36106 Managerial Decision Modeling
Modeling with Integer Variables – Part 1

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Reading and Excel Files

Reading (Powell and Baker):

- Chapter 11

Files used in this lecture:

- mpfpVanilla_key_int.xlsx
- capital_budget.xlsx
- capital_budget_key.xlsx
- `set_covering.xlsx`
- `set_covering_key.xlsx`

Lecture Outline

Basic Concepts

Capital Budgeting

Covering – Revisited

Learning Objectives

1. Learn to model go, no-go decisions.

2. Study applications where go, no-go decisions are critically important.

3. Learn to implement the go, no-go decisions into Solver.

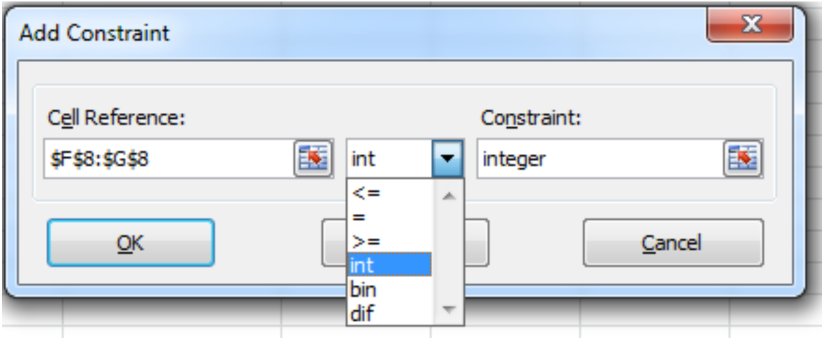
Basic Concepts – Integer Variables

Consider our cash flow matching example. We purchase a fractional number of bonds!

	A	B	C	D	E	F	G	H
1	**Multi Period Financial Planning**							
2								
3						Available Bonds		
4						One	Two	
5	Period	Cash_Req	Sav_Acct	Surplus		---------------	---------------	
6	0	$10	$4.804		Price ($000):	$0.980	$0.965	
7	1	$11	$5.604	$0.00	Coupon Rate:	6.00%	6.50%	
8	2	$12	$5.436	$0.00	No Purchased:	95.796	90.155	
9	3	$14	$3.262	$0.00	Maturity:	5	12	
10	4	$15	$0.000	$0.00				
11	5	$17	$90.404	$0.00	Savings Interest Rate:		4.00%	
12	6	$19	$80.880	($0.00)				
13	7	$20	$69.975	($0.00)	Initial Cost of Portfolio (000):			
14	8	$22	$56.634	($0.00)			$195.68	
15	9	$24	$40.760	($0.00)				
16	10	$26	$22.250	($0.00)				
17	11	$29	$0.000	($0.00)				
18	12	$31	$65.015	($0.00)				
19	13	$33	$34.615	$0.00				
20	14	$36	$0.000	$0.00				
21								

Basic Concepts – Integer Variables

How to require an integer number of bonds.



Note: Cells F8:G8 are the adjustable cells for Bond 1 and Bond 2.

Basic Concepts – Integer Variables

What will happen to the solution value?

Do we have more choices or fewer choices?

Sample Test Question: The old optimal solution value is \$195.68. The new optimal solution value will be:

- Exactly equal to \$195.68
- Strictly less than \$195.68
- Strictly greater than \$195.68

Basic Concepts – Integer Variables

The new solution

	A	B	C	D	E	F	G
1	**Multi Period Financial Planning**						
2							
3						Available Bonds	
4						One	Two
5	Period	Cash_Req	Sav_Acct	Surplus		---------------	---------------
6	0	$10	$4.797		Price ($000):	$0.980	$0.965
7	1	$11	$5.598	$0.00	Coupon Rate:	6.00%	6.50%
8	2	$12	$5.432	$0.00	No Purchased:	96.000	90.000
9	3	$14	$3.260	$0.00	Maturity:	5	12
10	4	$15	$0.000	$0.00			
11	5	$17	$90.610	$0.00	Savings Interest Rate:		4.00%
12	6	$19	$81.084	($0.00)			
13	7	$20	$70.178	($0.00)	Initial Cost of Portfolio (000):		
14	8	$22	$56.835	($0.00)			$195.73
15	9	$24	$40.958	($0.00)			
16	10	$26	$22.447	($0.00)			
17	11	$29	$0.194	($0.00)			
18	12	$31	$65.015	$0.04			
19	13	$33	$34.615	$0.00			
20	14	$36	$0.000	$0.00			
21							

Basic Concepts – Integer Variables

Compare the linear programming solution and the **integer programming solution.**

	Linear Solution	**Integer Solution**
Bond 1	95.796	96
Bond 2	90.155	90

The optimal solution value is $195.73 and the old value was $195.68. Not a big deal – **just round**! Well maybe not ...

Basic Concepts

Consider the following simple integer program.

```
T = NUMBER OF TOWNHOUSES PURCHASED;
A = NUMBER OF APARTMENT BUILDINGS PURCHASED;

OBJECTIVE:
MAX = 10*T + 15*A;

FUNDS AVAILABLE ($1000):
282*T + 400*A <= 2000;

MANAGERS'S TIME AVAILABILITY;
4*T + 40*A <= 140;

TOWNHOUSES AVAILABLE:
T <= 5;
```

Basic Concepts – Eastborne

Optimal *linear programming* solution is `T = 2.48` and `A = 3.25` for an optimal value of $73.57. So what should we do? Let's try and round:

T	A	Round	Value
3	4	Up Up	Infeasible
2	3	Down Down	$65,000
2	4	Down Up	Infeasible
3	3	Up Down	Infeasible

The optimal solution is `T = 4` and `A = 2` for an optimal value of $70,000 which is substantially better than the rounded solution value of $65,000.

Basic Concepts – Rounding

Bottom Line: When is rounding acceptable? Probably okay if variables take on large values and rounding does not have a big impact on feasibility or optimality. This was true with cash flow matching (but maybe not if there a lot of bond classes).

Rounding a **BIG problem** if optimal values are small. As we see next, they may even be 0 or 1.

Question: Why not always use the **int** constraint in Solver?

Answer: we get bitten by our old friend, the tradeoff between solvability and realism. *The int constraint makes the problem much harder.*

Capital Budgeting

Capital budgeting was one of the first application domains for optimization.

The basic construct goes back to **Jim Lorie** and **Leonard Savage**.

Jim Lorie – famous Booth Dean. Started Center for Research in Security Prices at Chicago.

Leonard Savage – famous U of C statistician. His son is Sam Savage author of *Flaw of Averages.*

Ideas for *Flaw of Averages* used later in quarter.

See the Marr Corporation on page 292 of the Powell and Baker text and `capital-budget.xlsx`. We modify the numbers slightly.

Capital Budgeting

The Marr Corporation in the text is a vanilla cone version, but is illustrative of the basic idea.

There are five potential projects to fund at time 0.

P1: Implement a new information system

P2: License a new technology from another firm

P3: Build a state-of-the-art recycling facility

P4: Install an automated machining center

P5: Move the receiving department to a new location

There is $160 available to fund the projects.

All projects are go/no-go (unlike Lajitas) and must be funded at either 0% or 100%.

Capital Budgeting

Here are the projected cash flows for the life of the project.

	A	B	C	D	E	F	G
1	Capital Budgeting - Powell and Baker page 293 -- with altered numbers						
2							
3		Interest Rate	0.04				
4		Time 0 Available Capital	160				
5							
6					Projects		
7			P1	P2	P3	P4	P5
8	Cash Flows	Year 1	35	30	20	15	12
9		Year 2	30	40	35	20	20
10		Year 3	30	40	40	20	30
11		Year 4	30	40	40	20	30
12		Net Present Value	$65.70	$39.58	$40.34	$35.79	$18.34
13							
14		Time 0 Cash Requirement	48	96	81	32	64

Ojective: *Pick which projects to fund in order to maximize net present value at time 0 without violating the $160 availability.*

Capital Budgeting – Algebraic Statement

Key Concept: Using **binary variables.**

Variable Definition: $X_j = 1$ if project j is accepted, 0 if not, for $j = 1, 2, 3, 4, 5$.

$$\max \; 65.70X_1 + 39.58X_2 + 40.34X_3 + 35.79X_4 + 18.34X_5$$

$$\begin{array}{lrcl} \text{subject to} & 48X_1 + 96X_2 + 81X_3 + 32X_4 + 64X_5 & \leq & 160 \\ & X_1, X_2, X_3, X_4, X_5 & \in & \{0, 1\} \end{array}$$

Capital Budgeting – Solver Model

Define a set of five adjustable cells for each X_j, $j = 1, 2, 3, 4, 5$

There is only one slack constraint saying you cannot violate the $160 available capital at time 0.

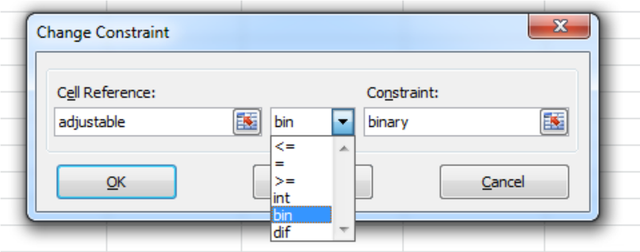
Define a best cell which is the `SUMPRODUCT` of the adjustable cells and the time 0 net present values.

Capital Budgeting – Solver Model

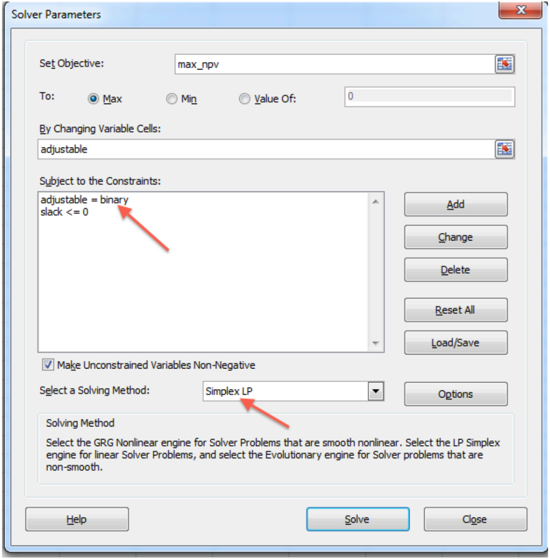
	A	B	C	D	E	F	G
1	Capital Budgeting - Powell and Baker page 293 -- with altered numbers						
2							
3		Interest Rate	0.04				
4		Time 0 Available Capital	160				
5							
6					Projects		
7			P1	P2	P3	P4	P5
8	Cash Flows	Year 1	35	30	20	15	12
9		Year 2	30	40	35	20	20
10		Year 3	30	40	40	20	30
11		Year 4	30	40	40	20	30
12		Net Present Value	$65.70	$39.58	$40.34	$35.79	$18.34
13							
14		Time 0 Cash Requirement	48	96	81	32	64
15							
16		Yes/No	1	0	0	1	1
17							
18							
19		Time 0 Capital Used	$ 144.00				
20		Time 0 Slack	$ (16.00)				
21		Total Net Present Value	$ 119.84				

Capital Budgeting – Solver Model

Make the adjustable cells **binary**, not **integer**.

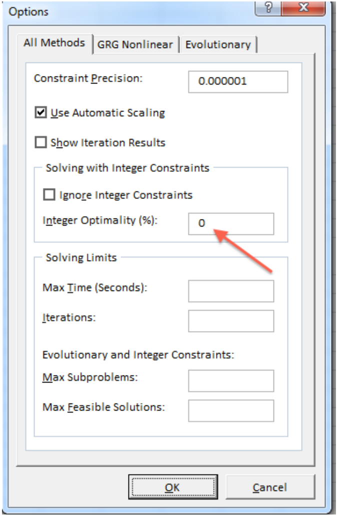


Capital Budgeting – Solver Model



Capital Budgeting – Solver Model

Note: Under options make sure you set the **Integer Optimality(%)** to zero. **Always** select this option.



Capital Budgeting

Optimal Solution: The optimal solution value is $119.84. The optimal solution is

```
P1 = 1
P2 = 0
P3 = 0
P4 = 1
P5 = 1
```

What happens if we just delete the **bin** constraint?

What happens if the **bin** constraint is replaced with the **int** constraint?

What is the optimal **continuous solution** value?

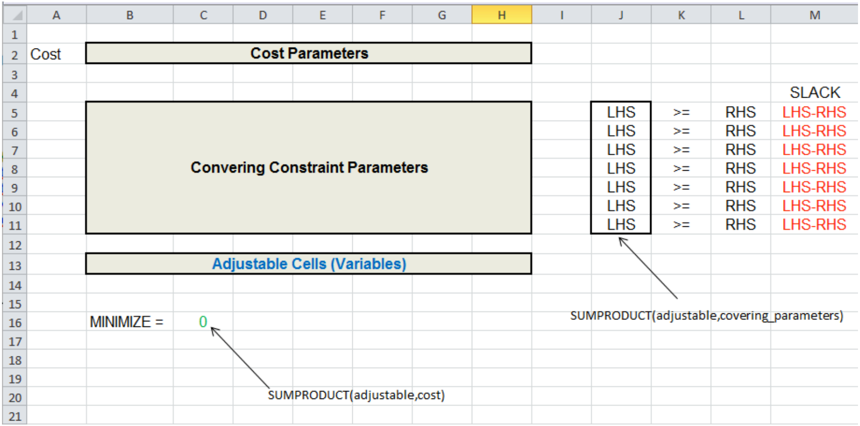
What is the value of an extra dollar of funding (i.e. a budget of 161)?

Capital Budgeting

What are some realistic extensions and variations?

Covering – Revisited

Recall our generic covering problem.



Covering – Revisited

Covering problems often have **fractional solutions.** Consider the following simple example.

$$
\begin{array}{rcccll}
\min & x_1 & +x_2 & +x_3 & & \\
 & x_1 & +x_2 & & \geq & 1 \\
 & & x_2 & +x_3 & \geq & 1 \\
 & x_1 & & +x_3 & \geq & 1
\end{array}
$$

The optimal linear programming solution is

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{1}{2}.$$

See the spreadsheet `simple` in the workbook `set_covering.xlsx`.

Covering – Revisited

Here is a good example of a covering problem. See page 295 of Powell and Baker.

- The city of Metropolis is divided into nine districts.
- Each district must be served by emergency fire vehicles that can reach any point in the district within three minutes.
- Metropolis is considering seven possible locations for fire stations.
- Each fire station can reach only a subset of the districts within the three-minute time limit.

Objective: meet the objective of **covering** all nine districts while building the minimum number of fire stations.

Covering – Revisited

In the table below a 1 indicates that a fire station at site *i* can cover district *j* within the three minute constraint, 0 if not. For examples, a fire station located at site S3 can reach Districts 4, 5, and 8 within the time limit, but none of the others.

Site	S1	S2	S3	S4	S5	S6	S7
District 1	0	1	0	1	0	0	1
District 2	1	0	0	0	0	1	1
District 3	0	1	0	0	0	1	1
District 4	0	1	1	0	1	1	0
District 5	1	0	1	0	1	0	0
District 6	1	0	0	1	0	1	0
District 7	1	0	0	0	0	0	1
District 8	0	0	1	1	1	0	0
District 9	1	0	0	0	1	0	0

See `set_covering.xlsx`

Covering – Revisited

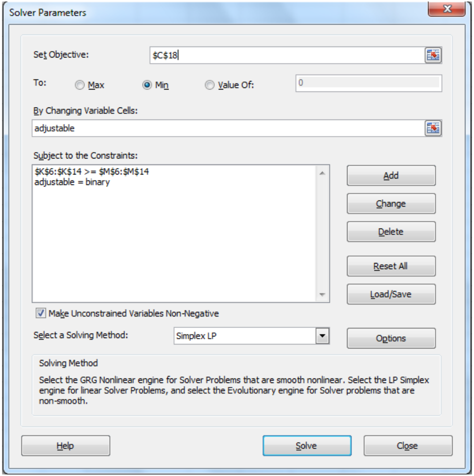
Here is the optimal covering.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2													
3	Cost		1	1	1	1	1	1	1				
4											Sites		
5		**Site**	**S1**	**S2**	**S3**	**S4**	**S5**	**S6**	**S7**		**Used**		
6		**District 1**	**0**	**1**	**0**	**1**	**0**	**0**	**1**		2	>=	1
7		**District 2**	**1**	**0**	**0**	**0**	**0**	**1**	**1**		1	>=	1
8		**District 3**	**0**	**1**	**0**	**0**	**0**	**1**	**1**		1	>=	1
9		**District 4**	**0**	**1**	**1**	**0**	**1**	**1**	**0**		1	>=	1
10		**District 5**	**1**	**0**	**1**	**0**	**1**	**0**	**0**		1	>=	1
11		**District 6**	**1**	**0**	**0**	**1**	**0**	**1**	**0**		1	>=	1
12		**District 7**	**1**	**0**	**0**	**0**	**0**	**0**	**1**		1	>=	1
13		**District 8**	**0**	**0**	**1**	**1**	**1**	**0**	**0**		2	>=	1
14		**District 9**	**1**	**0**	**0**	**0**	**1**	**0**	**0**		1	>=	1
15													
16			0	0	0	1	1	0	1				
17													
18		Minimize =	3										
19													

See `set_covering_key.xlsx`

Covering – Revisited

Here is the solver model.



See `set_covering_key.xlsx`