Readings on Demand Identification

Manu Navjeevan

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Starting with some readings on nonparametric identification of simultaneous equations for Demand Identification in Two Sided Markets

1 Matzkin Identification Chapter

Here mainly focus on section 3.5, Identification in Simultaneous Equation Models. Based off of Matzkin (2008), which should also be covered in these notes.

Focus is on the simultaneous equations model, where $Y \in \mathbb{R}^G$ denotes a vector of observable dependent variables, $X \in \mathbb{R}^K$ denotes a vector of observable explanatory variables, $\epsilon \in \mathbb{R}^G$ denotes a vector of unobservable explanatory variables and the relationship between these vectors is specified by a function $r^* : \mathbb{R}^G \times \mathbb{R}^K \to \mathbb{R}^G$ such that

$$\epsilon = r^*(Y, X)$$

The set S of $r^*, F_{\epsilon,X}$ that are considered consist of twice differentiable functions $r : \mathbb{R}^G \times \mathbb{R}^K \to \mathbb{R}^G$ and twice differentiable, strictly increasing distributions $F_{\epsilon,X} : \mathbb{R}^G \times \mathbb{R}^K \to \mathbb{R}$ such that (i) for all $F_{\epsilon,X}, \epsilon$ and X are distributed independently of each other (ii) for all r and y, x, $|\partial r(y,x)/\partial y| > 0$ (iii) for all r and all x, ϵ there exists a unique value of y such that $\epsilon = r(y, x)$, and (iv) for all r, all $F_{\epsilon,X}$ and all x, the distribution of Y given $X = x$, induced by r and $F_{\epsilon|X=x}$ has support $\mathbb{R}^G$.

For any $(r, F_{\epsilon,X}) \in S$ condition (iii) implies that there exists a function h such that for all ϵ, X

$$Y = h(X, \epsilon)$$

This is the reduced form system of the structural equations system determined by r. Will let h^* denote the reduced form function determined by r^* (the "true" value of r).

A special case of this model is the linear system of simultaneous equations, where for some invertible $G \times G$ matrix A and some $G \times K$ matrix B,

$$\epsilon = AY + BX$$

Premultiplication by A^{-1} yields the reduced form system

$$Y = \Pi X + \nu$$

where $\Pi = -A^{-1}B$ and $\nu = A^{-1}\epsilon$. The identification of the true values A^*, B^* is well studied (Koopmans (1949), Koopmans, Rubin, Leipnik (1950), and Fisher (1966) as well as most econometrics textbooks).

- My guess is that full nonparametric identification would amount to the conditions for identification of the linear system holding locally, evverywhere.

Main results here, assume that $\mathrm{E}(\epsilon) = 0$ and $\mathrm{Var}(\epsilon) = \Sigma^*$, an unkown matrix. Let W denote the varianve of ν. The identification of (A^*, B^*, Σ^*) is achieved when it can be uniqely recovered from Π and $\mathrm{Var}(\nu)$. A priori restrictions on A^*, B^*, Σ^* are typically used to determine the existence of a unique solution for any element of the above triple.

Analagoulsy, one can obstain necessary and suffecient conditions to uniquely recover r^* and F^*_ϵ from the distribution of the observable variables (Y, X), when the system of structural equations is nonparametric. The question of identification is whether we can uniquely recover the density f^*_ϵ and the function r^* from the conditional densities $f_{Y|X=x}$.

Following from the definition of observational equivalence, can state that two fucntions $r, \tilde{r}$ satisfying (i)-(iv) are obs. equivalent iff $\exists f_\epsilon, \tilde{f}_\epsilon$ such that $(f_\epsilon, r), (\tilde{f}_\epsilon, \tilde{r}) \in S$ and for all y, x

$$\tilde{f}_\epsilon(\tilde{r}(y,x)) \left| \frac{\partial \tilde{r}(y,x)}{\partial y} \right| = f_\epsilon(r(y,x)) \left| \frac{\partial r(y,x)}{\partial y} \right| \tag{1}$$

The function $\tilde{r}$ can be re-expressed as a transformation of (ϵ, x). To see this, define

$$g(\epsilon, x) = \tilde{r}(h(x, \epsilon), x)$$

where h is the reduced form equation corresponding to r above. Since

$$\left|\frac{\partial g(\epsilon, x)}{\partial \epsilon}\right| = \left|\frac{\partial \tilde{r}(h(x, \epsilon), x)}{\partial y}\right| \left|\frac{\partial h(x, \epsilon)}{\partial \epsilon}\right|$$

it follows from assumption (ii) that $\left|\frac{\partial g(\epsilon,x)}{\partial \epsilon}\right| > 0$ everywhere. Since, conditional on x, h is invertible in ϵ and $\tilde{r}$ is invertible in y, it follows that g is invertible in ϵ. Substituting into (1), we can see that $(\tilde{r}, \tilde{f}_\epsilon) \in S$ is observationally equivalent to $(r, f_\epsilon) \in S$ iff $\forall \epsilon, x$

$$\tilde{f}_\epsilon(g(\epsilon, x)) \left|\frac{\partial g(\epsilon, x)}{\partial \epsilon}\right| = f_\epsilon(\epsilon)$$

The following theorem provides conditions garunteeing a transformation g of ϵ does not generate an observationally equivalent pair $(\tilde{r}, \tilde{f}_\epsilon)$.

Theorem 1 (Matzkin, 2005)**.** *Let $(r, f_\epsilon) \in S$. Let $g(\epsilon, x)$ be such that $\tilde{r}(y, x) = g(r(y, x), x)$ and $\tilde{\epsilon} = g(\epsilon, x)$ are such that $(\tilde{r}, \tilde{f}_\epsilon) \in S$. If for some ϵ, x, the rank of the matrix*

$$\begin{pmatrix} \left(\frac{\partial g(\epsilon,x)}{\partial \epsilon}\right)' & \frac{\partial \log f_\epsilon(u)}{\partial \epsilon} - \frac{\partial \log\left|\frac{\partial g(\epsilon,x)}{\partial \epsilon}\right|}{\partial \epsilon} \\ \left(\frac{\partial g(\epsilon,x)}{\partial x}\right)' & -\frac{\partial \log\left|\frac{\partial g(\epsilon,x)}{\partial \epsilon}\right|}{\partial x} \end{pmatrix}$$

Alternatively, can express this as an identification result for the function r^*

Theorem 2 (Matzkin, 2005)**.** *Let $M \times \Gamma$ denote the set of pairs $(r, f_\epsilon) \in S$. The function r^* is identified in M if $r^* \in M$ and, for all $f_\epsilon \in \Gamma$ and all $\tilde{r}, r \in M$ such that $\tilde{r} \neq r$, there exist y, x such that the matrix*

$$\begin{pmatrix} \left(\frac{\partial \tilde{r}(y,x)}{\partial y}\right)' & \Delta_y(y, x; \partial r, \partial^2 r, \partial \tilde{r}, \partial^2 \tilde{r}) + \frac{\partial \log(f_\epsilon(r(y,x)))}{\partial \epsilon} \frac{\partial r(y,x)}{\partial y} \\ \left(\frac{\partial \tilde{r}(y,x)}{\partial x}\right)' & \Delta_y(y, x; \partial r, \partial^2 r, \partial \tilde{r}, \partial^2 \tilde{r}) + \frac{\partial \log(f_\epsilon(r(y,x)))}{\partial \epsilon} \frac{\partial r(y,x)}{\partial x} \end{pmatrix}$$

is strictly larger than G, where

$$\Delta_y(y, x; \partial r, \partial^2 r, \partial \tilde{r}, \partial^2 \tilde{r}) = \frac{\partial}{\partial y} \log\left|\frac{\partial r(y, x)}{\partial y}\right| - \frac{\partial}{\partial y} \log\left|\frac{\partial \tilde{r}(y, x)}{\partial y}\right|$$
$$\Delta_x(y, x; \partial r, \partial^2 r, \partial \tilde{r}, \partial^2 \tilde{r}) = \frac{\partial}{\partial x} \log\left|\frac{\partial r(y, x)}{\partial y}\right| - \frac{\partial}{\partial x} \log\left|\frac{\partial \tilde{r}(y, x)}{\partial y}\right|$$

Example 1. As a simple example, consider the simultaneous equations model analyzed by Matzkin (2007c), where for some unknown function g^* and some parameter values β^*, γ^*,

$$y_1 = g^*(y_2) + \epsilon_1$$
$$y_2 = \beta^* y_1 + \gamma^* x + \epsilon_2$$

Further, assume that (ϵ_1, ϵ_2) has an everywehre positive, differentiable desnsity $f^*_{\epsilon_1,\epsilon_2}$ such that, for two not necessarily known a-priori values $(\bar{\epsilon}_1, \bar{\epsilon}_2)$ and $(\epsilon_1'', \epsilon_2'')$

$$0 \neq \frac{\partial \log f^*_{\epsilon_1,\epsilon_2}(\bar{\epsilon}_1, \bar{\epsilon}_2)}{\partial \epsilon_1} \neq \frac{\partial \log f^*_{\epsilon_1,\epsilon_2}(\epsilon_1'', \epsilon_2'')}{\partial \epsilon_1} \neq 0$$
$$\frac{\partial \log f^*_{\epsilon_1,\epsilon_2}(\bar{\epsilon}_1, \bar{\epsilon}_2)}{\partial \epsilon_2} = \frac{\partial \log f^*_{\epsilon_1,\epsilon_2}(\epsilon_1'', \epsilon_2'')}{\partial \epsilon_2} = 0$$

The observable exogeneous variable x is assumed to be distributed independently of (ϵ_1, ϵ_2) and to possess support $\mathbb{R}$. In this model

$$\epsilon_1 = r_1^*(y_1, y_2, x) = y_1 - g^*(y_2)$$
$$\epsilon_2 = r_2^*(y_1, y_2, x) = -\beta^* y_1 + y_2 - \gamma^* x$$

The Jacobian determinant is

$$\left| \begin{pmatrix} 1 & -\frac{\partial g^*(y_2)}{\partial y_2} \\ -\beta^* & 1 \end{pmatrix} \right| = 1 - \beta^* \frac{\partial g^*(y_2)}{\partial y_2}$$

which will be positive so long as $1 > \beta^* \partial g^*(y_2)/\partial y_2$. Since the first element in the diagonal is positive, it follows from Gale and Nikaido (1965) that the function r^* is globally invertible if the conditionl $1 > \beta^* \frac{\partial g^*(y_2)}{\partial y_2}$ holds for all y[1]. Let $r, \tilde{r}$ be any two differentiable functions satisfying this condition and the other properties assumed about r^*. Suppose that at some y_2, $\frac{\partial \tilde{g}(y_2)}{\partial y_2} \neq \frac{\partial g(y_2)}{\partial y_2}$. Assume also that $\gamma \neq 0$ and $\tilde{\gamma} \neq 0$. Let $f_{\epsilon_1,\epsilon_2}$ denote any density satisfying the same properties that $f^*_{\epsilon_1,\epsilon_2}$ is assumed to satisfy. Denote by (ϵ_1, ϵ_2) and $(\epsilon_1', \epsilon_2')$ the two points such that

$$0 \neq \frac{\partial f_{\epsilon_1,\epsilon_2}(\epsilon_1, \epsilon_2)}{\partial \epsilon_1} \neq \frac{\partial \log f_{\epsilon_1,\epsilon_2}(\epsilon_1, \epsilon_2')}{\partial \epsilon_1} \neq 0$$
$$\frac{\partial \log f_{\epsilon_1,\epsilon_2}(\epsilon_1, \epsilon_2)}{\partial \epsilon_2} = \frac{\partial \log f_{\epsilon_1,\epsilon_2}(\epsilon_1', \epsilon_2')}{\partial \epsilon_2} = 0$$

Define

$$a_1(y_1, y_2, x) := \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_1} - \beta \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$$
$$a_2(y_1, y_2, x) := \left(\frac{\frac{\partial^2 g(y_2)}{\partial y_2^2}}{1 - \beta \frac{\partial g(y_2)}{\partial y_2}} - \frac{\frac{\partial^2 \tilde{g}(y_2)}{\partial y_2^2}}{1 - \beta \frac{\partial \tilde{g}(y_2)}{\partial y_2}} \right) - \frac{\partial g(y_2)}{\partial y_2} \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_1}$$
$$+ \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$$
$$a_3(y_1, y_2, x) := -\gamma \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$$

By Theorem 3.4, r and $\tilde{r}$ will not be observationally equivalent if for all $f_{\epsilon_1,\epsilon_2}$ there exists (y_1, x) such that the rank of the matrix

$$\begin{pmatrix} 1 & -\tilde{\beta} & a_1(y_1, y_2, x) \\ -\frac{\partial \tilde{g}(y_2)}{\partial y_2} & 1 & a_2(y_1, y_2, x) \\ 0 & -\tilde{\gamma} & a_3(y_1, y_2, x) \end{pmatrix}$$

[1] like a global extension to the implicit function theorem

is 3. Let

$$a_1'(y_1, y_2, x) := \left(\tilde{\beta} - \beta\right) \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$$

$$a_2'(y_1, y_2, x) := \left(\frac{\frac{\partial^2 g(y_2)}{\partial y_2^2}}{1 - \beta \frac{\partial g(y_2)}{\partial y_2}} - \frac{\frac{\partial^2 \tilde{g}(y_2)}{\partial y_2^2}}{1 - \beta \frac{\partial \tilde{g}(y_2)}{\partial y_2}} \right) + \left(\frac{\partial \tilde{g}(y_2)}{\partial y_2} - \frac{\partial g(y_2)}{\partial y_2} \right) \left(\frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_1} \right)$$

$$a_3'(y_1, y_2, x) = (\tilde{\gamma} - \gamma) \frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$$

Multiplying the first column of A by $-\frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_1}$ and adding it to the third column, and multiplying the second column by $\frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2}$ and adding it to the third column[2], one obtains the matrix

$$\begin{pmatrix} 1 & -\tilde{\beta} & a_1'(y_1, y_2, x) \\ -\frac{\partial \tilde{g}(y_2)}{\partial y_2} & 1 & a_2'(y_1, y_2, x) \\ 0 & -\tilde{\gamma} & a_3'(y_1, y_2, x) \end{pmatrix}$$

By assumption either

$$a_2'(\bar{y}_1, \bar{y}_2, \bar{x}) \neq 0 \quad \text{or} \quad a_2'(\tilde{y}_1, \tilde{y}_2, \tilde{x}) \neq 0$$

where $(\bar{y}_1, \bar{y}_2, \bar{x})$ correspond to an arbitrary (ϵ_1, ϵ_2), and $(\tilde{y}_1, \tilde{y}_2, \tilde{x})$ correspond to $(\epsilon_1'', \epsilon_2'')$ from above.[3].

Suppose the later. Let $y_1 = g(y_2 + \epsilon_1)$ and let $x = \frac{-\beta y_1 + y_2 - \epsilon_2'}{\gamma}$. Follows that

$$\frac{\partial \log f_{\epsilon_1,\epsilon_2}(y_1 - g(y_2), -\beta y_1 + y_2 - \gamma x)}{\partial \epsilon_2} = 0$$

At such a y_1, x the matrix above becomes the rank 3 matrix

$$\begin{pmatrix} 1 & -\tilde{\beta} & 0 \\ -\frac{\partial \tilde{g}(y_2)}{\partial y_2} & 1 & a_2'(y_1, y_2, x) \\ 0 & -\tilde{\gamma} & 0 \end{pmatrix}$$

so the derivatives of g^* are identified.

[2] Therse are standard row operations and do not change the invertibility

[3] Actually I'm a bit unsure here

2 An Almost Ideal Demand System

Here I read the 1980 Paper, *An Almost Ideal Demand System* By Angus Deaton and John Muelbauer. The paper goes over how to identify demand systems in large markets. The hope is to adapt this somehow to two sided markets under competition.

2.1 Introduction

- Richard Stone (1954) first estimated a system of demand equations derived explicitly from consumer theory
- Paper proposes and estimates a new model which is of comparable generality to the Rotterdam and translog models but has advantages over both
- Model (AIDS) gives an arbitrarty first order-approximation to any demand system, satisfies the axioms of choice exactly, aggregates perfectly over consumers without invoking parallel linear Engle curves, has a functional form which is consistent with known household-budget data, simple to estimate, largely avoids need for non-linear estimation.
- Model is estimated on postwar British data.

2.1.1 Specification of the AIDS

Generally, starting point has been the specification of a function general enough to act as a second-order approximation to any arbitrart direct or indirect utility function. It is possible to use a first order approximation to the demand functions themselves as in the Rotterdam model.

AIDS approach follows from these approaches but builds froma specific class of preferences by which the theorems of Muellbauer permit exact aggregation over consumers. Preferences are known as PIGLOG preferences and are represented via expenditure function[1].

[1] minimum expenditure necessary to obtain a specific utility level at given prices