Math 4510
Topology

Problem Set 2

1. (#5 in 2.5) Prove that the finite union of countable sets is countable. (See the hints given in the text for this problem).

   We wish to show that for some $k \in \mathbb{N}$, $\bigcup_{i=1}^{k} A_i$ is countable for some collection of countable sets $\{A_i\}_i^k$. Let's begin by showing the union of two countable sets is also countable. By definition of countable, there exist onto functions

   $$f_1 : \mathbb{N} \to A_1$$

   $$f_2 : \mathbb{N} \to A_2$$

   Let $B = A_1 \cup A_2$. We wish to show that there exists an onto function that maps $\mathbb{N} \to B$. Well, consider the function $f_B : \mathbb{N} \to B$

   $$f_B(n) = \begin{cases} f_1(\frac{n+1}{2}), \ n \text{ odd} \\ f_2(\frac{n}{2}), \ n \text{ even} \end{cases}$$

   Notice that f_B will alternate between mapping to elements of A_1 and A_2. That is, f_B is an onto map from $\mathbb{N}$ to B. That is, we have that B is a countable set.

   Now consider the union of the three countable sets $A_1 \cup A_2 \cup A_3 = B \cup A_3$. Since B is countable and A_3 is countable, we have from our work above that $B \cup A_3$ is also countable. Continuing this argument up to k, we have that the finite union of countable sets is countable.

2. (#4 in 3.2) Find all the topologies on the set $X = \{a, b, c\}$. There are 29 of them. (Hint: be extremely organized in how you write them down).

   - $\{\emptyset, X\}$
     Notice that this is the trivial topology.
   - $\{\emptyset, \{a\}, X\}$
   - $\{\emptyset, \{b\}, X\}$
   - $\{\emptyset, \{c\}, X\}$
     I will show the three sets above are topologies. Focus on $\{\emptyset, \{a\}, X\}$, and call this A. Notice the following:

     $$\emptyset \cup \{a\} = \{a\} \in A$$

     $$\emptyset \cap \{a\} = \emptyset \in A$$

     $$\{a\} \cup X = X \in A$$

     $$\{a\} \cap X = \{a\} \in A$$

     So A is a topology on X. The same holds for the other two sets have the same structure.

- $\{\emptyset, \{a, b\}, X\}$
- $\{\emptyset, \{a, c\}, X\}$
- $\{\emptyset, \{b, c\}, X\}$

  I will show the three sets above are also topologies. Let $B = \{\emptyset, \{a, b\}, X\}$ and notice the following:

$$\emptyset \cup \{a, b\} = \{a, b\} \in B$$
$$\emptyset \cap \{a, b\} = \emptyset \in B$$
$$X \cup \{a, b\} = X \in B$$
$$X \cap \{a, b\} = \{a, b\} \in B$$

  So B is a topology on X. The same holds because the other two sets because the collections have the same structure.

- $\{\emptyset, \{c\}, \{a, b\}, X\}$
- $\{\emptyset, \{b\}, \{a, c\}, X\}$
- $\{\emptyset, \{a\}, \{b, c\}, X\}$

  To show that the above three collections are topologies, let $\tau = \{\emptyset, \{c\}, \{a, b\}, X\}$ and notice the following:

$$\emptyset \in \tau$$
$$X \in \tau$$
$$\emptyset \cap \{a\} = \emptyset \in \tau$$
$$\emptyset \cap \{b, c\} = \emptyset \in \tau$$
$$\emptyset \cap X = \emptyset \in \tau$$
$$\emptyset \cup \{a\} = \{a\} \in \tau$$
$$\emptyset \cup \{b, c\} = \{b, c\} \in \tau$$
$$\emptyset \cup X = X \in \tau$$
$$\{a\} \cap \{b, c\} = \emptyset \in \tau$$
$$\{a\} \cap X = \{a\} \in \tau$$
$$\{b, c\} \cap X = \{b, c\} \in \tau$$
$$\{b, c\} \cup X = X \in \tau$$

  So τ is a topology on X. The other two collections are also topologies because they have the same structure.

- $\{\emptyset, \{a\}, \{a, b\}, X\}$
- $\{\emptyset, \{b\}, \{a, b\}, X\}$
- $\{\emptyset, \{a\}, \{a, c\}, X\}$

- $\{\emptyset, \{c\}, \{a, c\}, X\}$
- $\{\emptyset, \{b\}, \{b, c\}, X\}$
- $\{\emptyset, \{c\}, \{b, c\}, X\}$

  To show the above nine sets are topologies, let $C = \{\emptyset, \{a\}, \{a, b\}, X\}$ and notice the following:

$$\{a\} \cup \emptyset = \{a\} \in C$$
$$\{a\} \cap \emptyset = \emptyset \in C$$
$$\{a, b\} \cup \emptyset = \{a, b\} \in C$$
$$\{a, b\} \cap \emptyset = \emptyset \in C$$
$$\{a\} \cup X = X \in C$$
$$\{a\} \cap X = \{a\} \in C$$
$$\{a, b\} \cup X = X \in C$$
$$\{a, b\} \cap X = \{a, b\} \in C$$
$$\{a\} \cup \{a, b\} = \{a, b\} \in C$$
$$\{a\} \cap \{a, b\} = \{a\} \in C$$

  So C is a topology on X. The same holds for the other eight collections because the sets have the same structure.

- $\{\emptyset, \{a\}, \{a, b\}, \{b\}, X\}$
- $\{\emptyset, \{a\}, \{a, c\}, \{c\}, X\}$
- $\{\emptyset, \{b\}, \{b, c\}, \{c\}, X\}$

  To show the above three sets are topologies, let $D = \{\emptyset, \{a\}, \{a, b\}, \{b\}, X\}$ and notice the following:

$$\emptyset \cup \{a\} = \{a\} \in D$$
$$\emptyset \cap \{a\} = \emptyset \in D$$
$$X \cup \{a\} = X \in D$$
$$X \cap \{a\} = \{a\} \in D$$
$$\emptyset \cup \{b\} = \{b\} \in D$$
$$\emptyset \cap \{b\} = \emptyset \in D$$
$$X \cup \{b\} = X \in D$$
$$X \cap \{b\} = \{b\} \in D$$
$$\emptyset \cup \{a, b\} = \{a, b\} \in D$$
$$\emptyset \cap \{a, b\} = \emptyset \in D$$
$$X \cup \{a, b\} = X \in D$$
$$X \cap \{a, b\} = \{a, b\} \in D$$

$$\{a\} \cup \{a,b\} = \{a,b\} \in D$$
$$\{a\} \cap \{a,b\} = \{a\} \in D$$
$$\{b\} \cup \{a,b\} = \{a,b\} \in D$$
$$\{b\} \cap \{a,b\} = \{b\} \in D$$
$$\{a\} \cup \{b\} \cup \{a,b\} = \{a,b\} \in D$$
$$\{a\} \cap \{b\} \cap \{a,b\} = \emptyset \in D$$

So D is a topology on X. The same holds for the other two collections because they have the same structure.

- $\{\emptyset, \{a,b\}, \{a,c\}, \{a\}, X\}$
- $\{\emptyset, \{b,c\}, \{a,c\}, \{c\}, X\}$
- $\{\emptyset, \{b,c\}, \{a,b\}, \{b\}, X\}$

To show the above three sets are topologies, let $E = \{\emptyset, \{a,b\}, \{a,c\}, \{a\}, X\}$ and notice the following:

$$\emptyset \cup \{a\} = \{a\} \in E$$
$$\emptyset \cap \{a\} = \emptyset \in E$$
$$X \cup \{a\} = X \in E$$
$$X \cap \{a\} = \{a\} \in E$$
$$\emptyset \cup \{a,b\} = \{a,b\} \in E$$
$$\emptyset \cap \{a,b\} = \emptyset \in E$$
$$X \cup \{a,b\} = X \in E$$
$$X \cap \{a,b\} = \{a,b\} \in E$$
$$\emptyset \cup \{a,c\} = \{a,c\} \in E$$
$$\emptyset \cap \{a,c\} = \emptyset \in E$$
$$X \cup \{a,c\} = X \in E$$
$$X \cap \{a,c\} = \{a,c\} \in E$$
$$\{a\} \cup \{a,b\} = \{a,b\} \in E$$
$$\{a\} \cap \{a,b\} = \{a\} \in E$$
$$\{a\} \cup \{a,c\} = \{a,c\} \in E$$
$$\{a\} \cap \{a,c\} = \{a\} \in E$$
$$\{a,b\} \cup \{a,c\} = X \in E$$
$$\{a,b\} \cap \{a,c\} = \{a\} \in E$$
$$\{a\} \cup \{a,b\} \cup \{a,c\} = X \in E$$
$$\{a\} \cap \{a,b\} \cap \{a,c\} = \{a\} \in E$$

So E is a topology on X. The same result holds for the other two collections because the sets have the same structure.

- $\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}$
- $\{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\}$
- $\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$
- $\{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\}$
- $\{\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}, X\}$
- $\{\emptyset, \{b\}, \{c\}, \{a,c\}, \{b,c\}, X\}$

  To see that the above six sets are topologies, let $F = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}$ and notice the following:

$$\emptyset \cup \{a\} = \{a\} \in F$$
$$\emptyset \cap \{a\} = \emptyset \in F$$
$$X \cup \{a\} = X \in F$$
$$X \cap \{a\} = \{a\} \in F$$
$$\emptyset \cup \{b\} = \{b\} \in F$$
$$\emptyset \cap \{b\} = \emptyset \in F$$
$$X \cup \{b\} = X \in F$$
$$X \cap \{b\} = \{b\} \in F$$
$$\emptyset \cup \{a,b\} = \{a,b\} \in F$$
$$\emptyset \cap \{a,b\} = \emptyset \in F$$
$$X \cup \{a,b\} = X \in F$$
$$X \cap \{a,b\} = \{a,b\} \in F$$
$$\emptyset \cup \{b,c\} = \{b,c\} \in F$$
$$\emptyset \cap \{b,c\} = \emptyset \in F$$
$$X \cup \{b,c\} = X \in F$$
$$X \cap \{b,c\} = \{b,c\} \in F$$
$$\{a\} \cup \{b\} = \{a,b\} \in F$$
$$\{a\} \cap \{b\} = \emptyset \in F$$
$$\{a\} \cup \{a,b\} = \{a,b\} \in F$$
$$\{a\} \cap \{a,b\} = \{a\} \in F$$
$$\{a\} \cup \{b,c\} = X \in F$$
$$\{a\} \cap \{b,c\} = \emptyset \in F$$
$$\{b\} \cup \{a,b\} = \{a,b\} \in F$$
$$\{b\} \cap \{a,b\} = \{b\} \in F$$
$$\{b\} \cup \{b,c\} = \{b,c\} \in F$$

$$\{b\} \cap \{b, c\} = \{b\} \in F$$
$$\{a, b\} \cup \{b, c\} = X \in F$$
$$\{a, b\} \cap \{b, c\} = \{b\} \in F$$
$$\{a\} \cup \{b\} \cup \{a, b\} = \{a, b\} \in F$$
$$\{a\} \cap \{b\} \cap \{a, b\} = \emptyset \in F$$
$$\{a\} \cup \{b\} \cup \{b, c\} = X \in F$$
$$\{a\} \cap \{b\} \cap \{b, c\} = \emptyset \in F$$
$$\{a\} \cup \{b\} \cup \{a, b\} \cup \{b, c\} = X \in F$$
$$\{a\} \cap \{b\} \cap \{a, b\} \cap \{b, c\} = \emptyset \in F$$

- $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$
  Note that this is the discrete topology.

3. In the topology U on $\mathbb{R}$, give an example of an arbitrary intersection of open sets that is nonempty and not open.

   Consider the sequence of intervals $s_n = (a - \frac{1}{n}, a + \frac{1}{n}), a \in \mathbb{R},\ n \in \mathbb{N}$. Clearly, each $s_n \in \mathcal{U}$. Now consider the intersection

   $$\bigcap_{n=1}^{\infty} s_n$$

   Notice that $a \in (a - \frac{1}{n}, a + \frac{1}{n})$, for all n, and so a is in the intersection. To show a is the only element of the intersection, let

   $$b \in \bigcap_{n=1}^{\infty} s_n,\ b \neq a$$

   That is, $b \in s_n$ for all n. However, since $b \neq a$, we can find some natural number k sufficiently large enough such that $b \notin s_k$. Thus, if $b \neq a$, b is not in the intersection of s_n.

   So we have

   $$\bigcap_{n=1}^{\infty} s_n = \{a\}$$

   which is not open in $\mathbb{R}_{\mathcal{U}}$.

4. (#8 in 3.2) Prove that the set RR is a topology on $\mathbb{R}$.

   Proof: First consider the definition of the right ray topology: $\{V \subseteq \mathbb{R} | \text{for ever } x \in \mathbb{R}, \text{there exists a ray } (a, \infty) \text{ for some } a \in \mathbb{R} \text{ with } x \in (a, \infty) \subseteq V\}$.

   Notice that the empty set $\emptyset$ does not have any points that will contradict the requirements to be in RR, so $\emptyset \in RR$. To show that $\mathbb{R}$ is in RR, fix $x \in \mathbb{R}$. then for any

$a < x$, $x \in (a, \infty)$. Since this is true for any $a < x$, $(-\infty, \infty) = \mathbb{R}$ defines a right ray in $\mathbb{R}$, so $\mathbb{R}$ is in the right ray topology.

Let A_λ be an indexed collection of elements of $\mathcal{RR}$ where $A_\lambda = (a_\lambda, \infty)$. Consider their union:

$$\bigcup_{\lambda \in \Lambda} (a_\lambda, \infty)$$

Consider the case where $\{a_\lambda\}$ is bounded below. Let $a = \inf a_\lambda$. Then $\bigcup_{\lambda \in \Lambda} A_\lambda = (a, \infty) \in RR$.

Now consider the case where $\{a_\lambda\}$ is unbounded. Then $\bigcup_{\lambda \in \Lambda} A_\lambda = \mathbb{R} \in RR$ by our work above. Thus, we have an arbitrary union of elements of RR is also in RR.

Now we must show that a finite intersection of elements of RR is also in RR.

Begin by considering the set $\{b_n\} = \{b_1, b_2, \ldots, b_n\}$, $n \in \mathbb{N}$, each $b_i \in \mathbb{R}$. now consider

$$\bigcap_{k=1}^{n} (b_k, \infty)$$

Since $\{b_n\}$ is a finite set, it has a maximum, call it $b_m = \max\{b_1, b_2, \ldots, b_n\}$. Then

$$\bigcap_{k=1}^{n} (b_k, \infty) = (b_m, \infty) \in RR$$

So a finite intersection of elements of RR is also in RR. Thus, RR is a topology on $\mathbb{R}$.

Recommendation: Also write out the details for #12 in 3.2, showing that the set $\mathcal{FC}$ is a topology on $\mathbb{R}$.

By definition, $\emptyset \in \mathcal{FC}$. Also notice that $\mathbb{R} \in \mathcal{FC}$ since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ which is finite by definition. Now we wish to show that an arbitrary union of open sets in $\mathcal{FC}$ is in $\mathcal{FC}$. Let A_λ be an indexed collection of open sets in $\mathcal{FC}$ and consider

$$\bigcup_{\lambda \in \Lambda} A_\lambda$$

Notice that each $\mathbb{R} \setminus A_i$ is finite for all $A_i \in \{A_\lambda\}_{\lambda \in \Lambda}$. We wish to show that the above set is in $\mathcal{FC}$, so we wish to show

$$\mathbb{R} \setminus \bigcup_{\lambda \in \Lambda} A_\lambda$$

is finite. Notice by DeMorgan's laws:

$$\mathbb{R} \setminus \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda \in \Lambda} (\mathbb{R} \setminus A_\lambda)$$

And since each $\mathbb{R} \setminus A_\lambda$ is finite, we have an arbitrary intersection of finite sets, which must be finite.

Now we wish to show a finite intersection of open sets is also open. Let $B_1, B_,2, \ldots, B_n$ be open sets in $\mathcal{FC}$ for some natural number n. We wish to show that

$$\bigcap_{k=1}^{n} B_k \in \mathcal{FC}$$

That is, we wish to show that $\mathbb{R} \setminus \bigcap_{k=1}^{n} B_k$ is finite. Well, by DeMorgan's law:

$$\mathbb{R} \setminus \bigcap_{k=1}^{n} B_k = \bigcup_{k=1}^{n} (\mathbb{R} \setminus B_k)$$

So we have a finite union of finite sets, which by a previous homework assignment, a finite union of open sets is open, so $\mathcal{FC}$ is a topology on $\mathbb{R}$.