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Master's Thesis

Infinite Games: Algorithms and Reductions

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1. Introduction

2. Definitions

Foundational to our task are the different kinds of *Infinite Games* and how to determine the outcome of each such game. To do so, we also want a notion of *strategies* on such Infinite Games. To later reduce the games to one another, we furthermore want a definition of a *Reduction* in the computational complexity sense.

2.1. Infinite Games

Infinite Games are a category of games played by two players on a finite, directed graph. They are infinite in the sense that we require the out-degree of every vertex to be at least one. As such, regardless of the strategies chosen by the players, they never terminate.

2.1.1. Directed Graphs

Let V be a finite set of Vertices, let $E \subseteq V \times V$ be a set of Edges, then $G = (V, E)$ is a *Directed Graph*. We also define $src\colon E \to V$ as $src((u, v)) = u$ and $tgt\colon E \to V$ as $src((u, v)) = v$.

2.1.2. Arenas

An arena is an extension of Directed Graphs, where the set of Vertices, V, is partitioned into two disjunct subsets V_0 and V_1, respectively denoting the regions where player 0 and player 1 are to play. We also require that the out-degree of every vertex is at least one, so that any play on the Arena is infinite.
Formally, let (V, E) be a non-trivial Directed Graph, $V_0 \cup V_1 = V$, $V_0 \cap V_1 = \emptyset$ be a partition of V and $\forall v \in V : \exists e \in E : src(e) = v$, then $A = (V, V_0, V_1, E)$ is an Arena.

2.1.3. Play

2.1.4. Strategies

2.1.5. Parity Games

Parity Games are played by two players, *Even* or player 0, also represented by $\Box$ and *Odd* or player 1, also represented by $\bigcirc$.
A Parity Game, $PG = (A, p)$, is played on an Arena A with a priority function $p\colon V \to \{0, 1, ...|V|\}$.
Let $\pi_{\sigma,\tau}(PG) = \langle v_0, v_1, ...\rangle$ be the *play* resulting from applying the strategies σ and τ to Parity Game PG.
Let $\#_\infty(\pi_{\sigma,\tau}(PG)) = \{i \in \langle p(v_0), p(v_1), ...\rangle | \forall j \in \mathbb{N}\colon j < |\{v \in \langle v_0, v_1, ...\rangle | p(v) = i\}|\}$ be the set of priorities that appear infinitely often in the play. If $max(\#_\infty(\pi_{\sigma,\tau}(PG)))$ is Even, then *Even* wins and vice versa.

2.1.6. Mean Payoff Games

Mean Payoff Games are played by two players, *Max* or player 0 (□) and *Min* or player 1 (○).
A Mean Payoff Game, $MPG = (A, \nu, d, w)$, is played on an Arena A, with threshold $\nu \in \mathbb{Z}$ and an edge-weight function $w\colon E \to \{-d, ..., -1, 0, 1, ..., d\}$, $d \in \mathbb{N}_0$.
Max wins play $\pi_{\sigma,\tau}(MPG) = \langle v_0, v_1, ...\rangle$ if

$$\liminf_{n\to\infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} w((v_i, v_{i+1})) \right) \geq \nu.$$

2.1.7. Energy Games

Energy Games are played by two players, *Charging* or player 0 (□) and *Depleting* or player 1 (○).
An Energy Game, $EG = (A, c, d, w)$, is played on an Arena A, with credit $c \in \mathbb{N}_0$ and an edge-weight function $w\colon E \to \{-d, ..., -1, 0, 1, ..., d\}$, $d \in \mathbb{N}_0$.
Charging wins play $\pi_{\sigma,\tau}(EG) = \langle v_0, v_1, ...\rangle$ if

$$\forall k \in \mathbb{N}_0\colon \left(\sum_{i=0}^{k} w((v_i, v_{i+1})) \right) + c \geq 0.$$

2.1.8. Discounted Payoff Games

Discounted Payoff Games are played by two players, *Max* or player 0 (□) and *Min* or player 1 (○).
A Discounted Payoff Game, $DPG = (A, \nu, d, w, \lambda)$, is played on an Arena A, with threshold $\nu \in \mathbb{Z}$ and an edge-weight function $w\colon E \to \{-d, ..., -1, 0, 1, ..., d\}$, $d \in \mathbb{N}_0$ and a discount factor $0 < \lambda < 1$.
Max wins play $\pi_{\sigma,\tau}(DPG) = \langle v_0, v_1, ...\rangle$ if

$$(1-\lambda) \left(\sum_{i=0}^{\infty} \lambda^i \cdot w((v_i, v_{i+1})) \right) \geq \nu.$$

2.2. Simple Stochastic Games

Simple Stochastic Games are played by two players, *Max* or player 0 (□) and *Min* or player 1 (○).
A Simple Stochastic Game, $SSG = (G, (V_{max}, V_{min}, V_{avg}), V_0, V_1, p)$, is played on a Directed Graph G, with a partition $(V_{max}, V_{min}, V_{avg})$, two sink vertices V_0, V_1 and a probability function $p\colon E \supset (V_{avg}, V) \to (0, 1]$. We also require that the out-degree of every vertex is at least one, with the exception of V_0, V_1, for which it is zero. We say that a SSG is *stopping* if for every possible combination of strategies σ, τ every vertex has a path to one of the sink vertices.
Max wins if the V_1 sink is reached, *Min* wins if the V_0 sink is reached or the game doesn't terminate. Since the result of a play $\pi_{\sigma,\tau}(SSG)$ is probablistic, it is assigned a probabilty to reach the V_1 sink rather than a distinct fixed value.

2.3. Reductions

3. Reductions and Solutions in Theory

3.1. Reductions in Theory

3.1.1. PGs to MPGs

3.1.2. MPGs to DPGs

3.1.3. MPGs to EGs

3.1.4. DPGs to SSGs

3.2. Solutions in Theory

3.2.1. Value Iteration

3.2.2. Strategy Iteration

3.2.3. PGs

3.2.4. MPGs

3.2.5. DPGs

3.2.6. EGs

3.2.7. SSGs

4. Reductions and Solutions in Practise

4.1. Reductions in Practise

4.2. Solutions in Practise

5. Implementation

6. Evaluation

7. Conclusion and Future

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