Problem Set 5 (Astrophysics)

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- Mass of Earth = 5.974×10^{24} kg
- Radius of Earth = 6378 km
- Mass of Sun = 1.99×10^{30} kg
- Radius of Sun = 6.955×10^{5} km
- Mass of Moon = 7.349×10^{22} kg
- Radius of Moon = 1.737×10^{3} km
- Luminosity of Sun = 3.9×10^{26} watts
  - Semi-major axis of moon's orbit around Earth = 384.4×10^{3} km
  - Distance of Mercury from Sun = 0.387 AU = 5.79×10^{10} m
  - Distance of Earth from Sun = 1 AU = 1.5×10^{11} m
  - Distance of Mars from Sun = 228×10^{6} km
  - Distance of Jupiter from Sun = 5.2 AU = 778.3×10^{6} km
- Mass of Mercury = 3.3×10^{23} kg
- Mass of Io = 8.932×10^{22} kg
- Radius of Io = 1822 km
- Semi-major axis of Io's orbit around Jupiter = 421.6×10^{3} km
- Mass of Titan = 13.455×10^{22} kg
- Radius of Titan = 2.575×10^{3} km
- Mass of Jupiter = $317.8 \times M_{Earth} = 1.90 \times 10^{27}$ kg

1. The Hill radius is the maximum orbital radius for a satellite orbiting a planet.
   a. What is the Hill radius for the planet Mercury? (Maybe this is why Mercury has no moons!)
   b. What is the Hill radius for the planet Jupiter?

Recall the formula for the Hill radius:

$$r_{Hill} = (\frac{M_1}{2M_2})^{1/3} r_{1-2} \tag{0.1}$$

Where M_1 is the mass of the smaller body, M_2 is the mass of the larger body, and r_{1-2} is the distance between the two bodies.

a) From the table given at the beginning of the assignment, we have that

$$M_1 = M_{☿} = 3.3 \times 10^{23}\ kg$$

$$M_2 = M_{\odot} = 1.99 \times 10^{30}\ kg$$

$$r_{1-2} = r_{☿-\odot} = 5.79 \times 10^{10}\ m$$

Plugging these values into equation (1), we will find

$$r_{Hill} = (\frac{3.3 \times 10^{23}\ kg}{2 * 1.99 \times 10^{30}\ kg})^{1/3}(5.79 \times 10^{10}\ m)$$

$$\approx 2.52 \times 10^5 \; km$$

b) Also from the table above, we have that

$$M_1 = M_{♃} = 1.90 \times 10^{27} \; kg$$

$$M_2 = M_{\odot} = 1.99 \times 10^{30} \; kg$$

$$r_{1-2} = 7.783 \times 10^8 \; km$$

Plugging these into equation (1), we get

$$r_{Hill} = (\frac{1.90 \times 10^{27} \; kg}{1.99 \times 10^{30} \; kg})^{1/3}(7.783 \times 10^8 \; km)$$

$$\approx 6.083 \times 10^7 \; km$$

2. Compute the ratio of the differential tidal force on Jupiter's moon, Io, due to Jupiter to the differential tidal force on the Moon due to the Earth. (We often use ratios for tidal forces to avoid having to identify the mass of the test object. It also avoids questions like whether the radius or diameter of the moon should be used. Be careful to keep the masses straight for the objects.)

Recall that the tidal force on an object due to a massive object on another object with a test mass on its surface is given by

$$\Delta F = R_P \frac{2GMm}{r_0^3} \tag{0.2}$$

Where R_P is the radius of the body with the test mass, M is the mass of the body, m is some test mass, and r_0 is the center-to-center distance between the massive body and the test mass.

To find the tidal force of Jupiter on Io, using the data in the table above, we will find

$$\Delta F_{Io} = (1822 \; km)\frac{2G(1.90 \times 10^{27} \; kg)m}{(4.216 \times 10^8 \; m)^3}$$

And the tidal force of the moon on the Earth:

$$\Delta F_{Earth} = (6378 \; km)\frac{2G(7.349 \times 10^{22} \; kg)m}{(3.844 \times 10^8 \; m)^3}$$

And the ratio between the two:

$$\frac{\Delta F_{Io}}{\Delta F_{Earth}} = \frac{(1822 \; km)\frac{2G(1.90 \times 10^{27} \; kg)m}{(4.216 \times 10^8 \; m)^3}}{(6378 \; km)\frac{2G(7.349 \times 10^22 \; kg)m}{(3.844 \times 10^8 \; m)^3}}$$

$$= \frac{(1822)(1.90 \times 10^{27})(3.844 \times 10^8)^3}{(6378)(7.349 \times 10^{22})(4.216 \times 10^8)^3}$$

$$\approx 5600$$

So the tidal forces on Io from Jupiter are approximately 5600× stronger than that of the moon on Earth.

3. Let us explore Saturn's moon, Titan, which has a mass = 1.3 x 10^{23} kg, a radius of 2580 km, and a surface temperature of 94 K.

a. What is the gravitational acceleration (g) at the surface of Titan?

b. Would Titan be able to retain either H_2 or CO_2 or both in its atmosphere?

a) Recall that the gravitational acceleration of a body is given by

$$a = \frac{GM}{r^2} \tag{0.3}$$

Where M is the mass of the body and r is the radius of the body. For Titan,

$$a_{Titan} = \frac{(6.67 \times 10^{-11}\ m^3 kg^{-1} s^{-2})(1.3 \times 10^{23}\ kg)}{(2.58 \times 10^6\ m)^2}$$

$$\approx 1.303 \frac{m}{s^2}$$

So the gravitational acceleration at the surface of Titan is approximately 1.303 $\frac{m}{s^2}$

b) Recall the formula for the molecular mass that a body can hold:

$$\mu \geq 7.1(\frac{T}{1000\ K})(\frac{M}{M_\odot})(\frac{R}{R_\odot}) \tag{0.4}$$

For Titan, we have

$$\mu_{Titan} \geq 7.1(\frac{94\ K}{1000\ K})(\frac{1.3 \times 10^{23}\ kg}{5.974 \times 10^{24}\ kg})(\frac{2580\ km}{6378\ km})$$

$$\approx 0.00587$$

Since $\mu_{Hydrogen} = 1$, and $\mu_{Titan} \geq 0.00587$, it is safe to assume that Titan will be able to retain both H_2 and CO_2.

4. The asteroid, Eugenia, has a small natural satellite orbiting it with an orbital period of P = 4.76 days. The semimajor axis of its orbit is a = 1180 km. What is the mass of Eugenia? (It is OK to treat the mass of the satellite as very small compared to the mass of Eugenia.)

Recall Kepler's law:

$$P^2 \approx \frac{4\pi^2}{Gm} a^3 \tag{0.5}$$

where P is the period of the object's orbit, m is the parent object's mass and a is the semi-major axis of the orbit.

Solving for m in the above equation, we see that

$$m \approx \frac{4\pi^2}{GP^2} a^3 \tag{0.6}$$

So for the data we are given about Eugenia's satellite, we have that

$$m_{Eugenia} \approx \frac{4\pi^2}{(6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^2)(4.76 * 24 * 60 * 60s)^2}(1.18 \times 10^6\ m)$$

$$\approx 5.75 \times 10^{18}\ kg$$

So the mass of Eugenia is approximately $5.75 \times 10^{18}\ kg$.

5. A cometary nucleus rotates rapidly and has an albedo of 0.05. When its surface temperature reaches about 150 K, the ice of which it is mostly made will start to sublime and form a gaseous coma. How far is the cometary nucleus from the Sun when the coma starts to form?

Recall that we can estimate the temperature of a rapidly rotating object is given by

$$T_p = 279\ K(1-A)^{1/4}(\frac{r}{1\ AU})^{-1/2} \tag{0.7}$$

Solving for r and using the fact that the temperature is $150\ K$ and the albedo of the comet is 0.05, we will find that

$$\begin{aligned} r &= (\frac{279}{150})^2(0.95)^{1/2}(1\ AU) \\ &\approx 3.372\ AU \end{aligned}$$

So a comet's coma will start to form when the comet is approximately $3.372\ AU$ from the Sun.

6. Using the ideal gas law and assuming that the atmosphere of Venus is completely composed of carbon dioxide (CO_2), estimate the number density of molecules at the surface of Venus. How many times larger is this value than the density of Nitrogen at the surface of the Earth (2×10^{19} molecules/cm^3)? (The pressure at the surface of Venus is 90 atm and the temperature is 740 K).

Recall the ideal gas law:

$$PV = NkT \tag{0.8}$$

Where P is the pressure, V is the volume, N is the number of molecules, k is Boltzmann's constant ($k = 1.381 \times 10^{-23}\ \frac{Nm}{K}$, and T is the temperature. Plugging in the data given into this equation and solving for $\frac{N}{V}$, we have that

$$\begin{aligned} (\frac{N}{V})_{♀} &= \frac{9.119 \times 10^6\ \frac{N}{m^2}}{(1.381 \times 10^{-23}\ \frac{Nm}{K})(740\ K)} \\ &\approx 8.92 \times 10^{26}\ \frac{\text{molecules}}{m^3} \\ &= 8.92 \times 10^{20}\ \frac{\text{molecules}}{m^3} \end{aligned}$$

So the number density of molecules of CO_2 at the surface of Venus is approximately $8.92 \times 10^{20}\ \frac{\text{molecules}}{m^3}$.