Polytechnique Montréal
Department of Mechanical Engineering
Course Number : **MEC6215**
Course Name : **Méthodes Numériques en Ingénierie**
Professor : **Sébastien Leclaire**
Homework : **Hyperbolic Equations (30%)**
Date of Delivery : **December 1st 2021 23:55**

Pedagogical Guidelines

- Submit a concise report of 15 pages maximum on Moodle (excluding first page, table of contents, and references), letter format, no narrow fonts, minimum 12pt, and margin 1.9cm;
- Submit your code that can reproduce your numerical results;
- This work must be done in a team of two people;
- All data (report and code) must be submitted in a single zip file "YOURMATRICULE1-YOURMATRICULE2.zip" (less than 100MB);
- Your report must be perfectly understandable when printed in black and white.
- Prepare a maximum of 6 "PowerPoint" slides (required for the final presentation).

Objective

- Learn the basics of programming solvers adapted to hyperbolic equations.

1 Two-dimensional Advection Equation

Let's consider the two-dimensional advection equation:

$$u_t + au_x + bu_y = 0 \tag{1}$$

The Eq. (1) must be solve on a square computational domain $[-1/2, -1/2] \times [1/2, 1/2]$ using periodic boundary condition in the x- and y-direction. On that square domain, information is traveling in such a way that the time to complete periodic cycles is given by:

$$t_{\text{n-cycles}} = n\lambda/\sqrt{a^2+b^2} \tag{2}$$

where n is the number of cycles and the wavelength $\lambda = \min(|\sec(\text{atan2}(b,a))|, |\csc(\text{atan2}(b,a))|)$. Note that cycles are count here only for one dimension, either x or y. For example, if the absolute advection velocity $|a|$ in the x-direction is higher than the one $|b|$ in the y-direction, the cycle along x-direction will be complete in less time than the one for the y-direction. In that particular case, Eq. (2) will time the cycle in the x-direction.

Exercises Use the MS Visual Studio solution **"MAIN_ADVECTION2D.sln"** already provided in the homework statement. Calculations will need to be perform with the *Donor-Cell Upwind*, *Corner-Transport Upwind*, *Lax-Wendroff Scheme with dimensional splitting*, and the *two-dimensional Lax-Wendroff* schemes. Code using array programming technique is preferred. **Compare results visually** (i.e. 2D plots of u along x and y) as well as 1D slice plots along $x = 0$ at the final time. Use functions or C/C++ objects to organize the different calculations.

1.1 Bump function

The bump function is used as initial condition:

$$\mathring{u}(x, y) = \begin{cases} e^{1-\frac{(1/2)^2}{(1/2)^2-x^2-y^2}} & x^2 + y^2 \leq (1/2)^2 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The advection velocities in Eq. (1) are $a = 0.5$ and $b = -0.3$. The CFL number used is 0.9 and the number of cycles is $n = 1$. Perform a grid refinement study with $\Delta x = \Delta y = 1/20$, 1/40, 1/80, 1/160, 1/320, 1/640, and 1/1280. **Complete the Excel file "bumpError.xls"** to obtain the order of accuracy of each schemes. Insert this Excel table into your report document. Compare and discuss your results against the analytical solution. Comment and analyze the results you present.

1.2 Square wave function

The square wave function is used as initial condition:

$$\mathring{u}(x, y) = \begin{cases} 1 & 8|x| \leq 1 \text{ and } 8|y| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The advection velocities in Eq. (1) are $a = 0.5$ and $b = -0.5$. The CFL number used is 0.5 and the number of cycles is $n = 2$. Perform a grid refinement study with $\Delta x = \Delta y = 1/24$, 1/48, 1/96, 1/192, 1/320, 1/384, 1/768, and 1/1536. **Complete the Excel file "squareWaveError.xls"** to obtain the order of accuracy of each schemes. Insert this Excel table into your report document. Compare and discuss your results against the analytical solution. Comment and analyze the results you present. Also, discuss how and why the results differ from the previous "bump" function case (search in the scientific literature might be required).

2 Godunov method for the 1D Euler equations

Exercises Use the MS Visual Studio solution **"MAIN_GODUNOV1D.sln"** already provided in the homework statement. The main tasks are as follows:

- Implement for the 1D Euler equations an exact Riemann solver, *i.e.* the solution of $\mathrm{RP}(\mathbf{W}_\mathrm{L}, \mathbf{W}_\mathrm{R}, x/t)$; and
- Implement the Godunov method.

The test case problems to be solved are shown in Table (1). They consist of shock tube problems where the position of the diaphragm is at x_0 with constant states $\mathbf{W}_\mathrm{L} = (\rho_\mathrm{L}, u_\mathrm{L}, p_\mathrm{L})$ on the left side

Table 1: Parameters from Toro [1999].

Case	x_0	ρ_L	u_L	p_L	ρ_R	u_R	p_R	t_{final}
1	0.3	1	0.75	1	0.125	0	0.1	0.2
2	0.5	1	-2	0.4	1	2	0.4	0.15
3	0.5	1	0	1000	1	0	0.01	0.012
4	0.4	5.99924	19.5975	460.894	5.99242	-6.19633	46.095	0.035

and $\mathbf{W}_R = (\rho_R, u_R, p_R)$ on the right side. All problems need to be solved with $\gamma = 1.4$, CFL $= 0.9$, and within the domain $0 \leq x \leq 1$ discretized with $m = 100$ cells. It might be necessary and useful to consult the reference of Toro [1999]. **Compare results visually** (plots of density, pressure, velocity and internal energy) from Godunov's method against the solutions of an exact Riemann solver at t_{final}. Discuss your results as well as the theoretical and numerical framework regarding these "shock tube problems". Comment and analyze the results you present.

References

Eleuterio F. Toro. *Riemann solvers and numerical methods for fluid dynamics.* Springer-Verlag Berlin Heidelberg, 1999.