Optimization Midterm 2

Michael Nameika

April 2023

1. **Guaranteeing Convergence**

   Consider the function
   $$f(x) = 2(x_1 - 1)^2 + 10(x_2 - 2)^2,$$
   whose minimum clearly occurs at $(x_1, x_2) = (1, 2)$.
   The following questions related to choosing a line search update, x_1. Consider the initial guess $x_0 = (0, 0)^T$. All questions below use the search direction $p = (1, 1)^T$ and some step length $\alpha > 0$.

   (a) Show that this is a descent direction.

   To begin, let us find the gradient of f:
   $$\nabla f(x) = \begin{pmatrix} 4(x_1 - 1) \\ 20(x_2 - 2) \end{pmatrix}$$
   and so at $x_0 = (0, 0)^T$, $\nabla f(x_0) = (-4, -20)$. For p to be a direction of descent, we require $p^T \nabla f(x_0) < 0$. Notice
   $$\begin{aligned} p^T \nabla f(x_0) &= (1, 1) \begin{pmatrix} -4 \\ -40 \end{pmatrix} \\ &= -44 < 0. \end{aligned}$$
   So p is a descent direction.

   (b) For which value of ϵ does this direction satisfy the sufficient descent criteria?

   Recall that a search direction must satisfy the sufficient descent condition given by
   $$-\frac{p^T \nabla f(x_0)}{\|p\| \cdot \|\nabla f(x_0)\|} \geq \epsilon > 0$$
   Here, we have $p^T \nabla f(x_0) = -44$ and $\|p\| = \sqrt{2}$, $\|\nabla f(x_0)\| = 4\sqrt{101}$, so we require
   $$0 < \epsilon \leq \frac{11}{\sqrt{202}}$$

   (c) For which value of m does this direction satisfy the gradient related criteria?

   Recall that for the gradient related condition, we require
   $$\|p\| \geq m \|\nabla f(x_0)\| \quad \textit{for all } k \textit{ (with } m > 0\textit{)}$$
   Here, $\|p\| = \sqrt{2}$ and $\|\nabla f(x_0)\| = 4\sqrt{101}$, so we require
   $$\begin{aligned} \sqrt{2} &\geq m(4\sqrt{101}) \\ m &\leq \frac{\sqrt{2}}{4\sqrt{101}} \\ m &\leq \frac{1}{2\sqrt{202}} \end{aligned}$$

That is, we require

$$0 < m \leq \frac{1}{2\sqrt{202}}$$

(d) For what range of α values does this update satisfy the Armijo condition when $\mu = 2/44$?

Recall that the Armijo condition requires that $\alpha > 0$ must satisfy

$$f(x_0 + \alpha p) \leq f(x_0) + \mu \alpha p^T \nabla f(x_0)$$

Here, $f(x_0) = 42$, $p^T \nabla f(x_0) = -44$, and $f(x_0 + \alpha p) = f(\alpha p) = 12\alpha^2 - 44\alpha + 42$. That is, we require

$$\begin{aligned} 12\alpha^2 - 44\alpha + 42 &\leq 42 - 2\alpha \\ 12\alpha^2 - 42\alpha &\leq 0 \\ \alpha(2\alpha - 7) &\leq 0 \\ \alpha &\leq \frac{7}{2} \end{aligned}$$

That is, to satisfy the Armijo conditions when $\mu = 2/44$, we require

$$0 < \alpha \leq \frac{7}{2}$$

(e) For what range of α values does this update satisfy the Wolfe condition when $\eta = 40/44$?

Recall the Wolfe condition:

$$|p^T \nabla f(x_0 + \alpha p)| \leq \eta |p^T \nabla f(x_0)|$$

For $\eta = 40/44$, we have $\eta |p^T \nabla f(x_0)| = 40$ and notice that $|p^T \nabla f(x_0 + \alpha p)| = |24\alpha - 44|$. That is, we require

$$\begin{aligned} |24\alpha - 44| &\leq 40 \\ -40 \leq 24\alpha - 44 &\leq 40 \\ 4 \leq 24\alpha &\leq 84 \\ \frac{1}{6} \leq \alpha &\leq \frac{7}{2} \end{aligned}$$

That is, when $\eta = 40/44$, to satisfy the Wolfe condition, we require

$$\frac{1}{6} \leq \alpha \leq \frac{7}{2}$$

2. **Steepest-descent**

Consider the problem

$$\text{minimize} \quad f(x) = \frac{1}{2}x^T Qx - c^T x,$$

where Q is a positive-definite matrix. Let x_* be the minimizer of this function. Let v be an eigenvecctor of Q, and let λ be the associated eigenvalue. Suppose now that the starting point for the steepest-descent algorithm is $x_0 = x_* + v$.

(a) Show that the gradient at x_0 is $\nabla f(x_0) = \lambda v$.

Notice that the gradient of f is given as

$$\nabla f(x) = Qx - c$$

Since x_* is assumed to be the minimizer of f, we have $\nabla f(x_*) = 0$. Then plugging in x_0, we find

$$\begin{aligned}\nabla f(x_0) &= Qx_0 - c \\ &= Q(x_* + v) - c \\ &= Qx_* - c + Qv \\ &= \lambda v\end{aligned}$$

which is what we wanted to show.

(b) Prove that if the steepest-descent direction is taken, then the step length which minimizes f in this direction is $\alpha_0 = 1/\lambda$.

Proof: Recall that the steepest-descent direction is given by the negative gradient. That is, for a search direction p, $p = -\nabla f(x)$ is the direction of steepest descent. We wish to find the step length that minimizes f in this direction. That is, we wish to solve

$$\underset{\alpha > 0}{\text{minimize}} \quad f(x + \alpha p)$$

Let $F(\alpha) = f(x + \alpha p)$. Then we wish to find α such that $F'(\alpha) = 0$. Notice

$$F'(\alpha) = p^T \nabla f(x + \alpha p) = 0$$

and that

$$\begin{aligned}p^T \nabla f(x_0 + \alpha p) &= p^T (Q(x_0 + \alpha p) - c) \\ &= p^T Qx_0 - c + \alpha p^T Qp \\ &= p^T \nabla f(x_0) + \alpha p^T Qp \\ &= \lambda p^T v + \alpha p^T Qp \\ &= 0.\end{aligned}$$

Using $p = -\nabla f(x_0) = -\lambda v$, we find

$$\begin{aligned}\alpha(-\lambda v)^T Q(-\lambda v) &= \lambda^2 \|v\|^2 \\ \alpha &= \frac{\lambda^2 \|v\|^2}{\lambda^2 v^T Qv} \\ &= \frac{\lambda^2 \|v\|^2}{\lambda^2 v^T (\lambda v)} \\ &= \frac{\lambda^2 \|v\|^2}{\lambda^3 \|v\|^2} \\ &= \frac{1}{\lambda}.\end{aligned}$$

So $\alpha = 1/\lambda$, which is what we sought to show.

(c) Prove that the steepest-descent direction with an accurate step length will lead to the minimum of the function f in one iteration.

Proof: Using the initial guess of x_0, and steepest descent with step size $1/\lambda$ as given in part (a) and (b), we find

$$\begin{aligned} f(x_0 + \alpha p) &= f(x_* + v - 1/\lambda(\lambda v)) \\ &= f(x_* + v - v) \\ &= f(x_*) \end{aligned}$$

So we find the minimum of f in one iteration.

(d) Confirm this result for the function

$$f(x) = 3x_1^2 - 2x_1x_2 + 3x_2^2 + 2x_1 - 6x_2.$$

Suppose the starting point is $x_0 = (1, 2)^T$; compute the point obtained by one iteration of the steepest-descent algorithm. Prove that the point obtained is the unique minimum x_*. Verify that $x_0 - x_*$ is an eigenvector of the Hessian matrix.

Rewriting the function as a quadratic form $f(x) = \frac{1}{2}x^TQx - c^Tx$, we find

$$f(x) = \frac{1}{2}(x_1, x_2)\begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (-2, 6)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with gradient

$$\nabla f(x) = Qx - c$$

It can be easily shown that the eigenvalues of Q are $\lambda_1 = 4$ and $\lambda_2 = 8$. Use $\alpha = 1/\lambda_1$. The next value x_1 in the steepest descent algorithm will be given by

$$x_1 = x_0 - \frac{1}{4}\nabla f(x_0)$$

Notice that $\nabla f(x_0) = (4, 4)^T$ and we find

$$\begin{aligned} x_1 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

And notice that

$$\nabla f(0, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $x_1 = (1, 1)^T$ is the minimizer of f. To verify $x_0 - x_1 = (1, 1)^T$ is an eigenvector of the Hessian Q, notice

$$\begin{aligned} \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &= 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \lambda_1(x_0 - x_1) \end{aligned}$$

3. **Linear Equality Constraints**

Consider the problem of finding the minimum distance from a point r to a set $\{x : a^T x = b\}$. Take a^T to be a row vector. The problem can be expressed as

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}(x-r)^T(x-r) \\ \text{subject to} \quad & a^T x = b. \end{aligned}$$

(a) Prove that the solution is given by

$$x_* = r + \frac{b - a^T r}{a^T a} a.$$

Proof: For a stationary point x_* of f over the given constraint, we require

$$\nabla f(x_*) = a\lambda_*$$

where $\lambda_* \geq 0$ is a Lagrange multiplier. Notice that $\nabla f(x_*) = x_* - r$. That is,

$$\begin{aligned} x_* - r &= \lambda_* a \\ x_* &= \lambda_* a + r \end{aligned}$$

Additionally, from the constraint, we have $a^T x = b$, so using the fact that $x_* - r = \lambda_* a$, we find

$$\begin{aligned} a^T x_* &= \lambda_* a^T a + a^T r \\ &= b \\ \lambda_* a^T a &= b - a^T r \\ \lambda_* &= \frac{b - a^T r}{a^T a} \end{aligned}$$

And so we find that the minimizer is given by

$$x_* = r + \frac{b - a^T r}{a^T a} a$$

Which is what we sought to show.

(b) Prove that this point is a strict minimizer.

Notice that $\nabla^2 f(x) = I$, which is positive definite, so the second order sufficient condition is automatically satisfied, meaning that this point is a strict minimizer.

(c) Use part (a) to find where on the line $x_2 = 1 + x_1$ is closest to the point $r = (-1, 1)^T$.

Rewriting the equation of the line, we have

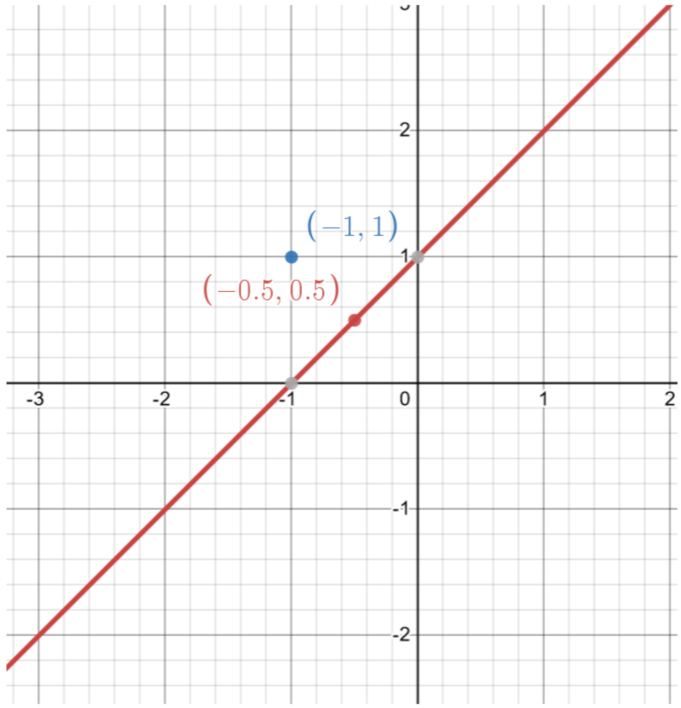
$$-x_1 + x_2 = 1$$

Which is our constraint $a^T x = b$ with $a = (-1, 1)^T$ and $b = 1$. Then using the formula we found in part (a), we have

$$\begin{aligned} x_* &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1-2}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \end{aligned}$$

So the point on the line $x_2 = 1 + x_1$ that is closest to $(-1, 1)^T$ is $x_* = (-1/2, 1/2)$.

(d) Make a sketch.

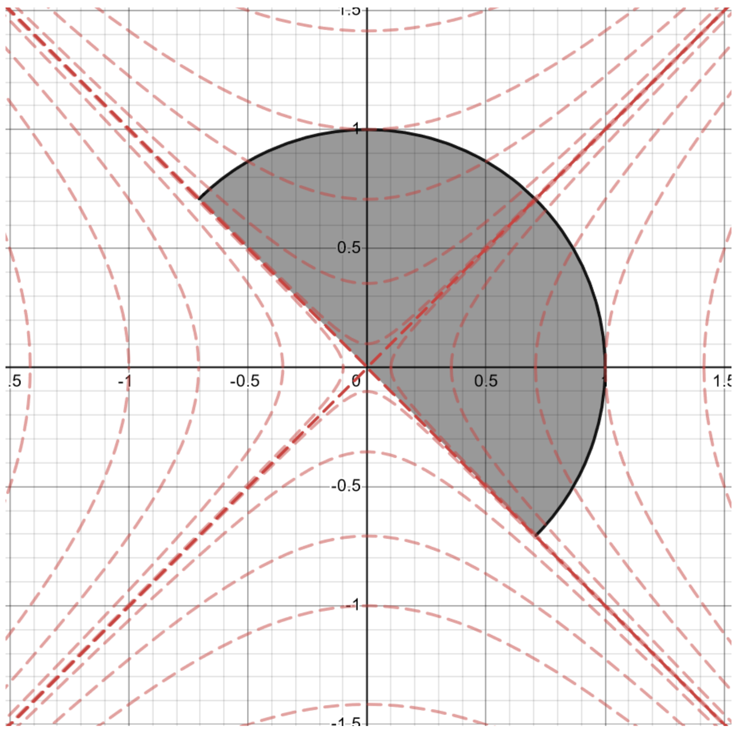


4. **Nonlinear Inequality Constraints**

Consider the problem

$$\begin{aligned} & f(x) = x_1^2 - x_2^2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 1 \\ & x_1 + x_2 \geq 0 \end{aligned}$$

(a) Make a sketch of the feasible region.



(b) Find all minimizers, maximizers, and saddle points.

Begin by rewriting the constraints in the "≥ 0" form:

$$\begin{aligned} g_1(x) &= 1 - x_1^2 - x_2^2 \geq 0 \\ g_2(x) &= x_1 + x_2 \geq 0 \end{aligned}$$

and define the Lagrangian

$$\begin{aligned} \mathscr{L}(x, \lambda) &= f(x) - \lambda^T g(x) \\ &= x_1^2 - x_2^2 - \lambda_1(1 - x_1^2 - x_2^2) - \lambda_2(x_1 + x_2) \end{aligned}$$

At a stationary point, we require $\nabla_x \mathscr{L}(x, \lambda) = 0$. That is,

$$\nabla_x \mathscr{L}(x, \lambda) = \begin{pmatrix} 2x_1 + 2\lambda_1 x_1 - \lambda_2 \\ -2x_2 + 2\lambda_1 x_2 - \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We consider the following cases:

Case 1: Both constraints are inactive. Then $\lambda_1 = \lambda_2 = 0$ and so $x_1 = x_2 = 0$, meaning the second constraint must be active, contradicting our assumption that both constraints were inactive.

Case 2: Both constraints are active.
Then $x_1 + x_2 = 0$ and $x_1^2 + x_2^2 = 1$. From this, we get two possible values for x:

$$x = \begin{pmatrix} \pm 1/\sqrt{2} \\ \mp 1/\sqrt{2} \end{pmatrix}$$

(i) Check $x = (-1/\sqrt{2}, 1/\sqrt{2})^T$:
From this, we find $\lambda_1 = 0$, $\lambda_2 = -\sqrt{2}$ meaning that this may be a local max. Now let us check the second order necessary conditions. At this point, the first constraint is degenerate, so we must find a null space matrix for the first row of the Jacobian of g at this point. The Jacobian of g is given by

$$\nabla g(x) = \begin{pmatrix} -2x_1 & -2x_2 \\ 1 & 1 \end{pmatrix}$$

so at the point $x = (-1/\sqrt{2}, 1/\sqrt{2})$,

$$\nabla g(-1/\sqrt{2}, 1/\sqrt{2}) = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

And a nullspace matrix for the first row is simply $Z = (1, 1)^T$. Now, the Hessian of $\mathscr{L}(x, \lambda)$ is given by

$$\nabla_{xx}^2 \mathscr{L}(x, \lambda) = \begin{pmatrix} 2 + 2\lambda_1 & 0 \\ 0 & -2 + 2\lambda_1 \end{pmatrix}$$

And since $\lambda_1 = 0$, the Hessian of $\mathscr{L}(x, \lambda)$ is

$$\nabla_{xx}^2 \mathscr{L}(x, \lambda) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Now checking $Z^T \nabla_{xx}^2 \mathscr{L}(x, \lambda) Z$, we have

$$\begin{aligned} Z^T \nabla_{xx}^2 \mathscr{L}(x, \lambda) Z &= (1, 1) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (1, 1) \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= 0 \end{aligned}$$

So the sufficiency conditions are not satisfied. Let us instead think about making a small movement interior to the domain to determine if this is a local maximum. Consider moving a small amount $\delta_1 > 0$ in the x_1 direction and δ_2 in the x_2 direction. To begin, let us consider the point $x = \left(-\frac{1}{\sqrt{2}} + \delta_1, \frac{1}{\sqrt{2}} + \delta_2\right)$. Factoring our objective function, we find

$$f(x) = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$$

at our perturbed point, we have

$$f(x + \delta) = (-\sqrt{2} + \delta_1 - \delta_2)(\delta_1 + \delta_2) < 0$$

for values of δ_1, δ_2 such that $x + \delta$ remains in the feasible region. Similarly, we can show the same for a perturbation δ_2 in the negative x_2 direction. That is, $x = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a local maximum.

(ii) Check $x = (1/\sqrt{2}, -1/\sqrt{2})^T$
Similar to part (i), we have $\lambda_2 = \sqrt{2}$ and $\lambda_1 = 0$. Notice now that

$$\nabla g(1/\sqrt{2}, -1/\sqrt{2}) = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

So similar to part (i), $Z = (1,1)^T$. Notice that the second order sufficiency condition $Z^T \nabla^2_{xx} \mathscr{L}(x,\lambda) Z$ is the same in this case, which is just zero, meaning that the sufficiency conditions are not satisfied. Similar as in part (i), we can show that the point $x = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is a local minimum.

Case 3: The first constraint is active.
Then $\lambda_2 = 0$ and $x_1^2 + x_2^2 = 1$.
From the gradient of $\mathscr{L}$, we have

$$\begin{aligned} x_2 &= \lambda_1 x_2 \\ -x_1 &= \lambda_1 x_1 \end{aligned}$$

Using this in combination with the first constraint being active, we find

$$\lambda_1 = \pm 1$$

i) Suppose $\lambda_1 = -1$. Then $x_2 = 0$ and $x_1 = \pm 1$. The point $x = (-1, 0)^T$ is infeasible, so $x_1 = 1$. At this point,

$$\begin{aligned} \nabla g(1,0) &= \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \\ \nabla^2_{xx} \mathscr{L}(x,\lambda) &= \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \end{aligned}$$

Notice that $Z = (1,-1)^T$ is a null space matrix for the degenerate constraint (constraint 2), so checking the reduced Hessian, we find

$$\begin{aligned} Z^T \nabla^2_{xx} \mathscr{L}(x,\lambda) Z &= (1,-1) \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= (1,-1) \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ &= -4 < 0 \end{aligned}$$

So $x = (1,0)^T$ is a strict local maximizer.

ii) Check $\lambda_1 = 1$. Then $x_1 = 0$ and $x_2 = 1$ by similar reasoning as in part i). At this point, we have for $\nabla g(x)$ and $\nabla^2_{xx} \mathscr{L}(x,\lambda)$:

$$\begin{aligned} \nabla g(0,1) &= \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \\ \nabla^2_{xx} \mathscr{L}(x,\lambda) &= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

And notice that $Z = (1,0)^T$ is a nullspace matrix for the degenerate constraint (constraint 1), so checking the reduced Hessian, we find

$$\begin{aligned} Z^T \nabla^2_{xx} \mathscr{L}(x,\lambda) Z &= (1,0) \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (1,0) \begin{pmatrix} 4 \\ 0 \end{pmatrix} \\ &= 4 > 0 \end{aligned}$$

So $x = (0,1)^T$ is a strict local minimizer.

Case 4: The second constraint is active.
Then $\lambda_1 = 0$ and $x_1 + x_2 = 0$.
From the reduced gradient, we find

$$\begin{aligned}\lambda_1 &= 2x_1 \\ \lambda_1 &= -2x_2\end{aligned}$$

which gives us the same information as the second constraint. That is, every point on the second constraint is a stationary point (?). A point of interest is the point $x = (0,0)^T$ since here, $\lambda_1 = 0$, and the first order sufficiency condition is satisfied. At this point, the gradient of the constraints and the reduced Hessian are as follows:

$$\begin{aligned}\nabla g(0) &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ \nabla^2_{xx}\mathscr{L}(0,\lambda) &= \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\end{aligned}$$

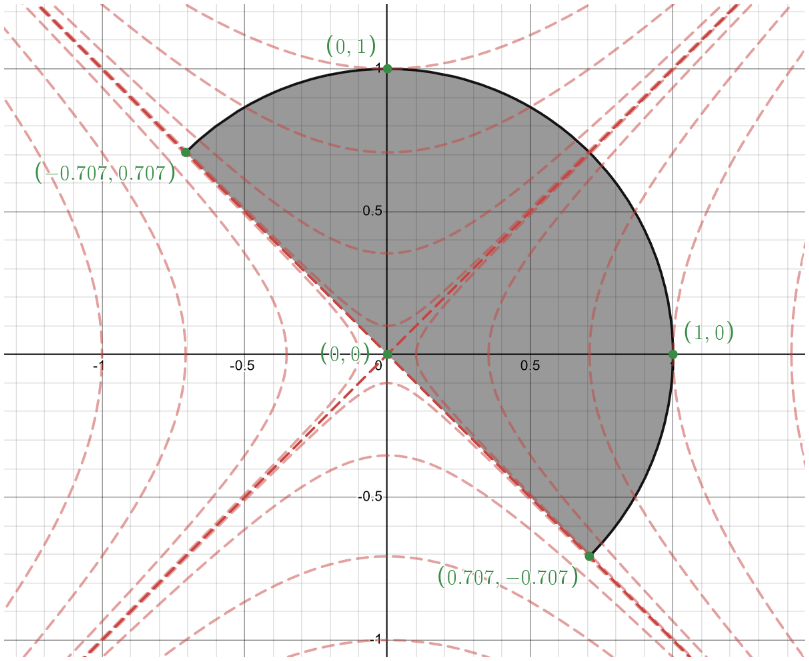
Since both constraints are degenerate at this point, we must find a null space matrix for $\nabla g(0)$, which is simply $Z = (1,-1)^T$. Checking the second order sufficiency condition:

$$\begin{aligned}Z^T\nabla^2_{xx}\mathscr{L}(0,\lambda)Z &= (1,-1)\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= 0\end{aligned}$$

So the second order conditions are not satisfied. Now, notice if a small perturbation is made in the positive x_1 direction, $f(\delta_1, 0) > f(0,0)$ and if a small perturbation is made in the positive x_2 direction, $f(0,\delta_2) < f(0,0)$, meaning that the point $(0,0)$ is a saddle point.

Finally, we found that the point $x = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$ is a local minimum with an associated objective value of $f(x) = 0$. Similarly, we found that the point $x = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ is a local maximum with an associated objective value of $f(x) = 0$.
Next, the points $x = (0,1)^T$ and $x = (0,1)^T$ were strict local minimizers and maximizers, respectfully, with associated objective values of $f(x) = -1$, $f(x) = 1$.
Finally, the point $x = (0,0)^T$ was shown to be a saddle point with an associate objective value of $f(x) = 0$.

(c) Add the points from part (b) to your sketch.



(0, 1)
(−0.707, 0.707)
(0, 0)
(1, 0)
(0.707, −0.707)
-1
-0.5
0
0.5
1
0.5
-0.5
-1

5. **Linear Objective, Quadratic Constraint** Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = c^T x \\ \text{subject to} \quad & x^T Q x \le 1, \end{aligned}$$

where Q is a positive-definite symmetric matrix.

(a) Solve the problem. Show that the optimality conditions are satisfied.

Begin by rewriting the constraint to be of the "≥ 0" type. That is, our constraint is $1 - x^T Q x \geq 0$. Building the Lagrangian, we have

$$\mathscr{L} = f(x) - \lambda g(x)$$

where λ is the problem's Lagrange multiplier and $g(x) = 1 - x^T Q x \geq 0$. For a stationary point to exist, we require $\nabla_x \mathscr{L} = 0$. That is,

$$\begin{aligned} \nabla_x \mathscr{L} &= c + \lambda Q x = 0 \\ \lambda Q x &= -c. \end{aligned}$$

Notice that if the constraint is inactive, $\lambda = 0$ and $\nabla_x \mathscr{L} = 0$ if and only if $c = 0$. If $c = 0$, the problem is not of interest since $f \equiv 0$. Then we require that the constraint is active. Thus, for a minimizer, we find (using the fact that Q is positive definite, so Q^{-1} exists):

$$x_* = -\frac{1}{\lambda} Q^{-1} c$$

Further, since the constraint is active, we have $x_*^T Q x_* = 1$. Using the above value for x_*, we find

$$\begin{aligned} \left(-\frac{1}{\lambda} Q^{-1} c\right)^T Q \left(-\frac{1}{\lambda} Q^{-1} c\right) &= 1 \\ \frac{1}{\lambda^2} c^T Q^{-1} Q Q^{-1} c &= 1 \\ \frac{1}{\lambda^2} c^T Q^{-1} c &= 1 \\ \lambda &= \pm\sqrt{c^T Q^{-1} c} \end{aligned}$$

At a minimum, we require $\lambda \geq 0$, so take $\lambda = \sqrt{c^T Q^{-1} c}$. Additionally, since Q is positive definite, Q^{-1} is also positive definite, so $c^T Q^{-1} c > 0$ for all $c \neq 0$. Now, notice that $\nabla^2_{xx} \mathscr{L} = \lambda Q$ and since $\lambda > 0$, $\nabla^2_{xx} \mathscr{L}$ is positive definite, so the second order sufficiency condition is automatically satisfied. (Alternatively, $\nabla g = Q$ which has only a trivial null space, so any basis for the null space is empty, also automatically satisfying the second order sufficiency condition.) Finally, our expression for x_* is

$$x_* = -\frac{Q^{-1} c}{\sqrt{c^T Q^{-1} c}}$$

(b) What is the optimal objective value?

Using the value for x_* we found in part (a), we find

$$\begin{aligned} f(x_*) &= c^T x_* \\ &= -\frac{c^T Q^{-1} c}{\sqrt{c^T Q^{-1} c}} \\ &= -\sqrt{c^T Q^{-1} c} \end{aligned}$$

(c) Use part (a) to find the minimizer of the function $f(x) = -x_1 + 3x_2$ subject to the constraint $2x_1^2 - 2x_1x_2 + 3x_2^2 \leq 1$.

For this problem, we have

$$c = (-1, 3)^T, \quad Q = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

From our work in part (a) and (b), we find the minimizer

$$\begin{aligned} x_* &= -\frac{Q^{-1}c}{\sqrt{c^TQ^{-1}c}} \\ &= \begin{pmatrix} 0 \\ 1/\sqrt{3} \end{pmatrix} \end{aligned}$$

with an associated function value of

$$\begin{aligned} f(x_*) &= -\sqrt{c^TQ^{-1}c} \\ &= -\sqrt{3} \end{aligned}$$