36106 Managerial Decision Modeling
Sensitivity Analysis

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Reading and Excel Files

Reading (Powell and Baker):

- Section 9.5
- Appendix 9.1

Files used in this lecture:

- `allocation_sens.xlsx`
- `allocation_sens_key.xlsx`
- `mpfpVanilla_.xlsx`

Lecture Outline

Motivation

Allowable Increase and Decrease

Cash Flow Matching Revisited

Objective Function Coefficient Sensitivity

Learning Objectives

Two big objectives:

- Learn how to price scarce resources.
- Learn how to read and understand the Solver sensitivity report.

This material is used in the next handout on **revenue management.**

Motivation

Important Reminder: We are working with **linear models!**

$2x_1 + 5x_2$	is linear
$2x_1 + 5x_1x_2$	is nonlinear
$2x_1^2 + x_2$	is nonlinear
x_1/x_2	is nonlinear

Some Excel functions are linear, others nonlinear.

SUM – a linear function

IF, **OR**, **AND**, **MAX**, **MIN** – nonlinear

SUMPRODUCT – could be either

Important: No integer constraints!

Motivation

The objective: how do we price a scarce resource?

Stated another way: what is the "fair" price of a scarce resource.

Disclaimer: extremely reasonable people may differ in terms of what is "fair."

Where we are headed: avoid politics, let Excel figure out what is fair.

Motivation

We are going to develop a very generic pricing mechanism with the following properties:

- if a resource is not actually scarce, i.e. supply exceeds demand, then the price is zero
- if you pay less than the "market" price of the resource, you make money
- if you pay more than the "market" price of the resource, you lose money

Resource Allocation – Pricing

Pricing Scarce Resources: Let's go back to our simple model with only a single constraint on the assembly time. That is, the model

$$
\begin{aligned}
\max 15C + 24D + 18T & \\
4C + 6D + 2T &\leq 1850
\end{aligned}
$$

	A	B	C	D	E	F	G	H
1		**Powell and Baker Resource Allocation Example -- Section 9.2**						
2								
3		Chair	Desk	Table				
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00				
5								
6						Available		
7						Hours		
8	Fabrication(Hours)	4	6	2		1850		
9								

How much would you pay to acquire **another hour** in the fabrication department?

Resource Allocation – Pricing

Key Idea: In the single constraint case the best bang-for-buck ratio is the *value of the resource*.

In this case the best bang for buck ratio is 9.0.

If one more hour is acquired in fabrication then

```
New Profit = Old Profit + 9.0
```

Why is this true?

Resource Allocation – Pricing

Dual Price: the value of an additional unit of resource. Think of it as a marginal price. Other terms are:

- Shadow Price
- Dual Value
- Lagrange Multiplier

Application: in portfolio optimization the dual price is the slope of the efficient frontier.

Regardless of the number of constraints, **Solver provides the dual price on all resources.**

Resource Allocation – Pricing

Here is the current optimal solution:

	A	B	C	D	E	F	G	H	I
1		Powell and Baker Resource Allocation Example -- Section 9.2							
2									
3		Chair	Desk	Table					
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00					
5									
6						Hours		Available	Slack
7						Used		Hours	Hours
8	Fabrication(Hours)	4	6	2		1850	<=	1850	0
9	Assembly(Hours)	3	5	7		2400	<=	2400	0
10	Shipping(Hours)	3	2	4		1153.1	<=	1500	346.88
11									
12									
13	Number Produced	0	254.6875	160.9375					
14									
15	Profit =	$ 9,009.38							

Resource Allocation – Pricing

Now let's make some changes to the available hours and make **four** Solver runs (one for each column).

	Available Hours			
Fabrication	1850	1851	1850	1850
Assembly	2400	2400	2401	2400
Shipping	1500	1500	1500	1501
Profit	$9,009.38	$9,011.81	$9,011.25	$9,009.38

Fabrication Dual Price = 9,011.81 - 9,009.38 = 2.43

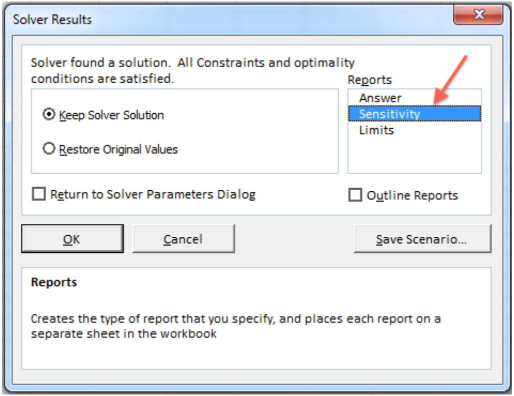
Assembly Dual Price = 9,011.25 - 9,009.38 = 1.87

Shipping Dual Price = 9,009.38 - 9,009.38 = 0

Solver makes these calculations **without** resolving the model.

Resource Allocation – Pricing

Pricing Scarce Resources:



Resource Allocation – Pricing

Important Note: When solving the model, I wrote the constraints as `F8:F10 <= H8:10`. I did not write the constraints in terms of nonnegative slack.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation.xlsx]model-done
Report Created: 1/5/2014 12:20:18 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-0.375	15	0.375	1E+30
C13	Number Produced Desk	254.6875	0	24	30	0.545454545
D13	Number Produced Table	160.9375	0	18	6	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F8	Fabrication(Hours) Used	1850	2.4375	1850	1030	1164.285714
F9	Assembly(Hours) Used	2400	1.875	2400	555	858.3333333
F10	Shipping(Hours) Used	1153.125	0	1500	1E+30	346.875

Resource Allocation – Pricing

Let's look at the information under **Constraints** in more detail.

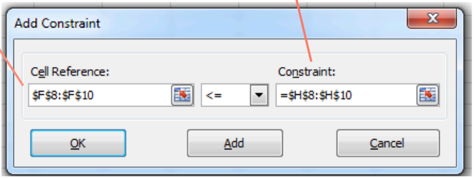
Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation_key.xlsx]model-done
Report Created: 9/28/2015 10:57:15 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-0.375	15	0.375	1E+30
C13	Number Produced Desk	254.6875	0	24	30	0.545454545
D13	Number Produced Table	160.9375	0	18	6	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F8	Fabrication(Hours) Used	1850	2.4375	1850	1030	1164.285714
F9	Assembly(Hours) Used	2400	1.875	2400	555	858.3333333
F10	Shipping(Hours) Used	1153.125	0	1500	1E+30	346.875



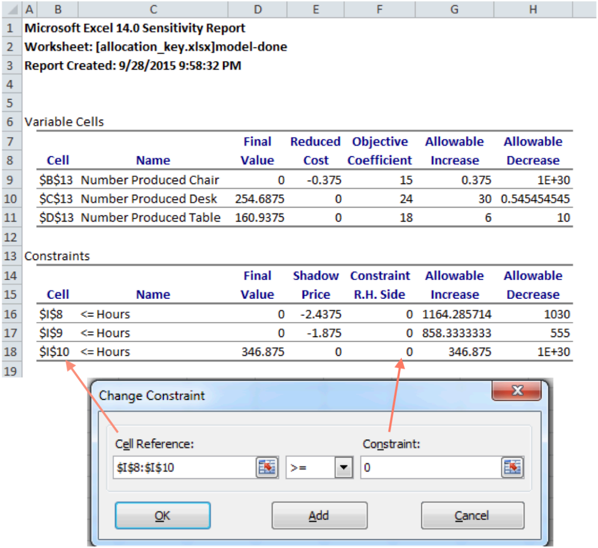
Resource Allocation – Pricing

Consider each column in the **Constraints** section:

1. **Cell:** this is the **Cell Reference** in the **Add Constraint** window.
2. **Name:** a string concatenation of the first text cell immediately above **Cell** and the first text cell to the left.
3. **Final Value:** the value of the formula in the **Cell** for the optimal solution.
4. **Shadow Price:** the marginal value of the right hand side.
5. **Constraint R.H. Side:** the **Constraint** in the **Add Constraint** window.
6. **Allowable Increase:** defined later.
7. **Allowable Decrease:** defined later.

Resource Allocation – Pricing

Let's look at the information under **Constraints** when we write the constraints as the slack nonnegative.



Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation_key.xlsx]model-done
Report Created: 9/28/2015 9:58:32 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-0.375	15	0.375	1E+30
C13	Number Produced Desk	254.6875	0	24	30	0.545454545
D13	Number Produced Table	160.9375	0	18	6	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
I8	<= Hours	0	-2.4375	0	1164.285714	1030
I9	<= Hours	0	-1.875	0	858.3333333	555
I10	<= Hours	346.875	0	0	346.875	1E+30

Resource Allocation – Pricing

Why has the shadow price for the fabrication department gone from 2.4375 to -2.4375?

Consider writing the constraint as

$$4C + 6D + 2T \leq 1850$$

versus

$$1850 - 4C - 6D - 2T \geq 0$$

The shadow price is giving the change in the optimal objective function value if I increase the right hand side by one unit.

See the handout on sensitivity analysis clarification.

`http://faculty.chicagobooth.edu/kipp.martin/root/htmls/coursework/36106/handouts/sensitivity_clarification.pdf`

Resource Allocation – Pricing

Now make a change by more than one unit. Let's start with the fabrication department. Each column corresponds to one of **four** runs.

	Available Hours			
Fabrication	1850	2350	2850	3000
Assembly	2400	2400	2400	2400
Shipping	1500	1500	1500	1500
Profit	\$9,009.38	\$10,228.13	\$11,446.88	\$11,700.00

$\$10,228.13 = \$9,009.38 + 2.4375*500$

$\$11,446.88 = \$9,009.38 + 2.4375*1000$

$\$11,700.00 < \$9,009.38 + 2.4375*1150 = \$11,812.5$

Argle Bargle! What happened?

Allowable Increase and Decrease

If life were fair:

- the dual price would be valid for any change in the right-hand-side
- we could change more than one right-hand-side simultaneously.

Fair has nothing to do with reality!

Life is not fair.

Fair is where animals are displayed in the summer.

Allowable Increase and Decrease

The dual price is valid only for its **allowable increase** and **allowable decrease** and for changing only **one** constraint at a time.

The allowable increase on a constraint right-hand-side is the maximum amount the right-hand-side can increase **without** the dual price changing.

The allowable decrease on a constraint right-hand-side is the maximum amount the right-hand-side can decrease **without** the dual price changing.

Allowable Increase and Decrease

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation.xlsx]model-done
Report Created: 1/5/2014 12:20:18 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-0.375	15	0.375	1E+30
C13	Number Produced Desk	254.6875	0	24	30	0.545454545
D13	Number Produced Table	160.9375	0	18	6	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F8	Fabrication(Hours) Used	1850	2.4375	1850	1030	1164.285714
F9	Assembly(Hours) Used	2400	1.875	2400	555	858.3333333
F10	Shipping(Hours) Used	1153.125	0	1500	1E+30	346.875

The allowable increase for the dual price of 2.4375 is 1030 and the allowable decrease is 1164.29.

Allowable Increase and Decrease

Two important rules in life:

Rule 1: Helping helps less and less!

Rule 2: Hurting hurts more and more!

These two rules will always tell you what happens as your scarce resource levels increase or decrease.

Allowable Increase and Decrease

When using the dual price in calculations we assume that **only one constraint** right-hand-side is changed.

If you change two or more constraint right-hand-sides you must use the **100% rule**.

The dual prices are valid (i.e. tell the actual change) as long as the percentage increases/decreases *used up* does not exceed 100%.

What do we mean by that?

Allowable Increase and Decrease

Case 1:

RHS	Allowable Increase	Change	Percentage Change
Fabrication	1030	515	50%
Assembly	555	111.0	20%
		Total	70%

Predicted new profit:

$$9,009.38 + 2.4375 * 515 + 1.875 * 111 = 10,472.82$$

Actual profit: 10,472.82

Allowable Increase and Decrease

Question: do increases and decreases "cancel" each other out?

Case 2: Assume we **increase** the right-hand-side for the Fabrication by 1000 from 1850 to 2850, and **decrease** the right-hand-side of the Assembly by 500 from 2400 to 1900. Then the *predicted* new optimal objective function value is:

$$9,009.38 + 2.4375 * 1000 - 1.875 * 500 = 10,509.38$$

If we actually run Solver with the new right-hand-sides we get, drum roll please: $9,366.67 < 10,509.38$. We are wrong!

RHS	Allowable Increase	Allowable Decrease	Change	Percentage Change
Fabrication	1030	1164.29	+1000	97%
Assembly	555	858.33	-500	-58%
			Total	155%

Allowable Increase and Decrease

Good midterm questions: be able to make these calculations from the sensitivity output report.

Important Modeling Ideas:

1. If a constraint has positive slack, then the value of the dual price is zero.
2. If we make the feasible region smaller we cannot improve the objective.
3. If we make the feasible region larger we cannot hurt the objective.

Allowable Increase and Decrease

Summary: what happens to the optimal objective function value under the following scenarios?

Constraint Type	maximum objective function		minimum objective function	
$\leq$	Increase	Decrease	Increase	Decrease
$\geq$	Increase	Decrease	Increase	Decrease

Sample Question: If we have a maximization problem, and a $\leq$ constraint, what is the effect on the objective function value of an increase in the right hand side?

Cash Flow Matching Revisited

Our Objective: understand how dual prices are useful in cash flow matching problems. Here is a sensitivity report.

27	Constraints						
28			Final	Shadow	Constraint	Allowable	Allowable
29	Cell	Name	Value	Price	R.H. Side	Increase	Decrease
30	D7	Surplus	$0.00	0.961538462	0	1E+30	4.996676602
31	D8	Surplus	$0.00	0.924556213	0	3.18822E+16	5.196543666
32	D9	Surplus	$0.00	0.888996359	0	1E+30	5.404405413
33	D10	Surplus	$0.00	0.854804191	0	3.93924E+16	3.392195867
34	D11	Surplus	$0.00	0.719062534	0	23.38344141	101.5435189
35	D12	Surplus	$0.00	0.691406283	0	24.31877906	94.01971971
36	D13	Surplus	$0.00	0.664813734	0	25.29153022	84.11496859
37	D14	Surplus	$0.00	0.639243975	0	26.30319143	72.77402743
38	D15	Surplus	$0.00	0.614657668	0	27.35531909	58.89944862
39	D16	Surplus	$0.00	0.591016989	0	28.44953185	42.38988666
40	D17	Surplus	$0.00	0.568285566	0	29.58751313	23.13994222
41	D18	Surplus	$0.00	0.410615185	0	33.5356203	96.0147929
42	D19	Surplus	$0.00	0.394822293	0	34.87704511	67.61538462
43	D20	Surplus	$0.00	0.37963682	0	36.27212691	36
44							

Cash Flow Matching Revisited

The dual price for the period 5 sources and uses constraint (`D11`) is 0.719062534. What is the interpretation of this number?

Why are the dual prices getting smaller over time?

What is the ratio of the period t dual price, to the period $t + 1$ dual price?

Will the ratio of the period t dual price, to the period $t + 1$ dual price be the same as t increases?

What is the smallest possible value for the ratio of the period t dual price, to the period $t + 1$ dual price?

With Solver, there is no need to pick an interest rate and do an NPV calculation. **Dual prices provide all of the NPV information!**

Objective Function Coefficient Sensitivity

Now, for the rest of the story! In addition to information about the right-hand-sides, Solver provides useful economic information about **objective function coefficients.**

Without resolving the linear program we can answer questions such as:

1. If the profit margin on tables went up by one dollar what would the new optimal production schedule and profit be?
2. If we could increase the profit margin of one of the products by one dollar which product should we choose?
3. How much would the profit margin of chairs have to increase in order for it to be optimal to produce chairs?

The Solver sensitivity report provides the answers!

Objective Function Coefficient Sensitivity

Objective function sensitivity analysis report.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation.xlsx]model-done
Report Created: 1/5/2014 12:20:18 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-0.375	15	0.375	1E+30
C13	Number Produced Desk	254.6875	0	24	30	0.545454545
D13	Number Produced Table	160.9375	0	18	6	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F8	Fabrication(Hours) Used	1850	2.4375	1850	1030	1164.285714
F9	Assembly(Hours) Used	2400	1.875	2400	555	858.3333333
F10	Shipping(Hours) Used	1153.125	0	1500	1E+30	346.875

Resource Allocation – Pricing

Consider each column in the **Variables** section:

1. **Cell:** this is the cell reference for each adjustable cell.
2. **Name:** a string concatenation of the first text cell immediately above **Cell** and the first text cell to the left.
3. **Final Value:** the optimal solution value of the adjustable cell.
4. **Reduced Cost:** how much we have to reduce the cost of the corresponding objective function coefficient in order for there to be an optimal solution with the corresponding adjustable cell positive.
5. **Objective Coefficient:** the coefficient of this adjustable cell in the objective function formula after simplification.
6. **Allowable Increase:** defined later.
7. **Allowable Decrease:** defined later.

Objective Function Coefficient Sensitivity

The allowable increase and allowable decrease on an objective function coefficient is the amount the objective function coefficient can increase or decrease without changing the optimal solution.

What are the implications?

1. If the profit margin of tables goes up by one dollar, the profit will go up by approximately $160.9375. Why?

2. We would be better off increasing the profit margin on desks by one dollar rather than tables by one dollar. Why?

3. If we could increase the profit margin on chairs by at least 37.5 cents (.375 dollars) then it would be optimal to produce chairs. Rerun the model by increasing the profit margin on chairs to $15.38. What happens?

Objective Function Coefficient Sensitivity

Reduced Cost: two ways to define a reduced cost.

- the dual price on the nonnegativity constraint
- for a maximization, the allowable increase on an objective function coefficient if the variable is currently in the solution at 0

For a maximization problem, if a variable is currently at zero in the optimal solution, the reduced cost is the amount by which the objective function coefficient must **increase** in order for there to be an optimal solution where the variable is positive.

For a minimization problem, if a variable is currently at zero in the optimal solution, the reduced cost is the amount by which the objective function coefficient must be **reduced** in order for there to be an optimal solution where the variable is positive.

Objective Function Coefficient Sensitivity

Reduced Cost: The reduced cost is also the dual price on constraints that place a limit on the variables (for example cannot sell more than a certain amount or must produce at least a certain amount).

Run the Veerman furniture example with

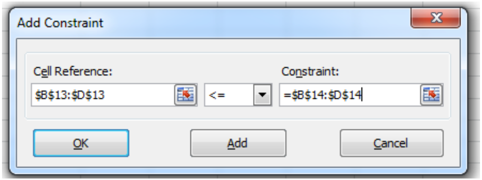
- the chair demand limit is 360
- the desk demand limit is 300
- the table demand limit is 100

and rerun Solver and generate a sensitivity report. What is the dual price on the constraint that the table demand limit is 100?

Objective Function Coefficient Sensitivity

Now for one last **argle bargle**! Add the following demand limit constraints.

$$C \leq 360$$
$$D \leq 300$$
$$T \leq 100$$



Objective Function Coefficient Sensitivity

Look at the sensitivity report.

What happened to our constraints on the demand limits?

How can we figure out the dual price on the $T \leq 100$ demand limit constraint?

Objective Function Coefficient Sensitivity

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation_key2.xlsx]model-done
Report Created: 1/13/2014 11:55:33 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-1	15	1	1E+30
C13	Number Produced Desk	275	0	24	30	1.5
D13	Number Produced Table	100	10	18	1E+30	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F8	Fabrication(Hours) Used	1850	4	1850	150	1650
F9	Assembly(Hours) Used	2075	0	2400	1E+30	325
F10	Shipping(Hours) Used	950	0	1500	1E+30	550

If we can sell 101 tables, what will the new profit be?

Allowable Increase and Decrease

Now rerun the model by introducing explicit constraints that say the demand limit slack cannot be zero.

	A	B	C	D	E	F	G	H	I
1		Powell and Baker Resource Allocation Example -- Section 9.2							
2									
3		Chair	Desk	Table					
4	Profit Margin	$ 15.00	$ 24.00	$ 18.00					
5									
6						Hours		Available	Slack
7						Used		Hours	Hours
8	Fabrication(Hours)	4	6	2		1850	<=	1850	0
9	Assembly(Hours)	3	5	7		2075	<=	2400	325
10	Shipping(Hours)	3	2	4		950	<=	1500	550
11									
12	Demand Limit Slack	360	25	0					
13	Number Produced	0	275	100					
14	Demand Limits	360	300	100					
15	Profit =	$ 8,400.00							
16									

Allowable Increase and Decrease

Here is the new sensitivity report with the constraints `B12:D12 >= 0`.

Microsoft Excel 14.0 Sensitivity Report
Worksheet: [allocation_sens.xlsx]model-done
Report Created: 1/10/2015 3:11:04 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B13	Number Produced Chair	0	-1	15	1	1E+30
C13	Number Produced Desk	275	0	24	30	1.5
D13	Number Produced Table	100	0	18	1E+30	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B12	Production Slack Chair	360	0	0	360	1E+30
C12	Production Slack Desk	25	0	0	25	1E+30
D12	Production Slack Table	0	-10	0	75	60.9375
F8	Fabrication(Hours) Used	1850	4	1850	150	1650
F9	Assembly(Hours) Used	2075	0	2400	1E+30	325
F10	Shipping(Hours) Used	950	0	1500	1E+30	550

Objective Function Coefficient Sensitivity

When the demand constraints **were not explicitly written** in terms of nonnegative slack, the reduced cost gave the dual price.

When the demand constraints **were explicitly written** in terms of nonnegative slack, the slack constraints gave the dual price.

Objective Function Coefficient Sensitivity

Important: the allowable increase and decrease is on the objective function coefficient, NOT on the right hand side for the relevant dual price.

Important: everything we said about the 100 percent rule for dual prices applies to the objective function!

Based on what we have seen, how might you spot alternative optima?