sinc^2 Integral

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Show

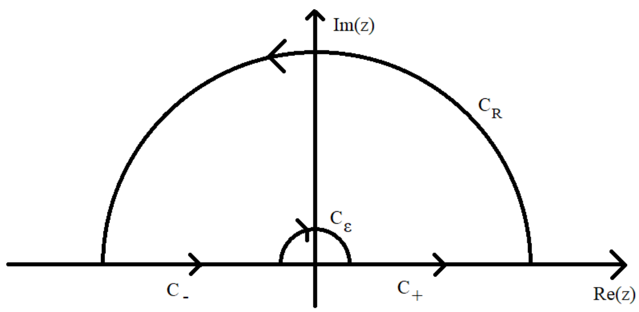
$$\int_0^\infty \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2}$$

Proof: We first show that the integral converges. Note that $\frac{\sin^2(x)}{x^2}$ has a removable singularity at $x = 0$ since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Thus, $\frac{\sin^2(x)}{x^2}$ is integrable near $x = 0$. Further notice, for $R > 0$

$$\begin{aligned}
\left|\int_R^\infty \frac{\sin^2(x)}{x^2}\right| &\le \int_R^\infty \left|\frac{\sin^2(x)}{x^2}\right| dx \\
&\le \int_R^\infty \frac{1}{x^2} dx \\
&= \frac{1}{R} < \infty.
\end{aligned}$$

Thus, the integral converges. Now, we approach this in two ways.

(a) **Approach 1.** Consider the function $f(z) = \frac{1-e^{2iz}}{z^2}$ over the indented semicircle contour.



By Cauchy's theorem, since $f(z)$ is analytic in and on C,

$$\oint_C f(z) = 0.$$

For C_R, parameterize $z = Re^{i\theta}$, $dz = Rie^{i\theta}d\theta$. Then

$$\begin{aligned}
\int_{C_R} \frac{e^{2iz}-1}{z^2} dz &= \int_0^\pi \frac{e^{2iRe^{i\theta}}-1}{R^2e^{2i\theta}} Rie^{i\theta} d\theta \\
&= \frac{i}{R}\int_0^\pi \frac{e^{2iR\cos(\theta)}e^{-2R\sin(\theta)}-1}{e^{i\theta}} d\theta \\
\implies \left|\int_{C_R} f\right| &\le \frac{1}{R}\int_0^\pi \left|\frac{e^{2iR\cos(\theta)}e^{-R\sin(\theta)}-1}{e^{i\theta}}\right| d\theta \\
&\le \frac{1}{R}\int_0^\pi \left(e^{-2R\sin(\theta)}+1\right) d\theta \\
&= \frac{\pi}{R} + \frac{1}{R}\int_0^\pi e^{-2R\sin(\theta)} d\theta \\
&= \frac{\pi}{R} + \frac{2}{R}\int_0^{\pi/2} e^{-2R\sin(\theta)} d\theta.
\end{aligned}$$

Note that for $\theta \in [0, \pi/2]$, $\sin(\theta) \geq \frac{2}{\pi}\theta \implies -2R\sin(\theta) \leq -\frac{4}{\pi}R\theta$ hence

$$\begin{aligned}
\left|\int_{C_R} f\right| &\leq \frac{\pi}{R} + \frac{2}{R}\int_0^\pi e^{-\frac{4}{\pi}R\theta}d\theta \\
&= \frac{\pi}{R} + \frac{2}{R}\left[-\frac{\pi}{4R}e^{-\frac{4}{\pi}R\theta}\right]\Big|_0^{\pi/2} \\
&= \frac{\pi}{R} - \frac{\pi}{2R^2}e^{-2R} + \frac{\pi}{4R^2} \\
&\to 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Now, on C_ε, Taylor expanding e^{2iz} gives

$$e^{2iz} = 1 + 2iz - 2z^2 - \frac{4i}{3}z^3 + \cdots$$

so that the integral becomes

$$\begin{aligned}
\int_{C_\varepsilon} \frac{e^{2iz} - 1}{z^2}dz &= \int_{C_R} \frac{1 + 2iz - 2z^2 - \frac{4i}{3}z^3 + \cdots - 1}{z^2}dz \\
&= \int_{C_\varepsilon} \frac{1}{z}\left(2i - 2z - \frac{4}{3}z^2 + \cdots\right)dz
\end{aligned}$$

now parameterize $z = \varepsilon e^{i\theta}$, $dz = \varepsilon i e^{i\theta}d\theta$. Then the above integral becomes

$$\begin{aligned}
\int_\pi^0 \frac{1}{\varepsilon e^{i\theta}}\left(2i - 2\varepsilon e^{i\theta} + \cdots\right)\varepsilon i e^{i\theta}d\theta = i\int_\pi^0 \left(2i - 2\varepsilon e^{i\theta} + \cdots\right)d\theta \\
= 2\pi + 4\varepsilon + \mathcal{O}(\varepsilon^2) \\
= 2\pi \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}$$

Further, as $\varepsilon \to 0$ and $R \to \infty$,

$$\begin{aligned}
\int_{C_-} f + \int_{C_+} f &\to \int_{-\infty}^\infty \frac{e^{2ix} - 1}{x^2}dx \\
&= \int_{-\infty}^\infty \frac{\cos(2x) - 1}{x^2}dx + i\int_{-\infty}^\infty \frac{\sin(2x)}{x^2}dx \\
&= -2\int_{-\infty}^\infty \frac{\sin^2(x)}{x^2}dx + i\int_{-\infty}^\infty \frac{\sin(2x)}{x^2}dx
\end{aligned}$$

and note that the imaginary part of the above integral converges in the principle value sense. Putting it together, we have

$$\begin{aligned}
-2\int_{-\infty}^\infty \frac{\sin^2(x)}{x^2}dx + i\int_{-\infty}^\infty \frac{\sin(2x)}{x^2}dx + 2\pi = 0 \\
\implies \int_{-\infty}^\infty \frac{\sin^2(x)}{x^2}dx = \pi
\end{aligned}$$

and since $\frac{\sin^2(x)}{x^2}$ has even parity and $\displaystyle\int_0^\infty \frac{\sin^2(x)}{x^2}dx$ converges, we have

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}$$

as desired.

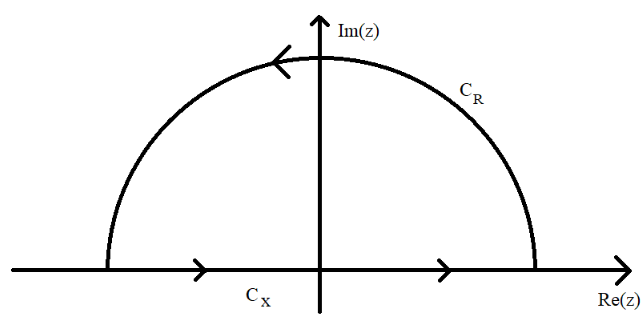
(b) **Approach 2.** Consider the function $f(z) = \frac{e^{2iz}-1-2iz}{z^2}$. Note that f has a removable singularity at $z = 0$. By L'Hopital's rule, we have

$$\begin{aligned}\lim_{z\to 0} \frac{e^{2iz}-1-2iz}{z^2} &= \lim_{z\to 0} \frac{2ie^{iz}-2i}{2z} \\ &= \lim_{z\to 0} \frac{-4e^{iz}}{2} \\ &= -2\end{aligned}$$

Define the auxillary function

$$\hat{f}(z) = \begin{cases} f(z) & z \neq 0 \\ -2 & z = 0. \end{cases}$$

We then integrate $\hat{f}(z)$ over the contour



By Cauchy's theorem, we have

$$\oint_C \hat{f} = 0.$$

On C_R, parameterize $z = Re^{i\theta}$, $dz = Rie^{i\theta}d\theta$. Then

$$\begin{aligned}\int_{C_R} \hat{f}(z)dz &= \int_0^\pi \frac{e^{2iRe^{i\theta}}-1-2iRe^{i\theta}}{R^2e^{2i\theta}} Rie^{i\theta}d\theta \\ &= i\int_0^\pi \frac{e^{2iRe^{i\theta}}}{Re^{i\theta}}d\theta - i\int_0^\pi \frac{1}{Re^{i\theta}}d\theta + 2\pi \\ &= i\int_0^\pi \frac{e^{2iRe^{i\theta}}}{Re^{i\theta}}d\theta - \frac{2}{R} + 2\pi.\end{aligned}$$

Now notice

$$\begin{aligned}\left| i\int_0^\pi \frac{e^{2iRe^{i\theta}}}{Re^{i\theta}}d\theta \right| &\leq \frac{1}{R}\int_0^\pi \left| e^{2iR\cos(\theta)-2R\sin(\theta)} \right| d\theta \\ &= \frac{1}{R}\int_0^\pi e^{-2R\sin(\theta)}d\theta \\ &= \frac{1}{R}\int_0^{\pi/2} e^{-2R\sin(\theta)}d\theta\end{aligned}$$

And notice that for $0 \leq \theta \leq \frac{\pi}{2}$, $\sin(\theta) \geq \frac{2}{\pi}\theta \implies -2R\sin(\theta) \leq -\frac{4R}{\pi}\theta$ hence

$$\begin{aligned}
\frac{1}{R}\int_0^{\pi/2} e^{-2R\sin(\theta)}d\theta &\leq \frac{1}{R}\int_0^{\pi/2} e^{-\frac{4}{\pi}R\theta}d\theta \\
&= \frac{1}{R}\left[-\frac{\pi}{4R}e^{-\frac{4}{\pi}R\theta}\right]\Bigg|_0^{\pi/2} \\
&= \frac{1}{R}\left[\frac{\pi}{4R} - \frac{\pi}{4R}e^{-2}\right] \\
&\to 0 \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Thus, as $R \to \infty$,

$$\int_{C_R} \hat{f} \to 2\pi.$$

Further, as $R \to \infty$,

$$\begin{aligned}
\int_{C_x} \hat{f} &\to \int_{-\infty}^{\infty} \frac{e^{2ix} - 1 - 2ix}{x^2}dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(2x) + i\sin(2x) - 1 - 2ix}{x^2}dx \\
&= \int_{-\infty}^{\infty} \frac{\cos(2x) - 1}{x^2}dx + i\int_{-\infty}^{\infty} \frac{\sin(2x) - 2x}{x^2}dx \\
\implies \operatorname{Re}\left(\int_{C_x} \hat{f}\right) &= -2\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}dx
\end{aligned}$$

Now,

$$\oint_C \hat{f} = \int_{C_x} \hat{f} + \int_{C_R} \hat{f} = 0$$

$$\begin{aligned}
\implies -2\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}dx + i\int_{-\infty}^{\infty} \frac{\sin(2x) - 2x}{x^2}dx + 2\pi = 0 \\
\implies \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}dx = \pi \\
\implies 2\int_0^{\infty} \frac{\sin^2(x)}{x^2}dx = \pi \\
\implies \int_0^{\infty} \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}
\end{aligned}$$

As desired.