Homework 6

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1. Determine the radius of convergence of the given power series

   (a) $\sum_{n=0}^{\infty}(x-3)^n$

   Soln. Notice that, for this power series, $a_n = 1$ for all n. By the ratio test, we have

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty} 1 \\ &= 1 \end{aligned}$$

   so that the radius of convergence is $R = 1$.

   (b) $\sum_{n=1}^{\infty}\frac{(x-x_0)^n}{n}$

   Soln. Notice that for this power series, $a_n = \frac{1}{n}$. By the ratio test, we find

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} \\ &= \lim_{n\to\infty}\frac{n+1}{n} \\ &= 1 \end{aligned}$$

   so that the radius of convergence is $R = 1$.

   (c) $\sum_{n=1}^{\infty}\frac{(-1)^n n^2(x+2)^n}{3^n}$

   Soln. Notice that $a_n = \frac{(-1)^n n^2}{3^n}$ and so, by the ratio test, we have

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^n n^2}{3^n}}{\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}}}\right| \\ &= \lim_{n\to\infty}\frac{3^{n+1}n^2}{3^n(n+1)^2} \\ &= 3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 \\ &= 3 \end{aligned}$$

   so that the radius of convergence is $R = 3$.

2. Find the power series solution about the given point x_0. Find the first four terms in each of the two solutions y_1 and y_2. By evaluating the Wronskian $W(y_1, y_2)(x_0)$ show that y_1 and y_2 form a fundamental set of solutions.

   (a) $y'' - xy' - y = 0, \qquad x_0 = 0$

   Soln. Assume that y may be expressed as a power series $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ that converges for $|x| < \rho$ for some $\rho > 0$. Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

   and so, by plugging this in to our differential equation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \implies \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+1) a_n x^n \end{aligned}$$

   and by re-indexing, we find

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

   so that the above series representation of the differential equation becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = \sum_{n=0}^{\infty} (n+1) a_n x^n$$

   which gives us the following recurrence relation:

$$a_{n+2} = \frac{a_n}{n+2}.$$

   For the first few even terms, notice

$$\begin{aligned} a_2 &= \frac{a_0}{2} \\ a_4 &= \frac{a_2}{4} = \frac{a_0}{4 \cdot 2} \\ a_6 &= \frac{a_4}{6} = \frac{a_0}{6 \cdot 4 \cdot 2} \\ a_8 &= \frac{a_6}{8} = \frac{a_0}{8 \cdot 6 \cdot 4 \cdot 2} \\ &\vdots \end{aligned}$$

and likewise for the odd terms,

$$\begin{aligned}
a_3 &= \frac{a_1}{3} \\
a_5 &= \frac{a_3}{5} = \frac{a_1}{5 \cdot 3} \\
a_7 &= \frac{a_5}{7} = \frac{a_1}{7 \cdot 5 \cdot 3} \\
a_9 &= \frac{a_7}{9} = \frac{a_1}{9 \cdot 7 \cdot 5 \cdot 3} \\
&\vdots
\end{aligned}$$

which gives us

$$a_{2k} = \frac{a_0}{(2k)!!}, \qquad a_{2k+1} = \frac{a_1}{(2k+1)!!}$$

where $(\cdot)!!$ denotes the double factorial. Thus, the solution to our differential equation is given by $y = a_0 y_1 + a_1 y_2$ with

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!!}, \qquad y_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!!}.$$

To see that y_1 and y_2 form a fundamental set of solutions, let us inspect the Wronskian $W(y_1, y_2)(x_0)$:

$$\begin{aligned}
W(y_1, y_2)(x_0) &= \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \\
&= y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0)
\end{aligned}$$

and notice that $y_2(x_0) = 0$, and that $y_1(x_0) = 1$, $y_2'(x_0) = 1$ so that

$$W(y_1, y_2)(x_0) = 1 \neq 0$$

so that y_1 and y_2 form a fundamental solution set. Also notice that both series converge for all real numbers since

$$\lim_{n \to \infty} \frac{(n+1)!!}{n!!} = \infty$$

(b) $(1-x)y'' + y = 0, \qquad x_0 = 0$

Soln. Assume that we may express y as a power series $y = \sum_{n=0}^{\infty} a_n x^n$ that converges for $|x| < \rho$ for some $\rho > 0$. Then

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and so, by plugging this into our differential equation, we find

$$\begin{aligned}
(1-x) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\implies \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n &= 0
\end{aligned}$$

which, after re-indexing, becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Equating terms, we find

$$a_2 = -\frac{1}{2}a_0$$

and in general,

$$a_{n+3} = \frac{n+1}{n+3}a_{n+2} - \frac{1}{(n+3)(n+2)}a_{n+1}$$

3. The Chebyshev differential equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant. (a) Determine two solutions in powers of x for $|x| < 1$ and show that the form a fundamental set of solutions. (b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are useful in problems that require a polynomial approximation defined on $-1 \leq x \leq 1$. (c) Find a polynomial solution for the cases $\alpha = n = 0, 1, 2, 3$.

Soln. (a) Assume that we may express the solution y of the differential equation in a power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

for $|x| < \rho$ for some $\rho > 0$. Now,

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

so that the differential equation becomes

$$\begin{aligned} (1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} n a_n x^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \implies \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n=0}^{\infty} a_n x^n [n(n-1) + n - \alpha^2] \\ \implies \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n=0}^{\infty} a_n x^n (n^2 - \alpha^2) \end{aligned}$$

and by re-indexing the left hand side, we find

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = \sum_{n=0}^{\infty} a_n x^n (n^2 - \alpha^2)$$

which gives us the recurrence relation

$$a_{n+2} = a_n \frac{n^2 - \alpha^2}{(n+2)(n+1)}.$$

Writing out the first few even terms, we find

$$\begin{aligned} a_2 &= a_0 \frac{-\alpha^2}{2!} \\ a_4 &= a_2 \frac{2^2 - \alpha^2}{4 \cdot 3} = a_0 \frac{(2^2-\alpha^2)(0^2-\alpha^2)}{4!} \\ a_6 &= a_4 \frac{4^2 - \alpha^2}{6 \cdot 5} = a_0 \frac{(4^2-\alpha^2)(2^2-\alpha^2)(0^2-\alpha^2)}{6!} \\ &\vdots \end{aligned}$$

and the first few odd terms:

$$\begin{aligned}
a_3 &= a_1\frac{1^2-\alpha^2}{3!}\\
a_5 &= a_3\frac{3^2-\alpha^2}{5\cdot 4} = a_1\frac{(3^2-\alpha^2)(1^2-\alpha^2)}{5!}\\
a_7 &= a_5\frac{5^2-\alpha^2}{7\cdot 6} = a_1\frac{(5^2-\alpha^2)(3^2-\alpha^2)(1^2-\alpha^2)}{7!}\\
&\vdots
\end{aligned}$$

which gives us the general forms

$$a_{2k} = \frac{a_0}{(2k)!}\prod_{n=1}^{k}[(2n-2)^2-\alpha^2]$$

$$a_{2k+1} = \frac{a_1}{(2k+1)!}\prod_{n=1}^{k}[(2n-1)^2-\alpha^2]$$

so that our solution takes the form $y = a_0 y_1 + a_1 y_2$ with

$$y_1 = 1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\prod_{k=1}^{n}[(2k-2)^2-\alpha^2]$$

$$y_2 = x + \sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\prod_{k=1}^{n}[(2k-1)^2-\alpha^2].$$

Now, to see that y_1 and y_2 form a linearly independent solution set, consider $W(y_1,y_2)(0)$:

$$W(y_1,y_2)(0) = y_1(0)y_2'(0) - y_1'(0)y_2(0)$$

and notice that $y_1(0) = 1$, $y_2(0) = 0$ and that $y_2' = 1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)!}\prod_{k=1}^{n}[(2k-1)^2-\alpha^2)]$ so that $y_2'(0) = 1$. Thus,

$$W(y_1,y_2)(0) = 1 \neq 0$$

and so y_1 and y_2 form a fundamental solution set.

(b) Let $\alpha = \ell \in \mathbb{Z}$ and consider the following cases:
Case I: ℓ is even. Then for $n \geq \ell/2+1$, the terms in the series expansion truncate due to the appearance of $\ell^2 - \alpha^2$ in the terms. That is,

$$y_1 = 1 + \sum_{n=1}^{\ell/2}\frac{x^{2n}}{(2n)!}\prod_{k=1}^{n}[(2k-2)^2-\alpha^2]$$

Case II: ℓ is odd. Same story for Case I, except now y_2 truncates for $n > (\ell-1)/2$:

$$y_2 = x + \sum_{n=1}^{(\ell-1)/2}\frac{x^{2n+1}}{(2n+1)!}\prod_{k=1}^{n}[(2k-1)^2-\alpha^2]$$

(c) For $\alpha = 0$, we find

$$y_1 = 1$$

and for $\alpha = 1$:

$$y_2 = x$$

and $\alpha = 2$:

$$y_1 = 1 - 2x^2$$

and finally for $\alpha = 3$:

$$y_2 = x - \frac{4x^3}{3}$$

4. Consider the Euler equations
$$x^2y'' + \alpha xy' + \beta y = 0$$
with repeated roots solutions, i.e. $(\alpha - 1)^2 = 4\beta$. Derive the general solution.

Soln. Begin with the ansatz $y = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ and plugging this into the equation yields

$$\begin{aligned} r(r-1) + \alpha r + \beta &= 0 \\ r^2 + (\alpha - 1)r + \beta &= 0 \\ \implies r &= \frac{1 - \alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2} \\ \implies r &= \frac{1-\alpha}{2} \end{aligned}$$

so that
$$y_1 = x^{\frac{1-\alpha}{2}}$$

Now, since the roots are repeated, we use reduction of order to seek a solution of the form

$$y_2 = c(x)y_1$$

and notice

$$\begin{aligned} y_2' &= c'(x)y_1 + c(x)y_1' \\ y_2'' &= c''(x)y_1 + 2c'(x)y_1' + c(x)y_1'' \end{aligned}$$

and plugging this into our differential equation yields

$$\begin{aligned} x^2c''(x)y_1 + 2x^2c'(x)y_1' + x^2c(x)y_1'' + \alpha c'(x)y_1 + \alpha c(x)y_1' + \beta c(x)y_1 &= 0 \\ x^2y_1c''(x) + 2x^2c'(x)(\tfrac{1-\alpha}{2})y_1x^{-1} + \alpha c'(x)y_1 + c(x)[x^2y_1'' + \alpha xy_1' + \beta y_1] &= 0 \end{aligned}$$

and since y_1 is a solution to the differential equation, we are left with

$$\begin{aligned} x^2y_1c''(x) + x(1-\alpha)y_1c'(x) + \alpha y_1c'(x) &= 0 \\ \implies x^2c''(x) + xc'(x) &= 0 \\ \implies xc''(x) + c'(x) &= 0 \end{aligned}$$

let $v = c'$ so that $v' = c''$ and the above differential equation

$$\begin{aligned} xv' + v &= 0 \\ \implies \int \frac{dv}{v} &= -\int \frac{1}{x}dx \\ \implies \ln(v) &= -\ln(x) + C \\ \implies v(x) &= \frac{C}{x} \\ \implies c(x) &= C\ln(x) \end{aligned}$$

so that the second solution is given by
$$y_2 = \ln(x)y_1.$$

Thus, the general solution for repeated roots is given by

$$y = a_1y_1 + a_2\ln(x)y_1.$$

5. Consider the differential equation

$$x^2y'' + xy' + (x-2)y = 0.$$

(a) Show that the differential equation has a regular singular point at $x = 0$.

Soln. Begin by dividing the differential equation through by x^2:

$$y'' + \frac{1}{x}y' + \frac{(x-2)}{x^2}y = 0$$

and notice that $p(x) = \frac{1}{x}$ and $q(x) = \frac{x-2}{x^2}$ and that

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$$
$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0}(x-2) = -2$$

so that $x = 0$ is a regular singular point of the differential equation.

(b) Determine the indicial equation, the recurrence relations, and the roots of the indicial equation.

Soln. Notice that the power series expansions for $xp(x)$ and $x^2q(x)$ are given as

$$xp(x) = 1, \qquad x^2q(x) = -2 + x$$

so that the corresponding Euler equation is

$$x^2y'' + xy' - 2y = 0.$$

Seek solutions of the form $y = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ so that, by plugging this into the differential equation, we find

$$x^r[r(r-1) + r - 2] = 0$$

which gives us the indicial equation

$$r^2 - 2 = 0$$

so that the roots of the indicial equation are $r = \pm\sqrt{2}$. Now seek the Frobenius solution

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}$$

so that $y' = \sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}$, $y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n-2}$, and by plugging these into our differential equation, we find

$$\begin{aligned} x^2y'' + xy' + (x-2)y &= \sum_{n=0}^{\infty}(r+n)(r+n-1)a_nx^{r+n} + \sum_{n=0}^{\infty}(r+n)a_nx^{r+n} + \sum_{n=1}^{\infty}a_{n-1}x^{r+n} - 2\sum_{n=0}^{\infty}a_nx^{r+n} \\ &= a_0x^r[r(r-r) + r - 2] + \sum_{n=1}^{\infty}x^{r+n}[(r+n)(r+n-1)a_n + (r+n)a_n - 2a_n + a_{n-1}] \\ &= a_0x^r[r^2 - 2] + \sum_{n=1}^{\infty}x^{r+n}[((r+n)^2 - 2)a_n + a_{n-1}] = 0 \end{aligned}$$

so that we find the recurrence relation for the coefficients:

$$a_n = -\frac{1}{(r+n)^2 - 2}a_{n-1}$$

which gives us

$$a_n = \frac{(-1)^n}{((r+n)^2-2)((r+n-1)^2-2)\cdots((r+1)^2-2)}a_0$$

(c) Find the two series solutions for $x > 0$.

Soln. We find the first solution from $r = \sqrt{2}$, so that

$$y_1 = x^{\sqrt{2}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\prod_{k=1}^{n}((k+\sqrt{2})^2 - 2)} \right)$$

and for $r = -\sqrt{2}$, we find

$$y_2 = x^{-\sqrt{2}} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\prod_{k=1}^{n}((k-\sqrt{2})^2 - 2)} \right)$$

6. (i) Show that $x = 0$ is a regular singular point of the given differential equation.
(ii) Find the exponents at the singularity point $x = 0$.
(iii) Find the first three nonzero terms in each of the two solutions about $x = 0$

(a) $xy'' + y' - y = 0$

Soln.

(i) Begin by dividing the differential equation by x:

$$y'' + \frac{1}{x}y' - \frac{1}{x}y = 0$$

and let $p(x) = \frac{1}{x}$, $q(x) = -\frac{1}{x}$ and notice

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} 1 = 1$$
$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x = 0$$

so that $x = 0$ is a regular singular point of the differential equation.

(ii) To find the exponents at the singularity $x = 0$, notice that the series expansions for $xp(x)$ and $x^2q(x)$ are

$$xp(x) = 1$$
$$x^2 q(x) = x$$

so that the corresponding Euler equation is

$$x^2y'' + xy' = 0$$

we assume solutions of the form $y = x^r$ and plug into the differential equation and find

$$r(r-1)x^r + rx^r = 0$$

and so $r^2 = 0 \implies r = 0$. Thus, the exponents at the singularity point $x = 0$ are $r_1 = r_2 = 0$.

(iii) We seek Frobenius solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ so that $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{n+r-2}$ and plugging this into the differential equation yields

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 7$$
$$\implies (r(r-1)+r)a_0 + \sum_{n=0}^{\infty}(n+r+1)^2 a_{n+1}x^{n+r} - a_n x^{n+r} = 0$$

which gives us the recurrence relation

$$a_{n+1} = \frac{a_n}{n+1+r}.$$

Hence, since $r = 0$, we find

$$a_n = \frac{a_0}{(n!)^2}$$

so that one solution to the differential equation is

$$y_1 = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

Since the roots of the indicial equation are repeated, we have that the other equation of the differential equation will be of the form

$$y_2 = \sum_{n=1}^{\infty} a_n'(0)x^n + \ln(x)y_1$$

where $a_n'(0) = \frac{da_n}{dr}\big|_{r=0}$. Notice

$$\begin{aligned}
a_1'(0) &= \left(\frac{d}{dr}\frac{a_0}{(1+r)^2}\right)\Bigg|_{r=0} \\
&= -2a_0 \\
a_2'(0) &= \left(\frac{d}{dr}\frac{a_0}{(1+r)^2(2+r)^2}\right)\Bigg|_{r=0} \\
&= a_0\left(\frac{-2}{2^2} - \frac{2}{2^3}\right) \\
&= a_0\left(-\frac{1}{2} - \frac{1}{4}\right) \\
&= -\frac{3}{4}a_0 \\
a_3'(0) &= \left(\frac{d}{dr}\frac{a_0}{(1+r)^2(2+r)^2(3+r)^2}\right)\Bigg|_{r=0} \\
&= a_0\left(-\frac{-2}{2^2\cdot 3^2} - \frac{2}{2^3\cdot 3^2} - \frac{2}{2^2\cdot 3^3}\right) \\
&= -2a_0\left(\frac{1}{36} + \frac{1}{72} + \frac{1}{108}\right) \\
&= -\frac{11}{108}a_0
\end{aligned}$$

so that the first three nonnegative terms of the two solutions are

$$\begin{aligned}
y_1 &= 1 + x + \frac{x^2}{4} + \cdots \\
y_2 &= -2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 + \ln(x)\left(1 + x + \frac{x^2}{4}\right) + \cdots
\end{aligned}$$

(b) $xy'' + 2xy' + 6e^x y = 0$

Soln.

(i) Begin by dividing the differential equation by x:

$$y'' + 2y' + \frac{6e^x}{x}y = 0$$

and let $p(x) = 2$, $q(x) = \frac{6e^x}{x}$ and notice

$$\begin{aligned}
\lim_{x\to 0} xp(x) &= \lim_{x\to 0} 2x = 0 \\
\lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} 6xe^x = 0
\end{aligned}$$

so that $x = 0$ is a regular singular point of the differential equation.

(ii) To find the exponents of the singularity $x = 0$, notice that the series expansions for $xp(x)$ and $x^2q(x)$ are given as

$$\begin{aligned} xp(x) &= 2x \\ x^2q(x) &= 6x + 6x^2 + 3x^3 + x^4 + \cdots \end{aligned}$$

so that the local Euler equation is

$$x^2y'' = 0.$$

Assume solutions of the form $y = x^r$. Then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ so that plugging this into the local Euler equation gives us

$$x^r[r(r-1)] = 0$$

which gives us

$$\begin{aligned} r(r-1) &= 0 \\ \implies r_1 &= 1, \qquad r_2 = 0 \end{aligned}$$

so that the exponents of the singularity $x = 0$ are $r_1 = 1$ and $r_2 = 0$.

(iii) Begin by noticing that $r_1 - r_2 = 1 \in \mathbb{Z}^+$. We first seek a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$. Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$ and plugging this into our differential equation yields

$$\begin{aligned} \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 6e^x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \implies r(r-1)a_0x^{r-1} + \sum_{n=0}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + 2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 6e^x \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \end{aligned}$$

by using the expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and collecting powers of x^{n+r}, we find the first few coefficients:

$$\begin{aligned} (r+1)(r)a_1 + 2ra_0 + 6a_0 &= 0 \\ \implies a_1 &= -a_0\frac{2r+6}{r(r+1)} \\ \implies a_1 &= -4a_0 && (r = 1) \\ (r+2)(r+1)a_2 + 2(r+1)a_1 + 6a_1 + 6a_0 &= 0 \\ \implies (r+2)(r+1)a_2 &= -2a_1(r+4) - 6a_0 \\ \implies a_2 &= a_0\frac{8r+26}{(r+2)(r+1)} \\ \implies a_2 &= \frac{17}{3}a_0 && (r = 1) \end{aligned}$$

Thus, the first few nonzero terms of y_1 are given by

$$y_1 = x\left(1 - 4x + \frac{17}{3}x^2 + \cdots\right) = x - 4x^2 + \frac{17}{3}x^3 + \cdots$$

Now since $r_1 - r_2 = 1$, we seek a second solution of the form

$$y_2 = a\ln(x)y_1 + \sum_{n=0}^{\infty} b_n x^n$$

where

$$a = \lim_{r\to 0} ra_1(r) = \lim_{r\to 0} -a_0\frac{2r+6}{r+1} = -6a_0$$

so

$$y_2 = -6\ln(x)y_1 + \sum_{n=0}^{\infty} b_n x^n.$$

We now seek to find b_n. Differentiating y_2 yields

$$y_2' = -\frac{6}{x}y_1 - 6\ln(x)y_1' + \sum_{n=0}^{\infty} nb_n x^{n-1}$$
$$y_2'' = \frac{6}{x^2}y_1 - \frac{12}{x}y_1' - 6\ln(x)y_1'' + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2}$$

plugging this into the differential equation gives us

$$\frac{6}{x}y_1 - 12y_1' - 6x\ln(x)y_1'' + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - 12y_1 - 12\ln(x)y_1' + \cdots$$
$$\cdots + 2\sum_{n=0}^{\infty} nb_n x^n - 36e^x\ln(x)y_1 + 6e^x\sum_{n=0}^{\infty} b_n x^n = 0$$
$$-6\ln(x)[xy_1'' + 2xy_1' + 6e^x y_1] + \frac{6}{x}y_1 - 12y_1' - 12y_1 + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} + \cdots$$
$$\cdots + 2\sum_{n=0}^{\infty} nb_n x^n + 6e^x\sum_{n=0}^{\infty} b_n x^n = 0$$

$-6\ln(x)[xy_1'' + 2xy_1' + 6e^x y_1] = 0$ since y_1 satisfies the ODE, so we are left with

$$\frac{6}{x}y_1 - 12y_1' - 12y_1 + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} + 2\sum_{n=0}^{\infty} nb_n x^n + 6e^x\sum_{n=0}^{\infty} b_n x^n = 0$$

using the series expansion $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the expansion for y_1 and collecting like powers of x, we find

$$b_0 = a_0$$
$$b_2 + 4b_1 = -33a_0$$
$$6b_3 + 10b_2 + 6b_1 = 119a_0$$