Homework 1

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Section 1.2 Problems

3. *Classify the following differential equations. Is the equation linear, autonomous? What is its order?*

(i) $y'(x) + y(x) = 0$.

(ii) $\frac{d^2}{dt^2}u(t) = t\sin(u(t))$.

(iii) $y(t)^2 + 2y(t) = 0$.

(iv) $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$.

(v) $\dot{x} = -y, \dot{y} = x$.

Soln.

(i) $y'(x) + y(x) = 0$ is a first order, linear, autonomous equation.

(ii) $\frac{d^2}{dt^2}u(t) = t\sin(u(t))$ is a second order, nonlinear, non-autonomous equation.

(iii) $y(t)^2 + 2y(t) = 0$ is a zeroth order, nonlinear, autonomous equation.

(iv) $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$ is a second order, linear, autonomous equation.

(v) $\dot{x} = -y, \dot{y} = x$ is a first order, linear, autonomous system of equations.

4. *Which of the following differential equations for $y(x)$ are linear?*

(i) $y' = \sin(x)y + \cos(y)$.

(ii) $y' = \sin(y)x + \cos(x)$.

(iii) $y' = \sin(x)y + \cos(x)$.

Soln.

(i) $y' = \sin(x)y + \cos(y)$ is nonlinear because of the appearance of the $\cos(y)$ term.

(ii) $y' = \sin(y)x + \cos(x)$ is nonlinear because of the appearance of the $\sin(y)$ term.

(iii) $y' = \sin(x)y + \cos(x)$ is linear since the y terms and all its derivative terms appear linearly.

6. *Transform the following differential equations into first-order systems.*

(i) $\ddot{x} + t\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -y$, $\ddot{y} = x$.

Soln.

(i) Let $y = \dot{x}$ so that the equation becomes the following system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - t\sin(y) \end{cases}$$

(ii) Let $z = \dot{x}$ and $w = \dot{y}$ so that the system becomes the new system

$$\begin{cases} \dot{x} = z \\ \dot{y} = w \\ \dot{z} = -y \\ \dot{w} = x \end{cases}$$

7. *Transform the following differential equations into autonomous first-order systems.*

(i) $\ddot{x} + t\sin(\dot{x}) = x$.

(ii) $\ddot{x} = -\cos(t)x$.

The last equation is linear. Is the corresponding autonomous system also linear?

Soln.

(i) Let $z_1 = t$, $z_2 = x$, and $z_3 = \dot{x}$ so that the equation becomes the new system

$$\begin{cases} \dot{z_1} = 1 \\ \dot{z_2} = z_3 \\ \dot{z_3} = z_2 - z_1\sin(z_3) \end{cases}$$

(ii) Let $z_1 = t$, $z_2 = x$, and $z_3 = \dot{x}$ so that the equation becomes the new system

$$\begin{cases} \dot{z_1} = 1 \\ \dot{z_2} = z_3 \\ \dot{z_3} = -\cos(z_1)z_2 \end{cases}$$

Notice that, while the original equation was linear, our transformed equation is nonlinear because of the appearance of the $\cos(z_1)$ term for the $\dot{z_3}$ equation.

Section 1.3 Problems

9. *Solve the following differential equations:*

(i) $\dot{x} = x^3$.

(ii) $\dot{x} = x(1-x)$.

Soln.

(i) Rewriting the differential equation, we have

$$\frac{dx}{dt} = x^3$$

via separation of variables, we find

$$\begin{aligned} \frac{dx}{x^3} &= dt \\ \int \frac{dx}{x^3} &= \int dt \\ -\frac{1}{2x^2} &= t + C \\ x^2 &= -\frac{1}{2(t+C)} \\ x(t) &= \pm\sqrt{-\frac{1}{2(t+C)}} \end{aligned}$$

where C is a constant of integration and $t + C < 0$.

(ii) Rewriting the differential equation, we have

$$\frac{dx}{dt} = x(1-x)$$

via separation of variables, we find

$$
\begin{aligned}
\frac{dx}{x(1-x)} &= dt \\
\int \frac{dx}{x(1-x)} &= \int dt \\
\int \frac{dx}{x} + \int \frac{dx}{1-x} &= t + C \\
\ln|x| - \ln|1-x| &= t + C \\
\ln\left|\frac{x}{1-x}\right| &= t + C \\
\frac{x}{1-x} &= C_1 e^t \\
x &= C_1 e^t - x C_1 e^t \\
x(1 + C_1 e^t) &= C_1 e^t \\
x(t) &= \frac{C_1 e^t}{1 + C_1 e^t}
\end{aligned}
$$

where $C_1 = e^C$ with C a constant of integration.

10. *Show that the solution of (1.20) is unique if $f \in C^1(\mathbb{R})$.*

Proof: Suppose by way of contradiction that there exist two distinct functions $\phi(t)$ and $\varphi(t)$ that satisfy (1.20) and further suppose that that I_t is an open interval containing t_0 such that the differential equation (1.20) is satisfied, and let I_x be the associated open interval that contains x_0. Since $f \in C_1(\mathbb{R})$, by the mean value theorem, for $(x_0, x) \subseteq I_x$ (likewise $(t_0, t) \in I_t$), there exists a $c(x) \in (x_0, x)$ such that

$$
f'(c(x)) = \frac{f(x) - f(x_0)}{x - x_0}
$$

but from (1.20), we have $\dot{\phi}(t) = f(x)$ and $\dot{\varphi}(t) = f(x)$, hence from the above equation, we find

$$
\frac{\dot{\phi}(t) - f(x_0)}{x - x_0} = \frac{\dot{\varphi}(t) - f(x_0)}{x - x_0}
$$

Hence,

$$
\begin{aligned}
\dot{\phi}(t) - f(x_0) &= \dot{\varphi}(t) - f(x_0) \\
\dot{\phi}(t) &= \dot{\varphi}(t) \\
\phi(t) &= \varphi(t) + C
\end{aligned}
$$

Plugging in $t = t_0$, we find

$$
\begin{aligned}
\phi(t_0) &= \varphi(t_0) + C \\
x_0 &= x_0 + C \\
C &= 0
\end{aligned}
$$

Hence, $\phi(t) = \varphi(t)$. And since $t \in I_t$ was chosen arbitrarily, we have that $\phi(t) = \varphi(t)$ for all t satisfying the differential equation, a contradiction.

12. *Solve the following differential equations:*

(i) $\dot{x} = \sin(t)x$.

(iii) $\dot{x} = \sin(t)e^x$.

Sketch the solutions. For which initial conditions (if any) are the solutions bounded?

Soln.

(i) Rewriting our differential equation, we have

$$\frac{dx}{dt} = \sin(t)x$$

via separation of variables, we find

$$\begin{aligned}\frac{dx}{x} &= \sin(t)dt \\ \int \frac{dx}{x} &= \int \sin(t)dt \\ \ln|x| &= -\cos(t) + C \\ x(t) &= C_1 e^{-\cos(t)}\end{aligned}$$

where $C_1 = e^C$ with C a constant of integration. Notice that since $|\cos(t)| \leq 1$ for all t,

$$C_1 e^{-1} \leq x(t) \leq C_1 e$$

so $x(t)$ is bounded for all t. Hence, for any initial condition, the solution remains bounded.

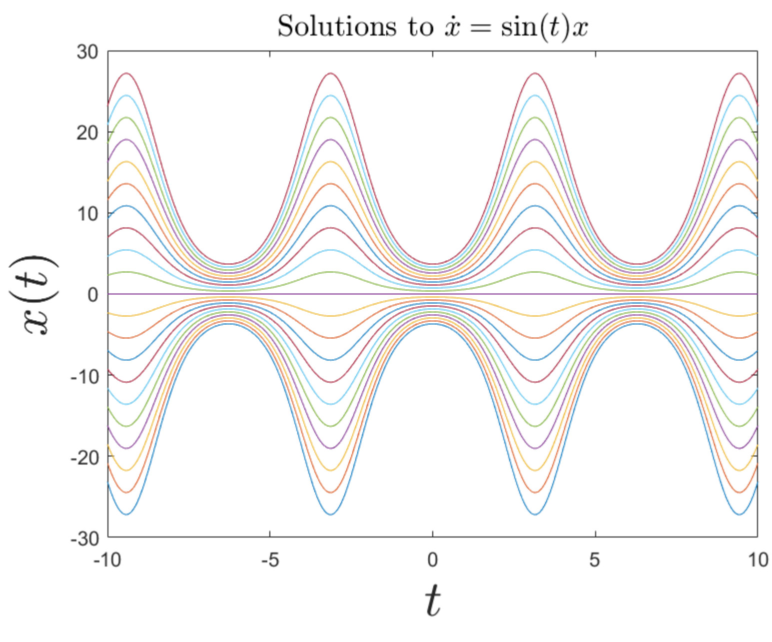


Figure 1: Various solutions to $\dot{x} = \sin(t)x$ for values of C_1 between -10 and 10.

(iii) Rewriting the differential equation, we have

$$\frac{dx}{dt} = \sin(t)e^x$$

via separation of variables, we find

$$\begin{aligned}
\frac{dx}{e^x} &= \sin(t)dt \\
\int e^{-x}dx &= \int \sin(t)dt \\
-e^{-x} &= -\cos(t) + C \\
e^{-x} &= \cos(t) - C \\
-x &= \ln(\cos(t) - C) \\
x(t) &= -\ln(\cos(t) - C)
\end{aligned}$$

Where C is a constant of integration. For some initial condition $x(0) = x_0$, from our above equation, we find

$$\begin{aligned}
x_0 &= -\ln(1 - C) \\
e^{-x_0} &= 1 - C \\
C &= 1 - e^{-x_0}
\end{aligned}$$

so that we may rewrite our solution as

$$x(t) = -\ln(\cos(t) - 1 + e^{-x_0})$$

Note that since $|\cos(t)| \leq 1$ for all t, the argument of the natural log will never go to infinity, and thus is bounded so long as

$$\cos(t) - 1 + e^{-x_0} \neq 0$$

or equivalently,

$$\cos(t) \neq 1 - e^{-x_0}$$

and since $|\cos(t)| \leq 1$ is continuous, $\cos(t)$ will attain every real value between -1 and 1. Hence, our above condition is equivalent to

$$|1 - e^{-x_0}| > 1$$

which means that either

$$1 - e^{-x_0} > 1$$

or

$$-1 + e^{-x_0} > 1.$$

Notice that the first condition is never satisfied, so turning our attention to the second condition, we have (using the fact that $\ln(\cdot)$ is monotone),

$$\begin{aligned}
e^{-x_0} &> 2 \\
-x_0 &> \ln(2) \\
x_0 &< -\ln(2)
\end{aligned}$$

Thus, our solution is bounded for all initial conditions $x_0 < -\ln(2)$.

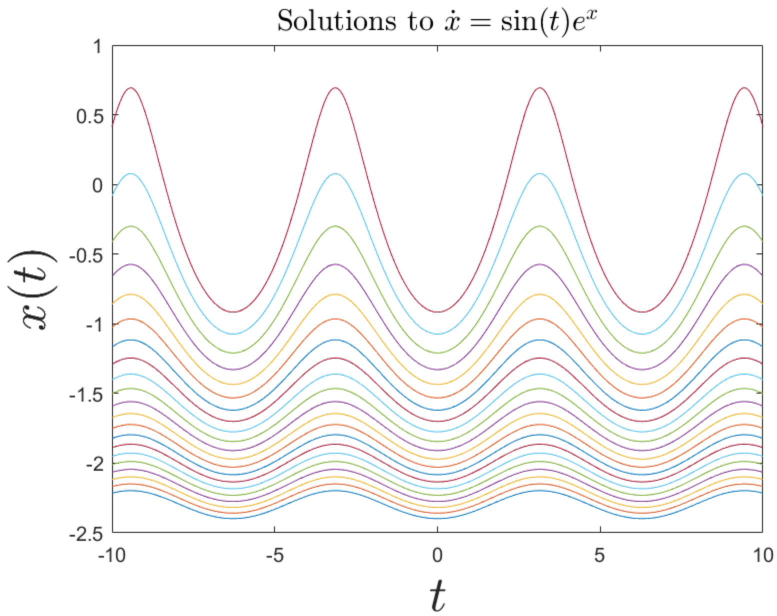


Figure 2: Various solutions to $\dot{x} = \sin(t)e^x$ for values of C between -10 and -1.5.

Section 1.4 Problems

18. *Try to find solutions of the following differential equations:*

(i) $\dot{x} = \frac{3x-2t}{t}$.

(ii) $\dot{x} = \frac{x-t+2}{2x+t+1} + 5$.

(iii) $y' = y^2 - \frac{y}{x} - \frac{1}{x^2}$.

(iv) $y' = \frac{y}{x} - \tan\left(\frac{y}{x}\right)$.

Soln.

(i) To begin, notice we may rewrite the differential equation as

$$\dot{x} = f\left(\frac{x}{t}\right)$$

for $f\left(\frac{x}{t}\right) = 3\frac{x}{t} - 2$. Let $y = \frac{x}{t}$. Then

$$\dot{y} = \frac{1}{t}f(y) - \frac{y}{t}$$

so that

$$\begin{aligned}
\int \frac{dy}{f(y) - y} &= \int \frac{dt}{t} \\
\int \frac{dy}{3y - 2 - y} &= \ln|t| + C_0 \\
\frac{1}{2}\int \frac{dy}{y-1} &= \ln|t| + C_0 \\
\ln|y-1| &= \ln(t^2) + C_1 \\
y - 1 &= C_2 t^2 \\
y &= C_2 t^2 + 1
\end{aligned}$$

Substituting back $y = \frac{x}{t}$, we find the solution to our differential equation is

$$x(t) = C_2 t^3 + t$$

(ii) Rewriting, we have

$$\begin{aligned}\dot{x} &= f(x,t)\\ &= \frac{11x + 4t + 7}{2x + t + 1}\\ &= \frac{ax + bt + c}{\alpha x + \beta t + \gamma}\end{aligned}$$

for $a = 11, b = 4, c = 7$ and $\alpha = 2, \beta = 1, \gamma = 1$. Note that $a\beta - \alpha b = 3 \neq 0$, so we will make the substitution $y = x - x_0$, $s = t - t_0$ where x_0, t_0 solve the following linear system:

$$\begin{pmatrix} 11 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \end{pmatrix}$$

which has the unique solution $(x_0, t_0)^T = (-1, 1)$. Hence, let $y = x + 1$ and $s = t - 1$ so that $f(y, s)$ becomes

$$\begin{aligned}f(y,s) &= \frac{11y + 4s}{2y + s}\\ &= \frac{11\frac{y}{s} + 4}{2\frac{y}{s} + 1}\\ &= f\left(\frac{y}{s}\right)\end{aligned}$$

Let $z = \frac{y}{s}$ so that $f\left(\frac{y}{s}\right) = f(z)$ and $\dot{z} = \frac{f(z) - z}{s} = \frac{1}{s}\left(\frac{-2z^2 + 10z + 4}{2z + 1}\right)$ and our equation is now separable. Then

$$\begin{aligned}\dot{z} &= \frac{1}{s}\left(\frac{-2z^2 + 10z + 4}{2z + 1}\right)\\ \int \frac{2z + 1}{-2z^2 + 10z + 4} dz &= \int \frac{1}{s} ds.\end{aligned}$$

Let us evaluate the integral on the left hand side of the above equation:

$$\begin{aligned}\int \frac{2z + 1}{-2z^2 + 10z + 4} dz &= -\frac{1}{2} \int \frac{2z + 1}{z^2 - 5z - 2} dz\\ &= -\frac{1}{2} \int \frac{2z - 5 + 6}{z^2 - 5z - 2} dz\\ &= -\frac{1}{2} \int \frac{2z - 5}{z^2 - 5z - 2} dz - 3 \int \frac{1}{z^2 - 5z - 2}\\ &= -\frac{1}{2} \ln|z^2 - 5z - 2| - 3 \int \frac{1}{\left(z - \frac{5}{2}\right)^2 - \frac{29}{4}} dz\end{aligned} \tag{1}$$

Now, inspecting the second integral, let $z - \frac{5}{2} = \frac{\sqrt{29}}{2} \tanh(\varphi)$ so that $dz = \frac{\sqrt{29}}{2} \operatorname{sech}^2(\varphi) d\varphi$ and the integral becomes

$$\begin{aligned}\int \frac{1}{\left(z - \frac{5}{2}\right)^2 - \frac{29}{4}} dz &= \int \frac{\frac{\sqrt{29}}{4} \operatorname{sech}^2(\varphi)}{\frac{29}{4} \operatorname{sech}^2(\varphi)} d\varphi\\ &= \frac{1}{\sqrt{29}} \varphi + C\\ &= \frac{1}{\sqrt{29}} \tanh^{-1}\left(\frac{2}{\sqrt{29}}(z - \tfrac{5}{2})\right) + C.\end{aligned}$$

Then (1) becomes

$$-\frac{1}{2}\ln|z^2-5z-2|-\frac{3}{\sqrt{29}}\tanh^{-1}\left(\frac{2}{\sqrt{29}}(z-\tfrac{5}{2})\right)+C.$$

Hence, our differential equation becomes

$$-\frac{1}{2}\ln|z^2-5z-2|-\frac{3}{\sqrt{29}}\tanh^{-1}\left(\frac{2}{\sqrt{29}}(z-\tfrac{5}{2})\right)=\ln|s|+C$$

Substituting back in $z=\frac{y}{s}$, we have

$$-\frac{1}{2}\ln\left|\left(\frac{y}{s}\right)^2-5\frac{y}{s}+2\right|-\frac{3}{\sqrt{29}}\tanh^{-1}\left(\frac{2}{\sqrt{29}}(z-\tfrac{5}{2})\right)=\ln|s|+C.$$

Now, substituting back in $y=x+1$ and $s=t-1$, we have

$$-\frac{1}{2}\ln\left|\left(\frac{x+1}{t-1}\right)^2-5\left(\frac{x+1}{t-1}\right)-2\right|-\frac{3}{\sqrt{29}}\tanh^{-1}\left(\frac{2}{\sqrt{29}}\left(\frac{x+1}{t-1}-\frac{5}{2}\right)\right)=\ln|t-1|+C.$$

So the solution to our differential equation satisfies the above implicit equation.

(iii) Notice we may rewrite the differential equation as

$$y'=\frac{(xy)^2-xy-1}{x^2}$$

Let $z=xy$. Then

$$y'=\frac{z^2-z-1}{x^2}$$

and $z'=y'x+y$ so that

$$z'=\frac{z^2-1}{x}$$

Solving via separation of variables, we find

$$\begin{aligned}
\int\frac{dz}{z^2-1}&=\int\frac{dx}{x}\\
\frac{1}{2}\int\frac{dz}{z-1}-\frac{1}{2}\int\frac{dz}{z+1}&=\ln|x|+C_0\\
\ln|z-1|-\ln|z+1|&=\ln(x^2)+C_1\\
\ln\left|\frac{z-1}{z+1}\right|&=\ln(x^2)+C_1\\
\frac{z-1}{z+1}&=C_2x^2\\
z&=C_2x^2z+C_2x^2+1\\
z(1-C_2x^2)&=C_2x^2+1\\
z&=\frac{C_2x^2+1}{1-C_2x^2}
\end{aligned}$$

Substituting xy back in for z, we find the solution to our differential equation as

$$y(x)=\frac{C_2x^2+1}{x-C_2x^3}$$

(iv) Let $z = \frac{y}{x}$ so that

$$y' = f(z)$$

with $f(z) = \frac{x}{y} - \tan\left(\frac{y}{x}\right)$ and $z' = \frac{f(z)-z}{x}$. Using separation of variables,

$$\begin{aligned}
\int \frac{dz}{f(z) - z} &= \int \frac{dx}{x} \\
\int \frac{dz}{-\tan(z)} &= \ln|x| + C_0 \\
-\ln|\sin(z)| &= \ln|x| + C_0 \\
\ln|\sin(z)| &= \ln\left|\frac{1}{x}\right| + C_1 \\
\sin(z) &= \frac{C_2}{x} \\
z &= \arcsin\left(\frac{C_2}{x}\right)
\end{aligned}$$

substituting back $\frac{y}{x} = z$, we have the solution to our differential equation as

$$y(x) = x \arcsin\left(\frac{C_2}{x}\right)$$

19. (Euler equation). *Transform the differential equation*

$$t^2\ddot{x} + 3t\dot{x} + x = \frac{2}{t}$$

to the new coordinates $y = x$, $s = \log(t)$. *(Hint: You are* not *asked to solve it.)*

Soln. Notice that since $y = x$, we have that

$$\frac{dy}{dt} = \frac{dx}{dt}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}$$

By the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt}$$

and since $s = \log(t)$,

$$\frac{ds}{dt} = \frac{1}{t} = e^{-s}$$

so that

$$\dot{x} = \dot{y}e^{-s}.$$

Additionally by the chain rule,

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{ds}{dt}\right)^2 + \frac{dy}{ds}\frac{d^2s}{dt^2}$$

and

$$\frac{d^2s}{dt^2} = -\frac{1}{t^2} = -e^{-2s}$$

so

$$\ddot{x} = \ddot{y}e^{-2s} - \dot{y}e^{-2s}$$

so that our differential equation becomes

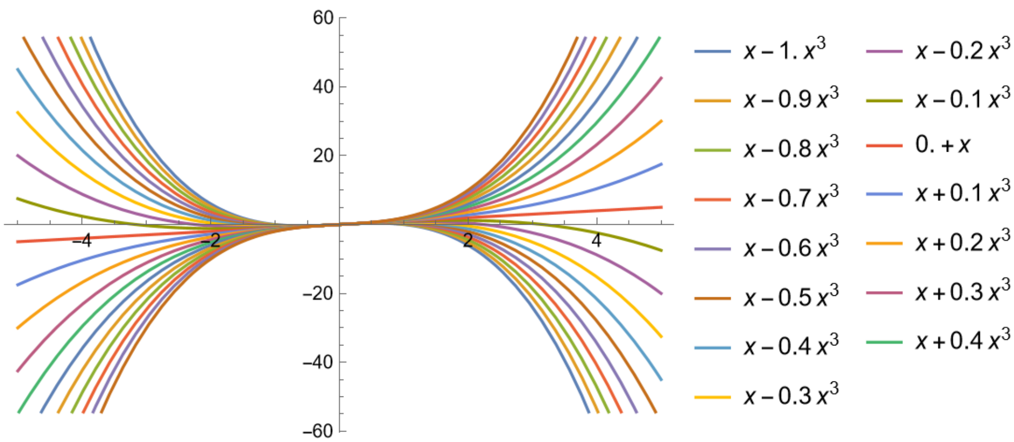
$$\begin{aligned} e^{2s}(e^{-2s}\ddot{y} - e^{-2s}\dot{y}) + 3e^{s}(e^{-s}\dot{y}) + y &= 2e^{-s} \\ \ddot{y} - \dot{y} + 3\dot{y} + y &= 2e^{-s} \\ \ddot{y} + 2\dot{y} + y &= 2e^{-s} \end{aligned}$$

Thus, our differential equation under the given transformation is

$$\ddot{y} + 2\dot{y} + y = 2e^{-s}$$

20. *Pick some differential equations from the previous problems and solve them using your favorite computer algebra system. Plot the solutions.*

Using Mathematica to solve the differential equation in problem 18 (a), and plotting for multiple values of C between -1 and 1, we find the following:



25. (Catenary). *Solve the differential equation describing the shape $y(x)$ of a hanging chain suspended at two points:*

$$y'' = a\sqrt{1 + (y')^2}, \qquad a > 0.$$

To begin, let $v = y'$ so that $v' = y''$ and the differential equation becomes

$$v' = a\sqrt{1 + v^2}$$

which is separable. Separating, we find

$$\begin{aligned} \int \frac{dv}{\sqrt{1+v^2}} &= \int a\,dx \\ &= ax + C_0. \end{aligned}$$

To evaluate the integral on the left hand side, make the substitution $v = \sinh(u)$ so that $dv = \cosh(u)du$ and the integral becomes

$$\begin{aligned} \int \frac{dv}{\sqrt{1+v^2}} &= \int \frac{\sinh(u)}{\sqrt{1+\cosh^2(u)}}du \\ &= \int \frac{\sinh(u)}{\sinh(u)}du \\ &= \int du \\ &= u \\ &= \sinh^{-1}(v). \end{aligned}$$

We now have

$$\begin{aligned}
\sinh^{-1}(v) &= ax + C_0 \\
v &= \sinh(ax + C_0) \\
\frac{dy}{dx} &= \sinh(ax + C_0) \\
\int dy &= \int \sinh(ax + C_0)dx \\
y &= \frac{1}{a}\cosh(ax + C_0) + C_1.
\end{aligned}$$