Problem Set 3 (Astrophysics)

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- Mass of Earth = 5.974×10^{24} kg
- Radius of Earth = 6378 km
- Mass of Sun = 1.99×10^{30} kg
- Radius of Sun = 6.955×10^{5} km
- Luminosity of Sun = 3.9×10^{26} watts
  - Distance of Venus from Sun = 0.723 AU
  - Distance of Earth from Sun = 1 AU = 1.5×10^{11} m
  - Distance of Mars from Sun = 228×10^{6} km
  - Distance of Jupiter from Sun = 5.2 AU = 778.3×10^{6} km
- Mass of Titan = 1.35×10^{23} kg
- Radius of Titan = 2575 km
- Mass of Jupiter = $317.8 \times M_{Earth} = 1.90 \times 10^{27}$ kg
- Mass of Uranus = $14.50 \times M_{Earth}$
- Radius of Uranus = $3.98 \times R_{Earth}$

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1. a) **Calculate the escape velocities for an object at the surface of the Earth, trying to escape from Earth's gravity and an object located at a distance from the Sun equal to Earth's average orbital distance and trying to escape from the Sun. b) If we launch an object from the surface of the Earth, will it be harder for it to escape from Earth or escape from the solar system ?**

Recall that the escape velocity for an object is given by

$$v_e = \sqrt{\frac{2GM}{r}} \tag{1.1}$$

Where M is the mass of the body you're trying to escape from, and r is the distance from the center of mass. a) We wish to find the escape velocity for the Earth. Using the values for the mass and distance from the center of the Earth:

$$M_{Earth} = 5.974 \times 10^{24}\ kg$$
$$r_{Earth} = 6.378 \times 10^{6}\ m$$

Plugging these values into (1.1), we will get the following:

$$v_e = \sqrt{\frac{2(6.67 \times 10^{-11}\ \cancel{kg^{-1}}\ m^{\cancel{3}\to 2}\ s^{-2})(5.974 \times 10^{24}\ \cancel{kg})}{6.378 \times 10^{6}\ \cancel{m}}}$$
$$= 11.18\frac{km}{s}$$

so the escape velocity at the surface of the Earth is approximately $11.18\frac{km}{s}$. Now we wish to find the escape velocity from the sun at the mean distance of the Earth from the Sun.

Using that the mass of the Sun is $M_{Sun} = 1.99 \times 10^{30}\ kg$ and the distance of the Earth from the Sun $1.5 \times 10^{11}\ m$ from the table at the beginning of the assignment, we find that the escape velocity from the Sun at Earth's distance is

$$v_e = \sqrt{\frac{2(6.67 \times 10^{-11}\ \cancel{kg}^{-1} m^{\cancel{3} \to 2} s^{-2})(1.99 \times 10^{30}\ \cancel{kg})}{1.5 \times 10^{11}\ \cancel{m}}}$$

$$= 42068.7\ \frac{m}{s}$$

$$= 42.1 \frac{km}{s}$$

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2. The sun's equator rotates with a period of about 25.4 days. The sun's pole rotates with a period of about 35 days. How many rotations (and how much time) does it take for the equatorial regions of the sun to "lap" the polar regions by one full rotation?

We wish to find t such that

$$\frac{t}{25.4\ days} = \frac{t}{35\ days} + 1$$

That is, we want to find t where the equator rotates one more time than the poles of the Sun. Solving, we get

$$t(\frac{1}{25.4\ days} - \frac{1}{35\ days}) = 1$$

$$t = \frac{1}{\frac{1}{25.4\ days} - \frac{1}{35\ days}}$$

$$\approx 92.6\ days$$

Now let's find the total amount of rotations that occurred during that time:

At the equator:

$$\Delta_{rot} = \frac{92.6\ \cancel{days}}{25.4 \frac{\cancel{days}}{rotation}} = 3.646\ rotations$$

At the pole:

$$\Delta_{rot} = \frac{92.6\ \cancel{days}}{35 \frac{\cancel{days}}{rotation}} = 2.646\ rotations$$

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3. What is the mean density of Saturn's largest satellite, Titan? What does this suggest about the composition of Titan?

From the data in the given table, we have that $M_{Titan} = 1.35 \times 10^{23}\ kg$ and $r_{Titan} = 2575\ km$, and assuming that Titan is nearly spherical, we can calculate the average density of Titan:

$$\rho_{Titan} = \frac{1.35 \times 10^{23}\ kg}{\frac{4}{3}\pi(2.575 \times 10^{6}\ m)^3}$$

$$\approx 1.888 \times 10^3 \frac{kg}{m^3}$$

$$= 1.888 \frac{g}{m^3}$$

Thus Titan has a density roughly twice that as water, and about a third of that of the Earth. So it wouldn't be unreasonable to assume that Titan is probably mostly made up of lighter materials than Earth, probably things like water ice.

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4. Mercury is a slowly rotating planet with no atmosphere and an albedo of A = 0.06. The orbit is more elliptical than many planets.
   a. What is the temperature of Mercury when it is closest to the sun (r = 0.307 AU)?
   b. What is the temperature of Mercury when it is farthest from the sun (r = 0.467 AU)?

Recall the formulas for the sub-solar temperature and power radiated temperature:

$$T_P = 279(1-A)^{1/4}(\frac{r_p}{1\ AU})^{-1/2} \tag{4.1}$$

$$T_{SS} = 395(1-A)^{1/4}(\frac{r_p}{1\ AU})^{-1/2} \tag{4.2}$$

I will inspect both of these temperatures.

a) We are given that the distance between the Sun and Mercury is $r_p = 0.307\,AU$ and that the albedo of Mercury is $A = 0.06$. Plugging this into equation (4.1) we find that the power radiated temperature is

$$T_P = 279(1-0.06)^{1/4}(\frac{0.307\,\cancel{AU}}{1\,\cancel{AU}})^{-1/2}$$

$$= 279(0.94)^{1/4}(0.307)^{-1/2}$$

$$\approx 495.8\ K$$

Now to find the sub-solar temperature, plug in the value of A and r_p into equation (4.2), we find

$$T_{SS} = 395(1-0.06)^{1/4}(\frac{0.307\,\cancel{AU}}{1\,\cancel{AU}})^{-1/2}$$

$$= 395(0.94)^{1/4}(0.307)^{-1/2}$$

$$\approx 701.96\ K$$

At Mercury's closest point to the Sun, we predict its temperature to lie in the range of $496 - 702\ K$

b) Now let's find the power radiated and sub-solar temperatures when Mercury is at its furthest point from the sun ($r_p = 0.467\ AU$).

$$T_P = 279(0.94)^{1/4}(0.467)^{-1/2}$$

$$\approx 402\ K$$

Now the sub-solar temperature:

$$T_{SS} = 395(0.94)^{1/4}(0.467)^{-1/2}$$

$$\approx 569.14\ K$$

At Mercury's furthest point, we predict that its temperature lies in the range of $402 - 569\ K$.

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5. Pure, solid water ice has an albedo A = 0.35. What is the minimum distance from the Sun at which a rapidly rotating ice cube would remain frozen? (It is useful to know that water sublimes from a solid to a gas at 200 K in vacuum.). Between the orbits of which two planets does this distance lie?

We can expect that using the power radiated temperature will give the smallest value for the radius. I will find the radius when the power radiated temperature of solid water ice is equal to 200 K.

$$T_P = 279(1-A)^{1/4}(\frac{r}{1\ AU})^{-1/2} = 200\ K$$

Solving for r, we find

$$\begin{aligned} r &= (\frac{279}{200}(1-A)^{1/4}(1\ AU))^2 \\ &= (\frac{279}{200}(0.65)^{1/4}(1\ AU))^2 \\ &\approx 1.57\ AU \end{aligned}$$

This outside a distance of approximately 1.57 AU, we can expect that solid water ice will remain frozen. Notice that this distance lies between the orbits of Mars (1.5 AU) and Jupiter (5.2 AU).

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6. Suppose that we moved the planet Uranus to the orbital distance of Jupiter. Would Uranus then retain its hydrogen-rich atmosphere? Many different ways to calculate a temperature for Uranus if it were located at Jupiter. Technically you should used the temperature and radius of the exobase. Since we do not know that, you can assume that $T_{Ex} = T_P$ and $R_{Ex} = R_P$.

Let's inspect the worst case scenario for Uranus at the distance of Jupiter. That is, assume that Uranus is a perfect blackbody, that is, $A = 0$ and use equation (4.2) to find the sub-solar temperature of Uranus at this distance:

$$\begin{aligned} T_{SS} &= 395\ K(\frac{r_{Uranus}}{1\ AU})^{-1/2} \\ &= 395\ K(5.2)^{-1/2} \\ &\approx 173.2\ K \end{aligned}$$

So our worst-case scenario temperature for Uranus is about 173.2 K. Now let's find the molecular mass of gasses that Uranus can retain. We will use the following formula:

$$\mu \geq 7.1(\frac{T_{Uranus}}{1000\ K})(\frac{M_{Uranus}}{M_{Earth}})^{-1}(\frac{R_{Uranus}}{R_{Earth}}) \tag{6.1}$$

From the table at the top of the assignment, we can see that the mass and radius of Uranus are as follows:

$$M_{Uranus} = 14.5\ M_{Earth}$$

$$R_{Uranus} = 3.98\ R_{Earth}$$

Plugging these values into (6.1), we find

$$\begin{aligned} \mu &\geq 7.1(\frac{173.2\ \cancel{K}}{1000\ \cancel{K}})(\frac{14.5\ \cancel{M_{Earth}}}{\cancel{M_{Earth}}})^{-1}(\frac{3.98\ \cancel{R_{Earth}}}{\cancel{R_{Earth}}}) \\ &= 7.1(.1732)(\frac{1}{14.5})(3.98) \\ &\approx 0.338 \end{aligned}$$

That is, Uranus can retain gasses with molecular masses of greater than 0.338. And since the molecular mass of hydrogen is 2, we can safely say that Uranus will retain its hydrogen-rich atmosphere if were located where Jupiter is.