EDOs (suite)
PVI

Runge-Kutta 4

$$a_{ij}$$

$$
c_i \left\{
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
\end{array}
\right.
$$

$$\underbrace{1/6 \quad 1/3 \quad 1/3 \quad 1/6}_{w_j}$$

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Runge-Kutta 4

$$k_1 = hf(x_i, y_i) \qquad = hf_1$$

$$k_2 = hf(x_i + c_2 h, y_i + a_{21}k_1) \qquad = hf_2$$

$$k_3 = hf(x_i + c_3 h, y_i + a_{31}k_1 + a_{32}k_2) \qquad = hf_3$$

$$k_4 = hf(x_i + c_4 h, y_i + a_{41}k_1 + a_{42}k_2 + a_{43}k_3) \qquad = hf_4$$

c_2: 1/2 $\quad a_{21}$: 1/2

c_3: 1/2 $\quad a_{31}$: 0 $\quad a_{32}$: 1/2

c_4: 1 $\quad a_{41}$: 0 $\quad a_{42}$: 0 $\quad a_{43}$: 1

$$y_{i+1} = y_i + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4$$

1/6 2/6 2/6 1/6

$$k_1 = hf\left(x_i, y_i\right)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right)$$

$$k_4 = hf\left(x_i + h, y_i + k_3\right)$$

Méthodes de R-K explicites

$$
\begin{array}{c|cccccc}
c_2 & a_{21} & & & & & \\
c_3 & a_{31} & a_{32} & & & & \\
c_4 & a_{41} & a_{42} & a_{43} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
c_m & a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m,m-1} & \\
\hline
 & w_1 & w_2 & w_3 & \cdots & w_{m-1} & w_m
\end{array}
$$

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Méthodes de R-K implicites

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & a_{12} & & & \cdots & a_{1,m} \\
c_2 & a_{21} & a_{22} & & & \cdots & a_{2,m} \\
c_3 & a_{31} & a_{32} & a_{33} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
c_m & \cdots & & & & \cdots & \cdots \\
 & a_{m1} & a_{m2} & & & & a_{m,m} \\
\hline
 & w_1 & w_2 & w_3 & \cdots & w_{m-1} & w_m
\end{array}
$$

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Méthode d'Euler explicite

$$f(X,t) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

La solution exacte est un cercle:

$$X(t) = \begin{pmatrix} r\cos(t+k) \\ r\sin(t+k) \end{pmatrix}$$

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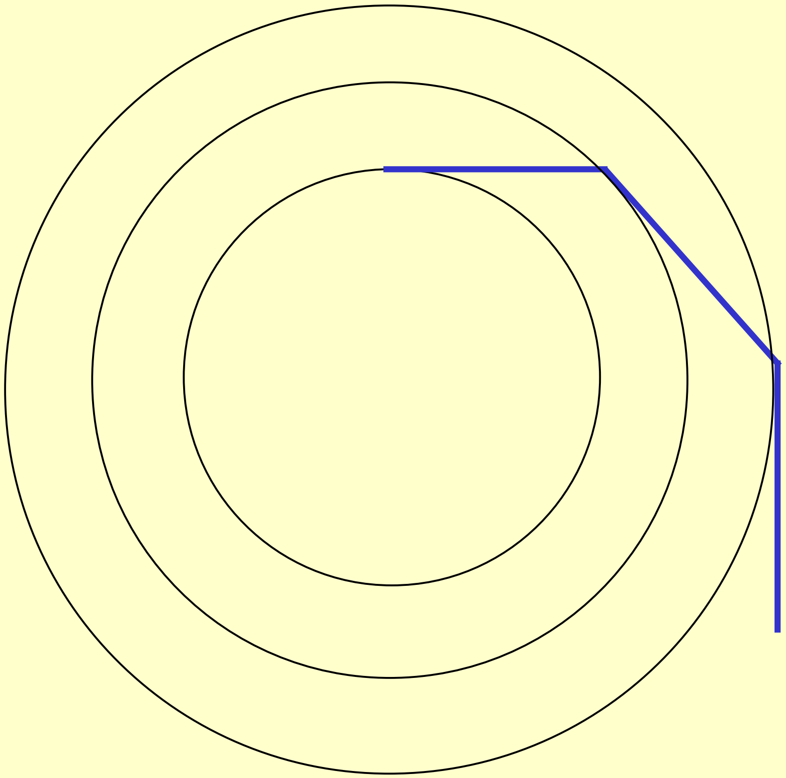
La Spirale d'Euler

La solution exacte est un cercle:



La méthode d'Euler engendre des spirales

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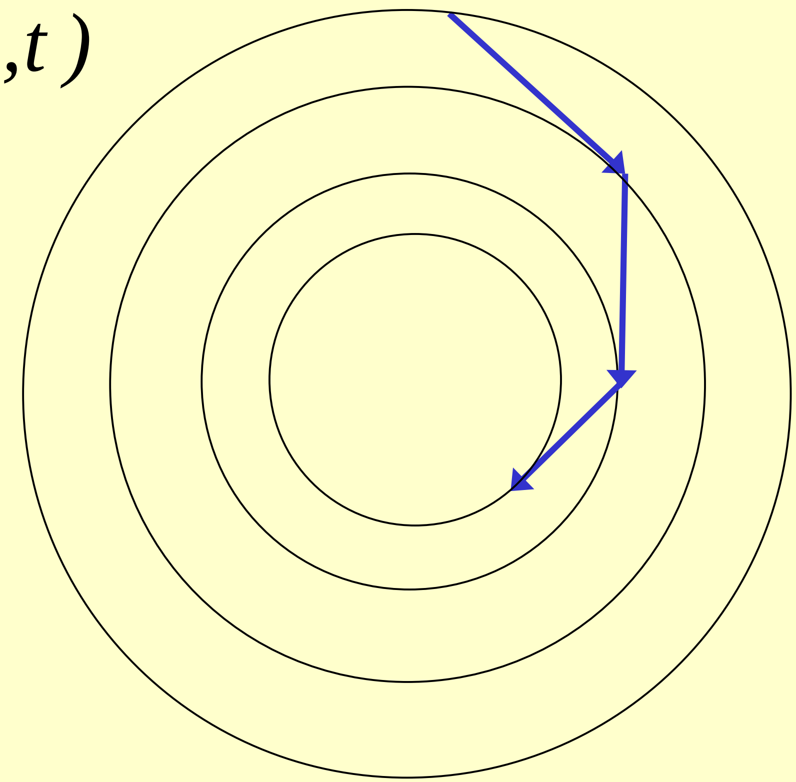
Euler Implicite

$$\frac{dX}{dt} = f(X,t)$$



$$f(X,t) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$X(t) = \begin{pmatrix} r\cos(t+k) \\ r\sin(t+k) \end{pmatrix}$$

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Euler: Explicite vs. Implicite

$$\frac{dy}{dt} = f(t, y)$$

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad \textbf{\textit{vs.}} \qquad y_{i+1} = y_i + f(\underline{x_{i+1}, y_{i+1}})h$$

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Euler Explicite $\frac{dx}{dt} = -kx$

$$x(t) = x_0 e^{-kt}$$

En général

$$x_{i+1} = x_i + hf(t+h, x_i)$$

Pour ce cas

$$x_{i+1} = x_i - hkx_i \quad \Rightarrow \quad \begin{aligned} x_1 &= (1-kh)x_0 \\ x_i &= (1-kh)^i x_0 \end{aligned}$$

$$|1 - hk| \le 1$$ Contrainte

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Stabilité

$$\left.\begin{aligned} \frac{d\boldsymbol{y}}{d\boldsymbol{t}} &= -\boldsymbol{a}\boldsymbol{y} \\ \boldsymbol{y}(0) &= \boldsymbol{y}_0 \end{aligned}\right\} \rightarrow y = y_0 \boldsymbol{e}^{-\boldsymbol{a}\boldsymbol{t}}$$

$$si\ y(0) = y_0 + \boldsymbol{\varepsilon},\quad alors\ \ y^* = (\boldsymbol{y}_0 + \boldsymbol{\varepsilon})\boldsymbol{e}^{-\boldsymbol{a}\boldsymbol{t}}$$

$$soit\ E(x) = y^*(t) - y(t) \ \Rightarrow \begin{cases} \boldsymbol{dE}/\boldsymbol{dt} = -\boldsymbol{a}\boldsymbol{E} \\ \boldsymbol{E}(0) = \boldsymbol{\varepsilon} \end{cases}$$

$$E = \boldsymbol{\varepsilon} e^{-\boldsymbol{a}\boldsymbol{t}} \Rightarrow \begin{cases} \boldsymbol{a} < 0\text{: l'erreur croit exponentiellement} \rightarrow \text{instable} \\ \boldsymbol{a} = 0\text{: stable (neutre)} \\ \boldsymbol{a} > 0\text{: l'erreur decroit exponentiellement} \rightarrow \text{stable} \end{cases}$$

Euler explicite

$$\frac{dy}{dt} = f(x,y) = -ay\,; \quad y(0) = y_0$$

$$y_{i+1} = y_i + \frac{dy_i}{dt}h = y_i + (-ay_i)h = (1-ah)y_i$$

Condition de stabilité

$$\left|\frac{y_{i+1}}{y_i}\right| \le 1 \quad ou \quad |1-ah| \le 1$$

Region de stabilité absolue

$$-1 \le 1 - ah \le 1 \qquad \Rightarrow \boxed{0 \le ah \le 2}$$

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Euler implicite

$$\frac{dy}{dt} = f(x,y) = -ay \;; \quad y(0) = y_0$$

$$y_{i+1} = y_i + \frac{dy_{i+1}}{dt} h \;; \quad \frac{dy_{i+1}}{dt} = -ay_{i+1}$$

$$y_{i+1} = \frac{y_i}{1+ah}$$

$$\left|\frac{y_{i+1}}{y_i}\right| = \left|\frac{1}{1+ah}\right| \leq 1 \quad pour\ tout\ h$$

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Inconditionellement stable !

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Régions de Stabilité

Euler explicite

$$0 \le ah \le 2$$

$$y_{i+1} = (1 - ah)\, y_i$$

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Questions

- Stabilité: la solution est bornée, mais est-elle semblable à la solution réelle?

- Précision: quel Δt (h) utiliser pour obtenir une erreur en dessous d'un certain seuil?

Euler Implicite $\dfrac{dy}{dt} = -ky$

$$y(t) = y_0 e^{-kt}$$

$$y_{i+1} = y_i - hky_{i+1} \Longrightarrow \boxed{y_{i+1} = \left(\frac{1}{1+hk}\right) y_i}$$

Stable, pas de limite pour h, mais on a la même précision que celle de la méthode explicite

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Euler implicite

En général

$$y_{i+1} = y_i + hf\,(t+h, y_i + \Delta y)$$

$$f\,(t+h, y_i + \Delta y) = f\,(t+h, y_i) + \frac{df}{dy}\Delta y \quad \text{Taylor}$$

$$\Delta y = y_{i+1} - y_i = h\left(f\,(t+h, y_i) + \frac{df}{dy}\Delta y \right)$$

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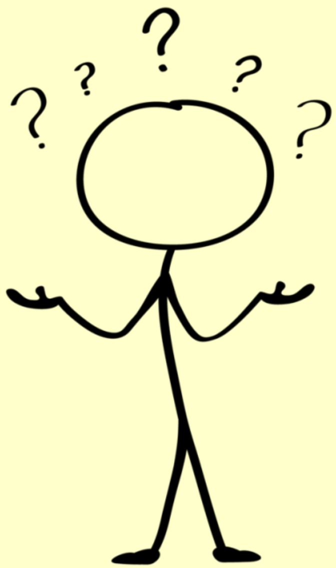
Euler implicite

$$\boxed{\Delta y = \frac{hf(t+h, y_i)}{1 - h\dfrac{df}{dy}}}$$



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Systèmes

$$y' = Ay \qquad f(t,y) = Ay$$

$$y_{k+1} = y_k + Ay_{k+1}h_k$$

$$\boxed{(I - h_k A)y_{k+1} = y_k}$$

On doit résoudre un système à chaque itération

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Méthodes R-K implicites

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & a_{12} & & & \cdots & a_{1,m} \\
c_2 & a_{21} & a_{22} & & & \cdots & a_{2,m} \\
c_3 & a_{31} & a_{32} & a_{33} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
c_m & \cdots & & & & \cdots & \cdots \\
 & a_{m1} & a_{m2} & & & & a_{m,m} \\
\hline
 & w_1 & w_2 & w_3 & \cdots & w_{m-1} & w_m
\end{array}
$$

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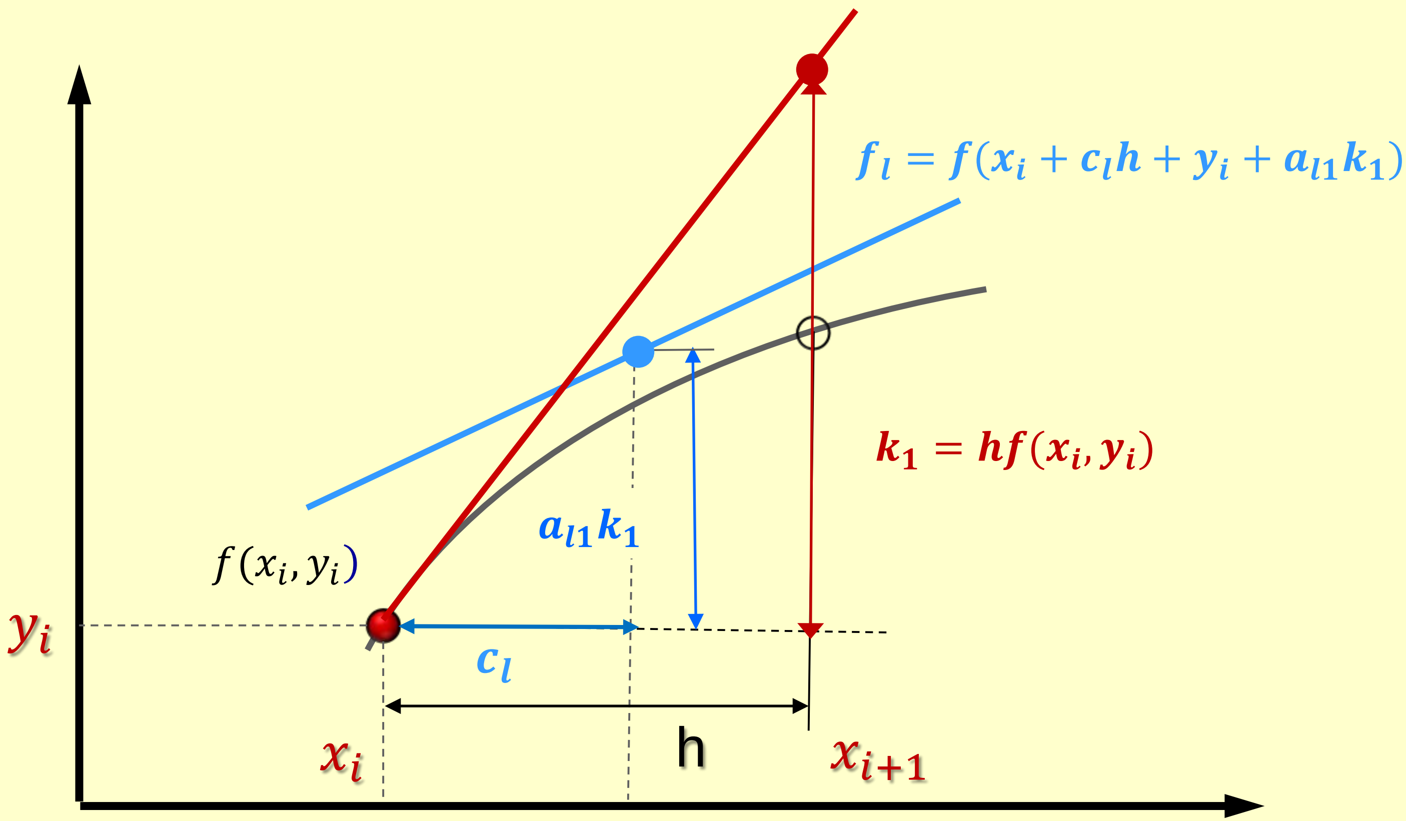
Position c_l



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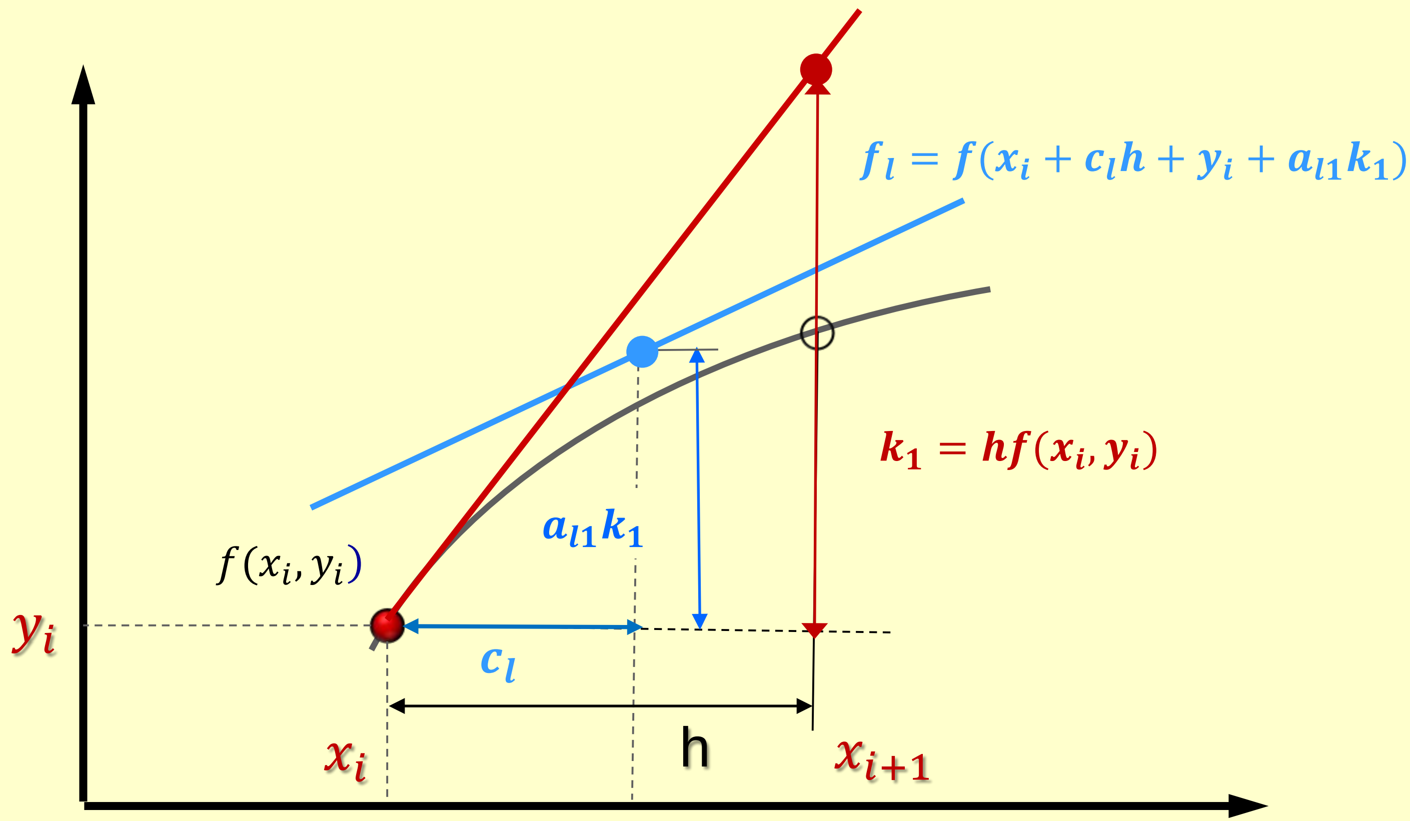
Position c_l



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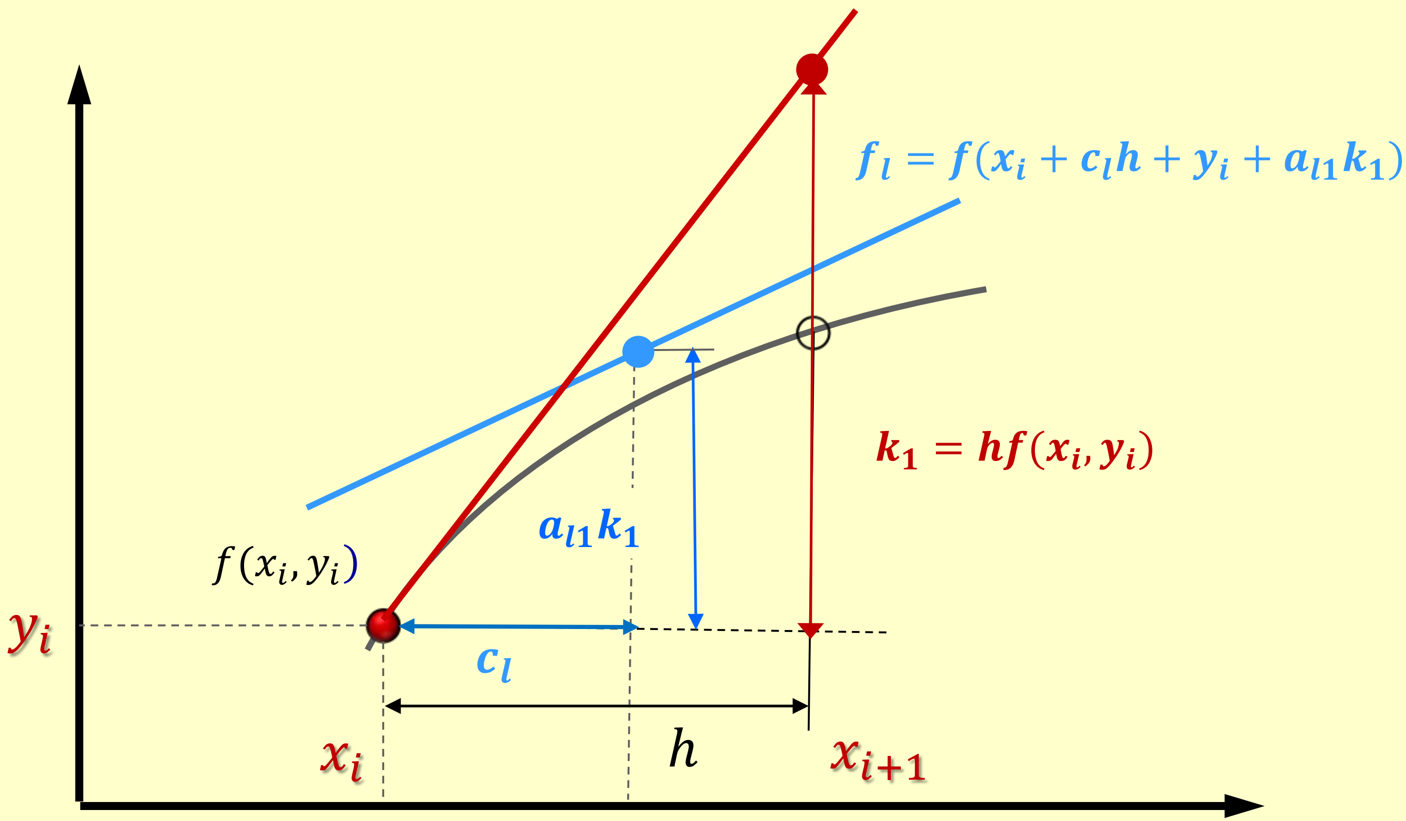
Deux pentes pondérées



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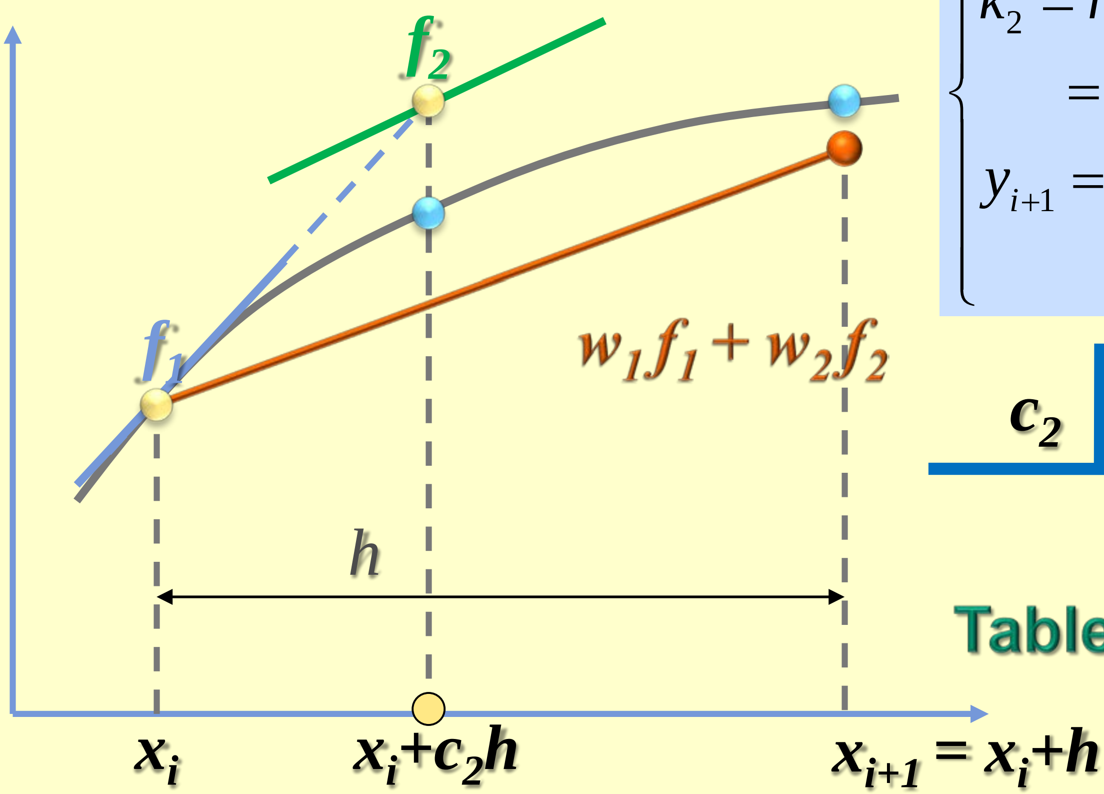
Deux pentes pondérées



$$\begin{cases} k_1 = hf(x_i, y_i) = hf_1 \\ k_2 = hf(x_i + c_2 h, y_i + a_{21} k_1) \\ \qquad = hf_2 \\ y_{i+1} = y_i + w_1 k_1 + w_2 k_2 \\ \qquad = y_i + h(w_1 f_1 + w_2 f_2) \end{cases}$$

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \end{array}$$

Tableau de Butcher

Formulation générale

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & a_{12} & & & \cdots & a_{1,m} \\
c_2 & a_{21} & a_{22} & & & \cdots & a_{2,m} \\
c_3 & a_{31} & a_{32} & a_{33} & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
c_m & \cdots & & & & \cdots & \cdots \\
 & a_{m1} & a_{m2} & & & & a_{m,m} \\
\hline
 & w_1 & w_2 & w_3 & \cdots & w_{m-1} & w_m
\end{array}
$$

$$\boxed{\begin{aligned} k_l &= hf\left(x_i + c_l h, \; y_i + \sum_{m=1}^{s} a_{lm} k_m \right) \\ y_{i+1} &= y_i + \sum_{l=1}^{s} b_l k_l \end{aligned}}$$

$$k_l = \Delta y_l$$

$$b_l = w_l$$

$$c_m = \sum_{j=1}^{s} a_{mj} \qquad m = 1,2 \ldots s$$

$$\sum_{l=1}^{s} b k_l = w_1 \Delta y_1 + w_2 \Delta y_2 + \cdots w_s \Delta y_s$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

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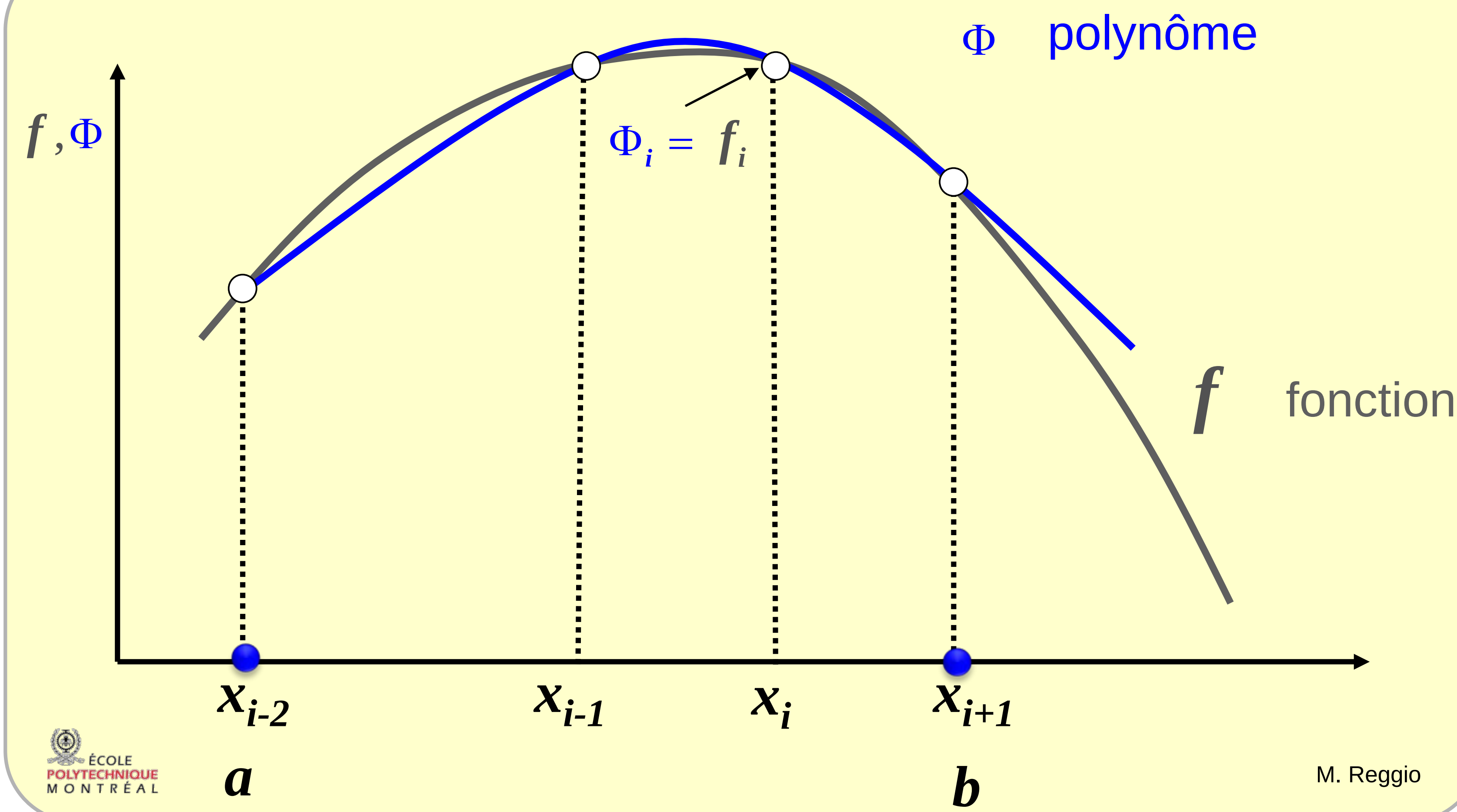
Approximation de l'intégrale



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Collocation

$$\underbrace{}_{}$$

quadrature

$$I = \int_a^b f(x)dx$$

$$\Phi_j = f_j$$

$$L_j = \prod_{i=1}^{s} \frac{(x - x_i)}{(x_j - x_i)}$$ ***polynôme de Lagrange***

$$w_j = \int_a^b L_j dx$$

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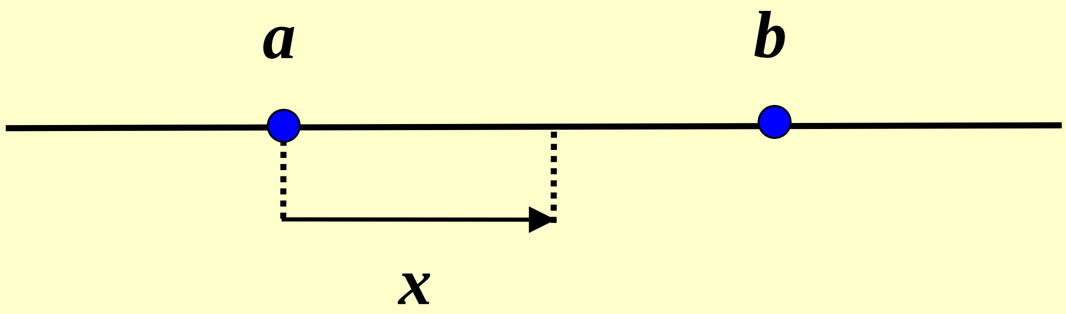
Collocation

$$I = \int_a^b f(x)dx \approx \sum_{j=1}^{s} w_j f(x_j)$$



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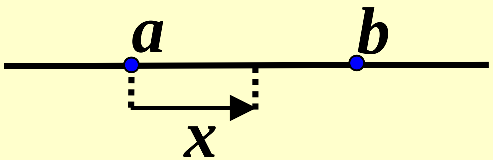
Collocation

$$L_j = \prod_{i=1}^{s} \frac{(x - x_i)}{(x_j - x_i)}$$

Exemple: polynôme linéaire



$$L_1 = \frac{(x-b)}{(a-b)} \qquad L_2 = \frac{(x-a)}{(b-a)}$$

$$w_1 = \int_a^b \frac{(x-b)}{(a-b)} dx = \left.\frac{(x-b)^2}{2(a-b)}\right|_a^b = \frac{1}{2}(b-a)$$

$$\boxed{I \approx w_1 f_1 + w_2 f_2 = \frac{b-a}{2}\left(f(a) + f(b)\right)}$$

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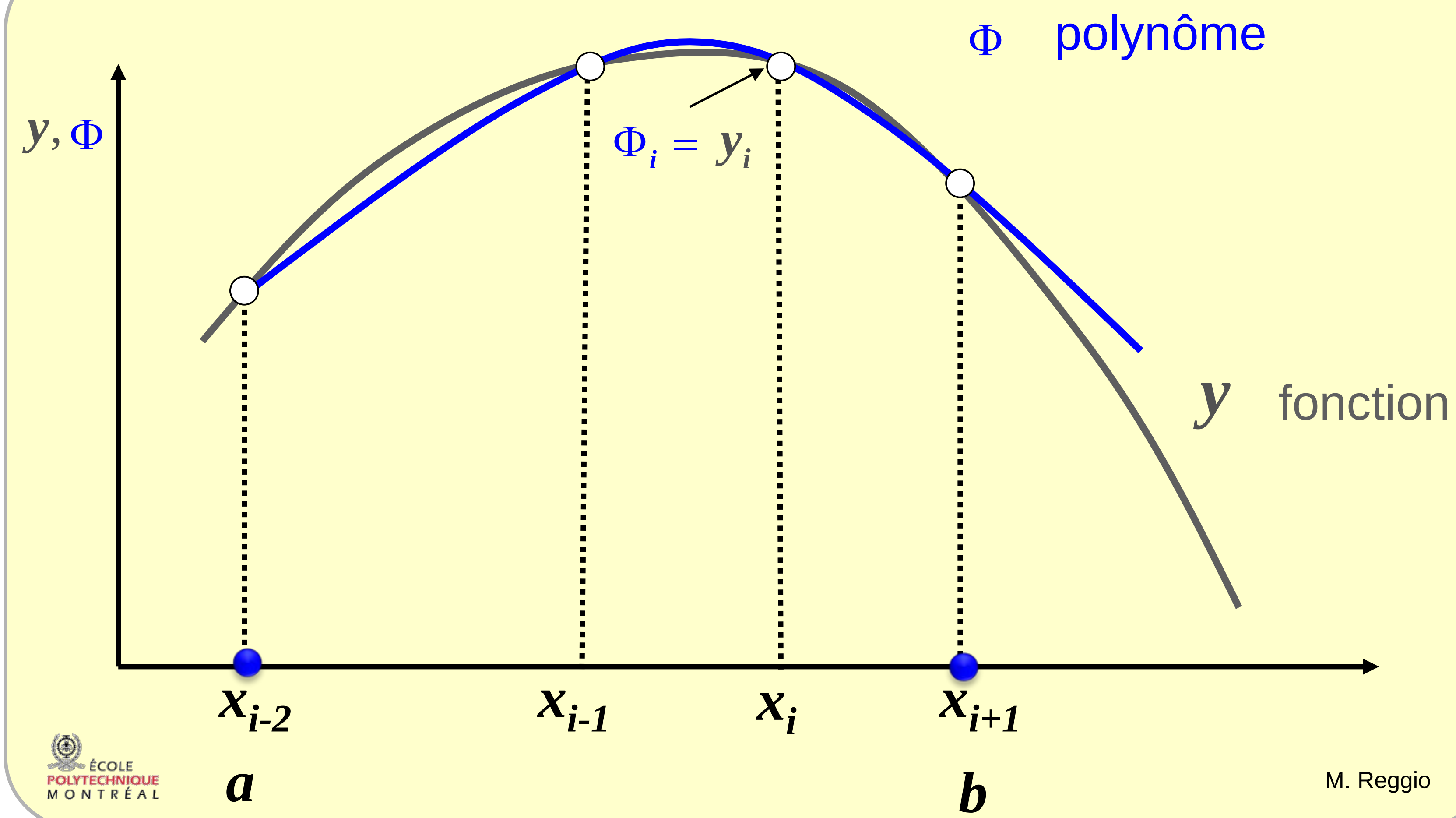

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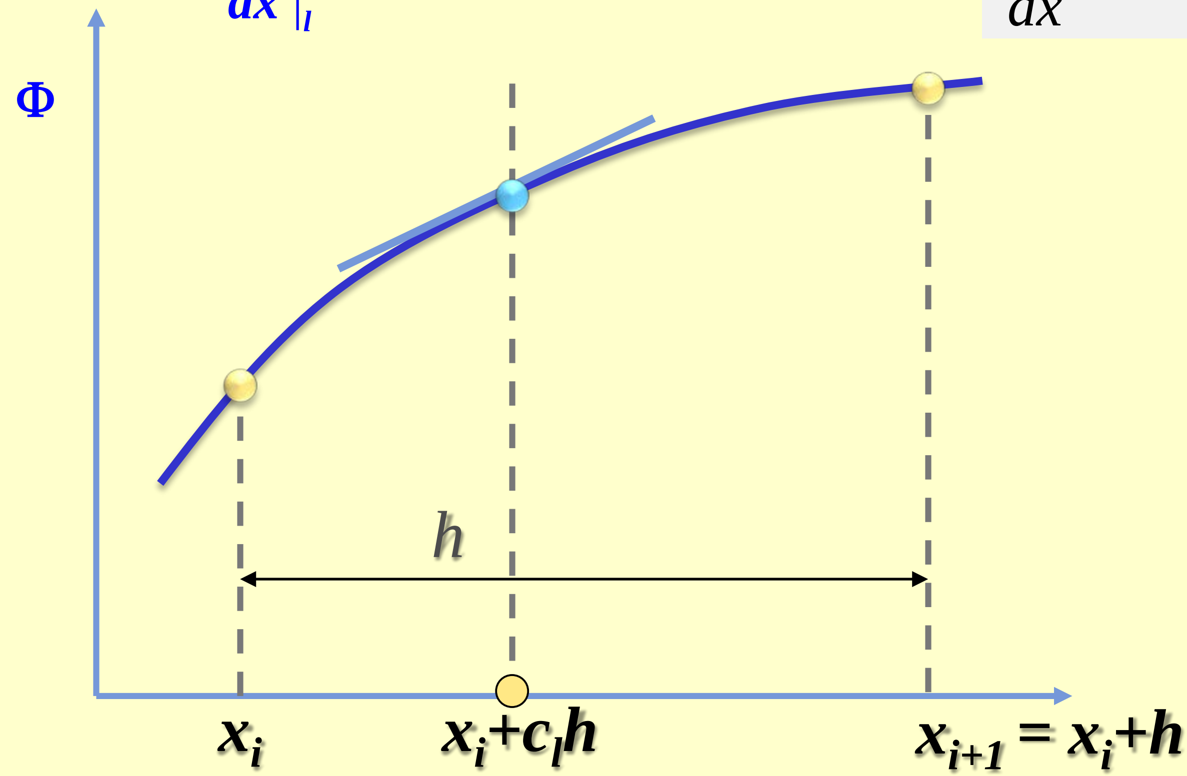
$$\Phi'_l = \left.\frac{d\Phi}{dx}\right|_l = f\left(\mathbf{x}_i + \mathbf{c}_l\mathbf{h},\ \Phi(\mathbf{x}_i + \mathbf{c}_l\mathbf{h})\right)$$

$$\frac{d\Phi}{dx} = f(x, \Phi)$$



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Quadrature et RK

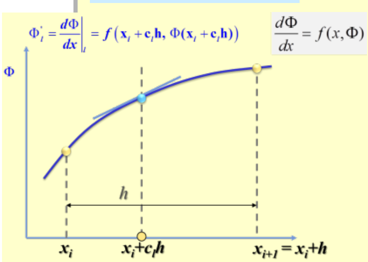
$$\frac{dy}{dx} = f(x,y)$$

$$\frac{d\Phi}{dx} = f(x,\Phi)$$

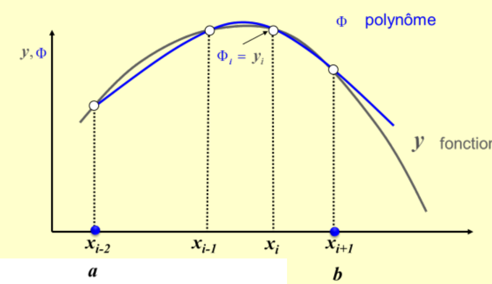
$$\phi(x_{i+1}) = y_{i+1}, \quad \phi(x_i) = y_i, \quad x_{i+1} = x_i + h$$

$$x_m$$

$$K_m = \phi'(x_i + c_m h) = f(x_i + c_m h, \quad \phi(x_i + c_m h)), m = 1, 2..$$



$$K_m = \phi'(x_i + c_m h)$$



$$\int \Rightarrow \quad \Phi'(x) = h \sum_{m=1}^{s} L_m(c) \, K_m$$

Quadrature de Lagrange pour *Φ'*

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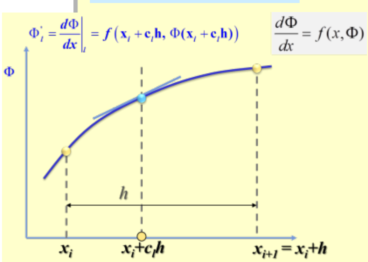
Quadrature et RK

$$\frac{dy}{dx} = f(x, y)$$

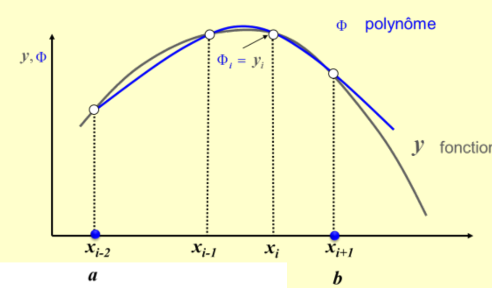
$$\frac{d\Phi}{dx} = f(x, \Phi)$$

$$\phi(x_{i+1}) = y_{i+1}, \quad \phi(x_i) = y_i, \quad x_{i+1} = x_i + h$$

$$K_m = \phi'(\; x_m \;) = f(\; x_m \; , \; \phi(\; x_m \;)), m = 1, 2..$$



$$K_m = \phi'(\; x_m \;)$$



$\int$ ➡

$$\Phi'(x) = \sum_{m=1}^{s} L_m(c) \quad K_m$$

Quadrature de Lagrange pour *Φ'*

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$$\Phi'(x) = \sum_{m=1}^{s} L_m(c) \quad K_m$$

x_m

$$K_m = \phi'(x_i + c_m h)$$

$$L_m(c) = \prod_{l \neq m} \frac{(c - c_l)}{(c_m - c_l)} \quad 1 \leq l \leq s$$

$$\phi(x_i + c_l h) - \phi(x_i) = \sum_{m=1}^{s} \left(\int_0^{c_l} L_m(c)\,dc \right) K_m$$

y_l $\quad y_i \quad a_{lm}$

$$\phi(x_{i+1}) - \phi(x_i) = \sum_{m=1}^{s} \left(\int_0^{1} L_m(c)\,dc \right) K_m$$

y_{i+1} $\quad y_i \quad b_l$

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$$\Phi'(x) = \sum_{m=1}^{s} L_m(c) \quad K_m$$

$$K_m = \phi'(\quad x_m \quad)$$

$$\int$$

$$L_m(c) = \prod_{l \neq m} \frac{(c - c_l)}{(c_m - c_l)} \quad 1 \leq l \leq s$$

$$\phi(\quad x_l \quad) - \phi(x_i) = \sum_{m=1}^{s} \left(\int_0^{c_l} L_m(c)dc \right) K_m$$

y_l y_i a_{lm}

$$\phi(x_{i+1}) - \phi(x_i) = \sum_{m=1}^{s} \left(\int_0^{1} L_m(c)dc \right) K_m$$

y_{i+1} y_i b_l

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$$a_{lm} = \int_0^{c_l} L_m(c)\,dc \qquad b_m = \int_0^{1} L_m(c)\,dc \qquad l, m = 1, \ldots$$

$$k_l = \Delta y_l$$

y_l

$x_l = x_i + c_l h$

❶ $$k_l = hf\left(x_l, \overbrace{y_i + \sum_{m=1}^{s} a_{lm} k_m}^{x_i + c_l h} \right)$$

❷ $$y_{i+1} = y_i + \sum_{l=1}^{s} b_l k_l$$

l: points d'intégration

Pondération interne

$$\sum_{l=1}^{s} bk_l = w_1 \Delta y_1 + w_2 \Delta y_2 + \cdots w_s \Delta y_s$$

Pondération finale des Δy_l

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Un point de Gauss:s=1,c=1/2

$L_1 = 1$

$$a_{11} = \int_0^{1/2} 1\, dx = \frac{1}{2} \qquad b_1 = \int_0^1 1\, dx = 1$$

$L_1 = 1$

$$\begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} c_1 & a_{11} \\ \hline & b_1 \end{array}$$

Méthode *implicite* du point milieu

$$a_{lm} = \int_0^{c_l} L_m(c)dc \qquad b_m = \int_0^1 L_m(c)dc$$

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Un point de Radau:s=1,c=1

$$a_{11} = \int_0^{1} 1dx = 1 \qquad b_1 = \int_0^1 1dx = 1$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} c_1 & a_{11} \\ \hline & b_1 \end{array}$$

$L_1 = 1$

Méthode d'Euler *implicite*

$$a_{lm} = \int_0^{c_l} L_m(c)dc \qquad b_m = \int_0^1 L_m(c)dc$$

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Un point de Radau:s=1,c=1

$$a_{11} = \int_0^1 1\, dx \;= 1 \qquad b_1 = \int_0^1 1\, dx \;= 1$$

$$L_1 = 1$$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} c_1 & a_{11} \\ \hline & b_1 \end{array}$$

Méthode d'Euler *implicite*

$$a_{lm} = \int_0^{c_l} L_m(c)dc \qquad b_m = \int_0^1 L_m(c)dc$$

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Deux points de Lobatto $c_1 = 0, c_2 = 1$

$$a_{21} = \int_0^{c_2=1} \underbrace{(1-x)}_{L_1} dx = 1/2$$

$$b_1 = \int_0^1 (1-x)dx = 1/2$$

$$a_{22} = \int_0^{c_2=1} \underbrace{x}_{L_2} dx = 1/2$$

$$b_2 = \int_0^1 xd = 1/2$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array} \Longleftarrow \begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

Méthode trapezoïdale *implicite*

$$a_{lm} = \int_0^{c_l} L_m(c)dc \qquad b_m = \int_0^1 L_m(c)dc$$

Euler implicite

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1})$$

$$F_i^k = \boldsymbol{y_{i+1}^k} - y_i^k - hf(\boldsymbol{y_{i+1}^k}, t_{i+1})$$

$$\frac{\partial F}{\partial y} = 1 - h\frac{\partial f}{\partial y}$$

MÉTHODE

de

NEWTON

$$\begin{aligned} \left(1 - h\frac{\partial f}{\partial y}\right)\delta y_{i+1} &= -F(y_{i+1}^k) \\ F_i^k = \boldsymbol{y_{i+1}^k} - y_i^k - hf(\boldsymbol{y_{i+1}^k}, t_{i+1}) & \\ y_{i+1}^{k+1} &= y_{i+1}^k + \delta y_{i+1} \end{aligned}$$

ÉCOLE POLYTECHNIQUE MONTRÉAL

M. Reggio

Méthode Trapezoïdale

$$y_{i+1} = y_i + \frac{h}{2} f(y_{i+1,} t_{i+1}) + \frac{h}{2} f(y_i, t_i)$$

Explicite

$$y_{i+1} - \frac{h}{2} f(\underbrace{y_i + hf(y_i, t_i)}_{\approx\, y_{i+1}}, t_{i+1}) = y_i + \frac{h}{2} f(y_i, t_i)$$

Implicite

$$y_{i+1} - \frac{h}{2} f(y_{i+1}, t_{i+1}) = y_i + \frac{h}{2} f(y_i, t_i)$$

$$F_i^k = y_{i+1}^k - y_i^k - \frac{h}{2} f(y_i^k, t_i) - \frac{h}{2} f(y_{i+1}^k, t_{i+1})$$

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Méthode Trapezoïdale

MÉTHODE

de

NEWTON

$$\left(1 - h\frac{\partial f}{2\partial y}\right)\delta y_i = -F_i^k$$

$$y_{i+1}^{k+1} = y_i^k + \delta y_i$$

$$F_i^k = y_{i+1}^k - y_i^k - \frac{h}{2} f(y_i^k, t_i) - \frac{h}{2} f(y_{i+1}^k, t_{i+1})$$

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Méthode du point milieu

Explicite

$$y_{i+1} = y_i + hf\left(\frac{y_i}{2} + \frac{1}{2}\underbrace{f(y_i + hf(y_i, t_i))}_{\approx y_{i+1}}, t_i + \frac{h}{2}\right)$$

Implicite

$$y_{i+1} = y_i + hf\left(\frac{y_i}{2} + \frac{y_{i+1}}{2}, t_i + \frac{h}{2}\right)$$

$$F_i^k = y_{i+1}^k - y_i^k - hf\left(\frac{y_i}{2} + \frac{y_{i+1}}{2}, t_i + \frac{h}{2}\right)$$

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Méthode du point milieu

$$y_{i+1} = y_i + hf\left(\frac{y_i}{2} + \frac{y_{i+1}}{2}, t_i + \frac{h}{2}\right)$$

$$\left[1 - h\frac{\partial f\left(\frac{y_i}{2} + \frac{y_{i+1}}{2}, t_i + \frac{h}{2}\right)}{\partial y}\right]\delta y_i = -F_i^k$$

$$y_{i+1} = y_i + \delta y_i$$

$$\boldsymbol{F_i^k = y_{i+1}^k - y_i^k - hf\left(\frac{y_i}{2} + \frac{y_{i+1}}{2}, t_i + \frac{h}{2}\right)}$$

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Deux points c_1 et c_2

$$Y_1 = y_i + ha_{11} f(x_i + c_1 h, Y_1) + ha_{12} f(x_i + c_2 h, Y_1)$$

$$Y_2 = y_i + ha_{21} f(x_i + c_1 h, Y_2) + ha_{22} f(x_i + c_2 h, Y_2)$$

$$\boxed{\begin{aligned} k_l &= hf\left(x_l , \boxed{y_i + \sum_{m=1}^{s} a_{lm} k_m} \right) \\ y_{i+1} &= y_i + \sum_{l=1}^{s} b_l k_l \end{aligned}}$$

$$Y_l, \quad l = 1, 2$$

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Système 2x2

$$F_1 = Y_1 - y_i - ha_{11} f(x_i + c_1 h, Y_1) - ha_{12} f(x_i + c_2 h, Y_2)$$

$$F_2 = Y_2 - y_i - ha_{21} f(x_i + c_1 h, Y_1) - ha_{22} f(x_i + c_2 h, Y_2)$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \Rightarrow \begin{aligned} \boldsymbol{J}\,\delta\boldsymbol{Y}^k &= -\boldsymbol{F}(\boldsymbol{Y}^k) \\ \boldsymbol{Y}^{k+1} &= \boldsymbol{Y}^k + \delta\boldsymbol{Y}^k \end{aligned} \Leftarrow F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

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$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \delta Y_1 \\ \delta Y_2 \end{bmatrix} = -\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Boucle de Newton

$$\begin{bmatrix} 1 - ha_{11} & -ha_{12} \\ -ha_{21} & 1 - ha_{22} \end{bmatrix} \begin{bmatrix} \delta Y_1 \\ \delta Y_2 \end{bmatrix} = -\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \qquad (1)$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^{(k+1)} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}^{(k)} + \begin{bmatrix} \delta Y_1 \\ \delta Y_2 \end{bmatrix} \qquad (2)$$

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Autres méthodes

Runge-Kutta

- un pas à la fois
- utilisent des <u>valeurs intermédiares entre</u> x_i et x_{i+1}
- plusieurs évaluations par pas

Multi-pas

- <u>utilisent plusieurs points</u> en x_i , x_{i-1} , $x_{i-2,}$ etc
- une seule évaluation par point

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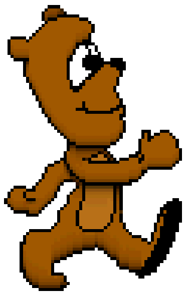
Méthodes Multi-pas



Méthodes multi-pas

L'idée derrière les mèthodes multi-pas est l'utilisation de valeurs de *y* et/ou de *f*, soit *dy/dx*, pour construir un polynôme Φ pour approcher la fonction

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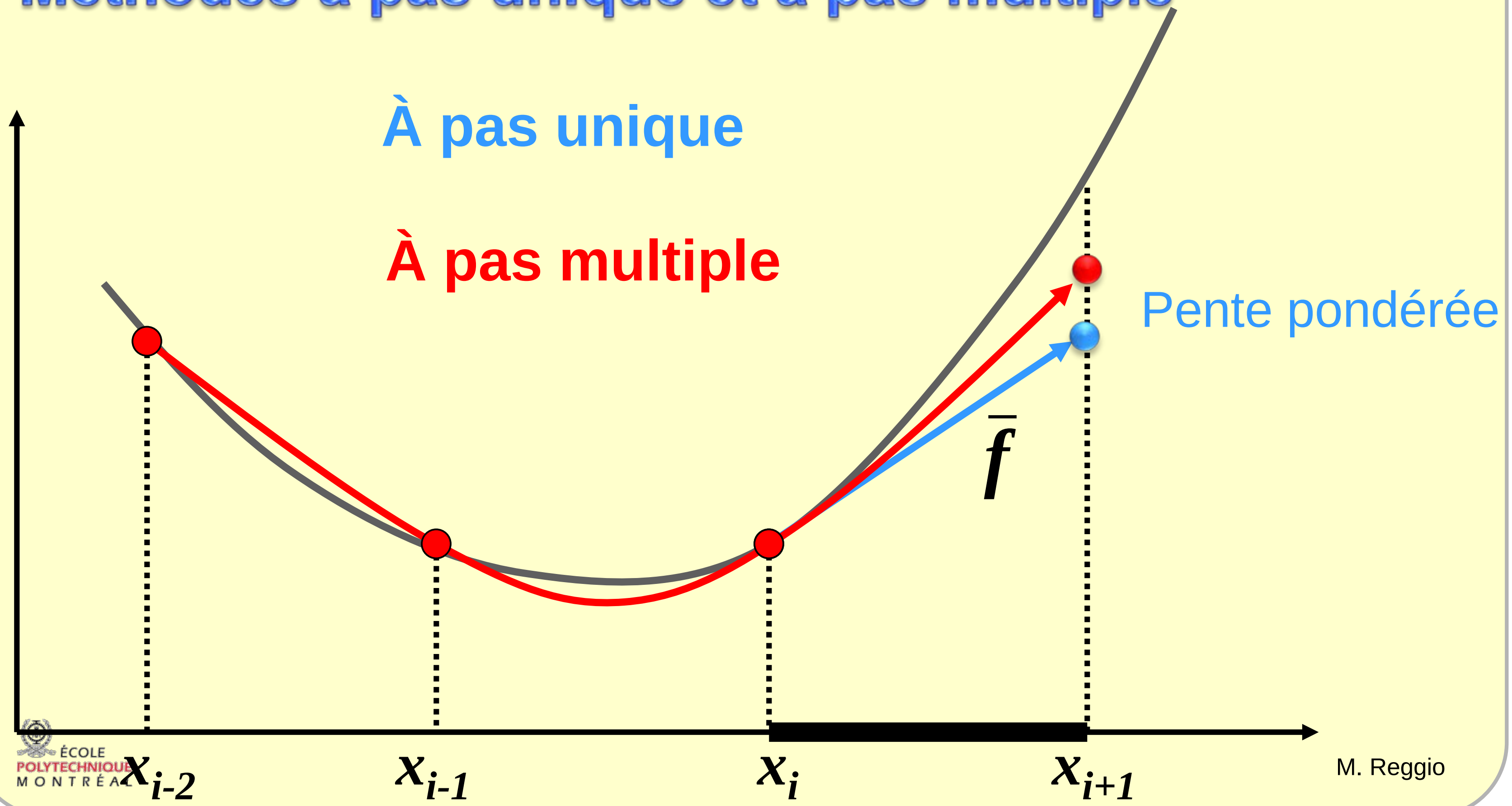
Comparaison

Méthodes à pas unique et à pas multiple



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Adams-Bashforth

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Développement classique

Série de Taylor

$$y_{i+1} = y_i + h \left.\frac{dy}{dx}\right|_i + \left.\frac{h^2}{2}\frac{d^2y}{dx^2}\right|_i + \cdots \left.\frac{h^n}{n!}\frac{d^ny}{dx^n}\right|_i$$

$$\Rightarrow$$

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2}f_i' + \cdots \frac{h^n}{n!}f_i^{n-1}$$

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Formule d'ordre deux

$$\boxed{y_{i+1} = y_i + hf_i + \frac{h^2}{2} f_i'} + \theta(h^3)$$

$$f_i' = \frac{f_i - f_{i-1}}{h} + \theta(h)$$

$$\boxed{y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})}$$

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Développement en arrière

$$y_i = y_{i+1} - hf_{i+1} + \frac{h^2}{2} f'_{i+1} - \frac{h^3}{6} f''_{i+1} + \cdots$$

$$y_{i+1} = y_i + hf_{i+1} - \frac{h^2}{2} f'_{i+1} + \frac{h^3}{6} f''_{i+1} + \cdots$$

Formule d'ordre deux

$$\boxed{y_{i+1} = y_i + h f_{i+1} - \frac{h^2}{2} f'_{i+1}} + \theta(h^3)$$

$$f'_{i+1} = \frac{f_{i+1} - f_i}{h} + \theta(h)$$

$$y_{i+1} = y_i + \frac{h^2}{2}(f_{i+1} + f_i)$$

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Formule générale

$$y_{i+1} = y_i + h\sum_{k=0}^{n} \beta_{nk} f_{i-k} + O(h^{n+2})$$

Adams-Moulton

Formule générale

$$y_{i+1} = y_i + h\sum_{k=0}^{n} \breve{\beta}_{nk}\, f_{i+1-k} + O(h^{n+2})$$

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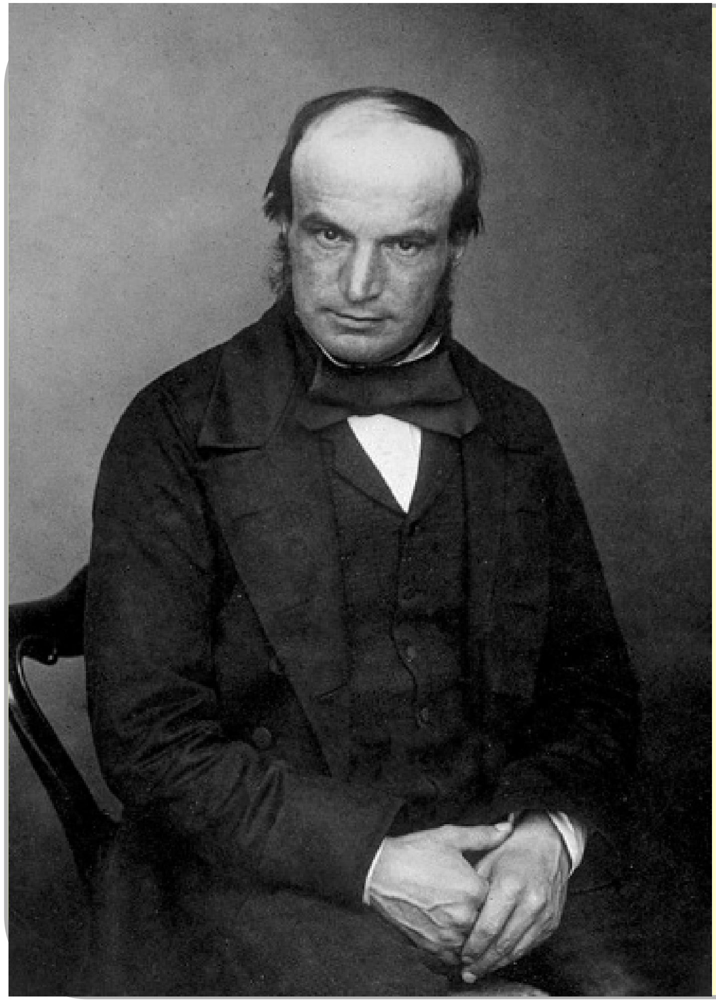
John C. Adams(1819-1892)



Francis Bashforth (1819-1912)

An Attempt to Test the Theories of Capillary Action By Comparing the Theoretical and Measured Forms of Drops of Fluid

Francis Bashforth 1819-1912

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THE THEORETICAL AND MEASURED FORMS OF DROPS OF FLUID,

BY

FRANCIS BASHFORTH, B.D.

LATE PROFESSOR OF APPLIED MATHEMATICS TO THE ADVANCED CLASS
OF ROYAL ARTILLERY OFFICERS, WOOLWICH,
AND FORMERLY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

WITH

AN EXPLANATION OF THE METHOD OF INTEGRATION
EMPLOYED IN CONSTRUCTING THE TABLES WHICH GIVE THE THEORETICAL
FORMS OF SUCH DROPS,

BY

J. C. ADAMS, M.A, F.R.S.

FELLOW OF PEMBROKE COLLEGE, AND LOWNDEAN PROFESSOR OF ASTRONOMY AND GEOMETRY
IN THE UNIVERSITY OF CAMBRIDGE.

Cambridge:
AT THE UNIVERSITY PRESS.
1883

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Adams-Bashforth

Méthode à deux pas

$$y_{n+1} = y_n + h\left[\frac{3}{2} f\left(t_n, y_n\right) - \frac{1}{2} f\left(t_{n-1}, y_{n-1}\right)\right]$$

Pour démarrer, y_1 doit être calculée par une autre méthode, avec celle de Runge-Kutta, par exemple

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Adams-Bashforth

À trois pas

$$y_{n+1} = y_n + \frac{h}{12}\left[23 f\left(t_n, y_n\right) - 16 f\left(t_{n-1}, y_{n-1}\right) + 5 f\left(t_{n-2}, y_{n-2}\right)\right]$$

À quatre pas

$$y_{n+1} = y_n + \frac{h}{24}\left[55 f\left(t_n, y_n\right) - 59 f\left(t_{n-1}, y_{n-1}\right) + 37 f\left(t_{n-2}, y_{n-2}\right) - 9 f\left(t_{n-3}, y_{n-3}\right)\right]$$

Adams-Bashforth

$$\begin{cases} 2\ pas : y_{i+1} = y_i + \dfrac{h}{2}(3 f_i - f_{i-1}) \\ 3\ pas : y_{i+1} = y_i + \dfrac{h}{12}(23 f_i - 16 f_{i-1} + 5 f_{i-2}) \\ 4\ pas : y_{i+1} = y_i + \dfrac{h}{24}(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}) \\ 5\ pas : y_{i+1} = y_i + \dfrac{h}{720}(1901 f_i - 2774 f_{i-1} + 2616 f_{i-2} - 1274 f_{i-3} + 251 f_{i-4}) \end{cases}$$

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Méthodes d'Adams-Bashforth

$ordre$	b_0	b_1	b_2	b_3	b_4	b_5	$err.\ de\ troncature$
1	1						$\frac{1}{2}h^2 f'(\eta)$
2	$\frac{3}{2}$	$-\frac{1}{2}$					$\frac{5}{12}h^3 f''(\eta)$
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$				$\frac{9}{24}h^4 f'''(\eta)$
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$			$\frac{251}{720}h^5 f^{(4)}(\eta)$
5	$\frac{1901}{720}$	$-\frac{2774}{720}$	$\frac{2616}{720}$	$-\frac{1274}{720}$	$\frac{251}{720}$		$\frac{475}{1440}h^6 f^{(5)}(\eta)$
6	$\frac{4277}{720}$	$-\frac{7923}{720}$	$\frac{9982}{720}$	$-\frac{7298}{720}$	$\frac{2877}{720}$	$-\frac{475}{720}$	$\frac{19087}{60480}h^7 f^{(6)}(\eta)$

Forest Ray Moulton (1872-1952)

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Adams-Moulton

Adams-Moulton à 1 pas (Euler modifiée: Heun)

$$\boldsymbol{y_{n+1}} = y_n + \frac{h}{2}[f(\boldsymbol{t_{n+1}}, \boldsymbol{y_{n+1}}) + f(t_n, y_n)]$$

Adams-Moulton à 2 pas

$$\boldsymbol{y_{n+1}} = y_n + \frac{h}{12}[5f(\boldsymbol{t_{n+1}}, \boldsymbol{y_{n+1}}) + 8f(t_n, y_n) - f(t_{n-1}, y_{n-1})]$$

Adams-Moulton

Méthode Implicite – on a besoin de f_{i+1}

$$\begin{cases} 1\ \textit{pas}: y_{i+1} = y_i + \dfrac{h}{2}(f_{i+1} + f_i) \\ 2\ \textit{pas}: y_{i+1} = y_i + \dfrac{h}{12}(5f_{i+1} + 8f_i - f_{i-1}) \\ 3\ \textit{pas}: y_{i+1} = y_i + \dfrac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}) \\ 4\ \textit{pas}: y_{i+1} = y_i + \dfrac{h}{720}(251f_{i+1} + 646f_i - 264f_{i-1} + 106f_{i-2} - 19f_{i-3}) \end{cases}$$

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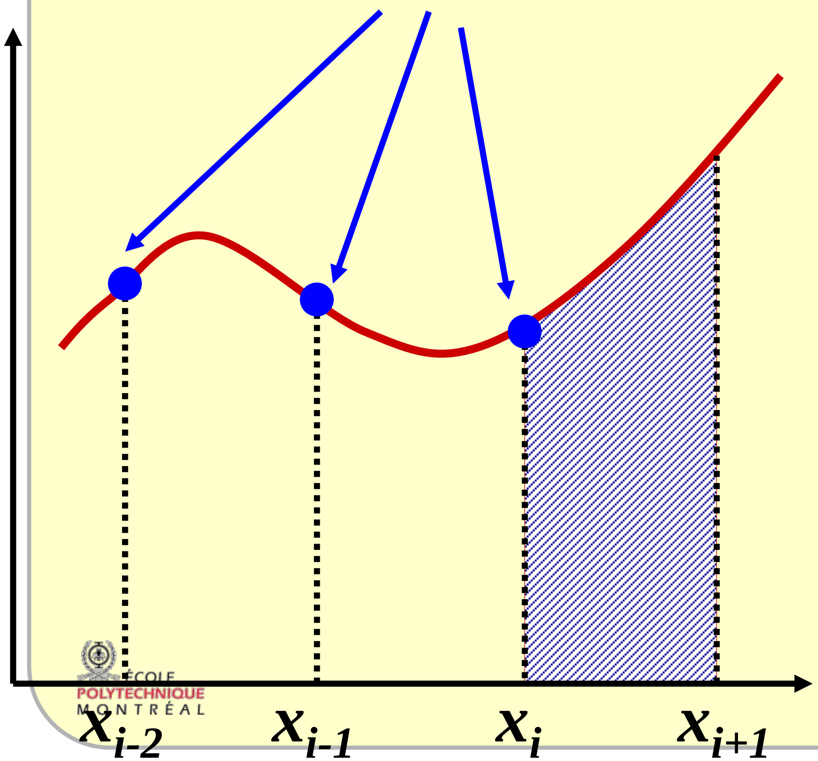
Méthodes d'Adams-Moulton

$ordre$	b_0	b_1	b_2	b_3	b_4	b_5	$err.\ de\ troncature$
2	$\frac{1}{2}$	$\frac{1}{2}$					$-\frac{1}{12}h^3 f''(\eta)$
3	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$				$-\frac{1}{24}h^4 f'''(\eta)$
4	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$			$-\frac{19}{720}h^5 f^{(4)}(\eta)$
5	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$		$-\frac{27}{1440}h^6 f^{(5)}(\eta)$
6	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$	$-\frac{863}{60480}h^7 f^{(6)}(\eta)$

Adams-Bashforth et Adams-Moulton

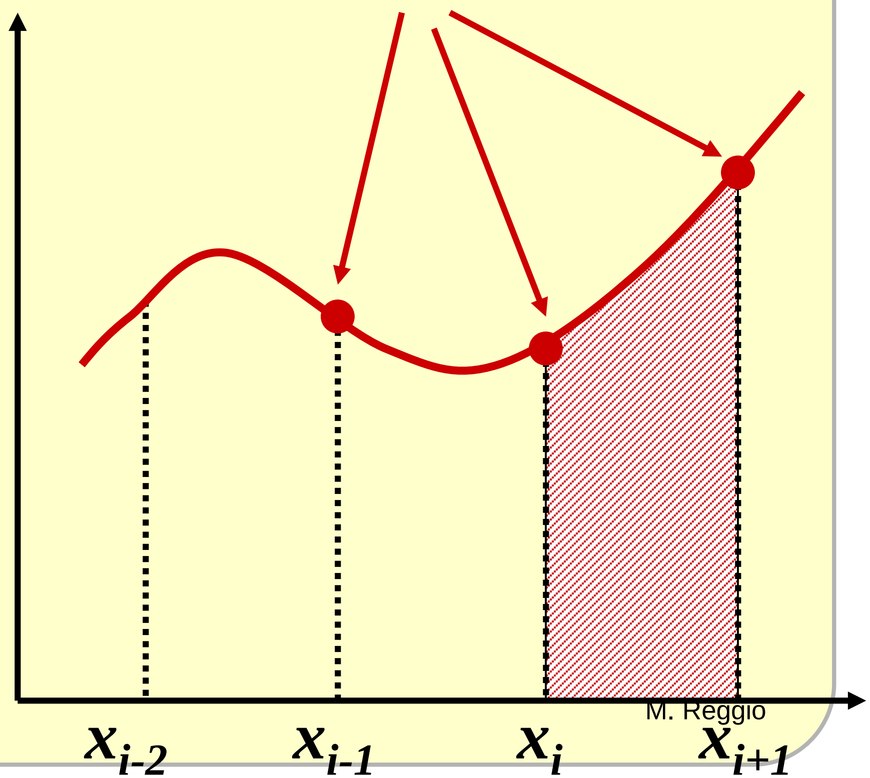
Adams-Bashforth

Ouvertes



Adams-Mouton

Fermés



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Schémas multi-pas

- Comment alléger le côté implicite d' Adams-Moulton ?
- On remplace y_{n+1} par une valeur estimée par Adams-Bashforth :

Adams-Bashforth-Moulton

$$\begin{cases} prédicteur : y^*_{i+1} = y_i + h(b_1 f + b_2 f_{i-1} + \ldots) \\ correcteur : y_{i+1} = y_i + h(\overline{b}_0 f^*_{i+1} + \overline{b}_1 f_i + \ldots);\ f^*_{i+1} = f(x_{i+1}, y^*_{i+1}) \end{cases}$$

Prédicteur-correcteur d'ordre 3

$$y^*_{i+1} = y_i + \frac{h}{12}(23 f_i - 16 f_{i-1} + 5 f_{i-2})$$

$$= y_i + \frac{h}{12}\left[23 f(x_i, y_i) - 16 f(x_{i-1}, y_{i-1}) + 5 f(x_{i-2}, y_{i-2})\right]$$

$$y_{i+1} = y_i + \frac{h}{12}(5 f^*_{i+1} + 8 f_i - f_{i-1})$$

$$= y_i + \frac{h}{12}\left[5 f(x_{i+1}, y^*_{i+1}) + 8 f(x_i, y_i) - f(x_{i-1}, y_{i-1})\right]$$

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Prediction-Correction

Adams- Bashforth à trois points pour un premier pas

$$\boldsymbol{y}^{*}_{\boldsymbol{i+1}} = y_i + \frac{\Delta h}{12}\left[23f_i - 16f_{i-1} + 5f_{i-2}\right]$$

Adams-Moulton à trois points pour un deuxième pas

$$\boldsymbol{y}_{\boldsymbol{i+1}} = y_i + \frac{\Delta h}{12}\left[\boldsymbol{5f^{*}_{i+1}} + 8f_i - f_{i-1}\right]$$

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Exemple: Adams-Bashforth

Solution Exacte

$$\frac{dy}{dx} = y - x^2$$

$$y = 2 + 2x + x^2 - e^x$$

$$y(0) = 1$$

Le pas: $\Delta h = 0.1$

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Adams-Basforth-Moulton (3points)

À partir de quantités obtenues par Runge-Kutta-4

$$f(0,1) = 1.0000 \qquad \frac{dy}{dx} = y - x^2 = f(x,y)$$

$$f(0.1, 1.104829) = \mathbf{1.094829}$$

$$f(0.2, \underbrace{\mathbf{1.218597}}_{\boldsymbol{y(0.2), RK4}}) = \mathbf{1.178597}$$

Pour calculer $y_{0.3}$ la formule d'**Adams-Bashforth** est:

$$\Delta y = \frac{0.1}{12}\left[23 f_{0.2} - 16 f_{0.1} + 5 f_{0.0}\right]$$

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Adams-Basforth-Moulton (3points)

Étape de **prédiction**: $y^*_{i+1} = y_i + \dfrac{\Delta h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}]$

$$\Delta y = \frac{0.1}{12}[23(\mathbf{1.178597}) - 16(\mathbf{1.094829}) + 5(1)]$$

$$= 0.121587$$

$$y^*(0.3) = \underbrace{\mathbf{1.218597}}_{\boldsymbol{y(0.2)}} + 0.121587 = \mathbf{1.340184}$$

$$f^*(0.3, \mathbf{1.340184}) = \mathbf{1.250184}$$

$$\frac{dy}{dx} = y - x^2 = f(x, y)$$

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Adams-Basforth-Moulton (3points)

Étape de **correction:** $\boldsymbol{y_{i+1}} = y_i + \dfrac{\Delta h}{12}[\boldsymbol{5f^*_{i+1}} + 8f_i - f_{i-1}]$

$$\Delta y = \frac{0.1}{12}[5(\mathbf{1.250184}) + 8(\mathbf{1.178597}) - 1(\mathbf{1.094829})]$$

$$= 0.121541$$

$$y(0.3) = \underbrace{\mathbf{1.218597}}_{\boldsymbol{y}(\mathbf{0.2})} + 0.121541 = 1.340138$$

$$f(0.3, 1.340184) = 1.250138$$

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Adams-Basforth-Moulton (3points)

$$\Delta y_{i+1} = |y_{i+1} - y^*_{i+1}| \leq tol$$

	Adams-Moulton Predicteur-Correcteur à Trois Points					
x	y	f		y*	f*	
0	1	1				
0.1	1.104829	1.094829				
0.2	1.218597	1.178597		1.340184	1.250184	
0.3	1.340138	1.250138		1.468219	1.308219	
0.4	1.468168	1.308168		1.601323	1.351323	
0.5	1.601266	1.351266		1.737925	1.377925	
0.6	1.737863	1.377863		1.876291	1.386291	
0.7	1.876222	1.386222		2.014502	1.374502	
0.8	2.014425	1.374425		2.150438	1.340438	
0.9	2.150353	1.340353		2.281757	1.281757	
1	2.281663	1.281663		2.405869	1.195869	

Interprétation intégrale

Les méthodes à multi-pas découlent de l'intégration de:

$$\frac{dy}{dt} = f(y,t) \qquad y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(y,t)\,dt$$

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} y'(t)\,dt = y(t_n) + \underbrace{\int_{t_i}^{t_{i+1}} P_i\,dt}$$

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Adams-Bashforth

L'intégrale **I** est calculée en remplaçant **f** par une interpolation polynomiale d'ordre **r** (avec les points t_n à t_{n-r})

$$f\left(t, y(t)\right) \approx \sum_{k=0}^{r} L_k(t)\, f\left(t_{n-k}, y\left(t_{n-k}\right)\right)$$

$$\boxed{I = h \sum_{k=0}^{r} b_k\, f\left(t_{n-k}, y\left(t_{n-k}\right)\right)}$$

$$\left(w_j = \int_a^b L_j\, dx \right) \qquad \left(b_k = \frac{1}{h} \int_{t_n}^{t_{n+1}} L_k(t)\, dt \right)$$

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Adams-Moulton

L'intégrale I et calculée en remplaçant **f** par une interpolation polynomiale d'ordre **r+1** (avec les points t_{n+1} à t_{n-r})

$$I = h\sum_{k=-1}^{r} b_k\, f\left(t_{n-k}, y\left(t_{n-k}\right)\right)$$

$$y_{n+1} = y_n + I$$

Méthode **implicite** : y_{n+1} dépend de $f(t_{n+1}, y_{n+1})$

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$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} P_i(x)dx$$

$$P_i^{(r)} = \sum_{k=0}^{r} f_{i-k} L_{i,k}^{(r)}(x)$$

$$L_{i,k}^{(r)}(x) = \prod_{j=0}^{r} \frac{(x - x_{i-j})}{(x_{i-k} - x_{i-j})}$$

k: indice du polynôme, **i** indice du dernier point d'un total de **r+1** points

$$y_{i+1} - y_i = \sum_{k=0}^{r} f_{i-k} \int_{x_i}^{x_{i+1}} L_{i,k}^{(r)}(x)dx$$

$$b_{i,k}^{(r)} = \frac{1}{h} \int_{x_i}^{x_{i+1}} L_{i,k}^{(r)}(x)dx$$

$$y_{i+1} = y_i + h \sum_{k=0}^{r} b_{i,k}^{(r)} f_{i-k}$$

$$b_{i,k} = w_{i-k}$$

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Cas linéaire

$$x = \boldsymbol{x_0}, \quad x = \boldsymbol{x_1}$$
$$y = \boldsymbol{y_0}, \quad y = \boldsymbol{y_1}$$

$$L_0(x) = \frac{(x - x_1)}{(\boldsymbol{x_0} - x_1)} \qquad L_1(x) = \frac{(x - x_0)}{(\boldsymbol{x_1} - x_0)}$$

$$P(x) = \boldsymbol{y_0} L_0(x) + \boldsymbol{y_1} L_1(x)$$

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Cas quadrtique

$x_1 = 2, x_2 = 4, x_3 = 5,$

$y_1 = 3, y_2 = 2, y_3 = 4,$

$$L_1(x) = \frac{(x-4)(x-5)}{(\mathbf{2}-4)(\mathbf{2}-5)}$$

$$L_2(x) = \frac{(x-2)(x-5)}{(\mathbf{4}-2)(\mathbf{4}-5)}$$

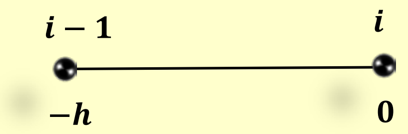
$$L_3(x) = \frac{(x-2)(x-4)}{(\mathbf{5}-2)(\mathbf{5}-4)}$$

$$P(x) = 3L_1(x) + 2L_2(x) + 4L_3(x)$$

Ex:Adams-Bashforth

$$L_0(x) = \frac{(x - x_1)}{(\boldsymbol{x_0} - x_1)} \qquad \boldsymbol{x_0 = -h} \qquad \boldsymbol{x_1 = 0} \qquad L_1(x) = \frac{(x - x_0)}{(\boldsymbol{x_1} - x_0)}$$

$$L_{i,0}^{(1)}(x) = -\frac{x}{h}$$



$$L_{i,1}^{(1)}(x) = \frac{(x+h)}{h}$$

$$b_{i,0}^{(1)} = \frac{1}{h}\int_0^h -\frac{x}{h}dx = -\frac{1}{2}$$

$$b_{i,1}^{(1)} = \frac{1}{h}\int_0^h \frac{x+h}{h}dx = \frac{3}{2}$$

$$\boldsymbol{y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})}$$

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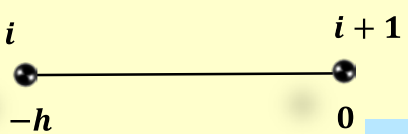
Ex:Adams-Moulton

$$L_{i,0}^{(1)}(x) = -\frac{x}{h}$$

$$L_{i,1}^{(1)}(x) = \frac{(x+h)}{h}$$

$$\hat{b}_{i,0}^{(1)} = \frac{1}{h}\int_{-h}^{0} -\frac{x}{h}dx = \frac{1}{2}$$

$$\hat{b}_{i,1}^{(1)} = \frac{1}{h}\int_{-h}^{0} \frac{x+h}{h}dx = \frac{1}{2}$$



$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1})$$

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$y'(x) = f(x,y)$ dans l'intervalle $[i, i+1]$

1. L'intégration

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x,y)dx$$

2. L'approximation de $f(x,y)$ par un polynôme $\phi(x,y)$

$$\phi^{(r)}(x) = \sum_{k=0}^{r} f_k L_k^{(r)}(x) \quad \text{avec} \quad L_k^{(r)}(x) = \prod_{j=0}^{r} \frac{(x - x_k)}{(x_j - x_k)}$$

Formulation Unique

$$y_{i+1} = y_i + h\sum_{k=1}^{r}\left(\int_{x_i}^{x_{i+1}} L_k(x)dx\right) f_k$$

Lorsque les f_k sont évaluées aux positions; $i+1, i, i-1, \cdots \rightarrow$ ADAMS.

Si on inclut, $i+1$, $\rightarrow$ Adams-Moulton.
Le cas contraire $\rightarrow$ Adams-Bashforth.

Lorsque les f_k sont évaluées dans l'intévalle $[i, i+1$, $\rightarrow$ Runge-Kutta.

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EDOs d'ordre supérieur

On peut les représenter par

$$\begin{cases} y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)}) \\ y(0) = \alpha_0, y'(0) = \alpha_1, \cdots, y^{(n-1)} = \alpha_{n-1} \end{cases}$$

et on les transforme

$$soit \begin{cases} u_1 = y \\ u_2 = y' \\ u_3 = y'' \\ \quad \vdots \\ u_n = y^{(n-1)} \end{cases} \Rightarrow \begin{cases} u_1' = u_2, & u_1(0) = \alpha_0 \\ u_2' = u_3, & u_2(0) = \alpha_1 \\ u_3' = u_4, & u_3(0) = \alpha_2 \\ \quad \vdots & \\ u_n' = f(x, u_1, u_2, \cdots, u_n), & u_n(0) = \alpha_{n-1} \end{cases}$$

Système de 1er ordre

Exemple

$$\begin{cases} y''' = f(x, y, y', y'') = x^2 + 4xy - 3y' + 5y'' \\ y(0) = 2,\ \ y'(0) = 4,\ \ y''(0) = 1 \end{cases}$$

$$avec\ \ \mathrm{u}_1 = \mathrm{y},\ \ \mathrm{u}_2 = \mathrm{y}',\ \ \ \mathrm{u}_3 = \mathrm{y}''$$

On trouve trois EDOs aux valeurs initiales

$$\begin{cases} u_1' = f_1(x, u_1, u_2, u_3) = u_2 \\ u_2' = f_2(x, u_1, u_2, u_3) = u_3 \\ u_3' = f_3(x, u_1, u_2, u_3) = x^2 + 4xu_1 - 3u_2 + 5u_3 \end{cases} \qquad \begin{cases} u_1(0) = 2 \\ u_2(0) = 4 \\ u_3(0) = 1 \end{cases}$$

$$\text{En Notation Vectorielle} \quad \Rightarrow \quad u' = f\,(x, u)$$

$$\begin{cases} y''' = f(x, y, y', y'') = x^2 + 4xy - 3y' + 5y'' \\ y(0) = 2, \qquad y'(0) = 4, \qquad y''(0) = 1 \end{cases}$$

Premier pas: x(0) = 0, x(1) = 0.5 (h = 0.5)

$$\begin{cases} u_1(1) = u_1(0) + hu_2(0) = 2 + (0.5)(4) = 4.0 \\ u_2(1) = u_2(0) + hu_3(0) = 4 + (0.5)(1) = 4.5 \\ u_3(1) = u_3(0) + h[x^2(0) + 4x(0)u_1(0) - 3u_2(0) + 5u_3(0)] \\ \qquad = 1 + 0.5[(0)^2 + 4(0)(2) - 3(4) + 5(1)] = -5/2 \end{cases}$$

$$\begin{cases} \boldsymbol{u_1(0) = 2} \\ \boldsymbol{u_2(0) = 4} \\ \boldsymbol{u_3(0) = 1} \end{cases}$$

$$\begin{cases} u_1' = f_1(x, u_1, u_2, u_3) = u_2 \\ u_2' = f_2(x, u_1, u_2, u_3) = u_3 \\ u_3' = f_4(x, u_1, u_2, u_3) = x^2 + 4xu_1 - 3u_2 + 5u_3 \end{cases} \qquad \begin{cases} u_1(0) = 2 \\ u_2(0) = 4 \\ u_3(0) = 1 \end{cases}$$

En Notation Vectorielle $\Rightarrow \quad u' = f(x, u)$

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Exemple: Euler

$$\begin{cases} u_1(2) = u_1(1) + hu_2(1) = 4.0 + (0.5)(4.5) = 6.25 \\ u_2(2) = u_2(1) + hu_3(1) = 4.5 + (0.5)(-5/2) = -13/4 \\ u_3(2) = u_3(1) + h[x^2(1) + 4x(1)u_1(1) - 3u_2(1) + 5u_3(1)] \\ \qquad = 2.5 + 0.5[(0.5)^2 + 4(0.5)(4.0) - 3(4.5) + 5(-5/2)] = \end{cases}$$

$$\begin{cases} u_1' = f_1(x, u_1, u_2, u_3) = u_2 \\ u_2' = f_2(x, u_1, u_2, u_3) = u_3 \\ u_3' = f_4(x, u_1, u_2, u_3) = x^2 + 4xu_1 - 3u_2 + 5u_3 \end{cases} \qquad \begin{cases} u_1(0.5) = 4 \\ u_2(0.5) = 4.5 \\ u_3(0.5) = -2.5 \end{cases}$$

En Notation Vectorielle $\Rightarrow \quad u' = f(x, u)$

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Deux EDOs Runge-Kutta

Méthode du point milieu

$$\begin{cases} k_1 = hf(x_i, y_i) \\ k_2 = hf(x_i + h/2, y_i + k_1/2) \\ y_{i+1} = y_i + k_2 \end{cases}$$

Deux EDOs

(u,v)

$$\begin{cases} k_1 = hf(x_i, u_i, v_i) \\ m_1 = hg(x_i, u_i, v_i) \\ k_2 = hf(x_i + h/2, u_i + k_1/2, v_i + m_1/2) \\ m_2 = hg(x_i + h/2, u_i + k_1/2, v_i + m_1/2) \\ u_{i+1} = u_i + k_2 \\ v_{i+1} = v_i + m_2 \end{cases}$$

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Runge-Kutta:systèmes

Runge-Kutta d’ordre deux

$$\begin{cases} k_1 = hf_1(x(i), u_1(i), u_2(i), u_3(i)) \\ k_2 = hf_2(x(i), u_1(i), u_2(i), u_3(i)) \\ k_3 = hf_3(x(i), u_1(i), u_2(i), u_3(i)) \end{cases}$$

$$\begin{cases} m_1 = hf_1(x(i) + h/2,\ u_1(i) + k_1/2,\ u_2(i) + k_2/2,\ u_3(i) + k_3/2) \\ m_2 = hf_2(x(i) + h/2,\ u_1(i) + k_1/2,\ u_2(i) + k_2/2,\ u_3(i) + k_3/2) \\ m_3 = hf_3(x(i) + h/2,\ u_1(i) + k_1/2,\ u_2(i) + k_2/2,\ u_3(i) + k_3/2) \end{cases}$$

$$\begin{cases} u_1(i+1) = u_1(i) + m_1 \\ u_2(i+1) = u_2(i) + m_2 \\ u_3(i+1) = u_3(i) + m_3 \end{cases}$$

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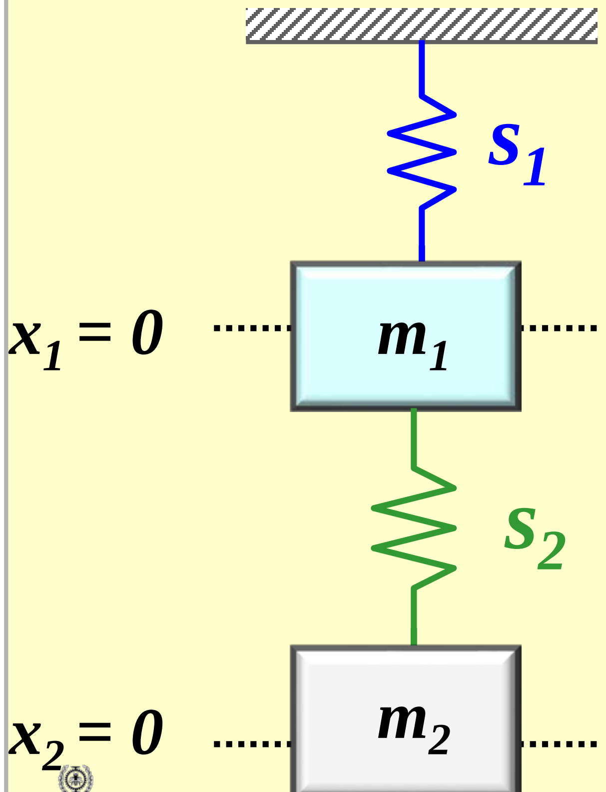
Système M-R-A

Constantes des ressorts (s_1, s_2)



$$m_1\ddot{x}_1 = -s_1 x_1 + s_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 = -s_2(x_2 - x_1)$$

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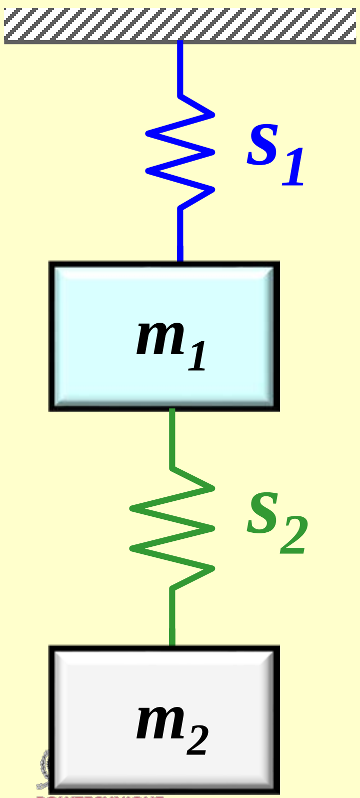
Système M-R-A

On transforme en un système de 4 EDOs de 1er ordre



$$soit\ \ u_1 = x_1,\ u_2 = x_1',\ u_3 = x_2,\ u_4 = x_2'$$

$$\begin{cases} u_1' = u_2 \\ u_2' = -\dfrac{s_1}{m_1}u_1 + \dfrac{s_2}{m_1}(u_3 - u_1) \\ u_3' = u_4 \\ u_4' = -\dfrac{s_2}{m_2}(u_3 - u_1) \end{cases} \qquad \begin{cases} u_1(0) = \alpha_1 \\ u_2(0) = \alpha_2 \\ u_3(0) = \alpha_3 \\ u_4(0) = \alpha_4 \end{cases}$$

$$m_2\ddot{x}_2 = -s_2(x_2 - x_1) \qquad m_1\ddot{x}_1 = -s_1x_1 + s_2(x_2 - x_1)$$

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Fin..............................

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