Scientific Computation HW3

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Exercise 3.1 *(code for Poisson problem)*

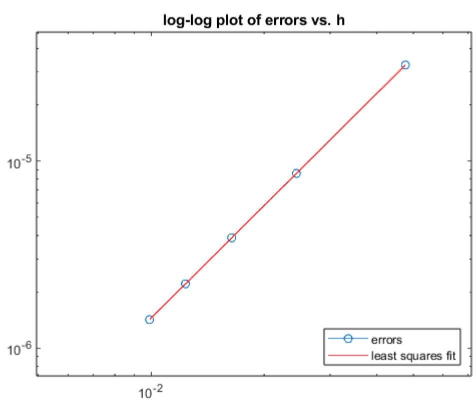
The MATLAB script **poisson.m** solves the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x,y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x,y) = 1.25\exp(x + y/2)$.

(a) Test this script by performing a grid refinement study to verify that it is second order accurate.

(b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.

(c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

(a) By introducing a for loop to loop through increasing values of m, ($m = 20, 40, 60, 80, 100$), and using the error_table.m and error_loglog.m files, we find the following:

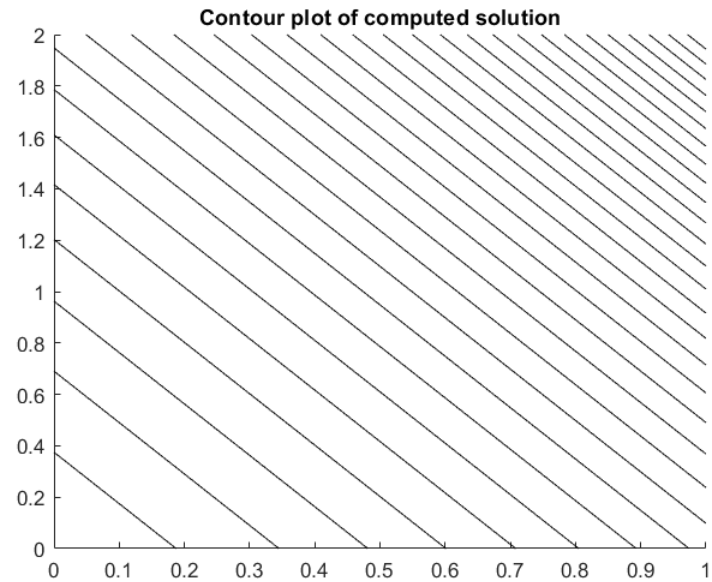


h	error	ratio	observed order
0.04762	3.27323e-05	NaN	NaN
0.02439	8.60139e-06	3.80547	1.99752
0.01639	3.88878e-06	2.21185	1.99805
0.01235	2.20538e-06	1.76332	2.00016
0.00990	1.41868e-06	1.55453	1.99923

Least squares fit gives E(h) = 0.0143668 * h^1.99835

which shows us the 5-point is approximately a second order method, which is what we wished to show.

(b) See the attached poisson_rectangular_Nameika.m for changes to poisson.m.
running the modified script results in the following:




```
>> poisson_rectangular_Nameika
Error relative to true solution of PDE =  7.654e-05
Error relative to true solution of PDE =  2.014e-05
Error relative to true solution of PDE =  9.111e-06
Error relative to true solution of PDE =  5.170e-06
```

as we can see, there is hardly any change in the error when decreasing the value of h. I suspect this may be due to the fact that h is the same for both x and y, which could lead to grid irregularities on a rectangular domain.

(c) For the case $\Delta y \neq \Delta x$, the five point method becomes

$$\frac{1}{(\Delta x)^2}(u_{i-1,j} + u_{i+1,j}) + \frac{1}{(\Delta y)^2}(u_{i,j-1} + u_{i,j-1}) - 2\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right) u_{i,j} = f_{i,j}$$

Writing out the matrix A using this equation, we find

$$A = \begin{bmatrix} T & Y & & & \\ Y & T & Y & & \\ & \ddots & \ddots & \ddots & \\ & & Y & T & Y \end{bmatrix}$$

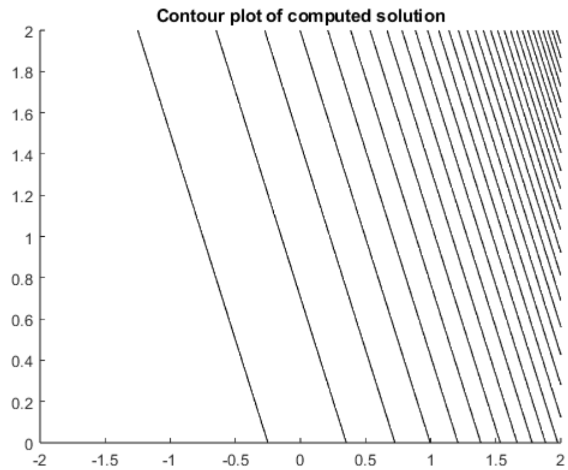
where

$$T = \begin{bmatrix} -2\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right) & \frac{1}{(\Delta x)^2} & & & & \\ \frac{1}{(\Delta x)^2} & -2\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right) & \frac{1}{(\Delta x)^2} & & & \\ & \frac{1}{(\Delta x)^2} & -2\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right) & \frac{1}{(\Delta x)^2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{(\Delta x)^2} & -2\left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right) & \frac{1}{(\Delta x)^2} \end{bmatrix}$$

and

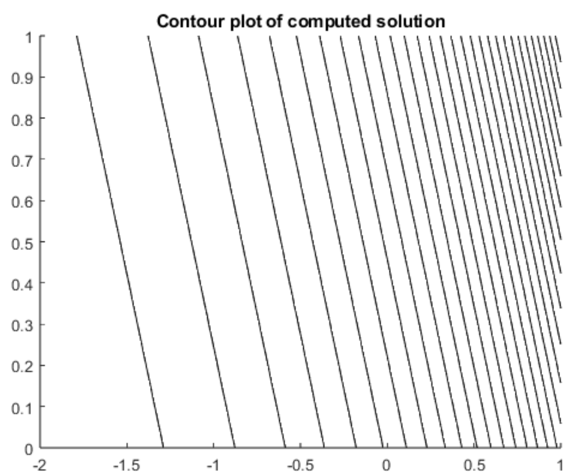
$$Y = \frac{1}{(\Delta y)^2} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Modifying the code to implement this modification, we find the following:




```
>> poisson
Error relative to true solution of PDE =  3.536e-04
```

where we defined $\Delta y = \frac{b_y - a_y}{m+1}$ and $\Delta x = \frac{b_x + a_x}{m+1}$. Running the script for the rectangular domain we used in part (b) gives us




```
>> poisson
Error relative to true solution of PDE =  3.291e-04
Error relative to true solution of PDE =  8.662e-05
Error relative to true solution of PDE =  3.924e-05
Error relative to true solution of PDE =  2.226e-05
```

As we can see, the error has significantly improved from part (b). See the attached code file for precisely what changes were made to the script.

Exercise 3.2 *(9-point Laplacian)*

(a) Show that the 9-point Laplacian (3.17) has the truncation error derived in Section 3.5. **Hint:** To simplify the computation, note that the 9-point Laplacian can be written as the 5-point Laplacian (with known truncation error) plus a finite difference approximation that models $\frac{1}{6}h^2 u_{xxyy} + O(h^4)$.

(b) Modify the MATLAB script `poisson.m` to use the 9-point Laplacian (3.17) instead of the 5-point Laplacian, and to solve the linear system (3.18) where f_{ij} is given by (3.19). Perform a grid refinement study to verify that fourth order accuracy is achieved.

(a) Using Taylor expansion (see page of work at the end of the document), we find the following:

$$\nabla_9^2 u_{ij} = \nabla^2 u_{ij} + \frac{h^2}{12}\left(\frac{\partial^4 u_{ij}}{\partial x^4} + 2\frac{\partial^4 u_{ij}}{\partial x^2 \partial y^2} + \frac{\partial^4 u_{ij}}{\partial y^4}\right) + \mathcal{O}(h^4)$$

Which is what we wanted to show.

(b) The nine point laplacian is given by the following formula:

$$\nabla_9^2 u_{ij} = \frac{1}{6h^2}(u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + 4u_{i,j-1} + 4u_{i,j+1} + 4u_{i-1,j} + 4u_{i+1,j} - 20u_{ij})$$

Writing this in matrix form, we see

$$A = \frac{1}{6h^2}\begin{bmatrix} W & T & & & & \\ T & W & T & & & \\ & T & W & T & & \\ & & \ddots & \ddots & \ddots & \\ & & & T & W & T \\ & & & & T & W \end{bmatrix}$$

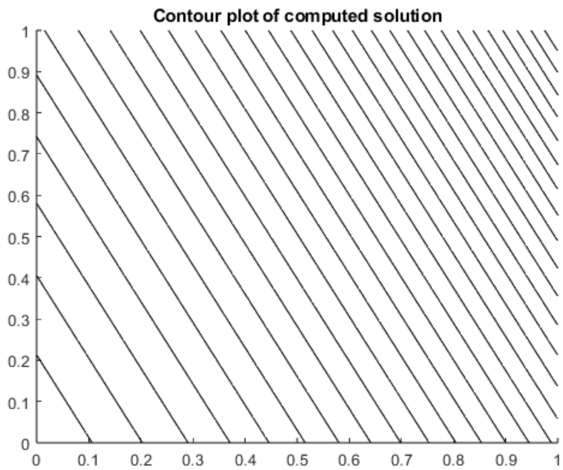
where

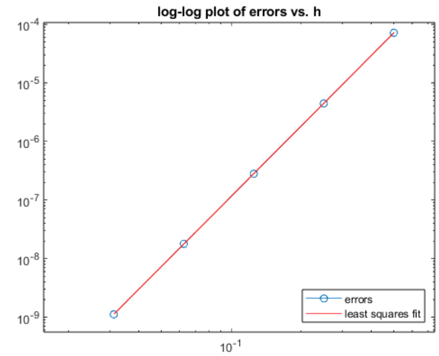
$$W = \begin{bmatrix} -20 & 4 & & & & \\ 4 & -20 & 4 & & & \\ & 4 & -20 & 4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 4 & -20 & 4 \\ & & & & 4 & -20 \end{bmatrix}$$

is an $m \times m$ matrix where m is the number of interior points in the x dimension. And

$$T = \begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix}$$

is an $n \times n$ matrix where n is the number of interior points in the y dimension. Implementing this change in the poisson.m script and accounting for the added boundary conditions introduced with the four extra points added by the nine point laplacian, we find the following on $[0,1] \times [0,1]$:






```
>> poisson
Error relative to true solution of PDE =  7.168e-05
Error relative to true solution of PDE =  4.453e-06
Error relative to true solution of PDE =  2.811e-07
Error relative to true solution of PDE =  1.781e-08
Error relative to true solution of PDE =  1.116e-09

Least squares fit gives E(h) = 0.00113317 * h^3.9908
```

Notice the changes made to the right hand side:

```
% adjust the rhs to include boundary terms:
rhs(:,1) = rhs(:,1) - (AAy*usoln(Iint,1))/(6*h^2);
rhs(:,my) = rhs(:,my) - (AAy*usoln(Iint,my+2))/(6*h^2);
rhs(1,:) = rhs(1,:) - (usoln(1,Jint)*AAx)/(6*h^2);
rhs(mx,:) = rhs(mx,:) - (usoln(mx+2,Jint)*AAx)/(6*h^2);

rhs(1,1) = rhs(1,1) - usoln(1,1)/(6*h^2);
rhs(1,my) = rhs(1, my) - usoln(1,my+2)/(6*h^2);
rhs(mx,1) = rhs(mx,1) - usoln(mx+2,1)/(6*h^2);
rhs(mx,my) = rhs(mx,my) - usoln(mx+2,my+2)/(6*h^2);
```

Where AAx is an $m \times m$ tridiagonal matrix with 1 on the super and sub diagonal and 4 on the main diagonal. Similarly, AAy is an $n \times n$ tridiagonal matrix with 1 on the super and sub diagonal and 4 on the main diagonal.

See attached code file for more details.

Work for 3.2 (a)

To begin, recall Taylor's expansion for a smooth function of two variables centered at (x_0, y_0):

$$f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{1}{i!j!}\frac{\partial^{i+j}f(x_0,y_0)}{\partial x^i \partial y^j}(x-x_0)^i(y-y_0)^j + \mathcal{O}(h^{n+1})$$

In our case, $x_0 = x_i$, $y_0 = y_j$ and $x_{i+1} = x_i + h$, $y_{j+1} = y_j + h$.
Now, let's find the Taylor expansion for $u_{i-1,j+1}, u_{i+1,j-1}, u_{i-1,j-1}, u_{i+1,j+1}, u_{i,j-1}, u_{i,j+1}, u_{i-1,j}$, and $u_{i+1,j}$:

$$u_{i-1,j} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} + \mathcal{O}(h^6)$$

$$u_{i+1,j} = u_{ij} + h\frac{\partial u_{ij}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} + \mathcal{O}(h^6)$$

then

$$u_{i-1,j} + u_{i+1,j} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} + \mathcal{O}(h^6) \tag{1}$$

Similarly,

$$u_{i,j-1} = u_{ij} - h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} + \mathcal{O}(h^6)$$

$$ui,j+1 = u_{ij} + h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} + \mathcal{O}(h^6)$$

Then

$$u_{i,j-1} + u_{i,j+1} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial y^2} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \mathcal{O}(h^6) \tag{2}$$

Now,

$$u_{i-1,j-1} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} - h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} - \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x^2\partial y} - \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial y^2\partial x} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} +$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x^3\partial y} + \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x\partial y^3} + \frac{h^4}{4}\frac{\partial^4 u_{ij}}{\partial x^2\partial y^2} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} - \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x^4\partial y} - \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x\partial y^4} - \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^3\partial y^2} +$$
$$- \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^2\partial y^3} + \mathcal{O}(h^6)$$

Similarly,

$$u_{i+1,j+1} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} + h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} + \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x^2\partial y} + \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial y^2\partial x} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} +$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x^3\partial y} + \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x\partial y^3} + \frac{h^4}{4}\frac{\partial^4 u_{ij}}{\partial x^2\partial y^2} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} - \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x^4\partial y} + \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x\partial y^4} + \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^3\partial y^2} +$$
$$+ \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^2\partial y^3} + \mathcal{O}(h^6)$$

Then

$$\begin{aligned} u_{i-1,j-1} + u_{i+1,j+1} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2} + h^2\frac{\partial^2 u_{ij}}{\partial y^2} + 2h^2\frac{\partial^2 u_{ij}}{\partial x\partial y} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{3}\frac{\partial^4 u_{ij}}{\partial x^3\partial y} + \\ + \frac{h^4}{3}\frac{\partial^4 u_{ij}}{\partial x\partial y^3} + \frac{h^4}{2}\frac{\partial^4 u_{ij}}{\partial x^2\partial y^2} + \mathcal{O}(h^6) \end{aligned} \tag{3}$$

Now,

$$u_{i-1,j+1} = u_{ij} - h\frac{\partial u_{ij}}{\partial x} + h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} - h^2\frac{\partial^2 u_{ij}}{\partial x \partial y} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} + \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x^2 \partial y} - \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x \partial y^2} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} +$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{2}\frac{\partial^4 u_{ij}}{\partial x^2 \partial y^2} - \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x \partial y^3} - \frac{h^3}{3!}\frac{\partial^4 u_{ij}}{\partial x^3 \partial y} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} +$$
$$- \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x \partial y^4} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^4 \partial y} - \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^3 \partial y^2} + \frac{h^5}{3!}\frac{\partial u_{ij}}{\partial x^2 \partial y^3} + \mathcal{O}(h^6)$$

Similarly,

$$u_{i+1,j-1} = u_{ij} + h\frac{\partial u_{ij}}{\partial x} - h\frac{\partial u_{ij}}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial x^2} + \frac{h^2}{2}\frac{\partial^2 u_{ij}}{\partial y^2} - h^2\frac{\partial^2 u_{ij}}{\partial x \partial y} + \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial x^3} - \frac{h^3}{3!}\frac{\partial^3 u_{ij}}{\partial y^3} - \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x^2 \partial y} + \frac{h^3}{2}\frac{\partial^3 u_{ij}}{\partial x \partial y^2} + \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} +$$
$$+ \frac{h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{2}\frac{\partial^4 u_{ij}}{\partial x^2 \partial y^2} - \frac{h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x \partial y^3} - \frac{h^3}{3!}\frac{\partial^4 u_{ij}}{\partial x^3 \partial y} + \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^5} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial y^5} +$$
$$+ \frac{h^5}{4!}\frac{\partial^5 u_{ij}}{\partial x \partial y^4} - \frac{h^5}{5!}\frac{\partial^5 u_{ij}}{\partial x^4 \partial y} + \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^3 \partial y^2} - \frac{h^5}{3!}\frac{\partial^5 u_{ij}}{\partial x^2 \partial y^3} + \mathcal{O}(h^6)$$

Then

$$\begin{aligned} u_{i-1,j+1} + u_{i+1,j-1} = 2u_{ij} + h^2\frac{\partial^2 u_{ij}}{\partial x^2} + h^2\frac{\partial^2 u_{ij}}{\partial y^2} - 2h^2\frac{\partial^2 u_{ij}}{\partial x \partial y} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial x^4} + \frac{2h^4}{4!}\frac{\partial^4 u_{ij}}{\partial y^4} + \frac{h^4}{2}\frac{\partial^2 u_{ij}}{\partial x^2 \partial y^2} + \\ - \frac{2h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x \partial y^3} - \frac{2h^4}{3!}\frac{\partial^4 u_{ij}}{\partial x^3 \partial y} + \mathcal{O}(h^6) \end{aligned} \tag{4}$$

Finally, adding up $\frac{1}{6h^2}(4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{ij})$, we find

$$\begin{aligned} \nabla_9^2 u_{ij} &= \frac{1}{6h^2}\left(6h^2\nabla^2 u_{ij} + \frac{h^4}{2}\left(\frac{\partial^4 u_{ij}}{\partial x^4} + 2\frac{\partial^4 u_{ij}}{\partial x^2 \partial y^2} + \frac{\partial^4 u_{ij}}{\partial y^4}\right) + \mathcal{O}(h^6)\right) \\ &= \nabla^2 u_{ij} + \frac{h^2}{12}\left(\frac{\partial^4 u_{ij}}{\partial x^4} + 2\frac{\partial^4 u_{ij}}{\partial x^2 \partial y^2} + \frac{\partial^4 u_{ij}}{\partial y^4}\right) + \mathcal{O}(h^4) \end{aligned}$$