Math 4510
Topology

Problem Set 9

1. (#2 in 6.5) Prove the general form of Theorem 6.5.3: If A_γ is a connected subspace of X_τ for every $\gamma \in \Lambda$ and if $\cap_{\gamma\in\Lambda} A_\gamma \neq \emptyset$, then $\cup_{\gamma\in\Lambda} A_\gamma$ is connected as a subspace of X.

   Proof: Let A_γ be a connected subspace of X_τ for every $\gamma \in \Lambda$ and assume that $\cap_{\gamma\in\Lambda} A_\gamma \neq \emptyset$. Suppose by way of contradiction that $\cup_{\gamma\in\Lambda} A_\gamma$ is disconnected. Then for two disjoint $\tau_{\cup_{\gamma\in\Lambda} A_\gamma}$ open sets U and V, $\cup_{\gamma\in\Lambda} A_\gamma = U \cup V$.

   Suppose that $U \cap (\cup_{\gamma\in\Lambda} A_\gamma)$ and $V \cap (\cup_{\gamma\in\Lambda} A_\gamma)$ are non-empty. For every $\gamma \in \Lambda$, $U \cap A_\gamma$ and $V \cap A_\gamma$ are nonempty. Well, since A_γ is connected for every $\gamma \in \Lambda$, we have that $U \cap A_\gamma$ or $V \cap A_\gamma$ is empty, or $U \cap A_\gamma$ is empty for all $\gamma \in \Lambda$.

   Suppose $U \cap A_\gamma$ is empty for $\gamma \in \Gamma \subseteq \Lambda$.

   For some $\gamma' \in \Lambda \setminus \Gamma$, we have $V \cap A_{\gamma'}$ is empty. Then since $\cap_{\gamma\in\Lambda} A_\gamma \neq \emptyset$, we have that there must exist a point $x \in \cap_{\gamma\in\Lambda} A_\gamma$ that is neither in U nor V. But since $U \cup V$ was assumed to be equal to $\cup_{\gamma\in\Lambda} A_\gamma$, we have a contradiction.

   Now suppose that $U \cap A_\gamma$ is empty for all $\gamma \in \Lambda$. Since we have $\bigcap_{\gamma\in\Lambda} A_\gamma \neq \emptyset$, and $\bigcup_{\gamma\in\Lambda} A_\gamma = U \cup V$, and $U \cap A_\gamma$ is empty, U and $\bigcup_{\gamma\in\Lambda} A_\gamma$ are disjoint. Then we must have $U = \emptyset$, and $\bigcup_{\gamma\in\Lambda} A_\gamma = V$. Equivalently, there does not exist a disconnection for $\bigcup_{\gamma\in\Lambda} A_\gamma$ for this case.

   So $\cup_{\gamma\in\Lambda} A_\gamma$ is connected for A_γ a connected subspace of X_τ for every $\gamma \in \Lambda$.

2. (#2 in 7.2) Prove Corollary 7.2.2: (i) If X_τ is compact and τ' is any topology on X that is coarser than τ, then $X_{\tau'}$ is also compact. (ii) If Y_ν is not compact and ν' is any topology on Y that is finer than ν, then $Y_{\nu'}$ is not compact. (Hint: use the identity maps on X and Y).

   Proof of (i): Let X_τ be a compact space and let $\tau' \subseteq \tau$ be a topology on X. Since $\tau' \subseteq \tau$, we have that $i_x : X_\tau \to X_{\tau'}$ the identity map is continuous. Since $\mathrm{Im}(i_x) = X$ and compactness is a strong topological property, we have that $X_{\tau'}$ is compact.

   Proof of (ii): Let Y_ν be not compact and let $\nu \subseteq \nu'$ be a topology on Y. Suppose by way of contradiction that $Y_{\nu'}$ is compact. Notice since $\nu \subseteq \nu'$, we have that $i_y : Y_{\nu'} \to Y_\nu$, the identity map, is continuous. Then since compactness is a strong topological property and $\mathrm{Im}(i_y) = Y$, we have that Y_ν is compact, a contradiction. Thus, if Y_ν is not compact, and $\nu \subseteq \nu'$, then $Y_{\nu'}$ is not compact.

3. (#6 in 7.2) Show from the definition that $\mathbb{R}^2_{\mathcal{U}^2}$ is not compact.

Proof: Let $C = \bigcup_{n\in\mathbb{N}}\{(-n,n)\times(-n,n)\}$ be a cover for $\mathbb{R}^2_{\mathcal{U}^2}$ where $\times$ denotes the Cartesian product. Observe that C does not have a finite subcover:

Suppose C did have a finite subcover. Then for some $k \in \mathbb{N}$, $\mathbb{R}^2 = \bigcup_{n=1}^{k}\{(-n,n)\times(-n,n)\}$. Notice that this is a union of nested intervals. That is, $\bigcup_{n=1}^{k}\{(-n,n)\times(-n,n)\} = (-k,k)\times(-k,k)$. But then for $k+1$

$$(-k,k)\times(-k,k) \subset (-k-1,k+1)\times(-k-1,k+1) \subset \mathbb{R}^2$$

So C does not have a finite subcover.

so by definition of compactness, $\mathbb{R}^2_{\mathcal{U}^2}$ is not compact.

4. (#4 in 7.3) Prove that a subspace of a Hausdorff space is Hausdorff.

Proof: Let X_τ be Hausdorff and let A be a subspace of X. We wish to show that A_{τ_A} is also Hausdorff. Well, if $A = X$, then we're done. Suppose $A \subset X$, Card$(A) \geq 2$, and let $a_1, a_2 \in A$. Then since $A \subset X$, $a_1, a_2 \in X$. And since X is Hausdorff, there exist disjoint τ-open sets U and V such that $a_1 \in U$ and $a_2 \in V$.

Notice that $A\cap U$ and $A \cap V$ are open in τ_A and that $a_1 \in A\cap U$ and $a_2 \in A \cap V$. We must show $A \cap U$ and $A\cap V$ are disjoint. Consider

$$(A\cap U)\cap(A\cap V) = A\cap U\cap A\cap V$$

$$= A\cap A\cap U\cap V$$
$$= A\cap(U\cap V)$$
$$= A\cap(\emptyset)$$
$$= \emptyset$$

That is, we have two disjoint τ_A open sets each containing one of a_1 and a_2. Then by definition, A is a Hausdorff space.

5. (#6 in 7.3) Prove Theorem 7.3.2: Let X_τ be compact and Hausdorff. If τ' is any topology on X with τ' strictly finer than τ, then $X_{\tau'}$ is not compact. If τ'' is any topology on X with τ'' strictly coarser than τ, then $X_{\tau''}$ is not Hausdorff. (Hint: consider the hint for problem #5 in 7.3).

Proof: Let X_τ be compact and Hausdorff and suppose by way of contradiction that $X_{\tau'}$ is compact for $\tau \subset \tau'$. Since τ is coarser than τ', we have that the identity map $i_x : X_{\tau'} \to X_\tau$ is continuous by theorem 3.5.1. Thus, i_x is a continuous bijection. So by theorem 7.3.3, we have that i_x is a homeomorphism between $X_{\tau'}$ and X_τ. But i_x can only be a homeomorphism if and only if $\tau = \tau'$, contradicting the fact that $\tau \subset \tau'$. Thus, if X_τ is compact and Hausdorff, $X_{\tau'}$ cannot be compact for $\tau \subset \tau'$.

Now let τ'' be a topology on X such that $\tau'' \subset \tau$. We wish to show $X_{\tau''}$ is not Hausdorff. Suppose by way of contradiction that $X_{\tau''}$ is Hausdorff. Well, since τ'' is coarser than τ, we have that $i_x : X_\tau \to X_{\tau''}$ is continuous by theorem 3.5.1. Thus, i_x is a continuous bijection between X_τ and $X_{\tau''}$. By theorem 7.3.3, we have that i_x is a homeomorphism between X_τ and $X_{\tau''}$. But i_x can only be a homeomorphism if and only if $\tau = \tau'$, which contradicts the fact that $\tau'' \subset \tau$. Thus, $X_{\tau''}$ is not Hausdorff.

6. (#2 in 7.5) Prove that any finite union of compact subsets of a topological space is compact.

   Proof: Let $A_1, A_2 \subseteq X$ be compact where X_τ is a topological space. We wish to show that $A_1 \cup A_2$ is also compact. Well, let C be a cover for $A_1 \cup A_2$. Notice that $C \cap A_1$ is a cover for A_1 and $C \cap A_2$ is a cover for A_2. Since A_1, A_2 are compact, we have that there exist finite subcovers $C_1, C_2 \subset C$ such that C_1 covers A_1 and C_2 covers A_2.

   Now $C' = C_1 \cup C_2 \subseteq C$ is a cover for $A_1 \cup A_2$. Since C_1, C_2 are finite, we have that C' is also finite. That is, we have a finite subcover of C. Hence, $A_1 \cup A_2$ is compact. To continue this argument, let $B = A_1 \cup A_2$ and $A_3 \subset X$ be compact. Following the same logic as above, we get that $B \cup A_3$ is compact. Continuing this argument up to some natural number n, we find that

$$\bigcup_{i=1}^{n} A_i$$

   is compact for $A_i \subseteq X$ compact for every $1 \leq i \leq n$.

Bonus (#7 in 7.3) Prove the Closed Graph Theorem: If $f : X_\tau \to Y_\nu$ is continuous, with Y both compact and Hausdorff, then the graph

$$G_f = \{(x, y) \in X \times Y | y = f(x)\}$$

is closed in $X_\tau \times Y_\nu$.