Learning latent variable models using tensor decompositions

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Computer Science Department & Data Science Institute
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Machine Learning Summer School
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El tema (subject matter)

Learning algorithms

for **latent variable models**

based on **decompositions of moment tensors**.

El tema (subject matter)

Learning algorithms (parameter estimation)

for **latent variable models**

based on **decompositions of moment tensors**.

"Method-of-moments" **(Pearson, 1894)**

Example #1: summarizing a corpus of documents

Observation: **documents express one or more thematic topics.**

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By Rory Smith, James Montague and Tariq Panja

June 24, 2018

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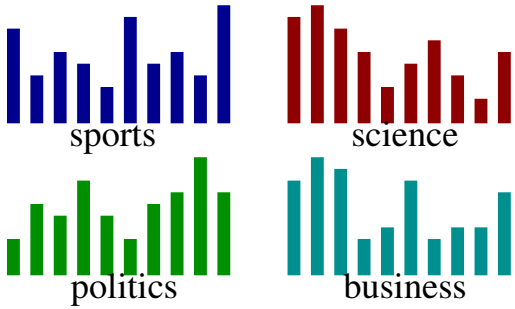
- What topics are expressed in a corpus of documents?
- How prevalent is each topic in the corpus?

Topic model (e.g., latent Dirichlet allocation)



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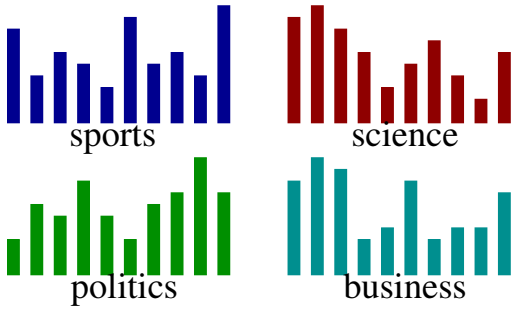

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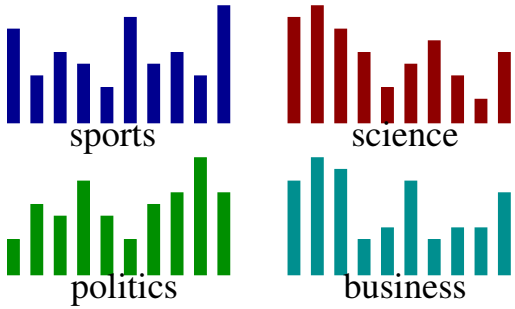

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Given corpus of documents (and "hyper-parameters", e.g., K), produce estimates of **model parameters**, e.g.:

- Distribution $\boldsymbol{P}_t$ over vocab words, for each $t \in [K]$.
- Weight w_t of topic t in document corpus, for each $t \in [K]$.

Labels / annotations

- Suppose each word token x in document is *annotated* with source topic $t_x \in \{1, 2, \ldots, K\}$.

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- Unfortunately, we often don't have such annotations (i.e., data are *unlabeled* / topics are *hidden*).

  "Direct" approach to estimation unavailable.

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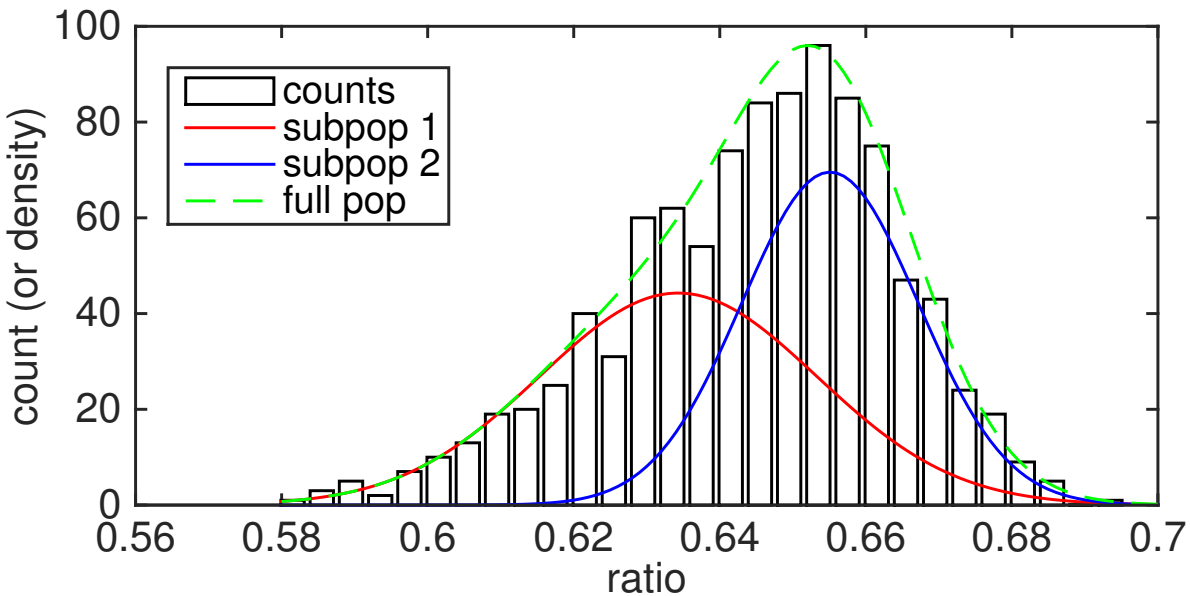
Sample may be comprised of different sub-species of crabs.



Gaussian mixture model

$$
\begin{aligned}
H &\sim \text{Categorical}(\pi_1, \pi_2, \ldots, \pi_K)\,; \\
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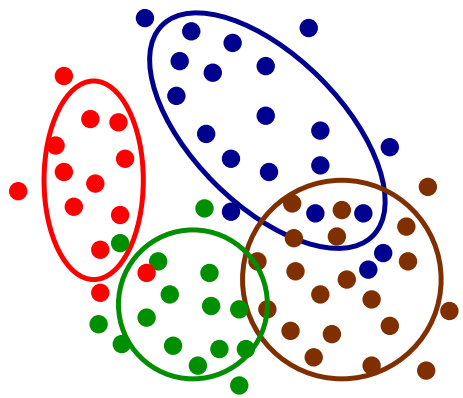

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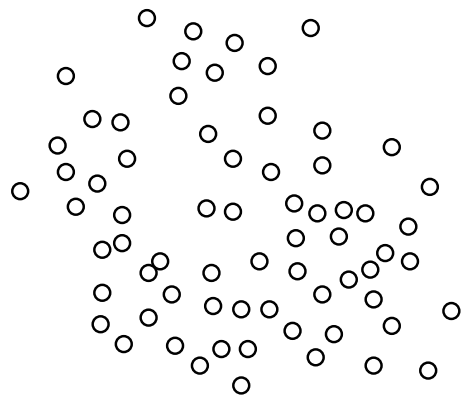

Estimate **mean vector**, **covariance matrix**, and **mixing weight** of each subpopulation from *unlabeled data*.

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  - **Note**: log-likelihood is not necessarily concave function of $\boldsymbol{\theta}$.
  - For latent variable models, often use local optimization, most notably via Expectation-Maximization (EM) (Dempster, Laird, & Rubin, 1977).

MLE for Gaussian mixture models

Given data $\{\boldsymbol{x}_i\}_{i=1}^n$, find $\{(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t, \pi_t)\}_{t=1}^K$ to maximize

$$\sum_{i=1}^{n} \log \left(\sum_{t=1}^{K} \pi_t \cdot \frac{1}{\det(\boldsymbol{\Sigma}_t)^{1/2}} \exp \left\{ -\frac{1}{2} (\boldsymbol{x}_i - \boldsymbol{\mu}_t)^\top \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}_t) \right\} \right) .$$

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- Sensible with restrictions on $\boldsymbol{\Sigma}_t$ (e.g., $\boldsymbol{\Sigma}_t \succeq \sigma^2 \boldsymbol{I}$).
- But NP-hard to maximize (Tosh and Dasgupta, 2018):
  Can't expect efficient algorithms to work for all data sets.

Parameter learning objective

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$$\Pr\left(\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\| \leq \epsilon\right) \geq 1 - \delta$$

with $\mathrm{poly}(p, 1/\epsilon, 1/\delta, \dots)$ sample size and running time.

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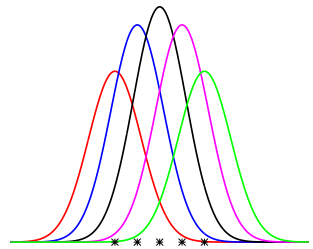

Cryptographic hardness
(e.g., Mossel & Roch, 2006)

Information-theoretic hardness
(e.g., Moitra & Valiant, 2010)

May require $2^{\Omega(K)}$ running time or $2^{\Omega(K)}$ sample size.

Ways around the barriers

- **Separation conditions**.

  E.g., assume mixture component distributions are far apart.
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This lecture: learning algorithms for non-degenerate instances via *method-of-moments*.

Method-of-moments at a glance

1. Determine function of model parameters $\boldsymbol{\theta}$ estimatable from observable data:

$$\mathbb{E}_{\boldsymbol{\theta}}[f(\boldsymbol{X})] \qquad (\text{“moments”})\,.$$

2. Form estimates of moments using data (e.g., iid sample):

$$\widehat{\mathbb{E}}[f(\boldsymbol{X})] \qquad (\text{“empirical moments”})\,.$$

3. Approximately solve equations for parameters $\boldsymbol{\theta}$:

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Method-of-moments at a glance

1. Determine function of model parameters $\boldsymbol{\theta}$ estimatable from observable data:

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Which moments? Often low-order moments suffice.

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How? Algorithms for tensor decomposition.

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- Form *estimates* of $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ from iid sample $\{x_i\}_{i=1}^n$: e.g.,

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We'll follow this same basic recipe for much richer models!

Outline

1. Topic model for single-topic documents.
    - Identifiability.
    - Parameter recovery via orthogonal tensor decomposition.
2. Moment decompositions for other models.
    - Mixtures of Gaussians and linear regressions.
    - Multi-view models (e.g., HMMs).
    - Other models (e.g., single-index models).
3. Error analysis (if time).

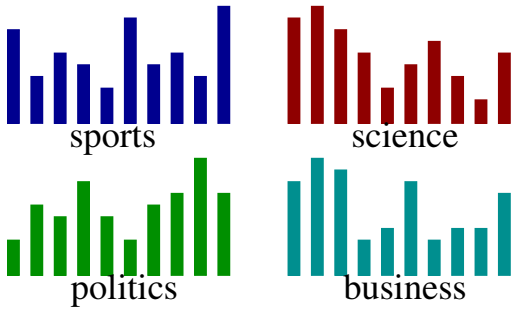
Other models amenable to moment tensor decomposition

- Models for independent components analysis (Comon, 1994; Frieze, Jerrum, & Kannan, 1996; Arora, Ge, Moitra & Sachdeva, 2012; Anandkumar, Foster, H., Kakade, & Liu, 2012, 2015; Belkin, Rademacher, & Voss, 2013; etc.)
- Latent Dirichlet Allocation (Anandkumar, Foster, H., Kakade, & Liu, 2012, 2015; Anderson, Goyal, & Rademacher, 2013)
- Mixed-membership stochastic blockmodels (Anandkumar, Ge, H., & Kakade, 2013, 2014)
- Simple probabilistic grammars (H., Kakade, & Liang, 2012)
- Noisy-or networks (Halpern & Sontag, 2013; Jernite, Halpern & Sontag, 2013; Arora, Ge, Ma, & Risteski, 2016)
- Indian buffet process (Tung & Smola, 2014)
- Mixed multinomial logit model (Oh & Shah, 2014)
- Dawid-Skene model (Zhang, Chen, Zhou, & Jordan, 2014)
- Multi-task bandits (Azar, Lazaric, & Brunskill, 2013)
- Partially obs. MDPs (Azizzadenesheli, Lazaric, & Anandkumar, 2016)
- ...

1. Topic model for single-topic documents

Topic model

General topic model (e.g., Latent Dirichlet Allocation)



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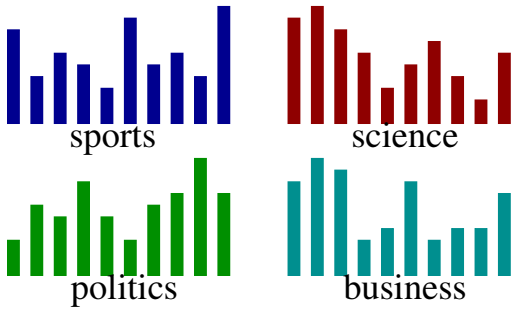

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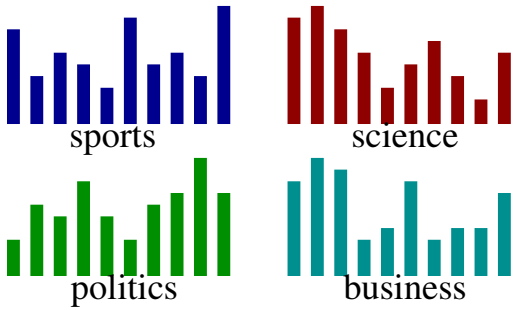

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Given iid sample of documents of length L,
produce estimates of **model parameters** $\{(\boldsymbol{P}_t, w_t)\}_{t=1}^K$.

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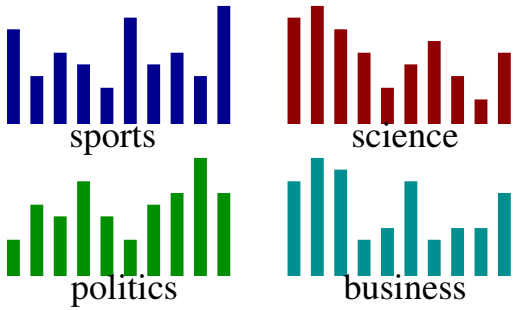

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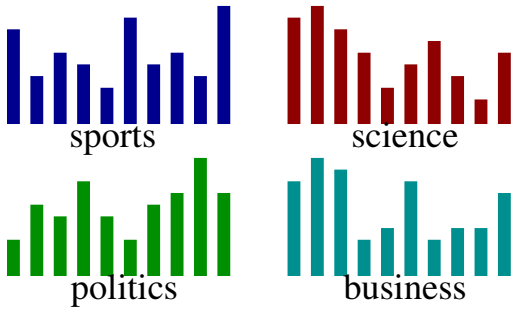
Given iid sample of documents of length L,
produce estimates of **model parameters** $\{(\boldsymbol{P}_t, w_t)\}_{t=1}^K$.

How long must the documents be?

Topic model

Topic model for single-topic documents



K topics (dists. over words) $\{\boldsymbol{P}_t\}_{t=1}^K$.
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(Answering this question leads to efficient algorithms for estimating parameters!)

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Generative process:
Pick $t \sim \text{Categorical}(w_1, w_2, \ldots, w_K)$.
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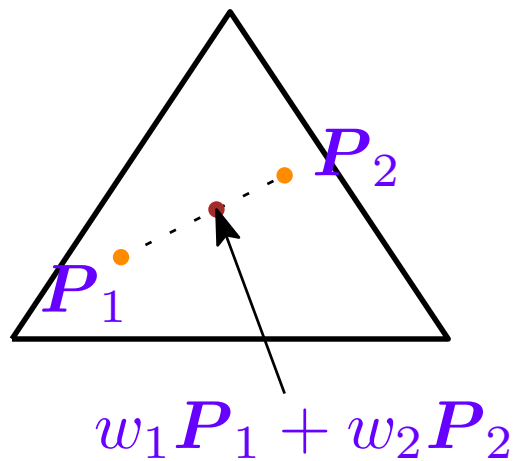

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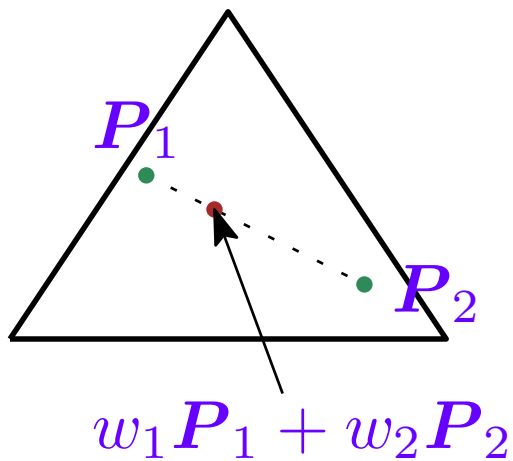

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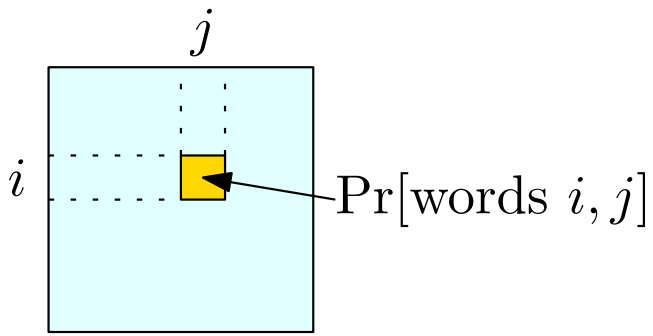

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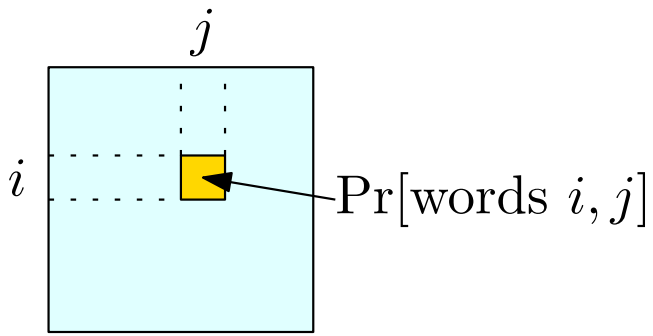

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Simple observation

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Then it can also be written as

$$\boldsymbol{M} \;=\; (\boldsymbol{A}\boldsymbol{R})(\boldsymbol{A}\boldsymbol{R})^\top$$

for any orthogonal matrix $\boldsymbol{R}$ (because $\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}$).

Identifiability: $L = 2$ counterexample

Parameters $\{(\boldsymbol{P}_1, w_1), (\boldsymbol{P}_2, w_2)\}$ and $\{(\widetilde{\boldsymbol{P}}_1, \tilde{w}_1), (\widetilde{\boldsymbol{P}}_2, \tilde{w}_2)\}$

$$(\boldsymbol{P}_1, w_1) = \left(\begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}, 0.5 \right), \quad (\boldsymbol{P}_2, w_2) = \left(\begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, 0.5 \right);$$

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Cannot identify parameters from length-two documents.

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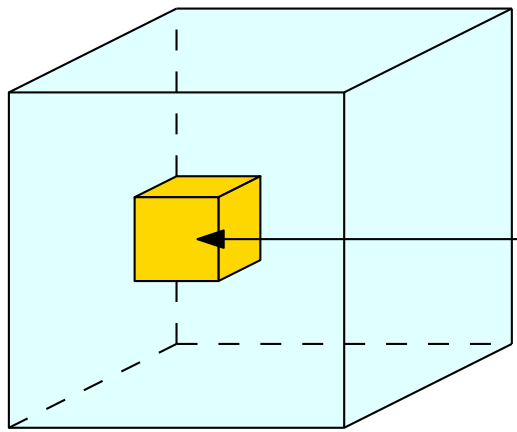

$$\Pr[\text{words } i, j, k]$$

Random document $\sim \sum_{t=1}^{K} w_t \boldsymbol{P}_t \otimes \boldsymbol{P}_t \otimes \boldsymbol{P}_t$.

Identifiability from documents of length three

Claim: If $\{\boldsymbol{P}_t\}_{t=1}^K$ are linearly independent & all $w_t > 0$, then parameters $\{(\boldsymbol{P}_t, w_t)\}_{t=1}^K$ are identifiable from word triples.

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Next: Brief overview of tensors.

Tensors of order two

Matrices (tensors of order two): $\boldsymbol{M} \in \mathbb{R}^{d\times d}$.

- Regard as *bi-linear function* $\boldsymbol{M} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$:

$$\begin{aligned} \boldsymbol{M}(a\boldsymbol{x} + \boldsymbol{x}', \boldsymbol{y}) &= a\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{M}(\boldsymbol{x}', \boldsymbol{y})\,; \\ \boldsymbol{M}(\boldsymbol{x}, a\boldsymbol{y} + \boldsymbol{y}') &= a\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{y}')\,. \end{aligned}$$

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Tensors are *multi-linear* generalization.

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Usual caveat

(Hillar & Lim, 2013)

Most Tensor Problems Are NP-Hard

CHRISTOPHER J. HILLAR, Mathematical Sciences Research Institute
LEK-HENG LIM, University of Chicago

We prove that multilinear (tensor) analogues of many efficiently computable problems in numerical linear algebra are NP-hard. Our list includes: determining the feasibility of a system of bilinear equations, deciding whether a 3-tensor possesses a given eigenvalue, singular value, or spectral norm; approximating an eigenvalue, eigenvector, singular vector, or the spectral norm; and determining the rank or best rank-1 approximation of a 3-tensor. Furthermore, we show that restricting these problems to symmetric tensors does not alleviate their NP-hardness. We also explain how deciding nonnegative definiteness of a symmetric 4-tensor is NP-hard and how computing the combinatorial hyperdeterminant is NP-, #P-, and VNP-hard.

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  For $\epsilon = 1/k$, have $\lim_{k\to\infty} \boldsymbol{T}_k = \boldsymbol{T}$.

Aside: eigenvalue decomposition

Recall: every symmetric matrix $\boldsymbol{M} \in \mathbb{R}^{d \times d}$ of rank K has an *eigen-decomposition* (which can be efficiently computed):

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For (symmetric) tensors of order $p \geq 3$:
an analogous decomposition is **not** guaranteed to exist.

Reduction to orthonormal case

Suppose we have (estimates of) moments of the form

$$\boldsymbol{M} = \sum_{t=1}^{K} \boldsymbol{v}_t \otimes \boldsymbol{v}_t \,, \qquad \text{(e.g., word pairs)}$$

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- $\boldsymbol{M}$ determines inner product system on $\operatorname{span}\{\boldsymbol{v}_t\}_{t=1}^K$ s.t. $\{\boldsymbol{v}_t\}_{t=1}^K$ are **orthonormal**:

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- $\therefore$ Can assume $d = K$ and $\{\boldsymbol{v}_t\}_{t=1}^d$ are orthonormal.
  (Similar to PCA; called "whitening" in signal processing context.)

Orthogonally decomposable tensors $(d = K)$

Goal: Given tensor $\boldsymbol{T} = \sum_{t=1}^{d} \lambda_t \cdot \boldsymbol{v}_t \otimes \boldsymbol{v}_t \otimes \boldsymbol{v}_t \in \mathbb{R}^{d \times d \times d}$ where:

- $\{\boldsymbol{v}_t\}_{t=1}^{d}$ are orthonormal;
- all $\lambda_t > 0$;

approximately recover $\{(\boldsymbol{v}_t, \lambda_t)\}_{t=1}^{d}$.

Exact orthogonally decomposable tensor

(Zhang & Golub, 2001)

Matching moments:

$$\{(\hat{\boldsymbol{v}}_t, \hat{\lambda}_t)\}_{t=1}^d := \underset{\{(\boldsymbol{x}_t, \sigma_t)\}_{t=1}^d}{\arg\min} \left\| \boldsymbol{T} - \sum_{t=1}^d \sigma_t \cdot \boldsymbol{x}_t \otimes \boldsymbol{x}_t \otimes \boldsymbol{x}_t \right\|_F^2 .$$

(Here, $\|\cdot\|_F$ is "Frobenius norm", just like for matrices.)

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- Greedy approach:
  - Find best rank-1 approximation:

$$(\hat{\boldsymbol{v}}, \hat{\lambda}) := \underset{\|\boldsymbol{x}\|=1,\sigma\geq 0}{\arg\min} \|\boldsymbol{T} - \sigma \cdot \boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}\|_F^2 .$$

  - "Deflate" $\boldsymbol{T} := \boldsymbol{T} - \hat{\lambda} \cdot \hat{\boldsymbol{v}} \otimes \hat{\boldsymbol{v}} \otimes \hat{\boldsymbol{v}}$ and repeat.

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Rank-1 approximation problem

Claim: Local maximizers of the function

$$\boldsymbol{x} \;\mapsto\; \boldsymbol{T}(\boldsymbol{x},\boldsymbol{x},\boldsymbol{x}) \;=\; \sum_{i,j,k} T_{i,j,k}\cdot x_i x_j x_k$$

(over the unit ball) are $\{\boldsymbol{v}_t\}_{t=1}^{d}$, and

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Corollary: decomposition of $\boldsymbol{T}$ as $\sum_{t=1}^K \lambda_t \cdot \boldsymbol{v}_t^{\otimes 3}$ is *unique*!

Proof

By linearity and orthogonality:

$$\boldsymbol{T}(\boldsymbol{v}_t, \boldsymbol{v}_t, \boldsymbol{v}_t) = \sum_{s=1}^{d} (\lambda_s \cdot \boldsymbol{v}_s^{\otimes 3})(\boldsymbol{v}_t, \boldsymbol{v}_t, \boldsymbol{v}_t)$$

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Uniqueness of orthogonal decompositions

What we have seen so far:

1. When components $\{\boldsymbol{v}_t\}_{t=1}^d$ are linearly independent:
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Algorithm: use gradient ascent to find all of the local maximizers, which are exactly $\boldsymbol{v}_t$.

(Can use "deflation" to remove components from $\boldsymbol{T}$ that you've already found.)

Application to topic model parameters

Probabilities of word triples as third-order tensor:

$$\boldsymbol{T} = \sum_{t=1}^{K} w_t \boldsymbol{P}_t \otimes \boldsymbol{P}_t \otimes \boldsymbol{P}_t = \sum_{t=1}^{K} \boldsymbol{v}_t \otimes \boldsymbol{v}_t \otimes \boldsymbol{v}_t$$

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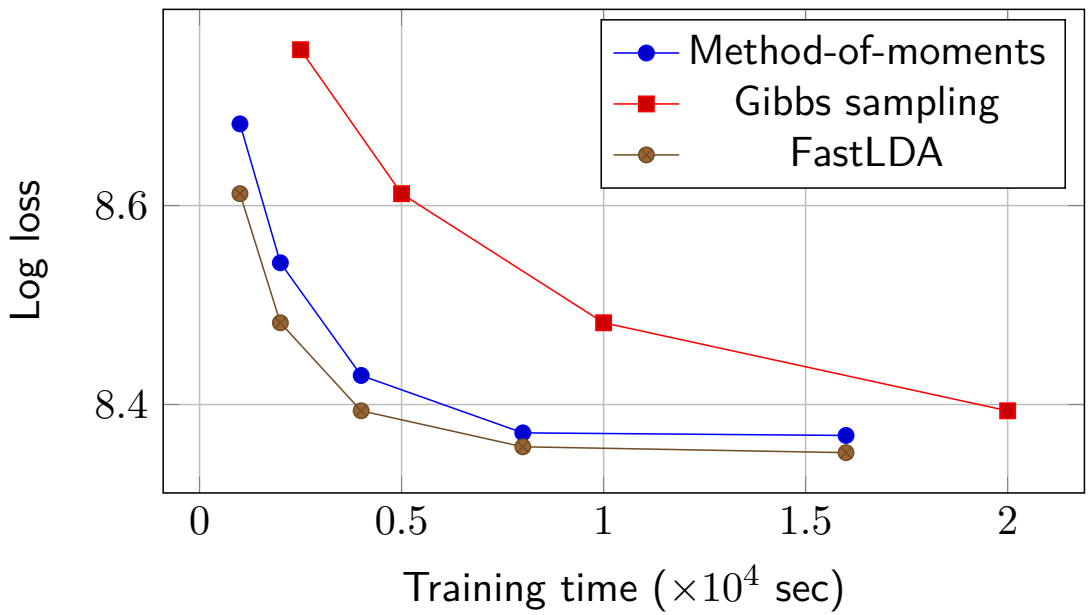
- Two-word documents not sufficient (without further assumptions).
- Variational characterization of orthogonally decomposable tensors leads to simple and efficient algorithms!

Illustrative empirical results

- Corpus: $300,000$ New York Times articles.
- Vocabulary size: $102,660$ words.
- Set number of topics $K := 50$.

Model predictive performance:
≈ 4–$8\times$ speed-up over Gibbs sampling for LDA;
comparable to "FastLDA" (Porteous, Newman, Ihler, Asuncion, Smyth, & Welling, 2008).



Illustrative empirical results

Sample topics: (showing top 10 words for each topic)

Econ.	**Baseball**	**Edu.**	**Health care**	**Golf**
sales	run	school	drug	player
economic	inning	student	patient	tiger_wood
consumer	hit	teacher	million	won
major	game	program	company	shot
home	season	official	doctor	play
indicator	home	public	companies	round
weekly	right	children	percent	win
order	games	high	cost	tournament
claim	dodger	education	program	tour
scheduled	left	district	health	right

Illustrative empirical results

Sample topics: (showing top 10 words for each topic)

Invest.	**Election**	**auto race**	**Child's Lit.**	**Afghan War**
percent	al_gore	car	book	taliban
stock	campaign	race	children	attack
market	president	driver	ages	afghanistan
fund	george_bush	team	author	official
investor	bush	won	read	military
companies	clinton	win	newspaper	u_s
analyst	vice	racing	web	united_states
money	presidential	track	writer	terrorist
investment	million	season	written	war
economy	democratic	lap	sales	bin

Illustrative empirical results

Sample topics: (showing top 10 words for each topic)

Web	**Antitrust**	**TV**	**Movies**	**Music**
com	court	show	film	music
www	case	network	movie	song
site	law	season	director	group
web	lawyer	nbc	play	part
sites	federal	cb	character	new_york
information	government	program	actor	company
online	decision	television	show	million
mail	trial	series	movies	band
internet	microsoft	night	million	show
telegram	right	new_york	part	album

etc.

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Learning algorithms

- Estimation via **method-of-moments**:
  1. *Estimate* distribution of three-word documents $\rightarrow$ $\widehat{\boldsymbol{T}}$ (*empirical moment tensor*).
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     Can more reliably estimate lower-order moments; distribution-specific sample complexity bounds.

  2. Robustness of *(approximate) tensor decomposition*?

     In some sense, more stable than matrix eigen-decomposition (Mu, H., & Goldfarb, 2015)!

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  2. Robustness of *(approximate) tensor decomposition*?

     In some sense, more stable than matrix eigen-decomposition (Mu, H., & Goldfarb, 2015)!

  3. *Generality* beyond simple topic models?

Next: Moment decompositions for other models.

2. Moment decompositions for other models

Moment decompositions

Some examples of usable moment decompositions.

1. Two classical mixture models.
2. Models with multi-view structure.
3. Single-index models.

Mixture model #1: Mixtures of spherical Gaussians

$$
\begin{aligned}
H &\sim \text{Categorical}(\pi_1, \pi_2, \ldots, \pi_K) \quad \text{(hidden)}\,; \\
\boldsymbol{X} \mid H = t &\sim \text{Normal}(\boldsymbol{\mu}_t, \sigma_t^2 \boldsymbol{I}_d)\,, \quad t \in [K]\,.
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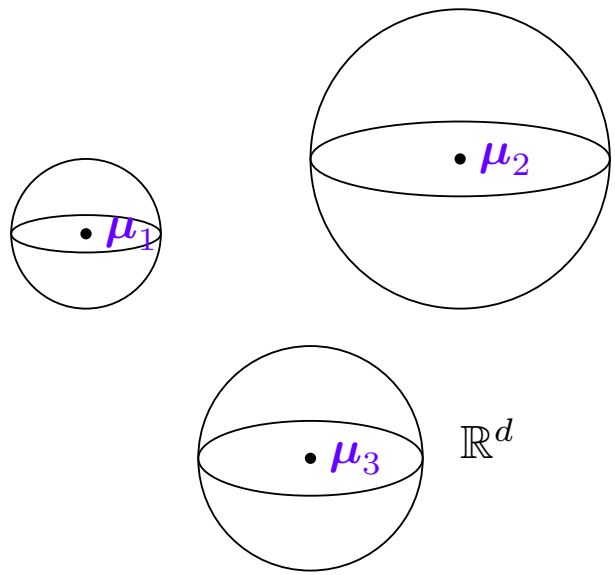

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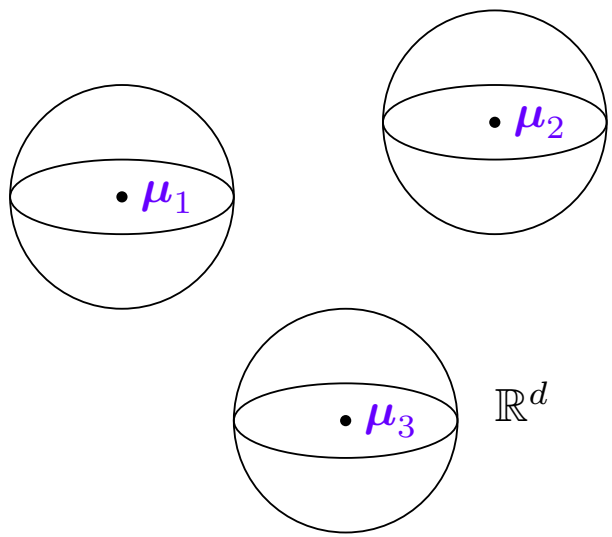

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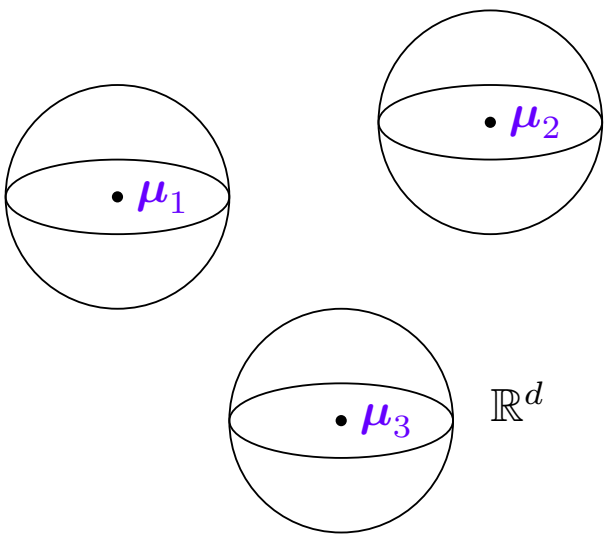

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Using moments for spherical Gaussian mixtures

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First- and second-order moments:

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Claim (Vempala & Wang, 2002):
Span of top K eigenvectors of $\mathbb{E}(\boldsymbol{X} \otimes \boldsymbol{X})$ contains $\{\boldsymbol{\mu}_t\}_{t=1}^K$.
(K-dimensional Principal Component Analysis (PCA) subspace.)

Proof

Key fact: k-dimensional PCA subspace (based on $\mathbb{E}(\boldsymbol{X} \otimes \boldsymbol{X})$) captures as much of overall variance as any other k-dim. subspace.

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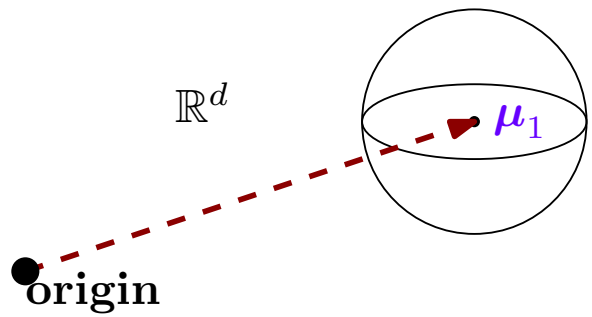

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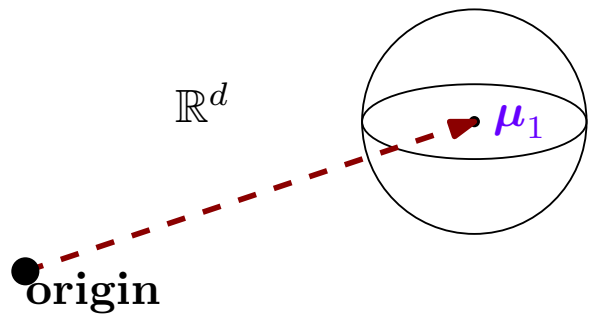

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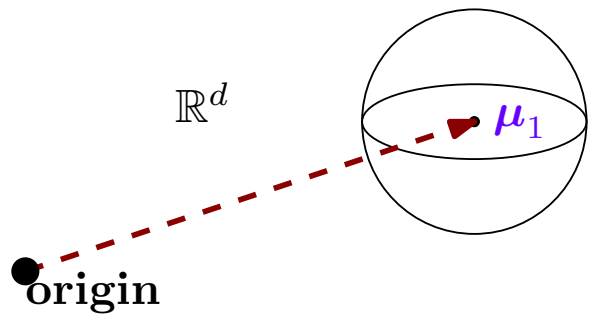

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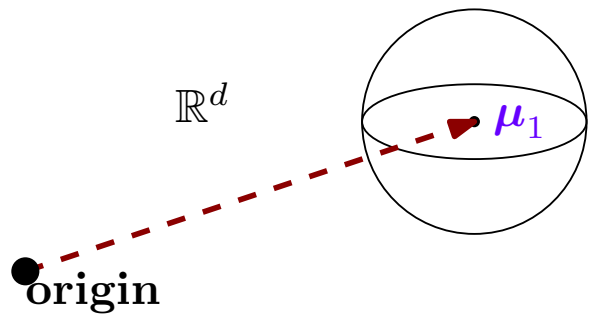

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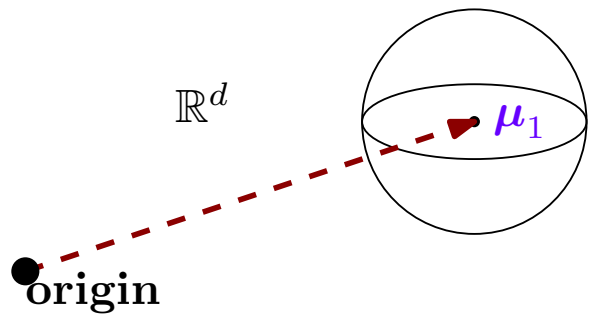

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Best direction (1-dim. PCA subspace): $\boldsymbol{v} = \pm \boldsymbol{\mu}_1 / \|\boldsymbol{\mu}_1\|$.

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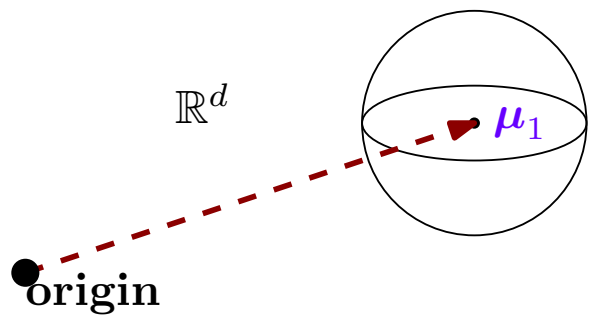

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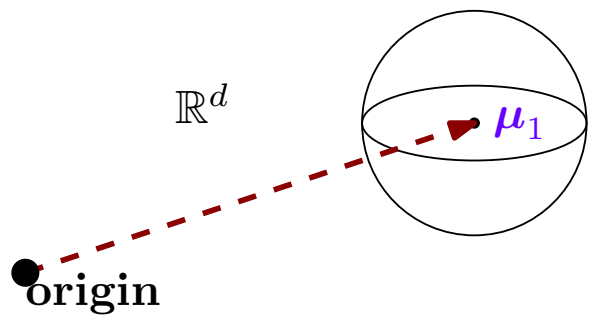

Answer: any k-dim. subspace containing $\boldsymbol{\mu}_1$.

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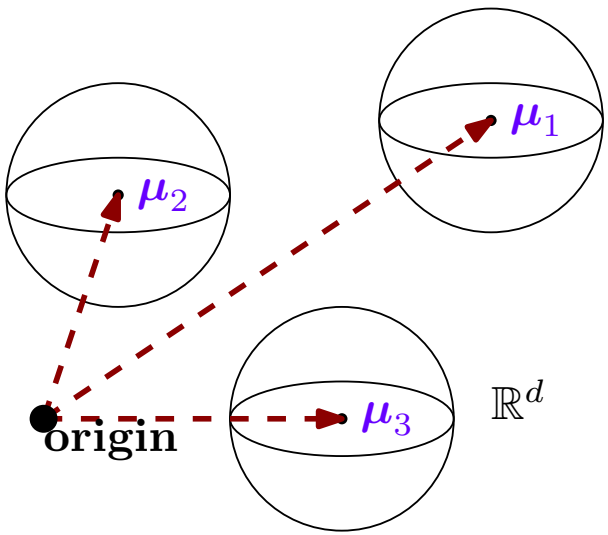

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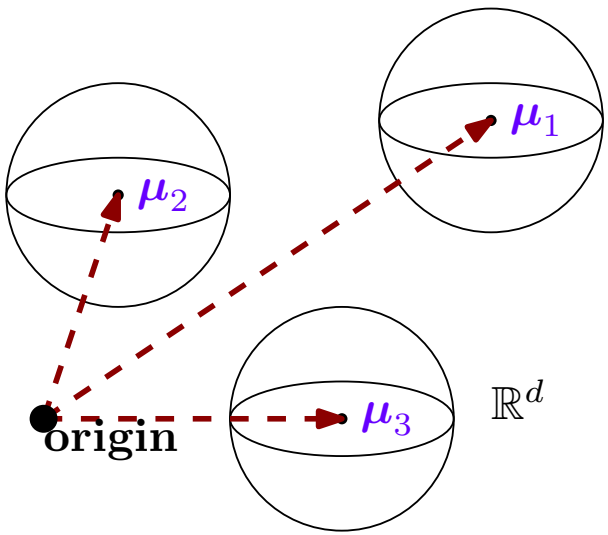

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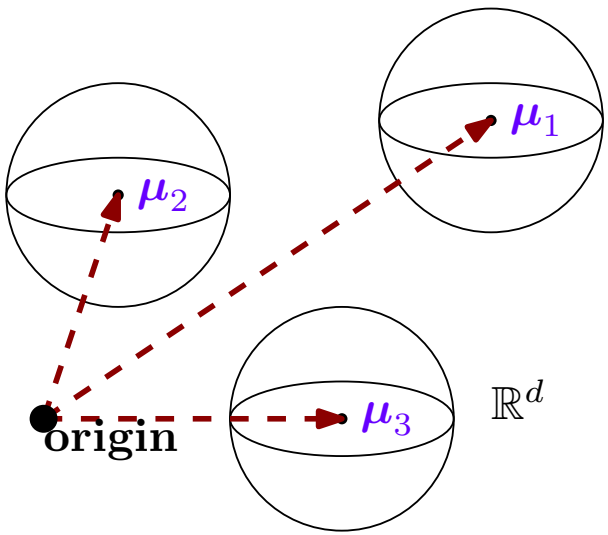

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How does this help with learning mixtures of Gaussians?

Use of moments for mixtures of spherical Gaussians

Separation (Dasgupta, 1999):

standard deviations between component means

$$\text{sep} := \min_{i \neq j} \frac{\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\|}{\sigma} .$$

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(Belkin & Sinha, 2010; Moitra & Valiant, 2010):
General Gaussians & no minimum sep, but Kth-order moments.

Third-order moments of spherical Gaussian mixtures

Generative process:

$$\boldsymbol{X} \;=\; \boldsymbol{Y} + \sigma \boldsymbol{Z}$$

where $\Pr(\boldsymbol{Y} = \boldsymbol{\mu}_t) = \pi_t$, and $\boldsymbol{Z} \sim \mathrm{Normal}(\mathbf{0}, \boldsymbol{I}_d)$, $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z}$.

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(Above, $\boldsymbol{\mu} = \mathbb{E}(\boldsymbol{X})$ and $\tau(\boldsymbol{\mu})$ is a third-order tensor involving only $\boldsymbol{\mu}$.)

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Exercise: find explicit formula for $\tau(\boldsymbol{\mu})$.

Tensor decomposition for spherical Gaussian mixtures

(H. & Kakade, 2013)

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Can use tensor decomposition to recover $\{(\boldsymbol{\mu}_t, \pi_t)\}_{t=1}^K$ from $\boldsymbol{T}$.

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  Mixtures of $d^{O(1)}$ Gaussians (w/ simple or known covariance) via **smoothed analysis** and $O(1)$-order moments.

- (Ge, Huang, & Kakade, 2015)
  Also with **unknown covariances of arbitrary shape**.

Mixture model #2: Mixtures of linear regressions

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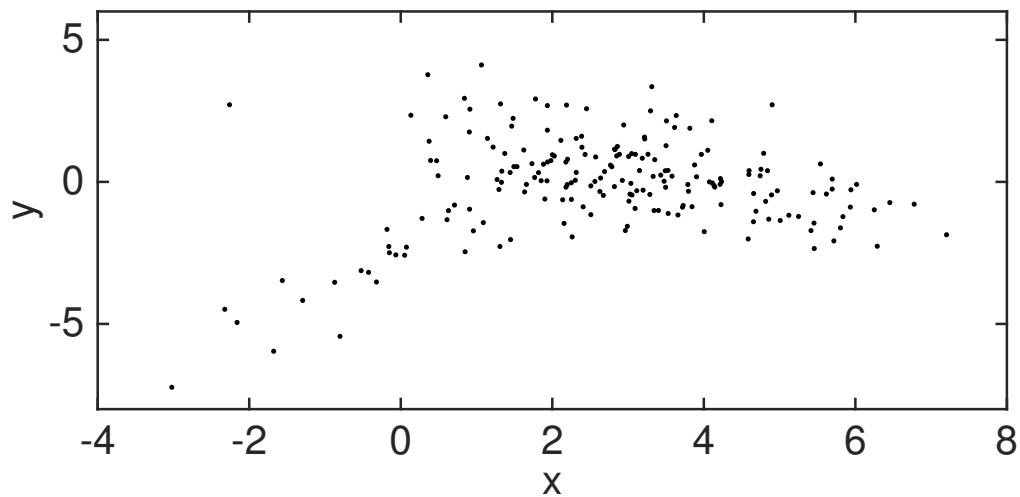

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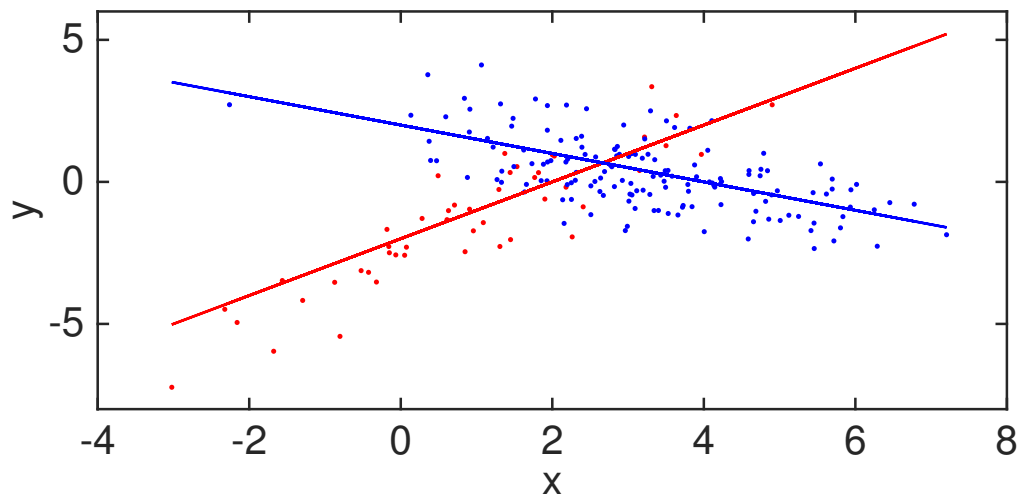

Use of moments for mixtures of linear regressions

Second-order moments (assume $\boldsymbol{X} \sim \mathrm{Normal}(\mathbf{0}, \boldsymbol{I}_d)$):

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Can recover parameters $\{(\boldsymbol{\beta}_t, \pi_t)\}_{t=1}^K$ with higher-order moments (Chaganty & Liang, 2013; Yi, Caramanis, & Sanghavi, 2014, 2016).

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Also for GLMs, via Stein's identity (Sedghi & Anandkumar, 2014).

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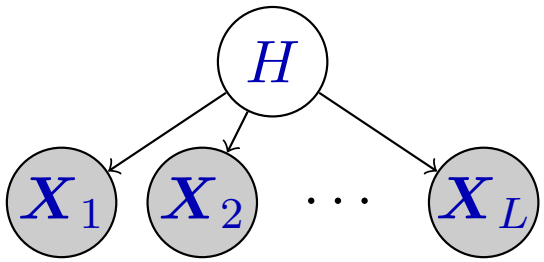
Next: Multi-view approach to finding usable moments.

Multi-view interpretation of topic model

Recall: Topic model for single-topic documents



K topics (dists. over words) $\{\boldsymbol{P}_t\}_{t=1}^K$.

Pick topic $H = t$ with prob. w_t (**hidden**).

Word tokens $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_L \overset{\text{iid}}{\sim} \boldsymbol{P}_H$.

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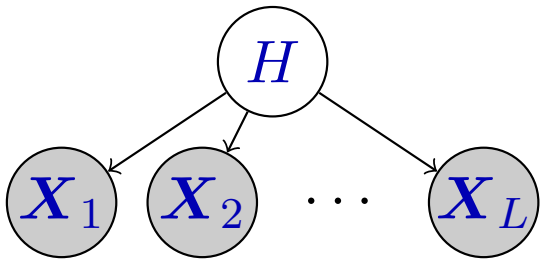

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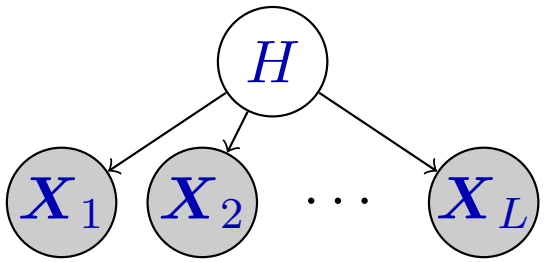

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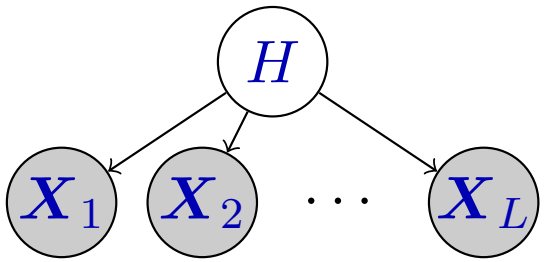

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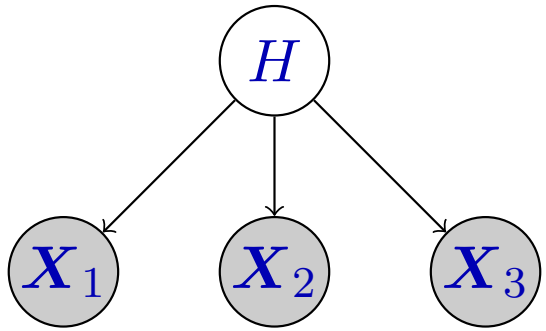
Some previous analyses:

- (Blum & Mitchell, 1998)
  Co-training in semi-supervised learning.
- (Chaudhuri, Kakade, Livescu, & Sridharan, 2009)
  Multi-view Gaussian mixture models.

Multi-view mixture model



View 1: $\boldsymbol{X}_1$ View 2: $\boldsymbol{X}_2$ View 3: $\boldsymbol{X}_3$

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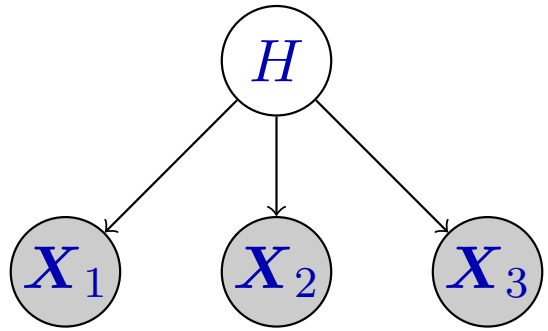

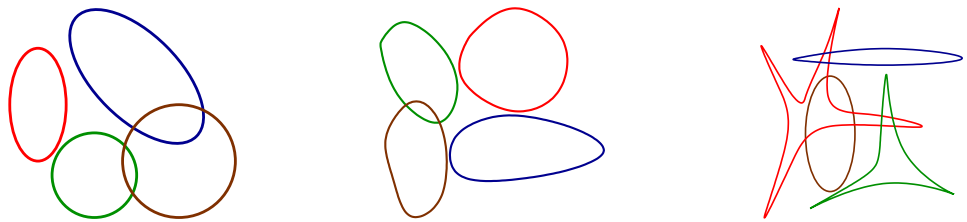

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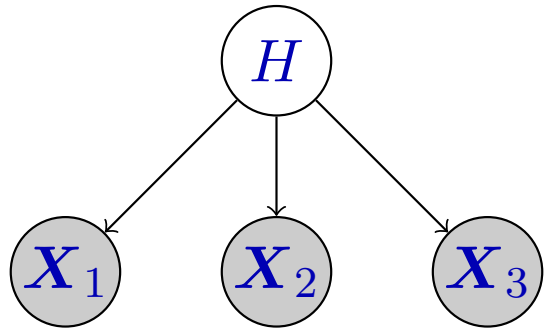

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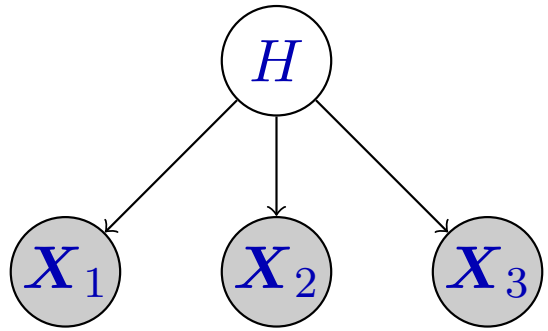

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Tensor decomposition approach works in this asymmetric case as long as $\{\boldsymbol{\mu}_t^{(j)}\}_{t=1}^{K}$ are lin. indpt. for each j, and all $\pi_t > 0$.

Examples of multi-view mixture models

(Mossel & Roch, 2006; Anandkumar, H., & Kakade, 2012)

1. Mixtures of high-dimensional product distributions.
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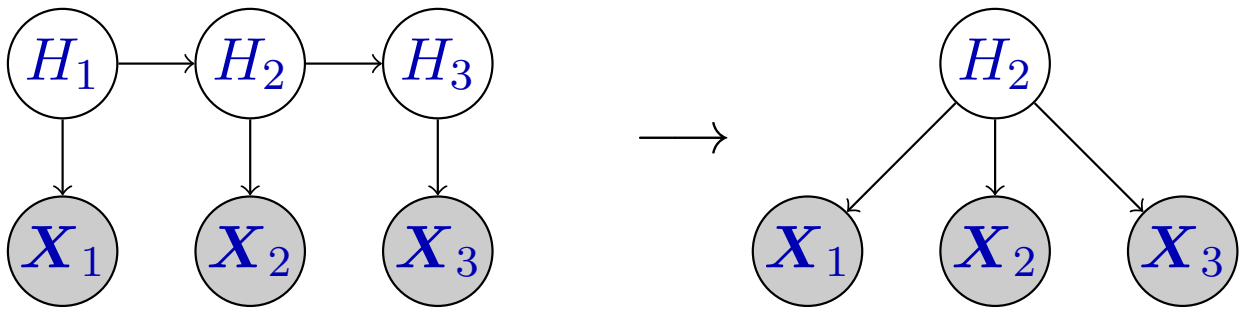

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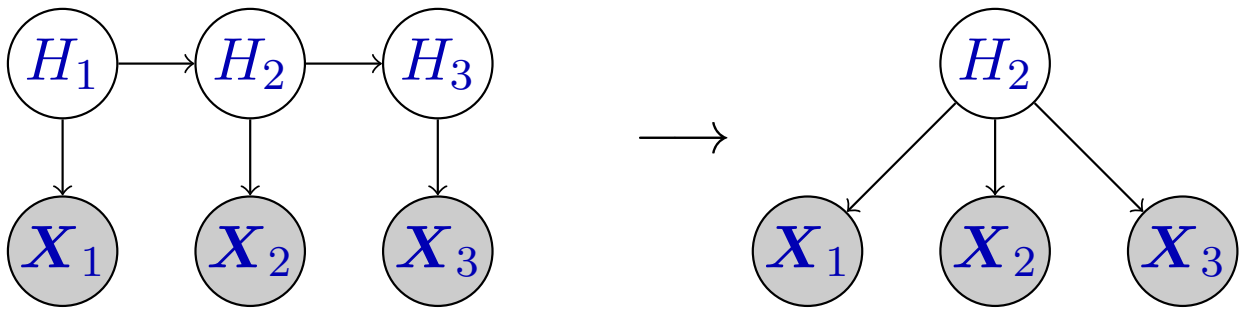

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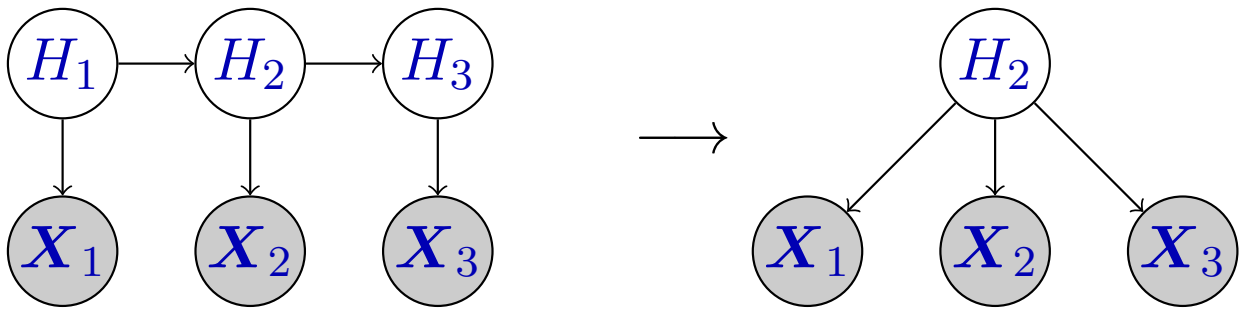

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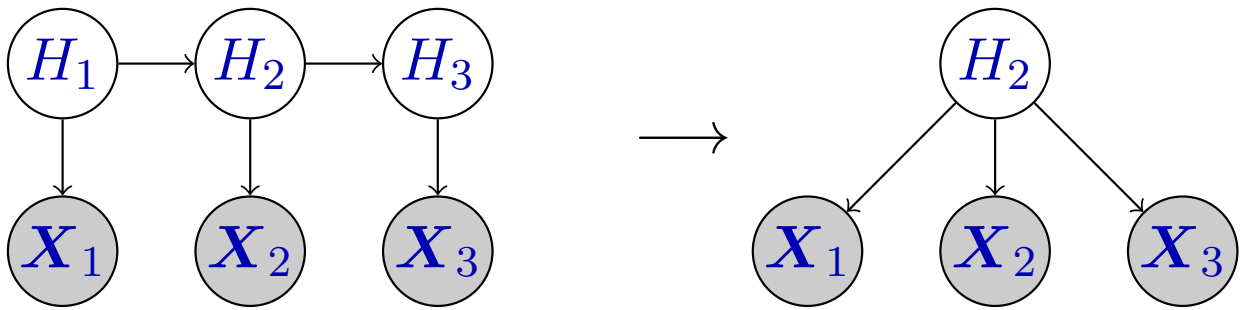

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Next: Single index models.

Single-index models

$$\begin{aligned} \boldsymbol{X} &\sim \text{Normal}(\mathbf{0}, \boldsymbol{I})\,; \\ Y \mid \boldsymbol{X} = \boldsymbol{x} &\sim \text{Normal}(g(\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle), \sigma^2)\,. \end{aligned}$$

Here, $g \colon \mathbb{R} \to \mathbb{R}$ is the *link function*.

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- **Phase retrieval** (real signals): assume $g(z) = z^2$.
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Semi-parametric estimation: regard g as nuisance parameter; focus on estimating $\boldsymbol{\beta}$.

Aside: symmetric tensors and homogeneous polynomials

Recall formula for tensor function value:

$$\boldsymbol{T}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}) \;=\; \sum_{i_1,\ldots,i_p} T_{i_1,\ldots,i_p} \cdot x^{(1)}_{i_1} \cdots x^{(p)}_{i_p} \,.$$

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p-th order symmetric tensors $\simeq$ degree-p homogeneous polynomials.

Using orthogonal polynomials

(Dudeja & H., 2018)

Let $H_p : \mathbb{R} \to \mathbb{R}$ denote the degree-p Hermite polynomial.

Assume (for $Z \sim \mathrm{Normal}(0, 1)$):

- $\mathbb{E}[g(Z)^2] = 1$ (normalization—this is WLOG);
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$\Rightarrow$ Get efficient algorithms for semi-parametric estimation of single-index model parameters, for very general link functions.

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- Estimation via **method-of-moments**:
  1. *Estimate* moments $\rightarrow$ empirical moment tensor $\widehat{\boldsymbol{T}}$.
  2. *Approximately decompose* $\widehat{\boldsymbol{T}}$ $\rightarrow$ parameter estimate $\hat{\boldsymbol{\theta}}$.

3. Error analysis (if time)

Moment estimates

Estimation of $\mathbb{E}[\boldsymbol{X}^{\otimes 3}]$ (say) from iid sample $\{\boldsymbol{x}_i\}_{i=1}^n$:

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Inevitably expect error of order $n^{-1/2}$ in some norm, e.g.,

$$\|\boldsymbol{T}\| \;:=\; \sup_{\|\boldsymbol{x}\|=\|\boldsymbol{y}\|=\|\boldsymbol{z}\|=1} \boldsymbol{T}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \quad \text{(injective/“spectral” norm)}\,,$$

$$\|\boldsymbol{T}\|_F \;:=\; \left(\sum_{i,j,k} T_{i,j,k}^2\right)^{1/2} \qquad \text{(Frobenius norm)}\,.$$

Nearly orthogonally decomposable tensor

(Mu, H., & Goldfarb, 2015)

Let $\varepsilon = \|\boldsymbol{E}\|$ for $\boldsymbol{E} := \widehat{\boldsymbol{T}} - \boldsymbol{T}$.

Claim: Let $\hat{\boldsymbol{v}} := \underset{\|\boldsymbol{x}\|=1}{\arg\max}\, \widehat{\boldsymbol{T}}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ and $\hat{\lambda} := \widehat{\boldsymbol{T}}(\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}})$.

Then

$$|\hat{\lambda} - \lambda_t| \;\leq\; \varepsilon\,, \qquad \|\hat{\boldsymbol{v}} - \boldsymbol{v}_t\| \;\leq\; O\left(\frac{\varepsilon}{\lambda_t} + \left(\frac{\varepsilon}{\lambda_t}\right)^2\right)$$

for some $t \in [d]$ with $\lambda_t \geq \max_{t'} \lambda_{t'} - 2\varepsilon$.

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Claim: Let $\hat{\boldsymbol{v}} := \underset{\|\boldsymbol{x}\|=1}{\arg\max}\, \widehat{\boldsymbol{T}}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x})$ and $\hat{\lambda} := \widehat{\boldsymbol{T}}(\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}})$.

Then

$$|\hat{\lambda} - \lambda_t| \leq \varepsilon\,, \qquad \|\hat{\boldsymbol{v}} - \boldsymbol{v}_t\| \leq O\left(\frac{\varepsilon}{\lambda_t} + \left(\frac{\varepsilon}{\lambda_t}\right)^2\right)$$

for some $t \in [d]$ with $\lambda_t \geq \max_{t'} \lambda_{t'} - 2\varepsilon$.

Many efficient algorithms for solving this approximately, when ε is small enough, like $1/d$ or $1/\sqrt{d}$ (e.g., Anandkumar, Ge, H., Kakade, & Telgarsky, 2014; Ma, Shi, & Steurer, 2016).

Recall: greedy decomposition

(Zhang & Golub, 2001)

Matching moments:

$$\{(\hat{\boldsymbol{v}}_t, \hat{\lambda}_t)\}_{t=1}^d := \underset{\{(\boldsymbol{x}_t, \sigma_t)\}_{t=1}^d}{\arg\min} \left\| \boldsymbol{T} - \sum_{t=1}^d \sigma_t \cdot \boldsymbol{x}_t \otimes \boldsymbol{x}_t \otimes \boldsymbol{x}_t \right\|_F^2 .$$

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- Greedy approach:
  - Find best rank-1 approximation:

$$(\hat{\boldsymbol{v}}, \hat{\lambda}) := \underset{\|\boldsymbol{x}\|=1,\sigma\geq 0}{\arg\min} \|\boldsymbol{T} - \sigma \cdot \boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}\|_F^2 .$$

  - "Deflate" $\boldsymbol{T} := \boldsymbol{T} - \hat{\lambda} \cdot \hat{\boldsymbol{v}} \otimes \hat{\boldsymbol{v}} \otimes \hat{\boldsymbol{v}}$ and repeat.

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Errors from deflation

(For simplicity, assume $\lambda_t = 1$ for all t, so $\boldsymbol{T} = \sum_t \boldsymbol{v}_t^{\otimes 3}$.)

First greedy step:
Rank-1 approx. $\hat{\boldsymbol{v}}_1^{\otimes 3}$ to $\widehat{\boldsymbol{T}}$ satisfies $\|\hat{\boldsymbol{v}}_1 - \boldsymbol{v}_1\| \leq \varepsilon$ (say).

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Deflation: To find next $\boldsymbol{v}_t$, use

$$\begin{aligned}\widehat{\boldsymbol{T}} - \hat{\boldsymbol{v}}_1^{\otimes 3} &= \boldsymbol{T} + \boldsymbol{E} - \hat{\boldsymbol{v}}_1^{\otimes 3} \\ &= \sum_{t=\mathbf{2}}^{d} \boldsymbol{v}_t^{\otimes 3} + \boldsymbol{E} + \left(\boldsymbol{v}_1^{\otimes 3} - \hat{\boldsymbol{v}}_1^{\otimes 3}\right).\end{aligned}$$

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Now error seems to have doubled (i.e., of size 2ε) ...

Effect of deflation errors

For any unit vector $\boldsymbol{x}$ orthogonal to $\boldsymbol{v}_1$:

$$\left\| \frac{1}{3} \nabla_{\boldsymbol{x}} \left\{ \left(\boldsymbol{v}_1^{\otimes 3} - \hat{\boldsymbol{v}}_1^{\otimes 3} \right) (\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) \right\} \right\| = \left\| \langle \boldsymbol{v}_1, \boldsymbol{x} \rangle^2 \boldsymbol{v}_1 - \langle \hat{\boldsymbol{v}}_1, \boldsymbol{x} \rangle^2 \hat{\boldsymbol{v}}_1 \right\|$$

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So effect of errors (original and from deflation) $\boldsymbol{E}+\left(\boldsymbol{v}_1^{\otimes 3}-\hat{\boldsymbol{v}}_1^{\otimes 3}\right)$ in directions orthogonal to $\boldsymbol{v}_1$ is $(1+o(1))\varepsilon$ rather than 2ε.

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- Deflation errors have lower-order effect on finding other $\boldsymbol{v}_t$.

  (Analogous statement for deflation with matrices does not hold.)

Summary

- Using method-of-moments with low-order moments, can efficiently estimate parameters for many models.
  - Exploit distributional properties, multi-view structure, and other structure to determine usable moments tensors.
  - Some efficient algorithms for carrying out the tensor decomposition to obtain parameter estimates.

Summary

- Using method-of-moments with low-order moments, can efficiently estimate parameters for many models.
  - Exploit distributional properties, multi-view structure, and other structure to determine usable moments tensors.
  - Some efficient algorithms for carrying out the tensor decomposition to obtain parameter estimates.
- Many issues to resolve!
  - Handle model misspecification, increase robustness.
  - General methodology.
  - Incorporate general prior knowledge and interactive feedback.

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Further reading:

- Anandkumar, Ge, H., Kakade, & Telgarsky.
  Tensor decompositions for learning latent variable models.
  Journal of Machine Learning Research, 15(Aug):2773–2831, 2014.
  `https://goo.gl/F8HudN`

¡Gracias!