Modern Algebra HW7

Michael Nameika

October 2022

Section 15 Problems

For the following, (1) compute the order of each element in the factor group and (2) determine whether or not the factor group is cyclic.

2. $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$
(1) Let us begin by computing $\langle(0,2)\rangle$:

$$\langle(0,2)\rangle = \{(0,0),(0,2)\}$$

So $\langle(0,2)\rangle$ has 2 elements, and since $\mathbb{Z}_2 \times \mathbb{Z}_4$ has 8 elements, $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$ has 4 elements. Now let's find those elements:

$$\begin{aligned}
(0,0) + \langle(0,2)\rangle &= \{(0,0),(0,2)\} \\
(1,0) + \langle(0,2)\rangle &= \{(1,0),(1,2)\} \\
(1,1) + \langle(0,2)\rangle &= \{(1,1),(1,3)\} \\
(0,1) + \langle(0,2)\rangle &= \{(0,1),(0,3)\}
\end{aligned}$$

And now let us calculate the order of each element:

$$\begin{aligned}
|(0,0) + \langle(0,2)\rangle| &= 1 \\
|(1,0) + \langle(0,2)\rangle| &= 2 \\
|(1,1) + \langle(0,2)\rangle| &= 2 \\
|(0,1) + \langle(0,2)\rangle| &= 2
\end{aligned}$$

(2) Since $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)\rangle$ has four elements and each element has order less than four, this factor group cannot be cyclic!

3. $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$
(1) Let us begin by computing $\langle(1,2)\rangle$:

$$\langle(1,2)\rangle = \{(0,0),(1,2)\}$$

Similar to problem 2, we have that $\langle(1,2)\rangle$ will have four elements. Let's list them:

$$\begin{aligned}
(0,0) + \langle(1,2)\rangle &= \{(0,0),(1,2)\} \\
(0,1) + \langle(1,2)\rangle &= \{(0,1),(1,3)\} \\
(1,0) + \langle(1,2)\rangle &= \{(1,0),(0,2)\} \\
(1,1) + \langle(1,2)\rangle &= \{(1,1),(0,3)\}
\end{aligned}$$

And now let us calculate the order of each element:

$$|(0,0)+\langle(1,2)\rangle| = 1$$
$$|(0,1)+\langle(1,2)\rangle| = 4$$
$$|(1,0)+\langle(1,2)\rangle| = 2$$
$$|(1,1)+\langle(1,2)\rangle| = 4$$

(2) Since $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$ has four elements and two elements have order 4, we have that $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$ is a cyclic group.

Bonus Problem!!!

Prove if G is a cyclic group and H is any subgroup of G then G/H is cyclic.

Proof: Let G be a cyclic group and $H \leq G$. We wish to show that G/H is cyclic. Well, by definition, the elements of G/H are cosets of H. Additionally, since G is cyclic, there exists some $a \in G$ such that $\langle a\rangle = G$. Now, I claim that aH generates G/H. Well, repeatedly composing aH with itself n times gives a^nH, and since a generates G, we will eventually find every coset of H. So aH generates G/H, so G/H is cyclic.