Math 4510
Topology

Problem Set 6

1. Prove Theorem 4.4.3: For any subset A of a topological space X_τ, $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bdy}(A)$.

   Let A be a subset of a topological space X_τ and consider the case where $A = \emptyset$. Then $\mathrm{Cl}(A) = \emptyset$ since $\emptyset$ is both open and closed in τ. And since $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Cl}(A)$, we have that $\mathrm{Int}(A) = \emptyset$. Additionally, since $A = \emptyset$, we have that there are no points $x \in A$ such that a neighborhood V_x of x does not satisfy the conditions for $x \in \mathrm{Bd}(A)$. That is, $\mathrm{Bd}(A) = \emptyset$.

   In this case, $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$

   Now consider the case where $A \neq \emptyset$ and let $x \in \mathrm{Cl}(A)$. By definition, for some neighborhood V_x in x, $V_x \cap A \neq \emptyset$. If $V_x \subseteq A$, then $x \in \mathrm{Int}(A)$. If V_x is not a subset of A, then $V_x \cap A \neq \emptyset$ and $V_x \cap (X \setminus A) \neq \emptyset$, and so $x \in \mathrm{Bd}(A)$. Thus,

   $$\mathrm{Cl}(A) \subseteq \mathrm{Int}(A) \cup \mathrm{Bd}(A)$$

   Now let $x \in \mathrm{Int}(A) \cup \mathrm{Bd}(A)$. If $x \in \mathrm{Int}(A)$, then $x \in \mathrm{Cl}(A)$ since $\mathrm{Int}(A) \subseteq A \subseteq \mathrm{Bd}(A)$. If $x \in \mathrm{Bd}(A)$, then any neighborhood V_x of x contains points in A and $X \setminus A$. That is, $V_x \cap A \neq \emptyset$ and $V_x \cap (X \setminus A) \neq \emptyset$. That is, x is a limit point of A, and thus $x \in \mathrm{Cl}(A)$.

   So we have

   $$\mathrm{Int}(A) \cup \mathrm{Bd}(A) \subseteq \mathrm{Cl}(A)$$

   By double inclusion, we have

   $$\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$$

2. (#4 in 4.5) Let $A \subseteq X_\tau$ and let $f : X_\tau \to Y_\nu$ be continuous. If x is a limit point of A, must $f(x)$ be a limit point of $f(A) \subseteq Y$? Explain.

   No. Consider the function $f : \mathbb{R}_\mathcal{U} \to \mathbb{R}_\mathcal{U}$ defined by

   $$f(x) = 4$$

   and let $A = [0, 1]$. Notice that $A' = [0, 1]$ and that $f(A) = \{4\}$. Also notice that $f(A)' = \emptyset$.

   Notice that $\frac{1}{2}$ is a limit point of A, and that $f(\frac{1}{2}) = 4 \notin f(A)'$. But f is continuous because it is a constant function. More specifically, if $U \subset \mathbb{R}$ and $\{4\} \in U$, $f^{-1}(U) = \mathbb{R}$ which is open, and if $\{4\} \notin U$, $f^{-1}(U) = \emptyset$ which is open.

   So a limit point of A is not a limit point of $f(A)$.

3. (#2 in 5.2) Consider the product space $\mathbb{R}_\mathcal{L} \times \mathbb{R}_\mathcal{L}$.

(a) Sketch a typical basis set in $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$.

(b) Sketch several open sets in $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$.

(c) Sketch several sets which are not open in $\mathbb{R}_{\mathcal{L}} \times \mathbb{R}_{\mathcal{L}}$.

See attached sketches.

4. (#5 in 5.2) Prove Theorem 5.2.3: If X_τ and Y_σ are any topological spaces, with base-points $x_0 \in X$ and $y_0 \in Y$, then the inclusion maps

$$i_X : X_\tau \hookrightarrow X_\tau \times Y_\sigma$$

and

$$i_Y : Y_\sigma \hookrightarrow X_\tau \times Y_\sigma$$

are both continuous, where $X_\tau \times Y_\sigma$ denotes the Cartesian product endowed with the product topology.

(Hint: these are maps into a product space).

Let $i_X : X_\tau \hookrightarrow X_\tau \times Y_\sigma$ and $i_Y : Y_\sigma \hookrightarrow X_\tau \times Y_\sigma$ where $X_\tau \times Y_\sigma$ is the Cartesian product with the product topology.

Let U be τ-open and consider $(P_X \circ i_X)^{-1}(U)$:

$$\begin{aligned}
&(P_X \circ i_X)^{-1}(U) \\
&= (i_X^{-1} \circ P_X^{-1})(U) \\
&= i_X^{-1}(U \times Y) \\
&= U
\end{aligned}$$

which is open by assumption, so $P_X \circ i_X$ is continuous. Now we wish to show that $P_Y \circ i_X$ is continuous. Let V be σ-open and consider

$$\begin{aligned}
&(P_Y \circ i_X)^{-1}(V) \\
&= (i_X^{-1} \circ P_Y^{-1})(V) \\
&= i_X^{-1}(X \times V) \\
&= X
\end{aligned}$$

which is open by definition. So i_X is continuous.

Now we wish to show that i_Y is continuous. Following the same logic above, let U be τ-open and consider

$$\begin{aligned}
&(P_X \circ i_Y)^{-1}(U) \\
&= (i_Y^{-1} \circ P_X^{-1})(U) \\
&= i_Y^{-1}(U \times Y) \\
&= Y
\end{aligned}$$

which is open by definition. Now let V be σ-open and consider

$$(P_Y \circ i_Y)^{-1}(V)$$

$$= (i_Y^{-1} \circ P_Y^{-1})(V)$$
$$= i_Y^{-1}(X \times V)$$
$$= V$$

which is open by assumption. So i_Y is continuous.

5. Let $f : A \to B$ and $g : C \to D$ be continuous functions. Define a map $f \times g : A \times C \to B \times D$ by the equation
$$(f \times g)(a, c) = (f(a), g(c)).$$
Show that $f \times g$ is continuous.

Proof: Let $f : A \to B$ and $g : C \to D$ be continuous functions and let β be open in the product topology on $B \times D$. Also consider the projection maps

$$P_B : B \times D \to B$$

$$P_D : B \times D \to D$$

To show $f \times g$ is continuous, it suffices to show that $P_B \circ (f \times g)$ and $P_D \circ (f \times g)$ are continuous.

To begin, we will show that

$$(f \times g)^{-1} = f^{-1} \times g^{-1}$$

where
$$f^{-1} \times g^{-1} :$$

Let $V_1 \subseteq B$ and $V_2 \subseteq D$. Since $V_1 \subseteq B$, for some $U_1 \subseteq A$, we have that $f^{-1}(V_1) = U_1$, and similarly for $V_2 \subseteq D$, for some $U_2 \subseteq C$, $g^{-1}(V_2) = U_2$.

Let $\beta = V_1 \times V_2$ and $\alpha = U_1 \times U_2$.

We wish to show that $((f \times g) \circ (f^{-1} \times g^{-1}))(\beta) = \beta$ and $((f^{-1} \times g^{-1}) \circ (f \times g))(\alpha) = \alpha$.

Notice that
$$((f \times g) \circ (f^{-1} \times g^{-1}))(\beta)$$
$$= (f \times g) \circ (f^{-1}(V_1) \times g^{-1}(V_2))$$
$$= (f \times g)(U_1 \times U_2)$$
$$= (f(U_1) \times g(U_2)) = V_1 \times V_2 = \beta$$

Also notice that
$$((f^{-1} \times g^{-1}) \circ (f \times g))(\alpha)$$
$$= (f^{-1} \times g^{-1}) \circ (f(U_1) \times g(U_2))$$

$$= (f^{-1} \times g^{-1})(V_1 \times V_2)$$
$$= (f^{-1}(V_1) \times g^{-1}(V_2)) = U_1 \times U_2 = \alpha$$

So $(f \times g)^{-1} = (f^{-1} \times g^{-1})$. Now we wish to show that $P_B \circ (f \times g)$ is continuous. Let $U_3 \subseteq B$ and consider

$$(P_B \circ (f \times g))^{-1}(U_3)$$
$$= ((f \times g)^{-1} \circ P_B^{-1})(U_3)$$
$$= (f^{-1} \times g^{-1})(U_3 \times D)$$
$$= (f^{-1}(U_3) \times g^{-1}(D))$$
$$= V_3 \times C \subseteq A \times C$$

for some $V_3 \subseteq A$. So $P_B \circ (f \times g)$ is continuous.

Now let $V_4 \subseteq D$ and consider

$$(P_D \circ (f \times g))^{-1}(V_4)$$
$$= ((f \times g)^{-1} \circ P_D^{-1})(V_4)$$
$$= (f^{-1} \times g^{-1})(B \times V_4)$$
$$= (f^{-1}(B) \times g^{-1}(V_4)) = A \times U_4 \subseteq A \times C$$

for some $U_4 \subseteq C$. So $P_D \circ (f \times g)$ is continuous, and so $f \times g$ is continuous.

Bonus (#1 in 5.3) Prove that the basis for the box topology on $\prod X_\alpha$, $\mathcal{B} = \prod \tau_\alpha$, is in fact a basis. That is, show that it satisfies the two conditions of Definition 4.2.1.