Numerical Methods for Hyperbolic Equations

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Based on the following references:
- Gary A. Sod (1985) Numerical Methods in Fluid Dynamics. [https://doi.org/10.1017/CBO9780511753138]
- Eleuterio F. Toro (1999) Riemann solvers and numerical methods for fluid dynamics. [https://doi.org/10.1007/b79761]
- Randall J. LeVeque (2002) Finite Volume Methods for Hyperbolic Problems. [https://doi.org/10.1017/CBO9780511791253]
- Course notes given by late Prof. Paul Arminjon for MAT6165 (2006) UdeM.

Hyperbolic equations

- Classification
- Scalar Advection Equation
  - Characteristics
  - Finite Difference Scheme
  - Local Truncation Error
  - Stability Analysis
  - Courant-Friedrichs-Lewy condition (CFL condition)
  - Upwind Scheme
  - Donor-Cell Scheme
  - Lax-Friedrichs Scheme
  - Lax-Wendroff Scheme
- Two-Dimensional Scalar Advection Equation
  - Characteristics
  - Donor-Cell Upwind Scheme (2D)
  - Corner-Transport Upwind Scheme (2D)
  - Lax-Wendroff Scheme with dimensional splitting
  - Lax-Wendroff Scheme (2D)
  - Examples:
    - Bump function
    - Square wave
- Linear Hyperbolic Systems
  - Domain of dependence
  - Characteristics
  - Discontinuous Initial Condition
    - Scalar Advection Equation
    - Linear hyperbolic systems
- Conservation Law
  - Domain of Determinacy and Range of Influence
  - Rankine-Hugoniot Conditions
  - One-Dimensional Integral Form
  - Euler equations
    - One-Dimensional Euler equations
- **The Riemann Problem for the Euler equations**
  - **Problem Description and Form of the Solution**
  - **Solution for the pressure star and velocity star**
  - **Solution for all Possible States**
  - **Complete Solution**
  - **Godunov Scheme**
  - **Examples**

Hyperbolic equations

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- Two-Dimensional Scalar Advection Equation
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Riemann Problem Description

- The Riemann problem for the Euler equations is an initial value problem:

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x \equiv \frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2 + p\\ u\,(E+p)\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \quad -\infty < x < \infty,\, t > 0$$

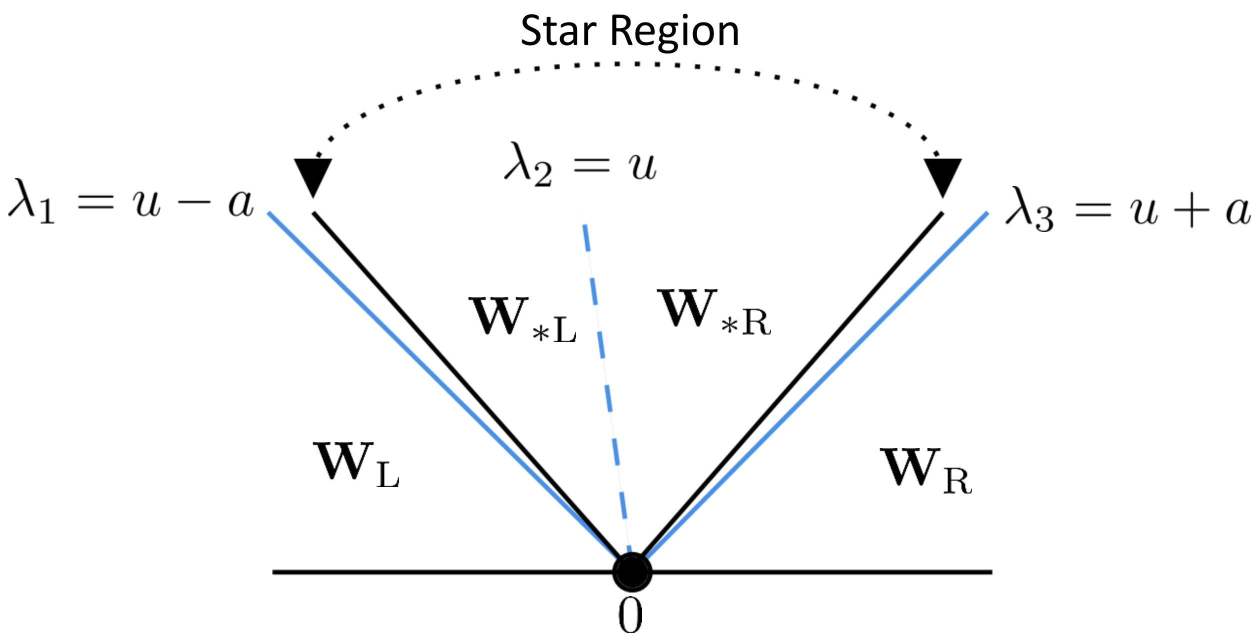
- The initial value is discontinuous:

$$\mathbf{U}(x, t=0) = \begin{cases} \mathbf{U}_l & x < 0 \\ \mathbf{U}_r & x > 0 \end{cases}$$

- Let's define the primitive variable:

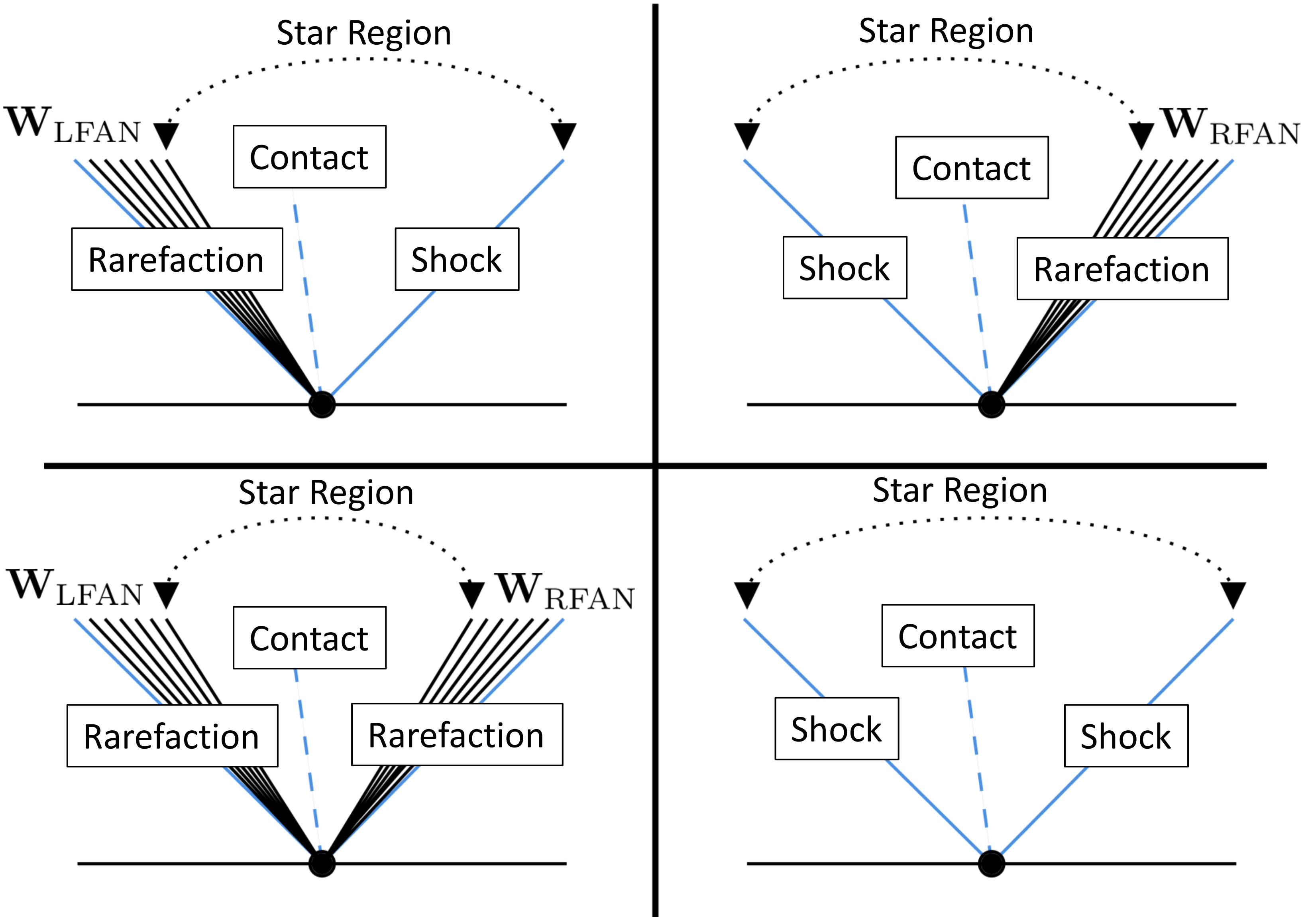
$$\mathbf{W} = (\rho, u, p)^{\mathsf{T}}$$

Form of the Solution



- Depending on the initial condition, the wave λ_1 and λ_3 could either be a shock wave or an expansion (rarefaction) wave.
- Between wave #1 and #3 there is the “star” region.
- The wave #2 always exist and is called a contact discontinuity.
- There is a jump in the solution at the contact discontinuity.

Form of the Solution – Principal Wave Patterns



Form of the Solution

- An analysis of the eigenstructure of the Euler equations could prove that the pressure as well as the velocity are continuous across the contact discontinuity:

$$\Rightarrow \quad \begin{aligned} p_{*\mathrm{L}} &= p_{*\mathrm{R}} = p_* \\ u_{*\mathrm{L}} &= u_{*\mathrm{R}} = u_* \end{aligned}$$

- We thus need to find the values for:
  - p_* ;
  - u_*;
  - $\rho_{*\mathrm{L}}$;
  - $\rho_{*\mathrm{R}}$; and if the case present
  - $\mathbf{W}_{\mathrm{LFAN}}$ and/or $\mathbf{W}_{\mathrm{RFAN}}$

Solution for the pressure star and velocity star

- It is difficult and long to study in detail the proof of the following statement, so we will take it as granted.
- The solution for p_* with an ideal gas equation of state is given by the root of the following pressure equation [1]:
  - $f\left(p, \mathbf{W}_{\mathrm{L}}, \mathbf{W}_{\mathrm{R}}\right) \equiv f_{\mathrm{L}}(p, \mathbf{W}_{\mathrm{L}}) + f_{\mathrm{R}}(p, \mathbf{W}_{\mathrm{R}}) + u_{\mathrm{R}} - u_{\mathrm{L}} = 0$ with:
    - $f_i(p, \mathbf{W}_i) = \begin{cases} (p - p_i)\left[\frac{A_i}{p+B_i}\right]^{\frac{1}{2}} & p > p_i \text{ (shock)} \\ \frac{2a_i}{\gamma-1}\left[\left(\frac{p}{p_i}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] & p \leq p_i \text{ (rarefaction)} \end{cases}, \ i=\mathrm{L,R}$
    - The left and right $(i=\mathrm{L,R})$ speed of sound: $a_i = \sqrt{\frac{\gamma p_i}{\rho_i}}$
    - The left and right $(i=\mathrm{L,R})$ coefficients: $A_i = \frac{2}{(\gamma+1)\rho_i}$ and $B_i = \frac{(\gamma-1)}{(\gamma+1)} p_i$
    - The value for $u_* = \frac{1}{2}\left(u_{\mathrm{L}} + u_{\mathrm{R}}\right) + \frac{1}{2}\left[f_{\mathrm{R}}(p_*, \mathbf{W}_{\mathrm{R}}) - f_{\mathrm{L}}(p_*, \mathbf{W}_{\mathrm{L}})\right]$

[1] Eleuterio F. Toro (1999) Riemann solvers and numerical methods for fluid dynamics. [https://doi.org/10.1007/b79761]

Numerical scheme to find the pressure star

- To solve the pressure equation for p_* a Newton-Raphson iterative procedure can be used:

$$p_{(k+1)} = p_{(k)} - \frac{f(p_k, \mathbf{W}_\mathrm{L}, \mathbf{W}_\mathrm{R})}{f'(p_k, \mathbf{W}_\mathrm{L}, \mathbf{W}_\mathrm{R})}$$

- A possible initial guess could be: $p_{(0)} = (p_\mathrm{L} + p_\mathrm{R})/2$
- Numerical difficulty may arise and pressure could become negative after the first iteration if the initial pressure guess is too large.
- If $p_{(k+1)} < 0$, one possibility is to overwrite $p_{(k+1)}$ with another smaller initial guess:

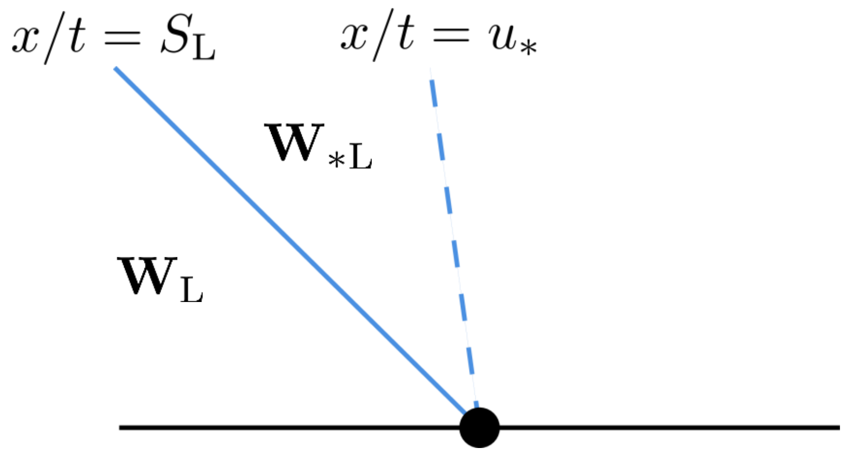
$$p_{(k+1)} \leftarrow \frac{1}{2^k}\left[\frac{a_\mathrm{L} + a_\mathrm{R} - (\gamma-1)(u_\mathrm{R} - u_\mathrm{L})/2}{\frac{a_\mathrm{L}}{(p_\mathrm{L})^{\frac{\gamma-1}{2\gamma}}} + \frac{a_\mathrm{R}}{(p_\mathrm{R})^{\frac{\gamma-1}{2\gamma}}}}\right]^{\frac{2\gamma}{\gamma-1}} \equiv \frac{1}{2^k} p_\mathrm{FAN}$$

- The pressure p_FAN is the exact solution for p_* when wave #1 and wave #3 are rarefaction. A small enough initial guess is sufficient to always converges to the real solution.

Solution for all Possible States – Left Shock Wave

Left shock wave

$p_* > p_L$



- The left shock speed: $S_L = u_L - a_L \left[\frac{(\gamma+1)}{2\gamma}\frac{p_*}{p_L} + \frac{(\gamma-1)}{2\gamma} \right]^{\frac{1}{2}}$

- The left density star for a shock: $\rho_{*L}^{\text{sho}} = \rho_L \left[\frac{\frac{p_*}{p_L} + \frac{(\gamma-1)}{(\gamma+1)}}{\frac{(\gamma-1)}{(\gamma+1)}\frac{p_*}{p_L} + 1} \right]$

Solution for all Possible States – Left Rarefaction Wave

- The left density star for a rarefaction:

$$\rho_{*\mathrm{L}}^{\mathrm{rar}} = \rho_{\mathrm{L}} \left(p_*/p_{\mathrm{L}}\right)^{\frac{1}{\gamma}}$$

- The left speed of sound star:

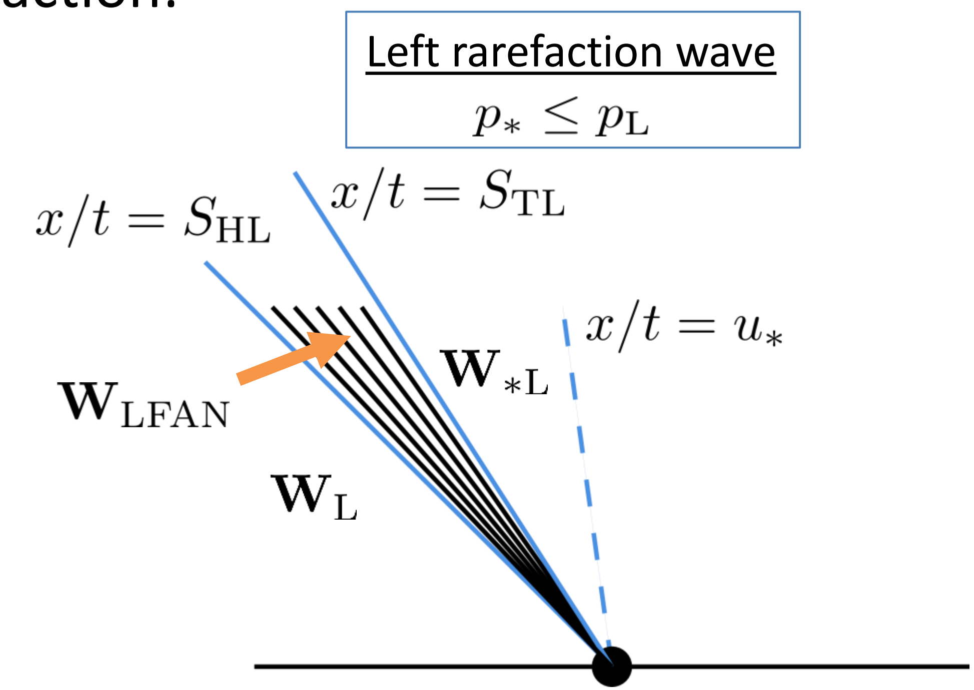
$$a_{*\mathrm{L}} = a_{\mathrm{L}} \left(p_*/p_{\mathrm{L}}\right)^{\frac{\gamma-1}{2\gamma}}$$

- The left tail speed:

$$S_{\mathrm{TL}} = u_* - a_{*\mathrm{L}}$$

- The left head speed:

$$S_{\mathrm{HL}} = u_{\mathrm{L}} - a_{\mathrm{L}}$$



- The values in the fan:

$$\mathbf{W}_{\mathrm{LFAN}} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_{\mathrm{LFAN}} = \begin{pmatrix} \rho_{\mathrm{L}} \left[\frac{2}{\gamma+1} + \frac{(\gamma-1)}{(\gamma+1)a_{\mathrm{L}}}\left(u_{\mathrm{L}} - \frac{x}{t}\right)\right]^{\frac{2}{\gamma-1}} \\ \frac{2}{\gamma+1}\left[a_{\mathrm{L}} + \frac{(\gamma-1)}{2}u_{\mathrm{L}} + \frac{x}{t}\right] \\ p_{\mathrm{L}} \left[\frac{2}{\gamma+1} + \frac{(\gamma-1)}{(\gamma+1)a_{\mathrm{L}}}\left(u_{\mathrm{L}} - \frac{x}{t}\right)\right]^{\frac{2\gamma}{\gamma-1}} \end{pmatrix}$$

Solution for all Possible States – Right Shock Wave

- Same formula as for the left shock wave but:
  - Need to replace left L with right R.
  - A sign in the right shock speed formula is different:

$$S_{\mathrm{R}} = u_{\mathrm{R}} + a_{\mathrm{R}} \left[\frac{(\gamma+1)}{2\gamma} \frac{p_*}{p_{\mathrm{R}}} + \frac{(\gamma-1)}{2\gamma} \right]^{\frac{1}{2}}$$

Solution for all Possible States – Right Rarefaction Wave

- Same formula as for the left rarefaction wave but:
  - Need to replace left L with right R .
  - Some signs in the formula are different:
    - The left tail speed: $S_{\mathrm{TR}} = u_* + a_{*\mathrm{R}}$
    - The left head speed: $S_{\mathrm{HR}} = u_{\mathrm{R}} + a_{\mathrm{R}}$
    - The values in the fan:

$$\mathbf{W}_{\mathrm{RFAN}} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_{\mathrm{RFAN}} = \begin{pmatrix} \rho_{\mathrm{R}} \left[\frac{2}{\gamma+1} - \frac{(\gamma-1)}{(\gamma+1)a_{\mathrm{R}}} \left(u_{\mathrm{R}} - \frac{x}{t} \right) \right]^{\frac{2}{\gamma-1}} \\ \frac{2}{\gamma+1} \left[-a_{\mathrm{R}} + \frac{(\gamma-1)}{2} u_{\mathrm{R}} + \frac{x}{t} \right] \\ p_{\mathrm{R}} \left[\frac{2}{\gamma+1} - \frac{(\gamma-1)}{(\gamma+1)a_{\mathrm{R}}} \left(u_{\mathrm{R}} - \frac{x}{t} \right) \right]^{\frac{2\gamma}{\gamma-1}} \end{pmatrix}$$

Complete Solution

- Let's define: $\mathbf{W}_{*\mathrm{i}}^{\mathrm{sho}} = [\rho_{*\mathrm{i}}^{\mathrm{sho}}, u_*, p_*]$, $\mathbf{W}_{*\mathrm{i}}^{\mathrm{rar}} = [\rho_{*\mathrm{i}}^{\mathrm{rar}}, u_*, p_*]$, i=L,R

- The complete solution is:

$$
\mathrm{RP}(\mathbf{W}_\mathrm{L}, \mathbf{W}_\mathrm{R}, x/t) = \left\{
\begin{array}{lll}
\left.\begin{array}{ll}
\mathbf{W}_{*\mathrm{L}}^{\mathrm{sho}} & S_\mathrm{L} \le \frac{x}{t} \le u_* \\
\mathbf{W}_\mathrm{L} & \frac{x}{t} \le S_\mathrm{L}
\end{array}\right\} & p_* > p_\mathrm{L} & \\
\left.\begin{array}{ll}
\mathbf{W}_\mathrm{L} & \frac{x}{t} \le S_\mathrm{HL} \\
\mathbf{W}_\mathrm{LFAN} & S_\mathrm{HL} \le \frac{x}{t} \le S_\mathrm{TL} \\
\mathbf{W}_{*\mathrm{L}}^{\mathrm{rar}} & S_\mathrm{TL} \le \frac{x}{t} \le u_*
\end{array}\right\} & p_* \le p_\mathrm{L} & \Bigg\} \; \frac{x}{t} \le u_* \\
\left.\begin{array}{ll}
\mathbf{W}_{*\mathrm{R}}^{\mathrm{sho}} & u_* \le \frac{x}{t} \le S_\mathrm{R} \\
\mathbf{W}_\mathrm{R} & \frac{x}{t} \ge S_\mathrm{R}
\end{array}\right\} & p_* > p_\mathrm{R} & \\
\left.\begin{array}{ll}
\mathbf{W}_\mathrm{R} & \frac{x}{t} \ge S_\mathrm{HR} \\
\mathbf{W}_\mathrm{RFAN} & S_\mathrm{TR} \le \frac{x}{t} \le S_\mathrm{HR} \\
\mathbf{W}_{*\mathrm{R}}^{\mathrm{rar}} & u_* \le \frac{x}{t} \le S_\mathrm{TR}
\end{array}\right\} & p_* \le p_\mathrm{R} & \Bigg\} \; \frac{x}{t} \ge u_*
\end{array}
\right.
$$

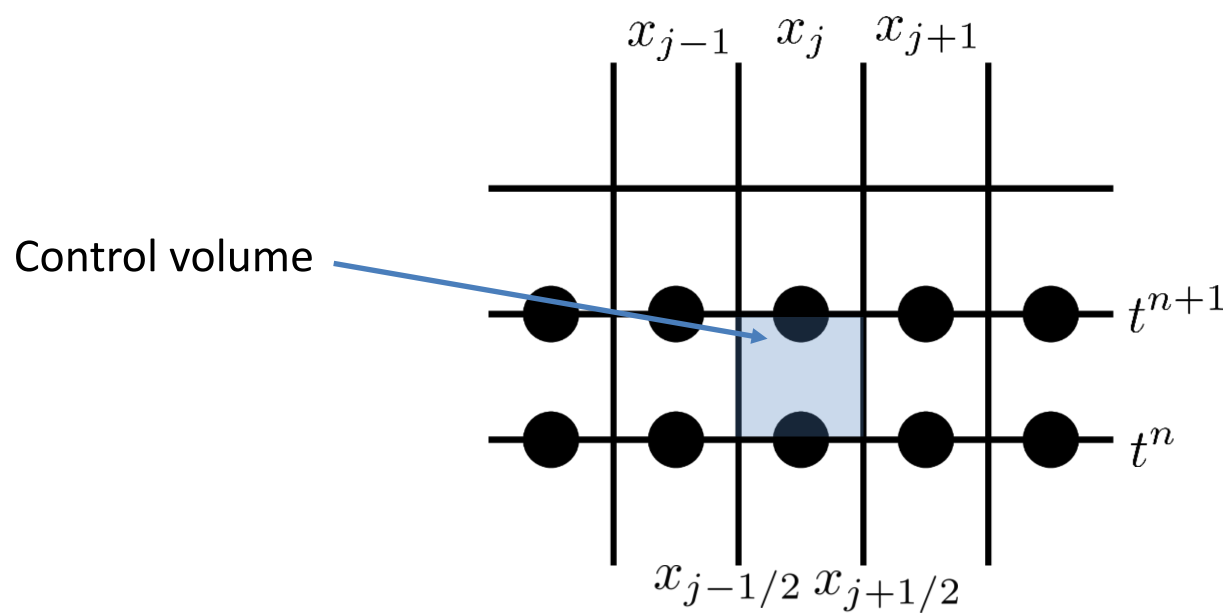
- Note that this solution assumed no vacuum (i.e. $\rho > 0$) and extended solution may be found for that case.

Godunov Scheme

- Initial-Boundary Value Problem (IBVP) over $[0, t_{\text{final}}] \times [0, L]$:

$$\begin{aligned} \text{PDE} &: \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \\ \text{IC} &: \mathbf{U}(x,0) = \mathring{\mathbf{U}}(x) \\ \text{BC} &: \mathbf{U}(0,t) = \mathbf{U}_l(t) \\ \text{BC} &: \mathbf{U}(L,t) = \mathbf{U}_r(t) \end{aligned}$$

- Discretization of $[0, L]$ with cells centered in x_j for $j = 1, \ldots, m$



Godunov Scheme

- The initial condition is approximated with a constant piecewise function such that:

$$\mathring{\mathbf{U}}(x) = \mathring{\mathbf{U}}(x_j),\ x_{j-1/2} < x < x_{j+1/2},\ \ j = 1, ..., m$$

- We recall the one-dimensional integral form:

$$\int_{x_l}^{x_r} \mathbf{U}(x, t_2) dx = \int_{x_l}^{x_r} \mathbf{U}(x, t_1) dx + \int_{t_1}^{t_2} \mathbf{F}\left(\mathbf{U}(x_l, t)\right) dt - \int_{t_1}^{t_2} \mathbf{F}\left(\mathbf{U}(x_r, t)\right) dt$$

- We apply this form to each control volume $j = 1, ..., m$:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^{n+1}) dx =$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^n) dx + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j-1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j+1/2}, t)\right) dt$$

Godunov Scheme

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^{n+1})dx =$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^n)dx + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j-1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j+1/2}, t)\right) dt$$

- Let's define the cell averages:

$$\mathbf{U}_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^n)dx$$

$$\mathbf{U}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}(x, t^{n+1})dx$$

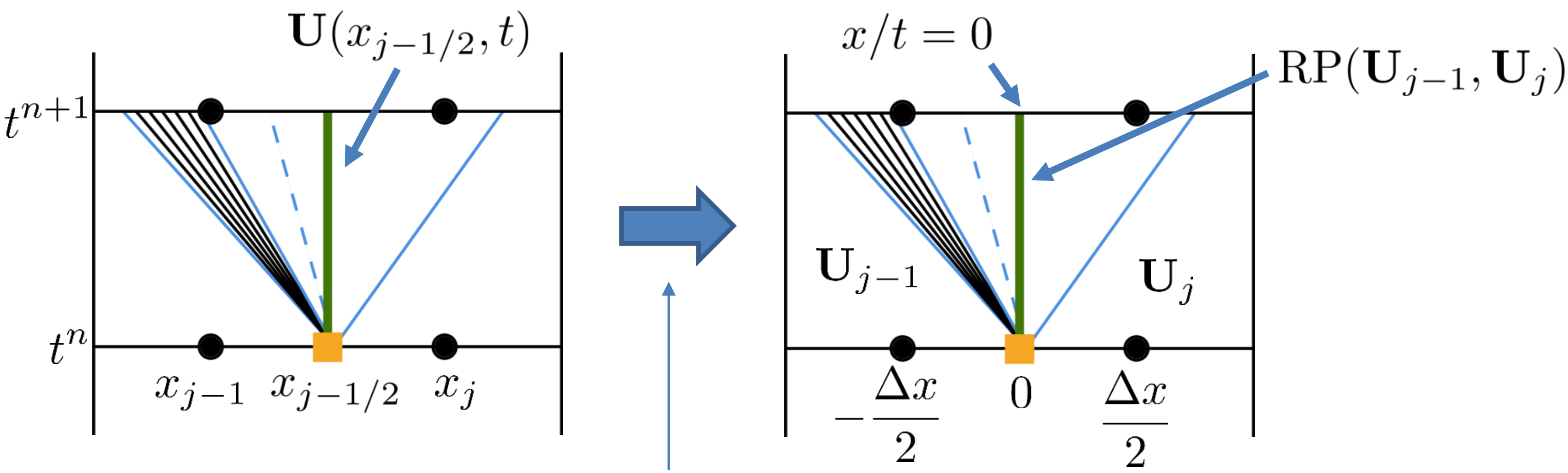
- Assuming a piecewise constant solution:

$$\mathbf{U}_j^{n+1}\Delta x = \mathbf{U}_j^n \Delta x + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j-1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j+1/2}, t)\right) dt$$

Godunov Scheme

$$\mathbf{U}_j^{n+1}\Delta x = \mathbf{U}_j^n \Delta x + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j-1/2},t)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j+1/2},t)\right) dt$$

- Assuming no wave interaction at $x_{j-1/2}$ for $t \in [t^n, t^{n+1}]$ imply:
  - $\mathbf{U}(x_{j-1/2},t) = \mathrm{RP}(\mathbf{U}_{j-1},\mathbf{U}_j, x/t = 0) = \mathrm{RP}(\mathbf{U}_{j-1},\mathbf{U}_j)$
  - $\mathbf{U}(x_{j+1/2},t) = \mathrm{RP}(\mathbf{U}_j,\mathbf{U}_{j+1}, x/t = 0) = \mathrm{RP}(\mathbf{U}_j,\mathbf{U}_{j+1})$



Change of coordinates

Godunov Scheme

$$\mathbf{U}_j^{n+1}\Delta x = \mathbf{U}_j^n\Delta x + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j-1/2},t)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathbf{U}(x_{j+1/2},t)\right) dt$$

$$\mathbf{U}_j^{n+1}\Delta x =$$
$$\mathbf{U}_j^n\Delta x + \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_{j-1}^n,\mathbf{U}_j^n)\right) dt - \int_{t^n}^{t^{n+1}} \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)\right) dt$$

- The integrant is piecewise constant:

$$\mathbf{U}_j^{n+1}\Delta x = \mathbf{U}_j^n\Delta x + \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_{j-1}^n,\mathbf{U}_j^n)\right)\Delta t - \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)\right)\Delta t$$

<u>Godunov scheme</u> : $\mathbf{U}_j^{n+1} = \mathbf{U}_j^n + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j-1/2}^n - \mathbf{F}_{j+1/2}^n\right)$

– with: $\mathbf{F}_{j-1/2}^n = \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_{j-1}^n,\mathbf{U}_j^n)\right)$

$\mathbf{F}_{j+1/2}^n = \mathbf{F}\left(\mathrm{RP}(\mathbf{U}_j^n,\mathbf{U}_{j+1}^n)\right)$

Godunov Scheme

- The CFL condition for this Godunov scheme is:

$$\Delta t \leq \frac{\Delta x}{S^n_{\max}}$$

  - where $S^n_{\max}$ is the fastest wave at time t^n on the whole computational domain.

- For the Euler equations, a good estimate for the fastest wave is:

$$S^n_{\max} \approx \max_{1 \leq j \leq m} \left\{ |u^n_j| + a^n_j \right\}$$

  - Stability may be improved by using a CFL number lower than 1:

$$\Delta t \leq \mathrm{CFL} \frac{\Delta x}{S^n_{\max}}$$

  - Typically $\mathrm{CFL} = 0.9$ is a practical choice.

Example

- Exact Riemann solution and Godunov method:
  - Domain: $0 \leq x \leq 1$
  - Diaphragm at $x_0 = 0.3$
  - 100 cells
  - $\gamma = 1.4$
  - $\mathrm{CFL} = 0.9$
  - Left initial condition:

$$\mathbf{W}_{\mathrm{L}} = (\rho_{\mathrm{L}}, u_{\mathrm{L}}, p_{\mathrm{L}})^{\mathsf{T}} = (1, 0.75, 1)^{\mathsf{T}}$$

  - Right initial condition:

$$\mathbf{W}_{\mathrm{R}} = (\rho_{\mathrm{R}}, u_{\mathrm{R}}, p_{\mathrm{R}})^{\mathsf{T}} = (0.125, 0, 0.1)^{\mathsf{T}}$$