Homework 1 (Analysis)

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January 2022

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(a) Let $\{a_n\}$ be a sequence of real numbers such that $|a_{n+1} - a_n| < 3^{-n}$ for all $n \in \mathbb{N}$. Prove that a_n is a convergent sequence.

Proof: We will show that a_n satisfies the Cauchy criterion. First, fix $\epsilon > 0$. Now take $m > n > N \in \mathbb{N}$ and consider $|a_m - a_n|$.

Notice that

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \ldots + a_{n+1} - a_n|$$

And by the triangle inequality,

$$|a_m - a_{m-1} + a_{m-1} - \ldots + a_{n+1} - a_n| \leq$$
$$|a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \ldots + |a_{n+1} - a_n|$$

$$< 3^{-(m-1)} + 3^{-(m-2)} + \ldots + 3^{-n}$$

$$= \sum_{k=n}^{m-1} \frac{1}{3^k}$$

$$= \sum_{k=0}^{m-1} \frac{1}{3^k} - \sum_{k=0}^{n-1} \frac{1}{3^k}$$

$$= \frac{1 - \frac{1}{3^m}}{1 - \frac{1}{3}} - \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}}$$

$$= \frac{3}{2}(1 - \frac{1}{3^m} - 1 + \frac{1}{3^n})$$

$$= \frac{3}{2}(\frac{1}{3^n} - \frac{1}{3^m})$$

$$= \frac{1}{2}(\frac{1}{3^{n-1}} - \frac{1}{3^{m-1}})$$

And since $m > n$,

$$3^{-(n-1)} > 3^{-(m-1)}$$

Then

$$\frac{1}{3^{n-1}} - \frac{1}{3^{m-1}} > 0$$

Additionally, since $m > n > N$,

$$3^{-(N-1)} > 3^{-(m-1)} > 3^{-(n-1)} > 0$$

So

$$0 < \frac{1}{2}(\frac{1}{3^{n-1}} - \frac{1}{3^{m-1}}) < \frac{1}{2}(\frac{1}{3^{n-1}}) < \frac{1}{2}(\frac{1}{3^{N-1}})$$

Now let $\epsilon = \frac{1}{2}(\frac{1}{3^{N-1}})$
Now we have

$$|a_m - a_n| < \epsilon$$

Satisfying the Cauchy criterion.
$\therefore a_n$ is a convergent sequence.

$\square$

(b) Let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $|a_n - b_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, and $a_n \to L$. Then prove that $b_n \to L$.

Proof: First note that by definition of convergence, for $\epsilon > 0$, and $n > N \in \mathbb{N}$,

$$|a_n - L| < \epsilon$$

We wish to show that for $n > N$,

$$|b_n - L| < \epsilon^*$$

for some $\epsilon^* > 0$

Begin by noticing that

$$|b_n - L| = |b_n - a_n + a_n - L|$$

And by the triangle inequality,

$$|b_n - a_n + a_n - L| \leq |b_n - a_n| + |a_n - L|$$

$$= |a_n - b_n| + |a_n - L|$$

And assume that $n > N$, then

$$|a_n - b_n| + |a_n - L| < |a_n - b_n| + \epsilon$$

$$\leq \frac{1}{n} + \epsilon$$

And since $n > N$, $\frac{1}{n} < \frac{1}{N}$, so

$$\frac{1}{n} + \epsilon < \frac{1}{N} + \epsilon$$

Let $\epsilon^* = \frac{1}{N} + \epsilon > 0$

Finally, we have

$$|b_n - L| < \epsilon^*$$

and since ϵ^* can be made arbitrarily small, by definition of convergence, $b_n \to L$.

$\square$

2

2. (a) A sequence of real numbers $\{a_n\}$ is defined by $a_1 = 0$ and $a_{n+1} = \sqrt{3a_n + 4}, n \geq 1$. Prove that a_n is a convergent sequence and find $\lim_{n\to\infty} a_n$. (Hint: Show that $a_n \leq 4$ for all $n \geq 1$).

Proof: First I will show $a_n \leq 4$ for all $n \geq 1$ by induction. The base case is obvious ($a_1 = 0 \leq 4$). Assume this relationship to be true up to some natural number k. We must show the relation also holds for $k + 1$.
By the induction assumption,

$$a_k \leq 4$$

$$3a_k \leq 12$$

$$3a_k + 4 \leq 16$$

Then by the definition of a_{n+1},

$$a_{k+1}^2 \leq 16$$

$$|a_{k+1}| \leq 4$$

Thus $a_n \leq 4$ for all $n \geq 1$.
Now I will show a_n is a decreasing sequence by induction.
Base case:

$$a_2 = \sqrt{3 * 0 + 4} = \sqrt{4} = 2 \geq 0 = a_1$$

Now assume this relationship to be true up to some natural number k. We must show the relationship also holds for $k + 1$. By the induction assumption,

$$a_k \geq a_{k-1}$$

$$3a_k \geq 3a_{k-1}$$

$$3a_k + 4 \geq 3a_{k-1} + 4$$

$$\sqrt{3a_k + 4} \geq \sqrt{3a_{k-1} + 4}$$

$$a_{k+1} \geq a_k$$

Which tells us that a_n is an increasing sequence, provided $a_k \geq -\frac{4}{3}$ for all k. Notice that $a_1 = 0$, and a_n is increasing for all non-negative terms, so we have that a_n is an increasing sequence. Now Since a_n is increasing, bounded above by 4, and clearly bounded below by 0, by the Monotone Convergence Theorem, a_n is a convergent sequence.

□

Now that we have established that a_n is a convergent sequence, let $\lim_{n\to\infty} a_n = a$. We must find a. Begin by applying the limit to the recursion relation:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3a_n + 4}$$

$$a = \sqrt{3a + 4}$$

$$a^2 = 3a + 4$$

$$a^2 - 3a - 4 = 0$$

$$(a-4)(a+1) = 0$$

So either $a = 4$ or $a = -1$. But since $a_n \geq 0$ for all n, we have that a=4. Thus,

$$\lim_{n\to\infty} a_n = 4$$

(b) Define a sequence $\{x_n\}$ by $x_{n+1} = 1 - \sqrt{1 - x_n}$, n = 0,1,2, ... where $0 < x_0 < 1$. Find x_2 and x_3 in terms of x_0 and prove that the sequence $\{x_n\}$ converges.

$$x_1 = 1 - \sqrt{1 - x_0}$$

$$x_2 = 1 - \sqrt{1 - x_1} = 1 - \sqrt{1 - (1 - \sqrt{1 - x_0})}$$

$$= 1 - (1 - x_0)^{\frac{1}{4}}$$

$$x_3 = 1 - \sqrt{1 - x_2} = 1 - \sqrt{1 - (1 - (1 - x_0)^{\frac{1}{4}})}$$

$$= 1 - (1 - x_0)^{\frac{1}{8}}$$

Proof: I will begin by showing $\{x_n\}$ is bounded. A simple induction argument will show

$$x_n = 1 - (1 - x_0)^{\frac{1}{2^n}}$$

Now consider $f_n(x_0) = 1 - (1 - x_0)^{\frac{1}{2^n}}$ on $0 < x_0 < 1$ and find its extreme values:

$$\frac{df_n}{dx_0} = \frac{-1}{2^n}(1 - x_0)^{\frac{1}{2^n} - 1}(-1)$$

$$= \frac{1}{2^n}(1-x_0)^{\frac{1}{2^n}-1}$$

Notice that $\frac{df_n}{dx_0}$ contains no zeros on (0,1), so by the Extreme Value Theorem, we know that the extreme values must be at $x_0 = 0$ and $x_0 = 1$. Plugging these values into f_n:

$$f_n(0) = 1 - \sqrt{1-0} = 1 - 1 = 0$$
$$f_n(1) = 1 - \sqrt{1-1} = 1$$

Now we have that $\sup f_n(x_0) = 1$ and $\inf f_n(x_0) = 0$, or in other words, $f_n(x_0)$ is bounded. Then $\{x_n\}$ is bounded. And notice that $\frac{df_n}{dx_0} \geq 0$, so $f_n(x_0)$ is increasing, then $\{x_n\}$ is increasing. Now by the Monotone Convergence Theorem, $\{x_n\}$ converges.

□

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3. (a) Let $\{a_k\}$ be a real sequence. Define $\sigma_n := \frac{a_1+a_2+\ldots+a_n}{n}$. If $\lim a_k = a$, prove that $\lim \sigma_n = a$. Show that the converse is false.

Proof:We have $\lim a_k = a$, so by the definition of limits, for $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that when $k > N$,

$$|a_k - a| < \epsilon$$

We wish to show for some $\hat{\epsilon} > 0$,

$$|\sigma_n - a| < \hat{\epsilon} \; whenever \; n > N$$

Well,

$$|\sigma_n - a| = |\frac{a_1+a_2+\ldots+a_n}{n} - a|$$
$$= |\frac{a_1+a_2+\ldots+a_n - na}{n}|$$
$$= \frac{1}{n}|(a_1-a)+(a_2-a)+\ldots+(a_n-a)|$$
$$\leq \frac{1}{n}(|a_1-a|+|a_2-a|+\ldots+|a_n-a|)$$
$$< \frac{1}{n}(|a_1-a|+|a_2-a|+\ldots+|a_N-a|+(n-N)\epsilon)$$
$$= \frac{1}{n}(|a_1-a|+|a_2-a|+\ldots+|a_N-a|)+\frac{n-N}{n}\epsilon$$
$$\leq \frac{1}{n}(|a_1-a|+|a_2-a|+\ldots+|a_N-a|)+\epsilon$$

Now let $A = \max\{|a_1-a|, |a_2-a|, \ldots, |a_N-a|\}$. Now we have

$$\frac{1}{n}(|a_1-a|+|a_2-a|+\ldots+|a_N-a|)+\epsilon \leq \frac{1}{n}(NA)+\epsilon$$

$$= \frac{N}{n}A + \epsilon$$

And by the Archimedean property, for n large, $\frac{1}{n} < \epsilon^*$ for some $\epsilon^* > 0$. Now we have

$$\frac{N}{n}A + \epsilon < NA\epsilon^* + \epsilon$$

Now let $\hat{\epsilon} = NA\epsilon^* + \epsilon$, which can be made arbitrarily small. We finally have

$$|\sigma_n - a| < \hat{\epsilon}$$

And by the definition of the limit, $\lim \sigma_n = a$.

$\square$

Now consider $\{a_n\} = (-1)^n$. Then $\sigma_n = \frac{-1+1-1+1-...+1}{n}$. Notice that

$$\sigma_n = 0\, n\, even$$

$$\sigma_n = \frac{-1}{n}\, n\, odd$$

Clearly,

$$\lim \sigma_n = 0$$

but $\lim a_n$ DNE. Thus, the converse is false.

$\square$

(b) For a real sequence $\{a_n\}$ define $d_n := a_{n+1} - a_n$ for $n \geq 1$. If $\lim nd_n = 0$ and the sequence $\{\sigma_n\}$ defined in part (a) converges, then prove that the sequence $\{a_n\}$ converges and $\lim a_n = \lim \sigma_n$. (Hint: Show that $\frac{1}{n}\sum_{k=1}^{n-1} kd_k = a_n - \sigma_n$ for $n > 1$).

Proof: I will begin by showing that

$$\frac{1}{n}\sum_{k=1}^{n-1} kd_k = a_n - \sigma_n \tag{1}$$

Using the definition of σ_n in part a),

$$a_n - \sigma_n = a_n - \frac{a_1 + a_2 + \ldots + a_n}{n}$$

$$= \frac{-a_1 - a_2 - \ldots + (n-1)a_n}{n}$$

Now let's expand the left side of equation (1):

$$\frac{1}{n}\sum_{k=1}^{n-1} kd_k = \frac{1}{n}(d_1 + 2d_2 + \ldots + (n-1)d_{n-1})$$

$$= \frac{1}{n}(a_2 - a_1 + 2a_3 - 2a_2 + \ldots + (n-1)a_n)$$

$$= \frac{-a_1 - a_2 - \ldots + (n-1)a_n}{n}$$

$$= a_n - \sigma_n$$

Now, since we are given that $\lim nd_n = 0$, from part a), we know that

$$\lim\left(\frac{d_1 + 2d_2 + \ldots + nd_n}{n}\right) = 0$$

Now let's add and subtract nd_n to the left side of (1):

$$\frac{1}{n}\sum_{k=1}^{n-1} kd_k + nd_n - nd_n = a_n - \sigma_n$$

Which simplifies to

$$\frac{1}{n}\sum_{k=1}^{n} kd_k - nd_n = a_n - \sigma_n$$

Now apply the limit to each side:

$$\lim\left(\frac{1}{n}\sum_{k=1}^{n} kd_k\right) - \lim nd_n = \lim\,(a_n - \sigma_n)$$

Notice that

$$\frac{1}{n}\sum_{k=1}^{n} kd_k = \frac{d_1 + 2d_2 + 3d_3 + \ldots + nd_n}{n}$$

And since

$$\lim \frac{d_1 + 2d_2 + \ldots + nd_n}{n} = 0,$$

$$\lim \frac{1}{n}\sum_{k=1}^{n} kd_k = 0$$

Now we have

$$-\lim nd_n = \lim\,(a_n - \sigma_n)$$

And since $\lim nd_n = 0$,

$$\lim\,(a_n - \sigma_n) = 0$$

It is not entirely clear that $\{a_n\}$ converges. Assume by contradiction that $\{a_n\}$ diverges. And since $\{\sigma_n\}$ converges, say to σ, for $\epsilon > 0$, $n > N \in \mathbb{N}$, such that

$$|\sigma_n - \sigma| < \epsilon$$

Or, alternatively,

$$\sigma - \epsilon < \sigma_n < \sigma + \epsilon$$

And notice that

$$a_n + \sigma - \epsilon > a_n - \sigma_n > a_n - (\sigma - \epsilon)$$

Now we have

$$\begin{aligned} 0 = \lim\left(a_n - \sigma_n\right) &> \lim\left(a_n - (\sigma + \epsilon)\right) \\ &= \lim a_n - \lim \sigma + \epsilon \\ &= \lim a_n - (\sigma - \epsilon) \\ &= +\infty \end{aligned}$$

So now we get

$$0 > +\infty$$

A contradiction! Thus, $\{a_n\}$ must converge. In fact,

$$\lim\left(a_n - \sigma_n\right) = 0$$

$$\lim a_n - \lim \sigma_n = 0$$

$$\lim a_n = \lim \sigma_n$$

□

4

4. (a) Let $\{a_n\}$ be a strictly decreasing sequence of positive numbers. Assume $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim n a_n = 0$. (Hint: Use Cauchy convergence criterion for series).

Proof: We have that $\sum_{n=1}^{\infty} a_n$ converges, so by Cauchy criterion for series, we have for $\epsilon > 0$, $\exists\, n > m > N \in \mathbb{N}$,

$$\left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| < \epsilon$$

Or, equivalently,

$$\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$$

We also know that $\{a_n\}$ is a strictly decreasing sequence, so for $k = m+1, m+2 \ldots, n$, $a_n < a_i$ for all $i = k$. Then

$$\begin{aligned} \left|\sum_{k=m+1}^{n} a_k\right| &\geq \left|\sum_{k=m+1}^{n} a_n\right| \\ &= |a_n(n-m)| \\ &= |na_n - ma_n| \end{aligned}$$

$$\geq |na_n| - |ma_n|$$

Additionally, by the test for divergence, we have that $\lim a_n = 0$, or by definition, for some $\epsilon^* > 0$, $n > N^* \in \mathbb{N}$,

$$|a_n - 0| < \epsilon^*$$

From above, we have that

$$|na_n| - |ma_n| < \epsilon$$

$$|na_n| < \epsilon + m|a_n|$$

Now take $N^{max} = max\{N, N^*\}$ and assume $n > N^{max}$. Now we have

$$|na_n| < \epsilon + m\epsilon^*$$

let $\hat{\epsilon} = \epsilon + m\epsilon^*$, which can get arbitrarily small. Now we have

$$|na_n| < \hat{\epsilon}$$

$$|na_n - 0| < \hat{\epsilon}$$

And finally, by definition of limits, we have that $\lim na_n = 0$.

$\square$

(b) Give an example of a strictly decreasing positive sequence $\{b_n\}$ such that $\lim_{n\to\infty} nb_n = 0$, but $\sum_{n=1}^{\infty} b_n$ *diverges*. You must show the divergence of your example series.

Consider $b_n = \frac{1}{n \ln n}$ for n = 2,3,... and $b_1 = 0$. Notice that

$$\lim_{n\to\infty} nb_n = \lim_{n\to\infty} \frac{n}{n \ln n}$$

$$= \lim_{n\to\infty} \frac{1}{\ln n} = 0$$

I will claim that $\sum_{n=1}^{\infty} b_n$ diverges. Quickly note that $\sum_{n=1}^{\infty} b_n = \sum_{n=2}^{\infty} b_n$ since $b_1 = 0$.

Proof: Consider $f(x) = \frac{1}{x \ln{(x)}}$ on $x \geq 2$ and note that

$$\int_2^{\infty} f(x)dx \leq \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Well,

$$\int_2^{\infty} f(x)dx = \lim_{a\to\infty} \int_2^a \frac{1}{x \ln x} dx$$

Now, let $u = \ln x$, then $du = \frac{1}{x}dx$ and the integral becomes

$$\lim_{a\to\infty} \int_2^a \frac{1}{u} du$$

$$= \lim_{a\to\infty} [\ln|u|]|_{\ln 2}^{\ln a}$$

$$= \lim_{a\to\infty} (\ln(\ln a) - \ln(\ln 2))$$

$$= +\infty$$

So by comparison,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \; diverges$$

□

5

5. (a) Use the Mean Value Theorem to show that $\frac{x}{1+x} \leq \ln(1+x) \leq x, \; x \geq 0$. Then set $x = \frac{1}{n}$ to obtain

$$\frac{1}{n+1} \leq \ln(1 + \frac{1}{n}) \leq \frac{1}{n}$$

Let $f(y) = \ln(1+y)$ and consider the interval $y \in [0, x], x > 0$. By the Mean Value Theorem, there exists a $c \in [0, x]$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

Well,

$$f'(c) = \frac{1}{1+c}$$

So now we have

$$\frac{\ln(1+x)}{x} = \frac{1}{1+c}$$

Notice since $c \geq 0$, $\frac{1}{1+c} \leq 1$. Now we have

$$\frac{\ln(1+x)}{x} \leq 1$$

$$\ln(1+x) \leq x$$

Also notice that since $c \in [0, x]$, $x \geq c$, and

$$\frac{1}{1+c} \geq \frac{1}{1+x}$$

Now,

$$\frac{1}{1+x} \leq \frac{\ln(1+x)}{x}$$

$$\frac{x}{1+x} \leq \ln(1+x)$$

Putting these inequalities together, we have

$$\frac{x}{1+x} \leq \ln(1+x) \leq x$$

Now replace x with $\frac{1}{n}$ to obtain

$$\frac{\frac{1}{n}}{1+\frac{1}{n}} \leq \ln(1+\frac{1}{n}) \leq \frac{1}{n}$$

Which simplifies to

$$\frac{1}{n+1} \leq \ln(1+\frac{1}{n}) \leq \frac{1}{n}$$

(b) Define $\gamma_n = (1+1/2+1/3+\ldots+1/n) - \ln n$. Use part (a) to show that $\gamma_n \geq 0$ and that $\{\gamma_n\}$ is a decreasing sequence.

Proof: Consider the sum

$$\sum_{k=1}^{n-1} \ln(1+\frac{1}{k})$$

And notice that

$$\begin{aligned}\ln(1+\frac{1}{k}) &= \ln(\frac{k+1}{k}) \\ &= \ln(k+1) - \ln k\end{aligned}$$

Substituting this into the sum above, we get

$$\sum_{k=1}^{n-1}(\ln(k+1) - \ln k) = \ln 2 - \ln 1 + \ln 3 - \ln 2 + \ldots + \ln n - \ln(n-1)$$

$$= \ln n$$

And by the inequality in part (a),

$$\sum_{k=1}^{n-1}(\ln(k+1) - \ln k) \leq \sum_{k=1}^{n-1}\frac{1}{k}$$

$$\ln n \leq 1 + \frac{1}{2} + \ldots + \frac{1}{n-1}$$

$$\ln n + \frac{1}{n} \leq 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} + \frac{1}{n}$$

$$0 \leq \frac{1}{n} \leq 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$$

So, from above, we have that

$$\gamma_n \geq 0$$

□

Now we must show $\{\gamma_n\}$ is a decreasing sequence. That is, we must show $\gamma_{n+1} - \gamma_n \leq 0$. By definition of γ_n,

$$\gamma_{n+1} - \gamma_n = (1 + \frac{1}{2} + \ldots + \frac{1}{n} + \frac{1}{n+1}) - \ln(n+1) - [(1 + \frac{1}{2} + \ldots + \frac{1}{n}) - \ln n]$$

$$= \frac{1}{n+1} - \ln(n+1) + \ln n$$

$$= \frac{1}{n+1} - \ln(1 + \frac{1}{n})$$

Notice by the inequality in part (a),

$$\frac{1}{n+1} \leq \ln(1 + \frac{1}{n})$$

$$\frac{1}{n+1} - \ln(1 + \frac{1}{n}) \leq 0$$

Then $\gamma_{n+1} - \gamma_n \leq 0$, meaning that $\{\gamma_n\}$ is a decreasing sequence.

$\square$

(c) Show that $\{\gamma_n\}$ converges. $\lim \gamma_n = \gamma$ is called Euler's constant.

Since $\{\gamma_n\}$ is a decreasing sequence, γ_1 will be an upper bound for $\{\gamma_n\}$.

$$\gamma_1 = 1 - \ln 1 = 1$$

And since we showed that $\gamma_n \geq 0$ for all n,

$$0 \leq \gamma_n \leq 1$$

Now we have that $\{\gamma_n\}$ is a bounded decreasing sequence, and so, by the Monotone Convergence Theorem, $\{\gamma_n\}$ converges.

$\square$