Homework 4 (Analysis)

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1. Let K be a nonempty sequentially compact subspace of a metric space (X, d).

   (a) Let p_0 be a point in K. Prove that there exists a number $M > 0$ such that K is contained in the open ball $\mathcal{B}_M(p_0)$ of radius M about the point p_0.
   Proof: Since K is sequentially compact, we have that K is compact. And since K is compact, K is totally bounded and is therefore bounded. Let

   $$M = \sup\{d(x_1, x_2) \mid x_1, x_2 \in K\}$$

   Essentially, M is the diameter of the set K. + Since K is bounded, $M < \infty$. Now consider $B_M(p_0)$, the open ball of radius M centered at p_0. By the definition of M above, we have that

   $$K \subseteq B_M(p_0)$$

   (b) Let $\mathcal{O}$ be an open set in X that contains K. Prove that there exists an $r > 0$ such that for every point p in K the open ball $\mathcal{B}_r(p)$ is contained in $\mathcal{O}$.
   Proof: Let $\mathcal{O}$ be an open set that contains K and suppose by way of contradiction that there does not exist an $r > 0$ such that for every point $p \in K$, $B_r(p) \subseteq \mathcal{O}$. Let $\{x_n\}$ be a sequence in K. Since K is sequentially compact, we have that there is a convergent subsequence of $\{x_n\}$, $\{x_{n_k}\}$ that converges to some $x_0 \in K$.
   Since there does not exist an $r > 0$ such that $B_r(x_0)$ is contained in $\mathcal{O}$, we have that for every n_k, $B_{1/n_k}(x_0)$ is not contained in $\mathcal{O}$. But since $\mathcal{O}$ is open, there exists some $\epsilon > 0$ such that $B_\epsilon(x_0) \subseteq \mathcal{O}$. But for $n_k > N \in \mathbb{N}$, the Archimedean property gives us that

   $$\frac{1}{n_k} < \epsilon$$

   That is, $B_\epsilon(x_0)$ is not contained in $\mathcal{O}$. Then since $\mathcal{O}$ is open, we have that $x_0 \notin \mathcal{O}$. But $x_0 \in K$ and $K \subseteq \mathcal{O}$, x_0 must be in $\mathcal{O}$, a contradiction.
   Thus, for every point $p \in K$, there exists an $r > 0$ such that $B_r(x_0) \subseteq \mathcal{O}$.

2. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $f(\mathbf{x}) \geq ||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$. (Here $|| \cdot ||$ denote the Euclidean norm on $\mathbb{R}^n$). Prove that the inverse image $f^{-1}[0, 1]$ is a compact subset of $\mathbb{R}^n$.
   Proof: Notice that $[0, 1]$ is a closed subset of $\mathbb{R}$, and since f is continuous, we have that $f^{-1}([0, 1])$ is a closed subset of $\mathbb{R}^n$.
   Let $A = f^{-1}([0, 1])$ and $A_i \in A$, $i \in I$ and let $a = \max\{||A_i|| \mid A_i \in A, i \in I\}$. Since $f(A) = [0, 1]$, we have $a \leq \max\{[0, 1]\}$, $0 \leq a \leq 1$, so A is bounded. Then by the Heine-Borel Theorem, we have that $f^{-1}([0, 1])$ is a compact subset of $\mathbb{R}^n$.

   (b) Prove that $A = \{(x, \tan(x)) : 0 \leq x < \pi/2\}$ is closed in $\mathbb{R}^2$, but A is not sequentially compact.
   Proof: Notice that as $x \to \pi/2$, $\tan(x) \to \infty$. That is, A is unbounded above. Since A is unbounded, the Heine-Borel Theorem gives us that A cannot be compact, and is thus not sequentially compact. Now we must show that A is closed in $\mathbb{R}^2$.
   Let $\{x_n\}$ be a convergent sequence on $[0, \pi/2)$. That is, $x_n \to x_0$ for some $x_0 \in [0, \pi/2)$. Since $\tan(x)$ is continuous on $[0, \pi/2)$, we have that $\tan(x_n) \to \tan(x_0)$. That is, $(x_0, \tan(x_0)) \in A$, so A is closed.

3. (a) Let (X, d) be a metric space. Prove that X is sequentially compact if and only if X satisfies *both* of the following properties:

(P1) X is a complete metric space.

(P2) Every sequence $\{x_n\}$ in X has a Cauchy subsequence.

Proof: First let X be sequentially compact. That is, every sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$ where $x_{n_k} \to x_{n_0} \in X$. Since $\{x_{n_k}\}$ converges, $\{x_{n_k}\}$ is a Cauchy sequence, and so (P2) is satisfied.

Now suppose that X does not have a convergent Cauchy subsequence. That is, suppose that X is compact but not complete.

Fix $\epsilon > 0$ and let $y \in X$ and $\{x_n\}$ a Cauchy sequence in X. Then $\{x_n\}$ does not converge to y, and so for $n > N \in \mathbb{N}$,

$$d(x_n, y) \geq \epsilon$$

That is, the open ball of radius ϵ contains finitely many points in $\{x_n\}$.

Now let $\epsilon_0 > 0$ depend on the choice for y. Then we have a cover for X:

$$X = \bigcup \{B_{\epsilon_0}(y) \mid y \in X\}$$

and since X is compact, we have that there exists a finite subcover for the above cover:

$$X = \bigcup_{i=1}^{n} \{B_{\epsilon_0}(y_i) \mid y_i \in X\}$$

And since each $B_{\epsilon_0}(y_i)$ contains finitely many points in $\{x_n\}$ and we have a finite subcover for X, X must contain a finite number of points in $\{x_n\}$. But since $\{x_n\}$ is a Cauchy sequence in X, this cannot happen.

So we have that X is complete.

Now suppose that X satisfies (P1) and (P2). By (P2) we have that every sequence in X contains a Cauchy subsequence, and by (P1), we have that X is complete, so we must have that every Cauchy sequence in X converges to some point in X. That is, by definition, X is sequentially compact.

(b) Let (X, d) be a sequentially compact metric space. Suppose $f : X \to \mathbb{R}$ is a continuous function with the property: for each $x \in X$, there exists $x' \in X$ such that $|f(x')| \leq \frac{1}{2}|f(x)|$. Prove that there exists a point $x_0 \in X$ such that $f(x_0) = 0$.

Proof: Let $\{x_n\}$ be a sequence in X such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$. Since X is sequentially compact, we have that there exists a convergent subsequence of $\{x_n\}$, call it $\{x_{n_k}\}$ that converges to some $x_0 \in X$. Since f is continuous, we have that $f(X)$ is sequentially compact, and so the sequence $f(x_{n_k})$ has a convergent subsequence. Since f is continuous, $f(x_{n_k})$ converges to x_0.

Now, using the recursion relation we defined above, notice the following:

$$|f(x_2)| \leq \frac{1}{2}|f(x_1)|$$

$$|f(x_3)| \leq \frac{1}{2}|f(x_2)| \leq \frac{1}{4}|f(x_1)|$$

And continuing up to some $n + 1$, we'll find

$$|f(x_{n+1})| \leq \frac{1}{2^n}|f(x_1)|$$

And letting $n \to \infty$, notice

$$|f(x_0)| \leq 0$$

Since $\frac{1}{2^n}|f(x_1)| \to 0$ as $n \to \infty$ since $f(x_1)$ is a fixed value. That is, we have for some x_0, $f(x_0) = 0$.

4. Let (X, d) be a metric space. Define the real valued function $f(x) := d(z_0, x), x \in X$ for any fixed $z_0 \in X$.

   (a) Prove that $f(x)$ is uniformly continuous on X.
   Proof: Fix $\epsilon > 0$ and let $x, y \in X$ such that $d(x, y) < \delta$ for some $\delta > 0$ and consider

   $$|f(x) - f(y)| = |d(z_0, x) - d(z_0, y)|$$

   by the reverse triangle inequality, we have

   $$|d(z_0, x) - d(z_0, y)| \leq d(x, y) < \delta$$

   choose $\delta = \epsilon$. Then we have

   $$|f(x) - f(y)| < \epsilon$$

   so $f(x)$ is uniformly continuous by definition.

   (b) Let $K \subset X$ be a non-empty, compact subset of the metric space (X, d). Using the basic properties of compactness and the result of part (a) prove that $\exists\, x_0 \in K$ such that $d(z_0, x_0) = \inf_{x \in K} d(z_0, x)$.
   Proof: We have $K \subset X$ is a compact subset. From part (a), we have that $f(x) = d(z_0, x), x \in X$ for any fixed $z_0 \in X$ is uniformly continuous. Since f is uniformly continuous, f is continuous. Then since f is continuous and K is compact, we have that f posses the extreme value property on K, and so for some $x_0 \in K$,

   $$f(x_0) = \inf_{x \in K} f(x)$$

   Or equivalently,

   $$d(z_0, x_0) = \inf_{x \in K} d(z_0, x)$$

5. (a) Prove that an open, connected subset of $\mathbb{R}^n$ is path connected.
   To start, I will prove as a lemma that an open ball is path connected.
   Lemma: For some $x_0 \in A$, $r > 0$, $B_r(x_0)$ is path connected.
   Proof: Let $r > 0$, $x_0, x_1 \in A$ and $B_r(x_0)$ be an open ball in A. Then the function

   $$f : [0, 1] \to B_r(x_0)$$

   given by

   $$f(t) = tx_1 + (1 - t)x_0$$

   is a path joining x_0 and x_1 in $B_r(x_0)$.

   Proof of (a): Let A be an open, connected subset of $\mathbb{R}^n$ and $x, y, z \in A$. Let $\Omega \subseteq A$ be the set of all points that can be connected to x with a path and let $y \in \Omega$. We wish to show that Ω is open. Since A is open, there exists an $r > 0$ such that $B_r(x) \subseteq A$. But from the lemma above, we have that $B_r(x)$ is path connected, so for any $z \in B_r(x)$, y can be joined to z by a path, and hence can be joined to x by a path. Since this holds for any $x \in \Omega$, we have that Ω is open. Now let $\Gamma = A \setminus \Omega$. We wish to show that Γ is also open. Well, let $w \in \Gamma$. Then for some $r > 0$, $B_r(w) \subseteq A$. Let $p \in B_r(w)$. Since $B_r(w)$ is path connected by the lemma, p cannot be joined to x with a path. However, p can be joined to w, and by similar logic above, we have that Γ is open. Clearly, we have

   $$\Omega \cap \Gamma = \emptyset$$

   and

   $$\Omega \cup \Gamma = A$$

   But since $x \in \Omega$, we have that $\Omega \neq \emptyset$ and since A is connected, $\Gamma = \emptyset$. Hence, $\Omega = A$ and A is path connected.
   Thus, any open, connected subset of $\mathbb{R}^n$ is connected.

   (b) Prove that a real continuous function on a closed interval $I \subset \mathbb{R}^2$ cannot be one-to-one.