Homework 2

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Section 1.5 Problems

31. Discuss the equation $\dot{x} = x^2 - t$.

Discussion: The equation $\dot{x} = x^2 - t$ is a Riccati type equation. Similar to the example in the text, we have that $f(x,t) = x^2 - t \in C^1(\mathbb{R}^2, \mathbb{R})$ and so a unique solution exists locally near (x_0, t_0). Notice that the equation has nullclines whenever $x(t) = \pm\sqrt{t}$. Note that

$$\begin{aligned} f(x,t) > 0 \quad &\text{when} \quad x(t) > \sqrt{t} \\ f(x,t) < 0 \quad &\text{when} \quad -\sqrt{t} < x(t) < \sqrt{t} \\ f(x,t) > 0 \quad &\text{when} \quad x(t) < -\sqrt{t} \end{aligned}$$

Thus, $x(t)$ can move from region I to region II, or move from region III to region II, but once in region II, will remain in region II, as can be seen in the following figure:

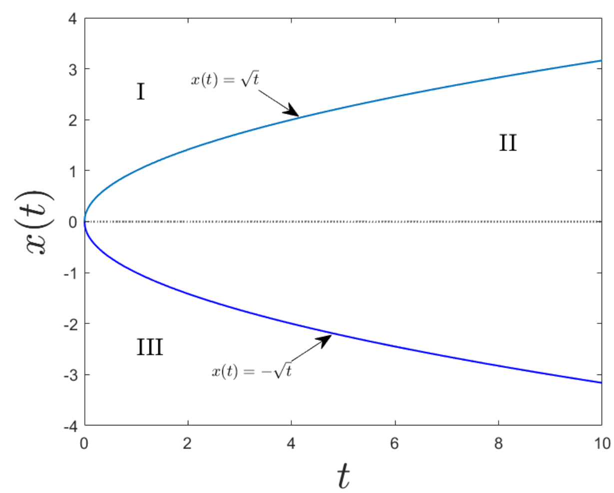

Figure 1: Region splitting :)

Note that as $t \to \infty$, we have that if $x(t)$ is in region I, $x(t)$ will either diverge to $+\infty$ or enter into region II. If $X(t)$ is in region II, then $x(t)$ will diverge to $-\infty$, but cannot diverge in finite time since $x(t)$ is bounded below by $-\sqrt{t}$. Finally, if $x(t)$ is in region III, then $x(t)$ will eventually cross over into region II and diverge to $-\infty$.

Section 2.1 Problems

2. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \to f$, $g_n \to g$, and $\alpha_n \to \alpha$, then $\|f_n\| \to \|f\|$, $f_n + g_n \to f + g$, and $\alpha_n f_n \to \alpha f$.

Proof: To begin, fix $\varepsilon > 0$. We will begin by showing $\|f_n\| \to \|f\|$. By definition of convergence in a normed space, we have for some natural number N_1, whenever $n > N_1$,

$$\|f_n - f\| < \varepsilon$$

but by the reverse triangle inequality (see previous submission), we have

$$|\|f_n\| - \|f\|| \leq \|f_n - f\|$$

hence

$$|\|f_n\| - \|f\|| < \varepsilon$$

so that $\|f_n\| \to \|f\|$. Now, for some other natural number N_2, whenever $n > N_2$, we have

$$\|f_n - f\| < \frac{\varepsilon}{2} \qquad \|g_n - g\| < \frac{\varepsilon}{2}$$

$$\begin{aligned}
\|(f_n + g_n) - (f - g)\| &= \|(f_n - f) + (g_n - g)\| \\
&\leq \|f_n - f\| + \|g_n - g\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, $f_n + g_n \to f + g$. Now consider

$$\begin{aligned}
\|\alpha_n f_n - \alpha f\| &= \|\alpha_n f_n - \alpha_n f + \alpha_n f - \alpha f\| \\
&\leq \|\alpha_n f_n - \alpha_n f\| + \|\alpha_n f - \alpha f\| \\
&= |\alpha_n|\|f_n - f\| + |\alpha_n - \alpha|\|f\| && \text{(Homogeneity of the norm)}
\end{aligned}$$

and since $\alpha_n \to \alpha$, $\{\alpha_n\}$ is a bounded sequence. That is, there exists some $M > 0$ such that

$$|\alpha_n| \leq M$$

for all n. Now, for some $N_3 \in \mathbb{N}$, we have that, whenever $n > N_3$[†],

$$\|f_n - f\| < \frac{\varepsilon}{2M} \qquad |\alpha_n - \alpha| < \frac{\varepsilon}{2\|f\|}$$

so that, from our work above, we have

$$\begin{aligned}
|\alpha_n|\|f_n - f\| + |\alpha_n - \alpha|\|f\| &< M\frac{\varepsilon}{2M} + \|f\|\frac{\varepsilon}{2\|f\|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence, $\alpha_n f \to \alpha f$.

Section 2.2 Problems

6. Are the following functions Lipschitz continuous near 0? If yes, find a Lipschitz constant for some interval containing 0.

(i) $f(x) = \frac{1}{1-x^2}$.

For any closed interval $[a, b]$ $(a > -1, b < 1)$ containing 0, since $f(x) \in C^1[a, b]$, using Taylor's theorem, we have

$$|f(x) - f(x_0)| = |f'(\xi)||x - x_0|$$

[†] If $\|f\| = 0$, we simply use $\|f_n - f\| < \frac{\varepsilon}{2M}$ since $\|f\||\alpha_n - \alpha| = 0$.

for $x, x_0 \in [a,b]$ and $\xi \in [x, x_0]$. Then since

$$f'(\xi) = \frac{2\xi}{(1-\xi^2)^2}$$

and since f' is monotonically increasing on, we have

$$|f'(\xi)| \leq \frac{2b}{(1-b^2)^2}$$

so that

$$|f(x) - f(x_0)| \leq \frac{2b}{(1-b^2)^2}(b-a).$$

Alternatively, on any compact interval $[a,b] \subset (-1,1)$, since $f \in C^1[a,b]$, f is Lipschitz over $[a,b]$.

(ii) $f(x) = |x|^{1/2}$.

I claim f is not Lipschitz near zero. To see this, take $[a,b]$ an interval that contains zero. If f is Lipschitz, then there exists some constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y|$$

In particular, take $y = 0$. Then we have

$$\begin{aligned} \sqrt{|x|} &\leq K|x| \\ \frac{1}{\sqrt{|x|}} &\leq K \end{aligned}$$

but as $x \to 0$, $\frac{1}{\sqrt{|x|}} \to \infty$, so K is unbounded. Hence f is not Lipschitz near zero.

(iii) $f(x) = x^2 \sin(\frac{1}{x})$.

(Can we assume that $f(0) = 0$ so the discontinuity is removed?) I claim $f(x)$ is globally Lipschitz. By the mean value theorem, for $x, x_0 \in \mathbb{R}$, we have that there exists some $\xi \in (x, x_0)$ such that

$$|f(x) - f(x_0)| = |f'(\xi)||x - x_0|$$

and

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

and so

$$\begin{aligned} |f'(x)| &= \left|2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right| \\ &\leq 2\left|x \sin\left(\frac{1}{x}\right)\right| + \left|\cos\left(\frac{1}{x}\right)\right| \\ &\leq 2\left|x \sin\left(\frac{1}{x}\right)\right| + 1 \\ &\leq 2 + 1 \\ &= 3 \end{aligned}$$

Hence, $|f'(x)| \leq 3$ for all $x \in \mathbb{R}$. Then

$$|f(x) - f(x_0)| \leq 3|x - x_0|$$

so that f is globally Lipschitz with Lipschitz constant 3.

Note: Since $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$ by the squeeze theorem, and

$$\begin{aligned}\lim_{x\to\pm\infty} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to\pm\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\ &= 1.\end{aligned}$$

8. Apply the Picard iteration to the first-order equation

$$\dot{x} = 2t - 2\sqrt{\max(0,x)}, \;\; x(0) = 0.$$

Does it converge?

Soln. Applying the Picard iteration to the above equation, we have $x_0(t) = 0$ and

$$\begin{aligned}x_1(t) &= 0 + \int_0^t (2s - 2\sqrt{\max(0,0)})ds \\ &= \int_0^t 2sds \\ &= t^2\end{aligned}$$

and

$$\begin{aligned}x_2(t) &= \int_0^t (2s - 2\sqrt{\max(0,s^2})ds \\ &= \int_0^t (2s-2s)ds \\ &= \int_0^t 0ds \\ &= 0.\end{aligned}$$

We now fall into a cycle. It is clear from here that our n^{th} Picard iteration will have the following form:

$$x_n(t) = \begin{cases} 0 & n \equiv 0 \mod 2 \\ t^2 & n \equiv 1 \mod 2\end{cases}$$

So that it does not converge.

Section 2.4 Problems

12. Show (2.38). (Hint: Introduce $\tilde{\psi}(t) = \psi(t) + \frac{\gamma}{\beta}$.)

Proof: Suppose

$$\psi(t) \le \alpha + \int_0^t (\beta\psi + \gamma)ds.$$

Let $\tilde{\psi}(t) = \psi(t) + \frac{\gamma}{\beta}$. Then the above inequality becomes

$$\begin{aligned}\tilde{\psi}(t) - \frac{\gamma}{\beta} &\le \alpha + \int_0^t \beta\tilde{\psi}(s)ds \\ \tilde{\psi}(t) &\le \alpha + \frac{\gamma}{\beta} + \int_0^t \beta\tilde{\psi}(s)ds\end{aligned}$$

Since $\alpha + \frac{\gamma}{\beta}$ is just a constant, call it $\tilde{\alpha}$. Then the above inequality becomes

$$\tilde{\psi}(t) \leq \tilde{\alpha} + \int_0^t \beta \tilde{\psi}(s) ds.$$

Then by (2.36), we have

$$\tilde{\psi}(t) \leq \tilde{\alpha} e^{\beta t}$$

which, using $\tilde{\alpha} = \alpha + \frac{\gamma}{\beta}$ and $\tilde{\psi} = \psi + \frac{\gamma}{\beta}$, we have

$$\begin{aligned} \psi(t) &\leq \left(\alpha + \frac{\gamma}{\beta}\right) e^{\beta t} - \frac{\gamma}{\beta} \\ &= \alpha e^{\beta t} + \frac{\gamma}{\beta}(e^{\beta t} - 1) \end{aligned}$$

so that

$$\psi(t) \leq \alpha e^{\beta t} + \frac{\gamma}{\beta}(e^{\beta t} - 1)$$

which is what we wanted to show.

Section 2.6 Problems

18. Show that Theorem 2.17 is false (in general) if the estimate is replaced by

$$|f(t,x)| \leq M(T) + L(T)|x|^{\alpha}$$

with $\alpha > 1$.

Proof: Consider the IVP

$$\dot{x} = 1 + x^2, \quad x(0) = 0$$

Clearly, the solution to this equation is $x(t) = \tan(t)$, and since $t_0 = 0$, we have that the maximum interval where the IVP is satisfied is $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ since $\tan(t)$ cannot be continuously extended beyond this interval. But notice

$$|f(t,x)| \leq 2 + |x|^2$$

so that theorem 2.17 is false in general if $\alpha > 1$.

Extra

Show that the supremum norm in Eq. (2.3) is indeed a norm.

Proof: Let $x(t) \in C(I)$. Then for a fixed $t \in I$, we have

$$0 \leq |x(t)|$$

hence,

$$0 \leq \sup_{t \in I} |x(t)|$$

and notice that $\sup_{t \in I} |x(t)| = 0$ only if $x(t) \equiv 0$, for if there exists some value of $s \in I$ where $x(s) \neq 0$, then $|x(s)| > 0$, so that $\sup_{t \in I} |x(t)| > 0$. Thus, nonnegativity holds. Let α be an arbitrary scalar. Then

$$\begin{aligned} \|x\| &= \sup_{t \in I} |\alpha x(t)| \\ &= \sup_{t \in I} |\alpha||x(t)| \end{aligned}$$

and since $\sup(aX) = a\sup(X)$ for $a > 0$, we have

$$\begin{aligned} \sup_{t\in I} |\alpha||x(t)| &= |\alpha| \sup_{t\in I} |x(t)| \\ &= |\alpha|\|x\| \\ \implies \|\alpha x\| &= |\alpha|\|x\| \end{aligned}$$

so homogeneity holds. To show the triangle inequality holds, let $x, y \in C(I)$ and notice, for a fixed $t \in I$,

$$\begin{aligned} |x(t) + y(t)| &\leq |x(t)| + |y(t)| \\ &\leq \sup_{t\in I} |x(t)| + \sup_{t\in I} |y(t)|. \end{aligned}$$

Then $\sup_{t\in I} |x(t)| + \sup_{t\in I} |y(t)| = \|x\| + \|y\|$ is an upper bound for $|x(t) + y(t)|$ for all t, hence

$$\begin{aligned} \sup_{t\in I} |x(t) + y(t)| &\leq \sup_{t\in I} |x(t)| + \sup_{t\in I} |y(t)| \\ \implies \|x + y\| &\leq \|x\| + \|y\| \end{aligned}$$

so the triangle inequality holds. Thus, the supremum norm is indeed a norm.