Trajectory Nonlinear H_∞ Tracking Control of Wheeled Mobile Robots with Kinematic Disturbances Observer

Jesús A. Rodríguez-Arellano, Roger Miranda-Colorado, Luis T. Aguilar, and Marco Antonio Negrete-Villanueva

***Abstract*— In practical applications, a Wheeled Mobile Robot (WMR) is always affected by kinematic disturbances, state estimation error, and measurement noise, which may diminish the performance of the closed-loop system. Hence, this work proposes a novel observer-based H_∞ controller that is robust against matched and unmatched disturbances. The proposed methodology employs an observer to estimate disturbances, while an H_∞ controller is designed to make the WMR follow a desired reference signal. A formal stability proof demonstrates the feasibility of the proposed control method. Numerical simulations show the superior performance of the proposed controller against feedback and a finite-time controller.**

***Index Terms*— Wheeled mobile robot, H_∞ control, Disturbance observer, Trajectory tracking control, Observer-based control.**

I. Introduction

Kinematic disturbances affecting a Wheeled Mobile Robot (WMR) occur when the robot's motion suddenly changes or gets perturbed due to various factors like wear and tear, changes in the terrain, sudden obstacles, or sudden changes in the robot's velocity or direction (generating undesirable effects such as skidding or slipping). These factors lead to a deviation from the desired path [1]. Also, WMRs have some features that may complicate the controller design stage, such as the nonlinear mathematical models describing their kinematics and dynamics with dynamic coupling, mechanical restrictions limiting the robot's movements, and the need to fulfill the rolling without slipping conditions [2].

The field of WMRs consists of the synergy of different research areas [3], [4], [5]. Nevertheless, to make the robot accomplish a given task, a control law methodology must be implemented, considering the task the robot will perform. Generally, every task is encompassed within three types: Regulation, Path Planning, and Trajectory Tracking [3].

Many researchers have reported linear control methodologies for controlling WMRs. Some examples include feedback schemes [3], Proportional Integral Derivative controllers [6], and Linear Quadratic Regulator-based methodologies [7], among others. These controllers are appealing because of their simplicity. However, when the vehicle is disturbed, the closed-loop performance may diminish. Then, to cope with the effect of the disturbances, robust nonlinear methods have been employed, such as sliding mode control (SMC) [1], [8], adaptive control [9], [10], neural networks [11], observer-based control [12], [13], and H_∞ control [14].

For trajectory tracking tasks, some works report the use of nonlinear disturbance observers [1], neural network-based adaptive tracking controllers [9], neural network-based adaptive dynamic programming schemes [11], fixed-time adaptive methodologies [10], fixed-time extended state observers [12], adaptive disturbance observers [13], time-invariant nonlinear controllers [15], and nonlinear dynamic methods [16]. However, these schemes have disadvantages, such as chattering, not considering the effect of disturbances, or low convergence rates. Moreover, most of them have a complex mathematical structure, which makes the controller tuning stage cumbersome. Besides, only matched disturbances are considered.

The H_∞ theory is an appealing choice that allows attenuating the effect of matched and unmatched disturbances. An H_∞ controller attenuates both matched and unmatched disturbances while ensuring the robust performance of complex mechanical systems (see, e.g., [17], [18], to name a few). WMRs are not free of any class of disturbances. Thus, this manuscript develops an observer-based global nonlinear H_∞ tracking controller for WMRs. The H_∞ controller is inspired by the passivity approaches of Isidori and Astolfi [18] and Orlov and Aguilar [17] for time-domain systems. Unlike the local solutions to the applications in [17], we propose a positive definite function, such as the Lyapunov function, ensuring the global verification of the Hamilton-Jacobi-Issacs inequality, thus avoiding the numerical computation of such a nonlinear partial differential equation. The new proposal attains asymptotic convergence of the tracking error in the free-disturbance case while ensuring that the $\mathscr{L}_2$-gain of the closed-loop system is less than an attenuation level γ. A formal theoretical analysis demonstrating the validity of the novel control scheme is included. Besides, in contrast with the local H_∞ control developments in [17], we synthesized a global nonlinear H_∞ controller, also avoiding the computational solution of the differential Riccati equation. Furthermore, we comprise an exhaustive numerical study, demonstrating the superior performance of the novel methodology when compared against feedback and a finite-time controller. The proposed controller has a simple structure, simplifying the controller tuning stage. In addition, low

This work was supported by CONACyT Project Cátedras 1537.

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power demands are obtained with the novel scheme.

II. Preliminaries and System Description

We first include some preliminaries related to nonlinear H_∞ control, which provide the mathematical tools required to solve the H_∞ control problem via a state-feedback controller. Subsequently, the description of the WMR is provided. Finally, the problem formulation is stated.

A. H_∞ control

A performance output $z(t)$, where the disturbances must be attenuated, is employed in the proposed scheme. The $\mathscr{L}_2$-space corresponds to the set of all piecewise-continuous functions $u(t)$ such that $\int_0^\infty u(t)^T u(t) dt = \int_0^\infty \|u(t)\|^2 dt < \infty$ [19]. Furthermore, the $\mathscr{L}_2$-norm of $u(t)$ is given by $\|u(t)\|_{\mathscr{L}_2} = \left(\int_0^\infty \|u(t)\|^2 dt\right)^{1/2}$. Let $w(t) \in \mathrm{I\!R}^n$ and $z(t) \in \mathrm{I\!R}^n$ denote the input and response of a given system, respectively. Then, the $\mathscr{L}$-gain is defined as the ratio between the $\mathscr{L}_2$-norm of the system's input-output, i.e., the $\mathscr{L}_2$-gain corresponds to the attenuation level $\gamma \in \mathrm{I\!R}^+$ such that $\|z(t)\|_{\mathscr{L}_2} / \|w(t)\|_{\mathscr{L}_2} \le \gamma$. The design of a nonlinear H_∞ controller considers nonautonomous nonlinear systems having the following structure

$$\begin{aligned} \dot{x}(t) &= f(x,t) + g_1(x,t)w(t) + g_2(x,t)u(t), \\ z(t) &= h_1(x,t) + k_{12}(x,t)u(t), \\ y(t) &= h_2(x,t) + k_{21}(x,t)w(t), \end{aligned} \tag{1}$$

where $t \ge 0$ is the time, $x(t) \in \mathrm{I\!R}^n$ represents the state-vector, $u(t) \in \mathrm{I\!R}^m$ corresponds to the control input, $z(t) \in \mathrm{I\!R}^l$ is the performance output to be controlled, $y(t) \in \mathrm{I\!R}^p$ is the system's output corresponding to the available measurements, $f(x,t) \in \mathrm{I\!R}^n$, $g_1(x,t) \in \mathrm{I\!R}^{n\times r}$, $g_2(x,t) \in \mathrm{I\!R}^{n\times m}$, $h_1(x,t) \in \mathrm{I\!R}^l$, $h_2(x,t) \in \mathrm{I\!R}^p$, $k_{12}(x,t) \in \mathrm{I\!R}^{l\times m}$, $k_{21}(x,t) \in \mathrm{I\!R}^{p\times r}$, $w(t) \in \mathrm{I\!R}^r$ denote the disturbances affecting the system, which are assumed to be uniformly bounded in t, i.e., $\|w(t)\|_\infty \le w^+$, where w^+ is a positive constant. The subsequent development considers the following assumptions inherent to H_∞ control theory [17].

Assumption 1: *Functions $f(x,t)$, $g_1(x,t)$, $g_2(x,t)$, $h_1(x,t)$, $h_2(x,t)$, $k_{12}(x,t)$, and $k_{21}(x,t)$ are piecewise-continuous in t for all x and continuously differentiable in x for all t.*

Assumption 2: *$f(\mathbf{0},t) = \mathbf{0}$, $h_1(\mathbf{0},t) = \mathbf{0}$, and $h_2(\mathbf{0},t) = \mathbf{0}$ for all t.*

Assumption 3: *$h_1 k_{12}^T = 0$, $k_{12}^T k_{12} = I$, $k_{21} g_1^T = \mathbf{0}$, and $k_{21}^T k_{21} = I$.*

A static state-feedback controller $u(t) = \mathscr{K}(x,t)$ is called *admissible* if the equilibrium point of the system (1), in closed-loop with $u(t)$, is asymptotically stable when $w(t) = \mathbf{0}$ [17]. Also, considering the attenuation level γ, the system (1) in closed-loop with an admissible controller has $\mathscr{L}_2$-gain less than γ if the response $z(t)$ resulting from $w(t)$ and the initial state $x(t_0) = \mathbf{0}$ fulfills $\int_{t_0}^{t_1} \|z(t)\|^2 dt < \gamma^2 \int_{t_0}^{t_1} \|w(t)\|^2 dt$, for all $t_1 > t_0$ and all piecewise continuous functions $w(t)$. In this work, the main result regarding nonlinear control theory is based on the following hypothesis and the theorem that provides the solution to the H_∞ control problem via a state-feedback controller [17].

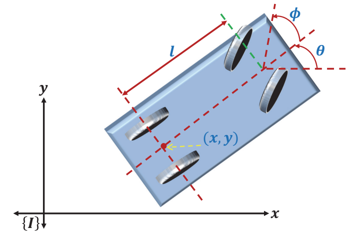

Fig. 1. Description of a WMR using the coordinates $q^T = [x, y, \theta, \phi]$.

Hypothesis 1: *Given $\gamma \in \mathrm{I\!R}^+$ and a positive definite function $F(x)$, there exists a locally Lipschitz continuous positive definite decrescent, radially unbounded solution $V(x,t)$ of the Hamilton-Jacobi-Isaacs (HJI) inequality*

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) + \gamma^2 \alpha_1^T(x,t)\alpha_1(x,t) \\ &-\alpha_2^T(x,t)\alpha_2(x,t) + h_1^T(x,t)h_1(x,t) + F(x) \le 0, \end{aligned} \tag{2}$$

where $\alpha_1(x,t) = (2\gamma^2)^{-1} g_1^T(x,t)(\partial V/\partial x)$, $\alpha_2(x,t) = -g_2^T(x,t)(\partial V/\partial x)/2$.

Theorem 1: (Solution of the H_∞ problem [17]). *Let the system (1) and consider that Assumptions 1–3 and Hypothesis 1 hold. Then, the static state-feedback controller*

$$u(t) = \alpha_2(x,t), \tag{3}$$

is globally admissible, and the $\mathscr{L}_2$-gain of the closed-loop system (1) and (3), concerning $z(t)$, is less than γ.

B. System description and problem formulation

A diagram describing the WMR under study is depicted in Fig. 1. By employing the coordinate vector $q^T = [x, y, \theta, \phi]$, the point (x, y) denotes the vehicle's position. The distance between the rear and front wheels is given by l, while $\theta(t)$ represents the vehicle orientation and $\phi(t)$ provides the steering angle. Also, the WMR control inputs are given by $v_1(t)$ and $v_2(t)$, denoting the WMR's linear and angular velocities. Then, the WMR's kinematic model is [3]

$$\dot{q} \;=\; S(q)v + d(t), S(q) = \begin{bmatrix} c_\theta & s_\theta & t_\phi / l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{4}$$

where $v^T(t) = [v_1(t), v_2(t)] \in \mathrm{I\!R}^2$, and $d(t) = [d_1(t), d_2(t), d_3(t), d_4(t)]^T \in \mathrm{I\!R}^4$ encompasses the kinematic disturbances affecting the WMR [20].

Recall that a WMR is a mechatronic system. Hence, in real applications, it is reasonable to assume the boundedness of position, velocity, acceleration, disturbances, and the corresponding time derivatives [20]. Furthermore, actuators provide bounded signals. Then, it is feasible to consider the following assumption.

Assumption 4: *Some positive constants $\bar{v}_i$, $\underline{d}_j$, $\bar{d}_j$, and $\bar{\phi}$ exist such that $v_1(t)$, $v_2(t)$, $d_i(t)$, $\dot{d}_i(t)$, and $\phi(t)$ fulfill the conditions $|v_i(t)| \le \bar{v}_i$, $i = 1, 2$, $|d_j(t)| \le \underline{d}_j$, $|\dot{d}_j(t)| \le \bar{d}_j$, $j = 1, 2, 3, 4$, $|\phi(t)| \le \bar{\phi} < \pi/2$.*

Model (4) has a singularity when $\phi(t) = \pm\pi/2\,[rad]$. However, this is not a possible configuration in a practical WMR vehicle [3], [2].

In the trajectory tracking problem, a reference signal must be designed to fulfill the WMR's kinematic restrictions considered in the kinematic model (4). Thus, a feasible reference signal must be described by

$$\dot{q}_d = S(q_d)v_d, S(q_d) = \begin{bmatrix} c_{\theta_d} & s_{\theta_d} & t_{\phi_d}/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T, \quad (5)$$

where $v_d(t) = [v_{d1}(t), v_{d2}(t)]^T$ and $q_d(t) = [x_d(t), y_d(t), \theta_d(t), \phi_d(t)]^T \in \mathbb{R}^4$ represent the references of $v(t)$ and $q(t)$, respectively.

Now, let us consider the output variable $\zeta(t) \in \mathbb{R}^2$, defined as

$$\zeta(\mathbf{t}) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = \begin{bmatrix} x + lc_\theta + \delta c_{\theta+\phi} \\ y + ls_\theta + \delta s_{\theta+\phi} \end{bmatrix}, \quad (6)$$

with $\delta \neq 0$. Then, by employing (4), the first and second time derivatives of $\zeta(t)$ are computed as follows

$$\dot{\zeta}(t) = A(\theta,\phi)v(t) + \Gamma_1(t), \quad (7)$$
$$\ddot{\zeta}(t) = A(\theta,\phi)\bar{u}(t) + \bar{A}(t)v(t) + \dot{\Gamma}_1(t), \quad (8)$$

where

$$A(\theta,\phi) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \bar{A}(t) = \begin{bmatrix} \dot{a}_{11} & \dot{a}_{12} \\ \dot{a}_{21} & \dot{a}_{22} \end{bmatrix},$$
$$\Gamma_1(\theta,\phi) = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}^T, \dot{\Gamma}_1(\theta,\phi) = \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{bmatrix},$$

with $a_{11} = c_\theta - t_\phi\left(s_\theta + \frac{\delta}{l}s_{\theta+\phi}\right)$, $a_{12} = -\delta s_{\theta+\phi}$, $a_{21} = s_\theta + t_\phi\left(c_\theta + \frac{\delta}{l}c_{\theta+\phi}\right)$, $a_{22} = \delta c_{\theta+\phi}$, $\gamma_1 = d_1(t) - ls_\theta d_3(t) - \delta s_{\theta+\phi}(d_3(t) + d_4(t))$, $\gamma_2 = d_2(t) + lc_\theta d_3(t) + \delta c_{\theta+\phi}(d_3(t) + d_4(t))$, $\bar{u} = [\bar{u}_1, \bar{u}_2]^T = [\dot{v}_1, \dot{v}_2]^T$.

Now, let us define the reference signal $\zeta_d(t)$ as follows

$$\zeta_d(t) = \begin{bmatrix} x_d + lc_{\theta_d} + \delta c_{\theta_d+\phi_d} \\ y_d + ls_{\theta_d} + \delta s_{\theta_d+\phi_d} \end{bmatrix}. \quad (9)$$

Then, by utilizing (5), the first and second time derivatives of $\zeta_d(t)$ are computed as follows

$$\dot{\zeta}_d(t) = A(\theta_d,\phi_d)v_d, \quad (10)$$
$$\ddot{\zeta}_d(t) = A(\theta_d,\phi_d)\bar{u}_d + \bar{A}_d(t)v_d, \quad (11)$$

where

$$A(\theta_d,\phi_d) = \begin{bmatrix} a_{d11} & a_{d12} \\ a_{d21} & a_{d22} \end{bmatrix}, \bar{A}_d(t) = \begin{bmatrix} \dot{a}_{d11} & \dot{a}_{d12} \\ \dot{a}_{d21} & \dot{a}_{d22} \end{bmatrix},$$

with $a_{d11} = c_{\theta_d} - t_{\phi_d}\left(s_{\theta_d} + \frac{\delta}{l}s_{\theta_d+\phi_d}\right)$, $a_{d12} = -\delta s_{\theta_d+\phi_d}$, $a_{d21} = s_{\theta_d} + t_{\phi_d}\left(c_{\theta_d} + \frac{\delta}{l}c_{\theta_d+\phi_d}\right)$, $a_{d22} = \delta c_{\theta_d+\phi_d}$, $\bar{u}_d = [\bar{u}_{d1}, \bar{u}_{d2}]^T = [\dot{v}_{d1}, \dot{v}_{d2}]^T$.

The subsequent observer-based H_∞ controller is based on the new output signal $\zeta(t)$. Then, we compute the tracking error $\tilde{\zeta}(t) = [\tilde{\zeta}_1(t), \tilde{\zeta}_2(t)]^T = \zeta(t) - \zeta_d(t)$. Finally, from equations (8) and (11), the following tracking error dynamics are obtained

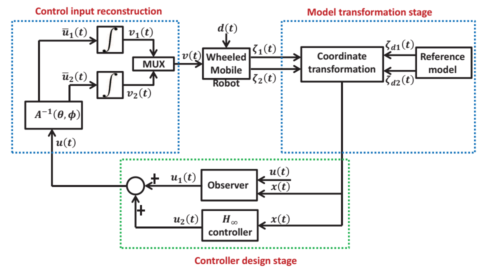

Fig. 2. Proposed observer-based H_∞-controller.

$$\ddot{\tilde{\zeta}}(t) = A(\theta,\phi)\bar{u} + \bar{A}(t)v + \dot{\Gamma}_1 - A(\theta_d,\phi_d)\bar{u}_d(t)$$
$$- \bar{A}_d(t)v_d. \quad (12)$$

To accomplish the trajectory tracking objective, we must ensure that the tracking error converges to zero asymptotically in the free-disturbance case. At the same time, the $\mathcal{L}_2$-gain of the perturbed nonlinear closed-loop system must be less than a constant positive level γ. Thus, now we define the control goal.

Problem formulation 1: *Consider the WMR's kinematic model (4) and the error signal $\tilde{\zeta}(t)$. Then, given a smooth reference signal $\zeta_d(t)$, design a control law $v(t)$ ensuring that, in the free-disturbance case, the condition*

$$\lim_{t\to\infty} \|\zeta(t) - \zeta_d(t)\| = 0, \quad (13)$$

holds. Besides, the $\mathcal{L}_2$-gain of the perturbed system is less than a positive constant level γ concerning a given performance output $z(t)$.

III. Controller Synthesis

The proposed control methodology is depicted in Fig. 2 and includes the following steps: **(1)** The output signals $\zeta(t)$ and $\zeta_d(t)$ are computed from equations (4) and (5). **(2)** Two decoupled second-order systems are obtained using a coordinate transformation, generating the new state $x(t)$. **(3)** The control input is split into two parts. The first corresponds to a disturbance observer, while the second part is designed as an H_∞ controller. The disturbance observer allows obtaining two decoupled second-order systems affected by uniformly bounded disturbances that converge to zero. The H_∞ controller is designed to accomplish the control objective (13). **(4)** Finally, the actual control input $v(t)$ is computed via an input reconstruction stage.

Let us consider the state vector $x(t) = [x_1, x_2, x_3, x_4]^T$, where the states $x_i(t)$, $i = 1,2,3,4$, are defined as $x_1(t) = \tilde{\zeta}_1(t)$, $x_2(t) = \dot{\tilde{\zeta}}_1(t)$, $x_3(t) = \tilde{\zeta}_2(t)$, $x_4(t) = \dot{\tilde{\zeta}}_2(t)$. Then, from equation (12), we obtain the following second-order systems

$$\begin{cases} \dot{x}_1(t) = x_2(t) + w_2(t), \\ \dot{x}_2(t) = u_1(t) + \sigma_1(t), \end{cases}, \begin{cases} \dot{x}_3(t) = x_4(t) + w_4(t), \\ \dot{x}_4(t) = u_2(t) + \sigma_2(t), \end{cases} \quad (14)$$

where $u(t) = [u_1(t), u_2(t)]^T = A(\theta,\phi)\bar{u}(t)$, $\sigma_1(t) = \bar{a}_{11}v_1 + \bar{a}_{12}v_2 + \dot{\gamma}_1 - a_{d11}\dot{v}_{d1} - a_{d12}\dot{v}_{d2} - \bar{a}_{d11}v_{d1} - \bar{a}_{d12}v_{d2}$, $\sigma_2(t) =$

$\bar{a}_{21}v_1 + \bar{a}_{22}v_2 + \dot{\gamma}_2 - a_{d21}\dot{v}_{d1} - a_{d22}\dot{v}_{d2} - \bar{a}_{d21}v_{d1} + \bar{a}_{d22}v_{d2}$, with $\sigma_1(t)$, $\sigma_2(t)$ denoting the system's disturbances. Also, the terms $w_2(t)$ and $w_4(t)$ represent the uncertainty due to state estimation errors.

The previous decoupled structure allows designing $u_1(t)$ and $u_2(t)$ independently, i.e., the control design stage is simplified. Also, in a real scenario, the measurements related to the car's position are usually noisy. Then, perturbations due to noisy measurements, together with the unmatched disturbances $w_2(t)$ and $w_4(t)$, may adversely affect the performance of the closed-loop system. Then, a possibility to attenuate the effect of the disturbances consists in utilizing an H_∞ approach, which is the purpose of the subsequent development.

For designing the proposed observer-based H_∞ controller, let us split $u_1(t)$ and $u_2(t)$ as $u_i(t) = u_{1i}(t) + u_{2i}(t), \quad i \in \{1,2\}$. Then, the control inputs $u_{1i}(t)$ are designed as

$$u_{1i}(t) = -\hat{e}_i(t), \tag{15}$$

where the signals $\hat{e}_i(t)$ are computed from the following equations [21]:

$$\frac{d}{dt}\begin{bmatrix} \hat{e}_i \\ \hat{\dot{e}}_i \\ x_{ci} \end{bmatrix} = \begin{bmatrix} k_{oi}\tilde{e}_{1i} + \hat{\dot{e}}_i \\ l_{oi}\mathrm{sign}(\tilde{e}_{1i}) \\ u_i \end{bmatrix}, \quad i \in \{1,2\}, \tag{16}$$

with $e_1(t) = x_2(t) - x_{c1}(t)$, $e_2(t) = x_4(t) - x_{c2}(t)$, $\tilde{e}_{1i}(t) = e_i(t) - \hat{e}_i(t)$, $\tilde{e}_{2i}(t) = \dot{e}_i(t) - \hat{\dot{e}}_i(t)$, and k_{oi}, l_{oi} are positive constants.

The control inputs $u_{1i}(t)$ allow estimating the disturbances $\sigma_i(t)$. To verify this assumption, note from (14), (16), that $\hat{\dot{e}}_i(t) = \sigma_i(t) - \tilde{e}_{2i}(t)$. Also, based on [21], if $|\dot{\sigma}(t)| \leq l_{oi}$, then $\tilde{e}_{2i}(t)$ is uniformly bounded and converges to zero asymptotically. Therefore, $u_{1i}(t)$ converges to $\sigma_i(t)$ asymptotically. Now, by employing (15) and (16), the systems (14) yield

$$\begin{cases} \dot{x}_1 = x_2 + w_2, \\ \dot{x}_2 = u_{21} + \tilde{e}_{21}, \end{cases}, \begin{cases} \dot{x}_3 = x_4 + w_4, \\ \dot{x}_4 = u_{22} + \tilde{e}_{22}. \end{cases} \tag{17}$$

Note that the closed-loop second-order systems (17) are now perturbed by the asymptotically convergent terms $\tilde{e}_{2i}(t)$, $i = 1,2$.

A. H_∞ control synthesis

Now, we proceed to design the H_∞ controllers for the systems (17). Because these systems are decoupled, the procedure for designing $u_{2i}(t)$ is the same for each second-order system. First, the system on the left-hand side of equation (17) is selected. Let us define

$$g_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, k_{211} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

and $x_{a1} = [x_1, x_2]^T$, $f_1(x_{a1}) = [x_2, 0]^T$, $g_{21}(x_{a1}) = [0,1]^T$, $h_{11}(x_{a1}) = \rho_1[0, x_{a1}]^T$, $k_{121} = [1, \mathbf{0}_{2\times1}]^T$, $h_{21}(x_{a1}) = x_{a1}$, $w_1^T = [w_2(t), \tilde{e}_{21}(t), w_3(t), w_4(t)]^T$. Then, the first second-order system in (17) can be expressed in its H_∞ standard form

$$\dot{x}_{a1} = f_1(x_{a1}) + g_{11}(x_{a1})w_1 + g_{21}(x_{a1})u_{21}, \tag{19}$$
$$z_1(t) = h_{11}(x_{a1}) + k_{121}(x_{a1})u_{21}, \tag{20}$$
$$y_1(t) = h_{21}(x_{a1}) + k_{211}(x_{a1})w_1, \tag{21}$$

where the terms $w_2(t), w_3(t), w_4(t)$ correspond to uncertainties due to state estimation errors, parameter uncertainties, and measurement noise.

Note that the system (19)–(21) has the same structure as that given in (1). Also, the elements $f_1(x_{a1})$, $g_{11}(x_{a1})$, $g_{21}(x_{a1})$, $h_{11}(x_{a1})$, $h_{21}(x)$, $k_{121}(x_{a1})$, and $k_{211}(x_{a1})$ are piecewise-continuous in t for all x_{a1}. Also, these functions are continuously differentiable in x_{a1} for all t. Thus, Assumption 1 holds. Besides, it can be verified that $f(\mathbf{0}) = \mathbf{0}$, $h_{11}(\mathbf{0}) = \mathbf{0}$, and $h_{21}(\mathbf{0}) = \mathbf{0}$. Therefore, Assumption 2 is fulfilled. Finally, it is easy to prove that $h_{11}^T(x_{a1})k_{121}(x) = 0$, $k_{121}^T(x_{a1})k_{121}(x_{a1}) = 1$, $k_{211}(x_{a1})g_{11}^T(x_{a1}) = \mathbf{0}_{2\times2}$, and $k_{211}(x_{a1})k_{211}^T(x_{a1}) = I_{2\times2}$, indicating the fulfillment of Assumption 3.

Now, to apply Theorem 1, it remains to verify the fulfillment of Hypothesis 1. To this end, we define functions $F_1(x)$ and $V_1(x)$ as follows

$$F_1(x_{a1}) = \varepsilon_1\left[x_1^2 + x_2^2\right], \tag{22}$$
$$V_1(x_{a1}) = \frac{1}{2}x_{a1}^T P_1 x_{a1}, \quad P_1 = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{bmatrix}, \tag{23}$$

where $\varepsilon_1 \in \mathbb{R}^+$, while β_1, β_2, and β_3 are chosen as

$$\beta_1 > 0, \beta_2 < \sqrt{\beta_1\beta_3} \tag{24}$$

to ensure that P_1 and $V_1(x_{a1})$ are positive definite. Besides, note that

$$\frac{\partial V_1}{\partial x_{a1}}^T = \begin{bmatrix} \beta_1x_1 + \beta_2x_2 \\ \beta_2x_1 + \beta_3x_2 \end{bmatrix}, \alpha_1 = \frac{1}{2\gamma^2}\begin{bmatrix} \beta_1x_1 + \beta_2x_2 \\ \beta_2x_1 + \beta_3x_2 \\ 0 \\ 0 \end{bmatrix},$$

$\alpha_2 = -[\beta_2x_1 + \beta_3x_2]/2$, $h_1^Th_1 = \rho_1^2[x_1^2 + x_2^2]$, $(\partial V_1 \partial x_{a1})^T f_1 = \beta_1x_1x_2 + \beta_2x_2^2$, $\gamma^2\alpha_1^T\alpha_1 = [(\beta_1x_1 + \beta_2x_2)^2 + (\beta_2x_1 + \beta_3x_2)^2]/(4\gamma^2)$, $\alpha_2^T\alpha_2 = (\beta_2x_1 + \beta_3x_2)^2/4$. Then, the left-hand side of the HJI inequality (2) can be expressed as $\mathscr{L}(x_{a1}) := -[\beta_2^2/4 - \beta_1^2/(4\gamma^2) - \beta_2^2/(4\gamma^2) - \rho_1^2 - \varepsilon_1]x_1^2 - [\beta_3^2/4 - \beta_2 - \beta_2^2/(4\gamma^2) - \beta_3^2/(4\gamma^2) - \rho_1^2 - \varepsilon_1]x_2^2 + [\beta_1 + \beta_1\beta_2/(4\gamma^2) + \beta_2\beta_3/(4\gamma^2) - \beta_2\beta_3/4]x_1x_2$. Hence, $\mathscr{L}(x_{a1})$ is negative definite if and only if

$$\beta_1 = \frac{n_1}{n_2}, \beta_2 > \sqrt{\frac{n_3}{n_4}}, \beta_3 > \sqrt{\frac{n_5}{n_4}}, \tag{25}$$

with $n_1 = \gamma^2\beta_2\beta_3(\gamma^2 - 1)$, $n_2 = 4(4\gamma^2 + \beta_2)$, $n_3 = \beta_1^2 + 4\gamma^2(\rho_1^2 + \varepsilon_1)$, $n_4 = \gamma^2 - 1$, $n_5 = \beta_2^2 + 4\gamma^2\beta_2 + 4\gamma^2(\rho_1^2 + \varepsilon_1)$, thus ensuring the fulfillment of Hypothesis 1. Then, conditions of Theorem 1 hold. Thus, the static feedback controller

$$u_{21}(t) = -\frac{1}{2}\left[\beta_2x_1 + \beta_3x_2\right], \tag{26}$$

is globally admissible and ensures that the $\mathscr{L}_2$-gain of the first second-order system from (17) in closed-loop with (26), concerning the performance output $z_1(x)$, is less than γ_1. Besides, the unperturbed closed-loop system is asymptotically stable.

A similar procedure can be followed with the second system in (17), ensuring that its $\mathscr{L}_2$-gain in closed-loop with

$$u_{22}(t) = -\frac{1}{2}\left[\beta_5x_3 + \beta_6x_4\right], \tag{27}$$

concerning the performance output $z_2(x) = h_{12}(x_{a2}) + k_{122}(x_{a2})u_{22}(t)$, is less than a given attenuation level γ_2 if the following conditions hold

$$\beta_4 = \frac{n_6}{n_7}, \sqrt{\beta_4\beta_6} > \beta_5 > \sqrt{\frac{n_8}{n_4}}, \beta_6 > \sqrt{\frac{n_9}{n_4}}, \quad (28)$$

with $n_6 = \gamma^2\beta_5\beta_6(\gamma^2-1)$, $n_7 = 4(4\gamma^2+\beta_5)$, $n_8 = \beta_4^2 + 4\gamma^2(\rho_2^2+\varepsilon_2)$, $n_9 = \beta_5^2 + 4\gamma^2\beta_5 + 4\gamma^2(\rho_2^2+\varepsilon_2)$, and

$$g_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, k_{212} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$V_2(x_{a2}) = \frac{1}{2}x_{a2}^T P_2 x_{a2}, \quad P_2 = \begin{bmatrix} \beta_4 & \beta_5 \\ \beta_5 & \beta_6 \end{bmatrix},$$

$x_{a2} = [x_3, x_4]^T$, $f_2(x_{a2}) = [x_4, 0]^T$, $g_{22}(x_{a2}) = [0,1]^T$, $h_{12}(x_{a2}) = \rho_2[0, x_{a2}]^T$, $k_{122}(x_{a2}) = [1, \mathbf{0}_{2\times 1}]^T$, $h_{22}(x_{a2}) = x_{a2}$, $w_2^T = [w_6(t), \tilde{e}_{22}(t), w_7(t), w_8(t)]^T$, $F_2(x_{a2}) = \varepsilon_2[x_3^2 + x_4^2]$, and $\rho_2 \in \mathrm{I\!R}^+$, $\varepsilon_2 \in \mathrm{I\!R}^+$, and β_4, β_5, β_6 ensure that $P_2 > 0$. Moreover, the unperturbed closed-loop system is asymptotically stable for the free-disturbance case. Therefore, the control objective (13) is accomplished. Thus, we can summarize the results given above via the following result.

Theorem 2: Let the perturbed second-order systems (14), and assume that Assumption 4 and conditions (24), (25), and (28) hold. Consider the controllers $u_{1i}(t)$, $u_{21}(t)$, and $u_{22}(t)$ given in equations (15), (26), and (27), respectively. Then, the controllers

$$u_i(t) = u_{1i}(t) + u_{2i}(t), i \in \{1,2\}, \quad (29)$$

ensure that the control objective (13) is accomplished for some positive constants β_2, β_3, β_5, and β_6.

IV. Controller Assessment

This Section evaluates the performance of the novel observer-based H_∞ controller. To this end, two controllers from state of the art were used: the feedback controller developed in [3] and the finite-time controller designed in [22]. In the following, the controllers from [3], [22], and the new proposal will be termed FC, FTC, and HC controllers. Also, we design a feasible reference by following the same procedure described in [2]. Then, we selected a sinusoidal signal described by $x_d(t) = t$ and $y_d(t) = \sin(\pi t)$. Besides, following the lines of [23], the effect of kinematic disturbances was included by employing $d_1 = 0.1 + 0.1\sin(2t)$, $d_2 = -0.1 - 0.1\cos(2t)$, $d_3(t) = 0.1$, and $d_4(t) = -0.1$. The mathematical description of the FC and FTC controllers can be consulted in [2]. The following numerical simulations used the values $k_1 = 3$, $k_2 = 3$, $k_3 = 3$, $k_4 = 5$ for the FTC technique. The FC scheme was tuned with $k_{a1} = 15$, $k_{a2} = 15$, $k_{v1} = 75$, $k_{v2} = 75$, $k_{p1} = 125$, and $k_{p2} = 125$. Finally, the gains $k_{o1} = 100$, $k_{o2} = 100$, $l_{o1} = 1$, $l_{o2} = 1$, $\beta_2 = 700$, $\beta_3 = 300$, $\beta_5 = 1200$, and $\beta_6 = 300$ were utilized with the HC methodology.

The controllers were numerically evaluated by utilizing Matlab-Simulink, `ode4` Runge-Kutta solver, and a sampling time of 1 ms. The WMR kinematic model was simulated by considering the initial conditions $x(0) = -2$, $y(0) = -1$, $\theta(0) = \arctan(\pi)\,[rad]$, and $\phi(0) = 0\,[rad]$. Besides, we considered two cases: **C1.** The kinematic model is simulated without including the effect of the kinematic disturbances $d(t)$. Furthermore, a perfect knowledge of signals $x(t)$, $y(t)$, $\theta(t)$, and $\phi(t)$ was assumed. **C2.** The same conditions from **C1**, but including the effect of $d(t)$. Besides, it is assumed that $x(t)$, $y(t)$, $\theta(t)$, and $\phi(t)$ measurements are corrupted by white noise. This effect was simulated via the Band-limited white noise block from Matlab-Simulink with noise power of 1×10^{-4} and sample time of $1\,[ms]$.

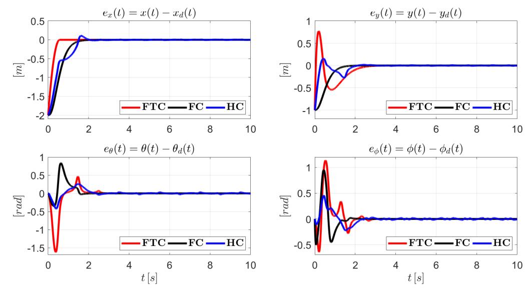

Fig. 3. Error signals for case **C1**.

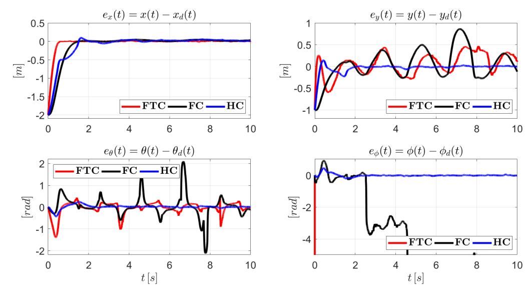

Fig. 4. Error signals for case **C2**.

Simulations for **C1** and **C2** are depicted in Fig. 3 and Fig. 4. For **C1**, the fastest response is obtained with the FTC controller in $x(t)$. However, note that the lowest overshoots are obtained with the HC controller. In case **C2**, the performance of the closed-loop system worsens when utilizing the FTC and FC controllers. In contrast, the performance of the closed-loop system is remarkable when employing the HC controller. These results can be numerically evaluated via the IAE, and ISV performance indexes [2], as depicted in Table I. For case **C1**, the fast response of the FTC controller in $x(t)$ is reflected in low error values in $e_x(t)$. Besides, the HC controller obtained the lowest error values for coordinates $y(t)$, $\theta(t)$, and $\phi(t)$. In case **C2**, all the error values increase considerably for the FTC and FC controllers. The FTC controller still exhibits the lowest error values for coordinate $x(t)$. However, the HC controller shows the lowest error values for the remaining coordinates. Moreover, the HC controller presents the minimum values for the ISV performance index.

Finally, verifying that the $\mathscr{L}_2$-gain of the closed-loop systems (17) is less than some given attenuation level γ is essential. This information is depicted in Fig. 5, which corroborates that $\int_0^T z_1^T z_1 dt \le \gamma^2 \int_0^T w_1^T w_1 dt$ and $\int_0^T z_2^T z_2 dt \le$

TABLE I

C2. NUMERICAL VALUES FOR *IAE* AND *ISV*.

	IAE				*ISV*
	e_x	$e_y\,(10^3)$	$e_\theta\,(10^3)$	$e_\phi\,(10^4)$	(10^5)
HC (**C1**)	1237.6	**0.3**	**0.4**	**0.050**	3.8
HC (**C2**)	1560.5	**0.5**	**0.6**	**0.060**	**3.9**
FTC (**C1**)	**378.1**	0.7	0.7	0.060	4.2
FTC (**C2**)	**519.6**	8.1	13.1	3280	9.3×10^7
FC (**C1**)	1201.0	0.6	0.5	0.051	**3.7**
FC (**C2**)	1576.4	5.6	5.1	8.3	7.7×10^2

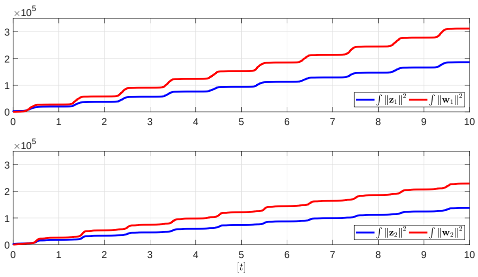

Fig. 5. $\mathscr{L}_2$-gain values of the closed-loop systems given in (17).

$\gamma^2 \int_0^T w_2^T w_2 dt$, for some $T \in \mathbb{R}^+$.

From the previous analysis, the proposed observer-based H_∞ methodology provides the best performance for the closed-loop system in the free-disturbance case and when affected by disturbances. Besides, the mathematical simplicity of the novel controller simplifies the controller tuning stage. Furthermore, the power demands for the HC controller are similar to those obtained with the FC controller in case **C1**, and the HC scheme is the only one that remains to have a good performance in case **C2**. Thus, the novel controller proposed in this work is an appealing choice for controlling WMRs.

V. CONCLUSION

A novel observer-based H_∞ controller was developed for controlling WMRs in trajectory-tracking tasks. Then, a formal analysis demonstrated that the new proposal accomplishes the trajectory tracking task despite disturbances. Numerical simulations showed the superior performance of the new control scheme, in the free-disturbance and disturbed cases, against feedback and a finite-time controller. Therefore, the new control scheme is appealing for controlling WMRs in trajectory tracking tasks.

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