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Homework V

Section 2.8 Problems

3. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

Soln. Let $x \in C[-1,1]$ and notice

$$\begin{aligned}
|f(x)| &= \left|\int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt\right| \\
&\leq \left|\int_{-1}^{0} x(t)dt\right| + \left|\int_{0}^{1} x(t)dt\right| \\
&\leq \int_{-1}^{0} |x(t)|dt + \int_{0}^{1} |x(t)|dt \\
&\leq \|x(t)\| \int_{-1}^{0} dt + \|x(t)\| \int_{0}^{1} dt \\
&= \|x(t)\| + \|x(t)\| \\
&= 2\|x(t)\|
\end{aligned}$$

so that $|f| \leq 2$. Now, define the sequence of functions $\{x_n(t)\}$ in $C[-1,1]$ by

$$x_n(t) = t^{\frac{1}{2n+1}}$$

and notice that $\|x_n(t)\| = 1$ for each n. Then we have

$$\begin{aligned}
f(x_n) &= \int_{-1}^{0} t^{\frac{1}{2n+1}}dt - \int_{0}^{1} t^{\frac{1}{2n+1}}dt \\
&= \frac{2n+1}{2n+2}\left[t^{\frac{2n+2}{2n+1}}\right]\Big|_{-1}^{0} - \frac{2n+1}{2n+2}\left[t^{\frac{2n+2}{2n+1}}\right]\Big|_{0}^{1} \\
&= -\frac{2n+1}{2n+2} - \frac{2n+1}{2n+2} \\
&= -2\left(\frac{2n+1}{2n+2}\right)
\end{aligned}$$

but since $\frac{2n+1}{2n+2} \to 1$ as $n \to \infty$ and $|f(x_n)| < 2$ for all n, we have for any positive number $\varepsilon > 0$, there exists a natural number N such that whenever $n > N$,

$$||f(x_n)| - 2| < \varepsilon$$

Thus,

$$|f| = 2$$

10. Show that in Prob. 9, two elements $x_1, x_2 \in X$ belong to the same element of the quotient space $X/\mathcal{N}(f)$ if and only if $f(x_1) = f(x_2)$; show that codim $\mathcal{N}(f) = 1$.

Proof: First suppose that $f(x_1) = f(x_2)$. By problem 9, we have that for a fixed $x_0 \in X\backslash\mathcal{N}(f)$, x_1, x_2 have the unique representations

$$x_1 = \alpha_1 x_0 + y_1$$
$$x_2 = \alpha_2 x_0 + y_2$$

where $y_1, y_2 \in \mathcal{N}(f)$. Then notice, since f is a linear functional

$$f(x_1) = \alpha_1 f(x_0)$$
$$f(x_2) = \alpha_2 f(x_0)$$

and since $f(x_1) = f(x_2)$, we have $\alpha_1 = \alpha_2$, so that x_1 and x_2 differ only by their null space component. Hence, x_1, x_2 belong to the coset

$$\alpha_1 x_0 + \mathcal{N}(f)$$

so that x_1, x_2 belong to the same element of the quotient space. Now suppose x_1, x_2 belong to the same element of the quotient space. That is, for some $x \in X\backslash\mathcal{N}(f)$, $x_1, x_2 \in x + \mathcal{N}(f)$. That is, there exists vectors $y_1, y_2 \in \mathcal{N}(f)$ such that

$$x_1 = x + y_1$$
$$x_2 = x + y_2$$

then we have

$$f(x_1) = f(x)$$
$$f(x_2) = f(x)$$

so that $f(x_1) = f(x_2)$.

We now wish to find the codimension of $\mathcal{N}(f)$, or $\dim(X/\mathcal{N}(f))$. Let $x \in X$. Then for a fixed $x_0 \in X\backslash\mathcal{N}(f)$, x has the unique representation

$$x = \alpha x_0 + y$$

for $y \in \mathcal{N}(f)$. Then x belongs to the element

$$\alpha x_0 + \mathcal{N}(f) = \{v \mid v = \alpha x_0 + y, y \in \mathcal{N}(f)\}$$

but any other element $z \in X$ can be written, for some β and $\tilde{y} \in \mathcal{N}(f)$,

$$z = \beta x_0 + \tilde{y}$$

Thus, $z \in \beta x_0 + \mathcal{N}(f)$. Then any vector in X is an element of a scalar multiple of the coset $x_0 + \mathcal{N}(f)$. Thus,

$$X/\mathcal{N}(f) = \text{span}\{x_0 + \mathcal{N}(f)\}$$

so that $\dim(X/\mathcal{N}(f)) = 1$.

Section 2.9 Problems

8. If Z is an $(n-1)$-dimensional subspace of an n-dimensional vector space X, show that Z is the null space of a suitable linear functional on X, which is uniquely determined to within a scalar multiple.

Proof: Let $E = \{e_1, \cdots, e_{n-1}\}$ be a basis for Z. Then since $\dim(X) = n$, there exists a vector $v \in X$ such that $V = \{e_1, \cdots, e_{n-1}, v\}$ forms a basis for V. Define the linear functional f_v with the property that $f_v(e_j) = 0$ for all $1 \leq j \leq n-1$ and $f_v(v) = 1$. Then for any $z = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{n-1} e_{n-1}$ in Z, we have that

$$\begin{aligned} f_v(z) &= f_v(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{n-1} e_{n-1}) \\ &= \alpha_1 f_v(e_1) + \alpha_2 f_v(e_2) + \cdots + \alpha_{n-1} f_v(e_{n-1}) \\ &= 0 + 0 + \cdots + 0 \\ &= 0. \end{aligned}$$

and for $x = \beta v$ in $X \backslash Z$, we have

$$f_v(x) = \beta f_v(v) = \beta$$

So that f_v is a linear functional on X with $\mathcal{N}(f) = Z$. Now suppose there exists some other linear function g with the property that $\mathcal{N}(g) = Z$. Then for any $x \in X \backslash Z$, we have $x = \beta v$ and so

$$\begin{aligned} g(x) &= g(\beta v) \\ &= \beta g(v) \end{aligned}$$

so that $g(x)$ differs from $f(x)$ by the scalar $g(v)$.

12. If $f_1, \cdots, f_p$ are linear functionals on an n-dimensional vector space X, where $p < n$, show that there is a vector $x \neq 0$ in X such that $f_1(x) = 0, \cdots, f_p(x) = 0$. What consequences does this result have with respect to linear equations?

Proof: To begin, we consider the mapping

$$T : x \mapsto (f_1(x), \cdots, f_p(x)) \in K^p$$

where K is the scalar field. Then $\mathcal{D}(T) = X$ by construction so that $\dim(\mathcal{D}(T)) = n$. Note that T is linear since each of $f_1, \cdots, f_p$ are linear. Suppose by way of contradiction that there exists no $x \neq 0$ in X such that $f_1(x) = 0, \cdots f_p(x) = 0$. Then the only vector where each $f_1, \cdots, f_p$ is equal to zero is the zero vector. Hence,

$$\mathcal{N}(T) = \{\mathbf{0}\}.$$

And so T is injective since its null space contains only the zero vector. And since T is injective, we have that T^{-1} exists. Then since $\dim(\mathcal{R}(T)) \leq p$, we have $\dim(\mathcal{D}(T)) = \dim(\mathcal{R}(T)) \leq p$. So we have

$$n \leq p < n$$

a contradiction.

What this tells us is, if we have $p < n$ linear functionals in an n-dimensional space, then the linear system corresponding to said functionals will have a nontrivial null space. That is, suppose we write $f_i = (\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_n^{(i)})$ for $1 \leq i \leq p$ and $x = (\xi_1, \xi_2, \cdots, \xi_n)$ where $Tx = 0$ so that the mapping T on x has the following form:

$$\begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} & \cdots & \alpha_n^{(1)} \\ \alpha_1^{(2)} & \alpha_2^{(2)} & \cdots & \alpha_n^{(2)} \\ \vdots & \vdots & & \vdots \\ \alpha_1^{(p)} & \alpha_2^{(p)} & \cdots & \alpha_n^{(p)} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \mathbf{0}$$

will have a nontrivial solution.

Section 2.10 Problems

6. If X is the space of ordered n-tuples of real numbers and $\|x\| = \max_j |\xi_j|$, where $x = (\xi_1, \cdots, \xi_n)$, what is the corresponding norm on the dual space X'?

Proof: Let f be a linear functional on X and suppose we write f as

$$f = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

we wish to find the norm on f. Let $x = (\xi_1, \xi_2, \cdots, \xi_n)$ be such that $\|x\| = 1$. Then

$$\begin{aligned} |f(x)| &= |\alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_n \xi_n| \\ &\leq |\alpha_1 \xi_1| + |\alpha_2 \xi_2| + \cdots + |\alpha_n \xi_n| \\ &= |\alpha_1||\xi_1| + |\alpha_2||\xi_2| + \cdots + |\alpha_n||\xi_n| \\ &\leq |\alpha_1| \max_j |\xi_j| + |\alpha_2| \max_j |\xi_j| + \cdots + |\alpha_n| \max_j |\xi_j| \\ &= |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \end{aligned}$$

so we have

$$|f(x)| \leq |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|.$$

For a lower bound, take $y = (\eta_1, \eta_2, \cdots, \eta_n)$ in X defined by

$$\eta_j = \begin{cases} 1, & \alpha_j \geq 0 \\ -1, & \alpha_j < 0 \end{cases}$$

and note that $\|y\| = 1$. Then

$$f(y) = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$$

so that

$$\|f\| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$$

Then the norm on the dual space is the "one norm," defined in the above equation.