Econ 103: Introduction to Simple Linear Regression

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Content Outline

Linear Regression

- Line of best fit and linear model
- Formulas for parameters

Estimation

- Using data to estimate parameters of interest
- Formulas for parameter estimates

Asymptotic Distribution

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- Estimating variance of parameter estimates

Hypothesis Testing and Confidence Intervals

- Using asymptotic distribution to test statements about underlying parameters
- Using asymptotic distribution to give a range of plausible underlying parameter values

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Linear Regression as Line of Best Fit

Suppose we have two variables, Y and X. We are interested in using data to learning about the relationship between Y and X.

Examples:

- How are education and wages related?
- How are unemployment and inflation related?
- What is the relationship between receiving a treatment and a health outcome?

Linear Regression as Line of Best Fit

One way to model the relationship between Y and X would be to try to find the line of best fit between the two variables.

By the line of best fit we mean finding the line, characterized by a slope and an intercept, that minimizes the distance between Y and $\tilde{\beta}_0 + \tilde{\beta}_1 \cdot X$.

Formally, we are interested in the parameters β_0 and β_1 that solve

$$\begin{aligned}
\beta_0, \beta_1 &= \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - (\tilde{\beta}_0 + \tilde{\beta}_1 \cdot X)\right)^2\right] \\
&= \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]
\end{aligned}$$

Linear Regression as Line of Best Fit

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$$\begin{aligned}\beta_0, \beta_1 &= \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - (\tilde{\beta}_0 + \tilde{\beta}_1 \cdot X)\right)^2\right] \\ &= \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]\end{aligned}$$

- By $\arg\min$ we just mean we are interested in the **arguments** β_0 and β_1 that minimize
$$\mathbb{E}[(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X)^2]$$
rather than the value $\mathbb{E}[(Y - \beta_0 - \beta_1 \cdot X)^2]$ itself.
- Another way of saying this is that
$$\mathbb{E}[(Y - \beta_0 - \beta_1 \cdot X)^2] < \mathbb{E}[(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X)^2]$$
for any $(\tilde{\beta}_0, \tilde{\beta}_1) \neq (\beta_0, \beta_1)$.

Linear Regression as Line of Best Fit

We are interested in the parameters β_0 and β_1 that solve

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

Why do we care about these parameters?

- Knowing the line of best fit will help us predict Y using X
  - Will provide the best linear prediction of Y using X.
  - Even though a linear model may seem too simple, ends up being tremendously useful in practice.
- We can also interpret the parameters β_0 and β_1 to learn (to a first order degree) about the relationship between Y and X
  - Is there a positive or negative relationship between Y and X? $\iff$ Is β_1 positive or negative?
  - How much can we expect Y to change if we see an increase in X of one unit? $\iff$ What is β_1?
  - What is the average value of Y when X is zero? $\iff$ What is β_0?
  - To a first order degree because β_0 and β_1 describe the line of best fit rather than the "true" relationship.
    - No need to worry about this difference for now though.

Linear Regression: The Parameters

We are interested in the parameters β_0 and β_1 that solve

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

Let's solve for β_0 and β_1 by taking first order conditions:

$$\frac{\partial}{\partial \tilde{\beta}_0} : \mathbb{E}\left[Y - \beta_0 - \beta_1 \cdot X\right] = 0$$

$$\frac{\partial}{\partial \tilde{\beta}_1} : \mathbb{E}\left[(Y - \beta_0 - \beta_1 \cdot X) \cdot X\right] = 0$$

We will return to these first order conditions shortly. For now, after rearranging we get that

$$\beta_1 = \frac{\mathbb{E}[YX] - \mathbb{E}[Y]\mathbb{E}[X]}{\mathbb{E}[X^2] - \mathbb{E}[X]\mathbb{E}[X]} = \frac{\mathrm{Cov}(Y, X)}{\mathrm{Var}(X)}$$

$$\beta_0 = \mathbb{E}[Y] - \beta_1 \mathbb{E}[X]$$

Exercise: Show this rearrangement.

Linear Regression: The Error Term

Let's define the random variable

$$\begin{aligned} \epsilon &= Y - (\beta_0 + \beta_1 \cdot X) \\ &= Y - \beta_0 - \beta_1 \cdot X \end{aligned}$$

We can then write

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

which is the linear regression equation you may have seen before. The random variable ϵ will be important later on as we try to do inference.

Linear Regression: The Error Term

Let's define the random variable

$$\begin{aligned}\epsilon &= Y - (\beta_0 + \beta_1 \cdot X)\\ &= Y - \beta_0 - \beta_1 \cdot X\end{aligned}$$

We call ϵ the linear regression error variable.

Recall that from the first order conditions for β_0 and β_1 we have that

$$\mathbb{E}\big[\underbrace{Y - \beta_0 - \beta_1 \cdot X}_{=\epsilon}\big] = 0$$

$$\mathbb{E}\big[\underbrace{(Y - \beta_0 - \beta_1 \cdot X)}_{=\epsilon} \cdot X\big] = 0$$

These give us the properties that

$$\mathbb{E}[\epsilon] = 0 \text{ and } \mathbb{E}[\epsilon X] = 0.$$

Linear Regression: Model Summary

In total our line of best fit parameters

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

generate a model betwen Y and X that can be written as

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon \tag{1}$$

where

$$\mathbb{E}[\epsilon] = 0 \text{ and } \mathbb{E}[\epsilon X] = 0.$$

- It is often convenient to work directly with this representation or make assumptions about ϵ.
- You may have seen this representation before, the prior slides go over where this model comes from

Linear Regression: Model Summary

Our line of best fit parameters

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

are useful for

- Making predictions about Y using X.
  - Predict Y when $X = x$ with $\beta_0 + \beta_1 \cdot x$
- Learning about the relationship between Y and X.
  - Interpret the signs and magnitudes of β_0 and β_1

Questions?

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Linear Regression: Estimation Introduction

As we went over in the last section we are interested in the line of best fit parameters

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

Problem: We do not know know the joint distribution of (Y, X), so we cannot to solve for β_0 and β_1 by evaluating the expectation above.

Solution: Use data to estimate the parameters β_0 and β_1.

Linear Regression: The Estimator

Solution: Use data to try and estimate the parameters β_0 and β_1.

How do we do this?

Intuition:

- Suppose we have access to n randomly collected samples $\{Y_i, X_i\}_{i=1}^n$
- We are interested in the line of best fit between Y and X in the population

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

- We estimate the line of best fit between Y and X in the population using the line of best fit between Y_i and X_i in our sample:

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^n (Y_i - b_0 - b_1 \cdot X_i)^2$$

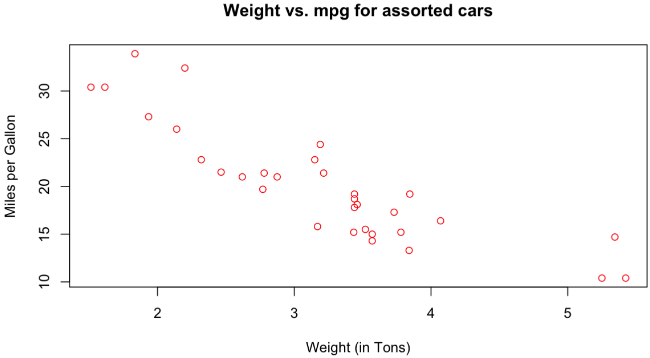
  - Same idea as using $\bar{X}$ to estimate $\mathbb{E}[X]$, etc.

- We estimate the line of best fit between Y and X in the population using the line of best fit between Y_i and X_i in our sample:

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \cancel{\frac{1}{n}} \sum_{i=1}^n (Y_i - b_0 - b_1 \cdot X_i)^2$$

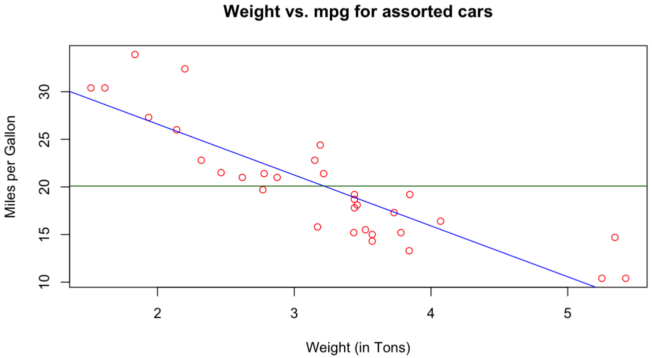
Let's see how this looks like in practice. Suppose we are interested in the relationship between X, a car's weight, and Y a car's miles per gallon (mpg).

We collect some data $\{Y_i, X_i\}_{i=1}^{n}$ where each (Y_i, X_i) pair represents the miles per gallon and weight of a particular vehicle in our dataset. We can represent our data using a scatterplot



Linear Regression: The Estimator

Now to estimate $\hat{\beta}_0, \hat{\beta}_1$ we simply find the line of best fit between the Y_i and X_i 's in our data.



The blue line represents the line of best fit whereas the green line represents a straight line through $\bar{Y}$. We can see that the blue line is much closer to the data than the green line.

Linear Regression: The Estimator

In this case we have that $\hat{\beta}_0 = 37.2851$ and $\hat{\beta}_1 = -5.3445$.

How do we interpret these estimates?

- $\hat{\beta}_0 = 37.2851$: We estimate that the average value of Y when $X = 0$ is 37.2851
  - In context: we estimate that the average mpg for a car that weights 0 tons is 37.2851 miles per gallon
- $\hat{\beta}_1 = -5.3445$: We estimate that, on average, a one unit increase in X is associated with a 5.3445 unit decrease in Y.
  - In context: we estimate that, on average, a one ton increase in car weight is associated with a 5.3445 unit decrease in miles per gallon.

Linear Regression: The Estimator

In this case we have that $\hat{\beta}_0 = 37.2851$ and $\hat{\beta}_1 = -5.3445$.

How can we use these estimates for prediction?

- Suppose we have a car that weighs 3.5 tons. Based on our estimates, what would we predict its miles per gallon to be?
  - Our estimated regression line is
$$\text{Predicted MPG} = 37.2851 - 5.3445 \cdot \text{Weight in Tons.}$$
  - Using this line and plugging in we get that
$$\text{Predicted MPG} = 37.2851 - 5.3445 \cdot 3.5 = 18.5793.$$
  - We denote this predicted MPG as $\widehat{\text{MPG}}$ and in general will denote our predictions as $\hat{Y}$ so that our estimated regression line can be written
$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot X.$$

Notice a couple things in the above interpretations

- The intercept is often uninterpretable (What car would weigh 0 tons?). For this reason we often focus our analysis on the slope coefficient.
- The interpretation is deliberately not causal. We use "associated with a decrease..." as opposed to "leads to a decrease..."

Linear Regression: Formulas

Now that we've gotten some intuition for what linear regression is doing and how to use our sample to estimate the parameters of interest, let's derive explicit formulas for $\hat{\beta}_0$ and $\hat{\beta}_1$.

Recall that

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} (Y_i - b_0 - b_1 \cdot X_i)^2 .$$

Taking first order conditions gives us that

$$\frac{\partial}{\partial b_0} : \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i) = 0$$

$$\frac{\partial}{\partial b_1} : \frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i) \cdot X_i = 0$$

Linear Regression: Formulas

Rearranging the first equality gives us

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} Y_i - \frac{1}{n}\sum_{i=1}^{n} \hat{\beta}_0 - \frac{1}{n}\sum_{i=1}^{n} \hat{\beta}_1 \cdot X_i &= 0 \\
\bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} X_i &= 0 \\
\bar{Y} - \hat{\beta}_0 - \hat{\beta}_1 \bar{X} &= 0 \\
\hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X}
\end{aligned}$$

So that what remains is to solve for $\hat{\beta}_1$.

Linear Regression: Formulas

Rearranging the second equality gives us

$$\frac{1}{n}\sum_{i=1}^{n} Y_i X_i - \hat{\beta}_0 \frac{1}{n}\sum_{i=1}^{n} X_i - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} X_i^2 = 0$$

Using the prior result that $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$ gives:

$$\frac{1}{n}\sum_{i=1}^{n} Y_i X_i - (\bar{Y} - \hat{\beta}_1 \bar{X})\bar{X} - \hat{\beta}_1 \frac{1}{n}\sum_{i=1}^{n} X_i^2 = 0$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} Y_i X_i - \bar{Y}\bar{X}\right) + \hat{\beta}_1 \left((\bar{X})^2 - \frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = 0$$

So, finally

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n} Y_i X_i - \bar{Y}\bar{X}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2}.$$

Linear Regression: Formulas

Let's make use of the following equalities to represent $\hat{\beta}_1$

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X}) = \frac{1}{n}\sum_{i=1}^{n} Y_i X_i - \bar{Y}\bar{X}$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\bar{X})^2$$

Then:

$$\hat{\beta}_1 = \frac{\overbrace{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}^{\text{Sample Covariance between } Y \text{ and } X}}{\underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}_{\text{Sample Variance of } X}}$$

This ties in nicely as, if we recall from earlier, we found that

$$\beta_1 = \frac{\text{Cov}(Y, X)}{\text{Var}(X)} = \frac{\mathbb{E}[(Y - \mu_Y)(X - \mu_X)]}{\mathbb{E}[(X - \mu_X)^2]}.$$

Linear Regression: Randomness

We have now gone over how use data to obtain estimates $\hat{\beta}_0, \hat{\beta}_1$ of our parameters of interest β_0, β_1.

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} (Y_i - b_0 - b_1 \cdot X_i)^2$$

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right]$$

Notice that, while the parameters of interest β_0 and β_1 are fixed quantities, the estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are functions of the data; they depend on the specific sample of data collected.

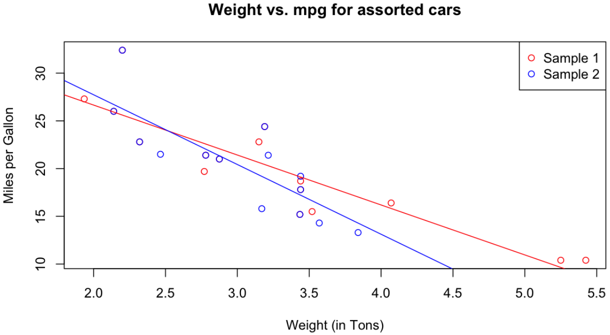
Some Questions to Consider:

1. What would happen to our estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ if we were to collect a different sample of data?
2. How can we model the distribution of our estimates $\hat{\beta}_0$ and $\hat{\beta}_1$?
3. What happens to this distribution as $n \to \infty$?

Linear Regression: Randomness

Question: What would happen to our estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ if we were to collect a different sample of data?

Let's return to the cars data and see how our regression lines look when we consider two different (random) samples.



- Sample 1: $\hat{\beta}_0 = 37.1285$ and $\hat{\beta}_1 = -5.2341$.
- Sample 2: $\hat{\beta}_0 = 42.352$ and $\hat{\beta}_1 = -7.307$.

Linear Regression: Randomness

Key Concept: Because the estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are functions of the random sample $\{Y_i, X_i\}_{i=1}^n$ they are themselves random variables.

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}$$

Problem: How do we connect $\hat{\beta}_0$ and $\hat{\beta}_1$ to the population parameters β_0 and β_1?

Fundamental Question: Given estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ what can we say about the underlying parameters of interest β_0 and β_1?

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Linear Regression: Motivating Idea

Suppose we are interested in the association between years of education and income. We collect a random sample of size $n = 100$, $\{Y_i, X_i\}_{i=1}^{100}$ and run a simple linear regression of $Y = INC$ against $X = EDU$.

That is, we are interested in the parameters β_0 and β_1 that dictate the line of best fit between income and education in the population

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[(INC - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot EDU)^2\right].$$

or equivalently the parameters from the linear model

$$INC = \beta_0 + \beta_1 \cdot EDU + \epsilon.$$

where $\mathbb{E}[\epsilon \cdot EDU] = 0$.

Linear Regression: Motivating Idea

Using our data $\{Y_i, X_i\}_{i=1}^{n}$ we find that $\hat{\beta}_1 = 0.5$.

$$\hat{\beta}_0 \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} \{Y_i - b_0 - b_1 \cdot X_i\}^2 .$$

Our friend, Prince Harry Estranged of England, however claims that there is no association between education and income, that is that $\beta_1 = 0$.

Question: How can we tell if he is right?

Answer: One way would be to find the probability that we would obtain $\hat{\beta}_1 = 0.5$ (or something more extreme) if the true value of β_1 was 0.

$$\Pr(|\hat{\beta}_1| \geq 0.5 | \beta_1 = 0).$$

If this probability is sufficently low, we can reject Former Prince Harry's claim. Otherwise he may be right.

To calculate this probability we will need to know something about the (approximate) distribution of $\hat{\beta}_1$ and how that is related to the true parameter β_1.

In order to connect the estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ to the population parameters, we will need to make some (light) assumptions about the underlying distribution of (Y, X) from which our sample $\{Y_i, X_i\}_{i=1}^n$ is drawn.

It will be helpful to recall the following definitions here

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[(Y - \tilde{\beta}_0 - \tilde{\beta}_1)^2\right]$$

$$\epsilon = Y - \beta_0 - \beta_1 \cdot X$$

And see that ϵ is itself a random variable.

Linear Regression: Assumptions

Make the following assumptions

1. Random Sampling: Assume that $\{Y_i, X_i\}$ are independently and identically distributed; $(Y_i, X_i) \overset{\text{i.i.d}}{\sim} (Y, X)$
   - Essentially this means that our random sample is "representative of the population"
   - Would be violated if say, we only sampled cars made in Los Angeles and we were trying to make inferences about all cars produced in the US
2. Homoskedasticity: Assume that $\mathbb{E}[\epsilon^2 | X = x] = \sigma_\epsilon^2$ for all possible values of x.
   - Since, ϵ is mean zero, this means that Y is equally spread around the regression line for all values of X.
   - This is a fairly strong assumption to make and we will relax it later on, but it is helpful for now to provide insight.
   - An important implication of this is that
     $$\mathrm{Var}(\epsilon(X - \mu_X)) = \mathrm{Var}(\epsilon)\,\mathrm{Var}(X) = \sigma_\epsilon^2 \sigma_X^2.$$

   Questions?
3. Rank Condition: There must be at least two distinct values of X that appear in the population.
   - Need at least two distinct points to make a line.
   - If there is only one distinct point then our minimization problem is undefined.

Linear Regression: Asymptotic Distribution

Given these assumptions (Random Sampling, Homoskedasticity, Rank Condition) let's try and figure out what the approximate distribution is of $\hat{\beta}_1$.

Recall that

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

By definition of $\epsilon = Y - \beta_0 - \beta_1 \cdot X$:

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon;$$

and that by the first order conditions of β_0 and β_1:

$$\mathbb{E}[\epsilon] = 0$$
$$\mathbb{E}[\epsilon \cdot X] = 0$$

We will also make use of the following results from our probability review. If Z is a random variables and we have i.i.d observations $Z_1, Z_2, .., Z_n$:

The Law of Large Numbers states that as $n \to \infty$:

$$\bar{Z} \to \mathbb{E}[Z]$$

or, equivalently, $\bar{Z} \approx \mathbb{E}[Z]$ for n large.

The Central Limit Theorem states that as $n \to \infty$, approximately,

$$\sqrt{n}\left(\bar{Z} - \mathbb{E}[Z]\right) \sim N\left(0, \mathrm{Var}(Z)\right)$$

or, equivalently, $\bar{Z} \sim N\left(\mathbb{E}[Z], \mathrm{Var}(Z)/n\right)$.

Linear Regression: Asymptotic Distribution

Starting with:

$$\sqrt{n}\hat{\beta}_1 = \frac{\sqrt{n}\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Expand $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$ and $\bar{Y} = \beta_0 + \beta_1 \bar{X} + \bar{\epsilon}$, where $\bar{\epsilon} = \frac{1}{n}\sum_{i=1}^n \epsilon_i$:

$$\sqrt{n}\hat{\beta}_1 = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(\beta_1(X_i - \bar{X}) + (\epsilon_i - \bar{\epsilon})\right)(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Distribute to get:

$$\sqrt{n}\hat{\beta}_1 = \sqrt{n}\beta_1 \frac{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} + \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (\epsilon_i - \bar{\epsilon})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Distribute to get:

$$\sqrt{n}\hat{\beta}_1 = \sqrt{n}\beta_1 \cancel{\frac{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}} + \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n (\epsilon_i - \bar{\epsilon})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Linear Regression: Asymptotic Distribution

So we have that:

$$\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\epsilon_i - \bar{\epsilon})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

Using Law of Large Numbers replace $\bar{\epsilon} \approx \mathbb{E}[\epsilon] = 0$, $\bar{X} \approx \mu_X$, and $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \approx \sigma_X^2$:

$$\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right) \approx \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\epsilon_i(X_i - \mu_X)}{\sigma_X^2}.$$

Linear Regression: Asymptotic Distribution

Finally, note that by Central Limit Theorem, since

$$\mathbb{E}[\epsilon(X_i - \mu_X)] = \mathbb{E}[\epsilon X_i] - \mathbb{E}[\epsilon]\mu_X = 0.$$

we have that (approximately for large n):

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i (X_i - \mu_X) \sim N\left(0, \mathrm{Var}\left(\epsilon(X - \mu_X)\right)\right).$$

Now note that by Homoskedaticity:

$$\mathrm{Var}(\epsilon(X - \mu_X)) = \sigma_\epsilon^2 \sigma_X^2$$

so that (approximately for large n):

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \epsilon_i (X_i - \mu_X) \sim N\left(0, \sigma_\epsilon^2 \sigma_X^2\right).$$

Linear Regression: Asymptotic Distribution

Putting this all together, we have that, approximately for n large;

$$\sqrt{n}\left(\hat{\beta}_1 - \beta_1\right) \sim \frac{N(0, \sigma_\epsilon^2 \sigma_X^2)}{\sigma_X^2} = N\big(0, \underbrace{\sigma_\epsilon^2 / \sigma_X^2}_{:=\sigma_{\beta_1}^2}\big).$$

where in the last equality we use the fact that $N(0, a)/b \sim N(0, a/b^2)$. Other ways of putting this are, approximately for n large:

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma_{\beta_1}^2 / n\right)$$

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0, 1)$$

where as a reminder $\sigma_{\beta_1} = \sigma_\epsilon / \sigma_X$. This last form is what we will use the most.

Following similar steps we can derive the approximate distribution of $\hat{\beta}_0$ as well as the covariance between $\hat{\beta}_0$ and $\hat{\beta}_1$:

$$\sqrt{n}\left(\hat{\beta}_1 - \hat{\beta}_1\right) \sim N\left(0, \frac{\sigma_\epsilon^2}{\sigma_X^2}\right)$$

$$\sqrt{n}\left(\hat{\beta}_0 - \beta_0\right) \sim N\left(0, \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{\sigma_X^2}\right)$$

$$\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_0) = -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}$$

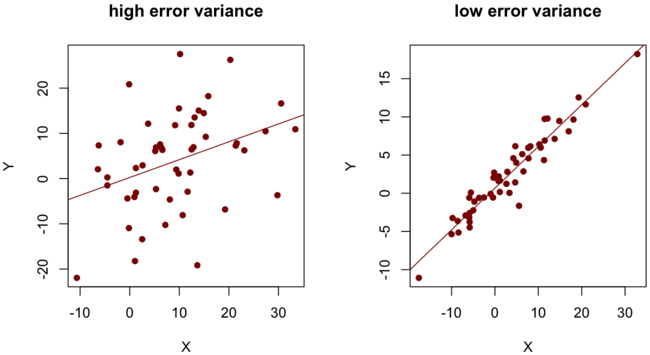
Important to remember these! The above is just providing intuition on how we get these results.

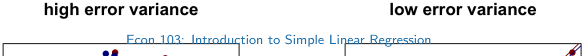
Linear Regression: Asymptotic Variances

For large n we have that

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{n \cdot \sigma_X^2}, \quad \mathrm{Var}(\hat{\beta}_0) = \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{n \cdot \sigma_X^2}, \quad \text{and} \quad \mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_0) = -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}.$$

First notice that these variances are increasing with σ_ϵ^2.



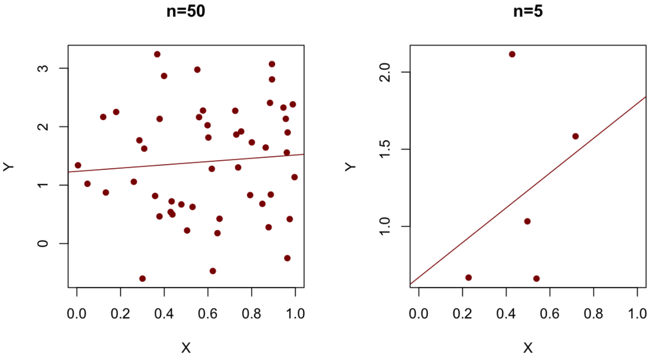


Linear Regression: Asymptotic Variances

For large n we have that

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{n \cdot \sigma_X^2}, \quad \mathrm{Var}(\hat{\beta}_0) = \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{n \cdot \sigma_X^2}, \quad \text{and} \quad \mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_0) = -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}.$$

These variances tend to zero as $n \to \infty$; as we collect more data we are closer to the true values β_0 and β_1.



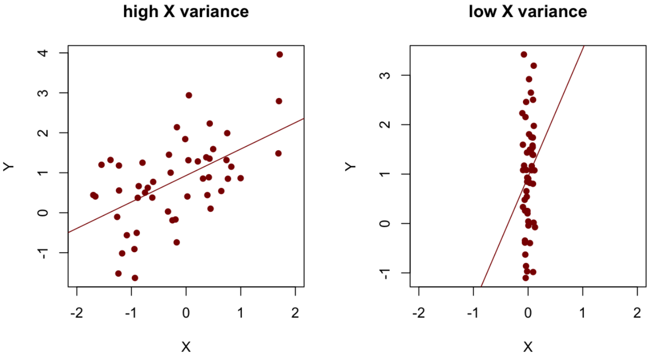
n=50
n=5

Linear Regression: Asymptotic Variances

For large n we have that

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma_\epsilon^2}{n \cdot \sigma_X^2}, \quad \mathrm{Var}(\hat{\beta}_0) = \sigma_\epsilon^2 \frac{\mathbb{E}[X^2]}{n \cdot \sigma_X^2}, \quad \text{and} \quad \mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_0) = -\sigma_\epsilon^2 \frac{\mathbb{E}[X]}{n \cdot \sigma_X^2}.$$

These variances decrase as σ_X^2 increases; as the spread of X increases we can make out the line more clearly.



high X variance
low X variance

Questions?

Linear Regression: Variance Estimation

Positive Result: Under homoskedasticity, for n large, we have (approximately)

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0,1).$$

where

$$\sigma_{\beta_1}^2 = \frac{\sigma_\epsilon^2}{\sigma_X^2}.$$

Problem: What is $\sigma_{\beta_1}^2$? How can we estimate it?

- By LLN we know how to esimate $\mathrm{Var}(X)$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \approx \mathrm{Var}(X).$$

- But what about $\mathrm{Var}(\epsilon) = \sigma_\epsilon^2$?

Linear Regression: Variance Estimation

To estimate $\mathrm{Var}(\epsilon)$ we first construct estimated residuals $\hat{\epsilon}_i$ via

$$\hat{\epsilon}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i.$$

Because $\hat{\beta}_1 \to \beta_1$ and $\hat{\beta}_0 \to \beta_0$ we can say that $\hat{\epsilon}_i \approx \epsilon_i = Y_i - \beta_0 - \beta_1 X_i$ (for n large).

Also by the first order conditions for $\hat{\beta}_0$ we have that

$$-\frac{1}{n}\sum_{i=1}^{n}(\underbrace{Y_i - \hat{\beta}_0 - \hat{\beta}_1 \cdot X_i}_{=\hat{\epsilon}_i}) = 0.$$

so that

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i = \bar{\hat{\epsilon}}_i = 0.$$

Linear Regression: Variance Estimation

Putting this together we can estimate $\mathrm{Var}(\epsilon) = \sigma_\epsilon^2$ by calculating the sample variance of $\hat{\epsilon}_i$:

$$\hat{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{i=1}^{n} \hat{\epsilon}_i^2 - \cancel{(\bar{\hat{\epsilon}}_i)^2}$$

By $\hat{\beta}_1 \to \beta_1$ and $\hat{\beta}_0 \to \beta_0$ as $n \to \infty$;

$$\approx \frac{1}{n}\sum_{i=1}^{n} \epsilon_i^2$$

By Law of Large Numbers;

$$\approx \mathbb{E}[\epsilon^2]$$

By $\mathbb{E}[\epsilon] = 0$;

$$= \mathrm{Var}(\epsilon) = \sigma_\epsilon^2$$

Putting all of this together, we can estimate $\sigma^2_{\beta_1} = \frac{\sigma^2_X}{\sigma^2_X}$ via;

$$\hat{\sigma}^2_{\beta_1} = \frac{\hat{\sigma}^2_\epsilon}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} \approx \sigma^2_{\beta_1}.$$

since for large n

$$\hat{\sigma}^2_\epsilon = \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_i^2 \approx \sigma^2_\epsilon$$

$$\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2 \approx \sigma^2_X.$$

Now, since we have that (approximately, for large n):

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma_{\beta_1}/\sqrt{n}} \sim N(0,1).$$

And since, as we have established above, $\hat{\sigma}_{\beta_1} \approx \sigma_{\beta_1}$, for large n we can say that (approximately)

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0,1).$$

Linear Regression: Variance Estimation

The quantity $\hat{\sigma}_{\beta_1}/\sqrt{n}$ is often referred to as the standard error of $\hat{\beta}_1$.

In general, if we have a parameter θ that we estimate with $\hat{\theta}$, the quantity $\hat{\sigma}_\theta/\sqrt{n}$ will be referred to as the standard error of $\hat{\theta}$ where

$$\hat{\sigma}_\theta/\sqrt{n} = \sqrt{\mathrm{Var}(\hat{\theta})} = \sqrt{\frac{\hat{\sigma}_\theta^2}{n}}$$

and σ_θ^2 is such that

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, \sigma_\theta^2).$$

Variance Estimation: Questions

Questions?

Linear Regression: Why Asymptotic Distribution?

Let's return to our example and see why this characterization is useful. Recall that in our example we are interested in the regression parameters from regression $Y = INC$ (income in thousands of dollars) against $X = EDU$ (years of education).

After collecting a sample size of 100, $\{Y_i, X_i\}_{i=1}^{100}$ we find that:

$$\hat{\beta}_1 = 0.5$$

$$\frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2 = 25$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = 16$$

Linear Regression: Why Asymptotic Distribution?

Our friend His Majesty Prince Harry claims there is no relationship between education and income, $\beta_1 = 0$. We claim that observing the magnitute of $|\hat{\beta}_1| = 0.5$ is evidence against this claim. Who is right?

- If $\beta_1 = 0$ we would expect $\hat{\beta}_1$ to be close to zero.
- But there is still some randomness in $\hat{\beta}_1$, maybe we got $\hat{\beta}_1 = 0.5$ by chance.

Want to use the (asymptotic) distribution of $\hat{\beta}_1$ to answer this question.

- First need to estimate σ_{β_1}.

Using $\hat{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{i=1}^n \epsilon_i^2 = 25$, and $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2 = 16)$ we calculate

$$\begin{aligned}\hat{\sigma}_{\beta_1}^2 &= \frac{\hat{\sigma}_\epsilon^2}{\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2} \\ &= \frac{25}{16}\end{aligned}$$

Using this, we find that $\hat{\sigma}_{\beta_1} = \sqrt{\hat{\sigma}_{\beta_1}^2} = \frac{5}{4}$.

Linear Regression: Why Asymptotic Distribution?

Now recall that for n large we have that (approximately)

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0,1).$$

If the true value of $\beta_1 = 0$ this means that

$$\frac{\hat{\beta}_1}{5/40} = \frac{\hat{\beta}_1}{0.125} \sim N(0,1).$$

Linear Regression: Why Asymptotic Distribution?

Given that if $\beta_1 = 0$, $\hat{\beta}_1/0.125 \sim N(0,1)$, what is the probability of us observing $|\hat{\beta}_1| \geq 0.5$?

$$\begin{aligned}\Pr\left(|\hat{\beta}_1| \geq 0.5\right) &= \Pr\left(|\hat{\beta}_1/0.125| \geq 0.5/0.125\right) \\ &= \Pr(|Z| \geq 4)\end{aligned}$$

where $Z \sim N(0,1)$

$$\begin{aligned}&= \Pr(Z \geq 4) + \Pr(Z \leq -4) \\ &= 2\Pr(Z \geq 4)\end{aligned}$$

By symmetry of the normal distribution

$$\approx 0.00006$$

Linear Regression: Why Asymptotic Distribution?

Using the asymptotic distribution result

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0,1),$$

we have found that if $\beta_1 = 0$, then $\Pr(|\hat{\beta}_1| \geq 0.5) \approx 0.0006$.

So, given that we observed $\hat{\beta}_1 = 0.5$, it seems very unlikely that $\beta_1 = 0$. We can conclude against Prince Harry's claim.

Asymptotic Distribution: Questions

Questions?

Table of Contents

Linear Regression: Hypothesis Testing

The last exercise where we tested whether Prince Harry's claim made sense was an example of a hypothesis test.

In this section we will formally discuss hypothesis testing.

Linear Regression: What is a Hypothesis Test?

Often in linear regression analysis, we are interested in using parameter estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$, to test some baseline or <u>null</u> hypothesis about the poulation against an opposite or <u>alternative</u> hypothesis.

- There is no association between years of education and income
  - Null Hypothesis: $\beta_1 = 0$.
  - Alternative Hypothesis: $\beta_1 \neq 0 \iff |\beta_1| > 0$
- Smoking has a negative effect on life expectancy
  - Null Hypothesis: $\beta_1 \leq 0$
  - Alternative Hypothesis: $\beta_1 > 0$
- There is a positive association between the miles per gallon of a car and its final sales price
  - Null Hypothesis: $\beta_1 \geq 0$
  - Alternative Hypothesis: $\beta_1 < 0$

We will denote the null hypothesis as H_0 and the alternative as H_1.

- There is no association between years of education and income
  - H_0: $\beta_1 = 0$.
  - H_1: $\beta_1 \neq 0 \iff |\beta_1| > 0$
- Smoking has a negative effect on life expectancy
  - H_0: $\beta_1 \leq 0$
  - H_1: $\beta_1 > 0$
- There is a positive association between the miles per gallon of a car and its final sales price
  - H_0: $\beta_1 \geq 0$
  - H_1: $\beta_1 < 0$

Linear Regression: What is a Hypothesis Test?

If H_1 contains a "$\neq$" sign, we call this a "two-sided" alternative.

Example: There is no association between years of education and income

- H_0: $\beta_1 = 0$
- H_1: $\beta_1 \neq 0$

If H_1 contains a "$>$" or a "$<$" sign, we call this a "one-sided" alternative.

Example: Cups of coffee drank has a negative association with hours of sleep

- H_0: $\beta_1 \leq 0$
- H_1: $\beta_1 > 0$

Linear Regression: How to Hypothesis Test

So, how do we use our data and parameter estimates $\hat{\beta}_1$ and $\hat{\beta}_0$ to test hypotheses? Given a null hypothesis H_0 and an alternative hypothesis, we have two options.

- We can reject the null hypothesis in favor of the alternative hypothesis.
  - Do this when the probability of obtaining our observed value of $\hat{\beta}$ (or something even further from the null hypothesis) under the null hypothesis is smaller than a pre-specified value α.
  - The value α is called the "level" or "significance level" of the test.
  - It is also the probability of a "Type 1" error, the probability that we will reject the null hypothesis when the null hypothesis is true.
- We can fail to reject the null hypothesis.
  - Do this when the probability of obtaining our observed value of $\hat{\beta}$ (or something even further from the null hypothesis) under the null hypothesis is larger than a pre-specified value α.

How do we calculate the probability, given that our null hypothesis is true, of observing our value of $\hat{\beta}$ or something even further from the null hypothesis?

Recall that, approximately for large n

$$\frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\beta_1}/\sqrt{n}} \sim N(0,1) \text{ and } \frac{\hat{\beta}_0 - \beta_0}{\hat{\sigma}_{\beta_0}/\sqrt{n}} \sim N(0,1).$$

where $\hat{\sigma}^2_{\beta_1} = \hat{\sigma}^2_{\epsilon}/\hat{\sigma}^2_X$ and $\hat{\sigma}^2_{\beta_0} = \frac{1}{n}\sum_{i=1}^n X_i^2 \cdot \hat{\sigma}^2_{\epsilon}/\hat{\sigma}^2_X$.

Linear Regression: How to Hypothesis Test

Let $Z \sim N(0,1)$. Using the distributions above, if we are testing $H_0 : \beta_1 = b$ against $H_1 : \beta_1 \neq b$ we can compute the probability (under the null hypothesis) that we observe our value of $\hat{\beta}_1$ or something even further from the null hypothesis by computing

$$\Pr\left(|Z| > \left|\frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1}/\sqrt{n}}\right|\right).$$

If we are testing $H_0 : \beta_1 \geq b$ against $H_1 : \beta_1 < b$ we can compute the probability (under the null hypothesis) that we observe our value of $\hat{\beta}_1$ or something even further from the null hypothesis by computing

$$\Pr\left(Z < \frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1}/\sqrt{n}}\right).$$

If we are testing $H_0 : \beta_1 \leq b$ against $H_1 : \beta_1 > b$ we can compute the probability (under the null hypothesis) that we observe our value of $\hat{\beta}_1$ or something even further from the null hypothesis by computing

$$\Pr\left(Z > \frac{\hat{\beta}_1 - b}{\hat{\sigma}_{\beta_1}/\sqrt{n}}\right).$$

This probability is called the p-value and we reject our null hypothesis if the

Linear Regression: How to Hypothesis Test

In summary, the test above can be conducted as follows. Suppose $H_0 : \beta \leq b$, $H_0 : \beta \geq b$, or $H_0 : \beta = b$

1. Compute the test statistic
$$t^* = \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}}.$$
2. Compute the p-value, the probability that we would obtain our observed value of $\hat{\beta}$, or something even further from the null hypothesis, if the null hypothesis was correct
   - If $H_0 : \beta = b$ and $H_1 : \beta \neq b$ compute
   $$p = \Pr(|Z| > |t^*|) = 2\Pr(Z > |t^*|).$$
   - If $H_0 : \beta \leq b$ and $H_1 : \beta > b$ compute
   $$p = \Pr(Z > t^*).$$
   - If $H_0 : \beta \geq b$ and $H_1 : \beta < b$ compute
   $$p = \Pr(Z < t^*).$$
3. Reject the null hypothesis in favor of the alternative hypothesis if $p < \alpha$. Otherwise fail to reject the null hypothesis.

Linear Regression: Hypothesis Testing Example

Let's see this work in practice. Our close personal friend Jason Derulo claims that there is a negative association between a car's miles per gallon, X, and it's sales price in thousands of dollars, Y.

We want to use data to test this claim. We collect a random (i.i.d) sample of size 64, $\{Y_i, X_i\}_{i=1}^{64}$ of cars and find

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X}) = 4$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = 16$$

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2 = 36$$

We will this data to test Derulo's claim, $H_0 : \beta_1 \leq 0$, against an alternate hypothesis, $H_1 : \beta_1 > 0$.

Linear Regression: Hypothesis Testing Example

In order to test this null hypothesis (against it's alternative) we need to calculate the test statistic $t^* = \frac{\hat{\beta}_1 - 0}{\hat{\sigma}_{\beta_1}/\sqrt{n}}$.

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{4}{16} = 0.25$$

$$\hat{\sigma}_{\beta_1} = \frac{\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{36}{16}$$

Using this, we compute the test statistic

$$t^* = \frac{0.25}{\sqrt{36/16}/\sqrt{64}} \approx 1.333.$$

Using this test statistic, $t^* \approx 1.333$, let's conduct the following test at level $\alpha = 0.1$

$$H_0 : \beta_1 \leq 0 \text{ and } H_1 : \beta_1 > 0.$$

Compute the p-value

$$p = \Pr(Z > 1.333) = 1 - \Pr(Z \leq 1.333) = 1 - 0.908 = 0.092.$$

Because the p-value, 0.092 is less than $\alpha = 0.1$, we reject the null hypothesis that there is a negative association between miles per gallon and sales price in favor of the alternative that there is a positive relationship between the two.

Linear Regression: Hypothesis Testing Example

Now given $t^* \approx 1.333$, suppose that we wanted to conduct a two sided test at level $\alpha = 0.1$. That is, suppose we wanted to test

$$H_0 : \beta_1 = 0 \text{ and } H_1 : \beta_1 \neq 0.$$

Compute the p value for a two-sided test

$$p = \Pr(|Z| > |t^*|) = 2\Pr(Z > |t^*|) = 2(1 - \Pr(Z \leq 1.333)) = 2 \cdot 0.092 \approx 0.194.$$

Given that $p = 0.194 > 0.1$ we fail to reject the null hypothesis that there is no relationship between miles per gallon and sales price.

Notice that the p-value for a two-sided test was twice the p-value for the one-sided test! The reverse is not necessarily true however.

Why?

- Suppose $t^* = 1.64$ so that the p-value for a two sided test is

$$\Pr(|Z| > 1.64) = 2\Pr(Z > 1.64) = 0.1.$$

- What is the p-value for the test $H_0 : \beta_1 \leq 0$ against $H_1 : \beta_1 > 0$?
- What is the p-value for the test $H_0 : \beta_1 \geq 0$ against $H_1 : \beta_1 < 0$?

Hypothesis Testing: Questions

Questions?

Linear Regression: How to Hypothesis Test

Conducting the test above can also follow another standard procedure. Suppose $H_0 : \beta \leq b$, $H_0 : \beta \geq b$, or $H_0 : \beta = b$

1. Compute the test statistic or "t-statistic"
$$t^* = \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}}.$$
2. For a given level α compute $z_{1-\alpha}$ for a one sided alternative or $z_{1-\alpha/2}$ for a 2 sided alternative, where $z_{1-\alpha}$ and $z_{1-\alpha/2}$ are such that
$$\Pr(Z > z_{1-\alpha}) = \alpha \text{ and } \Pr(Z > z_{1-\alpha/2}) = \frac{\alpha}{2}.$$
These are called the $1 - \alpha$ and $1 - \alpha/2$ quantiles of the standard normal distribution, respectively.
   - $z_{0.9} \approx 1.28$
   - $z_{0.95} \approx 1.64$
   - $z_{0.975} \approx 1.96$
   - $z_{0.99} \approx 2.32$
   - $z_{0.995} \approx 2.57$
3. Compare the test statistic t^* to the quantile $z_{1-\alpha}$ or $z_{1-\alpha/2}$.
   - If $H_0 : \beta = b$ and $H_1 : \beta \neq b$, reject if $|t^*| > z_{1-\alpha/2}$

Linear Regression: Hypothesis Testing Example

Let's return to the hypothesis testing example from earlier to verify that this procedure gives the same results as comparing p-values.

Recall that in this example our friend Jason Derulo has claimed that there is a negative association between miles per gallon of a car and sales price of a car. That is we want to test at level $\alpha = 0.1$

$$H_0 : \beta_1 \leq 0 \text{ vs. } H_1 : \beta_1 > 0.$$

After collecting data, we find that $t^* \approx 1.333$. To test this hypothesis, we will compare this value to $z_{1-0.1} = z_{0.9} = 1.28$. We are conducting a one sided alternative ($>$ sign) so we look to see if $t^* > z_{0.9}$.

Since $t^* \approx 1.3333 > z_{0.9} = 1.28$ we reject the null hypothesis that there is a negative association between miles per gallon of a car and sales price of a car in favor of the alternative hypothesis that there is a positive relationship.

- Same result as when using the p-value

Now let's use this procedure to test at level $\alpha = 0.1$

$$H_0 : \beta_1 = 0 \text{ vs. } H_1 : \beta_1 \neq 0.$$

Because we are dealing with a two sided alternative ($\neq$ sign) we have to compare $|t^*|$ to $z_{1-\alpha/2} = z_{1-0.1/2} = z_{0.95}$.

Since $t^* \approx 1.333 < z_{0.95} = 1.64$ we fail to reject the null hypothesis against a two-sided alternative.

Hypothesis Testing: Questions

Questions?

Linear Regression: Confidence Intervals

Given our data $\{Y_i, X_i\}_{i=1}^n$ we now know how to construct estimates, $\hat{\beta}_0, \hat{\beta}_1$ of the linear model parameters β_0, β_1 where

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 \cdot X\right)^2\right].$$

As a reminder, these parameters β_0, β_1 can equivalently be described as coming from a linear model

$$Y = \beta_0 + \beta_1 \cdot X + \epsilon.$$

where $\mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0$. The term ϵ is called the "linear regression error".

Also given our data $\{Y_i, X_i\}_{i=1}^n$ we know how to test hypothesis about the linear regression parameters β_0 and β_1 such as

$$H_0 : \beta_1 \geq 6 \text{ vs. } H_1 : \beta_1 < 6.$$

or

$$H_0 : \beta_0 = 0 \text{ vs. } H_1 : \beta_0 \neq 0.$$

Linear Regression: Confidence Intervals

Now, given our data $\{Y_i, X_i\}_{i=1}^n$ we want to do is construct a range of values that we are "confident" that the true parameter, β_0 or β_1 lies in.

We call this range of values a $100 \cdot (1-\alpha)\%$ Confidence Interval.

- e.j if $\alpha = 0.05$ we would want to construct a 95% confidence interval.

What values should we include in a $100 \cdot (1-\alpha)\%$ Confidence Interval?

- Any value b for which we would not reject $H_0 : \beta = b$ against a two sided alternative $H_1 : \beta \neq b$ at level α.

Linear Regression: Confidence Intervals

What values should we include in a $100 \cdot (1-\alpha)\%$ Confidence Interval?

- Any value b for which we would not reject $H_0 : \beta = b$ against a two sided alternative $H_1 : \beta \neq b$ at level α.

Recall that we reject $H_0 : \beta = b$ in favor of $H_1 : \beta \neq b$ if

$$|t^*| = \left| \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}} \right| > z_{1-\alpha/2}.$$

We fail to reject $H_0 : \beta = b$ in favor of $H_1 : \beta \neq b$ if

$$\left| \frac{\hat{\beta} - b}{\hat{\sigma}_\beta / \sqrt{n}} \right| \leq z_{1-\alpha/2}.$$

Linear Regression: Confidence Intervals

Equivalently we can say that we fail to reject $H_0 : \beta = b$ in favor of $H_1 : \beta \neq b$ if

$$\hat{\beta} - z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right) \leq b \leq \hat{\beta} + z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right).$$

Thus our $100 \cdot (1-\alpha)\%$ confidence interval is given

$$\left[\hat{\beta} - z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right), \hat{\beta} + z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right)\right].$$

This is interpreted as: we are $100 \cdot (1-\alpha)\%$ confident that the true value of β lies in the interval

$$\left[\hat{\beta} - z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right), \hat{\beta} + z_{1-\alpha/2} \cdot \left(\hat{\sigma}_\beta/\sqrt{n}\right)\right].$$

Linear Regression: Confidence Interval Example

Let's see this in practice. Suppose the government wants to know what the effect is of offering cash incentives to people to get vaccinated on their vaccination status.

To study this policy we randomly select 100 (unvaccinated) people from the population and offer them a random cash incentive (from $\$0$ to $\$100$) and then observe whether or not they get vaccinated.

Our data then looks like $\{Y_i, X_i\}_{i=1}^{100}$ where $Y_i \in \{0, 1\}$ denotes a person's vaccination status and $X_i \in [0, 100]$ denotes the cash incentive offered to people. We want to construct a confidence interval for the parameter β_1 from the linear model

$$Y = \beta_0 + \beta_1 \cdot X_i + \epsilon_i, \quad \mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0.$$

- As a reminder we can think of this model as generated by the line of best fit parameters
$$\beta_0, \beta_1 = \arg \min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[\left(Y - \tilde{\beta}_0 - \tilde{\beta}_1 X\right)^2\right].$$
- Important for the government, when considering a policy, to not only have a point estimate of the effect but also a measure of how confident we are in the point estimate.

Linear Regression: Confidence Interval Example

After collecting our data $\{Y_i, X_i\}_{i=1}^{100}$ we find that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = 6$$

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = 4$$

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_i^2 = 0.25$$

$$\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X}) = 0.1$$

Linear Regression: Confidence Interval Example

Using this data we compute

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{0.1}{4} = 0.025$$

$$\hat{\sigma}^2_{\beta_1} = \frac{\hat{\sigma}^2_{\epsilon}}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{0.25}{4} = 0.0625$$

Question: Given that $Y \in \{0, 1\}$, how do we interpret $\hat{\beta}_1$ in this context? How would we interpret $\hat{\beta}_0$ in this context?

Linear Regression: Confidence Interval Example

Now let's construct a 95% confidence interval for β_1. Recall that a $100 \cdot (1-\alpha)\%$ confidence interval for β_1 is given by

$$\hat{\beta}_1 \pm z_{1-\alpha/2} \cdot \frac{\hat{\sigma}_{\beta_1}}{\sqrt{n}}.$$

In this case $\alpha = 0.05$. From above we have that $z_{0.975} \approx 1.96$. Plugging in our values from above the 95% confidence interval for β_1 is given

$$0.025 \pm 1.96 \cdot \frac{\sqrt{0.0625}}{\sqrt{100}} = 0.025 \pm 1.96 \cdot \frac{0.25}{10} = [-0.024, 0.074].$$

Linear Regression: Confidence Interval Example

Plugging in our values from above the 95% confidence interval for β_1 is given

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Questions:

1. How do we interpret this confidence interval?
2. Suppose we wanted to test $H_0 : \beta_1 = 0$ vs $H_1 : \beta_1 \neq 0$ at level $\alpha = 0.05$. What would be the result?
   - What about if we wanted to test this hypothesis at level $\alpha = 0.025$?

Table of Contents

In this lecture we have introduced the line of best fit parameters

$$\beta_0, \beta_1 = \arg\min_{\tilde{\beta}_0, \tilde{\beta}_1} \mathbb{E}\left[(Y - \beta_0 - \beta_1 X)^2\right]$$

After taking $\epsilon = Y - \beta_0 - \beta_1 X$, these parameters generate the linear model

$$Y = \beta_0 + \beta_1 X + \epsilon, \quad \mathbb{E}[\epsilon] = \mathbb{E}[\epsilon X] = 0.$$

While the linear model is often easier to work with, it is useful to keep the line of best fit interpretation in the back of our mind. It provides our model interpretability even when the true relationship between Y and X is not linear.

Linear Regression: Final Notes

Since we do not know the joint distribution of (Y, X), we have to use data, $\{Y_i, X_i\}_{i=1}^n$ to estimate $\hat{\beta}_0$ and $\hat{\beta}_1$

$$\hat{\beta}_0, \hat{\beta}_1 = \arg\min_{b_0, b_1} \frac{1}{n} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2.$$

Taking first order conditions this gives

$$\begin{aligned} \hat{\beta}_0 &= \bar{Y} - \hat{\beta}_1 \bar{X} \\ \hat{\beta}_1 &= \frac{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} \end{aligned}$$

Linear Regression: Final Notes

We also derived the asymptotic distribution of our estimates. Using the law of large numbers and the central limit theorem we can say that, under homoskedasticity, approximately for large n,

$$\hat{\beta}_0 \sim N\left(\beta_0, \mathbb{E}[X^2]\frac{\hat{\sigma}_\epsilon^2}{n\sigma_X^2}\right)$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma_\epsilon^2}{n\sigma_X^2}\right)$$

Estimation of $\hat{\sigma}_\epsilon^2 = \frac{1}{n}\sum_{i=1}^n \hat{\epsilon}_i^2$

Linear Regression: Final Notes

Finally, we covered how to use these asymptotic distributions and our data to test various hypothesis about the underlying parameters such as

$$H_0 : \beta_0 = 5 \text{ vs. } H_1 : \beta_0 \neq 5$$

or

$$H_0 : \beta_1 \leq 0 \text{ vs. } H_1 : \beta_1 > 0$$

As well as construct confidence intervals for the parameters β_0 and β_1.

- These sorts of inferential results are important for policy analysis and separate the econometrics/statistics approaches from machine learning

Linear Regression: Final Notes

As a quick aside, in the above we used a lot of “approximations” to get the asymptotic distributions and then conduct inference:

- In the derivation of the asymptotic distribution of $\hat{\beta}_1$ used $\bar{Y} \approx \mu_Y$ and $\bar{X} \approx \mu_X$
- When we conduct inference on the parameters β_0 and β_1 used the fact that approximately for large n

$$\hat{\beta}_1 \sim N(\beta_1, \frac{\sigma_\epsilon^2}{\sigma_X^2}).$$

- When estimating σ_ϵ^2 used the fact that, since $\hat{\beta}_1 \to \beta_1$ and $\hat{\beta}_0 \to \beta_0$, $\hat{\epsilon}_i \approx \epsilon_i$

Linear Regression: Final Notes

It is natural to wonder, is this too much approximation?

- In general in this class we will ignore these approximation errors
- They tend to be second order and go away rather quickly with n (and get arbitrarily small as n increases)
- In practice, usually ok so long as $n \geq 50$. Otherwise have to rely on strong additional assumptions that are generally violated.