

Econ 103: Probability and Statistics

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Single Random Variable

- Discrete and Continuous random variables
- Mean, variance, and expectations

Multiple Random Variables

- Conditional probabilities and conditional means
- Covariance and independence

The Normal Distribution

- Properties and computing probabilities

The Law of Large Numbers and the Central Limit Theorem

- The sample mean as a random variable

Question: What is a random variable?

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While the outcome is random, the random variable does have a *distribution*. For any subset of the outcome space, the distribution describes the probability that the random variable takes a value in that subset.

- **Example:** We know that our flipped coin has a 50% probability of taking a value in the set $\{H\}$, a 50% probability of taking a value in the set $\{T\}$, and a 100% probability of taking a value in the set $\{H, T\}$.

Question: Why do we care about random variables? What does this have to do with econometrics?

Consider the population of California. Suppose we want to know about the education levels of people in the population.

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 - i.e 30% of people have high school diplomas, 40% of people have college degrees, etc.
- In general however, we may not know the exact distribution of the random variable. Econometrics is about using a random sample of data to make inferences about the underlying distribution of the random variable.

Single Random Variables: Outcome Spaces

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If the outcome space of X is *countable* (think finite), then we say that X is a **discrete random variable**. If the outcome space of X is *uncountable* (think infinite), we say that X is a **continuous random variable**.

- Flipping a coin and rolling a die would be discrete random variables
- The 100m sprint time of an Olympic athlete would be a continuous random variable.

Single Random Variables: Outcome Spaces and Probability

In general in this class, we will notate the outcome space of a random variable X as \mathcal{O}_X . Let $2^{\mathcal{O}_X}$ denote all the subsets of \mathcal{O}_X .

We will typically be interested in the probability that X takes values in some $A \in 2^{\mathcal{O}_X}$ (that is $A \subseteq \mathcal{O}_X$). This probability is a number between 0 and 1 and will be notated as $\mathbb{P}_X(A)$. We will require the probability $\mathbb{P}_X(\cdot)$ to satisfy certain properties:

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- If A_1, A_2, \dots are pairwise disjoint, then $\mathbb{P}_X(\cup_i A_i) = \sum_i \mathbb{P}_X(A_i)$.

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When we say we are interested in the *distribution* of the random variable X , we really mean we are interested in $\mathbb{P}_X(\cdot)$ as viewed as a map from $2^{\mathcal{O}_X}$ onto $[0, 1]$.

If X is a **discrete random variable** the distribution or probability function \mathbb{P}_X can be described by the *probability mass function* or *pmf*, $p_X(\cdot) : \mathcal{O}_X \rightarrow [0, 1]$.

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For each element a of the outcome space ($a \in \mathcal{O}_X$), the probability mass function evaluated at a , $p_X(a)$, describes the probability that X takes value a . That is $p_X(a) = \mathbb{P}_X(\{a\})$.

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By the last property of probability measures, the pmf can be used to recover the probability that X takes values in any subset A of the outcomes space \mathcal{O}_X

$$\mathbb{P}_X(A) = \sum_{a \in A} \mathbb{P}_X(\{a\}) = \sum_{a \in A} p_X(a).$$

Single Random Variables: Discrete Random Variables

Let's see an example of this. Let X denote the outcome of a fair dice roll. We can describe the distribution of X via the probability mass function

$$p_X(a) = \begin{cases} \frac{1}{6} & \text{if } a \in \{1, 2, 3, 4, 5, 6\} \\ 0 & \text{for any other value of } a \end{cases}$$

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Of course, this result is a bit obvious. However, if the die was not fair, we would follow the same procedure to compute this probability.

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$$\mathbb{P}_X(A) = \sum_{a \in A} \mathbb{P}_X(\{a\}) = \infty.$$

- So we must have $\mathbb{P}_X(\{a\}) = 0$ for all $a \in \mathcal{O}_X$.

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This rules out being able to use a pmf to describe the distribution of a continuous random variable.

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The identity above as well as the rules for the probability measure \mathbb{P}_X can be used to calculate $\mathbb{P}_X(A)$ for any set $A \subseteq \mathcal{O}_X$.

Example: Let X be a continuous random variable with pdf f_X given

$$f_X(a) = \begin{cases} 1 & \text{if } 0 \leq a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is called the *uniform distribution* on $[0, 1]$.

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Single Random Variables: Review

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Continuous Random Variables:

- Random variable whose outcome space is uncountable (think infinite)
- Distribution/Probability measure completely described by pdf
 - $\mathbb{P}_X([a, b]) = \int_a^b f_X(x) dx$

Questions?

One property of a random variable that we may be interested in is the **expectation** of a random variable.

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The expectation of X is denoted $\mathbb{E}[X]$ or μ_X and is calculated via the following

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- Note that the difference between discrete and continuous is just summation vs. integral.

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We can expect to win about \$150 by playing this lottery

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$$\begin{aligned} \mathbb{E}[X] &= \int_{\mathcal{O}_X} a \cdot f_X(a) da \\ &= \int_0^1 a \cdot 1 da \\ &= \left. \frac{a^2}{2} \right|_0^1 \end{aligned}$$

Single Random Variables: Expectation

Let's consider the uniform distribution from before. That is, let X be continuously distributed with outcome space $\mathcal{O}_X = [0, 1]$ and pdf given by

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So the expected value of X is $\frac{1}{2}$. There are many ways to interpret this expected value, the most straightforward for me is the average amount you can expect to win from a lottery whose payouts follow the distribution of X .

We can generalize this concept a bit further and consider the mean of any function $g(X)$, which is typically denoted $\mathbb{E}[g(X)]$.

- Note that $g(X)$ itself is a random variable. It's outcome is not deterministic but rather follows a distribution based on X .
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The formula for calculating $\mathbb{E}[g(X)]$ is basically the same as for calculating $\mathbb{E}[X]$.

Type	Discrete R.V	Continuous R.V
Formula	$\sum_{a \in \mathcal{O}_X} g(a) \cdot p_X(a)$	$\int_{\mathcal{O}_X} g(a) \cdot f_X(a) da$

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- All that has changed here is that we are multiplying the pmf/pdf by $g(a)$ instead of by a .

Single Random Variables: Expectation

The formulas from this generalization also gives us a nice property that we will use later. Recall from the last slide we can calculate $\mathbb{E}[g(X)]$ using:

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It is then straightforward to see the following:

$$\mathbb{E}[ag(X) + bh(X)] = a\mathbb{E}[g(X)] + b\mathbb{E}[h(X)]$$

for any $a, b \in \mathbb{R}$. We will refer to this property as the **linearity of the expectation**. Later on when we consider multiple or joint random variables, we will see that this can be naturally extended and for two different random variables X and Y :

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

Let's return to the uniform distribution from before. Suppose that X follows the pdf

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For an arbitrary random variable X , $\mathbb{E}[X^2]$ is typically referred to as the *second moment* of X .

Single Random Variables: Variance

Now that we can compute $\mathbb{E}[g(X)]$ for any random variable X and any function $g(\cdot)$, we are ready to talk about the variance.

The variance of a random variable X is given by the formula

$$\sigma_X^2 = \text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

The variance is a measure of the “spread” of the random variable X ; it represents how far on average X is from its mean. Using linearity of the expectation the expression above can be simplified:

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$$\begin{aligned} &= \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

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Typically this last expression is easiest to work with. However, the first expression gives us an important property: the variance is always ≥ 0 .

From the formula $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ as well as the linearity of the expectation we can see the variance has some nice properties. For any constants $a, b \in \mathbb{R}$ and any random variable X we get that

- $\text{Var}(X + a) = \text{Var}(X)$
- $\text{Var}(aX) = a^2 \text{Var}(X)$

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Putting these together gives us: $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Exercise: See if you can work this out yourselves. It shouldn't take more than a few minutes.

Single Random Variables: Variance

Intuitively, $\text{Var}(X + a) = \text{Var}(X)$ means that the spread around the mean is not effected by just shifting the mean of the random variable (while otherwise keeping the distribution the same).

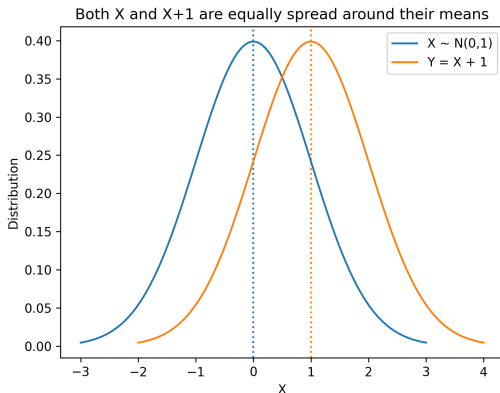


Figure 1: Variance remains the same for both variables

The square root of the variance is denoted

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\text{Var}(X)}.$$

and is called the **standard deviation** of X .

Let's see this in practice. Suppose we have a lottery that pays \$200 with probability $\frac{1}{2}$ and nothing (\$0) with probability $\frac{1}{2}$. The payout of this lottery is a random variable X with pmf

$$p_X(a) = \begin{cases} \frac{1}{2} & \text{if } a \in \{0, 200\} \\ 0 & \text{otherwise} \end{cases}.$$

It is straightforward to see that the expected payout of this lottery is \$100, $\mathbb{E}[X] = 100$.

Let's see this in practice. Suppose we have a lottery that pays \$200 with probability $\frac{1}{2}$ and nothing (\$0) with probability $\frac{1}{2}$. The payout of this lottery is a random variable X with pmf

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It is straightforward to see that the expected payout of this lottery is \$100, $\mathbb{E}[X] = 100$. However, we may want to know how much we can expect our winnings to deviate from the expected value. that is, we want to know what the variance of the payouts is.

Let's compute this two ways, first using the formula $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$ and the second using $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

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Second:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

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$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \sum_{a \in \{0, 200\}} a^2 \cdot p_X(a) - 100^2 \end{aligned}$$

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No matter how we compute it, the variance of this lottery is $100^2 = 10000$. This makes sense as no matter what happens (win or lose), we are \$100 away from the expected payout.

Questions?

Oftentimes we are not only interested in a single random variable, but the relationship between two random variables. In fact this will be the case through the rest of this course:

- We care about the relationship between education and income
- We care about the relationship between consumption of a medicine and a health outcome

Oftentimes we are not only interested in a single random variable, but the relationship between two random variables. In fact this will be the case through the rest of this course:

- We care about the relationship between education and income
- We care about the relationship between consumption of a medicine and a health outcome

In the examples above notice that, not only are the random variables themselves not deterministic

- not everyone has the same education/income

but the relationship between the two random variables may not be either

- not everyone who takes a medicine will have the same health outcome

Multiple Random Variables: Introduction

Question: How do we think about joint random variables?

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Answer: Much the same as before. Let (X, Y) be a pair of joint random variables. This means there is some outcome space \mathcal{O}_{XY} that (X, Y) can take values in and a probability measure $\mathbb{P}_{XY}(\cdot) : 2^{\mathcal{O}_{XY}} \rightarrow [0, 1]$ that takes in subsets of the outcome space $A \subseteq \mathcal{O}_{XY}$ and gives the probability of both X and Y taking values in the set A .

- For example if X is income and Y is age,

$$\mathbb{P}_{XY}(\{0 \leq X \leq 100000, 40 \leq Y \leq 42\})$$

is the probability that a randomly selected person from the population has an income between \$0 and \$100,000 and is between 40 and 42 years old.

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As before if \mathcal{O}_{XY} is countable (finite), we say that (X, Y) are jointly discrete random variables whereas if \mathcal{O}_{XY} is uncountable (infinite) we say that (X, Y) are jointly continuous random variables.

Let's quickly go over an example of a joint discrete random variables. As before, the distribution of jointly discrete random variables will be defined by a joint probability mass function.

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- Because we are considering two random variables, X, Y , we can represent the probability mass function as a table

Multiple Random Variables: Discrete Random Variables

Let X be a random variable that describes whether a person gets 4 hours of sleep a night, 8 hours a sleep a night, or 12 hours of sleep a night. Let Y be a random variable that describes whether a person drinks 1 or 2 cups of coffee a day.

The joint pmf of X and Y can be described with the table below

$p(x, y)$	1 cup	2 cups
4 hours	0	1/6
8 hours	1/3	1/3
12 hours	1/6	0

Using this table we can see that the probability that a randomly selected person gets 8 hours of sleep and drinks 1 cup of coffee a day is $1/3$.

Exercise: What is the probability that a randomly selected person gets 8 hours of sleep?

Now we'll go over an example of a joint continuously distributed random variable. Just like with single random variables, the distribution of a continuous random variable is defined by a probability density function, $f_{XY}(x, y)$.

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- As before, the joint pdf will be related to the joint probability measure $\mathbb{P}_{XY}(\cdot)$ via the relation

$$\mathbb{P}_{XY}(\{a \leq X \leq b, c \leq Y \leq d\}) = \int_a^b \int_c^d f_{XY}(x, y) dy dx.$$

Multiple Random Variables: Continuous Random Variables

Let's consider two sprinters in the 100m dash. Let X denote the finish time of the UCLA sprinter and Y denote the finish time of the USC sprinter. Suppose their times follow the following joint pdf

$$f_{XY}(x, y) = \begin{cases} 1 & \text{if } 9.5 \leq x \leq 10.5, 10 \leq y \leq 11 \\ 0 & \text{otherwise} \end{cases}.$$

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Let's try to find the probability that the the UCLA sprinter runs faster than 10 seconds and that the USC sprinter runs faster than 10.5 seconds. That is we want $\mathbb{P}_{XY}(\{X \leq 10, Y \leq 10.5\})$.

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$$\begin{aligned} \mathbb{P}_{XY}(\{X \leq 10, Y \leq 10.5\}) &= \int_{9.5}^{10} \int_{10}^{10.5} f_{XY}(x, y) dy dx \\ &= \int_{9.5}^{10} \int_{10}^{10.5} 1 dy dx \\ &= \int_{9.5}^{10} 0.5 dx \\ &= 0.5 \cdot 0.5 = 0.25 \end{aligned}$$

Just as before we may want to consider the average or expected value that some function, $g(X, Y)$, of our joint random variables may take. That is, we want to calculate $\mathbb{E}[g(X, Y)]$.

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Here are some examples of functions $g(x, y)$ for which we may be interested in $\mathbb{E}[g(X, Y)]$:

- $g(x, y) = x \implies \mathbb{E}[g(X, Y)] = \mathbb{E}[X]$, the expected value of X
- $g(x, y) = x - y \implies \mathbb{E}[g(X, Y)] = \mathbb{E}[X - Y]$, the average difference between X and Y .
- $g(x, y) = \mathbb{1}\{x \leq a, y \leq b\} \implies \mathbb{E}[g(X, Y)] = \mathbb{P}_{XY}(\{X \leq a, Y \leq b\})$.
- $g(x, y) = (x - \mu_X)(y - \mu_Y) \implies \mathbb{E}[g(X, Y)] = \text{Cov}(X, Y)$, the covariance between X and Y .

Multiple Random Variables: Expectations

The formula for calculating expected value is the same as before:

Type	Discrete R.V	Continuous R.V
Formula	$\sum_{a,b \in \mathcal{O}_{XY}} g(a,b) p_{XY}(a,b)$	$\int_X \int_Y g(a,b) f_{XY}(a,b) db da$

Intuition: We are evaluating the function at each point in the outcome space and weighting by the probability of that outcome occurring.

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Again note that we have the following linearity of the expectation. For any two functions $g(x, y)$ and $h(x, y)$ and any $a, b \in \mathbb{R}$:

$$\mathbb{E}[a \cdot g(X, Y) + b \cdot h(X, Y)] = a\mathbb{E}[g(X, Y)] + b\mathbb{E}[h(X, Y)].$$

in particular, $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.

Multiple Random Variables: Expectations

Let's return to the 100m dash example from before, with X representing the finishing time of the UCLA and Y representing the finishing time of the USC sprinter.

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Multiple Random Variables: Expectations

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As mentioned before, a particular expectation we may be interested in is the **covariance** between X and Y , defined as

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

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As before, we can simplify the expression for covariance:

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}[XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

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Let's calculate the covariance between X and Y from the example before:

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So $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 105 - 105 = 0$. At least to a first order, there is no association between the UCLA sprinters times and the USC sprinters times.

Another quantity of interest here is the **correlation coefficient** defined as

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

where we recall that

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\text{Var}(X)} = \sqrt{\mathbb{E}[(X - \mathbb{E}[X])^2]}$$

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Calculating the correlation is no more difficult than calculating the covariance, but it is a bit tedious so we won't go over an example right now.

Multiple Random Variables: Conditioning and Independence

Finally, given two joint random variables, X and Y , we may be interested in characteristics of the distribution of Y *conditional* on X taking a certain value.

Typically this will be denoted $\mathbb{E}[Y|X = x]$ and will be called the **conditional expectation** of Y given $X = x$.

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In all of these note that knowing the conditional expectation is useful for making predictions as we typically observe the X variable before we observe the Y variable.

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Calculating the conditional expectation essentially involves fixing the X variable at the point $X = x$ and then integrating.

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$\mathbb{E}[Y X = x]$	$\sum_y y \cdot p_{XY}(y, x)$	$\int_Y y \cdot f_{XY}(x, y) dy$

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Remark: Note that $X = x$ is fixed in the above equations while we allow Y to vary.