \cap U_2) \\
&= A \cap A \cap U_1 \cap U_2 \\
&= A \cap (U_1 \cap U_2) \\
&= A \cap \emptyset \\
&= \emptyset
\end{aligned}
$$

So we have two τ_A-open sets V_1, V_2 such that $V_1 \cup V_2 = A$ and $V_1 \cap V_2 = \emptyset$. So by definition, we have that A is disconnected, which contradicts the hypothesis that A is connected, and thus, B must be a connected space of X_τ.

2. Let X_τ be a space with the property that given any $x \in X$ and any neighborhood U of x, there is a neighborhood V of x such that $\mathrm{Cl}(V)$ is a proper subset of U.

   (a) Give an example that shows that X need not be connected.

   Consider a set X equipped with the discrete topology and let $x \in X$. Then any subset $U \subseteq X$ that contains x is a neighborhood of x. Then for any proper subset V of U, $\mathrm{Cl}(V) = V$ in the discrete topology, and so we have $\mathrm{Cl}(V) \subset U$. So $X_{\mathcal{D}}$ satisfies these conditions. Additionally, as long as $\mathrm{Card}(X) \geq 2$, we have that $X_{\mathcal{D}}$ is disconnected.

   (b) If we add the condition that for every $x \in X$ and every neighborhood U of x there is a connected neighborhood V_x of x such that $x \in V_x \subseteq \mathrm{Cl}(V_x) \subseteq U$, is this sufficient to make X connected? Explain.

   No. Let X be a space equipped with the discrete topology where $\mathrm{Card}(X) > 2$. Let $x \in X$ and U be a neighborhood of x. Let $V_x = \{x\}$ be the neighborhood of x that contains only x. Since X is equipped with the discrete topology, $\mathrm{Cl}(V_x) = V_x$ and notice that V_x is trivially connected and $x \in \{x\} \subseteq U$. However, since X is a discrete topological space, we have that X is disconnected.

3. (#6 in 6.4) Prove that if X_τ is path-connected and $\tau' \subseteq \tau$, then $X_{\tau'}$ is also path-connected.

   Proof: Let X_τ and $X_{\tau'}$ be topological spaces where $\tau' \subseteq \tau$ and X_τ is path-connected. Since $\tau' \subseteq \tau$, we have that the identity map $i_x : X_\tau \to X_{\tau'}$ is continuous. And since X_τ is path-connected, we have that $X_{\tau'}$ is also path-connected since $X_{\tau'}$ is the continuous image of a path-connected space.

4. Prove that any quotient space of a path-connected space is path-connected. That is, if X_τ is a path-connected space and $\sim$ is an equivalence relation on X, then the quotient space $X/\sim$ is path-connected.

   Proof: Let X_τ be a path-connected space, $\sim$ be an equivalence relation of X and let $X/\sim$ be the quotient space. Let $\nu : X_\tau \to X_{/\sim}$ be the natural map. We have that ν is continuous, and we wish to show that ν is surjective. Suppose by way of contradiction that ν is not surjective. That is, there exists some element $[x] \in X_{/\sim}$ such that $\nu^{-1}([x]) = \emptyset$. But from the definition of the natural map, $\nu^{-1}([x]) = x \in X$. So ν is surjective. Since path-connectedness is a strong topological property and ν is continuous, we have that $X_{/\sim}$ is also path-connected.

Bonus Show that if U is an open connected subset of $\mathbb{R}^2$, then U is path-connected. (Hint: Show that given a point $x_0 \in U$, the set of points in U that can be joined to x_0 by a path in U is both open and closed.)