Modern Algebra HW 12

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Section 27 Problems

18. Is $\mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$ a field? Why?

$\mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$ is not a field because we may factor $x^2 - 5x + 6$ as $(x - 3)(x - 2)$ which shows us that $x^2 - 5x + 6$ is reducible in $\mathbb{Q}[x]$, and thus, $\mathbb{Q}[x]/\langle x^2 - 5x + 6\rangle$ is not a field.

19. Is $\mathbb{Q}[x]/\langle x^2 - 6x + 6\rangle$ a field? Why?

$\mathbb{Q}[x]/\langle x^2 - 6x + 6\rangle$ is a field. By the Eisenstein principle, let $p = 3$ and notice that 3 does not divide $a_2 = 1$ and 3 divides $a_1 = -6$ and $a_0 = 6$ but $3^2 = 9$ does not divide $a_0 = 6$. Thus, by the Eisenstein principle, $(x^2 - 6x + 6)$ is irreducible in $\mathbb{Q}$ and so $\mathbb{Q}[x]/\langle x^2 - 6x + 6\rangle$ is a field.

Section 29 Problems

18. a. Show that the polynomial $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.

By the low degree test, it suffices to show that any element of $\mathbb{Z}_3$ is not a zero for $x^2 + 1$ in $\mathbb{Z}_3[x]$. Denote $f(x) = x^2 + 1$ and notice the following:

$$f(0) = 0^2 + 1 = 1$$
$$f(1) = 1^2 + 1 = 1 + 1 = 2$$
$$f(2) = 2^2 + 1 = 1 + 1 = 2$$

So $x^2 + 1$ is irreducible in $\mathbb{Z}_3[x]$.
b. Let α be a zero of $x^2 + 1$ in an extension field of $\mathbb{Z}_3$. As in Example 29.19, give the multiplication and addition tables for the nine elements of $\mathbb{Z}_3(\alpha)$, written in the order 0,1 2, α, 2α, $1+\alpha$, $1+2\alpha$, $2+\alpha$, and $2+2\alpha$.

Quickly note that in $\mathbb{Z}_3(\alpha)$, $\alpha^2 = \overline{-1}$. Now let us inspect the addition and multiplication tables.

$+$	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
0	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
1	1	2	0	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$	α	2α
2	2	0	1	$2+\alpha$	$2+2\alpha$	α	2α	$1+\alpha$	$1+2\alpha$
α	α	$1+\alpha$	$2+\alpha$	2α	0	$1+2\alpha$	1	$2+2\alpha$	2
2α	2α	$1+2\alpha$	$2+2\alpha$	0	α	1	$1+\alpha$	2	$2+\alpha$
$1+\alpha$	$1+\alpha$	$2+\alpha$	α	$1+2\alpha$	1	$2+2\alpha$	2	2α	0
$1+2\alpha$	$1+2\alpha$	$2+2\alpha$	2α	1	$1+\alpha$	2	$2+\alpha$	0	α
$2+\alpha$	$2+\alpha$	α	$1+\alpha$	$2+2\alpha$	2	2α	0	$1+2\alpha$	1
$2+2\alpha$	$2+2\alpha$	2α	$1+2\alpha$	2	$2+\alpha$	0	α	1	$1+\alpha$

$\cdot$	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
2	0	2	1	2α	α	$2+2\alpha$	$2+\alpha$	$1+2\alpha$	$1+\alpha$
α	0	α	2α	2	1	$2+\alpha$	$1+\alpha$	$2+2\alpha$	$1+2\alpha$
2α	0	2α	α	1	2	$1+2\alpha$	$2+2\alpha$	$1+\alpha$	$2+\alpha$
$1+\alpha$	0	$1+\alpha$	$2+2\alpha$	$2+\alpha$	$1+2\alpha$	2α	2	1	α
$1+2\alpha$	0	$1+2\alpha$	$2+\alpha$	$1+\alpha$	$2+2\alpha$	2	α	2α	1
$2+\alpha$	0	$2+\alpha$	$1+2\alpha$	$2+2\alpha$	$1+\alpha$	1	2α	α	2
$2+2\alpha$	0	$2+2\alpha$	$1+\alpha$	$1+2\alpha$	$2+\alpha$	α	1	2	2α