Homework IX

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Section 4.7 Problems

2. Of what category is the set of all integers (a) in $\mathbb{R}$, (b) in itself (taken with the metric induced from $\mathbb{R}$)?

Soln. (a) In $\mathbb{R}$, for any $n \in \mathbb{Z}$, $\{n\}$ is not an open set since, for <u>any</u> $r > 0$, the open ball of radius r centered at n, $B_r(n)$ contains elements in $\mathbb{R}$ not in $\{n\}$. But $\overline{\{n\}} = \{n\}$ so that $\{n\}$ is rare for each $n \in \mathbb{Z}$. Now notice,

$$\mathbb{Z} = \bigcup_{n=-\infty}^{\infty} \{n\}$$

so that $\mathbb{Z}$ is a countable union of rare sets in $\mathbb{R}$, hence $\mathbb{Z}$ is meager in $\mathbb{R}$.

(b) In $\mathbb{Z}$, any subset $S \subseteq \mathbb{Z}$ contains an open set since, if $n \in S$ (the case $S = \emptyset$ is itself trivially open), the open ball of radius $1/2$, $B_{1/2}(n) = \{n\} \subseteq S$. Thus, $\mathbb{Z}$ is nonmeager in itself.

6. Show that the complement M^c of a meager subset M of a complete metric space X is nonmeager.

We begin by proving the following lemma:
Lemma: The union of two meager sets is meager.

Proof: Let $A, B \subseteq X$ be meager. Then there exist countable collections of rare sets $(a_k)_{k\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}}$ such that

$$A = \bigcup_{k\in\mathbb{N}} a_k, \qquad B = \bigcup_{j\in\mathbb{N}} b_k$$

thus,

$$A \cup B = \left(\bigcup_{k\in\mathbb{N}} a_k\right) \cup \left(\bigcup_{j\in\mathbb{N}} b_k\right)$$

and notice that the right hand side is the union of two countable unions, which is itself countable.

$\square$

Now for the main problem:

Proof: Suppose by way of contradiction that M^c is meager. Then, by definition of set complement, we may express X as

$$X = M^c \cup M$$

but since M and M^c are meager, X is meager in itself by the above proof. But this contradicts Baire's Category Theorem, where, since X is complete, X is nonmeager in itself. Thus, M^c is nonmeager, which is what we sought to show.

8. Show that the completeness of X is essential in Theorem 4.7-3 and cannot be omitted. [Consider the subspace $x \subset \ell^\infty$ consisting of all $x = (\xi_j)$ such that $\xi_j = 0$ for $j \geq J \in \mathbb{N}$, where J depends on x, and let T_n be defined by $T_n x = f_n(x) = n\xi_n$.]

Proof: We first show that X is incomplete. Consider the sequence $\{x_n\}$ in X defined by $x_n =$

$(1, \frac{1}{\sqrt{2}}, \cdots, \frac{1}{\sqrt{n}}, 0, 0, \cdots)$. Notice that this defines a Cauchy sequence in X since, for natural numbers $n > m$,

$$\|x_n - x_m\| = \frac{1}{\sqrt{m}}$$

and so, for any any $\varepsilon > 0$, by the Archimedean property of $\mathbb{R}$, there exists an index N such that whenever $n > m > N$,

$$\frac{1}{\sqrt{m}} < \varepsilon$$
$$\implies \|x_n - x_m\| < \varepsilon.$$

Now, notice that, as $n \to \infty$, $x_n \to (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \cdots) \notin X$. Thus, X is not a complete space. Now, suppose that T_n is uniformly bounded. That is, there exists a c such that $\|T_n\| \leq c$ for all n. But notice

$$\|T_n x_n\| = \sqrt{n}$$

so that, for $n > c^2$, $\|T_n x_n\| > c$, a contradiction. Thus, the completeness of X in Theorem 4.7-3 is essential, by the above example.

10. **(Space $\mathbf{c_0}$)** Let $y = (\eta_j)$, $\eta_j \in \mathbb{C}$, be such that $\sum \xi_j \eta_j$ converges for every $x = (\xi_j) \in c_0$, where $c_0 \in \ell^\infty$ is the subspace of all complex sequences converging to zero. Show that $\sum |\eta_j| < \infty$. (Use 4.7-3.)

Proof: Define the sequence of linear functionals $\{f_n\}$ by

$$f_n(x) = \sum_{j=1}^{n} \xi_j \eta_j \qquad (x = (\xi_1, \xi_2, \cdots))$$

and since $\sum_{j=1}^{\infty} \xi_j \eta_j$ converges for all x, the sequence $\{s_n\}$ defined by $s_n = \sum_{j=1}^{n} \xi_j \eta_j$ is bounded by some c_x (depending on x), $|s_n| \leq c_x$. Thus,

$$|f_n(x)| \leq c_x.$$

Note also that each f_n is bounded since

$$\begin{aligned} |f_n(x)| &\leq \sum_{j=1}^{n} |\xi_j||\eta_j| && \text{(Triangle Inequality)} \\ &\leq \sup_{j \geq 1} |\xi_j| \sum_{j=1}^{n} |\eta_j| \\ &= \|x\| \sum_{j=1}^{n} |\eta_j| \\ \implies \|f_n\| &\leq \sum_{j=1}^{n} |\eta_j| \end{aligned}$$

which is bounded since $\sum_{j=1}^{n} |\eta_j|$ is a finite sum. Thus, since c_0 is a complete space, by the uniform boundedness theorem, there exists some $c > 0$ such that

$$\|f_n\| \leq c.$$

Now, notice that as $n \to \infty$,

$$f_n \to f = \sum_{j=1}^{\infty} \xi_j \eta_j$$

and so, by continuity of the norm,

$$\|f\| \leq c.$$

Now, define the sequence $\{x_n\}$ where $x_n = (\xi_1, \xi_2, \cdots, \xi_n, 0, 0, \cdots)$ with

$$\xi_j = \begin{cases} \frac{\overline{\eta_j}}{|\eta_j|}, & \text{if } \eta_j \neq 0 \\ 0, & \text{if } \eta_j = 0. \end{cases}$$

Notice that $\|x_n\| = 1$ for all n and that

$$\begin{aligned} f(x_n) &= \sum_{j=1}^{n} \xi_j \eta_j \\ &= \sum_{j=1}^{n} |\eta_j| \\ \implies |f(x_n)| &= \sum_{j=1}^{n} |\eta_j| \\ \implies \|f\| &\geq \sum_{j=1}^{n} |\eta_j| \end{aligned}$$

but since f is a bounded linear functional ($\|f\| \leq c$), we have

$$\sum_{j=1}^{n} |\eta_j| \leq c$$

and since $|\eta_j| \geq 0$ for all j, so that by the monotone convergence theorem, $\sum_{j=1}^{\infty} |\eta_j|$ converges, as was desired.