Midterm Exam

Michael Nameika

1. (20) Consider complex number sequences $x = (\xi_1, \xi_2, \dots)$ with the usual addition and complex scalar multiplication. Define $X = \{(\xi_1, \xi_2, \dots) \mid \sum_{k=1}^{\infty} |\xi_k|^2/k \text{ converges}\}$. For each $x \in X$, define $\|x\| = (\sum_{k=1}^{\infty} |\xi_k|^2/k)^{1/2}$.

(a) First show that X is a vector subspace of all complex sequences. Second, show that the given norm satisfies $\|x\| = \sqrt{\langle x, x\rangle}$ for a certain inner product $\langle x, y\rangle$ on elements $x, y \in X$. Verify that your inner product is well defined.

Proof: We will begin by showing that X is a vector subspace of $V = \{$ all complex sequences $\}$. To begin, notice that $\mathbf{0} = (0, 0, \dots) \in X$ since $\sum_{k=1}^{\infty} 0/k = 0$. Now let $x, y \in X$ where $x = (\xi_1, \xi_2, \dots)$, $y = (\eta_1, \eta_2, \dots)$, and let α be an arbitrary scalar. We will show $\alpha x = (\alpha\xi_1, \alpha\xi_2, \dots) \in X$. Notice

$$\sum_{k=1}^{\infty} \frac{|\alpha\xi_k|^2}{k} = \sum_{k=1}^{\infty} \frac{|\alpha|^2|\xi_k|^2}{k}$$
$$|\alpha|^2 \sum_{k=1}^{\infty} \frac{|\xi|^2}{k}$$

which converges, hence $\alpha x \in X$. We will now show that $x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \dots) \in X$. Notice

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{|\xi_k + \eta_k|^2}{k} &= \sum_{k=1}^{\infty} \frac{(\xi_k + \eta_k)(\overline{\xi_k} + \overline{\eta_k})}{k} \\
&= \sum_{k=1}^{\infty} \frac{|\xi_k|^2 + \xi_k\overline{\eta_k} + \overline{\xi_k}\eta_k + |\eta_k|^2}{k} \\
&= \sum_{k=1}^{\infty} \frac{|\xi|^2}{k} + \sum_{k=1}^{\infty} \frac{|\eta_k^2|}{k} + \sum_{k=1}^{\infty} \frac{\xi_k\overline{\eta_k} + \overline{\xi_k}\eta_k}{k}
\end{aligned}$$

and since $x, y \in X$, $\sum_{k=1}^{\infty} |\xi|^2/k, \sum_{k=1}^{\infty} |\eta_k|^2/k$ both converge, so that the problem of showing $x+y \in X$ comes down to showing $\sum_{k=1}^{\infty} \frac{\xi_k\overline{\eta_k} + \overline{\xi_k}\eta_k}{k}$ converges. To show this, we will show the series converges absolutely. Notice

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} \frac{\xi_k\overline{\eta_k} + \overline{\xi_k}\eta_k}{k}\right| &\le \sum_{k=1}^{\infty} \left|\frac{\xi_k\overline{\eta_k} + \overline{\xi_k}\eta_k}{k}\right| \\
&\le 2\sum_{k=1}^{\infty} \frac{|\xi_k||\eta_k|}{k}.
\end{aligned}$$

Now, by Hölder's inequality, we have

$$\begin{aligned}
2\sum_{k=1}^{\infty} \frac{|\xi_k||\eta_k|}{k} &= 2\sum_{k=1}^{\infty} \left(\frac{|\xi_k|}{\sqrt{k}}\right)\left(\frac{|\eta_k|}{\sqrt{k}}\right) \\
&\le 2\left(\sum_{k=1}^{\infty} \frac{|\xi_k|^2}{k}\right)^{1/2}\left(\sum_{k=1}^{\infty} \frac{|\eta_k|^2}{k}\right)^{1/2} \\
&= 2\|x\|\|y\|
\end{aligned}$$

so that $\sum_{k=1}^{\infty} |\xi_k||\eta_k|/k$ is bounded above, hence, by direct comparison, converges. Thus, $x + y \in X$. Hence, X is a vector subspace of V.

Now, I claim that the inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \frac{\xi_k \overline{\eta_k}}{k}.$$

We begin by verifying that this is indeed an inner product. Let $x, y, z \in X$ and α an arbitrary scalar where $x = (\xi_1, \xi_2, \dots)$, $y = (\eta_1, \eta_2, \dots)$, and $z = (\zeta_1, \zeta_2, \dots)$ and begin by considering $\langle x + y, z \rangle$:

$$\begin{aligned}
\langle x + y, z \rangle &= \sum_{k=1}^{\infty} \frac{(\xi_k + \eta_k)\overline{\zeta_k}}{k} \\
&= \sum_{k=1}^{\infty} \frac{\xi_k \overline{\zeta_k} + \eta_k \overline{\zeta_k}}{k} \\
&= \sum_{k=1}^{\infty} \frac{\xi_k \overline{\zeta_k}}{k} + \sum_{k=1}^{\infty} \frac{\eta_k \overline{\zeta_k}}{k} \\
&= \langle x, z \rangle + \langle y, z \rangle
\end{aligned}$$

so that linearity holds. Now, consider $\langle \alpha x, y \rangle$:

$$\begin{aligned}
\langle \alpha x, y \rangle &= \sum_{k=1}^{\infty} \frac{(\alpha \xi_k)\overline{\eta_k}}{k} \\
&= \alpha \sum_{k=1}^{\infty} \frac{\xi_k \overline{\eta_k}}{k} \\
&= \alpha \langle x, y \rangle
\end{aligned}$$

so that homogeneity holds. Now, notice

$$\begin{aligned}
\overline{\langle y, x \rangle} &= \overline{\sum_{k=1}^{\infty} \frac{\eta_k \overline{\xi_k}}{k}} \\
&= \sum_{k=1}^{\infty} \frac{\xi_k \overline{\eta_k}}{k} \\
&= \langle x, y \rangle
\end{aligned}$$

so that conjugate symmetry holds. Finally,

$$\langle x, x \rangle = \sum_{k=1}^{\infty} \frac{|\xi_k|^2}{k} \geq 0$$

for all $x \in X$. And notice that $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$ since, if x contains at least one nonzero element, $\langle x, x \rangle > 0$. So now we have established $\langle \cdot, \cdot, \rangle$ is an inner product. Now, by our above work, $\langle x, y \rangle$ is well defined since $\sum_{k=1}^{\infty} |\xi_k||\eta_k|/k$ converges and $|\sum_{k=1}^{\infty} \xi_k \overline{\eta_k}/k| \leq \sum_{k=1}^{\infty} |\xi_k||\eta_k|/k$ and the value of the series is unique by uniqueness of limits.

(b) Suppose that Y is a proper, dense subspace of X. Prove that $Y^{\perp} = \{\mathbf{0}\}$.

Proof: Let $x \in Y^{\perp}$. Since Y is dense in X, $x \in \overline{Y}$. Thus, there exists a sequence $\{x_n\}$ in Y converging to x. Since $x_n \in Y$,

$$\langle x_n, x \rangle = 0$$

for all n. By continuity of the inner product, we have

$$\lim_{n \to \infty} \langle x_n, x \rangle = \langle x, x \rangle = 0$$

so that $x = 0$. Since x was chosen arbitrarily, we have that $Y^\perp = \{\mathbf{0}\}$.

(c) Define $Y = \{(\xi_1, \xi_2, \dots) \in X \mid \sum_{k=1}^\infty |\xi_k|^2 \text{ converges}\}$. Prove that Y forms a proper, dense subspace of X.

Proof: First consider the sequence $x = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots\right)$ and notice that

$$\begin{aligned}\sum_{k=1}^\infty \frac{|\xi_k|^2}{k} &= \sum_{k=1}^\infty \frac{1/k}{k} \\ &= \sum_{k=1}^\infty \frac{1}{k^2}\end{aligned}$$

which converges by the p-series test ($p = 2$). However,

$$\sum_{k=1}^\infty |\xi_k|^2 = \sum_{k=1}^\infty \frac{1}{k}$$

which diverges (harmonic series). Moreover, if $x \in Y$, then $x \in H$ since

$$\sum_{k=1}^\infty \frac{|\xi_k|^2}{k} \leq \sum_{k=1}^\infty |\xi_k|^2$$

Additionally, since each element of Y is an element of X, Y is closed under addition and scalar multiplication since X is. Additionally, $\mathbf{0} \in Y$ since clearly,

$$\sum_{k=1}^\infty 0 = 0.$$

Thus, Y is a proper subspace of X. Now, to show Y is dense in X, fix $x \in X$. Then since $\sum_{k=1}^\infty |\xi_k|^2/k$ converges, for any $\varepsilon > 0$, there exists an index N such that, whenever $n > m > N$,

$$\left|\sum_{k=m+1}^{n} \frac{|\xi_k|^2}{k}\right| < \frac{\varepsilon^2}{4}$$

letting $n \to \infty$, we have

$$\left|\sum_{k=m+1}^{\infty} \frac{|\xi_k|^2}{k}\right| \leq \frac{\varepsilon^2}{4}.$$

Now, define $y = (\xi_1, \xi_2, \cdots, \xi_m, 0, 0, \cdots)$ and notice that, since y contains finitely many nonzero elements, $\sum_{k=1}^\infty |\eta_k|^2$ converges where $\eta_k = y_k$ so that $y \in Y$. Now, notice

$$\begin{aligned}\|x - y\| &= \|(0, 0, \cdots, 0, \xi_{m+1}, \xi_{m+1}, \cdots)\| \\ &= \left(\sum_{k=m+1}^\infty \frac{|\xi_k|^2}{k}\right)^{1/2} \\ &\leq \left(\frac{\varepsilon^2}{4}\right)^{1/2} \\ &= \frac{\varepsilon}{2} \\ &< \varepsilon\end{aligned}$$

so that Y is dense in X.

2. (10) Let X be the subspace of $C[-1,1]$ consisting of continuous functions $x(t)$ on $[-1,1]$ that are also differentiable on $(-1,1)$. The norm on X is the subspace norm: $\|x\| = \max_{-1\le t\le 1} |x(t)|$.

(a) Consider the sequence (x_n) in X defined by $x_n(t) = \sqrt{t^2 + 1/n}$, for $-1 \le t \le 1$. Verify that the definition of completeness of X fails by the example of this subsequence.

Proof: We will show that (x_n) is a Cauchy sequence in X with no limit in X. Let $n > m$ and inspect $|x_n - x_m|$:

$$\begin{aligned}
|x_n - x_m| &= |\sqrt{t^2+1/n} - \sqrt{t^2+1/m}| \\
&= \left| \frac{t^2 + 1/n - t^2 - 1/m}{\sqrt{t^2+1/n} + \sqrt{t^2+1/m}} \right| \\
&= \left| \frac{1/n - 1/m}{\sqrt{t^2+1/n} + \sqrt{t^2+1/m}} \right| \\
&\le \left| \frac{1}{n} - \frac{1}{m} \right| \\
&\le \frac{1}{n} + \frac{1}{m} \\
&\le \frac{2}{m}.
\end{aligned}$$

We have that $\frac{2}{m}$ is an upper bound for $|x_n - x_m|$ for all $t \in [-1,1]$, so that

$$\|x_n - x_m\| \le \frac{2}{m}$$

Hence, (x_n) is Cauchy. Now notice, by continuity of $\sqrt{\cdot}$,

$$\begin{aligned}
\lim_{n\to\infty} x_n(t) &= \lim_{n\to\infty} \sqrt{t^2+1/n} \\
&= |t|
\end{aligned}$$

which is continuous, but not differentiable at $t = 0$. Then (x_n) is a Cauchy sequence in X with no limit in X. Hence X is an incomplete space, as desired.

(b) Prove that the linear functional $f(x) = x'(0)$ is unbounded on X.
Hint: Consider bounded trigonometric functions with small period.

Proof: Let $\{x_n\}$ be a sequence in $C[-1,1]$ where $x_n(t) = \sin(nt)$. Notice that $\|x_n(t)\| = \max_{-1\le t\le 1} |\sin(nt)| = 1$ and that $x_n'(t) = n\cos(nt)$ so that $x_n'(0) = n$. Thus, f is unbounded on X for if it were bounded, there exists some $M \ge 0$ such that $\|f\| \le M$. But for any $M \ge 0$, take $n = \lceil M \rceil + 1$ so that

$$\begin{aligned}
\frac{\|f(x_n)\|}{\|x_n\|} &= \|f(x_n)\| \\
&= n \\
&> M
\end{aligned}$$

so that f is unbounded.

3. (15) For parts (a) and (b), let f be a nonzero bounded linear functional on a real Banach space X.
(a) Show that the set $M = \{x \in X \mid f(x) \leq 1\}$ is complete and convex in X.

Proof: To show that M is convex in M, let $x, y \in M$ and consider $tx + (1-t)y$ where $t \in [0,1]$. Then since $x, y \in M$, $f(x), f(y) \leq 1$ and so

$$\begin{aligned} f(tx + (1-t)y) &= f(tx) + f((1-t)y) \\ &= tf(x) + (1-t)f(y) \\ &\leq t + (1-t) \\ &= 1 \\ \implies tx + (1-t)y \in M \end{aligned}$$

so that M is convex in X. To see that M is complete, we must show that M is closed. Let x be a limit point of M. Then there exists a sequence $\{x_n\}$ in M converging to x. If $f(x) < 1$, then it is clear that $x \in M$. The interesting case is when $f(x) = 1$. Since $\{x_n\}$ is in M, for every n, we have

$$f(x_n) \leq 1$$

and since f is a bounded linear functional, f is continuous, hence

$$\lim_{n \to \infty} f(x_n) = f(x) \leq 1$$

so that $x \in M$, hence M is closed and is thus complete.

(b) Define the set $M_0 = \{x \in X \,|\, f(x) < 1\}$. Prove that the closure of M_0 is $\overline{M_0} = M$, for M of part (a).

Proof: Let $x \in X$ be such that $f(x) = 1$ and define the sequence $\{x_n\}$ where $x_n = (1 - \frac{1}{n})x$. Notice

$$\begin{aligned} f(x_n) &= (1 - \tfrac{1}{n})f(x) \\ &= 1 - \tfrac{1}{n} \\ &< 1 \end{aligned}$$

so that $x_n \in M_0$ for all $n \in \mathbb{N}$. Further, fix $\varepsilon > 0$. Then there exists an index N such that whenever $n > N$,

$$\frac{1}{n} < \frac{\varepsilon}{\|x\|}.$$

And so, whenever $n > N$,

$$\begin{aligned} \|x_n - x\| &= \|x - \tfrac{1}{n}x - x\| \\ &= \tfrac{1}{n}\|x\| \\ &< \frac{\varepsilon}{\|x\|}\|x\| \\ &= \varepsilon \\ \implies \|x_n - x\| &< \varepsilon \end{aligned}$$

so that $x_n \to x$. And so, letting $n \to \infty$, by continuity of f,

$$\lim_{n \to \infty} f(x_n) = f(x)$$

so that x is a limit point of M_0. Hence, the closure of M_0. $\overline{M_0} = M$ as in part (a).

(c) Let now $X = C[0,1]$, with $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$, and define the linear functional f by $f(x) = \int_0^1 x(t)dt$. Fix the element x_0 of $C[0,1]$ defined by $x_0(t) = 8|t - \frac{1}{2}|$, $0 \leq t \leq 1$, and define M by part (a) for the present (definite integral) functional f. Show that

(i) $f(x_0) = 2$.

Soln. Begin by noticing, by definition of absolute value,

$$8\left|t - \frac{1}{2}\right| = \begin{cases} -8\left(t - \frac{1}{2}\right), & 0 \le t \le \frac{1}{2} \\ 8\left(t - \frac{1}{2}\right), & \frac{1}{2} \le t \le 1 \end{cases}$$

so that

$$\begin{aligned} f(x_0) &= \int_0^1 8|x - \tfrac{1}{2}|dt \\ &= -8\int_0^{1/2} (t - \tfrac{1}{2})dt + 8\int_{1/2}^1 (t - \tfrac{1}{2})dt \\ &= -8\left[\frac{1}{2}t^2 - \frac{1}{2}t\right]\Bigg|_0^{1/2} + 8\left[\frac{1}{2}t^2 - \frac{1}{2}\right]\Bigg|_{1/2}^1 \\ &= -8\left[-\frac{1}{8}\right] - 8\left[-\frac{1}{8}\right] \\ &= -8\left[-\frac{1}{4}\right] \\ &= 2. \end{aligned}$$

(ii) for all $\tilde{x} \in M$ we have $\|x_0 - \tilde{x}\| \ge 1$. *Hint*: first show $\|f\| \le 1$.

We first show $\|f\| \le 1$ so that $|f(x)| \le \|x\|$ hence $|f(x_0 - \tilde{x})| \le \|x_0 - \tilde{x}\|$. To begin, let $x \in X$ be such that $\|x\| = 1$. Notice

$$\begin{aligned} \left|\int_0^1 x(t)dt\right| &\le \int_0^1 |x(t)|dt \\ &\le \int_0^1 dt \\ &= 1. \end{aligned}$$

Hence, $\|f\| \le 1$. Now, since f is linear, $|f(x_0 - \tilde{x})| = |f(x_0) - f(\tilde{x})|$:

$$\begin{aligned} |f(x_0) - f(\tilde{x})| &= |2 - f(\tilde{x})| \\ &\ge 2 - f(\tilde{x}) \end{aligned}$$

and since $f(\tilde{x}) \le 1$, $-f(\tilde{x}) \ge -1$ so that

$$2 - f(\tilde{x}) \ge 2 - 1 = 1.$$

Hence,

$$\|x_0 - \tilde{x}\| \ge 1$$

as desired.

(iii) Find $x_1 \in M$ such that $\|x_0 - x_1\| = 1$.

Soln. Take $x_1 = x_0 - \mathbf{1}$, that is, $x_1(t) = 8|t - \frac{1}{2}| - 1$. Here we denote the constant function having value one on $[0,1]$ by $\mathbf{1}$. Then note that

$$\begin{aligned}
f(x_1) &= \int_0^1 x_1(t)dt \\
&= \int_0^1 (8|t-\tfrac{1}{2}|-1)dt \\
&= \int_0^1 8|t-\tfrac{1}{2}|dt - \int_0^1 1dt \\
&= f(x_0) - 1 \\
&= 2 - 1 \\
&= 1
\end{aligned}$$

so that $f(x_1) = 1$ and so $x_1 \in M$. Then notice that

$$\begin{aligned}
\|x_0 - x_1\| &= \|x_0 - (x_0 - \mathbf{1})\| \\
&= \|\mathbf{1}\| \\
&= 1
\end{aligned}$$

as desired.

4. (10) Suppose that H is a Hilbert space. Let Y be a closed subspace of H. Suppose in turn that W is a closed subspace of H with $W \subset Y$, and $W \neq Y$. Let $x \in H$. Let $y = P_Y x$ and $w = P_W x$ be the orthogonal projections of x onto the subspaces Y and W respectively.
(a) Prove that $(y - w) \in W^\perp$.

Proof: Since Y, W are closed subspaces of a Hilbert space, we may decompose H in the following ways:

$$H = W \oplus W^\perp; \qquad H = Y \oplus Y^\perp.$$

That is, for $x \in H$, we may uniquely express x as

$$x = w + w_p \quad \text{or} \quad x = y + y_p$$

where $w = P_W x$, $w_p \in W^\perp$ and $y = P_Y x$, $y_p \in Y^\perp$. Subtracting these two expressions, we have

$$\begin{aligned} x - x &= w + w_p - y - y_p \\ 0 &= (w - y) + (w_p - y_p) \\ w - y &= y_p - w_p. \end{aligned}$$

Now, since $W \subset Y$, we have that $Y^\perp \subseteq W^\perp$, so that $y_p \in W^\perp$ and since $W^\perp$ is a subspace, we have $y_p - w_p \in W^\perp$. Thus, $w - y \in W^\perp$, as desired.

(b) Prove that $\|x\|^2 = \|w\|^2 + \|y - w\|^2 + \|x - y\|^2$. Illustrate for specific points x, y and z of $H = \mathbb{R}^3$, where $Y =$ a plane through the origin, $W =$ a line through the origin, and where $x \notin Y$ and $w \notin Y$.

Proof: To begin, as in part a), we have the following representations for x:

$$\begin{aligned} x &= w + w_p \\ x &= y + y_p \end{aligned}$$

where $w = P_W x$, $y = P_Y x$, $w_p \in W^\perp$, and $y_p \in Y^\perp$. Now, consider $\|x - y\|^2$:

$$\begin{aligned} \|x - y\|^2 &= \langle x - y, x - y\rangle \\ &= \langle x, x\rangle - \langle y, x\rangle - \langle x, y\rangle + \langle y, y\rangle \\ &= \|x\|^2 - \langle y, x\rangle - \langle x, y\rangle + \|y\|^2 \end{aligned}$$

and notice

$$\begin{aligned} \langle x, y\rangle &= \langle y + y_p, y\rangle \\ &= \langle y, y\rangle + \langle y_p, y\rangle \\ &= \|y\|^2 \end{aligned}$$

and so $\langle y, x\rangle = \overline{\langle x, y\rangle} = \overline{\|y\|^2} = \|y\|^2$. Thus,

$$\begin{aligned} \|x - y\|^2 &= \|x\|^2 - \|y\|^2 - \|y\|^2 + \|y\|^2 \\ &= \|x\|^2 - \|y\|^2. \end{aligned}$$

Now, let us consider $\|y - w\|^2$:

$$\begin{aligned} \|y - w\|^2 &= \langle y - w, y - w\rangle \\ &= \langle y, y\rangle - \langle w, y\rangle - \langle y, w\rangle + \langle w, w\rangle \\ &= \|y\|^2 - \langle w, y\rangle - \langle y, w\rangle + \|w\|^2. \end{aligned}$$

Now notice

$$\begin{aligned}\langle w, y\rangle &= \langle w, x - w_p\rangle \\ &= \langle w, x\rangle - \langle w, w_p\rangle \\ &= \|w\|^2\end{aligned}$$

so that $\langle y, w\rangle = \overline{\langle w, y\rangle} = \overline{\|w\|^2} = \|w\|^2$. Hence

$$\begin{aligned}\|y - w\|^2 &= \|y\|^2 - \|w\|^2 - \|w\|^2 + \|w\|^2 \\ &= \|y\|^2 - \|w\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\|w\|^2 + \|y - w\|^2 + \|x - y\|^2 &= \|w\|^2 + \|y\|^2 - \|w\|^2 + \|x\|^2 - \|y\|^2 \\ &= \|x\|^2\end{aligned}$$

which is what we sought to show.

For an example, take $Y =$ the $x - y$ plane and $W =$ the $x-$axis and let $x = (1, 1, 1)$. Then

$$\begin{aligned}y &= P_Y x = (1, 1, 0) \\ w &= P_W x = (1, 0, 0)\end{aligned}$$

and so

$$\begin{aligned}x - y &= (0, 0, 1) \\ y - w &= (0, 1, 0)\end{aligned}$$

and so

$$\begin{aligned}\|w\|^2 + \|x - y\|^2 + \|y - w\|^2 &= 1 + 1 + 1 \\ &= 3.\end{aligned}$$

Further, we have $\|x\|^2 = 3$ and so

$$\|x\|^2 = \|w\|^2 + \|x - y\|^2 + \|y - w\|^2.$$

5. (10) Let H be a Hilbert space and let (e_k) be an orthonormal sequence in H. Let f be a bounded linear functional on H. Denote $\gamma_k = f(e_k)$, for all $k \in \mathbb{N}$.
(a) Prove that

(i) for every $n \in \mathbb{N}$, $\|f\| \geq (\sum_{k=1}^{n} |\gamma_k|^2)^{1/2}$
Proof: Let $x_n = \overline{\gamma_1}e_1 + \overline{\gamma_2}e_2 + \cdots + \overline{\gamma_n}e_n$. Then, since f is linear,

$$\begin{aligned} f(x_n) &= \overline{\gamma_1}f(e_1) + \overline{\gamma_2}f(e_2) + \cdots + \overline{\gamma_n}f(e_n) \\ &= \overline{\gamma_1}\gamma_1 + \overline{\gamma_2}\gamma_2 + \cdots + \overline{\gamma_n}\gamma_n \\ &= |\gamma_1|^2 + |\gamma_2|^2 + \cdots + |\gamma_n|^2. \end{aligned}$$

And also notice

$$\begin{aligned} \|x\|^2 &= \langle x, x\rangle \\ &= \left\langle \sum_{k=1}^{n} \overline{\gamma_k}e_k, \sum_{j=1}^{n} \overline{\gamma_j}e_j \right\rangle \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\gamma_k}\gamma_j\langle e_k, e_j\rangle \\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} \overline{\gamma_k}\gamma_j\delta_{jk} \\ &= \sum_{k=1}^{n} \overline{\gamma_k}\gamma_k \\ &= \sum_{k=1}^{n} |\gamma_k|^2. \end{aligned}$$

Hence,

$$\begin{aligned} |f(x)| &= \|x\|^2 \\ &= \|x\|\|x\| \\ &= \left(\sum_{k=1}^{n} |\gamma_k|^2\right)^{1/2} \|x\| \end{aligned}$$

hence

$$\|f\| \geq \left(\sum_{k=1}^{n} |\gamma_k|^2\right)^{1/2}$$

which is what we sought to show.

(ii) $\lim_{n\to\infty} \gamma_n = 0$.
Proof: By part (i), we have that

$$\left(\sum_{k=1}^{n} |\gamma_k|^2\right)^{1/2} \leq \|f\|$$

for all $n \in \mathbb{N}$. And so, since f is a bounded linear functional, we have the sequence $\{s_n\}$ defined by $s_n = \sum_{k=1}^{n} |\gamma_k|^2$ is a bounded monotonically increasing sequence so that $\{s_n\}$ is a convergent sequence.

And since $\{s_n\}$ is convergent, we have that

$$\begin{aligned}
&\lim_{n\to\infty} |\gamma_k|^2 = 0 \\
\implies &\lim_{n\to\infty} |\gamma_k| = 0 \\
\implies &\lim_{n\to\infty} \gamma_k = 0
\end{aligned}$$

which is what we sought to show.

(b) Suppose that for some scalars $\alpha_1, \alpha_2, \ldots,$ we have that $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to an element x_0 of H. Prove that $|f(x_0)| \leq \|x_0\| (\sum_{k=1}^{\infty} |\gamma_k|^2)^{1/2}$.

Proof: First note that

$$\begin{aligned}
\|x_0\|^2 &= \lim_{n\to\infty} \left\langle \sum_{k=1}^{n} \alpha_k e_k, \sum_{j=1}^{n} \alpha_j e_j \right\rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \overline{\alpha_j} \langle e_k, e_j \rangle \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \overline{\alpha_j} \delta_{jk} \\
&= \sum_{k=1}^{\infty} |\alpha_k|^2 \\
\implies \|x_0\| &= \left(\sum_{k=1}^{\infty} |\alpha_k|^2 \right)^{1/2}.
\end{aligned}$$

And notice,

$$\begin{aligned}
f(x_0) &= f\left(\sum_{k=1}^{\infty} \alpha_k e_k \right) \\
&= \sum_{k=1}^{\infty} \alpha_k f(e_k) \\
&= \sum_{k=1}^{\infty} \alpha_k \gamma_k
\end{aligned}$$

and so

$$\begin{aligned}
|f(x_0)| &= \left| \sum_{k=1}^{\infty} \alpha_k \gamma_k \right| \\
&\leq \sum_{k=1}^{\infty} |\alpha_k| |\gamma_k|
\end{aligned}$$

and by Hölder's inequality, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |\alpha_k||\gamma_k| &\leq \left(\sum_{k=1}^{\infty} |\alpha_k|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\gamma_k|^2\right)^{1/2} \\
&= \|x_0\| \left(\sum_{k=1}^{\infty} |\gamma_k|^2\right)^{1/2} \\
\implies |f(x_0)| &\leq \|x_0\| \left(\sum_{k=1}^{\infty} |\gamma_k|^2\right)^{1/2}
\end{aligned}$$

which is what we sought to show.

6. (10) Let H be a Hilbert space, let (e_n) be an orthonormal sequence in H, and let $x \in H$ be fixed. Define the sequence (x_n) in H by $x_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k$, for all $n \in \mathbb{N}$.
(a) Prove by direct computation that $\|x - x_n\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$.

Proof: By definition,

$$\begin{aligned}\|x - x_n\| &= \langle x - x_n, x - x_n \rangle \\ &= \langle x, x \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x_n, x_n \rangle\end{aligned}$$

and notice

$$\begin{aligned}\langle x_n, x \rangle &= \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, x \right\rangle \\ &= \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, x \rangle \\ &= \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle} \\ &= \sum_{k=1}^{n} |\langle x, e_k \rangle|^2\end{aligned}$$

and that

$$\begin{aligned}\langle x, x_n \rangle &= \left\langle x, \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\rangle \\ &= \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} \langle x, e_k \rangle \\ &= \sum_{k=1}^{n} |\langle x, e_k \rangle|^2.\end{aligned}$$

Now,

$$\begin{aligned}\langle x_n, x_n \rangle &= \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \sum_{j=1}^{n} \langle x, e_k \rangle e_k \right\rangle \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \langle e_k, e_j \rangle \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \delta_{jk} \\ &= \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle} \\ &= \sum_{k=1}^{n} |\langle x, e_k \rangle|^2.\end{aligned}$$

Finally, we have

$$\begin{aligned}\|x - x_n\|^2 &= \|x\|^2 - 2 \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 + \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \\ &= \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2\end{aligned}$$

as was desired.

(b) First explain why the infinite series $\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ converges to some element x_0 in H. Second, suppose in addition that the following holds: (†) whenever $u \in H$ satisfies $\langle u, e_j\rangle = 0$, for all $j \in \mathbb{N}$, then in fact $u = 0$. Prove directly (and not by quoting theory from the text) that we have $x_0 = x$.

Soln & Proof: Using the result from part (a), we have

$$\|x - x_n\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \geq 0$$
$$\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

hence, the sequence $\{\sigma_n\}$ defined by $\sigma_n = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2$ is bounded above by $\|x\|^2$ and is strictly increasing since for each k, $|\langle x, e_k\rangle|^2 \geq 0$. Hence, by the monotone convergence theorem, we have that $\{\sigma_n\}$ converges. Now, note that $\{x_n\}$ is Cauchy in the norm of H if and only if $\{\sigma_n\}$ is Cauchy in $\mathbb{R}$ since, for $n > m$,

$$\begin{aligned}
\|x_n - x_m\|^2 &= \left\| \sum_{k=m+1}^{n} \langle x, e_k\rangle e_k \right\|^2 \\
&= \left\langle \sum_{k=m+1}^{n} \langle x, e_k\rangle e_k, \sum_{j=m+1}^{n} \langle x, e_j\rangle e_j \right\rangle \\
&= \sum_{k=m+1}^{n} \sum_{j=m+1}^{n} \langle x, e_k\rangle \overline{\langle x, e_j\rangle} \langle e_k, e_j\rangle \\
&= \sum_{k=m+1}^{n} \sum_{j=m+1}^{n} \langle x, e_k\rangle \overline{\langle x, e_j\rangle} \delta_{jk} \\
&= \sum_{k=m+1}^{n} |\langle x, e_k\rangle|^2.
\end{aligned}$$

Thus, since $\{\sigma_n\}$ converges in $\mathbb{R}$, $\{x_n\}$ is Cauchy in the norm of H and hence converges to some $x_0 \in H$. Now, let $u = x - x_0$ and consider $\langle u, e_j\rangle$ for some $e_j \in (e_n)$:

$$\begin{aligned}
\langle u, e_j\rangle &= \langle x - x_0, e_j\rangle \\
&= \langle x, e_j\rangle - \langle x_0, e_j\rangle \\
&= \langle x, e_j\rangle - \left\langle \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k, e_j \right\rangle \\
&= \langle x, e_j\rangle - \sum_{k=1}^{\infty} \langle x, e_k\rangle \langle e_k, e_j\rangle \\
&= \langle x, e_j\rangle - \sum_{k=1}^{\infty} \langle x, e_k\rangle \delta_{kj} \\
&= \langle x, e_j\rangle - \langle x, e_j\rangle \\
&= 0
\end{aligned}$$

and since e_j was chosen arbitrarily, we have $\langle x - x_0, e_j\rangle = 0$ for all $j \in \mathbb{N}$, hence

$$\begin{aligned}
&x - x_0 = 0 \\
\implies &x = x_0
\end{aligned}$$

as was desired.

7. (c) Find $\|T\|$. Justify your assertion.

Soln. From part (a) it can be shown that $\|T\| \leq \pi$. Now, for the lower bound, take $x = e_1 + \tilde{e}_1$ so that

$$\begin{aligned}\|x\|^2 &= \langle e_1 + \tilde{e}_1, e_1 + \tilde{e}_1\rangle \\ &= \langle e_1, e_1\rangle + \langle \tilde{e}_1, e_1\rangle + \langle e_1, \tilde{e}_1\rangle + \langle \tilde{e}_1, \tilde{e}_1\rangle \\ &= 2\end{aligned}$$

and

$$\begin{aligned}\tilde{T}x &= \pi(\langle e_1 + \tilde{e}_1, e_1\rangle e_1 + \langle e_1 + \tilde{e}_1, \tilde{e}_1\rangle \tilde{e}_1) \\ &= \pi(e_1 + \tilde{e}_1) \\ &= \pi e_1 + \pi \tilde{e}_1\end{aligned}$$

so that

$$\begin{aligned}\|\tilde{T}x\|^2 &= \langle \pi e_1 + \pi \tilde{e}_1, \pi e_1 + \pi \tilde{e}_1\rangle \\ &= \pi^2(\langle e_1, e_1\rangle + \langle e_1, \tilde{e}_1\rangle + \langle \tilde{e}_1, e_1\rangle + \langle \tilde{e}_1, \tilde{e}_1\rangle) \\ &= 2\pi^2\end{aligned}$$

hence,

$$\frac{\|\tilde{T}x\|}{\|x\|} = \pi$$

so that

$$\pi \leq \|\tilde{T}\| \leq \pi$$

hence $\|\tilde{T}\| = \pi$ and since $\|T\| = \|\tilde{T}\|$, we have

$$\|T\| = \pi.$$