Optimization HW 8

Michael Nameika

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Section 14.2 Problems

1. Consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1^2 + x_1^2x_3^2 + 2x_1x_2 + x_2^4 + 8x_2 \\ \text{subject to} \quad & 2x_1 + 5x_2 + x_3 = 3. \end{aligned}$$

(i) Determine which of the following points are stationary points: (a) $(0,0,2)^T$, (b) $(0,0,3)^T$; (c) $(1,0,1)^T$.

Let us begin by computing the gradient of f:

$$\nabla f(x) = \begin{pmatrix} 2x_1 + 2x_1x_3^2 + 2x_2 \\ 2x_1 + 4x_2^3 + 8 \\ 2x_1^2x_3 \end{pmatrix}$$

And we have our constraint matrix: $A = (2,5,1)$. We wish to find a basis for the null space of A. Well, solving $Ax = 0$ yields the following expression for a null space vector:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -2x_1 - 5x_2 \end{pmatrix}$$

And so a basis for the null space of A is given by

$$\text{basis}(\text{N}(A)) = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \right\}$$

So choose $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -5 \end{pmatrix}$. Now we may determine if the above points are stationary points of f:

(a) Notice

$$\nabla f(0,0,2) = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}$$

and so

$$\begin{aligned} Z^T \nabla f(0,0,2) &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 8 \end{pmatrix} \neq 0 \end{aligned}$$

So $(0,0,2)^T$ is not a stationary point of f.

(b) Notice

$$\nabla f(0,0,3) = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}$$

So from our work from part (a), we have that $(0,0,3)^T$ is not a stationary point of f.

(c) Notice

$$\nabla f(1,0,1) = \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$$

And so

$$\begin{aligned} Z^T \nabla f(1,0,1) &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So $(1,0,1)^T$ is a stationary point of f.

(ii) Determine whether each stationary point is a local minimizer, a local maximizer, or a saddle point.

From part (i), we have the only stationary point of (a), (b), and (c) is $(1,0,01)^T$. To determine if this point is a local minimizer, maximizer, or saddle point, we must inspect the reduced Hessian $Z^T \nabla^2 f(1,0,1)Z$. Well,

$$\nabla^2 f(x) = \begin{pmatrix} 2+2x_3^2 & 2 & 4x_1x_3 \\ 2 & 12x_2^2 & 0 \\ 4x_1x_3 & 0 & 2x_1^2 \end{pmatrix}$$

and so

$$\nabla^2 f(1,0,1) = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 0 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

Finally for the reduced Hessian:

$$\begin{aligned} Z^T \nabla^2 f(1,0,1) Z &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 0 & 0 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} -4 & -18 \\ 2 & 0 \\ 0 & -10 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 2 \\ 2 & 50 \end{pmatrix} \end{aligned}$$

Clearly, the reduced Hessian is not positive definite. Row reducing (simply add 1/2 of the first row to the second), we get the reduced Hessian in row echelon form:

$$(Z^T \nabla^2 f(1,0,1)Z)_{REF} = \begin{pmatrix} -4 & 2 \\ 0 & 49 \end{pmatrix}$$

So the reduced Hessian is indefinite. That is, the stationary point $(1,0,1)^T$ is a saddle point of f.

3. Find all the values of the parameters a and b such that $(0,0)^T$ minimizes or maximizes the following function subject to the given constraint:

$$f(x_1,x_2) = (a+2)x_1 - 2x_2 \quad \text{subject to} \quad a(x_1 + e^{x_1}) + b(x_2 + e^{x_2}) = 1.$$

Define $g(x) = a(x_1 + e^{x_1}) + b(x_2 + e^{x_2}) - 1 = 0$. Building the Lagrangian, we have

$$\begin{aligned}\mathscr{L}(x,\lambda) &= g(x) - \lambda g(x) \\ &= (a+2)x_1 - 2x_2 - \lambda(a(x_1 + e^{x_1}) + b(x_2 + e^{x_2}) - 1)\end{aligned}$$

At a stationary point, we require $\nabla_x \mathscr{L}(x,\lambda) = 0$. Well, since we wish to have a stationary point at $(0,0)^T$, we have

$$\nabla_x(x,\lambda) = \begin{pmatrix} a + 2 - 2\lambda a \\ -2 - 2\lambda b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice $\lambda \neq 0$ since if $\lambda = 0$, $\nabla_x \mathscr{L}(x,\lambda) \neq 0$. Then we find

$$\lambda = -\frac{1}{b}$$

and so

$$a + 2 = -\frac{2a}{b}$$

And since $g(0) = 0$, we have $a + b = 1$ and we find

$$\begin{aligned} 3 - b &= -2\left(\frac{1}{b} - 1\right) \\ 1 - b &= -\frac{2}{b} \\ b^2 - b - 2 &= 0 \\ (b-2)(b+1) &= 0 \end{aligned}$$

So $b = 2$ or $b = -1$. Then for $b = 2$, $a = -1$ and for $b = -1$, $a = 2$. Additionally, for $b = 2$, $\lambda = -1/2$ so $(a,b) = (-1,2)$ corresponds to a local maximum. For $b = -1$, $\lambda = 1$, so $(a,b) = (2,-1)$ corresponds to a local minimum. Clearly, the min/max value at $(0,0)^T$ is $f(0,0) = 0$.

7. Let A be a matrix of full row rank. Find the point in the set $Ax = b$ which minimizes $f(x) = \frac{1}{2}x^T x$.

Let $A \in \mathbb{R}^{m \times n}$. We approach this problem using Lagrange multipliers. Recall that at an optimal point x_*, $\nabla f(x_*) = A^T \lambda_*$ where $\lambda_* \in \mathbb{R}^m$ are the Lagrange multipliers. The gradient of f is given as

$$\nabla f(x) = x$$

and so $\nabla f(x_*) = x_*$. Thus, we have

$$x_* = A^T \lambda_*$$

From the constraints $Ax = b$, we can see by multiplying the above equation on each side by A:

$$\begin{aligned} Ax_* &= AA^T \lambda_* \\ b &= AA^T \lambda_*. \end{aligned}$$

Since A is full row rank, we have AA^T is nonsingular, so

$$\lambda_* = (AA^T)^{-1} b.$$

Multiplying the above equation on each side by A^T we have

$$x_* = A^T (AA^T)^{-1} b$$

But since $\nabla^2 f(x) = I$, which is positive definite.

Section 14.4 Problems

1. Solve the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \tfrac{1}{2}x_1^2 + x_2^2 \\ \text{subject to} \quad & 2x_1 + x_2 \geq 2 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0. \end{aligned}$$

To begin, let us find the gradient and Hessian of f as well as the constraint matrix A:

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix}, \qquad \nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

We must consider all possible combinations for the complementary slackness condition.
Case 1: All constraints are active. Then there are no feasible points.
Case 2: Suppose the first and second constraints are active. Then $\lambda_3 = 0$ and

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda_2$$

And since the first and second constraints are active, it is easy to see that $x_1 = 1$ and $x_2 = 0$. For our Lagrange multipliers, we have $\lambda_1 = \lambda_2 = 1/3$. Additionally, $(1, 0)^T$ is not a strict local minimizer since $Z_+ = 0$.

Case 3: Suppose the first and third constraints are active. Then $x_1 = 0$, $x_2 = 2$, and $\lambda_2 = 0$. Then

$$\nabla f(x) = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \lambda_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda_3$$

And so $\lambda_1 = 4$, $\lambda_3 = -8$, so this point is not optimal.

Case 4: Suppose the second and last constraints are active. Then $x_1 = 0$ and $x_2 = -1$, which is infeasible.

Case 5: Suppose the first constraint is the only active constraint. Then $2x_1 + x_2 = 2$ and $\lambda_2 = \lambda_3 = 0$. We have

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \lambda_1$$

Now we have $2\lambda_1 = x_1$ and $2x_2 = \lambda_1$, so using our active constraint, we find $x_1 = 8/9$ and $x_2 = 2/9$. At this point, two of our constraints are degenerate, so the submatrix $\hat{A}_+$ corresponding to the nondegenerate constraint is $\hat{A}_+ = (2, 1)$. A basis Z_+ for the nullspace of $\hat{A}_+$ is $Z_+ = (1, -2)^T$. Checking the second order sufficiency condition, we have

$$\begin{aligned} Z_+^T \nabla^2 f(\tfrac{8}{9}, \tfrac{2}{9}) Z_+ &= (1, -2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= (1, -2) \begin{pmatrix} 1 \\ -4 \end{pmatrix} \\ &= 9 \geq 0 \end{aligned}$$

So the point $x = (\frac{8}{9}, \frac{2}{9})^T$ is a strict local minimizer.

Case 6: Suppose the second constraint is the only active constraint. Then $x_1 - x_2 = 1$, $\lambda_1 = \lambda_3 = 0$, and

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda_2$$

So $x_1 = \lambda_2$ and $2x_2 = -\lambda_2$ and so we find $x_1 = 2/3$ and $x_2 = -1/3$ which is infeasible.

Case 7: Suppose the third constraint is the only active constraint. Then $x_1 = 0$, $\lambda_1 = \lambda_2 = 0$ and

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda_3$$

Then $x_2 = 0$ which is infeasible.

Case 8: Now suppose all constraints are inactive. Then $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and

$$\nabla f(x) = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So $x_1 = x_2 = 0$ which is infeasible.

Then the minimizer to this problem is $x_* = (8/9, 2/9)^T$ with an associated minimum value of $f(x_*) = 4/9$.

4. Consider the linear program

$$\begin{array}{rl} \text{minimize} & f(x) = c^T x \\ \text{subject to} & Ax \geq b. \end{array}$$

(i) Write the first- and second-order necessary conditions for a local solution.

We require

- $Ax_* \geq b$
- $\nabla f(x_*) = A^T \lambda_*$ or equivalently, $c = A^T \lambda_*$
- $\lambda_* \geq 0$
- $\lambda_*^T (Ax_* - b) = 0$
- $Z^T \nabla^2 f(x_*) Z$ is positive semidefinite for Z a nullspace matrix of the active constraints at x_*. Notice for this problem, $\nabla^2 f(x) = 0$, so this condition is trivially satisfied.

(ii) Show that the second-order sufficiency conditions do not hold anywhere, but that any point x_* satisfying the first-order necessary conditions is a global minimizer. (*Hint*: Show that there are no feasible directions of descent at x_*, and that this implies that x_* is a global minimizer.)

The second order sufficiency condition states that $Z^T \nabla^2 f(x) Z$ is positive definite. However, from above, we have that $\nabla^2 f(x) = 0$, so $Z^T \nabla^2 f(x) Z = 0$ for all x, Z. Then the second order sufficiency condition is never satisfied.
Suppose that x_* is a point satisfying the first-order necessary conditions and suppose by way of contradiction that p is a direction of descent at x_*. That is, $f(x_* + p) < f(x_*)$. Then notice that

$$f(x_* + p) = c^T (x_* + p) = c^T x_* + c^T p = f(x_*) + c^T p$$

then

$$f(x_* + p) - f(x_*) = c^T p$$

That is, we must have that $c^T p < 0$. Additionally, for p to be a feasible direction of descent, we must have

$$\begin{aligned} A(x_* + p) &\geq b \\ Ax_* + Ap &\geq b \\ Ap &\geq b - Ax_* \geq 0 \end{aligned}$$

But from the first order conditions, we have $c = A^T\lambda_*$, or equivalently, $c^T = \lambda_*^T A$. Putting it together, we find

$$\lambda^T Ap < 0$$

but since $Ap \geq 0$, that means that there must be some element λ_i in λ_* that is less than zero, contradicting out necessary condition $\lambda \geq 0$.

5. Consider the quadratic problem

$$\begin{array}{ll} \text{minimize} & f(x) = \frac{1}{2}x^T Qx - c^T x \\ \text{subject to} & Ax \geq b. \end{array}$$

Where Q is a symmetric matrix.

(i) Write the first- and second-order necessary optimality conditions. State all assumptions that you are making.

- $Ax_* \geq b$
- $\nabla f(x_*) = A^T\lambda_*$ or equivalently, $Qx_* - c = A^T\lambda_*$
- $\lambda_* \geq 0$
- $\lambda_*^T(Ax_* - b) = 0$
- $Z^T\nabla^2 f(x_*)Z$ is positive semi definite for Z a nullspace matrix of the active constraints at x_*.

Notice $\nabla^2 f(x_*) = Q$, so we require $Z^T QZ$ to be positive semidefinite.

(ii) Is it true that any local minimum to the problem is also a global minimium?

No, consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) = -x^2 \\ \text{subject to} & x \geq -1 \end{array}$$

Clearly, the problem has a local minimum of -1 at $x = -1$ but no global minimizer. The problem is unbounded below!

Section 14.5 Problems

3. Solve the problem

$$\begin{array}{ll} \text{minimize} & f(x) = x_1 + x_2 \\ \text{subject to} & \log(x_1) + 4\log(x_2) \geq 1. \end{array}$$

Let $g(x) = \log(x_1) + 4\log(x_2) - 1 \geq 0$ so that our constraint is in the "≥ 0" form. Define the Lagrangian $\mathscr{L}(x, \lambda)$:

$$\mathscr{L}(x, \lambda) = f(x) - \lambda g(x)$$

For a stationary point, we require $\nabla_x \mathscr{L}(x, \lambda) = \nabla f(x) - \lambda\nabla g(x) = 0$. Notice

$$\nabla f(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\nabla g(x) = \begin{pmatrix} 1/x_1 \\ 4/x_2 \end{pmatrix}$$

so

$$\nabla_x \mathscr{L} = \begin{pmatrix} 1 - \lambda/x_1 \\ 1 - 4\lambda/x_2 \end{pmatrix}$$

We now consider the following cases.

Case 1: The constraint is inactive. Then $\lambda = 0$ and $\nabla_x \mathscr{L} \neq 0$, so no stationary points exist in this case.

Case 2: The constraint is active. Then $\lambda \neq 0$ and from $\nabla_x \mathscr{L}(x, \lambda) = 0$, we have

$$\begin{aligned} 1 &= \frac{\lambda}{x_1} \\ 1 &= \frac{4\lambda}{x_2} \end{aligned}$$

From this, we can see $4x_1 = x_2$. And since the constraint is active, we have $\log(x_1) + 4\log(x_2) = 1$, or equivalently, $\log(x_1) + 4\log(4x_1) = 1$. Solving,

$$\begin{aligned} 5\log(x_1) + 4\log(4) &= 1 \\ 5\log(x_1) &= 1 - 4\log(4) \\ \log(x_1) &= \frac{1 - 4\log(4)}{5} \\ x_1 &= \exp\left(\frac{1 - 4\log(4)}{5}\right) \end{aligned}$$

and so

$$x_2 = 4\exp\left(\frac{1 - 4\log(4)}{5}\right)$$

And since $g(x) = 0$ at this point, and $\lambda = x_1 > 0$, the second order sufficiency condition is vacuously satisfied, so this point is a minimizer for f with an associated minimal value of

$$f\left(\exp\left(\frac{1 - 4\log(4)}{5}\right), \exp\left(\frac{1 - 4\log(4)}{5}\right)\right) = 5\exp\left(\frac{1 - 4\log(4)}{5}\right)$$

6. Let Q be an $n \times n$ symmetric matrix.

(i) Find all stationary points of the problem

$$\begin{aligned} \text{maximize} \quad & f(x) = x^T Q x \\ \text{subject to} \quad & x^T x = 1 \end{aligned}$$

Notice we may rewrite the constraint as $g(x) = x^T x - 1 = 0$. Using this, define the Lagrangian $\mathscr{L}(x, \lambda) = f(x) - \lambda^T g(x)$. Since there is only one constraint function $g(x)$, we have $\lambda \in \mathbb{R}$, so $\mathscr{L}(x, \lambda) = f(x) - \lambda g(x)$. For a stationary point of f over the given constraint, we require $\nabla_x \mathscr{L} = 0$. So

$$\begin{aligned} \nabla_x \mathscr{L}(x_*, \lambda_*) = \nabla f(x_*) - \lambda_* \nabla g(x_*) &= 0 \\ \nabla f(x_*) &= \lambda_* \nabla g(x_*) \\ Q x_* &= \lambda_* x_* \end{aligned}$$

That is, the stationary points of f over the constraint g are the (normalized) eigenvectors of Q.

(ii) Determine which of the stationary points are global maximizers.

Notice

$$\begin{aligned} f(x_*) &= x_*^T Q x_* \\ &= x_*^T (\lambda_*) x^* \\ &= \lambda_* x_*^T x_* \\ &= \lambda_* \end{aligned}$$

Then the maximizer of f is the eigenvector corresponding to the maximum eigenvalue of Q. Let v be the (normalized) eigenvector of Q that corresponds to the maximum eigenvalue of Q. Then v and $-v$ are maximizers to the optimization problem since $v^T v = 1$ and $(-v)^T(-v) = v^T v = 1$.

(iii) How do your results in part (i) change if the constraint is replaced by

$$x^T A x \leq 1,$$

where A is positive definite?

Since A is positive definite, we have that A is invertible. Let $g(x) = 1 - x^T A x = 0$ be the constraint function. Building our Lagrangian, we have

$$\mathscr{L}(x, \lambda) = f(x) - \lambda g(x)$$

and we require $\nabla_x \mathcal{L}(x, \lambda) = 0$ for a stationary point. Then

$$\begin{aligned} \nabla_x \mathscr{L}(x, \lambda) = Qx + \lambda A x &= 0 \\ Qx &= -A(\lambda x) \\ -A^{-1} Q x &= \lambda x \end{aligned}$$

That is, the eigenvectors of $-A^{-1}Q$ are stationary points for the problem with the new constraint.

7. Use the optimality conditions to find all local solutions to the problem

$$\begin{aligned} \text{minimize} \quad & f(x) = x_1 + x_2 \\ \text{subject to} \quad & (x_1 - 1)^2 + x_2^2 \leq 2 \\ & (x_1 + 1)^2 + x_2^2 \geq 2. \end{aligned}$$

To begin, let us rewrite the constraints to be of the "≥ 0" type. That is, the first constraint, call it $g_1(x)$, is

$$g_1(x) = 2 - (x_1 - 1)^2 - x_2^2 \geq 0$$

Similarly, for the second constraint, calling it $g_2(x)$, we have

$$g_2(x) = (x_1 + 1)^2 + x_2^2 - 2 \geq 0$$

Define the Lagrangian

$$\mathscr{L}(x, \lambda) = f(x) - \lambda^T g(x)$$

For a stationary point of f to exist on the given constraint, we require $\nabla_x \mathscr{L}(x, \lambda) = 0$. That is,

$$\begin{pmatrix} 1 + 2\lambda_1(x_1 - 1) - 2\lambda_2(x_1 + 1) \\ 1 + 2\lambda_1 x_2 - 2\lambda_2 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We now consider the following cases:

Case 1: Both constraints are inactive.
Then $\lambda_1 = \lambda_2 = 0$ and so $\nabla_x \mathscr{L} \neq 0$, so no stationary points exist in this case.

Case 2: The first constraint is active.
Then $\lambda_2 = 0$ and from $\nabla_x \mathscr{L}(x, \lambda) = 0$, we have

$$\begin{aligned} 1 &= -2\lambda_1(x_1 - 1) \\ 1 &= -2\lambda_1 x_2 \end{aligned}$$

From this, we have $x_2 = x_1 - 1$. Since the first constraint is active,

$$\begin{aligned} (x_1 - 1)^2 + x_2^2 &= 2 \\ 2x_2^2 &= 2 \\ x_2 &= \pm 1 \end{aligned}$$

Then we find the following points: $x = (2, 1)^T$ and $x = (0, -1)^T$. For $x = (2, 1)^T$, $\lambda_1 = -1/2 < 0$ so $(2, 1)^T$ is not optimal. Additionally, from the second constraint, we can see that $x = (0, -1)^T$ is infeasible.

Case 3: The second constraint is active.
Then $\lambda_1 = 0$ and from $\nabla_x \mathscr{L}(x, \lambda) = 0$, we have

$$\begin{aligned} 1 &= 2\lambda_2(x_1 + 1) \\ 1 &= 2\lambda_2 x_2 \end{aligned}$$

From this, we can see $x_1 + 1 = x_2$ and since the second constraint is active, we have

$$\begin{aligned} (x_1 + 1)^2 + x_2^2 &= 2 \\ 2x_2^2 &= 2 \\ x_2 &= \pm 1 \end{aligned}$$

Which gives us the following points: $x = (0, 1)^T$ and $x = (-2, -1)^T$. From the first constraint, $x = (-2, -1)^T$ is infeasible. For the first point, we find $\lambda_2 = 1/2$, so now we need to check the sufficient minimum condition. Notice that

$$\nabla^2_{xx} \mathscr{L}(x, \lambda) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\nabla g(x) = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

Since $\lambda_1 = 0$, the first constraint is degenerate, so we must find a null space basis matrix Z_+ for $(2, -2)^T$. Clearly, $Z_+ = (1, 1)^T$ will work. Now, let us check the second order sufficiency condition:

$$\begin{aligned} Z_+^T \nabla^2_{xx} \mathscr{L}(x, \lambda) Z_+ &= (1, 1) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (1, 1) \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ &= -2 < 0 \end{aligned}$$

So $x = (0, 1)^T$ is not a minimizer of f.

Case 4: Both constraints are active.
Then

$$(x_1 - 1)^2 + x_2^2 = 2$$
$$(x_1 + 1)^2 + x_2^2 = 2$$

Subtracting the second from the first, we have

$$(x_1 - 1)^2 = (x_1 + 1)^2$$

so

$$x_1 - 1 = \pm(x_1 + 1).$$

If $x_1 - 1 = x_1 + 1$, we find $2 = 0$, a contradiction. Then $x_1 - 1 = -x_1 - 1$, which gives us $x_1 = 0$. Then $x_2 = \pm 1$. From case 3, we saw $x = (0, 1)^T$ is not a minimizer so we must check $x = (0, -1)^T$. For this point, and the fact $\nabla_x \mathscr{L}(x, \lambda) = 0$, we find $\lambda_1 = 1/2$, $\lambda_2 = 0$. Then the second constraint is degenerate. With these values of λ, we have

$$\nabla^2_{xx} \mathscr{L}(x, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the second constraint is degenerate, we must find a null space matrix Z_+ for the second row of $\nabla g(x)$:

$$\nabla g(x) = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$$

Then $Z_+ = (1, 1)^T$, the same as in case 3. Finally, we must check the second order sufficiency condition:

$$\begin{aligned} Z_+ \nabla^2_{xx} \mathscr{L}(x, \lambda) Z &= (1, 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= (1, 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 2 > 0 \end{aligned}$$

So the second order sufficiency conditions are satisfied. So $x = (0, -1)^T$ is a minimizer for f with respect to the given constraints with an associated minimum value of

$$f(0, -1) = -1$$