Approximation Methods HW 4

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6.4.1 Use the method of steepest descent to find the leading asymptotic behavior as $k \to \infty$ of

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{te^{ik\left(\frac{t^3}{3}+t\right)}}{1+t^4}dt$

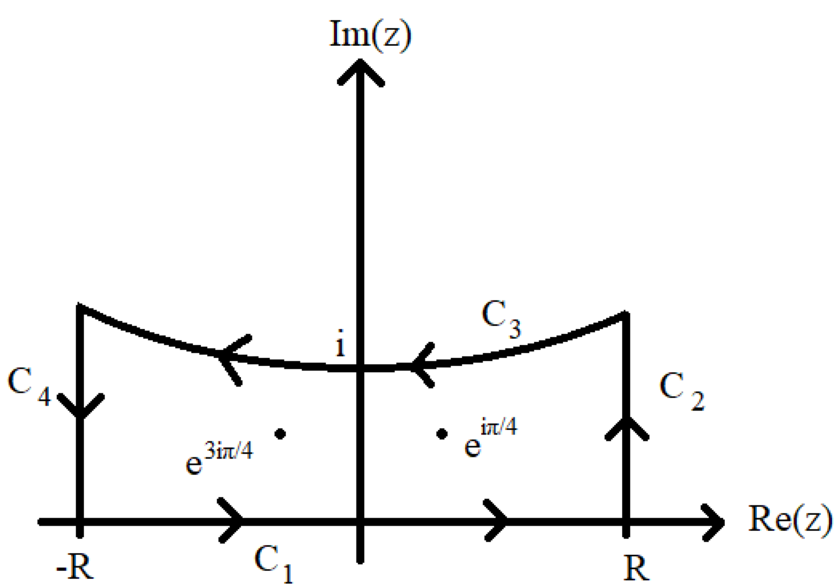
Soln. Let $\phi(z) = i\left(\frac{z^3}{3} + z\right)$ and notice $\phi'(z) = i(z^2+1) = 0 \implies z = \pm i$ so that $\pm i$ are saddle points of ϕ. Notice

$$\begin{aligned}\phi(i) &= i\left(-\frac{i}{3} + i\right)\\ &= -\frac{2}{3}\\ \phi(-i) &= i\left(\frac{i}{3} - i\right)\\ &= \frac{2}{3}\end{aligned}$$

and since $\text{Re}(\phi(i)) < 0$, $\text{Re}(\phi(-i)) > 0$, we expect the main contribution of the integral to be near $z = i$. Now, notice

$$\begin{aligned}\phi(z) &= i\left(\frac{(x+iy)^3}{3} + x + iy\right)\\ &= i\left(\frac{1}{3}x^3 + ix^2y - xy^2 - \frac{1}{3}iy^3 + x + iy\right)\\ &= -\left(x^2y - \frac{1}{3}y^3 + y\right) + i\left(\frac{1}{3}x^3 - xy^2 + x\right)\end{aligned}$$

so that at $z = i$, the steepest descent contour is given by $\frac{1}{3}x^3 - xy^2 + x = 0 \implies x\left(\frac{1}{3}x^2 - y^2 + 1\right) = 0 \implies x = 0$ or $y^2 - x^2 = 1$. We then deform the contour:



Notice that $z = e^{i\pi/4}$ and $z = e^{3i\pi/4}$ are simple poles of $\frac{z}{1+z^4}$ since

$$(1+z^4) = (z - e^{i\pi/4})(z - e^{3i\pi/4})(z - e^{5i\pi/4})(z - e^{7i\pi/4}).$$

Further note that the poles at $z = e^{i\pi/4}$ and $e^{3i\pi/4}$ are contained in our deformed contour for sufficiently large R, so that by the Residue theorem, we have

$$\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 2\pi i \left(\operatorname*{Res}_{z=e^{i\pi/4}} \frac{ze^{ik\left(\frac{z^3}{3}+z\right)}}{1+z^4} + \operatorname*{Res}_{z=e^{3i\pi/4}} \frac{ze^{ik\left(\frac{z^3}{3}+z\right)}}{1+z^4} \right).$$

Computing the residues yields

$$\begin{aligned}
\operatorname*{Res}_{z=e^{i\pi/4}} \frac{ze^{ik\left(\frac{z^3}{3}+z\right)}}{1+z^4} &= \lim_{z\to e^{i\pi/4}} \frac{z(z-e^{i\pi/4})e^{ik\left(\frac{z^3}{3}+z\right)}}{(z-e^{i\pi/4})(z-e^{3i\pi/4})(z-e^{5i\pi/4})(z-e^{7i\pi/4})} \\
&= \lim_{z\to e^{i\pi/4}} \frac{ze^{ik\left(\frac{z^3}{3}+z\right)}}{(z-e^{3i\pi/4})(z-e^{5i\pi/4})(z-e^{7i\pi/4})} \\
&= \frac{e^{i\pi/4}e^{ik\left(e^{ik\left(\frac{e^{3i\pi/4}}{3}+e^{i\pi/4}\right)}\right)}}{(e^{i\pi/4}-e^{3i\pi/4})(e^{i\pi/4}-e^{5i\pi/4})(e^{i\pi/4}-e^{7i\pi/4})} \\
&= -\frac{i}{4}e^{ik\left(\frac{e^{3i\pi/4}}{3}+e^{i\pi/4}\right)}.
\end{aligned}$$

Similarly,

$$\operatorname*{Res}_{z=e^{3i\pi/4}} \frac{ze^{ik\left(\frac{z^3}{3}+z\right)}}{1+z^4} = \frac{i}{4}e^{ik\left(\frac{e^{i\pi/4}}{3}+e^{i\pi/4}\right)}.$$

We now argue that the sides of the contour tend to zero as $R\to\infty$. We inspect C_2 and note that a similar argument will hold for C_4. On C_2, parameterize $z = R + iy$ with $0 \le y \le \sqrt{1+\frac{R^2}{3}}$. Then

$$\begin{aligned}
\int_{C_2} &= i\int_0^{\sqrt{1+\frac{R^2}{3}}} \frac{(R+iy)}{1+(R+iy)^4} e^{ik\left(\frac{(R+iy)^3}{3}+(R+iy)\right)}dy \\
&= i\int_0^{\sqrt{1+\frac{R^3}{3}}} \frac{(R+iy)}{1+(R+iy)^4} e^{ik\left(\frac{1}{3}R^3+iR^2y-Ry^2-\frac{i}{3}y^3+R+iy\right)}dy \\
\implies \left|\int_{C_2}\right| &\le \int_0^{\sqrt{1+\frac{R^3}{3}}} \frac{R+1}{R^4-1} e^{-\left(R^2y-\frac{1}{3}y^3+y\right)}dy
\end{aligned}$$

where we used $|z| \ge |\mathrm{Re}(z)|$. Now notice for sufficiently large R, $R > \sqrt{1+\frac{R^2}{3}}$ since $\sqrt{1+\frac{R^2}{3}} \sim R/\sqrt{3}$ as $R\to\infty$. And also note that $R^2y - \frac{1}{3}y^3 + y > 0$ for $0 \le y \le R$, hence we can bound the exponential part of the integrand by 1. Hence

$$\begin{aligned}
\left|\int_{C_2}\right| &\le \frac{R(R+1)}{R^4-1} \\
&\to 0 \quad \text{as} \quad R\to\infty.
\end{aligned}$$

A similar argument shows $\left|\int_{C_4}\right| \to 0$ as $R\to\infty$.

And near $z = i$, notice

$$\phi(z) \sim -\frac{2}{3} - (z-i)^2.$$

Then the steepest descent directions are given by

$$\begin{aligned} \theta &= \frac{2m+1}{2}\pi + \frac{\pi}{2}, \qquad m = 0, 1 \\ \implies \theta &= 0, \pi. \end{aligned}$$

Thus the top contour is asymptotically equivalent to (after parameterizing with $z = x + i$ and using $\frac{z}{1+z^4} \sim \frac{i}{2}$ as $z \to i$)

$$\begin{aligned} \int_{C_2} &\sim \frac{i}{2}\int_{-\infty}^{\infty} e^{k\left(-\frac{2}{3}-x^2\right)}dz \\ &= \frac{ie^{-2k/3}}{2}\int_{-\infty}^{\infty} e^{-kx^2}dx \\ &= \frac{ie^{-2k/3}}{2}\sqrt{\frac{\pi}{k}}. \end{aligned}$$

Finally we have

$$\int_{-\infty}^{\infty} \frac{te^{ik\left(\frac{t^3}{3}+t\right)}}{1+t^4}dt \sim \frac{ie^{-2k/3}}{2}\sqrt{\frac{\pi}{k}} + \frac{\pi}{2}\left(e^{ik\left(\frac{e^{i3\pi/4}}{3}+e^{i\pi/4}\right)} - e^{ik\left(\frac{e^{i\pi/4}}{3}+e^{3i\pi/4}\right)}\right).$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ik\left(\frac{t^5}{5}+t\right)}}{1+t^2}dt$

Soln. Begin by considering the function $f(z) = \frac{e^{ik\left(\frac{z^5}{5}+z\right)}}{1+z^2}$. Note that $f(z)$ has simple poles at $z = \pm i$ since $1 + z^2 = (i - z)(i + z)$. Let $\phi(z) = i\left(\frac{z^5}{5} + z\right)$. Then $\phi'(z) = i(z^4 + 1) = 0 \implies z = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$ are saddle points of ϕ. Note that

$$\begin{aligned} \phi(z) &= i\left(\frac{(x+iy)^5}{5} + x + iy\right) \\ &= i\left(\frac{1}{5}x^5 + ix^4y - 2x^3y^2 - 2ix^2y^3 + xy^4 + \frac{i}{5}y^5 + x + iy\right) \\ &= -\left(x^4y - 2x^2y^3 + \frac{1}{5}y^5 + y\right) + i\left(\frac{1}{5}x^5 - 2x^3y^2 + xy^4 + x\right) \\ \implies \operatorname{Re}(\phi(e^{i\pi/4})) &= -\left(\frac{3}{4\sqrt{2}} + \frac{1}{20\sqrt{2}}\right) < 0 \\ \operatorname{Re}(\phi(e^{3i\pi/4})) &= -\left(\frac{3}{4\sqrt{2}} + \frac{1}{20\sqrt{2}}\right) < 0 \\ \operatorname{Re}(\phi(e^{5i\pi/4})) &= -\left(-\frac{3}{4\sqrt{2}} - \frac{1}{20\sqrt{2}}\right) > 0 \\ \operatorname{Re}(\phi(e^{7i\pi/4})) &= -\left(-\frac{3}{4\sqrt{2}} - \frac{1}{20\sqrt{2}}\right) > 0 \end{aligned}$$

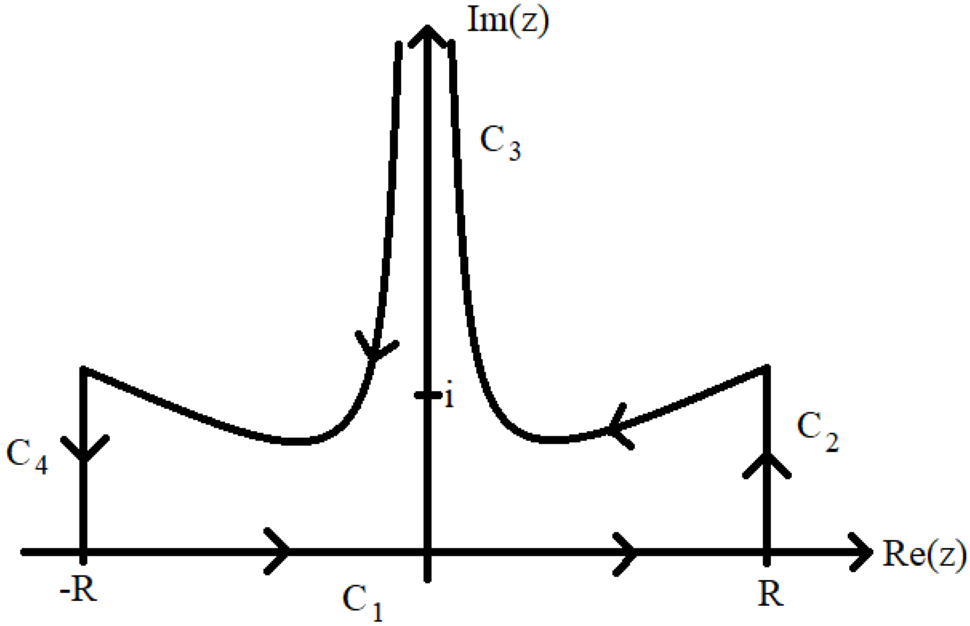
which gives that the saddle point contributions will come from $z = e^{i\pi/4}, e^{3i\pi/4}$. Now, at $z = e^{i\pi/4}, e^{3i\pi/4}$ we find

$$\begin{aligned} \operatorname{Im}(\phi(e^{i\pi/4})) &= \frac{4}{5\sqrt{2}} \\ \operatorname{Im}(\phi(e^{3i\pi/4})) &= -\frac{4}{5\sqrt{2}}. \end{aligned}$$

Thus the curves of steepest descent are given by

$$\frac{x^5}{5} - 2x^3y^2 + xy^4 + x = \frac{4}{5\sqrt{2}} \quad (z = e^{i\pi/4})$$
$$\frac{x^5}{5} - 2x^3y^2 + xy^4 + x = -\frac{4}{5\sqrt{2}} \quad (z = e^{3i\pi/4}).$$

Deforming the contour gives us



Thus, by the residue theorem, we have

$$\begin{aligned}
\int_{C_1} + \int_{C_2} + \int_{C_3} &= 2\pi i \operatorname*{Res}_{z=i} \frac{e^{ik\left(\frac{z^5}{5}+z\right)}}{1+z^2} \\
&= 2\pi i \lim_{z\to i} \frac{(z-i)e^{ik\left(\frac{z^5}{5}+z\right)}}{(z-i)(z+i)} \\
&= 2\pi i \lim_{z\to i} \frac{e^{ik\left(\frac{z^5}{5}+z\right)}}{z+i} \\
&= 2\pi i \frac{e^{ik\left(\frac{i}{5}+i\right)}}{2i} \\
&= \pi e^{-\frac{6k}{5}}
\end{aligned}$$

We now argue that the side contours go to zero as $R \to \infty$. We show $\int_{C_2} \to 0$ as $R \to \infty$ and argue similarly for C_4. On C_2 parameterize $y = R + iy$. From the parameterization for the steepest descent path, we have

$$\begin{aligned}
&\frac{R^5}{5} - 2R^3y^2 + Ry^4 + R = \frac{4}{5\sqrt{2}} \\
\implies &\frac{R^5}{5} - 2R^3 s + Rs^2 + R = \frac{4}{5\sqrt{2}} \quad (s = y^2) \\
&\implies s = \frac{2R^2 \pm \sqrt{4R^4 - 4\left(\frac{R^4}{5} + 1 - \frac{4}{5\sqrt{2}R}\right)}}{2} \\
&\implies y = \pm\sqrt{R^2 \pm \sqrt{\frac{4}{5}R^4 - 1 + \frac{4}{5\sqrt{2}R}}}.
\end{aligned}$$

The upper bound on y is then given as

$$\begin{aligned}
y &= \sqrt{R^2 - \sqrt{\frac{4}{5}R^4 - 1 + \frac{4}{5\sqrt{2}R}}} \\
&\leq R.
\end{aligned}$$

Thus, on C_2 we have

$$\begin{aligned}
\left|\int_{C_2}\right| &\leq \int_0^{\sqrt{R^2 - \sqrt{\frac{4}{5}R^4 - 1 + \frac{4}{5\sqrt{2}R}}}} \left|\frac{R+iy}{1+(R+iy)^4} e^{ik\left(\frac{(R+iy)^3}{3} + R + iy\right)}\right| |idy| \\
&\leq \int_0^{\sqrt{R^2 - \sqrt{\frac{4}{5}R^4 - 1 + \frac{4}{5\sqrt{2}R}}}} \frac{2R}{R^4 - 1} e^{-k\left(R^2 - \frac{1}{3}y^3 + y\right)} dy.
\end{aligned}$$

Note that since $y \leq R$, $y^3 \leq R^3$ and so $R^2 - \frac{1}{3}y^3 + y = R^2 + \frac{2}{3}R^3 > 0$ so we can bound the exponential in the integrand by 1:

$$\left|\int_{C_2}\right| \leq \frac{2R^2}{R^4 - 1} \to 0 \quad \text{as} \quad R \to \infty.$$

A similar argument shows $\left|\int_{C_4}\right| \to 0$ as $R \to \infty$.

We now wish to find the descent directions near the saddle points on C_3. Expanding $\phi(z)$ near $z = e^{i\pi/4}, e^{3i\pi/4}$ gives

$$\begin{aligned}
\phi(z) &\approx -\frac{4}{5\sqrt{2}} + i\frac{4}{5\sqrt{2}} - 2e^{3i\pi/4}(z - e^{i\pi/4}) \quad \text{near} \quad z = e^{i\pi/4} \\
\phi(z) &\approx -\frac{4}{5\sqrt{2}} - i\frac{4}{5\sqrt{2}} - 2e^{i\pi/4}(z - e^{3i\pi/4}) \quad \text{near} \quad z = e^{3i\pi/4}.
\end{aligned}$$

Thus, the descent directions at $z = e^{i\pi/4}$ are

$$\begin{aligned}
\theta &= \frac{2m+1}{2}\pi - \frac{5\pi}{8} \quad m = 0, 1 \\
\implies \theta &= -\frac{\pi}{8}, \frac{7\pi}{8}
\end{aligned}$$

and at $z = e^{3i\pi/4}$,

$$\begin{aligned}
\theta &= \frac{2m+1}{2}\pi - \frac{3\pi}{8} \quad m = 0, 1 \\
\implies \theta &= \frac{\pi}{8}, \frac{9\pi}{8}.
\end{aligned}$$

And also note that

$$\frac{1}{1+z^2} \sim \frac{1}{1+i} \quad \text{near} \quad z = e^{i\pi 4}$$
$$\frac{1}{1+z^2} \sim \frac{1}{1-i} \quad \text{near} \quad z = e^{3i\pi/4}.$$

Denote C_3' as the steepest descent curve in the first quadrant and C_3'' as the steepest descent curve in the second quadrant. Parameterize $z = re^{-i\pi/8} + e^{i\pi/4}$ on C_3' and $z = re^{i\pi/8} + e^{3i\pi/4}$ on C_3''. Then

$$\begin{aligned}
\int_{C_3'} &\sim -\frac{e^{-i\pi/8}}{1+i} \int_{-\infty}^{\infty} e^{k\left(-\frac{4}{5\sqrt{2}}+i\frac{4}{5\sqrt{2}}\right)} e^{-2kr^2}\, dr \\
&= -\sqrt{\frac{\pi}{2k}} \frac{e^{-i\pi/8}}{1+i} e^{k\left(-\frac{4}{5\sqrt{2}}+i\frac{4}{5\sqrt{2}}\right)} \\
&= -\sqrt{\frac{\pi}{k}} \frac{e^{-i\pi/8}}{2e^{i\pi/4}} e^{k\left(-\frac{4}{5\sqrt{2}}+i\frac{4}{5\sqrt{2}}\right)} \\
&= -\sqrt{\frac{\pi}{k}} \frac{e^{-3i\pi/8}}{2} e^{k\left(-\frac{4}{5\sqrt{2}}+i\frac{4}{5\sqrt{2}}\right)} \\
\int_{C_3''} &\sim -\frac{e^{i\pi/8}}{1-i} \int_{-\infty}^{\infty} e^{k\left(-\frac{4}{5\sqrt{2}}-i\frac{4}{5\sqrt{2}}\right)} e^{-2kr^2}\, dr \\
&= -\sqrt{\frac{\pi}{2k}} \frac{e^{i\pi/8}}{1-i} e^{k\left(-\frac{4}{5\sqrt{2}}-i\frac{4}{5\sqrt{2}}\right)} \\
&= -\sqrt{\frac{\pi}{k}} \frac{e^{i\pi/8}}{2e^{-i\pi/4}} e^{k\left(-\frac{4}{5\sqrt{2}}-i\frac{4}{5\sqrt{2}}\right)} \\
&= -\sqrt{\frac{\pi}{k}} \frac{e^{3i\pi/8}}{2} e^{k\left(-\frac{4}{5\sqrt{2}}-i\frac{4}{5\sqrt{2}}\right)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{C_3} &\sim -\sqrt{\frac{\pi}{k}} \left(\frac{e^{3i\pi/8}}{2} e^{k\left(-\frac{4}{5\sqrt{2}}-i\frac{4}{5\sqrt{2}}\right)} + \frac{e^{-3i\pi/8}}{2} e^{k\left(-\frac{4}{5\sqrt{2}}+i\frac{4}{5\sqrt{2}}\right)} \right) \\
&= -\sqrt{\frac{\pi}{k}} e^{-k\frac{4}{5\sqrt{2}}} \left(\frac{e^{i\left(\frac{4}{5\sqrt{2}}k-\frac{3\pi}{8}\right)} + e^{-i\left(\frac{4}{5\sqrt{2}}k-\frac{3\pi}{8}\right)}}{2} \right) \\
&= -e^{-\frac{4}{5\sqrt{2}}k} \sqrt{\frac{\pi}{k}} \cos\left(\frac{4}{5\sqrt{2}}k - \frac{3\pi}{8} \right).
\end{aligned}$$

Putting it together, we have

$$\int_{-\infty}^{\infty} \frac{e^{ik\left(\frac{t^5}{5}+t\right)}}{1+t^2} dt \sim e^{-\frac{4}{5\sqrt{2}}k} \sqrt{\frac{\pi}{k}} \cos\left(\frac{4}{5\sqrt{2}}k - \frac{3\pi}{8} \right) + \pi e^{-\frac{6}{5}k}$$

6.4.4 In this problem we will find the "complete" asymptotic behavior of

$$I(k) = \int_0^{\frac{\pi}{4}} e^{ikt^2} \tan(t)dt \quad \text{as} \quad k \to \infty$$

(a) Show that the steepest descent paths are given by

$$x^2 - y^2 = C; \quad C = \text{ constant}$$

Proof: Let $\phi(z) = iz^2 = i(x+iy)^2 = i(x^2 - y^2 + 2ixy) = -2xy + i(x^2 - y^2)$. Thus at a point $z_0 = x_0 + iy_0$, $\text{Im}(\phi(z_0)) = x_0^2 - y_0^2 := C$. Thus at z_0, the steepest descent paths are given by

$$x_0^2 - y_0^2 = C$$

where C is constant, as desired.

(b) Show that the steepest descent/ascent paths that go through $z = 0$ are given by

$$x = \pm y$$

and the steepest paths that go through $z = \frac{\pi}{4}$ are given by

$$x = \pm\sqrt{\left(\frac{\pi}{4}\right)^2 + y^2}$$

Proof: From part (a), the steepest descent path through $z = 0$ is given by

$$\begin{aligned} &x^2 - y^2 = 0 \\ &\implies x = \pm y. \end{aligned}$$

Similarly, the steepest paths through $z = \frac{\pi}{4}$ are given by

$$\begin{aligned} &x^2 - y^2 = \left(\frac{\pi}{4}\right)^2 \\ &\implies x = \pm\sqrt{\left(\frac{\pi}{4}\right)^2 + y^2} \end{aligned}$$

as desired.

(c) Note that the steepest descent paths in the first quadrant are $C_1 : x = y$ and $C_3 : x = \sqrt{\left(\frac{\pi}{4}\right)^2 + y^2}$. Construct a path C_2 as shown in the Figure 6.4.12, and therefore
$I(k)$ can be written as $I_1 + I_2 + I_3$ where I_i refers to the integral along contour C_i, for $i = 1, 2, 3$. As $k \to \infty$: show that

$$\begin{aligned} I_1 &\sim \frac{i}{2k} - \frac{1}{6k^2}, \\ I_2 &\sim 0, \\ I_3 &\sim \frac{-2i}{k\pi} e^{ik(\pi/4)^2}. \end{aligned}$$

Proof: Note that $\tan(z)e^{ikz^2}$ is holomorphic in and on C, so by Cauchy's theorem, we have

$$\int_C = I_1 + I_2 + I_3.$$

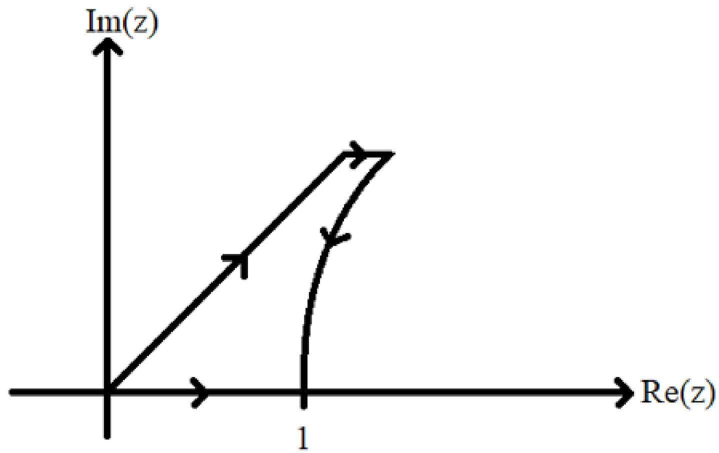


Figure 1: Contour for 6.4.4

For I_1, we begin by showing $\tan(z) \sim z + \frac{z^3}{3}$ as $z \to 0$. Notice

$$\begin{aligned}
\tan(z) &= \frac{\sin(z)}{\cos(z)} \\
&= \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \mathcal{O}(z^7)}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \mathcal{O}(z^6)} \\
&= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \mathcal{O}(z^7)\right)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \mathcal{O}(z^6)\right) \quad \text{near} \quad z = 0 \\
&= z + \left(\frac{z^3}{2} - \frac{z^3}{6}\right) + \left(\frac{5z^5}{24} - \frac{z^5}{12} + \frac{z^5}{120}\right) + \mathcal{O}(z^7) \\
&= z + \frac{z^3}{3} + \frac{2z^5}{15} + \mathcal{O}(z^7) \\
\implies \tan(z) &\sim z + \frac{z^3}{3} \quad \text{as} \quad z \to 0.
\end{aligned}$$

Now parameterize $z = re^{i\pi/4}$ and so

$$\begin{aligned}
I_1 &\sim e^{i\pi/4} \int_0^R re^{i\pi/4} e^{-kr^2} dr + \frac{e^{i\pi/4} e^{3i\pi/4}}{3} \int_0^R r^3 e^{-kr^2} dr \\
&\to i \int_0^\infty re^{-kr^2} dr - \frac{1}{3} \int_0^\infty r^3 e^{-kr^2} dr \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Let $u = kr^2$, $du = 2krdr$ so that the above integrals become

$$\begin{aligned}
I_1 &\sim \frac{i}{2k} \int_0^\infty e^{-u} du - \frac{1}{6k^2} \int_0^\infty ue^{-u} du \\
&= \frac{i}{2k} - \frac{1}{6k^2}
\end{aligned}$$

as desired. Now, on C_2, parameterize $z = x + iR$ and so

$$\begin{aligned}
\int_{C_2} &= \int_R^{\sqrt{\left(\frac{\pi}{4}\right)^2 + R^2}} \tan(x + iR) e^{ik(x+iR)^2} dx \\
\implies \left|\int_{C_2}\right| &\le \int_R^{\sqrt{\left(\frac{\pi}{4}\right)^2 + R^2}} |\tan(x + iR)| e^{-2kRx} dx.
\end{aligned}$$

We now wish to find an upper bound on $\tan(x+iR)$. Well,

$$\begin{aligned}|\tan(x+iR)| &= \left| -i\frac{e^{ik}e^{-R}-e^{-ix}e^{R}}{e^{ix}e^{-R}+e^{-ix}e^{R}} \right| \\ &\leq \frac{e^R+e^{-R}}{e^R-e^{-R}}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{C_2}\right| &\leq \frac{e^R+e^{-R}}{e^R-e^{-R}}\int_R^{\sqrt{\left(\frac{\pi}{4}\right)^2+R^2}} e^{-2kRx}dx \\ &= -\frac{1}{2kR}\left(\frac{e^R+e^{-R}}{e^R-e^{-R}}\right)\left[e^{-2kRx}\right]\Big|_R^{\sqrt{\left(\frac{\pi}{4}\right)^2+R^2}} \\ &\to 0 \quad \text{as} \quad R\to\infty.\end{aligned}$$

Finally, on C_3, parameterize $x=\sqrt{\left(\frac{\pi}{4}\right)^2+y^2}$. Then $z^2=\left(\frac{\pi}{4}\right)^2+2iy\sqrt{\left(\frac{\pi}{4}\right)^2+y^2}$. Let $s=2iy\sqrt{\left(\frac{\pi}{4}\right)^2+y^2}$. Then

$$\begin{aligned}z^2 &= \left(\frac{\pi}{4}\right)^2+is \\ \implies z &= \sqrt{\left(\frac{\pi}{4}\right)^2+is} \\ \implies dz &= \frac{i}{2\sqrt{\left(\frac{\pi}{4}\right)^2+is}}ds.\end{aligned}$$

Further, $\tan(z)=1+2\left(z-\frac{\pi}{4}\right)+\mathcal{O}\left(\left(z-\frac{\pi}{4}\right)^2\right)$ and so $\tan(z)\sim 1$ as $z\to\frac{\pi}{4}$. Using this, I_3 becomes

$$\begin{aligned}I_3 &\sim -\frac{i}{2}\int_0^\infty \left(\left(\frac{\pi}{4}\right)^2+is\right)^{-1/2} e^{ik\left(\left(\frac{\pi}{4}\right)^2+is\right)}ds \\ &= -\frac{2i}{\pi}e^{ik(\pi/4)^2}\int_0^\infty \left(1+i\left(\frac{4}{\pi}\right)^2 s\right)^{-1/2} e^{-2ks}ds\end{aligned}$$

And near $s=0$, the binomial expansion gives

$$\begin{aligned}\left(1+i\left(\frac{4}{\pi}\right)^2 s\right)^{-1/2} &= \sum_{n=0}^{\infty}\binom{-1/2}{n}\left[i\left(\frac{4}{\pi}\right)^2 s\right]^k \\ &\sim 1 \quad \text{as} \quad s\to 0.\end{aligned}$$

Thus

$$\begin{aligned}I_3 &\sim -\frac{2i}{\pi}e^{ik(\pi/4)^2}\int_0^\infty e^{-ks}ds \\ &= -\frac{2i}{\pi k}e^{ik(\pi/4)^2}\end{aligned}$$

as desired.

6.4.5 Consider the integral

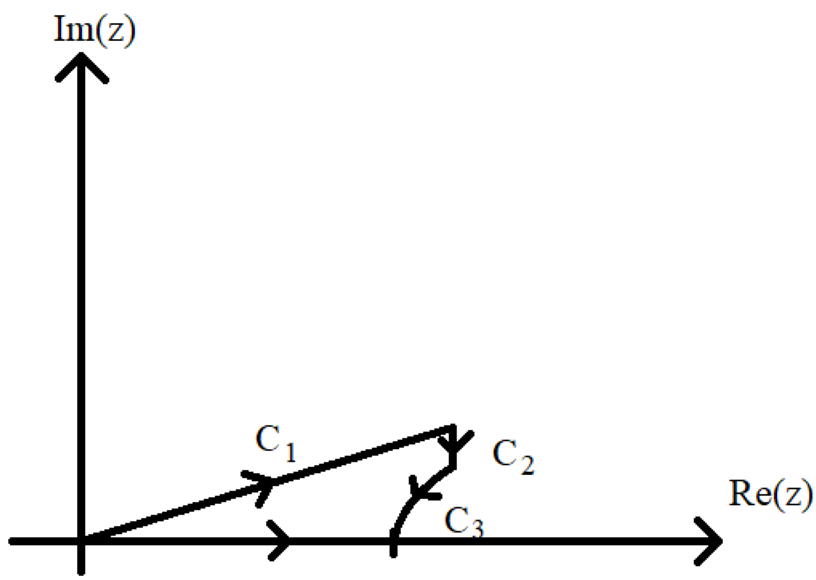
$$I(k) = \int_0^1 e^{ikt^3} dt \quad \text{as} \quad k \to \infty$$

Show that $I(k) = I_1 + I_2 + I_3$ where as $k \to \infty$:

$$I_1 \sim \frac{e^{i\pi/6}}{3k^{1/3}} \Gamma\left(\frac{1}{3}\right),$$
$$I_2 \sim 0,$$
$$I_3 \sim -\frac{ie^{ik}}{3k} - \frac{2e^{ik}}{9k^2}$$

and the contour associated with I_1 is the steepest descent contour $y = x/\sqrt{3}$ passing through $z = 0$; the contour I_3 is the steepest descent contour $x^3 - 3xy^2 = 1$ passing through $z = 1$; and the contour associated with I_2 is parallel to the $x-$axis and is at a large distance from the origin.

Proof: We consider I_2 to be along the path connecting C_1 and C_3 parallel to the imaginary axis.



On C_1, parameterize $z = re^{i\pi/6}$. Then

$$\begin{aligned}
\int_{C_1} &= \int_0^{\sqrt{2}/3R} e^{-kr^3} e^{i\pi/6} dr \\
&= e^{i\pi/6} \int_0^{\sqrt{2}/3R} e^{-kr^3} dr \\
&\to e^{i\pi/6} \int_0^{\infty} e^{-kr^3} dr \\
&= \frac{e^{i\pi/6}}{3k^{1/3}} \Gamma\left(\frac{1}{3}\right).
\end{aligned}$$

On I_2, parameterize $z = R + iy$. Then the lower bound on y is given as $y = \sqrt{\frac{R^3-1}{3R}}$. Then

$$\begin{aligned}
I_2 &= i\int_R^{\sqrt{\frac{R^3-1}{3R}}} e^{ik(R+iy)^3}\,dy \\
\implies |I_2| &\leq \int_{\sqrt{\frac{R^3-1}{3R}}}^{R} e^{-(3R^2y-y^3)}dy \\
&\leq e^{R^3}\int_{\sqrt{\frac{R^3-1}{3R}}}^{R} e^{-3R^2y}dy \\
&= -\frac{e^{R^3}}{3R^2}\left[e^{-3R^2y}\right]\Big|_{\sqrt{\frac{R^3-1}{3R}}}^{R} \\
&= -\frac{1}{3R^2}e^{R^3}\left(e^{-3R^3} - e^{-3R^2\sqrt{\frac{R^3-1}{3R}}}\right) \\
&\to 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Now, let $\phi(z) = iz^3 = i(x+iy)^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3) = -(3x^2y - y^3) + i(x^3 - 3xy^2)$. At $z = 1$, $\text{Im}(\phi(1)) = 1$, so the steepest descent curve at $z = 1$ is given by $x^3 - 3xy^2 = 1$. Solving for y yields $y = \sqrt{\frac{x^3-1}{3x}}$. Let $x = 1 + \varepsilon$. For $\varepsilon \ll 1$, notice

$$\begin{aligned}
y &= \frac{1}{\sqrt{3}}\left(\frac{(1+\varepsilon)^3 - 1}{1+\varepsilon}\right)^{1/2} \\
&= \frac{1}{\sqrt{3}}\left(\frac{3\varepsilon + 3\varepsilon^2 + \varepsilon^3}{1+\varepsilon}\right)^{1/2} \\
&= \frac{1}{\sqrt{3}}\left(\frac{3\varepsilon(1+\varepsilon)}{1+\varepsilon} + \frac{\varepsilon^3}{1+\varepsilon}\right)^{1/2} \\
&= \sqrt{\varepsilon}\left(1 + \frac{\varepsilon^2}{3(1+\varepsilon)}\right)^{1/2} \\
&= \sqrt{\varepsilon}\left(1 + \frac{\varepsilon^2}{3}(1 - \varepsilon + \varepsilon^2 - \mathcal{O}(\varepsilon^3))\right)^{1/2} \\
&= \sqrt{\varepsilon}\left(1 - \frac{\varepsilon^2}{6} + \mathcal{O}(\varepsilon^3)\right) \\
\implies y &\sim \sqrt{\varepsilon} \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Further, expanding $\phi(z)$ around $z = 1$ yields

$$\phi(z) = i\left(1 + 3(z-1) + 3(z-1)^2 + \frac{1}{4}(z-1)^3\right)$$

parameterize $z = 1 + \varepsilon + i\sqrt{\varepsilon}$, $dz = \left(1 + \frac{i}{2\varepsilon}\right)d\varepsilon$. Using this parameterization for the expansion of $\phi(z)$ above, we find

$$\begin{aligned}
\phi(z) &= i\left(1 + 3(\varepsilon + i\sqrt{\varepsilon}) + 3(1 + i\sqrt{\varepsilon})^2 + \frac{1}{4}(\varepsilon + i\sqrt{\varepsilon})^3\right) \\
&= i\left(1 + 3\varepsilon + 3i\sqrt{\varepsilon} + 3\varepsilon^2 + 6i\varepsilon\sqrt{\varepsilon} - 3\varepsilon + \frac{1}{4}\varepsilon^3 + \frac{3}{4}i\varepsilon^2\sqrt{\varepsilon} - \frac{3}{4}\varepsilon^2 - i\varepsilon^{3/2}\right) \\
&= i\left(1 + 3i\sqrt{\varepsilon} + 3\varepsilon^2 + 6i\varepsilon\sqrt{\varepsilon} + \frac{1}{4}\varepsilon^3 + \frac{3}{4}i\varepsilon^2\sqrt{\varepsilon} - \frac{3}{4}\varepsilon^2 - i\varepsilon^{3/2}\right) \\
&\sim i\left(1 + 3i\sqrt{\varepsilon}\right) \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Thus, using these approximations, we have

$$\begin{aligned} I_3 &\sim -\int_0^\infty e^{ik(1+3i\sqrt{\varepsilon})}\left(1+\frac{i}{2\sqrt{\varepsilon}}\right)d\varepsilon \\ &= -e^{ik}\int_0^\infty e^{-3k\sqrt{\varepsilon}}\left(1+\frac{i}{2\sqrt{\varepsilon}}\right)d\varepsilon \\ &= -e^{ik}\left(\int_0^\infty e^{-3k\sqrt{\varepsilon}}d\varepsilon + i\int_0^\infty \frac{1}{2\sqrt{\varepsilon}}e^{-3\sqrt{\varepsilon}}d\varepsilon\right) \end{aligned}$$

letting $u = 3k\sqrt{\varepsilon}$, $du = \frac{3k}{2\sqrt{\varepsilon}}d\varepsilon$ yields

$$\begin{aligned} \implies I_3 &\sim -e^{ik}\left(\frac{2}{9k^2}\int_0^\infty ue^{-u}du + \frac{i}{3k}\int_0^\infty e^{-u}du\right) \\ &= -e^{ik}\left(\frac{2}{9k^2}+\frac{i}{3k}\right) \\ &= -\frac{2e^{ik}}{9k^2} - \frac{ie^{ik}}{3k} \end{aligned}$$

as desired.