36106 Managerial Decision Modeling
Nonlinear Models

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Reading and Excel Files

Reading (Powell and Baker):

- Chapter 8 (especially 8.7)

Files used in this lecture:

- `nonConvex.xlsx`
- `markowitz.xlsx`
- `markowitz_key.xlsx`

Lecture Outline

Learning Objectives

1. Understand the different categories of optimization models
2. Understand the tradeoff between model realism and model solvability
3. Learn to implement nonlinear models in Excel
4. Build and implement a portfolio optimization model in Excel

Motivation

There are many points in the early history of a firm where winning one order or being a little ahead of somebody can make these very nonlinear differences.

Bill Gates Business Week, 1992

Motivation

Many business processes behave in a very **nonlinear** way.

- the price of a bond is a nonlinear function of interest rates
- the price of a stock option is a nonlinear function of the price of the underlying stock
- the quantity demanded for a product is a nonlinear function of the price that product
- marginal costs are usually not constant over a wide range of production quantities
- revenue is the product of price and quantity sold

Basic Concepts

Optimization Types:

- Linear program – objective function and all constraints purely linear.
- Nonlinear program – at least one nonlinear term appears somewhere. These problems may be **convex** or **concave**.
- Linear integer program – objective function and all constraints purely linear, but at least one variable is required to be an integer (binary) variable.
- Nonlinear integer program – at least one nonlinear term appears somewhere and at least one variable is required to be an integer (binary) variable.
- Stochastic (could be nonlinear and/or have integer variables) – we actually treat a stochastic parameter as a probability distribution.

Types of Constrained Optimization Models

Don't worry about nonconvex versus convex for now, just linear versus nonlinear.

$$\begin{array}{ll} 2x_1 + 5x_2 & \text{is linear} \\ 2x_1 + 5x_1x_2 & \text{is nonlinear} \\ 2x_1^2 + x_2 & \text{is nonlinear} \\ x_1/x_2 & \text{is nonlinear} \end{array}$$

Some Excel functions are linear, others nonlinear.

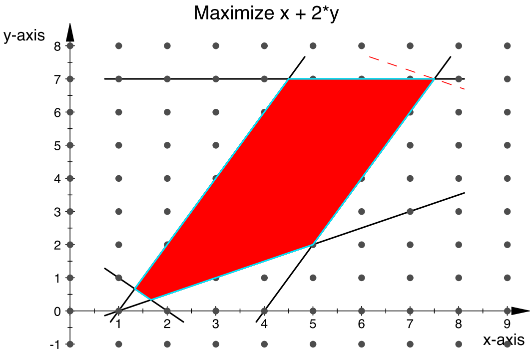
SUM – a linear function

IF, **OR**, **AND**, **MAX**, **MIN** – nonlinear

SUMPRODUCT – could be either

Basic Concepts

Linear programs are the *easiest to solve.* The feasible region looks like this:



Historical Notes

Big Breakthrough: George Dantzig found a way to solve constrained optimization problems when the constraints and objective function are linear. He showed that if there was an optimal solution then there was an optimal **extreme point** solution.

Extreme point: a point on the boundary of the feasible region where the constraints intersect.

Dantzig's extreme point result led to a very effective solution algorithm for linear programs (linear objective and linear constraints), called the **Simplex Algorithm**.

Historical Notes

A few technical points:

1. every extreme points correspond to a square, nonsingular matrix (basis matrix)
2. the Simplex algorithm works on finding the inverse to the basis matrix
3. the sensitivity report in Solver is generated from the basis matrix inverse that corresponds to the optimal extreme point

It is now possible to solve linear programs with quite literally millions of variables and constraints.

Basic Concepts

All Nonlinear Model Are Not Created Equal:

- There are "difficult" nonlinear problems
- There are "easy" nonlinear problems

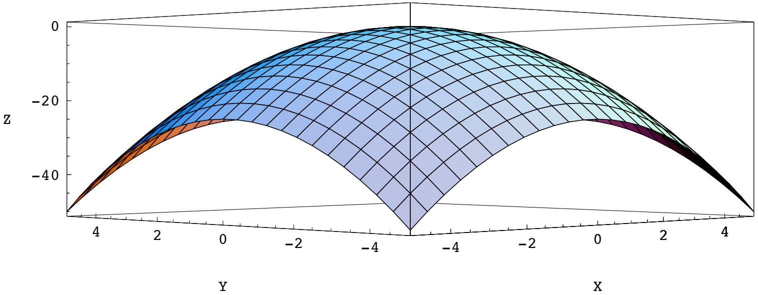
We illustrate one of each type.

Difficult: The model in `nonConvex.xlsx`

Easy: Portfolio optimization `markowiz.xlsx`

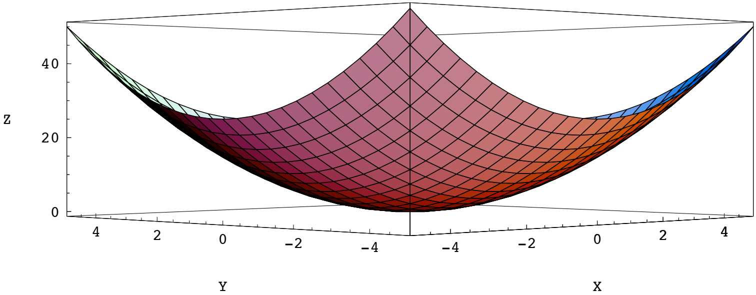
Basic Concepts

A nice function $f(x,y) = -x^2 - y^2$. This is called **concave**. It is "bowl shaped down."



Basic Concepts

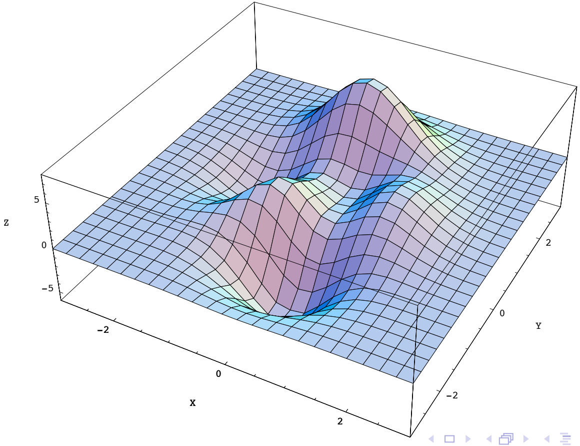
A nice function $f(x, y) = x^2 + y^2$. This is called **convex**. It is "bowl shaped up."



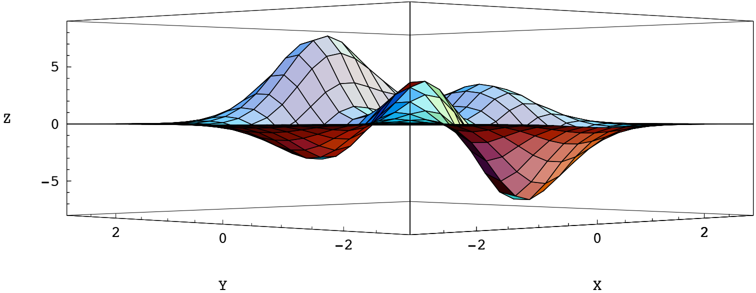
Basic Concepts

A seriously mean function:

$$
\begin{aligned}
f(X,Y) &= 3(1-X)^2 e^{\left(-X^2-(Y+1)^2\right)} \\
&\quad -10(X/5-X^3-Y^5)e^{\left(-X^2-Y^2\right)} - e^{\left(-(X+1)^2-Y^2\right)}/3.
\end{aligned}
$$



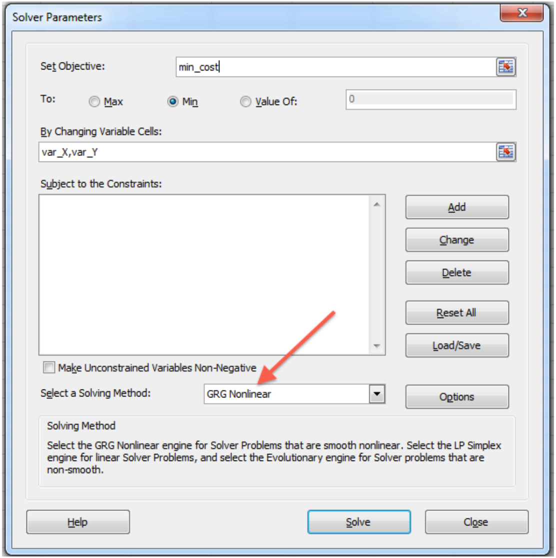
Basic Concepts



Using an **IF** function in Excel that references adjustable cells is a great way to create problems that look like this.

Basic Concepts

See Workbook `nonConvex.xlsx`. Pick the **GRG Nonlinear** solver.



Basic Concepts

The GRG Nonlinear solver uses techniques based on Calculus.

The GRG Nonlinear solver calculates **gradients** and tries to move in directions of steepest ascent or decent.

It can "get stuck" down in a valley or at the top of a hill.

The valley or hill at termination may not be the deepest (minimization case) valley or highest (maximization case) hill.

A convex function has only one valley and a concave function only one hill.

Basic Concepts

These Calculus-based algorithms for the nonlinear problems are based on the work of Karush, Kuhn, Tucker and others.

Karush was a student of Lawrence Graves in the U of C math department.

Lawrence Graves was the father of Robert Graves.

Robert Graves was a Booth Deputy Dean for many years and professor in the OM group.

Basic Concepts

Karush home at 1050 North Damen.

Basic Concepts

Open workbook **nonConvex.xlsx**. The adjustable cells are named `var_X` and `var_Y`.

Experiment One: Run Solver starting with initial values of (0, 0) for the adjustable cells. What is the optimal objective function value?

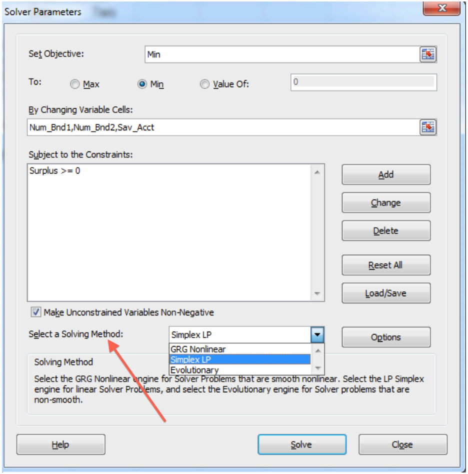
Experiment Two: Run Solver starting with initial values of (0, -1) for the adjustable cells. What is the optimal objective function value?

When solving nonlinear problems, how do you know if you have the true optimal solution?

Welcome to the real world!

Types of Constrained Optimization Models

When working with solver it is important to know the kind of optimization problem you have. Use the **Select a Solving Method** judiciously.



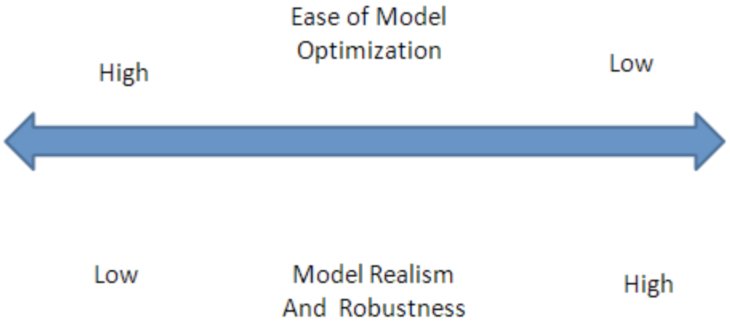
Types of Constrained Optimization Models

Model Category	Solver Solving Method
Linear program	Simplex LP
Integer linear program	Simplex LP
Convex nonlinear program	GRG Nonlinear
Nonconvex nonlinear program	GRG Nonlinear
Integer nonlinear program	GRG Nonlinear/Evolutionary
Stochastic program	@Risk, RiskSolver

Solving a stochastic, nonlinear, nonconvex, integer programming model – you are kidding, right?

Model and Solution Tradeoff

Linear programs are the easiest to solve. However, most real problems usually have some nonlinearities and discrete aspects.



Model and Solution Tradeoff

Even now, the most commonly solved problems are deterministic mixed-integer linear programs.

Is the world linear? I don't think so!

Question: So why all the solving of mixed integer linear programs?

Answer: Because of the proliferation of great software!

Portfolio Optimization

A good application of nonlinear programming is **portfolio optimization.** (See Powell and Baker, Section 8.7) We use Excel (`markowitz.xlsx`) to:

- Download stock prices from the Web – spreadsheet `aapl`
- aggregate stock prices into a single table – spreadsheet `prices`
- Calculate the stock returns – spreadsheet `returns`
- Calculate the matrix of excess returns – spreadsheet `returns`
- Build a variance-covariance matrix – spreadsheet `model`
- Minimize portfolio variance subject to a return constraint `model`

This is an excellent example of how Excel is a multi-talented player.

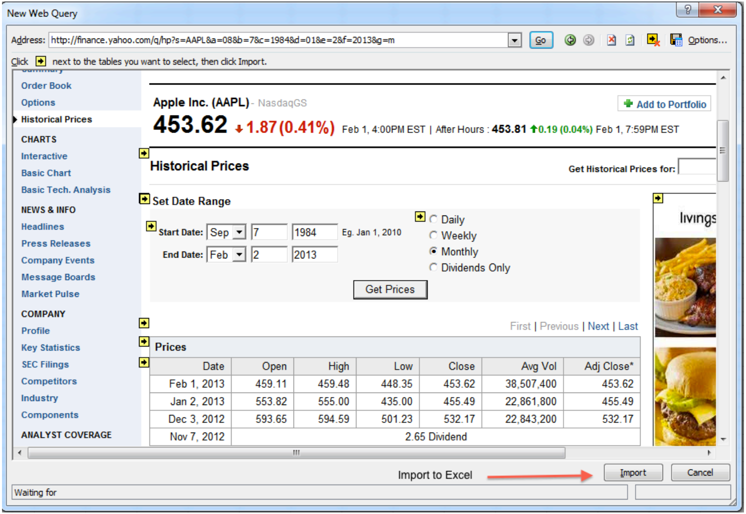
Portfolio Optimization

Getting stock prices:

- Many Web sites provide stock prices. We use **finance.yahoo.com**.
- You can download prices directly into Excel.
- Go to the **Data** tab and then **FromWeb** in the **Get External Data** tab.
- Paste in the appropriate URL

Portfolio Optimization

Get the prices for Apple and import into `aapl` spreadsheet.



Portfolio Optimization

Now calculate the stock returns.

- Returns are calculated in the spreadsheet `returns`
- The prices are in spreadsheet `prices`
- We assume continuous compounding
- For example, to calculate the first period Apple returns

```
=LOG(prices!B3/prices!B2, base_e)
```

Note: **base_e** is the range that holds the irrational number e used as the logarithm base.

Portfolio Optimization

Now calculate the variance-covariance matrix.

Basic Stat: The **covariance**, σ_{12} between two random variables, r_1 and r_2, is:

$$\sigma_{12} = \sum_{t=1}^{n} (r_{1t} - \mu_1)(r_{2t} - \mu_2)/n.$$

$$\mu_i = \sum_{t=1}^{n} r_{it}/n$$

Calculate the matrix of excess returns. See the range **excess_returns** in the spreadsheet `returns`. Each element in this range is $r_{it} - \mu_i$.

Portfolio Optimization

Now calculate the variance-covariance matrix (continued)

Use the Excel function **MMULT** to multiply the transpose of the excess returns matrix by the excess returns matrix. See:

`http://www.youtube.com/watch?v=ZfJW3ol2FbA`

See the range named **variance_covariance** in the spreadsheet `model`.

The formula is:

```
=MMULT(TRANSPOSE(excess_returns),excess_returns)/ROWS(excess_returns)
```

Important: When entering the `MMULT` function hit control, shift, return, not just return.

Portfolio Optimization

The Markowitz model: Let X_i be the fraction of the portfolio invested in stock i. If

$$X = [X_1, X_2, \ldots, X_n]$$

$$\begin{aligned}
\min XCX^\top & & & (1)\\
\sum_{i=1}^{n} \mu_i X_i &\geq R & & (2)\\
\sum_{i=1}^{n} X_i &= 1 & & (3)\\
X_i &\leq 1, & i = 1, \ldots, n & \quad (4)\\
X_i &\geq 0, & i = 1, \ldots, n & \quad (5)
\end{aligned}$$

Portfolio Optimization

The Markowitz model (continued): The objective is to *minimize portfolio variance*. This is $XCX^\top$ where C is the variance-covariance matrix of returns. In other words:

$$Var(r_1X_1 + r_2X_2 + \cdots + r_nX_n) = XCX^\top$$

In our implementation $XCX^\top$ is in the cell (range) named **portfolio_variance**. The formula is

```
=MMULT(MMULT(investment_vars,variance_covariance),TRANSPOSE(investment_vars))
```

For taking the product of the three matrices X, C, and $X^\top$, using **MMULT** see:

http://www.youtube.com/watch?v=py8yBL052Jg

Another Approach: .

```
=SUMPRODUCT(MMULT(investment_vars,variance_covariance), investment_vars)
```

This works because

```
MMULT(investment_vars,variance_covariance)
```

is a $1 \times n$ matrix and

```
investment_vars
```

is a $1 \times n$ matrix.

Portfolio Optimization

The Markowitz model (continued): The variance of the portfolio is minimized subject to a constraint that the expected return must meet or exceed a required return of R.

$$EXP(r_1X_1 + r_2X_2 + \cdots + r_nX_n) = \sum_{i=1}^{n} \mu_i X_i \geq R$$

The sum of the investment fractions must sum to 1.0.

$$\sum_{i=1}^{n} X_i = 1$$

Finally, the investment fractions are between 0 and 1.

$$0 \leq X_i \leq 1, \quad i = 1, 2, \ldots, n$$

Portfolio Optimization

The Markowitz model spreadsheet (see `markowitz.xlsx`).

C22 f_x

	A	B	C	D
1	**Markowitz Portfolio Optimization**			
2				
3		**Variance Covariance Matrix**		
4		AAPL	AMD	ORCL
5	AAPL	0.0047	-0.0111	0.0058
6	AMD	-0.0111	0.4341	0.1120
7	ORCL	0.0058	0.1120	0.2134
8	Mean Returns	-0.0501	0.0359	0.1425
9	Investment Level	0.116920575	0	0.883079
10				
11	Expected Return	0.1200		
12	Required Return	0.1200		
13	Expected Return Slack	0.0000		
14				
15	Portfolio Variance	0.1677		
16				
17	Investment Level Sum	1		
18	Unity Constraint	0		
19				

Generating a variance-covariance matrix:

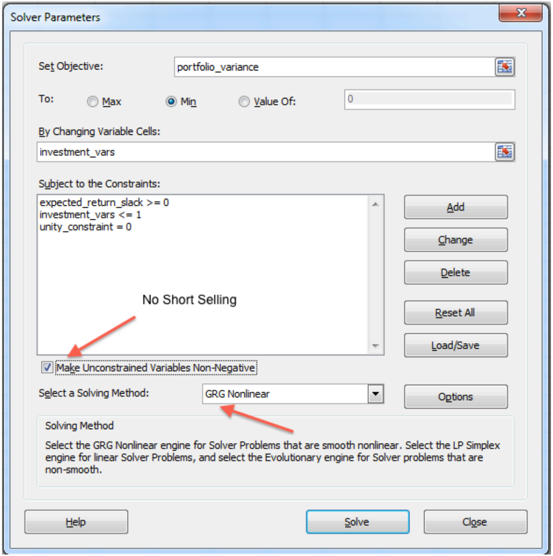
http://www.youtube.com/watch?v=ZfJW3ol2FbA

Multiply Three Matrices

http://www.youtube.com/watch?v=py8yBL052Jg

Portfolio Optimization

The Markowitz solver model (see `markowitz.xlsx`).



Portfolio Optimization

Sanity Checks:

- What should the objective function value be if all of the portfolio is invested in Oracle stock?
- If the required return, R, is set to 0.1425 what is the optimal solution?
- If the required return, R, is set to 0.1500 what is the optimal solution?
- Apple has a negative expected return, so why is it present in the optimal solution?

Portfolio Optimization

Here is the sensitivity analysis report. Instead of a shadow price or dual price we now have a **Lagrange Multiplier.**

Why is there no allowable increase or decrease?

	A	B	C	D	E	F
1	**Microsoft Excel 14.0 Sensitivity Report**					
2	**Worksheet: [markowitz_key.xlsx]model**					
3	**Report Created: 10/23/2015 11:53:21 PM**					
4						
5						
6	Variable Cells					
7				Final	Reduced	
8		Cell	Name	Value	Gradient	
9		B9	Investment Level AAPL	0.116921314	0	
10		C9	Investment Level AMD	0	0.020063003	
11		D9	Investment Level ORCL	0.883079686	0	
12						
13	Constraints					
14				Final	Lagrange	
15		Cell	Name	Value	Multiplier	
16		B13	expected_return_slack	-2.24584E-13	1.90514085	
17		B18	unity_constraint	-1E-06	-0.106733397	
18						
19						

Portfolio Optimization

What is the interpretation of the dual price or Lagrange multiplier on the return constraint?

Hint: think *efficient frontier*.

Required Return	Portfolio Variance
0.06	0.0734
0.07	0.0863
0.08	0.1004
0.09	0.1155
0.10	0.1318
0.11	0.1492
0.12	0.1677

Portfolio Optimization

The **efficient frontier:**