Problem Set 2 (Analysis)

Michael Nameika

February 16 2022

1. Determine if each of the following pairs (X, d) define a metric space. a) $X = \mathbb{R}$, $d(x, y) = \sqrt{|x-y|}$, $x, y \in \mathbb{R}$
   Let us first check non-negativity:
   Notice that $|x-y| \geq 0$, and so $\sqrt{|x-y|} \geq 0$. Now we must show that $\sqrt{|x-y|} = 0$ iff $x = y$. Start off by assuming $\sqrt{|x-y|} = 0$. Then
   $$|x-y| = 0$$
   $$x - y = 0$$
   $$x = y$$
   Now assume that $x = y$. Then
   $$\sqrt{|x-y|} = \sqrt{|y-y|}$$
   $$= \sqrt{0}$$
   $$= 0$$
   Now we will show symmetry holds:
   $$\sqrt{|x-y|} = \sqrt{|(-1)(y-x)|}$$
   $$= \sqrt{|y-x|}$$
   Thus symmetry holds.
   Now we will show that the triangle inequality holds.
   Let $x, y, z \in \mathbb{R}$. Start by considering $|x-y|$. Notice
   $$|x-y| = |x-z+z-y|$$
   $$\leq |x-z| + |z-y|$$
   $$\sqrt{|x-y|} \leq \sqrt{|x-z| + |z-y|}$$
   Now we need to show that $\sqrt{|x-z| + |z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$.
   Consider $a, b \in \mathbb{R}^+ \cup \{0\}$. we wish to show that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. First consider
   $$(\sqrt{a+b})^2 = a + b$$
   Now consider
   $$(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{a}\sqrt{b} + b$$
   Now take the difference between these two quantities:
   $$(\sqrt{a+b})^2 - (\sqrt{a} + \sqrt{b})^2 = a + b - (a + 2\sqrt{a}\sqrt{b} + b)$$
   $$= -2\sqrt{a}\sqrt{b}$$

And since $0 \leq a, b$, $-2\sqrt{a}\sqrt{b} \leq 0$, which tells us that

$$(\sqrt{a+b})^2 \leq (\sqrt{a}+\sqrt{b})^2$$

$$\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$$

Finally, we have

$$\sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$$

$$d(x,y) \leq d(x,z) + d(z,y)$$

Thus, non-negativity, symmetry, and the triangle inequality all hold, sol (X, d) forms a metric space.

b)$X = \mathbb{R}$, $d(x,y) = |x| + |x-y| + |y|$ when $x \neq y$, and $d(x,y) = 0$ when $x = y$, $x, y \in \mathbb{R}$

We will begin by showing non-negativity. Let $x, y \in \mathbb{R}$. Then by definition of absolute value, $|x| \geq 0$, $|x-y| \geq 0$, and $|y| \geq 0$. Then

$$|x| + |x-y| + |y| \geq 0$$

Now we must show that $d(x,y) = 0$ iff $x = y$. Begin by assuming $x = y$. Then by the definition above, $d(x,y) = 0$. Now assume $d(x,y) = 0$. Then

$$|x| + |x-y| + |y| = 0$$

Since $|x|, |y|, |x-y| \geq 0$, we must have that $x = y = 0$. Thus, non-negativity holds. Now we will show that symmetry holds. Notice that

$$|x| + |x-y| + |y| = |y| + |(-1)(y-x)| + |x|$$

$$= |y| + |y-x| + |x|$$

Thus, $d(x,y) = d(y,x)$, so symmetry holds. Now we will show that the triangle inequality holds. Let $x, y, z \in \mathbb{R}$. Notice

$$|x| + |x-y| + |y| = |x| + |x-z+z-y| + |z| + |z| + |y| - 2|z|$$

$$\leq |x| + |x-z| + |z| + |z| + |z-y| + |y| - 2|z|$$

$$= d(x,z) + d(z,y) - 2|z|$$

Since $|z| \geq 0$,

$$d(x,z) + d(z,y) - 2|z| \leq d(x,z) + d(z,y)$$

Finally, we have

$$d(x,y) \leq d(x,z) + d(z,y)$$

So non-negativity, symmetry, and the triangle inequality hold. Thus, (X, d) forms a metric space.

c) $X =$ space of all Riemann integrable functions on $[a,b]$, $d(f,g) = \int_a^b |f(x) - g(x)| dx$, $f, g \in X$.

I will claim that this is not a metric space. In particular, I will show that $d(f,g) = 0$ for some $f \neq g$, $f, g \in X$. Let

$$f(x) = 0, x \in [a,b]$$

$$g(x) = \begin{cases} 0, x \neq x_i \\ 1, x = x_i \end{cases} \quad x_i \in [a,b]$$

Clearly, $f(x) \neq g(x)$

We will show using Darboux sums that $\int_a^b |f(x) - g(x)| dx = 0$. Consider a partition P of $[a,b]$, $P = \{a, a+\frac{1}{n}, a+\frac{2}{n}, \ldots, b-\frac{1}{n}, b\}$ and let $x_i \in P$. Now consider the upper Darboux sum of $|f-g|$ on P:

$$U(|f-g|, P) = \sum_{k=1}^{n-1} \sup_{x \in [x_k, x_{k+1}]} (|f(x) - g(x)|)(x_{k+1} - x_k)$$

Since $f(x) = 0$, and $g(x) = 0$ except at $x = x_i$, the above sum reduces to

$$U(|f-g|, P) = \frac{1}{n}$$

and in the limit, $U(|f-g|) = \lim_{n\to\infty} U(|f-g|, P) = \lim_{n\to\infty} \frac{1}{n} = 0$ Now we wish to show that the lower sum also goes to zero.

$$L(|f-g|, P) = \sum_{k=1}^{n-1} \inf_{x\in[x_k, x_{k+1}]} (|f(x) - g(x)|)(x_{k+1} - x_k)$$

Right away, $\inf(|f(x) - g(x)|) = 0$, so $L(|f-g|, P) = 0$. And in the limit, $L(|f-g|) = 0$. So we have

$$U(|f-g|) = L(|f-g|) = 0$$

So by definition of Darboux integrability,

$$\int_a^b |f(x) - g(x)|dx = 0$$

So (X, d) does not form a metric space.

d) Let (U, d_U), (V, d_V) be metric spaces. Define $X = U \times V$ as the set of ordered pairs (u, v) with $u \in U$, $v \in V$, and $d((u_1, v_1), (u_2, v_2)) = \max(d_U(u_1, u_2), d_V(v_1, v_2))$.

We will begin by showing that non-negativity holds:

Notice that since $d((u_1, v_1), (u_2, v_2)) = \max(d_U(u_1, u_2), d_V(v_1, v_2))$, and (U, d_U) and (V, d_V) form metric spaces, so

$$d_U(u_1, u_2), d_V(v_1, v_2) \geq 0$$

then

$$d((u_1, v_1), (u_2, v_2)) \geq 0$$

Now we must show that

$$d((u_1, v_1), (u_2, v_2)) = 0$$

if and only if

$$(u_1, v_1) = (u_2, v_2)$$

Begin by assuming $d((u_1, v_1), (u_2, v_2)) = 0$. Then by the definition of our metric, $d_U(u_1, u_2) = d_V(v_1, v_2) = 0$, which implies $u_1 = u_2$ and $v_1 = v_2$. So $(u_1, v_1) = (u_2, v_2)$.

Now assume that $(u_1, v_1) = (u_2, v_2)$. That is, $u_1 = u_2$ and $v_1 = v_2$. Then $d_U(u_1, u_2) = d_V(v_1, v_2) = 0$. So $\max(d_U(u_1, u_2), d_V(v_1, v_2)) = 0$, or equivalently, $d((u_1, v_1), (u_2, v_2)) = 0$.

Now we must show that symmetry holds. Notice that

$$d_U(u_1, u_2) = d_U(u_2, u_1)$$

and

$$d_V(v_1, v_2) = d_V(v_2, v_1)$$

So

$$\max(d_U(u_1, u_2), d_V(v_1, v_2)) = \max(d_U(u_2, u_1), d_V(v_2, v_1))$$

or

$$d((u_1, v_1), (u_2, v_2)) = d((u_2, v_2), (u_1, v_1))$$

Now we must show that the triangle inequality holds.

Notice that for $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in X$, either

$$\max(d_U(u_1, u_2), d_V(v_1, v_2)) = d_U(u_1, u_2) \le d_U(u_1, u_3) + d_U(u_3, u_2)$$

or

$$\max(d_U(u_1, u_2), d_V(v_1, v_2)) = d_V(v_1, v_2) \le d_V(v_1, v_3) + d_V(v_3, v_2)$$

Additionally, notice that

$$d_U(u_1, u_3) + d_U(u_3, u_2) \le \max(d_U(u_1, u_3), d_V(v_1, v_3)) + \max(d_U(u_3, u_2), d_V(v_3, v_2))$$

and

$$d_V(v_1, v_3) + d_V(v_3, v_2) \le \max(d_U(u_1, u_3), d_V(v_1, v_3)) + \max(d_U(u_3, u_2), d_V(v_3, v_2))$$

Now, by transitivity, we have

$$\max(d_U(u_1, u_2), d_V(v_1, v_2)) \le \max(d_U(u_1, u_3), d_V(v_1, v_2)) + \max(d_U(u_3, u_2), d_V(v_3, v_2))$$

$$d((u_1, v_1), (u_2, v_2)) \le d((u_1, v_1), (u_3, v_3)) + d((u_3, v_3), (u_2, v_2))$$

Thus the triangle inequality holds, so (X, d) forms a metric space.

2. Let (X, d) be a metric space.

a) Let E be a nonempty subset of X. Define the distance of $x \in X$ to E by $\rho_E(x) := \inf_{y \in E} d(x, y)$. Prove that (i) $\rho_E(x) = 0$ if and only if $x \in E^c$ (ii) $\rho_E : X \to \mathbb{R}$ is uniformly continuous on X.

ii) We wish to show that $\rho_E(x)$ is uniformly continuous on (X, d). Let $x, z \in X$ and $y \in E$. Fix $\delta > 0$ independent of x be such that $d(x, z) < \delta$. We wish to show $|\rho_E(x) - \rho_E(z)| < \epsilon$ for $\epsilon > 0$. Start with $d(x, y)$:

$$d(x, y) \le d(x, z) + d(z, y)$$

$$d(x, y) - d(z, y) \le d(x, z)$$

$$d(y, x) - d(y, z) \le d(x, z)$$

Now, we can see

$$-d(x, z) \le d(y, x) - d(y, z)$$

So we can say

$$|d(y, x) - d(y, z)| \le d(x, z)$$

$$|\rho_E(x) - \rho_E(z)| \le d(x, z) < \delta$$

$$|\rho_E(x) - \rho_E(z)| < \delta$$

Now let $\epsilon = \delta$. So we have

$$|\rho_E(x) - \rho_E(z)| < \epsilon$$

So by definition of uniform continuity, $\rho_E(x)$ is uniformly continuous.

b) Suppose $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences in X. Show that the sequence $a_n = d(x_n, y_n)$ converges in $\mathbb{R}$.

Proof: By definition of Cauchy sequences in a metric space, for any $\epsilon > 0$, there exist natural numbers $n, m > N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{\epsilon}{2}$$

$$d(y_n, y_m) < \frac{\epsilon}{2}$$

We wish to show that a_n converges in $\mathbb{R}$. It suffices to show that a_n is Cauchy in $\mathbb{R}$. That is, we wish to show that $|a_n - a_m| < \epsilon$.

$$|a_n - a_m| = |d(x_n, y_n) - d(x_m, y_m)|$$

By the triangle inequality,

$$\begin{aligned} d(x_n, y_n) &\le d(x_n, x_m) + d(x_m, y_n) \\ &\le d(x_n, x_m) + d(y_n, y_m) + d(x_m, y_m) \end{aligned}$$

Then

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &\le |d(x_n, x_m) + d(y_n, y_m) + d(x_m, y_m) - d(x_m, y_m)| \\ &= |d(x_n, x_m) + d(y_n, y_m)| \\ &\le |d(x_n, x_m)| + |d(y_n, y_m)| = d(x_n, x_m) + d(y_n, y_m) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

So we have

$$|a_n - a_m| < \epsilon$$

which converges since $\mathbb{R}$ is complete.

3. Let X be the space of all bounded, real sequences, $\mathbf{x} = \{x_n\}$, $\mathbf{y} = \{y_n\} \in X$ with metric $d(\mathbf{x}, \mathbf{y}) = \sup_{n \ge 1} |x_n - y_n|$. Prove that each Cauchy sequence $\{\mathbf{x}^{(\mathbf{m})}\} \subset X$ converges to some $\mathbf{x} \in X$.

   By definition of Cauchy sequences, we have for any $\epsilon > 0$, there exist natural numbers $n > m \ge N \in \mathbb{N}$ (fix $m = N$) such that

   $$d(\mathbf{x}^{(n)}, \mathbf{x}^{(N)}) < \epsilon$$

   By the definition of our metric,

   $$\sup_{k \ge 1} |x_k^{(n)} - x_k^{(N)}| < \epsilon$$

   Notice that

   $$|x_k^{(n)} - x_k^{(N)}| \le \sup_{k \ge 1} |x_k^{(n)} - x_k^{(N)}| < \epsilon$$

   That is, $\{x_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}$ for every element k. Since $\mathbb{R}$ is complete, $x_k^{(n)}$ converges to some real number x_k. Define $\mathbf{x} = \{x_k\}$. Since $\{\mathbf{x}^{(m)}\}$ is Cauchy, and each element converges to $\{x_k\}$, we have for any $\epsilon > 0$, there exists natural numbers n, N such that whenever $n \ge N$,

   $$|x_n^{(m)} - x_n| < \epsilon$$

   Since this is true for all $n > N$,

   $$\sup |x_n^{(m)} - x_n| < \epsilon$$

   or, by the definition of our metric,

   $$d(\mathbf{x}^{(m)}, \mathbf{x}) < \epsilon$$

   So $\mathbf{x}^{(m)}$ converges to $\mathbf{x}$.

   We have that $\mathbf{x}$ is a real sequence, but we need to show that it's bounded.

   Notice from the work above and the triangle inequality that

   $$|x_k^{(n)}| < |x_k^{(N)}| + \epsilon$$

   Now since each $x_k^{(i)}$ is a bounded sequence, let $M_i \in \mathbb{R}$ be such that

   $$|x_k^{(i)}| \le M_i$$

   then

   $$|x_k^{(N)}| \le M_N \in \mathbb{R}$$

   so we have

   $$|x_k^{(n)}| < M_N + \epsilon$$

That is, $|x_k^{(n)}|$ is also bounded above for all n. Call this bound M^*. That is,

$$|x_k^{(n)}| \leq M^*$$

Now since $\{\mathbf{x}^{(k)}\}$ converges to a sequence $\mathbf{x}$, we have for $k > N$

$$|x_k - x_k^{(n)}| < \epsilon$$

by the reverse triangle inequality,

$$|x_k| < |x_k^{(n)}| + \epsilon$$

by our work above, we have

$$|x_k| < M^* + \epsilon$$

That is, each x_k is bounded. So we have that $\mathbf{x}$ is a bounded sequence. And since $\mathbf{x}$ is real and bounded, $\mathbf{x} \in X$.

4. Let $M_n(\mathbb{R})$ denote the set of all real, $n \times n$ matrices. For $A := (a_{ij})$, $B = (b_{ij}) \in M_n(\mathbb{R})$, define $d(A, B) = \max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}|$. a) Show that $d(A, B)$ is a metric on $M_n(\mathbb{R})$.

   We will begin by showing that non-negativity holds. Notice that

$$0 \leq |a_{ij} - b_{ij}| \leq \max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}|$$

   Now we must show that $d(A, B) = 0$ iff $A = B$. First assume that $d(A, B) = 0$. Then we have

$$\max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}| = 0$$

   since $0 \leq \max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}|$, we must have that

$$|a_{ij} - b_{ij}| = 0$$

   for every i, j. Thus, we have

$$a_{ij} = b_{ij}, \forall i, j$$

   or $A = B$.

   Now assume $a_{ij} = b_{ij} \forall i, j$ That is, $A = B$. Then $a_{ij} - b_{ij} = 0 \forall i, j$. So clearly, $\max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}| = 0$. So $d(A, B) = 0$. Thus, we have shown non-negativity to hold. Now we will show symmetry holds:

$$\max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}| = \max_{1 \leq i,j \leq n} |(-1)(b_{ij} - a_{ij})|$$

$$= \max_{1 \leq i,j \leq n} |b_{ij} - a_{ij}|$$

   Or, equivalently,

$$d(A, B) = d(B, A)$$

   so symmetry holds. Now we will show that the triangle inequality holds:

   Let $A, B, C \in M_n(\mathbb{R})$ where $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$. We wish to show that $d(A, B) \leq d(A, C) + d(C, B)$.

   Consider

$$|a_{ij} - b_{ij}| = |a_{ij} - c_{ij} + c_{ij} - b_{ij}|$$
$$\leq |a_{ij} - c_{ij}| + |c_{ij} - b_{ij}|$$

   Clearly

$$\max_{1 \leq i,j \leq n} |a_{ij} - b_{ij}| \leq \max_{1 \leq i,j \leq n} |a_{ij} - c_{ij}| + \max_{1 \leq i,j \leq n} |c_{ij} - b_{ij}|$$

so

$$d(A,B) \le d(A,C) + d(C,B)$$

b) Let $\{A^{(k)}\}$ be a sequence in $M_n(\mathbb{R})$. Prove that $\{A^{(k)}\}$ is a convergent sequence if and only if $\{a_{ij}^{(k)}\}$ is a convergent sequence in $\mathbb{R}$.

Proof: First assume that $\{a_{ij}^{(k)}\}$ is a convergent sequence in $\mathbb{R}$. That is, there exists a real number (since $\mathbb{R}$ is complete) a_{ij} such that $a_{ij}^{(k)} \to a_{ij}$. Now let A be a matrix defined by $A = a_{ij}$. Since each $\{a_{ij}^{(k)}\}$ converges to a_{ij}, we have that $\{A^{(k)}\}$ converges component wise to A.

Now assume that $\{A^{(k)}\}$ converges element wise to some matrix $A = (a_{ij})$. Let the elements of $\{A^{(k)}\}$ be defined by $a_{ij}^{(k)}$. Since $\{A^{(k)}\}$ converges element wise to A, each $\{a_{ij}^{(k)}\}$ must converge. Since $\mathbb{R}$ is complete, we have that $\{a_{ij}^{(k)}\}$ converges to a real number, so $\{a_{ij}^{(k)}\}$ is convergent in $\mathbb{R}$.

c) Show that $(M_n(\mathbb{R}), d)$ is a complete metric space.

Proof: We wish to show for the Cauchy sequence $\{A^{(k)}\}$ in $M_n(\mathbb{R})$, that there exists a matrix $A \in M_n(\mathbb{R})$ such that $d(A^{(k)}, A) \to 0$ as $k \to \infty$.

We have that $\{A^{(k)}\}$ is a Cauchy sequence. That is, for any $\epsilon > 0$, there exist natural numbers $n, m > N \in \mathbb{N}$ such that

$$d(A^{(n)}, A^{(m)}) < \epsilon$$

by the definition of our metric:

$$d(A^{(n)}, A^{(m)}) = \max_{1 \le i,j \le n} |a_{ij}^{(n)} - a_{ij}^{(m)}| < \epsilon$$

Notice that for all i, j,

$$|a_{ij}^{(n)} - a_{ij}^{(m)}| \le \max_{1 \le i,j \le n} |a_{ij}^{(n)} - a_{ij}^{(m)}| < \epsilon$$

then

$$|a_{ij}^{(n)} - a_{ij}^{(m)}| < \epsilon$$

That is, for all i, j, $\{a_{ij}^{(k)}\}$ is a convergent sequence in $\mathbb{R}$ since it is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists some $a_{ij} \in \mathbb{R}$ such that

$$a_{ij}^{(k)} \to a_{ij}$$

By our work in part b), since $\{a_{ij}^{(k)}\}$ is convergent in $\mathbb{R}$, we have that $\{A^{(k)}\}$ is a convergent sequence, say it converges to some matrix $A = (a_{ij})$. Now since every $a_{ij}^{(k)}$ converges to a real number, $\{A^{(k)}\}$ converges component wise to A, and so every component of A is real, so $A \in M_n(\mathbb{R})$. Thus by definition, $(M_n(\mathbb{R}), d)$ is a complete metric space.

5. a) Suppose $\{f_n : [0,1] \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to f on $[0,1]$. Let $g_n(x) = [f_n(x)]^2$. Prove that $\{g_n\}$ converges uniformly to $g(x) = [f(x)]^2$ on $[0,1]$.

   Proof: Recall that the uniform limit of continuous functions is also continuous. That is, every f_n and f is continuous. Now since f_n and f are continuous on a closed, bounded interval, by the extreme value theorem, we have that each f_n and f is bounded, say by

$$|f_n| \le M_n \in \mathbb{R}$$

$$|f| \le M \in \mathbb{R}$$

   By the Cauchy criterion of uniform continuity, fix $\epsilon > 0$ independent of x and take $n > m \ge N \in \mathbb{N}$. In particular, take $m = N$. Now let $M' = \max\{M_1, M_2, \ldots, M_N\}$ be such that

$$|f_n - f_N| < \frac{\epsilon}{M' + M}$$

By the reverse triangle inequality, notice

$$|f_n| < |f_N| + \epsilon$$

$$|f_n| < M' + \epsilon$$

Which tells us that $|f_n|$ is bounded above. Call this bound M^*.

Now, we wish to show that f_n^2 converges to f^2 uniformly. Begin by considering

$$|f_n^2 - f^2| = |f_n - f||f_n + f|$$

$$\leq |f_n - f|(|f_n| + |f|)$$
$$\leq |f_n - f|(M^* + M)$$
$$< \epsilon(M^* + M)$$

So we have

$$|f_n^2 - f^2| < \epsilon(M^* + M)$$

Thus, $g_n = f_n^2$ converges uniformly to f^2 on $[0, 1]$.

b) Proof: Fix $\epsilon > 0$. By definition of uniform continuity, there exists a $\delta > 0$ such that whenever $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

Consider the interval $(0, \delta)$ and fix $y \in (0, \delta)$. Notice for any $x \in (0, \delta)$, $|x - y| < \delta$. Then by the definition above,

$$|f(x) - f(y)| < \epsilon$$

by the reverse triangle inequality,

$$|f(x)| < |f(y)| + \epsilon$$

That is, $f(x)$ is bounded on $(0, \delta)$, say by a real number M_1.

Consider the interval $[\delta, 1]$. Since f is uniformly continuous on $(0, 1]$, f is continuous on $[\delta, 1]$. By the extreme value theorem, we have that f is bounded on $[\delta, 1]$, say by a real number M_2.

Now let $M = \max\{M_1, M_2\}$. Since f is bounded by M_1 on $(0, \delta)$ and by M_2 on $[\delta, 1]$, f is bounded by M on $(0, 1]$.