

Figure 9.1: Three-Dimensional Cube

Equation (9.1.2) is a model of the temperature u as a function of time and at points within the interior of the body. As usual, to complete the model we need to specify boundary conditions, and for this purpose it is simplest for exposition to treat the corresponding problem in two space dimensions:

$$u_t = c(u_{xx} + u_{yy}). \quad (9.1.3)$$

We can consider (9.1.3) to be the mathematical model of the temperature in a flat, thin plate as shown in Figure 9.2, where we have taken the plate to be the unit square.

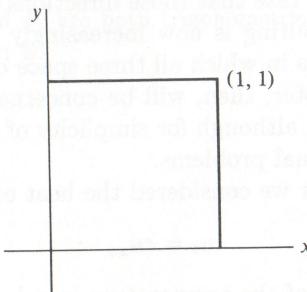


Figure 9.2: Flat, Thin Plate

The simplest boundary conditions occur when the temperature is prescribed on the four sides of the plate:

$$u(t, x, y) = g(x, y), \quad (x, y) \text{ on boundary}, \quad (9.1.4)$$

9.1 TWO AND THREE SPACE DIMENSIONS

where g is a given function. Another possibility is to assume that one of the sides, say $x = 0$, is perfectly insulated; thus, there is no heat loss across that side and no change in temperature, so the boundary condition is

$$u_x(t, 0, y) = 0, \quad 0 \leq y \leq 1, \quad (9.1.5)$$

combined with the specification (9.1.4) on the other sides. A boundary condition of the form (9.1.5) is usually called a *Neumann condition*, and that of the form (9.1.4) is a *Dirichlet condition*. Clearly, various other such combinations are possible, including a specified temperature change (other than zero) across a boundary. Boundary conditions for the three-dimensional problem can be given in a similar fashion. We also must specify a temperature distribution at some time which we take to be $t = 0$; such an initial condition for (9.1.3) is of the form

$$u(0, x, y) = f(x, y). \quad (9.1.6)$$

Given the initial condition (9.1.6) and boundary conditions of the form (9.1.4) and/or (9.1.5), it is intuitively clear that the temperature distribution should evolve in time to a final steady state that is determined only by the boundary conditions. In many situations it is this steady-state solution that is of primary interest, and since it no longer depends on time it should satisfy the equation (9.1.3) with $u_t = 0$:

$$u_{xx} + u_{yy} = 0. \quad (9.1.7)$$

This is Laplace's equation and, as mentioned in the previous chapter, is the prototype of an elliptic equation. If we wish only the steady-state solution of the temperature distribution problem that we have been discussing, we can proceed, in principle, in two ways: solve equation (9.1.3) for u as a function of time until convergence to a steady state is reached, or solve (9.1.7) only for the steady-state solution.

Finite Differences for Poisson's Equation

We will return to the time-dependent problem shortly, after considering the finite difference method for (9.1.7) and, more generally, Poisson's equation

$$u_{xx} + u_{yy} = f, \quad (9.1.8)$$

where f is a given function of x and y . We assume that the domain of the problem is the unit square $0 \leq x, y \leq 1$, and that Dirichlet boundary conditions

$$u(x, y) = g(x, y), \quad (x, y) \text{ on boundary} \quad (9.1.9)$$

are given, where g is a known function. We impose a mesh of grid points on the unit square with spacing h between the points in both the horizontal and vertical directions; this is illustrated in Figure 9.3.

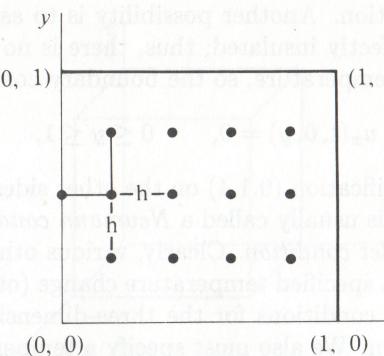


Figure 9.3: Mesh Points on the Unit Square and Discretization Stencil

The interior grid points are given by

$$(x_i, y_j) = (ih, jh), \quad i, j = 1, \dots, N, \quad (9.1.10)$$

where $(N+1)h = 1$. Now consider a typical grid point (x_i, y_j) . We approximate u_{xx} and u_{yy} at this point by the centered difference approximations

$$u_{xx}(x_i, y_j) \doteq \frac{1}{h^2} [u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)], \quad (9.1.11a)$$

$$u_{yy}(x_i, y_j) \doteq \frac{1}{h^2} [u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})]. \quad (9.1.11b)$$

If we put these approximations into the differential equation (9.1.8), we obtain

$$\begin{aligned} u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j) \\ \doteq h^2 f(x_i, y_j), \end{aligned} \quad (9.1.12)$$

which is an approximate relationship that the exact solution u of (9.1.8) satisfies at any grid point in the interior of the domain.

We now define approximations u_{ij} to the exact solution $u(x_i, y_j)$ at the N^2 interior grid points by requiring that they satisfy exactly the relationship (9.1.12); that is,

$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{ij} = -h^2 f_{ij}, \quad i, j = 1, \dots, N, \quad (9.1.13)$$

where we have multiplied (9.1.12) by -1 . This is a linear system of equations in the $(N+2)^2$ variables u_{ij} . Note, however, that the variables $u_{0,j}$, $u_{N+1,j}$, $j = 0, \dots, N+1$, and $u_{i,0}$, $u_{i,N+1}$, $i = 0, \dots, N+1$, correspond to the grid points on the boundary and thus are given by the boundary condition (9.1.9):

$$\begin{aligned} u_{0,j} &= g(0, y_j), & u_{N+1,j} &= g(1, y_j), & j &= 0, 1, \dots, N+1, \\ u_{i,0} &= g(x_i, 0), & u_{i,N+1} &= g(x_i, 1), & i &= 0, 1, \dots, N+1. \end{aligned} \quad (9.1.14)$$

Therefore (9.1.13) is a linear system of N^2 equations in the N^2 unknowns u_{ij} , $i, j = 1, \dots, N$, corresponding to the interior grid points. The stencil in Figure 9.3 shows how u_{ij} is coupled to its north, south, east, and west neighbors in (9.1.13). It is easy to show (Exercise 9.1.1) that the local discretization error in the u_{ij} is $O(h^2)$. Note that (9.1.13) is the natural extension to two space variables of the discrete equations

$$-u_{i+1} + 2u_i - u_{i-1} = -h^2 f_i, \quad i = 1, \dots, N,$$

obtained in Chapter 3 for the “one-dimensional Poisson equation” $u'' = f$.

We now wish to write the system (9.1.13) in matrix-vector form, and for this purpose we will number the interior grid points in the manner shown in Figure 9.4, which is called the *natural* or *row-wise ordering*. Corresponding to this ordering of the grid points, we order the unknowns $\{u_{ij}\}$ into the vector

$$(u_{11}, \dots, u_{N1}, u_{12}, \dots, u_{N2}, \dots, u_{1N}, \dots, u_{NN}), \quad (9.1.15)$$

and write the system of equations in the same order. We illustrate this for $N = 2$ (Exercise 9.1.2):

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \\ u_{22} \end{bmatrix} = -h^2 \begin{bmatrix} f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \end{bmatrix} + \begin{bmatrix} u_{01} + u_{10} \\ u_{20} + u_{31} \\ u_{02} + u_{13} \\ u_{32} + u_{23} \end{bmatrix}, \quad (9.1.16)$$

in which we have put the known boundary values on the right-hand side of the equation.

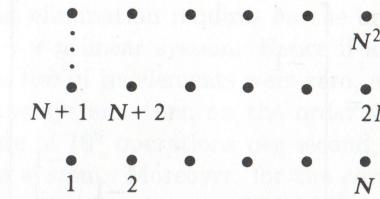


Figure 9.4: Natural Ordering of the Interior Grid Points

The equations (9.1.16) begin to illustrate the structure of the linear system. For general N , a typical row of the matrix will be

$$-1 \ 0 \ \cdots \ 0 \ -1 \ 4 \ -1 \ 0 \ \cdots \ 0 \ -1$$

where $N-2$ zeros separate the -1 's in both directions. The equations corresponding to an interior grid point adjacent to a boundary point will contain a known boundary value, and this value will be moved to the right side of the

$$\begin{matrix}
 4 & -1 & & -1 \\
 -1 & 4 & -1 & & -1 \\
 & -1 & 4 & -1 & & -1 \\
 & & -1 & 4 & -1 & & -1 \\
 & & & 4 & -1 & & -1 \\
 & & & -1 & 4 & -1 & & -1 \\
 & & & & -1 & 4 & -1 & & -1 \\
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 & & & & & & & & & & & & & -1 & 4 & & -1 \\
 & & & & & & & & & & & & & & -1 & 4 & & -1
 \end{matrix}$$

Figure 9.5: Coefficient Matrix of (9.1.13) for $N = 4$

equation, eliminating the corresponding -1 from the matrix. This happened in each equation in (9.1.16) because of the size of N . We show in Figure 9.5 the coefficient matrix for $N = 4$ (see Exercise 9.1.2).

Although Figure 9.5 illustrates the structure of the coefficient matrix, this form is cumbersome for large N ; it is much easier to write it in a block matrix form. To do this we define the $N \times N$ tridiagonal matrix

$$T_N = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & & \\ & \ddots & \ddots & -1 \\ & & \ddots & -1 \\ & & & -1 & 4 \end{bmatrix} \quad (9.1.17)$$

and let I_N denote the $N \times N$ identity matrix. Then the $N^2 \times N^2$ coefficient matrix of (9.1.13) is the *block tridiagonal matrix*

$$A = \begin{bmatrix} T_N & -I_N & & & \\ -I_N & T_N & \ddots & & \\ & -I_N & \ddots & & \\ & & \ddots & -I_N & \\ & & & -I_N & T_N \end{bmatrix}. \quad (9.1.18)$$

9.1 TWO AND THREE SPACE DIMENSIONS

The matrix of (9.1.16) is the special case of (9.1.18) for $N = 2$, and Figure 9.5 shows the matrix for $N = 4$.

If we also define the vectors

$$\begin{aligned}
 \mathbf{u}_i &= (u_{1i}, \dots, u_{Ni})^T, & \mathbf{f}_i &= (f_{1i}, \dots, f_{Ni})^T, & i &= 1, \dots, N, \\
 \mathbf{b}_1 &= (u_{01} + u_{10}, u_{20}, \dots, u_{N-1,0}, u_{N,0} + u_{N+1,1})^T, \\
 \mathbf{b}_i &= (u_{0i}, 0, \dots, 0, u_{N+1,i})^T, & i &= 2, \dots, N-1, \\
 \mathbf{b}_N &= (u_{0,N} + u_{1,N+1}, u_{2,N+1}, \dots, u_{N-1,N}, u_{N,N+1} + u_{N+1,N})^T,
 \end{aligned}$$

then we can write the system (9.1.13) in the compact form

$$\begin{bmatrix} T_N & -I_N & & \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & -I_N \\ & & -I_N & T_N \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_N \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 - h^2 \mathbf{f}_1 \\ \mathbf{b}_2 - h^2 \mathbf{f}_2 \\ \vdots \\ \mathbf{b}_N - h^2 \mathbf{f}_N \end{bmatrix}. \quad (9.1.19)$$

We now make several comments about this system of equations. If N is of moderate size, say $N = 100$, then there are $N^2 = 10^4$ unknowns, and the matrix in (9.1.19) is $10,000 \times 10,000$. In each row of the matrix there are at most five nonzero elements, regardless of the size of N , so the distribution of nonzero to zero elements is very “sparse” if N is at all large. Such matrices are called *large sparse matrices* and arise in a variety of ways besides the numerical solution of partial differential equations.

It is the property of being sparse that allows such large systems of equations to be solved on today’s computers with relative ease. Recall that in Chapter 4, we saw that Gaussian elimination requires on the order of n^3 arithmetic operations to solve an $n \times n$ linear system. Hence if a $10^4 \times 10^4$ linear system were “dense,” that is, few of its elements were zero, and Gaussian elimination were used to solve the system, then on the order of 10^{12} operations would be required. At a rate of 10^6 operations per second, it would require several hours to solve such a system. Moreover, for the corresponding three dimensional problem the size of the system would be 10^6 , requiring 10^{18} operations, which is completely beyond the capacity of the fastest computers. However, by utilizing the special structure and sparsity of systems such as (9.1.19), we shall see in the next two sections that they can be accurately solved relatively quickly and accurately, despite their large size.

The Heat Equation

We end this section by applying the discretization of Poisson’s equation to the heat equation (9.1.3) in two space variables where, again for simplicity in exposition, we will assume that the x, y domain is the unit square of Figure 9.3