

# Econ 103: Introduction to Simple Linear Regression

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July 24, 2021

## Linear Regression

- Line of best fit and linear model
- Formulas for parameters

## Estimation

- Using data to estimate parameters of interest
- Formulas for parameter estimates

## Asymptotic Distribution

- Approximate distribution of parameter estimates for “large  $n$ ”
- Estimating variance of parameter estimates

## Hypothesis Testing and Confidence Intervals

- Using asymptotic distribution to test statements about underlying parameters
- Using asymptotic distribution to give a range of plausible underlying parameter values

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## The Basic Model

## Estimation

## Asymptotic Distribution

## Hypothesis Testing and Confidence Intervals

## Conclusion

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- How are unemployment and inflation related?

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### Examples:

- How are education and wages related?
- How are unemployment and inflation related?
- What is the relationship between receiving a treatment and a health outcome?

One way to model the relationship between  $Y$  and  $X$  would be to try to find the **line of best fit** between the two variables.

By the **line of best fit** we mean finding the line, characterized by a slope and an intercept, that minimizes the distance between  $Y$  and  $\tilde{\beta}_0 + \tilde{\beta}_1 \cdot X$ .



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Formally, we are interested in the parameters  $\beta_0$  and  $\beta_1$  that solve

$$\begin{aligned}\beta_0, \beta_1 &= \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - (\tilde{\beta}_0 + \tilde{\beta}_1 \cdot X) \right)^2 \right] \\ &= \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X \right)^2 \right]\end{aligned}$$

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- 
- By arg min we just mean we are interested in the **arguments**  $\beta_0$  and  $\beta_1$  that minimize

$$\mathbb{E}[(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X)^2]$$

rather than the value  $\mathbb{E}[(Y - \beta_0 - \beta_1 \cdot X)^2]$  itself.

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- Another way of saying this is that

$$\mathbb{E}[(Y - \beta_0 - \beta_1 \cdot X)^2] < \mathbb{E}[(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X)^2]$$

for any  $(\tilde{\beta}_0, \tilde{\beta}_1) \neq (\beta_0, \beta_1)$ .

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Why do we care about these parameters?

- Knowing the line of best fit will help us predict  $Y$  using  $X$ 
  - Will provide the **best linear prediction** of  $Y$  using  $X$ .
  - Even though a linear model may seem to simple, ends up being tremendously useful in practice.

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  - What is the average value of  $Y$  when  $X$  is zero?  $\iff$  What is  $\beta_0$ ?
  - To a first order degree because  $\beta_0$  and  $\beta_1$  describe the line of best fit rather than the “true” relationship.
    - No need to worry about this difference for now though.

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Let's solve for  $\beta_0$  and  $\beta_1$  by taking first order conditions:

$$\frac{\partial}{\partial \tilde{\beta}_0} : \mathbb{E} [Y - \beta_0 - \beta_1 \cdot X] = 0$$

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We will return to these first order conditions shortly. For now, after rearranging we get that

$$\beta_1 = \frac{\mathbb{E}[YX] - \mathbb{E}[Y]\mathbb{E}[X]}{\mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[X]} = \frac{\text{Cov}(Y, X)}{\text{Var}(X)}$$

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Exercise: Show this rearrangement.

Let's define the random variable

$$\begin{aligned}\epsilon &= Y - (\beta_0 + \beta_1 \cdot X) \\ &= Y - \beta_0 - \beta_1 \cdot X\end{aligned}$$

We can then write

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

which is the linear regression equation you may have seen before. The random variable  $\epsilon$  will be important later on as we try to do inference.

## Linear Regression: The Error Term

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We call  $\epsilon$  the **linear regression error** variable.

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Recall that from the first order conditions for  $\beta_0$  and  $\beta_1$  we have that

$$\begin{aligned}\mathbb{E}\left[\underbrace{Y - \beta_0 - \beta_1 \cdot X}_{=\epsilon}\right] &= 0 \\ \mathbb{E}\left[\underbrace{(Y - \beta_0 - \beta_1 \cdot X) \cdot X}_{=\epsilon X}\right] &= 0\end{aligned}$$

These give us the properties that

$$\mathbb{E}[\epsilon] = 0 \quad \text{and} \quad \mathbb{E}[\epsilon X] = 0.$$



In total our **line of best fit** parameters

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X \right)^2 \right]$$

generate a model between  $Y$  and  $X$  that can be written as

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon \tag{1}$$

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- It is often convenient to work directly with this representation or make assumptions about  $\epsilon$ .
- You may have seen this representation before, the prior slides go over where this model comes from

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are useful for

- Making predictions about  $Y$  using  $X$ .
  - Predict  $Y$  when  $X = x$  with  $\beta_0 + \beta_1 \cdot x$

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are useful for

- Making predictions about  $Y$  using  $X$ .
  - Predict  $Y$  when  $X = x$  with  $\beta_0 + \beta_1 \cdot x$
- Learning about the relationship between  $Y$  and  $X$ .
  - Interpret the signs and magnitudes of  $\beta_0$  and  $\beta_1$

Questions?

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As we went over in the last section we are interested in the line of best fit parameters

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X \right)^2 \right]$$

**Problem:** We do not know the joint distribution of  $(Y, X)$ , so we cannot solve for  $\beta_0$  and  $\beta_1$  by evaluating the expectation above.

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**Intuition:**

- Suppose we have access to  $n$  randomly collected samples  $\{Y_i, X_i\}_{i=1}^n$
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- We estimate the line of best fit between  $Y$  and  $X$  in the population using the line of best fit between  $Y_i$  and  $X_i$  in our sample:

$$\hat{\beta}_0, \hat{\beta}_1 = \arg \min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^n (Y_i - b_0 - b_1 \cdot X_i)^2$$

- Same idea as using  $\bar{X}$  to estimate  $\mathbb{E}[X]$ , etc.

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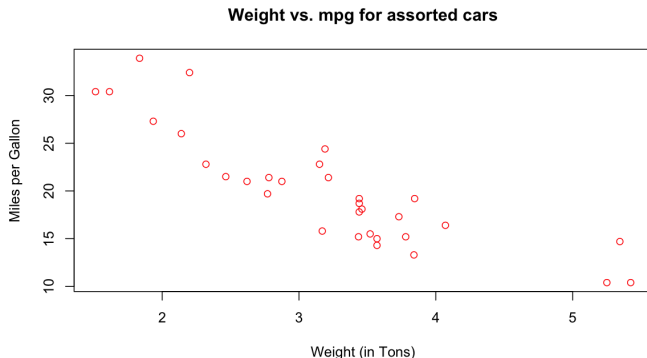
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## Linear Regression: The Estimator

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Let's see how this looks like in practice. Suppose we are interested in the relationship between  $X$ , a car's weight, and  $Y$  a car's miles per gallon (mpg).

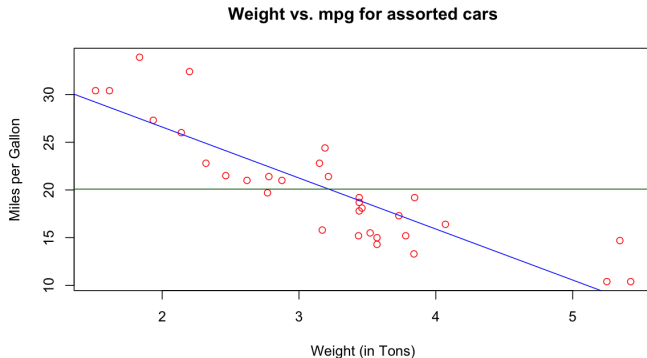
We collect some data  $\{Y_i, X_i\}_{i=1}^n$  where each  $(Y_i, X_i)$  pair represents the miles per gallon and weight of a particular vehicle in our dataset. We can represent our data using a scatterplot



## Linear Regression: The Estimator

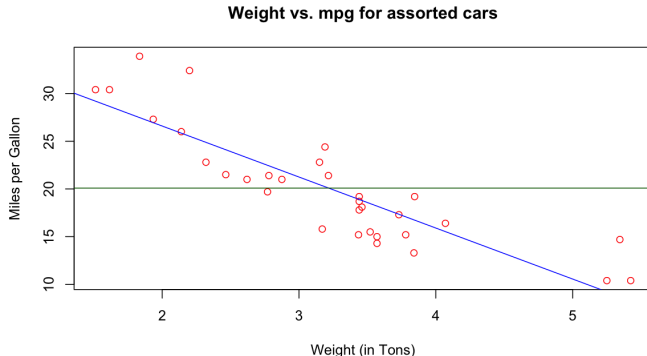
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Now to estimate  $\hat{\beta}_0, \hat{\beta}_1$  we simply find the line of best fit between the  $Y_i$  and  $X_i$  's in our data.



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The **blue** line represents the line of best fit whereas the **green** line represents a straight line through  $\bar{Y}$ . We can see that the **blue** line is much closer to the data than the **green** line.



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  - In context: we estimate that the average mpg for a car that weights 0 tons is 37.2851 miles per gallon
- $\hat{\beta}_1 = -5.3445$ : We estimate that, on average, a one unit increase in  $X$  is associated with a 5.3445 unit **decrease** in  $Y$ .
  - In context: we estimate that, on average, a one ton increase in car weight is associated with a 5.3445 unit decrease in miles per gallon.

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How can we use these estimates for prediction?

- Suppose we have a car that weighs 3.5 tons. Based on our estimates, what would we predict its miles per gallon to be?

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$$\text{Predicted MPG} = 37.2851 - 5.3445 \cdot \text{Weight in Tons.}$$

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- We denote this predicted MPG as  $\hat{MPG}$  and in general will denote our predictions as  $\hat{Y}$  so that our estimated regression line can be written

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot X.$$

Notice a couple things in the above interpretations

- The intercept is often uninterpretable (What car would weigh 0 tons?). For this reason we often focus our analysis on the slope coefficient.
- The interpretation is deliberately not causal. We use “associated with a decrease...” as opposed to “leads to a decrease...”



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Taking first order conditions gives us that

$$\frac{\partial}{\partial b_0} : \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i) = 0$$

$$\frac{\partial}{\partial b_1} : \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i) \cdot X_i = 0$$

Rearranging the first equality gives us

$$\frac{1}{n} \sum_{i=1}^n Y_i - \frac{1}{n} \sum_{i=1}^n \hat{\beta}_0 - \frac{1}{n} \sum_{i=1}^n \hat{\beta}_1 \cdot X_i = 0$$

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$$\frac{1}{n} \sum_{i=1}^n Y_i - \frac{1}{n} \sum_{i=1}^n \hat{\beta}_0 - \frac{1}{n} \sum_{i=1}^n \hat{\beta}_1 \cdot X_i = 0$$

$$\bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 \frac{1}{n} \sum_{i=1}^n X_i = 0$$

$$\bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{X} = 0$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

So that what remains is to solve for  $\hat{\beta}_1$ .

Rearranging the second equality gives us

$$\frac{1}{n} \sum_{i=1}^n Y_i X_i - \hat{\beta}_0 \frac{1}{n} \sum_{i=1}^n X_i - \hat{\beta}_1 \frac{1}{n} \sum_{i=1}^n X_i^2 = 0$$

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Using the prior result that  $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$  gives:

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n Y_i X_i - (\bar{Y} - \hat{\beta}_1 \bar{X}) \bar{X} - \hat{\beta}_1 \frac{1}{n} \sum_{i=1}^n X_i^2 &= 0 \\ \left( \frac{1}{n} \sum_{i=1}^n Y_i X_i - \bar{Y} \bar{X} \right) + \hat{\beta}_1 \left( (\bar{X})^2 - \frac{1}{n} \sum_{i=1}^n X_i^2 \right) &= 0 \end{aligned}$$



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So, finally

$$\hat{\beta}_1 = \frac{\frac{1}{n} \sum_{i=1}^n Y_i X_i - \bar{Y} \bar{X}}{\frac{1}{n} \sum_{i=1}^n X_i^2 - (\bar{X})^2}.$$

Let's make use of the following equalities to represent  $\hat{\beta}_1$

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X}) &= \frac{1}{n} \sum_{i=1}^n Y_i X_i - \bar{Y} \bar{X} \\ \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 &= \frac{1}{n} \sum_{i=1}^n X_i^2 - (\bar{X})^2\end{aligned}$$

## Linear Regression: Formulas

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Then:

$$\hat{\beta}_1 = \frac{\overbrace{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}^{\text{Sample Covariance between } Y \text{ and } X}}{\underbrace{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}_{\text{Sample Variance of } X}}$$

## Linear Regression: Formulas

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This ties in nicely as, if we recall from earlier, we found that

$$\beta_1 = \frac{\text{Cov}(Y, X)}{\text{Var}(X)} = \frac{\mathbb{E}[(Y - \mu_Y)(X - \mu_X)]}{\mathbb{E}[(X - \mu_X)^2]}.$$

We have now gone over how use data to obtain estimates  $\hat{\beta}_0, \hat{\beta}_1$  of our parameters of interest  $\beta_0, \beta_1$ .

$$\hat{\beta}_0, \hat{\beta}_1 = \arg \min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^n (Y_i - b_0 - b_1 \cdot X_i)^2$$
$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X \right)^2 \right]$$

Notice that, while the parameters of interest  $\beta_0$  and  $\beta_1$  are fixed quantities, the estimators  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are functions of the data; they depend on the specific sample of data collected.

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### Some Questions to Consider:

1. What would happen to our estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  if we were to collect a different sample of data?

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1. What would happen to our estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  if we were to collect a different sample of data?
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2. How can we model the distribution of our estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$ ?
3. What happens to this distribution as  $n \rightarrow \infty$ ?



## Linear Regression: Randomness

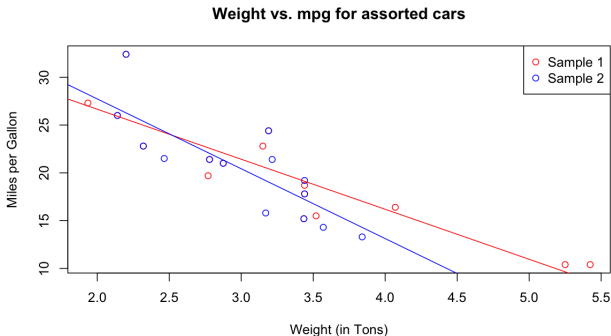
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## Linear Regression: Randomness

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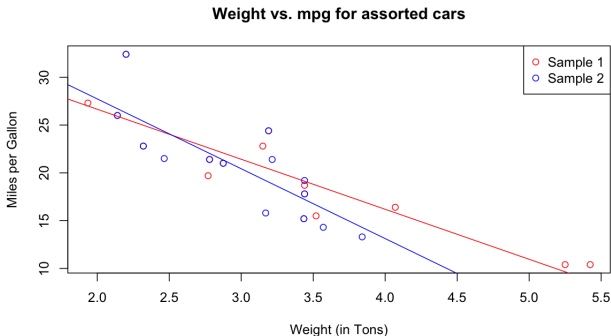
Let's return to the cars data and see how our regression lines look when we consider two different (random) samples.



## Linear Regression: Randomness

**Question:** What would happen to our estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  if we were to collect a different sample of data?

Let's return to the cars data and see how our regression lines look when we consider two different (random) samples.



- **Sample 1:**  $\hat{\beta}_0 = 37.1285$  and  $\hat{\beta}_1 = -5.2341$ .
- **Sample 2:**  $\hat{\beta}_0 = 42.352$  and  $\hat{\beta}_1 = -7.307$ .

**Key Concept:** Because the estimators  $\hat{\beta}_0$  and  $\hat{\beta}_1$  are functions of the random sample  $\{Y_i, X_i\}_{i=1}^n$  they are themselves random variables.

$$\begin{aligned}\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X} \\ \hat{\beta}_1 &= \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}\end{aligned}$$

**Problem:** How do we connect  $\hat{\beta}_0$  and  $\hat{\beta}_1$  to the population parameters  $\beta_0$  and  $\beta_1$ ?

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**Problem:** How do we connect  $\hat{\beta}_0$  and  $\hat{\beta}_1$  to the population parameters  $\beta_0$  and  $\beta_1$ ?

**Fundamental Question:** Given estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  what can we say about the underlying parameters of interest  $\beta_0$  and  $\beta_1$ ?

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The Basic Model

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Asymptotic Distribution

Hypothesis Testing and Confidence Intervals

Conclusion

Suppose we are interested in the association between years of education and income. We collect a random sample of size  $n = 100$ ,  $\{Y_i, X_i\}_{i=1}^{100}$  and run a simple linear regression of  $Y = INC$  against  $X = EDU$ .

Suppose we are interested in the association between years of education and income. We collect a random sample of size  $n = 100$ ,  $\{Y_i, X_i\}_{i=1}^{100}$  and run a simple linear regression of  $Y = INC$  against  $X = EDU$ .

That is, we are interested in the parameters  $\beta_0$  and  $\beta_1$  that dictate the line of best fit between income and education in the population

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ (INC - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot EDU)^2 \right].$$

or equivalently the parameters from the linear model

$$INC = \beta_0 + \beta_1 \cdot EDU + \epsilon.$$

where  $\mathbb{E}[\epsilon \cdot EDU] = 0$ .



Using our data  $\{Y_i, X_i\}_{i=1}^n$  we find that  $\hat{\beta}_1 = 0.5$ .

$$\hat{\beta}_0 \hat{\beta}_1 = \arg \min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^n \{Y_i - b_0 - b_1 \cdot X_i\}^2.$$

Our friend, Prince Harry Estranged of England, however claims that there is no association between education and income, that is that  $\beta_1 = 0$ .

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**Answer:** One way would be to find the probability that we would obtain  $\hat{\beta}_1 = 0.5$  (or something more extreme) if the true value of  $\beta_1$  was 0.

$$\Pr(|\hat{\beta}_1| \geq 0.5 | \beta_1 = 0).$$

If this probability is sufficiently low, we can reject Former Prince Harry's claim. Otherwise he may be right.

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---

To calculate this probability we will need to know something about the (approximate) distribution of  $\hat{\beta}_1$  and how that is related to the true parameter  $\beta_1$ .

In order to connect the estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  to the population parameters, we will need to make some (light) assumptions about the underlying distribution of  $(Y, X)$  from which our sample  $\{Y_i, X_i\}_{i=1}^n$  is drawn.

In order to connect the estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  to the population parameters, we will need to make some (light) assumptions about the underlying distribution of  $(Y, X)$  from which our sample  $\{Y_i, X_i\}_{i=1}^n$  is drawn.

It will be helpful to recall the following definitions here

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ (Y - \tilde{\beta}_0 - \tilde{\beta}_1)^2 \right]$$

$$\epsilon = Y - \beta_0 - \beta_1 \cdot X$$

And see that  $\epsilon$  is itself a random variable.

Make the following assumptions

1. **Random Sampling:** Assume that  $\{Y_i, X_i\}$  are independently and identically distributed;  $(Y_i, X_i) \stackrel{\text{i.i.d}}{\sim} (Y, X)$ 
  - Essentially this means that our random sample is “representative of the population”
  - Would be violated if say, we only sampled cars made in Los Angeles and we were trying to make inferences about all cars produced in the US

## Linear Regression: Assumptions

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Make the following assumptions

1. **Random Sampling:** Assume that  $\{Y_i, X_i\}$  are independently and identically distributed;  $(Y_i, X_i) \stackrel{\text{i.i.d}}{\sim} (Y, X)$
2. **Homoskedasticity:** Assume that  $\text{Var}(\epsilon \mid X = x) = \sigma_\epsilon^2$  for all possible values of  $x$ .
  - This means that  $Y$  is equally spread around the regression line for all values of  $X$ .
  - This is a fairly strong assumption to make and we will relax it later on, but it is helpful for now to provide insight.
  - Conditional variance is similar to the conditional expectation that we went over in our Econ 41 review

$$\text{Var}(\epsilon \mid X = x) = \mathbb{E}[\epsilon^2 \mid X = x] - (\mathbb{E}[\epsilon \mid X = x])^2.$$

- An important implication of this is that

$$\text{Var}(\epsilon(X - \mu_X)) = \text{Var}(\epsilon) \text{Var}(X) = \sigma_\epsilon^2 \sigma_X^2.$$

Questions?



Make the following assumptions

1. **Random Sampling:** Assume that  $\{Y_i, X_i\}$  are independently and identically distributed;  $(Y_i, X_i) \stackrel{\text{i.i.d}}{\sim} (Y, X)$
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3. **Rank Condition:** There must be at least two distinct values of  $X$  that appear in the population.
  - Need at least two distinct points to make a line.
  - If there is only one distinct point then our minimization problem is undefined.

Make the following assumptions

1. **Random Sampling:** Assume that  $\{Y_i, X_i\}$  are independently and identically distributed;  $(Y_i, X_i) \stackrel{\text{i.i.d.}}{\sim} (Y, X)$
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And that's it!

Given these assumptions (Random Sampling, Homoskedasticity, Rank Condition) let's try and figure out what the approximate distribution is of  $\hat{\beta}_1$ .

Recall that

$$\hat{\beta}_1 = \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}$$

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By definition of  $\epsilon = Y - \beta_0 - \beta_1 \cdot X$ :

$$Y = \beta_0 + \beta_1 \cdot X;$$

and that by the first order conditions of  $\beta_0$  and  $\beta_1$ :

$$\mathbb{E}[\epsilon] = 0$$

$$\mathbb{E}[\epsilon \cdot X] = 0$$

We will also make use of the following results from our probability review. If  $Z$  is a random variables and we have i.i.d observations  $Z_1, Z_2, \dots, Z_n$ :

The **Law of Large Numbers** states that as  $n \rightarrow \infty$ :

$$\bar{Z} \rightarrow \mathbb{E}[Z]$$

or, equivalently,  $\bar{Z} \approx \mathbb{E}[Z]$  for  $n$  large.

The **Central Limit Theorem** states that as  $n \rightarrow \infty$ , approximately,

$$\sqrt{n} (\bar{Z} - \mathbb{E}[Z]) \sim N(0, \text{Var}(Z))$$

or, equivalently,  $\bar{Z} \sim N(\mathbb{E}[Z], \text{Var}(Z)/n)$ .

Starting with:

$$\sqrt{n}\hat{\beta}_1 = \frac{\sqrt{n} \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}.$$

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Expand  $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$  and  $\bar{Y} = \beta_0 + \beta_1 \bar{X} + \bar{\epsilon}$ , where  $\bar{\epsilon} = \frac{1}{n} \sum_{i=1}^n \epsilon_i$ :

$$\sqrt{n}\hat{\beta}_1 = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n (\beta_1(X_i - \bar{X}) + (\epsilon_i - \bar{\epsilon})) (X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}.$$

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Distribute to get:

$$\sqrt{n}\hat{\beta}_1 = \sqrt{n}\beta_1 \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} + \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n (\epsilon_i - \bar{\epsilon})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}.$$



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So we have that:

$$\sqrt{n} \left( \hat{\beta}_1 - \beta_1 \right) = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n (\epsilon_i - \bar{\epsilon})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}.$$

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Using **Law of Large Numbers** replace  $\bar{\epsilon} \approx \mathbb{E}[\epsilon] = 0$ ,  $\bar{X} \approx \mu_X$ , and  $\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \approx \sigma_X^2$ :

$$\sqrt{n} \left( \hat{\beta}_1 - \beta_1 \right) \approx \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i (X_i - \mu_X)}{\sigma_X^2}.$$

Finally, note that by **Central Limit Theorem**, since

$$\mathbb{E}[\epsilon(X_i - \mu_X)] = \mathbb{E}[\epsilon X_i] - \mathbb{E}[\epsilon]\mu_X = 0.$$

we have that (approximately for large  $n$ ):

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i(X_i - \mu_X) \sim N\left(0, \text{Var}(\epsilon(X - \mu_X))\right).$$

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Putting this all together, we have that, approximately for  $n$  large;

$$\sqrt{n} \left( \hat{\beta}_1 - \beta_1 \right) \sim \frac{N(0, \sigma_\epsilon^2 \sigma_X^2)}{\sigma_X^2} = N\left(0, \underbrace{\sigma_\epsilon^2 / \sigma_X^2}_{:= \sigma_{\beta_1}^2}\right).$$

where in the last equality we use the fact that  $N(0, a)/b \sim N(0, a/b^2)$ .

Putting this all together, we have that, approximately for  $n$  large;

$$\sqrt{n} \left( \hat{\beta}_1 - \beta_1 \right) \sim \frac{N(0, \sigma_\epsilon^2 \sigma_X^2)}{\sigma_X^2} = N\left(0, \underbrace{\sigma_\epsilon^2 / \sigma_X^2}_{:= \sigma_{\beta_1}^2}\right).$$

where in the last equality we use the fact that  $N(0, a)/b \sim N(0, a/b^2)$ . Other ways of putting this are, approximately for  $n$  large:

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma_{\beta_1}^2/n\right)$$
$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0, 1)$$

where as a reminder  $\sigma_{\beta_1} = \sigma_\epsilon / \sigma_X$ . This last form is what we will use the most.

Following similar steps we can derive the approximate distribution of  $\hat{\beta}_0$  as well as the covariance between  $\hat{\beta}_0$  and  $\hat{\beta}_1$ :

$$\begin{aligned}\sqrt{n} \left( \hat{\beta}_1 - \beta_1 \right) &\sim N \left( 0, \frac{\sigma_\epsilon^2}{\sigma_X^2} \right) \\ \sqrt{n} \left( \hat{\beta}_0 - \beta_0 \right) &\sim N \left( 0, \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{\sigma_X^2} \right) \\ \text{Cov}(\hat{\beta}_1, \hat{\beta}_0) &= -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}\end{aligned}$$



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Important to remember these! The above is just providing intuition on how we get these results.

For large  $n$  we have that

$$\text{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{n \cdot \sigma_X^2}, \quad \text{Var}(\hat{\beta}_0) = \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{n \cdot \sigma_X^2}, \quad \text{and} \quad \text{Cov}(\hat{\beta}_1, \hat{\beta}_0) = -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}.$$

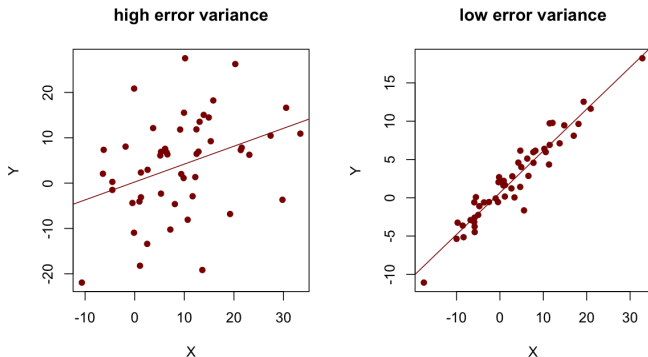
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## Linear Regression: Asymptotic Variances

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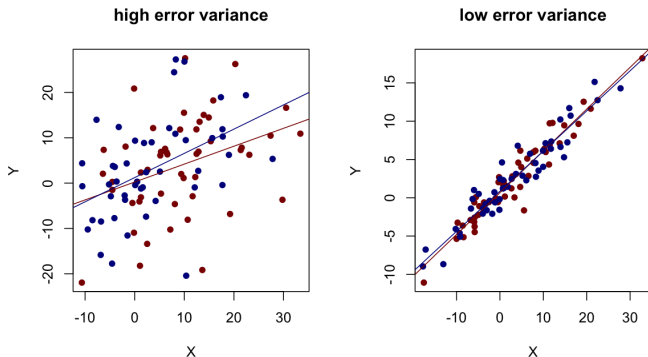


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**Intuition:** If points are more tightly distributed around the regression line it is easier to tell what the regression line is.

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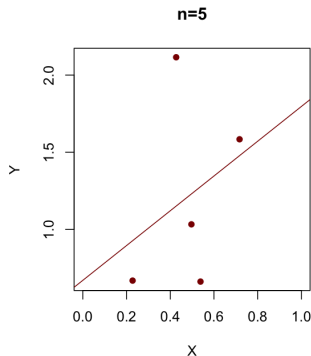
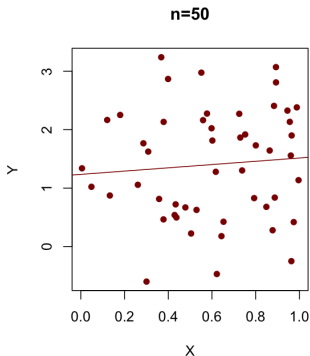
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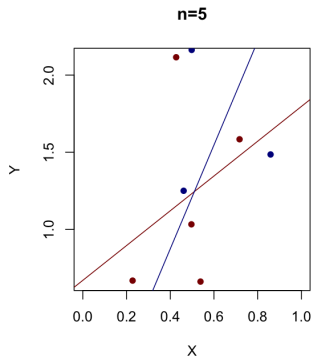
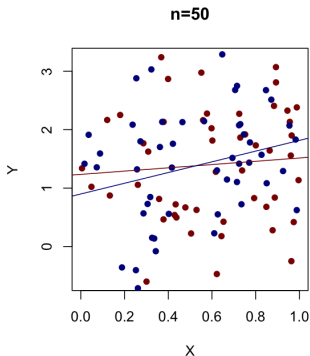


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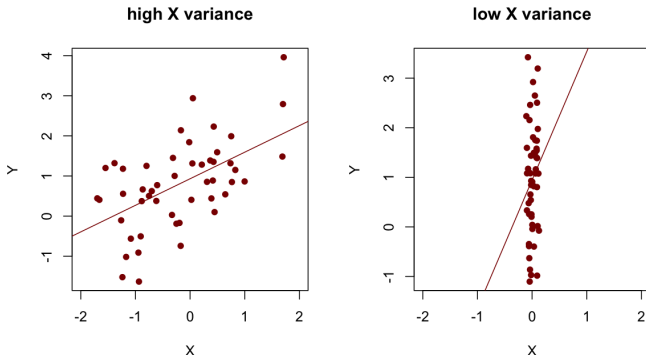
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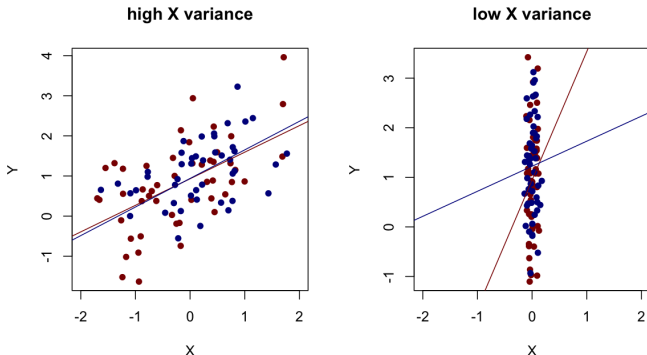


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Questions?

**Positive Result:** Under homoskedasticity, for  $n$  large, we have (approximately)

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0, 1).$$

where

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$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \approx \text{Var}(X).$$

- But what about  $\text{Var}(\epsilon) = \sigma_{\epsilon}^2$ ?



To estimate  $\text{Var}(\epsilon)$  we first construct estimated residuals  $\hat{\epsilon}_i$  via

$$\hat{\epsilon}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i.$$

Because  $\hat{\beta}_1 \rightarrow \beta_1$  and  $\hat{\beta}_0 \rightarrow \beta_0$  we can say that  $\hat{\epsilon}_i \approx \epsilon_i = Y_i - \beta_0 - \beta_1 X_i$  (for  $n$  large).

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Also by the first order conditions for  $\hat{\beta}_0$  we have that

$$-\frac{1}{n} \sum_{i=1}^n \underbrace{(Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i)}_{=\hat{\epsilon}_i} = 0.$$

so that

$$\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i = \bar{\hat{\epsilon}}_i = 0.$$

## Linear Regression: Variance Estimation

---

Putting this together we can estimate  $\text{Var}(\epsilon) = \sigma_\epsilon^2$  by calculating the sample variance of  $\hat{\epsilon}_i$ :

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By  $\hat{\beta}_1 \rightarrow \beta_1$  and  $\hat{\beta}_0 \rightarrow \beta_0$  as  $n \rightarrow \infty$ ;

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By **Law of Large Numbers**;

$$\approx \mathbb{E}[\epsilon^2]$$

By  $\mathbb{E}[\epsilon] = 0$ ;

$$= \text{Var}(\epsilon) = \sigma_\epsilon^2$$

Putting all of this together, we can estimate  $\sigma_{\beta_1}^2 = \frac{\sigma_X^2}{\sigma_X^2}$  via;

$$\hat{\sigma}_{\beta_1}^2 = \frac{\hat{\sigma}_\epsilon^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \approx \sigma_{\beta_1}^2.$$

since for large  $n$

$$\hat{\sigma}_\epsilon^2 = \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2 \approx \sigma_\epsilon^2$$

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \approx \sigma_X^2.$$

Now, since we have that (approximately, for large  $n$ ):

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0, 1).$$

And since, as we have established above,  $\hat{\sigma}_{\beta_1} \approx \sigma_{\beta_1}$ , for large  $n$  we can say that (approximately)

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0, 1).$$



The quantity  $\hat{\sigma}_{\beta_1}/\sqrt{n}$  is often referred to as the **standard error** of  $\hat{\beta}_1$ .

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In general, if we have a parameter  $\theta$  that we estimate with  $\hat{\theta}$ , the quantity  $\hat{\sigma}_{\theta}/\sqrt{n}$  will be referred to as the **standard error** of  $\hat{\theta}$  where

$$\hat{\sigma}_{\theta} = \sqrt{\text{Var}(\hat{\theta})}.$$

Questions?

Let's return to our example and see why this characterization is useful. Recall that in our example we are interested in the regression parameters from regression  $Y = INC$  (income in thousands of dollars) against  $X = EDU$  (years of education).

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After collecting a sample size of 100,  $\{Y_i, X_i\}_{i=1}^{100}$  we find that:

$$\hat{\beta}_1 = 0.5$$

$$\frac{1}{n} \sum_{i=1}^n \epsilon_i^2 = 25$$

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = 16$$

Our friend His Majesty Prince Harry claims there is no relationship between education and income,  $\beta_1 = 0$ . We claim that observing the magnitude of  $|\hat{\beta}_1| = 0.5$  is evidence against this claim. Who is right?



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Want to use the (asymptotic) distribution of  $\hat{\beta}_1$  to answer this question.

- First need to estimate  $\sigma_{\beta_1}$ .

Using  $\hat{\sigma}_\epsilon^2 = \frac{1}{n} \sum_{i=1}^n \epsilon_i^2 = 25$ , and  $\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = 16$ ) we calculate

$$\begin{aligned}\hat{\sigma}_{\beta_1}^2 &= \frac{\hat{\sigma}_\epsilon^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} \\ &= \frac{25}{16}\end{aligned}$$

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Using this, we find that  $\hat{\sigma}_{\beta_1} = \sqrt{\hat{\sigma}_{\beta_1}^2} = \frac{5}{4}$ .

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If the true value of  $\beta_1 = 0$  this means that

$$\frac{\hat{\beta}_1}{5/40} = \frac{\hat{\beta}_1}{0.125} \sim N(0, 1).$$



## Linear Regression: Why Asymptotic Distribution?

---

Given that if  $\beta_0 = 0$ ,  $\hat{\beta}_1/0.125 \sim N(0, 1)$ , what is the probability of us observing  $|\hat{\beta}_1| \geq 0.5$ ?

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$$\begin{aligned}&= \Pr(Z \geq 4) + \Pr(Z \leq -4) \\ &= 2\Pr(Z \geq 4)\end{aligned}$$

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$$\approx 0.00006$$

Using the asymptotic distribution result

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0, 1),$$

we have found that if  $\beta_1 = 0$ , then  $\Pr(|\hat{\beta}_1| \geq 0.5) \approx 0.0006$ .

Using the asymptotic distribution result

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we have found that if  $\beta_1 = 0$ , then  $\Pr(|\hat{\beta}_1| \geq 0.5) \approx 0.0006$ .

So, given that we observed  $\hat{\beta}_1 = 0.5$ , it seems very unlikely that  $\beta_1 = 0$ . We can conclude against Prince Harry's claim.

Questions?



# Table of Contents

---

The Basic Model

Estimation

Asymptotic Distribution

Hypothesis Testing and Confidence Intervals

Conclusion

The last exercise where we tested whether Prince Harry's claim made sense was an example of a **hypothesis test**.

In this section we will formally discuss hypothesis testing.

## Linear Regression: What is a Hypothesis Test?

---

Often in linear regression analysis, we are interested in using parameter estimates,  $\hat{\beta}_0$  and  $\hat{\beta}_1$ , to test some baseline or null hypothesis about the population against an opposite or alternative hypothesis.

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  - Alternative Hypothesis:  $\beta_1 \neq 0 \iff |\beta_1| > 0$
- Smoking has a negative effect on life expectancy
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Often in linear regression analysis, we are interested in using parameter estimates,  $\hat{\beta}_0$  and  $\hat{\beta}_1$ , to test some baseline or null hypothesis about the population against an opposite or alternative hypothesis.

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  - Null Hypothesis:  $\beta_1 = 0$ .
  - Alternative Hypothesis:  $\beta_1 \neq 0 \iff |\beta_1| > 0$
- Smoking has a negative effect on life expectancy
  - Null Hypothesis:  $\beta_1 \leq 0$
  - Alternative Hypothesis:  $\beta_1 > 0$
- There is a positive association between the miles per gallon of a car and its final sales price
  - Null Hypothesis:  $\beta_1 \geq 0$
  - Alternative Hypothesis:  $\beta_1 < 0$

We will denote the null hypothesis as  $H_0$  and the alternative as  $H_1$ .

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  - $H_0: \beta_1 \geq 0$
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If  $H_1$  contains a “ $\neq$ ” sign, we call this a “two-sided” alternative.

**Example:** There is no association between years of education and income

- $H_0: \beta_1 = 0$
- $H_1: \beta_1 \neq 0$

If  $H_1$  contains a “ $\neq$ ” sign, we call this a “two-sided” alternative.

**Example:** There is no association between years of education and income

- $H_0: \beta_1 = 0$
- $H_1: \beta_1 \neq 0$

If  $H_1$  contains a “ $>$ ” or a “ $<$ ” sign, we call this a “one-sided” alternative.

**Example:** Cups of coffee drank has a negative association with hours of sleep

- $H_0: \beta_1 \leq 0$
- $H_1: \beta_1 > 0$

## Linear Regression: How to Hypothesis Test

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So, how do we use our data and parameter estimates  $\hat{\beta}_1$  and  $\hat{\beta}_0$  to test hypotheses? Given a null hypothesis  $H_0$  and an alternative hypothesis, we have two options.

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- We can **reject** the null hypothesis in favor of the alternative hypothesis.
  - Do this when the probability of obtaining our observed value of  $\hat{\beta}$  (or something even further from the null hypothesis) under the null hypothesis is smaller than a pre-specified value  $\alpha$ .

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  - The value  $\alpha$  is called the “level” or “significance level” of the test.
  - It is also the probability of a “Type 1” error, the probability that we will reject the null hypothesis when the null hypothesis is true.
- We can **fail to reject** the null hypothesis.
  - Do this when the probability of obtaining our observed value of  $\hat{\beta}$  (or something even further from the null hypothesis) under the null hypothesis is larger than a pre-specified value  $\alpha$ .



How do we calculate the probability, given that our null hypothesis is true, of observing our value of  $\hat{\beta}$  or something even further from the null hypothesis?

Recall that, approximately for large  $n$

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0, 1) \quad \text{and} \quad \frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}_{\beta_0}/\sqrt{n}} \sim N(0, 1).$$

where  $\hat{\sigma}_{\beta_1}^2 = \hat{\sigma}_\epsilon^2 / \hat{\sigma}_X^2$  and  $\hat{\sigma}_{\beta_0}^2 = \frac{1}{n} \sum_{i=1}^n X_i^2 \cdot \hat{\sigma}_\epsilon^2 / \hat{\sigma}_X^2$ .

Let  $Z \sim N(0, 1)$ . Using the distributions above, if we are testing  $H_0 : \beta_1 = b$  against  $H_1 : \beta_1 \neq b$  we can compute the probability (under the null hypothesis) that we observe our value of  $\hat{\beta}_1$  or something even further from the null hypothesis by computing

$$\Pr \left( |Z| > \left| \frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1} / \sqrt{n}} \right| \right).$$

This probability is called the **p-value** and we reject our null hypothesis if the **p-value**,  $p$ , is less than  $\alpha$ .

If we are testing  $H_0 : \beta_1 \geq b$  against  $H_1 : \beta_1 < b$  we can compute the probability (under the null hypothesis) that we observe our value of  $\hat{\beta}_1$  or something even further from the null hypothesis by computing

$$\Pr \left( Z < \frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1} / \sqrt{n}} \right).$$

This probability is called the **p-value** and we reject our null hypothesis if the **p-value**,  $p$ , is less than  $\alpha$ .

If we are testing  $H_0 : \beta_1 \leq b$  against  $H_1 : \beta_1 > b$  we can compute the probability (under the null hypothesis) that we observe our value of  $\hat{\beta}_1$  or something even further from the null hypothesis by computing

$$\Pr \left( Z > \frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1} / \sqrt{n}} \right).$$

This probability is called the **p-value** and we reject our null hypothesis if the **p-value**,  $p$ , is less than  $\alpha$ .

In summary, the test above can be conducted as follows. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

1. Compute the test statistic

$$t^* = \frac{\hat{\beta} - b}{\hat{\sigma}_{\beta} / \sqrt{n}}.$$

In summary, the test above can be conducted as follows. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

2. Compute the p-value, the probability that we would obtain our observed value of  $\hat{\beta}$ , or something even further from the null hypothesis, if the null hypothesis was correct

- If  $H_0 : \beta = b$  and  $H_1 : \beta \neq b$  compute

$$p = \Pr(|Z| > |t^*|) = 2 \Pr(Z > |t^*|).$$

- If  $H_0 : \beta \leq b$  and  $H_1 : \beta > b$  compute

$$p = \Pr(Z > t^*).$$

- If  $H_0 : \beta \geq b$  and  $H_1 : \beta < b$  compute

$$p = \Pr(Z < t^*).$$

In summary, the test above can be conducted as follows. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

3. **Reject** the null hypothesis in favor of the alternative hypothesis if  $p > \alpha$ .  
Otherwise **fail to reject** the null hypothesis.

## Linear Regression: Hypothesis Testing Example

---

Let's see this work in practice. Our close personal friend Jason Derulo claims that there is a negative association between a car's miles per gallon,  $X$ , and its sales price in thousands of dollars,  $Y$ .



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We want to use data to test this claim. We collect a random (i.i.d) sample of size 64,  $\{Y_i, X_i\}_{i=1}^{64}$  of cars and find

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X}) = 4$$

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = 16$$

$$\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2 = 36$$

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$$\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2 = 36$$

We will this data to test Derulo's claim,  $H_0 : \beta_1 \leq 0$ , against an alternate hypothesis,  $H_1 : \beta_1 > 0$ .

## Linear Regression: Hypothesis Testing Example

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In order to test this null hypothesis (against it's alternative) we need to calculate the test statistic  $t^* = \frac{\hat{\beta}_1 - 0}{\hat{\sigma}_{\beta_1} / \sqrt{n}}$ .

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$$\hat{\beta}_1 = \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} = \frac{4}{16} = 0.25$$
$$\hat{\sigma}_{\beta_1} = \frac{\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} = \frac{36}{16}$$

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Using this, we compute the test statistic

$$t^* = \frac{0.25}{\sqrt{36/16}/\sqrt{64}} \approx 1.333.$$

## Linear Regression: Hypothesis Testing Example

---

Using this test statistic,  $t^* \approx 1.333$ , let's conduct the following test at level  $\alpha = 0.1$

$$H_0 : \beta_1 \leq 0 \quad \text{and} \quad H_1 : \beta_1 > 0.$$

Using this test statistic,  $t^* \approx 1.333$ , let's conduct the following test at level  $\alpha = 0.1$

$$H_0 : \beta_1 \leq 0 \text{ and } H_1 : \beta_1 > 0.$$

Compute the p-value

$$p = \Pr(Z > 1.333) = 1 - \Pr(Z \leq 1.333) = 1 - 0.908 = 0.092.$$

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$$p = \Pr(Z > 1.333) = 1 - \Pr(Z \leq 1.333) = 1 - 0.908 = 0.092.$$

Because the  $p$ -value, 0.092 is less than  $\alpha = 0.1$ , we **reject** the null hypothesis that there is a negative association between miles per gallon and sales price in favor of the alternative that there is a positive relationship between the two.



Now given  $t^* \approx 1.333$ , suppose that we wanted to conduct a two sided test at level  $\alpha = 0.1$ . That is, suppose we wanted to test

$$H_0 : \beta_1 = 0 \quad \text{and} \quad H_1 : \beta_1 \neq 0.$$

Now given  $t^* \approx 1.333$ , suppose that we wanted to conduct a two sided test at level  $\alpha = 0.1$ . That is, suppose we wanted to test

$$H_0 : \beta_1 = 0 \quad \text{and} \quad H_1 : \beta_1 \neq 0.$$

Compute the  $p$  value for a two-sided test

$$p = \Pr(|Z| > |t^*|) = 2 \Pr(Z > |t^*|) = 2(1 - \Pr(X \leq 1.333)) = 2 \cdot 0.092 \approx 0.194.$$

Now given  $t^* \approx 1.333$ , suppose that we wanted to conduct a two sided test at level  $\alpha = 0.1$ . That is, suppose we wanted to test

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Given that  $p = 0.194 > 0.1$  we **fail to reject** the null hypothesis that there is no relationship between miles per gallon and sales price.

Notice that the p-value for a two-sided test was twice the p-value for the one-sided test! The reverse is not necessarily true however.

Why?

- Suppose  $t^* = 1.64$  so that the p-value for a two sided test is

$$\Pr(|Z| > 1.64) = 2 \Pr(Z > 1.64) = 0.1.$$

- What is the p-value for the test  $H_0 : \beta_1 \leq 0$  against  $H_1 : \beta_1 > 0$ ?
- What is the p-value for the test  $H_0 : \beta_1 \geq 0$  against  $H_1 : \beta_1 < 0$ ?

Questions?

Conducting the test above can also follow another standard procedure. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

1. Compute the test statistic or “t-statistic”

$$t^* = \frac{\hat{\beta} - b}{\hat{\sigma}_{\beta} / \sqrt{n}}.$$

Conducting the test above can also follow another standard procedure. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

- For a given level  $\alpha$  compute  $z_{1-\alpha}$  for a one sided alternative or  $z_{1-\alpha/2}$  for a 2 sided alternative, where  $z_{1-\alpha}$  and  $z_{1-\alpha/2}$  are such that

$$\Pr(Z > z_{1-\alpha}) = \alpha \quad \text{and} \quad \Pr(Z > z_{1-\alpha/2}) = \frac{\alpha}{2}.$$

These are called the  $1 - \alpha$  and  $1 - \alpha/2$  **quantiles** of the standard normal distribution, respectively.

- $z_{0.9} \approx 1.28$
- $z_{0.95} \approx 1.64$
- $z_{0.975} \approx 1.96$
- $z_{0.99} \approx 2.32$
- $z_{0.995} \approx 2.57$

Conducting the test above can also follow another standard procedure. Suppose  $H_0 : \beta \leq b$ ,  $H_0 : \beta \geq b$ , or  $H_0 : \beta = b$

3. Compare the test statistic  $t^*$  to the quantile  $z_{1-\alpha}$  or  $z_{1-\alpha/2}$ .

- If  $H_0 : \beta = b$  and  $H_1 : \beta \neq b$ , **reject** if  $|t^*| > z_{1-\alpha/2}$
- If  $H_0 : \beta \geq b$  and  $H_1 : \beta < b$ , **reject** if  $t^* < -z_{1-\alpha}$
- If  $H_0 : \beta \leq b$  and  $H_1 : \beta > b$ , **reject** if  $t^* > z_{1-\alpha}$

Otherwise, **fail to reject** the null hypothesis.



Let's return to the hypothesis testing example from earlier to verify that this procedure gives the same results as comparing p-values.

Recall that in this example our friend Jason Derulo has claimed that there is a negative association between miles per gallon of a car and sales price of a car. That is we want to test at level  $\alpha = 0.1$

$$H_0 : \beta_1 \leq 0 \text{ vs. } H_1 : \beta_1 > 0.$$

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$$H_0 : \beta_1 \leq 0 \text{ vs. } H_1 : \beta_1 > 0.$$

After collecting data, we find that  $t^* \approx 1.333$ . To test this hypothesis, we will compare this value to  $z_{1-0.1} = z_{0.9} = 1.28$ . We are conducting a one sided alternative ( $>$  sign) so we look to see if  $t^* > z_{0.9}$ .

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After collecting data, we find that  $t^* \approx 1.333$ . To test this hypothesis, we will compare this value to  $z_{1-0.1} = z_{0.9} = 1.28$ . We are conducting a one sided alternative ( $>$  sign) so we look to see if  $t^* > z_{0.9}$ .

Since  $t^* \approx 1.3333 > z_{0.9} = 1.28$  we **reject** the null hypothesis that there is a negative association between miles per gallon of a car and sales price of a car in favor of the alternative hypothesis that there is a positive relationship.

- Same result as when using the p-value

Now let's use this procedure to test at level  $\alpha = 0.1$

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0.$$

Because we are dealing with a two sided alternative ( $\neq$  sign) we have to compare  $|t^*|$  to  $z_{1-\alpha/2} = z_{1-0.1/2} = z_{0.95}$ .

Now let's use this procedure to test at level  $\alpha = 0.1$

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0.$$

Because we are dealing with a two sided alternative ( $\neq$  sign) we have to compare  $|t^*|$  to  $z_{1-\alpha/2} = z_{1-0.1/2} = z_{0.95}$ .

Since  $t^* \approx 1.333 < z_{0.95} = 1.64$  we **fail to reject** the null hypothesis against a two-sided.

Questions?

Given our data  $\{Y_i, X_i\}_{i=1}^n$  we now know how to construct estimates,  $\hat{\beta}_0, \hat{\beta}_1$  of the linear model parameters  $\beta_0, \beta_1$  where

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ \left( Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X \right)^2 \right].$$

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As a reminder, these parameters  $\beta_0, \beta_1$  can equivalently be described as coming from a linear model

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

where  $\mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0$ . The term  $\epsilon$  is called the “linear regression error”.



Also given our data  $\{Y_i, X_i\}_{i=1}^n$  we know how to test hypothesis about the linear regression parameters  $\beta_0$  and  $\beta_1$  such as

$$H_0 : \beta_1 \geq 6 \text{ vs. } H_1 : \beta_1 < 6.$$

or

$$H_0 : \beta_0 = 0 \text{ vs. } H_1 : \beta_0 \neq 0.$$

## Linear Regression: Confidence Intervals

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Now, given our data  $\{Y_i, X_i\}_{i=1}^n$  we want to do is construct a range of values that we are “confident” that the true parameter,  $\beta_0$  or  $\beta_1$  lies in.

Now, given our data  $\{Y_i, X_i\}_{i=1}^n$  we want to do is construct a range of values that we are “confident” that the true parameter,  $\beta_0$  or  $\beta_1$  lies in.

We call this range of values a  $100 \cdot (1 - \alpha)\%$  **Confidence Interval**.

- e.g if  $\alpha = 0.05$  we would want to construct a 95% confidence interval.

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- e.g if  $\alpha = 0.05$  we would want to construct a 95% confidence interval.

What values should we include in a  $100 \cdot (1 - \alpha)\%$  Confidence Interval?

- Any value  $b$  for which we would not reject  $H_0 : \beta = b$  against a two sided alternative  $H_1 : \beta \neq b$  at level  $\alpha$ .

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Recall that we reject  $H_0 : \beta = b$  in favor of  $H_1 : \beta \neq b$  if

$$|t^*| = \left| \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}} \right| > z_{1-\alpha/2}.$$

What values should we include in a  $100 \cdot (1 - \alpha)\%$  Confidence Interval?

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$$|t^*| = \left| \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}} \right| > z_{1-\alpha/2}.$$

We fail to reject  $H_0 : \beta = b$  in favor of  $H_1 : \beta \neq b$  if

$$\left| \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}} \right| \leq z_{1-\alpha/2}.$$

Equivalently we can say that we fail to reject  $H_0 : \beta = b$  in favor of  $H_1 : \beta \neq b$  if

$$\hat{\beta} - z_{1-\alpha/2} \cdot \left( \hat{\sigma}_{\beta} / \sqrt{n} \right) \leq b \leq \hat{\beta} + z_{1-\alpha/2} \cdot \left( \hat{\sigma}_{\beta} / \sqrt{n} \right).$$



Equivalently we can say that we fail to reject  $H_0 : \beta = b$  in favor of  $H_1 : \beta \neq b$  if

$$\hat{\beta} - z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}) \leq b \leq \hat{\beta} + z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}).$$

Thus our  $100 \cdot (1 - \alpha)\%$  confidence interval is given

$$\left[ \hat{\beta} - z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}), \hat{\beta} + z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}) \right].$$

Equivalently we can say that we fail to reject  $H_0 : \beta = b$  in favor of  $H_1 : \beta \neq b$  if

$$\hat{\beta} - z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}) \leq b \leq \hat{\beta} + z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}).$$

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$$\left[ \hat{\beta} - z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}), \hat{\beta} + z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}) \right].$$

This is interpreted as: we are  $100 \cdot (1 - \alpha)\%$  confident that the true value of  $\beta$  lies in the interval

$$\left[ \hat{\beta} - z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}), \hat{\beta} + z_{1-\alpha/2} \cdot (\hat{\sigma}_\beta / \sqrt{n}) \right].$$

## Linear Regression: Confidence Interval Example

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Our data then looks like  $\{Y_i, X_i\}_{i=1}^{100}$  where  $Y_i \in \{0, 1\}$  denotes a person's vaccination status and  $X_i \in [0, 100]$  denotes the cash incentive offered to people. We want to preform construct a confidence interval for the parameter  $\beta_1$  from the linear model

$$Y = \beta_0 + \beta_1 \cdot X_i + \epsilon_i, \quad \mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0.$$

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- As a reminder we can think of this model as generated by the line of best fit parameters

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ (Y - \tilde{\beta}_0 - \tilde{\beta}_1 X)^2 \right].$$

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- Important for the government, when considering a policy, to not only have a point estimate of the effect but also a measure of how confident we are in the point estimate.

After collecting our data  $\{Y_i, X_i\}_{i=1}^{100}$  we find that

$$\frac{1}{n} \sum_{i=1}^n X_i^2 = 6$$

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 = 4$$

$$\frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2 = 0.25$$

$$\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X}) = 0.1$$



Using this data we compute

$$\hat{\beta}_1 = \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} = \frac{0.1}{4} = 0.025$$

$$\hat{\sigma}_{\beta_1}^2 = \frac{\hat{\sigma}_\epsilon^2}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2} = \frac{0.25}{4} = 0.0625$$

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**Question:** Given that  $Y \in \{0, 1\}$ , how do we interpret  $\hat{\beta}_1$  in this context? How would we interpret  $\hat{\beta}_0$  in this context?

## Linear Regression: Confidence Interval Example

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Now let's construct a 95% confidence interval for  $\beta_1$ .

## Linear Regression: Confidence Interval Example

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Now let's construct a 95% confidence interval for  $\beta_1$ . Recall that a  $100 \cdot (1 - \alpha)\%$  confidence interval for  $\beta_1$  is given by

$$\hat{\beta}_1 \pm z_{1-\alpha/2} \cdot \frac{\hat{\sigma}_{\beta_1}}{\sqrt{n}}.$$

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In this case  $\alpha = 0.05$ . From above we have that  $z_{0.975} \approx 1.96$ . Plugging in our values from above the 95% confidence interval for  $\beta_1$  is given

$$0.025 \pm 1.96 \cdot \frac{\sqrt{0.0625}}{\sqrt{100}} = 0.025 \pm 1.96 \cdot \frac{0.25}{10} = [-0.024, 0.074].$$

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Questions:

1. How do we interpret this confidence interval?
2. Suppose we wanted to test  $H_0 : \beta_1 = 0$  vs  $H_1 : \beta_1 \neq 0$  at level  $\alpha = 0.05$ . What would be the result?



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  - What about if we wanted to test this hypothesis at level  $\alpha = 0.025$ ?

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The Basic Model

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Hypothesis Testing and Confidence Intervals

Conclusion

In this lecture we have introduced the line of best fit parameters

$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E} \left[ (Y - \beta_0 - \beta_1 X)^2 \right]$$

After taking  $\epsilon = Y - \beta_0 - \beta_1 X$ , these parameters generate the linear model

$$Y = \beta_0 + \beta_1 X + \epsilon, \quad \mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0.$$

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While the linear model is often easier to work with, it is useful to keep the line of best fit interpretation in the back of our mind. It provides our model interpretability even when the true relationship between  $Y$  and  $X$  is not linear.

Since we do not know the joint distribution of  $(Y, X)$ , we have to use data,  $\{Y_i, X_i\}_{i=1}^n$  to estimate  $\hat{\beta}_0$  and  $\hat{\beta}_1$

$$\hat{\beta}_0, \hat{\beta}_1 = \arg \min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^n (Y_i - b_0 - b_1 X_i)^2.$$

Taking first order conditions this gives

$$\begin{aligned}\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X} \\ \hat{\beta}_1 &= \frac{\frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2}\end{aligned}$$

We also derived the asymptotic distribution of our estimates. Using the law of large numbers and the central limit theorem we can say that, under homoskedasticity, approximately for large  $n$ ,

$$\hat{\beta}_0 \sim N\left(\beta_0, \mathbb{E}[X^2] \frac{\hat{\sigma}_\epsilon^2}{n\sigma_X^2}\right)$$
$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma_\epsilon^2}{n\sigma_X^2}\right)$$

Estimation of  $\hat{\sigma}_\epsilon^2 = \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2$

Finally, we covered how to use these asymptotic distributions and our data to test various hypothesis about the underlying parameters such as

$$H_0 : \beta_0 = 5 \text{ vs. } H_1 : \beta_0 \neq 5$$

or

$$H_0 : \beta_1 \leq 0 \text{ vs. } H_1 : \beta_1 > 0$$

As well as construct confidence intervals for the parameters  $\beta_0$  and  $\beta_1$ .

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As well as construct confidence intervals for the parameters  $\beta_0$  and  $\beta_1$ .

- These sorts of inferential results are important for policy analysis and separate the econometrics/statistics approaches from machine learning