Exam 2 Practice Questions

1. Consider the 1D Advection Equation
$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$
with $c > 0$. The Lax-Wendroff scheme for this equation is given as
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = c^2\frac{\Delta t}{2}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$
   (a) Determine the order of accuracy of this scheme.
   (b) Verify that the scheme is conditionally stable for $c\frac{\Delta t}{\Delta x} \leq 1$

2. Consider the trapezoidal method
$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$
   (a) Determine if the method is zero stable.
   (b) Determine if the method is consistent.
   (c) Is the method convergent?

3. Consider the so-called Inviscid Burger Equation
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$
Note that this of the form of the advection equation whose speed is given by the function itself.
   (a) Using the product rule, rewrite the $u\frac{\partial u}{\partial x}$ term as a perfect derivative involving u^2.
   (b) Write down a first order in time and second order in space scheme for the above equation.

4. Recall that the forward time, central space scheme for the advection equation is unconditionally unstable:
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0.$$
Now consider the Crank-Nicolson scheme, which involves replacing the discretization of the spatial derivative at time step n with the average of the spatial discretization at the n and $n+1$ time steps:
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2}\left(\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right) = 0.$$
Determine if the Crank-Nicolson scheme is stable.

5. Consider the 1D advection equation with initial condition:
$$u_t + au_x = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}.$$
Using the method of characteristics, find the characteristic lines and show that the general solution is given by
$$u(t, x) = u_0(x - at).$$