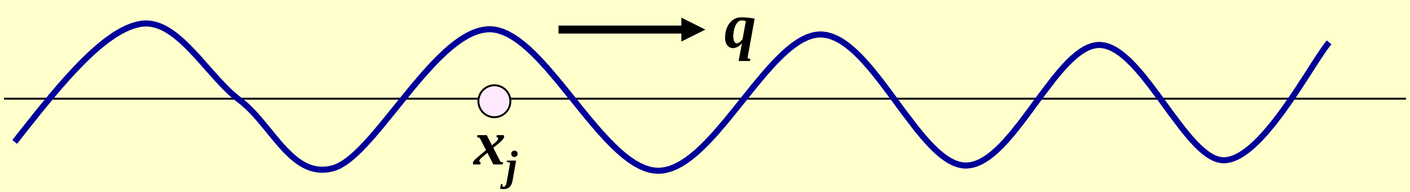
Précision de la représentation



$$f(x,t) = \cos[m(x - qt)]$$

$$\left\{ \text{Nombre d'onde : } m = \frac{2\pi}{\lambda}; \quad \text{longueur d'onde } \lambda_m = \frac{2\pi}{m} \right.$$

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Derivées Exactes

$$\overline{T} = \cos\left[m(x_j - qt_n) \right] \Rightarrow \begin{cases} \dfrac{\partial \overline{T}}{\partial x} = -m \sin\left[m(x - qt)\right] \\ \dfrac{\partial^2 \overline{T}}{\partial x^2} = -m^2 \cos\left[m(x - qt)\right] \end{cases}$$

$$\begin{cases} \left[\dfrac{\partial \overline{T}}{\partial x}\right]_j^n = -m \sin\left[m(x_j - qt_n)\right] \\ \left[\dfrac{\partial^2 \overline{T}}{\partial x^2}\right]_j^n = -m^2 \cos\left[m(x_j - qt_n)\right] \end{cases}$$

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Derivées exactes

$$
\begin{cases}
\left[\dfrac{\partial \overline{T}}{\partial x}\right]_j^n = -m\sin\left[m(x_j - qt_n)\right] = -m\sin\alpha \\[2ex]
\left[\dfrac{\partial^2 \overline{T}}{\partial x^2}\right]_j^n = -m^2\cos\left[m(x_j - qt_n)\right] = -m^2\cos\alpha
\end{cases}
$$

$$\alpha = m(x_j - qt_n)$$

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Différences centrées

$$\bar{T} = \cos\left[m(x_j - qt_n)\right], \quad \alpha = m(x_j - qt_n), \quad \beta = m\Delta x$$

$$\left[\frac{\partial \bar{T}}{\partial x}\right]_j^n \cong \frac{\bar{T}_{j+1}^n - \bar{T}_{j-1}^n}{2\Delta x}$$

$$= \frac{1}{2\Delta x}\left\{ \cos[m(x_j + \Delta x - qt_n)] - \cos[m(x_j - \Delta x - qt_n)]\right\}$$

$$= \frac{1}{2\Delta x}\left\{ \cos(\alpha + \beta) - \cos(\alpha - \beta)\right\}$$

$$= \frac{1}{2\Delta x}(-2\sin\alpha\sin\beta) = -\frac{m\sin[m(x_j - qt_n)]\sin(m\Delta x)}{m\Delta x}$$

$$\boxed{= -m\sin\alpha\frac{\sin\beta}{\beta}}$$

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$$\bar{T} = \cos\left[m(x_j - qt_n)\right], \quad \alpha = m(x_j - qt_n), \quad \beta = m\Delta x$$

$$\left[\frac{\partial^2 \bar{T}}{\partial x^2}\right]_j^n \cong \frac{\bar{T}_{j+1}^n - 2\bar{T}_j^n + \bar{T}_{j-1}^n}{\Delta x^2}$$

$$= \frac{1}{\Delta x^2}\{\cos[m(x_j + \Delta x - qt_n)] - 2\cos[m(x_j - qt_n)]$$

$$+ \cos[m(x_j - \Delta x - qt_n)]\}$$

$$= \frac{1}{\Delta x^2}\left\{\cos(\alpha + \beta) + \cos(\alpha - \beta) - 2\cos\alpha\right\}$$

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$$\bar{T} = \cos\left[m(x_j - qt_n)\right], \quad \alpha = m(x_j - qt_n), \quad \beta = m\Delta x$$

$$\left[\frac{\partial^2 \bar{T}}{\partial x^2}\right]_j^n \cong \frac{\bar{T}_{j+1}^n - 2\bar{T}_j^n + \bar{T}_{j-1}^n}{\Delta x^2}$$

$$= \frac{1}{\Delta x^2}(2\cos\alpha\cos\beta - 2\cos\alpha) = \frac{2}{\Delta x^2}\cos\alpha(\cos\beta - 1)$$

$$= \frac{2}{\Delta x^2}\cos\alpha\left(-2\sin^2\frac{\beta}{2}\right) = -\frac{4}{\Delta x^2}\cos\alpha\sin^2\frac{\beta}{2}$$

$$= -\frac{4m^2}{\beta^2}\cos\alpha\sin^2\frac{\beta}{2} = \boxed{-m^2\cos\alpha\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2}$$

Amplification

Rapport d' amplification pour la m$^{\text{ième}}$-derivée

$$A(m) = \frac{\left[\dfrac{\partial^m \overline{T}}{\partial \xi^m}\right]_j^n}{\left[\dfrac{\partial^m \overline{T}}{\partial \xi^m}\right]_{exacte}} = \frac{Solution\ Numérique}{Solution\ Exacte}$$

Dérivée première

$$A(1) = \frac{-m \sin\alpha \sin\beta / \beta}{-m \sin\alpha} = \frac{\sin\beta}{\beta} = \frac{\sin(m\Delta x)}{m\Delta x}$$

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Amplification

Rapport d' amplification pour la m-ième-derivée

$$A(m) = \frac{\left[\dfrac{\partial^m \overline{T}}{\partial x^m}\right]_j^n}{\left[\dfrac{\partial^m \overline{T}}{\partial x^m}\right]_{exact}} = \frac{\text{Solution Numérique}}{\text{Solution Exacte}}$$

Dérivée seconde

$$A(2) = \frac{-m^2 \cos\alpha[\sin(\beta/2)/(\beta/2)]^2}{-m^2\cos\alpha} = \left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

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Amplitude

Amplification pour la première dérivée

Longueur d'onde λ, nombre d'onde *m* = 2π/λ

$$\beta = m\Delta x$$

$$AR(1) = \frac{\sin\beta}{\beta} \leq 1$$

amortissement

$$\begin{cases} \lambda = 2000\Delta x,\ \beta = m\Delta x = \dfrac{2\pi}{2000} = 0.001\pi,\ AR(1) = \dfrac{\sin\beta}{\beta} = 1.000000 \\ \lambda = 20\Delta x,\ \beta = m\Delta x = \dfrac{2\pi}{20} = 0.1\pi,\ AR(1) = \dfrac{\sin\beta}{\beta} = 0.9836 \end{cases}$$

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Amplification

Amplification pour la première dérivée

Longueur d'onde λ, nombre d'onde m = $2\pi/\lambda$

$$AR(1) = \frac{\sin\beta}{\beta} \leq 1$$

Amortissement

$$\begin{cases} \lambda = 4\Delta x, \ \beta = m\Delta x = \dfrac{2\pi}{4} = 0.5\pi, \ AR(1) = \dfrac{\sin\beta}{\beta} = 0.6366 \\ \lambda = 2\Delta x, \ \beta = m\Delta x = \dfrac{2\pi}{2} = \pi, \ AR(1) = \dfrac{\sin\beta}{\beta} = 0 \end{cases}$$

ondes courtes

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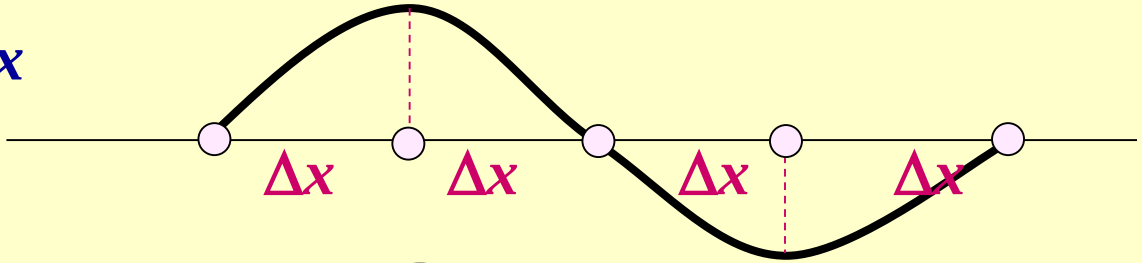
Attenuation

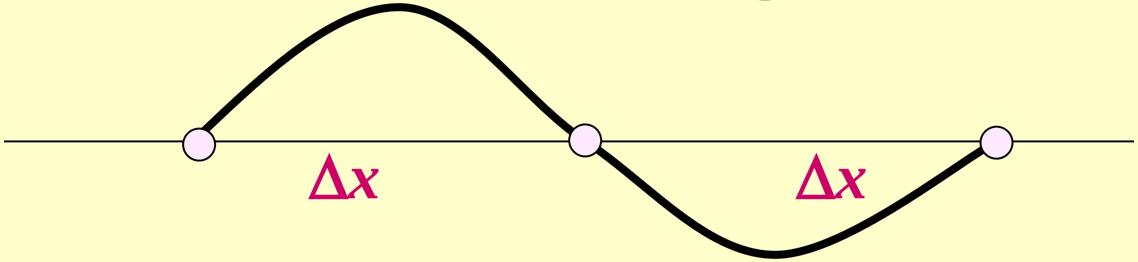
Les méthodes numériques filtrent les fréquences élevées

$$\begin{cases} \text{Onde très longue:} & \lambda = 2000\Delta x, & AR(1) = 1.000000 \\ \text{Onde longue:} & \lambda = 20\Delta x, & AR(1) = 0.9836 \\ \text{Onde courte:} & \lambda = 4\Delta x, & AR(1) = 0.6366 \\ \text{Très courte:} & \lambda = 2\Delta x, & AR(1) = 0 \end{cases}$$

$\lambda = 4\ \Delta x$



$\lambda = 2\ \Delta x$



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Rapport d'Amplitude

Amplification pour la dérivée seconde

$$AR(2) = \left(\frac{\sin \beta / 2}{\beta / 2} \right)^2 \leq 1$$ **amortissement**

$$\begin{cases} \lambda = 2000\Delta x, \ \beta = \dfrac{m\Delta x}{2} = \dfrac{\pi}{2000}, \ AR(2) = 1.000000 \\ \lambda = 20\Delta x, \ \beta = \dfrac{m\Delta x}{2} = \dfrac{\pi}{20} =, \ AR(2) = 0.9918 \\ \lambda = 4\Delta x, \ \beta = \dfrac{m\Delta x}{2} = \dfrac{\pi}{4}, \ AR(2) = 0.8106 \\ \lambda = 2\Delta x, \ \beta = \dfrac{m\Delta x}{2} = \dfrac{\pi}{2}, \ AR(2) = 0.4053 \end{cases}$$

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Différence en avant

Erreur de phase

$$\left[\frac{\partial \bar{T}}{\partial x}\right]_j^n \cong \frac{\bar{T}_{j+1}^n - \bar{T}_j^n}{\Delta x}$$

$$\alpha = m(x_j - qt_n) \text{ et } \beta = m\Delta x$$

$$= \frac{1}{\Delta x}\left\{\cos[m(x_j + \Delta x - qt_n)] - \cos[m(x_j - qt_n)]\right\}$$

$$= \frac{1}{\Delta x}\left\{\cos\left[(\alpha + \frac{\beta}{2}) + \frac{\beta}{2}\right] - \cos\left[(\alpha + \frac{\beta}{2}) - \frac{\beta}{2}\right]\right\}$$

$$= \frac{-2}{\Delta x}\sin(\alpha + \frac{\beta}{2})\sin\frac{\beta}{2}$$

$$= -\frac{2}{\Delta x}\sin\left[m(x_j + \frac{\Delta x}{2}) - mqt_n\right]\sin\left(\frac{m\Delta x}{2}\right)$$

$$\left[\frac{\partial \bar{T}}{\partial x}\right]_j^n = -m\sin\left[m(x_j - qt_n)\right] = -m\sin\alpha$$

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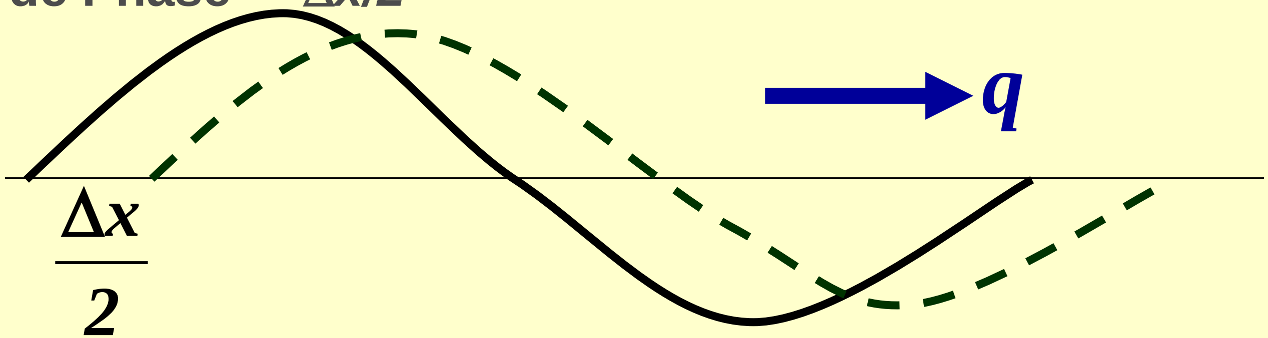
Différence en avant

Amplitude

$$AR(1) = \frac{\sin(\beta/2)}{\beta/2} = \frac{\sin(m\Delta x/2)}{m\Delta x/2}$$

Erreur de Phase = $\Delta x/2$



L'amplitude pourrait être acceptable mais il y a une erreur de phase

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Différence en arrière

$$\left[\frac{\partial \bar{T}}{\partial x}\right]_j^n = -m\sin\left[m(x_j - qt_n)\right] = -m\sin\alpha$$

$$\left[\frac{\partial \bar{T}}{\partial x}\right]_j^n \cong \frac{\bar{T}_j^n - \bar{T}_{j-1}^n}{\Delta x}$$

$$= \frac{1}{\Delta x}\left\{\cos[m(x_j - qt_n)] - \cos[m(x_j - \Delta x - qt_n)]\right\}$$

$$= \frac{1}{\Delta x}\left\{\cos\left[(\alpha - \frac{\beta}{2}) + \frac{\beta}{2}\right] - \cos\left[(\alpha - \frac{\beta}{2}) - \frac{\beta}{2}\right]\right\}$$

$$= \frac{-2}{\Delta x}\sin(\alpha - \frac{\beta}{2})\sin\frac{\beta}{2}$$

$$= -\frac{2}{\Delta x}\sin\left[m(x_j - \frac{\Delta x}{2}) - mqt_n\right]\sin\left(\frac{m\Delta x}{2}\right)$$

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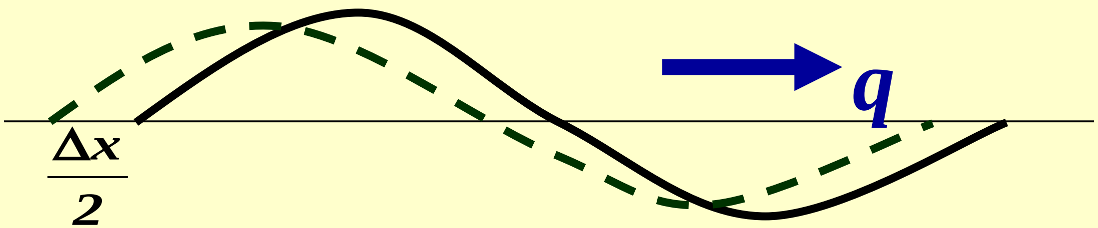
Différence en arrière

Amplitude

$$AR(1) = \frac{\sin(\beta/2)}{\beta/2} = \frac{\sin(m\Delta x/2)}{m\Delta x/2}$$

Erreur de Phase = -Δ*x/2*



$$\begin{cases} \text{Phase exacte :} & \alpha = m(x_j - q_j t_n) \\ \text{Diff. en avant :} & \alpha + \dfrac{\beta}{2} = m[(x_j + \dfrac{\Delta x}{2}) - q_j t_n] \\ \text{Diff. en arrière :} & \alpha - \dfrac{\beta}{2} = m[(x_j - \dfrac{\Delta x}{2}) - q_j t_n] \end{cases}$$

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Formule d'orde supérieur

$$\left[\frac{\partial \overline{T}}{\partial x}\right]_j^n \cong \frac{\overline{T}_{j-2}^{\,n} - 8\overline{T}_{j-1}^{\,n} + 8\overline{T}_{j+1}^{\,n} - \overline{T}_{j+2}^{\,n}}{12\Delta x}$$

$$= \frac{1}{12\Delta x}[\cos(\alpha - 2\beta) - 8\cos(\alpha - \beta) + 8\cos(\alpha + \beta) - \cos(\alpha + 2\beta)]$$

$$= \frac{1}{12\Delta x}(2\sin\alpha\sin 2\beta - 16\sin\alpha\sin\beta)$$

$$= \frac{1}{12\Delta x}(4\sin\alpha\sin\beta\cos\beta - 16\sin\alpha\sin\beta)$$

$$= \frac{1}{3\Delta x}\sin\alpha\sin\beta(\cos\beta - 4)$$

$$\boxed{AR(1) = \frac{\frac{1}{3\Delta x}\sin\alpha\sin\beta(\cos\beta - 4)}{-m\sin\alpha} = \frac{\sin\beta}{\beta}\left(\frac{4}{3} - \frac{1}{3}\cos\beta\right)}$$

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Formule d'ordre supérieur

$$\left[\frac{\partial^2 \bar{T}}{\partial x^2}\right]_j^n \cong \frac{-\bar{T}_{j-2}^n + 16\bar{T}_{j-1}^n - 30\bar{T}_j^n + 16\bar{T}_{j+1}^n - \bar{T}_{j+2}^n}{12\Delta x^2}$$

$$= \frac{1}{12\Delta x^2}[-\cos(\alpha - 2\beta) + 16\cos(\alpha - \beta) - 30\cos\alpha + 16\cos(\alpha + \beta) - \cos(\alpha + 2\beta)]$$

$$= \frac{1}{12\Delta x^2}(-2\cos\alpha\cos 2\beta + 32\cos\alpha\cos\beta - 30\cos\alpha)$$

$$= \frac{1}{12\Delta x^2}(-\cos 2\beta + 16\cos\beta - 15) = \frac{2\cos\alpha}{12\Delta x^2}[-(\cos 2\beta - 1) + 16(\cos\beta - 1)]$$

$$= \frac{2\cos\alpha}{12\Delta x^2}(2\sin^2\beta - 32\sin^2\frac{\beta}{2}) = \frac{2\cos\alpha}{12\Delta x^2}[2(\sin\frac{\beta}{2}\cos\frac{\beta}{2})^2 - 32\sin^2\frac{\beta}{2}]$$

$$= \frac{2\cos\alpha}{12\Delta x^2}\sin^2\frac{\beta}{2}(\cos^2\frac{\beta}{2} - 4)$$

$$AR(2) = \frac{\frac{4\cos\alpha}{3\Delta x^2}\sin^2\frac{\beta}{2}(\cos^2\frac{\beta}{2} - 4)}{-m^2\cos\alpha} = \frac{4}{3}(1 - \frac{1}{4}\cos^2\frac{\beta}{2})\frac{\sin^2\frac{\beta}{2}}{(\frac{\beta}{2})^2}$$

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