Discrétisations et stabilité

Problème de propagation

Considérons le schéma explicite pour l'équation parabolique:

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial u}{\partial t} = 0$$

Toutes les dérivées spatiales sont évaluées au temps t (aucune au temps t+Δt)

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“Molécule” de calcul

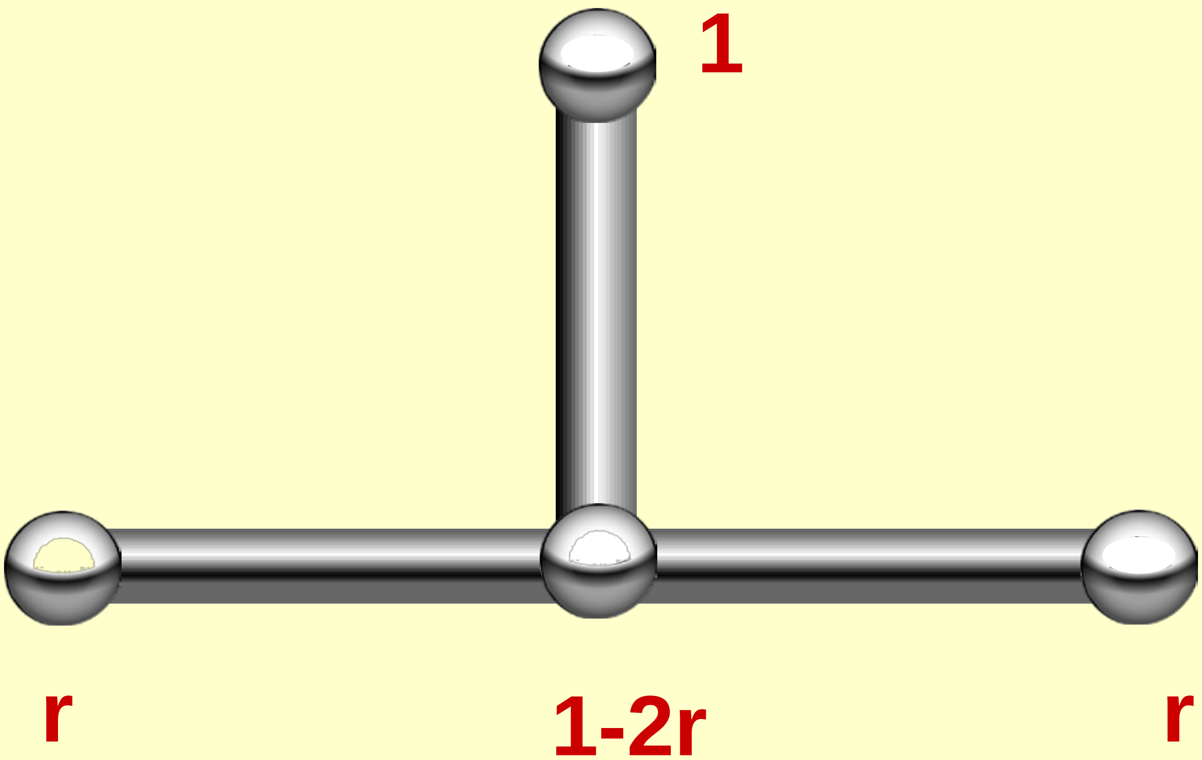

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Schéma Explicite

Ce type de discrétisation entraîne une contrainte de *stabilité*

$$0 < r = \frac{\Delta t}{a \Delta x^2} \leq \frac{1}{2}$$

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Stabilité

Si la contrainte n'est pas respectée , la solution oscille et croît sans limite

On ne retrouve pas cette limite lorsqu'on utilise une méthode implicite

Stabilité matricielle

$$u_i^{n+1} = r u_{i-1}^n + (1-2r) u_i^n + r u_{i+1}^n$$

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_{m-1}^{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & r & 1-2r & r & \\ & & \ddots & \ddots & -r \\ & & & r & 1-2r \end{bmatrix}}_{A} \begin{bmatrix} u_1^{n} \\ u_2^{n} \\ u_3^{n} \\ \vdots \\ u_{m-1}^{n} \end{bmatrix} + \begin{bmatrix} B \\ C \\ \\ \vdots \\ \\ \end{bmatrix}$$

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Évolution

$$\left.\begin{aligned} u^{(1)} &= Au^{(0)} \\ u^{(2)} &= Au^{(1)} \end{aligned}\right\} \Rightarrow \quad u^{(2)} = A^2 u^{(0)}$$

$$\vdots$$

$$u^{(n)} = A^n u^{(0)}$$

$$\Downarrow$$

$$e^{(n)} = u^{(n)} - u^{*} \Rightarrow \boxed{e^{(n)} = A^n e^{(0)}}$$

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Rayon spectral

$$||A^n e^0|| \leq ||A^n|| \, ||e^0|| \quad \rightarrow ||A||^n \leq 1$$

$$\rho(\mathrm{A})^n \leq 1$$

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Stabilité

Valeurs propres(r, 1-2r,r)

$$\mu_i = 1 - 4r\left(\sin\frac{i\pi}{2m}\right)^2$$

$$i = 1,2,3\ldots m-1$$

$$\rho(A) = \max_{1\le i\le m-1}\left|1 - 4r\left(\sin\frac{i\pi}{2m}\right)^2\right| \le 1$$

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Stabilité

$$\rho(A) = \max_{1 \le i \le m-1} \left| 1 - 4r\left(\sin \frac{i\pi}{2m} \right)^2 \right| \le 1$$

$$0 \le r\left(\sin \frac{i\pi}{2m} \right)^2 \le \frac{1}{2} \quad i = 1,2,3....m-1$$

$$\lim_{m \to \infty} \left(\sin \frac{i\pi}{2m} \right)^2 = 1$$

$$0 \le r \le \frac{1}{2}$$

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Conditionnellement stable

$$r = \frac{a\Delta t}{\Delta x^2}$$

$$\boxed{0 < r \leq \frac{1}{2}}$$

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Stabilité

$$0 \le r \le \frac{1}{2}$$

$$0 \le a\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

$$a = 1, \quad \Delta t = 0.0005, \quad \Delta x = 0.1$$

$$a\frac{\Delta t}{\Delta x^2} = 0.05 \rightarrow Stable$$

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Stabilité

$$0 \le r \le \frac{1}{2}$$

$$0 \le a\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$$

$$a = 1, \quad \Delta t = 0.01, \quad \Delta x = 0.1$$

$$a\frac{\Delta t}{\Delta x^2} = 1 \to Instable$$

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Los tres amigos

Consitance

Un schéma numérique est dit **consistant** avec une équation aux dérivées partielles, si **l'erreur de troncature** $\boldsymbol{E_T}$ tend vers zéro lorsque tous les pas de discrétisation tendent vers zéro

Stabilité

Un schéma est **stable** si la solution du problème discret reste bornée

Convergence

Un schéma est **convergent** si la différence *u-U* entre la **solution exacte *u*** et **la solution numérique *U*** tend vers zéro quand les pas de discrétisation tendent vers zéro

Théorème de Lax

Pour un problème **linéaire**, la consistance et la stabilité sont nécessaires et suffisantes pour assurer la convergence

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Exemple

$$L(u) = \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n - \theta(\Delta t^2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} - \frac{\Delta x^2}{12}\left(\frac{\partial u^4}{\partial x^4}\right) - \theta(\Delta x^4)$$

Exemple

Équa-diff

$$L(u) = \frac{u_i^{n+1} - u_i^n}{\Delta t} \quad - \quad \alpha \left(\frac{u_{i-1}^i - 2u_i^n + u_{i+1}^n}{\Delta x^2} \right)$$

$$\boxed{\begin{aligned} & - \quad \frac{\Delta t}{2} \left(\frac{\partial^2 u}{\partial t^2} \right) + \alpha \frac{\Delta x^2}{12} \left(\frac{\partial u^4}{\partial x^4} \right) \\ & - \quad \theta(\Delta t^2) - \theta(\Delta x^4) = 0 \end{aligned}}$$

Schéma

$$L(U) = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} \right)$$

Exemple

Erreur

$$\boxed{E_T = \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) - \alpha\frac{\Delta x^2}{12}\left(\frac{\partial u^4}{\partial x^4}\right) + \theta(\Delta t^2) - \theta(\Delta x^4)}$$

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Stabilité (ODE)

$$\frac{du}{dt} = au$$

$$u(0) = u_0$$

$$u(t) = u_0 e^{at}$$

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Exemple

$$u_{k+1} = u_k + a u_k \Delta t$$

$$u_1 = u_0 (1 + a\Delta t)$$

$$u_2 = u_0 (1 + a\Delta t)^2$$

$$\vdots$$

$$u_k = u_0 (1 + a\Delta t)^k$$

$$u_k = u_0 G^k \quad avec \quad G = (1 + a\Delta t)$$

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Exemple

$$G = (1 + a\Delta t)$$

Si a> et G>1, la solution croit avec le temps, et elle suit la solution analytique. La valeur du pas Δt n'influe que sur la précision des résultats

$$lorsque \quad a = -a$$

$$G = (1 - a\Delta t)$$

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Exemple

Trois situations possibles

$$G, < -1 \qquad -1 \le G < 0 \qquad 0 < G \le 1$$

$$Si \quad G \le -1$$

$$G = (1 - a\Delta t)$$

Soit G=-2

$$u_1 = -2u_0$$
$$u_2 = \quad 4u_0$$
$$u_3 = -8u_0$$

Exemple

$$-1 \le G < 0 \qquad -1 \le 1 - a\Delta t \qquad \Delta t \le 2/a$$

Soit G=-0.5

$$G = (1 - a\Delta t)$$

$$u_1 = -0.5u_0$$
$$u_2 = \quad 0.25u_0$$
$$u_3 = -0.125u_0$$

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Exemple

$$0 \le G < 1 \qquad 0 \le 1 - a\Delta t \qquad \Delta t \le 1/a$$

$$G = (1 - a\Delta t)$$

Le schéma est *absolument stable*; la solution reste bornée, et sans osciller, elle s'approche de la solution analytique

Soit G=0.5

$$u_1 = 0.5 u_0$$

$$u_2 = 0.25 u_0$$

$$u_3 = 0.125 u_0$$

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Cas général explicite

$$\frac{dy}{dt} = f(t,y) \qquad y(t=0) = y_0$$

$$y_{n+1} = y_n \equiv y(n\Delta t), \quad f_n \equiv f(y_n)$$

$$\left.\frac{dy}{dt}\right|_n = \frac{\Delta y_n}{\Delta t} + O(\Delta t) \qquad \Delta y_n = (y_{n+1} - y_n)$$

$$\frac{\Delta y_n}{\Delta t} = f_n + O(\Delta t) \qquad \boxed{y_{n+1} = y_n + f_n \Delta t + O(\Delta t^2)}$$

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Cas vectoriel

$$f_n = L y_n \Rightarrow y_{n+1} = y_n + f_n \Delta t$$

$$y_{n+1} = y_n + L y_n \Delta t$$

$$y_{n+1} = \underbrace{(I + L\Delta t)}_{G} y_n$$

$$\boxed{e_{n+1} = (I + L\Delta t) e_n}$$

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Cas général implicite

$$\left.\frac{dy}{dt}\right|_{n+1} = \frac{\nabla y_{n+1}}{\Delta t} + O(\Delta t) \qquad (\nabla y_{n+1} = y^{n+1} - y^{n})$$

$$y_{n+1} = y_n + f_{n+1}\Delta t + O(\Delta t^2)$$

$$f_{n+1} = L y_{n+1}$$

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Cas général implicite

$$y_{n+1} = y_n + L y_{n+1} \Delta t + O(\Delta t^2)$$

$$y_{n+1} = (I - L\Delta t)^{-1} y_n + O(\Delta t^2)$$

$$e_{n+1} = \underbrace{(I - L\Delta t)^{-1}}_{G} e_n$$

$$e_{n+1} = G e_n$$

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Cas particulier implicite

$$\frac{dy}{dt} = -ay \qquad f = -ay$$

Schéma implicite

$$L = -a$$

$$G = (I - L\Delta t)^{-1} \quad \Rightarrow \quad |G| = \left| \frac{1}{(1 + a\Delta t)} \right|$$

stable !

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Méthode trapezoïdale

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left(f(y_{i+1}, t_{i+1}) + f(y_i, t_i) \right)$$

$$f_{i+1} = L y_{i+1} \qquad f_i = L y_i$$

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left(L y_{i+1} + L y_i \right) + O(\Delta t^3)$$

$$\left(I - \frac{\Delta t}{2} L \right) y_{i+1} = \left(I + \frac{\Delta t}{2} L \right) y_i + O(\Delta t^3)$$

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Méthode trapezoïdale

$$y_{i+1} = \left(I - \frac{\Delta t}{2} L \right)^{-1} \left(I + \frac{\Delta t}{2} L \right) y_i + O(\Delta t^3)$$

$$G = \left(I - \frac{\Delta t}{2} L \right)^{-1} \left(I + \frac{\Delta t}{2} L \right)$$

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Méthode trapezoïdale

$$L = -a$$

$$G = \left(I + \frac{\Delta t}{2} a \right)^{-1} \left(I - \frac{\Delta t}{2} a \right)$$

$$\left| G \right| = \left| \frac{1 - a\Delta t / 2}{1 + a\Delta t / 2} \right| \leq 1$$

stable !

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Stabilité de Von Neumann

Pour analyser la stabilité d'un schéma on peut également utiliser l'analyse de Von Neumann.

On écrit l'erreur en fonction de séries de Fourier:

$$u(x,t)=\sum_{m=1}^{\infty} A_m e^{-\left(\frac{m\pi}{L}\right)^2 t}\sin\left(\frac{m\pi x}{L}\right)$$

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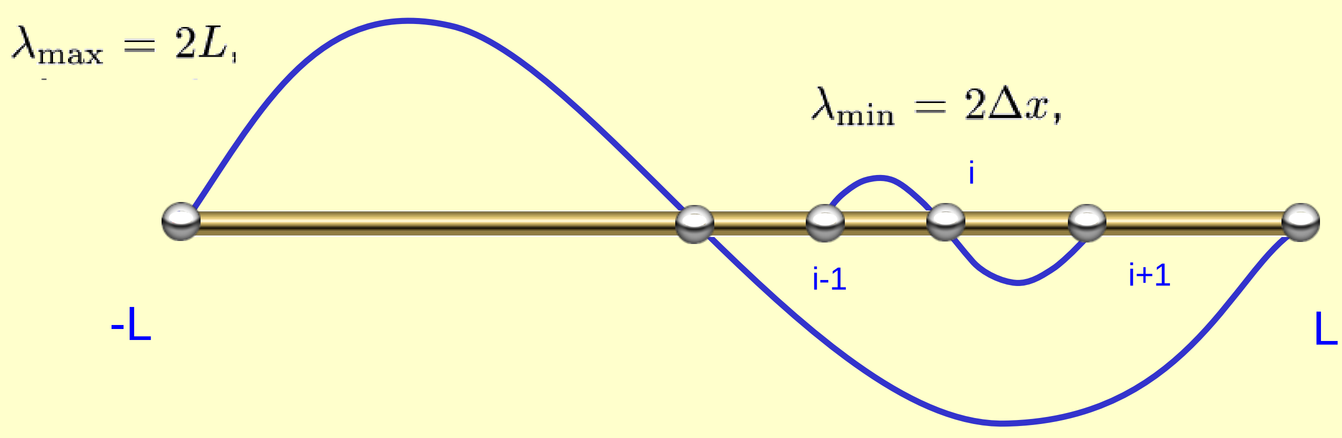
Les fréquences

$$u(x,t) = \sum_{m=1}^{\infty} A_m e^{-\left(\frac{m\pi}{L}\right)^2 t} \sin\left(\frac{m\pi x}{L}\right)$$



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Stabilité deVon Neumann

$$k_m = m\frac{\pi}{N\Delta x} \qquad m = 1,2\ldots\ldots N$$

N intervalles, entre 0 et L $\Delta x = L/N$,

$$u(x,\mathbf{0}) = \sum_{m=1}^{N} A_m \sin\left(\frac{m\pi x}{L}\right) = \sum_{m=1}^{N} A_m \sin(k_m x)$$

$$\left.\begin{aligned} k_{\max} &= \frac{\pi}{\Delta x} \\ k_{\min} &= \frac{\pi}{L} \end{aligned}\right\} \quad \lambda = \frac{2\pi}{k}$$

$$\lambda_{\max} = 2L$$

$$\lambda_{\min} = 2\Delta x$$

Stabilité deVon Neumann

$$u(x,0) = \sum_{m=1}^{N} A_m \sin\left(\frac{m\pi x}{L}\right) = \sum_{m=1}^{N} A_m \sin(k_m x)$$

$u(x,0)$ est considérée périodique,

$$u(x,0) = \sum_{m=-N}^{N} c_m^{(0)} e^{i(k_m x)}$$

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Stabilité deVon Neumann

$$u_j^0 \quad avec \quad x_j = j\Delta x$$

$$u_j^0 = \sum_{m=-N}^{N} c_m^{(0)} e^{i(k_m x_j)} = \sum_{m=-N}^{N} c_m^{(0)} e^{i(k_m j \Delta x)}$$

$$\beta = k_m \Delta x = \frac{m\pi}{L}\Delta x = \pi\left(\frac{m}{N}\right)$$

$\beta = \pi \rightarrow$ la fréquence la plus élévée $\qquad \lambda_{min} = 2\Delta x$

Si on ne considère qu'une seule harmonique

$$u_j^0 = c^{(o)} e^{ij\beta}$$

$$k_m = m\frac{\pi}{N\Delta x}$$

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Stabilité deVon Neumann

$$
\begin{aligned}
c^{(1)} &= Gc^{(0)} \\
c^{(2)} &= Gc^{(1)} \\
c^{(2)} &= G^2c^{(0)} \\
&\vdots \\
c^{(n)} &= G^nc^{(0)} \\
u_j^n &= G^n e^{ij\beta}
\end{aligned}
$$

$$\boxed{|G| \leq 1}$$

$$\beta = k_m \Delta x = \frac{m\pi}{L}\Delta x = \pi\left(\frac{m}{N}\right)$$

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Stabilité deVon Neumann

$$u_j^n = G^n e^{ij\beta}$$

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Analyse deVon Neumann

Après développement, l'analyse de Von Neumann se résume à l'utilisation de l'expression

$$\boxed{u_j^n = G^n e^{ij\beta}}$$

dans un schéma numérique pour évaluer la progression d'une composante au point générique ***j***, avec

$$\beta = \frac{m\pi}{l}\Delta x = \pi\left(\frac{m}{L}\right)$$

$$i: indice\ pour\ l'imaginaire$$

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Exemple:schéma explicite

$$u_j^n = G^n e^{ij\beta}$$

$$u_j^{n+1} = r u_{j-1}^n + (1-2r) u_j^n + r u_{j+1}^n$$

$$G^{n+1} e^{i\,\beta J} = r G^n e^{i\,\beta(J-1)} + (1-2r) G^n e^{i\,\beta J} + r G^n e^{i\,\beta(J+1)}$$

$$G^{n+1} e^{i\,\beta J} = G^n [r e^{i\,\beta J} e^{-i\beta} + (1-2r) e^{i\,\beta J} + r e^{i\,\beta J} e^{i\beta}]$$

$$G = r e^{-i\beta} + (1-2r) \; + r e^{i\beta}$$

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Exemple

$$\cos\beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$G = 1 + r[e^{-i\beta} + e^{i\beta} - 2]$$

$$G = 1 + 2r[\cos\beta - 1]$$

$$G = 1 - 4r\sin^2\left(\frac{\beta}{2}\right) \Longleftarrow |G| \leq 1$$

$$\Downarrow$$

$$|1 - 4r| \leq 1$$

$$\boxed{0 \leq r \leq \frac{1}{2}}$$

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Exemple:schéma implicite

$$u_j^n = G^n e^{ij\beta}$$

$$r u_{j-1}^{n+1} - (1+2r) u_j^{n+1} + r u_{j+1}^{n+1} = -u_j^n$$

$$rG^{n+1} e^{i\,\beta(J-1)} - (1+2r) G^{n+1} e^{i\,\beta J} + rG^{n+1} e^{i\,\beta(J+1)} = G^n e^{i\,\beta J}$$

$$G^{n+1} e^{i\,\beta J} [r e^{-i\,\beta} - (1+2r) + r e^{i\,\beta}] = -G^n e^{i\,\beta J}$$

$$\mathrm{G}\left[r\left(e^{-i\,\beta} + e^{i\,\beta}\right) - (1+2r)\right] = -1$$

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Exemple:schéma implicite

$$cos\beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$G[r(e^{-i\beta} + e^{i\beta}) - (1 + 2r)] = -1$$

$$\mathrm{G} = \frac{1}{1 + 2r(1 - cos\beta)} = \frac{1}{1 + 4rsin^2\left(\frac{\beta}{2}\right)}$$

$|G| \leq 1$ ⇒ Stable

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Équation de convection

Nous regardons brièvement l'équation de convection pour illustrer le concept de viscosité numérique.

Cette équation sera présentée en détaille plus tard

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

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Schéma de Lax

Schéma centré en espace

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$u_j^n \approx \frac{u_{j+1}^n + u_{j-1}^n}{2}$$

$$u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2} = -a\Delta t\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

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Viscosité Numérique

$$u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2} = -a\Delta t \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta t}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \frac{\Delta x^2}{2\Delta t}\boxed{\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} \qquad -\mathbf{a}\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \qquad \sim \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$$

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Resumé

Schéma de Lax : $$G = \cos m\Delta x - i\,C \sin m\Delta x$$

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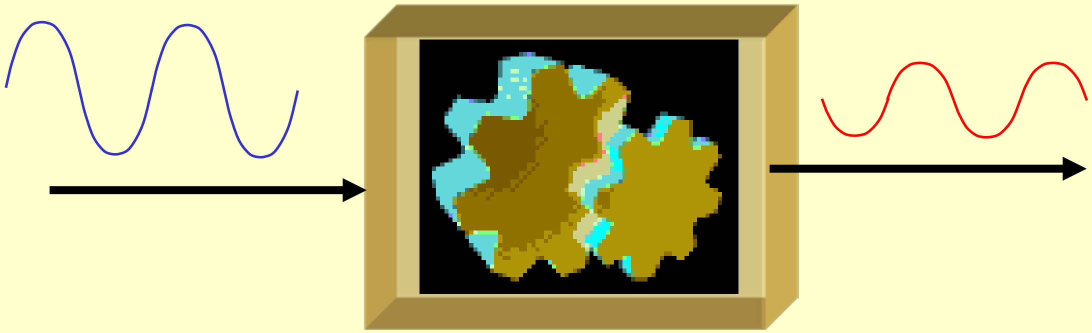
Signal numérique

$A \sin \omega t$

$B \sin(\omega t + \phi)$



Schéma

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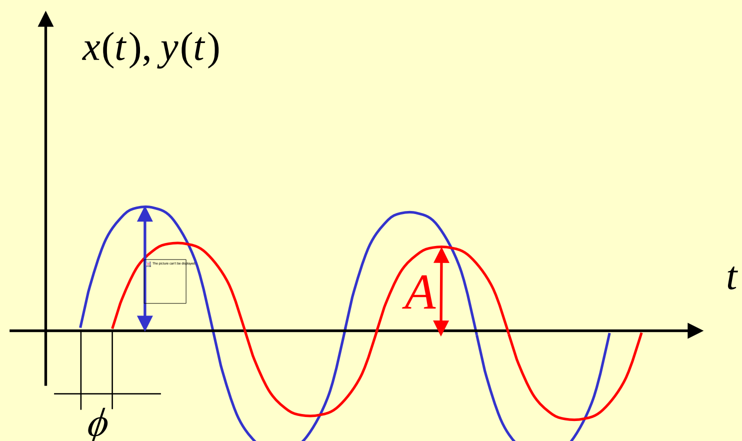
$$x(t) = A\sin(\omega t) \qquad y(t) = B\sin(\omega t + \phi)$$



$$B / A = rapport\ d'amplitude\ (RA)$$

$$\phi = déphasage$$

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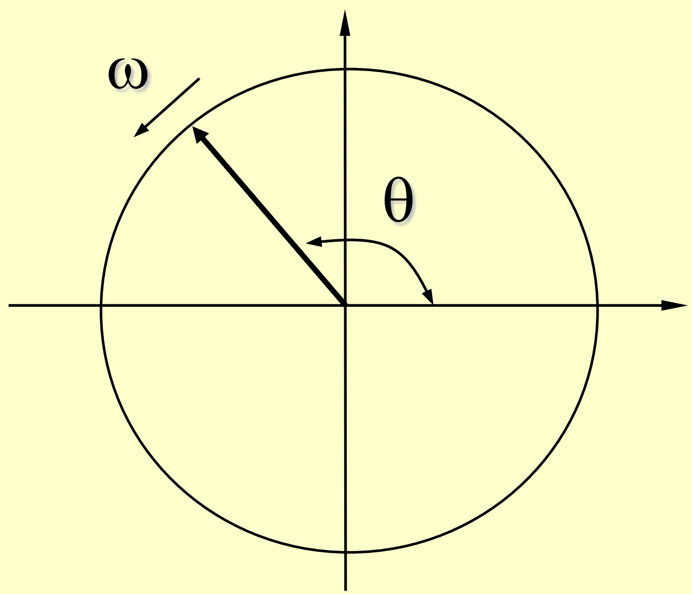
Diagramme polaire

$\theta = \omega t$



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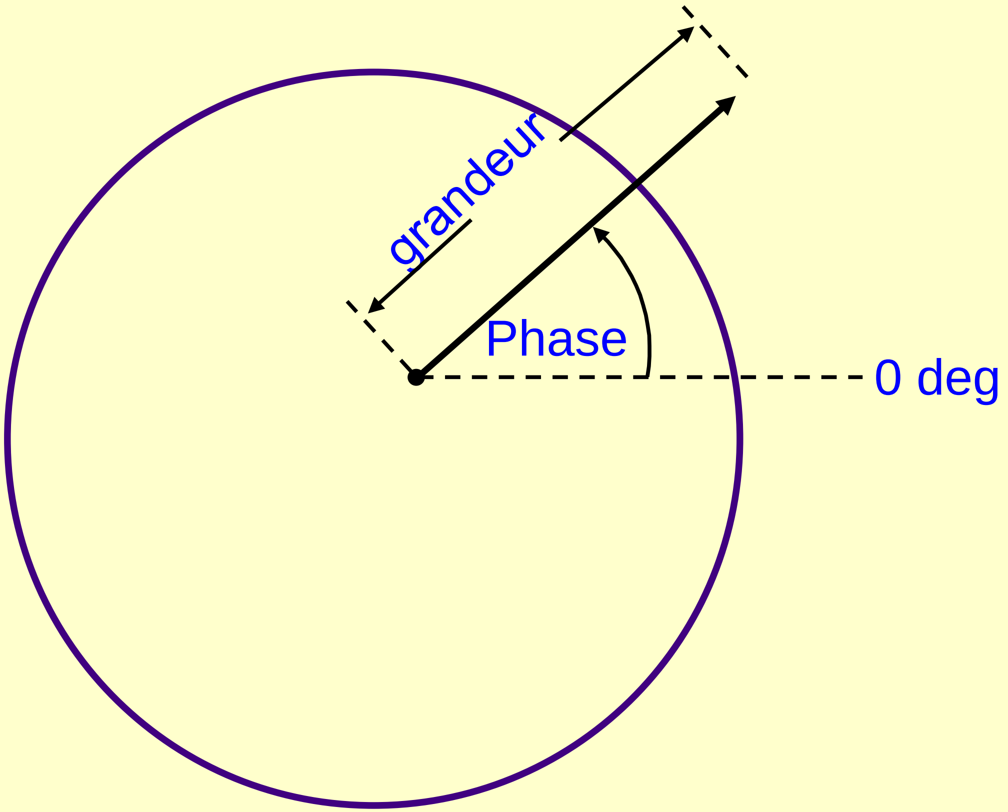
Grandeur & Phase

- La grandeur est une quantité absolue
- La phase est une quantité relative



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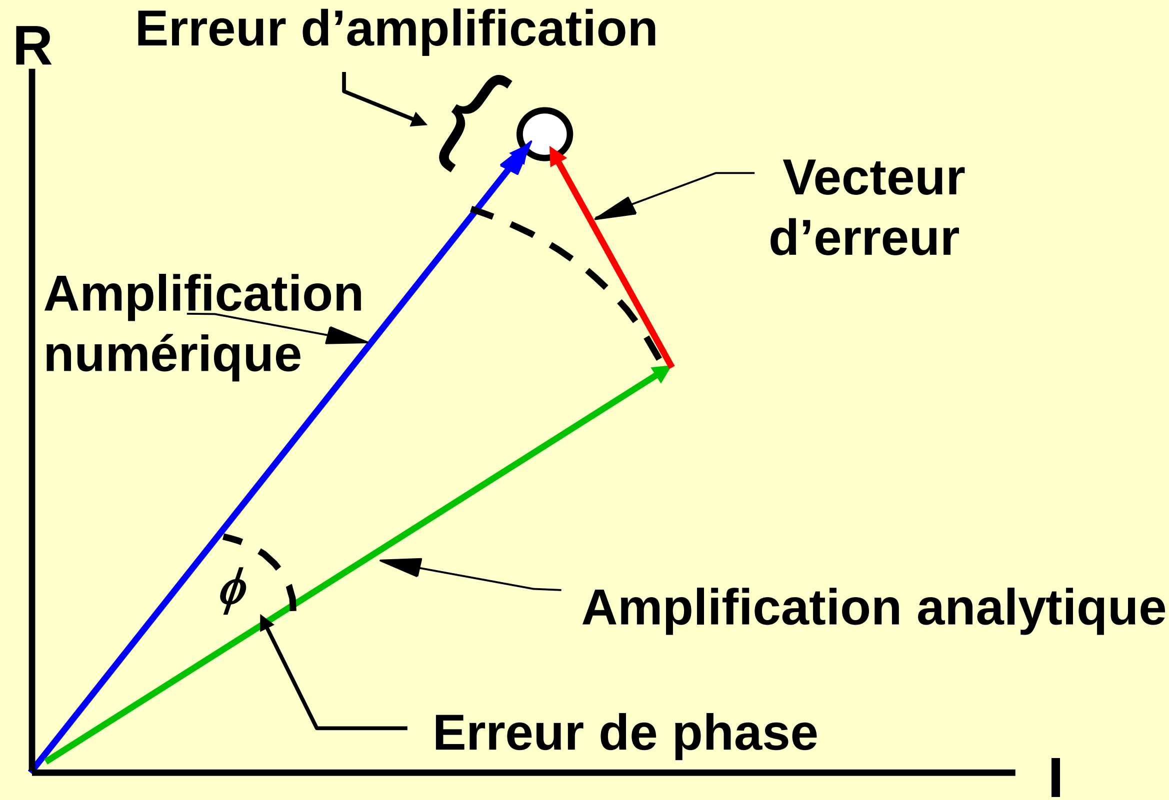
Ensemble d'erreurs



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Amplitude et fréquence

$$G = A(\beta) + iB(\beta)$$

$$G = |G|e^{i\Phi}$$

Φ, le *déphasage*. Dans le temps $\rightarrow$ *retard de phase*.

$$\Phi = tan^{-1}\left(\frac{Im(G)}{Re(G)}\right) = tan^{-1}\left(\frac{B}{A}\right)$$

Erreur de déphasage= *dispersion*.

Erreur relative : $e_\Phi = \Phi/\Phi_e$

Si $e_\Phi < 1$ la solution est en retard

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Amplitude et fréquence

$$|G|/G_e$$

$|G|$: le facteur d'amplification du schéma
G_e: le facteur d'amplification de la solution exacte.

L'erreur d'amplitude est appelée erreur de *diffusion*.

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L'équation modifiée

$$\begin{cases} espace & u_{j\pm1}^n = \boxed{u_j^n \pm \dfrac{\partial u}{\partial x}\Delta x + \dfrac{\partial^2 u}{\partial x^2}\dfrac{\Delta x^2}{2} \pm \dfrac{\Delta x^3}{6}\dfrac{\partial^3 u}{\partial x^3}} \\ temps & u_j^{n+1} = \boxed{u_j^n + \dfrac{\partial u}{\partial t}\Delta t + \dfrac{\partial^2 u}{\partial t^2}\dfrac{\Delta t^2}{2} + \dfrac{\Delta t^3}{6}\dfrac{\partial^3 u}{\partial t^3}} \end{cases}$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad \boxed{u_j^{n+1} = r u_{j-1}^n + (1-2r)u_j^n + r u_{j+1}^n} \qquad r = \frac{\alpha \Delta t}{\Delta x^2}$$

Schéma explicite

ÉCOLE POLYTECHNIQUE MONTRÉAL

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L'équation modifiée

$$u_j^{n+1} = r u_{j-1}^n + (1-2r) u_j^n + r u_{j+1}^n$$

$$u_j^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^3}{6}\cdots$$

$$r\left(u_j^n - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} - \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6}\right)$$

$$r\left(u_j^n + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6}\right)\cdots$$

ÉCOLE POLYTECHNIQUE MONTRÉAL

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L'équation modifiée

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

$$\boxed{\boxed{\frac{\partial u}{\partial t}} + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^2}{6} \cdots \alpha \left(-\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4}\frac{\Delta x^2}{12} + \frac{\partial^6 u}{\partial x^6}\frac{\Delta x^4}{360} \right)}$$

Méthodologie I

$$F = \frac{\partial u}{\partial t} + A_2 \frac{\partial^2 u}{\partial t^2} + A_3 \frac{\partial^3 u}{\partial t^3} + \cdots B_1 \frac{\partial u}{\partial x} + B_2 \frac{\partial^2 u}{\partial x^2} \cdots = 0$$

$$\frac{\partial u}{\partial t} + C_1 \frac{\partial u}{\partial x} + C_2 \frac{\partial^2 u}{\partial x^2} + C_3 \frac{\partial^3 u}{\partial x^3} + C_4 \frac{\partial^4 u}{\partial x^4} + \cdots \quad = \quad 0$$

$$\frac{\partial F}{\partial t} \quad = \quad 0, \qquad \frac{\partial F}{\partial x} \quad = \quad 0, \qquad \frac{\partial^2 F}{\partial t^2} \quad = \quad 0, \qquad \frac{\partial^2 F}{\partial t \partial x} \quad =$$

$$\frac{\partial^2 F}{\partial x^2} \quad = \quad 0, \qquad \frac{\partial^3 F}{\partial t^3} \quad = \quad 0, \qquad \frac{\partial^3 F}{\partial^2 t \partial x} \quad = \quad 0 \qquad \cdots$$

ÉCOLE POLYTECHNIQUE MONTRÉAL

M. Reggio

Combinaison linéaire

$$\boxed{F + p_1 \frac{\partial F}{\partial t}} + p_2 \frac{\partial F}{\partial x} + p_3 \frac{\partial^2 F}{\partial t^2} + p_4 \frac{\partial^2 F}{\partial t \partial x} + p_5 \frac{\partial^2 F}{\partial x^2} + \cdots = 0$$

$$F = \frac{\partial u}{\partial t} + A_2 \frac{\partial^2 u}{\partial t^2} + \ldots (A_2 + p_1) \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \cdot \left(p_1 B_1 + p_2\right) \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = 0$$

$$= 0$$

$$p_1 \frac{\partial F}{\partial t} = p_1 \frac{\partial^2 u}{\partial t^2} + \ldots$$

$$p_1 \frac{\partial F}{\partial t} = \ldots p_1 B_1 \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \ldots$$

$$F = \frac{\partial u}{\partial t} + A_2 \frac{\partial^2 u}{\partial t^2} + A_3 \frac{\partial^3 u}{\partial t^3} + \cdots \boxed{B_1 \frac{\partial u}{\partial x}} + B_2 \frac{\partial^2 u}{\partial x^2} \cdots = 0$$

Combinaison linéaire

$$F+p_1\frac{\partial F}{\partial t}+p_2\frac{\partial F}{\partial x}+p_3\frac{\partial^2 F}{\partial t^2}+p_4\frac{\partial^2 F}{\partial t\partial x}+p_5\frac{\partial^2 F}{\partial x^2}+\cdots=0$$

$$\begin{aligned}
A_2+p_1 &= 0\\
p_1B_1+p_2 &= 0\\
A_3+p_1A_2+p_3 &= 0\\
&\vdots\\
p_1B_2+p_4B_1+p_5 &= 0\\
&\vdots\\
p_1B_3+p_4B_2+p_8B_1+p_9 &= 0
\end{aligned}$$

$$F=\frac{\partial u}{\partial t}+A_2\frac{\partial^2 u}{\partial t^2}+A_3\frac{\partial^3 u}{\partial t^3}+\cdots B_1\frac{\partial u}{\partial x}+B_2\frac{\partial^2 u}{\partial x^2}\cdots=0$$

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$$F = \frac{\partial u}{\partial t} + A_2\frac{\partial^2 u}{\partial t^2} + A_3\frac{\partial^3 u}{\partial t^3} + \cdots B_1\frac{\partial u}{\partial x} + B_2\frac{\partial^2 u}{\partial x^2}\cdots = 0$$

		1	p_1	p_2	p_3	p_4	p_5	$\cdots$
		F	F_t	F_x	F_{tt}	F_{tx}	F_{xx}	$\cdots$
	$\partial u/\partial t$	1						
C_1	$\boxed{\partial u/\partial x}$	B_1						
	$\partial^2 u/\partial t^2$	A_2	1					
	$\partial^2 u/\partial t\partial x$		B_1	1				
C_2	$\boxed{\partial^2 u/\partial x^2}$	B_2		B_1				
	$\partial^3 u/\partial t^3$	A_3	A_2		1			
	$\partial^3 u/\partial t^2\partial x$			A_2	B_1	1		
	$\partial^3 u/\partial t\partial x^2$		B_2			B_1	1	
C_3	$\boxed{\partial^3 u/\partial x^3}$	B_3		B_2			B_1	$\cdots$
	$\vdots$							

$$A_2 + p_1 = 0$$

$$p_1 B_1 + p_2 = 0$$

$$A_3 + p_1 A_2 + p_3 = 0$$

$$\begin{aligned}
A_2 + p_1 &= 0\\
p_1 B_1 + p_2 &= 0\\
A_3 + p_1 A_2 + p_3 &= 0\\
&\vdots\\
p_1 B_2 + p_4 B_1 + p_5 &= 0\\
&\vdots\\
p_1 B_3 + p_4 B_2 + p_8 B_1 + p_9 &= 0
\end{aligned}$$

Matrice triangulaire

$$
\begin{aligned}
p_1 &= -A_2 \\
p_2 &= A_2 B_1 \\
p_3 &= -A_3 + A_2^2 \\
&\vdots \\
p_5 &= A_2 B_2 + B_1^2(2A_2^2 - A_3) \\
&\vdots \\
p_9 &= A_2 B_3 + B_1(4A_2^2 - 2A_3)B_2 + B_1^3(5A_2^3 - 5A_2A_3 + A_4
\end{aligned}
$$

$$
\begin{aligned}
A_2 + p_1 &= 0 \\
p_1 B_1 + p_2 &= 0 \\
A_3 + p_1 A_2 + p_3 &= 0 \\
&\vdots \\
p_1 B_2 + p_4 B_1 + p_5 &= 0 \\
&\vdots \\
p_1 B_3 + p_4 B_2 + p_8 B_1 + p_9 &= 0
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= B_1 \\
C_2 &= B_2 + p_2 B_1 \\
C_3 &= B_3 + p_2 B_2 + p_5 B_1
\end{aligned}
$$

$$
\frac{\partial u}{\partial t} + C_1 \frac{\partial u}{\partial x} + C_2 \frac{\partial^2 u}{\partial x^2} + C_3 \frac{\partial^3 u}{\partial x^3} + C_4 \frac{\partial^4 u}{\partial x^4} + \cdots = 0
$$

ÉCOLE POLYTECHNIQUE MONTRÉAL

gio

Cas 1: Éq.modifiée

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

$$u_j^{n+1} = r u_{j-1}^n + (1-2r) u_j^n + r u_{j+1}^n$$

$$\boxed{(\alpha \Delta t)/\Delta x^2) = 1/6}$$

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = \left(-\frac{1}{2}\alpha^2 \Delta t + \frac{\alpha \Delta x^2}{12} \right) \frac{\partial^4 u}{\partial x^4}$$

$$+ \left(\frac{1}{3}\alpha^3 \Delta t^2 - \frac{1}{12}\alpha^2 \Delta t \Delta x^2 + \frac{1}{360}\alpha \Delta x^4 \right) \frac{\partial^6 u}{\partial x^6} +$$

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Cas 2: Éq.modifiée

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \boxed{\frac{a\Delta x}{2}(1-\mu)}\frac{\partial^2 u}{\partial x^2} + \boxed{a\frac{\Delta x^2}{6}(2\mu^2 - 3\mu + 1)}\frac{\partial^3 u}{\partial x^3}$$

$$\boxed{- \quad a\frac{\Delta x^3}{24}(1-\mu)(1+6\mu^2-6\mu)}\frac{\partial^4 u}{\partial x^4} + \dots = 0$$

avec $\mu = \dfrac{a\Delta t}{\Delta x}.$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} \boxed{- \quad \frac{a\Delta x}{2}(1-\mu)\frac{\partial^2 u}{\partial x^2}} + \boxed{a\frac{\Delta x^2}{6}(2\mu^2 - 3\mu + 1)\frac{\partial^3 u}{\partial x^3}}$$

$$- \quad a\frac{\Delta x^3}{24}(1-\mu)(1+6\mu^2-6\mu)\frac{\partial^4 u}{\partial x^4} + \ldots. = 0$$

$$\frac{\partial u}{\partial t} + C_1\frac{\partial u}{\partial x} + \boxed{C_2\frac{\partial^2 u}{\partial x^2}} + \boxed{C_3\frac{\partial^3 u}{\partial x^3}} + C_4\frac{\partial^4 u}{\partial x^4} + \cdots \quad = \quad 0$$

Méthodologie II

On utilise des développements en série de Taylor et on les combine de manière séquentielle et structurée dans un tableau

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Deux dévéloppments

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_j^{n+1} = u_j^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 u}{\partial t^2}\frac{\Delta t^3}{6} + \ldots$$

Taylor en avant dans le temps

$$u_{j-1}^n = u_j^n - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} - \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6}\ldots$$

Taylor en arrière dans l'espace

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^2} - \ldots$$

$$\frac{\partial u}{\partial x} = \frac{u_j^n - u_{j-1}^n}{\Delta x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\ldots$$

$\Rightarrow$

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Le schéma en amont

$$L(u)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0$$

$$+\begin{cases}\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}}=\dfrac{\boldsymbol{u}_j^{n+1}-\boldsymbol{u}_j^n}{\Delta \boldsymbol{t}}-\dfrac{\Delta \boldsymbol{t}}{2}\dfrac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2}-\dfrac{\Delta \boldsymbol{t}^2}{6}\dfrac{\partial^3 \boldsymbol{u}}{\partial \boldsymbol{t}^2}-\dots\\[2ex] \boldsymbol{a}\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}=\boldsymbol{a}\dfrac{\boldsymbol{u}_j^{n}-\boldsymbol{u}_{j-1}^n}{\Delta \boldsymbol{x}}+\dfrac{\boldsymbol{a}\Delta \boldsymbol{x}}{2}\dfrac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}-\dfrac{\boldsymbol{a}\Delta \boldsymbol{x}^2}{6}\dfrac{\partial^3 \boldsymbol{u}}{\partial \boldsymbol{x}^3}\dots\end{cases}$$

$$\Downarrow$$

$$\boldsymbol{L}(\boldsymbol{u})=\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{t}}+\boldsymbol{a}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}=\frac{\boldsymbol{u}_j^{n+1}-\boldsymbol{u}_j^n}{\Delta \boldsymbol{t}}+\boldsymbol{a}\frac{\boldsymbol{u}_j^{n}-\boldsymbol{u}_{j-1}^n}{\Delta \boldsymbol{x}}-\frac{\Delta \boldsymbol{t}}{2}\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2}-\frac{\Delta \boldsymbol{t}^2}{6}\frac{\partial^3 \boldsymbol{u}}{\partial \boldsymbol{t}^2}+\frac{\boldsymbol{a}\Delta \boldsymbol{x}}{2}\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}-\frac{\boldsymbol{a}\Delta \boldsymbol{x}^2}{6}\frac{\partial^3 \boldsymbol{u}}{\partial \boldsymbol{x}^3}\dots=0$$

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Exemple

$$L(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \boxed{\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x}}$$

Le schéma

$$\underbrace{\boxed{-\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^2} + \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} \ldots = 0}}_{\text{L'erreur de troncature}}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x}$$

$$= \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^2} + \ldots$$

Exemple

$$\overbrace{\frac{u_j^{n+1}-u_j^{n}}{\Delta t}+a\frac{u_j^{n}-u_{j-1}^{n}}{\Delta x}}^{\text{le schéma}}$$

$$=\underbrace{\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}}_{\text{l'équation}}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}+\underbrace{\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^2}-\frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+\frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^2}+\dots}_{\text{l'erreur}}$$

$$E_T(u)=\left(\frac{a^2\Delta t}{6}-\frac{a\Delta x}{2}\right)\frac{\partial^2 u}{\partial x^2}+\left(\frac{a\Delta x^2}{6}+\frac{a^3\Delta t^2}{6}\right)\frac{\partial^3 u}{\partial x^2}+\cdots$$

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Pour l'analyse, on élimine toute dérivée faisant intervenir le temps

$$(1)\ L_a(u) = 0 = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^2} + \ldots$$

$$(2)\ \frac{\partial L_a(u)}{\partial t} = 0 = \frac{\partial^2 u}{\partial t^2} + a\frac{\partial^2 u}{\partial t\partial x} + \frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3} - \frac{a\Delta x}{2}\frac{\partial^3 u}{\partial t\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t\partial x^3} + \ldots$$

$$(3)\ \frac{\partial L_a(u)}{\partial x} = 0 = \frac{\partial^2 u}{\partial t\partial x} + a\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^2\partial x} - \frac{a\Delta x}{2}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^3\partial x} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial x^4} + \ldots$$

$$(4)\ \frac{\partial^2 L_a(u)}{\partial t^2} = 0 = \frac{\partial^3 u}{\partial t^3} + a\frac{\partial^3 u}{\partial t^2\partial x} + \frac{\Delta t}{2}\frac{\partial^4 u}{\partial t^4} - \frac{a\Delta x}{2}\frac{\partial^4 u}{\partial t^2\partial x^2} + \ldots$$

$$(5)\ \frac{\partial^2 L_a(u)}{\partial x\partial t} = 0 = \frac{\partial^3 u}{\partial x\partial t^2} + a\frac{\partial^3 u}{\partial t\ \partial x^2} + \frac{\Delta t}{2}\frac{\partial^4 u}{\partial t^3\partial x} - \frac{a\Delta x}{2}\frac{\partial^4 u}{\partial t\partial x^3} + \ldots$$

$$(6)\ \frac{\partial^2 L_a(u)}{\partial x^2} = 0 = \frac{\partial^2 u}{\partial t\partial x^2} + a\frac{\partial^3 u}{\partial x^3} + \frac{\Delta t}{2}\frac{\partial^4 u}{\partial t^2\partial x^2} - \frac{a\Delta x}{2}\frac{\partial^4 u}{\partial x^4} + \ldots.\frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial x} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial x^4} + \ldots$$

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Pour l’analyse, on élimine toute dérivée faisant intervenir le temps

$$(1)\; L_a(u) = 0 = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \boxed{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \boxed{\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^2} + \ldots$$

$$\boxed{-\frac{\Delta t}{2}} \qquad (2)\; \frac{\partial L_a(u)}{\partial t} = 0 = \boxed{\frac{\partial^2 u}{\partial t^2}} + a\frac{\partial^2 u}{\partial t \partial x} + \boxed{\frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3}} - \frac{a\Delta x}{2}\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t \partial x^3} + \ldots$$

$$(2^*)\; 0 = \boxed{-\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}} + -\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t \partial x} - \boxed{\frac{\Delta t^2}{4}\frac{\partial^3 u}{\partial t^3}} - \frac{a\Delta x \Delta t}{4}\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t \partial x^3} + \ldots$$

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \dots$$

Pour l'analyse, on élimine toute dérivée faisant intervenir le temps

$$(1)\ L_a(u) = 0 = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \boxed{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \boxed{\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^2} + \dots$$

$$(2^*)\ 0 = \boxed{-\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}} + -\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t \partial x} - \boxed{\frac{\Delta t^2}{4}\frac{\partial^3 u}{\partial t^3}} - \frac{a\Delta x \Delta t}{4}\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t \partial x^3} + \dots$$

$$\left(\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^2}{4}\frac{\partial^3 u}{\partial t^3}\right) = \boxed{-\frac{\Delta t^2}{12}\frac{\partial^3 u}{\partial t^3}}$$

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$$L_a(u) = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + a\frac{\partial u}{\partial x} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Coefficient	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
(1)	1	a	$\frac{\Delta t}{2}$		$-\frac{a\Delta x}{2}$	$\frac{\Delta t^2}{6}$			$\frac{a\Delta x^2}{6}$
$-\frac{\Delta t}{2}\times(2)$			$-\frac{\Delta t}{2}$	$-\frac{a\Delta t}{2}$		$-\frac{\Delta t^2}{4}$		$\frac{a\Delta x\Delta t}{4}$	
	1	a	0						0

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \underbrace{\frac{a\Delta x}{2}(c-1)\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3}}_{E_T} + \ldots$$

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Pour l'analyse, on élimine toute dérivée faisant intervenir le temps

$$(2)\ \frac{\partial L_a(u)}{\partial t} = 0 = \frac{\partial^2 u}{\partial t^2} - \boxed{\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t\partial x}} + \frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3} - \frac{a\Delta x}{2}\frac{\partial^3 u}{\partial t\partial x^2} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t\partial x^3} + \ldots$$

$$\boxed{a\frac{\Delta t}{2}}\quad (3)\ \frac{\partial L_a(u)}{\partial x} = 0 = \boxed{\frac{\partial^2 u}{\partial t\partial x}} + a\frac{\partial^2 u}{\partial x^2} + \frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^2\partial x} - \frac{a\Delta x}{2}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^3\partial x} + \frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial x^4} + \ldots$$

$$(3^*)\quad 0 = \boxed{\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t\partial x}} + \boxed{\frac{a^2\Delta t}{4}\frac{\partial^2 u}{\partial x^2}} + \boxed{\frac{a\Delta t^2}{4}\frac{\partial^3 u}{\partial t^2\partial x}} - \boxed{\frac{a^2\Delta x\Delta t}{4}\frac{\partial^3 u}{\partial x^3}} + \frac{a\Delta t^3}{12}\frac{\partial^4 u}{\partial t^3\partial x} + \frac{a^2\Delta t\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \ldots$$

$$L_a(u)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}-\frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}+\ldots$$

Pour l'analyse, on élimine toute dérivée faisant intervenir le temps

$$(2)\ \frac{\partial L_a(u)}{\partial t}=0=\frac{\partial^2 u}{\partial t^2}-\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t\partial x}+\frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^3}-\frac{a\Delta x}{2}\frac{\partial^3 u}{\partial t\partial x^2}+\frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^4}+\frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial t\partial x^3}+\ldots$$

$$(3^*)\quad 0=\frac{a\Delta t}{2}\frac{\partial^2 u}{\partial t\partial x}+\frac{a^2\Delta t}{4}\frac{\partial^2 u}{\partial x^2}+\frac{a\Delta t^2}{4}\frac{\partial^3 u}{\partial t^2\partial x}-\frac{a^2\Delta x\Delta t}{4}\frac{\partial^3 u}{\partial x^3}+\frac{a\Delta t^3}{12}\frac{\partial^4 u}{\partial t^3\partial x}+\frac{a^2\Delta t\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}+\ldots$$

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$$L_a(u) = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + a\frac{\partial u}{\partial x} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Coefficient	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
(1)	1	a	$\frac{\Delta t}{2}$		$-\frac{a\Delta x}{2}$	$\frac{\Delta t^2}{6}$			$\frac{a\Delta x^2}{6}$
$-\frac{\Delta t}{2}\times(2)$			$-\frac{\Delta t}{2}$	$-\frac{a\Delta t}{2}$		$-\frac{\Delta t^2}{4}$		$\frac{a\Delta x\Delta t}{4}$	
$a\frac{\Delta t}{2}\times(3)$				$\frac{a\Delta t}{2}$	$\frac{a^2\Delta t}{2}$		$\frac{a\Delta t^2}{4}$		$-\frac{a^2\Delta x\Delta t}{4}$
									0
	1	a	0	0					

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \underbrace{\frac{a\Delta x}{2}(c-1)\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3}}_{E_T} + \ldots$$

$$L_a(u)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}-\frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^3}+\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^2}+\frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}+\ldots$$

Pour l'analyse, on doit éliminer toute dérivée faisant intervenir le temps

$a\frac{\Delta t}{2}$ (3) $$\frac{\partial L_a(u)}{\partial x}=0=\frac{\partial^2 u}{\partial t\partial x}+a\frac{\partial^2 u}{\partial x^2}+\frac{\Delta t}{2}\frac{\partial^3 u}{\partial t^2\partial x}-\frac{a\Delta x}{2}\frac{\partial^3 u}{\partial x^3}+\frac{\Delta t^2}{6}\frac{\partial^4 u}{\partial t^3\partial x}+\frac{a\Delta x^2}{6}\frac{\partial^4 u}{\partial x^4}+\ldots$$

$\frac{\Delta t^2}{12}$ (4) $$\frac{\partial^2 L_a(u)}{\partial t^2}=0=\frac{\partial^3 u}{\partial t^3}+a\frac{\partial^3 u}{\partial t^2\partial x}+\frac{\Delta t}{2}\frac{\partial^4 u}{\partial t^4}-\frac{a\Delta x}{2}\frac{\partial^4 u}{\partial t^2\partial x^2}+\ldots$$

(4*) $$0=\frac{\Delta t^2}{12}\frac{\partial^3 u}{\partial t^3}+\frac{a\Delta t^2}{12}\frac{\partial^3 u}{\partial t^2\partial x}+\frac{\Delta t^3}{24}\frac{\partial^4 u}{\partial t^4}-\frac{a\Delta x\Delta t^2}{24}\frac{\partial^4 u}{\partial t^2\partial x^2}+\ldots$$

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$$L_a(u) = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + a\frac{\partial u}{\partial x} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Coefficient	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
(1)	1	a	$\frac{\Delta t}{2}$		$-\frac{a\Delta x}{2}$	$\frac{\Delta t^2}{6}$			$\frac{a\Delta x^2}{6}$
$-\frac{\Delta t}{2}\times(2)$			$-\frac{\Delta t}{2}$	$-\frac{a\Delta t}{2}$		$-\frac{\Delta t^2}{4}$		$\frac{a\Delta x\Delta t}{4}$	
$a\frac{\Delta t}{2}\times(3)$				$\frac{a\Delta t}{2}$	$\frac{a^2\Delta t}{2}$		$\frac{a\Delta t^2}{4}$		$\frac{a^2\Delta x\Delta t}{4}$
$\frac{\Delta t^2}{12}\times(4)$						$\frac{\Delta t^2}{12}$	$\frac{a\Delta t^2}{12}$		
	1	a	0	0	$\frac{a\Delta x}{2}(c-1)$	0			

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \underbrace{\frac{a\Delta x}{2}(c-1)\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3}}_{E_T} + \ldots$$

$$\boxed{-a\frac{\Delta t^2}{3}}\ (5)\ \frac{\partial^2 L_a(u)}{\partial x \partial t} = 0 = \frac{\partial^3 u}{\partial x \partial t^2} + a\frac{\partial^3 u}{\partial t\ \partial x^2} + \frac{\Delta t}{2}\frac{\partial^4 u}{\partial t^3 \partial x} - \frac{a\Delta x}{2}\frac{\partial^4 u}{\partial t \partial x^3} + \ldots$$

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$$L_a(u) = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} + a\frac{\partial u}{\partial x} - \frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \ldots$$

Coefficient	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
(1)	1	a	$\frac{\Delta t}{2}$		$-\frac{a\Delta x}{2}$	$\frac{\Delta t^2}{6}$			$\frac{a\Delta x^2}{6}$
$-\frac{\Delta t}{2}\times(2)$			$-\frac{\Delta t}{2}$	$-\frac{a\Delta t}{2}$		$-\frac{\Delta t^2}{4}$		$\frac{a\Delta x\Delta t}{4}$	
$a\frac{\Delta t}{2}\times(3)$				$\frac{a\Delta t}{2}$	$\frac{a^2\Delta t}{2}$		$\frac{a\Delta t^2}{4}$		$\frac{a^2\Delta x\Delta t}{4}$
$\frac{\Delta t^2}{12}\times(4)$						$\frac{\Delta t^2}{12}$	$\frac{a\Delta t^2}{12}$		
$-\frac{a\Delta t^2}{3}\times(5)$							$-\frac{a\Delta t^2}{3}$	$-\frac{a^2\Delta t^2}{3}$	
	1	a	0	0	$\frac{a\Delta x}{2}(c-1)$	0	0		

$$L_a(u) = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \underbrace{\frac{a\Delta x}{2}(c-1)\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3}}_{E_T} + \ldots$$

$$L_a(u)=\frac{\partial u}{\partial t}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}+a\frac{\partial u}{\partial x}-\frac{a\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+\frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}+\ldots$$

Coefficient	u_t	u_x	u_{tt}	u_{tx}	u_{xx}	u_{ttt}	u_{ttx}	u_{txx}	u_{xxx}
(1)	1	a	$\frac{\Delta t}{2}$	0	$-\frac{a\Delta x}{2}$	$\frac{\Delta t^2}{6}$	0	0	$\frac{a\Delta x^2}{6}$
$-\frac{\Delta t}{2}\times(2)$			$-\frac{\Delta t}{2}$	$-\frac{a\Delta t}{2}$	0	$-\frac{\Delta t^2}{4}$	0	$\frac{a\Delta x\Delta t}{4}$	0
$a\frac{\Delta t}{2}\times(3)$				$\frac{a\Delta t}{2}$	$\frac{a^2\Delta t}{2}$	0	$\frac{a\Delta t^2}{4}$	0	$\frac{a^2\Delta x\Delta t}{4}$
$\frac{\Delta t^2}{12}\times(4)$						$\frac{\Delta t^2}{12}$	$\frac{a\Delta t^2}{12}$	0	0
$-\frac{a\Delta t^2}{3}\times(5)$							$-\frac{a\Delta t^2}{3}$	$-\frac{a^2\Delta t^2}{3}$	0
$\left(\frac{a^2\Delta t^2}{3}-\frac{a\Delta x\Delta t}{4}\right)\times(6)$								$\left(\frac{a^2\Delta t^2}{3}-\frac{a\Delta x\Delta t}{4}\right)$	$\left(\frac{a^3\Delta t^2}{3}-\frac{a^2\Delta x\Delta t}{4}\right)$
	1	a	0	0	$\frac{a\Delta x}{2}(c-1)$	0	0	0	$\frac{a\Delta x^2}{6}\left(2c^2-3c+1\right)$

$$L_a(u)=\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+\underbrace{\frac{a\Delta x}{2}(c-1)\frac{\partial^2 u}{\partial x^2}+\frac{a\Delta x^2}{6}(1-c)(1-2c)\frac{\partial^3 u}{\partial x^3}}_{E_T}+\ldots$$