Modern Algebra HW5

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Section 11 Problems

16. Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

Yes, $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_3 \times \mathbb{Z}_6$ are isomorphic. To see this, notice that $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ and that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. Then we have

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$$

and

$$\begin{aligned} \mathbb{Z}_4 \times \mathbb{Z}_6 &\cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \\ &= \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \end{aligned}$$

So we have

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_6$$

24. Find all abelian groups up to isomorphism, of order 720.

Notice that the prime factorization of 720 is $720 = 2^4 3^2 5$ and so all abelian groups up to isomorphism of order 720 are given by the following:

$$\begin{aligned}
&\mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \\
&\mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5
\end{aligned}$$

Section 13 Problems

6. Determine whether the following is a homomorphism: $\phi : \mathbb{R} \to \mathbb{R}^*$, where $\mathbb{R}$ is additive and $\mathbb{R}^*$ is multiplicative, be given by $\phi(x) = 2^x$.

I claim that ϕ is a homomorphism.
Proof: Let $x, y \in \mathbb{R}$ and consider $\phi(xy)$:

$$\begin{aligned}\phi(x+y) &= 2^{x+y} \\ &= 2^x 2^y \\ &= \phi(x)\phi(y)\end{aligned}$$

So ϕ is a homomorphism.

8. Let G be any group and let $\phi : G \to G$ be given by $\phi(g) = g^{-1}$ for $g \in G$.

I claim that ϕ is a homomorphism if and only if G is abelian.
Proof: Begin by supposing G is abelian and let $x, y \in G$ and consider $\phi(xy)$:

$$\begin{aligned}\phi(xy) &= (xy)^{-1} \\ &= y^{-1}x^{-1} \\ &= x^{-1}y^{-1} \\ &= \phi(x)\phi(y)\end{aligned}$$

Now suppose ϕ is a homomorphism. We wish to show G is abelian. Well, since ϕ is a homomorphism, we have $\phi(xy) = \phi(x)\phi(y)$. That is,

$$\begin{aligned}\phi(xy) &= x^{-1}y^{-1} \\ &= (xy)^{-1} \\ &= y^{-1}x^{-1}\end{aligned}$$

So we have

$$y^{-1}x^{-1} = x^{-1}y^{-1}$$

which holds so long as G is Abelian. So ϕ is a homomorphism if and only if G is abelian.

29. Prove that for G a group, $g \in G$, define $\phi_g : G \to G$ be defined by $\phi_g(x) = gxg^{-1}$ for $x \in G$.
Proof: Let G be a group, $g \in G$ and $\phi_g : G \to G$ defined by $\phi_g(x) = gxg^{-1}$. Let $x, y \in G$, $e \in G$ be the identity element and consider $\phi(xy)$:

$$\begin{aligned}\phi_g(xy) &= gxyg^{-1} \\ &= gxeyg^{-1} \\ &= gxg^{-1}gyg^{-1} \\ &= (gxg^{-1})(gyg^{-1}) \\ &= \phi_g(x)\phi_g(y)\end{aligned}$$

So ϕ_g is a homomorphism.

47. Show that any group homomorphism $\phi : G \to G'$ where $|G|$ is prime must be either the trivial homomorphism or a one-to-one map.

Proof: Let $\phi : G \to G'$ be a group homomorphism where $|G|$ is prime. Let us begin by inspecting $\text{Ker}(\phi)$. Since ϕ is a group homomorphism, we have that $\text{Ker}(\phi)$ is a normal subgroup of G. Then by Lagrange's theorem, we have $|\text{Ker}(\phi)|$ divides $|G|$. Then since $|G|$ is prime, either $|\text{Ker}(\phi)| = 1$ or $|\text{Ker}(\phi)| = |G|$. Let us inspect each of these cases:

Case 1: $|\text{Ker}(\phi)| = 1$
Then we must have $\text{Ker}(\phi) = \{e\}$ and so is one-to-one by corollary 13.18.

Case 2: $|\text{Ker}| = |G|$
Then for all $g \in G$, $\phi(g) = e'$ where $e' \in G'$ is the identity. So by definition, ϕ is the trivial homomorphism.

Additional Problem: Let ϕ be a homomorphism from G to G'. Prove that $\text{Ker}(\phi)$ is a subgroup of G.

Proof: Let ϕ be a group homomorphism from G to G'. We wish to show that $\text{Ker}(\phi)$ is a subgroup of G. To begin, let $x, y \in \text{Ker}(\phi)$ and consider xy. Since $x, y \in \text{Ker}(\phi)$, we have $\phi(x) = \phi(y) = e'$ where e' is the identity of G'. Now, since ϕ is a homomorphism, we have

$$\begin{aligned} \phi(xy) &= \phi(x)\phi(y) \\ &= e'e' \\ &= e' \end{aligned}$$

So we have $xy \in \text{Ker}(\phi)$. Now we must show that $e \in \text{Ker}(\phi)$. Well, for any $g \in G$,

$$g = ge$$

so

$$\begin{aligned} \phi(g) &= \phi(ge) \\ &= \phi(g)\phi(e) \end{aligned}$$

then it must be the case that $\phi(e) = e'$, so $e \in \text{Ker}(\phi)$. Finally, we must show for any $g \in \text{Ker}(\phi)$, $g^{-1} \in \text{Ker}(\phi)$. Well,

$$\begin{aligned} \phi(g^{-1}) &= (\phi(g))^{-1} \\ &= e'^{-1} \\ &= e' \end{aligned}$$

So $g^{-1} \in \text{Ker}(\phi)$. So by the subgroup theorem, $\text{Ker}(\phi)$ is a subgroup of G.