Nonlinear Waves Problems 2.9 & 2.11

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2.9 Consider the equation

$$u_t + uu_x = 1, \quad -\infty < x < \infty, \quad t > 0.$$

(a) Find the general solution.

Soln. By method of characteristics, we wish to solve the following system of ODEs:

$$\frac{dx}{ds} = u$$
$$\frac{dt}{ds} = 1$$
$$\frac{du}{ds} = 1.$$

From these equations, we can immediately see $u(s) = s + c_1$ and $t(s) = s + c_2$. From these two equations, we have $s = t - c_2$ and so $u = t - c_2 + c_1 = \tilde{c}$ with $\tilde{c} = c_1 - c_2$. That is, $u - t =$ constant. And from $\frac{dx}{ds}$, we find $x(s) = \frac{t^2}{2} + \tilde{c}t + c_3$, where $c_1, c_2, c_3 \in \mathbb{R}$. Further, adding and subtracting $\frac{s^2}{2}$ from $x(s)$ gives us $x(s) = -\frac{s^2}{2} + su + c_3 \implies x + \frac{s^2}{2} - su = c_3$. Relating the constants $\tilde{c}$ and c_3 with some arbitrary function g, we have $g(c_3) = \tilde{c} \implies u - t = g\left(x + \frac{t^2}{2} - tu\right)$. Assuming initial condition $u(x, 0) = f(x)$, it is easy to see

$$u = t + f\left(x + \frac{t^2}{2} - tu\right).$$

(b) Discuss the solution corresponding to: $u = \frac{1}{2}t$ when $t^2 = 4x$.

Soln. Not a solution.

(c) Discuss the solution corresponding to: $u = t$ when $t^2 = 2x$.

Soln. From our work in determining the characteristics in part (a), we can see that the case $u = t$ corresponds to $\tilde{c} = 0$ and $t^2 = 2x$ implies $c_3 = 0$. This also corresponds to the case $f(x) = 0$ and so we can conclude $u = t$ on the curve $t^2 = 2x$.

2.11 Find the solution of the equation

$$yu_x - xu_y = 0,$$

corresponding to the data $u(x, 0) = f(x)$. Explain what happens if we give $u(x(s), y(s)) = f(s)$ along the curve defined by $\{s : x^2(s) + y^2(s) = a^2\}$.

Soln. By method of characteristics, we wish to solve the following set of differential equations:

$$\frac{dx}{ds} = y$$
$$\frac{dy}{ds} = -x$$
$$\frac{du}{ds} = 0.$$

From this system, we conclude $u = c_1 \in \mathbb{R}$, $x = A_1\cos(s) + A_2\sin(s)$, and $y = B_1\cos(s) + B_2\sin(s)$. From the initial condition $u(x, 0) = f(x)$, we take $x_0 = A_1$, $y_0 = 0$. Solving for the constants A_1, A_2, B_1, B_2 yields $A_1 = t$, $A_2 = 0$, $B_1 = 0$, $B_2 = t$. Giving $x = A_1\cos(s)$ and $y = A_1\sin(s)$.

Thus $x^2 + y^2 = A_1^2 =$ constant. Therefore the characteristics are circles centered at the origin, and $u =$ constant on these characteristics. Further, we can relate the constants t^2 and c_1 by an arbitrary function $g(A_1^2) = c_1$, so that

$$u(x, y) = g(x^2 + y^2).$$

By our initial condition $u(x, 0) = f(x)$, we have

$$\begin{aligned} g(x^2) &= f(x) \\ \implies f\left(\pm\sqrt{x}\right) &= g(x) \\ \implies f\left(\pm\sqrt{x^2+y^2}\right) &= g(x^2 + y^2). \end{aligned}$$

Thus the general solution to this PDE is given as

$$u(x, y) = f\left(\sqrt{x^2 + y^2}\right).$$

Now, on the characteristic $\{s : x^2(s) + y^2(s) = a^2\}$, we have $u(x(s), y(s)) = f(s) = f(a)$.