Homework VI

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Section 3.1 Problems

4. If an inner product space X is real, show that the condition $\|x\| = \|y\|$ implies $\langle x+y, x-y\rangle = 0$. What does this mean geometrically if $X = \mathbb{R}^2$? What does the condition imply if X is complex?

Proof: Begin with $\langle x+y, x-y\rangle$:

$$\begin{aligned}\langle x+y, x-y\rangle &= \langle x,x\rangle + \langle y,x\rangle - \langle x,y\rangle - \langle y,y\rangle \\ &= \|x\|^2 + \langle y,x\rangle - \langle x,y\rangle - \|y\|^2\end{aligned}$$

Since X is a real space, we have that $\langle y,x\rangle = \langle x,y\rangle$, so that the above equation becomes

$$\begin{aligned}\|x\|^2 + \langle y,x\rangle - \langle x,y\rangle - \|y\|^2 &= \|x\|^2 - \|y\|^2 \\ &= 0\end{aligned}$$

since $\|x\| = \|y\| \implies \|x\|^2 = \|y\|^2$. If $X = \mathbb{R}^2$, this relationship geometrically means that the diagonals of the parallelogram formed by two vectors of equal length are orthogonal.

If X is complex, we have that $\langle x,y\rangle = \overline{\langle x,y\rangle}$, so that $\overline{\langle x,y\rangle} - \langle x,y\rangle = -2i\mathrm{Im}(\langle x,y\rangle)$. Thus,

$$\langle x+y, x-y\rangle = -2i\mathrm{Im}(\langle x,y\rangle)$$

Section 3.2 Problems

8. Show that in an inner product space, $x \perp y$ if and only if $\|x+\alpha y\| \geq \|x\|$ for all scalars α.

Proof: If $y = 0$, the result is immediate. Let $y \neq 0$ and first suppose $\langle x,y\rangle = 0$. Let α be an arbitrary scalar and notice

$$\begin{aligned}\|x+\alpha y\|^2 &= \langle x+\alpha y, x+\alpha y\rangle \\ &= \langle x,y\rangle + \langle x,\alpha y\rangle + \langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle \\ &= \|x\|^2 + \overline{\alpha}\langle x,y\rangle + \alpha\langle y,x\rangle + |\alpha|^2\|y\|^2\end{aligned}$$

but since $\langle x,y\rangle = 0$, the above equation becomes

$$\|x\|^2 + \overline{\alpha}\langle x,y\rangle + \alpha\langle y,x\rangle + |\alpha|^2\|y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2$$

and since $|\alpha|^2, \|y\|^2 \geq 0$, $|\alpha|^2\|y\|^2 \geq$, so that

$$\begin{aligned}&\|x\|^2 + |\alpha|^2\|y\|^2 \geq \|x\|^2 \\ \implies &\|x+\alpha y\|^2 \geq \|x\|^2 \\ \implies &\|x+\alpha y\| \geq \|x\|.\end{aligned}$$

We must now show that for any scalar α, $\|x+\alpha y\| \geq \|y\|$ implies $\langle x,y\rangle = 0$. Notice

$$\begin{aligned}\|x+\alpha y\|^2 &= \langle x+\alpha y, x+\alpha y\rangle \\ &= \langle x,x\rangle + \overline{\alpha}\langle x,y\rangle + \alpha\langle y,x\rangle + \overline{\alpha}\alpha\langle y,y\rangle \\ &= \|x\|^2 + \overline{\alpha}\langle x,y\rangle + \alpha[\langle y,x\rangle + \overline{\alpha}\langle y,y\rangle]\end{aligned}$$

in particular, take $\overline{\alpha} = \frac{-\langle y,x\rangle}{\langle y,y\rangle}$. Then the above equation becomes

$$\begin{aligned}
\|x\|^2 + \overline{\alpha}\langle x,y\rangle + \alpha[\langle y,x\rangle + \overline{\alpha}\langle y,y\rangle] &= \|x\|^2 - \frac{\langle y,x\rangle}{\langle y,y\rangle}\langle x,y\rangle - \frac{\langle x,y\rangle}{\langle y,y\rangle}\left[\langle y,x\rangle - \frac{\langle y,x\rangle}{\langle y,y\rangle}\langle y,y\rangle\right] \\
&= \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2} \geq \|x\|^2 \\
\implies -\frac{|\langle x,y\rangle|^2}{\|y\|^2} \geq 0
\end{aligned}$$

but since $\frac{|\langle x,y\rangle|^2}{\|y\|^2} \geq 0$, we have

$$\begin{aligned}
0 \leq \frac{|\langle x,y\rangle|^2}{\|y\|^2} \leq 0 \\
\implies \frac{|\langle x,y\rangle|^2}{\|y\|^2} = 0 \\
\implies |\langle x,y\rangle|^2 = 0
\end{aligned}$$

hence, $\langle x,y\rangle = 0$, and so $x \perp y$, which is what we sought to show.

10. **(Zero Operator)** Let $T : X \to X$ be a bounded linear operator on a complex inner product space X. If $\langle Tx,x\rangle = 0$ for all $x \in X$, show that $T = 0$. Show that this does not hold in the case of a *real* inner product space.

Proof: Let $x, y \in X$ and consider $v = x + iy$. Then $\langle Tv,v\rangle = 0$ and notice

$$\begin{aligned}
\langle Tv,v\rangle &= \langle T(x+iy), x+iy\rangle \\
&= \langle Tx,x\rangle + i\langle Ty,x\rangle - i\langle Tx,y\rangle + \langle Ty,y\rangle \\
&= i\langle Ty,x\rangle - i\langle Tx,y\rangle = 0
\end{aligned}$$

so that we have $\langle Ty,x\rangle - \langle Tx,y\rangle = 0$. Now consider $z = x + y$. Then $\langle Tz,z\rangle = 0$ and we have

$$\begin{aligned}
\langle Tz,z\rangle &= \langle T(x+y), x+y\rangle \\
&= \langle Tx,x\rangle + \langle Ty,x\rangle + \langle Tx,y\rangle + \langle T,y\rangle \\
&= \langle Ty,x\rangle + \langle Tx,y\rangle = 0
\end{aligned}$$

and so we have $\langle Ty,x\rangle = 0$ for any $x, y \in X$ since our initial choice of x, y was arbitrary. Then in particular, take $x = Ty$ so that

$$\langle Ty,Ty\rangle = \|Ty\|^2 = 0$$

Thus, for any $y \in X$, $Ty = 0$, so that $T = 0$.

To see that this result does not hold for real inner product spaces, consider the real inner product space $X = \mathbb{R}^2$ and the linear operator T represented in matrix form as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that T rotates vectors in $\mathbb{R}^2$ by $\frac{\pi}{2}$ radians counter clockwise. Then for any $x = (x_1, x_2)^T \in \mathbb{R}^2$, notice

$$\begin{aligned}
Tx &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}
\end{aligned}$$

and that

$$\begin{aligned}\langle Tx, x\rangle &= (-x_2, x_1)\begin{pmatrix}x_1\\x_2\end{pmatrix}\\ &= -x_2x_1 + x_1x_2\\ &= 0\end{aligned}$$

but clearly, $T \neq 0$.

Section 3.3 Problems

8. Show that the annihilator $M^\perp$ of a set $M \neq \emptyset$ in an inner product space X is a closed subspace of X.

Proof: Let $\{x_n\}$ be a sequence in $M^\perp$ converging to $x \in X$. That is, for any $y \in M$,

$$\langle x_n, y\rangle = 0$$

for all n. Then notice

$$\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle = 0$$

hence $x \in M^\perp$, so that $M^\perp$ is closed.

10. If $M \neq \emptyset$ is any subset of a Hilbert space H, show that $M^{\perp\perp}$ is the smallest closed subspace of H which contains M, that is, $M^{\perp\perp}$ is contained in any closed subspace $Y \subset H$ such that $Y \supset M$.

Proof: Let Y be an arbitrary closed subspace containing M. That is,

$$M \subseteq Y.$$

By problem 7 b)*, we have

$$Y^\perp \subseteq M^\perp$$

and by problem 7 b) again,

$$M^{\perp\perp} \subseteq Y^{\perp\perp}$$

and since Y is a closed subspace of a Hilbert space, we have that

$$Y = Y^{\perp\perp}$$

and also, since $M \subseteq M^{\perp\perp}$, we have

$$M \subseteq M^{\perp\perp} \subseteq Y$$

thus, since Y was chosen arbitrarily, $M^{\perp\perp}$ is the smallest closed subset containing M.

(*) Proof of problem 7 b):
Let A and B be nonempty subsets of an inner product space X where $A \subseteq B$. Let $x \in B^\perp$. Then $x \perp B$ by definition, and since $A \subseteq B$, we have that $x \perp A$. Thus $x \in A^\perp$ so that

$$B^\perp \subseteq A^\perp$$

which is what we sought to show.