Numerical Methods for Hyperbolic Equations

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Based on the following references:
- Gary A. Sod (1985) Numerical Methods in Fluid Dynamics. [https://doi.org/10.1017/CBO9780511753138]
- Eleuterio F. Toro (1999) Riemann solvers and numerical methods for fluid dynamics. [https://doi.org/10.1007/b79761]
- Randall J. LeVeque (2002) Finite Volume Methods for Hyperbolic Problems. [https://doi.org/10.1017/CBO9780511791253]
- Course notes given by late Prof. Paul Arminjon for MAT6165 (2006) UdeM.

Hyperbolic equations

- **Classification**
- **Scalar Advection Equation**
  - **Characteristics**
  - **Finite Difference Scheme**
  - **Local Truncation Error**
  - **Stability Analysis**
  - **Courant-Friedrichs-Lewy condition (CFL condition)**
  - **Upwind Scheme**
  - **Donor-Cell Scheme**
  - **Lax-Friedrichs Scheme**
  - **Lax-Wendroff Scheme**
- Two-Dimensional Scalar Advection Equation
  - Characteristics
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  - Lax-Wendroff Scheme (2D)
  - Examples:
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Hyperbolic equations

- Classification
- Scalar Advection Equation
- Two-Dimensional Scalar Advection Equation
- Linear Hyperbolic Systems
- Conservation Law
- The Riemann Problem for the Euler equations

Classification

- Quasi-linear partial differential equations (PDE):

$$\begin{aligned} &A(x,y,u,u_x,u_y)u_{xx}+\\ &2B(x,y,u,u_x,u_y)u_{xy}+\\ &C(x,y,u,u_x,u_y)u_{yy}=F(x,y,u,u_x,u_y) \end{aligned}$$

- Linear PDE:

$$\begin{aligned} &A(x,y)u_{xx}+\\ &2B(x,y)u_{xy}+\\ &C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y) \end{aligned}$$

- Semi-linear PDE:

$$\begin{aligned} &A(x,y)u_{xx}+\\ &2B(x,y)u_{xy}+\\ &C(x,y)u_{yy}=D(x,y,u,u_x,u_y) \end{aligned}$$

Classification – PDE type

- For linear and semi-linear PDE
  - The discriminant can be calculated:

$$\Delta(x,y) = B^2(x,y) - A(x,y)C(x,y)$$

Discriminant at point (x,y)	PDE type at point (x,y)
$\Delta(x,y) > 0$	Hyperbolic
$\Delta(x,y) = 0$	Parabolic
$\Delta(x,y) < 0$	Elliptic

Hyperbolic equations

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  - Upwind Scheme
  - Donor-Cell Scheme
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Scalar advection equation

- Scalar advection equation:

$$u_t + au_x = 0,\ -\infty < x < \infty,\, t > 0$$

- Initial condition:

$$u(x, t = 0) = \mathring{u}(x),\ -\infty < x < \infty$$

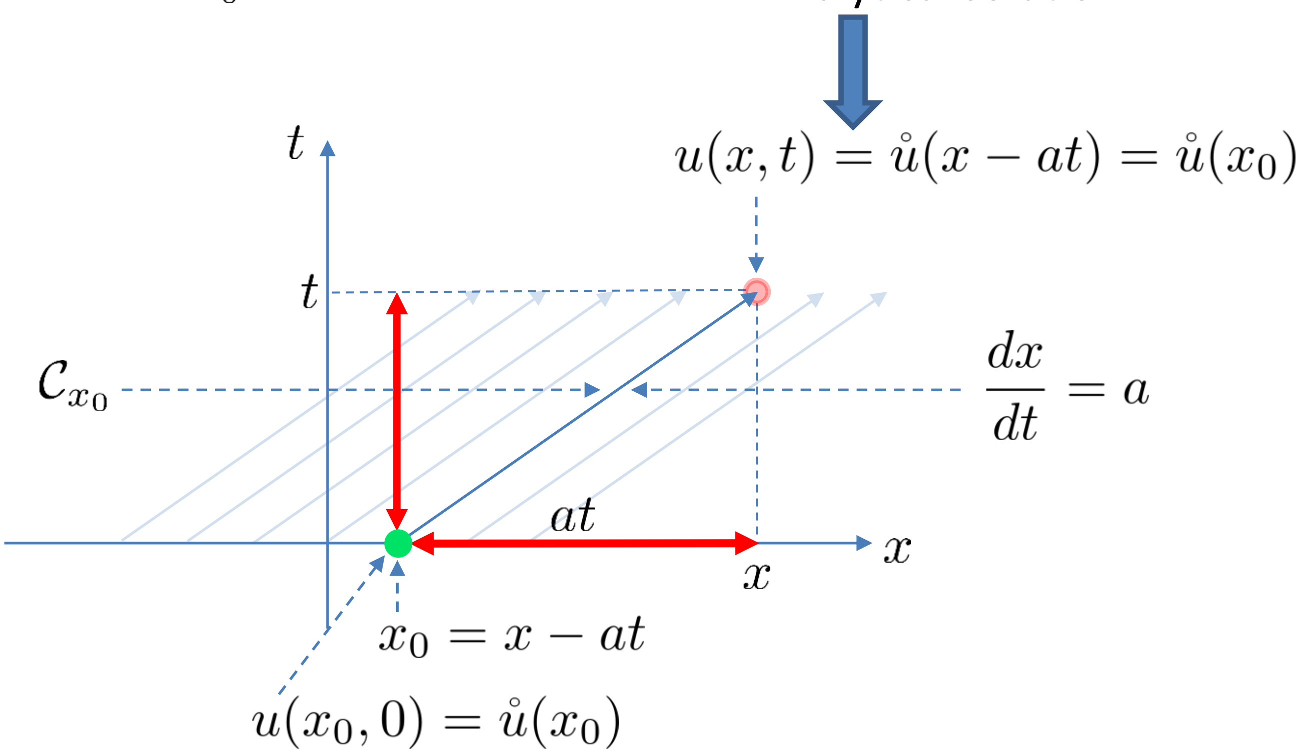
Characteristics

- Family of integral curves of the differential equation :

$$\frac{dx}{dt} = a$$

- Characteristics $\mathcal{C}_{x_0}$: $x = x_0 + at$

Analytical solution



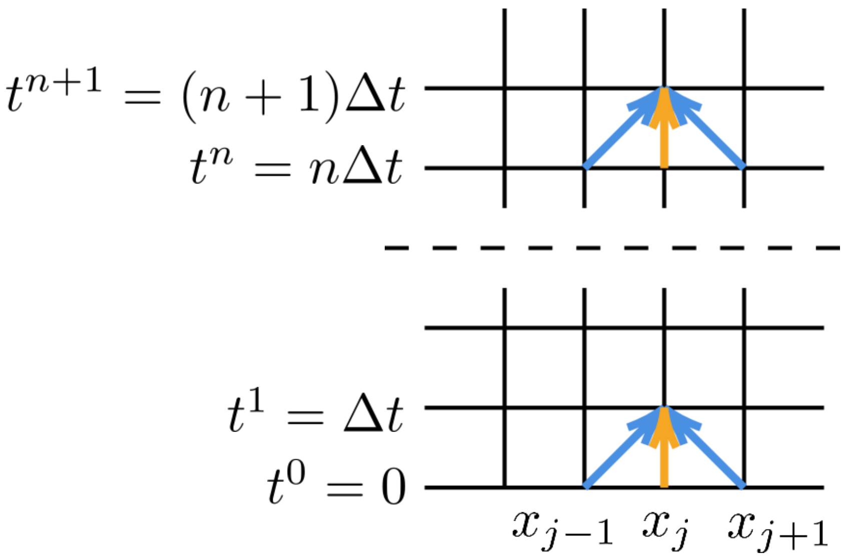
Finite Difference Scheme

- First order explicit in time, second order centered in space:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

- Explicit form :

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$



Local Truncation Error

- Local Truncation Error:

$$\tau_i^n = \frac{u(x_j, t^{n+1}) - u(x_j, t^n)}{\Delta t} + a\frac{u(x_{j+1}, t^n) - u(x_j - 1, t^n)}{2\Delta x}$$

- After second order Taylor's expansion and simplification:

$$\tau_j^n = \frac{\Delta t}{2}\partial_t^2 u\Big|_{x_j}^{t^{n_p}} + \frac{(\Delta x)^2}{12}\left(\partial_x^3 u\Big|_{x_{j_p}}^{t^n} + \partial_x^3 u\Big|_{x_{j_m}}^{t^n}\right)$$

with $t^n < t^{n_p} < t^{n+1}$ and $x_{j-1} < x_{j_m} < x_j < x_{j_p} < x_{j+1}$.

- Scheme has a local truncation error of first order in time and second order in space:

$$\tau_j^n \propto \mathcal{O}[\Delta t] + \mathcal{O}[(\Delta x)^2]$$

Stability Analysis

- Fourier-von Neumann analysis with single Fourier harmonic:

$$\hat{u}_j^n = c_k(n)\mathrm{e}^{ikx_j}$$

- With $\mu = a\Delta t/\Delta x$, substitution in the explicit form lead to:

$$c_k(n+1) = \big(1 - i\mu\sin(k\Delta x)\big)c_k(n) \equiv \rho(k,\Delta x)c_k(n)$$

- Amplification factor:

$$\rho(k,\Delta x) = 1 - i\mu\sin(k\Delta x)$$

- Unconditionally unstable scheme because :

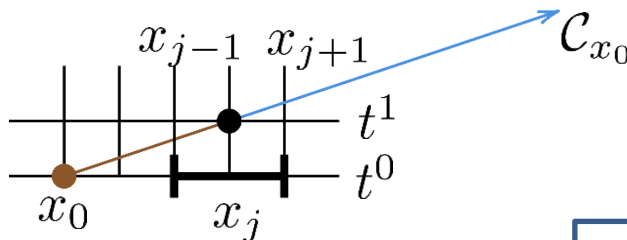
$$|\rho(k,\Delta x)| = \sqrt{1 + \mu^2\sin^2(k\Delta x)} > 1$$

- Warning:
  - Fourier-von Neumann analysis is a necessary condition for stability, but not sufficient.

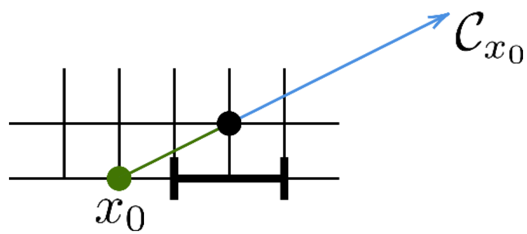
Courant-Friedrichs-Lewy Condition

- Domain of dependency of $u(x_j, t^1)$ and u_j^1
  - For the scalar advection equation; and
  - For the numerical scheme (first order explicit and second order centered).

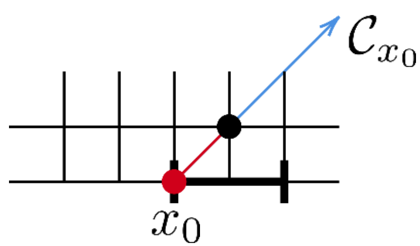
$$a = \frac{3\Delta x}{\Delta t}$$



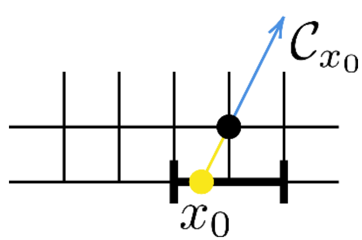
$$a = \frac{2\Delta x}{\Delta t}$$



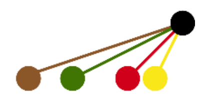
$$a = \frac{\Delta x}{\Delta t}$$



$$a = \frac{\Delta x}{2\Delta t}$$



Legend



The position x_0 is the theoretical domain of dependency

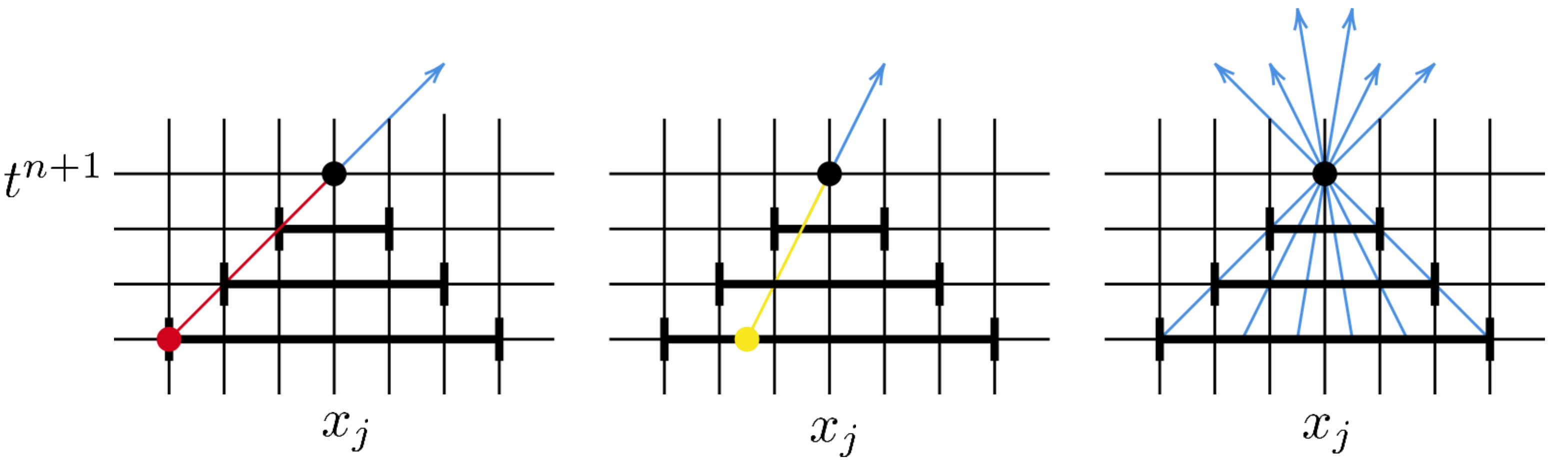
The numerical domain of dependency for the previous time step is contained in the interval containing the stencil of the scheme

Courant-Friedrichs-Lewy Condition

- Numerical domain of dependency: $D_h(x_j, t^n) \subseteq [x_{j-n}, x_{j+n}]$

- Theoretical domain of dependency: $D(x_j, t^n) = \{x_j - at^n\}$

$$a = \frac{\Delta x}{\Delta t} \qquad a = \frac{\Delta x}{2\Delta t} \qquad \frac{|a|\Delta t}{\Delta x} \leq 1$$



Courant-Friedrichs-Lewy Condition

Courant-Friedrichs-Lewy (CFL) stability condition:

A numerical method satisfies the CFL condition if for any point in the grid, the numerical dependency domain contains the theoretical dependency domain in the sense of the inclusion of intervals.

- For the scalar advection equation and the first order explicit and second order centered scheme, the CFL condition is:

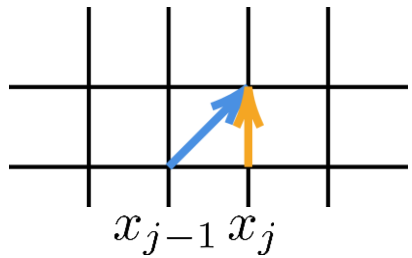
$$\frac{|a|\Delta t}{\Delta x} \leq 1$$

- Warning:
  - The Courant-Friedrichs-Lewy (CFL) condition is a necessary condition for stability, but not sufficient. Indeed, we saw with the Fourier-von Neumann analysis that this scheme is unconditionally stable.
  - The CFL condition depends on the theoretical equations and the numerical scheme.
  - The theoretical domain of dependency is not always a single point.

Upwind Scheme

$$a > 0$$
$$u_t + a u_x = 0 \quad \Rightarrow \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$$\Rightarrow \quad u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$



$$t^{n+1} = (n+1)\Delta t$$
$$t^n = n\Delta t$$

- Local Truncation Error (first order scheme):

$$\tau_j^n = \frac{\Delta t}{2} u_{tt}\Big|_j^{n_p} - \frac{a\Delta x}{2} u_{xx}\Big|_{j_m}^n$$

Upwind Scheme

- The approximated equation solved by the scheme :

$$u_t + a u_x = \left(\frac{1}{2}a\Delta x - \frac{1}{2}a^2\Delta t\right) u_{xx}$$

- For non-negative diffusion coefficient, it implicate that :

$$\frac{1}{2}a\Delta x - \frac{1}{2}a^2\Delta t \geq 0 \quad \Rightarrow \quad \mu = \frac{a\Delta t}{\Delta x} \leq 1$$

Same as CFL condition!

- Stability analysis:

$$0 < \mu \leq 1$$

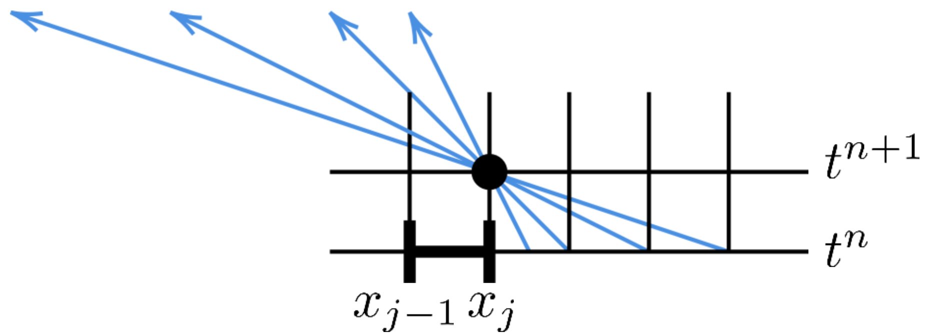
$$\rho(k,\Delta x) = 1 - \mu + \mu e^{-ik\Delta x} \leq |1-\mu| + |\mu| = 1 - \mu + \mu = 1$$

$\Rightarrow$ Conditionally stable under CFL condition

Upwind Scheme

- Problem with $a < 0$:

The numerical domain of dependency NEVER contains the theoretical domain of dependency.

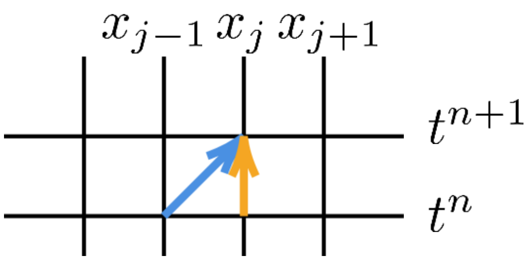


Solution → Donor-Cell upwind scheme

Donor-Cell Upwind Scheme

- Upwind scheme:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$



CFL condition

$$\frac{a\Delta t}{\Delta x} \leq 1$$

- Donor-Cell upwind scheme:

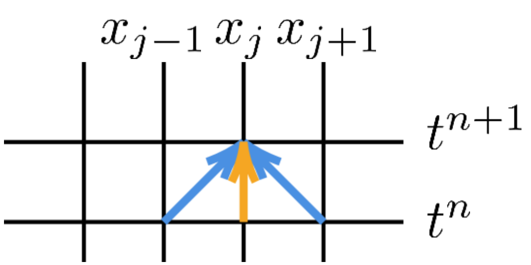
$$u_j^{n+1} =$$

$$u_j^n - \frac{\Delta t}{\Delta x}\left[a^+\left(u_j^n - u_{j-1}^n\right) + a^-\left(u_{j+1}^n - u_j^n\right)\right]$$

  – With:

$$a^+ = \max(a, 0) = \frac{a + |a|}{2}$$

$$a^- = \min(a, 0) = \frac{a - |a|}{2}$$



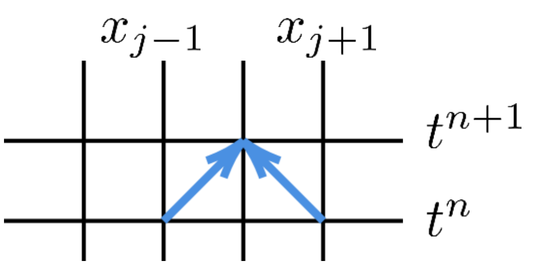
CFL condition

$$\frac{|a|\Delta t}{\Delta x} \leq 1$$

Lax-Friedrichs Scheme

- Lax-Friedrichs Scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{a\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$



- Local Truncation Error:

$$\tau_j^n = \left(\frac{a^2\Delta t}{2} - \frac{(\Delta x)^2}{2\Delta t}\right) u_{xx}\Big|_j^n + \mathcal{O}[(\Delta t)^2] + \mathcal{O}[(\Delta x)^2]$$

  - The scheme is first order in time and space if $\Delta t \propto \mathcal{O}[\Delta x]$.

- The approximated equation solved by the scheme :

$$u_t + au_x = \left(\frac{(\Delta x)^2}{2\Delta t} - \frac{a^2\Delta t}{2}\right) u_{xx}$$

- For non-negative diffusion coefficient, it implicate that :

$$\Rightarrow \quad \frac{|a|\Delta t}{\Delta x} \leq 1$$

Same as CFL condition!

Lax-Wendroff Scheme

- Lax-Wendroff Scheme

  - Based on a Taylor series expansion of u around (x, t^n) evaluated in (x, t^{n+1}):

$$u(x, t^{n+1}) = u(x, t^n) + \Delta t u_t(x, t^n) + \frac{\Delta t^2}{2} u_{tt}(x, t^n) + \ldots$$

  - Deriving with respect to time the advection equation ($u_t = -a u_x$):

$$u_{tt} = -a u_{xt} = -a u_{tx} = -a(u_t)_x = a^2 u_{xx} \Rightarrow u_{tt} = a^2 u_{xx}$$

  - Substituting in the Taylor series and keeping only terms of order 2 or less:

$$u(x, t^{n+1}) = u(x, t^n) + \Delta t \left(-a u_x(x, t^n)\right) + \frac{\Delta t^2}{2} a^2 u_{xx}(x, t^n)$$

Lax-Wendroff Scheme

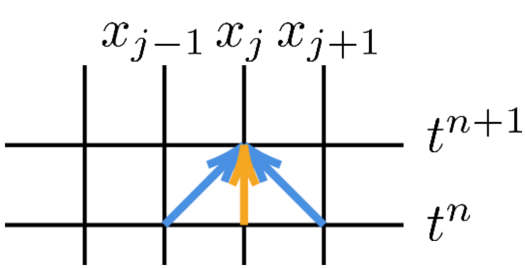
- Lax-Wendroff Scheme

$$\Rightarrow \quad u(x,t^{n+1}) = u(x,t^n) - a\Delta t u_x(x,t^n) + \frac{a^2\Delta t^2}{2} u_{xx}(x,t^n)$$

  - By performing spatial derivatives using centered finite differences:

$$u_j^{n+1} = u_j^n - a\Delta t\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + \frac{a^2\Delta t^2}{2}\left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}\right)$$

  - The Lax-Wendroff scheme is second order in time and space.
    - Intuitively:
      - Terms of the Taylor series in time preserved to the second order.
      - Second order centered derivatives in space.



CFL condition

$$\frac{|a|\Delta t}{\Delta x} \leq 1$$