# Homework III

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## Section 2.3 Problems

**2.** Show that $c_0$ in Prob. 1 is a *closed* subspace of $\ell^\infty$, so that $c_0$ is complete by 1.5-2 and 1.4-7.

*Proof*: Let $x$ be a limit point of $c_0$. We will show that $x \in c_0$. Since $x$ is a limit point of $c_0$, there exists a sequence $\{x_n\}$ of points in $c_0$ such that for any $\varepsilon > 0$, there exists a natural number $N_0$ such that

$$||x_n - x|| < \frac{\varepsilon}{2}.$$

whenever $n > N_0$. Similarly, since $x_n \in c_0$, there exists a natural number $N_1$ such that

$$|x_n^{(m)}| < \frac{\varepsilon}{2}$$

whenever $m > N_1$ where the superscript $(m)$ denotes the $m^{\text{th}}$ element of $x_n$. Now notice, for $m > N_1$, $n > N_0$,

$$\begin{aligned}
|x^{(m)}| &= |x^{(m)} - x_n^{(m)} + x_n^{(m)}| \\
&\leq |x^{(m)} - x_n^{(m)}| + |x_n^{(m)}| \\
&\leq \sup_{m>N_1} |x^{(m)} - x_n^{(m)}| + |x_n^{(m)}| \\
&= ||x^{(m)} - x_n^{(m)}|| + |x_n^{(m)}| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \\
&= \varepsilon
\end{aligned}$$

So $|x^{(m)}| < \varepsilon$ for $m > N_1$. Since $\varepsilon$ is an upper bound for $|x^{(m)}|$ for all $m > N_1$,

$$\sup_{m>N_1} |x^{(m)}| < \varepsilon$$

Thus, $x \to 0$, so $x \in c_0$ and $c_0$ is a closed subspace of $\ell^\infty$.

**5.** Show that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$. Show that $\alpha_n \to \alpha$ and $x_n \to x$ implies $\alpha_n x_n \to \alpha x$.

*Proof*: To start, let $x_n \to x$ and $y_n \to y$. Fix $\varepsilon > 0$. Then there exist indices $N_1, N_2 \in \mathbb{N}$ such that, whenever $n > N_1$,

$$||x_n - x|| < \frac{\varepsilon}{2}$$

and similarly, when $n > N_2$,

$$||y_n - y|| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$, notice

$$\begin{aligned}
||(x_n + y_n) - (x + y)|| &= ||(x_n - x) + (y_n - y)|| \\
&\leq ||x_n - x|| + ||y_n - y|| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

That is, $||(x_n + y_n) - (x + y)|| < \varepsilon$ for $n > N$, so $x_n + y_n \to x + y$.

Now let $\alpha_n \to \alpha$. We wish to show that $\alpha_n x_n \to \alpha x$. Notice the following:

$$\begin{aligned}\|x_n\alpha_n - x\alpha\| &= \|x_n\alpha_n - x\alpha_n + x_n\alpha - x\alpha\| \\ &\leq \|x_n\alpha_n - x_n\alpha\| + \|x_n\alpha - x\alpha\| \\ &= |\alpha_n - \alpha|\|x_n\| + |\alpha|\|x_n - x\|. && \text{(Homogeneity of the norm)}\end{aligned}$$

Since $x_n$ converges, $\{x_n\}$ is a bounded sequence, hence there exists some positive number $M$ such that

$$\|x_n\| \leq M$$

for all $n$. Now, fix $\varepsilon > 0$. Since $\alpha_n \to \alpha$, there exists a natural number $N_1$ such that

$$|\alpha_n - \alpha| < \frac{\varepsilon}{2M}$$

whenever $n > N_1$. Similarly, since $x_n \to x$, there exists a natural number $N_2$ such that

$$\|x_n - x\| < \frac{\varepsilon}{2|\alpha|}$$

whenever $n > N_2$. Take $N = \max\{N_1, N_2\}$ so that whenever $n > N$,

$$\begin{aligned}\|x_n\||\alpha_n - \alpha| + |\alpha|\|x_n - x\| &\leq M|\alpha_n - \alpha| + |\alpha|\|x_n - x\| \\ &< M\frac{\varepsilon}{2M} + |\alpha|\frac{\varepsilon}{2|\alpha|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Thus, $\alpha_n x_n \to \alpha x$.

**10.** **(Schauder Basis)** Show that if a normed space has a Schauder basis, it is separable.

*Proof*: Let $X = (X, \|\cdot\|)$ be a normed space that has a Schauder basis. That is, for each $x \in X$, there exists a unique sequence of scalars $\{\alpha_n\}$ and a sequence of "basis" vectors $\{e_n\}$ such that $\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0$ as $n \to \infty$. Now, define the set

$$M = \{x \in X \mid x = q_1 e_1 + \cdots + q_k e_k\}.$$

where $q_i \in \mathbb{Q}$ if the scalar field is $\mathbb{R}$, or $q_i = q_i^R + iq_i^I$, $q_i^R, q_i^I \in \mathbb{R}$ if the scalar field is $\mathbb{C}$. That is, $M$ is the set of vectors in $X$ that can be expressed as a linear combination of basis vectors whose coefficients are dense (but countable) in the scalar field. We will begin by showing that $M$ is dense in $X$.

Fix $\varepsilon > 0$ and let $y \in X$. Since $X$ has a Schauder basis, there exists a unique sequence of scalars $\{\alpha_n\}$ and an index $N \in \mathbb{N}$ such that whenever $n > N$,

$$\|y - (\alpha_1 e_1 + \cdots + \alpha_2 e_n)\| < \frac{\varepsilon}{2}$$

Now, for each $i$, we may find $q_i$ such that

$$|\alpha_i - q_i| < \frac{\varepsilon}{2n||e_i||}$$

Then notice

$$\begin{aligned}
\|y-(q_1e_1+\cdots+q_ne_n)\| &= \|y-(\alpha_1e_1+\cdots+\alpha_ne_n)+(\alpha_1e_1+\cdots+\alpha_ne_n)-(q_1+\cdots+q_ne_n\| \\
&\leq \|y-(\alpha_1e_1+\cdots+\alpha_ne_n)\|+\|(\alpha_1-q_1)e_1+\cdots+(\alpha_n-q_n)e_n\| \\
&\leq \|y-(\alpha_1e_1+\cdots+\alpha_ne_n)\|+|\alpha_1-q_1|\|e_1\|+\cdots+|\alpha_n+q_n|\|e_n\| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2n}+\cdots+\frac{\varepsilon}{2n} \\
&= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $(q_1e_1+\cdots+q_ne_n)\in M$, we have that $M$ is dense in $X$. Also, since $M$ has a countable basis, namely, $\{e_1,\cdots,e_n\}$ and since the scalars are pulled from a countable and dense subset of $\mathbb{R}$ or $\mathbb{Q}$, it follows that $M$ is countable.

Thus, if $X$ has a Schauder basis, then $X$ is a separable space.

# Section 2.4 Problems

**6.** Theorem 2.4-5 implies that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ in Prob. 8, Sec. 2.2, are equivalent Give a direct proof of this fact.

*Proof*: Let $X$ be the set of $n-$tuples of numbers and $x=(\xi_1,\cdots,\xi_n)\in X$. We wish to show that the 2 and infinity norms defined below are equivalent.

$$\begin{aligned}
\|x\|_2 &= \sqrt{(\xi_1)^2+\cdots+(\xi_n)^2} \\
\|x\|_\infty &= \max\{|\xi_1|,\cdots,|\xi_n|\}
\end{aligned}$$

To begin, since $\{|\xi_1|,\cdots,|\xi_n|\}$ is a finite set, there exists an index $i$ such that

$$|\xi_i|=\max\{|\xi_1|,\cdots,|\xi_n|\}=\|x\|_\infty$$

Now notice that

$$\begin{aligned}
(\xi_i)^2 &\leq (\xi_1)^2+\cdots+(\xi_i)^2+\cdots+(\xi_n)^2 \\
|\xi_i| &\leq \sqrt{(\xi_1)^2+\cdots+(\xi_i)^2+\cdots+(\xi_n)^2} \\
\implies \|x\|_\infty &\leq \|x\|_2.
\end{aligned}$$

Now, notice that since $|\xi_i|=\max\{|\xi_1|,\cdots,|\xi_n|\}$,

$$\begin{aligned}
(\xi_1)^2+\cdots+(\xi_n)^2 &\leq (\xi_i)^2+\cdots+(\xi_i)^2 \\
&= n(\xi_i)^2
\end{aligned}$$

hence,

$$\begin{aligned}
\sqrt{(\xi_1)^2+\cdots+(\xi_n)^2} &\leq \sqrt{n}|\xi_i| \\
\implies \|x\|_2 &\leq \sqrt{n}\|x\|_\infty.
\end{aligned}$$

Putting the two inequalities together, we have

$$\|x\|_\infty\leq\|x\|_2\leq\sqrt{n}\|x\|_\infty$$

Thus, $\|x\|_2$ and $\|x\|_\infty$ are equivalent norms.

# Assigned Exercises

**III.1** Let $c_0$ be the subspace of $\ell^\infty$ consisting of all sequences that converge to zero. Prove that $c_0$ has the Schauder basis $(e_n)$, where $e_n = (\delta_{nj})$ is the $n-$th unit coordinate sequence.

*Proof*: Let $x \in c_0$. We wish to show that there exists a unique sequence of scalars $\{\alpha_n\}$ such that

$$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Well, take $\alpha_n = x^{(n)}$, where the superscript $(n)$ denotes the $n^{\text{th}}$ element of the sequence. Additionally, since $x \in c_0$, for any $\varepsilon > 0$, there exists a natural number $N$ such that

$$|x^{(n)}| < \frac{\varepsilon}{2}$$

whenever $n > N$. That is, $\frac{\varepsilon}{2}$ is an upper bound for all $x^{(n)}$ so that $\sup_{i>n} |x^{(i)}| \leq \frac{\varepsilon}{2} < \varepsilon$. Now, let $y = x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)$ so that

$$\|y\| = \|(0, 0, \cdots, 0, x^{(n+1)}, \cdots)\|$$

but since each $y^{(i)} = 0$ for $1 \leq i \leq n$,

$$\begin{aligned} \sup_{i \geq 1} |y^{(i)}| &= \sup_{i>n} |y^{(i)}| \\ &= \sup_{i>n} |x^{(i)}|. \end{aligned}$$

Hence, for $n > N$ as above,

$$\sup_{i>n} |x^{(i)}| < \varepsilon.$$

Hence,

$$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0 \quad \text{as} \quad n \to \infty$$

so that $c_0$ has the Schauder basis $\{e_n\}$.