Math 4510
Topology

Problem Set 7

1. (#1 in 5.4) Sketch some open sets in the quotient space $\mathbb{R}/\sim$ of Example 5.2. (If you are writing your solutions in LaTeX, you may sketch these on paper, scan and email them to me, or upload in Canvas as a second file). Be sure to show that the sets in $\mathbb{R}_{\mathcal{U}}$ that project to these sets under the quotient map $\nu : \mathbb{R} \to \mathbb{R}/\sim$.

   See attached for sketches.

2. Find a subspace X of $\mathbb{R}^2$ and an equivalence relation $\sim$ on X so that $X/\sim \cong S^2$, where S^2 is the unit sphere centered at the origin in $\mathbb{R}^3$. Illustrate typical open sets in the quotient space and in X. You do not have to give an explicit homeomorphism between $X/\sim$ and S^2, but you should describe a function between the two and explain why it is a homeomorphism.

   Consider the disk of radius one centered at the origin in $\mathbb{R}^2$ given by $D^2 = \{(x,y)|x^2 + y^2 \leq 1, x, y \in \mathbb{R}\}$. Define the equivalence relation on D^2 by $\mathbf{x} \sim \mathbf{x}$ for $\mathbf{x} = (x,y)$ if $x^2 + y^2 < 1$ and $\mathbf{x}_0 \sim \mathbf{x}_1$ for $\mathbf{x}_0 = (x_0, y_0)$, $\mathbf{x}_1 = (x_1, y_1)$ if $x_0^2 + y_0^2 = 1 = x_1^2 + y_1^2$. That is, define all the points on the boundary of D^2 to be equivalent to each other.

   Let $f : D^2_{/\sim} \to S^2$ map such that $f(\partial D^2) = (0,0,1) \in S^2$, $f([(0,0)]) = (0,0,-1) \in S^2$, and for some concentric circle of radius $0 < r < 1$ in D^2, f maps the equivalence classes of points on the concentric circle to a circle on S^2. See sketches for more details. .

   Also see attached for sketches of open sets in the quotient space and X.

3. (#1 in 5.5) A *path* in a space X_τ is a continuous function $\alpha : [0,1]_{\mathcal{U}} \to X_\tau$. If α and β are two paths in X_τ such that $\alpha(1) = \beta(0)$, then the map $\alpha \star \beta : [0,1] \to X$ defined by

$$(\alpha \star \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

   is continuous. (*Hint*: Draw two such paths, then consider the Pasting Lemma.)

   We first wish to show that $\alpha(2t)$ is continuous on $0 \leq t \leq 1/2$ and that $\beta(2t-1)$ is continuous on $1/2 \leq t \leq 1$. Well, since $\alpha(t)$ and $\beta(t)$ are paths in X_τ and paths are defined to be continuous, we have that $\alpha(t)$ and $\beta(t)$ are continuous on $0 \leq t \leq 1$.

   Let $f : [0,1/2]_{\mathcal{U}} \to [0,1]_{\mathcal{U}}$ map $t \mapsto 2t$. Let $(a,b) \subseteq [0,1]$ be open in $[0,1]_{\mathcal{U}}$. Notice that $f^{-1}((a,b)) = (a/2, b/2)$ which is also open in $[0,1/2]_{\mathcal{U}}$. So by definition, f is continuous.

   Now let $g : [1/2,1] \to [0,1]$ where $t \mapsto 2t-1$. Let (a,b) be as above. Notice that $g^{-1}((a,b)) = (\frac{a+1}{2}, \frac{b+1}{2})$ which is open in $[1/2,1]_{\mathcal{U}}$. So by definition, g is continuous.

   Now $\alpha(2t)$ which maps $[0,1/2] \to [0,1]$ is continuous since $f = 2t$ and α are continuous.

   Additionally, $\beta(2t-1)$ which maps $[1/2,1] \to [0,1]$ is continuous since $g = 2t-1$ and β are continuous.

Let $A = [0, 1/2]$ and $B = [1/2, 1]$. Notice that $A \cup B = [0, 1]$, the domain of $\alpha \star \beta$. Now since $[0, 1]$ is equipped with the usual topology, notice that $[0, 1] \setminus A = (1/2, 1]$ which is open in $[0, 1]_{\mathcal{U}}$ and $[0, 1] \setminus B = [0, 1/2)$, which is open in $[0, 1]_{\mathcal{U}}$, so A and B are closed subsets of $[0, 1]$.

Additionally, notice that $A \cap B = \{1/2\}$ and that $\alpha(2(1/2)) = \alpha(1)$ and $\beta(2(1/2)-1) = \beta(1-1) = \beta(0)$. And from the definition of α and β, we have that $\alpha(1) = \beta(0)$. So by the pasting lemma, we have that $(\alpha \star \beta)(t)$ is continuous.

4. (# 7a in 6.2) A space X_τ is said to be *totally disconnected* if every subspace of X with more than one element is disconnected (in the subspace topology). Show that every discrete space is totally disconnected.

   Let $X_{\mathcal{D}}$ be a discrete space and $A \subseteq X$ be a subspace of X such that $\text{card}(A) \geq 2$. Since A is a subspace of a discrete space, $\mathcal{D}_A = \mathcal{P}(A)$. Since $\text{card}(A) \geq 2$, there exist subsets B, C of A such that $B \cap C = \emptyset$, $B \cup C = A$. For example, let $A = \{a_1, a_2\}$. Take $B = \{a_1\}$ and $C = \{a_2\}$. Notice that $B \cup C = A$ and that $B \cap C = \emptyset$. For $\text{Card}(A) > 2$, take $B = \{a_1\}$ and $C = A \setminus \{a_1\}$, and the same argument will hold. Note that B and C are in $\mathcal{D}_A$.

   By definition, $A_{\mathcal{D}_A}$ is disconnected. Since A is an arbitrary subset of X with cardinality greater than two, $X_{\mathcal{D}}$ is totally disconnected.