Homework VII

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Section 3.4 Problems

8. Show that an element x of an inner product space X cannot have "too many" Fourier coefficients $\langle x, e_k\rangle$ which are "big"; here (e_k) is a given orthonormal sequence; more precisely, show that the number n_m of $\langle x, e_k\rangle$ such that $|\langle x, e_k\rangle| > 1/m$ must satisfy $n_m < m^2\|x\|^2$.

Proof: Let $\{e_l\}$ be the subset of elements of $\{e_k\}$ such that $|\langle x, e_k\rangle| > 1/m$. Suppose that there are n_m of such elements. We will show that $n_m < \infty$. Notice, by Bessel's inequality,

$$\sum_{l=1}^{n_m} |\langle x, e_l\rangle|^2 \leq \|x\|^2$$

and that, since $\langle x, e_l\rangle > 1/m$, we have

$$\frac{n_m}{m^2} < \sum_{l=1}^{n_m} |\langle x, e_l\rangle|^2.$$

Using this inequality along with Bessel's inequality above, we have

$$\begin{aligned} \frac{n_m}{m^2} &< \|x\|^2 \\ n_m &< m^2\|x\|^2 \end{aligned}$$

so that n_m is bounded and hence finite.

9. Orthonormalize the first three terms of the sequence $(x_0, x_1, x_2, \cdots)$, where $x_j(t) = t^j$, on the interval $[-1, 1]$, where

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)dt.$$

Soln. Applying the Gram-Schmidt process, let $v_0 = x_0 = 1$. Then taking $e_0 = \frac{v_0}{\|v_0\|}$, we have

$$\begin{aligned} \|v_0\| &= \sqrt{\int_{-1}^{1} dt} \\ &= \sqrt{2} \end{aligned}$$

so that $e_1 = \frac{1}{\sqrt{2}}$. Now, $v_1 = x_1 - \langle x_1, e_0\rangle e_0$ for $x_1 = t$. Then

$$\begin{aligned} \langle x_1, e_0\rangle &= \int_{-1}^{1} \frac{1}{\sqrt{2}} t dt \\ &= 0 \end{aligned}$$

and so $v_1 = x_1 = t$. Normalizing,

$$\begin{aligned} \|v_1\| &= \sqrt{\int_{-1}^{1} t^2 dt} \\ &= \sqrt{\frac{2}{3}} \end{aligned}$$

so that
$$e_1 = \frac{v_1}{\|v_1\|} = \sqrt{\frac{3}{2}}t.$$
Finally, finding $v_2 = x_2 - \langle x_2, e_1\rangle e_1 - \langle x_2, e_0\rangle e_0$:
$$\begin{aligned}\langle x_2, e_1\rangle &= \int_{-1}^{1} \sqrt{\frac{3}{2}}t(t^2)dt \\ &= \sqrt{\frac{3}{2}}\int_{-1}^{1} t^3 dt \\ &= 0\end{aligned}$$
and
$$\begin{aligned}\langle x_2, e_0\rangle e_0 &= \left(\int_{-1}^{1} \frac{1}{\sqrt{2}}t^2 dt\right) e_0 \\ &= \frac{1}{\sqrt{2}}\frac{2}{3}e_0 \\ &= \frac{\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{3}.\end{aligned}$$
We now have
$$v_2 = t^2 - \frac{1}{3}.$$
Normalizing,
$$\begin{aligned}\|v_2\| &= \sqrt{\int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt} \\ &= \sqrt{\int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right) dt} \\ &= \sqrt{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} \\ &= \sqrt{\frac{2}{5} - \frac{2}{9}} \\ &= \sqrt{\frac{8}{45}} \\ &= \frac{2\sqrt{2}}{3\sqrt{5}}\end{aligned}$$
so that
$$\begin{aligned}e_2 &= \frac{3\sqrt{5}}{2\sqrt{2}}\left(t^2 - \frac{1}{3}\right) \\ &= \frac{3\sqrt{5}}{2\sqrt{2}}t^2 - \frac{\sqrt{5}}{2\sqrt{2}}.\end{aligned}$$
Then the first few orthonormal terms are
$$\{e_0, e_1, e_2\} = \left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}t,\ \frac{3\sqrt{5}}{2\sqrt{2}}t^2 - \frac{\sqrt{5}}{2\sqrt{2}}\right\}$$

Section 3.5 Problems

6. Let (e_j) be an orthonormal sequence in a Hilbert space H. Show that if

$$x = \sum_{j=1}^{\infty} \alpha_j e_j, \qquad y = \sum_{j=1}^{\infty} \beta_j e_j, \qquad \text{then} \qquad \langle x, y\rangle = \sum_{j=1}^{\infty} \alpha_j \overline{\beta}_j,$$

the series being absolutely convergent.

Proof: First notice

$$\begin{aligned}
\langle x, y\rangle &= \left\langle \sum_{j=1}^{\infty} \alpha_j e_j, \sum_{k=1}^{\infty} \beta_j e_j \right\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \alpha_j \overline{\beta}_k \langle e_j, e_k\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \alpha_j \overline{\beta}_k \delta_{jk} \\
&= \sum_{j=1}^{\infty} \alpha_j \overline{\beta}_j
\end{aligned}$$

where δ_{jk} is the Kronecker delta. Now, the norm of x and y are given by the following (since (e_j) is orthonormal):

$$\begin{aligned}
\|x\|^2 &= \sum_{j=1}^{\infty} |\alpha_j|^2 \\
\|y\|^2 &= \sum_{j=1}^{\infty} |\beta_j|^2
\end{aligned}$$

each of which is convergent. Then notice

$$\begin{aligned}
\left| \sum_{j=1}^{\infty} \alpha_j \overline{\beta_j} \right| &\leq \sum_{j=1}^{\infty} |\alpha_j \beta_j|^2 \\
&\leq \sum_{j=1}^{\infty} |\alpha_j|^2 \sum_{k=1}^{\infty} |\beta_k|^2 \\
&= \|x\|^2 \|y\|^2
\end{aligned}$$

so that the series is absolutely convergent.

8. Let (e_k) be an orthonormal sequence in a Hilbert space H, and let $M = \operatorname{span}(e_k)$. Show that for any $x \in H$ we have $x \in \overline{M}$ if and only if x can be represented by (6) with coefficients $\alpha_k = \langle x, e_k\rangle$.

Proof: First suppose that $x \in H$ can be represented by

$$\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k.$$

Then we have that the sequence (s_n) defined by

$$\sum_{k=1}^{n} \langle x, e_k\rangle e_k$$

is a Cauchy sequence in M which converges to x. Hence x is a limit point of M and so $x \in \overline{M}$. Now suppose $x \in \overline{M}$. We wish to show that x can be represented by

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k.$$

Section 3.6 Problems

10. Let M be a subset of a Hilbert space H, and let $v, w \in H$. Suppose that $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$, show that M is total in H.

Proof: To begin, note that for any $x \in H$, $\langle 0, x \rangle = 0$. Now let $v \in M^{\perp}$. That is,

$$\langle v, x \rangle = 0$$

for all $x \in M$. Thus,

$$\begin{aligned} \langle v, x \rangle &= \langle 0, x \rangle \\ \implies v &= 0 \end{aligned}$$

Thus, $M^{\perp} = \{0\}$, so that the span of M is dense in H, and thus, M is total in H.

Extra Credit Exercise VII.1

(a) Let (x_j) be an *orthogonal* sequence in an inner product space X, meaning $\langle x_i, x_j \rangle = 0$ for all $i \neq j$, and suppose that the series $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 + \cdots$ converges. Show that (s_n) is a Cauchy sequence, where $s_n = x_1 + \cdots + x_n$.

Proof: Let (x_j) be an orthogonal sequence in an inner product space X and suppose $M = \|x_1\|^2 + \|x_2\|^2 + \cdots$ converges. We will show that (s_n) is a Cauchy sequence. To begin, note that since M converges, the sequence of partial sums (M_n) of M is Cauchy. Fix $\varepsilon > 0$. Then there exists an index N such that for all $n > m > N$,

$$\sum_{j=m+1}^{n} \|x_j\|^2 < \varepsilon^2.$$

Now let us inspect $\|s_n - s_m\|^2$:

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\langle \sum_{j=m+1}^{n} x_j, \sum_{k=m+1}^{n} x_k \right\rangle \\ &= \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \langle x_j, x_k \rangle \\ &= \sum_{j=m+1}^{n} \|x_j\|^2 \end{aligned}$$

since (x_j) is orthogonal. Then

$$\begin{aligned} \|s_n - s_m\|^2 &= \sum_{j=m+1}^{n} \|x_j\|^2 \\ &< \varepsilon^2 \\ \implies \|s_n - s_m\| &< \varepsilon. \end{aligned}$$

Hence, (s_n) is a Cauchy sequence, as desired.

(b) Remove the orthogonality assumption from part (a), but assume instead the more stringent series condition that $\|x_1\| + \|x_2\| + \|x_3\| + \cdots$ converges. Show that (s_n) is a Cauchy sequence, where $s_n = x_1 + \cdots + x_n$.

Proof: Suppose $M = \|x_1\| + \|x_2\| + \cdots$ converges. Then M is a Cauchy sequence, hence, for any $\varepsilon > 0$, there exists an index N such that whenever $n > m > N$, we have

$$\sum_{k=m+1}^{n} \|x_k\| < \varepsilon.$$

Now, define

$$s_n := x_1 + x_2 + \cdots + x_n.$$

We will show that (s_n) is Cauchy. Let $n > m > N$ as above, and notice

$$\begin{aligned}\|s_n - s_m\|^2 &= \left\langle \sum_{j=m+1}^{n} x_j, \sum_{k=m+1}^{n} x_k \right\rangle \\ &= \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \langle x_j, x_k \rangle\end{aligned}$$

and by the Schwarz inequality, we have

$$\begin{aligned}\|s_n - s_m\|^2 &= \left| \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \langle x_j, x_k \rangle \right| \\ &\leq \sum_{j=m+1}^{n} \sum_{k=m+1}^{n} \|x_j\| \|x_k\| \\ &= \left(\sum_{j=m+1}^{n} \|x_j\| \right) \left(\sum_{k=m+1}^{n} \|x_k\| \right) \\ &< \varepsilon \cdot \varepsilon \\ &= \varepsilon^2.\end{aligned}$$

So now we have $\|s_n - s_m\|^2 < \varepsilon^2$ for all $n > m > N$, hence

$$\|s_n - s_m\| < \varepsilon$$

so that (s_n) is a Cauchy sequence.