Nonlinear Waves Problems 3.10

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10. Solve the linear one-dimensional linear Schrödinger equation with quadratic potential (the "simple harmonic oscillator")

$$iu_t = u_{xx} - V_0 x^2 u,$$

with $V_0 > 0$ constant and $u(x, 0) = f(x)$ where $f(x)$ decays rapidly as $|x| \to \infty$. In what sense is the "ground state" (i.e., the lowest eigenvalue) the most important solution in the long-time limit?

Soln. Using separation of variables, we assume

$$u(x, t) = T(t)X(x)$$

and putting this into the differential equation gives

$$\begin{aligned} iT'(t)X(x) &= T(t)X''(x) - V_0 x^2 T(t)X(x) \\ \implies i\frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)} - V_0 x^2 \end{aligned}$$

since the left hand side is a function of t and the right hand side is a function of x, there exists a constant μ such that

$$\begin{aligned} i\frac{T'(t)}{T(t)} &= \mu \\ \implies iT' &= \mu T \\ \implies T(t) &= C_1 e^{-i\mu t} \\ \frac{X''(x)}{X(x)} - V_0 x^2 &= \mu \\ \implies X''(x) - (\mu + V_0 x^2)X(x) &= 0. \end{aligned}$$

Note that the above ODE is a Sturm-Liouville type equation with weighting function $w(x) = 1$. Now make the transformation $X(x) = e^{-\sqrt{V_0}x^2/2}y(x)$ and so

$$\begin{aligned} X'(x) &= -\sqrt{V_0}xe^{-\sqrt{V_0}x^2/2}y(x) + e^{-\sqrt{V_0}x^2/2}y'(x) \\ X''(x) &= -\sqrt{V_0}e^{-\sqrt{V_0}x^2/2}y' - 2\sqrt{V_0}xe^{-\sqrt{V_0}x^2/2}y + V_0 x^2 e^{-\sqrt{V_0}x^2/2} + e^{-\sqrt{V_0}x^2/2}y''. \end{aligned}$$

Plugging this into the ODE gives

$$\begin{aligned} -\sqrt{V_0}e^{-\sqrt{V_0}x^2/2}y - 2\sqrt{V_0}e^{-\sqrt{V_0}x^2/2} + V_0 x^2 e^{-\sqrt{V_0}}y + e^{-\sqrt{V_0}x^2/2}y'' - (\mu + V_0 x^2)e^{-\sqrt{V_0}x^2/2}y &= 0 \\ -\sqrt{V_0}y - 2\sqrt{V_0}xy' + y'' - \mu y &= 0 \\ y'' - 2\sqrt{V_0}xy' - (\mu + \sqrt{V_0})y &= 0. \end{aligned}$$

For the above ODE, we assume a series solution

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y'(x) &= \sum_{n=1}^{\infty} a_n n x^{n-1} \\ y''(x) &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}. \end{aligned}$$

Plugging this into the ODE, we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2\sqrt{V_0}\sum_{n=1}^{\infty} a_n n x^n - (\mu+\sqrt{V_0})\sum_{n=0}^{\infty} a_n x^n = 0 \\
\implies & 2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots \\
& - 2\sqrt{V_0}(a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots) \\
& - (\mu+\sqrt{V_0})(a_0 + a_1 x + a_2 x^2 + \cdots) = 0
\end{aligned}
$$

collecting terms order-by-order gives

$$
\begin{aligned}
\mathcal{O}(1) : 2a_2 - (\mu+\sqrt{V_0})a_0 &= 0 \\
\implies a_2 &= \frac{\mu+\sqrt{V_0}}{2}a_0 \\
\mathcal{O}(x) : 6a_3 - 2\sqrt{V_0}a_1 - (\mu+\sqrt{V_0})a_1 &= 0 \\
\implies a_3 &= \frac{\mu+3\sqrt{V_0}}{6}a_1 \\
\mathcal{O}(x^2) : 12a_4 - 4\sqrt{V_0}a_2 - (\mu+\sqrt{V_0})a_2 &= 0 \\
\implies a_4 &= \frac{\mu+5\sqrt{V_0}}{12}a_2 \\
&= \left(\frac{\mu+5\sqrt{V_0}}{12}\right)\left(\frac{\mu+\sqrt{V_0}}{2}\right)a_0.
\end{aligned}
$$

Then the two solutions to the differential equation are

$$
\begin{aligned}
y_1(x) &= a_0\left(1 + \frac{\mu+\sqrt{V_0}}{2}x^2 + \left(\frac{\mu+\sqrt{V_0}}{2}\right)\left(\frac{\mu+5\sqrt{V_0}}{12}\right)x^4 + \cdots\right) \\
y_2(x) &= a_1\left(x + \frac{\mu+3\sqrt{V_0}}{6}x^3 + \left(\frac{\mu+3\sqrt{V_0}}{6}\right)\left(\frac{\mu+7\sqrt{V_0}}{20}\right)x^5 + \cdots\right).
\end{aligned}
$$

Rewriting as

$$
\begin{aligned}
y_1(x) &= a_0\left(1 + b_2 x^2 + b_4 x^4 + \cdots\right) \\
y_2(x) &= a_1(x + b_3 x^3 + b_5 x^5 + \cdots)
\end{aligned}
$$

where

$$
\begin{aligned}
b_{2n} &= \frac{1}{(2n)!}\prod_{k=1}^{n}\left(\mu + (4k-3)\sqrt{V_0}\right) \\
b_{2n+1} &= \frac{1}{(2n+1)!}\prod_{k=1}^{n}\left(\mu + (4k-2)\sqrt{V_0}\right).
\end{aligned}
$$

Now consider the following cases:

$$
\begin{aligned}
(\mu_0 = -\sqrt{V_0}) &: y_1(x) = a_0 \\
(\mu_1 = -3\sqrt{V_0}) &: y_2(x) = a_1 x \\
(\mu_2 = -5\sqrt{V_0}) &: y_1(x) = a_0(1 - \sqrt{V_0}x^2) \\
(\mu_3 = -7\sqrt{V_0}) &: y_2(x) = a_1(x - \frac{2\sqrt{V_0}}{3}x^3) \\
&\vdots
\end{aligned}
$$

and note that we find one of the solutions of the ODE related to the Hermite polynomials. Let $H_n(x)$ be the n^{th} Hermite polynomial generated from the differential equation for eigenvalues $\mu_n = -(2n-1)\sqrt{V_0}$. Now let $\psi_n(x) = e^{-\sqrt{V_0}x^2/2}H_n(x)$. Then a general solution to the PDE is given as

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-i\mu_n t}\psi_n(x).$$

From the initial condition, we have

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x)$$

and by orthogonality, we find

$$c_n = \frac{\int_{-\infty}^{\infty} f(x)\psi_n(x)dx}{\int_{-\infty}^{\infty} \psi_n^2(x)dx}.$$

In the long time limit, note that all the temporal nodes cause oscillatory behavior, and for the smallest eigenvalue μ_0, the first term in the expansion will have the smallest temporal and spatial oscillatory behavior.