Optimization HW 4

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Section 2.2 Problems

2. Solve the following linear programs using the simplex method. If the problem is two dimensional, graph the feasible region, and outline the progess of the algorithm.

(i)

$$\begin{aligned} \text{minimize} \quad & z = -5x_1 - 7x_2 - 12x_3 + x_4 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 2x_3 + x_4 \leq 38 \\ & 3x_1 + 2x_2 + 4x_3 - x_4 \leq 55 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Adding the slack variables $x_5, x_6 \geq 0$ to the first and second constraints, respectively, we have the problem in standard form with

$$A = \begin{pmatrix} 2 & 3 & 2 & 1 & 1 & 0 \\ 3 & 2 & 4 & -1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 38 \\ 55 \end{pmatrix} c = \begin{pmatrix} -5 \\ -7 \\ -12 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Choosing the initial basis corresponding to (x_5, x_6) and running a script that implements the simplex method, we find the following:

```
Iteration number: 0

B =

     1     0
     0     1

N =

     2     3     2     1
     3     2     4    -1

cb =

     0
     0

cn =

    -5
    -7
   -12
     1

y =

     0
     0
```

```
Iteration Number: 1

Entering Variable: x3
Leaving Variable: x6

B =

     1     2
     0     4

N =

     2     3     0     1
     3     2     1    -1

cb =

     0
   -12

cn =

     4
    -1
     3
    -2

y =

     0
    -3
```

```
Iteration Number: 2

Entering Variable: x4
Leaving Variable: x5

B =

     1     2
    -1     4

N =

     2     3     0     1
     3     2     1     0

cb =

     1
   -12

cn =

    4.6667
    1.6667
    2.3333
    1.3333

y =

   -1.3333
   -2.3333
```

```
funcVal =

  -179
```

From the above, we see that the minimal value is $z = -179$.

(ii)

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & 4x_1 + 5x_2 + 2x_3 + x_4 \leq 20 \\ & 3x_1 + 4x_2 - x_3 + x_4 \leq 30 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

Converting the problem to a minimization problem and adding the slack variables $x_5, x_6 \geq 0$ to the first and second constraints, respectively, we have the problem in standard form with

$$A = \begin{pmatrix} 4 & 5 & 2 & 1 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 20 \\ 30 \end{pmatrix} \quad c = \begin{pmatrix} -5 \\ -3 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

Choosing the initial basis corresponding to (x_5, x_6) and running the simplex script, we find

```
Iteration number: 0

B =

     1     0
     0     1

N =

     4     5     2     1
     3     4    -1     1

cb =

     0
     0

cn =

    -5
    -3
    -2
     0

y =

     0
     0
```

```
Iteration Number: 1

Entering Variable: x1
Leaving Variable: x5

B =

     4     0
     3     1

N =

     1     5     2     1
     0     4    -1     1

cb =

    -5
     0

cn =

    1.2500
    3.2500
    0.5000
    1.2500

y =

   -1.2500
         0
```

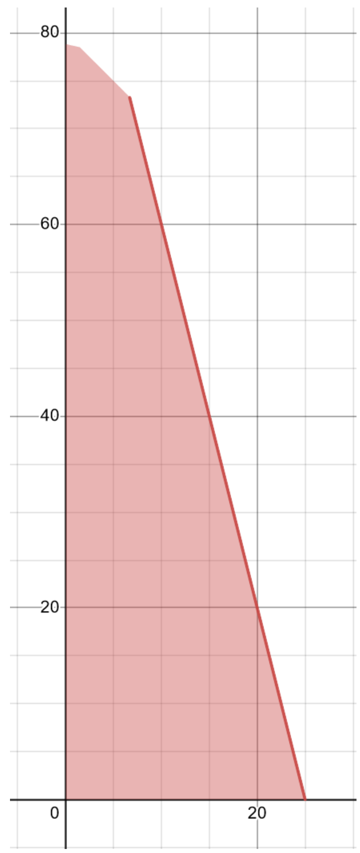
```
funcVal =

   -25
```

From the above, we can see the minimal value is $-z = -25$, and so the maximal value is $z = 25$.

(v)

$$
\begin{aligned}
\text{maximize} \quad & z = 7x_1 + 8x_2 \\
\text{subject to} \quad & 4x_1 + x_2 \leq 100 \\
& x_1 + x_2 \leq 80 \\
& x_1 \leq 40 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Since this problem is two dimensional, we may plot the feasible region:



Converting the problem to a minimization problem and adding the slack variables $x_3, x_4, x_5 \geq 0$ to the first, second, and third constraints, respectively, we have the problem in standard form with

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 100 \\ 80 \\ 40 \end{pmatrix} \quad c = \begin{pmatrix} -7 \\ -8 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Choosing the initial basis corresponding to (x_4, x_5, x_6) and running the simplex script, we find

```
Iteration number: 0


B =

     1     0     0
     0     1     0
     0     0     1


N =

     4     1
     1     1
     1     0


cb =

     0
     0
     0


cn =

    -7
    -8


y =

     0
     0
     0
```

```
Iteration Number: 1

Entering Variable: x2
Leaving Variable: x4


B =

     1     1     0
     0     1     0
     0     0     1


N =

     4     0
     1     1
     1     0


cb =

     0
    -8
     0


cn =

     1
     8


y =

     0
    -8
     0
```

```
funcVal =

  -640
```

From the above, we can see the minimal value is $z = -640$, and so the maximal value is $z = 640$.

3. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = x_1 - x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Derive an expression for the set of optimal solutions to this problem, and show that this set is unbounded.

To begin, introduce slack variables $x_3, x_4 \geq 0$ so that the problem is in standard form with

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

From the system above, performing Gaussian elimination on $(A|b)$ (adding row 1 to row 2 and putting it back into row 2), we find

$$\left(\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 & 3 \end{array}\right)$$

From this, we see that

$$\begin{aligned} x_2 &= x_3 + x_4 - 3 \\ x_1 &= 2x_3 + x_4 - 4 \end{aligned}$$

Then our solution set takes the following form:

$$x = \begin{pmatrix} -4 \\ -3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Notice that $(x_1, x_2)^T = (-4, -3)$ lies on the first constraint, so we may "shift" the point to $(0, 1)$ so that our solution set takes the form

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$x_3, x_4 \geq 0$$

. Let $d_1 = (2, 1, 1, 0)^T$ and $d_2 = (1, 1, 0, 1)^T$ and consider Ad_1 and Ad_2:

$$\begin{aligned} Ad_1 &= \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} Ad_2 &= \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we have that both d_1 and d_2 are directions of unboundedness, meaning that this problem is unbounded.

11. Solve the linear programs in Exercise 2.2 using the tableau.

See attached work.