Incompressible Flow HW 3

Michael Nameika

Kundu 4.1) Let a one-dimensional velocity field be $u = u(x,t)$ with $v = 0$ and $w = 0$. The density varies as $\rho = \rho_0(2 - \cos(\omega t))$. Find an expression for $u(x,t)$ if $u(0,t) = U$.

Soln. By conservation of mass, we have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

But notice $\frac{\partial \rho}{\partial t} = \rho_0 \frac{\partial}{\partial t}(2 - \cos(\omega t)) = \rho_0 \omega \sin(\omega t)$. Then the conservation of mass equation gives us

$$\begin{aligned}
\rho_0 \omega \sin(\omega t) + \rho_0(2 - \cos(\omega t))\frac{\partial u}{\partial x} &= 0 \\
\implies \frac{\partial u}{\partial x} &= -\frac{\omega \sin(\omega t)}{2 - \cos(\omega t)} \\
\implies u(x,t) &= -\frac{\omega \sin(\omega t)}{2 - \cos(\omega t)}x + f(t) \\
u(0,t) &= f(t) = U \\
\implies u(x,t) &= -\frac{\omega \sin(\omega t)}{2 - \cos(\omega t)}x + U.
\end{aligned}$$

Kundu 4.2) Consider the one-dimensional Cartesian velocity field: $\mathbf{u} = (\alpha x/t, 0, 0)$ where α is a constant. Find a spatially uniform, time-dependent density field, $\rho = \rho(t)$, that renders this flow field mass conserving when $\rho = \rho_0$ at $t = t_0$.

Soln. We again use conservation of mass. By the assumption $\rho = \rho(t)$, we have

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} = 0 \\
\implies &\frac{\partial \rho}{\partial t} + \rho \frac{\alpha}{t} = 0 \\
\implies &\int \frac{d\rho}{\rho} = -\alpha \int \frac{dt}{t} \\
\implies &\log(|\rho|) = -\alpha \log(|t|) + c_0 \\
\implies &\rho(t) = ce^{1/t^\alpha} \\
&\rho(t_0) = ce^{1/t_0^\alpha} = \rho_0 \\
\implies &c = \rho_0 e^{-1/t_0^\alpha} \\
\implies &\rho(t) = \rho_0 e^{1/t^\alpha - 1/t_0^\alpha}.
\end{aligned}
$$

Kundu 4.4) A proposed conservation law for ξ, a new fluid property, takes the following form: $\frac{d}{dt}\int_{V(t)}\rho\xi dV + \int_{A(t)} \mathbf{\Theta}\cdot\mathbf{n}dS = 0$, where $V(t)$ is a material volume that moves with the fluid velocity $\mathbf{u}$, $A(t)$ is the surface of $V(t)$, ρ is the fluid density, and $\mathbf{\Theta} = -\rho\gamma\nabla\xi$.

a) What partial differential equation is implied by the above conservation statement?

Soln. By Reynold's Transport Theorem, we have

$$\frac{d}{dt}\int_{V(t)}\rho\xi dV = \int_{V(t)}\frac{\partial(\rho\xi)}{\partial t}dV + \int_{A(t)}\rho\xi\mathbf{u}\cdot\mathbf{n}dA$$

and by the Divergence Theorem, we may rewrite the area integral as a volume integral:

$$\frac{d}{dt}\int_{V(t)}\rho\xi dV = \int_{V(t)}\left(\frac{\partial(\rho\xi)}{\partial t} + \nabla\cdot(\rho\xi\mathbf{u})\right)dV$$

so that our conservation law may be rewritten as

$$\int_{V(t)}\left(\frac{\partial(\rho\xi)}{\partial t} + \nabla\cdot(\rho\xi\mathbf{u}) - \nabla\cdot(\rho\gamma\nabla\xi)\right)dV = 0$$

and since this holds for any $V(t)$, we may conclude

$$\frac{\partial(\rho\xi)}{\partial t} + \nabla\cdot(\rho\xi\mathbf{u}) - \nabla\cdot(\rho\gamma\nabla\xi) = 0.$$

b) Use the part a) result and the continuity equation to show: $\frac{\partial\xi}{\partial t} + \mathbf{u}\cdot\nabla\xi = \frac{1}{\rho}\nabla\cdot(\rho\gamma\nabla\xi)$.

Soln. Recall the continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

Using this along with the result we found in part a) gives us

$$\begin{aligned}
\xi\frac{\partial\rho}{\partial t} + \rho\frac{\partial\xi}{\partial t} + \nabla\cdot(\rho\xi\mathbf{u}) - \nabla\cdot(\rho\gamma\nabla\xi) &= 0\\
\rho\frac{\partial\xi}{\partial t} - \nabla\cdot(\rho\mathbf{u})\xi + \nabla\cdot(\rho\xi\mathbf{u}) - \nabla\cdot(\rho\gamma\nabla\xi) &= 0
\end{aligned}$$

using the product rule for divergence, we use the following identities:

$$\begin{aligned}
\xi\nabla\cdot(\rho\mathbf{u}) &= \xi\nabla\rho\cdot\mathbf{u} + \xi\rho\nabla\cdot\mathbf{u}\\
\nabla\cdot(\rho\xi\mathbf{u}) &= \nabla(\rho\xi)\cdot\mathbf{u} + \rho\xi\nabla\cdot\mathbf{u}\\
&= \xi\nabla\rho\cdot\mathbf{u} + \rho\nabla\xi\cdot\mathbf{u} + \rho\xi\nabla\cdot\mathbf{u}.
\end{aligned}$$

Substituting these into our above differential equation yields

$$-\xi\nabla\rho\cdot\mathbf{u} - \xi\rho\nabla\cdot\mathbf{u} + \rho\frac{\partial\xi}{\partial t} + \xi\nabla\rho\cdot\mathbf{u} + \rho\nabla\xi\cdot\mathbf{u} + \rho\xi\nabla\cdot\mathbf{u} - \nabla\cdot(\rho\gamma\nabla\xi) = 0$$

$$\begin{aligned}
\implies \rho\frac{\partial\xi}{\partial t} + \rho\nabla\xi\cdot\mathbf{u} &= \nabla\cdot(\rho\gamma\nabla\xi)\\
\implies \frac{\partial\xi}{\partial t} + \nabla\xi\cdot\mathbf{u} &= \frac{1}{\rho}\nabla\cdot(\rho\gamma\nabla\xi)
\end{aligned}$$

as desired.

Kundu 4.5) The components of a mass flow vector $\rho\mathbf{u}$ are $\rho u = 4x^2y$, $\rho v = xyz$, $\rho w = yz^2$.

a) Compute the net mass outflow through the closed surface formed by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Soln. The net mass outflow through the surface is given by the surface integral:

$$\oiint_S \rho\mathbf{u} \cdot d\mathbf{A}.$$

Splitting the cube into 6 seperate surfaces (see figure at end of problem), we may split the integral into 6 parts:

$$\oiint_S \rho\mathbf{u} \cdot d\mathbf{A} = \iint_1 \rho\mathbf{u} \cdot d\mathbf{A} + \iint_2 \rho\mathbf{u} \cdot d\mathbf{A} + \iint_3 \rho\mathbf{u} \cdot d\mathbf{A} + \iint_4 \rho\mathbf{u} \cdot d\mathbf{A} + \iint_5 \rho\mathbf{u} \cdot d\mathbf{A} + \iint_6 \rho\mathbf{u} \cdot d\mathbf{A}.$$

For surface 1, note that $z = 0$ and $\mathbf{n} = -\mathbf{z}$, thus $\rho\mathbf{u} \cdot \mathbf{n} = -\rho w = yz^2 = 0$. Thus

$$\iint_1 \rho\mathbf{u} \cdot d\mathbf{A} = 0.$$

On surface 2, we have $z = 1$, and $\mathbf{n} = \mathbf{z}$, so $\rho\mathbf{u} \cdot \mathbf{n} = \rho w = yz^2 = y$. Thus

$$\begin{aligned}\iint_2 \rho\mathbf{u} \cdot d\mathbf{A} &= \int_0^1 \int_0^1 y dx dy \\ &= \frac{1}{2}.\end{aligned}$$

On surface 3, $y = 1$ and $\mathbf{n} = \mathbf{y}$ and so $\rho\mathbf{u} \cdot \mathbf{n} = \rho v = xyz = xz$. Thus

$$\begin{aligned}\iint_3 \rho\mathbf{u} \cdot d\mathbf{A} &= \int_0^1 \int_0^1 xz dx dz \\ &= \int_0^1 x dx \int_0^1 z dz \\ &= \frac{1}{4}.\end{aligned}$$

On surface 4, $y = 0$ and $\mathbf{n} = -\mathbf{y}$, so $\rho\mathbf{u} \cdot \mathbf{n} = \rho v = xyz = 0$. Thus

$$\iint_4 \rho\mathbf{u} \cdot d\mathbf{A} = 0.$$

On surface 5, $x = 0$ and $\mathbf{n} = -\mathbf{x}$, so $\rho\mathbf{u} \cdot \mathbf{n} = -\rho u = -4x^2y = 0$. Thus

$$\iint_5 \rho\mathbf{u} \cdot d\mathbf{A} = 0.$$

And on surface 6, $x = 1$ and $\mathbf{n} = \mathbf{x}$, so $\rho\mathbf{u} \cdot \mathbf{n} = \rho u = 4x^2y = 4y$. Thus

$$\begin{aligned}\iint_6 \rho\mathbf{u} \cdot d\mathbf{A} &= \int_0^1 \int_0^1 4y dy dz \\ &= 4\int_0^1 y dy \\ &= 2.\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}\oiint_S \rho\mathbf{u} \cdot d\mathbf{A} &= \frac{1}{2} + \frac{1}{4} + 2 \\ &= \frac{11}{4}.\end{aligned}$$

b) Compute $\nabla \cdot (\rho \mathbf{u})$ and integrate over the volume bounded by the surface defined inpart a).

Soln. Notice

$$\nabla \cdot (\rho \mathbf{u}) = 8xy + xz + 2yz$$

so

$$\begin{aligned}
\iiint_V \nabla \cdot (\rho \mathbf{u}) dV &= \int_0^1 \int_0^1 \int_0^1 (8xy + xz + 2yz) dxdydz \\
&= \int_0^1 \int_0^1 \left[4x^2 y + \frac{x^2}{2} z + 2xyz\right] \Big|_0^1 dydz \\
&= \int_0^1 \int_0^1 \left(4y + \frac{1}{2} z + 2yz\right) dydz \\
&= \int_0^1 \left[2y^2 + \frac{1}{2} yz + y^2 z\right] \Big|_0^1 dz \\
&= \int_0^1 \left(2 + \frac{1}{2} z + z\right) dz \\
&= \left[2z + \frac{3}{4} z^2\right] \Big|_0^1 \\
&= \frac{11}{4}.
\end{aligned}$$

c) Explain why the results for parts a) and b) should be equal or unequal.

Soln. Note that the results in part a) and b) are equal, which is what we expect since this result is precisely what is guaranteed by the Divergence Theorem.

Kundu 4.7) The definition of the stream function for two-dimensional, constant-density flow in the $x - y$ plane is: $\mathbf{u} = -\mathbf{e}_z \times \nabla\psi$, where $\mathbf{e}_z$ is the unit vector perpendicular to the $x - y$ plane that determines a right-handed coordinate system.

a) Verify that this vector definition is equivalent to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ in Cartesian coordinates.

Soln. By definition of the cross product, we have

$$\begin{aligned}
-\mathbf{e}_z \times \nabla\psi &= \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ 0 & 0 & -1 \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial\psi}{\partial y}\right)\mathbf{x} - \left(\frac{\partial\psi}{\partial x}\right)\mathbf{y} \\
\implies u &= \frac{\partial\psi}{\partial y} \\
v &= -\frac{\partial\psi}{\partial x}
\end{aligned}$$

as desired.

b) Determine the velocity components in $r - \theta$ polar coordinates in terms of $r - \theta$ derivatives of ψ.

Soln. Recall that we may express the Cartesian basis vectors in the following way:

$$\begin{aligned}
\mathbf{x} &= \cos(\theta)\mathbf{r} - \sin(\theta)\boldsymbol{\theta} \\
\mathbf{y} &= \sin(\theta)\mathbf{r} + \sin(\theta)\boldsymbol{\theta}.
\end{aligned}$$

And since $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$, we have

$$\begin{aligned}
\mathbf{u} &= \frac{\partial\psi}{\partial y}\mathbf{x} - \frac{\partial\psi}{\partial x}\mathbf{y} \\
&= \frac{\partial\psi}{\partial y}(\cos(\theta)\mathbf{r} - \sin(\theta)\boldsymbol{\theta}) - \frac{\partial\psi}{\partial x}(\sin(\theta)\mathbf{r} + \cos(\theta)\boldsymbol{\theta}) \\
&= \left(\cos(\theta)\frac{\partial\psi}{\partial y} - \sin(\theta)\frac{\partial\psi}{\partial x}\right)\mathbf{r} - \left(\sin(\theta)\frac{\partial\psi}{\partial y} + \cos(\theta)\frac{\partial\psi}{\partial x}\right)\boldsymbol{\theta}.
\end{aligned}$$

Now, using the chain rule, notice

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} \\
\frac{\partial}{\partial \theta} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta}.
\end{aligned}$$

And by $x = r\cos(\theta)$, $y = r\sin(\theta)$, the above equation becomes

$$\begin{aligned}
\frac{\partial}{\partial r} &= \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \theta} &= -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y}.
\end{aligned}$$

Then we may rewrite the expression for $\mathbf{u}$ in polar coordinates as

$$\mathbf{u} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{r} - \frac{\partial\psi}{\partial r}\boldsymbol{\theta}.$$

c) Determine an equation for the $z-$component of the vorticity in terms of ψ.

Soln. Recall that vorticity is defined as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

Then

$$\begin{aligned}
\boldsymbol{\omega} &= \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} \\
&= -\frac{\partial v}{\partial z}\mathbf{x} + \frac{\partial u}{\partial z}\mathbf{y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{z}
\end{aligned}$$

and since the flow is two-dimensional, u and v are independent of z, hence the above equation becomes

$$\boldsymbol{\omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{z}.$$

Now, by the streamfunction relation, we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) \\
&= \frac{\partial^2 \psi}{\partial y^2} \\
\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) \\
&= -\frac{\partial^2 \psi}{\partial x^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\boldsymbol{\omega} &= \left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\mathbf{z} \\
\implies \boldsymbol{\omega} &= -\nabla^2 \psi \mathbf{z}.
\end{aligned}$$

That is, the vorticity is simply the negative of the Laplacian of the streamfunction ψ.

Kundu 4.10) A jet of water with a diameter of 8 cm and a speed of 25 m/s impinges normally on a large stationary flat plate. Find the force required to hold with the plate stationary. Compare the average pressure on the plate with the stagnation pressure if the plate is 20 times the area of the jet.

Soln. Consider the control volume depicted in the figure below. We use conservation of momentum to find the pressure force from the jet of water:

$$\frac{d}{dt}\int_V \rho\mathbf{u}dV + \int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \int_V \rho\mathbf{g}dV + \int_A \mathbf{f}dA.$$

We assume the flow is steady, so that the time derivative term drops out, and we neglect the body forces. Then we are left with

$$\int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \int_A \mathbf{f}dA.$$

The contributions due to the curved part of the control surface are zero since the flow is parallel to the surface. At the surface of the plate, the velocity of the fluid is zero, so there will be no contribution from the plate. At the sides of the control volume, the contributions are in equal and opposite directions, so they cancel out. Thus the only contribution on the left hand side of the momentum equation is from the inlet.

Further, considering gauge pressure, we have that the pressure term $\int_{A_{\text{out}}} \mathbf{f}dA = -\mathbf{F}_{\text{pressure}}$. At the inlet, we have $\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) = -U^2\mathbf{x}$ and so the momentum equation gives

$$-\rho U^2 A\mathbf{x} = -\mathbf{F}_{\text{pressure}}$$
$$\mathbf{F}_{\text{pressure}} = \rho U^2 A\mathbf{x}.$$

Using $\rho = 1000\frac{\text{kg}}{\text{m}^3}$, $A = \pi(0.04)^2 m^2$ and $U = 25\frac{\text{m}}{\text{s}}$, we have the force required to hold the plate in place is

$$|\mathbf{F}_{\text{pressure}}| \approx 3.142 \times 10^3 \text{ N}.$$

Now, to find the stagnation pressure, consider the streamline coaxial with the jet of water and consider the two points at the inlet of the control volume and the point coincident with the plate. By Bernoulli's equation, we have

$$\frac{1}{2}U_1^2 + gz + \frac{p_1}{\rho} = \frac{1}{2}U_2^2 + gz + \frac{p_2}{\rho}.$$

Here, U_1 is the given velocity, $U_2 = 0$ since (2) is a stagnation point, $p_1 = 0$, and p_2 is the stagnation pressure. Then

$$p_2 = \frac{\rho}{2}U^2.$$

Consider the area of the plate to be $A_{\text{plate}} = 20A$, we have that the average pressure on the plate is

$$P_{\text{plate}} = \frac{\rho}{20}U^2.$$

That is, the stagnation pressure is 10 times the average pressure on the plate.

Kundu 4.14) The pressure rise $\Delta p = p_2 - p_1$ that occurs for flow through a sudden pipe-cross-sectional-area expansion can depend on the average upstream flow speed U_{avg}, the upstream pipe diameter d_1, the downstream pipe diameter d_2, and the fluid density ρ and viscosity μ. Here p_2 is the pressure downstream of the expansion where the flow is first fully adjusted to the larger pipe diameter.

c) Use a control volume analysis to determine Δp in terms of U_{ave}, d_1, d_2, and ρ for the high Reynolds number limit. [*Hints*: 1) a streamline drawing might help in determining or estimating the pressure on the vertical surfaces of the area transition, and 2) assume uniform flow profiles wherever possible.]

Soln. We apply conservation of momentum:

$$\frac{d}{dt}\int_V \rho \mathbf{u} dV + \int_A \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \int_V \rho \mathbf{g} dV + \int_A \mathbf{f} dA.$$

Here, we consider the cylindrical control volume that coincides with the larger diameter pipe (see figure at the end of the problem). Since the fluid is assumed to be incompressible, the time derivative equals zero. We also assume the body forces are negligible. Then the above equation becomes

$$\int_A \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \int_A \mathbf{f} dA.$$

We assume the flow at the inlet and outlet of the control volume is uniform and neglect body forces. On the cylindrical shell, the velocity will be tangent to the control volume, so the shell contributions will be zero (also the case if we consider no slip conditions). On the inlet , we have $\mathbf{u} = U_{\text{ave}}\mathbf{x}$ for the smaller pipe area, 0 otherwise, so

$$\int_{A_{\text{in}}} \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = -\rho U_{\text{ave}}^2 A_1 \mathbf{x}$$

and at the outlet,

$$\int_{A_{\text{out}}} \rho \mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = \rho U_{\text{out}}^2 A_2 \mathbf{x}.$$

And since we assume that the pressure on the left side of the pipe is equal on the inside and outside, we have

$$\int_{A_{\text{in}}} \mathbf{f} dA = p_1 A_2 \mathbf{x}$$

and at right end:

$$\int_{A_{\text{out}}} \mathbf{f} dA = -p_2 A_2 \mathbf{x}.$$

Putting this into the conservation of momentum equation yields

$$-\rho U_{\text{ave}}^2 A_1 + U_{\text{out}}^2 A_2 = A_2(p_1 - p_2).$$

Now we wish to determine U_{out} in terms of U_{ave}. We do so by using conservation of mass:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{u}\cdot\mathbf{n} dA = 0.$$

The time derivative term vanishes, so we are simply left with the area integral above. From our work above, we have

$$\int_{A_{\text{in}}} \mathbf{u}\cdot\mathbf{n} dA = -A_1 U_{\text{ave}}$$

$$\int_{A_{\text{out}}} \mathbf{u}\cdot\mathbf{n} dA = A_2 U_{\text{out}}$$

thus

$$\int_A \mathbf{u} \cdot \mathbf{n} dA = A_2 U_{\text{out}} - A_1 U_{\text{ave}} = 0$$
$$\implies U_{\text{out}} = \frac{A_1}{A_2} U_{\text{ave}}.$$

Thus our equation for the pressure difference becomes

$$\begin{aligned} -\Delta p &= -\rho U_{\text{ave}}^2 \frac{A_1}{A_2} + \frac{A_1^2}{A_2^2} \rho U_{\text{ave}}^2 \\ &= \rho \frac{A_1}{A_2} U_{\text{ave}}^2 \left(\frac{A_1}{A_2} - 1 \right) \\ \implies \Delta p &= \rho \frac{A_1}{A_2} U_{\text{ave}}^2 \left(1 - \frac{A_1}{A_2} \right) \end{aligned}$$

d) Compute the ideal flow value for Δp using the Bernoulli equation (4.19) and compare this to the result from part c) for a diameter ratio of $d_1/d_2 = 1/2$. What fraction of the maximum possible pressure rise does the sudden expansion achieve?

Soln. Consider the streamline that is coaxial with the pipe and consider the points at the inlet of the diameter increase and the end of the control volume. By Bernoulli's equation, we have

$$\begin{aligned} \frac{1}{2} U_{\text{ave}}^2 + gz + \frac{p_1}{\rho} &= \frac{1}{2} U_{\text{out}}^2 + gz + \frac{p_2}{\rho} \\ \implies \frac{1}{2} \left(U_{\text{ave}}^2 - U_{\text{out}}^2 \right) &= \frac{p_2 - p_1}{\rho} \\ \implies \Delta p_{\text{Bernoulli}} &= \frac{\rho}{2} U_{\text{ave}}^2 \left(1 - \frac{A_1^2}{A_2^2} \right). \end{aligned}$$

Note we may express $A_1 = \frac{\pi}{4} d_1^2$ and $A_2 = \frac{\pi}{4} d_2^2$, and plugging these into the equations for Δp we found above, we have

$$\begin{aligned} \Delta p_{\text{Bernoulli}} &= \frac{\rho}{2} U_{\text{ave}}^2 \left(1 - \frac{d_1^4}{d_2^4} \right) \\ \Delta p_{\text{CV}} &= \rho \frac{d_1^2}{d_2^2} U_{\text{ave}}^2 \left(1 - \frac{d_1^2}{d_2^2} \right). \end{aligned}$$

Using $d_2 = 2d_1$, we have

$$\begin{aligned} \Delta p_{\text{Bernoulli}} &= \frac{15\rho}{32} U_{\text{ave}}^2 \\ \Delta p_{CV} &= \frac{3\rho}{16} U_{\text{ave}}^2 \end{aligned}$$

and their ratio gives

$$\frac{\Delta p_{CV}}{\Delta p_{\text{Bernoulli}}} = \frac{2}{5}.$$

Thus, the sudden expansion achieves $\frac{2}{5}$ of the maximum possible pressure rise.

–

Kundu 4.16) Wind strikes the side of a simple residential structure and is deflected up over the top of the structure. Assume the following: two-dimensional steady inviscid constant-density flow, uniform upstream velocity profile, linear gradient in the downstream velocity profile (velocity U at the upper boundary and zero velocity at the lower boundary as shown), no flow through the upper boundary of the control volume, and constant pressure on the upper boundary of the control volume. Use the control volume shown.

a) Determine h_2 in terms of U and h_1.

Soln. To simplify the analysis, we assume the flow of the air is incompressible and steady. We also assume that the control volume extends into the page length ℓ. Using the given control volume, we apply conservation of mass:

$$\frac{d}{dt}\int_V \rho dV + \int_A \rho \mathbf{u}\cdot\mathbf{n} dA = 0.$$

Since the flow is incompressible and the control volume is fixed, the time derivative term is zero, and all we're left with is

$$\int_A \mathbf{u}\cdot\mathbf{n} dA = 0.$$

On the top boundary, the velocity field is tangent to the control volume, so $\mathbf{u}\cdot\mathbf{n} = 0$ on the top, and there is no contribution from the top boundary. Then the left and right faces of the control volume contribute to the area integral. On the left side of the control volume, we have $\mathbf{u} = U\mathbf{x}$ and $\mathbf{n} = -\mathbf{x}$ and so $\mathbf{u}\cdot\mathbf{n} = -U$ and so

$$\int_{A_{\text{in}}} \mathbf{u}\cdot\mathbf{n} dA = -U\ell h_1.$$

On the right boundary, since we have a linear velocity gradient, $\mathbf{U}(y) = \frac{U}{h_2} y\mathbf{x}$. We also have $\mathbf{n} = \mathbf{x}$ on the right boundary, so

$$\begin{aligned}\int_{A_{\text{out}}} \mathbf{u}\cdot\mathbf{n} dA &= \frac{U}{h_2}\ell\int_0^{h_2} y dy\\ &= \frac{U}{h_2}\ell\frac{h_2^2}{2}\\ &= \frac{U\ell}{2}h_2.\end{aligned}$$

Thus conservation of mass gives us

$$\begin{aligned}-\rho U\ell h_1 + \rho\frac{U\ell}{2}h_2 &= 0\\ \implies h_2 &= 2h_1.\end{aligned}$$

b) Determine the *direction* and *magnitude* of the horizontal force on the house per unit depth into the page in terms of the fluid density ρ, the upstream velocity U, and the height of the house h_1.

Soln. Here we use conservation of momentum:

$$\frac{d}{dt}\int_V \rho\mathbf{u} dV + \int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) dA = \int_V \rho\mathbf{g} dV + \int_A \mathbf{f} dA.$$

The time derivative of the first integral is zero, and we ignore body forces. Then the above equation becomes

$$\int_A \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) dA = \int_A \mathbf{f} dA.$$

As in part a), the contributions of the area integral on the top and bottom of the control volume, leaving us with only the contributions from the left and right surfaces. On the left surface, we have $\mathbf{u} = U\mathbf{x}$, so $\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) = -U^2\mathbf{x}$. Then

$$\int_{A_{\text{in}}} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA = -\rho U^2 \ell h_1$$

and on the right boundary, we have $\mathbf{u}(\mathbf{u}\cdot\mathbf{n}) = \frac{U^2}{h_2^2}y^2\mathbf{x}$, $\mathbf{n} = \mathbf{x}$, so

$$\begin{aligned}\int_{A_{\text{out}}} \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})dA &= \ell\rho\frac{U^2}{h_2^2}\int_0^{h_2} y^2 dy \\ &= \rho\ell\frac{U^2}{3h_2^2}h_2^3 \\ &= \rho\ell\frac{U^2}{3}h_2.\end{aligned}$$

The right hand side gives us $\int_A \mathbf{f}dA = \mathbf{F}$ where $\mathbf{F}$ is the pressure force on the house. Then the conservation of momentum equation gives

$$\begin{aligned}\left(-U^2\rho\ell h_1 + \frac{2}{3}U^2\rho\ell h_1\right)\mathbf{x} &= \mathbf{F} \\ \implies \mathbf{F} &= \frac{1}{3}\rho\ell U^2 h_1\mathbf{x}\end{aligned}$$

c) Evaluate the magnitude of the force for a house that is 10 m tall and 20 m long in wind of 22 m/sec (approximately 50 miles per hour).

Soln. For air of density $\rho = 1.293\frac{\text{kg}}{\text{m}^3}$, and a house of height $h_1 = 10$ m, length of $\ell = 20$ m, and wind speed of $U = 22\frac{\text{m}}{\text{s}}$, using the equation in part b) gives us

$$\begin{aligned}|\mathbf{F}| &= \frac{1}{3}\left(1.293\frac{\text{kg}}{\text{m}^3}\right)\left(22^2\frac{\text{m}^2}{\text{s}^2}\right)(20\text{ m})(10\text{ m}) \\ &\approx 41.79\times 10^3\text{ N}.\end{aligned}$$