Modern Algebra HW6

Michael Nameika

October 2022

Section 14 Problems

12. Find the order of $(3,1) + \langle(1,1)\rangle$ in $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\langle(1,1)\rangle$.

To begin, notice that $\langle(1,1)\rangle = \{(0,0),(1,1),(2,2),(3,3)\}$. To find the order of $(3,1) + \langle(1,1)\rangle$, it is sufficient to compute $(3,1) + (3,1) + \cdots$ until we reach an element in $\langle(1,1)\rangle$. Well,

$$(3,1) + (3,1) = (2,2) \in \langle(1,1)\rangle$$

We only needed to add $(3,1)$ to itself once to find an element in $\langle(1,1)\rangle$. That is, $(3,1) + \langle(1,1)\rangle$ has order 2 in $(\mathbb{Z}_4 \times \mathbb{Z}_4)/\langle(1,1)\rangle$.

21. A student is asked to show that if H is a normal subgroup of an abelian group G, then G/H is abelian. The student's proof starts as follows:
We must show that G/H is abelian. Let a and b be two elements of G/H.

a. Why does the instructor reading this proof expect to find nonsense from here on in the student's paper?

0 The instructor can expect to find nonsense from here because a and b should be elements of G. Also, the student should reiterate what G, H and G/H are.

b. What should the student have written?

Let H be a normal subgroup of an abelian group G and let $a, b \in G$. We wish to show that G/H is abelian.

c. Complete the proof.

Continuing off of part **c**, since $a, b \in G$, aH and bH are cosets of H and thus, $aH, bH \in G/H$. We must show $(aH)(bH) = (bH)(aH)$. Well, from the definition of the binary operation on G/H, we have $(aH)(bH) = (ab)H$. Since G is abelian, we have $ab = ba$ so $(ab)H = (ba)H$. Again by the definition of the binary operation on G/H, we have $(ba)H = (bH)(aH)$, thus $(aH)(bH) = (bH)(aH)$.

40. Use the properties $\det(AB) = \det(A)\det(B)$ and $\det(I_n) = 1$ for $n \times n$ matrices to show the following:

a. The $n \times n$ matrices with determinant 1 form a normal subgroup of $GL(n, \mathbb{R})$.

Let H be the set of $n \times n$ matrices with real entries having determinant 1. To begin, we must show $H \leq Gl(n, \mathbb{R})$. Let $A, B \in H$ and consider AB. Since A, B both have real entries, by definition of matrix multiplication and closure of $\mathbb{R}$, we have AB has real entries. Additionally, $\det(AB) = \det(A)\det(B) = (1)(1) = 1$, so $AB \in H$. That is, H is closed. Now we must show that $I_n \in H$. Well, I_n has real entries and $\det(I_n) = 1$, so $I_n \in H$. Finally, we must show for any

$A \in H$, $A^{-1} \in H$. Well, since $\det(A) = 1$, A^{-1} exists and $\det(A^{-1}) = 1/\det(A) = 1/1 = 1$, so $A^{-1} \in H$. So H is a subgroup of $GL(n, \mathbb{R})$.
Now, to show H is a normal subgroup of $GL(n, \mathbb{R})$, let $g \in GL(n, \mathbb{R})$, $h \in H$ and consider ghg^{-1}. Well, $det(ghg^{-1}) = \det(g)\det(h)\det(g^{-1}) = \det(g)(1)(1/\det(g)) = 1$, so $ghg^{-1} \in H$. Then by theorem 14.13, H forms a normal subgroup of $GL(n, \mathbb{R})$.

Bonus Problems!!!

1. Let K denote the subgroup $\langle \rho_1 \rangle$ in the group D_4.

(a) True or false? For every $a \in D_4$ and every $k \in K$ the equation $ak = ka$ is valid.

False. Consider $k = \rho_1$ and $a = \mu_1$. Notice

$$ak = \mu_1 \rho_1 = \delta_2$$

and

$$ka = \rho_1 \mu_1 = \delta_1$$

so $ak \neq ka$.

(b) List all the right cosets of K in D_4.

Since $\langle \rho_1 \rangle = \{\rho_0, \rho_1, \rho_2, \rho_3\}$, $|\langle \rho_1 \rangle| = 4$, so the index of K in D_4 is 2, since $|D_4| = 8$. Now, $K = K\rho_0$ is one of the cosets, so we need only find the other. Notice

$$K\mu_1 = \{\mu_1, \delta_1, \mu_2, \delta_2\}$$

So the two right cosets of K in D_4 are

$$\{\mu_1, \delta_1, \mu_2, \delta_2\}\,; \quad \{\rho_0, \rho_1, \rho_2, \rho_3\}$$

(c) Prove that K is a normal subgroup of D_4.

Proof: To do so, it suffices to show that each left coset is also a right coset. To begin, it is clear that $\rho_0 K = K\rho_0$. Let us inspect the remaining cosets:

$$\rho_1 K = \{\rho_1, \rho_2, \rho_3, \rho_0\}$$
$$K\rho_1 = \{\rho_1, \rho_2, \rho_3, \rho_0\} = \rho_1 K$$

$$\rho_2 K = \{\rho_2, \rho_3, \rho_0, \rho_1\}$$
$$K\rho_2 = \{\rho_2, \rho_3, \rho_0, \rho_1\} = \rho_2 K$$

$$\rho_3 K = \{\rho_3, \rho_0, \rho_1, \rho_2\}$$
$$K\rho_3 = \{\rho_3, \rho_0, \rho_1, \rho_2\} = \rho_3 K$$

$$\mu_1 K = \{\mu_1, \delta_2, \mu_2, \delta_1\}$$
$$K\mu_1 = \{\mu_1, \delta_1, \mu_2, \delta_2\} = \mu_1 K$$

$$\mu_2 K = \{\mu_2, \delta_1, \mu_1, \delta_2\}$$
$$K\mu_2 = \{\mu_2, \delta_2, \mu_1, \delta_1\} = \mu_2 K$$

$$\delta_1 K = \{\delta_1, \mu_1, \delta_2, \mu_2\}$$
$$K\delta_1 = \{\delta_1, \mu_2, \delta_2, \mu_1\} = \delta_1 K$$

$$\delta_2 K = \{\delta_2, \mu_2, \delta_1, \mu_1\}$$
$$K\delta_2 = \{\delta_2, \mu_1, \delta_1, \mu_2\} = \delta_2 K$$

So all left cosets are also right cosets. Then by definition, K is a normal subgroup of D_4.

(d) Give the group table of the factor group D_4/K.

Let the coset K be denoted by $\rho_0 K$ and likewise the coset $\{\mu_1, \delta_1, \mu_2, \delta_2\}$ be denoted by $\mu_1 K$. Then the group table for the factor group D_4/K is as follows:

D_4/K	$\rho_0 K$	$\mu_1 K$
$\rho_0 K$	$\rho_0 K$	$\mu_1 K$
$\mu_1 K$	$\mu_1 K$	$\rho_0 K$

(e) Find the order of the element $K\delta_1$ in the group D_4/K.

Well, notice from the work in part (c) that $K\delta_1 = \mu_1 K$, and from the group table in part (d), we can see that $(\mu_1 K)(\mu_1 K) = \rho_0 K$, so the order of $K\delta_1$ is 2 in D_4/K.

(f) To what "known" group is the group D_4/K isomorphic? Justify appropriately.

$D_4/K \cong \mathbb{Z}_2$. To see this, let us inspect the group table for $\mathbb{Z}_2$:

$\mathbb{Z}_2$	0	1
0	0	1
1	1	0

Notice from the group tables that D_4/K and $\mathbb{Z}_2$ have the same structure. Thus, $D_4 \cong \mathbb{Z}_2$.

2. Let H denote the subgroup $\langle \rho_2 \rangle$ in the group D_4.

(a) List all the right cosets of H in D_4.

Notice $H = \{\rho_0, \rho_2\}$ and so $|H| = 2$, thus we will have 4 cosets since $|D_4| = 8$. Then the right cosets of H in D_4 are

$$\{\rho_0, \rho_2\}\ ;\quad \{\rho_1, \rho_2\}\ ;\quad \{\mu_1, \mu_2\}\ ;\quad \{\delta_1, \delta_2\}$$

where $\{\rho_0, \rho_2\} = H\rho_0$ $\{\rho_1, \rho_2\} = H\rho_1$, $\{\mu_1, \mu_2\} = H\mu_1$, and $\{\delta_1, \delta_2\} = H\delta_1$.

(b) Give the group table of the factor group D_4/H.

D_4/H	$H\rho_0$	$H\rho_1$	$H\mu_1$	$H\delta_1$
$H\rho_0$	$H\rho_0$	$H\rho_1$	$H\mu_1$	$H\delta_1$
$H\rho_1$	$H\rho_1$	$H\rho_0$	$H\delta_1$	$H\mu_1$
$H\mu_1$	$H\mu_1$	$H\delta_1$	$H\rho_0$	$H\rho_1$
$H\delta_1$	$H\delta_1$	$H\mu_1$	$H\rho_1$	$H\rho_0$

(c) Find the order of the element $H\delta_1$ in the group D_4/H.

From the group table above, we can see that $(H\delta_1)(H\delta_1) = H\rho_0$, so the order of $H\delta_1$ in D_4/H is 2.

(d) To what 'known' group is the group D_4/H isomorphic? Justify appropriately.

D_4/H is isomorphic to V, the Klein 4-group. To see this, let us inspect the group table of V:

V	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

and note that V and D_4/H have the same structure.

3. Let L denote the subgroup $\langle \mu_1 \rangle$ in the group D_4.

(a) List all the right cosets of L in D_4.

Notice $\langle \mu_1 \rangle = \{\rho_0, \mu_1\}$ has order 2, so there will be 4 right cosets of $\langle \mu_1 \rangle$ in D_4. Then the right cosets are

$$\begin{aligned} L\rho_0 &= \{\rho_0, \mu_1\} \\ L\rho_1 &= \{\rho_1, \delta_1\} \\ L\rho_2 &= \{\rho_2, \mu_2\} \\ L\rho_3 &= \{\rho_3, \delta_2\} \end{aligned}$$

(b) Prove that L is NOT a normal subgroup of D_4.

Proof: We need only find a left coset of L that is not also a right coset of L in D_4. From part (a), we have $L\rho_1 = \{\rho_1, \delta_1\}$. Now let us compute $\rho_1 L$:

$$\rho_1 L = \{\rho_1, \delta_2\} \neq L\rho_1$$

So L is not a normal subgroup of D_4.

(c) Give examples of specific elements a, b, c, d in D_4 which have the properties:

$$La = Lb \text{ and } Lc = Ld, \quad \text{but } Lac \neq Lbd$$

Notice $L\delta_2 = \{\delta_2, \rho_3\} = L\rho_3$ and $L\mu_1 = \{\mu_1, \rho_0\} = L\rho_0$ and that $\delta_2\mu_1 = \rho_1$ and $\rho_3\rho_0 = \rho_3$, so

$$L\delta_2\mu_1 = L\rho_1 = \{\rho_1, \delta_1\}$$

and

$$L\rho_3\rho_0 = L\rho_3 = \{\rho_3, \delta_2\} \neq L\delta_2\mu_1$$

(d) Give examples of specific elements x and y in D_4 for which the product of cosets $Lx * Ly$ is NOT a right coset of L.

Let $x = \rho_1$ and $y = \rho_2$ and consider $Lx = \{\rho_1, \delta_1\}$ and $Ly = \{\rho_2, \mu_2\}$. Now consider $Lx * Ly$:

$$\begin{aligned} Lx * Ly &= \{\rho_1, \delta_1\} * \{\rho_2, \mu_2\} \\ &= \{\rho_1\rho_2, \delta_1\mu_2\} \\ &= \{\rho_3, \rho_1\} \end{aligned}$$

Which is not a right coset of L in D_4.