$\mathbb{R}_{>0}$-normed spaces

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1 Introduction

Let X be an $\mathbb{R}$-vector space, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu\colon \mathcal{A} \to [0,\infty]$ be a function such that

1. μ is **normalized** meaning $\mu(\varnothing) = 0$.
2. μ is **monotone** (or has the **monotonicity** property) meaning $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{A}$.
3. μ is **finitely subadditive** (or has the **finite subadditivit**y property) meaning $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}$

The map $\mathrm{d}_\mu\colon \mathcal{A} \times \mathcal{A} \to [0,\infty)$ defined by

$$\mathrm{d}_\mu(A,B) = \mu(A\Delta B)$$

for all $A, B \in \mathcal{A}$ gives $\mathcal{A}$ the structure of a pseudometric space (where the pseudometric d_μ is allowed to take infinite values). If μ is understood from context, we simplify notation and write $\mathrm{d} = \mathrm{d}_\mu$ and we refer to the psuedometric space $(\mathcal{A}, \mathrm{d})$ via its underlying set $\mathcal{A}$. The reason d is a pseudometric and not a metric is because we not have identity of indiscernibles: we may have $\mu(A\Delta B) = 0$ with $A \neq B$. All is not lost however as every pseudometric space induces a metric space in a natural way. Let us briefly describe the metric space induced by the pseudometric space $\mathcal{A}$. We introduce an equivalence relation $\sim$ on $\mathcal{A}$ as follows: let $A, B \in \mathcal{A}$. Then we say

$$A \sim B \text{ if and only if } \mathrm{d}(A,B) = 0. \tag{1}$$

One checks that $\sim$ is an equivalence relation on $\mathcal{A}$ and so we may consider quotient space

$$[\mathcal{A}] := \mathcal{A}/\!\sim .$$

We denote $[A]$ to be the coset in $[\mathcal{A}]$ with $A \in \mathcal{A}$ as a particular representative. We define a metric $[\mathrm{d}]$ on $[\mathcal{A}]$ by

$$[\mathrm{d}]([A],[B]) = \mathrm{d}(A,B) \tag{2}$$

One checks that (**??**) is well-defined and satisfies all of the properties required for it to be a metric. Also the difference operator Δ induces a map $[\Delta]\colon [\mathcal{M}] \times [\mathcal{M}] \to [\mathcal{M}]$ defined by

$$[\Delta]([A],[B]) = [A\Delta B]. \tag{3}$$

One checks that (3) is well-defined and gives $[A]$ the structure of an abelian group. To clean notation in what follows, we will simply write $\mathcal{A}$ instead of $[\mathcal{A}]$ with the understanding that elements in $\mathcal{A}$ are really equivalence classes via the equivalence relation (1). Similarly we drop the brackets around $[A]$, $[\mathrm{d}]$, and $[\Delta]$ and simply write A, d, and Δ. In particular, a map $f\colon \mathcal{A} \to Y$ is well-defined only if it respects the equivalence relation ($A \sim B$, then $f(A) = f(B)$).

1.1 Giving $\mathcal{A}$ the structure of an $\mathbb{R}_{>0}$-module

We want to give $\mathcal{A}$ the structure of a normed vector space over $\mathbb{R}$ with μ being the norm. In fact, we won't be able to do this since $\mathcal{A}$ is an $\mathbb{F}_2$-vector space ($2 \cdot A = A\Delta A = \varnothing$). However we can still get very close to doing this.

Definition 1.1. An $\mathbb{R}_{>0}$-**module** V is an abelian group equipped with a map $\mathbb{R}_{>0} \times V \to V$ denoted $(\alpha, v) \mapsto \alpha v$ such that the following identities hold:

1. $1v = v$ for all $v \in V$;
2. $(\alpha\beta)v = \alpha(\beta v)$ for all $\alpha, \beta \in \mathbb{R}_{>0}$ and $v \in V$;
3. $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in \mathbb{R}_{>0}$ and $v, w \in V$.

Remark 1. In particular, an $\mathbb{R}_{>0}$-module V is like an $\mathbb{R}$-vector space except we don't require $(\alpha+\beta)v = \alpha v + \beta v$.

Definition 1.2. Let V be an $\mathbb{R}_{>0}$-module. An $\mathbb{R}_{>0}$**-norm** on V (or norm for short) is a function $\|\cdot\|\colon V \to [0,\infty]$ which satisfies the following properties:

1. (positive-definiteness) $\|v\| = 0$ if and only if $v = 0$;
2. (homogeneity) $\|\alpha v\| = \alpha\|v\|$ for all $\alpha \in \mathbb{R}_{>0}$ and $v \in V$;
3. (subadditivity) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

In the case where $\|\cdot\|$ is only homogeneous and subadditive (but not necessarily positive-definite), then we call $\|\cdot\|$ an $\mathbb{R}_{>0}$**-pseudonorm**. An $\mathbb{R}_{>0}$**-normed space** is an $\mathbb{R}_{>0}$-module V equipped with an $\mathbb{R}_{>0}$-norm.

Remark 2. If V is an $\mathbb{R}_{>0}$-normed space, then we need not have the identity $(\alpha+\beta)v = \alpha v + \beta v$, however one can still show that $\|(\alpha+\beta)v\| \geq \|\alpha v + \beta v\|$.

We now wish to give $\mathcal{A}$ the structure of an $\mathbb{R}_{>0}$-module. In order to this, we will need to make further assumptions on $\mathcal{A}$ and μ. For all $\alpha \in \mathbb{R}_{>0}$ and $A \in \mathcal{M}$, we set

$$\alpha A = \{\alpha a \mid a \in A\},$$

By replacing $\mathcal{A}$ with a larger algebra if necessary, we assume that $\alpha A \in \mathcal{A}$ for all $\alpha \in \mathbb{R}_{>0}$ and $A \in \mathcal{A}$. We also assume that $\mu(\alpha A) = \alpha\mu(A)$ for all $\alpha \in \mathbb{R}_{>0}$ and $A \in \mathcal{A}$. With these assumptions in place, we claim that the map $\mathbb{R}_{>0} \times \mathcal{A} \to \mathcal{A}$ given by $(\alpha, A) \mapsto \alpha A$ gives $\mathcal{A}$ the structure of an $\mathbb{R}_{>0}$-vector space which, when equipped with μ, is further given the structure of an $\mathbb{R}_{>0}$-normed space. Before showing this, we record the following lemma:

Lemma 1.1. *Let A and B be subsets of X and let $\alpha \in \mathbb{R}_{>0}$. The following identities hold.*

1. $\alpha(A \cup B) = \alpha A \cup \alpha B$.
2. $\alpha(A \backslash B) = \alpha A \backslash \alpha B$.
3. $\alpha(A \Delta B) = (\alpha A)\Delta(\alpha B)$.

Proof. 1. Let $\alpha x \in \alpha(A \cup B)$ where $x \in A \cup B$. Without loss of generality, we may assume $x \in A$. Then clearly $\alpha x \in \alpha A \cup \alpha B$. Thus

$$\alpha(A \cup B) \subseteq \alpha A \cup \alpha B.$$

Conversely, suppose $y \in \alpha A \cup \alpha B$. Without loss of generality, we may assume $y \in \alpha A$. Then $y = \alpha x$ for some $x \in A$. Thus $y = \alpha x \in \alpha(A \cup B)$. Thus

$$\alpha(A \cup B) \supseteq \alpha A \cup \alpha B.$$

2. Let $\alpha a \in \alpha(A \backslash B)$ where $a \in A \backslash B$. Then observe that $\alpha a \in \alpha A$ but $\alpha a \notin \alpha B$. Indeed, if $\alpha a = \alpha b$ for some $b \in B$, then $a = b$ since $\alpha \neq 0$, which contradicts the assumption that $a \notin B$. Thus $\alpha a \in \alpha A \backslash \alpha B$ which implies

$$\alpha(A \backslash B) \subseteq \alpha A \backslash \alpha B.$$

Conversely, suppose $x \in \alpha A \backslash \alpha B$. Since $x \in \alpha A$, we have $x = \alpha a$ for some $a \in A$. Since $x \notin \alpha B$, it follows that $a \notin B$. Thus $x = \alpha a \in \alpha(A \backslash B)$ which implies

$$\alpha(A \backslash B) \supseteq \alpha A \backslash \alpha B.$$

3. By 2 and 3, we have

$$\begin{aligned}
\alpha(A \Delta B) &= \alpha((A \backslash B) \cup (B \backslash A)) \\
&= \alpha(A \backslash B) \cup \alpha(B \backslash A) \\
&= \alpha A \backslash \alpha B \cup \alpha B \backslash \alpha A \\
&= (\alpha A)\Delta(\alpha B).
\end{aligned}$$

□

Theorem 1.2. *The map $\mathbb{R}_{>0} \times [\mathcal{A}] \to [\mathcal{A}]$ given by $(\alpha, [A]) \mapsto [\alpha A]$ gives $[\mathcal{A}]$ the structure of an $\mathbb{R}_{>0}$-module. Furthermore, the map $[\mu]\colon [\mathcal{A}] \to [0,\infty]$ given by $[A] \mapsto \mu(A)$ is an $\mathbb{R}_{>0}$-norm on $[\mathcal{A}]$.*

Proof. Lemma (1.1) shows that the action $\mathbb{R}_{>0} \times \mathcal{A} \to \mathcal{A}$ given by $(\alpha, A) \mapsto \alpha A$ gives $\mathcal{A}$ the structure of an $\mathbb{R}_{>0}$-module. It is straightforward to check that the map $\mu\colon \mathcal{A} \to [0, \infty]$ is an $\mathbb{R}_{>0}$-pseudonorm. Thus we just need to check that the induced maps $\mathbb{R}_{>0} \times [\mathcal{A}] \to [\mathcal{A}]$ and $[\mu]\colon [\mathcal{A}] \to [0, \infty]$ are well-defined. Suppose $A \sim B$ (so $\mu(A \Delta B) = 0$). Then

$$\begin{aligned} \mu(\alpha A \Delta \alpha B) &= \mu\alpha(A \Delta B)) \\ &= \alpha\mu(A \Delta B) \\ &= 0, \end{aligned}$$

shows that $\alpha A \sim \alpha B$. Thus the map $\mathbb{R}_{>0} \times [\mathcal{A}] \to [\mathcal{A}]$ is well-defined. A similar argument shows that $[\mu]$ is well-defined as well. □

1.2 Marcel Riesz Extension Theorem

Definition 1.3. Let V be an $\mathbb{R}_{>0}$-module. A set $P \subseteq V$ is said to be a **convex cone** if

1. $x, y \in P$ implies $x + y \in P$.
2. if $x \in P$ and $\alpha \in \mathbb{R}_{>0}$, then $\alpha x \in P$.

Theorem 1.3. *(Marcel Riesz Extension Theorem) Let V be an $\mathbb{R}_{>0}$-module, let $W \subseteq V$ be a subspace of V, and let $P \subseteq V$ be a convex cone. Suppose $V = W + P$ and $\psi\colon W \to \mathbb{R}$ is a linear functional such that $\psi|_{P \cap W} \geq 0$. Then there exists a linear functional $\widetilde{\psi}\colon V \to \mathbb{R}$ such that $\widetilde{\psi}|_W = \psi$ and $\widetilde{\psi}|_P \geq 0$.*

Proof. Let $v \in V \backslash W$. We will first show that we can extend ψ to a linear functional $\widetilde{\psi}\colon W + \mathbb{R}v \to \mathbb{R}$ such that $\widetilde{\psi}$ preserves the positivity condition. Define two sets $A = \{x \in W \mid -x \leq_P v\}$ and $B = \{y \in W \mid v \leq_P y\}$. Note that A and B are nonempty since $V = W + P$. We claim that

$$\sup\{-\psi(x) \mid x \in A\} \leq \inf\{\psi(y) \mid y \in B\}. \tag{4}$$

Indeed, let $x \in A$ and let $y \in B$. Then note that $-x \leq_P v \leq_P y$ implies $x + y \in C$. It follows that

$$\begin{aligned} 0 &\leq \psi(x + y) \\ &= \psi(x) + \psi(y). \end{aligned}$$

In other words, $-\psi(x) \leq \psi(y)$, which implies (4).

We set $\widetilde{\psi}(v)$ to be any number between $\sup\{-\psi(x) \mid x \in A\}$ and $\inf\{\psi(y) \mid y \in B\}$ and we define we define $\widetilde{\psi}\colon W + \mathbb{R}v \to \mathbb{R}$ by

$$\widetilde{\psi}(w + \lambda v) = \psi(w) + \lambda\widetilde{\psi}(v) \tag{5}$$

for all $w + \lambda v \in W + \mathbb{R}v$. Note that (5) is well-defined since v is linearly independent from W. It is easy to check that (5) gives us a linear functional $\widetilde{\psi}\colon W + \mathbb{R}v \to \mathbb{R}$ such that $\widetilde{\psi}|_W = \psi$. Furthermore we have

$$-\psi(x) \leq \widetilde{\psi}(v) \leq \psi(y)$$

for all $x \in A$ and $y \in B$. The only thing left is to check that $\widetilde{\psi}$ satisfies the positivity condition. Let $w + \lambda v \in P \cap (W + \mathbb{R}v)$. We consider the following cases:

Case 1: Assume that $\lambda > 0$. Then note that $(1/\lambda)w + v = (1/\lambda)(w + \lambda v) \in P$ since P is a convex cone. This implies $(1/\lambda)w \in A$. Thus

$$\begin{aligned} 0 &\leq \lambda(\psi((1/\lambda)w) + \widetilde{\psi}(v)) \\ &= \psi(w) + \lambda\widetilde{\psi}(v) \\ &= \widetilde{\psi}(w + \lambda v). \end{aligned}$$

Case 2: Assume that $\lambda < 0$. Then note that $(-1/\lambda)w - v = (-1/\lambda)(w + \lambda v) \in P$ since P is a convex cone. This implies $(-1/\lambda)w \in B$. Thus

$$\begin{aligned} 0 &\leq -\lambda(\psi((-1/\lambda)w) - \widetilde{\psi}(v)) \\ &= \psi(w) + \lambda\widetilde{\psi}(v) \\ &= \widetilde{\psi}(w + \lambda v). \end{aligned}$$

Case 3: Assume that $\lambda = 0$. Then $w \in P \cap W$, and hence $0 \leq \psi(w) = \widetilde{\psi}(w)$.

In all three cases, we see that the positivity condition is satisfied. Thus we can extend ψ to a linear functional on $W + \mathbb{R}v$ while preserving the positivity condition.

Now to extend ψ to all of V, we must appeal to Zorn's Lemma. More specifically, we define a partially ordered set $(\mathcal{F}, \leq)$ as follows: the underlying set $\mathcal{F}$ is given by

$$\mathcal{F} = \{\text{linear functionals } \psi' \colon W' \to \mathbb{R} \mid W' \supseteq W,\ \psi'|_W = \psi, \text{ and } \psi'|_{W' \cap C = P} \geq 0\}.$$

A member of $\mathcal{F}$ is denoted by an ordered pair: (ψ', W'). If (ψ_1, W_1) and (ψ_2, W_2) are two members of $\mathcal{F}$ then we say $(\psi_1, W_1) \leq (\psi_2, W_2)$ if $W_1 \subseteq W_2$ and $\psi_2|_{W_1} = \psi_1$. Observe that every totally ordered subset in $(\mathcal{F}, \leq)$ has an upper bound. Indeed, suppose $\{(\psi_i, W_i)\}_{i \in I}$ is a totally ordered subset in $(\mathcal{F}, \leq)$. Then if we set $W' = \bigcup_{i \in I} W_i$ and if we define $\psi' \colon W \to \mathbb{R}$ as follows: if $x \in W$, then $x \in W_i$ for some i and we set $\psi'(x) = \psi_i(x)$. Then it is easy to check that (ψ', W') is a member of $\mathcal{F}$ and that it is an upper bound of $\{(\psi_i, W_i)\}_{i \in I}$. Since $\mathcal{F}$ is nonempty (it contains (ψ, W)) and every totally ordered subset of $\mathcal{F}$ has an upper bound, we can apply Zorn's Lemma to obtain a *maximal* element in $(\mathcal{F}, \leq)$. This maximal element *must* be defined on all of V, otherwise we can extend it to a larger subspace as shown above and obtain a contradiction. $\square$

Theorem 1.4. *Let V be an $\mathbb{R}_{>0}$-normed space and let U be a subspace of V. Furthermore, let $\ell \colon U \to \mathbb{R}_{\geq 0}$ be a positive linear functional which satisfies $\ell(u) \leq \|u\|$ for all $u \in U$. Then there exists a positive linear functional $\widetilde{\ell} \colon V \to \mathbb{R}_{\geq 0}$ such that $\widetilde{\ell}|_U = \ell$ and $\widetilde{\ell}(v) \leq \|v\|$ for all $v \in V$.*