Problem Set 6 (Astrophysics)

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1. As we discuss atoms, clouds of gas, and stars, it is useful to have a feel for the relative values of atomic energies and thermal energies. Show that at room temperature, the thermal energy (kT) is approximately 1/40 eV. At what temperature is kT equal to 1 eV ? to 13.6 eV ?

Using the value of $T = 68°F = 293K$ for room temperature, we find that

$$kT = (8.617 \times 10^{-5}\frac{eV}{K})(293K) \approx 0.0252eV \approx \frac{1}{40}eV$$

Now we wish to find the temperature when $kT = 1eV$. That is,

$$T = \frac{1}{K}eV$$
$$= \frac{1}{8.617 \times 10^{-5}\frac{eV}{K}}eV$$
$$\approx 11605K$$

Thus, at a temperature of approximately $11605K$, $kT = 1eV$. Now we wish to find the temperature when $kT = 13.6eV$. Notice if we multiply the previous answer by 13.6, we will find the temperature. Thus,

$$T \approx 157828K$$

So at a temperature of approximately $157828K$, $kT = 13.6eV$

2. Using a little calculus (finding the maximum of a function), show that the Maxwell-Boltzmann distribution, $F(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} v^2 exp\left(\frac{-mv^2}{2kT}\right)$, has its maximum at a speed of $v_p = \left(\frac{2kT}{m}\right)^{1/2}$.

We wish to show that $F(v)$ attains its maximum at $v = (\frac{2kT}{m})^{1/2}$. Recall that a function attains a minimum or maximum at a critical value, or when that function's first derivative is zero. Notice that

$$F'(v) = 8\pi(\frac{m}{2\pi kT})^{1/2}v\exp(\frac{-mv^2}{2kT}) - 4\pi(\frac{m}{2\pi kT})^{1/2}v^2(\frac{mv}{kT})\exp(\frac{-mv^2}{2kT})$$

And setting $F'(v)$ equal to zero, we find

$$8\pi(\frac{m}{2\pi kT})^{1/2}v\exp(\frac{-mv^2}{2kT}) - 4\pi(\frac{m}{2\pi kT})^{1/2}v^2(\frac{mv}{kT})\exp(\frac{-mv^2}{2kT}) = 0$$
$$8\pi(\frac{m}{2\pi kT})^{1/2}v\exp(\frac{-mv^2}{2kT}) = 4\pi(\frac{m}{2\pi kT})^{1/2}v^2(\frac{mv}{kT})\exp(\frac{-mv^2}{2kT})$$

which reduces to

$$2 = v^2(\frac{m}{kT})$$

and clearly,

$$v = (\frac{2kT}{m})^{1/2}$$

which is what we wished to show.

3. Collisional ionization:
   a. A neutral sodium atom has an ionization potential of χ = 5.1 eV. Calculate the speed of a free electron that has just enough kinetic energy to collisionally ionize a sodium atom in its ground state.
   b. Calculate the speed of a free proton that has just enough kinetic energy to collisionally ionize a sodium atom in its ground state.
   c. What is the temperature, T, of a gas in which the average particle kinetic energy, $\langle E \rangle = \frac{3}{2}kT$, is just sufficient to ionize a sodium atom in its ground state.

a) We are given that $\chi = 5.1\, eV$, and we wish to find the kinetic energy of a free electron that has just enough kinetic energy to collisionally ionize a sodium atom. That is, if we consider an elastic collision where all of the kinetic energy from the electron is transferred to the sodium atom, we can consider an electron with a kinetic energy of $5.1\, eV$ and find the speed of said electron.

Ignoring relativistic effects, recall that the classical formula for kinetic energy is given by

$$KE = \frac{1}{2}mv^2$$

and we have that $KE = 5.1\, eV = 8.175 \times 10^{-19}\, J$, so

$$\frac{1}{2}m_e v_e^2 = 8.175 \times 10^{-19}\, J$$

$$\frac{1}{2}(9.1 \times 10^{-31}\, kg)v_e^2 = 8.175 \times 10^{-19}\, J$$

$$v_e = (\frac{2 \times 8.175 \times 10^{-19}\, J}{9.1 \times 10^{-31}\, kg})^{1/2}$$

$$v_e = 1.34 \times 10^6\, \frac{m}{s}$$

So a free electron would need to have a minimum kinetic energy of approximately $1.34 \times 10^6\, \frac{m}{s}$ to ionize a neutral sodium atom.

b) We wish to find the minimum speed for a free proton to collisionally ionize a neutral sodium atom. Following the same process as above, we find

$$v_p = (\frac{2 \times 8.175 \times 10^{-19}\, J}{1.67 \times 10^{-27}\, kg})^{1/2}$$

$$v_p = 3.13 \times 10^4\, \frac{m}{s}$$

so a free proton would require a speed of approximately $3.13 \times 10^4\, \frac{m}{s}$ to ionize a neutral sodium atom.

c) Now we wish to find the temperature of a gas where the average particle kinetic energy is enough to ionize a neutral sodium atom. That is, we want to find the temperature of a gas such that its kinetic energy is at least $5.1\, eV$. That is,

$$\frac{3}{2}(8.617 \times 10^{-5}\frac{eV}{K})T = 5.1\, eV$$

$$T = \frac{10.2}{3 \times 8.617 \times 10^{-5}}K$$

$$T = 3.95 \times 10^4\, K$$

So for a gas whose average kinetic energy is given by $< E >= \frac{3}{2}kT$, having a temperature of approximately $3.95 \times 10^4\ K$ is enough to ionize neutral sodium.

4. A slab of glass 0.2 meters thick absorbs 50% of the light passing through it. Using the equation of radiative transfer, $I = I_0 e^{-n\sigma x}$, how thick must a slab of identical glass be in order to absorb 90% of the light passing through it? How thick to absorb 99% of the light?

We are given that at a thickness of $0.2\,m$, a slab of glass absorbs 50% of the light passing through it. We wish to find how thick a slab of identical glass must be to absorb 90% and 99% of light. To begin, we know that

$$0.5 I_0 = I_0 e^{-n\sigma(0.2)}$$
$$-\ln(2) = -n\sigma(0.2)$$

so we have

$$n\sigma = 5\ln(2)$$

Now we wish to find the thickness of a slab of glass that absorbs 90% of the light passing through it. That is, we wish to solve

$$0.1 I_0 = I_0 e^{-5\ln(2)x}$$

for x. Well,

$$-\ln(10) = -5\ln(2)x$$
$$x = \frac{\ln(10)}{5\ln(2)}$$
$$x \approx 0.66\,m$$

That is, a slab of glass approximately $0.66\,m$ thick will absorb 90% of light passing through it.

Now we wish to find the thickness of a slab that absorbs 99% of light passing through it. Similar to the process above, we have

$$0.01 I_0 = I_0 e^{-5\ln(2)x}$$
$$-\ln(100) = -5\ln(2)x$$
$$x = \frac{\ln(100)}{5\ln(5)}$$
$$x \approx 1.33\,m$$

That is, a slab of glass would require a thickness of approximately $1.33\,m$ to absorb 99% of the light passing through it.

5. The Boltzmann equation is useful for determining the number of atoms in excited states in a gas. Consider a gas of neutral hydrogen atoms. At what temperature will equal numbers of atoms have electrons in the ground state and in the second excited state (n=3) ?

To begin, recall the Boltzmann equation:

$$\frac{n_B}{n_A} = \frac{g_B}{g_A}\exp\left(\frac{E_A - A_B}{kT}\right)$$

For neutral hydrogen, we have that $g_n = 2n^2$, and wish to find the temperature when the number of atoms in the ground state ($n = 1$ state) is equal to the number of atoms in the second excited state ($n = 3$ state). Also recall that the energy states of neutral hydrogen are given by $E_n = \frac{-13.6}{n^2}\,eV$. That is, we wish to find T whenever $\frac{n_3}{n_1} = 1$. To begin, let's first find E_1 and E_3:

$$E_1 = \frac{-13.6}{1^2}\,eV$$

$$= -13.6\, eV$$

and E_3:

$$E_3 = \frac{-13.6}{3^2}\, eV$$
$$= -1.51\, eV$$

Now plugging these values into the Boltzmann equation, we find

$$1 = 9 \exp\left(\frac{-13.6\, eV + 1.51\, eV}{(8.617 \times 10^{-5}\, eV/K)T}\right)$$
$$\ln(9) = \frac{12.09\, eV}{(8.617 \times 10^{-5} eV/K)T}$$
$$T = \frac{12.09}{(8.617 \times 10^{-5}) \ln(9)} K$$
$$T \approx$$