Nonlinear Waves Problem 4.1

Michael Nameika

4.1

(a) Obtain the nonlinear change in the frequency given by (4.19) by applying the frequency-shift method to the equation

$$\frac{d^2y}{dt^2} + y - \varepsilon y^3 = 0, \qquad |\varepsilon| \ll 1.$$

Soln. Introduce the new time variable $\tau = \omega t$ so that $\frac{d}{dt} = \omega \frac{d}{d\tau}$. Then the ODE becomes

$$\omega^2 \frac{dy^2}{d\tau^2} + y - \varepsilon y^3 = 0.$$

Assume that $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ and $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$. Plugging this into the differential equation yields

$$(1 + \varepsilon\omega_1 + \cdots)^2 \frac{d}{d\tau}(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots - \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)^3 = 0.$$

Collecting the order 1 and ε terms gives

$$\begin{aligned}
\mathcal{O}(1): \quad & y_0'' + y_0 = 0 \\
\implies & y_0 = Ae^{i\tau} + \text{c.c.} \\
\mathcal{O}(\varepsilon): \quad & y_1'' + \omega_1 y_0'' + y_1 - y_0^3 = 0 \\
\implies & y_1'' + y_1 = 2\omega_1 Ae^{i\tau} + 3A^2A^*e^{i\tau} + A^3e^{3i\tau} + \text{c.c.}
\end{aligned}$$

For the $\mathcal{O}(\varepsilon)$ equation, notice the secular terms are $2\omega_1 e^{i\tau} + 3A^2A^*e^{i\tau}$, so we make the restriction

$$\begin{aligned}
2\omega_1 A + 3A^2A^* = 0 \\
\implies \omega_1 = -\frac{3}{2}|A|^2.
\end{aligned}$$

And for y_1, we wish to solve the equation

$$y_1'' + y_1 = A^3e^{3i\tau} + \text{c.c.}$$

Using the ansatz $y_1 = Be^{3i\tau} + \text{c.c.}$, we find

$$\begin{aligned}
-8Be^{3i\tau} + \text{c.c.} = A^3e^{3i\tau} + \text{c.c.} \\
\implies B = -\frac{A^3}{8}
\end{aligned}$$

and so

$$y_1 = -\frac{A^3}{8}e^{3i\tau} + \text{c.c.}$$

Putting it together, we have

$$y \sim Ae^{it\left(1 - \frac{3}{2}|A|^2\right)} - \varepsilon\frac{A^3}{8}e^{3it\left(1 - \frac{3}{2}|A|^2\right)} + \text{c.c.}$$

and so we find the nonlinear change in the frequency is given by

$$\Omega = 1 - \frac{3}{2}|A|^2$$

as desired.

(b) Use the multiple-scales method to find the leading-order approximation to the solution of

$$\frac{d^2y}{dt^2} + y - \varepsilon\left(y^3 + \frac{dy}{dt}\right) = 0, \qquad 0 < \varepsilon \ll 1.$$

Soln. See part (c).

(c) Find the next-order (first-order) approximation, valid for times $t = o(1/\varepsilon^2)$, to the solution of part (b).

Soln. Since we need to find the leading-order behavior to find the first-order approximation, we complete part (b) here. Introduce the fast time and slow time variables $t_1 = t$ and $t_2 = \varepsilon t$. Via the chain rule, we find

$$\frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}.$$

We also assume $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ and plugging these into the differential equation given in part (b) gives

$$\begin{aligned}&\left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)^2 (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots + \\ &- \varepsilon\left((y_0 + \varepsilon y_1 + \varepsilon^2 y_2)^3 + \left(\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}\right)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)\right) = 0\end{aligned}$$

which yields, to $\mathcal{O}(\varepsilon)$:

$$\frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2} + y_1 - y_0^3 - \frac{\partial y_0}{\partial t_1} = 0.$$

Solving the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$:

$$\begin{aligned}\mathcal{O}(1): \quad & \frac{\partial^2 y_0}{\partial t_1^2} + y_0 = 0 \\ \implies & y_0 = A(t_2)e^{i2_1} + \text{c.c.} \\ \mathcal{O}(\varepsilon): \quad & \frac{\partial^2 y_1}{\partial t_1^2} + 2\frac{\partial^2 y_0}{\partial t_1 \partial t_2} + y_1 - y_0^3 - \frac{\partial y_0}{\partial t+1} = 0 \\ \implies & \frac{\partial^2 y_1}{\partial t_1^2} + y_1 = A^3 e^{3it_1} + A^2 A^* e^{it_1} + iAe^{it_1} - 2iA' e^{it_1} + \text{c.c.}\end{aligned}$$

In the above differential equation for y_1, note that the secular term on the right hand side are $e^{it_1}(A^2A^* + iA - 2iA')$ and so we set the secular term to zero:

$$\begin{aligned}& A^2A^* + iA - 2iA' = 0 \\ \implies & 2iA' - iA = A|A|^2\end{aligned}$$

multiplying by A^* yields

$$\begin{aligned}& 2iA'A^* - i|A|^2 = |A|^4 \in \mathbb{R} \\ \implies & 2iA'A^* + 2i(A')^*A - 2i|A|^2 = 0 \\ \implies & \frac{d}{dt_2}\left(|A|^2\right) = |A|^2 \\ \implies & |A(t_2)|^2 = Ce^{t_2}\end{aligned}$$

where C is a constant of integration. For $t_2 = 0$, $|A(0)|^2 = C$, so call $C = |A_0|^2$. Thus

$$|A(t_2)|^2 = |A_0|^2 e^{t_2}.$$

By our choice of A, we have the differential equation for y_1 becomes

$$\frac{\partial^2 y_1}{\partial t_1^2} + y_1 = A^3 e^{3it_1} + \text{c.c.}$$

and using the ansatz $y_1 = Be^{3it_1} + \text{c.c.}$ yields $B = -A^3/8$ and so

$$y_1 = -\frac{A^3}{8} e^{3it_1} + \text{c.c.}$$

Now let us find a more explicit formula for $A(t_2)$. From the differential equation for A,

$$\begin{aligned}
&2iA' = |A|^2 A + iA \\
\implies &2iA' = |A_0|^2 e^{t_2} A + iA \\
\implies &2iA' = A\left(|A_0|^2 e^{t_2} + i\right) \\
\implies &A' = A\left(-\frac{i}{2}|A|_0^2 e^{t_2} + \frac{1}{2}\right) \\
\implies &A = Ce^{\left(t_2/2 - i/2|A_0|^2 e^{t_2}\right)}
\end{aligned}$$

and using $A(0) = A_0$, we have $C = A_0 e^{i/2|A_0|^2}$ and so

$$A(t_2) = A_0 e^{t2/2} e^{i/2\left(|A_0|^2(1-e^{t_2})\right)}.$$

Putting it together, we have

$$y \sim A_0 e^{\varepsilon t/2} e^{i/2|A_0|^2(1-e^{\varepsilon t})+it} - \varepsilon \frac{A_0^3}{8} e^{3/2\varepsilon t} e^{3i/2|A_0|^2(1-e^{\varepsilon t})+3it} + \text{c.c.}$$