Rudin Functional Analysis

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Contents

1 Chapter 1: Topological Vector Spaces

1.1 Introduction and Normed Spaces

Many problems require analysis of classes of functions. Most interesting classes are vector spaces with normed.

A vector space X is said to be a *normed space* if, for every $x \in X$ there is a nonnegative real number $\|x\|$, called the *norm* of x, such that

1. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$
2. $\|\alpha x\| = |\alpha| \|x\|$, for $x \in X$ and $\alpha \in \mathbb{R}$
3. $\|x\| > 0$ if $x \neq 0$

The *norm* is the *function* that maps x to $\|x\|$. Every normed space can be regarded as a metric space, in which the distance is determined by $d(x, y) = \|x - y\|$. The relevant properties of $d(x, y)$ between x and y is $\|x - y\|$. The relevant properties of d are

1. $0 \leq d(x, y) < \infty$ for all x and y
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ for all x, y
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z

In any metric space, the *open ball* with center at x and radius r is the set

$$B_r(x) = \{y : d(x, y) < r\}$$

In particular, if X is a normed space, the sets

$$B_1(0) = \{x : \|x\| < 1\} \quad \text{and} \quad \bar{B}_1(0) = \{x : \|x\| \leq 1\}$$

are the open unit ball and closed unit ball of X, respectively. We can form a *topology* on X by declaring a set open if and only if it is a (possibly empty) union of open balls.

Comment. It is easy to verify that the vector spece operations (addition and scalar multiplication) are continuous in this topology. A continuous function maps open sets to open sets. The set of open balls forms a basis of the topology. To verify continuity, we only need to verify this property for elements of the basis.

1. For $\alpha \in \mathbb{R}$, $\alpha(B_r(x)) = B_{\alpha r}(\alpha x)$
2. Notate $x + y$ as a function $+ : X \times X \to X$. For x, y we have $+(B_r(x) \times B_{r'}(y)) = B_{r+r'}(x + y)$.

A *Banach space* is a normed space which is *complete*, which means that all Cauchy Sequences converge. Most of the best-known function spaces are Banach spaces. For example,

- The set of all continuous functions on compact spaces
- L^p spaces that occur in integration theorem
- Hilbert spaces

1.2 Vector Spaces

The letters $\mathbb{R}$ and $\mathbb{C}$ will denote the real and complex numbers, respectively. For the moment, let Φ stand for either $\mathbb{R}$ and $\mathbb{C}$. A scalar is a member of the scalar field Φ. A vector space of Φ is a set X whose elements are called vectors and in which two operations, *addition* and *scalar multiplication* are defined with the following algebraic properties

1. To every pair of vectors x and y corresponds a vector $x+y$ in such a way that

$$x+y=y+x \quad \text{and} \quad x+(y+z)=(x+y)+z$$

2. To every pair $(\alpha, x) \in \Phi \times X$, there corresponds a vector αx in such a way that

$$1x=x \quad \text{and} \quad \alpha(\beta x)=(\alpha\beta)x$$

   and so that the two dsitributive laws hold

$$\alpha(x+y)=\alpha x+\alpha y \quad \text{and} \quad (\alpha+\beta)x=\alpha x+\beta y$$

The symbol 0 will also be used for the zero element of the scalar field. A real vector space is one for which $\Phi=\mathbb{R}$ and a complex vector space is one for which $\Phi=\mathbb{C}$.

If X is a vector space, $A, B \subset X$, $x \in X$ and $\lambda \in \Phi$, the following notation is used

$$\begin{aligned} x+A &= \{x+a : a \in A\} \\ x-A &= \{x+(-1a) : a \in A\} \\ A+B &= \{a+b : a \in A, b \in B\} \\ \lambda A &= \{\lambda a : a \in A\} \end{aligned}$$

Not from these conventions that it need not be that $2A=A+A$.[1]

A set $Y \subset X$ is called a subspace of X if Y is itself a vector space. Importantly this means closed under scalar multiplication and addition. One can check that this happens if and only if $0 \in Y$ and

$$\alpha Y+\beta Y \subset Y$$

for all scalars $\alpha, \beta \in \Phi$.

A set $C \subset X$ is said to be *convex* if

$$tC+(1-t)C \subseteq C, (0 \leq t \leq 1)$$

A set $B \subset X$ is said to be balanced if $\alpha B \subset B$ for every $\alpha \in \Phi$ with $|\alpha| \leq 1$.[2] A vector space has *dimension* n if it has a basis $\{u_1, \ldots, u_n\}$.[3].

1.3 Topological Spaces

A *topological space* is a set S in which a collection τ of subsets (call *open sets*) has been specified with the following properties

1. $S, \emptyset$ are open
2. The intersection of any two open sets is open
3. THe union of every collection of open sets is open

A set $E \subset S$ is *closed* if and only if it's complement is open. The *closure*, $\bar{E}$, of E is the intersection of all closed sets that contain $\bar{E}$[4]

[1] For example, if $A=\{1,3\}$, then $2A=\{2,6\} \neq \{2,4,6\}=A+A$. We should, however, always have that $2A \subset A+A$.
[2] This would be like decreasing returns to scale on a production possibilities set.
[3] Every $x \in X$ can be written as a linear combination of basis elements
[4] An equivalent definition of this is the smallest closed set that contains E. The intersection of an arbitrary collection of closed sets is closed, this is clear from taking the "complement" of the fact that an arbitrary collection of open sets is open.

The *interior* of E is the union of of all open sets that are subsets of E[5]. A *neighborhood* of a point $p \in S$ is any open set that contains p. (S, τ), read S equipped with the topology τ, is a *Hausdorff space* and τ is a *Hausdorff Topology* if distinct points of S have disjoint neighborhoods[6]. A set $K \subset S$ is *compact* if every open cover of K has a finite subcover.[7]

A collection $\tau' \subset \tau$ is a *base* for τ if every member of τ is a union of members of τ'. A collection γ of neighborhoods of a point $p \in S$ if every neighbothood of p contains a member of γ.

If $E \subset S$ and if σ is the collection of all intersection $\sigma\{E \cap V : V \in \tau\}$, then we call this the topology E inherits from S. It is easy to verify that this is a valid topology.

If a topology τ is induced by a metric d, we say d and τ are *compatible.*

Topological Vector Spaces Suppose τ is a topology on X such that every point of X is a closed set[8] and the vector space operations are continuous with respect τ. Under these condition, τ is said to be a *vector topology* on X and X is a *topological vector space.*

A subset E is said to be *bounded* if, for every neighborhood V of 0 in X corresponds to a number $s > 0$ such that $E \subset tV$ for every $t > s$.

Let X be a topological vector space. Associate to each $a \in X$ and each $\lambda \neq 0$ the translation operator T_a and multiplication operator M_λ, by the formulas

$$T_a(x) = a + x \quad \text{and} \quad M_\lambda(x) = \lambda x$$

The following proposition is important

Proposition 1. *A homeomorphism is a continuous function with a continuous inverse. T_a and M_λ are homeomorphisms of X onto X.*

Proof. The cector space axioms imply that T_a and M_λ are one to one, that they map X onto X, and that their inverses are T_{-a} and $M_{1/\lambda}$, respectively. We have already seen that all of these are continuous. $\square$

One consequence of this is that every vector topology τ is translation invariant. That is a set $E \subset X$ is open iff each of its translates $a + E$ is open. So τ is completely determined by any local base. In the cector space contest, the term local base will always mean a local base at 0. A local base of a topological vector space X is thus a collection $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{B}$. A metric d on a vector space X will be called invariant if

$$d(x + x, y + x) = d(x, y)$$

for all $x, y, z \in X$.

Types of topological vector spaces

1. X is locally convex if there is a local base $\mathcal{B}$ whose members are convex
2. X is locally bounded if 0 has a bounded neighborhood
3. X is locally compact if 0 has a neighborhood whose closure is compact
4. X is metrizable if τ is compatible with some metric d.
5. X is an F-space is it's topology is induced by a complete invariant metric d.

[5] Again, equivalently, this is the largest open set contained in E

[6] Because there are real numbers between every real number, the standard topology on the reals has this property

[7] Formally, if $K \subset \bigcup_{i \in \mathcal{I}} U_i$ for a collection of open sets $\{U_i\}_{i \in \mathcal{I}}$ then there exists a subset $J \subset \mathcal{I}$ with $|J| < \infty$ such that $K \subset \bigcup_{j \in J} U_j$

[8] This is the case for Hausdorff spaces

6. X is a Frechet space if X is a locally convex F-space.
7. X is normable if a norm exists on X such that the metric induced by the norm is compatible with τ.
8. X Normed spaces and Banach spaces are defined above.
9. X has the Heine-Borel property if every closed and bounded subset of X is compact.

1.4 Seperation Properties

Theorem 1. *Suppse K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \emptyset$*

2 Chapter 2: Completeness

Chapter one looks like mainly a review of topology and elements of analysis. Chapter 2 starts on studying completness.