Homework XI

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Section 7.1 Problems

7. **(Inverse)** Show that the inverse A^{-1} of a square matrix exists if and only if all the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A are different from zero. If A^{-1} exists, show that it has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

   Proof: First suppose that A^{-1} exists. Then we have $\det(A) \neq 0$, and notice $\det(A) = \det(A - 0 \cdot I) \neq 0$ so that 0 is not an eigenvalue by definition. Now suppose that 0 is not an eigenvalue. Then $\det(A - 0 \cdot I) = \det(A) \neq 0$ so that A is invertible.

   Now, suppose A^{-1} exists and let λ be an eigenvalue of A and let v be an associated eigenvector of λ. Then
$$Av = \lambda v$$
   multiplying the above equation on the left by A^{-1}, we have
$$\begin{aligned} A^{-1}Av &= A^{-1}(\lambda v) \\ v &= \lambda A^{-1} v \end{aligned}$$
   and since $\lambda \neq 0$ by the above proof, we may divide each side of the above equation by λ:
$$A^{-1}v = \frac{1}{\lambda} v$$
   so that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} by definition. Thus, if λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Thus, $\frac{1}{\lambda_1}, \cdots \frac{1}{\lambda_n}$ are eigenvalues of A^{-1}. We must now show that these are precisely the eigenvalues of A^{-1}. By the fundamental theorem of algebra and factorization theorem, we have
$$\det(A - \lambda I) = \prod_{k=1}^{n} (\lambda - \lambda_k)$$
   where $\lambda_k, 1 \leq k \leq n$ are the eigenvalues of A. But then
$$\det(A^{-1} - \lambda I) = \prod_{k=1}^{n} (\lambda - \lambda'_k)$$
   where $\lambda'_k, 1 \leq k \leq n$ are the eigenvalues of A^{-1}. But since $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ are eigenvalues of A^{-1}, we have
$$\det(A^{-1} - \lambda I) = \prod_{k=1}^{n} (\lambda - \tfrac{1}{\lambda_k})$$
   so that $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ are precisely the eigenvalues of A^{-1}.

14. Show that the geometric multiplicity of an eigenvalue cannot exceed the algebraic multiplicity.

    Proof: Let X be a normed space and $\dim(X) = n$. Let $T : X \to X$ be a linear transformation and λ_0 an eigenvalue of T with algebraic multiplicity ℓ and geometric multiplicity m. Let $\{e_1, \cdots, e_m\}$ be a basis for the eigenspace corresponding to λ_0. We may extend this to a basis
$$e = \{e_1, \cdots, e_m, e_{m+1}, \cdots, e_n\}$$

of X. We now find the matrix representation of T with respect to the basis e, $[T]_e$. Suppose

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C, D are themselves matrices. Let v be an eigenvector corresponding to λ_0 and since $Tv = \lambda_0 v$, we see by the block matrix representation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} v = \lambda_0 v$$

and definition of matrix-vector multiplication we see that $A = \lambda_0 I_m$ where I_m is the identity matrix relative to the basis $\{e_1, \cdots, e_m\}$. We then have that $C = \mathbf{0}$, so that

$$[T]_e = \begin{bmatrix} \lambda_0 I_m & B \\ \mathbf{0} & D \end{bmatrix}.$$

Now, finding the characteristic equation of $[T]_e$, notice

$$\begin{aligned} \det(T - \lambda I_n) &= \det(\lambda_0 I_m - \lambda I_m)\det(D - \lambda I_{n-m-1}) \\ &= (\lambda_0 - \lambda)^m \det(D - \lambda I_{n-m-1}) \end{aligned}$$

here I_n is the identity matrix relative to the basis e and I_{n-m-1} is the identity matrix relative to $\{e_{m+1}, \cdots, e_n\}$. Notice that the above expression gives us that the algebraic multiplicity ℓ of λ_0 is greater than or equal to m, that is

$$\ell \geq m$$

which is what we sought to show.

Section 7.2 Problems

3. **(Invariant subspace)** A subspace Y of a normed space X is said to be invariant under a linear operator $T : X \to X$ if $T(Y) \subset Y$. Show that an eigenspace of T is invariant under T. Give examples.

Proof: Let λ be an eigenvalue of T and let E be the eigenspace of T corresponding to λ. Let $v \in E$. Then by definition, we have

$$Tv = \lambda v.$$

Since E is itself a vector space and λ is a scalar, we have that $\lambda v \in E$. Thus, since $v \in E$ was chosen arbitrarily, we have

$$T(E) \subseteq E$$

so that E is invariant under T by definition.

For examples, considering the matrices in problems 12 and 13 from section 7.1, we have
12:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which has eigenvalues $\lambda_1 = \lambda_2 = 1$ with associated eigenvector $v = (1, 0)^T$. Now consider $x = a \cdot v$ for some scalar a. Notice

$$\begin{aligned} Ax &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} \end{aligned}$$

so that A maps elements of the eigenspace to the eigenspace.
13: Let

$$A_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

be an $n \times n$ matrix. We prove by induction that $\lambda_1 = \cdots = \lambda_n = 1$ for $n \geq 2$. For the case $n = 2$, see the above example. Now suppose this holds up to some integer k. We wish to show that A_{k+1} has eigenvalues of only 1. Well, notice

$$\det(A_{k+1} - \lambda I_{k+1}) = (1 - \lambda)\det(A_k - \lambda I_k)$$

(this follows from the cofactor definition of the determinant and the fact that the last row of $A_{k+1} = (0, 0, \cdots, 0, 1)$) and since, by the induction hypothesis, A_k has eigenvalues that are all equal to 1, we see that the eigenvalues of A_{k+1} are all equal to one.
It is also easy to see that the associated eigenvector of A_n is $v = (1, 0, \cdots, 0)^T$. Thus,

$$A_n(av) = av$$

so that A_n maps elements of its eigenspace to its eigenspace.

Section 7.3 Problems

4. Let $T : \ell^2 \to \ell^2$ be defined by $y = Tx$, $x = (\xi_j)$, $y = (\eta_j)$, $\eta_j = \alpha_j \xi_j$, where (α_j) is dense in $[0, 1]$. Find $\sigma_p(T)$ and $\sigma(T)$.

   Proof: I claim that $\sigma_p(T) = \{\alpha_j \mid j \geq 1\}$. Notice that if $x_j = (0, 0, \cdots, 1, 0, \cdots)$ (all zeros except a 1 in the j $^{\text{th}}$ position), we have that

   $$\begin{aligned} Tx_j &= (0, 0, \cdots, \alpha_j, 0, \cdots) \\ &= \alpha_j x_j \end{aligned}$$

   so that x_j is an eigenvector with associated eigenvalue α_j. Now, suppose that there exists eigenvalues $\lambda \notin \{\alpha_j \mid j \geq 1\}$. Then by definition, we have

   $$Tv = \lambda v$$

   with $v = (\nu_1, \nu_2, \cdots)$. Note that $v \neq x_j$, $j \geq 1$ since if it were equal to an x_j, α_j would be an eigenvalue, contrary to our assumption. But

   $$\begin{aligned} Tv &= (\alpha_1 \nu_1, \alpha_2 \nu_2, \cdots) \\ &= (\lambda \nu_1, \lambda \nu_2, \cdots) \end{aligned}$$

   which gives us either $v \equiv 0$ or $\lambda = \alpha_j$ for all α_j such that $\nu_j \neq 0$. Since λ is a single scalar, it must be the case that $v \equiv 0$ which is not an eigenvector by definition. Thus, if $\lambda \notin \{\alpha_j \,|\, j \geq 1\}$, then $\lambda \notin \sigma_p(T)$.

   We now note that T is bounded since

   $$\begin{aligned} \|Tx\|^2 &= \sum_{j=1}^{\infty} \alpha_j^2 |\xi_j|^2 \\ &\leq \sum_{j=1}^{\infty} |\xi_j|^2 = \|x\|^2 \\ \implies \|Tx\| &\leq \|x\|. \end{aligned}$$

We now show that if $\lambda \in \mathbb{C}$ such that $\lambda \notin [0,1]$, we have that $\lambda \in \rho(T)$.
We must show that λ is a regular value of T. Consider $T_\lambda = T - \lambda I$. We must first show that T_λ^{-1} exists. To do so, suppose $x \in \ell^2$ such that $T_\lambda x = 0$. That is,

$$\begin{aligned} T_\lambda x &= (T - \lambda I)x = 0 \\ \implies Tx &= \lambda x \end{aligned}$$

which, if $x \neq 0$, gives us that λ is an eigenvalue by definition. But since $\lambda \notin [0,1]$, we have that λ is not an eigenvalue by our above work, so that it must be the case that $x = 0$. Thus, $\mathcal{N}(T_\lambda) = \{0\}$ so that T_λ^{-1} exists. We now show that T_λ^{-1} is surjective. Let $y \in \ell^2$, $y = (\eta_1, \eta_2, \cdots)$ and consider $x = (\xi_1, \xi_2, \cdots)$ with $\xi_j = \frac{\eta_j}{\alpha_j - \lambda}$. Note that there exists some $\varepsilon > 0$ such that $|\alpha_j - \lambda| \geq \varepsilon$ for all $j \in \mathbb{N}$, for if there were no such ε, for some $j \in \mathbb{N}$, we would have

$$|\alpha_j - \lambda| < \varepsilon$$

for all $\varepsilon > 0$. But then λ is a limit point of (α_j), and since (α_j) is dense in $[0,1]$, we have $\lambda \in [0,1]$, a contradiction. Thus,

$$\frac{1}{\alpha_j - \lambda} \leq \frac{1}{\varepsilon}$$

and so

$$\begin{aligned} \sum_{j=1}^{\infty} |\xi_j|^2 &= \sum_{j=1}^{\infty} \frac{|\eta_j|^2}{|\alpha_j - \lambda|^2} \\ &\leq \sum_{j=1}^{\infty} \frac{|\eta_j|^2}{\varepsilon^2} < \infty \end{aligned}$$

since $y \in \ell^2$. Hence $x \in \ell^2$ and T_λ^{-1} is surjective and is hence bijective since T is injective. Since T is bounded, T_λ is bounded, and by the bounded inverse theorem, we have that T_λ^{-1} is bounded. Thus, $\lambda \in \rho(T)$.

Now, since T is bounded and ℓ^2 is a Banach space, we have that $\sigma(T)$ is closed, thus

$$\sigma_p(T) \subseteq \sigma(T) \subseteq [0,1]$$

and since $\sigma_p(T)$ is dense in $[0,1]$, and $\sigma(T)$ is closed, we have

$$\overline{\sigma_p(T)} = [0,1] \subseteq \overline{\sigma(T)} = \sigma(T)$$

so

$$[0,1] \subseteq \sigma(T) \subseteq [0,1]$$

hence

$$\sigma(T) = [0,1].$$