FLOATING POINT

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Overview

- This lecture
  - Hardware for division
  - MIPS instructions for division
  - Signed division
  - Floating point numbers

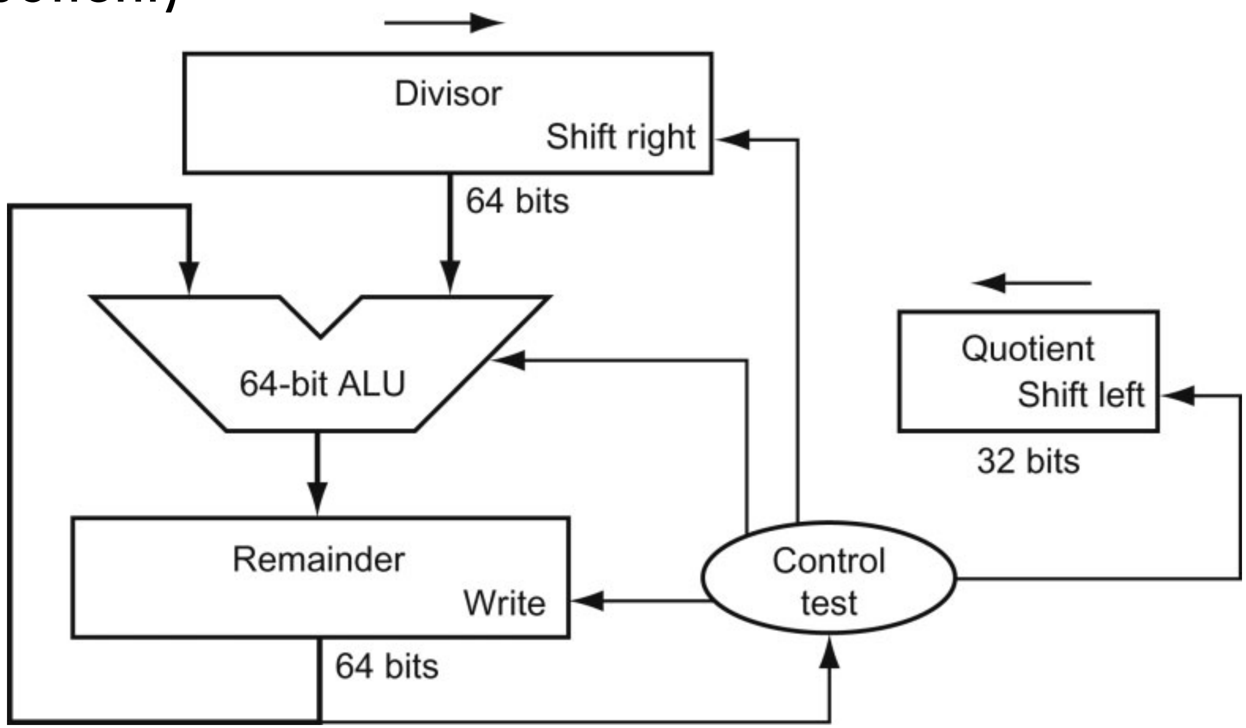
Recall: Division Example

- Divide 7_{ten} ($0000\ 0111_{two}$) by 2_{ten} (0010_{two})

Iter	Step	Quot	Divisor	Remainder
0	Initial values	0000	0010 0000	0000 0111
1	Rem = Rem – Div	0000	0010 0000	1110 0111
	Rem < 0 → +Div, shift 0 into Q	0000	0010 0000	0000 0111
	Shift Div right	0000	0001 0000	0000 0111
2	Same steps as 1	0000	0001 0000	1111 0111
		0000	0001 0000	0000 0111
		0000	0000 1000	0000 0111
3	Same steps as 1	0000	0000 0100	0000 0111
4	Rem = Rem – Div	0000	0000 0100	0000 0011
	Rem >= 0 → shift 1 into Q	0001	0000 0100	0000 0011
	Shift Div right	0001	0000 0010	0000 0011
5	Same steps as 4	0011	0000 0001	0000 0001

Hardware for Division (pp. 189-193)

- A comparison requires a subtract; the sign of the result is examined; if the result is negative, the divisor must be added back
- Similar to multiply, results are placed in Hi (remainder) and Lo (quotient)



MIPS Instructions for Division

- MIPS ignores overflow in division
- Signed division (div)

```
div     $s2, $s3
```

computes the division and stores it in two "internal" registers that can be referred to as hi and lo

```
mfhi     $s0
mflo     $s1
```

moves the remainder into $s0

moves the quotient into $s1

- Unsigned division (divu)

```
divu    $s2, $s3

mfhi     $s0
mflo     $s1
```

Signed Division

- Simplest solution: convert to positive and adjust sign later
- Note that multiple solutions exist for the equation
  - Dividend = Quotient x Divisor + Remainder

+7	div	+2	Quo =	Rem =
-7	div	+2	Quo =	Rem =
+7	div	-2	Quo =	Rem =
-7	div	-2	Quo =	Rem =

Signed Division

- Simplest solution: convert to positive and adjust sign later
- Note that multiple solutions exist for the equation
  - Dividend = Quotient x Divisor + Remainder

+7	div	+2	Quo = +3	Rem = +1
-7	div	+2	Quo = -3	Rem = -1
+7	div	-2	Quo = -3	Rem = +1
-7	div	-2	Quo = +3	Rem = -1

Signed Division

- Simplest solution: convert to positive and adjust sign later
- Note that multiple solutions exist for the equation
  - Dividend = Quotient x Divisor + Remainder

+7 div +2	Quo = +3	Rem = +1
-7 div +2	Quo = -3	Rem = -1
+7 div -2	Quo = -3	Rem = +1
-7 div -2	Quo = +3	Rem = -1

- Convention
  - Dividend and remainder have the same sign
  - Quotient is negative if signs disagree
  - These rules fulfil the equation above

Floating Point Numbers

- Example: how to represent 32.5
  - weights smaller than 1

$\mathtt{100000.100}_{two}$

Floating Point Numbers

- Example: how to represent 32.5
  - weights smaller than 1

$$100000.100_{two}$$

- Normalized scientific notation: leave a single non-zero digit to the left of the point
  - 3.25×10^1

$$1.00000100_{two} * 2^5 = (1*2^0 + 1*2^{-6}) * 2^5$$

- The IEEE 754 standard

Sign-Magnitude Representation

- Since we are only representing normalized numbers, we are guaranteed that the number is of the form 1.xxxx.

- Every 32-bit number has three fields: sign (S), exponent (E), and fraction (F)
  - value = $(-1)^{S} \times (1+F) \times 2^{E-127}$

1 bit	8 bits	23 bits
S	E	F

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0	10000100	00000100000000000000000

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- Using more bits
  - Increase E bits to represent a wider range of numbers
  - Increase F bits to represent more precision
- Double precision format with 64 bits

1 bit	11 bits	52 bits
S	E	F

Single Precision Floating Point

1 bit	8 bits	23 bits
S	E	F

- Largest number that can be represented

- Smallest number that can be represented

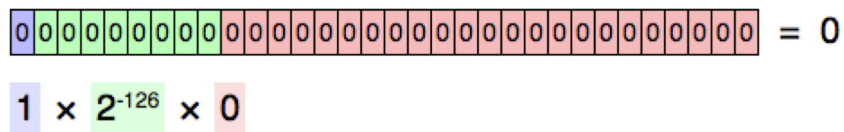
Single Precision Floating Point

1 bit	8 bits	23 bits
S	E	F

- Largest number that can be represented
  - $2.0 \times 2^{128} = 2.0 \times 10^{38}$
- Smallest number that can be represented
  - $1.0 \times 2^{-127} = 2.0 \times 10^{-38}$
- Overflow
  - representing a number larger than the one above
- Underflow
  - representing a number smaller than the one above

Special Notes

- The number "0" has a special code so that the implicit 1 does not get added
  - see discussion of denorms (pg. 222) in the textbook



- The largest exponent value (with zero fraction) represents +/- infinity

- The largest exponent value (with non-zero fraction) represents NaN (not a number) – for the result of 0/0 or (infinity minus infinity)

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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	= 0

$1 \times 2^{-126} \times 0$

- The largest exponent value (with zero fraction) represents +/- infinity

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	= -Infinity

$-1 \times 2^{128} \times 1$

- The largest exponent value (with non-zero fraction) represents NaN (not a number) – for the result of 0/0 or (infinity minus infinity)

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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

= 0

$1 \times 2^{-126} \times 0$

- The largest exponent value (with zero fraction) represents +/- infinity

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

= -Infinity

$-1 \times 2^{128} \times 1$

- The largest exponent value (with non-zero fraction) represents NaN (not a number) – for the result of 0/0 or (infinity minus infinity)

1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

= NaN

$-1 \times 2^{128} \times 1.0000001$

Special Notes

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00000000000000000000000000000000 = 0

1 × 2^{-126} × 0

- The largest exponent value (with zero fraction) represents +/- infinity

11111111100000000000000000000000 = -Infinity

-1 × 2^{128} × 1

- The largest exponent value (with non-zero fraction) represents NaN (not a number) – for the result of 0/0 or (infinity minus infinity)

11111111100000000000000000000001 = NaN

-1 × 2^{128} × 1.0000001

These choices impact the smallest and largest numbers that can be represented

Example 1

- Represent -0.75_{ten} in single- and double-precision formats

$$\text{value} = (-1)^{S} \times (1 + F) \times 2^{(E - 127)}$$

- Single precision

1 bit	8 bits	23 bits
S	E	F

Example 1

- Represent -0.75_{ten} in single- and double-precision formats

$$-0.75 = -1 \times 0.11_{two} = -1 \times 1.1_{two} \times 2^{-1} = (-1)^{1} \times (1 + F) \times 2^{(E - 127)}$$

- Single precision

1 bit	8 bits	23 bits
1	01111110	10000000000000000000000
	126	0.5

Example 1

- Represent -0.75_{ten} in single- and double-precision formats

$$-0.75 = -1 \times 0.11_{two} = -1 \times 1.1_{two} \times 2^{-1} = (-1)^{1} \times (1 + F) \times 2^{(E - 127)}$$

- Single precision

1 bit	8 bits	23 bits
1	01111110	10000000000000000000000000
	126	0.5

- Double precision

$$-1 \times 1.1_{two} \times 2^{-1} = (-1)^{1} \times (1 + F) \times 2^{(E - 1023)}$$

1 bit	11 bits	52 bits
1	01111111110	10 000000000000000000000000000000000...000
	1022	0.5

Example 2

- Represent 3.40625_{ten} in single- and double-precision formats

$$\text{value} = (-1)^{S} \times (1 + F) \times 2^{(E - 127)}$$

- Single precision

1 bit	8 bits	23 bits
S	E	F

Example 2

- Represent 3.40625_{ten} in single- and double-precision formats

  $3.40625 = 11.01101_{two} = 1.101101_{two} \times 2^1 = (-1)^0 \times (1 + F) \times 2^{(E - 127)}$

- Single precision

1 bit	8 bits	23 bits
0	10000000	10110100000000000000000
	128	0.703125

Example 2

- Represent 3.40625_{ten} in single- and double-precision formats

$$3.40625 = 11.01101_{two} = 1.101101_{two} \times 2^1 = (-1)^0 \times (1 + F) \times 2^{(E - 127)}$$

- Single precision

1 bit	8 bits	23 bits
0	10000000	10110100000000000000000
	128	0.703125

- Double precision

$$1.101101_{two} \times 2^1 = (-1)^0 \times (1 + F) \times 2^{(E - 1023)}$$

1 bit	11 bits	52 bits
0	10000000000	1011010000000000000000000000000…000
	1024	0.703125