CFD Homework 2

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9/12/24

1. a) Classify the given PDE below.

$$2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$$

Soln. This is a PDE of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

with $A = 2 = C$, $B = -2$, $D = E = 0$, and $F = 3$. To classify the PDE, we must inspect $B^2 - AC$. Well, $B^2 - AC = (-2)^2 - (2)(2) = 4 - 4 = 0$. Thus, by definition, the PDE is **parabolic**.

b) Convert the PDE into a system of first order equations while keeping it the same. Write this system in matrix form.

Soln. Let $v = u_x$ and $w = u_y$. Then the PDE in part a) becomes

$$2v_x - 4v_y + 2w_y + 3u = 0. \tag{1}$$

Assuming a sufficiently smooth solution u, we also have

$$v_y = w_x \tag{2}$$

and after dividing (1) by the scalar 2, we have the system

$$\begin{aligned} v_x - 2v_y + w_y + \frac{3}{2}u &= 0 \\ w_x - v_y &= 0. \end{aligned} \tag{3}$$

Writing (3) as a matrix system, we have

$$\frac{\partial}{\partial x}\begin{pmatrix} v \\ w \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \frac{3}{2}u \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

c) Classify the system of equations found in part b.

Soln. To classify the system in part b), we must inspect the eigenvalues of $A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$. That is,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \\ &= \lambda(\lambda + 2) + 1 \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda + 1)^2 = 0 \\ \implies \lambda &= -1 \end{aligned}$$

with algebraic multiplicity 2. Thus, since we have a repeated eigenvalue, the system is **parabolic**.

2. Derive the fourth-order-accurate central finite-difference for $\frac{\partial^2 u}{\partial x^2}$:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12(\Delta x)^2} + \mathcal{O}(\Delta x^4)$$

This is actually the same scheme in Anderson (4.18). In the derivation of the above equation, we have assumed the mesh is uniform, that is, Δx is constant.

Soln. We approach this by method of undetermined coefficients. Let $x_i = x_0$ and $x_{i\pm1} = x_0 \pm \Delta x$. We seek a scheme of the form

$$u_{xx}(x_0) \approx au(x_0 - 2\Delta x) + bu(x_0 - \Delta x) + cu(x_0) + du(x_0) + \Delta x) + eu(x_0 + 2\Delta x).$$

Taylor expanding each of the terms above, we find

$$\begin{aligned}
u(x_0 + 2\Delta x) &= u(x_0) + 2\Delta x u_x(x_0) + 2(\Delta x)^2 u_{xx}(x_0) + \frac{8(\Delta x)^3}{3!} u_{xxx}(x_0) + \frac{16(\Delta x)^4}{4!} u_{xxxx}(x_0) + \mathcal{O}((\Delta x)^5) \\
u(x_0 + \Delta x) &= u(x_0) + \Delta x u_x(x_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0) + \frac{(\Delta x)^3}{3!} u_{xxx}(x_0) + \frac{(\Delta x)^4}{4!} u_{xxxx}(x_0) + \mathcal{O}((\Delta x)^5) \\
u(x_0 - \Delta x) &= u(x_0) - \Delta x u_x(x_0) + \frac{(\Delta x)^2}{2} u_{xx}(x_0) - \frac{(\Delta x)^3}{3!} u_{xxx}(x_0) + \frac{(\Delta x)^4}{4!} u_{xxxx}(x_0) + \mathcal{O}((\Delta x)^5) \\
u(x_0 - 2\Delta x) &= u(x_0) - 2\Delta x u_x(x_0) + 2(\Delta x)^2 u_{xx}(x_0) - \frac{8(\Delta x)^3}{3!} u_{xxx}(x_0) + \frac{16(\Delta x)^4}{4!} u_{xxxx}(x_0) + \mathcal{O}((\Delta x)^5))
\end{aligned}$$

By matching coefficients, we find the following linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{(\Delta x)^2} \\ 0 \\ 0 \end{pmatrix}$$

To solve the problem, we first perform the following row operations: $R_2 + 2R_1 \to R_2$, $R_2 - 2R_1 \to R_3$,

$R_4 + \frac{4}{3}R_1 \to R_4$, and $R_5 - \frac{2}{3}R_1 \to R_5$. Then our system becomes (in augmented form)

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & -\frac{3}{2} & -2 & -\frac{3}{2} & 0 & \frac{1}{(\Delta x)^2} \\ 0 & \frac{7}{6} & \frac{4}{3} & \frac{3}{2} & \frac{8}{3} & 0 \\ 0 & -\frac{5}{8} & -\frac{2}{3} & -\frac{5}{8} & 0 & 0 \end{array}\right) \begin{array}{l} R_3 - \frac{3}{2}R_2 \to R_3 \\ \\ R_4 - \frac{7}{6}R_2 \to R_4 \\ \\ R_5 + \frac{5}{8}R_2 \to R_5 \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & \frac{1}{(\Delta x)^2} \\ 0 & 0 & -1 & -2 & -2 & 0 \\ 0 & 0 & \frac{7}{12} & \frac{5}{4} & \frac{5}{2} & 0 \end{array}\right) \begin{array}{l} R_4 + R_3 \to R_4 \\ \\ R_5 - \frac{7}{12}R_4 \to R_5 \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & \frac{1}{(\Delta x)^2} \\ 0 & 0 & 0 & 1 & 4 & \frac{1}{(\Delta x)^2} \\ 0 & 0 & 0 & -\frac{1}{2} & -1 & -\frac{7}{12(\Delta x)^2} \end{array}\right) \; R_5 + \frac{1}{2}R_4 \to R_5$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & \frac{1}{(\Delta x)^2} \\ 0 & 0 & 0 & 1 & 4 & \frac{1}{(\Delta x)^2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{12(\Delta x)^2} \end{array}\right).$$

Back substitution gives us $e = -\frac{1}{12(\Delta x)^2}$, $d = \frac{16}{12(\Delta x)^2}$, $c = -\frac{30}{12(\Delta x)^2}$, $b = \frac{16}{12(\Delta x)^2}$, $a = -\frac{1}{12(\Delta x)^2}$. Further, by symmetry of the coefficients and Taylor expansions, we find that the fifth order terms cancel (and the sixth order terms do not), so we are left with

$$\begin{aligned} \left(\frac{\partial u}{\partial x}\right)_i &= \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12(\Delta x)^2} + \frac{1}{12(\Delta x)^2}\mathcal{O}((\Delta x)^6) \\ &= \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12(\Delta x)^2} + \mathcal{O}((\Delta x)^4) \end{aligned}$$

as desired.

3. Anderson Problems 4.6. Derive the third-order-accurate one-sided difference for $\frac{\partial u}{\partial y}$

$$\left(\frac{\partial u}{\partial y}\right)_{i,j} = \frac{1}{6\Delta y}(-11u_{i,j} + 18u_{i,j+1} - 9u_{i,j+2} + 2u_{i,j+3})$$

Soln. We approach this by method of undetermined coefficients. Letting $x_i = x_0$, $y_i = y_0$, and $y_{i\pm1} = y_0 \pm \Delta y$, we seek a scheme of the form

$$\left(\frac{\partial u}{\partial y}\right)_i \approx au(x_0, y_0) + bu(x_0, y_0 + \Delta y) + cu(x_0, y + 2\Delta y) + du(x_0, y_0 + 3\Delta y).$$

Taylor expanding the above terms, we find

$$\begin{aligned}
u(x_0, y_0 + \Delta y) &= u(x_0, y_0) + \Delta y u_y(x_0, y_0) + \frac{(\Delta y)^2}{2} u_{yy}(x_0, x_0) + \frac{(\Delta y)^3}{3!} u_{yyy}(x_0, y_0) + \mathcal{O}((\Delta y)^4) \\
u(x_0, y_0 + 2\Delta y) &= u(x_0, y_0) + 2\Delta y u_y(x_0, y_0) + 2(\Delta y)^2 u_{yy}(x_0, y_0) + \frac{8(\Delta y)^3}{3!} u_{yyy}(x_0, y_0) + \mathcal{O}((\Delta y)^4) \\
u(x_0, y_0 + 3\Delta y) &= u(x_0, y_0) + 3\Delta y u_y(x_0, y_0) + \frac{9(\Delta y)^2}{2} u_{yy}(x_0, y_0) + \frac{27(\Delta y)^3}{3!} u_{yyy}(x_0, y_0) + \mathcal{O}((\Delta y)^4).
\end{aligned}$$

Matching coefficients, we find the following linear system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & \frac{1}{2} & 2 & \frac{9}{2} \\ 0 & \frac{1}{6} & \frac{4}{3} & \frac{9}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\Delta y} \\ 0 \\ 0 \end{pmatrix}$$

Performing the following row operations, $R_3 - \frac{1}{2}R_2 \to R_3$ and $R_4 - \frac{1}{6}R_2 \to R_4$, the system becomes (in augmented form)

$$\begin{aligned}
&\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & \frac{1}{\Delta y} \\ 0 & \frac{1}{2} & 2 & \frac{9}{3} & 0 \\ 0 & \frac{1}{6} & \frac{4}{3} & \frac{9}{2} & 0 \end{array}\right) \quad \begin{matrix} R_3 - \frac{1}{2}R_2 \to R_3 \\ \\ R_4 - \frac{1}{6}R_2 \to R_4 \end{matrix} \\
\sim &\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & \frac{1}{\Delta y} \\ 0 & 0 & 1 & 3 & -\frac{1}{2\Delta y} \\ 0 & 0 & 0 & 1 & \frac{1}{3\Delta y} \end{array}\right).
\end{aligned}$$

Back substitution gives $d = \frac{1}{3\Delta y}$, $c = -\frac{3}{2\Delta y}$, $b = \frac{3}{\Delta y}$, $a = -\frac{11}{6\Delta y}$ so that our scheme is

$$\begin{aligned}
\left(\frac{\partial u}{\partial y}\right)_{i,j} &= \frac{1}{6\Delta y}(-11u_{i.j} + 18u_{i,j+1} - 9u_{i,j+2} + 24u_{i,j+3}) + \frac{1}{6\Delta y}\mathcal{O}((\Delta y)^4) \\
&= \frac{1}{6\Delta y}(-11u_{i,j} + 18u_{i,j+1} - 9u_{i,j+2} + 24u_{i,j+3}) + \mathcal{O}((\Delta y)^3)
\end{aligned}$$

as desired.

4. Consider first order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

(1) The finite difference equation with forward difference in time, central difference in space can be written as:

$$\frac{u_i^{t+\Delta t} - u_i^t}{\Delta t} = -c\frac{u_{i+1}^t - u_{i-1}^t}{2\Delta x}$$

Using von Neumann stability analysis, find out the amplification factor, and verify the above scheme is unconditional unstable.

Soln. Set $\varepsilon_i^n = e^{at+ikx}$. We have that ε_i^n satisfies the scheme above so that

$$\frac{e^{a(t+\Delta t)+ikx} - e^{at+ikx}}{\Delta t} = -c\frac{e^{at+ik(x+\Delta x)} - e^{at+ik(x-\Delta x)}}{2\Delta x}.$$

Dividing through by e^{at+ikx}, the above equation becomes

$$\begin{aligned} \frac{e^{a\Delta t} - 1}{\Delta t} &= -c\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \\ \implies e^{a\Delta t} &= 1 - c\frac{\Delta t}{2\Delta x} i \sin(k\Delta x) \\ \implies \left|e^{a\Delta t}\right|^2 &= \left|1 - c\frac{\Delta t}{2\Delta x} i \sin(k\Delta x)\right|^2 \\ &= 1 + \left(c\frac{\Delta t}{2\Delta x}\right)^2 \sin^2(k\Delta x). \end{aligned}$$

So we have the amplification factor $1 + \left(c\frac{\Delta t}{2\Delta x}\right)^2 \sin^2(k\Delta x)$, which we require to be greater than or equal to 1. That is, we require

$$1 + \left(c\frac{\Delta t}{2\Delta x}\right)^2 \sin^2(k\Delta x) \geq 1$$

and since $\sin^2(k\Delta x) \geq 0$, we have that the scheme is unconditionally unstable, as desired.

(2) Using Lax method, the finite difference equation can be written as:

$$u_i^{n+1} = \frac{u_{i+1}^n + u_{i-1}^n}{2} - c\frac{\Delta t}{\Delta x}\frac{u_{i+1}^n - u_{i-1}^n}{2}$$

Using von Neumann stability analysis, find out the amplification factor, and find the requirement for the above scheme to be stable.

Soln. Following the process as in part (1), setting $\varepsilon_i^n = e^{at+ikx}$, we have

$$e^{a(t+\Delta t)+ikx} = \frac{e^{at+ik(x+\Delta x)} + e^{at+ik(x-\Delta x)}}{2} - c\frac{\Delta t}{\Delta x}\frac{e^{at+ik(x+\Delta x)} - e^{at+ik(x-\Delta x)}}{2}.$$

Dividing through by e^{at+ikx} gives

$$\begin{aligned}
e^{a\Delta t} &= \frac{e^{ik\Delta x}+e^{-ik\Delta x}}{2} - c\frac{\Delta t}{\Delta x}\,\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2} \\
&= \cos(k\Delta x) - ic\frac{\Delta t}{\Delta x}\sin(k\Delta x) \\
\implies \left|e^{a\Delta t}\right|^2 &= \left|\cos(k\Delta x) - ic\frac{\Delta t}{\Delta x}\sin(k\Delta x)\right|^2 \\
&= \cos^2(k\Delta x) + \left(c\frac{\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) \\
&= 1 - \sin^2(k\Delta x) + \left(c\frac{\Delta t}{\Delta x}\right)^2 \sin^2(k\Delta x) \\
&= 1 + \left[\left(\frac{\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x).
\end{aligned}$$

Since we require $\left|e^{a\Delta t}\right|^2 \leq 1$, we have

$$\begin{aligned}
1 + \left[\left(c\frac{\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x) &\leq 1 \\
\implies \left[\left(c\frac{\Delta t}{\Delta x}\right)^2 - 1\right]\sin^2(k\Delta x) &\leq 0
\end{aligned}$$

and since $\sin^2(k\Delta x) \geq 0$, we require

$$\begin{aligned}
\left(c\frac{\Delta t}{\Delta x}\right)^2 - 1 &\leq 0 \\
\implies \left(c\frac{\Delta t}{\Delta x}\right)^2 &\leq 1 \\
\implies \left|c\frac{\Delta t}{\Delta x}\right| &\leq 1.
\end{aligned}$$

Thus, the scheme is stable whenever $\left|c\frac{\Delta t}{\Delta x}\right| \leq 1$ and has amplification factor $\cos(k\Delta x) - ic\frac{\Delta t}{\Delta x}\sin(k\Delta x)$.

5. Consider first order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

The forward difference in time and central difference in space can be written as:

$$\frac{u_i^{t+\Delta t} - u_i^n}{\Delta t} = -c\frac{u_{i+1}^t - u_{i-1}^t}{2\Delta x}$$

Derive the modified equation for the scheme above:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{c\Delta x}{2}\nu\frac{\partial^2 u}{\partial x^2} - \frac{c(\Delta x)^2}{6}(1+2\nu^2)\frac{\partial^3 u}{\partial x^3} + \cdots$$

and verify the scheme is unconditionally unstable.

Soln. Let u satisfy the finite difference equation. Taylor expanding, we have

$$\begin{aligned}
u_i^{n+1} &= u_i^n + \Delta t\left(\frac{\partial u}{\partial t}\right)_i^t + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_i^n + \mathcal{O}((\Delta t)^4)\\
u_{i+1}^n &= u_i^n + \Delta x\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \mathcal{O}((\Delta x)^4)\\
u_{i-1}^n &= u_i^n - \Delta x\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n - \frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i^n + \mathcal{O}((\Delta x)^4)\\
\implies \frac{u_i^{n+1} - u_i^n}{\Delta t} &= \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + \mathcal{O}((\Delta t)^3)\\
\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} &= \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}((\Delta x)^4)
\end{aligned}$$

So the finite difference scheme becomes

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} - c\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3} + \mathcal{O}((\Delta x)^4 + (\Delta t)^2). \tag{4}$$

To derive the modified equation, we wish to express $\frac{\partial^2 u}{\partial t^2}, \frac{\partial^3 u}{\partial t^3}$ in terms of $\frac{\partial^n u}{\partial x^n}$. First differentiate (4) with respect to t:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\partial x} = -\frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3} - \frac{(\Delta t)^2}{6}\frac{\partial^4 u}{\partial t^4} - c\frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial t\partial x^3} + \cdots .$$

We wish to keep up to second order terms in (4), so we truncate after the linear terms, giving

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial^2 u}{\partial t\partial x} = -\frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3} + \cdots . \tag{5}$$

Differentiating (4) with respect to t, we find (to leading order)

$$\frac{\partial^3 u}{\partial t^3} = -c\frac{\partial^3 u}{\partial t^2\partial x} + \cdots . \tag{6}$$

Now, we need to express $\frac{\partial^3 u}{\partial t\partial x^2}, \frac{\partial^3 u}{\partial t^2\partial x}$ in terms of $\frac{\partial^n u}{\partial x^n}$. Assuming a sufficiently smooth solution u, we have

$$\begin{aligned}
\frac{\partial^3 u}{\partial x^2\partial t} &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)\\
\frac{\partial^3 u}{\partial x\partial t^2} &= \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right)
\end{aligned}$$

Notice that we have (to leading order)

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) &= \frac{\partial^2}{\partial x^2}\left(-c\frac{\partial u}{\partial x}+\cdots\right)\\ &= -c\frac{\partial^3 u}{\partial x^3}+\cdots.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial t^2}\right) &= \frac{\partial}{\partial x}\left(-c\frac{\partial^2 u}{\partial x\partial t}+\cdots\right)\\ &= -c\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}+\cdots\right)\\ &= -c\frac{\partial^2}{\partial x^2}\left(-c\frac{\partial u}{\partial x}+\cdots\right)\\ &= c^2\frac{\partial^3 u}{\partial x^3}+\cdots\end{aligned}$$

which gives us

$$\frac{\partial^3 u}{\partial t^3} = -c^3\frac{\partial^3 u}{\partial x^3}+\cdots$$

and

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}+c^3\Delta t\frac{\partial^3 u}{\partial x^3}+\cdots.$$

Putting these equations into (4), we find

$$\begin{aligned}\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x} &= -\frac{\Delta t}{2}\left(c^2\frac{\partial^2 u}{\partial x^2}+c^3\Delta t\frac{\partial^3 u}{\partial x^3}+\cdots\right)-\frac{(\Delta t)^2}{6}\left(-c^3\frac{\partial^3 u}{\partial x^3}+\cdots\right)-c\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}+\cdots\\ &= -c^2\frac{\Delta t}{2}\frac{\partial^2 u}{\partial x^2}-c^3\frac{(\Delta t)^2}{2}\frac{\partial^3 u}{\partial x^3}+c^3\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial x^3}-c\frac{\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}+\cdots\\ &= -c\nu\frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}-c\frac{(\Delta x)^2}{6}(1+2\nu^2)\frac{\partial^3 u}{\partial x^3}+\cdots\end{aligned}$$

where we used $\nu = c\frac{\Delta t}{\Delta x}$, as desired.

Now, to verify that the scheme is unconditionally unstable, we apply von Neumann stability analysis: let $u_i^n = e^{at+ikx}$. Then the scheme becomes

$$\begin{aligned}\frac{e^{a(t+\Delta t)+ikx}-e^{at+ikx}}{\Delta t} &= -c\frac{e^{at+ik(x+\Delta x)}-e^{at+ik(x-\Delta x)}}{2\Delta x}\\ \implies \frac{e^{a\Delta t}-1}{\Delta t} &= -c\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2\Delta x}\\ \implies e^{a\Delta t} &= 1-\frac{c\Delta t}{\Delta x}\left(\frac{e^{ik\Delta x}-e^{-ik\Delta x}}{2}\right)\\ \implies e^{a\Delta t} &= 1-i\frac{c\Delta t}{\Delta x}\sin(k\Delta x).\end{aligned}$$

For stability, we require $|e^{a\Delta t}|^2 \le 1$. From above, we have

$$\left|e^{a\Delta t}\right|^2 = 1+\left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x)$$

and since $\left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x) \ge 0$, we have $|e^{a\Delta t}|^2 \ge 1$, hence the scheme is unconditionally unstable, as desired.

6. Consider first order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

The Lax-Wendroff scheme for this wave equation is given as:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{c^2 \Delta t}{2}\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Derive the associated modified equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{c(\Delta x)^2}{6}(1-\nu^2)\frac{\partial^3 u}{\partial x^3} - \frac{c(\Delta x)^3}{8}\nu(1-\nu^2)\frac{\partial^4 u}{\partial x^4} + \cdots$$

and verify that the scheme is conditionally stable for $\nu \leq 1$.

Soln. We begin by showing the method is conditionally stable for $\nu \leq 1$. Replace $u_i^n = e^{at+ikx}$ so that the scheme becomes

$$\frac{e^{a(t+\Delta t)+ikx} - e^{at+ikx}}{\Delta t} + c\frac{e^{at+ik(x+\Delta x)} - e^{at+ik(x-\Delta x)}}{2\Delta x} = \frac{c^2\Delta t}{2}\frac{e^{at+ik(x+\Delta x)} - 2e^{at+ikx} + e^{at+ik(x-\Delta x)}}{(\Delta x)^2}$$

$$\implies \frac{e^{a\Delta t} - 1}{\Delta t} + c\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} = \frac{c^2\Delta t}{2}\frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{(\Delta x)^2}$$

$$\implies e^{a\Delta t} = 1 - \frac{c\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

$$\implies e^{a\Delta t} = 1 - i\frac{c\Delta t}{\Delta x}\sin(k\Delta x) + \left(\frac{c\Delta t}{\Delta x}\right)^2(\cos(k\Delta x) - 1)$$

$$\implies \left|e^{a\Delta t}\right|^2 = \left(1 + \left(\frac{c\Delta t}{\Delta x}(\cos(k\Delta x) - 1)\right)\right)^2 + \left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x)$$

$$\begin{aligned}
\implies \left|e^{a\Delta t}\right|^2 &= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)(\cos(k\Delta x) - 1) + \left(\frac{c\Delta t}{\Delta x}\right)^2(\cos(k\Delta x) - 1)^2 + \left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x) \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)(\cos(k\Delta x) - 1) + \left(\frac{c\Delta t}{\Delta x}\right)^2(\cos^2(k\Delta x) - 2\cos(k\Delta x) + 1) + \left(\frac{c\Delta t}{\Delta x}\right)^2\sin^2(k\Delta x) \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\cos(k\Delta x) - 2\left(\frac{c\Delta t}{\Delta x}\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2(\cos^2(k\Delta x) + \sin^2(k\Delta x)) + \\
&\quad - 2\left(\frac{c\Delta t}{\Delta x}\right)^2\cos(k\Delta x) + \left(\frac{c\Delta t}{\Delta x}\right)^2 \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\cos(k\Delta x) - 2\left(\frac{c\Delta t}{\Delta x}\right) + \left(\frac{c\Delta t}{\Delta x}\right)^2 - 2\left(\frac{c\Delta t}{\Delta x}\right)^2\cos(k\Delta x) + \left(\frac{c\Delta t}{\Delta x}\right)^2 \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\cos(k\Delta x) - 2\frac{c\Delta t}{\Delta x} + 2\left(\frac{c\Delta t}{\Delta x}\right)^2 - 2\left(\frac{c\Delta t}{\Delta x}\right)^2\cos(k\Delta x) \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\left[\frac{c\Delta t}{\Delta x} - 1 - \frac{c\Delta t}{\Delta x}\cos(k\Delta x) + \cos(k\Delta x)\right] \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\left[\left(\frac{c\Delta t}{\Delta x} - 1\right) - \cos(k\Delta x)\left(\frac{c\Delta t}{\Delta x} - 1\right)\right] \\
&= 1 + 2\left(\frac{c\Delta t}{\Delta x}\right)\left(\frac{c\Delta t}{\Delta x} - 1\right)[1 - \cos(k\Delta x)] \leq 1 \\
\implies 2&\left(\frac{c\Delta t}{\Delta x}\right)\left(\frac{c\Delta t}{\Delta x} - 1\right)[1 - \cos(k\Delta x)] \leq 0
\end{aligned}$$

which is satisfied whenever $0 \leq \frac{c\Delta t}{\Delta x} \leq 1$. Thus the scheme is conditionally stable whenever

$$\nu := \frac{c\Delta t}{\Delta x} \leq 1$$

as desired.