Approximation Methods Homework 2

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1.29. Let $\mathbf{A}$ and $\mathbf{D}$ be (real) $n \times n$ matrices.

(a) Suppose $\mathbf{A}$ is symmetric and has n distinct eigenvalues. Find a two-term expansion of the eigenvalues of the perturbed matrix $\mathbf{A} + \varepsilon \mathbf{D}$, where $\mathbf{D}$ is positive definite. What you are finding is known as a Rayleigh-Schrödinger series for the eigenvalues.

Soln. We begin by noting that since A is real symmetric and has n distinct eigenvalues, the eigenvectors of A for a complete orthonormal basis for $\mathbb{R}^n$. Now, consider the perturbed eigenvalue problem

$$(\mathbf{A} + \varepsilon \mathbf{D})v = \lambda v$$

and assume

$$\lambda \sim \lambda_0 + \varepsilon^{\alpha} \lambda_1 + \cdots$$

and

$$v \sim v_0 + \varepsilon^{\beta} v_1 + \cdots .$$

Plugging these expansions into our eigenvalue problem, we find

$$(\mathbf{A} + \varepsilon \mathbf{D})(v_0 + \varepsilon^{\beta} v_1 + \cdots) = (\lambda_0 + \varepsilon^{\alpha}\lambda_1 + \cdots)(v_0 + \varepsilon^{\beta} v_1 + \cdots)$$
$$\implies \mathbf{A}(v_0 + \varepsilon^{\beta} v_1 + \cdots) + \varepsilon \mathbf{D}(v_0 + \varepsilon^{\beta} v_1 + \cdots) = \lambda_0 v_0 + \varepsilon^{\alpha}\lambda_1 v_0 + \cdots + \varepsilon^{\beta}\lambda_0 v_1 + \varepsilon^{\alpha+\beta}\lambda_1 v_1 + \cdots$$

From the $\mathcal{O}(1)$ terms, we have the system

$$\mathbf{A}v_0 = \lambda_0 v_0$$

which gives us that λ_0 is an eigenvalue of $\mathbf{A}$ with associated eigenvector v_0. Further, we see that from the $\mathcal{O}(\varepsilon)$ term, either $\alpha = 1$ or $\beta = 1$. Consider the case $\beta = 1$ and $\alpha < \beta$. If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then from balancing terms on the right hand side of the above equation, we have $\lambda_1 = \lambda_2 = \cdots = 0$ since the powers on the right hand side are of the form $\alpha + k$, $k \in \mathbb{N}$ so there is no perturbation of λ. If $\alpha \in \mathbb{Q}$, then there is an integer n such that $\lambda_1, \cdots, \lambda_{n-1} = 0$ and $\lambda_n \neq 0$, which will recover the case $\alpha = 1$ with reordering. The cases $\alpha > \beta$ or $\alpha = 1$ with $\alpha < \beta$ or $\alpha > \beta$ follow similarly leading us to conclude $\alpha = \beta = 1$. Thus the $\mathcal{O}(\varepsilon)$ terms give

$$\mathbf{A}v_1 + \mathbf{D}v_0 = \lambda_1 v_0 + \lambda_0 v_1.$$

Now project onto v_0:

$$\langle v_0, \mathbf{A}v_1\rangle + \langle v_0, \mathbf{D}v_0\rangle = \langle v_0, \lambda_1 v_0\rangle + \lambda_0\langle v_0, v_1\rangle. \tag{1}$$

Since $\mathbf{A}$ is real symmetric, A is self-adjoint, so

$$\begin{aligned}\langle v_0, \mathbf{A}v_1\rangle &= \langle \mathbf{A}v_0, v_1\rangle \\ &= \lambda_0\langle v_0, v_1\rangle.\end{aligned}$$

Then equation (1) becomes

$$\begin{aligned}\langle v_0, \mathbf{D}v_0\rangle &= \langle v_0, \lambda_1 v_0\rangle \\ &= \lambda_1^*\end{aligned}$$

and since $\mathbf{D}$ is positive definite, we have $\langle v_0, \mathbf{D}v_0\rangle > 0$ so that $\lambda_1^* = \lambda_1$ giving us

$$\lambda_1 = \langle v_0, \mathbf{D}v_0\rangle$$

and so our two term expansion for λ is

$$\lambda \sim \lambda_0 + \varepsilon\langle v_0, \mathbf{D}v_0\rangle$$

with λ_0 an eigenvalue of $\mathbf{A}$ and v_0 the associated eigenvector.

(b) Suppose $\mathbf{A}$ is the identity and $\mathbf{D}$ is symmetric. Find a two-term expansion of the eigenvalues for the matrix $\mathbf{A} + \varepsilon\mathbf{D}$.

Soln. Consider the perturbed eigenvalue problem

$$(\mathbf{A} + \varepsilon\mathbf{D})v = \lambda v.$$

Following the same argument in part (a), we can asssume

$$\lambda \sim \lambda_0 + \varepsilon\lambda_1 + \cdots$$

and

$$v \sim v_0 + \varepsilon v_1 + \cdots .$$

Note that, since $\mathbf{A}$ is the identity matrix, $\mathbf{A}$ has only one eigenvalue, $\lambda_{\mathbf{A}} = 1$. Plugging these expansions into our eigenvalue problem, we find

$$\begin{aligned} &(\mathbf{A} + \varepsilon\mathbf{D})(v_0 + \varepsilon v_1 + \cdots) = (\lambda_0 + \varepsilon\lambda_1 + \cdots)(v_0 + \varepsilon v_1 + \cdots) \\ \implies &\mathbf{A}(v_0 + \varepsilon v_1 + \cdots) + \varepsilon\mathbf{D}(v_0 + \varepsilon v_1 + \cdots) = \lambda_0 v_0 + \varepsilon\lambda_1 v_0 + \cdots + \varepsilon\lambda_0 v_1 + \cdots .\end{aligned}$$

The $\mathcal{O}(1)$ terms gives

$$\mathbf{A}v_0 = \lambda_0 v_0$$

and since $\mathbf{A}$ is the identity, $\lambda_0 = 1$ and v_0 is to be determined. The $\mathcal{O}(\varepsilon)$ terms gives

$$\begin{aligned} &\mathbf{D}v_0 + \mathbf{A}v_1 = \lambda_1 v_0 + \lambda_0 v_1 \\ \implies &\mathbf{D}v_0 + v_1 = \lambda_1 v_0 + v_1 \\ \implies &\mathbf{D}v_0 = \lambda_1 v_0 \end{aligned}$$

which gives us λ_1 is an eigenvalue of $\mathbf{D}$ with associated eigenvector v_0. Further, since $\mathbf{D}$ is symmetric, we have λ_1 is real, and so our two term expansion for λ is

$$\lambda \sim 1 + \varepsilon a$$

where a is an eigenvalue of $\mathbf{D}$.

(c) Considering

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ \mathbf{D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

show that $\mathcal{O}(\varepsilon)$ perturbation of a matrix need not result in a $\mathcal{O}(\varepsilon)$ perturbation of the eigenvalues. This example also demonstrates that a smooth perturbation of a matrix need not result in a smooth perturbation of the eivgenvalues.

Proof: Notice

$$\mathbf{A} + \varepsilon\mathbf{D} = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}.$$

Computing the eigenvalues, we find

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 \\ \varepsilon & -\lambda \end{vmatrix} &= 0 \\
\implies \lambda^2 - \varepsilon &= 0 \\
\implies \lambda^2 &= \varepsilon \\
\implies \lambda &= \pm\sqrt{\varepsilon}.
\end{aligned}$$

And note that the eigenvalues of $\mathbf{A}$ are $\lambda = 0$. Thus a $\mathcal{O}(\varepsilon)$ perturbation in the matrix results in a $\mathcal{O}(\varepsilon^{1/2})$ perturbation of the eigenvalues.

1.33. Find a two-term asymptotic expansion, for small ε, of the solution of the following problems. Also, comment on how the boundary conditions help determine the form of the expansion.

(c) $y'' - y + \varepsilon y^3 = 0$, where $y(0) = 0$ and $y(1) = \varepsilon$.

Soln. Assume $y \sim \varepsilon^\alpha(y_0 + \varepsilon^\beta y_1 + \cdots)$ and that

$$y'' \sim \varepsilon^\alpha(y_0'' + \varepsilon^\beta y_1'' + \cdots)$$

and

$$\begin{aligned} y^3 &= \varepsilon^{3\alpha}\left(y_0 + \varepsilon^\beta y_1 + \cdots\right)^3 \\ &= \varepsilon^{3\alpha}\left(y_0^3 + 3\varepsilon^\beta y_0^2 y_1 + \cdots\right). \end{aligned}$$

Applying our boundary conditions to this expansion gives

$$\begin{aligned} y(0) &= \varepsilon^\alpha(y_0(0) + \varepsilon^\beta y_1(0) + \cdots) \\ &= 0 \\ \implies y_0(0) &= y_1(0) = \cdots = 0 \end{aligned}$$

and

$$\begin{aligned} y(1) &= \varepsilon^\alpha(y_0(1) + \varepsilon^\beta y_1(1) + \cdots) \\ &= \varepsilon \\ \implies y_1(1) &= y_2(1) = \cdots = 0 \end{aligned}$$

and balancing gives $\alpha = 1$ with $y_0(1) = 1$. Plugging this expansion into our differential equation yields

$$\varepsilon y_0'' + \varepsilon^{1+\beta} y_1'' + \cdots - \varepsilon y_0 - \varepsilon^{1+\beta} y_1 - \cdots + \varepsilon^4 y_0^3 + 3\varepsilon^{\beta+2} y_0^2 y_1 + \cdots = 0.$$

Now, the $\mathcal{O}(\varepsilon)$ term gives the ODE

$$y_0'' - y_0 = 0, \qquad y_0(0) = 0 \quad \text{and} \quad y_0(1) = 1.$$

which has general solution

$$y_0(t) = a_0\cosh(t) + b_0\sinh(t)$$

From the boundary conditions, we find $y_0(0) = a_0 = 0$ and $y_0(1) = b_0\sinh(1) = 1 \implies b_0 = \frac{1}{\sinh(1)}$. Balancing the next lowest order terms in the differential equation gives us $\beta = 3$. Thus the $\mathcal{O}(\varepsilon^4)$ terms yield

$$y_1'' - y_1 + \frac{\sinh^3(t)}{\sinh^3(1)} = 0, \qquad y_1(0) = 0 \quad \text{and} \quad y_1(1) = 0.$$

The general solution to the homogeneous equation is again $y_{1,h} = a\cosh(t) + b\sinh(t)$. To get the particular solution, we apply variation of parameters. Let $g(t) = -\frac{\sinh^3(t)}{\sinh^3(1)}$. Then from variation of parameters, we have $y_{1,p} = u_1 x_1 + u_2 x_2$ with $x_1 = \cosh(t)$ and $x_2 = \sinh(t)$ and

$$\begin{aligned} u_1 &= -\int_0^t \frac{g(s)x_2}{W}ds \\ u_2 &= \int_0^t \frac{g(s)x_1}{W}ds \end{aligned}$$

where $W = W(\cosh(t), \sinh(t)) = 1$ so

$$\begin{aligned} u_1 &= \frac{1}{\sinh^3(1)} \int_0^t \sinh^4(s) ds \\ u_2 &= -\frac{1}{\sinh^3(1)} \int_0^t \cosh(s)\sinh^3(s) ds. \end{aligned}$$

Let's now compute the above integrals. For u_2, let $v = \sinh(s)$ so that $dv = \cosh(s)ds$ and

$$\begin{aligned} \int_0^t \cosh(s)\sinh^3(s) ds &= \int_0^{\sinh(t)} v^3 dv \\ &= \frac{\sinh^4(t)}{4} \\ \implies u_2 &= -\frac{\sinh^4(t)}{4\sinh^3(1)}. \end{aligned}$$

For u_1, notice

$$\begin{aligned} \int_0^t \sinh^4(s) ds &= \int_0^t \sinh^2(s)(\cosh^2(s) - 1) ds \\ &= \int_0^t \sinh^2(s)\cosh^2(s) ds - \int_0^t \sinh^2(s) ds. \end{aligned}$$

For the second integral, using the identity $\sinh^2(s) = \frac{\cosh(2s)-1}{2}$ so that

$$\begin{aligned} \int_0^t \sinh^2(s) ds &= \frac{1}{2}\int_0^t (\cosh(2s) - 1) ds \\ &= \frac{\sinh(2t)}{4} - \frac{t}{2}. \end{aligned}$$

For the first integral, we use the above identity for $\sinh^2(s)$ with the identity $\cosh^2(s) = \frac{\cosh(2s)+1}{2}$ so that

$$\begin{aligned} \int_0^t \sinh^2(s)\cosh^2(s) ds &= \int_0^t \left(\frac{\cosh(2s) - 1}{2}\right)\left(\frac{\cosh(2s) + 1}{2}\right) ds \\ &= \frac{1}{4}\int_0^t (\cosh^2(2s) - 1) ds \\ &= \frac{1}{8}\int_0^t (\cosh(4s) + 1) - \frac{t}{4} \\ &= \frac{\sinh(4t)}{32} + \frac{t}{8} - \frac{t}{4} \\ &= \frac{\sinh(4t)}{32} - \frac{t}{8}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^t \sinh^4(s) ds &= \frac{\sinh(4t)}{32} - \frac{\sinh(2t)}{4} - \frac{t}{8} + \frac{t}{2} \\ &= \frac{\sinh(4t)}{32} - \frac{\sinh(2t)}{4} + \frac{3t}{8}. \end{aligned}$$

Thus our solution to this differential equation is

$$y_1(t) = a\cosh(t) + b\sinh(t) + \frac{\cosh(t)}{\sinh^3(1)}\left(\frac{\sinh(4t)}{32} - \frac{\sinh(2t)}{4} + \frac{3t}{8}\right) - \frac{\sinh^5(t)}{4\sinh^3(1)}.$$

And from our boundary conditions, we have

$$
\begin{aligned}
&y_1(0) = a = 0 \\
&y_1(1) = b\sinh(1) + \frac{\cosh(1)}{\sinh^3(1)}\left(\frac{\sinh(4)}{32} - \frac{\sinh(2)}{4} + \frac{3}{8}\right) - \frac{\sinh^2(1)}{4} = 0 \\
\implies\ & b = -\frac{\cosh(1)}{\sinh^4(1)}\left(\frac{\sinh(4)}{32} - \frac{\sinh(2)}{4} + \frac{3}{8}\right) + \frac{\sinh(1)}{4}.
\end{aligned}
$$

Thus our two term expansion for the differential equation is

$$
y(t) \sim \varepsilon\frac{\sinh(t)}{\sinh(1)} + \varepsilon^4\left(b\sinh(t) + \frac{\cosh(t)}{\sinh^3(1)}\left(\frac{\sinh(4t)}{32} - \frac{\sinh(2t)}{4} + \frac{3t}{8}\right) - \frac{\sinh^5(t)}{4\sinh^3(1)}\right).
$$

1.36. The eigenvalue problem for the vertical displacement, $y(x)$, of an elastic string with variable density is

$$y'' + \lambda^2 \rho(x, \varepsilon) y = 0, \qquad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. For small ε assume $\rho \sim 1 + \varepsilon\mu(x)$, where $\mu(x)$ is positive and continuous. In this case the solution $y(x)$ and eigenvalue λ depend on ε, and the appropriate expansions are $y \sim y_0(x) + \varepsilon y_1(x)$ and $\lambda \sim \lambda_0 + \varepsilon\lambda_1$ (better expansions will be discussed in Sect. 3.6).

(a) Find y_0 and λ_0.

Soln. Plugging the above expansions into our differential equation, we have

$$y_0'' + \varepsilon y_1'' + (\lambda_0 + \varepsilon\lambda_1)^2(1 + \varepsilon\mu(x))(y_0 + \varepsilon y_1) = 0$$

which, up to $\mathcal{O}(\varepsilon)$ becomes

$$y_0'' + \varepsilon y_1'' + \lambda_0^2 y_0 + \varepsilon(\lambda_0^2 y_1 + \lambda_0^2 y_0 \mu(x) + 2\lambda_0\lambda_1 y_0) = 0$$

For the $\mathcal{O}(1)$ terms, we have

$$y_0'' + \lambda_0^2 y_0 = 0, \qquad y_0(0) = 0, y_0(1) = 0$$

which has general solution $y_0 = a_0 \cos(\lambda_0 x) + b_0 \sin(\lambda_0 x)$. From the boundary conditions, we have

$$\begin{aligned} y_0(0) &= a_0 = 0 \\ y_0(1) &= b_0 \sin(\lambda_0). \end{aligned}$$

Since this is an eigenvalue problem, $b_0 \neq 0$ giving $\lambda_0 = n\pi$ $n \in \mathbb{Z}$, $n \neq 0$. Thus

$$\begin{aligned} y_0 &= b_0 \sin(n\pi x) \\ \lambda_0 &= n\pi. \end{aligned}$$

(b) Find y_1 and λ_1.

Soln. From the $\mathcal{O}(\varepsilon)$ terms in the expansion in part (a), we find

$$y_1'' + \pi^2 n^2 y_1 + \sin(n\pi x)(\mu(x) n^2\pi^2 + 2\lambda_0\lambda_1) = 0.$$

Which admits the homogeneous solution $y_{1,h} = a_1 \cos(n\pi x) + b_1 \sin(n\pi x)$. Let $g(x) = -\sin(n\pi x)(\mu(x)n^2\pi^2 + 2\lambda_0\lambda_1)$ and $x_1 = \cos(n\pi x)$ and $x_2 = \sin(n\pi x)$. From variation of parameters, we seek functions u_1, u_2 such that $y_p = u_1 x_1 + u_2 x_2$ where

$$\begin{aligned} u_1 &= -\int_0^x \frac{g(s) x_2(s)}{W(x_1, x_2)} ds \\ u_2 &= -\int_0^x \frac{g(x) x_1(s)}{W(x_1, x_2)} ds. \end{aligned}$$

Now,

$$\begin{aligned} W(x_1, x_2) &= \begin{vmatrix} \cos(n\pi x) & \sin(n\pi x) \\ -n\pi \sin(n\pi x) & n\pi \cos(n\pi x) \end{vmatrix} \\ &= n\pi(\cos^2(n\pi x) + \sin^2(n\pi x)) \\ &= n\pi. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= \frac{1}{n\pi}\left(b_0 \int_0^x \sin^2(n\pi s)\mu(x) ds + 2b_0 n\pi\lambda_1 \int_0^x \sin^2(n\pi s) ds \right) \\ &= n\pi b_0 \int_0^x \sin^2(n\pi x)\mu(s) ds + \lambda_1 b_0 \left(x - \frac{\sin(2n\pi x)}{2\pi n} \right) \end{aligned}$$

and

$$\begin{aligned}
u_2 &= -\frac{1}{n\pi}\left(n\pi a_0\int_0^x \sin(n\pi s)\cos(n\pi s)\mu(s)ds - 2\lambda_1 b_0\int_0^x \sin(n\pi s)\cos(n\pi s)ds\right)\\
&= -\frac{b_0 n\pi}{2}\int_0^x \sin(2n\pi s)\mu(s)ds - \lambda_1 b_0\int_0^x \sin(2n\pi s)ds\\
&= -\frac{b_0 n\pi}{2}\int_0^x \sin(2n\pi s)\mu(s)ds + \frac{\lambda_1 b_0}{2n\pi}(\cos(2n\pi x)-1)
\end{aligned}$$

so that our general solution for y_1 takes the form

$$\begin{aligned}
y_1 &= n\pi b_0\cos(n\pi x)\int_0^x \sin^2(n\pi s)\mu(s)ds + n\pi\lambda_1 b_0\cos(n\pi x)\left(x - \frac{\sin(2n\pi x)}{2n\pi}\right)\\
&\quad - \frac{n\pi b_0}{2}\sin(n\pi x)\int_0^x \sin(2n\pi s)\mu(s)ds + \frac{\lambda_1 b_0}{2}\sin(n\pi x)(\cos(2n\pi x)-1) + a\cos(n\pi x) + b\sin(n\pi x).
\end{aligned}$$

From the boundary conditions, notice $y_1(0) = a = 0$ and

$$\begin{aligned}
y_1(1) &= n\pi a_0\cos(n\pi)\int_0^1 \sin^2(n\pi s)\mu(s)ds + \lambda_1 b_0\cos(n\pi) = 0\\
\implies \lambda_1 &= -n\pi\int_0^1 \sin^2(n\pi s)\mu(s)ds
\end{aligned}$$

and so

$$\begin{aligned}
y_1 &= n\pi a_0\cos(n\pi x)\int_0^x \sin^2(n\pi s)\mu(s)ds + \lambda_1 b_0\cos(n\pi x)\left(x - \frac{\sin(2n\pi x)}{2n\pi}\right)\\
&\quad - \frac{n\pi}{2}\sin(n\pi x)\int_0^x \sin(2n\pi s)\mu(s)ds + \frac{\lambda_1 b_0}{2n\pi}\sin(n\pi x)(\cos(2n\pi x)-1) + b_1\sin(n\pi x).
\end{aligned}$$

1.41. In quantum mechanics, the perturbation theory for bound states involves the time-independent (normalized) Schrödinger equation

$$\Psi'' - [V_0(x) + \varepsilon V_1(x)]\Psi = -E\Psi, \qquad \text{for} \quad -\infty < x < \infty,$$

where $\psi(-\infty) = \psi(\infty) = 0$. In this problem the eigenvalue E is the energy, V_1 i s the perturbing potential, and ε is called the coupling constant. The potentials V_0 and V_1 are given continuous functions. This exercise examines what is known as a logarithmic perturbation expansion to fin dthe corrections to the energy. To do this, it is assumed that the unperturbed ($\varepsilon = 0$) state is nonzero (more specifically, it is a nondegenerate ground state).

(a) Assuming $\psi \sim \psi_0(x) + \varepsilon\psi_1(x) + \varepsilon^2\psi_2(x)$ and $E \sim E_0 + \varepsilon E_1 + \varepsilon^2 E_2$, find what problem the first term in these expansions satisfies. In this problem assume

$$\int_{-\infty}^{\infty} \psi_0^2 dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} |V_1(x)| dx < \infty.$$

Soln. Plugging these expansions into our differential equation, we find the following:

$$\psi_0'' + \varepsilon\psi_1'' + \varepsilon^2\psi_2'' - [V_0(x) + \varepsilon V_1(x)](\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2) = -(E_0 + \varepsilon E_1 + \varepsilon^2 E_2)(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2)$$

and notice the $\mathcal{O}(1)$ terms give us the following equation:

$$\psi_0'' - V_0(x)\psi_0 = -E_0\psi_0.$$

That is, the first terms of the expansions satisfy the standard time independent Schrödinger equation.

(b) Letting $\psi = e^{\varphi(x)}$, find the problem $\varphi(x)$ satisfies.

Soln. For $\psi = e^{\varphi(x)}$, we have

$$\begin{aligned} \psi' &= \varphi' e^{\varphi} \\ \psi'' &= \varphi'' e^{\varphi} + (\varphi')^2 e^{\varphi}. \end{aligned}$$

Plugging this into our equation, we have

$$\begin{aligned} &\varphi'' e^{\varphi} + (\varphi')^2 e^{\varphi} - [V_0 + \varepsilon V_1]e^{\varphi} = -Ee^{\varphi} \\ \implies &\varphi'' + (\varphi')^2 - [V_0 + \varepsilon V_1] = -E. \end{aligned}$$

(c) Expand $\varphi(x)$ for small ε, and from this find E_1 and E_2 in terms of ψ_0 and the perturbing potential.

Soln. Assume $\varphi \sim \varphi_0 + \varepsilon^\alpha\varphi_1 + \varepsilon^{2\alpha}\varphi_2$ and $\varphi'' \sim \varphi_0'' + \varepsilon^\alpha\varphi_1'' + \varepsilon^{2\alpha}\varphi_2''$. Plugging this expansion into our differential equation gives

$$\varphi_0'' + \varepsilon^\alpha\varphi_1'' + \varepsilon^{2\alpha}\varphi_2'' + \cdots + \left(\varphi_0' + \varepsilon^\alpha\varphi_1' + \varepsilon^{2\alpha}\varphi_2' + \cdots\right)^2 - V_0 - \varepsilon V_1 = -E_0 - \varepsilon E_1 - \varepsilon^2 E_2.$$

From the $\mathcal{O}(1)$ terms, we have

$$\varphi_0'' + (\varphi_0')^2 - V_0 = -E_0$$

and from the $\mathcal{O}(\varepsilon)$ term, we have $\alpha = 1$ and so

$$\varphi_1'' + 2\varphi_0'\varphi_1' - V_1 = -E_1$$

and finally the $\mathcal{O}(\varepsilon^2)$ term gives

$$\varphi_2'' + (\varphi_1')^2 + 2\varphi_0'\varphi_2' = -E_2.$$

Now, relating the asymptotic expansion of ψ with the asymptotic expansion for φ, notice

$$\begin{aligned}
\psi &\sim e^{\varphi_0+\varepsilon\varphi_1+\varepsilon^2\varphi_2+\cdots} \\
&= e^{\varphi_0}e^{\varepsilon\varphi_1+\varepsilon^2\varphi_2+\cdots} \\
&= e^{\varphi_0}\left(1+\varepsilon(\varphi_1+\varepsilon\varphi_2+\cdots)+\frac{\varepsilon^2}{2}(\varphi_1+\varepsilon\varphi_2+\cdots)^2+\cdots\right) \\
&= \psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots \\
\implies \psi_0 &= e^{\varphi_0} \\
\psi_1 &= e^{\varphi_0}\varphi_1 = \psi_0\varphi_1 \\
\psi_2 &= e^{\varphi_0}\left(\frac{\varphi_1^2}{2}+\varphi_2\right) = \psi_0\left(\frac{\varphi_1^2}{2}+\varphi_2\right).
\end{aligned}$$

Now, we wish to solve for E_1 and E_2, so for the $\mathcal{O}(\varepsilon)$ ODE for φ_1, multiply each side by $e^{2\varphi_0}$ so that the equation becomes

$$\begin{aligned}
& e^{2\varphi_0}\varphi_1'' + 2\varphi_0'e^{2\varphi_0}\varphi_1' - V_1e^{2\varphi_0} = -E_1e^{2\varphi_0} \\
\implies & \frac{d}{dx}\left(e^{2\varphi_0}\varphi_1'\right) = e^{2\varphi_0}(V_1-E_1) \\
\implies & \int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{2\varphi_0}\varphi_1'\right)dx = \int_{-\infty}^{\infty}e^{2\varphi_0}(V_1-E_1)dx \\
\implies & \psi_0^2(\infty)\varphi_1'(\infty)-\psi_0^2(-\infty)\varphi_1'(-\infty) = \int_{-\infty}^{\infty}V_1\psi_0^2dx - E_1\int_{-\infty}^{\infty}\psi_0^2dx.
\end{aligned}$$

And since $\varphi_1 = \frac{\psi_1}{\psi_0}$, we have $\varphi_1' = \frac{\psi_1'\psi_0-\psi_0'\psi_1}{\psi_0^2}$ so that the above equation becomes

$$\psi_1'(\infty)\psi_0(\infty)-\psi_0'(\infty)\psi_1(\infty)-\psi_1'(-\infty)\psi_0(\infty)+\psi_0'(-\infty)\psi_1(-\infty) = \int_{-\infty}^{\infty}V_1\psi_0^2dx - E_1.$$

Now using the fact that $\psi_0(\pm\infty) = \psi_1(\pm\infty) = 0$, it follows that (see lemma 1) $\psi_0'(\pm\infty) = \psi_1'(\pm\infty) = 0$ so that the above equation becomes

$$E_1 = \int_{-\infty}^{\infty}V_1\psi_0^2dx.$$

For E_2, multiply the $\mathcal{O}(\varepsilon^2)$ ODE for φ_2 by $e^{2\varphi_0}$:

$$\begin{aligned}
& e^{2\varphi_0}\varphi_2'' + e^{2\varphi_0}(\varphi_1')^2 + 2e^{2\varphi_0}\varphi_0'\varphi_2' = -E_2e^{2\varphi_0} \\
\implies & \frac{d}{dx}\left(\psi_0^2\varphi_2'\right) = -\psi_0^2(E_2+(\varphi_1')^2) \\
\implies & \int_{-\infty}^{\infty}\frac{d}{dx}(\psi_0^2\varphi_2')dx = -\int_{-\infty}^{\infty}\psi_0^2(E_2+(\varphi_1')^2)dx. \\
\implies & \psi_0^2(\infty)\varphi_2'(\infty)-\psi_0^2(-\infty)\varphi_2'(-\infty) = -\int_{-\infty}^{\infty}\psi_0^2(E_2+(\varphi_1')^2)dx.
\end{aligned}$$

Now notice

$$\begin{aligned}
\varphi_2'\psi_0^2 &= \psi_2'\psi_0 - \psi_0'\psi_2 \\
\implies \varphi_2'(\pm\infty)\psi_0^2(\pm\infty) &= \psi_2'(\pm\infty)\psi_0(\pm\infty) - \psi_0'(\pm\infty)\psi_2(\pm\infty) \\
&= 0
\end{aligned}$$

by lemma 1. Thus

$$\begin{aligned} 0 &= -E_2\int_{-\infty}^{\infty}\psi_0^2dx - \int_{-\infty}^{\infty}\psi_0^2(\varphi_1')^2dx \\ \implies E_2 &= -\int_{-\infty}^{\infty}\psi_0^2(\varphi_1')^2dx. \end{aligned}$$

And from out work in finding E_1, recall

$$\begin{aligned} \frac{d}{dx}(\psi_0^2\varphi_1') &= \psi_0^2(V_1 - E_1) \\ \implies \varphi_1'(x) &= \psi_0^{-2}\int_{-\infty}^{x}\psi_0^2(V_1 - E_1)dt \\ \implies (\varphi_1'(x))^2 &= \psi_0^{-4}\left(\int_{-\infty}^{x}\psi_0^2(V_1 - E_1)dt\right)^2. \end{aligned}$$

Thus

$$E_2 = \int_{-\infty}^{\infty}\psi_0^{-2}\left(\int_{-\infty}^{x}\psi_0^2(V_1 - E_1)dt\right)^2dx.$$

with

$$E_1 = \int_{-\infty}^{\infty}V_1\psi_0^2dx.$$

Lemma 1: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$\lim_{x\to\pm\infty}|f(x)| = M < \infty$$

then

$$\lim_{x\to\pm\infty}|f'(x)| = 0.$$

The purpose of this lemma is to show that $\lim_{x\to\pm\infty} f(x)f'(x) \neq 0$ cannot be the case.

Proof: It suffices to show that if $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} f'(x) = 0$. The case where $x \to -\infty$ will follow similarly. Let $\{x_n\}$ be a sequence of positive real numbers such that $x_n \to \infty$ as $n \to \infty$. In particular, suppose x_n satisfies the relation $x_n = 2x_{n-1}$. By the mean value theorem, there exists a $c(x_n) \in (x_{n-1}, x_n)$ such that

$$\begin{aligned} f'(c) &= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}. \\ \implies |f'(c)| &\le \frac{|f(x_n)| + |f(x_{n-1})|}{|x_n - x_{n-1}|} \\ \implies |f'(c(x))| &\le \frac{4M}{x_n} \end{aligned}$$

and as $n \to \infty$, $\frac{4M}{x_n} \to 0$, so that $\lim_{x\to\infty} f'(x) = 0$, as desired.

(d) For a harmonic oscillator (thus, $V_0 = \lambda^2x^2$ with $\lambda > 0$) with perturbing potential $V_1 = \alpha x e^{-\gamma x^2}$ (where α and γ are positive) show that

$$E \sim \lambda - \frac{1}{4}\left(\frac{\varepsilon\alpha}{\gamma + \lambda}\right)^2\sqrt{\frac{\lambda}{\lambda + 2\gamma}}.$$

Proof: We assume $\psi \sim \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2$, $E \sim E_0 + \varepsilon E_1 + \varepsilon^2 E_2$ and $\psi_0 = ae^{-\lambda x^2/2}$ where a is a scaling constant to be determined that allows ψ_0 to satisfy the normalization condition

$$\int_{-\infty}^{\infty} \psi_0^2 dx = 1.$$

Notice

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi_0^2 dx &= \int_{-\infty}^{\infty} a^2 e^{-\lambda x^2} dx. \\
&= \frac{a^2}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= a^2 \sqrt{\frac{\pi}{\lambda}} = 1 \\
\implies a &= \left(\frac{\lambda}{\pi}\right)^{1/4}.
\end{aligned}$$

With $u = \sqrt{\lambda}x$. To get E_0, we plug this equation into the equation found in part (a):

$$\begin{aligned}
&\frac{d^2}{dx^2}\left(ae^{-\lambda x^2/2}\right) - \lambda^2 x^2 a e^{-\lambda x^2/2} = -E_0 e^{-\lambda x^2/2} \\
&\implies -\lambda e^{-\lambda x^2/2} + \lambda^2 x^2 e^{-\lambda x^2/2} - \lambda^2 x^2 e^{-\lambda x^2/2} = -E_0 e^{-\lambda x^2/2} \\
&\implies E_0 = \lambda.
\end{aligned}$$

Now, notice

$$\begin{aligned}
E_1 &= \int_{-\infty}^{\infty} V_1 \psi_0^2 dx \\
&= \int_{-\infty}^{\infty} \alpha \sqrt{\frac{\lambda}{\pi}} x e^{-(\gamma+\lambda)x^2} dx.
\end{aligned}$$

And since the integrand is odd and the integral converges $(*)$, we have that $E_1 = 0$. Now, solving for E_2, let us first evaluate $\int_{-\infty}^{x} \psi_0^2 V_1 dt$:

$$\begin{aligned}
\int_{-\infty}^{x} \psi_0^2 V_1 dt &= \alpha \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{x} t e^{-(\gamma+\lambda)t^2} dt \\
&= \frac{\alpha}{2(\gamma+\lambda)} \sqrt{\frac{\lambda}{\pi}} \int_{\infty}^{(\gamma+\lambda)x^2} e^{-u} du \\
&= \frac{\alpha}{2(\gamma+\lambda)} \sqrt{\frac{\lambda}{\pi}} [-e^{-u}]\Big|_{\infty}^{(\gamma+\lambda)x^2} \\
&= -\frac{\alpha}{2(\gamma+\lambda)} \sqrt{\frac{\lambda}{\pi}} \left(e^{-(\gamma+\lambda)x^2}\right) \\
\implies \left(\int_{-\infty}^{x} \psi_0^2 V_1 dt\right)^2 &= \frac{\lambda}{4\pi}\left(\frac{\alpha}{\gamma+\lambda}\right)^2 e^{-2(\gamma+\lambda)x^2}
\end{aligned}$$

with $u = (\gamma+\lambda)t^2$. Thus

$$\begin{aligned}
E_2 &= \frac{\lambda}{4\pi}\left(\frac{\alpha}{\gamma+\lambda}\right)^2\sqrt{\frac{\pi}{\lambda}}\int_{-\infty}^{\infty} e^{-(\lambda+2\gamma)x^2}dx \\
&= \frac{1}{4}\sqrt{\frac{\lambda}{\pi(\lambda+2\gamma)}}\left(\frac{\alpha}{\gamma+\lambda}\right)^2\int_{-\infty}^{\infty} e^{-u^2}du \\
&= \frac{1}{4}\sqrt{\frac{\lambda}{\pi(\lambda+2\gamma)}}\left(\frac{\alpha}{\gamma+\lambda}\right)^2\sqrt{\pi} \\
&= \frac{1}{4}\left(\frac{\alpha}{\gamma+\lambda}\right)^2\sqrt{\frac{\lambda}{2\gamma+\lambda}}.
\end{aligned}$$

With $u = \sqrt{2\gamma+\lambda}x$. Hence

$$E \sim \lambda - \frac{1}{4}\left(\frac{\varepsilon\alpha}{\gamma+\lambda}\right)^2\sqrt{\frac{\lambda}{2\gamma+\lambda}}$$

as desired.

Proof of $(*)$: By a change of variables, it suffices to inspect the integral

$$\int_{-\infty}^{\infty} xe^{-x^2}dx.$$

We split the integral from $-\infty$ to 0 and 0 to ∞:

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx.$$

Further, it is sufficients to show $\int_0^\infty xe^{-x^2}dx < \infty$ since $\int_{-\infty}^0 xe^{-x^2}dx = -\int_0^\infty xe^{-x^2}dx$. From letting $u = x^2$, we find

$$\begin{aligned}
\int_0^\infty xe^{-x^2}dx &= \frac{1}{2}\int_0^\infty e^{-u}du \\
&= -\frac{1}{2}[e^{-u}]\Big|_0^\infty \\
&= \frac{1}{2}
\end{aligned}$$

Thus, the integral converges, as desired.