Nonlinear Waves HW 1

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1. Recreate Figure 1.6.

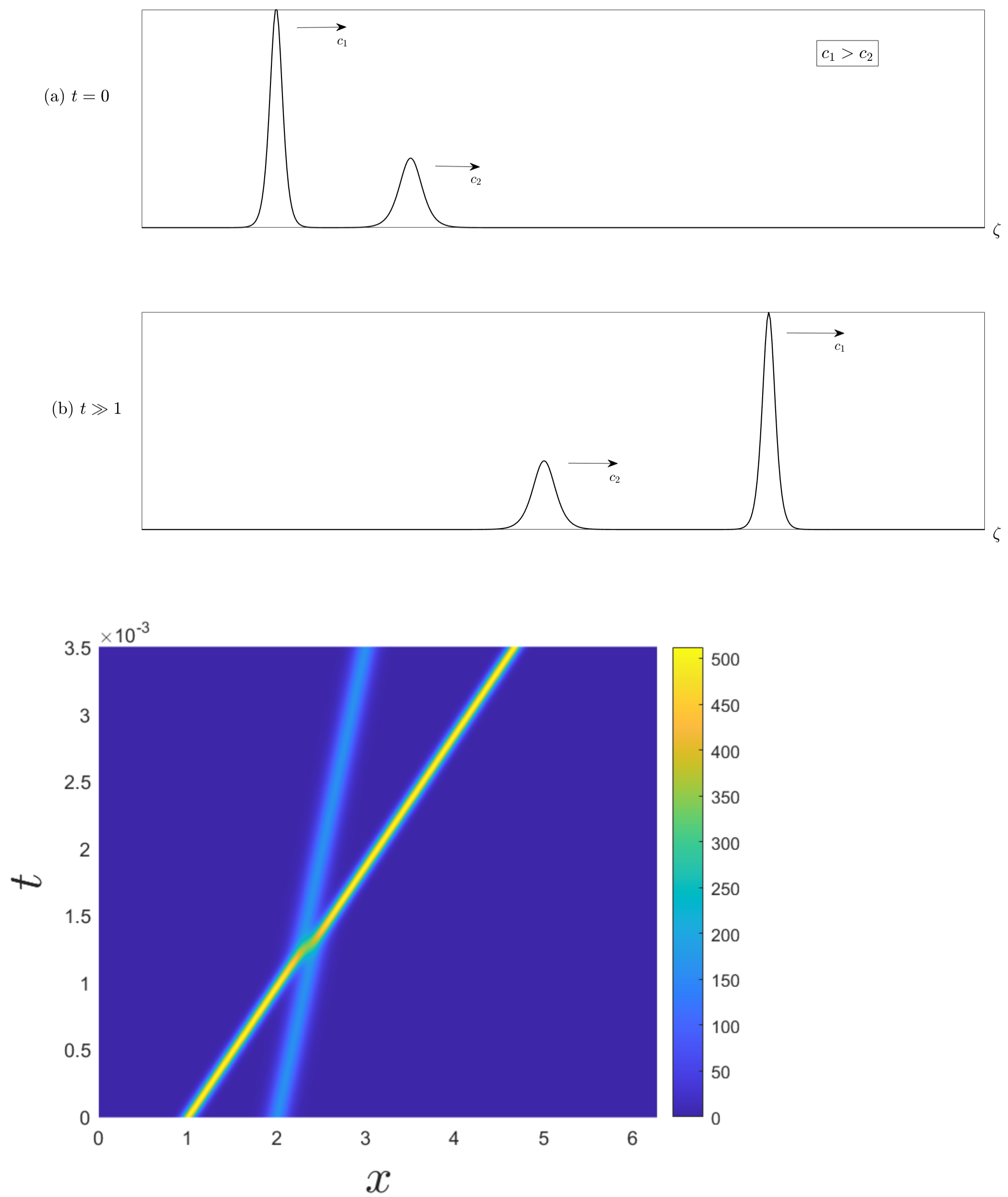


Figure 1: Above: elastic soliton interaction. Below: waterfall plot of soliton interactions

2. Given the modificed KdV (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0,$$

reduce the problem to an ODE by investigating traveling wave solutions of the form: $u = U(x - ct)$.

Soln. Suppose $u = U(x - ct)$ and let $\zeta = x - ct$. Then

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{dU}{d\zeta}\frac{\partial \zeta}{\partial t} = -c\frac{dU}{d\zeta} \\ \frac{\partial u}{\partial x} &= \frac{dU}{d\zeta}\frac{\partial \zeta}{\partial x} = \frac{dU}{d\zeta} \\ \implies \frac{\partial^3 u}{\partial x^3} &= \frac{d^3 U}{d\zeta^3}. \end{aligned}$$

Thus the mKdV equation becomes

$$-c\frac{dU}{d\zeta} + 6U^2\frac{dU}{d\zeta} + \frac{d^3U}{d\zeta^3} = 0.$$

Notice that $6u^2\frac{dU}{d\zeta} = 2\frac{d}{d\zeta}\left(U^3\right)$ so the above equation becomes

$$-c\frac{dU}{d\zeta} + 2\frac{d}{d\zeta}\left(U^3\right) + \frac{d^3U}{d\zeta} = 0.$$

Integrating once yields

$$-cU + 2U^3 + \frac{d^2U}{d\zeta} = E_1$$

where E_1 is a constant of integration. Multiply each side by $\frac{dU}{d\zeta}$ and integrate again to get

$$\begin{aligned} -\frac{c}{2}U^2 + \frac{1}{2}U^4 + \frac{1}{2}\left(\frac{\partial U}{\partial \zeta}\right)^2 &= E_1 u + E_2 \\ -cU^2 + U^4 + \left(\frac{\partial u}{\partial \zeta}\right)^2 &= \tilde{E}_1 U + \tilde{E}_2 \end{aligned}$$

where $\tilde{E}_{1,2} = 2E_{1,2}$. Isolating $\frac{dU}{d\zeta}$ gives

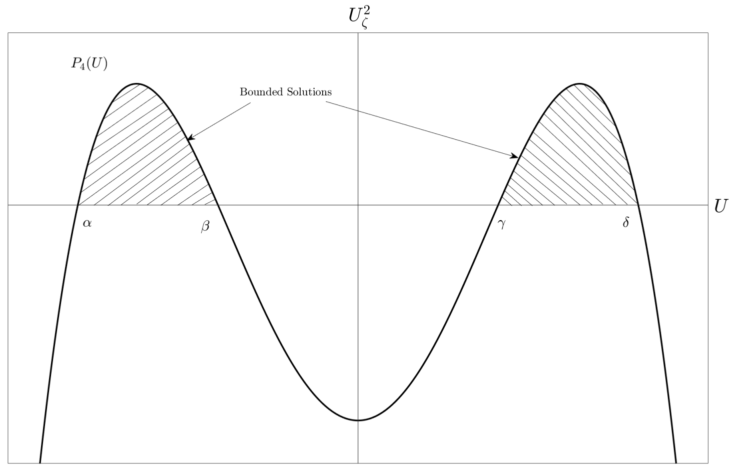
$$\begin{aligned} \left(\frac{\partial U}{\partial \zeta}\right)^2 &= -U^4 + cU^2 + \tilde{E}_1 U + \tilde{E}_2 \\ &= P_4(U) \end{aligned}$$

where $P_4(U) = -U^4 + cU^2 + \tilde{E}_1 U + \tilde{E}_2$.

(a) Express the bounded periodic solutions in terms of Jacobi elliptic functions.

Soln. We now assume that P_4 splits, that is, there exist roots $\alpha \leq \beta \leq \gamma \leq \delta$ such that

$$P_4(U) = -(U - \alpha)(U - \beta)(U - \gamma)(U - \delta).$$



The above figure shows the general shape of $P_4(U)$. To find bounded solutions, we restrict $U \in (\alpha, \beta)$ or $U \in (\gamma, \delta)$. To this end, consider $U \in (\alpha, \beta)$. Then from our work above, we have

$$\begin{aligned} \left(\frac{\partial U}{\partial \zeta}\right)^2 &= -(U-\alpha)(U-\beta)(U-\gamma)(U-\delta) \\ \implies \frac{\partial U}{\partial \zeta} &= \pm\sqrt{(U-\alpha)(U-\beta)(U-\gamma)(\delta-U)} \end{aligned}$$

We consider the positive case and note that this is a separable equation. Separating gives

$$\begin{aligned} \int_\alpha^u \frac{dU}{\sqrt{(U-\alpha)(U-\beta)(U-\gamma)(\delta-U)}} &= \int_0^\zeta d\zeta \\ \implies \int_\alpha^u \frac{dU}{\sqrt{(U-\alpha)(U-\beta)(U-\gamma)(\delta-U)}} &= \zeta. \end{aligned}$$

Above we assumed $U(0) = \alpha$. This may be achieved by a simple shift of ζ. For the integral on the left hand side, make the substitution $y = A\sqrt{\frac{U-\alpha}{\beta-U}}$ where A is a constant to be determined. Solving for U gives

$$\begin{aligned} \frac{y^2}{A^2} &= \frac{U-\alpha}{\beta-U} \\ \implies (\beta-U)\frac{y^2}{A^2} &= U-\alpha \\ \implies \alpha+\beta\frac{y^2}{A^2} &= \left(1+\frac{y^2}{A^2}\right)U \\ \implies U &= \frac{\alpha+\beta\frac{y^2}{A^2}}{1+\frac{y^2}{A^2}}. \end{aligned}$$

Then

$$dU = \frac{2\frac{\beta}{A^2}y\left(1+\frac{y^2}{A^2}\right) - 2\frac{1}{A^2}y\left(\alpha+\beta\frac{y^2}{A^2}\right)}{\left(1+\frac{y^2}{A^2}\right)^2}dy$$
$$= \frac{\frac{2\beta}{A^2}y + \frac{2\beta}{A^4}y^3 - \frac{2\alpha}{A^2}y - \frac{2\beta}{A^4}y^3}{\left(1+\frac{y^2}{A^2}\right)^2}dy$$
$$= \frac{\frac{2}{A^2}(\beta-\alpha)y}{\left(1+\frac{y^2}{A^2}\right)^2}dy.$$

Thus the integral becomes

$$\frac{2}{A^2}\int_0^{A\sqrt{\frac{u-\alpha}{\beta-u}}} \frac{(\beta-\alpha)y\left(1+\frac{y^2}{A^2}\right)^{-2}}{\sqrt{\left(\frac{\alpha+\beta y^2/A^2}{1+y^2/A^2}-\alpha\right)\left(\frac{\alpha+\beta y^2/A^2}{1+y^2/A^2}-\beta\right)\left(\frac{\alpha+\beta y^2/A^2}{1+y^2/A^2}-\gamma\right)\left(\delta-\frac{\alpha+\beta y^2/A^2}{1+y^2/A^2}\right)}}dy =$$
$$= \frac{2}{A^2}\int_0^{A\sqrt{\frac{u-\alpha}{\beta-u}}} \frac{(\beta-\alpha)y}{\sqrt{\left(\alpha+\beta\frac{y^2}{A^2}-\alpha-\alpha\frac{y^2}{A^2}\right)\left(\alpha+\beta\frac{y^2}{A^2}-\beta-\beta\frac{y^2}{A^2}\right)\left(\alpha+\beta\frac{y^2}{A^2}-\gamma-\gamma\frac{y^2}{A^2}\right)\left(\delta+\delta\frac{y^2}{A^2}-\alpha-\beta\frac{y^2}{A^2}\right)}}dy$$
$$= \frac{2}{A^2}\int_0^{A\sqrt{\frac{u-\alpha}{\beta-u}}} \frac{(\beta-\alpha)y}{(\beta-\alpha)y/A\sqrt{([\gamma-\alpha]+[\gamma-\beta]y^2/A^2)([\delta-\alpha]+[\delta-\beta]y^2/A^2)}}dy$$
$$= \frac{2}{A\sqrt{(\gamma-\alpha)(\delta-\beta)}}\int_0^{A\sqrt{\frac{u-\alpha}{\beta-u}}} \frac{dy}{\sqrt{\left(1+\frac{\gamma-\beta}{\gamma-\alpha}\frac{y^2}{A^2}\right)\left(1+\frac{\delta-\beta}{\delta-\alpha}\frac{y^2}{A^2}\right)}}.$$

From here, we take A such that $\frac{\delta-\beta}{\delta-\alpha}\frac{1}{A^2} = 1 \implies A = \sqrt{\frac{\delta-\beta}{\delta-\alpha}}$. Then the above integral becomes

$$\frac{2\sqrt{\delta-\alpha}}{\sqrt{\gamma-\alpha}\sqrt{\delta-\alpha}\sqrt{\delta-\beta}}\int_0^{\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}} \frac{dy}{\sqrt{(1+y^2)\left(1+\frac{\gamma-\beta}{\gamma-\alpha}\frac{\delta-\alpha}{\delta-\beta}y^2\right)}} =$$
$$= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}} \frac{dy}{\sqrt{(1+y^2)(1+m'y^2)}}$$

where $m' = \left(\frac{\gamma-\beta}{\gamma-\alpha}\right)\left(\frac{\delta-\alpha}{\delta-\beta}\right)$. Now let $y = \tan(\theta)$, $dt = \sec^2(\theta)d\theta$. Then the above integral becomes

$$\begin{aligned}
&\frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\arctan\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right)} \frac{\sec^2(\theta)d\theta}{\sqrt{(1+\tan^2(\theta))(1+m'\tan^2(\theta))}} \\
&= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\arctan\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right)} \frac{\sec(\theta)}{\sqrt{(1+m'\tan^2(\theta))}}d\theta \\
&= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\arctan\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right)} \frac{\sec(\theta)}{\sqrt{(1+\tan^2(\theta)-m\tan^2(\theta))}}d\theta \\
&= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\arctan\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right)} \frac{\sec(\theta)}{\sqrt{\sec^2(\theta)-m\tan^2(\theta)}}d\theta \\
&= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}\int_0^{\arctan\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right)} \frac{d\theta}{\sqrt{1-m\sin^2(\theta)}} \\
&= \frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}F\left(\tan^{-1}\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right);m\right)
\end{aligned}$$

where $F(y, m)$ is the first elliptic integral with modulus $m = 1 - m'$. Using this, the solution to the ODE is given as

$$\begin{aligned}
\frac{2}{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}F\left(\tan^{-1}\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right);m\right) &= \zeta \\
F\left(\tan^{-1}\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right);m\right) &= \frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta \\
\implies \tan^{-1}\left(\sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}}\right) &= \mathrm{am}\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right) \\
\implies \sqrt{\frac{(\delta-\beta)(u-\alpha)}{(\delta-\alpha)(\beta-u)}} &= \frac{\mathrm{sn}\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)} \\
\implies \frac{(\delta-\alpha)(u-\alpha)}{(\delta-\alpha)(\beta-u)} &= \frac{\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)} \\
\implies \frac{u-\alpha}{\beta-u} &= \frac{\delta-\alpha}{\delta-\beta}\frac{\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)} \\
\implies u-\alpha &= (\beta-u)\frac{\delta-\alpha}{\delta-\beta}\frac{\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\alpha}}{2}\zeta;m\right)} \\
\implies u\left(1+\frac{\delta-\alpha}{\delta-\beta}\right)\frac{\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)} &= \alpha+\beta\frac{\delta-\alpha}{\delta-\beta}\frac{\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)} \\
\implies u &= \frac{\alpha\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)+\beta\frac{\delta-\alpha}{\delta-\beta}\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}{\mathrm{cn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)+\frac{\delta-\alpha}{\delta-\beta}\mathrm{sn}^2\left(\frac{\sqrt{\gamma-\alpha}\sqrt{\delta-\beta}}{2}\zeta;m\right)}
\end{aligned}$$

where $\mathrm{am}(\cdot;\cdot)$ is the elliptic amplitude function, $\mathrm{sn}(\cdot;\cdot)$ is the elliptic sine function, and $\mathrm{cn}(\cdot;\cdot)$ is the elliptic cosine function

(b) Find all bounded solitary wave solutions.

Soln. In the limiting case $\gamma \to \beta$, notice $m \to 1$. And as $m \to 1$, $\mathrm{cn}(y;m) \to \mathrm{sech}(y)$ and $\mathrm{sn}(y;m) \to \tanh(y)$, so that as $m \to 1$,

$$u \to \frac{\alpha \mathrm{sech}^2\left(\frac{\sqrt{\beta-\alpha}\sqrt{\delta-\beta}}{2}\zeta\right) + \beta\frac{\delta-\alpha}{\delta-\beta}\tanh^2\left(\frac{\sqrt{\beta-\alpha}\sqrt{\delta-\beta}}{2}\zeta\right)}{\mathrm{sech}^2\left(\frac{\sqrt{\beta-\alpha}\sqrt{\delta-\beta}}{2}\zeta\right) + \frac{\delta-\alpha}{\delta-\beta}\tanh^2\left(\frac{\sqrt{\beta-\alpha}\sqrt{\delta-\beta}}{2}\zeta\right)}$$

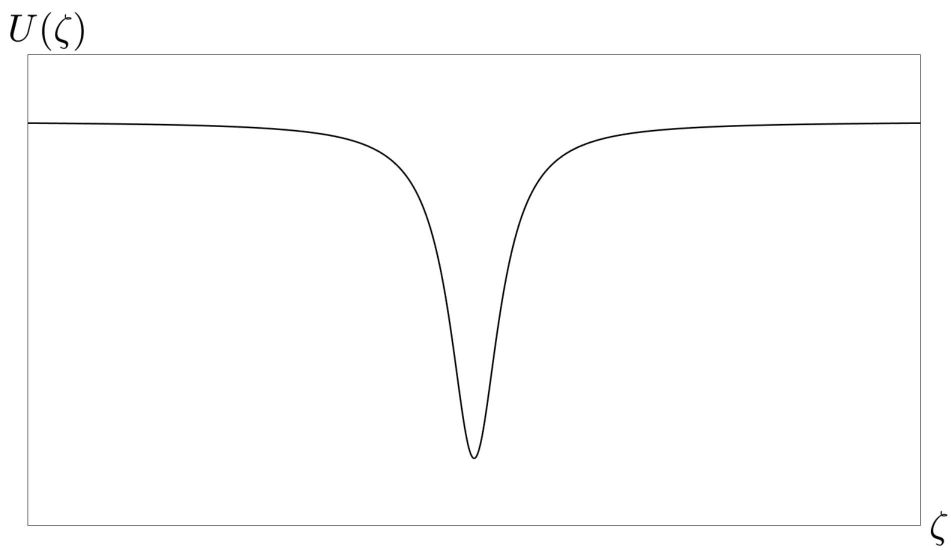


Figure 2: Example displaying the shape of the above solution.