Homework 1

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1.1 Problems

6. Show that d in **1.1-6** satisfies the triangle inequality.

Proof: Let X be the set of all bounded sequences and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$$

where x_i, y_i are the i^{th} elements of x, $y \in X$, respectively.

Let $x, y, z \in X$. Then for any i, using the triangle inequality for $|\cdot|$, we have

$$\begin{aligned} |x_i - y_i| &= |x_i - z_i + z_i - y_i| \\ &\leq |x_i - z_i| + |z_i - y_i|. \end{aligned}$$

Notice by definition of supremum,

$$\begin{aligned} |x_i - z_i| &\leq \sup_{i\in\mathbb{N}} |x_i - z_i| \\ |z_i - y_i| &\leq \sup_{i\in\mathbb{N}} |z_i - y_i|. \end{aligned}$$

So then

$$\begin{aligned} |x_i - y_i| &\leq \sup_{i\in\mathbb{N}} |x_i - z_i| + \sup_{i\in\mathbb{N}} |z_i - y_i| \\ &= d(x,z) + d(z,y) \end{aligned}$$

then $|x_i - y_i|$ is bounded above by $d(x,z) + d(z,y)$ for all i, hence

$$\begin{aligned} \sup_{i\in\mathbb{N}} |x_i - y_i| &\leq d(x,z) + d(z,y) \\ d(x,y) &\leq d(x,z) + d(z,y). \end{aligned}$$

Hence, d satisfies the triangle inequality.

12. **(Triangle inequality)** The triangle inequality has several usefull consequences. For instance, using (1), show that

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$$

Proof: Let (X,d) be a metric space and $x, y, z, w \in X$. By the triangle inequality, we have

$$d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$$

so that

$$\begin{aligned}|d(x,y)-d(z,w)| &= |d(x,z)+d(z,w)+d(w,y)-d(z,w)| \\ &= |d(x,z)+d(y,w)|.\end{aligned}$$

Since $d(x,z) \geq 0$, $d(y,w) \geq 0$,

$$|d(x,z)+d(w,y)| = d(x,z)+d(y,w)$$

hence,

$$|d(x,y)-d(z,w)| \leq d(x,z)+d(y,w).$$

1.2 Problems

4. **(Space l^p)** Find a sequence which converges to 0, but is not in any space l^p, where $1 \leq p < +\infty$.

Consider the sequence of real numbers $\{x_n\}$ defined by

$$x_n = \frac{1}{\ln(n+1)}.$$

Notice since $\ln(n+1) \to \infty$ as $n \to \infty$, so $\frac{1}{\ln(1+n)} \to 0$ as $n \to \infty$. We will show that the series

$$\sum_{n=1}^{\infty} \frac{1}{|\ln(1+n)|^p}$$

diverges for all natural numbers $1 \leq p < +\infty$. Recall that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

for all natural numbers n. Hence, there exists some real number x_0 such that for all $x > x_0$,

$$x^n < e^x.$$

Take $x = \ln(y+1) > x_0$ for $y+1 > e^{x_0} = y_0$. Then by the above inequality, we have

$$(\ln(y+1))^n < y+1$$

so

$$\frac{1}{1+y} < \frac{1}{(\ln(1+y))^n}$$

for all $y > y_0$. Then

$$\sum_{k=\lceil y_0 \rceil}^{\infty} \frac{1}{1+y} < \sum_{k=\lceil y_0 \rceil}^{\infty} \frac{1}{(\ln(1+y))^n} < \sum_{k=1}^{\infty} \frac{1}{(\ln(1+y))^n}.$$

Since $\sum_{k=\lceil k_0 \rceil}^{\infty} \frac{1}{1+y}$ diverges, by direct comparison, we have

$$\sum_{k=1}^{\infty} \frac{1}{(\ln(1+y))^n}$$

diverges for all natural numbers n.

1.3 Problems

8. Show that the closure $\overline{B(x_0;r)}$ of an open ball $B(x_0;r)$ in a metric space can differ from the closed ball $\overline{B}(x_0;r)$.

Proof: Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ denotes the set of rational real numbers and $d(x,y) = |x-y|$. Let $x \in \mathbb{Q}$, $r > 0$ and consider the open ball of radius r centered at x,

$$B_r(x) = \{y \in \mathbb{Q} \mid d(x,y) < r\}.$$

Then the closed ball of radius r centered at x is given by

$$\overline{B}_r(x) = \{y \in \mathbb{Q} \mid d(x,y) \leq r\}.$$

However, since the set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$, the closure of the open ball of radius r is given by

$$\overline{B_r(x)} = \{y \in \mathbb{R} \mid d(x,y) \leq r\}.$$

Hence, $\overline{B_r(x)} \neq \overline{B}_r(x)$ since $\overline{B_r(x)}$ contains all irrational numbers in the interval $[x-r, x+r]$, but $\overline{B}_r(x)$ contains no irrational numbers in the interval $[x-r, x+r]$.

1.4 Problems

6. If (x_n) and (y_n) are Cauchy sequences in a metric space (X,d), show that (a_n), where $a_n = d(x_n, y_n)$, converges. Give illustrative examples.

Proof: Let (X,d) be a metric space and $\{x_n\}$, $\{y_n\}$ be Cauchy sequences in (X,d). Define the sequence of real numbers $\{a_n\}$ by $a_n = d(x_n, y_n)$. Since $\mathbb{R}$ is a complete metric space, we will show that $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$ and is therefore convergent.
Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,d), there exist natural numbers N_1, N_2 such that whenever $n,m > N_1$,

$$d(x_n, x_m) < \frac{\epsilon}{2}$$

and similarly, whenever $n,m > N_2$,

$$d(y_n, y_m) < \frac{\epsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$ and $n,m > N$ and consider

$$\begin{aligned}
|a_n - a_m| &= |d(x_n, y_n) - d(x_m, y_m)| \\
&\leq |d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) - d(x_m, y_m)| \\
&= |d(x_n, x_m) + d(y_n, y_m)| \\
&= d(x_n, x_m) + d(y_n, y_m) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

We then have

$$|a_n - a_m| < \epsilon.$$

Hence, $\{a_n\}$ is a Cauchy sequence in $\mathbb{R}$.

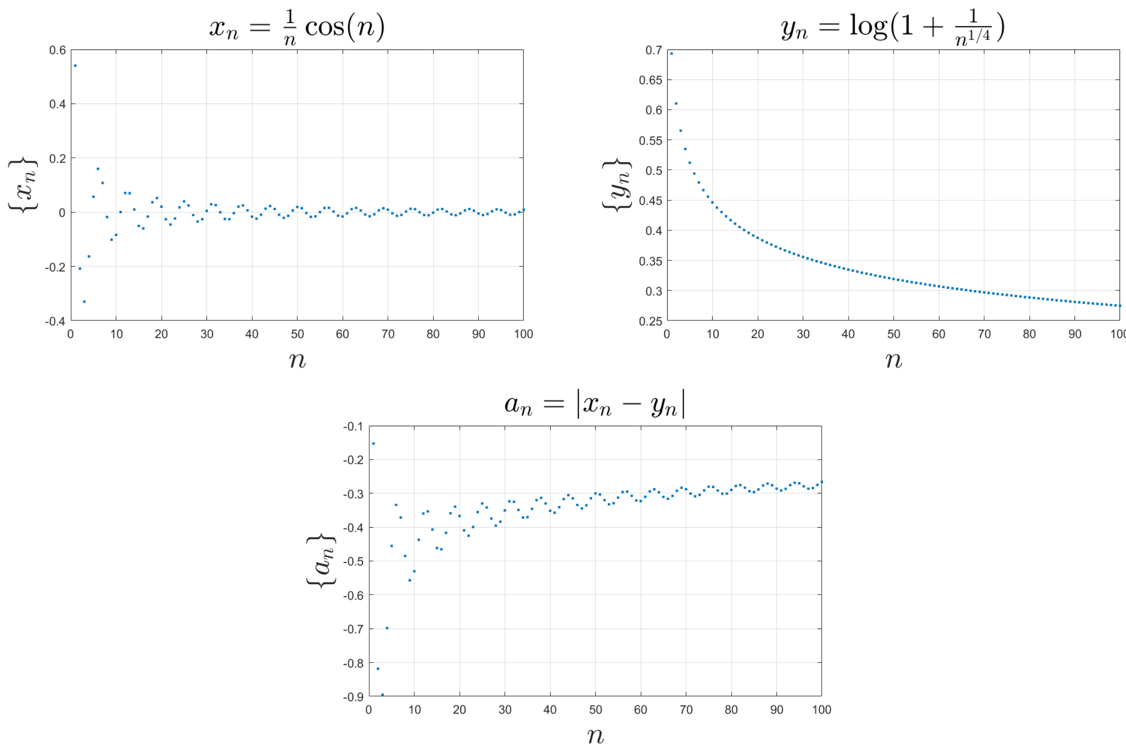
As an example, consider the metric space $(\mathbb{R}, |\cdot|)$ and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{aligned}
x_n &= \frac{1}{n}\cos(n) \\
y_n &= \log\left(1 + \frac{1}{n^{1/4}}\right)
\end{aligned}$$

Clearly, these sequences converge in $\mathbb{R}$ and are therefore Cauchy by the completeness of $\mathbb{R}$. Define the sequence $\{a_n\}$ by

$$a_n = |x_n - y_n| = \left| \frac{1}{n}\cos(n) - \log\left(1 + \frac{1}{n^{1/4}}\right) \right|$$

From our work above, we have that $\{a_n\}$ converges in $\mathbb{R}$. An illustration of these sequences can be found in the following figure:



For an example in an incomplete metric space, consider the space of polynomials on the interval $[-1, 1]$ equipped with the sup metric, denote this space by $P[-1, 1]$. Consider the Cauchy sequences $\{y_n\}$ and $\{z_n\}$ defined by

$$y_n = \sum_{k=1}^{n} \frac{x^k}{k!}$$

and

$$z_n = \sum_{k=0}^{n} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

Clearly, $y_n \to e^x$ and $z_n \to \sin(\pi x)$ in $C[-1, 1]$, but e^x and $\sin(\pi x)$ not in $P[-1, 1]$. Hence, $P[-1, 1]$ is an incomplete metric space. Now define the sequence $\{a_n\}$ of real numbers by $a_n = d(y_n, z_n) = \sup_{x \in [-1,1]} |y_n - z_n|$. By our work above, we know that $\{a_n\}$ converges. For a visualization, see the figure below.

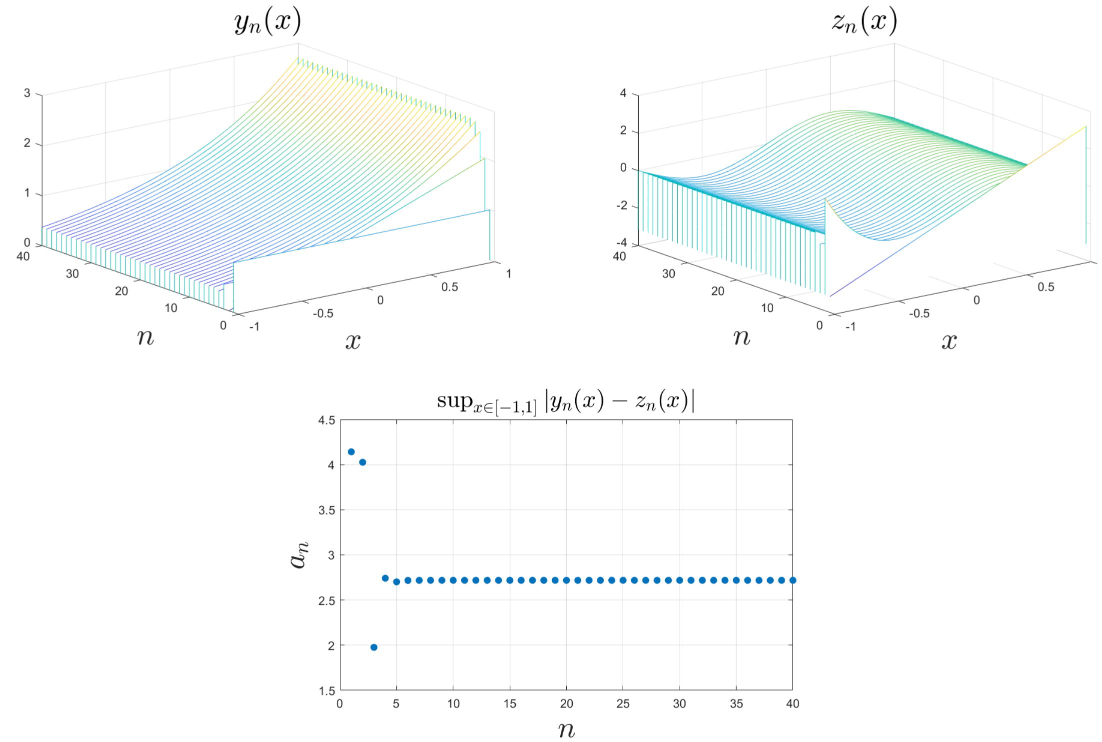


Figure 1: Plots for $y_n = \sum_{k=0}^{n} \frac{x^k}{k!}$, $z_n = \sum_{k=0}^{n} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$ and the sequence of their maximum difference, a_n, for various values of n.