Final Exam

Michael Nameika

1. (10 pts) Let H be a Hilbert space and let $T : H \to H$ be a bijective bounded linear operator. Prove that
(a) $(T^{-1})^*$ exists on H.
(b) $(T^*)^{-1}$ exists on H and $(T^*)^{-1} = (T^{-1})^*$.

Proof: (a) Since H is a Hilbert space and T is bijective and bounded, we have that T^{-1} exists, and by the bounded inverse theorem, T^{-1} is bounded. Then by the existence theorem of Hilbert-adjoint operators, we have that $(T^{-1})^*$ exists.

(b) We have that T^* exists by the existence theorem of Hilbert-adjoint operators. To show that $(T^*)^{-1}$ exists, we show that $\mathcal{N}(T^*) = \{\mathbf{0}\}$. Let $x, y \in H$ with $x \neq 0$ and $y \in \mathcal{N}(T^*)$. Then

$$\begin{aligned} \langle Tx, y\rangle &= \langle x, T^*y\rangle \\ &= \langle x, 0\rangle \\ &= 0 \end{aligned}$$

so that $Tx \perp y$ hence $y \in \mathcal{R}(T)^\perp$. But since T is bijective, we have $\overline{\mathcal{R}(T)} = \mathcal{R}(T) = H$ and by the direct sum decomposition for Hilbert spaces, we have

$$\begin{aligned} H &= \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp \\ &= H \oplus \mathcal{R}(T)^\perp \\ \implies \mathcal{R}(T)^\perp &= \{\mathbf{0}\} \end{aligned}$$

so that $y = \mathbf{0}$, and so $\mathcal{N}(T^*) = \{\mathbf{0}\}$ and T^* is hence injective and so $(T^*)^{-1}$ exists. We now show that $(T^*)^{-1} = (T^{-1})^*$. Let $x \in H$. We wish to show $T^*(T^{-1})^*x = x$. Well, notice

$$\begin{aligned} \langle T^*(T^{-1})^*x, x\rangle &= \langle T^*x, T^{-1}x\rangle \\ &= \langle x, TT^{-1}x\rangle \\ &= \langle x, x\rangle \end{aligned}$$

thus $(T^*)^{-1} = (T^{-1})^*$.

2. (15 pts) (a) Let X and Y be Banach spaces. Let $T_n : X \to Y$ be a sequence of bounded linear operators. Assume that $(T_n x)$ converges for every $x \in X$.

   (i) Prove that the sequence of operator norms $(\|T_n\|)$ is bounded.

   Proof: Since $(T_n x)$ converges for every $x \in X$, we have that for each x, $(T_n x)$ is bounded, that is,
$$\|T_n x\| \leq c_x$$
for some $c_x = c(x) > 0$. Then by the bounded inverse theorem, we have that $(\|T_n\|)$ is bounded.

   (ii) Prove that $Tx = \lim_{n\to\infty} T_n x$ defines a bounded linear operator $T : X \to Y$.

   Proof: By hypothesis, since $T_n x$ converges for every $x \in X$, we have that $Tx = \lim_{n\to\infty} T_n x$ exists and is well-defined by uniqueness of limits. We now show that T is linear. Let $x, y \in X$ and α be any scalar. Now notice
$$\begin{aligned}
T(\alpha x + y) &= \lim_{n\to\infty} T_n(\alpha x + y) & \\
&= \lim_{n\to\infty} T_n(\alpha x) + \lim_{n\to\infty} Ty & \text{(Linearity of Limits)} \\
&= \lim_{n\to\infty} \alpha T_n x + Ty & \text{(Linearity of } T_n\text{)} \\
&= \alpha \lim_{n\to\infty} T_n x + Ty & \text{(Linearity of Limits)} \\
&= \alpha Tx + Ty & \text{(Def. of } T\text{)}
\end{aligned}$$
so that T is linear. What remains is to show that T is bounded. Well, since the sequence $(\|T_n\|)$ is bounded by part (a), we have that T is bounded since for some $M > 0$,
$$\|T_n\| \leq M$$
hence, letting $n \to \infty$, we have
$$\|T\| \leq M.$$

(b) Let $T_n : \ell^2 \to \ell^2$ be defined by $T_n x = (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$, for $x = (\xi_j)$. Prove that the hypothesis of part (a) is satisfied, yet for the limit T of part (a)(ii), we have $\|T_n - T\| = 1$ for all n.

Proof: Note that ℓ^2 is a Banach space and T_n is bounded since
$$\begin{aligned}
\|T_n x\| &= \|(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)\| \\
&\leq \|x\| \\
\implies \|T_n\| &\leq 1.
\end{aligned}$$
Now, $(T_n x)$ clearly converges for any $x \in \ell^2$ since, as $n \to \infty$,
$$\lim_{n\to\infty} T_n x = (\xi_1, \xi_2, \xi_3, \ldots) = x.$$
Now, notice that
$$\|(T_n - T)x\| = \|(0, 0, \ldots, \xi_{n+1}, \ldots)\| \leq \|x\|$$
so that
$$\|T_n - T\| \leq 1.$$
For the lower bound, take, for each n, $x_n = (0, 0, \ldots, 1, 0, \ldots)$, all zeros except a 1 in the $(n+1)^{\text{th}}$ position. Then notice that $T_n x_n = 0$ and $Tx = x$ and that $\|x\| = 1$ so that
$$\|(T_n - T)x\| = \|x\| = 1$$
and we have
$$\|T_n - T\| = 1.$$

3. (20 pts) (a) Let $T_n : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $T_n x = \left(0, \frac{\xi_1}{1}, \frac{\xi_2}{2}, \dots, \frac{\xi_{n-1}}{n-1}\right)$, where $x = (\xi_1, \dots, \xi_n)$. Find all eigenvalues and eigenvectors of T_n and their algebraic and geometric multiplicities.
(b) Let $T : \ell^2 \to \ell^2$ be defined by $Tx = \left(0, \frac{\xi_1}{1}, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right)$, where $x = (\xi_1, \xi_2, \dots)$. Show that T has no eigenvalues, and that $\lambda = 0$ is a spectral value.
(c) Prove that the operator $T : \ell^2 \to \ell^2$ of part (b) is a compact linear operator but is *not* a self-adjoint linear operator.

Proof: (a) Let λ be an eigenvalue of T_n and $x = (\xi_1, \xi_2, \dots, \xi_n)$ be an associated eigenvector. Then

$$\begin{aligned} T_n x &= \lambda x \\ \implies \left(0, \xi_1, \frac{\xi_2}{2}, \dots, \frac{\xi_{n-1}}{n-1}\right) &= (\lambda\xi_1, \lambda\xi_2, \dots, \lambda\xi_n) \end{aligned}$$

which yields

$$\begin{aligned} 0 &= \lambda\xi_1 \\ \xi_1 &= \lambda\xi_2 \\ \frac{\xi_2}{2} &= \lambda\xi_3 \\ &\vdots \\ \frac{\xi_{n-1}}{n-1} &= \lambda\xi_n \end{aligned}$$

which, for $\lambda \neq 0$ gives us $x = \mathbf{0}$, which is not an eigenvector by definition. Now, if $\lambda = 0$, the above system of equations is satsified with $x = (0, 0, \dots, 0, \xi_n)$. Now, since $\dim \mathbb{C}^n = n$, we have that T_n has n eigenvalues (counting multiplicity), so that $\lambda = 0$ has algebraic multiplicity n with geometric multiplicity 1 since $x = (0, 0, \dots, \xi_n)$ is the only associated eigenvector.

(b) Suppose that T has an eigenvalue λ and let $x = (\xi_1, \xi_2, \dots)$ be an associated eigenvector. Then we have

$$\begin{aligned} Tx &= \lambda x \\ \implies \left(0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right) &= (\lambda\xi_1, \lambda\xi_2, \lambda\xi_3, \dots) \\ \implies 0 &= \lambda\xi_1 \\ \xi_1 &= \lambda\xi_2 \\ \frac{\xi_2}{2} &= \lambda\xi_3 \\ &\vdots \end{aligned}$$

which, if $\lambda \neq 0$ yields $x = \mathbf{0}$ which is not an eigenvector by definition. If $\lambda \neq 0$, we get $\xi_1 = 0 \implies \xi_2 = 0 \implies \xi_3 = 0 \dots$ which yields $x = \mathbf{0}$, which is not an eigenvector by definition. Thus, T has no eigenvalues. Now, to show that $\lambda = 0$ is a spectral value, we show that $\lambda = 0$ is an approximate eigenvalue, and then by problem **II.1**(b) from homework 12, we have that $\lambda \in \sigma(T)$. Define the sequence (x_n) by $x_n = (\xi_1, \xi_2, \dots)$ with $\xi_k = \delta_{kn}$, the Dirac delta. Notice

$$\begin{aligned} \|x_n\| &= \left(\sum_{k=1}^{\infty} |\delta_{kn}|^2\right)^{1/2} \\ &= 1 \end{aligned}$$

and that

$$\begin{aligned} \|Tx_n - \lambda x_n\| &= \|Tx_n\| \\ &= \frac{1}{n+1} \to 0 \end{aligned}$$

so that $\lambda = 0$ is an approximate eigenvalue and is thus in the spectrum of T.

(c) Define the sequence of linear operators (T_n) by $T_n x = \left(0, \xi_1, \frac{\xi_2}{2}, \dots, \frac{\xi_n}{n}, 0, 0, \dots\right)$ and notice that $\dim(\mathcal{R}(T_n)) = n$ so that each T_n is a compact linear operator. Now, notice that

$$\begin{aligned} \|(T - T_n)x\| &= \left\|\left(0, 0, \dots, 0, \frac{\xi_{n+1}}{n+1}, \dots\right)\right\| \\ &\le \frac{\|x\|}{n+1} \\ &\to 0 \end{aligned}$$

so that (T_n) defines a sequence of compact linear operators that converge uniformly to T in the operator norm, hence T is a compact linear operator.

4. (15 pts) Let (q_j) be a bounded sequence of real numbers. Define $T : \ell^2 \to \ell^2$ by $y = Tx$, $x = (\xi_j)$, $y = (\eta_j)$, $\eta_j = q_j\xi_j$, $j = 1, 2, \ldots$. Verify the hypothesis of Theorem 9.1-2 for T. Then apply the criterion (2) of this theorem, that characterizes the resolvent set, to prove that the spectrum of the operator T is the closure of its set of eigenvalues. What are these eigenvalues?

Proof: Note that ℓ^2 is a Hilbert space. Notice that T is a bounded linear operator since (q_j) is a bounded sequence, that is there exists some $M > 0$ such that $|q_j| \leq M$ for all $j \in \mathbb{N}$ and

$$\begin{aligned} \|Tx\|^2 &= \sum_{n=1}^{\infty} |q_j\xi_j|^2 \\ &\leq M^2 \sum_{n=1}^{\infty} |\xi_j|^2 \\ &= M^2\|x\|^2 \\ \implies \|Tx\| &\leq M\|x\|. \end{aligned}$$

Additionally, T is self-adjoint since

$$\begin{aligned} \langle Tx, y\rangle &= \sum_{n=1}^{\infty} (q_j\xi_j)\overline{\eta_j} \\ &= \sum_{n=1}^{\infty} \xi_j(q_j\overline{\eta_j}) \\ &= \sum_{n=1}^{\infty} \xi_j\overline{(q_j\eta_j)} && (q_j \in \mathbb{R}) \\ &= \langle x, Ty\rangle. \end{aligned}$$

Thus, the hypothesis of theorem 9.1-2 is satisfied. We now find the eigenvalues of T.
I claim that the eigenvalues of T are simply q_j for $j \in \mathbb{N}$ since, for $x_j = (\xi_1, \xi_2, \ldots)$, with $\xi_k = \delta_{jk}$, we have that

$$\begin{aligned} Tx_j &= (0, 0, \ldots, q_j, 0, \ldots) \\ &= q_jx_j. \end{aligned}$$

To see that there are no other eigenvalues, suppose there exists $\lambda \neq q_j$ for $j \in \mathbb{N}$ such that $Tx = \lambda x$, $x \neq x_j$, which gives us

$$(q_1\xi_1, q_2\xi_2, \ldots) = (\lambda\xi_1, \lambda\xi_2, \ldots)$$

which yields

$$\begin{aligned} q_1\xi_1 &= \lambda\xi_1 \\ q_2\xi_2 &= \lambda\xi_2 \\ &\vdots \end{aligned}$$

hence, either $x = \mathbf{0}$ which is not an eigenvector by definition, or $\lambda = q_1 = q_2 = \cdots$ so that (q_j) must be a constant sequence with $\lambda = q_1$ contrary to our hypothesis. Thus, the only eigenvalues are q_j. We now use (2) of the same theorem to show that the spectrum is the closure of the eigenvalues. Define $Q = \{q_j \mid j \in \mathbb{N}\}$ be the set of all elements of the given sequence and let $\lambda \notin \overline{Q}$. In particular, λ is not a limit point of Q, so there exists some $\varepsilon > 0$ such that

$$|\lambda - q| \geq \varepsilon, \qquad q \in Q.$$

Now let $x \in \ell^2$, $x = (\xi_1, \xi_2, \dots)$ and consider $(T - \lambda I)x$:

$$\begin{aligned}
\|(T-\lambda)x\|^2 &= \|Tx - \lambda x\|^2 \\
&= \|((q_1 - \lambda)\xi_1, (q_2 - \lambda)\xi_2, \dots)\|^2 \\
&= \sum_{n=1}^{\infty} |(q_n - \lambda)\xi_n|^2 \\
&\geq \varepsilon^2 \sum_{n=1}^{\infty} |\xi_n|^2 \\
\implies \|(T - \lambda I)\| &\geq \varepsilon \|x\|
\end{aligned}$$

so that by theorem 9.1-2, $\lambda \in \rho(T)$. We note that the above inequality does not hold if λ is a limit point of Q, so that $\sigma(T) = \overline{Q}$, as desired.

5. (parts (a)-(c), 15 pts) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H.
(a) Show that $\mathcal{R}(T) \perp \mathcal{N}(T^*)$, meaning that, for all y in the range of T and all z in the null-space of the Hilbert adjoint T^* we have $y \perp z$.
(b) Suppose in addition that T is self-adjoint and the range $\mathcal{R}(T)$ is dense in H. Prove that T is injective.
(c) Verify that the linear operator $T : \ell^2 \to \ell^2$ defined by $T(\xi_j) = (\xi_j/j)$ is bounded, self-adjoint, injective, and has dense range, but is *not* surjective.
(d) (extra credit, 5 pts) Suppose T is bounded and self-adjoint but not injective. Let $x \in \mathcal{N}(T)$ with $x \neq \mathbf{0}$, and let $\epsilon > 0$ be given. Suppose there exists $u \in H$ such that $z = x - Tu$ satisfies $\|z\| \leq \epsilon$. Prove that then $\|x\| \leq \epsilon$ as well. Show that this result offers a means to prove part (b).
Hint: Consider $\langle x, x - z\rangle$.

Proof: (a) Let $y \in \mathcal{R}(T)$. Then there exists some $x \in H$ such that $Tx = y$. Let $z \in \mathcal{N}(T^*)$ and consider

$$\begin{aligned}\langle y, z\rangle &= \langle Tx, z\rangle \\ &= \langle x, T^*z\rangle \\ &= \langle x, 0\rangle \\ &= 0\end{aligned}$$

since y and z were chosen arbitrarily, we have that $\mathcal{R}(T) \perp \mathcal{N}(T^*)$, as desired.

(b) Suppose that T is not injective. Then let $x \in \mathcal{N}(T)$ with $x \neq \mathbf{0}$. Take $y \in H$, $y \notin \mathcal{N}(T)$ and consider

$$\langle Tx, y\rangle = \langle 0, y\rangle = 0$$

but since T is self-adjoint,

$$\begin{aligned}\langle Tx, y\rangle &= \langle x, Ty\rangle \\ &= 0\end{aligned}$$

so that $x \perp Ty$. But since $\mathcal{R}(T)$ is dense in H, we have that $\mathcal{R}(T)^\perp = \{\mathbf{0}\}$. Additionally, by part (a), we have that, along with the fact that T is self-adjoint,

$$\mathcal{R}(T) \perp \mathcal{N}(T).$$

Thus, $\mathcal{N}(T) \subseteq \mathcal{R}(T)^\perp = \{\mathbf{0}\} \implies \mathcal{N}(T) = \{\mathbf{0}\}$ a contradiction. Hence T is injective.

(c) Let $x \in \ell^2$ with $x = (\xi_1, \xi_2, \dots)$ and notice $Tx = \left(\xi_1, \frac{\xi_2}{2}, \dots\right)$. Then note that, for any $j \in \mathbb{N}$,

$$\left|\frac{\xi_j}{j}\right| \leq |\xi_j|.$$

Thus

$$\|Tx\| \leq \|x\|$$

and so T is bounded. Now let $x \in \mathcal{N}(T)$, $x = (\xi_1, \xi_2, \dots)$ and notice

$$\begin{aligned} Tx &= \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots\right) \\ &= (0, 0, \dots) \\ \implies \xi_1 &= 0 \\ \xi_2 &= 0 \\ &\vdots \end{aligned}$$

so that $x = \mathbf{0}$ and so $\mathcal{N}(T) = \{\mathbf{0}\}$. Thus T is injective. We now show that T is self-adjoint. Let $x, y \in \ell^2$ with $x = (\xi_1, \xi_2, \dots)$, $y = (\eta_1, \eta_2, \dots)$. Recall that the inner product on ℓ^2 is defined by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} \xi_k \overline{\eta_k}$$

and so

$$\begin{aligned}\langle Tx, y\rangle &= \sum_{k=1}^{\infty} \frac{\xi_k}{k}\overline{\eta_k} \\ &= \sum_{k=1}^{\infty} \xi_k \frac{\overline{\eta_k}}{k} \\ &= \sum_{k=1}^{\infty} \xi_k \overline{\left(\frac{\eta_k}{k}\right)} \\ &= \langle x, Ty\rangle\end{aligned}$$

so that T is self-adjoint. To see that T is not surjective, notice that $x = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \in \ell^2$ since

$$\|x\|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

But note that the preimage of x is $y = (1, 1, 1, \dots) \notin \ell^2$. Hence, $x \notin \mathcal{R}(T)$ and T is thus not surjective. Finally, we show that $\mathcal{R}(T)$ is dense in ℓ^2. Let $x \in \ell^2$ with $x = (\xi_1, \xi_2, \dots)$. Define $x_n = (\xi_1, 2\xi_2, 3\xi_3, \dots, n\xi_n, 0, 0, \dots)$ and notice

$$Tx_n = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)$$

and that

$$\sum_{k=1}^{n} |k\xi_k|^2 < \infty$$

so that $x_n \in \ell^2$ for each $n \in \mathbb{N}$. Notice that

$$\begin{aligned}\|T(x - x_n)\|^2 &= \|(0, 0, \dots, 0, \xi_{n+1}, \dots)\|^2 \\ &= \sum_{k=n+1}^{\infty} |\xi_k|^2\end{aligned}$$

and since $x \in \ell^2$, the sequence of partial sums $\left(s_n = \sum_{k=1}^{n} |\xi_k|^2\right)$ converges, so that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that whenever $n > N$, $|s_n - s|^2 < \varepsilon$ where $s = \lim_{n\to\infty} s_n$. And notice that $|s_n - s| = \|T(x - x_n)\|$ so that for $n > N$,

$$\|T(x - x_n)\|^2 < \varepsilon$$

and so $\mathcal{R}(T)$ is dense in ℓ^2, as desired.

6. (15 pts) Let X be a Banach space and let $T : X \to X$ be a bounded linear operator such that $T^2 = T$. Such an operator is said to be *idempotent*. Assume $T \neq \mathbf{0}$ and $T \neq I$. Show that the spectrum of T is the two-point set $\sigma(T) = \{0, 1\}$. To proceed, find an explicit closed formula for $(T - \lambda I)^{-1}$ whenever $\lambda \neq 0, 1$ by formally applying (9) of Sec. 7.3 for $\lambda > 1$.
Hint: (9) collapses to a form $A_\lambda I + B_\lambda T$ for explicit expressions A_λ and B_λ in λ. Verify algebraically that this linear combination represents $(T - \lambda I)^{-1}$ for any $\lambda \neq 0, 1$. For instance, find that $(T + I)^{-1} = I - \frac{1}{2}T$.

Proof: We first consider the case $\lambda > 1$. By (9) of section 7.3-4, we have

$$R_\lambda = -\frac{1}{\lambda}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}T\right)^j$$

which yields

$$\begin{aligned}
R_\lambda &= -\frac{1}{\lambda}\left(I + \sum_{j=1}^{\infty}\left(\frac{1}{\lambda}T\right)^j\right) \\
&= -\frac{1}{\lambda}\left(I + T\sum_{j=1}^{\infty}\left(\frac{1}{\lambda}\right)^j\right) && (T \text{ idempotent}) \\
&= -\frac{1}{\lambda}\left(I + T\left(\frac{\frac{1}{\lambda}}{1 - \frac{1}{\lambda}}\right)\right) \\
&= -\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}T\right).
\end{aligned}$$

This formula suggests a form for R_λ for $\lambda \neq 1, 0$. We show that this does indeed define R_λ for $\lambda \neq 1, 0$. Notice

$$\begin{aligned}
-\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}T\right)(T - \lambda I) &= -\frac{1}{\lambda}T + \frac{1}{\lambda - \lambda^2}T + I - \frac{1}{1 - \lambda}T \\
&= -\frac{1}{\lambda}T + I + T\left(\frac{1 - \lambda}{\lambda - \lambda^2}\right) \\
&= -\frac{1}{\lambda}T + I + \frac{1}{\lambda}T \\
&= I
\end{aligned}$$

and

$$\begin{aligned}
(T - \lambda I)\left(-\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}T\right)\right) &= -\frac{1}{\lambda}(T - \lambda I)\left(I + \frac{1}{\lambda - 1}T\right) \\
&= -\frac{1}{\lambda}\left(T - \lambda I + \frac{1}{\lambda - 1}T - \frac{\lambda}{\lambda - 1}T\right) \\
&= -\frac{1}{\lambda}\left(T - \lambda I + \left(\frac{1 - \lambda}{\lambda - 1}\right)T\right) \\
&= -\frac{1}{\lambda}(T - \lambda I - T) \\
&= -\frac{1}{\lambda}(-\lambda I) \\
&= I
\end{aligned}$$

hence

$$R_\lambda = -\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}T\right)$$

for $\lambda \neq 1, 0$. Note that R_λ is bounded since

$$\begin{aligned}\|R_\lambda x\| &= \left\| -\frac{1}{\lambda}\left(I + \frac{1}{\lambda - 1}T\right)x \right\| \\ &\leq \frac{1}{|\lambda|}\|x\| + \frac{1}{|\lambda - 1|}\|T\|\|x\|.\end{aligned}$$

Further, R_λ is defined on all of X since $\mathcal{D}(T) = \mathcal{D}(I) = X$. Thus, whenever $\lambda \neq 1, 0$, $\lambda \in \rho(T)$. Hence,

$$\sigma(T) = \{0, 1\}$$

as desired.