Modern Algebra HW 11

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Section 26 Problems

12. Give an example to show that a factor ring of an integral domain may be a field.

Let p be prime and note that $\mathbb{Z}$ is an integral domain. Then $\mathbb{Z}/p\mathbb{Z}$ is a field.

13. Give an example to show that a factor ring of an integral domain may have divisors of 0.

Note that $\mathbb{Z}$ is an integral domain and consider the factor ring $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$. Notice that $2 \cdot 2 = 0$ in $\mathbb{Z}_4$, so the factor ring $\mathbb{Z}/4\mathbb{Z}$ has a divisor of 0.

1)

Prove: If I is an ideal of R, and 1_R is in I, then $I = R$.

Proof: Let I be an ideal of R and suppose $1_R \in I$. We wish to show that $I = R$. Well, since I is an ideal, the absorption property holds. Using the fact that $1_R \in I$ and the absorption property, we have for $r \in R$, $r \cdot 1_R \in I$ and $1_R \cdot r \in I$ and further, $r \cdot 1_R = 1_R \cdot r = r$ since 1_R is the multiplicative identity of R. That is, we have for any $r \in R$, $r \in I$. And since I is an ideal of R, we have $\langle I, +\rangle \leq \langle R, +\rangle$ and so $I = R$.

2)

Prove: Let F be a field. Then the only ideals of F are $\{0\}$ and F.

Proof: To begin, I will show that $\{0\}$ and F are ideals of F. For $\{0\}$, for any $f \in F$, we have $0 \cdot f = f \cdot 0 = 0$, so $\{0\}$ is clearly an ideal for F. On a similar vein, for any $f_1, f_2 \in F$, $f_1 \cdot f_2, f_2 \cdot f_1 \in F$ since F is a field. Thus, F is also an ideal of F. Now suppose by way of contradiction that there exists some ideal I of F where $I \neq \{0\}$ and $I \neq F$. Since I is assumed to be an ideal, the absorption property holds. But F is also a field, so $\langle F^{\neq 0}, \cdot\rangle$ is a group and so every element in the multiplicative group has an inverse. And since I is an ideal of F, we have for any $i \in I$, $i \in F$. Then $i^{-1} \in F$ is the multiplicative inverse of i and so

$$i^{-1} \cdot i = 1_F \in I$$

And by problem 1), we have that $I = F$, a contradiction.

3)

a) For any pair of integers a, b in $\mathbb{Z}$, it turns out that $a\mathbb{Z} \cap b\mathbb{Z} = c\mathbb{Z}$ for some integer c. What is c?

$c = \text{lcm}(a, b)$ where $\text{lcm}(a, b)$ denotes the least common multiple between a and b. See scratch work below.

By inspection, if a does not divide b, we have $a\mathbb{Z} \cap b\mathbb{Z} = \{\cdots, -2ab, -ab, 0, ab, 2ab, \cdots\}$. However, if a

does divide b, then the smallest non-zero value in $a\mathbb{Z} \cap b\mathbb{Z}$ is $\operatorname{lcm}(a, b)$. Notice in the case where a and b are relatively prime, $\operatorname{lcm}(a, b) = ab$.

b) For any pair of integers a, b in $\mathbb{Z}$, it turns out that $a\mathbb{Z} + b\mathbb{Z} = c\mathbb{Z}$ for some integer c. What is c?

   $c = \gcd(a, b)$. See scratch work below.

   Begin with $3\mathbb{Z} + 2\mathbb{Z}$:

$$\begin{aligned} 3\mathbb{Z} &= \{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\} \\ 6\mathbb{Z} &= \{\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots\} \\ 3\mathbb{Z} + 6\mathbb{Z} &= \{\cdots, 0, 1, 2, 3, 4, 5, 6, 7, \cdots\} \\ &= \mathbb{Z} \end{aligned}$$

   and notice $\gcd(3, 2) = 1$.
   Now inspect $6\mathbb{Z} + 8\mathbb{Z}$:

$$\begin{aligned} 6\mathbb{Z} &= \{\cdots, -18, -12, -6, 0, 6, 12, 18, \cdots\} \\ 8\mathbb{Z} &= \{\cdots, -24, -16, -8, 0, 8, 16, 24, \cdots\} \\ 6\mathbb{Z} + 8\mathbb{Z} &= \{\cdots, 0, 2, 4, 6, 8, 10, \cdots\} \\ &= 2\mathbb{Z} \end{aligned}$$

   and notice $\gcd(6, 8) = 2$.