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PROBABILITY AND STATISTICS
QUIZ-4.

Q. Let $X \sim \frac{1}{2} \exp(-|x_i - \mu|)$, iid $i = 1, \ldots n$

Show that the maximum likelihood estimator of μ is the median.

Then following the proof of theorem 12.8, Show that the median of the posterior distribution is the Bayes rule for L1 loss.

ANS. Likelihood function $L_n(\mu) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \mu|}$

Log likelihood function $l_n(\mu) = \log\left[\prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \mu|}\right]$

$$= \log\left[\frac{1}{2^n} \prod_{i=1}^{n} e^{-|x_i - \mu|}\right]$$

$$= \log \frac{1}{2^n} + \log\left[\prod_{i=1}^{n} e^{-|x_i - \mu|}\right]$$

$$= \log \frac{1}{2^n} + \sum_{i=1}^{n} \log e^{-|x_i - \mu|}$$

$$= \log \frac{1}{2^n} + \sum_{i=1}^{n} -|x_i - \mu|$$

$$= \log \frac{1}{2^n} - \sum_{i=1}^{n} |x_i - \mu|.$$

For $l_n(\mu)$ to be maximum.

$$\frac{d l_n(\mu)}{d\mu} = 0.$$

$$\therefore \frac{d}{d\mu} - \sum_{i=1}^{n} |x_i - \mu| = 0.$$

For this to be true, we need equal numbes of positive numbers and negative numbers.

$\therefore x_i - \mu$ is positive for half of the value

$x_i - \mu$ is negative for half of the values.

$\therefore \mu$ is the median of ~~the~~ x_i

Since $|x_i - \mu|$ is minimum when μ is the median, $-|x_i - \mu|$ will be maximum if μ is the median.

$\therefore$ MLE(μ) is the median $(x_1, x_2, x_3 \ldots x_n)$

Risk of an estimator $\hat{\theta}$ is:

$$R(\theta, \hat{\theta}) = E_\theta(L(\theta, \hat{\theta})) = \int L(\theta, \hat{\theta}(x)) f(x.\theta)\, dx$$

Bayes risk

$$r(f, \hat{\theta}) = \int R(\theta, \hat{\theta}) f(\theta)\, d\theta.$$

where $f(\theta)$ is prior of θ

Theorem 12.8:

If $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ then the Bayes estimator is $\hat{\theta}(x) = \int \theta f(\theta/x)\, d\theta = E(\theta/X = x)$.

If $L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$ then the Bayes estimator is the median of the posterior $f(\theta/x)$.

If $L(\theta, \hat{\theta})$ is zero-one loss, then the Bayes estimator is the mode of the posterior $f(\theta/x)$

Posterior risk:

$$r(\hat{\theta}/x) = \int L(\theta, \hat{\theta}(x)) \cdot f(\theta/x)\, d\theta$$

Bayes risk in terms of posterior risk:

$$r(f, \hat{\theta}) = \int r(\hat{\theta}/x) \left(f(x/\theta) \cdot f(\theta)\, d\theta \right) dx$$

$$r(f, \hat{\theta}) = \int r(\hat{\theta}/x)\, m(x)\, dx.$$

For L1 loss:

$$r(\hat{\theta}/x) = \int |\hat{\theta} - \theta)\, f(\theta/x)\, d\theta$$

$$\frac{d(r(\hat{\theta}/x))}{d\hat{\theta}} = \frac{d}{d\hat{\theta}} \int |\hat{\theta} - \theta|\, f(\theta/x)\, d\theta.$$

$$= \int \text{sign}(\hat{\theta} - \theta) \cdot f(\theta/x)\, d\theta.$$

For minimum,

$$\frac{d(r(\hat{\theta}/x))}{d\hat{\theta}} = 0.$$

$$\therefore \int \text{sign}(\hat{\theta} - \theta) \cdot f(\theta/x)\, d\theta = 0$$

$$\therefore \text{sign}(\hat{\theta} - \theta) = \begin{cases} 1 & \text{if } \hat{\theta} > \theta \\ -1 & \text{if } \hat{\theta} < \theta \\ 0 & \text{if } \hat{\theta} = \theta \end{cases}$$

$\cancel{\int f(\theta/x)\, d\theta = 0}$ $\int \text{sign}(\hat{\theta}-\theta) f(\theta/x)\, d\theta = 0$

$$\int_{\theta<\hat{\theta}} f(\theta/x)\, d\theta + \int_{\theta>\hat{\theta}} -1 \cdot f(\theta/x)\, d\theta = 0.$$

$$\therefore \int_{\theta<\hat{\theta}} f(\theta/x)\, d\theta = \int_{\theta>\hat{\theta}} f(\theta/x)\, d\theta$$

By theorem 12.8
Median of the posterior $f(\theta/x)$:

$$F(\hat{\theta}(x)/x) = \frac{1}{2}$$

Hence if L1 loss $L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$ then the median of the posterior distribution is the Baye's rule for L1 loss.