Homework X

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Section 4.12 Problems

6. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator Show that $T^{-1} : \mathfrak{R}(T) \to X$ is bounded if and only if $\mathfrak{R}$ is closed in Y.

   Proof: To begin, since T is injective, we have that $T^{-1} : \mathfrak{R}(T) \to X$ exists. Now, recall that $\mathfrak{R}(T)$ is a vector space since T is a linear operator. First suppose that $\mathfrak{R}(T)$ is closed. Then since $\mathfrak{R}(T)$ is a subspace of Y and is closed, $\mathfrak{R}(T)$ is a Banach space. Then since T is surjective onto $\mathfrak{R}(T)$, by the bounded inverse theorem, T^{-1} is bounded.

   Now suppose that T^{-1} is bounded. We wish to show that $\mathfrak{R}(T)$ is closed. Let $\{y_n\}$ be a Cauchy sequence in $\mathfrak{R}(T)$. That is, for any $\varepsilon > 0$, there exists an index N such that whenever $n > m > N$, we have
$$\|y_n - y_m\| < \frac{\varepsilon}{\|T^{-1}\|}$$
   but since each $y_n \in \mathfrak{R}(T)$, there exists an associated $x_n \in X$ such that $x_n = T^{-1}y_n$. Now notice for $n > m > N$,
$$\begin{aligned}
\|x_n - x_m\| &= \|T^{-1}y_n - T^{-1}y_m\| \\
&= \|T^{-1}(y_n - y_m)\| \\
&\leq \|T^{-1}\|\|y_n - y_m\| \\
&< \|T^{-1}\|\frac{\varepsilon}{\|T^{-1}\|} \\
&= \varepsilon \\
\implies \|x_n - x_m\| &< \varepsilon
\end{aligned}$$
   so that $\{x_n\}$ is Cauchy in X. Since X is a Banach space, $x_n \to x$ for some $x \in X$. But then since $\mathfrak{D}(T) = X$, $Tx = y$ for some $y \in \mathfrak{R}(T)$. Now, since T is bounded, T is continuous, so that $Tx_n \to Tx \implies y_n \to y$. Since $\{y_n\}$ was an arbitrary Cauchy sequence, we have that $\mathfrak{R}(T)$ is closed.

8. **(Equivalent Norms)** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X such that $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ are complete. If $\|x_n\|_1 \to 0$ always implies $\|x_n\|_2 \to 0$, show that convergence in X_1 implies convergence in X_2 and conversely, and there are positive numbers a and b such that for all $x \in X$,
$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1.$$
   Proof: First suppose that $\{x_n\}$ is a sequence that converges to some $x \in X_1$. Then by assumption,
$$\begin{aligned}
\|x_n - x\|_1 \to 0 &\quad \text{as} \quad n \to \infty \\
\implies \|x_n - x\|_2 \to 0 &\quad \text{as} \quad n \to \infty.
\end{aligned}$$
   Thus convergence in X_1 implies convergence in X_2. Now suppose $\{x_n\}$ is a sequence converging to $x \in X_2$. Define the linear operator $T : X_1 \to X_2$ by
$$Tx = x$$
   that is, we are sending $x \in X_1$ to its associated element in X_2. Notice that T is bounded since
$$\|Tx\| = \|x\|.$$

Hence $\|T\| = 1$. Notice that T is surjective by definition. Since X_1 and X_2 are complete spaces, by the bounded inverse theorem, we have that T^{-1} is bounded. Now notice $\{x_n\}$ and x as elements of X_1, we have $x_n = T^{-1}x_n$, $x = T^{-1}x$ and so

$$\begin{aligned}\|x_n - x\|_1 &= \|T^{-1}x_n - T^{-1}x\|_2 \\ &\leq \|T^{-1}\|\|x_n - x\|_2 \to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

so that convergence in X_2 implies convergence in X_1. Notice from above, we have

$$\frac{1}{\|T^{-1}\|}\|x\|_1 \leq \|x\|_2 \leq \|T\|\|x\|_1.$$

Section 4.13 Problems

8. Let X and Y be normed spaces and let $T : X \to Y$ be a closed linear operator. (a) Show that the image A of a compact subset $C \subset X$ is closed in Y. (b) Show that the inverse image B of a compact subset $K \subset Y$ is closed in X.

   Proof: (a) Let $\{a_n\}$ be a sequence in A that converges to some $a \in Y$. We wish to show that $a \in A$. Since $a_n \in A$ and $A = T(C)$, there exists, for each n, $c_n \in C$ such that $Tc_n = a_n$. And since C is compact, it is sequentially compact, so $\{c_n\}$ admits a convergent subsequence, $c_{n_k} \to c \in C$. Now, since T is closed, and $c_{n_k} \to c$, $Tc_{n_k} = a_{n_k} \to a$, we have that $a = Tc$, so that $a \in A$. Hence A is closed.

   (b) Let $\{x_n\}$ be a sequence in B such that $x_n \to x \in X$. We wish to show that $x \in B$. By definition of preimage, there exists, for each n, $k_n \in K$ such that $T^{-1}k_n = x_n$. Thus, since K is compact, K is sequentially compact, so $\{k_n\}$ admits a convergent subsequence, say $k_{n_\ell} \to k \in K$. But since $k \in K$, there exists some $z \in B$ such that $z = T^{-1}k$. But this says that $(z, k) \in \mathcal{G}(T)$ and so, since T is closed, $z = x$, so that B is closed.

Assigned Exercise X.1

Let X and Y be normed spaces and let $T : X \to Y$ be a closed linear operator. Suppose that for every convergent sequence (x_n) in X, the sequence $(y_n = Tx_n)$ admits a convergent subsequence (y_{n_k}). Prove that T is bounded.

Proof: Let $M \subseteq Y$ be a closed subset of Y, and let $A = T^{-1}(M)$, the preimage of M under T. Let x be a limit point of M. Then there exists a sequence $\{x_n\}$ in A such that $x \to x \in X$. We must show $x \in A$. But by hypothesis, we have that $\{y_n = Tx_n\}$ admits a convergent subsequence, $\{y_{n_k}\}$ converging to some y. Since M is closed, we have that $y \in M$. And since T is a closed linear operator, we have that $x_{n_k} \to x' \in A$ where $Tx' = y$. But since $x_n \to x$ and $x_{n_k} \to x'$, we have $x' = x$ by uniqueness. Thus, $x \in A$ as desired. Hence, A is closed in X, and by problem 1.3 #14, we have that T is continuous and is therefore bounded.