Problem Set 3 (Analysis)

Michael Nameika

March 2022

1. Let (X, d) be a metric space and $A, B \subseteq X$. A point $p \in X$ is called an **exterior point** of A provided there is an open ball $B_r(p)$ contained in $X \setminus A$.

   A point $p \in X$ is called a **boundary point** of A provided every open ball $B_r(p)$ contains a point in A and a point in $X \setminus A$.

   Denote respectively, by A°, A', $\text{ext}(A)$, and $\text{bd}(A)$, the set of interior, limit, exterior, and boundary points of A.

   (a) Prove the following:

      i. $(A \cap B)^\circ = A^\circ \cap B^\circ$
      Proof: First let $x \in (A \cap B)^\circ$. By definition, there exists an open ball of radius $r > 0$ around x that is contained complete in $A \cap B$. That is,

      $$B_r(x) \subseteq (A \cap B)$$

      Then by definition of set intersection, we have that $B_r(x) \subseteq A$ and $B_r(x) \subseteq B$. Then by definition, x is in the interior of A and B. Thus, $x \in A^\circ \cap B^\circ$. So we have

      $$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$$

      Now let $x \in A^\circ \cap B^\circ$. That is, there exists an $r_1, r_2 > 0$ such that $B_{r_1}(x) \subseteq A$ and $B_{r_2}(x) \subseteq B$. Let $r = \min\{r_1, r_2\}$. Then $B_r(x) \subseteq A$ and $B_r(x) \subseteq B$. So $B_r(x) \subseteq A \cap B$ and by definition of interior points, $x \in (A\ capB)^\circ$. So we have

      $$A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$$

      And by double inclusion, we have that

      $$(A \cap B)^\circ = A^\circ \cap B^\circ$$

      ii. $(A \cup B)' = A' \cup B'$
      Proof: First let $x \in (A \cup B)'$. That is, x is a limit point of $A \cup B$. That is, for any $r > 0$, the open ball $B_r(x)$ contains at least one point in $A \cup B$ other than x. without loss of generality, assume that this point is in A. Then by definition, x is a limit point for A, and thus, $x \in A'$. By definition of set union, we also have that $x \in A' \cup B'$. Now we have

      $$(A \cup B)' \subseteq A' \cup B'$$

      Now let $x \in A' \cup B'$. That is, x is either a limit point of A or a limit point of B. Without loss of generality, assume that x is a limit point for A. That is, for any $r > 0$, the open ball $B_r(x)$ contains some point $y \neq x$, $y \in A$. By definition of set union, $y \in A \cup B$, so by definition, $x \in (A \cup B)'$. Then we have

      $$A' \cup B' \subseteq (A \cup B)'$$

      By double inclusion, we have

      $$(A \cup B)' = A' \cup B'$$

iii. $A \setminus \text{bd}(A) = A^\circ$

Proof: First let $x \in A \setminus \text{bd}(A)$. By definition of set difference, $x \in A$ but $x \notin \text{bd}(A)$. By definition of boundary points, since x is not a boundary point, we have that every open ball of x will not contain points in and out of A. So either every open ball around x is entirely contained in A or entirely contained in $X \setminus A$. Well, since $x \in A$, every open ball of x is in A. So by definition, x is an interior point of A. Thus, $x \in A^\circ$, so

$$A \setminus \text{bd}(A) \subseteq A^\circ$$

Now let $x \in A^\circ$. Then by definition, every open ball around x is contained in A. So $x \in A$, and since every open ball of x is in A, x cannot be a boundary point, hence $x \in A \setminus \text{bd}(A)$. So we have

$$A^\circ \subseteq A \setminus \text{bd}(A)$$

And by double inclusion,

$$A \setminus \text{bd}(A) = A^\circ$$

iv. $\text{bd}(A)$ is a closed set in X

Proof: Consider $B = (\text{bd}(A))^c$. By definition, we have for $x \in B$, $B_r(x)$ will not contain points in both A and $X \setminus A$ for some $r > 0$. That is, $B_r(x)$ is either in A or $X \setminus A$. Without loss of generality, assume that $B_r(x) \subseteq A$ for some $r > 0$. Now if $B_r(x)$ contains some $y \in \text{bd}(A)$, consider the open ball $B_{r/2}(x)$. If this open ball contains a boundary point of A, continue reducing the radius by half until no boundary points are contained. Then we will have an open ball contained only in A, and so B is open.

Thus, $\text{bd}(A)$ is a closed set in X.

v. $A^\circ \cup \text{bd}(A) = A \cup A'$. Both define the closure $\overline{A}$.

Proof: First let $x \in A^\circ \cup \text{bd}(A)$. Consider the following cases:

Case 1: $x \in A^\circ$. Then $x \in A$ by definition, so by definition of set union, $x \in A \cup A'$ and $x \in A^\circ \cup \text{bd}(A)$. So we have

$$A^\circ \cup \text{bd}(A) \subseteq A \cup A'$$

Case 2: $x \in \text{bd}(A)$. By definition, for any $\epsilon > 0$, $B_\epsilon(x)$ contains points both in A and $X \setminus A$. If $x \in A$, then we have $A^\circ \cup \text{bd}(A) \subseteq A \cup A'$. If $x \notin A$,by definition, x is also a limit point of A, and so $x \in A'$. By definition of set unions, $x \in A \cup A'$. So $A^\circ \cup \text{bd}(A) \subseteq A \cup A'$.

Now let $x \in A \cup A'$. Consider the following cases:

Case 1: $x \in A$. If A is open, we have $x \in A^\circ$. If A is closed, then $x \in A^\circ$ or $x \in \text{bd}(A)$. If A is neither open nor closed, we still have $x \in A^\circ$ or $x \in \text{bd}(A)$ since possibly $\text{bd}(A) \subset A$. Then $x \in A^\circ \cup \text{bd}(A)$, so

$$A \cup A' \subseteq A^\circ \cup \text{bd}(A)$$

Case 2: $x \in A'$. That is, for any $r > 0$, $B_r(x)$ contains points in A different from x. Notice if $x \in \text{bd}(A)$, by definition of the boundary, this statement is satisfied. Additionally, if $x \in A^\circ$, the above statement is also satisfied.

So we have

$$A \cup A' \subseteq A^\circ \cup \text{bd}(A)$$

And by double inclusion,

$$A \cup A' = A^\circ \cup \text{bd}(A)$$

(b) Prove that if either A is open or it is closed, then $(bd(A))^\circ = \emptyset$. Give an example which shows that this assertion does not hold if A is neither open nor closed.

Proof: Let A be a set that is either open, or closed. Assume by way of contradiction that $(bd(A))^\circ \neq \emptyset$. Let $x \in (\text{bd}(A))^\circ$. Then by definition of interior, there exists an open ball of radius r such that $B_r(x) \subseteq \text{bd}(A)$. By definition of boundary, we have that $B_r(x)$ contains at least one point $p \in X \setminus A$. Let $d(x,p) = r' < r$ and consider $R = r - r'$, and form an open ball of radius R around p, $B_R(p) \subseteq B_r(x)$, which contains no points in the boundary, contradicting that $B_r(x) \subseteq \text{bd}(A)$.

Now consider $\mathbb{Q}$, which is neither open nor closed in $\mathbb{R}$ on the standard metric. Notice that $\text{bd}(\mathbb{Q}) = \mathbb{R}$, and so $(\text{bd}(\mathbb{Q}))^\circ = (\mathbb{R})^\circ = \mathbb{R}$.

2. Let (X, d) be a metric space and $A \subset X$.

   (a) Prove that $\overline{A}$ is the closure of A if and only if $\overline{A}$ is the intersection of all closed subsets of X containing A.

   Proof: Let K_λ be the collection of all X-closed sets containing A and consider the intersection of all K_λ:

   $$K = \bigcap_\lambda K_\lambda$$

   Since each K_λ is closed, K is also closed. And since $A \subseteq K_\lambda$ for all λ, $\overline{A} \subseteq K$.

   Now, since $\overline{A}$ is by definition a closed set that contains A, we have that $\overline{A} \in \{K_\lambda\}$. Thus,

   $$K \subseteq \overline{A}$$

   And by double inclusion, we have that

   $$\overline{A} = \bigcap_\lambda K_\lambda$$

   (b) Show that $x \in \overline{A}$ if and only if $\inf_{y \in A} d(x, y) = 0$.

   Proof: Let $x \in \overline{A}$. We wish to show $\inf_{y \in A} d(x, y)$. Since $x \in \overline{A}$, $x \in A \cup A'$. If $x \in A$, $\inf_{y \in A} d(x, y)$ is obvious.

   Now consider the case where $x \in A'$. By definition, for every $\epsilon > 0$, $B_\epsilon(x)$ contains a point in A. Then $0 < d(x, y) < \epsilon$.

   $$0 \leq \inf_{y \in A} d(x, y) \leq \epsilon$$

   Assume by way of contradiction that $inf_{y \in A} d(x, y) = \epsilon$ But since x is a limit point of A, we have that the open ball of radius $\epsilon/2$ will contain some $x_0 \in A$, thus

   $$d(x, x_0) < \epsilon/2$$

   A contradiction so we have

   $$\inf_{y \in A} d(x, y) = 0$$

   Now assume that $\inf_{y \in A} d(x, y) = 0$. This is obvious if $x \in A$. We wish to show this to be true for $x \in A'$.

   By definition of a limit point, for any $\epsilon > 0$, we have that $B_\epsilon(x)$ contains one point other than x in A. That is, $B_\epsilon(x)$ contains a point in A other than x. Assume by way of contradiction that $inf_{y \in A} d(x, y) = \epsilon$. But since x is a limit point, we have that $B_{\epsilon/2}(x)$ contains some point $x_0 \in A$. That is, $d(x, x_0) < \epsilon/2$, a contradiction. So if $x \in A'$, we have that $inf_{y \in A} d(x, y) = 0$.

   (c) Define the diameter $d(A) = \sup_{x, y \in A} d(x, y)$. Note that $d(A) < \infty$ if A is bounded and $d(A) = \infty$ if A is unbounded. Show that $d(A) = d(\overline{A})$.

   Proof: Consider the case where $d(A) = \infty$. Then $d(\overline{A}) = \infty$ and so $d(A) = d(\overline{A})$. Now consider the case where A is closed. Then A is equal to its own closure, so $d(\overline{A}) = d(A)$. Now consider the case where A is not closed. Then there exists a limit point $x \in A'$ such that $x \notin A$. Then for any $\epsilon > 0$, there exists $y \in A$ such that $d(x, y) < \epsilon$. Then ϵ is an upper bound for $d(x, y)$. That is,

   $$\sup d(x, y) \leq \epsilon$$

   Define an open ball of radius $d(A)$ and let $z \in B_{d(A)})(x)$. Then $d(x, z) < d(A)$. So $\sup d(x, y) \leq d(A)$, and thus $d(\overline{A}) \leq d(A)$, and clearly, $d(A) \leq d(\overline{A})$, so we have

   $$d(\overline{A}) = d(A)$$

3. For a metric space (X, d), determine in each of the following cases whether the given subset $A \subseteq X$ is open, closed, or neither open nor closed in X. Rigorously justify your answer.

(a) The set of integers $\mathbb{Z} \subset \mathbb{R}$
$\mathbb{Z}$ is closed. To see this, notice that $\mathbb{R} \setminus \mathbb{Z} = \ldots \cup (-1, 0) \cup (0, 1) \cup \ldots$. For any $x \in \mathbb{R} \setminus \mathbb{Z}$, we can find an open ball around x that is contained in $\mathbb{R} \setminus \mathbb{Z}$.

(b) $A = \{(x, y) \in \mathbb{R}^2 | y = x^2, x \in \mathbb{Q}\}$ A is neither open nor closed. Notice that $(0, 0) \in A$ and consider an open ball around $(0, 0)$ of radius $1 > \epsilon > 0$. By density of the irrationals, there exists an irrational number $p \in (-\epsilon, \epsilon)$. Since $\epsilon < 1$, $p < 1$, so $p^2 < 1$, and so $(p, p^2) \in B_\epsilon((0, 0))$.
That is, there exists an element in $B_\epsilon((0, 0))$ not in A. Then A is not open.
Now consider $A^c = \{(x, y) \in \mathbb{R}^2 | y = x^2, x \in \mathbb{R} \setminus \mathbb{Q}\}$. Using the density of the rationals, and the same argument as above, we can see that A^c is not open. That is, A is neither open nor closed.

(c) $A = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 + 2z = 0\}$ A is closed.
Notice that $x^2 + y^2 + z^2 + 2z = 0$ can be rewritten as $x^2 + y^2 + (z + 1)^2 = 1$. That is,

$$A = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + (z + 1)^2 = 1\}$$

Notice that A is a sphere of radius 1 centered around the point $(0, 0, -1)$. Consider the closed ball of radius 1 centered at $(0, 0, -1)$, $\overline{B}_1((0, 0, -1))$. Since $\overline{B}_1$ is closed, $\mathbb{R}^3 \setminus \overline{B}_1$ is open. Now, consider $(\overline{B}_1)^\circ$ which is open by definition. Notice that $A = \overline{B}_1 \setminus (\overline{B}_1)^\circ$.
So A is closed.

(d) $X = C[a, b]$ with $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. $A = \{f \in X | 0 < f(x) < 1, x \in [a, b]\}$
X is open. Let $f \in X$ and consider an open ball of radius r centered at f:

$$B_r(f)$$

and let $g \in B_r(f)$. That is, $d(f, g) < r$. Let $r' = r - d(f, g) > 0$ and let $h \in B_{r'}(g)$. Notice

$$d(f, h) \leq d(f, g) + d(g, h) < d(f, g) + r' = r$$

so

$$d(f, h) < r$$

and thus $h \in B_r(f)$. Then by definition, X is open.

(e) (X, d) same as in (d). $A = \{f \in X | \int_a^b f(x)dx = 0\}$.
X is closed. Let $\{f_n\}$ be a Cauchy sequence in X, that converges to some f. We wish to show that $f \in X$. Since $\{f_n\}$ is a Cauchy sequence, we have that for any $\epsilon > 0$, there exists a natural number $n > N \in \mathbb{N}$ such that

$$d(f_n, f) < \epsilon$$

By the definition of this metric,

$$d(f_n, f) = \sup_{x \in [a,b]} |f_n(x) - f(x)| < \epsilon$$

Since $|f_n(x) - f(x)| < \epsilon$ for any $x \in [a, b]$, we have that f_n converges to f uniformly on $[a, b]$. Then by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b (\lim_{n \to \infty} f_n(x))dx$$

And since $\{f_n\}$ is a sequence in X, we have that

$$\int_a^b f_n(x)dx = 0$$

thus

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \lim_{n \to \infty} (0) = 0$$

so

$$\int_a^b (\lim_{n \to \infty} f_n(x))dx = \int_a^b f(x)dx = 0$$

so $f \in X$. Thus, X is closed.

(f) $X = \mathbb{R}^2$ and let $f : \mathbb{R} \to \mathbb{R}$ be continuous. $A = \{(x, y) \in X | y = f(x)\}$.

A is closed. Consider a Cauchy sequence in $\mathbb{R}$, $\{x_n\}$. Since $\mathbb{R}$ is complete, we have that $\{x_n\} \to x \in \mathbb{R}$. Define a sequence of real numbers $\{y_n\} = \{f(x_n)\}$. Since f is continuous, we have that

$$y = \lim y_n = \lim f(x_n) = f(\lim x_n) = f(x)$$

That is,

$$\lim y_n = f(x)$$

We have that $(x, y) \in A$, so A is closed.

4. (a) Suppose (X, d) is a complete metric space and $Y \subset X$ is nonempty. Prove that (Y, d) is complete if and only if Y is a closed subset of X.

Proof: First assume that (Y, d) is complete. We wish to show that Y is closed. Assume by way of contradiction that a limit point y of Y is not in Y. That is, $y \in Y'$ but $y \notin Y$. By definition of limit points, we have that for any $\epsilon > 0$, the open ball $B_\epsilon(y)$ contains at least one other point in Y. Consider a sequence of open balls of radius $\frac{1}{n}$, and let $y_n \in B_{1/n}(y)$ be such that $y_n \in Y$. Then we have $d(y, y_n) < \frac{1}{n}$. Thus, the sequence $\{y_n\}$ converges to y. And since (Y, d) is complete, we must have that $y \in Y$, contradicting our assumption that $y \notin Y$. Thus, Y is closed.

Now assume that Y is a closed subset of X. We wish to show that (Y, d) is complete. Since Y is closed, Y contains all of its limit points. Now consider a Cauchy sequence in Y, $\{y_n\}$. Since Y is a subset of X and (X, d) is a closed metric space, we have that $y_n \to y$ for some $y \in X$. And since Y is closed, Y contains all of its limit points, and by definition, y is a limit point of Y, so $y \in Y$. That is, any Cauchy sequence in Y converges to some value in Y. Thus, (Y, d) is complete.

(b) Suppose (X, d) and (Y, d') are metric spaces and $f : X \to Y$, $g : X \to Y$ are continuous functions. Prove that the set $A = \{x \in X | f(x) = g(x)\}$ is closed.

Proof: First note that if $f(x) \neq g(x)$ for all $x \in X$, $A = \emptyset$ and is closed by definition. If A contains finitely many points, then each point in A is an isolated point, and so A has no limit points, and so contains its limit points vacuously, so A is closed. Now assume that $A \neq \emptyset$ and let $x_0 \in X$ and define a sequence $\{x_n\}$ in A such that $d(x, x_0) < \frac{1}{n}$. Then $\{x_n\}$ converges to x_0, and since f and g are continuous, we have that $\lim f(x_n) = f(\lim x_n) = f(x_0)$ and $\lim g(x_n) = g(\lim x_n) = g(x_0)$. And since $\{x_n\}$ is a sequence in A, we have that $g(x_0) = f(x_0)$, so $x_0 \in A$. Thus, A is closed by definition.

5. Let (X, d) be a metric space and $Y \subset X$.

(a) Prove that $Z \subseteq Y$ is closed if and only if there exists a closed subset $A \subseteq X$ such that $Z = A \cap Y$.

Proof: To prove this, I will prove the following two lemmas. Also denote $B_{r,Y}(x)$ as an open ball of radius r centered at x in the set Y.

Lemma 1: Let (X, d) be a metric space and Y a subspace of X. Let $z \in Y$ and $r > 0$. Then $B_{r,Y}(z) = B_{r,X}(z) \cap Y$

Proof of Lemma 1: Let $z \in Y$ and $B_{r,Y}(z)$ and $B_{r,X}(z)$ be open balls of radius r centered at z in Y and X, respectively. By definition of open balls,

$$B_{r,X}(z) = \{x \in X | d(x, z) < r\}$$

Now consider $B_{r,X}(z) \cap Y$:

$$B_{r,X}(z) \cap Y = \{x \in X | d(x, z) < r\} \cap Y$$

$$= \{x \in Y | d(x, z) < r\}$$

$$= B_{r,Y}(z)$$

Lemma 2: Z is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$

Proof of Lemma 2: Let Z be open in Y. We wish to show that $Z = G \cap Y$ for some open set G in X. Since Z is open, by definition, Z is the union of all open sets contained in Z. Consider an open ball around a point $z \in Z$. Since $Z \subseteq Y$, this is also an open ball in Y, call it $B_{r,Y}(z)$ where r depends on z. Then

$$Z = \bigcup_{z \in Z} B_{r,Y}(z)$$

Since $Y \subseteq X$, each $B_{r,Y}(z) \subseteq X$, and by Lemma 1, we have that $B_{r,Y}(z) = B_{r,X}(z) \cap Y$. Thus

$$Z = \bigcup_{z \in Z} (B_{r,X}(z) \cap X)$$

$$= (\bigcup_{z \in Z} B_{r,X}(z)) \cap X$$

Let $G = \bigcup_{z \in Z} B_{r,X}(z)$. Since G is a union of open balls in X, G is also an open set in X. Thus

$$Z = G \cap X$$

Now assume that $Z = G \cap X$ for some open set $G \subseteq X$. Let $z \in G$, then $z \in Z$, and so there exists an open ball $B_{r,X}(z)$ such that, by Lemma 1,

$$B_{r,Y}(z) = B_{r,X} \cap Y$$

$$\subseteq G \cap X = Z$$

for an arbitrary point in Z, there exists an open ball around z contained in Z. Thus, Z is open. Now, to prove the main problem, first let $Z \subseteq Y$ be closed. Then by definition, $Y \setminus Z$ is open. Thus, for some open set $G \in X$, we have by Lemma 2 that

$$Y \setminus Z = G \cap Y$$

Now take the complement of the above with respect to X:

$$X \setminus (Y \setminus Z) = (X \setminus G) \cup (X \setminus Y)$$

$$Z \cup (X \setminus Y) = (X \setminus G) \cup (X \setminus Y)$$

Now intersect each side with Y:

$$(Z \cup (X \setminus Y)) \cap Y = ((X \setminus G) \cup (X \setminus Y)) \cap Y$$

$$Z \cap Y = ((X \setminus G) \cap Y) \cup ((X \setminus Y) \cap Y)$$

$$Z = ((X \setminus G) \cap Y) \cup \emptyset$$

$$Z = (X \setminus G) \cap Y$$

Since G is open in X, $X \setminus G$ is closed in X. Let $A = X \setminus G$. Then

$$Z = A \cap Y$$

for a closed set $A \subseteq X$.

(b) Show that every subset $Z \subseteq Y$ that is closed in Y is also closed in X if and only if Y is a closed subset of X.

Proof: First suppose that every closed subset of Y is closed in X. Well, Y is a closed subset of itself, so by assumption, Y must be closed in X. Now assume that Y is closed in X and consider a closed subset $Z \subseteq Y$. By part (a), we have that for some closed subset $A \subseteq X$,

$$Z = A \cap Y$$

Since both A and Y are closed in X, and Z is an intersection of closed sets in X, Z must also be closed in X.

(c) Let $X = \mathbb{R}^2$ with the Euclidean metric, $Y = \{(x,0)|x \in \mathbb{R}\} \subset X$ with the induced metric, and $Z = \{(x,0)|0 < x < 1\} \subset Y$. Show that Z is an open subset of Y but is *not* an open subset of X.

Proof: Let $x \in Z$ and $r > 0$. Consider the open ball of radius $1/2$ in X centered at $(1/2, 0)$ denoted by $B_{1/2}((1/2,0))$. Notice that

$$B_{1/2} \cap Y = Z$$

and so by Lemma 2, we have that Z is an open set in Y.

To see that Z is not open in X, consider the point $(x,y) = (1/2,0) \in Y$ and consider an open ball of radius $r > 0$, denoted by $B_r((1/2,0))$. Notice that the point $(1/2, r/2) \in B_r((1/2,0))$, but $(1/2, r/2) \notin Y$, so Y is not open in X.