36106 Managerial Decision Modeling
Modeling with Integer Variables – Part 2

Kipp Martin
University of Chicago
Booth School of Business

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Reading and Excel Files

Reading (Powell and Baker):

- Section 10.2
- Section 10.4
- Chapter 11, Sections 11.4 and 11.5

Files used in this lecture:

- `allocationFC_if.xlsx`
- `allocationFC_if_key.xlsx`
- `allocationFC_int.xlsx`
- `allocationFC_int_key.xlsx`
- `supply_chain_trans.xlsx`
- `supply_chain_trans_key.xlsx`
- `supply_chain.xlsx`
- `supply_chain_if.xlsx`
- `supply_chain_key.xlsx`

Lecture Outline

Modeling Fixed Costs

Supply Chain Management

Modeling Logical Conditions

Sensitivity Analysis

Conclusion

Learning Objectives

1. Understand the distinction between fixed and sunk costs.
2. Learn why IF statements cause problems when modeling fixed costs.
3. Learn how to effectively model fixed costs in Excel.
4. Learn how to model supply chain management problems.
5. Learn how to model logical conditions.

Modeling Fixed Costs

In Powell and Baker see:

- Section 11.5

Motivation: recall the Shelby Shelving example.

- I only considered **variable overhead** in the objective function.
- I assumed the **fixed costs** were **sunk costs**.

How do we treat a fixed cost if it is not a sunk cost?

Modeling Fixed Costs

Consider the Veerman Furniture Company from Section 9.2. We can produce chairs, desks, and tables.

In our earlier model there were zero fixed costs.

New Key Assumptions:

- if we produce a nonzero quantity of chairs there is a fixed cost of $3000
- if we produce a nonzero quantity of desks there is a fixed cost of $3000
- if we produce a nonzero quantity of tables there is a fixed cost of $3000

Modeling Fixed Costs

How do we model this?

First, use an Excel **IF** function.

For example,

```
IF(number chairs produced > 0, 3000, 0)
IF(number desks produced > 0, 3000, 0)
IF(number tables produced > 0, 3000, 0)
```

Then put this IF function into the profit calculation.

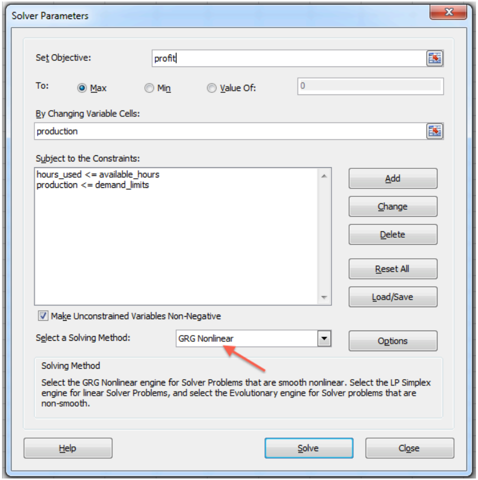
Modeling Fixed Costs

In the workbook `allocationFC_if.xlsx` put the fixed costs into the profit function and use Solver to find the optimal profit.

	A	B	C	D	E	F	G	H
1		Powell and Baker Resource Allocation Example -- Section 9.2						
2								
3		Chair	Desk	Table				
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00				
5	Fixed Costs	$ 3,000.00	$ 3,000.00	$ 3,000.00				
6						Hours		Available
7						Used		Hours
8	Fabrication(Hours)	4	6	2		0	<=	1850
9	Assembly(Hours)	3	5	7		0	<=	2400
10	Shipping(Hours)	3	2	4		0	<=	1500
11								
12								
13	Number Produced	0	0	0				
14	Demand Limits	360	300	100				
15	Fixed Cost							
16								
17	Total Fixed Cost							
18	Profit =	$ -						

Modeling Fixed Costs

Important: since the **IF** function will contain an *adjustable cell* the model is NOT linear!
Select **GRG Nonlinear** in the Solver window.



Modeling Fixed Costs

The optimal Solver solution is:

- number chairs = 0
- number desks = 0
- number tables = 0

for a zero profit!

Argh! Can this be optimal?

Rerun the model with a starting solution of 50 chairs, 50 desks, and 50 tables. Now what is the profit?

Modeling Fixed Costs

Summary:

- When the model is given an initial solution of 0 desks, 0 chairs, and 0 tables it produces an "optimal" solution of 0 chairs, 0 desks, and 0 tables for a 0 profit.
- When the model is given an initial solution of 50 desks, 50 chairs, and 50 tables it produces an "optimal" solution of 0 chairs, 275 desks, and 100 tables for a $2,400 profit.
- The true optimal solution is to produce 0 chairs, 300 desks, and 0 tables for a $4,200 profit.

How did we find the optimal solution?

Modeling Fixed Costs

There is a serious problem when you try to use an **IF** function in Excel when the **IF** function references adjustable cells.

All is not lost, we can do a much better job with **binary variables**!

Introduce the following binary variables.

- $Y_C = 1$ if a nonzero quantity of chairs are produced, 0 otherwise
- $Y_D = 1$ if a nonzero quantity of desks are produced, 0 otherwise
- $Y_T = 1$ if a nonzero quantity of tables are produced, 0 otherwise

Modeling Fixed Costs

Recall the production quantity variables:

- X_C is the number of chairs produced (and there is a demand limit of 360 chairs)
- X_D is the number of desks produced (and there is a demand limit of 300 desks)
- X_T is the number of tables produced (and there is a demand limit of 100 tables)

Now how do we **link** the Y and X variables?

Modeling Fixed Costs

The new **profit function** is

$$15X_C + 24X_D + 18X_T - 3000Y_C - 3000Y_D - 3000Y_T$$

and we **add the fixed-cost constraints**

- $X_C \leq 360Y_C$
- $X_D \leq 300Y_D$
- $X_T \leq 100Y_T$

Modeling Fixed Costs

Key Idea: Consider, for example, the chairs fixed-cost constraint.

$$X_C \leq 360Y_C$$

If $X_C > 0$ this **forces** $Y_C = 1$ since Y_C can only take the values 0 or 1. This accomplishes two things.

1. When $Y_C = 1$ we incur the fixed cost of \$3000 in the profit function.
2. When $Y_C = 1$ the constraint $X_C \leq 360Y_C$ becomes $X_C \leq 360$ and we cannot produce more than 360 chairs.

Is $X_C = 0$ and $Y_C = 1$ feasible? Would Solver do this?

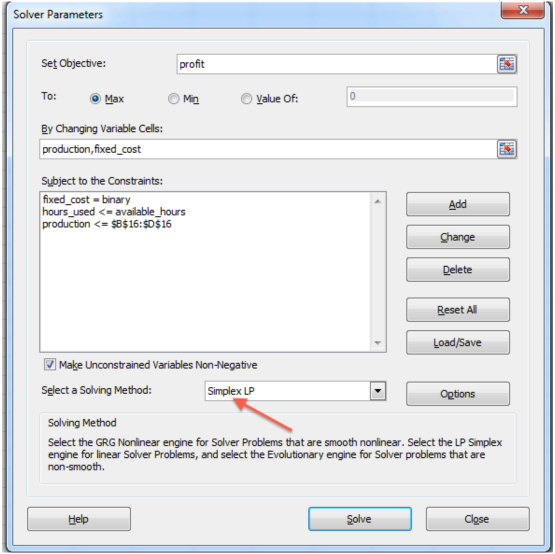
Modeling Fixed Costs

Open the workbook `allocationFC_int.xlsx` and implement the new profit function and constraints. Implement the fixed cost constraints in the range `B16:D16`.

	A	B	C	D	E	F	G	H
1		Powell and Baker Resource Allocation Example -- Section 9.2						
2								
3		Chair	Desk	Table				
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00				
5	Fixed Costs	$ 3,000.00	$ 3,000.00	$ 3,000.00				
6						Hours		Available
7						Used		Hours
8	Fabrication(Hours)	4	6	2		0	<=	1850
9	Assembly(Hours)	3	5	7		0	<=	2400
10	Shipping(Hours)	3	2	4		0	<=	1500
11								
12								
13	Number Produced	0	0	0				
14	Demand Limits	360	300	100				
15	Fixed Cost Variables	0	0	0				
16	Fixed Cost Constraint							
17								
18	Total Fixed Cost							
19	Profit =	$ -						

Modeling Fixed Costs

Here is the Solver model. Note that we have selected **Simplex LP**.



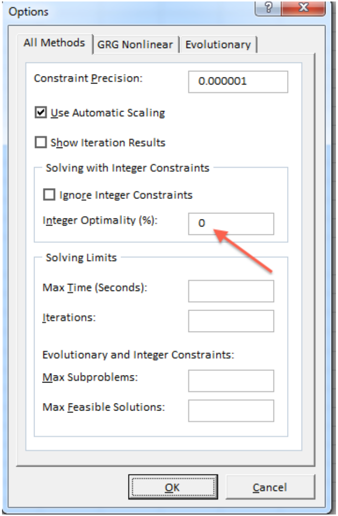
Modeling Fixed Costs

Here is the optimal solution. See the Key Excel spreadsheet.

	A	B	C	D	E	F	G	H	I
1		Powell and Baker Resource Allocation Example -- Section 9.2							
2									
3		Chair	Desk	Table					
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00					
5	Fixed Costs	$ 3,000.00	$ 3,000.00	$ 3,000.00					
6						Hours		Available	Slack
7						Used		Hours	Hours
8	Fabrication(Hours)	4	6	2		1800	<=	1850	50
9	Assembly(Hours)	3	5	7		1500	<=	2400	900
10	Shipping(Hours)	3	2	4		600	<=	1500	900
11									
12									
13	Number Produced	0	300	0					
14	Demand Limits	360	300	100					
15	Fixed Cost Variables	0	1	0					
16	Fixed Cost Constraint	0	300	0					
17									
18	Total Fixed Cost	$ 3,000.00							
19	Profit =	$ 4,200.00							

Modeling Fixed Costs

Note: Under options make sure you set the **Integer Optimality(%)** to zero.



Supply Chain Management

We now develop a typical supply chain management model.

These models can result in very large savings.

The Procter & Gamble company used a model similar to what we develop next and reduced the number of North American plants by almost 20 percent, saving over $200 million in pretax costs per year.

See `http://pubsonline.informs.org/doi/abs/10.1287/inte.27.1.128?journalCode=inte`

Supply Chain Management

In Powell and Baker see

- Section 10.2
- Section 10.4
- Section 11.5

In this model we add together:

Transportation Problem + Fixed Cost Problem

We have:

- Demand constraints
- Supply constraints
- Fixed cost constraints (to capture the cost of opening the new warehouses)

Supply Chain Management

First, just consider a **transportation problem**. See Section 10.2 of Powell and Baker. See Workbook `supply_chain_trans.xlsx`.

Levinson Food Company -- See Section 11.5 in Powell and Baker

		Distribution Center										
		Alb	Boise	Dallas	Denver	Houston	Okla	Phoenix	Salt	San Ant	Wichita	**Capacity**
Warehouse Location	Albuquerque	0.00	47.00	32.00	22.00	42.50	27.00	23.00	30.00	36.50	29.50	16,000
	Dallas	32.00	79.50	0.00	39.00	12.50	10.50	50.00	63.00	13.50	17.00	20,000
	Denver	21.00	42.00	39.00	0.00	51.50	31.50	40.50	24.00	47.50	26.00	10,000
	Houston	42.50	91.00	12.50	51.50	0.00	23.00	58.00	72.00	10.00	31.00	10,000
	Phoenix	23.00	49.00	50.00	40.50	58.00	49.00	0.00	32.55	50.00	52.00	12,000
	San Antonio	36.50	83.50	13.50	47.50	10.00	24.00	50.50	66.50	0.00	32.00	10,000
	Demand	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700	

Basic Idea: Supply the distribution centers from warehouses. Meet the distribution demand without exceeding warehouse capacity. Do so at minimum cost.

Supply Chain Management

Model Formulation:

Parameters: (stochastic or deterministic?)

- C_{ij} = the marginal transportation cost for shipping a unit from warehouse i to distribution center j
- D_j = the demand at distribution center j
- S_i = the supply at warehouse i

Variable Definition:

X_{ij} = number of units shipped from warehouse i to distribution center j

Supply Chain Management

Model Formulation:

What is the interpretation of $X_{34} = 5000$?

What is the interpretation of $X_{48} = 7000$?

We want to minimize **shipping costs**.

$$\min 47X_{12} + 32X_{13} + \cdots + 32X_{6,10}$$

In Excel the **shipping costs** are

`SUMPRODUCT(shipping_costs,shippingVars)`

Supply Chain Management

Supply Constraints: One for each warehouse.

$$\sum_{j=1}^{10} X_{ij} \leq \mathrm{S}_i, \quad i = 1, \ldots, 6$$

Sample Test Question: write out the San Antonio supply constraint.

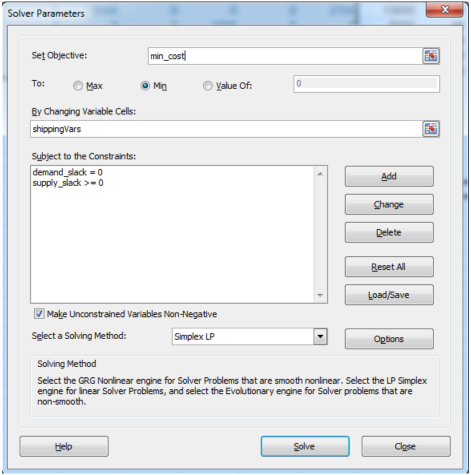
Demand Constraints: One for each distribution center.

$$\sum_{i=1}^{6} X_{ij} = \mathrm{D}_j, \quad j = 1, \ldots, 10$$

Sample Test Question: write out the Boise demand constraint.

Supply Chain Management

Build a Solver model.



Supply Chain Management

The optimal solution value is $230,850.00.

Levinson Food Company -- See Section 11.5 in Powell and Baker

		Alb	Boise	Dallas	Denver	Houston	Okla	Phoenix	Salt	San Ant	Wichita	**Capacity**
						Distribution Center						
	Albuquerque	0.00	47.00	32.00	22.00	42.50	27.00	23.00	30.00	36.50	29.50	16,000
	Dallas	32.00	79.50	0.00	39.00	12.50	10.50	50.00	63.00	13.50	17.00	20,000
Warehouse	Denver	21.00	42.00	39.00	0.00	51.50	31.50	40.50	24.00	47.50	26.00	10,000
Location	Houston	42.50	91.00	12.50	51.50	0.00	23.00	58.00	72.00	10.00	31.00	10,000
	Phoenix	23.00	49.00	50.00	40.50	58.00	49.00	0.00	32.55	50.00	52.00	12,000
	San Antonio	36.50	83.50	13.50	47.50	10.00	24.00	50.50	66.50	0.00	32.00	10,000
	Demand	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700	

	Plant	Alb	Boise	Dallas	Denver	Houston	Okla	Phoenix	Salt	San Ant	Wichita	**Row Sum**		**Capacity**
						Distribution Center								
	Albuquerque	3200	0	0	0	0	0	0	0	0	0	3200	<=	16,000
	Dallas	0	0	6800	0	0	3500	0	0	0	2700	13000	<=	20,000
Warehouse	Denver	0	2500	0	4000	0	0	0	1800	0	0	8300	<=	10,000
Location	Houston	0	0	0	0	9600	0	0	0	0	0	9600	<=	10,000
	Phoenix	0	0	0	0	0	0	5000	0	0	0	5000	<=	12,000
	San Antonio	0	0	0	0	0	0	0	0	7400	0	7400	<=	10,000
	Column Sum	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700			
		=	=	=	=	=	=	=	=	=	=			
	Demand	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700			
	Demand Slack	0	0	0	0	0	0	0	0	0	0			

X_{IJ} Variables

Min Cost = $ 230,850.00

Supply Chain Management

Realistic Modification: The **fixed cost** problem. If a warehouse is open, a fixed cost of operating the warehouse is charged. This cost is 0 if the warehouse is not open.

Hence we have: `IF(OPEN, Fixed Cost, 0)`

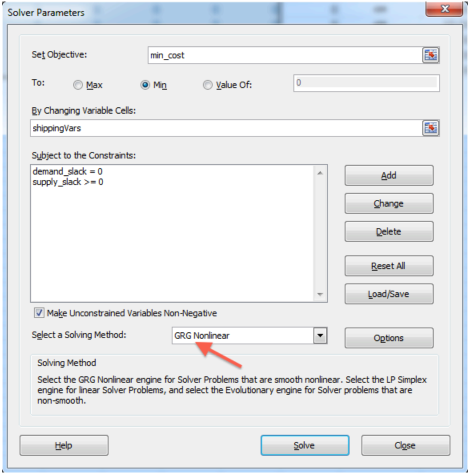
However, *we do not know before solving the model which warehouses are open.*

Indeed, we want the model to determine which warehouses to open!

Therefore, whether or not a warehouse should be open is a *decision variable*.

Supply Chain Management

First, open the workbook supply_chain_if.xlsx and solve the model using **IF** functions. The optimal value is $975,850. It takes some time to solve.



Supply Chain Management

Now model without the **IF** functions. Use **binary variables**.

Variable Definition:

$Y_i = 1$ if warehouse i is open, 0 if not.

Additional Parameters:

$F_i =$ the fixed cost of operating warehouse i

What is the interpretation of $Y_3 = 1$?

In Excel the fixed costs are:

`SUMPRODUCT(fixed_costs,warehouseVars)`

Supply Chain Management

Supply/Forcing Constraints:

Consider, for example, San Antonio (index 6). Here is the fixed cost/supply constraint

$$X_{61} + X_{62} + X_{63} + \cdots + X_{6,10} \leq 10000 * Y_6$$

This replaces our IF statement.

If there are any products sent from the San Antonio warehouse, i.e. $X_{61} > 0$, or $X_{62} > 0$, etc., then $Y_6 = 1$ or the constraint is not satisfied.

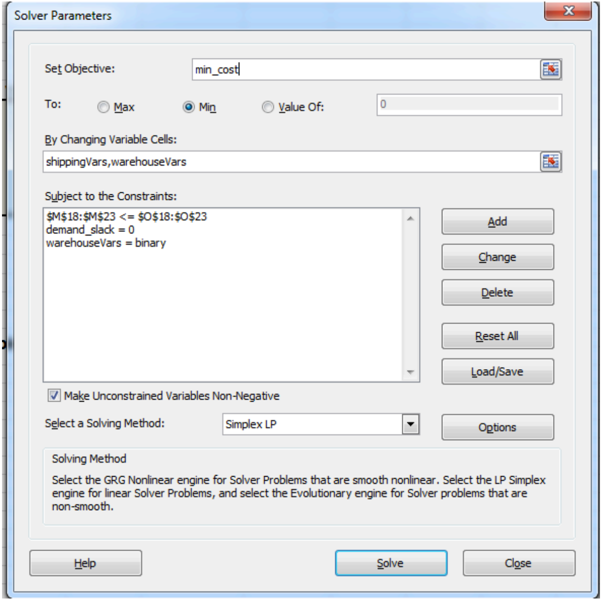
Sample Test Question: write out the Denver supply constraint.

Supply Chain Management

The algebraic model:

$$
\begin{aligned}
\min \sum_{i=1}^{6} \sum_{j=1}^{10} C_{ij} X_{ij} \quad &+ \quad \sum_{i=1}^{6} F_i Y_i \\
\sum_{i=1}^{6} X_{ij} \quad &= \quad D_j, \quad j = 1, \ldots, 10 \\
\sum_{j=1}^{10} X_{ij} \quad &\leq \quad S_i Y_i, \quad i = 1, \ldots, 6 \\
X_{ij} \quad &\geq \quad 0 \\
Y_i \quad &\in \quad \{0, 1\}
\end{aligned}
$$

Supply Chain Management – Solver Model



Supply Chain Management – Solver Model

How much money do we save by using integer variables instead of the **IF** function for our model?

Levinson Food Company -- See Section 11.5 in Powell and Baker

						Distribution Center							**Fixed**
		Alb	Boise	Dallas	Denver	Houston	Okla	Phoenix	Salt	San Ant	Wichita	**Capacity**	**Cost**
	Albuquerque	0.00	47.00	32.00	22.00	42.50	27.00	23.00	30.00	36.50	29.50	16,000	$ 140,000.00
	Dallas	32.00	79.50	0.00	39.00	12.50	10.50	50.00	63.00	13.50	17.00	20,000	$ 150,000.00
Warehouse	Denver	21.00	42.00	39.00	0.00	51.50	31.50	40.50	24.00	47.50	26.00	10,000	$ 100,000.00
Location	Houston	42.50	91.00	12.50	51.50	0.00	23.00	58.00	72.00	10.00	31.00	10,000	$ 110,000.00
	Phoenix	23.00	49.00	50.00	40.50	58.00	49.00	0.00	32.55	50.00	52.00	12,000	$ 125,000.00
	San Antonio	36.50	83.50	13.50	47.50	10.00	24.00	50.50	66.50	0.00	32.00	10,000	$ 120,000.00
	Demand	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700		

						Distribution Center									**Open or**
	Plant	Alb	Boise	Dallas	Denver	Houston	Okla	Phoenix	Salt	San Ant	Wichita	**Row Sum**		**Capacity**	**Closed**
	Albuquerque	0	0	0	0	0	0	0	0	0	0	0	<=	0	0
	Dallas	0	0	6800	0	0	3500	0	0	7000	2700	20000	<=	20,000	1
Warehouse	Denver	1700	2500	0	4000	0	0	0	1800	0	0	10000	<=	10,000	1
Location	Houston	0	0	0	0	9600	0	0	0	400	0	10000	<=	10,000	1
	Phoenix	1500	0	0	0	0	0	5000	0	0	0	6500	<=	12,000	1
	San Antonio	0	0	0	0	0	0	0	0	0	0	0	<=	0	0
	Column Sum	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700				
		=	=	=	=	=	=	=	=	=	=				
	Demand	3200	2500	6800	4000	9600	3500	5000	1800	7400	2700				
	Demand Slack	0	0	0	0	0	0	0	0	0	0				

Fixed Cost =	$ 485,000.00
Min Cost =	$ 884,550.00

X_{IJ} Variables

Y_I Variables

Modeling Logical Conditions

See Section 11.4 of the Powell and Baker text. As we have seen, binary variables are good for modeling **logical conditions.**

Example 1: Consider the Marr Corporation capital budgeting.

In capital budgeting, projects can be "linked." For example, assume that if project 4 is undertaken ($X_4 = 1$) then project 5 must be undertaken ($X_5 = 1$). This is represented as

$$X_4 \leq X_5$$

Example 2: Consider Marr Corporation again. Assume that, in addition to the condition in Example 1, if project 5 is undertaken ($X_5 = 1$), then project 4 must be undertaken ($X_4 = 1$). This is represented as

$$X_4 = X_5$$

Modeling Logical Conditions

Assume in the following examples $X_i = 1$ if project i is selected, 0 otherwise.

Example 3: If project 3 is accepted, project 4 is rejected.

$$X_4 \leq 1 - X_3$$

Example 4: At most three of the projects 1 through 5 can be accepted.

$$X_1 + X_2 + X_3 + X_4 + X_5 \leq 3$$

Modeling Logical Conditions

Assume in the following examples $X_i = 1$ if project i is selected, 0 otherwise.

Example 5: Exactly one of the first three projects must be accepted.

$$X_1 + X_2 + X_3 = 1$$

Sample Test Question: If project 1 is accepted, then at least one of projects 2, 3, or 4 must be accepted.

Modeling Logical Conditions

Example 6 (Minimum Batch Size): Let X represent the number of units of a product produced.

Model the following condition:

If the production level is strictly greater than 0, then at least 100 units must be produced.

Define $Y = 1$ if there is nonzero production and $Y = 0$ if 0 units are produced.

Assume the value of X will never exceed 500.

Add the constraints:

$$\begin{aligned} X &\leq 500 * Y \\ X &\geq 100 * Y \end{aligned}$$

Sensitivity Analysis

Bottom Line: There is no such thing as a dual price for integer programming. Consider the following example which we first solve as a linear program:

$$\max \quad 40X_1 + 60X_2 + 70X_3 + 160X_4$$

Subject To:

$$16X_1 + 35X_2 + 45X_3 + 85X_4 \leq 100$$

$$X_1, X_2, X_3, X_4 \leq 1 \text{ and nonnegative}$$

The optimal solution is $X_1 = 1$, $X_2 = 0$, $X_3 = 0$, and $X_4 = 84/85$ for a solution value of $198.1176.

The dual price is 160*(1/85).

Sensitivity Analysis

Now assume that the variables must be binary.

Optimal solution $X_1 = 1$, $X_2 = 1$, $X_3 = 1$, and $X_4 = 0$ for a solution value of \$170.

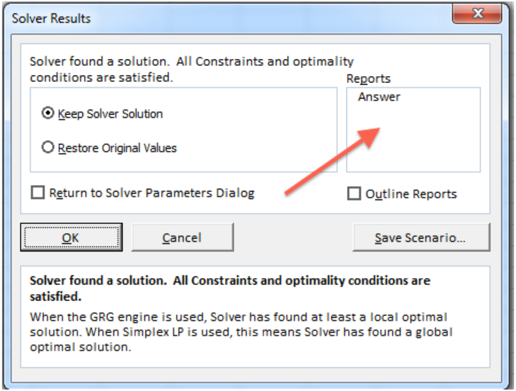
What is an extra penny worth?

What is an extra 10 cents worth?

What is an extra dollar worth?

Sensitivity Analysis

Solver does not provide sensitivity information when solving integer programs.



When to Use Integer/Binary Variables

- Model go, no-go decisions
- Model fixed costs
- Model logical conditions
- When rounding can be expensive