Nonlinear Waves Problem 2.1

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2.1 Solve the following equations using Fourier transforms.

(a) $u_t + u_{5x} = 0$, $u(x,0) = f(x)$.

(b) $u_t + \int K(x-\xi)u(\xi,t)d\xi = 0$, $u(x,0) = f(x)$, $\hat{K}(k) = e^{-k^2}$.

(c) $u_{tt} + u_{4x} = 0$, $u(x,0) = f(x)$. $u_t(x,0) = g(x)$.

(d) $u_{tt} - c^2 u_{xx} - m^2 u(x,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Contrast this solution with that of the standard Klein-Gordon equation.

Soln. Throughout, let $\hat{f}(k) = \mathcal{F}(f(x))$, $\hat{g}(k) = \mathcal{F}(g(x))$, and $\hat{u}(k,t) = \mathcal{F}(u(x,t))$. We also use Leibniz's Integral rule which allows us to interchange the temporal derivative with the integral.

(a) Taking the Fourier transform of the differential equation yields

$$\begin{aligned} \mathcal{F}(u_t) + \mathcal{F}(u_{5x}) &= 0 \\ \frac{\partial}{\partial t}\hat{u} + (ik)^5\hat{u} &= 0 \\ \frac{\partial \hat{u}}{\partial t} + ik^5\hat{u} &= 0. \end{aligned}$$

Note that this is an ODE in t, and separating variables gives

$$\begin{aligned} \int \frac{\partial \hat{u}}{\hat{u}} &= -ik^5 \int dt \\ \implies \log(\hat{u}) &= -ik^5 t + C_0(k) \\ \implies \hat{u}(k,t) &= C_1(k)e^{-ik^5 t}. \end{aligned}$$

Where $C_0(k)$ is a function of integration independent of t. Setting $t = 0$ in the above equality yields

$$\begin{aligned} \hat{u}(k,0) &= C_1(k) \\ \implies \hat{f}(k) &= C_1(k). \end{aligned}$$

Thus

$$\begin{aligned} \hat{u}(k,t) &= \hat{f}(k)e^{-ik^5 t} \\ \implies u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{i(kx - k^5 t)}dk. \end{aligned}$$

(b) Taking the Fourier transform of the differential equation and invoking the convolution theorem gives

$$\begin{aligned} \mathcal{F}(u_t) + \mathcal{F}\left(\int_{-\infty}^{\infty} K(x-\xi)u(\xi,t)d\xi\right) &= 0 \\ \frac{\partial \hat{u}}{\partial t} + \hat{K}(k)\hat{u}(k) &= 0 \\ \frac{\partial \hat{u}}{\partial t} + e^{-k^2}\hat{u}(k) &= 0. \end{aligned}$$

As in part (a), separation of variables gives

$$\int \frac{d\hat{u}}{\hat{u}} = -\int e^{-k^2} dt$$
$$\implies \log(\hat{u}) = -e^{-k^2}t + C_0(k)$$
$$\implies \hat{u}(k,t) = C_1(k)e^{-te^{-k^2}}$$

and setting $t = 0$ yields

$$\hat{u}(k,0) = C_1(k)$$
$$\implies \hat{f}(k) = C_1(k)$$

so that

$$\hat{u}(k,t) = \hat{f}(k)e^{-te^{-k^2}}$$
$$\implies u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx-te^{-k^2}}\,dk.$$

(c) Taking the Fourier transform of the differential equation yields

$$\mathcal{F}(u_{tt}) + \mathcal{F}(u_{xx}) = 0$$
$$\frac{\partial^2 \hat{u}}{\partial t^2} + (ik)^4\hat{u} = 0$$
$$\implies \frac{\partial^2 \hat{u}}{\partial t^2} + k^4\hat{u} = 0.$$

From here, we seek solutions of the form

$$\hat{u}(k,t) = h(k)e^{rt}$$

so that, putting this ansatz into the above differential equation gives

$$r^2h(k)e^{rt} + k^4h(k)e^{rt} = 0$$
$$\implies r = \pm ik^2.$$

Then

$$\hat{u}(k,t) = h_1(k)e^{ik^2t} + h_2(k)e^{-ik^2t}$$
$$\implies \hat{u}_t(k,t) = ik^2h_1(k)e^{ik^2t} - ik^2h_2(k)e^{-ik^2t}.$$

From our initial conditions, we find

$$h_1(k) + h_2(k) = \hat{f}(k)$$
$$h_1(k) - h_2(k) = -\frac{i}{k^2}\hat{g}(k)$$
$$\implies h_1(k) = \frac{1}{2}\left(\hat{f}(k) - \frac{i}{k^2}\hat{g}(k)\right)$$
$$h_2(k) = \frac{1}{2}\left(\hat{f}(k) + \frac{i}{k^2}\hat{g}(k)\right).$$

Thus

$$\hat{u}(k,t) = \frac{1}{2}\left[\left(\hat{f}(k) - \frac{i}{k^2}\hat{g}(k)\right)e^{ik^2t} + \left(\hat{f}(k) + \frac{i}{k^2}\hat{g}(k)\right)e^{-ik^2t}\right]$$

and taking the inverse Fourier transform gives us the solution to the PDE:

$$u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\hat{f}(k) - \frac{i}{k^2}\hat{g}(k)\right)e^{i(kx+k^2t)}dk + \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\hat{f}(k) + \frac{i}{k^2}\hat{g}(k)\right)e^{i(kx-k^2t)}dk$$

(d) Taking the Fourier transform of the PDE gives

$$\begin{aligned}
\mathcal{F}(u_{tt}) - c^2\mathcal{F}(u_{xx}) - m^2\mathcal{F}(u) &= 0 \\
\frac{\partial^2 \hat{u}}{\partial t^2} + c^2k^2\hat{u} - m^2\hat{u} &= 0 \\
\implies \frac{\partial^2 \hat{u}}{\partial t^2} + (c^2k^2 - m^2)\hat{u} &= 0.
\end{aligned}$$

As in part (c), consider the ansatz $\hat{u}(k,t) = h(k)e^{rt}$. Plugging this ansatz into our above differential equation yields

$$\begin{aligned}
h(k)r^2e^{rt} + (c^2k^2 - m^2)h(k)e^{rt} &= 0 \\
\implies r &= \pm i\sqrt{c^2k^2 - m^2}.
\end{aligned}$$

Let $\omega(k) = \sqrt{c^2k^2 - m^2}$. Then The general solution to our ODE in $\hat{u}$ is

$$\hat{u}(k,t) = h_1(k)e^{i\omega(k)t} + h_2(k)e^{-i\omega(k)t}.$$

From our initial conditions, we have

$$\begin{aligned}
h_1(k) + h_2(k) &= \hat{f}(k) \\
i\omega(k)h_1(k) - i\omega(k)h_2(k) &= \hat{g}(k) \\
\implies h_1(k) &= \frac{1}{2}\left(\hat{f}(k) - \frac{i}{\omega(k)}\hat{g}(k)\right) \\
h_2(k) &= \frac{1}{2}\left(\hat{f}(k) + \frac{i}{\omega(k)}\hat{g}(k)\right).
\end{aligned}$$

Thus

$$\hat{u}(k,t) = \frac{1}{2}\left(\hat{f}(k) - \frac{i}{\omega(k)}\hat{g}(k)\right)e^{i\omega(k)t} + \frac{1}{2}\left(\hat{f}(k) + \frac{i}{\omega(k)}\hat{g}(k)\right)e^{-i\omega(k)t}.$$

Taking the inverse transform yields

$$u(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\hat{f}(k) - \frac{i}{\omega(k)}\hat{g}(k)\right)e^{i(kx+\omega(k)t)}dk + \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\hat{f}(k) + \frac{i}{\omega(k)}\hat{g}(k)\right)e^{i(kx-\omega(k)t)}dk$$

with $\omega(k) = \sqrt{c^2k^2 - m^2}$. Note that if $c^2k^2 \geq m^2$, $\omega(k) \in \mathbb{R}$, and for $c^2k^2 < m^2$, $\omega(k) \in \mathbb{C}$. In the latter case, we have exponential behavior. Note that this differs from the Klein-Gordon equation since $\omega(k) \in \mathbb{R}$ for all cases.