Modern Algebra HW4

Michael Nameika

October 2022

Section 10 Problems

6. Find all left cosets of the subgroup $\{\rho_0, \mu_2\}$ of the group D_4 given by Table 8.12.

Let us begin by finding $\rho_0\{\rho_0, \mu_2\}$, $\rho_1\{\rho_0, \mu_2\}$, $\rho_2\{\rho_0, \mu_2\}$, and $\rho_3\{\rho_0, \mu_2\}$:

$$\begin{aligned}
\rho_0\{\rho_0, \mu_2\} &= \{\rho_0\rho_0, \rho_0\mu_2\} = \{\rho_0, \mu_2\} \\
\rho_1\{\rho_0, \mu_2\} &= \{\rho_1\rho_0, \rho_1\mu_2\} = \{\rho_1, \delta_2\} \\
\rho_2\{\rho_0, \mu_2\} &= \{\rho_2\rho_0, \rho_2\mu_2\} = \{\rho_2, \mu_1\} \\
\rho_3\{\rho_0, \mu_2\} &= \{\rho_3\rho_0, \rho_3\mu_2\} = \{\rho_3, \delta_1\}
\end{aligned}$$

Recall that cosets of a subgroup H of G partition G, so we may observe that we have found all partitions of G under the cosets of H. That is, the left cosets of $\{\rho_0, \mu_2\}$ of D_4 are the following:

$$\{\rho_0, \mu_2\} \quad \{\rho_1, \delta_2\} \quad \{\rho_2, \mu_1\} \quad \{\rho_3, \delta_1\}$$

To verify this, let's find $\mu_1\{\rho_0, \mu_2\}$, $\mu_2\{\rho_0, \mu_2\}$, $\delta_1\{\rho_0, \mu_2\}$, and $\delta_2\{\rho_0, \mu_2\}$:

$$\begin{aligned}
\mu_1\{\rho_0, \mu_2\} &= \{\mu_1\rho_0, \mu_1\mu_2\} = \{\mu_1, \rho_2\} \\
\mu_2\{\rho_0, \mu_2\} &= \{\mu_2\rho_0, \mu_2\mu_2\} = \{\mu_2, \rho_0\} \\
\delta_1\{\rho_0, \mu_2\} &= \{\delta_1\rho_0, \delta_1\mu_2\} = \{\delta_1, \rho_3\} \\
\delta_2\{\rho_0, \mu_2\} &= \{\delta_2\rho_0, \delta_2\mu_2\} = \{\delta_2, \rho_1\}
\end{aligned}$$

Which we notice to be the same as the cosets created by ρ_0, ρ_1, ρ_2, and ρ_3.

7. Repeat the preceding exercise, but find the right cosets this time. Are they the same as the left cosets?

As in problem 6, let us begin by finding the right cosets of $\{\rho_0, \mu_2\}$ for each ρ:

$$\begin{aligned}
\{\rho_0, \mu_2\}\rho_0 &= \{\rho_0\rho_0, \mu_2\rho_0\} = \{\rho_0, \mu_2\} \\
\{\rho_0, \mu_2\}\rho_1 &= \{\rho_0\rho_1, \mu_2\rho_1\} = \{\rho_1, \delta_1\} \\
\{\rho_0, \mu_2\}\rho_2 &= \{\rho_0\rho_2, \mu_2\rho_2\} = \{\rho_2, \mu_1\} \\
\{\rho_0, \mu_2\}\rho_3 &= \{\rho_0\rho_3, \mu_2\rho_3\} = \{\rho_3, \delta_2\}
\end{aligned}$$

So the right cosets of $\{\rho_0, \mu_2\}$ in D_4 are

$$\{\rho_0, \mu_2\} \quad \{\rho_1, \delta_1\} \quad \{\rho_2, \mu_1\} \quad \{\rho_3, \delta_2\}$$

These are NOT the same cosets as in problem 6. For example, the right coset $\{\rho_1, \delta_1\}$ is not a left coset of $\{\rho_0, \mu_2\}$ in D_4.

9. Repeat Exercise 6 for the subgroup $\{\rho_0, \rho_2\}$ of D_4.

Let us begin by finding $\mu_1\{\rho_0, \rho_2\}$, $\rho_1\{\rho_0, \rho_2\}$, $\delta_1\{\rho_0, \rho_2\}$, and $\rho_0\{\rho_0, \rho_2\}$:

$$\begin{aligned}
\mu_1\{\rho_0, \rho_2\} &= \{\mu_1\rho_0, \mu_1\rho_2\} = \{\mu_1, \mu_2\} \\
\rho_1\{\rho_0, \rho_2\} &= \{\rho_1\rho_0, \rho_1\rho_2\} = \{\rho_1, \rho_3\} \\
\delta_1\{\rho_0, \rho_2\} &= \{\delta_1\rho_0, \delta_1\rho_2\} = \{\delta_1, \delta_2\} \\
\rho_0\{\rho_0, \rho_2\} &= \{\rho_0\rho_0, \rho_0\rho_2\} = \{\rho_0, \rho_2\}
\end{aligned}$$

So the left cosets of $\{\rho_0, \rho_2\}$ in D_4 are

$$\{\rho_0, \rho_2\} \quad \{\mu_1, \mu_2\} \quad \{\rho_1, \rho_3\} \quad \{\delta_1, \delta_2\}$$

10. Repeat the preceding exercise, but find the right cosets this time. Are they the same as the left coests?

Repeating problem 9 to find the right cosets, we find

$$\begin{aligned}
\{\rho_0, \rho_2\}\mu_1 &= \{\rho_0\mu_1, \rho_2\mu_1\} = \{\mu_1, \mu_2\} \\
\{\rho_0, \rho_2\}\rho_1 &= \{\rho_0\rho_1, \rho_2\rho_1\} = \{\rho_1, \rho_3\} \\
\{\rho_0, \rho_2\}\delta_1 &= \{\rho_0\delta_1, \rho_2\delta_1\} = \{\delta_1, \delta_2\} \\
\{\rho_0, \rho_2\}\rho_0 &= \{\rho_0\rho_0, \rho_2\rho_0\} = \{\rho_0, \rho_2\}
\end{aligned}$$

So the right cosets of $\{\rho_0, \rho_2\}$ in D_4 are

$$\{\rho_0, \rho_2\} \quad \{\mu_1, \mu_2\} \quad \{\rho_1, \rho_3\} \quad \{\delta_1, \delta_2\}$$

which are the same as the left cosets we fond in exercise 9.

28. Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset of gH is the same as the right coset Hg.

Proof: Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$. We will proceed by double inclusion. That is, we will show that $gH \subseteq Hg$ and $Hg \subseteq gH$. Begin by considering $gh \in gH$. Notice that we may rewrite gh as $ghg^{-1}g$ and further as $((g^{-1})^{-1}hg^{-1})g$. Then let $a = g^{-1} \in G$. We find

$$(g^{-1})^{-1}hg^{-1} = a^{-1}ha$$

substituting, we have

$$((g^{-1})^{-1}hg^{-1})g = (a^{-1}ha)g$$

and since $a \in G$, $a^{-1}ha \in H$ by hypothesis, and by definition, $(a^{-1}ha)g \in Hg$. That is, $gh \in Hg$, so we have

$$gH \subseteq Hg$$

Now consider $hg \in Hg$. Similar to above, we may rewrite hg as $hg = gg^{-1}hg = g(g^{-1}hg)$. By hypothesis, $g^{-1}hg \in H$, so $g(g^{-1}hg) \in gH$ by definition. Then we have

$$Hg \subseteq gH$$

Finally, by double inclusion, we have $Hg = gH$.

40. Show that if a group G with identity e has finite order n, then $a^n = e$ for all $a \in G$.

Proof: Let G be a group with identity e and suppose $|G| = n$ and let $a \in G$. Consider $\langle a \rangle$ and suppose $|\langle a \rangle| = m$ for some $m \in \mathbb{Z}$. That is, m is the smallest integer such that $a^m = e$. By Lagrange's theorem, we have have that $m \Big| n$ since $\langle a \rangle \leq G$. Then for some $k \in \mathbb{Z}$, we have $n = mk$. Now consider a^n:

$$a^n = a^{mk} = (a^m)^k = e^k = e$$

Which is what we sought to show.

Section 11 Problems

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each of the elements. Is this group cyclic?

The elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$ are as follows:

$$(0,0)\ \ (0,1)\ \ (0,2)\ \ (0,3)$$

$$(1,0)\ \ (1,1)\ \ (1,2)\ \ (1,3)$$

This group is not cyclic. In light of theorem 11.5, we may observe that $\gcd(2,4) = 2$, so $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic. Alternatively, notice that there are eight elements in $\mathbb{Z}_2 \times \mathbb{Z}_4$, and that each element has the following orders:

$$\begin{array}{c|c} |(0,0)| = 1 & |(1,0)| = 2 \\ |(0,1)| = 2 & |(1,1)| = 4 \\ |(0,2)| = 2 & |(1,2)| = 2 \\ |(0,3)| = 4 & |(1,3)| = 4 \end{array}$$

Since there are eight elements and the highest order of an element of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is four, we cannot generate the group from a single element, so $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic.

2. Repeat Exercise 1 for the group $\mathbb{Z}_3 \times \mathbb{Z}_4$

The elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$ are as follows:

$$(0,0)\ \ (0,1)\ \ (0,2)\ \ (0,3)$$

$$(1,0)\ \ (1,1)\ \ (1,2)\ \ (1,3)$$

$$(2,0)\ \ (2,1)\ \ (2,2)\ \ (2,3)$$

By theorem 11.5, since $\gcd(3,4) = 1$, we have that $\mathbb{Z}_3 \times \mathbb{Z}_4$ is cyclic. Alternatively, notice that the elements have the following orders:

$$\begin{array}{c|c|c|c} |(0,0)| = 1 & |(0,1)| = 4 & |(0,2)| = 2 & |(0,3)| = 4 \\ |(1,0)| = 2 & |(1,1)| = 12 & |(1,2)| = 6 & |(1,3)| = 12 \\ |(2,0)| = 3 & |(2,1)| = 12 & |(2,2)| = 6 & |(2,3)| = 12 \end{array}$$

Notice four elements have order 12, and since $\mathbb{Z}_3 \times \mathbb{Z}_4$ has 12 elements, we can see that $\mathbb{Z}_3 \times \mathbb{Z}_4$ is cyclic.

14. Fill in the blanks.

a. The cyclic subgroup of $\mathbb{Z}_{24}$ generated by 18 has order 4.
To begin, we must find the greatest common divisor of 24 and 18. Well, $\gcd(24,18) = 6$, so the cyclic subgroup of $\mathbb{Z}_{24}$ generated by 18 has order $24/6 = 4$.

b. $\mathbb{Z}_3 \times \mathbb{Z}_4$ is of order 12. See problem 2.

c. The element $(4, 2)$ of $\mathbb{Z}_{12} \times \mathbb{Z}_8$ has order 12.
We must find the least common multiple of the greatest common divisors of 4 and 12, and 2 and 8, respectively. Begin with 4 and 12:

$$\gcd(4, 12) = 4$$

So 4 has order $12/4 = 3$ for $\mathbb{Z}_{12}$. Now for 2 and 8:

$$\gcd(2, 8) = 2$$

So 2 has order $8/2 = 4$ for $\mathbb{Z}_8$. Now we must find the least common multiple between 3 and 4:

$$\operatorname{lcm}(3, 4) = \frac{3 \cdot 4}{\gcd(3, 4)} = \frac{12}{1} = 12$$

d. The Klein 4-group is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.