Homework VIII

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Section 3.8 Problems

7. Show that the dual space H' of a Hilbert space H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f_z, f_v \rangle = \overline{\langle z, v \rangle} = \langle v, z \rangle$$

where $f_z(x) = \langle x, z \rangle$, etc.

Proof: Let $\{f_n\}$ be a Cauchy sequence in H'. By the Riesz representation, for each $n \in \mathbb{N}$, there exists a unique $x_n \in H$ such that

$$f_n(z) = \langle z, x_n \rangle.$$

Fix $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy, there exists an index N such that whenever $n > m > N$,

$$\|f_n - f_m\| < \varepsilon.$$

Now notice

$$\begin{aligned}
\|f_n - f_m\|^2 &= \langle f_n - f_m, f_n - f_m \rangle \\
&= \langle f_n, f_n \rangle - \langle f_m, f_n \rangle - \langle f_n, f_m \rangle + \langle f_m, f_m \rangle \\
&= \langle x_n, x_n \rangle - \langle x_n, x_m \rangle - \langle x_m, x_n \rangle + \langle x_m, x_m \rangle \\
&= \langle x_n - x_m, x_n \rangle - \langle x_n - x_m, x_m \rangle \\
&= \langle x_n - x_m, x_n - x_m \rangle \\
&= \|x_n - x_m\|^2 \\
\implies \|x_n - x_m\|^2 &< \varepsilon^2 \\
\implies \|x_n - x_m\| &< \varepsilon
\end{aligned}$$

so that $\{x_n\}$ is Cauchy in H. Since H is a Hilbert space, $\{x_n\}$ converges to some element $x \in H$. Now define the bounded linear functional $f \in H'$ by

$$f(z) := \langle z, x \rangle.$$

Now, for $\varepsilon > 0$ above, since $\{x_n\}$ converges to x, there exists an index M such that whenever $n > M$,

$$\|x_n - x\| < \varepsilon.$$

But from our work above, we have

$$\begin{aligned}
\|f_n - f\|^2 &= \|x_n - x\|^2 \\
&< \varepsilon^2 \\
\implies \|f_n - f\| &< \varepsilon
\end{aligned}$$

so that $f_n \to f$. Thus, H' is complete and is thus a Hilbert space.

Section 3.9 Problems

10. **(Right shift operator)** Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the *right shift operator* to be the linear operator $T : H \to H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert adjoint operator of T.

Soln. This operator is appropriately called the right shift operator since it "shifts" index of a given e_n in (e_n) by 1 to the right.
Since H is a separable Hilbert space and (e_n) is a total orthonormal sequence, any $x \in H$ has a unique representation

$$x = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Now, notice that if $x \neq \mathbf{0}$, there exists at least one $\alpha_k \neq 0$, and so $T(\alpha_k e_k) = \alpha_k e_{k+1} \neq \mathbf{0}$ so that $Tx \neq \mathbf{0}$. Thus,

$$\mathcal{N}(T) = \{\mathbf{0}\}.$$

For the range space, notice that for, since we are shifting each element of the orthonormal sequence to the right by one index, there does not exist an element e_k in (e_n) such that $Te_k = e_1$. Hence, any element in the range has the form $x = \sum_{k=2}^{\infty} \alpha_k e_k$ so that

$$\mathcal{R}(T) = \left\{ x \in H \;\middle|\; x = \sum_{k=2}^{\infty} \alpha_k e_k \right\}$$

Section 3.10 Problems

6. If $T : H \to H$ is a bounded self-adjoint linear operator and $T \neq 0$, then $T^n \neq 0$. Prove this (a) for $n = 2, 4, 8, 16, \cdots$,(b) for every $n \in \mathbb{N}$.

Proof: (a) We proceed by induction. First consider the case $n = 2$. Then since T is self-adjoint, we have $T = T^*$ so that

$$\begin{aligned} T^2 &= T^*T \\ &\neq 0 \end{aligned}$$

since $T \neq 0$. Now, notice that T^2 is self-adjoint since

$$(T^2)^* = (T^*T)^* = T^*(T^*)^* = T^*T = T^2.$$

Now assume that $T^n \neq 0$ (and is self-adjoint) for all $n = 2, 4, \cdots, 2^k$ for some $k \in \mathbb{N}$. We wish to show that $T^{n+1} \neq 0$ for $n = 2^{k+1}$. By the induction hypothesis, we have

$$T^{2^k} \neq 0$$

and so, since T^{2^k} is self-adjoint by assumption,

$$\begin{aligned} T^{2^{k+1}} &= \left(T^{2^k}\right)^2 \\ &= \left(T^{2^k}\right)\left(T^{2^k}\right)^* \\ &\neq 0 \end{aligned}$$

since $T^{2^k} \neq 0$.

(b) We will show by induction that T^n is self adjoint so that since $T \neq 0$, $T^n \neq 0$. We proved the case $n = 2$ in part (a). Now suppose this holds up to some integer k. We must show it holds for $k+1$. By the induction hypothesis, we have T^k is self adjoint. Then

$$\begin{aligned} T^{k+1} &= T^k T \\ \implies (T^k T)^* &= T^*(T^k)^* \\ &= TT^k \\ &= T^{k+1} \end{aligned}$$

so that T^{k+1} is self-adjoint. Thus, since $T \neq 0$, $T^n \neq 0$ for all n, and since T^n is self adjoint, $T^n \neq 0$ for all $n \in \mathbb{N}$.

Extra Credit Problems

3.9.8 Let $S = I + T^*T : H \to H$, where T is linear and bounded. Show that $S^{-1} : S(H) \to H$ exists.

Proof: We will show S is injective. To do so, we will show $\mathcal{N}(S) = \{\mathbf{0}\}$. Let $x \in H$ such that $Sx = 0$. That is,

$$\begin{aligned} Sx &= Ix + (T^*T)x \\ &= x + (T^*T)x \\ &= 0. \end{aligned}$$

Then we have $\|Sx\| = \|x + (T^*T)x\| = \|\mathbf{0}\| = 0$. Thus,

$$\begin{aligned} \|x + (T^*T)x\|^2 &= \langle x + (T^*T)x, x + (T^*T)x\rangle \\ &= \langle x, x\rangle + \langle x, (T^*T)x\rangle + \langle (T^*T)x, x\rangle + \langle (T^*T)x, (T^*T)x\rangle \\ &= \|x\|^2 + \langle Tx, Tx\rangle + \langle Tx, Tx\rangle + \|(T^*T)x\|^2 \qquad (\langle x, T^*y\rangle = \langle Tx, y\rangle) \\ &= \|x\|^2 + 2\|Tx\|^2 + \|(T^*T)x\|^2 \\ &= 0 \end{aligned}$$

but since $\|x\|^2, 2\|Tx\|^2, \|(T^*T)x\|^2 \geq 0$, it must be the case that $\|x\|^2 = \|Tx\|^2 = \|(T^*T)x\|^2 = 0$. Hence $x = \mathbf{0}$. Since x was chosen arbitrarily, we have that

$$\mathcal{N}(S) = \{\mathbf{0}\}$$

so that S is injective and hence invertible.

2.10.8 Show that the dual space of the space c_0 is ℓ^1.

Proof: Note that since c_0 is a subspace of ℓ^∞ and ℓ^∞ admits the standard Schauder basis $e_k = \delta_{jk}$, for any $x \in c_0$ there exist scalars $\xi_1, \xi_2, \cdots$ such that

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots .$$

Now, let $f \in c_0'$. Then since f is bounded and linear, f is continuous so that

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \xi_k \gamma_k \qquad (\gamma_k = f(e_k)) \\ \implies |f(x)| &\leq \max_{k \geq 1} |\xi_k| \sum_{k=1}^{\infty} |\gamma_k| \\ &= \|x\| \sum_{k=1}^{\infty} |\gamma_k| \end{aligned}$$

so that $\|f\| \leq \sum_{k=1}^{\infty} \gamma_k$. Now, for a lower bound, consider the sequence $x = (\xi_1, \xi_2, \cdots)$ in c_0 given by

$$\xi_k = \begin{cases} \frac{\overline{\gamma_k}}{|\gamma_k|} & \gamma_k \neq 0 \\ 1 & \gamma_k = 0 \end{cases} \quad k \leq n \\ \begin{cases} 0 & \end{cases} \quad k > n$$

for some $n \in \mathbb{N}$. Then notice $\|x\| = 1$ since $\left|\frac{\overline{\gamma_k}}{|\gamma_k|}\right| = 1$ for all k. Then notice

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} \frac{\overline{\gamma_k}}{|\gamma_k|}\gamma_k \\ &= \sum_{k=1}^{n} |\gamma_k| \end{aligned}$$

(note that for $\gamma_k = 0$, $\xi_k \gamma_k = 1 \cdot 0 = 0 = |\gamma_k|$ so that above holds for all k). And since $\|x\| = 1$, we have

$$\|f\| \geq \sum_{k=1}^{\infty} |\gamma_k|.$$

Since f is a bounded linear functional, and $|\gamma_k| \geq 0$ for all k, the sequence of partial sums $s_n = \sum_{k=1}^{n} |\gamma_k|$ is a monotonically increasing bounded sequence, so by the monotone convergence theorem, $\{s_n\}$ converges and, moreover,

$$\sum_{k=1}^{\infty} |\gamma_k| \leq \|f\|.$$

Since $\sum_{k=1}^{\infty} |\gamma_k|$ converges, the sequence $g_n = |\gamma_n| \in \ell^1$. Now, by the above two inequalities for $\|f\|$,

$$\|f\| = \sum_{k=1}^{\infty} |\gamma_k|.$$

so that f is norm preserving. Now, for any $b \in \ell^1$, $b = (\beta_1, \beta_2, \cdots)$, we may define an associated bounded linear functional $g(x)$ for $x \in c_0$:

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k.$$

Then the mapping $f \mapsto (g_n)$ where $g_n = \gamma_n = f(e_n)$ is norm preserving and bijective, so that $c_0' \cong \ell^1$. Hence, the dual space of c_0 is ℓ^1.

VIII.1

Let $T : H \to H$ be the right shift operator of Prob. 3.9 # 10, where (e_n) is a total orthonormal sequence in a separable Hilbert space H. By definition, a scalar λ and a nonzero vector $x \in H$ is an eigenvalue-eigenvector pair for a linear operator $T : H \to H$ if

$$Tx = \lambda x \qquad (\lambda \text{ a scalar, } x \neq \mathbf{0}).$$

(a) Show that T has no eigenvalue-eigenvector pairs.

Proof: Suppose that T has at least one eigenvalue-eigenvector pair. Then for some $x \in H$, $x = \xi_1 e_1 + \xi_2 e_2 + \cdots \neq \mathbf{0}$,

$$\begin{aligned} Tx &= \xi_1 e_2 + \xi_2 e_3 + \cdots \\ &= \lambda \xi_1 e_1 + \lambda \xi_2 e_2 + \cdots \end{aligned}$$

But then $Tx - \lambda x = 0$, so that

$$\begin{aligned} Tx - \lambda x &= (\xi_1 e_2 + \xi_2 e_3 + \cdots) - (\lambda \xi_1 e_1 + \lambda \xi_2 e_2 + \cdots) \\ &= -\lambda \xi_1 e_1 + (\xi_1 - \lambda \xi_2) e_2 + (\xi_2 - \lambda \xi_3) e_3 + \cdots \\ &= \mathbf{0} \end{aligned}$$

so that $\lambda = 0$ by the e_1 term, which then gives us that $\xi_j = 0$ for $j > 1$. But since $x \neq \mathbf{0}$, $\xi_1 \neq 0$, so that $\lambda x \neq 0$ since $Tx = \xi_1 e_2 \neq 0$, we have a contradiction.

(b) Show that the adjoint $T^* : H \to H$ has an eigenvalue-eigenvector pair for every scalar λ with $|\lambda| < 1$.

Proof: