Homework IV

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Section 2.5 Problems

10. Let X and Y be metric spaces, X compact, and $T : X \to Y$ bijective and continuous. Show that T is a homeomorphism.

Proof: Since T is bijective, we have that T^{-1} exists. Let $\{y_n\}$ be a sequence in Y that converges to a point $y \in Y$. We wish to show $T^{-1}(y_n) \to T^{-1}(y)$. Well, since T is bijective, we have that for each $y_n \in Y$, there exists $x_n \in X$ such that $T(x_n) = y_n$ (or, equivalently, $x_n = T^{-1}(y_n)$). Since X is compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Call the limit of this subsequence x. Since T is continuous, we have

$$T(x_{n_k}) \to T(x).$$

By the definition of our sequence $\{x_n\}$, we have $T(x_{n_k}) = y_{n_k}$ and since $y_n \to y$, $y_{n_k} \to y$ and since the limit of convergent sequences is unique, we have that $T(x) = y$.

Now, rewriting each x_{n_k} as $x_{n_k} = T^{-1}(y_{n_k})$, we have

$$T^{-1}(y_{n_k}) \to T^{-1}(y)$$

and since $\{y_{n_k}\}$ has the same limit as $\{y_n\}$, we have

$$T^{-1}(y_n) \to T^{-1}(y).$$

Thus, T^{-1} is continuous and so T is a homeomorphism.

Section 2.6 Problems

12. Does the inverse of T in 2.6-4 exit?

No, recall that the inverse of a linear operator T exists if and only if T is injective. I claim that T is not injective. Let $x(t) = t^2 + 1$ and $y(t) = t^2 + 2$ be in the space of polynomials on $[a, b]$. Then notice

$$Tx(t) = x'(t) = 2x$$

and

$$Ty(t) = y'(t) = 2x$$

so $Ty(t) = Tx(t)$, but $x(t) \neq y(t)$ for all $t \in [a, b]$. Thus, T is not injective, and so does not have an inverse.

Section 2.7 Problems

6. **(Range)** Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \to Y$ need not be closed in Y.

Proof: Consider the sequence $\{x_n\}$ in ℓ^∞ defined by $x_n = (1, 1, \cdots, 1, 0, 0, \cdots)$. Then $Tx_n = (1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots)$. So as $n \to \infty$, $Tx_n \to x = (1, \frac{1}{2}, \cdots, \frac{1}{n}, \frac{1}{n+1}, \cdots) \in \ell^\infty$. However, the preimage of x is given by

$$(1, 1, \cdots, 1, 1, \cdots) \notin \ell^\infty.$$

So $\{Tx_n\}$ is a sequence in $R(T)$ which converges to $x = (1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots) \in \ell^\infty$, but the preimage x is not in ℓ^∞ (that is, not in the domain of T), so $\{Tx_n\}$ does not converge in $R(T)$, hence $R(T)$ is not closed.

8. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded.

Proof: Consider the linear bounded operator $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \xi_j/j$, $x = (\xi_j)$ as in problem 5. Consider the sequence of vectors $x_n \in \ell^\infty$, $x_n = (1, 1, 1, \cdots, 1, 0, 0, \cdots)$ with n ones followed by zeros. Then

$$T^{-1}x_n = (1, 2, 3, \cdots, n, 0, 0, \cdots)$$

so that

$$\|T^{-1}x_n\| = n$$

which is not bounded below since, if m were a lower bound, there exists a natural number $N > m$ such that

$$\|T^{-1}x_N\| = N > m.$$

Hence, T^{-1} is unbounded

Assigned Exercise IV.1

(a) Let $X = C[0,1]$ be the continuous real-valued functions on $[0,1]$ with the usual sup-norm:

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define $T : X \to X$ by

$$y = Tx, \;\; y(t) = \int_0^t s \cdot x(s)ds, \quad \text{all } t \in [0,1].$$

Prove that T is a bounded linear operator and determine the value of the operator norm; $\|T\| = ?$ Justify your assertion.

Proof: We must first verify that T is a linear operator. Let α, β be arbitrary scalars and let $x, y \in C[0,1]$ and notice the following:

$$\begin{aligned}
T(\alpha x + \beta y) &= \int_0^t s \cdot (\alpha x(s) + \beta y(s))ds \\
&= \int_0^t (s \cdot (\alpha x(s)) + s \cdot (\beta y(s)))ds \\
&= \int_0^t (\alpha s \cdot x(t) + \beta s \cdot y(s))ds \\
&= \int_0^t \alpha s \cdot x(s)ds + \int_0^t \beta s \cdot y(s)ds \\
&= \alpha \int_0^t s \cdot x(s) + \beta \int_0^t s \cdot y(s)ds \\
&= \alpha Tx + \beta Ty
\end{aligned}$$

so T is a linear operator. We must now show that T is bounded, that is, there exists a real number c such that $\|Tx\| \le c\|x\|$ for all $x \in X$.

Notice

$$\begin{aligned}\left|\int_0^t s\cdot x(s)ds\right| &\le \int_0^t |s\cdot x(s)|ds\\ &\le \|x\|\int_0^t sds\\ &= \|x\|\frac{t^2}{2}\\ &\le \frac{1}{2}\|x\|.\end{aligned}$$

Then $\frac{1}{2}\|x\|$ is an upper bound for $|\int_0^t s\cdot x(s)ds|$, hence

$$\max_{t\in[0,1]}\left|\int_0^t s\cdot x(s)ds\right| \le \frac{1}{2}\|x\|.$$

Hence,

$$\|Tx\| \le \frac{1}{2}\|x\|$$

so that T is a bounded linear operator. For a lower bound, take $x(t) = 1$ on $[0,1]$. Clearly, $\|x(t)\| = 1$ and

$$\begin{aligned}\|Tx\| &= \max_{t\in[0,1]}\left|\int_0^t s\cdot 1ds\right|\\ &= \max_{t\in[0,1]}\left|\frac{t^2}{2}\right|\\ &= \frac{1}{2}\\ &= \frac{1}{2}\|x\|.\end{aligned}$$

Hence, $\|T\| \ge \frac{1}{2}$, so that we have

$$\|T\| = \frac{1}{2}$$

(b) Let X be the complex sequence space ℓ^2 with the usual norm $\|x\| = \sqrt{\sum_{j=1}^\infty |\xi_j|^2}$, where $x = (\xi_1, \xi_2, \dots)$. Fix $y = (\eta_1, \eta_2, \dots) \in \ell^2$, with $y \neq 0$, and define $f : X \to X$ by

$$f(x) = \sum_{j=1}^\infty \xi_j\overline{\eta}_j,\ \ x = (\xi_1, \xi_2, \dots) \in X; \quad \overline{\eta}_i \text{ is the complex conjugate of } \eta_i \in \mathbb{C}.$$

Prove that f is bounded as a linear operator (functional), and determine its operator norm $\|f\|$ in terms of y. Justify your assertion.

Proof: We will first verify that f is a linear operator. Let α, β be arbitrary scalars and let $x, z \in \ell^\infty$,

$x = (\xi_1, \xi_2, \cdots,)$ $z = (\zeta_1, \zeta_2, \cdots)$ and consider $f(\alpha x + \beta z)$:

$$\begin{aligned} f(\alpha x + \beta z) &= \sum_{j=1}^{\infty} (\alpha \xi_j + \beta \zeta_j) \overline{\eta}_j \\ &= \sum_{j=1}^{\infty} (\alpha \xi_j \overline{\eta}_j + \beta \zeta_j \overline{\eta}_j) \\ &= \sum_{j=1}^{\infty} \alpha \xi_j \overline{\eta}_j + \sum_{j=1}^{\infty} \beta \zeta_j \overline{\eta}_j \\ &= \alpha \sum_{j=1}^{\infty} \xi_j \overline{\eta}_j + \beta \sum_{j=1}^{\infty} \zeta_j \overline{\eta}_j \\ &= \alpha f(x) + \beta f(z) \end{aligned}$$

so f is a linear operator. We will now show that f is bounded. That is, we must find a $c \in \mathbb{R}$ such that $|f(x)| \leq c\|x\|$ for all x. Let $x \in \ell^2$, $x = (\xi_1, \xi_2, \cdots)$. Notice

$$\begin{aligned} |f(x)| &= \left| \sum_{j=1}^{\infty} \xi_j \overline{\eta}_j \right| \\ &\leq \sum_{j=1}^{\infty} |\xi_j \overline{\eta}_j| \\ &= \sum_{j=1}^{\infty} |\xi_j||\eta_j| \\ &\leq \left(\sum_{j=1}^{\infty} |\xi_j|^2 \right)^2 \left(\sum_{j=1}^{\infty} |\eta_j|^2 \right)^2 && \text{(Hölder's Inequality)} \\ &= \|y\|\|x\| \end{aligned}$$

Thus,

$$|f(x)| \leq \|y\|\|x\|.$$

and since $y \in \ell^2$ is fixed, we have that f is bounded. We must now find the operator norm. I claim that

$$|f| = \|y\|.$$

To see this, take $x = y \in \ell^2$. Then notice

$$\begin{aligned} |f(x)| &= \left| \sum_{j=1}^{\infty} \eta_j \overline{\eta}_j \right| \\ &= \left| \sum_{j=1}^{\infty} |\eta_j|^2 \right| \\ &= \|y\|^2 \\ &= \|y\|\|y\| \\ &= \|y\|\|x\|. \end{aligned}$$

Hence, $\|y\|$ is a lower bound for $|f|$, so

$$|f| = \|y\|$$