Homework 4

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2D Linear Systems

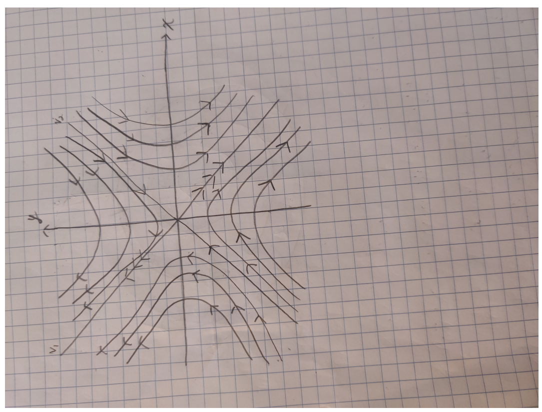
1. Consider the initial value problem $\dot{\mathbf{x}} = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for the following matrices. Characterize the stability of these systems. What type is it? Is it stable or unstable? Also, make a sketch of the system near the critical point. Note: these are the same systems you solved in Homework 3.

   (a) $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

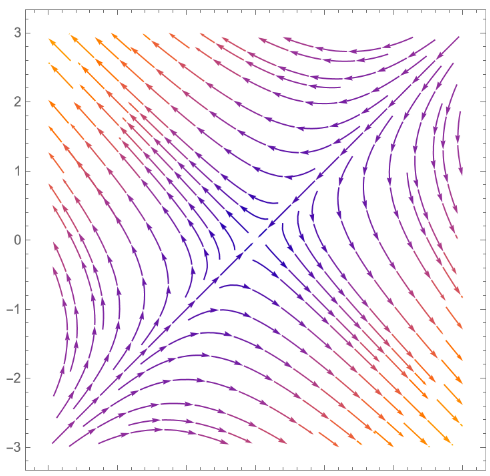
   Notice that the origin, $\mathbf{x} = \mathbf{0}$ is the critical point of the system. (As will be the case for parts (b) and (c).) From Homework 3, we have $\lambda_1 = 3$, $\lambda_2 = -1$ with associated eigenvectors

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Since $\lambda_1 > 0$, $\lambda_2 < 0$, we have that the origin is a saddle.
   Additionally, for an initial condition on the line spanned by v_1, since $\lambda_1 > 0$, the solution will be repulsed away from the origin. However, for an initial condition on the line spanned by v_2, since $\lambda_2 < 0$, the solution will be attracted to the origin. Off these lines, the solution will be "attracted" to the origin and then repulsed as it nears, as we can see in the following figure (the hand drawn figures got rotated for some reason, apologies):



   Which is supported by the following figure generated by Mathematica:

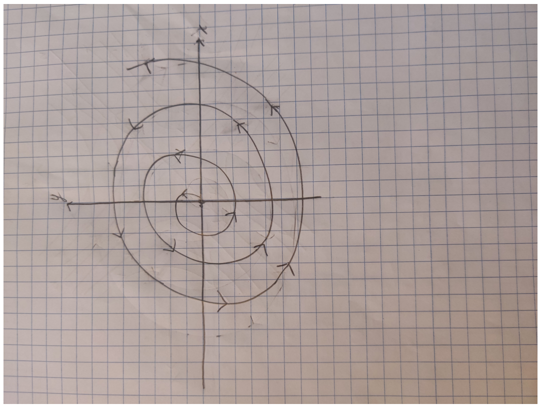


(b) $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

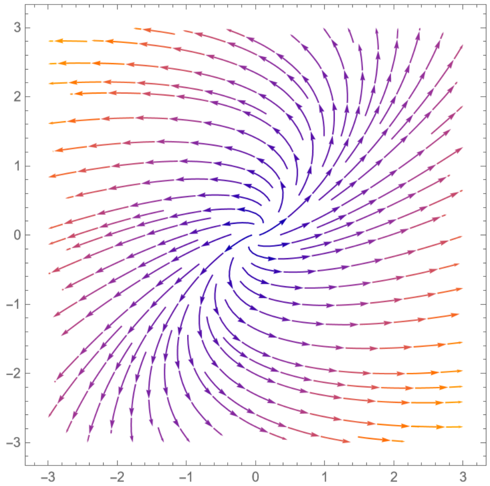
From Homework 3, we have $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$ so that the origin is an unstable spiral since $\text{Re}(\lambda) = 2 > 0$. To determine the orientation, let us write down the system associated with the above matrix:

$$\begin{aligned} \dot{x} &= 3x - 2y \\ \dot{y} &= x + y \end{aligned}$$

at $(x, y) = (1, 0)$, $\dot{x} = 3$, $\dot{y} = 1$, and at $(x, y) = (-1, 0)$, $\dot{x} = -3$, $\dot{y} = -1$, so that the spiral has counter-clockwise orientation, as we can see in the following figure:



Which is supported by the following figure generated by Mathematica:



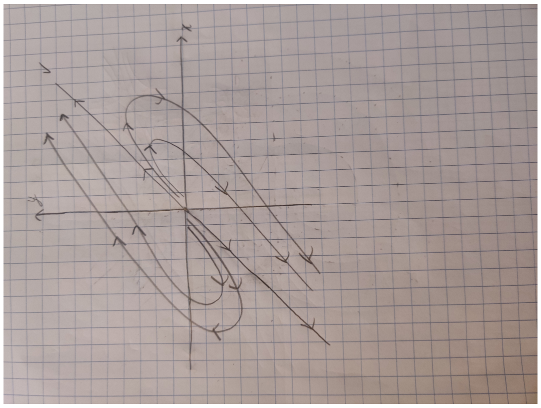
(c) $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$

From Homework 3, we have $\lambda_1 = \lambda_2 = 1$, so that we have real repeated eigenvalues. The associated eigenvector is

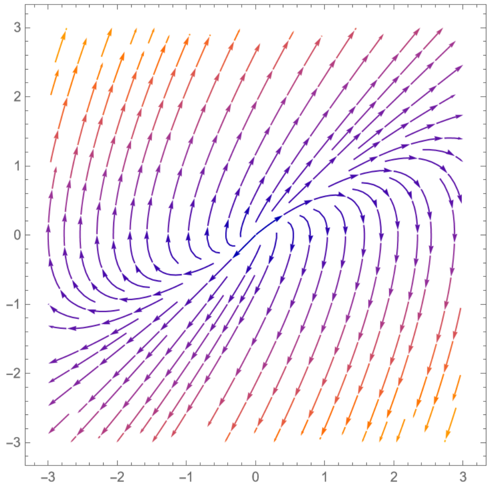
$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So that the origin is an unstable critical point, where any solution $\mathbf{x}(t) \to \pm\infty$ as $t \to \infty$. Additionally, for an initial condition on the line spanned by v, the solution will remain on that

line and will go to $\pm\infty$ as $t \to \infty$.
Now, if $y = 0$ and $x > 0$, notice $\dot{x} = 0$ and $\dot{y} < 0$ and if $y = 0$ and $x < 0$, $\dot{x} = 0$ and $\dot{y} > 0$. Additionally, taking $y = 1/2$, $x = 1$ (so that we are above the $x-$axis but below the line spanned by v), we have $\dot{x} = 1/2$, $\dot{y} = 0$. By a similar analysis for when $x, y < 0$, we have that the solutions near the line spanned by v "curve around" and go off to $\pm\infty$, as we can see in the below figures:



which is supported by the following vector plot generated by Mathematica:



Nonlinear Stability

2. Characterize the stability of the following *nonlinear* systems near the critical points. Make a sketch of the vector field near the critical points.

   (a)

$$\begin{aligned}\dot{x} &= x + x^2\\ \dot{y} &= x + y^2\end{aligned}$$

To begin, let us find the critical points of the system. Beginning with $\dot{x}$:

$$\begin{aligned}\dot{x} &= 0\\ \implies x + x^2 &= 0\\ x(1+x) &= 0\\ \implies x = 0, &\quad x = -1\end{aligned}$$

and now for $\dot{y}$:

$$\begin{aligned}\dot{y} &= 0\\ \implies x + y^2 &= 0\\ y^2 &= -x\end{aligned}$$

so that if $x = -1$, $y^2 = 1 \implies y = \pm 1$ and if $x = 0$, $y = 0$. Hence the critical points of the system are

$$(-1,-1), \quad (-1,1), \quad (0,0)$$

Now let us linearize the system. Let $f(x,y) = x + x^2$ and $g(x,y) = x + y^2$ so that

$$\begin{aligned}f_x(x,y) &= 1 + 2x\\ f_y(x,y) &= 0\\ g_x(x,y) &= 1\\ g_y(x,y) &= 2y\end{aligned}$$

so that, at the critical point $(-1,-1)$, we have $f_x(-1,-1) = -1$, $f_y = 0$, $g_x = 1$, and $g_y(-1,-1) = -2$. That is, we wish to find the behaviour of the linear system described by the following matrix:

$$A = \begin{pmatrix} -1 & 0\\ 1 & -2\end{pmatrix}$$

finding the eigenvalues:

$$\begin{aligned}\begin{vmatrix} -1-\lambda & 0\\ 1 & -2-\lambda\end{vmatrix} &= (\lambda+2)(\lambda+1)\\ &= 0\\ \implies \lambda_1 = -2, &\quad \lambda_2 = -1\end{aligned}$$

so that this critical point is an improper attractive node. Finding the associated eigenvectors:

$$\begin{pmatrix} -1-\lambda_1 & 0\\ 1 & -2-\lambda_1\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 1 & 0\end{pmatrix}.$$

Hence $v_1 = (0,1)^T$ (when letting the second component be 1). For λ_2,

$$\begin{pmatrix} -1-\lambda_2 & 0\\ 1 & -2-\lambda_2\end{pmatrix} = \begin{pmatrix} 0 & 0\\ 1 & -1\end{pmatrix}.$$

And so $v_2 = (1,1)^T$.
Now, for the critical point $(-1,1)$, $f_x(-1,1) = -1$, $f_y(-1,1) = 0$, and $g_x(-1,1)1$, $g_y(-1,1) = 2$ and so, finding the eigenvalues, we have

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} &= (\lambda-2)(\lambda+1) \\ &= 0 \\ \implies \lambda_1 = 2, &\quad \lambda_2 = -1 \end{aligned}$$

since $\lambda_1 > 0$ and $\lambda_2 < 0$, we have that this critical point is a saddle. Now, finding the eigenvectors for λ_1 and λ_2, we have, respectively:

$$\begin{pmatrix} -1-\lambda_1 & 0 \\ 1 & 2-\lambda_1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1-\lambda_2 & 0 \\ 1 & 2-\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}$$

so that

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Finally, for the critical point $(0,0)$, we have $f_x(0,0) = 1$, $f_y(0,0) = 0$, and $g_x(0,0) = 1$, $g_y(0,0) = 0$ so that the matrix that represents the linearized system near this critical point is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and so, finding the eigenvalues,

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} &= \lambda(1-\lambda) \\ &= 0 \\ \implies \lambda_1 = 0, &\quad \lambda_2 = 1. \end{aligned}$$

Thus, this critical point is unstable. Finding the eigenvectors:

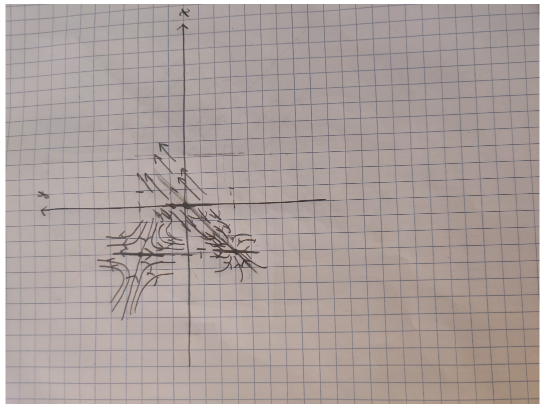
$$\begin{pmatrix} 1-\lambda_1 & 0 \\ 1 & -\lambda_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda_2 & 0 \\ 1 & -\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

so that our associated eigenvectors are, respectively,

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

All together, the behavior of this nonlinear system looks approximately like:



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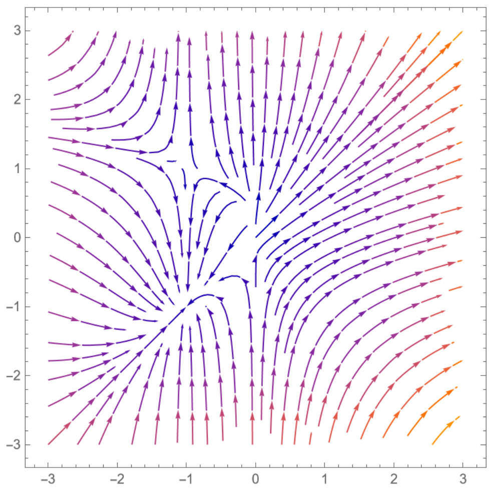


(b)

$$\dot{x} = x(3 - x - y)$$
$$\dot{y} = y(x - y - 1)$$

Let us begin by finding the critical points:

$$\begin{aligned} \dot{x} &= 0 \\ \implies x(3 - x - y) &= 0 \\ \implies x = 0 \quad &\text{or} \quad x = 3 - y \end{aligned}$$

and

$$\begin{aligned} \dot{y} &= 0 \\ \implies y(x - y - 1) &= 0 \\ \implies y = 0 \quad &\text{or} \quad y = x - 1 \end{aligned}$$

thus, if $x = 0$, we have that either $y = 0$ or $y = -1$. Additionally, if $y = 0$, we have $x = 3$. Finally, if $x = 3 - y$ and $y = x - 1$, we have $y = 1$ and $x = 2$ so that our critical points are

$$(0, 0), \quad (0, -1), \quad (3, 0), \quad (2, 1).$$

Now, let $f(x,y) = x(3-x-y)$ and $g(x,y) = y(x-y-1)$ so that

$$\begin{aligned} f_x(x,y) &= 3-2x-y \\ f_y(x,y) &= -x \end{aligned}$$

and

$$\begin{aligned} g_x(x,y) &= y \\ g_y(x,y) &= x-2y-1. \end{aligned}$$

Now, linearizing near the critical point $(0,0)$, we have $f_x(0,0)=3$, $f_y(0,0)=0$, and $g_x(0,0)=0$, $g_y(0,0)=-1$. Thus we wish to find the behavior of the system described by

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Clearly, $\lambda_1 = 3$ and $\lambda_2 = -1$ with associated eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and since $\lambda_1 > 0$, $\lambda_2 < 0$, we have that this critical point is a saddle.

Now, at the critical point $(0,-1)$, $f_x(0,-1)=4$, $f_y(0,-1)=0$, and $g_x(0,-1)=-1$, $g_y(0,1)=1$. Thus we wish to find the behavior of the system described by

$$A = \begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix}.$$

Beginning with the eigenvectors, we have

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 0 \\ -1 & 1-\lambda \end{vmatrix} &= (4-\lambda)(1-\lambda) \\ &= 0 \\ \implies \lambda_1 = 4, \quad \lambda_2 = 1. \end{aligned}$$

Hence this critical point is an improper, repulsive node. Finding the associated eigenvectors for λ_1, λ_2, respectively, we find

$$\begin{pmatrix} 4-\lambda_1 & 0 \\ -1 & 1-\lambda_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & -3 \end{pmatrix}$$

and

$$\begin{pmatrix} 4-\lambda_2 & 0 \\ -1 & 1-\lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -1 & 0 \end{pmatrix}$$

so that

$$v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

At the critical point $(3,0)$, we have $f_x(3,0)=-3$, $f_y(3,0)=-3$, and $g_x(3,0)=2$, $g_y(3,0)=2$. Thus we wish to find the behavior of the system described by

$$A = \begin{pmatrix} -3 & -3 \\ 0 & 2 \end{pmatrix}.$$

Finding the eigenvalues:

$$\begin{vmatrix} -3-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda-2)(\lambda+3)$$
$$= 0$$
$$\implies \lambda_1 = 2, \quad \lambda_2 = -3$$

thus this critical point is a saddle. Finding the associated eigenvectors:

$$\begin{pmatrix} -3-\lambda_1 & -3 \\ 0 & 2-\lambda_1 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -3-\lambda_2 & -3 \\ 0 & 2-\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 0 & 5 \end{pmatrix}$$

we have

$$v_1 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Finally, at the critical point $(2,1)$, we have $f_x(2,1) = -2$, $f_y(2,1) = -2$, and $g_x(2,1) = 1$, $g_y(2,1) = -1$. Thus we wish to find the behavior of the system described by

$$A = \begin{pmatrix} -2 & -2 \\ 1 & -1 \end{pmatrix}.$$

Finding the eigenvalues:

$$\begin{vmatrix} -2-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+2)+2$$
$$= \lambda^2 + 3\lambda + 4 = 0$$

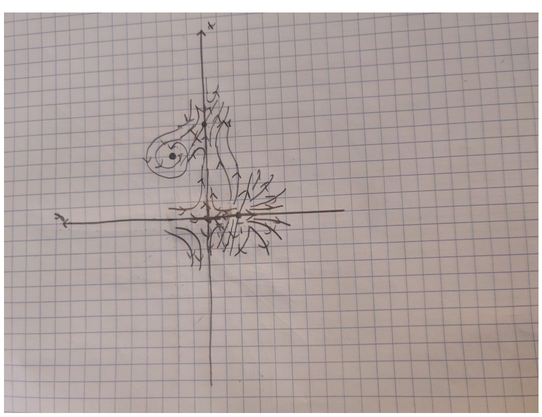
so

$$\lambda = \frac{-3 \pm i\sqrt{7}}{2}.$$

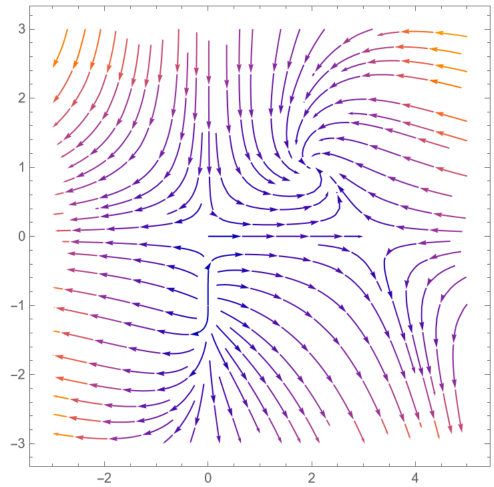
So we have complex eigenvalues with negative real parts, meaning that this critical point is a stable spiral. To determine the orientation of the spiral, check the points $(3,1)$ and $(1,1)$:

$$\dot{y}|_{(3,1)} = 1$$
$$\dot{y}|_{(1,1)} = -1$$

so that the orientation is counter-clockwise.
Putting it together, we have the phase portrait for the nonlinear system:



which is supported by the following vector plot generated by Mathematica:



Liapunov Stability

3. Consider the system

$$\dot{x} = -x + y + xy$$
$$\dot{y} = x - y - x^2$$

Find an appropriate Liapunov function to show that the origin is asymptotically stable and every initial condition converges to the origin. *Soln.* Consider the function $C(t) = x(t)^2 + y(t)^2$. Then

$$\begin{aligned}
\dot{C}(t) &= 2x(t)\dot{x}(t) + 2y(t)\dot{y}(t) \\
&= 2x(-x + y + xy) + 2y(x - y - x^2) \\
&= -2x^2 + 2xy + 2x^2y + 2xy - 2y^2 - 2x^2y \\
&= -2x^2 + 4xy - 2y^2 \\
&= -2(x^2 - 2xy + y^2) \\
&= -2(x - y)^2 \leq 0
\end{aligned}$$

hence,

$$0 \leq C(t) \leq C_0$$

so that solutions remain stable and bounded.

4. Show that $V(x,y) = x^2 + y^2$ is a Liapunov function for the system

$$\dot{x} = -\xi x + y, \qquad \dot{y} = -x - \eta y$$

where $\xi, \eta > 0$ and investigate the stability of $(x_0, y_0) = (0,0)$.

Proof: Begin by noticing that $V(x,y)$ is positive definite. Now,

$$\nabla V(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

and so

$$\begin{aligned}
\nabla V \cdot \dot{\vec{x}} &= (2x, 2y)\begin{pmatrix} -\xi x + y \\ -x - \eta y \end{pmatrix} \\
&= -2\xi x^2 + 2xy - 2xy - 2\eta y^2 \\
&= -2\xi x^2 - 2\eta y^2 \\
&\leq -2\min\{\xi, \eta\}(x^2 + y^2) \\
&= -2\min\{\xi, \eta\} V \\
\implies \nabla V \cdot \dot{\vec{x}} &\leq kV
\end{aligned}$$

where $k = -2\min\{\xi, \eta\} < 0$. Hence, V is a Liapunov function for the system. I claim that the point $(x_0, y_0) = (0,0)$ is asymptotically stable. To see this, notice

$$\begin{aligned}
\nabla V \cdot \dot{\vec{x}} - kV &\leq 0 \\
e^{-kt}\nabla V \cdot \dot{\vec{x}} - ke^{-kt}V &\leq 0 \\
\frac{d}{dt}\left(e^{-kt}V(\vec{x})\right) &\leq 0 \\
\implies e^{-kt}V(\vec{x}) &\leq V_0 e^{-kt_0} \\
\implies V(\vec{x}) &\leq V_0 e^{k(t-t_0)}
\end{aligned}$$

and since $k < 0$, as $t \to \infty$, $V \to 0$ and hence the origin is asymptotically stable.

Section 3.3 (from textbook)

5. Solve the following differential equations:

   (i) $\ddot{x} + 3\dot{x} + 2x = \sinh(t)$.

   Soln. Let us begin by solving the homogeneous problem:

$$\ddot{x_h} + 3\dot{x_h} + 2x_h = 0.$$

   Since this is a linear second order homogeneous equation, from the characteristic equation

$$\begin{aligned} r^2 + 3r + 2 &= 0 \\ (r+2)(r+1) &= 0 \\ \implies r &= -2, -1 \end{aligned}$$

   so that the solution to the homogeneous equation is

$$x_h = C_1 e^{-2t} + C_2 e^{-t}.$$

   For the particular solution, notice $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$ and so, since e^{-t} appears in the homogeneous equation, we will use the ansatz

$$x_p = Ae^t + Bte^{-t}.$$

   Then

$$\begin{aligned} \dot{x_p} &= Ae^t + Be^{-t} - Bte^{-t} \\ \ddot{x_p} &= Ae^t - 2Be^{-t} + Bte^{-t} \end{aligned}$$

   so that

$$\begin{aligned} \ddot{x_p} + 3\dot{x_p} + 2x_p &= 6Ae^t + 2Be^{-t} \\ &= \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \implies A = \frac{1}{12}, \quad B &= -\frac{1}{2}. \end{aligned}$$

   Hence, the solution to this differential equation is

$$x(t) = C_1 e^{-2t} + (C_2 - \tfrac{t}{2})e^{-t} + \tfrac{1}{12}e^t$$

   (ii) $\ddot{x} + 2\dot{x} + 2x = \exp(t)$.

   Soln. Let us begin by solving the homogeneous equation:

$$\ddot{x_h} + 2\dot{x_h} + 2x_h = 0.$$

   Since this is a second order linear homogeneous equation, we have from the characteristic equation

$$\begin{aligned} r^2 + 2r + 2 &= 0 \\ \implies r &= \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i \end{aligned}$$

so that the homogeneous equation is given by

$$x_h(t) = e^{-t}(C_1 \cos(t) + C_2 \sin(t)).$$

Now, for the particular solution, use the ansatz $x_p(t) = Ae^t$, so that

$$\begin{aligned} \dot{x_p} &= Ae^t \\ \ddot{x_p} &= Ae^t \end{aligned}$$

and so

$$\begin{aligned} \ddot{x_p} + 2\dot{x_p} + 2x_p &= 5Ae^t \\ \implies A &= \frac{1}{5} \end{aligned}$$

so that the solution to the differential equation is

$$x(t) = e^{-t}(C_1 \cos(t) + C_2 \sin(t)) + \frac{1}{5}e^t.$$

(iii) $\ddot{x} + 2\dot{x} + x = t^2$.

Soln. To begin, let us solve the homogeneous equation:

$$\ddot{x_h} + 2\dot{x_h} + x_h = t^2.$$

Since this is a linear second order homogeneous equation, from the characteristic equation, we have

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ (r+1)^2 &= 0 \\ \implies r &= -1 \end{aligned}$$

so that the homogeneous solution is

$$x_h(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Now, for the particular solution, use the ansatz $x_p(t) = At^2 + Bt + C$. Then

$$\begin{aligned} \dot{x_p} &= 2At + B \\ \ddot{x_p} &= 2A \end{aligned}$$

so that

$$\begin{aligned} \ddot{x_p} + 2\dot{x_p} + 2x_p &= At^2 + Bt + C + 4At + 2B + 2A \\ &= At^2 + (4A + B)t + (2A + 2B + C) \\ &= t^2 \\ \implies A &= 1 \\ B &= -4 \\ C &= 6 \end{aligned}$$

so that the solution to the differential equation is

$$x(t) = C_1 e^{-t} + C_2 t e^{-t} + t^2 - 4t + 6.$$

6. (Resonance catastrophe). Solve the equation

$$\ddot{x} + \omega_0^2 x = \cos(\omega t), \quad \omega_0, \omega > 0.$$

Discuss the behavior of the solutions as $t \to \infty$. The inhomogeneous term is also known as a forcing term. It is **resonant** if $\omega = \omega_0$. What happens in this case?

Soln. We first consider the case $\omega \neq \omega_0$. We will first solve the homogeneous equation:

$$\ddot{x_h} + \omega_0^2 x_h = 0.$$

Since this is a linear second order homogeneous equation, we have from the characteristic equations:

$$\begin{aligned} r^2 + \omega_0^2 &= 0 \\ \implies r &= \pm i\omega_0 \end{aligned}$$

so that

$$x_h(t) = C_1 \cos(\omega_0) + C_2 \sin(\omega_0).$$

Now, for a particular solution, we try the ansatz $x_p(t) = A\cos(\omega t)$:

$$\ddot{x_p} = -A\omega^2 \cos(\omega t)$$

so that

$$\begin{aligned} \ddot{x_p} + \omega_0^2 x_p &= -A\omega^2 \cos(\omega t) + \omega_0^2 A \cos(\omega t) \\ &= \cos(\omega t) \\ \implies A &= \frac{1}{\omega_0^2 - \omega^2} \end{aligned}$$

so that our solution to the differential equation is

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{1}{\omega_0^2 - \omega^2} \cos(\omega t).$$

Now, for the case $\omega_0 = \omega$, the inhomogeneous part of the equation appears in the homogeneous solution, so we now use the ansatz $x_p(t) = At\sin(\omega t)$. Then we have

$$\begin{aligned} \dot{x_p} &= A\sin(\omega t) + \omega At\cos(\omega t) \\ \ddot{x_p} &= 2\omega A\cos(\omega t) - \omega^2 At\sin(\omega t) \end{aligned}$$

so that

$$\begin{aligned} \ddot{x_p} + \omega_0^2 x &= \ddot{x_p} + \omega^2 x \\ &= 2\omega A\cos(\omega t) - \omega^2 At\sin(\omega t) + \omega^2 At\sin(\omega t) \\ &= 2\omega A\cos(\omega t) \\ &= \cos(\omega t) \\ \implies A &= \frac{1}{2\omega} \end{aligned}$$

so that our solution to the differential equation is

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{t}{2\omega} \cos(\omega t).$$

Notice that, as $t \to \infty$, $x(t) \to \frac{t}{2\omega}\cos(\omega t)$ which oscillates without bound with period $\frac{2\pi}{\omega}$.

7. Derive Taylor's formula with remainder

$$x(t) = \sum_{j=0}^{n} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{n!}\int_{t_0}^{t} x^{(n+1)}(s)(t-s)^n ds$$

for $x \in C^{n+1}$ from (3.57).

Proof: We will proceed by induction. For the base case $n = 1$, notice by the fundamental theorem of Calculus, we have

$$\begin{aligned}
x(t) &= x(t_0) + \int_{t_0}^{t} x'(s)ds \\
&= x(t_0) - \int_{t_0}^{t} x'(s)\frac{d}{ds}(t-s)ds \\
&= x(t_0) - [(t-s)x'(s)]|_{t_0}^{t} + \int_{t_0}^{t} x''(s)(t-s)ds \\
&= x(t_0) + x'(t_0)(t-t_0) + \int_{t_0}^{t} x''(s)(t-s)ds.
\end{aligned}$$

Assume that this relationship holds for $n = 1, 2, \ldots, k$. We must show this relationship holds for $k+1$. From the induction hypothesis, we have

$$\begin{aligned}
x(t) &= \sum_{j=0}^{k} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{k!}\int_{t_0}^{t} x^{(k+1)}(s)(t-s)^k ds \\
&= \sum_{j=0}^{k} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j - \frac{1}{k!}\int_{t_0}^{t} x^{(k+1)}(s)\frac{d}{ds}(t-s)^{k+1} ds \\
&= \sum_{j=0}^{k} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j - \frac{1}{(k+1)!}\left[x^{(k+1)}(s)(t-s)^{k+1}\right]\Big|_{t_0}^{t} + \frac{1}{(k+1)!}\int_{t_0}^{t} x^{(k+2)}(s)(t-s)^{k+1} ds \\
&= \sum_{j=0}^{k} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{(k+1)!}x^{(k+1)}(t_0)(t-t_0)^{k+1} + \frac{1}{(k+1)!}\int_{t_0}^{t} x^{(k+2)}(s)(t-s)^{k+1} ds \\
&= \sum_{j=0}^{k+1} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{(k+1)!}\int_{t_0}^{t} x^{(k+2)}(s)(t-s)^{k+1} ds
\end{aligned}$$

concluding the proof.