Problem Set 3 (Topology)

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Problem Set 3

1. In the topology $\mathcal{U}$ on $\mathbb{R}$, give an example of an infinite union of closed sets that is open (and bounded).

   Consider the collection of sets $\{a+x\}$ where $a \in \mathbb{R}$ and $x \in (0,1)$. Notice that $\{a+x\}$ is closed in $\mathbb{R}_{\mathcal{U}}$ since $\mathbb{R} \setminus \{a+x\} = (-\infty, a+x) \cup (a+x, \infty)$, which is open in $\mathbb{R}_{\mathcal{U}}$.

   Now consider the union
$$\bigcup_{x \in (0,1)} \{a+x\} = (a, a+1)$$
   Which is bounded and open in $\mathbb{R}_{\mathcal{U}}$.

2. (#2 in 3.3) Prove that no nonempty proper subset of $\mathbb{R}_{\mathcal{FC}}$ is simultaneously open and closed.

   Proof: Assume by way of contradiction that there exists a set $A \in \mathcal{FC}$ that is both open and closed, and consider $\mathbb{R} \setminus A$. Since A is assumed to be both open and closed, we have that $\mathbb{R} \setminus A \in \mathcal{FC}$.

   Since A is open, we have that $\mathbb{R} \setminus A$ is finite. That is, $\mathbb{R} \setminus A \sim \{1, 2, \ldots, n\}$ for some natural number n. Say $\mathbb{R} \setminus A = \{a_1, a_2, \ldots, a_n\}$. Now, if elements of $\mathbb{R} \setminus A$ are listed in ascending order, we have
$$\mathbb{R} \setminus (\mathbb{R} \setminus A) = (-\infty, a_i) \cup (a_i, a_j) \cup \ldots \cup (a_k, \infty)$$
   Since $\mathbb{R} \setminus (\mathbb{R} \setminus A)$ is a union of intervals, and intervals are uncountable, we have $\mathbb{R} \setminus (\mathbb{R} \setminus A)$ is infinite, contradicting our assumption that $\mathbb{R} \setminus A \in \mathcal{FC}$.

   Thus, we have that no set in $\mathcal{FC}$ can be both open and closed.

3. Find the closure and the interior of the interval [1,3] in $\mathbb{R}_{\mathcal{FC}}$.

   $\text{Int}([1,3]) = \emptyset$

   The largest set contained in $[1,3]$ that is open in $\mathbb{R}_{\mathcal{FC}}$ is the empty set. To see this, consider some non-empty subset $A \subseteq [1,3]$. Notice that $\mathbb{R} \setminus [1,3] \subseteq \mathbb{R} \setminus A$ and that $\mathbb{R} \setminus [1,3] = (-\infty, 1) \cup (3, \infty)$ which is uncountable since it is a union of open sets in $\mathbb{R}$. Thus, we have
$$(-\infty, 1) \cup (3, \infty) \subseteq \mathbb{R} \setminus A$$

so A is infinite. Thus, the interior of $[1,3]$ is the empty set.

$\text{Cl}([1,3]) = \mathbb{R}$

The smallest set that's closed in $\mathbb{R}_{\mathcal{FC}}$ that contains $[1,3]$ is $\mathbb{R}$. In other words, any proper subset of $\mathbb{R}$ that contains $[1,3]$ will not be open in $\mathbb{R}_{\mathcal{FC}}$. To see this, suppose by contradiction that $\text{Cl}([1,3]) = B$ for some $B \subset \mathbb{R}$. That is, $\mathbb{R} \setminus B$ is open in $\mathcal{FC}$, and so by definition, $\mathbb{R} \setminus (\mathbb{R} \setminus B)$ is open. Notice that $\mathbb{R} \setminus (\mathbb{R} \setminus B) = B$, so B must be finite. But since $\text{Cl}([1,3]) = B$, we have $[1,3] \subseteq B$. But $[1,3]$ is uncountable, so we must have that B is also uncountable, contradicting the fact that B must be finite from our assumption.

4. Let X and Y be topological spaces, and let $f : X \to Y$ be any function. Show that the following two conditions are equivalent:

   (a) If U is open in Y, then $f^{-1}(U)$ is open in X.

   (b) If F is closed in Y, then $f^{-1}(F)$ is closed in X.

   (Note: Statements A and B are *equivalent* if $A \implies B$ and $B \implies A$.)

   First I will prove as a lemma that $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.

   By definition of pre-image:

$$f^{-1}(Y \setminus F) = \{x \in X | f(x) \in Y \setminus F\}$$

$$X \setminus (f^{-1}(F)) = X \setminus \{x \in X | f(x) \in F\}$$

   We wish to show that these two are equal. Notice in the definition of $X \setminus (f^{-1}(F))$ that we are removing from X its elements that map into F. That is, we will be left with the elements in X that map into $Y \setminus F$. So

$$X \setminus (f^{-1}(F)) = \{x \in X | f(x) \in Y \setminus F\} = f^{-1}(Y \setminus F)$$

   Now begin by assuming that statement (a) is true and that F is a closed set in Y. We wish to show that $f^{-1}(F)$ is closed in X.

   Well, since F is a closed set in Y, $Y \setminus F$ is open in Y. And by (a), we have that $f^{-1}(Y \setminus F)$ is open in X. Then $X \setminus (f^{-1}(Y \setminus F))$ is closed in X. By the lemma, $f^{-1}(Y \setminus F) = X \setminus (f^{-1}(F))$, so $X \setminus f^{-1}(F)$ is open in X. Then we have $X \setminus (X \setminus f^{-1}(F)) = f^{-1}(x)$ is closed in X.

   So $f^{-1}(F)$ is closed in X.

   Now assume that statement (b) is true and that U is open in Y. We have that $Y \setminus U$ is closed in Y. Then, by (b), we have that $f^{-1}(Y \setminus U)$ is closed in X. By the lemma, we have that $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$. And since $X \setminus f^{-1}(U)$ is closed in X, we have that $X \setminus (X \setminus f^{-1}(U)) = f^{-1}(U)$ is open in X.

   So $f^{-1}(U)$ is open in X.

5. (#9 in 3.3) Prove Theorem 3.3.4: For a topological space X_τ and a subset $A \subseteq X_\tau$,

$$\mathrm{Int}(A) = \bigcup_{V \subseteq A, V \in \tau} V.$$

That is, show that the interior of a subset A of X_τ is the union of all τ-open sets contained in A.

Proof: Let $I = \bigcup_{V \subseteq A, V \in \tau} V$ and let $x \in I$ be an arbitrary element. Notice that since I is a union of open sets, that I is also open. Since every $V \subseteq A$, we have that $x \in A$. Thus, $I \subseteq A$, and since I is open, $I \subseteq \mathrm{Int}(A)$.

First consider the case that $\mathrm{Int}(A) = \emptyset$. Clearly, $\mathrm{Int}(A) \subseteq I$. Now consider the case $\mathrm{Int}(A) \neq \emptyset$. By definition, $\mathrm{Int}(A) \subseteq A$ is an open set in τ, and thus is by definition of I, $\mathrm{Int}(A) \subseteq I$. Finally, we have that

$$\mathrm{Int}(A) = \bigcup_{V \subseteq A, V \in \tau} V$$

6. (#11 in 3.3) Give an example of two subsets A and B of $\mathbb{R}_\mathcal{U}$ such that $\mathrm{Cl}(A \cap B) = \emptyset$ and $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathbb{R}$. Does your example work in $\mathbb{R}_\mathcal{L}$?

Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$. Notice that $A \cap B = \emptyset$, so $\mathrm{Cl}(A \cap B) = \emptyset$.

However, notice that $\mathrm{Cl}(A) = \mathbb{R}$ and $\mathrm{Cl}(B) = \mathbb{R}$ by density of the rationals and irrationals, so $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathbb{R}$.

This example does work in $\mathbb{R}_\mathcal{L}$. Consider any interval $(a, b) \in \mathbb{R}$.

We wish to show that $\mathrm{Cl}(A) = \mathbb{R}$. Assume by way of contradiction that $\mathrm{Cl}(A) \neq \mathbb{R}$. Then there exists some interval $[a, b) \subset \mathbb{R}$ such that $[a, b)$ contains no rational numbers. To see this, assume that $\mathrm{Cl}(\mathbb{Q}) = B \subset \mathbb{R}$. Then we have $\mathbb{R} \setminus B$ is open, and thus for every $x \in \mathbb{R} \setminus B$, there exists an interval $[a, b)$ such that $x \in [a, b) \subseteq \mathbb{R} \setminus B$. That is, $[a, b)$ must contain no rational numbers.

But by density of the rationals, we can find some rational number $c \in [a, b)$, contradicting the fact that $[a, b)$ contains no rational numbers. Then $\mathrm{Cl}(A) = \mathbb{R}$. A similar argument holds for $B = \mathbb{R} \setminus \mathbb{Q}$ using density of the irrationals. So we can see that $\mathrm{Cl}(A \cap B) = \emptyset$ and $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathbb{R}$.