CFD Homework 1

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9/5/24

1. Determine the mathematical character (hyperbolic, parabolic, or elliptic) of the system of the equations below.

$$\begin{aligned}\beta^2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0\\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0\end{aligned}$$

Soln. Begin by dividing the first equation by β^2:

$$\frac{\partial u}{\partial x} - \frac{1}{\beta^2}\frac{\partial v}{\partial y} = 0.$$

Writing the system of PDEs as a linear system, we find

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{\beta^2} \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 0 & -\frac{1}{\beta^2} \\ -1 & 0 \end{pmatrix}.$$

We can classify our system of PDEs by inspecting the eigenvalues of A. Let λ an eigenvalue of A. Then we want

$$\det(A - \lambda I) = 0$$

where I is the 2×2 identity matrix. That is,

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} -\lambda & -\frac{1}{\beta^2} \\ -1 & -\lambda \end{vmatrix}\\ &= \lambda^2 - \frac{1}{\beta^2} = 0\\ \implies \lambda &= \pm\frac{1}{\beta}.\end{aligned}$$

Thus, $\lambda_1 = \frac{1}{\beta}, \lambda_2 = -\frac{1}{\beta} \in \mathbb{R}$ hence the system of PDEs is **hyperbolic**.

2. a) Verify the solution of the following heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(t,0) = 0$, $u(t,1) = 0$ and initial distribution

$$u(0,x) = \sin(2\pi x)$$

is

$$u(x,t) = 2\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi x)\sin(n\pi x)dx\right)e^{-n^2\pi^2 t}\sin(n\pi x).$$

Note this problem only asks to verify the solution provided.

Soln. Note that since $u(x,t)$ is a Fourier series, care must be taken when differentiating $u(x,t)$. That is to say, we must prove why we can term-by-term differentiate the series expansion for $u(x,t)$. Notice

$$\begin{aligned}|u(x,t)| &= 2\left|\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi x)\sin(n\pi x)dx\right)e^{-n^2\pi^2 t}\sin(n\pi x)\right| \\ &\leq 2\sum_{n=1}^{\infty}\left(\int_0^1 |\sin(2\pi x)\sin(n\pi x)|dx\right)e^{-n^2\pi^2 t}|\sin(n\pi x)| \\ &\leq 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}.\end{aligned}$$

Note that for a fixed $t > 0$, we have that the series converges uniformly in x by geometric series. To show that the series converges uniformly in t, notice that on the interval $t \in [\varepsilon, T]$ for some $\varepsilon > 0$, T sufficiently large,

$$\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}$$

converges uniformly on $[\varepsilon, T]$. Taking $\varepsilon \to 0$ gives uniform convergence for all $t > 0$. Thus, we may term-by-term differentiate in t. Now, differentiating:

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2\frac{\partial}{\partial x}\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)e^{-n^2\pi^2 t}\sin(n\pi x) \\ &= 2\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)e^{-n^2\pi^2 t}(n\pi)\cos(n\pi x) \\ \implies \frac{\partial^2 u}{\partial x^2} &= 2\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)e^{-n^2\pi^2 t}(-n^2\pi^2)\sin(n\pi x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial t} &= 2\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)(-n^2\pi^2 t)e^{-n^2\pi^2 t}\sin(n\pi x) \\ &= \frac{\partial^2 u}{\partial x^2}.\end{aligned}$$

Further, notice

$$\begin{aligned}u(0,t) &= 2\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)e^{-n^2\pi^2 t}\sin(0) \\ &= 0\end{aligned}$$

and

$$u(1,t) = 2\sum_{n=1}^{\infty}\left(\int_0^1 \sin(2\pi y)\sin(n\pi y)dy\right)e^{-n^2\pi^2 t}\sin(n\pi)$$
$$= 0.$$

Now, using the fact that

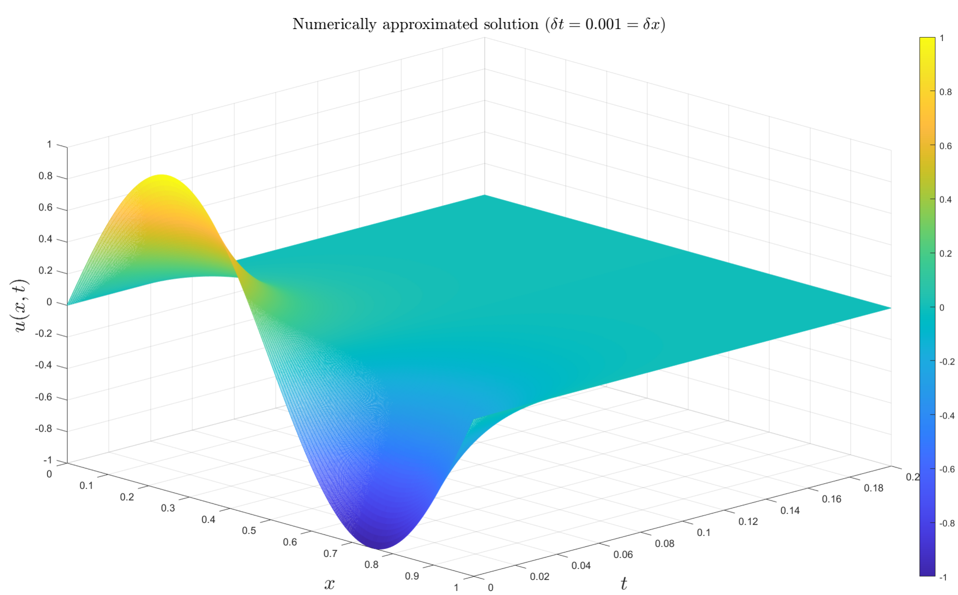
$$\int_0^1 \sin(2\pi y)\sin(n\pi y)dy = \begin{cases} 0 & n \neq 2 \\ \frac{1}{2} & n = 2 \end{cases}$$

so that

$$u(x,0) = \sin(2\pi x).$$

Thus, the given Fourier series satisfies the PDE.

b) Graph the above solution by numerically approximating the integral in the interval of $0 \leq x \leq 1$, $0 < t < 0.2$.



To numerically approximate the integral $\int_0^1 \sin(2\pi x)\sin(n\pi x)dx$, since the integrand $f(x) = \sin(2\pi x)\sin(n\pi x)$ is periodic on $[0,1]$, we use the trapezoidal rule for spectral accuracy. Further, 100 terms in the Fourier series were kept.