Homework 2

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Section 1.5 Problems

4. Show that M in Prob. 3 is not complete by applying Theorem 1.4-7.

Proof: Consider the sequence $\{x_n\} \in M$ defined by $x_n = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, 0, 0, \cdots\right)$. I claim that $x_n \to x = \left(1, \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots\right)$. Note that for any element $x^{(i)}$ of x, $|x^{(i)}| \leq 1$, hence $x \in \ell^\infty$. Then notice

$$\begin{aligned} d(x_n, x) &= \sup_{i \geq 1} |x_n^{(i)} - x^{(i)}| \\ &= \frac{1}{2^{n+1}}. \end{aligned}$$

Fix $\varepsilon > 0$ and take $N = \lfloor \log_2 \left(\frac{1}{\epsilon}\right) - 1 \rfloor$. Then for $n > N$, we have

$$d(x_n, x_m) < \varepsilon.$$

Hence, $\{x_n\}$ converges to x in ℓ^∞, however, notice that x contains only nonzero elements, hence $x \notin M$. That is, x is a limit point of M, but $x \notin M$. Hence, M is not closed, and by theorem 1.4-7, M is not a complete subspace of ℓ^∞.

Section 2.1 Problems

9. On a fixed interval $[a, b] \subset \mathbb{R}$, consider the set X consisting of all polynomials with real coefficients and of degree not exceeding a given n, and the polynomial $x = 0$ (for which a degree is not defined in the usual discussion of degree). Show that X, with the usual addition and the usual multiplication by real numbers, is a real vector space of dimension $n + 1$. Find a basis for X. Show that we can obtain a complex vector space $\tilde{X}$ in a similar fashion if we let those coefficients be complex. Is X a subspace of $\tilde{X}$?

Proof: (of X being a vector space with dimension $n + 1$.) Let $x = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n, y = \beta_0 + \beta_1 t + \cdots + \beta_n t^n, z = \gamma_0 + \gamma_1 t + \cdots + \gamma_n t^n \in X$. Since x, y, z are polynomials of real numbers, it follows that, since $\mathbb{R}$ is closed under addition, for any fixed $t \in [a, b]$,

$$\begin{aligned} x(t) + y(t) &= \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n + \beta_0 + \beta_1 t + \cdots + \beta_n t^n \\ &= (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)t + \cdots + (\alpha_n + \beta_n)t^n \\ &= \delta_0 + \delta_1 t + \cdots + \delta_n t^n \end{aligned}$$

with $\delta_i = \alpha_i + \beta_i$. Hence, X is closed under addition. Similarly, since $\mathbb{R}$ is closed under multiplication, for any $c \in \mathbb{R}$ and $t \in [a, b]$,

$$\begin{aligned} cx(t) &= c\alpha_0 + c\alpha_1 t + \cdots + c\alpha_n t^n \\ &= \tau_0 + \tau_1 t + \cdots + \tau_n t^n \end{aligned}$$

with $\tau_i = c\alpha_i$. Hence, X is closed under scalar multiplication. Now, since addition in $\mathbb{R}$ is commutative and associative, for $t \in [a,b]$, we have

$$x(t) + y(t) = y(t) + x(t)$$
$$(x(t) + y(t)) + z(t) = x(t) + (y(t) + z(t))$$

Hence, commutativity and associativity hold in X. Now, since $0 \in X$, we have

$$x + 0 = x$$

and

$$x + (-x) = 0.$$

It is also easy to verify that

$$1x = x.$$

Let $c, d \in \mathbb{R}$ and consider the following:

$$\begin{aligned} c(dx) &= c(d\alpha_0 + d\alpha_1 t + \cdots + d\alpha_n t^n) \\ &= cd\alpha_0 + cd\alpha_1 t + \cdots + cd\alpha_n t^n \\ &= d(c\alpha_0 + c\alpha_1 t + \cdots + c\alpha_n t^n) \\ &= d(cx) \end{aligned}$$

So multiplication by scalars is associative in X. Finally, checking distributivity, we find

$$\begin{aligned} c(x+y) &= c([\alpha_0 + \beta_0] + [\alpha_1 + \beta_1]t + \cdots + [\alpha_n + \beta_n]t^n) \\ &= [c\alpha_0 + c\beta_0] + [c\alpha_1 + c\beta_1]t + \cdots + [c\alpha_n + c\beta_n]t^n \\ &= cx + cy \end{aligned}$$

$$\begin{aligned} (c+d)x &= (c+d)\alpha_0 + (c+d)\alpha_1 t + \cdots + (c+d)\alpha_n t^n \\ &= c\alpha_0 + \cdots + c\alpha_n t^n + d\alpha_0 + \cdots + d\alpha_n t^n \\ &= cx + dx. \end{aligned}$$

Hence, distributivity holds. Thus, X is a vector space.
Finally, notice that $\{1, t, t^2, \cdots, t^n\}$ is a basis for X and has dimension $n+1$. Thus, since any basis of a vector space has the same cardinality, X has dimension $n+1$.

Note that if we replace $\mathbb{R}$ with $\mathbb{C}$ for our arguments involving scalar multiples above, we may show that $\tilde{X}$ is a vector space. However, X is not a subspace of $\tilde{X}$ since for any complex coefficient $\tilde{c}$ and element $x \in X$, $\tilde{c}x \notin X$.

10. If Y and Z are subspaces of a vector space X, show that $Y \cap Z$ is a subspace of X, but $Y \cup Z$ need not be one. Give examples.

Proof: Let X be a vector space and $Y, Z \subseteq X$ be subspaces and suppose $Y \cap Z \neq \emptyset$. Let $x, y, z \in Y \cap Z$. Since Y and Z are subspaces, it follows that $x + y \in Y$ and $x + y \in Z$ so $x + y \in Y \cap Z$. Also,

$$x + y = y + x$$
$$x + (y + z) = (x + y) + z$$

in $Y \cap Z$. Additionally, $0 \in Y \cap Z$ and $-x \in Y \cap Z$ for any $x \in Y \cap Z$. Let α, β be any scalars. Then $\alpha x \in Y \cap Z$ and the distributive laws hold in $Y \cap Z$ since they hold for both Y and Z. Hence $Y \cap Z$ is a subspace. To see that $Y \cup Z$ is not necessarily a subspace, consider the subspaces of $\mathbb{R}^2$ given by

$$Y = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}, \qquad Z = \text{span}\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$$

Then $Y \cup Z$ is the set of lines given by $y = \pm x$ in graphical form. To see why $Y \cup Z$ is not a subspace, consider $(1,1)^T \in Y$ and $(-1,1)^T \in Z$. Then

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin Y \cup Z.$$

Hence, $Y \cup Z$ is not closed under addition, so $Y \cup Z$ is not a subspace.

Assigned Exercise

II.1 Let M be a nonempty subset of a metric space (X,d) and define the closure of M as the smallest closed set containing M, that is $\overline{M} = \bigcap_{K \text{ closed }, M \subseteq K} K$. This definition is an alternative to the one in the text.

(a). Prove Theorem **1.4-6(a)** using the above definition of closure only, and not by using the equivalence stated on p. 21 of the text that the smallest closed set containing M is the same as the union of M with its accumulation points.

Proof: Let (X,d) be a metric space and $M \subseteq X$ be nonempty and let $\{K_\lambda \mid \lambda \in \Lambda\}$ be an indexed collection of closed sets in X that contain M. That is, $M \subseteq K_\lambda$ for all λ. First suppose that there is a sequence of points $\{x_n\}$ in M converging to x. We wish to show that $x \in \overline{M}$. Suppose by way of contradiction that $x \notin \overline{M}$. Then necessarily,

$$x \in X \backslash \overline{M} = \bigcup_{\lambda \in \Lambda} X \backslash K_\lambda$$

and quickly note that for any $m \in M$,

$$M \bigcap \left(\bigcup_{\lambda \in \Lambda} X \backslash K_\lambda \right) = \emptyset$$

since $M \subseteq K_\lambda$ for all λ.

Since each K_λ is a closed set, $\cap_{\lambda \in \Lambda} K_\lambda$ is closed and so $\cup_{\lambda \in \Lambda} X \backslash K_\lambda$ is open. Then there exists some $r > 0$ such that the open ball of radius r centered at x is completely contained in $\cup_{\lambda \in \Lambda} X \backslash K_\lambda$. That is,

$$B_r(x) \subseteq \bigcup_{\lambda \in \Lambda} X \backslash K_\lambda$$

But since $\{x_n\}$ converges to x, there exists an index N such that for all $n > N$,

$$d(x_n, x) < \frac{r}{2}$$

Meaning that for all $n > N$, $x_n \in B_r(x)$. But then $x_n \notin M$ for all $n > N$, contradicting the fact that $\{x_n\}$ is a sequence in M. Thus, $x \in \overline{M}$.

Now suppose $x \in \overline{M}$. We wish to show that there exists a sequence of points in M converging to x. Suppose by way of contradiction that there does not exist such a sequence. Then there exists some $r > 0$ such that the open ball $B_r(x)$ shares no points in common with M. If there was no such r, then we could select a point $x_n \in B_{1/n}(x)$ such that $x_n \in M$ for each n, which would contradict our assumption that there is no sequence in M converging to x. Now, since each K_λ is closed,

$$X \backslash \bigcap_{\lambda \in \Lambda} K_\lambda = \bigcup_{\lambda \in \Lambda} X \backslash K_\lambda$$

is open in X. Then define

$$I = B_r(x) \cup \left(\bigcup_{\lambda \in \Lambda} X \backslash K_\lambda \right)$$

is open in X, hence $X\backslash I$ is closed in X and contains M, since $M \subseteq K_\lambda$ for all λ. So then

$$X\backslash I \in \{K_\lambda \mid \lambda \in \Lambda\}$$

But since $x \notin X\backslash I$, $x \notin \overline{M}$, a contradiction.

(b) Prove the equivalence between the two definitions of closure stated on p. 21 of the text.

Proof: We wish to show that $M \cup M' = \cap_{\lambda\in\Lambda} K_\lambda$, where M' is the set of limit points of M. To begin, let $x \in M \cup M'$. If $x \in M$, then $x \in \cap_{\lambda\in\Lambda} K_\lambda$ since $M \subseteq K_\lambda$ for all λ. If $x \in M'$, then for any open ball $B_r(x)$, $B_r(x) \cap M \neq \emptyset$, hence, create the sequence $\{x_n\}$ by selecting $x_n \in B_{1/n}(x)$ such that $x_n \in M$ for each n. Then $d(x_n, x) < \frac{1}{n}$, hence, $\{x_n\}$ is a sequence in M converging to x, and so by our work in part (a), $x \in \cap_{\lambda\in\Lambda} K_\lambda$. Hence,

$$M \cup M' \subseteq \bigcap_{\lambda\in\Lambda} K_\lambda$$

Now let $x \in \cap_{\lambda\in\Lambda} K_\lambda$. Then by our work in part (a), there exists a sequence $\{x_n\}$ in M converging to x. That is, x is a limit point of M, so $x \in M'$, and so $x \in M \cup M'$. Then

$$\bigcap_{\lambda\in\Lambda} K_\lambda \subseteq M \cup M'$$

By double inclusion, we have

$$M \cup M' = \bigcap_{\lambda\in\Lambda} K_\lambda$$