Numerical Methods for Hyperbolic Equations

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Based on the following references:
- Gary A. Sod (1985) Numerical Methods in Fluid Dynamics. [https://doi.org/10.1017/CBO9780511753138]
- Eleuterio F. Toro (1999) Riemann solvers and numerical methods for fluid dynamics. [https://doi.org/10.1007/b79761]
- Randall J. LeVeque (2002) Finite Volume Methods for Hyperbolic Problems. [https://doi.org/10.1017/CBO9780511791253]
- Course notes given by late Prof. Paul Arminjon for MAT6165 (2006) UdeM.

Hyperbolic equations

- Classification
- Scalar Advection Equation
  - Characteristics
  - Finite Difference Scheme
  - Local Truncation Error
  - Stability Analysis
  - Courant-Friedrichs-Lewy condition (CFL condition)
  - Upwind Scheme
  - Donor-Cell Scheme
  - Lax-Friedrichs Scheme
  - Lax-Wendroff Scheme
- **Two-Dimensional Scalar Advection Equation**
  - **Characteristics**
  - **Donor-Cell Upwind Scheme (2D)**
  - **Corner-Transport Upwind Scheme (2D)**
  - **Lax-Wendroff Scheme with dimensional splitting**
  - **Lax-Wendroff Scheme (2D)**
  - **Examples:**
    - **Bump function**
    - **Square wave**
- Linear Hyperbolic Systems
  - Domain of dependence
  - Characteristics
  - Discontinuous Initial Condition
    - Scalar Advection Equation
    - Linear hyperbolic systems
- Conservation Law
  - Domain of Determinacy and Range of Influence
  - Rankine-Hugoniot Conditions
  - One-Dimensional Integral Form
  - Euler equations
    - One-Dimensional Euler equations
- The Riemann Problem for the Euler equations
  - Problem Description and Form of the Solution
  - Solution for the pressure star and velocity star
  - Solution for all Possible States
  - Complete Solution
  - Godunov Scheme
  - Examples

Hyperbolic equations

- Classification
- Scalar Advection Equation
- Two-Dimensional Scalar Advection Equation
  - Characteristics
  - Donor-Cell Upwind Scheme (2D)
  - Corner-Transport Upwind Scheme (2D)
  - Lax-Wendroff Scheme with dimensional splitting
  - Lax-Wendroff Scheme (2D)
  - Examples:
    - Bump function
    - Square wave
- Linear Hyperbolic Systems
- Conservation Law
- The Riemann Problem for the Euler equations

Characteristics

- Two-dimensional scalar advection equation:

$$u_t + au_x + bu_y = 0,\ -\infty < x < \infty,\ -\infty < y < \infty,\ t > 0$$

- Initial condition:

$$u(x, y, t = 0) = \mathring{u}(x, y),\ -\infty < x < \infty,\ -\infty < y < \infty$$

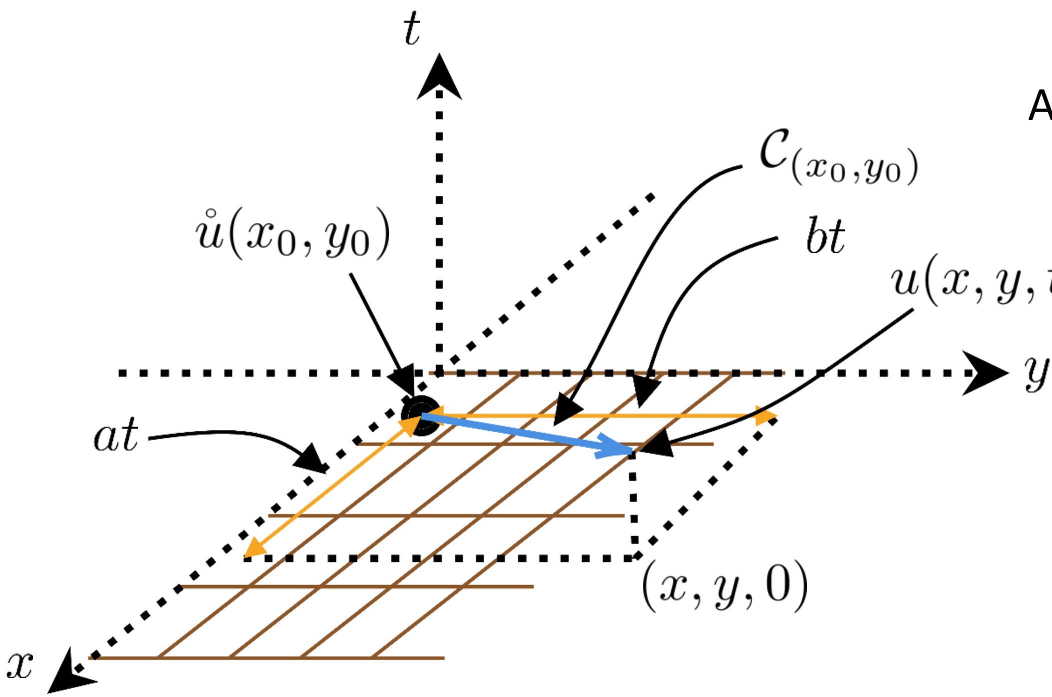
- Characteristics generated by the velocity vector $[a, b]$.

Characteristics

- Family of integral curves of the differential equation : $\dfrac{dx}{dt} = a$, $\dfrac{dy}{dt} = b$
- Characteristics $\mathcal{C}_{(x_0,y_0)}$: $x = x_0 + at$, $y = y_0 + bt$

Analytical solution

$$u(x,y,t) = \mathring{u}(x-at, y-bt) = \mathring{u}(x_0,y_0)$$



Donor-Cell Upwind Scheme (2D)

- Donor-Cell Upwind Scheme (2D):

$$u_{ij}^{n+1} = u_{ij}^{n} - \frac{\Delta t}{\Delta x}\left[a^{+}\left(u_{ij}^{n} - u_{i-1,j}^{n}\right) + a^{-}\left(u_{i+1,j}^{n} - u_{ij}^{n}\right)\right] \quad \text{X-Component}$$

$$-\frac{\Delta t}{\Delta y}\left[b^{+}\left(u_{ij}^{n} - u_{i,j-1}^{n}\right) + b^{-}\left(u_{i,j+1}^{n} - u_{ij}^{n}\right)\right] \quad \text{Y-Component}$$

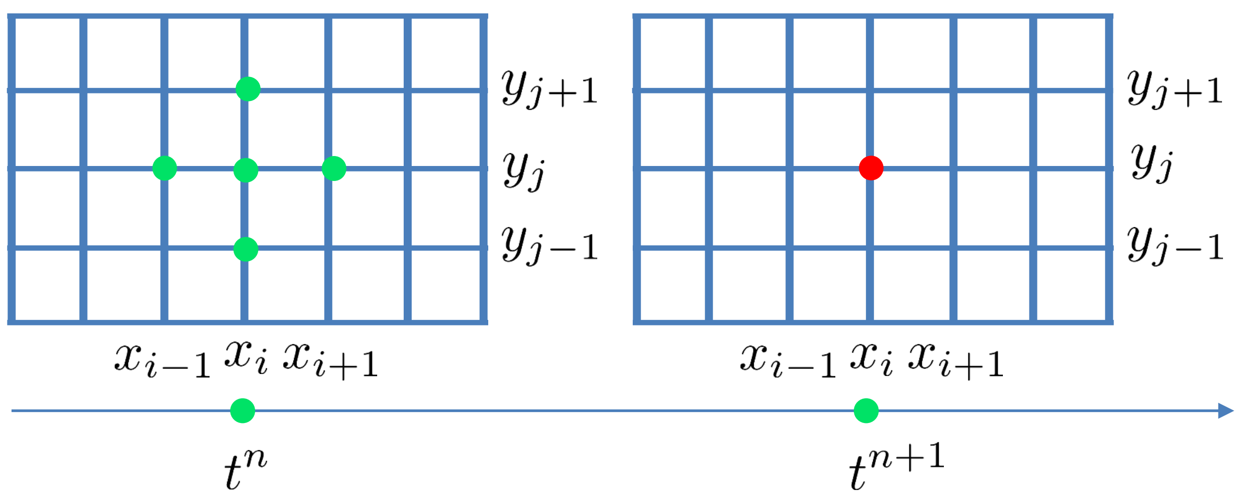
Error $\propto$

$$O(\Delta t) + O(\Delta x) + O(\Delta y)$$

CFL condition

$$\mathrm{CFL} \equiv \frac{|a|\Delta t}{\Delta x} + \frac{|b|\Delta t}{\Delta y} \leq 1$$

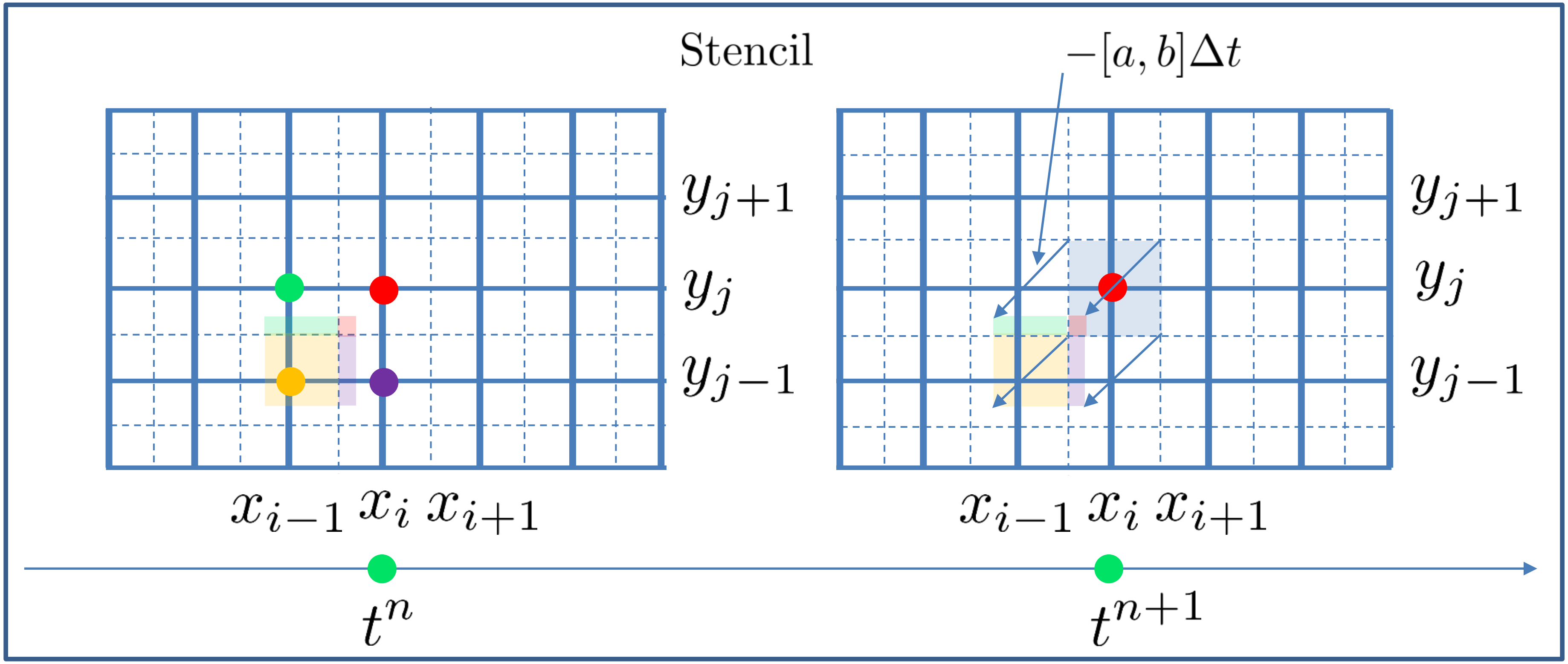
Stencil



Corner-Transport Upwind Scheme (2D)

- Corner-Transport Upwind Scheme (2D) with $a \geq 0$ and $b \geq 0$:

$$u_{ij}^{n+1} = \left[+(\Delta x - a\Delta t)(\Delta y - b\Delta t)u_{ij}^{n} + (\Delta x - a\Delta t)(b\Delta t)u_{i,j-1}^{n} \right.$$
$$\left. +(\Delta y - b\Delta t)(a\Delta t)u_{i-1,j}^{n} + (a\Delta t)(b\Delta t)u_{i-1,j-1}^{n}\right] / (\Delta x \Delta y)$$



Corner-Transport Upwind Scheme (2D)

- Corner-Transport Upwind Scheme (2D):

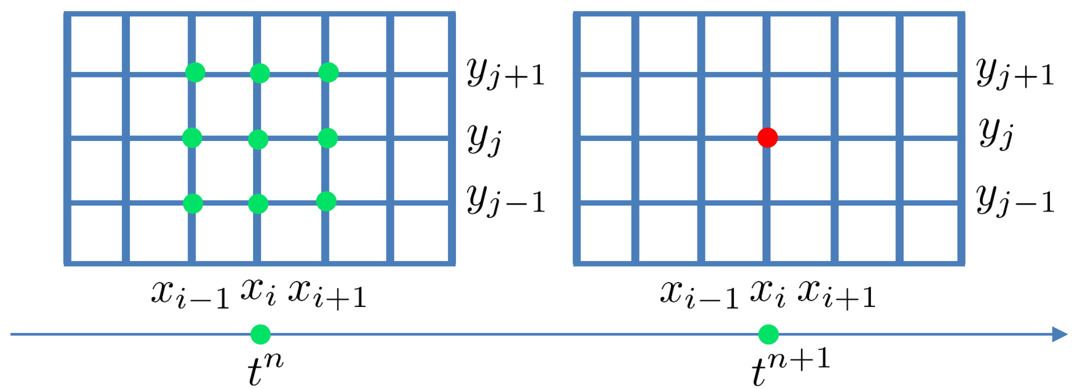
$$
\begin{aligned}
u_{ij}^{n+1} = u_{ij}^{n} &- \frac{\Delta t}{\Delta x}\left[a^{+}\left(u_{ij}^{n}-u_{i-1,j}^{n}\right)+a^{-}\left(u_{i+1,j}^{n}-u_{ij}^{n}\right)\right]-\frac{\Delta t}{\Delta y}\left[b^{+}\left(u_{ij}^{n}-u_{i,j-1}^{n}\right)+b^{-}\left(u_{i,j+1}^{n}-u_{ij}^{n}\right)\right]\\
&+\frac{\Delta t^2}{2}\left(+\frac{a^+}{\Delta x}\left(+\frac{b^+}{\Delta y}(u_{ij}-u_{i,j-1})-\frac{b^-}{\Delta y}(u_{ij}-u_{i,j+1})\right)-\frac{a^-}{\Delta x}\left(+\frac{b^+}{\Delta y}(u_{ij}-u_{i,j-1})-\frac{b^-}{\Delta y}(u_{ij}-u_{i,j+1})\right)\right)\\
&+\frac{\Delta t^2}{2}\left(+\frac{b^+}{\Delta y}\left(+\frac{a^+}{\Delta x}(u_{ij}-u_{i-1,j})-\frac{a^-}{\Delta x}(u_{ij}-u_{i+1,j})\right)-\frac{b^-}{\Delta y}\left(+\frac{a^+}{\Delta x}(u_{ij}-u_{i-1,j})-\frac{a^-}{\Delta x}(u_{ij}-u_{i+1,j})\right)\right)\\
&+\frac{\Delta t^2}{2}\left(+\frac{b^+}{\Delta y}\left(-\frac{a^+}{\Delta x}(u_{i,j-1}-u_{i-1,j-1})+\frac{a^-}{\Delta x}(u_{i,j-1}-u_{i+1,j-1})\right)-\frac{b^-}{\Delta y}\left(-\frac{a^+}{\Delta x}(u_{i,j+1}-u_{i-1,j+1})+\frac{a^-}{\Delta x}(u_{i,j+1}-u_{i+1,j+1})\right)\right)\\
&+\frac{\Delta t^2}{2}\left(+\frac{a^+}{\Delta x}\left(-\frac{b^+}{\Delta y}(u_{i-1,j}-u_{i-1,j-1})+\frac{b^-}{\Delta y}(u_{i-1,j}-u_{i-1,j+1})\right)-\frac{a^-}{\Delta x}\left(-\frac{b^+}{\Delta y}(u_{i+1,j}-u_{i+1,j-1})+\frac{b^-}{\Delta y}(u_{i+1,j}-u_{i+1,j+1})\right)\right)
\end{aligned}
$$

$$\text{Error} \propto O(\Delta t)+O(\Delta x)+O(\Delta y)$$

CFL condition

$$\mathrm{CFL} \equiv \max\left(\left|\frac{a\Delta t}{\Delta x}\right|,\left|\frac{b\Delta t}{\Delta y}\right|\right) \le 1$$

Stencil



Lax-Wendroff Scheme with dimensional splitting

- Lax-Wendroff Scheme with dimensional splitting:

$$u^*_{ij} = u^n_{ij} - a\Delta t\left(\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2\Delta x}\right) + \frac{a^2\Delta t^2}{2}\left(\frac{u^n_{i-1,j}-2u^n_{ij}+u^n_{i+1,j}}{\Delta x^2}\right)$$

X-Component

$$u^{n+1}_{ij} = u^*_{ij} - b\Delta t\left(\frac{u^*_{i,j+1}-u^*_{i,j-1}}{2\Delta y}\right) + \frac{b^2\Delta t^2}{2}\left(\frac{u^*_{i,j-1}-2u^*_{ij}+u^*_{i,j+1}}{\Delta y^2}\right)$$

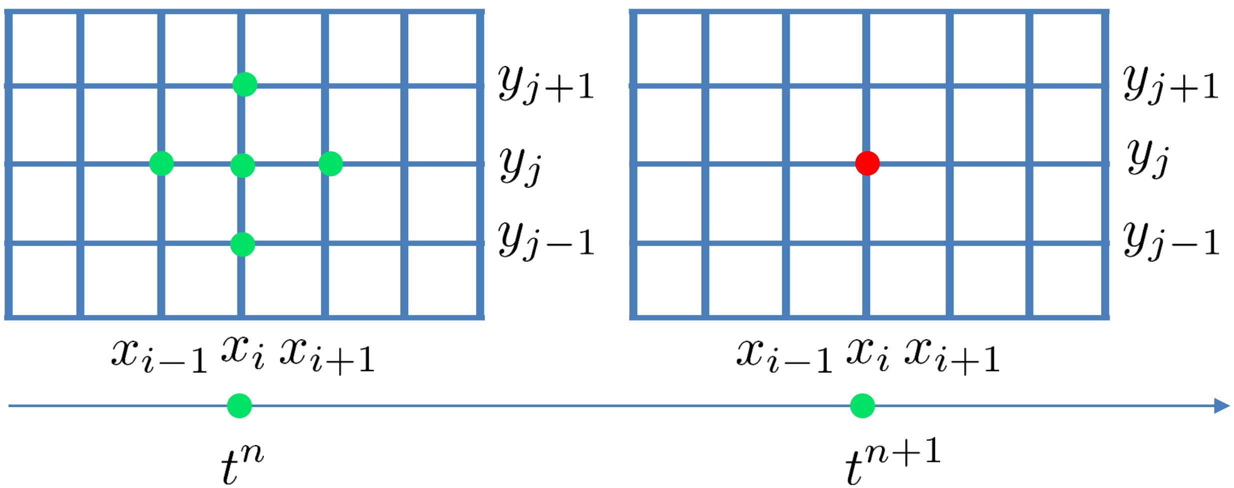
Y-Component

Error $\propto$ $O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$

CFL condition

$$\mathrm{CFL} \equiv \max\left(\left|\frac{a\Delta t}{\Delta x}\right|, \left|\frac{b\Delta t}{\Delta y}\right|\right) \leq 1$$

Stencil



Lax-Wendroff Scheme (2D)

- Lax-Wendroff Scheme (2D):

$$u_{ij}^{n+1} = u_{ij}^{n} \underbrace{- a\Delta t\left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x}\right) + \frac{a^2\Delta t^2}{2}\left(\frac{u_{i-1,j}^n - 2u_{ij}^n + u_{i+1,j}^n}{\Delta x^2}\right)}_{\text{X-Component}}$$

$$\underbrace{-b\Delta t\left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta y}\right) + \frac{b^2\Delta t^2}{2}\left(\frac{u_{i,j-1}^n - 2u_{ij}^n + u_{i,j+1}^n}{\Delta y^2}\right)}_{\text{Y-Component}}$$

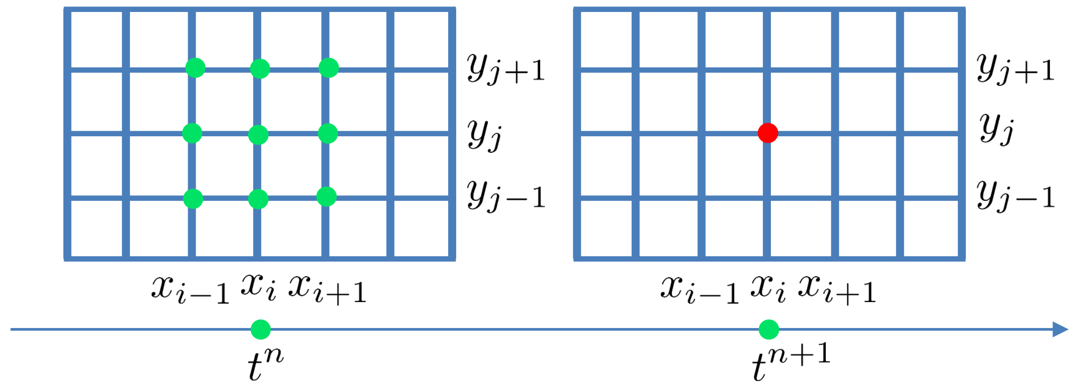
$$\underbrace{+\frac{ab\Delta t^2}{4\Delta x\Delta y}\left(\left(u_{i+1,j+1}^n - u_{i-1,j+1}^n\right) - \left(u_{i+1,j-1}^n - u_{i-1,j-1}^n\right)\right)}_{\text{Coupled Component}}$$

$$\text{Error} \propto O(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2)$$

CFL condition

$$\mathrm{CFL} \equiv \frac{2}{\sqrt{2}}\frac{\Delta t\sqrt{a^2+b^2}}{\min(\Delta x, \Delta y)} \leq 1$$

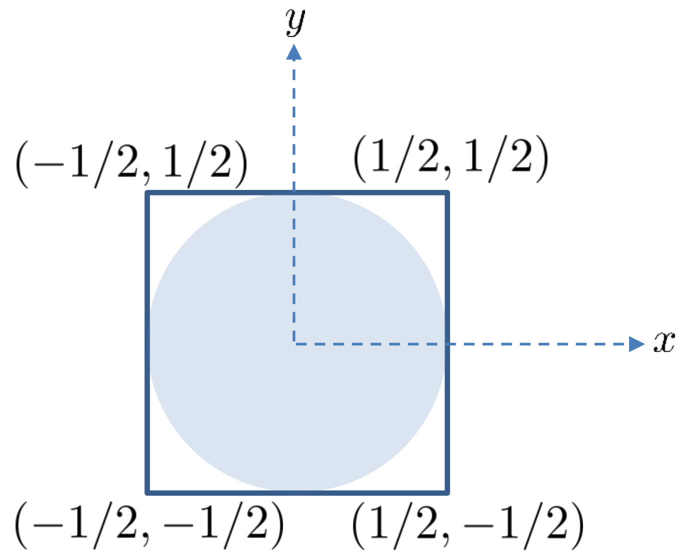
Stencil



Example: Bump Function

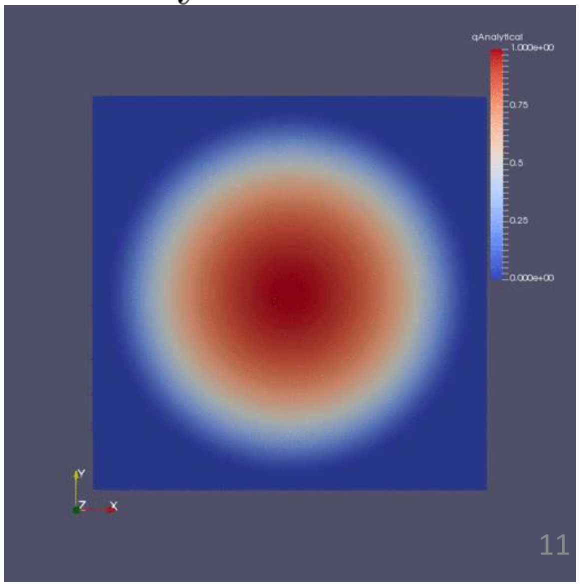
- Smooth initial condition:

$$u(x,y,0) = \begin{cases} e^{1-\frac{(1/2)^2}{(1/2)^2-x^2-y^2}} & x^2+y^2 \leq (1/2)^2 \\ 0 & \text{otherwise} \end{cases}$$



- Periodic boundary condition in X and Y.
- Constant advection velocity:
  - $a = 0.5$
  - $b = -0.3$
- Total time (n=1 cycle):
  - $t_{end} = n\lambda/\sqrt{a^2+b^2} = 2$
- With the wavelength:
  - $\lambda = \min(|\sec(\text{atan2}(b,a))|, |\csc(\text{atan2}(b,a))|)$
- CFL = 0.9

Analytical Solution



Example: Bump Function

- Bump Function – order of accuracy

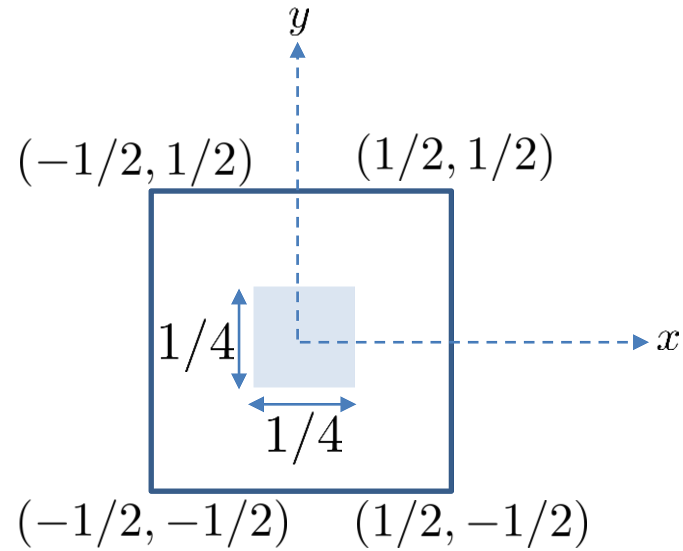
$$\epsilon = ||u_n - u_a||_{l_1} \equiv \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |u_n - u_a| dx dy \qquad p = \frac{\log(\epsilon_{\mathrm{coarser}}/\epsilon_{\mathrm{finer}})}{\log(\Delta x_{\mathrm{coarser}}/\Delta x_{\mathrm{finer}})}$$

		Donor-Cell Upwind		Corner-Transport Upwind		Lax-Wendroff Splitting		Lax-Wendroff	
Final time t_{end}	Spatial step ($\Delta x = \Delta y$)	Error l_1	Order p	Error l_1	Order p	Error l_1	Order p	Error l_1	Order p
2	1/20								
2	1/40								
2	1/80								
2	1/160								
2	1/320								
2	1/640								
2	1/1280								

Example: Square Wave

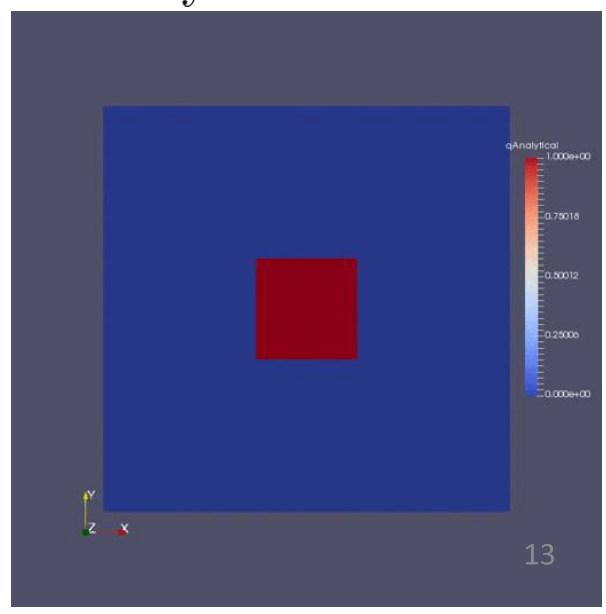
- Discontinuous initial condition:

$$u(x,y,0) = \begin{cases} 1 & 8|x| \le 1 \text{ and } 8|y| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



- Periodic boundary condition in X and Y.
- Constant advection velocity:
  - $a = 0.5$
  - $b = -0.5$
- Total time (n=2 cycles):
  - $t_{end} = n\lambda/\sqrt{a^2+b^2} = 4$
- With the wavelength:
  - $\lambda = \min(|\sec(\text{atan2}(b,a))|, |\csc(\text{atan2}(b,a))|)$
- CFL = 0.5

Analytical Solution



Example: Square Wave

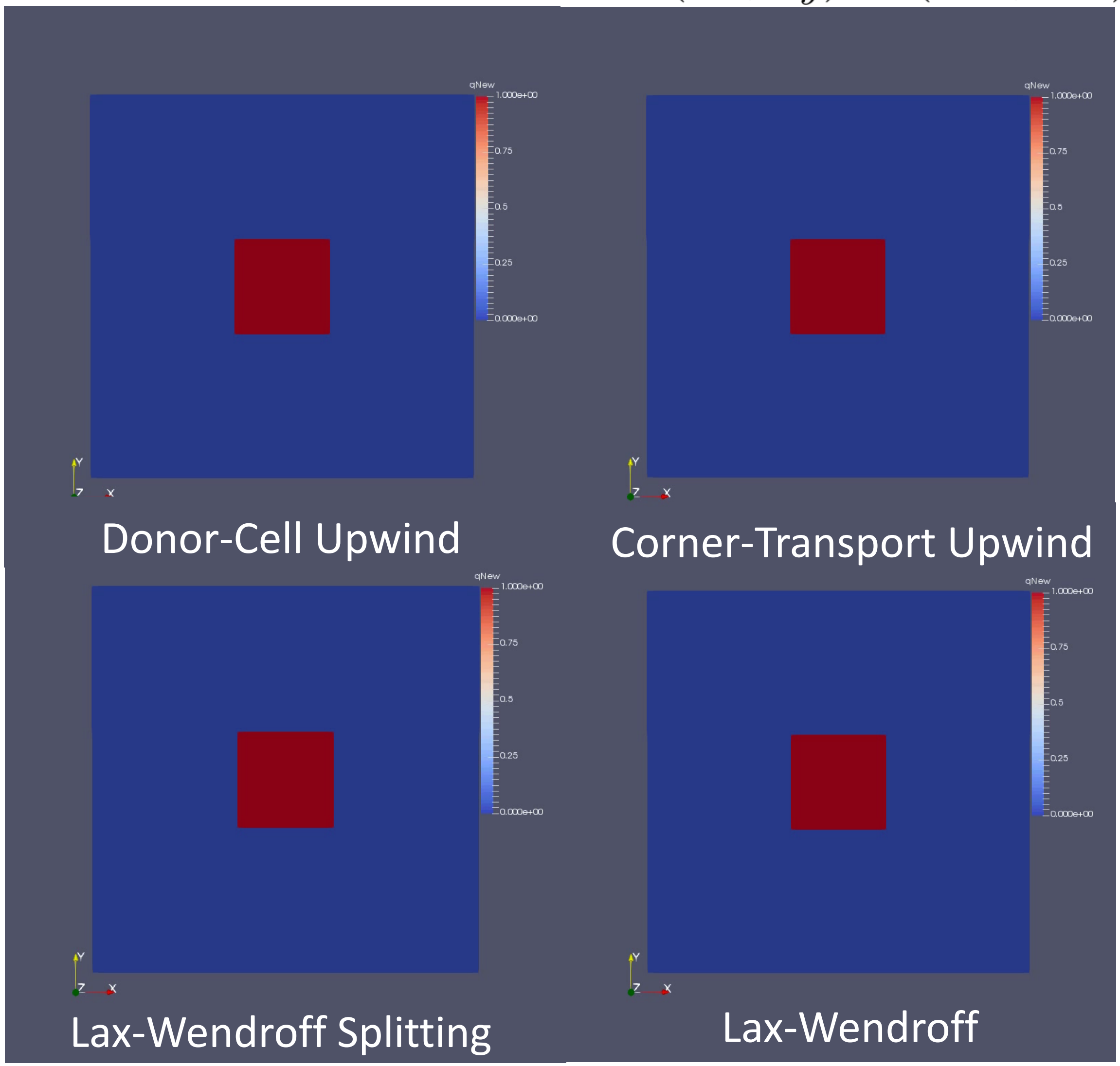
- Square Wave – order of accuracy

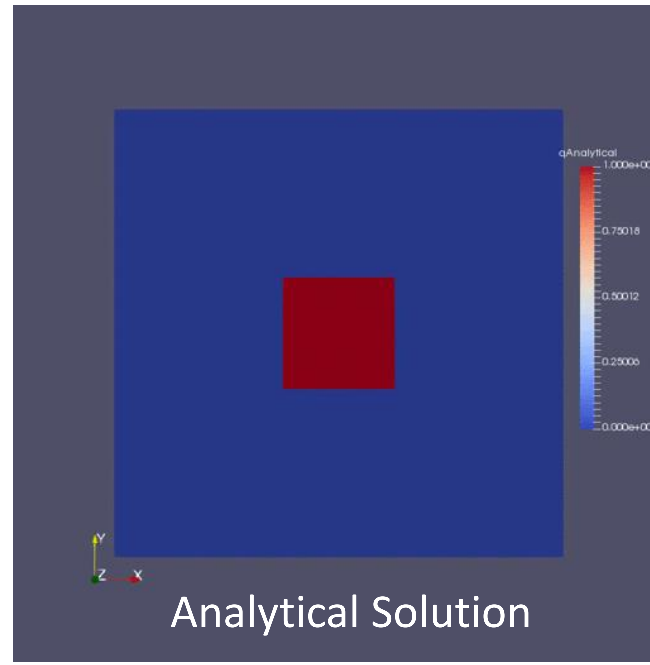
$$\epsilon = ||u_n - u_a||_{l_1} \equiv \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |u_n - u_a| dx dy \qquad p = \frac{\log(\epsilon_{\text{coarser}}/\epsilon_{\text{finer}})}{\log(\Delta x_{\text{coarser}}/\Delta x_{\text{finer}})}$$

Advection 2D		**Donor-Cell Upwind**		**Corner-Transport Upwind**		**Lax-Wendroff Splitting**		**Lax-Wendroff**	
Final time t_{end}	Spatial step ($\Delta x = \Delta y$)	Error l_1	Order p	Error l_1	Order p	Error l_1	Order p	Error l_1	Order p
4	1/24								
4	1/48								
4	1/96								
4	1/192								
4	1/384								
4	1/768								
4	1/1536								

Example: Square Wave

$$(n_x, n_y) = (384, 384)$$





Example: Square Wave

$$(n_x, n_y) = (1536, 1536)$$