Homework 3

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Section 2.6 Problems

1. Determine if the following functions satisfy a local or uniform Lipchitz condition.

   (a) $|y|$

   Soln. $f(y) = |y|$ is globally Lipschitz. To see this, let $x, y \in \mathbb{R}$ and notice

   $$||x| - |y|| \leq |x - y|$$

   by the reverse triangle inequality. Thus, $|y|$ is globally Lipschitz with Lipschitz constant $L = 1$.

   (b) $\tan^{-1}(y)$

   Soln. $f(y) = \tan^{-1}(y)$ is globally Lipschitz. To see this, let $x, y \in \mathbb{R}$. By the Mean Value Theorem, we have that there exists a $c \in (x, y)$ (or (y, x)) such that

   $$|\tan^{-1}(x) - \tan^{-1}(y)| = |f'(c)||x - y|.$$

   but

   $$f'(c) = \frac{1}{1 + c^2}$$

   which is clearly bounded above by 1 for all $c \in \mathbb{R}$. That is,

   $$|\tan^{-1}(x) - \tan^{-1}(y)| \leq |x - y|$$

   so that $\tan^{-1}(y)$ is globally Lipschitz with Lipschitz constant $L = 1$.

   (c) $\frac{t^2 y}{1+y^2}$

   Soln. Notice that $f(t, y) = \frac{t^2 y}{1+y^2}$ is not globally Lipschitz in t since for any fixed y, $f(t, y) = at^2$ with $a = \frac{y}{1+y^2}$, at^2 is not a globally Lipchitz function. However, on a compact interval in t, t^2 is Lipschitz since t^2 is continuous. Now, $f(t, y)$ is globally Lipschitz in y. To see this, by the Mean Value Theorem, we have that

   $$|f(t, b) - f(t, a)| = |f_y(t, c)||b - a|$$

   for some $c \in (a, b)$ (or (b, a)). We must find a bound for $f_y(t, c)$. To do so, we will find the critical points of $f_y(t, c)$. Notice

   $$f_{yy}(t, y) = \frac{2y(y^2 - 3)}{(1 + y^2)^3} = 0$$
   $$\implies 2y(y^2 - 3) = 0$$

   thus, $y = 0, \pm\sqrt{3}$ are the critical points of f_y. Plugging these in, we find

   $$f_y(t, 0) = t^2,\ f_y(t, \sqrt{3}) = \frac{\sqrt{3}}{16}t^2,\ f_y(t, -\sqrt{3}) = -\frac{\sqrt{3}}{16}t^2$$

   so that for any fixed t, $|f_y(t, y)| \leq t^2$. Hence, f is globally Lipschitz.

2. Find the maximal interval of existence for the differential equation

$$y' = \frac{1}{1+y^2}, \;\; y(0) = 0.$$

Show that the exact solution supports your conclusion.

Soln. I claim that $f(y) = \frac{1}{1+y^2}$ is globally Lipschitz.

Proof: Let $a, b \in \mathbb{R}$. By the mean value theorem, we have that there exists some $c \in [a, b]$ (or $[b, a]$) such that

$$|f(b) - f(a)| = |f'(c)||b - a|.$$

We wish to find a bound for $f'(y)$ for all $y \in \mathbb{R}$. To do so, let us first find any critical points:

$$f''(y) = \frac{1-3y^2}{(1+y^2)^3} = 0$$

which gives us $x = \pm\frac{1}{\sqrt{3}}$. Note that

$$f'\left(\frac{1}{\sqrt{3}}\right) = -f'\left(-\frac{1}{\sqrt{3}}\right) = -\frac{3\sqrt{3}}{8}$$

and since

$$|f'(y)| \to 0 \text{ as } y \to \infty$$

we have that

$$|f'(y)| \leq \frac{3\sqrt{3}}{8}$$

for all $y \in \mathbb{R}$. Thus,

$$|f(b) - f(a)| \leq \frac{3\sqrt{3}}{8}|b - a|$$

so that f is globally Lipschitz with Lipschitz constant $L = \frac{3\sqrt{3}}{8}$. Hence, a unique global solution exists.

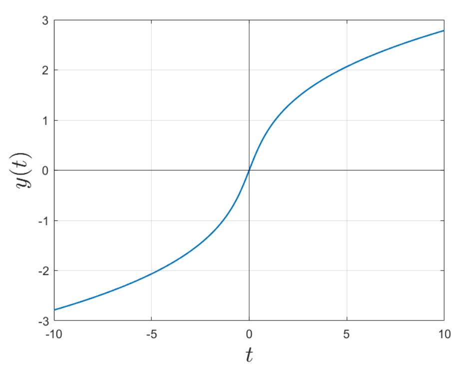
Inspecting the explicit solution, via separation of variables, we find

$$y^3 + 3y - 3t = 0$$

and by Cardano's formula, we have

$$y = \sqrt[3]{\frac{3}{2}t + \sqrt{\frac{9}{4}t^2 + 1}} + \sqrt[3]{\frac{3}{2}t - \sqrt{\frac{9}{4}t^2 + 1}}$$

which is clearly defined for all t. Below is a plot of the solution on $-10 \leq t \leq 10$:



Section 2.7 Problems

3. Compute the solution of the initial value problem $\dot{x} = t - x, x(0) = 1$, using the Euler and Runge-Kutta algorithms. Compare the results with the exact solution. Starting with $h = 0.1$, decrease h and show that the methods are converging in an expected manner.

   Soln. Implementing Euler's method and RK4 into MATLAB to solve the system on the interval $[0, 2]$, we find the following error plots for decreasing step sizes:

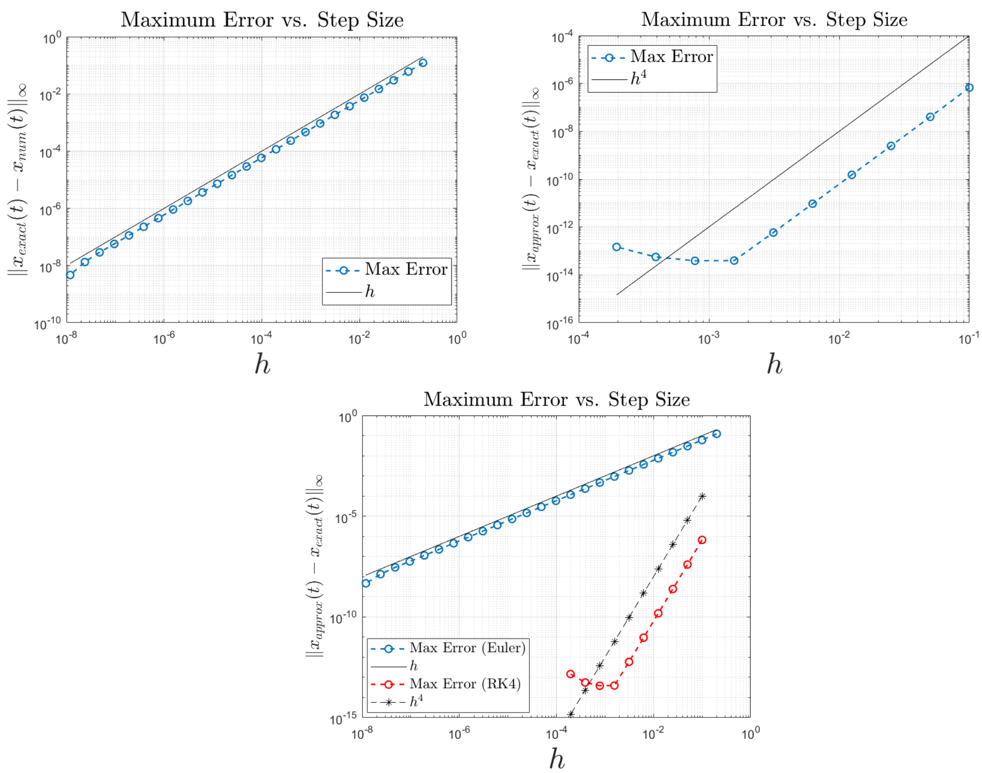


Figure 1: Maximum errors versus step size for Euler and Rk4 methods, with them plotted together.

   and the following plots of the numerical solution with the exact solution:

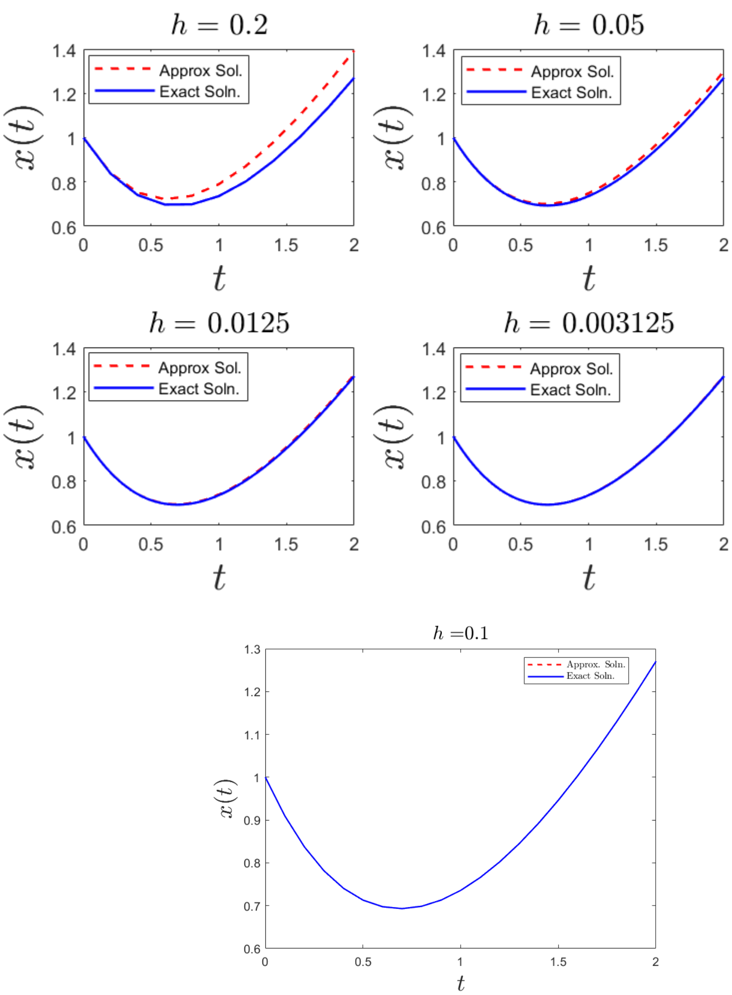


Figure 2: Top: Convergence of the Euler method for decreasing step sizes. Bottom: Approximate and Exact solution with RK4.

Section 3.1 Problems

4. **Grads only:** Show that the space of $n \times n$ matrices $\mathbb{C}^{n\times n}$ together with the matrix norm is a Banach space. Show (3.9).

   Proof: To begin, we will show that the operator norm is indeed a norm. Let X be a normed space and let $T : X \to X$ be a bounded linear operator on X. Define

$$\|T\| = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\|.$$

   Note that $\|T\| \geq 0$ by definition of a norm on X. Now suppose $\|T\| = 0$. Then

$$\sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\| = 0$$

so that

$$0 \le \|Tx\| \le 0$$

for all unit vectors $x \in X$. Thus, $T = 0$. Now let α be an arbitrary scalar and consider $\|\alpha T\|$. Notice

$$\begin{aligned}
\|\alpha T\| &= \sup_{\substack{x\in X\\ \|x\|=1}} \|\alpha(Tx)\| && \text{(Def. scalar mult. of ops.)}\\
&= \sup_{\substack{x\in X\\ \|x\|=1}} |\alpha|\|Tx\| && \text{(Homogeneity)}\\
&= |\alpha| \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\|\\
&= |\alpha|\|T\|
\end{aligned}$$

so homogeneity holds. Now, let $T, S : X \to X$ be bounded linear operators on X. Consider $\|T + S\|$:

$$\begin{aligned}
\|T+S\| &= \sup_{\substack{x\in X\\ \|x\|=1}} \|(T+S)(x)\| && \text{(Def. add. of linear op.)}\\
&= \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx+Sx\| && \text{(Linearity)}\\
&\le \sup_{\substack{x\in X\\ \|x\|=1}} (\|Tx\|+\|Sx\|)\\
&\le \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\| + \sup_{\substack{x\in X\\ \|x\|=1}} \|Sx\|\\
&= \|T\|+\|S\|
\end{aligned}$$

so that the triangle inequality holds. Thus, the operator norm is indeed a norm.

Now we will show that the operator norm on an $n \times n$ matrix over $\mathbb{C}$ satisfies the following inequality:

$$\max_{j,k} |A_{jk}| \le \|A\| \le n \max_{j,k} |A_{jk}|$$

Let $x = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in $\mathbb{C}^n$ be such that $\|x\| = 1$. Then Notice

$$\begin{aligned}
\|Ax\| &= \left\| \left(\sum_{k=1}^{n} A_{1k}\alpha_k, \sum_{k=1}^{n} A_{2k}\alpha_k, \cdots, \sum_{k=1}^{n} A_{nk}\alpha_k \right) \right\|\\
&\le \sum_{k=1}^{n} \|(A_{1k}\alpha_k, A_{2k}\alpha_k, \cdots, A_{nk}\alpha_k)\|\\
&\le \max_{j,k} |A_{jk}| \sum_{k=1}^{n} \|x\|\\
&= n \max_{j,k} |A_{jk}|.
\end{aligned}$$

And similarly, notice

$$\begin{aligned}
\|Ax\| &= \left\| \left(\sum_{k=1}^{n} A_{1k}\alpha_k, \sum_{k=1}^{n} A_{2k}\alpha_k, \cdots, \sum_{k=1}^{n} A_{nk}\alpha_k \right) \right\|\\
&\ge \max_{j,k} |A_{jk}| \|(\alpha_1, \alpha_2, \cdots, \alpha_n)\|\\
&= \max_{j,k} |A_{jk}| \|x\|\\
&= \max_{j,k} |A_{jk}|
\end{aligned}$$

so we have

$$\max_{j,k} |A_{j,k}| \leq \|Ax\| \leq n \max_{j,k} |A_{j,k}|$$

hence, for A to converge in $\mathbb{C}^{n\times n}$, it must be the case that each element of A converges in $\mathbb{C}$. That is, if we have a Cauchy sequence of matrices $\{A_n\}$ in $\mathbb{C}^{n\times n}$, since $\mathbb{C}$ is complete, we have that each element-wise sequence of $\{A_n\}$ converges in $\mathbb{C}$, hence $A_n \to A \in \mathbb{C}^{n\times n}$.

5. Find a Jordan canonical form for the following matrix

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$$

Soln. Let us begin by finding the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 1 & 0 \\ 1 & -3-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} \\ &= (-3-\lambda)\begin{vmatrix} -3-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 0 & -3-\lambda \end{vmatrix} \\ &= (\lambda+3)[-((\lambda+3)^2+1)+1] \\ &= (\lambda+3)[-(\lambda+3)^2-1+1] \\ &= -(\lambda+3)^3 \\ &= 0 \\ \implies \lambda_1, \lambda_2, \lambda_3 &= -3. \end{aligned}$$

So we have our Jordan matrix:

$$J = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$$

and we must now find the generalized eigenvectors to build the U and U^{-1} matrices. Let us begin by finding the true eigenvector associated with $\lambda_1 = -3$:

$$(A - \lambda_1)v_1 = 0 \iff \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} v_1 = 0$$

Denote $v_1 = (v_1^{(1)}, v_1^{(2)}, v_1^{(3)})^T$, it is clear from the above system that $v_1^{(2)} = 0$ and $v_1^{(1)} = v_1^{(3)}$. Set $v_1^{(3)} = 1$ so that

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We must now find the generalized eigenvectors, w_2, w_3. Finding w_2, we have

$$\begin{aligned} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_2^{(1)} \\ w_2^{(2)} \\ w_2^{(3)} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &\equiv \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \end{aligned}$$

which gives us $w_2^{(2)} = 1$ and $w_2^{(1)} = w_2^{(3)}$. Set $w_2^{(3)} = 1$ so that

$$
\begin{aligned}
w_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_1
\end{aligned}
$$

so take $w_2 = (0, 1, 0)^T$. We must now find the final generalized eigenvector, w_3:

$$
\begin{aligned}
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_3^{(1)} \\ w_3^{(2)} \\ w_3^{(3)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&\equiv \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right)
\end{aligned}
$$

which gives us $w_3^{(2)} = 0$ and $w_3^{(1)} = 1 + w_3^{(3)}$. Set $w_3^{(3)} = 1$ so that $w_3^{(1)} = 2$ and so

$$
w_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.
$$

Setting $U = [v_1 \mid w_2 \mid w_3]$,

$$
U = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
$$

Inverting, we find

$$
U^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}
$$

so that, finally, we have

$$
A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}
$$

6. Write down all 3×3 Jordan matrices that have eigenvalues 2 and 5 (and no others).

   Soln. We can have either 5 or 2 as a repeated eigenvalue, so the only 3×3 Jordan matrices that have 2 and 5 as eigenvalues are

$$
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}
$$

   If the repeated eigenvalues have distinct eigenvectors, that is, the geometric multiplicity is equal to the algebraic multiplicity, we have

$$
\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}
$$

7. Compute the matrix exponential $\exp(A)$ for the matrix in question 5.

Soln. Recall that $\exp(UJU^{-1}) = U\exp(J)U^{-1}$ so that $\exp(A)$ from question 5 has the form

$$\exp(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \exp\begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

So all we need to find is $\exp(J)$. Notice we may rewrite

$$J = -3I + N$$

where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and that since I commutes with any other matrix, $\exp(J) = \exp(-3I+N) = \exp(-3I)\exp(N)$. Notice that

$$N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

$$\exp(N) = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$\exp(J) = \begin{pmatrix} e^{-3} & e^{-3} & \frac{1}{2}e^{-3} \\ 0 & e^{-3} & e^{-3} \\ 0 & 0 & e^{-3} \end{pmatrix}$$

and so

$$\begin{aligned}
\exp(A) &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-3} & e^{-3} & \frac{1}{2}e^{-3} \\ 0 & e^{-3} & e^{-3} \\ 0 & 0 & e^{-3} \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}e^{-3} & e^{-3} & \frac{3}{2}e^{-3} \\ e^{-3} & e^{-3} & -e^{-3} \\ e^{-3} & 0 & -e^{-3} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{2}e^{-3} & e^{-3} & -\frac{1}{2}e^{-3} \\ e^{-3} & e^{-3} & -e^{-3} \\ \frac{1}{2}e^{-3} & e^{-3} & \frac{1}{2}e^{-3} \end{pmatrix}
\end{aligned}$$

Section 3.2 Problems

8. Solve the initial value problem $\dot{\mathbf{x}} = A\mathbf{x}(t)$, $\mathbf{x}(0) = \mathbf{x}_0$ for the following matrices. Be as explicit as possible in your representation of the solution. What is the solution behavior as $t \to \infty$?

   (a) $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

Soln. We begin by finding the eigenvalues of A:

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} &= (1-\lambda)^2 - 4 = 0 \\ \implies (\lambda-1)^2 &= 4 \\ \lambda - 1 &= \pm 2 \\ \lambda &= 1 \pm 2 \end{aligned}$$

so $\lambda_1 = 3$, $\lambda_2 = -1$ which are real, distinct eigenvalues. Finding the associated eigenvectors, we have, for λ_1,

$$(A - \lambda_1 I)u_1 = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = 0$$

solving the above system, we have $u_1^{(1)} = -u_1^{(2)}$, and letting $u_1^{(2)} = 1$, we have

$$u_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Now, for λ_2, we have

$$(A - \lambda_2 I)u_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} u_2^{(1)} \\ u_2^{(2)} \end{pmatrix} = 0$$

Solving, we have $u_2^{(1)} = u_2^{(2)}$, and letting $u_2^{(2)} = 1$, we find

$$u_2^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

and so

$$U^{-1} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus,

$$A = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

And since the solution to our system has the form

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 = U \exp(tJ) U^{-1} \mathbf{x}_0$$

we have

$$\begin{aligned} x(t) &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -e^{3t} & e^{3t} \\ e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -e^{3t} + e^{-t} & e^{-t} - e^{3t} \\ e^{-t} - e^{3t} & e^{3t} + e^{-t} \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}. \end{aligned}$$

Notice, if $x_0 = u_1$, the above system becomes

$$\begin{aligned} x(t) &= \frac{1}{2} \begin{pmatrix} e^{3t} + e^{-t} & e^{-t} - e^{3t} \\ e^{-t} - e^{3t} & e^{3t} + e^{-t} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2e^{3t} \\ 2e^{3t} \end{pmatrix} \\ &= e^{3t} u_1 \end{aligned}$$

so that, as $t \to \infty$, $x_1(t) \to -\infty$ and $x_2(t) \to +\infty$. Similarly, if $x_0 = u_2$, we have

$$\begin{aligned} x(t) &= \frac{1}{2}\begin{pmatrix} e^{3t}+e^{-t} & e^{-t}-e^{3t} \\ e^{-t}-e^{3t} & e^{3t}+e^{-t} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2e^{-t} \\ 2e^{-t} \end{pmatrix} \\ &= e^{-t}u_2 \end{aligned}$$

so that, as $t \to \infty$, $x(t) \to 0$. Note that if an initial condition is a linear combination of u_1 and u_2, the solution behavior from the u_1 component will dominate as $t \to \infty$ and the solution will tend to $\pm\infty$.

(b) $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

Let us begin by computing the eigenvalues for A:

$$\begin{aligned} \begin{vmatrix} 3-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} &= (3-\lambda)(1-\lambda)+2 = 0 \\ \implies \lambda^2 - 4\lambda + 5 &= 0 \\ \lambda^2 - 4\lambda + 4 &= -1 \\ (\lambda-2)^2 &= -1 \\ \lambda &= 2 \pm i \end{aligned}$$

so we have a complete set of complex eigenvalues. Now let us find the eigenvector corresponding to the eigenvalue $\lambda_1 = 2 + i$:

$$\begin{aligned} &\begin{pmatrix} 1-i & -2 \\ 1 & -1-i \end{pmatrix} R_1 - R_2 \to R_1 \\ \sim &\begin{pmatrix} -i & -1+i \\ 1 & -1-i \end{pmatrix} R_1 + iR_2 \to R_1 \\ \sim &\begin{pmatrix} 0 & 0 \\ 1 & -1-i \end{pmatrix} \end{aligned}$$

So we have $u_1^{(1)} = (1+i)u_1^{(2)}$. Setting $u_1^{(2)} = 1$, we see

$$u_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

then setting $U = [\mathrm{Re}(u_1)|\mathrm{Im}(u_1)]$:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and so

$$U^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus, by setting

$$R = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

we have

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

and since our solution has the form $\mathbf{x}(t) = \exp(tA)\mathbf{x}_0$, we have

$$\begin{aligned}\mathbf{x}(t) &= e^{2t}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x_{01}\\ x_{02}\end{pmatrix}\\ &= e^{2t}\begin{pmatrix}\sin(t)+\cos(t) & -2\sin(t)\\ \sin(t) & \cos(t)-\sin(t)\end{pmatrix}\begin{pmatrix}x_{01}\\ x_{02}\end{pmatrix}\end{aligned}$$

Note that, as $t \to \infty$, $\mathbf{x}(t) \to \infty$ for any (nonzero) initial condition since the solution will be driven by the factor of e^{2t}.

(c) $A = \begin{pmatrix}0 & 1\\ -1 & 2\end{pmatrix}$

Let us begin by finding the eigenvalues of A:

$$\begin{aligned}\begin{vmatrix}-\lambda & 1\\ -1 & 2-\lambda\end{vmatrix} &= -\lambda(2-\lambda)+1 = 0\\ \implies \lambda^2 - 2\lambda + 1 &= 0\\ (\lambda-1)^2 &= 0\end{aligned}$$

so $\lambda_1 = \lambda_2 = 1$. Finding the associated eigenvalue, we have

$$\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}u_1^{(1)}\\ u_1^{(2)}\end{pmatrix} = 0$$

solving the above system, we have $u_1^{(1)} = u_1^{(2)}$. Setting $u_1^{(2)} = 1$, we have

$$u_1 = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Now, finding the generalized eigenvector, we have

$$\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}u_2^{(1)}\\ u_2^{(2)}\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$$

so that we find $-u_2^{(1)} = 1 - u_2^{(2)}$. Setting $u_2^{(2)} = 1$, we find

$$u_2 = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

Then for $U = [u_1|u_2]$, we have

$$U = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

and so

$$U^{-1} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Thus,

$$A = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

And since the solution to our linear system has the form $\mathbf{x}(t) = \exp(tA)\mathbf{x}_0$, we have

$$\begin{aligned}\mathbf{x}(t) &= \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\exp\begin{pmatrix}t & t\\ 0 & t\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_{0,1}\\ x_{0,2}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^t & te^t\\ 0 & e^t\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}x_{0,1}\\ x_{0,2}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^t - te^t & te^t\\ -e^t & e^t\end{pmatrix}\begin{pmatrix}x_{0,1}\\ x_{0,2}\end{pmatrix}\\ &= \begin{pmatrix}e^t(1-t) & te^t\\ -te^t & e^t(1+t)\end{pmatrix}\begin{pmatrix}x_{0,1}\\ x_{0,2}\end{pmatrix}.\end{aligned}$$

If $\mathbf{x}_0 = u_1$, notice

$$\begin{aligned}
\mathbf{x}(t) &= \begin{pmatrix} e^t(1-t) & te^t \\ -te^t & e^t(1+t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} e^t \\ e^t \end{pmatrix} \\
&= e^t u_1
\end{aligned}$$

so that, as $t \to \infty$, $\mathbf{x}(t) \to \infty$. Now, if $\mathbf{x}_0 = u_2$, notice

$$\begin{aligned}
\mathbf{x}(t) &= \begin{pmatrix} e^t(1-t) & te^t \\ -te^t & e^t(1+t) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} te^t \\ e^t(1+t) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ e^t \end{pmatrix} + \begin{pmatrix} te^t \\ te^t \end{pmatrix} \\
&= e^t u_2 + te^t u_1
\end{aligned}$$

note the addition of the linear correction term with the first eigenvector. It is also clear in this case that as $t \to \infty$, $\mathbf{x}(t) \to \infty$.