Homework XII

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Section 8.1 Problems

14. Show that $T : \ell^\infty \to \ell^\infty$ with T defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j/j$, is compact.

Proof: Note that T is linear. Consider the sequence of linear operators $T_n : \ell^\infty \to \ell^\infty$ defined by

$$T_n x = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \cdots, \frac{\xi_n}{n}, 0, 0, \cdots\right)$$

we now inspect $\|T_n x - Tx\|$:

$$\|T_n x - Tx\| = \left\|\left(0, 0, \cdots, 0, \frac{\xi_{n+1}}{n+1}, \frac{\xi_{n+2}}{n+2}, \cdots\right)\right\|$$

and notice that since $|\xi_j| \leq \|x\|$ $(j \geq 1)$ by the sup norm on ℓ^∞, and so $\left|\frac{\xi_{n+1}}{n+1}\right| \leq \frac{\|x\|}{n+1}$ and $\left|\frac{\xi_{n+2}}{n+2}\right| \leq \frac{\|x\|}{n+2} \leq \frac{\|x\|}{n+1}$. Thus for any ξ_j/j for $j \geq n+1$, $|\xi_j/j| \leq \frac{\|x\|}{n+1}$. Hence

$$\|T_n x - Tx\| \leq \frac{\|x\|}{n+1}$$

thus

$$\|T_n - T\| \leq \frac{1}{n+1}.$$

Thus $T_n \to T$ uniformly in the operator norm so that T is compact.

Assigned Exercises

XII.1. For parts (a) - (b) let X be a complex Banach space and let $T : X \to X$ be a bounded linear operator.

(a) Using Theorem 4.12-2 and Lemma 7.2-3 of the text, prove that $\lambda \in \sigma(T)$ if and only if $T - \lambda I$ is *not* bijective.

Proof: First suppose that $\lambda \in \sigma(T)$. We wish to show that $T - \lambda I$ is not bijecctive. Suppose by way of contradiction that $T - \lambda I$ is bijective. In particular $T - \lambda I$ is injective so that $R_\lambda(T) = (T - \lambda)^{-1}$ exists, and by the bounded inverse theorem, $R_\lambda(T)$ is bounded. Additionally, $T - \lambda I$ is surjective, so that $\mathcal{D}(R_\lambda(T)) = X$, so that by definition, $\lambda \in \rho(T)$, contradicting the fact that $\rho(T) \cap \sigma(T) = \emptyset$.

Now suppose that $T - \lambda I$ is not bijective. We wish to show that $\lambda \in \sigma(T)$. Well, by Lemma 7.2-3, we have that since T is bounded, if $\lambda \in \rho(T)$, then $R_\lambda(T)$ is defined on X and is bounded, so that $T - \lambda I$ is bijective. Hence, it must be the case that $\lambda \in \sigma(T)$.

(b) We say that λ is an *approximate eigenvalue* if there exists a sequence (x_n) in X with $\|x_n\| = 1$ such that $Tx_n - \lambda x_n \to \mathbf{0}$. Note that an eigenvalue is an approximate eigenvalue. Prove that an approximate eigenvalue λ belongs to $\sigma(T)$. If such a λ is *not* an eigenvalue, can $T - \lambda I$ be surjective?

Proof: Let λ be an approximate eigenvalue of T and suppose that λ is not an eigenvalue of T since if λ is an eigenvalue of T, $\lambda \in \sigma_p(T) \subseteq \sigma(T)$. Suppose by way of contradiction that $\lambda \in \rho(T)$. Then $R_\lambda(T) = (T - \lambda I)^{-1}$ is bounded. Define

$$y_n = (T - \lambda I)(x_n)$$

where (x_n) is a sequence in X such that $Tx_n - \lambda x_n \to 0$. Then $y_n \to 0$ since $y_n = Tx_n - \lambda x_n$ and $(T - \lambda I)^{-1}$ is bijective by part (a),

$$x_n = (T - \lambda I)^{-1}(y_n)$$

and so

$$\|(T - \lambda I)^{-1}(y_n)\| = 1$$

thus

$$1 \leq \|(T - \lambda I)^{-1}\|\|y_n\|$$

and since $\lambda \in \rho(T)$, $(T - \lambda I)^{-1}$ is bounded, say $\|(T - \lambda I)^{-1}\| \leq M$ so

$$1 \leq M\|y_n\| \to 0$$

a contradiction. Thus, $\lambda \notin \rho(T)$, so that $\lambda \in \sigma(T)$ by definition.

(c) (extra credit, 2 pts.) By considering $\lambda = 0$ for the linear operator $T : \ell^2 \to \ell^2$ defined by $(\xi_1, \xi_2, \xi_3, \cdots) \mapsto (\xi_2, \xi_3, \cdots)$, show that there exists a bounded linear operator $T : X \to X$ on a complex Banach space X and an eigenvalue λ for T such that $T - \lambda I$ is surjective.

XII.2. For $1 \le p < \infty$, let $T : \ell^p \to \ell^p$ be defined by $(\xi_1, \xi_2, \xi_3, \cdots) \mapsto (\xi_2, \xi_3, \cdots)$. Note that T is a bounded linear operator on a complex Banach space. Prove that if $|\lambda| = 1$, that is λ is on the unit circle of $\mathbb{C}$, then
(a) λ is *not* an eigenvalue of T,

Proof: Suppose by way of contradiction that λ with $|\lambda| = 1$ is an eigenvalue of T. Then for some $x = (\xi_1, \xi_2, \cdots)$, we have

$$Tx = \lambda x$$

so that

$$\|Tx\| = |\lambda|\|x\| = \|x\|.$$

Then

$$\begin{aligned} Tx &= (\xi_2, \xi_3, \cdots) \\ \implies \|Tx\| &= \left(\sum_{k=2}^{\infty} |\xi_k|^p\right)^{1/p} \end{aligned}$$

and since $\|x\| = \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p}$, we have that

$$\begin{aligned} \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} &= \left(\sum_{k=2}^{\infty} |\xi_k|^p\right)^{1/p} \\ \implies \sum_{k=1}^{\infty} |\xi_k|^p &= \sum_{k=2}^{\infty} |\xi_k|^p \\ \implies |\xi_1|^p + |\xi_2|^p + |\xi_3|^p + \cdots &= |\xi_2|^p + |\xi_3|^p + \cdots \\ \implies |\xi_1|^p &= 0 \\ \implies |\xi_1| &= 0. \end{aligned}$$

Now, using $Tx = \lambda x$, we find

$$(\xi_2, \xi_3, \cdots) = (0, \lambda\xi_2, \lambda\xi_3, \cdots)$$

which gives us $\xi_2 = 0$, which likewise gives us $\xi_3 = 0$ and continuing, we find $\xi_n = 0$ for all $n \in \mathbb{N}$. Thus, $x = \mathbf{0}$ which is not an eigenvector by definition, so that λ satisfying $|\lambda| = 1$ is not an eigenvalue of T.

Proof: Begin by noting that $\lambda = 0$ is indeed an eigenvalue for T defined above since, for $x = (\xi_1, 0, 0, \cdots) \in \ell^2$, we have that

$$Tx = (0, 0, \cdots = 0 \cdot x.$$

Now, for $\lambda = 0$, we have that $T - \lambda I = T$ so we need to show that T is surjective. Let $y = (\eta_1, \eta_2, \cdots) \in \ell^2$. And notice that for $w = (0, \eta_1, \eta_2, \eta_3, \cdots)$ in ℓ^2 (since $y \in \ell^2$), we have

$$Tw = (\eta_1, \eta_2, \cdots) = y$$

so that T is surjective.

(b) λ is an approximate eigenvalue of T (cf. Exercise XII.1(b)).
Hint: For part (b) consider $x_n = c_n(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, \ldots)$ for an appropriate sequence $c_n = c_{n,p} > 0$.

Proof: Notice that $\|(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, \ldots)\| = n^{1/p}$, so define $c_n = 1/n^{1/p}$ so that $\|x_n\| = 1$.

Now, notice

$$\begin{aligned}
Tx_n - \lambda x_n &= c_n(\lambda, \lambda^2, \dots, \lambda^{n-1}, 0, 0, 0, \dots) - c_n(\lambda, \lambda^2, \dots, \lambda^{n-1}, \lambda^n, 0, 0, \dots) \\
&= \frac{1}{n^{1/p}}(0, 0, \dots, 0, \lambda^n, 0, 0, \dots) \\
\implies \|Tx_n - \lambda x_n\| &= \frac{|\lambda^n|^{1/p}}{n^{1/p}} \\
&= \frac{1}{n^{1/p}}.
\end{aligned}$$

Thus $Tx_n - \lambda x_n \to 0$, and so λ is an approximate eigenvalue by definition.

XII.3. (a) Let H be a complex Hilbert space, let $T : H \to H$ be a bounded linear operator and let $T^* : H \to H$ be the Hilbert-adjoint of T. Prove that $\lambda \in \rho(T)$ if and only if $\overline{\lambda} \in \rho(T^*)$, and therefore that $\sigma(T)$ and $\sigma(T^*)$ are complex conjugates of one another.
Hint: $\lambda \in \rho(T)$ if and only if $T - \lambda I$ is bijective if and only if there exists a bijective bounded linear operator $R : H \to H$ such that $R(T - \lambda I) = (T - \lambda I)R = I$. Apply the Hilbert-adjoint operator to the products.

Proof: Let $\lambda \in \rho(T)$. We wish to show that $\overline{\lambda} \in \rho(T^*)$. Well, since $\lambda \in \rho(T)$, we have that $R_\lambda = (T - \lambda I)^{-1}$ exists. Then by definition of the inverse of an operator, we have

$$(T - \lambda I)^{-1}(T - \lambda I) = I$$
$$(T - \lambda I)(T - \lambda I)^{-1} = I$$

applying the Hilbert adjoint to each side of the above two equations, we find

$$[(T - \lambda I)^{-1}]^*(T^* - \overline{\lambda} I) = I$$

and

$$(T^* - \overline{\lambda} I)[(T - \lambda I)^{-1}]^* = I$$

from this, we see

$$[(T - \lambda I)^{-1}]^* = (T^* - \overline{\lambda} I)^{-1}$$

and is bounded since $R_\lambda(T)$ is bounded and is similarly defined over a dense subset of H. Thus, $\overline{\lambda} \in \rho(T^*)$.

The case $\overline{\lambda} \in \rho(T^*)$ follows similarly using $(T^*)^* = T$.

(b) Let $T : \ell^2 \to \ell^2$ be the *right-shift operator* defined by $(\xi_1, \xi_2, \dots) \mapsto (0, \xi_1, \xi_2, \dots)$. Recall from Sec. 3.9 Prob. 10 that the Hilbert adjoint of T is the operator $T^* : \ell^2 \to \ell^2$ defined by $(\xi_1, \xi_2, \xi_3, \dots) \mapsto (\xi_2, \xi_3, \dots)$, that is the *left-shift operator*. Prove in this example that both spectra $\sigma(T)$ and $\sigma(T^*)$ are the same and equal $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. (Cf. Sec. 10.5 Probs. 7-10.)

Proof: Notice $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and by part (a), since $\sigma(T^*) = \overline{\sigma(T)}$, we have $\sigma(T^*) = \{\overline{\lambda} \in \mathbb{C} : |\overline{\lambda}| \leq 1\}$. Since $|\lambda| = |\overline{\lambda}|$, we see that $\sigma(T) = \sigma(T^*)$.