Problem Set 8 (Astrophysics)

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1. Consider the two stars whose properties are described below:

Star	V	$B-V$	M_V	T_{eff} (K)	Spectral class	BC
Betelgeuse	0.45	1.50	−0.60	3370	M2 Ib	−1.62
Gliese 887	7.35	1.48	9.76	3520	M2 V	−1.89

   a. Using the following two expressions for the bolometric magnitude, determine the ratio of the luminosities of Betelgeuse to Gliese 887.

   $M_{bol} = M_V + BC$ $\qquad$ $M_{bol} = 4.74 - 2.5\log(L_{star}/L_{sun})$

   b. Now use the following relationship to determine the ratio of the radii of Betelgeuse to Gliese 887.

   $$\frac{R_1}{R_2} = \left(\frac{L_1}{L_2}\right)^{1/2}\left(\frac{T_{eff-1}}{T_{eff-2}}\right)^{-2}$$

a) Using the first equation for the bolometric magnitude to calculate the bolometric magnitude for Betelgeuse and Gliese 887, we find

$$M_{Bol,B} = -0.60 - 1.62 = -2.22$$

$$M_{Bol,G} = 9.76 - 1.89 = 7.87$$

And now, using the second equation for bolometric magnitude, we can solve for the luminosity of Betelgeuse and Gliese 887. Notice that

$$L_{Star} = L_{Sun}(10^{\frac{4.74 - M_{Bol}}{2.5}})$$

and so

$$L_{Betelgeuse} = L_{Sun}(10^{\frac{4.74+2.22}{2.5}})$$
$$= L_{Sun}(10^{\frac{6.96}{2.5}})$$

and

$$L_{Gliese\ 887} = L_{Sun}(10^{\frac{4.74-7.87}{2.5}})$$
$$= L_{Sun}(10^{\frac{-3.13}{2.5}})$$

And so the ratio between the luminosities of Betelgeuse and Gliese 887 is given by

$$\frac{L_{Betelgeuse}}{L_{Gliese\ 887}} = \frac{L_{Sun}(10^{6.96/2.5})}{L_{Sun}(10^{-3.13/2.5})}$$
$$= 10^{\frac{10.09}{2.5}}$$
$$\approx 10860$$

So Betelgeuse is approximately 10,860 times more luminous than Gliese 887.

b) Using the values given in the table for the temperature of Betelgeuse and Gliese 887, and the values for the ratio of their luminosities, we find that

$$\frac{R_{Betelgeuse}}{R_{Gliese\ 887}} = \left(\frac{L_{Betelgeuse}}{L_{Gliese\ 887}}\right)^{1/2}\left(\frac{3370}{3520}\right)^{-2}$$

$$\approx (10860)^{1/2}\left(\frac{3370}{3520}\right)^{-2}$$

$$\approx 113.7$$

So Betelgeuse is approximately 113.7 times larger than Gliese 887 in radius.

2. Approximately half of the original hydrogen in the Sun's core has now been converted to helium. Remember the gas is ionized in the Sun For the center of the Sun, you can assume that the center originally had a composition which is the same as the current photosphere composition. Calculate the mean molecular mass (μ) at:
   a. The surface of the sun (given standard abundances of X = 0.734, Y = 0.250, and Z = 0.016).
   b. The center of the sun.

a)

$$\mu = (2X + \frac{3Y}{4} + \frac{Z}{2})^{-1}$$

$$= (1.468 + \frac{0.750}{4} + \frac{Z}{2})^{-1}$$

$$\approx 0.601$$

b) Since the composition is assumed to be the same in the center of the sun, we will find the same result, $\mu \approx 0.601$.

3. Star lifetimes: (If you are looking at Ryden's book, do NOT use equation 15.55 which is based on different assumptions that are not valid in this problem.)
   a. If a star has M = 100 M_{sun} and L = 10^6 L_{sun}, how long can it generate that luminosity if it started as pure hydrogen and is able to convert ALL of its H into He?
   b. If a star has M = 0.5 M_{sun} and L = 0.1 $L_{su,}$ how long can it generate that luminosity under the same conditions?

Recall that the lifetime of the sun is estimated to be approximately 10 billion years and that the lifetime can be approximated using the following formula:

$$\tau = \frac{E}{L}$$

where E is the available energy of the star and L is the luminosity. Also recall that $E \propto M$.

a) Given that $M = 100\, M_{Sun}$ and $L = 10^6\, L_{Sun}$, we have that

$$\tau = \frac{100E_{Sun}}{10^6 L_{Sun}}$$

$$= 10^{-4}(10 \times 10^9 \text{ years})$$

$$= 10^6 \text{ years}$$

b) Given that $M = 0.5\, M_{Sun}$ and $L = 0.1\, L_{Sun}$, we have that

$$\tau = \frac{0.5\, E_{Sun}}{0.1\, L_{Sun}}$$

$$= 5(10 \times 10^9\ years)$$

$$= 50 \text{ billion years}$$

4. Suppose the mass density of a star is $\rho(r) = \rho_0 \left[1 - \left(\frac{r}{R}\right)^2\right]$ where R is the radius of the star.
   a. Find the total mass M of the star in terms of ρ_0 and R by integrating the density equation to find M(r) and then setting r = R.
   b. Find the average density of the star in terms of ρ_0 using the total mass from part a.
   c. By integrating the equation of hydrostatic equilibrium using the given density equation and the M(r) that you found, show that the central pressure in the star is $P_C = \frac{15}{16\pi}\frac{GM^2}{R^4}$. Note that once we assume a mathematical form for the density, we do not need to make the approximations we did before.

a) Notice that the mass of the star as a function of r is given by

$$M(r) = \int\int\int_{\Omega} \rho(r) dV$$

which becomes

$$M(r) = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0 \left[1 - \left(\frac{r}{R}\right)^2\right] r^2 \sin(\phi) dr d\phi d\theta$$

Solving we find

$$M(r) = 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]$$

and plugging in $r = R$ to find the total mass:

$$M = \frac{8\pi\rho_0}{15} R^3$$

b) To find the average density, we must divide the total mass by the volume of the star (assuming spherical volume):

$$\frac{M}{V} = \left(\frac{8\pi\rho_0}{15} R^3\right) / \left(\frac{4}{3}\pi R^3\right)$$

and we find that

$$\rho_{avg} = \frac{2}{5}\rho_0$$

c) Recal the equation for hydrostatic equilibrium is given by

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Integrating this equation from $r = 0$ to $r = R$, we find that

$$P(R) - P(0) = -G\int_0^R \frac{M(r)\rho(r)}{r^2} dr$$

which becomes

$$P_C = G \int_0^R \frac{M(r)\rho(r)}{r^2} dr$$

and plugging in the expressions we found for $M(r)$ and $\rho(r)$, we find

$$P_C = G \int_0^R 4\pi\rho_0^2 \left[\frac{r^3}{3} - \frac{r^3}{5R^2} \right]$$

$$= 4\pi G\rho_0^2 \int_0^R \left[\frac{r}{3} - \frac{8r^3}{15R^2} + \frac{r^5}{5R^4} \right] dr$$

$$= 4\pi G\rho_0^2 \left(\frac{R^2}{6} - \frac{2R^2}{15} + \frac{R^2}{30} \right)$$

$$= 4\pi G\rho_0^2 \left(\frac{R^2}{15} \right)$$

And notice from our work above that

$$\rho_0 = \frac{15M}{8\pi R^3}$$

and so

$$P_C = \frac{4\pi}{15} G \left(\frac{15^2 M^2}{64\pi^2 R^6} \right) R^2$$

$$= \frac{15M^2G}{16\pi R^4}$$

Which is the result we wished to show.