Geometry

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Part I
Sheaves and Locally Ringed Spaces

1 Presheaves and Sheaves

Let X be a topological space.

1.1 Presheaves

A **presheaf** $\mathcal{F}$ on X assigns to each open set U in X a set $\mathcal{F}(U)$, and to every pair of nested open subsets $U \subseteq V$ of X, a function $\mathrm{res}^V_U \colon \mathcal{F}(V) \to \mathcal{F}(U)$, called the **restriction map**, such that

1. $\mathcal{F}(\emptyset) = 0$,
2. res^U_U is the identity map for all open sets U in X,
3. $\mathrm{res}^V_U \circ \mathrm{res}^W_V = \mathrm{res}^W_U$ for all open sets $U \subseteq V \subseteq W$ in X.

The elements $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U; elements of $\mathcal{F}(X)$ are called **global sections**. The restriction maps res^V_U are written as $f \mapsto f|_U$. Very often we will also write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$.

1.1.1 Morphism of Presheaves

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A **morphism** of presheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a family of maps $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets U of X such that for all pairs of open sets V of X such that $U \subseteq V$ the diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}^V_U"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^V_U"] \\ \mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{G}(U) \end{tikzcd}$$

is commutative. The composite of morphisms $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{G} \to \mathcal{H}$ is defined in the obvious way, namely we set $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$ for all open sets U of X. We obtain the category of presheaves on X which we denote by $\mathbf{Psh}(X)$. We often simplify our notation by denoting the composite of φ and ψ by $\varphi\psi$ instead of $\varphi \circ \psi$. Furthermore, we often drop U from subscript in φ_U and simply write φ whenever context is clear.

1.1.2 Category Theory

Using the language of category theory, we can define presheaves in a very concise way. Let $\mathbf{O}(X)$ be the category whose objects are open sets U of X and whose morphisms are the inclusion maps. Then a presheaf $\mathcal{F}$ is just a contravariant functor from $\mathbf{O}(X)$ to $\mathbf{Set}$, and morphisms of presheaves are natural transformations between functors. Alternatively, we can view $\mathcal{F}$ as a covariant functor from $\mathbf{O}(X)^{\mathrm{op}}$ to $\mathbf{Set}$. We can also replace the category $\mathbf{Set}$ with any other category $\mathbf{C}$ to obtain the notion of a presheaf with values in $\mathbf{C}$. This signifies that $\mathcal{F}(U)$ is an object in $\mathbf{C}$ for every open subset U of X and that the restriction maps are morphisms in $\mathbf{C}$. Similarly, we can replace the category $\mathbf{O}(X)$ with a category $\mathbf{C}$ to obtain the notion of a presheaf defined on a category $\mathbf{C}$.

1.2 Sheaves

Presheaves on X are top-down constructions; we can restrict information from larger to smaller sets. However, many objects in mathematics are bottom-up constructions; they are defined locally, which we then piece together to obtain something global. Presheaves do not provide the means to deduce global properties from the properties we find locally in the open sets of X. This is where the idea of sheaves come in.

Definition 1.1. A **sheaf** on X is a presheaf $\mathcal{F}$ on X which satisfies the following **sheaf axiom**:

- Suppose $\{U_i\}_{i\in I}$ is an open covering of an open subset U and suppose that for each $i \in I$ a section $s_i \in \mathcal{F}(U_i)$ is given such that for each pair $U_{i_1}, U_{i_2} \in \{U_i\}_{i\in I}$ we have

  $$s_{i_1}|_{U_{i_1}\cap U_{i_2}} = s_{i_2}|_{U_{i_1}\cap U_{i_2}}.$$

  Then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A **morphism of sheaves** is a morphism of presheaves. We denote by $\mathbf{Sh}(X)$ to be the category whose objects are sheaves and whose morphisms are morphism of sheaves. Note that $\mathbf{Sh}(X)$ is a faithfully full subcategory of $\mathbf{Psh}(X)$.

Proposition 1.1. *The sheaf axioms imply that any sheaf has exactly one section of the empty set.*

Proof. The empty set $\emptyset$ can be written as the union of an empty family (that is, the indexing set I is $\emptyset$). The condition given for the sheaf property is vacuously true. So there must exist a unique section in $F(\emptyset)$. □

Example 1.1. Let E be a set. A presheaf of functions on X with values in E is a presheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ consists of functions from U to E for all open sets U of X. Given such a presheaf $\mathcal{F}$, note he only thing preventing $\mathcal{F}$ from being a sheaf is the *existence* of global functions since *uniqueness* is already guarenteed. Indeed, suppose $\{U_i\}_{i\in I}$ is an open covering of an open set U of X, and suppose that for all $i \in I$ we have $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all $i, j \in I$ (here we use the notation $U_{ij} = U_i \cap U_j$). Then if $f, g \in \mathcal{F}(U)$ satisfy $f|_{U_i} = f_i = g|_{U_i}$ for all $i \in I$, then we must have $f = g$. This is because $f = g$ if and only if $f(x) = g(x)$ for all $x \in U$, and this is true since $x \in U_{i(x)}$ for some $i(x) \in I$ (depending on x), hence $f(x) = f_{i(x)}(x) = g(x)$.

1.2.1 Reformulating the sheaf axiom

We give a reformulation of the sheaf axiom in terms of arrows. Let $\mathcal{F}$ be a presheaf on X, let U be an open set of X, and let $\{U_i\}_{i\in I}$ be an open covering of U. We define maps

$$\begin{aligned}
\rho &: \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), && s \mapsto (s|_{U_i})_i \\
\sigma &: \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), && (s_i)_i \mapsto (s_i|_{U_i\cap U_j})_{(i,j)} \\
\sigma' &: \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), && (s_i)_i \mapsto (s_j|_{U_i\cap U_j})_{(i,j)}
\end{aligned}$$

The presheaf $\mathcal{F}$ is a sheaf, if it satisfies for all U and all open coverings $\{U_i\}_{i\in I}$ the following condition: The diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho"] & \prod_{i\in I} \mathcal{F}(U_i) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\sigma'"'] & \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)
\end{tikzcd}$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_i \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

For presheaves of abelian groups (or with values in any abelian category) we can reformulate the definition of a sheaf as follows: A presheaf $\mathcal{F}$ is a sheaf if and only if for all open subsets U and all coverings $\{U_i\}$ of U the sequence of abelian groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \prod_i \mathcal{F}(U_i) & \longrightarrow & \prod_{i,j} \mathcal{F}(U_i \cap U_j) \\
 & & s & \longmapsto & (s|_{U_i})_i & & \\
 & & & & (s_i)_i & \longmapsto & (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}
\end{array}$$

is exact.

1.3 Examples of Sheaves

1.3.1 Sheaf of Continuous Functions

Let X and Y be topological spaces. For each open subset U of X, we define

$$\mathcal{C}_{X;Y}(U) := \{f : U \to Y \mid f \text{ is continuous}\}.$$

Then $\mathcal{C}_{X;Y}$ is a presheaf of Y-valued functions on X. In fact, more is true: $\mathcal{C}_{X;Y}$ is a sheaf. Indeed, let $\{U_i\}$ be an open covering of U. If $f : U \to Y$ is a continuous function, then by restriction to U_i, we get continuous maps $f_i : U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j. Conversely, if we are given continuous maps $f_i : U_i \to Y$ that agree on the overlaps (that is, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j) then there is a unique set-theoretic map $f : X \to Y$ satisfying $f|_{U_i} = f_i$ for all i and it is continuous. Indeed, for any open $V \subseteq Y$ we have that $f^{-1}(V)$ is open in U because $f^{-1}(V) \cap U_i = f_i^{-1}(V)$ is open in U_i for every i.

1.3.2 Sheaf of C^α Functions

Let V and W be finite-dimensional $\mathbb{R}$-vector spaces and let X be an open subspace of V. Let $\alpha \in \widehat{\mathbb{N}}_0$. For each open subset U of X, we define

$$\mathcal{C}^\alpha_{X;W}(U) := \{f : U \to W \mid f \text{ is } C^\alpha \text{ map}\}.$$

Then $\mathcal{C}^\alpha_{X;W}$ is a sheaf of functions on X. It is a sheaf of $\mathbb{R}$-vector spaces. If $W = \mathbb{R}$, then we simply write $\mathcal{C}^\alpha_X$.

1.3.3 Sheaf of Holomorphic Functions

Let V and W be finite-dimensional $\mathbb{C}$-vector spaces and let X be an open subspace of V. For each open subset U of X, we define

$$\mathcal{O}_{X;W}(U) := \mathcal{O}^{\text{hol}}_{X;W}(U) := \{f\colon U \to W \mid f \text{ is holomorphic}\}.$$

Then $\mathcal{O}_{X;W}$ (with the usual restriction maps) is a sheaf of $\mathbb{C}$-vector spaces.

1.3.4 Constant Sheaf

Definition 1.2. Let X be a topological space and let E be a set. We define a sheaf on X, denote $\underline{E}$, called the **constant sheaf on X with value** E, by setting

$$\underline{E}(U) = \begin{cases} E & \text{if } U \text{ non-empty} \\ 0 & \text{else} \end{cases}$$

for all open $U \subseteq X$ and letting the restriction maps be the identity map.

1.4 Sheaves are determined by their values on a basis

Let $\mathcal{F}$ be a sheaf on X and let $\mathscr{B}$ be a basis for the topology on X. If we know what $\mathcal{F}(U)$ is fo of a sheaf on every element U of $\mathcal{B}$, then we can use the sheaf property to determine $\mathcal{F}(V)$ on an arbitrary open set V of X. We simply cover V by elements of $\mathcal{B}$. Here is a more systematic way of saying this:

$$\begin{aligned} \mathcal{F}(V) &= \left\{ (s_U)_U \in \prod_{\substack{U \in \mathscr{B} \\ U \subseteq V}} \mathcal{F}(U) \mid \text{for } U' \subseteq U \text{ both in } \mathcal{B} \text{ we have } s_U|_{U'} = s_{U'} \right\} \\ &= \lim_{\substack{U \in \mathscr{B} \\ U \subseteq V}} \mathcal{F}(U). \end{aligned} \tag{1}$$

Using this observation, we see that it suffices to define a sheaf on a basis $\mathcal{B}$ of open sets of the topology of a topological space X: Consider $\mathcal{B}$ as a full subcategory of $\mathbf{O}(X)$, then a presheaf on $\mathcal{B}$ is a contravariant functor $\mathcal{F} : \mathcal{B} \to \mathbf{Set}$. Every such presheaf $\mathcal{F}$ on $\mathcal{B}$ can be extended to a presheaf $\mathcal{F}'$ on X by using (1) as a definition.

Example 1.2. Let $\mathcal{F}$ be the presheaf of bounded continuous functions on $\mathbb{R}$ with values in $\mathbb{R}$. Then $\mathcal{F}$ is not a sheaf. Indeed, for each $i \in \mathbb{Z}$ let $U_i = (i, i+2)$ and $f_i = x|_{U_i}$. Then $\{U_i\}$ is a covering of $\mathbb{R}$ and there is no bounded continuous function f on $\mathbb{R}$ such that $f|_{U_i} = f_i$ for all i. The sheafification of $\mathcal{F}$ is isomorphic to $\mathcal{C}_{\mathbb{R};\mathbb{R}}$.

1.5 Gluing Sheaves

Proposition 1.2. *Let $\{U_i\}_{i\in I}$ be an open cover of X. For all $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i. Assume that for each pair (i,j) of indices we are given isomorphisms $\varphi_{ij}\colon \mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$ satisfying for all $i,j,k \in I$ the "cocycle condition" $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ on U_{ijk}. Then there exists a sheaf $\mathcal{F}$ on X and for all $i \in I$ isomorphisms $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ such that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i,j \in I$. Moreover, $\mathcal{F}$ and ψ_i are uniquely determined up to unique isomorphism by these conditions.*

Proof. Let U be an open subset of X. We define $\mathcal{F}(U)$ to be the set of collections of sections which are locally compatible:

$$\mathcal{F}(U) = \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap U) \mid s_i|_{U_{ij}\cap U} = \varphi_{ij}(s_j)|_{U_{ij}\cap U} \text{ for all } i,j \in I. \right\} \tag{2}$$

The restriction maps are defined pointwise. Thus if V is an open subset of U, then we set $(s_i)|_V = (s_i|_{U_i\cap V})$. The cocycle ensures that (2) is well-defined. By replacing U_i with $U_i \cap U$ if necessary, we may assume that $U_i \subseteq U$ for all $i \in I$. In this case, (2) has the slightly simpler description:

$$\mathcal{F}(U) = \left\{ (s_i) \in \prod_{i\in I} \mathcal{F}_i(U_i) \mid s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}} \text{ for all } i,j \in I. \right\}$$

Let us verify that $\mathcal{F}$ is a sheaf. Let $\{U_{i'}\}_{i'\in I'}$ be an open cover of U and for each $i' \in I'$ let $(s_{i,i'})_{i\in I} \in \mathcal{F}(U_{i'})$ such that

$$(s_{i,i'})|_{U_{ii'j'}} = (s_{i,j'})|_{U_{ii'j'}} \tag{3}$$

for all $i', j' \in I'$. We want to show that there exists a unique element $(s_i) \in \mathcal{F}(U)$ such that $(s_i)|_{U_{i'}} = (s_{i,i'})$ for all $i' \in I'$.

Note that (3) says for each $i \in I$, we have $s_{i,i'}|_{U_{ii'j'}} = s_{i,j'}|_{U_{ii'j'}}$ for all $i', j' \in I'$. Thus for each $i \in I$, since $\mathcal{F}_i$ is a sheaf and $\{U_{ii'}\}_{i'\in I'}$ is an open cover of U_i, we use the fact that $\mathcal{F}_i$ is a sheaf to obtain a unique element $s_i \in \mathcal{F}_i(U_i)$ such that $s_i|_{U_{ii'}} = s_{i,i'}$ for all $i' \in I$. Thus we obtain a unique sequence of sections $(s_i) \in \prod_{i\in I} \mathcal{F}_i(U_i)$ such that $(s_i)|_{U_{i'}} = (s_{i,i'})$ for all $i' \in I'$. This establishes uniqueness, so the only thing left to do is to check that $(s_i) \in \mathcal{F}(U)$. For each $i,j \in I$, note that $\{U_{iji'}\}_{i'\in I'}$ is an open cover of U_{ij} and

$$\begin{aligned}
s_i|_{U_{iji'}} &= s_{i,i'}|_{U_{iji'}} \\
&= \varphi_{ij}(s_{j,i'})|_{U_{iji'}} \\
&= \varphi_{ij}(s_{j,i'}|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j)|_{U_{iji'}}
\end{aligned}$$

for all $i' \in I'$. Thus by the uniqueness part in the sheaf axiom (for $\mathcal{F}_i$), we must have $s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}}$. It follows that $(s_i) \in \mathcal{F}(U)$ as claimed.

Now fix $i \in I$. We define the map $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. Let U be an open subset of U_i. Then for $s \in \mathcal{F}_i(U)$, we set $\psi_i(s) = (\varphi_{ji}(s|_{U_j\cap U}))_{j\in I}$. Conversely, if $(s_j)_{j\in I} \in \mathcal{F}|_{U_i}(U)$, then we set $\psi_i^{-1}((s_j)_{j\in I}) = s_i$. It is clear that ψ_i is a bijection with inverse ψ_i^{-1}. Furthermore, if V is an open subset of U, then $\psi_i(s|_V) = \psi_i(s)|_V$. Thus, ψ_i is an isomorphism of sheaves $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. We repeat this construction for all $i \in I$ to get an isomorpism $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ for all $i \in I$. Finally, that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i,j \in I$ follows from a direct calculation: for $s \in \mathcal{F}_j(U_{ij})$, we have

$$\begin{aligned}
(\psi_i \circ \varphi_{ij})(s) &= \psi_i(\varphi_{ij}(s)) \\
&= (\varphi_{ki}(\varphi_{ij}(s)|_{U_{ijk}}))_{k\in I} \\
&= (\varphi_{ki}(\varphi_{ij}(s))|_{U_{ijk}})_{k\in I} \\
&= (\varphi_{kj}(s)|_{U_{ijk}})_{k\in I} \\
&= (\varphi_{kj}(s|_{U_{ijk}}))_{k\in I} \\
&= \psi_j(s).
\end{aligned}$$

□

1.6 Stalks

Let $\mathcal{F}$ be a presheaf on X. Suppose that for each $x \in X$, there exists a smallest neighborhood containing x, say U_x. Then we can determine the sheaf completely by computing the values of the sheaf on these open sets. Indeed, then if U is an open subset of X, then $U = \bigcup_{x \in X} U_x$ where the union is disjoint. Therefore the sheaf axiom tells us

$$\mathcal{F}(U) \cong \prod_{x \in U} \mathcal{F}(U_x).$$

The problem of course is that almost all of the topolgical spaces we are interested in won't have a smallest open neighborhood of x. Another way of saying this is that the limit of the diagram which consists of all open neighborhoods of x will not exist. On the other hand, colimits in **Set** do exist, so there's nothing stopping us from looking at colimits in the diagram of $\mathcal{F}$-images of neighborhoods of x.

Definition 1.3. Let S be a subset of X. Define $\mathbf{N}(S)$ to be the full subcategory of $\mathbf{O}(X)$ whose objects are open neighborhoods of S and whose morphisms are inclusions. If x is a point of X, then we will denoted $\mathbf{N}(\{x\})$ by $\mathbf{N}(x)$. By restricting $\mathcal{F}$ to $\mathbf{N}(S)$, we obtain a contravariant functor (which we again denote by $\mathcal{F}$) from $\mathbf{N}(S)$ to **Set**. Note that the category $\mathbf{N}(S)^{\mathrm{op}}$ is filtered since for any two neighborhoods U_1 and U_2 of S there exists a neighborhood U of S with $U_1 \cap U_2 \supseteq U$ (namely $U = U_1 \cap U_2$) and since there is only one morphism between any two objects.

1. For each $x \in X$, we define the **stalk** of $\mathcal{F}$ at x, denoted $\mathcal{F}_x$, to be the filtered colimit

   $$\mathcal{F}_x = \mathrm{colim}_{U \in \mathbf{N}(x)} \mathcal{F}(U).$$

   More concretely, one has

   $$\mathcal{F}_x = \{(U, s) \mid U \text{ is an open neighborhood of } x \text{ and } s \in \mathcal{F}(U)\} \,/ \sim,$$

   where two pairs (U_1, s_1) and (U_2, s_2) are equivalent if there exists an open neighborhood U of x such that $U \subseteq U_1 \cap U_2$ and $s_1|_U = s_2|_U$. The equivalence class corresponding to (U, s) at x is denoted $[U, s]_x$, or even more simply by $[s]_x$ if U is understood from context. Elements in $\mathcal{F}_x$ are called **germs** at x. When we don't need to choose a particular representative for a germ at x, then we often use Greek letters like σ_x to denote germs at x.

2. For each open subset $U \subseteq X$ we obtain a canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ given by $s \mapsto [s]_x$. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. Then for each $x \in X$, we obtain a map $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ where $\varphi_x := \mathrm{colim}_{U \in \mathbf{U}(x)}$. In particular, φ_x is defined by

   $$\varphi_x([U, s]_x) = [U, \varphi_U(s)]_x$$

   This map is well-defined since φ commutes with restriction maps. We obtain a functor $\mathcal{F} \to \mathcal{F}_x$ from the category of presheaves on X to the category of sets.

Remark 1. If $\mathcal{F}$ is a presheaf of functions, one should think of the stalk $\mathcal{F}_x$ as the set of functions defined in some unspecified open neighborhood of x.

Remark 2. If $\mathcal{F}$ is a presheaf on X with values in $\mathbf{C}$, where $\mathbf{C}$ is any category in which filtered colimits exist (for instance the category of groups, of rings, of R-modules, or R-algebras, etc...), then the stalk $\mathcal{F}_x$ is an object in $\mathbf{C}$ and we obtain a functor $\mathcal{F} \mapsto \mathcal{F}_x$ from the category of presheaves on X with values in $\mathbf{C}$ to the category $\mathbf{C}$. Let us make this more precise for a sheaf $\mathcal{G}$ of groups. The group law of $\mathcal{G}_x$ is defined as follows: Let $g, h \in \mathcal{G}$ be represented by (U, s) and (V, t). Choose an open neighborhood W of x with $W \subseteq U \cap V$. Then $(U, s) \sim (W, s|_W)$ and $(V, t) \sim (W, t_W)$ and the product gh is the equivalence class of $(W, (s|_W)(t|_W))$. In the same way addition and multiplication is defined on the stalk for a sheaf of rings.

1.6.1 Examples of Stalks

Example 1.3. Let $\mathcal{O}$ be the sheaf of real analytic functions on $\mathbb{R}^n$ and let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a point in $\mathbb{R}^n$. Suppose $[U, f]_{\boldsymbol{x}}$ is a germ at $\boldsymbol{x}$. Since f is analytic at $\boldsymbol{x}$, we there exists an open neighborhood V of $\boldsymbol{x}$ such that $V \subseteq U$ and such that $f|_V$ is equal to its Taylor series at $\boldsymbol{x}$: for each $\boldsymbol{y} \in V$, we have

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \sum_i \partial_{x_i} f(\boldsymbol{x})(y_i - x_i) + \cdots + \frac{1}{k!} \sum_{i_1, \dots, i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f(\boldsymbol{x})(y_{i_1} - x_{i_1}) \cdots (y_{i_k} - x_{i_k}) + \cdots$$

Two real analytic functions f_1 and f_2 defined in open neighborhoods U_1 and U_2, respectively, of p agree on some open neighborhood $V \subseteq U_1 \cap U_2$ if and only if they have the same Taylor expansion around p. So we have a well-defined map $\mathcal{O}_{\mathbb{R}^n, p} \to$

Example 1.4. (Stalk of the sheaf of continuous functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U, f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$\mathrm{ev}_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(\mathrm{ev}_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. Let $s \in \mathcal{C}_{X,x} \backslash \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X \backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$. Therefore the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. This shows that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

Example 1.5. (Stalk of the sheaf of C^α functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U, f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$e_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(e_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. We claim that this is the unique maximal ideal of $\mathcal{C}_{X,x}$, i.e. that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

To prove this, we need to show that the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. Let $s \in \mathcal{C}_{X,x} \backslash \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X \backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$.

Example 1.6. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let X be an open subspace of V, and let $\mathcal{O}$ denote the sheaf $\mathcal{C}^\alpha_{X;W}$. We claim that $\mathcal{O}_x$ is a local ring. Inded, let $s \in \mathcal{O}_x$ be a germ and let (f, U) be a representative of s. By the very same argument as in the example above, we may assume that f does not vanish on U so that $1/f$ exists on U. It remains to show that $1/f$ is C^α on X. This follows from the stability of the C^α property under composition and the fact that $x \mapsto 1/x$ is a C^α map from $\mathbb{R}^\times$ to $\mathbb{R}^\times$.

1.6.2 Working With Stalks

The following result will be used very often.

Proposition 1.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X, and let $\varphi, \psi \colon \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves.*

1. *Assume that $\mathcal{F}$ is a sheaf. Then $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$ if and only if $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subseteq X$.*

2. *If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, then φ_x is bijective for all $x \in X$ if and only if φ_U is bijective for all open subsets $U \subseteq X$.*

3. *If $\mathcal{G}$ is a sheaf, then the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.*

Proof. 1. Suppose that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subseteq X$. Let $x \in X$ and suppose $[s]_x = [U^x, s]_x$ and $[t]_x = [V^x, t]_x$ are two germs in at x such that $\varphi_x([s]_x) = \varphi_x([t]_x)$. Then $[\varphi_{U^x}(s)]_x = [\varphi_{V^x}(t)]_x$ which implies there exists an open neighborhood W^x of x such that $W^x \subseteq U^x \cap V^x$ and

$$\varphi_{W^x}(s|_{W^x}) = \varphi_{W^x}(t|_{W^x}).$$

Since φ_{W^x} is injective, we see that $s|_{W^x} = t|_{W^x}$ which implies $[s]_x = [t]_x$. It follows that φ_x is injective for all $x \in X$. Conversely, suppose $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Let U be an open subset of X and suppose s and t are two sections in $\mathcal{F}(U)$ such that $\varphi_U(s) = \varphi_U(t)$. Then for each $x \in U$, we have $\varphi_x([s]_x) = \varphi_x([t]_x)$, and since φ_x is injective, this implies $[s]_x = [t]_x$. Thus for each $x \in U$, there exists an open neighborhood U^x of x such that $s|_{U^x} = t|_{U^x}$. This implies $s = t$ since $\mathcal{F}$ is a sheaf. It follows that φ_U is injective for all open sets U of X.

2. Suppose that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective for all open sets U of X . By 1, it suffices to show that φ_x is surjective for all $x \in X$. Let $x \in X$ and let $[t]_x = [U, t]_x$ be a germ at x. Since φ_U is surjective, there exists a section s in $\mathcal{F}(U)$ such that that $\varphi_U(s) = t$. In particular, this implies $\varphi_x([s]_x) = \varphi_x([t]_x)$. It follows that φ_x is surjective for all $x \in X$. Conversely, suppose that φ_x is bijective for all $x \in X$. By 1, it suffices to show that φ_U is surjective for all open sets U of X. Let $U \subseteq X$ be open and let t be a section over U. For each $x \in U$, since φ_x is surjective, there exists a germ $[s^x]_x = [U^x, s^x]_x$ at x such that $\varphi_x([s^x]_x) = [t]_x$. By replacing U^x with a smaller open set if necessary, we may assume that $U^x \subseteq U$ and that $\varphi_{U^x}(s^x) = t|_{U^x}$ for each $x \in U$. For each $x, y \in U$, denote $U^{xy} = U^x \cap U^y$ and observe that

$$
\begin{aligned}
\varphi_{U^{xy}}(s^x|_{U^{xy}}) &= \varphi_{U^x}(s^x)|_{U^{xy}} \\
&= (t|_{U^x})|_{U^{xy}} \\
&= t|_{U^{xy}} \\
&= (t|_{U^y})|_{U^{xy}} \\
&= \varphi_{U^y}(s^y)|_{U^{xy}} \\
&= \varphi_{U^{xy}}(s^y|_{U^{xy}}),
\end{aligned}
$$

and hence $s^x|_{U^{xy}} = s^y|_{U^{xy}}$ since $\varphi_{U^{xy}}$ is injective. Since $\mathcal{F}$ is a sheaf and $\{U^x\}_{x \in U}$ is an open cover of U, this implies there exists a unique section s over U such that $s|_{U^x} = s^x$ for all $x \in U$. In particular, this implies that $\varphi_U(s)|_{U^x} = \varphi_{U^x}(s^x) = t|_{U^x}$ for all $x \in U$. Since $\mathcal{G}$ is a sheaf and $\{U^x\}_{x \in U}$ is an open cover of U, this implies $\varphi_U(s) = t$. It follows that φ_U is surjective for all $x \in X$.

3. Suppose that $\varphi = \psi$. Let $x \in X$ and let $[s]_x = [U, s]_x$ be a germ at x. Then since $\varphi_U(s) = \psi_U(s)$, we see that

$$
\begin{aligned}
\varphi_x([s]_x) &= [\varphi_U(s)]_x \\
&= [\psi_U(s)]_x \\
&= \psi_x([s]_x).
\end{aligned}
$$

It follows that $\varphi_x = \psi_x$ for all $x \in X$. Conversely, suppose $\varphi_x = \psi_x$ for all $x \in X$. Let U be an open set of X and let s be a section over U. For each $x \in U$, since $[\varphi_U(s)]_x = [\psi_U(s)]_x$, there exists an open neighborhood U^x of x such that $U^x \subseteq U$ and $\varphi_U(s)|_{U^x} = \psi_U(s)|_{U^x}$. Since $\mathcal{G}$ is a sheaf and $\{U^x\}_{x \in U}$ is an open cover of U, we must have $\varphi_U(s) = \psi_U(s)$. It follows that $\varphi = \psi$. □

Definition 1.4. We call a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of sheaves **injective** (respectively **surjective**, respectively **bijective**) if $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective (respectively surjective, respectively bijective) for all $x \in X$. A sequence

$$
\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}
$$

of morphisms of sheaves of groups is called **exact** if for all $x \in X$ the induced sequence of stalks

$$
\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x
$$

is an exact sequence of groups.

Thus Proposition (1.3) tells us that φ is injective (respectively bijective) if and only if $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (respectively bijective) for all open subsets U of X. On ther other hand, φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathcal{G}(U)$ there exist an open cover $\{U_i\}_{i \in I}$ of U (depending on t) and sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$ for all $i \in I$. In other words, φ is surjective if locally we can find a preimage of t. In particular, surjectivity of φ does *not* imply that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets U of X. Indeed, in the proof of Proposition (1.3), we needed injectivity of $\varphi_{U^{xy}}$ in order to patch up the various local sections. Here is an example from complex analysis which demonstrates this:

Example 1.7. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on an open set X of $\mathbb{C}$. For every open set U of X and for every $f \in \mathcal{O}_X(U)$ we let $\mathrm{D}_U(f) = f'$ be the derivative. . We obtain a morphism $\mathrm{D} \colon \mathcal{O}_X \to \mathcal{O}_X$ of sheaves of $\mathbb{C}$-vector spaces. Note that D is surjective because locally every holomorphic function has a primitive. On the other hand, there exists open sets U of X and functions f on U which have no primitive on U. For instance, take $X = \mathbb{C}$, let $U = \mathbb{C} \backslash \{0\}$, and let $f(z) = 1/z$. Assume for a contradiction that F is a primitive of f defined on U.

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the path defined by $\gamma(t) = e^{2\pi i t}$. Then observe that

$$\begin{aligned}
0 &= F(1) - F(1) \\
&= F(\gamma(1)) - F(\gamma(0)) \\
&= \int_\gamma f(z)\mathrm{d}z \\
&= \int_0^1 f(\gamma(t))\gamma'(t)\mathrm{d}t \\
&= \int_0^1 \frac{1}{e^{2\pi i t}} 2\pi i \cdot e^{2\pi i t}\mathrm{d}t \\
&= \int_0^1 2\pi i \mathrm{d}t \\
&= 2\pi i,
\end{aligned}$$

which is obviously a contradiction. The issue here comes from the fact that $\pi_1(U) \cong \mathbb{Z}$ where $\pi_1(U)$ is the fundamental group of U. More generally, by complex analysis we know that D_U is surjective if and only if every connected component of U is simply connected (meaning $\pi_1(U) = 0$). The sufficiency of this condition will also be an immediate application of cohomologial methods developed later. We obtain an exact sequence of sheaves of $\mathbb{C}$-vector spaces

$$0 \longrightarrow \mathbb{C}_X \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\mathrm{D}} \mathcal{O}_X \longrightarrow 0$$

where $\mathbb{C}_X$ denotes the sheaf of locally constant $\mathbb{C}$-valued functions on X and where ι_U is the inclusion for all $U \subseteq X$ open.

Let $\{U_i\}_{i\in I}$ be an open cover of X. A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective (respectively surjective, respectively bijective) if and only if its restriction $\varphi|_{U_i}\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ to morphisms of sheaves on U_i is injective (respectively surjective, respectively bijective) for all $i \in I$. Indeed, this is because these notions are defined via the stalks. However note that the existence of the morphism φ is crucial: there exists sheaves $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{G}|_{U_i}$ for all i and such that $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic.

1.7 Sheafification

There is a functorial way to attach to a presheaf a sheaf:

Proposition 1.4. *Let $\mathcal{F}$ be a presheaf on X. There exists a pair $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ where $\widetilde{\mathcal{F}}$ is a sheaf on X and where $\iota_{\mathcal{F}}\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is a morphism of presheaves which satisfies the following universal mapping property: for all sheaves $\mathcal{G}$ on X and morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \mathcal{G}$ with $\widetilde{\varphi} \circ \iota_{\mathcal{F}} = \varphi$, in other words, the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota_{\mathcal{F}}"] \arrow[dr, "\varphi"'] & \widetilde{\mathcal{F}} \arrow[d, "\widetilde{\varphi}"] \\
& \mathcal{G}
\end{tikzcd} \tag{4}$$

Moreover, the following properties hold:

1. *For all $x \in X$, the map of stalks $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x$ is bijective.*

2. *For every presheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ making the diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota_{\mathcal{F}}"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{F}} \arrow[d, "\widetilde{\varphi}"] \\
\mathcal{G} \arrow[r, "\iota_{\mathcal{G}}"] & \widetilde{\mathcal{G}}
\end{tikzcd} \tag{5}$$

commutative. In particular, $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ is a functor from the category of presheaves on X to the category of sheaves on X.

*The sheaf $\widetilde{\mathcal{F}}$ (equipped with the canonical morphism $\iota_{\mathcal{F}}$) is called the **sheafification** of $\mathcal{F}$. We obtain a functor $\mathcal{F} \to \widetilde{\mathcal{F}}$ from $\mathbf{Psh}(X)$ to $\mathbf{Sh}(X)$ which we call the **sheafification** functor.*

Let's discuss why we are justified in saying $\widetilde{\mathcal{F}}$ is *the* sheafification of $\mathcal{F}$. The point is that $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is unique up to unique isomorphism. Indeed, let us simplify notation by writing $\iota = \iota_{\mathcal{F}}$. Now if $(\mathcal{F}', \iota')$ is another pair which satisfies the universal mapping property, then there exists a unique morphism $\widetilde{\iota'}\colon \widetilde{\mathcal{F}} \to \mathcal{F}'$ such that $\widetilde{\iota'} \circ \iota = \iota'$. Similarly there is a unique morphism $\widetilde{\iota}\colon \mathcal{F}' \to \widetilde{\mathcal{F}}$ such that $\widetilde{\iota} \circ \iota' = \iota$. We claim that $\widetilde{\iota'}$ is an isomorphism whose inverse is $\widetilde{\iota}$. Indeed, $\widetilde{\iota} \circ \widetilde{\iota'}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ is a morphism which satisfies

$$\begin{aligned}(\widetilde{\iota} \circ \widetilde{\iota'}) \circ \iota &= \widetilde{\iota} \circ (\widetilde{\iota'} \circ \iota) \\ &= \widetilde{\iota} \circ \iota' \\ &= \iota.\end{aligned}$$

In particular, we must have $\widetilde{\iota} \circ \widetilde{\iota'} = 1_{\widetilde{\mathcal{F}}}$ by the uniqueness part of the universal mapping property. A similar argument shows $\widetilde{\iota'} \circ \widetilde{\iota} = 1_{\mathcal{F}'}$. Thus if $\mathcal{F}$ is already a sheaf, then we can identify $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ with $(\mathcal{F}, 1_{\mathcal{F}})$ (using the unique isomorphism) and say $\mathcal{F}$ is the sheafification of $\mathcal{F}$ since the pair $(\mathcal{F}, 1_{\mathcal{F}})$ clearly satisfies the universal mapping property. A similar line of reasoning also justifies our calling the functor $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ *the* sheafification functor (since this functor is unique up to unique natural isomorphism). In the proof of Proposition (1.4) below, we will give a construction of $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$.

Proof. First we define $\widetilde{\mathcal{F}}$. Let $U \subseteq X$ be open. We define $\widetilde{\mathcal{F}}(U)$ to be the families of elements in the stalks of $\mathcal{F}$, which locally give rise to sections of $\mathcal{F}$:

$$\widetilde{\mathcal{F}}(U) := \left\{ (\sigma_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c}\text{for all } y \in U \text{ there exists an open neighborhood } U_y \subseteq U \text{ of } y \text{ and an } s_y \in \mathcal{F}(U_y) \\ \text{such that for all } x \in U_y \text{ the germ } \sigma_x \text{ can be represented by } (U_y, s_y)\end{array} \right\}$$

The restriction maps are defined pointwise. Thus if $V \subseteq U$ is a smaller open set, then we set

$$((\sigma_x)_{x \in U})|_V = (\sigma_x)_{x \in V}.$$

Clearly $\widetilde{\mathcal{F}}$ is a presheaf. To see why it is a sheaf, let $\{U_i\}_{i \in I}$ be an open cover of U and for each $i \in I$ let $(\sigma_x)_{x \in U_i} \in \widetilde{\mathcal{F}}(U_i)$ such that

$$((\sigma_x)_{x \in U_i})|_{U_{ij}} = (\sigma_x)_{x \in U_{ij}} = ((\sigma_x)_{x \in U_j})|_{U_{ij}}$$

for all $i, j \in I$. Then it is easy to see that $(\sigma_x)_{x \in U}$ is the unique element in $\mathcal{F}(U)$ such that

$$((\sigma_x)_{x \in U})|_{U_i} = (\sigma_x)_{x \in U_i}$$

for all $i \in I$. Thus $\widetilde{\mathcal{F}}$ is clearly a sheaf (note that the same proof also shows that presheaf given by $U \mapsto \prod_{x \in U} \mathcal{F}_x$ is a sheaf). Next we define the map $\iota\colon \mathcal{F} \to \widetilde{\mathcal{F}}$. Let $U \subseteq X$ be open and let $s \in \mathcal{F}(U)$. We set

$$\iota(s) = ([U, s]_x)_{x \in U}.$$

Let us check that the pair $(\widetilde{\mathcal{F}}, \iota)$ satisfies the universal mapping property.
For all $U \subseteq X$ open, we define

$$\iota_{\mathcal{F},U}(s) = ([U, s]_x)_{x \in U}$$

for all $s \in \mathcal{F}(U)$. We simplify notation by writing ι instead of $\iota_{\mathcal{F}}$ whenever context is clear. Now fix a point $y \in X$. We want to show that the induced map $\iota_y\colon \mathcal{F}_y \to \widetilde{\mathcal{F}}_y$ is a bijection. We do this in two steps:

ι_y **is injective**: let $[s]_y = [U, s]_y$ and $[t]_y = [V, t]_y$ be two germs in $\mathcal{F}_y$ such that

$$[([s]_x)_{x \in U})]_y = [([t]_x)_{x \in V})]_y.$$

Then there exists an open neighborhood $W \subseteq U \cap V$ of y such that $([s]_x)_{x \in W} = ([t]_x)_{x \in W}$, or in other words, such that $[s]_x = [t]_x$ for all $x \in W$. In particular, $[s]_y = [t]_y$. It follows that ι_y is injective.

ι_y **is surjective**: let $[U, (\sigma_x)_{x \in U}]_y$ be a germ in $\widetilde{\mathcal{F}}_y$. By definition of $\widetilde{\mathcal{F}}(U)$, there exists an open neighborhood $U^y \subseteq U$ of y and a section s^y in $\mathcal{F}(U^y)$ such that $\sigma_x = [U^y, s^y]_x$ for all $x \in U^y$. In particular, we see that

$$\begin{aligned}\iota_y([U^y, s^y]_y) &= [U^y, ([U^y, s^y]_x)_{x \in U^y}]_y \\ &= [U^y, (\sigma_x)_{x \in U^y}]_y \\ &= [U, (\sigma_x)_{x \in U}]_y.\end{aligned}$$

It follows that ι_y is surjective.

Now we prove the two properties stated in the proposition:

1. Let $x_0 \in X$. We want to show that $\iota_{\mathcal{F},x_0}\colon \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is bijective. First we show that $\iota_{\mathcal{F},x_0}$ is injective. Let $[s]_{x_0} = [U,s]_{x_0}$ and $[t]_{x_0} = [V,t]_{x_0}$ be two germs in $\mathcal{F}_{x_0}$ such that

$$[([s]_x)_{x\in U})]_{x_0} = [([t]_x)_{x\in V})]_{x_0}.$$

Then there exists an open neighborhood $W \subseteq U \cap V$ of x_0 such that $([s]_x)_{x\in W} = ([t]_x)_{x\in W}$, or in other words, such that $[s]_x = [t]_x$ for all $x \in W$. In particular, $[s]_{x_0} = [t]_{x_0}$.

Next we show that $\iota_{\mathcal{F},x_0}$ is surjective. To see this, let $[([s^x]_x)_{x\in U})]_{x_0}$ be a germ in $\widetilde{\mathcal{F}}_{x_0}$; so U is an open neighborhood of x_0 and $([s^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}(U)$. In fact, using the construction of $\widetilde{\mathcal{F}}$, we can find a better representative for this germ: choose an open neighborhood U_0 of x_0 and choose a section $s_0 \in \mathcal{F}(U_0)$ such that $[s^x]_x = [s_0]_x$ for all $x \in U_0$. Then clearly we have $[([s^x]_x)_{x\in U})]_{x_0} = [([s_0]_x)_{x\in U_0})]_{x_0}$. In particular, note that

$$\iota_{\mathcal{F},x_0}([s_0]_{x_0}) = [([s_0]_x)_{x\in U_0})]_{x_0}.$$

It follows that $\iota_{\mathcal{F},x_0}$ is surjective.

2. Let $\mathcal{G}$ be a presheaf on X and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism. We define $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ as follows: let U be an open set of X and define the map $\widetilde{\varphi}_U\colon \widetilde{\mathcal{F}}(U) \to \widetilde{\mathcal{G}}(U)$ by

$$\widetilde{\varphi}_U\left(([s^x]_x)_{x\in U}\right) = ([\varphi_{U^x}(s^x)]_x)_{x\in U}$$

for all $([s^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}$. It is straightforward to check that this gives rise a morphism $\widetilde{\varphi}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ of presheaves. Furthermore, observe that

$$\begin{aligned}
\widetilde{\varphi}_U \iota_{\mathcal{F},U}(s) &= \widetilde{\varphi}_U\left(([s]_x)_{x\in U}\right) \\
&= ([\varphi_U(x)]_x)_{x\in U} \\
&= \iota_{\mathcal{G},U}(\varphi_U(s)) \\
&= \iota_{\mathcal{G},U}\varphi_U(s).
\end{aligned}$$

It follows that $\widetilde{\varphi}\iota_{\mathcal{F}} = \iota_{\mathcal{G}}\varphi$, thus $\widetilde{\varphi}$ makes the diagram (5) commutative. We claim that $\widetilde{\varphi}$ is the unique morphism making the diagram (5) commutative. Indeed, if $\widetilde{\psi}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ is another morphism making the diagram commute, then $\widetilde{\psi}\iota_{\mathcal{F}} = \iota_{\mathcal{G}}\varphi = \widetilde{\varphi}\iota_{\mathcal{F}}$ implies $\widetilde{\psi}_x\iota_{\mathcal{F},x} = \widetilde{\varphi}_x\iota_{\mathcal{F},x}$ for all $x \in X$. Since $\iota_{\mathcal{F},x}$ is a bijection, it follows that $\widetilde{\varphi}_x = \widetilde{\psi}_x$ for all $x \in X$. Therefore $\widetilde{\varphi} = \widetilde{\psi}$ by Proposition (1.3).

If we assume in addition that $\mathcal{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathcal{G}}\colon \mathcal{G} \to \widetilde{\mathcal{G}}$, which is bijective on stalks, is an isomorphism by Proposition (1.3). Thus we can define $\widetilde{\varphi}' = \iota_{\mathcal{G}}^{-1}\widetilde{\varphi}$ and it is easily seen that $\widetilde{\varphi}'$ is the unique morphism such that $\widetilde{\varphi}'\iota_{\mathcal{F}} = \varphi$. Finally, the uniqueness of $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is a formal consequence from the universal mapping property. □

1.7.1 Sheafification is left adjoint to the forgetful functor

Lemma 1.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. Then there is a bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Psh}(X)}(\mathcal{F}, \mathcal{G}) \tag{6}$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Thus the sheafification functor from $\mathbf{Psh}(X)$ to $\mathbf{Sh}(X)$ is the left adjoint to the forgetful functor from $\mathbf{Sh}(X)$ to $\mathbf{Psh}(X)$. In particular, the sheafification functor preserves all colimits whereas the forgetful functor preserves all limits.

Proof. If $\varphi\colon \mathcal{F} \to \mathcal{G}$, then there exists a unique morphism $\widetilde{\varphi}'\colon \widetilde{F} \to \mathcal{G}$ such that $\widetilde{\varphi}'\iota_{\mathcal{F}} = \varphi$. Conversely, if $\widetilde{\varphi}'\colon \widetilde{\mathcal{F}} \to \mathcal{G}$, then we *define* $\varphi\colon \mathcal{F} \to \mathcal{G}$ by $\varphi := \widetilde{\varphi}'\iota_{\mathcal{F}}$. Functoriality in $\mathcal{F}$ and $\mathcal{G}$ is an easy exercise. □

Note that the bijection (6) restricts to a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Sh}(X)}(\mathcal{F}, \mathcal{G}),$$

which is again functorial in $\mathcal{F}$ and $\mathcal{G}$. Thus, viewing the sheafification/forgetful functors as functors from $\mathbf{Sh}(X)$ to itself, we see that sheafification preserves all colimits and the forgetful functor preserves all limits. In particular, if D is any diagram in $\mathbf{Sh}(X)$, then $\lim(D)$ is a sheaf!

1.7.2 Sheafification of a presheaf of functions

Proposition 1.5. *Let E be a set and let $\mathcal{F}$ be a presheaf of functions on X with values in E. Define a sheaf $\widehat{\mathcal{F}}$ on X by*

$$\widehat{\mathcal{F}}(U) = \left\{ f\colon U \to E \mid \text{there exists an open covering } \{U_i\}_{i\in I} \text{ of } U \text{ such that } f|_{U_i} \in \mathcal{F}(U_i) \text{ for all } i \in I \right\}.$$

for all open sets $U \subseteq X$ with the restriction maps of $\widehat{\mathcal{F}}$ being the usual ones. Then $\widehat{\mathcal{F}}$ (equipped with the inclusion map $\iota\colon \mathcal{F} \to \widehat{\mathcal{F}}$) is the sheafification of $\mathcal{F}$.

Proof. We just need to show that $\widehat{\mathcal{F}}$ satisfies the universal mapping property. Let $\mathcal{G}$ be a sheaf on X and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism. Let $U \subseteq X$ be open and let $f \in \widehat{\mathcal{F}}(U)$. Choose an open covering $\{U_i\}_{i\in I}$ of U such that $f|_{U_i} \in \mathcal{F}(U_i)$ for all $i \in I$. Then $\varphi_{U_i}(f|_{U_i}) \in \mathcal{G}(U_i)$ for all $i \in I$, furthermore we have

$$\varphi_{U_i}(f|_{U_i})|_{U_{ij}} = \varphi_{U_{ij}}(f|_{U_{ij}}) = \varphi_{U_j}(f|_{U_j})|_{U_{ij}}$$

for all $i, j \in I$. Thus since $\mathcal{G}$ is a sheaf, there exists a unique $\widetilde{\varphi}(f) \in \mathcal{G}(U)$ such that $\widetilde{\varphi}(f)|_{U_i} = \varphi_{U_i}(f|_{U_i})$ for all $i \in I$. This construction does not depend on the choice of an open covering $\{U_i\}$ of U since $\mathcal{G}$ is a sheaf. In particular, we have a well defined map $f \mapsto \widetilde{\varphi}(f)$ from $\widehat{\mathcal{F}}(U)$ to $\mathcal{G}(U)$. Furthermore, it is easy to see that $\widetilde{\varphi}$ is the unique morhpism which satisfies $\widetilde{\varphi} \circ \iota = \varphi$. □

Let's construct an explicit isomorphism from $\widetilde{\mathcal{F}}$ to $\widehat{\mathcal{F}}$. Let $([U^x, f^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}(U)$. For each $x \in U$, choose an open neighborhood V^x of x together with a section $g^x \in \mathcal{F}(V^x)$ such that $V^x \subseteq U$ and $[f^y]_y = [g^x]_y$ for all $y \in V^x$. For each $y \in V^x$, choose an open neighborhood $W^{x,y}$ of y such that $W^{x,y} \subseteq U^y \cap V^x$ and $f^y|_{W^{x,y}} = g^x|_{W^{x,y}}$. Define a function $g\colon U \to E$ by setting $g(x) = g^x(x)$ for all $x \in U$. Note that the function g is independent of our choice of the triple $(W^{x,y}, V^x, g^x)$ for if $(\widetilde{W}^{x,y}, \widetilde{V}^x, \widetilde{g}^x)$ were another such triple, then we'd have $\widetilde{g}^x(x) = f^x(x) = g^x(x)$. Observe that $\{W^{x,y}\}$ forms an open cover of U as we vary $x \in X$ and $y \in V^x$. Moreover, we have $g|_{W^{x,y}} = g^x|_{W^{x,y}}$ since

$$\begin{aligned} g(z) &= g^z(z) \\ &= f^z(z) \\ &= g^x(z) \end{aligned}$$

for all $z \in W^{x,y}$. Therefore $g|_{W^{x,y}} = f^y|_{W^{x,y}} \in \mathcal{F}(W^{x,y})$ for all $x \in U$ and $y \in V^x$. It follows that $g \in \widehat{\mathcal{F}}(U)$. Therefore we obtain map from $\widetilde{\mathcal{F}}(U) \to \widehat{\mathcal{F}}(U)$ given by $([f^x]_x)_{x\in U} \mapsto g$. It is straightforward to check that this map induces an isomorphism $\widetilde{\mathcal{F}} \cong \widehat{\mathcal{F}}$ of sheaves on X.

Definition 1.5. Let E be a set and let $\mathcal{F}$ be the constant presheaf with values in E. We denote by E_X to be the sheaf of locally constant functions with value E defined by

$$E_X(U) = \{f\colon U \to E \mid f \text{ is locally constant}\}.$$

for all open sets U of X. This is the sheafification of the presheaf of constant functions with values in E. The sheaf E_X is called the **constant sheaf** with values in E.

1.8 Direct and Inverse Images of Sheaves

Throughout this subsection, let $f\colon X \to Y$ be a continuous map.

1.8.1 Direct Image

Definition 1.6. Let $\mathcal{F}$ be a presheaf on X. We define $f_*\mathcal{F}$ to be the presheaf on Y whose values on opens $V \subseteq Y$ is given by

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

and whose restriction maps are given by the restriction maps of $\mathcal{F}$. We call $f_*\mathcal{F}$ the **direct image of $\mathcal{F}$ along f**. If $\varphi\colon \mathcal{F} \to \mathcal{F}'$ is a morphism of presheaves on X, then we obtain a morphism $f_*(\varphi) : f_*\mathcal{F} \to f_*\mathcal{F}'$ of presheaves on Y by defining

$$f_*(\varphi)_V = \varphi_{f^{-1}(V)}$$

for all open subsets $V \subseteq Y$. It is straightforward to check that we obtain a functor $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$, called the **direct image functor along** f. Note that if $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is a sheaf on Y. Indeed, let $\{V_i\}_{i\in I}$ be an open covering of an open set V of Y and let $s_i \in f_*\mathcal{F}(V_i) = \mathcal{F}(f^{-1}(V_i))$ such that

$$s_i|_{f^{-1}(V_{ij})} = s_j|_{f^{-1}(V_{ij})}$$

for all $i, j \in I$. Then $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of $f^{-1}(V)$, and so by the sheaf property of $\mathcal{F}$, there exists a unique $s \in \mathcal{F}(f^{-1}(V))$ such that $s|_{f^{-1}(V_i)} = s_i$. Thus, the direct image functor restricts to a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Example 1.8. Let $p\colon X \to Y$ be a continuous map, let E be a set, and let E_X and E_Y be the sheaf of locally constant E-valued functions on X and Y respectively. Define a morphism of sheaves $p^\star\colon E_Y \to p_*E_X$ by setting

$$p^\star_V(g) = g \circ p.$$

for all open sets V of Y and for all locally constant functions $g\colon V \to E$. Note that this definition makes sense because $g \circ p\colon p^{-1}(V) \to E$ is locally constant. Now assume that p is surjective, that Y has the induced quotient topology, and that p has connected fibers. Note that for all open sets V of Y, a locally constant map $h\colon p^{-1}(V) \to E$ is the same as a continuous map if we endow E with the discrete topology. The restriction of h to the fibers of p is constant and hence by the universal property of the quotient topology there exists a unique continuous map $g\colon V \to E$ such that $g \circ p = h$. In particular, we see that $p^\star$ is an isomorphism in htis case.

Example 1.9. Let $\varphi\colon A \to B$ be a ring homomorphism between commutative rings A and B. The ring homomorphism φ induces a scheme morphism $f\colon Y \to X$ where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. The underlying map of topological spaces $f\colon Y \to X$ is defined by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for all primes $\mathfrak{q}$ of B, and the morphism of sheaves $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ on X is defined by $f^\flat_{\mathrm{D}(a)} = \varphi_a$ for all principal opens $\mathrm{D}(a) \subseteq X$, where $\varphi_a\colon A_a \to B_a$ is given by $\varphi_a($

Proposition 1.6. *Assume that $f\colon X \to Y$ is a homeomorphism. Let $x \in X$ and let $\mathcal{F}$ be a presheaf on X. Then $(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x$.*

Proof. Let $\pi_{\mathcal{F},x}\colon (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ be the map given by

$$\pi_{\mathcal{F},x}\left([V,s]_{f(x)}\right) = [f^{-1}(V), s]_x$$

where V is an open neighborhood of $f(x)$ and $s \in \mathcal{F}(f^{-1}(V))$. We need to show that this map is well-defined. Let (V', s') be another representative of the equivalence class $[V,s]_{f(x)}$. Then there exists an open neighborhood V'' of $f(x)$ such that $V'' \subseteq V \cap V'$ and $s|_{f^{-1}(V'')} = s'|_{f^{-1}(V'')}$. Since $f^{-1}(V'')$ is an open neighborhood of x such that $f^{-1}(V'') \subseteq f^{-1}(V) \cap f^{-1}(V')$, we have

$$\begin{aligned}\pi_{\mathcal{F},x}\left([V',s']_{f(x)}\right) &= [f^{-1}(V'), s']_x \\ &= [f^{-1}(V), s]_x.\end{aligned}$$

Thus this map is well-defined.

To show that $\pi_{\mathcal{F},x}$ is bijective, we simply describe the inverse map: Let $\pi^{-1}_{\mathcal{F},x} : \mathcal{F}_x \to (f_*\mathcal{F})_{f(x)}$ be the map given by $\pi^{-1}_{\mathcal{F},x}([U,s]_x) = [f(U), s]_{f(x)}$. Note that we need f to be an injective open map for this it to be well-defined (It is not enough that f is an open mapping. We also need $s \in \mathcal{F}(f^{-1}(f(U)))$, so we must have $f^{-1}(f(U)) = U$. However in general we only have $f^{-1}(f(U)) \supseteq U$). Clearly $\pi_{\mathcal{F},x}$ and $\pi^{-1}_{\mathcal{F},x}$ are inverse to each other. □

1.8.2 Inverse Image

Let $\mathcal{G}$ be a presheaf on Y. We define a presheaf $f^+\mathcal{G}$ on X, called the **pre-pullback** of $\mathcal{G}$ by f, whose value at an open set $U \subseteq X$ is

$$f^+\mathcal{G}(U) = \operatorname{colim}_{V \in \mathbf{N}(f(U))} \mathcal{G}(V)$$

and whose restriction maps are the ones induced by the restriction maps on $\mathcal{G}$. More concretely, we have

$$f^+\mathcal{G}(U) = \{(V,t) \mid \text{ is an open neighborhood of } f(U) \text{ in } Y \text{ and } t \in \mathcal{G}(V)\}/\sim,$$

where two pairs (V,t) and (V',t') are equivalent if there exists $V'' \subseteq Y$ open such that $f(U) \subseteq V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. The equivalence class corresponding to (V,t) will be denoted $[V,t]_{f(U)}$, or even more simply by $[t]_{f(U)}$ when it is clear from context that t is a section over V. Note that $[t]_{f(U)}$ is a section that lives over U whereas t is a section that lives over V. If $U' \subseteq X$ is open such that $U' \subseteq U$, then

$$[t]_{f(U)}|_{U'} = [t]_{f(U')}.$$

The sheafification of $f^+\mathcal{G}$ is denoted $f^{-1}\mathcal{G}$. We call $f^{-1}\mathcal{G}$ the **pullback** of $\mathcal{G}$ by f. The construction of $f^+\mathcal{G}$ and hence of $f^{-1}\mathcal{G}$ is functorial in $\mathcal{G}$; hence we obtain a functor $f^{-1}\colon \mathbf{Psh}\, Y \to \mathbf{Sh}\, X$.

Proposition 1.7. *Suppose $f\colon X \to Y$ is an open continuous map and let $\mathcal{G}$ be a presheaf on Y. Define a presheaf $f^*\mathcal{G}$ on X by setting $f^*\mathcal{G}(U) = \mathcal{G}(f(U))$ for all open $U \subseteq X$ and letting the restrictions maps on $f^*\mathcal{G}$ be the ones induced from $\mathcal{G}$. Then we have isomorphisms*

$$f^+\mathcal{G} \simeq f^*\mathcal{G} \simeq f^{-1}\mathcal{G},$$

all of which are natural in $\mathcal{G}$.

Proof. Let $[t]_{f(U)} = [V,t]_{f(U)}$ be a germ at $U \subseteq X$ where t is a section over $V \supseteq f(U)$. We define $\rho\colon f^+\mathcal{G} \to f^*\mathcal{G}$ by

$$\rho([t]_{f(U)}) = t|_{f(U)}.$$

Similarly, let t_0 be a section over $f(U)$. We define $\iota\colon f^*\mathcal{G} \to f^+\mathcal{G}$ by

$$\iota(t_0) = [t_0]_{f(U)}.$$

It is straightforward to check that ρ and ι are both morphisms of presheaves and that they are inverse to each other. Furthermore, $\rho_{\mathcal{G}} = \rho$ and $\iota_{\mathcal{G}} = \iota$ are clearly both natural in $\mathcal{G}$. ∎

If f is the inclusion of a subspace X of Y, we also write $\mathcal{G}|_X$ instead of $f^{-1}\mathcal{G}$. Moreover, if $\mathcal{G}$ is a sheaf, then $f^+\mathcal{G}$ is a sheaf and hence $f^+\mathcal{G} = f^{-1}\mathcal{G}$. In particular, if f is the inclusion of an open subspace $U = X$ of Y, then for every sheaf $\mathcal{G}$ on Y and $U' \subseteq U$ open, we have $\mathcal{G}|_U(U') = \mathcal{G}(U')$

Proposition 1.8. *Let $x \in X$ and let $\mathcal{G}$ be a presheaf on Y. Then $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$.*

Proof. It suffices to show that $f^+\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ by Proposition (1.4). Define $\lambda_{\mathcal{G},x}\colon (f^+\mathcal{G})_x \to \mathcal{G}_{f(x)}$ as follows: let $[[t]_{f(U)}]_x = [U,[V,t]_{f(U)}]_x$ be an element in $(f^+\mathcal{G})_x$. We set

$$\lambda_{\mathcal{G},x}([[t]_{f(U)}]_x) = [t]_{f(x)}.$$

It is straightforward to check that $\lambda_{\mathcal{G},x}$ is well-defined. The inverse map is defined by

$$\lambda^{-1}_{\mathcal{G},x}([t]_{f(x)}) = [[t]_{f(f^{-1}(U))}]_x.$$

∎

Example 1.10. Let $\phi\colon A \to B$ be a ring homomorphism and let M be an A-module. Let $X = \operatorname{Spec} A$, let $Y = \operatorname{Spec} B$, and let $f =^{\mathrm{a}} \phi$. Recall that the

1.8.3 Inverse-Direct Image Adjointness

Let $\mathcal{F}$ be a presheaf on X and let $\mathcal{G}$ be a preasheaf on Y. Given a morphism $\varphi\colon f^+\mathcal{G} \to \mathcal{F}$ of presheaves on X, we define a morphism $\varphi^\flat\colon \mathcal{G} \to f_*\mathcal{F}$ of presheaves on Y as follows: let $V \subseteq Y$ be open and let t be a section in $\mathcal{G}$ over V. We set

$$(\varphi^\flat)t = \varphi([t]_{ff^{-1}V}), \tag{7}$$

where we denoted $ff^{-1}V = f(f^{-1}(V))$ as well as $(\varphi^\flat)t = (\varphi^\flat_V)(t)$ and $\varphi([t]_{ff^{-1}V}) = \varphi_{f^{-1}V}([t]_{ff^{-1}V})$ in order to suppress notation. This gives rise to a map

$$(-)^\flat\colon \operatorname{Hom}_{\mathbf{Psh}\, X}(f^+\mathcal{G},\mathcal{F}) \to \operatorname{Hom}_{\mathbf{Psh}\, Y}(\mathcal{G}, f_*\mathcal{F}).$$

The idea behind our notation is that one thinks of (7) as applying the "associative law" by pushing the parenthesis forward (which matches up with the idea that $(-)^\flat$ takes the morphism φ on X and pushes it forward to a morphism $\varphi^\flat$ on Y). Applying the associative law in this case has the effect of removing $\flat$ in the superscript and introducing $ff^{-1}V$ in the superscript. f We now want to define an inverse to $(-)^\flat$ which we will denote by

$$(-)^\#\colon \mathrm{Hom}_{\mathbf{Psh}\,Y}(\mathcal{G}, f_*\mathcal{F}) \to \mathrm{Hom}_{\mathbf{Psh}\,X}(f^+\mathcal{G}, \mathcal{F}).$$

Given a morphism $\psi\colon \mathcal{G} \to f_*\mathcal{F}$ of presheaves on Y, we define a morphism $\psi^\#\colon f^+\mathcal{G} \to \mathcal{F}$ of presheaves on X as follows: let $U \subseteq X$ and $V \subseteq Y$ be open such that $f(U) \subseteq V \subseteq Y$ and let t be a section in $\mathcal{G}$ over V. We set

$$\psi^\#([t]_{fU}) = (\psi t)|_U \tag{8}$$

where we denoted $\psi t = \psi(t)$ as well as $fU = f(U)$ and $\psi^\# = \psi^\#_U$ in order to suppress notation. Note that ψt lives in $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$, so restricting this section to $U \subseteq f^{-1}(V)$ is meaningful, however keep in mind that we can't bring $|_U$ inside the parenthesis since ψ and t and are associative with Y (not X). Again, the idea behind here is that one thinks of (**??**) as applying the "associative law" by pulling the parenthesis back (which matches up with the idea that $(-)^\#$ takes the morphism ψ on Y and pulls it back to a morphism $\psi^\#$ on X). The associative law in this case has the effect of removing # in the superscript and replacing the ${}_{fU}$ with $|_U$. One of the main benefits that we get with our notation is that it is very easy to see that $(-)^\flat$ and $(-)^\#$ are inverse to each other. For instance, given U, V, φ, ψ, and t as above, we have

$$\begin{aligned}
(\varphi^\flat)^\#([t]_{fU}) &= ((\varphi^\flat)t)|_U \\
&= (\varphi([t]_{ff^{-1}V}))|_U \\
&= \varphi([t]_{ff^{-1}V})|_U) \\
&= \varphi([t]_{fU}),
\end{aligned}$$

where the last part follows from the fact that φ is a morphism of presheaves on X (so we could bring the $|_U$ inside the parenthesis) and $[t]_{ff^{-1}V}|_U = [t]_{fU}$ since $f^{-1}V \supseteq U$. Similarly, we have

$$\begin{aligned}
((\psi^\#)^\flat)t &= \psi^\#([t]_{ff^{-1}V}) \\
&= (\psi t)|_{f^{-1}V} \\
&= \psi t,
\end{aligned}$$

where the last part follows from the fact that ψt lives over $f^{-1}V$. It is straightforward to check that $(-)^\flat$ and $(-)^\#$ are natural in $\mathcal{F}$ and $\mathcal{G}$. We therefore have

Proposition 1.9. *Let $\mathcal{F}$ be a presheaf on X and let $\mathcal{G}$ be a presheaf on Y. Then there is a bijection*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Psh}(X)}(f^+\mathcal{G}, \mathcal{F}) & \longleftrightarrow & \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\
\varphi & \to & \varphi^\flat \\
\psi^\# & \leftarrow & \psi
\end{array}$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Moreover, if $\mathcal{F}$ is a sheaf, then this restricts to a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \quad\longleftrightarrow\quad \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. In particular, the pullback functor $f^{-1}\colon \mathbf{Psh}\,Y \to \mathbf{Sh}\,X$ is left adjoint to the pushforward functor $f_*\colon \mathbf{Sh}\,X \to \mathbf{Psh}\,Y$, and hence f^{-1} preserves colimits while f_* preserves limits.

Remark 3. We will almost never use the concrete description of $f^{-1}\mathcal{G}$ in the sequel. Very often we are given f, $\mathcal{F}$, and $\mathcal{G}$, and a morphism of sheaves $f^\flat\colon \mathcal{G} \to f_*\mathcal{F}$. Then usually it is sufficient to understand for each $x \in X$ the map

$$f^\#_x\colon \mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x \cong \mathcal{F}_x,$$

induced by $f^\#\colon f^{-1}\mathcal{G} \to \mathcal{F}$ on stalks.

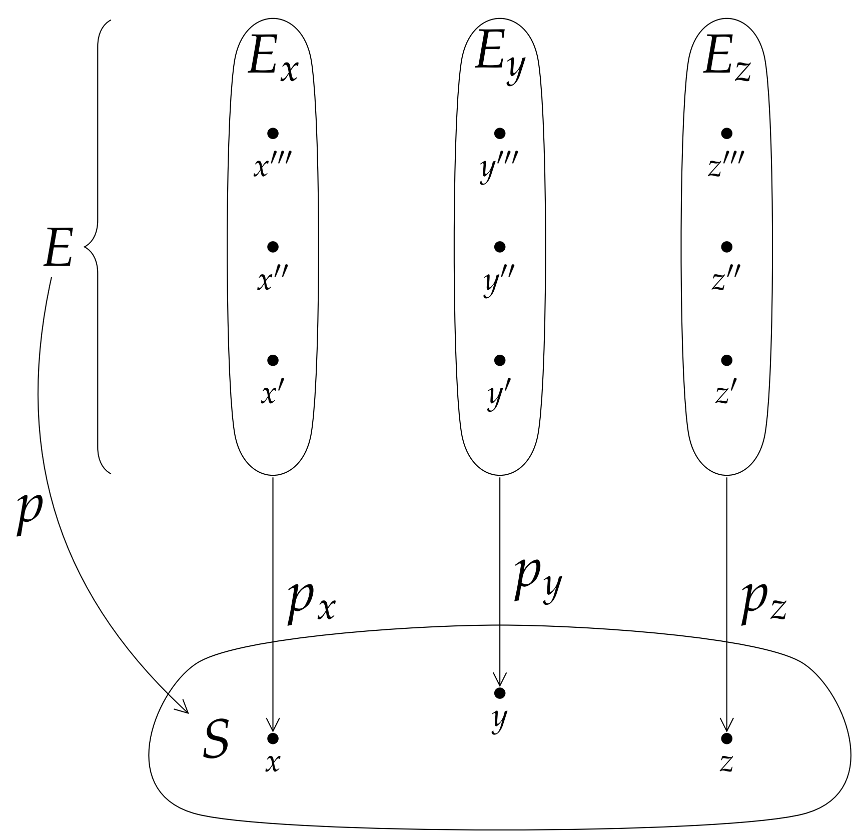
1.9 Sheaves and Etale Spaces

1.9.1 Bundles

Definition 1.7. The **slice category** $\mathbf{C}/c$ of a category $\mathbf{C}$ over an object $c \in \mathbf{C}$ is the category whose objects are morphisms $f : d \to c$ and whose morphisms from $f : d \to c$ to $f' : d' \to c$ are the morphisms $g : d \to d'$ such that $f' \circ g = f$:

<u>**Objects**</u>	<u>**Morphisms**</u>
$d \xrightarrow{f} c$	$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$
object (d, f)	morphism g

Example 1.11. Let S be a set. An object $(E, p) \in \mathbf{Set}/S$ can be pictured like this:



Let's take a moment to reflect on this image, because it will serve as a nice visualization tool for other categories. For each element $s \in S$, there is an associated set E_s, which is just the inverse image of s under p, i.e. $p^{-1}(s) = E_s$. E_s is called the **fiber** of p over s. Notice that for any distinct $s, s' \in S$, $E_s \cap E_{s'} = \emptyset$, and that $\bigcup_s E_s = E$. We also have functions p_s, which is just the restriction of p to E_s. The commutativity condition for morphisms in the slice category tells us that a morphism $f : (E, p) \to (E', p')$ satisfies $f(E_s) \subseteq E'_s$ for all $s \in S$. The whole structure is called a **bundle** of sets over the **base space** S, with E being called the **total space** and p being called the **projection**.

1.9.2 Etale Spaces

Definition 1.8. Let E and X be topological spaces. A **local homeomorphism** is a continuous map $\pi : E \to X$ with the additional property that for each point $e \in E$ there exists an open neighborhood U_e in E such that $\pi(U_e)$ is open in X, and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to \pi(U_e)$.

Intuitively, a local homeomorphism preserves "local structure". For example, B is locally compact if and only if $\pi(B)$ is.

Proposition 1.10. *Let $\pi : E \to X$ be a local homeomorphism, then π is an open map.*

Proof. Let U be open in E, we need to show that $\pi(U)$ is open in X. For each $e \in U$, choose an open neighborhood U_e of e such that $\pi(U_e)$ is open in X and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to p(U_e)$. Then $U \cap U_e$ is open in U_e, and since homeomorphisms are open maps, $\pi|_{U_e}(U \cap U_e)$ is open in $\pi(U_e)$ in the subspace topology. Since $\pi(U_e)$ is open in X, $\pi(U \cap U_e)$ is open in X too. Finally, since

$$\bigcup_{e \in U} \pi(U \cap U_e) = \pi(U),$$

$\pi(U)$ is open in X.

□

Example 1.12. If X is a topological space and Y is a discrete space, then the projection $Y \times X \to X$ is a local homeomorphism. On the other hand, the projection map $\mathbb{R} \times X \to X$ is never a local homeomorphism, because no product neighborhood is projected homeomorphically into X. For much the same reason, a nontrivial vector bundle is never a locally homeomorphism either.

Definition 1.9. An **etale space** over X is an object $(E, \pi) \in \mathbf{Top}/X$ such that π is a local homeomorphism. We denote by $\mathbf{Etale}(X)$ to be the full subcategory of $\mathbf{Top}/X$ whose objects are etale spaces over X and whose morphisms being the same as in $\mathbf{Top}/X$.

1.9.3 An equivalence of categories

We start with the main theorem:

Theorem 1.2. *For any topological space X there is a pair of adjoint functors*

$$\mathbf{Top}/X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set}^{O(X)^{op}}$$

where Γ assigns to each object in $(E, \pi) \in \mathbf{Top}/X$, the sheaf of all sections $\mathcal{F}_\pi$ of π, while its left adjoint Λ assigns to each preasheaf $\mathcal{F}$, the etale space $(E_\mathcal{F}, \pi_\mathcal{F})$. There are natural transformations

$$\eta_\mathcal{F} : \mathcal{F} \to \Gamma\Lambda\mathcal{F} \qquad \epsilon_E : \Lambda\Gamma E \to E$$

for a presheaf $\mathcal{F}$ and an object $(E, \pi) \in \mathbf{Top}/X$, which are unit and counit making Λ a left adjoint for Γ. If $\mathcal{F}$ is a sheaf, $\eta_\mathcal{F}$ is an isomorphism, while if (E, π) is etale, ϵ_E is an isomorphism.

1.9.4 From $\mathbf{Top}/X$ to $\mathbf{Sh}(X)$

Given an object $(E, \pi) \in \mathbf{Top}/X$, we can associate a sheaf $\mathcal{F}_\pi$ as follows: for all open subsets U of X, we define

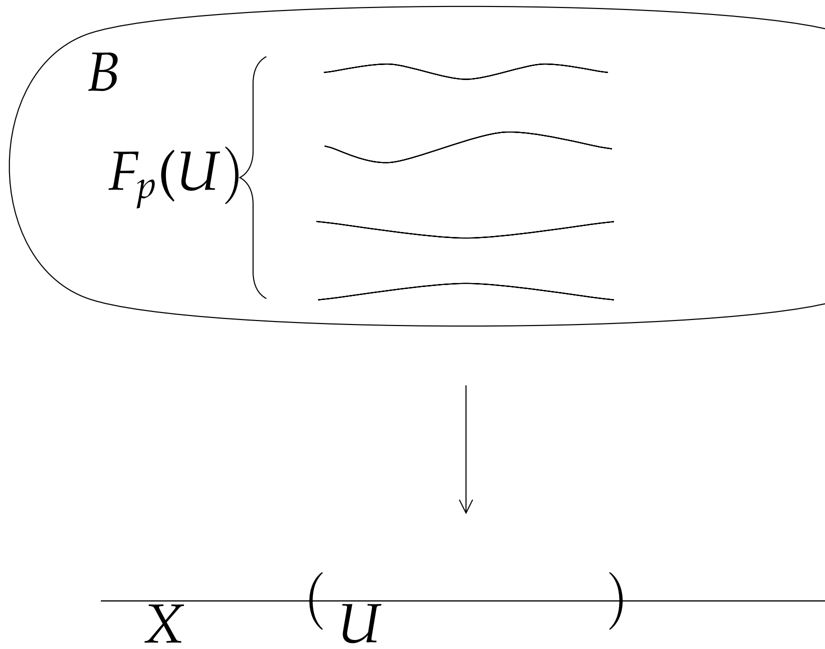
$$\mathcal{F}_\pi(U) := \{s : U \to E \mid s \text{ is continuous and } \pi|_U \circ s = \mathrm{id}_U\}.$$

For all inclusions of open sets $U \subseteq V$, we use the obvious restriction maps: if $s \in \mathcal{F}_\pi(V)$ then $s|_U \in \mathcal{F}_\pi(U)$. We claim that $\mathcal{F}_\pi$ is a sheaf (and not just a presheaf).

Indeed, let $\{U_i\}_{i \in I}$ be an open covering of an open subset U of X, and let $s_i \in \mathcal{F}_\pi(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all $i, j \in I$. We can construct an $s \in \mathcal{F}_\pi(U)$ such that $s|_{U_i} = s_i$ as follows: if $x \in U$, choose some U_i that has $x \in U_i$, and set $s(x) = s_i(x)$. We need to check that this is well-defined (i.e. independent of the choice of neighborhood of x). Suppose $x \in U_j$ for some $j \neq i$. Then because $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, we have $s(x) = s_j(x) = s_i(x)$. Thus, this construction is well-defined. Moreover, s is continous since if V is an open subset of E, then

$$s^{-1}(V) = \bigcup_{i \in I} s_i^{-1}(V)$$

is open in X. Finally, uniqueness of s is guaranteed since $\mathcal{F}_\pi$ is a presheaf of functions. We call $\mathcal{F}_\pi$ the **sheaf of sections of** π.



Let $f : (E, \pi) \to (E', \pi')$ be a morphism in $\mathbf{Top}/X$. Then for each open subset U of X, we define $f_U : \mathcal{F}_\pi(U) \to \mathcal{F}_{\pi'}(U)$ to be the function that maps a section $s \in \mathcal{F}_\pi(U)$ to $f \circ s$. The maps f_U are the components of a natural transformation from $\mathcal{F}_\pi \to \mathcal{F}_{\pi'}$. Thus, we have constructed a functor $\Gamma : \mathbf{Top}/X \to \mathbf{Sh}(X)$.

1.9.5 From $\mathbf{Psh}(X)$ to $\mathbf{Etale}(X)$

Let $\mathcal{F}$ be a presheaf on X. Define

$$E_\mathcal{F} := \coprod_{x \in X} \mathcal{F}_x = \bigcup_{x \in X} \{(x, s_0) \mid s_0 \in \mathcal{F}_x\}.$$

and let $\pi_\mathcal{F} : E_\mathcal{F} \to X$ be the obvious projection map (i.e. $(x, s) \mapsto x$). For each open subset U of X and section $s \in \mathcal{F}(U)$, let $[U, s] = \{(x, s_x) \mid x \in U\}$. Let τ be the topology on $E_\mathcal{F}$ with the collection of all $[U, s]$ as a subbasis. We claim that the collection of all $[U, s]$ is actually a basis for this topology. Indeed, let $[U, s], [V, t] \in \mathcal{B}$ and suppose $(x_0, s_{x_0}) \in [U, s] \cap [V, t]$. Then $x_0 \in U \cap V$ and $s_{x_0} = t_{x_0}$. This implies that there exists a neighborhood U_0 of x such that $U_0 \subseteq U \cap V$ and $s|_{U_0} = t|_{U_0}$. Hence $s_x = t_x$ for all $x \in U_0$. In particular, $[U_0, s|_{U_0}] \subseteq [U, s] \cap [V, t]$.

Finally, we want to show that $\pi_\mathcal{F} : E_\mathcal{F} \to X$ a local homeomorphism with respect to this topology. To see why, note that $\pi_\mathcal{F}$ maps basis elements to basis elements (i.e. $[U, s] \mapsto U$). Thus, it must be an open mapping. Also, if $(x, s_0) \in E_\mathcal{F}$, then after choosing a representative of s_0, say (U, s), we see that $(x, s_0) \in [U, s]$ and $\pi|_{[U,s]} : [U, s] \to U$ is a homeomorphism. Indeed, $\pi|_{[U,s]}$ is an open mapping and a bijection, hence its inverse must be continuous.

1.9.6 co-unit

Let $p : B \to X$ be any local homeomorphism, F_p its sheaf of sections, and $p_{F_p} : B_{F_p} \to X$ the associated sheaf of germs. Define a map $k : B \to B_{F_p}$ as follow: If $b \in B$, there exist a local section s of p through b, defined on an open set V, i.e. $b \in s(V)$ (we proved this earlier). Let $k(b) = (f(b), [s]_{f(b)})$ be the germ of s at $f(b)$. The definition of $k(b)$ does not depend on which section through b is chosen (we proved this earlier too). This gives us the following commutative diagram:

$$\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "p"'] & & B_{F_p} \arrow[dl, "p_{F_p}"] \\ & X & \end{tikzcd}$$

And k is a $\mathbf{Etale}(X)$-arrow from p to p_{F_p}, in fact, it is an iso.

1.9.7 unit

Define $\tau_U : F(U) \to F_{p_F}(U)$ by putting, for $s \in F(U)$, $\tau_U(s) = s_U$, where $s_U : U \to A_F$ is defined by putting $s_U(x) = (x, [s]_x)$ for all $x \in U$...

2 Ringed Spaces

2.1 Definition of a Ringed Space and a Locally Ringed Space

Throughout the rest of this article, let R be a commutative ring and let $\alpha \in \widehat{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty, \omega\}$. Ringed spaces formalize the idea of giving a geometric object by specifying its underlying topological space and the "functions" on all open subsets of this space.

Definition 2.1.

1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. If $R = \mathbb{Z}$, then we simplify our notation and write "ringed space" instead of "$\mathbb{Z}$-ringed space".

2. A **locally** R**-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_{X,x}$ (or simply $\mathfrak{m}_x$ if X is understood from context) the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa_{X,x} := \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ (or simply κ_x if X is understood from context) its residue field. If $s \in \mathcal{O}_X(U)$, then its image in κ_x is denoted $s(x)$. Note that if U is a fixed open subset of X, then it is not necessarily true that every element in κ_x can be expressed as $s(x)$ for some $s \in \mathcal{O}_X(U)$. On the other hand, it is the case that every element in κ_x as $t(x)$ for some $t \in \mathcal{O}_X(V)$ for some open neighborhood V of x (perhaps V needs to be smaller than U).

Usually we will denote a (locally) R-ringed space $(X, \mathcal{O}_X)$ simply by X. Also if we write "let X be a (locally) R-ringed space", then it will be understood that its structure sheaf is denoted $\mathcal{O}_X$ (unless otherwise specified of course). Sometimes we may write "let $(X, \mathcal{O})$ be a (locally) R-ringed space", and in this case the structure sheaf of X is denoted $\mathcal{O}$.

Example 2.1. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by $\mathcal{C}_X^\alpha$ the sheaf of C^α functions: For all open subsets U of X, we have

$$\mathcal{C}_X^\alpha(U) = \{f : U \to \mathbb{R} \mid f \text{ is } C^\alpha\}.$$

Then $\mathcal{C}_X^\alpha$ is a sheaf of $\mathbb{R}$-algebras. The same argument as for sheaves of continuous functions yields the following observation: For all $x \in X$ the stalk $C_{X,x}^\alpha$ is a local ring. In particular $(X, \mathcal{C}_X^\alpha)$ is a locally $\mathbb{R}$-ringed space.

Another example comes from Algebraic Geometry:

Example 2.2. Let k be an algebraically closed field and let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. The space X is equipped with the Zariski topology. Recall that a function $\varphi : U \to k$ from an open subset U of X to the field k is called **regular** at the point $x_0 \in X$ if there exists an open neighborhood U_0 of x_0 such that $U_0 \subseteq U$ and there are polynomials $f, g \in k[T_1, \dots, T_n]$ with $g(x) \neq 0$ and $\varphi(x) = \frac{f(x)}{g(x)}$ for all $x \in X$. With this in mind, we define the structure sheaf $\mathcal{O}_X$ of X as follows: for all open subsets U of X, we define

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

Proposition 2.1. *Let $X = (X, \mathcal{O})$ be a locally R-ringed space and let $u \in \mathcal{O}(X)$ such that $u_x \notin \mathfrak{m}_x$ for each $x \in X$. Then u is a unit in $\mathcal{O}(X)$.*

Proof. First note that $u_x \notin \mathfrak{m}_x$ is equivalent to saying u_x is a unit in $\mathcal{O}_x$ since $\mathcal{O}_x$ is a local ring. Therefore if $u_x \notin \mathfrak{m}_x$ for all $x \in X$, then u is locally a unit, meaning there exists an open covering $X = \bigcup_{i \in I} U_i$ such that $u|_{U_i}$ is a unit in $\mathcal{O}(U_i)$. Denote the inverse of $u|_{U_i}$ in $\mathcal{O}(U_i)$ by $v_i \in \mathcal{O}(U_i)$, so $(u|_{U_i})v_i = 1$ in $\mathcal{O}(U_i)$. Since the restriction maps are algebra homomorphisms we have

$$\begin{aligned}
(u|_{U_{ij}})(v_i|_{U_{ij}}) &= ((u|_{U_i})v_i)|_{U_{ij}} \\
&= 1|_{U_{ij}} \\
&= ((u|_{U_j})v_j)|_{U_{ij}} \\
&= (u|_{U_{ij}})(v_j|_{U_{ij}})
\end{aligned}$$

for all i, j. Since inverses are unique, it follows that $v_i|_{U_{ij}} = v_j|_{U_{ij}}$ for all i, j. Thus there exists a unique $v \in \mathcal{O}(X)$ such that $v|_{U_i} = v_i$ for all i. But then locally we have $uv - 1 = 0$, so the sheaf condintion again implies $uv - 1 = 0$. Thus u is a unit. $\square$

2.2 Morphisms of (Locally) Ringed Spaces

Definition 2.2. Let X and Y be R-ringed spaces.

1. A **morphism of R-ringed spaces** $X \to Y$ is a pair $(f, f^\flat)$, where $f\colon X \to Y$ is a continuous map of the underlying topological spaces and where $f^\flat\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of R-algebras on Y. The datum of $f^\flat$ is equivalent to the datum of a homomorphism of sheaves of R-algebras $f^\#\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X by Proposition (1.9). In particular, usually we simply write f instead of $(f, f^\#)$ or $(f, f^\flat)$.

2. Suppose X and Y are locally ringed spaces. For each $x \in X$, we obtain a homomorphism of R-algebras $f_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ which is defined by
$$f_x([t]_{f(x)}) = [f^\flat(t)]_x$$
for all $[t]_{f(x)} \in \mathcal{O}_{Y,f(x)}$. We say f is a **morphism of locally R-ringed spaces** if each f_x is **local**, meaning
$$f_x(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x.$$
In particular, this implies f_x induces a nonzero homomorphism of residue fields $\iota_x\colon \kappa_{Y,f(x)} \to \kappa_{X,x}$ where
$$\iota_x(t(f(x))) = f^\flat(t)(x)$$
where $t(f(x)) \in \kappa_{Y,f(x)}$. The map ι_x is necessarily an injective map: it realizes $\kappa_{Y,f(x)}$ as a field extension of $\kappa_{X,x}$. (**??**).

Notice that if f_x wasn't local, then $\iota_x\colon \kappa_Y(f(x)) \to \kappa_X(x)$ would just be the zero map. Thus the local condintion on f_x allows to identify $\kappa_Y(f(x))$ with the subfield $\iota_x\kappa_Y(f(x))$ of $\kappa_X(x)$. Recall that if $t \in \mathcal{O}_Y(V)$ is a section, then we can view it as a function whose value at $y \in V$ is given by $t(y) \in \kappa_Y(y)$ (if $y \neq y'$, then $\kappa_Y(y)$ may not be the same field as $\kappa_Y(y')$, so t really takes values in different fields), where $t(y)$ is the image of t under the composite map
$$\mathcal{O}_Y(V) \to \mathcal{O}_{Y,y} \to \kappa_Y(y).$$
Intuitively, the section $f^\flat(t) \in \mathcal{O}_X(f^{-1}(V))$ is equal to $f^*(t)$ as functions, meaning
$$f^\flat(t)(x) = t(f(x)) = f^*(t)(x) \tag{9}$$
for all $x \in f^{-1}(V)$. However we need to be careful, because $f^\flat(t)(x) \in \kappa_X(x)$ and $f^*(t)(x) \in \kappa_Y(f(x))$ belong to different fields in general. The key is that f_x being a local homomorphism allows us to identify $\kappa_Y(f(x))$ with the subfield $\iota_x\kappa_Y(f(x))$ of $\kappa_X(x)$. With this identification in mind (for each $x \in X$) we can make sense of the identity (**??**).

In general there exist locally ringed spaces and morphisms of ringed spaces between them that are not morphisms of *locally* ringed spaces. For spaces with functions of C^α functions such as the premanifolds defined below, we will see that every morphism of ringed spaces is automatically a morphism of locally ringed spaces.

Remark 4. The composition of morphisms of (locally) R-ringed spaces is defined in the obvious way using the compatibility of direct images with composition (i.e. $(g \circ f)_* = g_* \circ f_*$. We obtain the category of (locally) R-ringed spaces.

In general, $f^\flat$ (or $f^\#$) is an additional datum for a morphism. For instance it might happen that f is the identity but $f^\flat$ is not an isomorphism of sheaves. We will usually encounter the simpler case that the structure sheaf is a sheaf of functions on open subsets of X and that $f^\flat$ is given by composition with f. The following special case and its globalization is the main example.

Example 2.3. Let $X \subseteq V$ and $Y \subseteq W$ be open subsets of finite-dimensional $\mathbb{R}$-vector spaces V and W. Every C^α map $f : X \to Y$ defines by composition a morphism of locally $\mathbb{R}$-ringed spaces $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ by
$$\begin{array}{rcl} f^\flat_U : \mathcal{C}^\alpha_Y(U) & \longrightarrow & f_*(\mathcal{C}^\alpha_X)(U) = \mathcal{C}^\alpha_X(f^{-1}(U)) \\ t & \mapsto & t \circ f \end{array}$$
for $U \subseteq Y$ open.

The induced map on stalks $f_x\colon \mathcal{C}^\alpha_{Y,f(x)} \to \mathcal{C}^\alpha_{X,x}$ is then also given by composing an $\mathbb{R}$-valued C^α function t, defined in some neighborhood of $f(x)$, with f, which yields an $\mathbb{R}$-valued C^α function $t \circ f$ defined in some neighborhood of x. Conversely, let $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ be any morphism of $\mathbb{R}$-ringed spaces. We claim:

1. $(f, f^\flat)$ is automatically a morphism of *locally* $\mathbb{R}$-ringed spaces.

2. We have $f^\flat = f^\star$.

To show 1, let $x \in X$, set $\varphi = f_x$, set $B = \mathcal{C}^\alpha_{X,x}$, and set $A = \mathcal{C}^\alpha_{Y,f(x)}$. Then $\varphi\colon A \to B$ is a homomorphism of local $\mathbb{R}$-algebras such that $A/\mathfrak{m}_A = \mathbb{R}$ and $B/\mathfrak{m}_B = \mathbb{R}$. We claim that φ is automatically local, or equivalently that $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal of A. Indeed, φ induces an injective homomorphism of $\mathbb{R}$-algebras

$$A/\varphi^{-1}(\mathfrak{m}_B) \hookrightarrow B/\mathfrak{m}_B = \mathbb{R},$$

and as a homomorphism of $\mathbb{R}$-algebras, it is automatically surjective (indeed 1 maps to 1), hence $A/\varphi^{-1}(\mathfrak{m}_B) \cong \mathbb{R}$ is a field and hence $\varphi^{-1}(\mathfrak{m}_B)$ is the maximal ideal of A.

Let us show 2. Let V be an open set of Y and let $x \in f^{-1}(V)$. Consider the commutative diagram of $\mathbb{R}$-algebra homomorphisms

$$\begin{tikzcd}
\mathcal{C}^\alpha_Y(V) \arrow[rrr, "f^\flat_V"] \arrow[ddd] & & & \mathcal{C}^\alpha_X(f^{-1}(V)) \arrow[ddd] \\
& t \arrow[r, mapsto] \arrow[d, mapsto] & f^\flat_V(t) \arrow[d, mapsto] & \\
& {[V, t]_{f(x)}} \arrow[r, mapsto] & {[f^{-1}(V), f^\flat_V(t)]_x} & \\
\mathcal{C}^\alpha_{Y,f(x)} \arrow[rrr, "f_x"] \arrow[ddd] & & & \mathcal{C}^\alpha_{X,x} \arrow[ddd] \\
& {[V, t]_{f(x)}} \arrow[r, mapsto] \arrow[d, mapsto] & {[f^{-1}(V), f^\flat_V(t)]_x} \arrow[d, mapsto] & \\
& t(f(x)) & f^\flat_V(t)(x) & \\
\mathbb{R} & & & \mathbb{R}
\end{tikzcd}$$

The evaluation maps are surjective. Hence there exists a homomorphism of $\mathbb{R}$-algebras $\iota\colon \mathbb{R} \to \mathbb{R}$ making the lower rectangle commutative if and only if one has $f_x(\ker(\mathrm{ev}_{f(x)})) \subseteq \ker(\mathrm{ev}_x)$, but this latter condition is satisfied because f_x is local by 1. Moreover, as a homomorphism of $\mathbb{R}$-algebras, one must have $\iota = \mathrm{id}_{\mathbb{R}}$. Therefore we find $f^\flat_V(t)(x) = t(f(x)) = f^\star_V(t)(x)$, which shows 2.

Remark 5. A morphism $f : X \to Y$ of R-ringed spaces is an isomorphism in the category of R-ringed spaces if and only if f is a homeomorphism and $f_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism of R-algebras for all $x \in X$. Indeed, $(f, f^\flat)$ is an isomorphism if and only if f is a homeomorphism and $f^\flat$ is an isomorphism of sheaves of rings. We claim that if f is a homeomorphism, then $f^\flat$ is an isomorphism if and only if f_x is an isomorphism for all $x \in X$. To see this, note that since f is a homeomorphism, we have $f_x = \pi_{\mathcal{O}_X,x} \circ f^\flat_x$, where $\pi_{\mathcal{O}_X,x}$ is the isomorphism constructed in Proposition (1.6).

2.2.1 Open embedding

Let X be a locally R-ringed space and let $U \subseteq X$ be an open set. Then $(U, \mathcal{O}_{X|U})$ is a locally R-ringed space, where $\mathcal{O}_{X|U}$ is defined by

$$\mathcal{O}_{X|U}(U') = \mathcal{O}_X(U')$$

for all open subsets U' of U and where the restrictions maps are the ones incuded by $\mathcal{O}_X$. Such a locally ringed R-space is called an **open subspace** of X. There is an $\iota\colon U \to X$ of locally R-ringed spaces, where the continuous map $\iota\colon U \to X$ is the inclusion of the underlying topological spaces and where $\iota^\flat\colon \mathcal{O}_X \to \iota_*\mathcal{O}_{X|U}$ is given by the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ for all open subsets V of X. Thus

$$\iota^\flat_V(s) = s|_{U \cap V}$$

for all $s \in \mathcal{O}_X(V)$. Notice that $\iota^\#: \iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X|U}$ is the identity. In particular ι_x is the identity for all $x \in U$. Given any morphism $f: X \to Y$, we denote by $f|_U: U \to Y$ to be the composition $f \circ \iota$ of morphisms of locally R-ringed spaces.

Definition 2.3. Let $f: X \to Y$ and $i: Z \to X$ be morphisms of locally ringed R-spaces.

1. We say i is an **open embedding** if $i(Z)$ is an open subset of X and i induces an isomorphism $Z \cong i(Z)$ of locally ringed R-spaces.
2. We say f is a **local isomorphism** if there exists an open cover $\{U_i\}_{i\in I}$ of X such that $f|_{U_i}: U_i \to Y$ is an open embedding for all $i \in I$. In other words, $V_i := f(U_i)$ is an open subspace of Y and $f|_{U_i}: U_i \to V_i$ is an isomorphism of locally ringed R-spaces for each $i \in I$. Note that f is a local isomorphism if and only if f is a local homeomorphism and $f_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for all $x \in X$.

2.2.2 Closed Immersions

Let $i: Z \to X$ be a morphism of locally ringed spaces. We say that i is a **closed immersion** if

1. The map of topological spaces $i: Z \to X$ is a homeomorphism of Z onto a closed subset of X.
2. The morphism of sheaves $i^\flat: \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective (meaning for each $z \in Z$, the map $\iota_z: \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ is surjective, where $x = i(z)$).
3. The $\mathcal{O}_X$-module $\mathcal{I} = \ker(i^\flat)$ is locally generated by sections.

2.3 Gluing Ringed Spaces

Let X_1 and X_2 be locally R-ringed spaces, let $X_{1,2}$ be a nonempty open subset of X_1 and let $X_{2,1}$ be a nonempty open subset of X_2, and let $f: X_{1,2} \to X_{2,1}$ be an isomorphism of locally R-ringed spaces. We construct a locally R-ringed space X, obtained by gluing X_1 and X_2 using f as follows:

- The underlying set X is given by
$$X = X_1 \coprod X_2 / \sim$$
where $X_1 \coprod X_2$ is the disjoint union of X_1 and X_2 and where $\sim$ is the equivalence relation defined by $x \sim f(x)$ for all $x \in X_{1,2}$. We give X the structure of a topological space using the quotient topology with respect to $\sim$. Thus a set $U \subseteq X$ is open if and only if $U \cap U_1 \subseteq U_1$ and $U \cap U_2 \subseteq U_2$ are both open subsets of U_1 and U_2 respectively, where $U_1 = i_1(X_1)$ and $U_2 = i_2(X_2)$ with $i_1: X_1 \to X$ and $i_2: X_2 \to X$ being the obvious inclusion maps.
- We given X the structure of a locally R-ringed space by defining the structure sheaf $\mathcal{O}_X$ by
$$\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2), \text{ and } f^\flat(s_2)|_{U\cap X_{2,1}} = s_1|_{U \cap X_{1,2}})\}$$
for all open subsets U of X.

Let's go over specific examples of this construction:

Example 2.4. Let $X_1 = X_2 = \mathbb{A}^1$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $x \mapsto \frac{1}{x}$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $K = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$.
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X.

Example 2.5. Let X be the complex affine curve
$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = 1/x$, the equation of the curve becomes

$$y^2\widetilde{x}^4 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x},\widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x,y) \mapsto (\widetilde{x},\widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right)$$

$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x},\widetilde{y}) \mapsto (x,y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^n}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

2.4 $\mathcal{O}_X$-modules

Definition 2.4. Let X be a ringed space

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. We define the product presheaf $\mathcal{F} \times \mathcal{G}$ on X with respect to $\mathcal{F}$ and $\mathcal{G}$ by setting
$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$
for all open subsets U of X where the restriction maps are the products of the restriction maps for $\mathcal{F}$ and $\mathcal{G}$. Clearly this is a sheaf if $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

2. An $\mathcal{O}_X$**-module** is a sheaf $\mathcal{F}$ on X equipped with two morphisms of sheaves
$$\mathcal{F} \times \mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$
called addition and scalar-multiplication respectively, such that for each open subset U of X, addition and scalar-multiplication by $\mathcal{O}_X(U)$ gives $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module.

3. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say φ is an $\mathcal{O}_X$**-module homomorphism** if φ_U is an $\mathcal{O}_X(U)$-module homomorphism for each open subset U of X. The composition of two $\mathcal{O}_X$-module homomorphisms is again an $\mathcal{O}_X$-module homomorhism. We obtain a category of $\mathcal{O}_X$-modules and $\mathcal{O}_X$-module homomorphisms which we denote by $\mathbf{Mod}_{\mathcal{O}_X}$.

4. Assume that X is a locally ringed space. Let $x \in X$ and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Note that the $\mathcal{O}_X$-module structure on $\mathcal{F}$ induces an $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$. The **fiber** of $\mathcal{F}$ at x, denoted $\mathcal{F}(x)$, is the $\kappa(x)$-vector space
$$\mathcal{F}(x) := \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x).$$
If s is a section of $\mathcal{F}$ over an open neighborhood U of x, we denote by $s(x)$ the image of the germ $[s]_x \in \mathcal{F}_x$ in $\mathcal{F}(x)$.

3 Sheaves of Modules

Definition 3.1. Let $X = (X, \mathcal{O})$ be a ringed space and let $\mathcal{F}$ be an $\mathcal{O}$-module.

1. We define the **tensor algebra** of $\mathcal{F}$ to be the sheaf of noncommutative graded $\mathcal{O}$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}}(\mathcal{F}) = \bigoplus_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$, and for $n \geq 2$ we have $\mathrm{T}^n(\mathcal{F}) = \mathcal{F}^{\otimes n}$ where the tensor product is over $\mathcal{O}$. Thus $\mathrm{T}^n(\mathcal{F})(U)$ consists of all finite sums of the form

$$\sum_i u_{1,i} \otimes \cdots \otimes u_{i,n} = \sum_i \boldsymbol{u}_i$$

where $u_{1,i}, \ldots, u_{n,i} \in \mathcal{F}(U)$ for all i and where we used the notation $\boldsymbol{u}_i := u_{1,i} \otimes \cdots \otimes u_{i,n}$. If $\boldsymbol{u} \in \mathrm{T}^m(\mathcal{F})(U)$ and $\boldsymbol{v} \in \mathrm{T}^n(\mathcal{F})(U)$ where $\boldsymbol{u} = u_1 \otimes \cdots \otimes u_m$ and $\boldsymbol{v} = v_1 \otimes \cdots \otimes v_n$ are elementary tensors, then we set

$$\boldsymbol{uv} = u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n \in \mathrm{T}^{m+n}(\mathcal{F})(U).$$

This gives $\mathrm{T}(\mathcal{F})(U)$ the structure of a graded $\mathcal{O}(U)$-algebra.

2. We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal $\mathcal{I}$ of $\mathrm{T}(\mathcal{F})$ where $\mathcal{I}$ is generated by local sections $u \otimes u$ of $\mathrm{T}^2(\mathcal{F})$ where u is a local section of $\mathcal{F}$. Note that $\wedge(\mathcal{F})$ inherits the structure of graded $\mathcal{O}$-module from the grading on $\mathrm{T}(\mathcal{F})$. We denote the coset $\overline{\boldsymbol{u}}$ in $\wedge^n(\mathcal{F})(U)$ by

$$\overline{\boldsymbol{u}} = \wedge(\boldsymbol{u}) = u_1 \wedge \cdots \wedge u_n,$$

where in this notation we have $\wedge(\boldsymbol{u}) = 0$ if $u_i = u_j$ for some $i < j$. Notice that we have

$$\begin{aligned} 0 &= (u+v)(u+v) \\ &= u^2 + uv + vu + v^2 \\ &= uv + vu \end{aligned}$$

implies $uv = -vu$. In particular, $\wedge(\mathcal{F})$ is a graded-commutative $\mathcal{O}$-algebra.

3. We define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal $\mathcal{J}$ of $\mathrm{T}(\mathcal{F})$ where $\mathcal{J}$ is generated by local sections $u \otimes v - v \otimes u$ of $\mathrm{T}^2(\mathcal{F})$ where u, v are local sections of $\mathcal{F}$. Note that $\mathrm{Sym}(\mathcal{F})$ inherits the structure of graded $\mathcal{O}$-module from the grading on $\mathrm{T}(\mathcal{F})$. We denote the coset $\overline{\boldsymbol{u}}$ in $\mathrm{Sym}^n(\mathcal{F})(U)$ by

$$\overline{\boldsymbol{u}} = u_1 \cdots u_n$$

where in this notation we $uv = vu$. In particular, $\mathrm{Sym}(\mathcal{F})$ is a commutative $\mathcal{O}$-algebra.

Lemma 3.1. *With the notation above, $\wedge^n \mathcal{F}$ and $\mathrm{Sym}^n(\mathcal{F})$ are sheaves.*

Proof. It suffices to show that $\wedge^n \mathcal{F}$ is a sheaf since a similar works for $\mathrm{Sym}^n(\mathcal{F})$. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of U and let $\omega_\lambda \in \wedge^n \mathcal{F}(U_\lambda)$ where $\omega_\lambda|_{U_{\lambda\mu}} = \omega_\mu|_{U_{\lambda\mu}}$ for all λ, μ. To show that $\wedge^n \mathcal{F}$ is a sheaf, we need to find a unique $\omega \in \wedge^n \mathcal{F}(U)$ such that $\omega|_{U_\lambda} = \omega_\lambda$ for all λ. Note that the restriction maps $\wedge^n \mathcal{F}(U) \to \wedge^n \mathcal{F}(U_\lambda)$ are all injective. Thus we only need to show existence of ω since uniqueness is already satisfied. If $\omega_\lambda = 0$ for each λ, then $\omega = 0$ is the unique element in $\wedge^n \mathcal{F}(U)$ such that $\omega|_{U_\lambda} = \omega_\lambda$. Thus assume that $\omega_\lambda \neq 0$ for some λ. Write

$$\omega_\lambda = \sum_{i=1}^{k_\lambda} u_{1,\lambda,i} \wedge \cdots \wedge u_{n,\lambda,i} = \sum_{i=1}^{k_\lambda} \boldsymbol{u}_{\lambda,i},$$

Now if $U_{\lambda\mu} \neq \emptyset$, then $\omega_\lambda|_{U_{\lambda\mu}} = \omega_\mu|_{U_{\lambda\mu}}$ implies $k_\lambda = k = k_\mu$ and (perhaps after reordering) implies $\boldsymbol{u}_{\lambda,i} = \boldsymbol{u}_{\mu,i}$ which further implies (perhaps after reordering) $u_{j,\lambda,i} = u_{j,\mu,i}$. In other words, for each i, j we have a compatible sequence $(u_{j,\lambda,i})_{\lambda \in \Lambda}$ where $u_{j,\lambda,i} \in \mathcal{F}(U_\lambda)$. Thus there exists a unique $u_{j,i} \in \mathcal{F}(U)$ such that $u_{j,i}|_{U_\lambda} = u_{j,\lambda,i}$ for all λ. Then clearly $\omega = \sum \boldsymbol{u}_i = \sum u_{1,i} \wedge \cdots \wedge u_{k,i}$ is the unique element such that $\omega|_{U_\lambda} = \omega_\lambda$. $\square$

4 Sheaf Cohomology

Throughout this section, let $X = (X, \mathcal{O}_X)$ be a ringed space. We set $R = \mathcal{O}_X(X)$. Note that if K is a commutative ring such that X is K-ringed, then R is a K-algebra. Consider the functor $\Gamma(X, -)$ from the category of $\mathcal{O}_X$-

modules to the category of R-modules, given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X)$. Note that $\Gamma(X, -)$ is left exact. This means that if

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$$

is an exact sequence of $\mathcal{O}_X$-modules, then

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3)$$

is an exact sequence of R-modules.[[

Part II
Differential Geometry

5 Euclidean Spaces

The Euclidean space $\mathbb{R}^n$ is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. A good understanding of $\mathbb{R}^n$ is essential in generalizing differential and integral calculus to a manifold.

Definition 5.1. Let k be a nonnegative integer and U be an open subset in $\mathbb{R}^n$. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_j}} f$$

of all orders $j \leq k$ exist and are continuous at p. The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k at p for all $k \geq 0$. A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its component functions $f_1, \dots, f_n$ are C^k at p. We say that $f : U \to \mathbb{R}^m$ is C^k on U if it is C^k at every point in U. A similar defintion holds for a C^∞ function on an open set U. We treat the terms "C^∞" and "smooth" as synonymous.

Example 5.1.

1. A C^0 function on U is a continuous function on U.

2. The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

   $$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

   Thus the function f is C^0 but not C^1 at $x = 0$. On the other hand, f is C^1 on the open subset $\{x \in \mathbb{R} \mid x \neq 0\} \subseteq \mathbb{R}$. Now let $g : \mathbb{R} \to \mathbb{R}$ be defined by

   $$\int_0^x f(t)dt = \int_0^x t^{1/3} dt = \frac{3}{4}x^{4/3}.$$

   Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is smooth and even bijective with inverse f^{-1} given by $f^{-1}(x) = x^{1/3}$, but f^{-1} is not smooth, as shown above.

5. Continuity of a function can often be seen by inspection, but the smoothness of a function always requires a formula. The graph of $y = x^{5/3}$ looks perfectly smooth, but it is in fact not smooth at $x = 0$, since its second derivative $y'' = (10/9)x^{-1/3}$ is not defined there.

6. Consider the norm function from $\mathbb{R}^n$ to $\mathbb{R}$, given by sending $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ to $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \in \mathbb{R}$. We will do this in detail. First we claim that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial and $k \geq 1$. We prove this by induction on k. The base case is trivial:

   $$\partial_{x_n}(\|x\|) = \frac{x_n}{\|x\|}$$

Now suppose that $\partial_{x_n}^k(\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$ where $f(x)$ is a polynomial. Then

$$\begin{aligned}\partial_{x_n}^{k+1}(\|x\|) &= \partial_{x_n}\left(\frac{f(x)}{\|x\|^{2k-1}}\right)\\ &= \frac{(\partial_{x_n}f)(x)}{\|x\|^{2k-1}} + \frac{(1-2n)x_n f(x)}{\|x\|^{2k+1}}\\ &= \frac{(\partial_{x_1}f)(x)\left(x_1^2+\cdots+x_n^2\right)+(1-2n)x_n f(x)}{\|x\|^{2n+1}}.\end{aligned}$$

This establishes our claim. Now given that $\partial_{x_n}^k(\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial, it is clear that $\partial_{x_1}^{k_1}\cdots\partial_{x_n}^{k_n}(\|x\|)$ has the form $g(x)/\|x\|^{2(k_1+\cdots+k_n)-1}$, where $g(x)$ is a polynomial (just use the same induction proof). Now since $\|x\|=0$ if and only if $x=\mathbf{0}$, we see that the norm function is smooth in $\mathbb{R}^n\backslash\{\mathbf{0}\}$.

Proposition 5.1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be smooth. Then $g\circ f=\mathbb{R}^n\to\mathbb{R}$ is smooth.*

Proof. We will only sketch the proof here. By the chain rule, we have

$$\partial_{x_n}(g\circ f)=(g'\circ f)\cdot\partial_{x_n}f$$

By the product rule we have

$$\partial_{x_n}^2(g\circ f)=(g''\circ f)\cdot(\partial_{x_n}f)^2+(g'\circ f)\cdot\partial_{x_n}^2 f$$

Similarly, we have

$$\partial_{x_n}^3(g\circ f)=(g'''\circ f)(\partial_{x_n}f)^3+3(g''\circ f)(\partial_{x_n}f)(\partial_{x_n}^2 f)+(g'\circ f)\partial_{x_n}^3 f.$$

More generally, we will have a pattern which involves stirling numbers. □

Definition 5.2. Let $p=(p_1,\ldots,p_n)$ be a point in $\mathbb{R}^n$. A **neighborhood** of p in $\mathbb{R}^n$ is an open set containing p. The function f is **real-analytic** at p if in some neighborhood of p it is equal to its Taylor series at p:

$$f(x)=f(p)+\sum_i\partial_{x_i}f(p)(x_i-p_i)+\frac{1}{2!}\sum_{i,j}\partial_{x_i}\partial_{x_j}f(p)(x_i-p_i)(x_j-p_j)+\cdots+\frac{1}{k!}\sum_{i_1,\ldots,i_k}\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}f(p)(x_{i_1}-p_{i_1})\cdots(x_{i_k}-p_{i_k})+\cdots$$

A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x)=\sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots$$

then term-by-term differentiation gives

$$f'(x)=\cos x=1-\frac{1}{2!}x^2-\frac{1}{4!}x^4-\cdots$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0.

Example 5.2. (A C^∞ function very flat at 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x)=\begin{cases}e^{-1/x} & \text{for } x>0\\ 0 & \text{for } x\le 0\end{cases}$$

Clearly $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(0)=0$. Also,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-1/x}\right)=e^{-1/x}\left(\sum_{i=1}^n(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}\right)$$

Where $L(n,i)$ are the Lah numbers. Both $e^{-1/x}$ and $\sum_{i=1}^n(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ are well defined for $x>0$ and $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-1/x}\right)\to 0$ as $x\to 0$ (since $e^{-1/x}$ approaches 0 much faster than $\sum_{i=1}^n(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ approaches ∞), so this function is clearly C^∞ on $\mathbb{R}$. On the other hand, the Taylor series of this function at the origin is identically zero in any neighborhood of the origin since $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(0)=0$ for all $n\ge 1$. Therefore $f(x)$ cannot be equal to its Taylor series and thus $f(x)$ is not real-analytic at 0.

5.1 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. We say that a subset S of $\mathbb{R}^n$ is **star-shaped** with respect to a point p in S if for every x in S, the line segment from p to x lies in S. The line segment from p to x is parametrized by $\gamma : [0,1] \to \mathbb{R}^n$ where $\gamma(t) = (1-t)p + tx$. S is star-shaped with respect to p if for every x in S, $(1-t)p + tx$ is in S for all $t \in (0,1)$.

Lemma 5.1. *(Taylor's theorem with remainder). Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p_1, \ldots, p_n)$ in U. Then there are functions $g_1(x), \ldots, g_n(x) \in C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n} (x_i - p_i) g_i(x) \qquad g_i(p) = \partial_{x_i} f(p)$$

Remark 6. The idea behind this proof is to differentiate $f(p + t(x-p))$ and then integerate it.

Proof. Since U is star-shaped with respect to p, for any $x \in U$ the line segment $p + t(x-p)$, $0 \le t \le 1$ lies in U. So $f(p + t(x-p))$ is defined for $0 \le t \le 1$. By the chain rule

$$\begin{aligned}
\frac{df}{dt}(p + t(x-p)) &= \frac{df}{dt}(p_1 + t(x_1 - p_1), \ldots, p_n + t(x_n - p_n)) \\
&= (\partial_{x_1} f)(p + t(x-p)) \partial_t (p_1 + t(x_1 - p_1)) + \cdots + (\partial_{x_n} f)(p + t(x-p)) \partial_t (p_n + t(x_n - p_n)) \\
&= \sum_{i=1}^{n} (x_i - p_i) \partial_{x_i} f(p + t(x-p)).
\end{aligned}$$

If we integrate both sides with respect to t from 0 to 1, we get

$$f(p + t(x-p))|_0^1 = \sum (x_i - p_i) \int_0^1 \partial_{x_i} f(p + t(x-p)) dt \tag{10}$$

Now let $g_i(x) = \int_0^1 \partial_{x_i} f(p + t(x-p)) dt$. □

Example 5.3. We want to apply this proof to the function $f(x)$ on $\mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Let $p = 0$ and let $g(x) = \int_0^1 \frac{df}{dx}(p + t(x-p)) dt$. Then

$$\begin{aligned}
g(x) &= \int_0^1 \frac{df}{dx}(tx) dt \\
&= \int_0^1 \frac{-e^{-1/tx}}{tx^2} dt \\
&= \frac{e^{-1/tx}}{x} |_0^1 \\
&= \frac{e^{-1/x}}{x}.
\end{aligned}$$

Thus,

$$f(x) = f(0) + x \left(\frac{e^{-1/x}}{x} \right).$$

5.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

In elementary calculus we normally represent a vector at a point p in $\mathbb{R}^3$ algebraically as a column of numbers

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

or geometrically as an arrow emanating from p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p. Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a Euclidean space, and so would not apply to the projective plane, for example, which does not sit inside an $\mathbb{R}^n$ in any natural way.

5.2.1 The Directional Derivative

Let $p = (p_1, \ldots, p_n)$ be a point with direction $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$. The line through the point p in the direction v can be parametized by $\ell := (\ell_1, \ldots, \ell_n) : \mathbb{R} \to \mathbb{R}^n$, where

$$\ell(t) := p + tv = (p_1 + tv_1, \ldots, p_i + tv_i, \ldots, p_n + tv_n) =: (\ell_1(t), \ldots, \ell_i(t), \ldots, \ell_n(t)).$$

Now let f be a C^∞ in a neighborhood of p in $\mathbb{R}^n$. The **directional derivative** of f in the direction of v at p is defined to be

$$\begin{aligned} D_v f &:= \lim_{t \to 0} \left(\frac{f(\ell(t)) - f(p)}{t} \right) \\ &= \partial_t f(\ell(t))|_{t=0} \\ &= \sum_{i=1}^n \partial_{x_i} f(\ell(0)) \cdot \partial_t \ell_i(0) \\ &= \sum_{i=1}^n v_i \partial_{x_i} f(p) \end{aligned}$$

In the notation $D_v f$, it is understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v_i \partial_{x_i}|_p$$

for the map that sends a function f to the number $D_v f$. To simplify the notation we often omit the subscript p if it is clear from the context.

5.2.2 Germs of Functions

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We introduce a relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. It is easy to check that this is an equivalence relation by showing it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(\mathbb{R}^n)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Remark 7. What happens if we weaken the relation a bit? Say $(f_1, U_1) \sim (f_2, U_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$. In this case, we no longer have an equivalence relation. The reason is because this relation is not transitive: Suppose $(f_1, U_1) \sim (f_2, U_2)$ and $(f_2, U_2) \sim (f_3, U_3)$. Then $f_1 = f_2$ on $U_1 \cap U_2$ and $f_2 = f_3$ on $U_2 \cap U_3$, but this merely implies that $f_1 = f_3$ on $U_1 \cap U_2 \cap U_3$.

Example 5.4. The functions

$$f(x) = \frac{1}{1-x}$$

with domain $\mathbb{R} \setminus \{1\}$ and

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

with domain the open interval $(-1, 1)$ have the same germ at any point p in the open interval $(-1, 1)$.

The addition and multiplication of functions induce corresponding operations on C_p^∞ making it into an $\mathbb{R}$-algebra. Indeed, let (f_1, U_1) and (f_2, U_2) be two representatives. Then multiplication is given by

$$(f_1, U_1) \cdot (f_2, U_2) = (f_1 f_2, U_1 \cap U_2).$$

We need to check that this is well-defined, so let (f_1', U_1') and (f_2', U_2') be two different represenatives respectively. Then

$$f_1 = f_1' \text{ on } W_1 \subset U_1 \cap U_1' \text{ and } f_2 = f_2' \text{ on } W_2 \subset U_2 \cap U_2'$$

This implies

$$f_1 f_2 = f_1' f_2' \text{ on } W_1 \cap W_2 \subset U_1 \cap U_2,$$

and thus

$$(f_1 f_2, U_1 \cap U_2) \sim (f_1' f_2', U_1 \cap U_2)$$

and hence this is well-defined. Similarly, addition is given by

$$(f_1, U_1) + (f_2, U_2) = (f_1 + f_2, U_1 \cap U_2).$$

Example 5.5. This example requires some knowledge of Algebraic Geometry. Let X be an affine algebraic set over an algebraically closed field K, let $R = A(X)$ be its coordinate ring, let p be a point in X, and let $\mathfrak{m}$ be the maximal ideal in R given by the set of all $f \in R$ which vanish at p. There are two equivalent ways to define the local ring $O_{X,p}$ at p.

One way is to define $O_{X,p}$ to be the local ring $R_{\mathfrak{m}}$. Elements in $R_{\mathfrak{m}}$ are equivalence classes of elements of the form f/g, where $f, g \in R$ and $g \notin \mathfrak{m}$. We say f_1/g_1 is equivalent to f_2/g_2 if there is an $h \in R$ such that $h \notin \mathfrak{m}$ and $h(g_2 f_1 - g_1 f_2) = 0$.

The other way is to define $O_{X,p}$ to be the ring of all germs of polynomial functions defined on a neighborhood of p. A "polynomial function defined on a neighborhood of p" is of the form f/g where $f, g \in R$ and $g(p) \neq 0$. We can think of f/g here as being the germ $(f/g, D(g))$, where $D(g)$ is the set of all points such that $g \neq 0$. Two such polynomial functions f_1/g_1 (or germ $(f_1/g_1, D(g_1))$) and f_2/g_2 (or germ $(f_2/g_2, D(g_2))$) represent the same germ if they agree on some small neighborhood of p. A small open neighborhood of p in the Zariski topology is simply something of the form $D(h)$ where h does not vanish at p. Thus, we need $f_1/g_1 = f_2/g_2$ on $D(h) \cap D(g_1) \cap D(g_2)$. Another way of saying this is $g_1 g_2 h(f_1/g_1 - f_2/g_2) = 0$ as a function on X; this matches precisely the criterion for f_1/g_1 and f_2/g_2 to be equal in the local ring $R_{\mathfrak{m}}$.

5.2.3 Derivations at a Point

We claim that D_v gives a map from C_p^∞ to $\mathbb{R}$. Indeed we just need to check that it is well-defined: suppose $(f, U) \sim (g, V)$. Then $f|_W = g|_W$ for some open set $W \subseteq U \cap V$. In particular,

$$\partial_{x_i} f(p) = \lim_{h \to 0} \frac{f(p_1, \dots, p_i + h, \dots, p_n)}{h} = \lim_{h \to 0} \frac{g(p_1, \dots, p_i + h, \dots, p_n)}{h} = \partial_{x_i} g(p).$$

for all $i = 1, \dots, n$, which implies

$$\begin{aligned} D_v f &= \sum_{i=1}^n v_i \partial_{x_i} f(p) \\ &= \sum_{i=1}^n v_i \partial_{x_i} g(p) \\ &= D_v g. \end{aligned}$$

Thus $D_v : C_p^\infty \to \mathbb{R}$ is a well-defined map. In fact, D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = (D_v f) g(p) + f(p) D_v g, \tag{11}$$

precisely because the partial derivatives $\partial_{x_i}|_p$ have these properties.

In general, any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (11) is called a **derivation at** p or a **point-derivation** of C_p^∞. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiplication of a derivation at p are again derivations at p.

Thus far, we know that directional derivatives at p are all derivations at p, so there is a map

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n),$$

where a vector $v = (v_1, \dots, v_n)$ in $T_p(\mathbb{R}^n)$ is mapped to the point-derivation $D_v = \sum_{i=1}^n v_i \partial_{x_i}|_p$. Since D_v is clearly linear in v, the map ϕ is a linear map of vector spaces.

Lemma 5.2. *If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.*

Proof. By $\mathbb{R}$-linearity, $D(c) = cD(1)$, so it suffices to prove that $D(1) = 0$. By the the Leibniz rule (11), we have

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Substracting $D(1)$ from both sides gives $D(1) = 0$. □

The **Kronecker delta** δ is a useful notation that we frequently call upon:

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{if } i \neq j. \end{cases}$$

Theorem 5.3. *The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined above is an isomorphism of vector spaces.*

Proof. To prove injectivity, suppose $D_v = 0$ for $v = (v_1, \dots, v_n) \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x_j gives

$$\begin{aligned} 0 &= D_v x_j \\ &= \sum_i v_i \partial_{x_i} x_j \mid_p \\ &= v_j. \end{aligned}$$

Hence $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, V) be a representative of a germ in C_p^∞. Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped. By Taylor's theorem with remainder, there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x),$$

where $g_i(p) = \partial_{x_i} f(p)$. Applying D to both sides and noting that $Df(p) = 0$ and $D(p_i) = 0$ by Lemma (5.2), we get by the Leibniz rule (11)

$$\begin{aligned} Df &= D(f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x)) \\ &= D(f(p)) + \sum_{i=1}^n D\left((x_i - p_i) g_i(x)\right) \\ &= \sum_{i=1}^n D\left((x_i - p_i) g_i(x)\right) \\ &= \sum_{i=1}^n (D(x_i - p_i) g_i(p) + (p_i - p_i) D g_i) \\ &= \sum_{i=1}^n (D(x_i) - D(p_i)) g_i(p) \\ &= \sum_{i=1}^n D(x_i) g_i(p) \\ &= \sum_{i=1}^n D(x_i) \partial_{x_i} f(p) \end{aligned}$$

This proves that $D = D_v$ for $v = (Dx_1, \dots, Dx_n)$. □

5.2.4 Vector Fields

A **vector field** $\vec{v}$ on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector $\vec{v}(p)$ in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\{\partial_{x_i}|_p\}$, the vector $\vec{v}(p)$ is a linear combination

$$\vec{v}(p) = \sum_{i=1}^n \vec{v}_i(p) \partial_{x_i}(p),$$

where $\vec{v}_i(p) \in \mathbb{R}$. Thus we may write $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$, where the $\vec{v}_i$ are now functions on U. We say that a vector field $\vec{v}$ is C^∞ on U if the coefficient functions $\vec{v}_i$ are all C^∞ on U.

Example 5.6.

1. On $\mathbb{R}^2 \backslash \{0\}$, we have the vector field

$$\vec{v} = \frac{-y}{\sqrt{x^2 + y^2}} \partial_x + \frac{x}{\sqrt{x^2 + y^2}} \partial_y.$$

2. On $\mathbb{R}^2$, we have the vector field

$$\vec{v} = x\partial_x - y\partial_y.$$

The ring of C^∞ on an open set U is commonly denoted by $C^\infty(U)$. Multiplication of vector fields by functions on U is defined pointwise:

$$(f\vec{v})(p) = f(p)\vec{v}(p).$$

Clearly if $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$ is a C^∞ vector field and f is a C^∞ function on U, then

$$f\vec{v} = \sum_{i=1}^n f\vec{v}_i \partial_{x_i}$$

is a C^∞ vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $\mathrm{Vec}(U)$, is a $C^\infty(U)$-module.

5.3 Vector Fields as Derivations

If $\vec{v}$ is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function on U by

$$(\vec{v}f)(p) = \vec{v}(p)f$$

for all $p \in U$. Writing $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$, we get

$$(\vec{v}f)(p) = \sum_{i=1}^n \vec{v}_i(p)\partial_{x_i} f(p)$$

or $\vec{v}f = \sum_{i=1}^n \vec{v}_i \partial_{x_i} f$, which shows that $\vec{v}f$ is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$C^\infty(U) \to C^\infty(U), \qquad f \mapsto \vec{v}f.$$

Proposition 5.2. *(Leibniz rule for a vector field) If $\vec{v}$ is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $\vec{v}(fg)$ satisfies the Leibniz rule:*

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

Proof. At each point $p \in U$, the vector $\vec{v}(p)$ satisfies the Leibniz rule:

$$\vec{v}(p)(fg) = \vec{v}(p)(f) \cdot g(p) + f(p) \cdot \vec{v}(p)(g),$$

as p varies over U, this becomes an inequality of functions:

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

□

If A is an algebra over a field K, a **derivation** of A is a K-linear map $D : A \to A$ such that

$$D(ab) = (Da)b + a(Db),$$

for all $a, b \in A$. The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted by $\mathrm{Der}(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\varphi : \mathrm{Vec}(U) \to \mathrm{Der}(C^\infty(U)), \qquad \vec{v} \mapsto (f \mapsto \vec{v}f).$$

Just as the tangent vectors at a point p can be identified with the point-derivations of C_p^∞, so the vector fields on an open set U can be identified with the derivations of the algebra $C^\infty(U)$, i.e. the map φ is an isomorphism of vector spaces.

5.4 The Exterior Algebra of Multicovectors

The basic principle of manifold theory is the linearization principle, according to which every manifold can be locally approximated by its tangent space at a point, a linear object. In this way linear algebra enters into manifold theory.

Instead of working with tangent vectors, it turns out to be more fruitful to adopt the dual point of view and work with linear functions on a tangent space. After all, there is only so much that one can do with tangent vectors, which are essentially arrows, but functions, far more flexible, can be added, multiplied, and composed with other maps.

5.5 Dual Spaces

Definition 5.3. Let V and W be two $\mathbb{R}$-vector spaces. We denote by $\mathrm{Hom}_{\mathbb{R}}(V, W)$ the vector space of all linear maps $\varphi : V \to W$. Define the **dual space** $V^\vee$ of V to be the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The elements of $V^\vee$ are called **covectors** or 1-**covectors** on V.

Let V be a finite dimensional $\mathbb{R}$-vector space with basis $\{e_1, \dots, e_n\}$. Then every $v \in V$ can be uniquely expressed as $\sum_{i=1}^n v_i e_i$ with $v_i \in \mathbb{R}$. Let $\underline{e}_i \in V^\vee$ be the linear function that picks out the ith coordinate, $\underline{e}_i(v) = v_i$. Note that $\underline{e}_i$ is characterized by

$$\underline{e}_i(e_j) = \delta^i_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Proposition 5.3. *The functions $\underline{e}_1, \dots, \underline{e}_n$ form a basis for $V^\vee$.*

Proof. We first show that $\underline{e}_1, \dots, \underline{e}_n$ span $V^\vee$. Suppose $\ell \in V^\vee$. For all $v \in V$, we have

$$\begin{aligned} \ell(v) &= \ell\left(\sum_{i=1}^n v_i e_i\right) \\ &= \sum_{i=1}^n v_i \ell(e_i) \\ &= \sum_{i=1}^n \underline{e}_i(v)\ell(e_i) \\ &= \sum_{i=1}^n \ell(e_i)\underline{e}_i(v) \\ &= \left(\sum_{i=1}^n \ell(e_i)\underline{e}_i\right)(v) \end{aligned}$$

Therefore $\ell = \sum_{i=1}^n \ell(e_i)\underline{e}_i \in \mathrm{Span}(\{\underline{e}_1, \dots, \underline{e}_n\})$. Next we show the set $\{\underline{e}_1, \dots, \underline{e}_n\}$ is linearly independent over $\mathbb{R}$. Suppose

$$\sum_{i=1}^n c_i \underline{e}_i = 0, \tag{12}$$

where $e_i \in \mathbb{R}$. By applying e_i to both sides of equation (12), we obtain $c_i = 0$, for all $i = 1, \dots, n$. □

Remark 8. We say $\{\underline{e}_1, \dots, \underline{e}_n\}$ is the **dual basis** of $\{e_1, \dots, e_n\}$.

Proposition 5.4. *Let V be a finite-dimensional vector space and let $\ell \in V^\vee$. Then $Ker(\ell)$ is a hyperplane in V.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of V and let $\{\underline{e}_1, \dots, \underline{e}_n\}$ be its dual basis. Write ℓ in terms of the dual basis:

$$\ell = \sum_{i=1}^n a_i \underline{e}_i,$$

where $a_i \in \mathbb{R}$. A vector $\sum_{i=1}^n x_i e_i$ belongs to the kernel of ℓ if and only if $\sum_{i=1}^n x_i a_i = 0$. Thus

$$\mathrm{Ker}(\ell) = V\left(\sum_{i=1}^n a_i X_i\right) = \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i x_i = 0\right\}.$$

□

Proposition 5.5. *Let V be an n-dimensional vector space and let $\ell_1, \dots, \ell_k \in V^\vee$. Then $\{\ell_1, \dots, \ell_k\}$ is linearly independent if and only if*

$$dim\left(\bigcap_{1 \le i \le k} Ker(\ell_i)\right) = n - k.$$

Proof. Suppose $\{\ell_1, \dots, \ell_k\}$ is linearly independent. We may assume that we are working in $(\mathbb{R}^n)^\vee$ and that $\ell_i = \underline{e}_i$. Then

$$\begin{aligned} \dim\left(\bigcap_{1 \le i \le k} \mathrm{Ker}(\ell_i)\right) &= \dim\left(\bigcap_{1 \le i \le k} V(X_i)\right) \\ &= \dim\left(V(X_1, \dots, X_k)\right) \\ &= n - k. \end{aligned}$$

The converse is trivial. □

5.6 Differential Forms on $\mathbb{R}^n$

The **cotangent space** to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$. In parallel with the definition of a vector field, a **covector field** or **differential** 1**-form** on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p\in U} T_p^*(\mathbb{R}^n), \qquad p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

We call a differential 1-form a 1**-form** for short.

5.7 Jacobian

Let $f = (f_1, \dots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open subset U of $\mathbb{R}^n$. The **Jacobian** of f at a point $p \in U$ is the $m \times n$ matrix

$$J(f)(p) := \begin{pmatrix} (\partial_{x_1} f_1)(p) & \cdots & (\partial_{x_n} f_1)(p) \\ \vdots & \ddots & \vdots \\ (\partial_{x_1} f_m)(p) & \cdots & (\partial_{x_n} f_m)(p) \end{pmatrix}.$$

The Jacobian satisfies the following property: for all $p \in U$, we have

$$\frac{\|f(p+\varepsilon) - f(p) - J(f)_p(\varepsilon)\|}{\|\varepsilon\|} \to 0$$

as $\varepsilon \to 0$ in $\mathbb{R}^n$. One can view the Jacobian as a smooth linear map

$$J(f)(p) = (J(f)(p)_1, \dots, J(f)(p)_m) : \mathbb{R}^n \to \mathbb{R}^m,$$

where the ith component $J(f)(p)_i$ is given by

$$J(f)(p)_i(x_1, \dots, x_n) = \sum_{j=1}^n (\partial_{x_j} f_i)(p) x_j.$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$.

If $m = n$, then f is a function from $\mathbb{R}^n$ to itself and the Jacobian matrix is a square matrix. In particular, we can compute its determinant, known as the **Jacobian determinant**. The Jacobian determinant at a given point gives important information about the behavior of f near that point. For instance, the inverse function theorem tells us that f is invertible near a point $p \in \mathbb{R}^n$ if and only if the Jacobian determinant is non-zero. Furthermore, if the Jacobian determinant at p is positive, then f preserves orientation near p.

Example 5.7.

1. Consider $f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and where $U = \{(x,y) \in \mathbb{R}^2 \mid xy \in (-\pi/2, \pi/2) \text{ and } x + y \in (0, \infty)\}$ and where $f_1(x,y) = \tan(xy)$ and $f_2(x,y) = \ln(x+y)$ for all $(x,y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in U$ is given by

$$J(f)(x_0, y_0) = \begin{pmatrix} y_0 \sec^2(x_0 y_0) & x_0 \sec^2(x_0 y_0) \\ \frac{1}{x_0+y_0} & \frac{1}{x_0+y_0} \end{pmatrix}.$$

The Jacobian determinant is then

$$\det(J(f)(x_0, y_0)) = \frac{(y_0 - x_0)\sec^2(x_0 y_0)}{x_0 + y_0}.$$

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = x^2 + xy + y$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by

$$J(f)(x_0, y_0) = \begin{pmatrix} 2x_0 + y_0 \\ x_0 + 1 \end{pmatrix}.$$

Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that

$$\begin{aligned} f(x_0 + \varepsilon_1, y_0 + \varepsilon_2) &= (x_0 + \varepsilon_1)^2 + (x_0 + \varepsilon_1)(y_0 + \varepsilon_2) + (y_0 + \varepsilon_2) \\ &= x_0^2 + x_0 y_0 + y_0 + (2x_0 + y_0)\varepsilon_1 + (x_0 + 1)\varepsilon_2 + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \\ &= f(x_0, y_0) + J(f)(x_0, y_0)(\varepsilon_1, \varepsilon_2) + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \end{aligned}$$

3. Consider $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ where $f_1(x, y) = x^2 y$ and $f_2(x, y) = y^2 + x$ for all $(x, y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by

$$J(f)(x_0, y_0) = \begin{pmatrix} 2x_0 y_0 & x_0^2 \\ 1 & 2y_0 \end{pmatrix}.$$

Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that

$$\begin{aligned} f(x_0 + \varepsilon_1, y_0 + \varepsilon_2) &= ((x_0 + \varepsilon_1)^2 (y_0 + \varepsilon_2), (y_0 + \varepsilon_2^2) + (x_0 + \varepsilon_1)) \\ &= (x_0^2 y_0, y_0^2 + x_0) + (2x_0 y_0 \varepsilon_1 + x_0^2 \varepsilon_2, \varepsilon_1 + 2y_0 \varepsilon_2) + (y_0 \varepsilon_1^2 + \varepsilon_1^2 \varepsilon_2, \varepsilon_2^2) \\ &= f(x_0, y_0) + J(f)(x_0, y_0)(\varepsilon_1, \varepsilon_2) + (y_0 \varepsilon_1^2 + \varepsilon_1^2 \varepsilon_2, \varepsilon_2^2) \end{aligned}$$

Proposition 5.6. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map from an open subset U of $\mathbb{R}^m$ and let p be a point in $\mathbb{R}^m$. Then*

$$f(p + \varepsilon) = f(p) + J(f)_p(\varepsilon) + \psi(\varepsilon),$$

where ψ is a smooth map such that $\|\psi(\varepsilon)\| / \|\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Proof. Define $\psi : U \to \mathbb{R}^n$ by

$$\psi(\varepsilon) := f(p + \varepsilon) - f(p) - J(f)_p(\varepsilon).$$

□

Theorem 5.4. *Let W be the finite-dimensional $\mathbb{R}$-vector space of symmetric bilinear forms on V, endowed with its natural topology as a finite-dimensional $\mathbb{R}$-vector space. The subset of elements that are positive-definite inner products is open and connected.*

Proof. We first prove connectedness, and then we prove openness. There is a natural left action of $\mathrm{GL}(V)$ on W: given $T \in \mathrm{GL}(V)$ and $B \in W$, we define a symmetric bilinear form

$$T \cdot B = B \circ (T^{-1} \otimes T^{-1}).$$

Let $\boldsymbol{e} = (e_1, \dots, e_n)$ be a basis of V and let $\boldsymbol{x} = (x_1, \dots, x_n)$ be its corresponding dual basis. Identify $\boldsymbol{e}$ with the standard basis of $\mathbb{R}^n$. Then it is easy to see that the matrix representation of $T \cdot B$ with respect to $\boldsymbol{e}$ is given by

$$[T \cdot B] = [T^{-1}]^\top [B][T^{-1}].$$

By fixing a basis of V and computing in linear coordinates we see that the resulting map $\mathrm{GL}(V) \times W \to W$ is continuous. In particular, if we fix $B_0 \in W$ then the map $\mathrm{GL}(V) \to W$ given by $T \mapsto T \cdot B_0$ is continuous. Restricting to the connected subgroup $\mathrm{GL}^+(V)$, it follows from continuity that the $\mathrm{GL}^+(V)$-orbit of any B_0 is connected in W. But if we take B_0 to be an inner product then form the definition of the action we see that $T \cdot B_0$ is an inner product for every $T \in \mathrm{GL}^+(V)$ (even for $T \in \mathrm{GL}(V)$), and it can be shown that every inner product on V is obtained froma single B_0 by means of $T \in \mathrm{GL}^+(V)$. This gives the connectivity. □

6 Higher Derivatives and Taylor's Formula Via Multilinear Maps

6.1 Differentiability

Definition 6.1. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let $U \subseteq V$ be open, let $u \in U$, and let $f \colon U \to W$ be a function. We say f is **differentiable** at u if there exists a (necessarily unique) linear map $\mathrm{D}f(u) \colon V \to W$ such that

$$\frac{\|f(u + h) - f(u) - \mathrm{D}f(u)h\|}{\|h\|} \to 0 \tag{13}$$

as $h \to 0$ in V (where the norms on the top and bottom are on V and W, and the choices do not impact the definition since any two norms on a finite-dimensional $\mathbb{R}$-vector space are bounded by a constant positive multiple of each other). The linear map $\mathrm{D}f(u) \colon V \to W$ is called the **total derivative** of f at u. We say f is **differentiable** on U (or more simply just **differentiable** if U is understood from context) if f is differentiable at all points $u \in U$. The map $\mathrm{D}f \colon U \to \mathrm{Hom}(V, W)$, which sends a $u \in U$ to the linear map $\mathrm{D}f(u) \colon V \to W$, is called the **total derivative** of f on U (or more simply just the total derivative if context is clear).

Suppose V has dimension m and W has dimension n. If we fix ordered bases for V and W, then we implicitly identify V with $\mathbb{R}^m$ and W with $\mathbb{R}^n$ using these bases as follows: suppose $\boldsymbol{v} = v_1, \dots, v_m$ is an ordered basis for V and let $\boldsymbol{x} = x_1, \dots, x_m$ denote the corresponding dual basis. Then a vector $v = \sum x_i v_i$ in V is identified with the point $\boldsymbol{x} = (x_1, \dots, x_m)$ in $\mathbb{R}^m$ via the canonical linear isomorphism $[\cdot]_{\boldsymbol{v}} \colon V \simeq \mathbb{R}^m$ which sends the ith basis element v_i in V to the ith standard basis element $\boldsymbol{e}_i = (0,\dots,1,\dots 0)$ in $\mathbb{R}^m$. Similarly, suppose $\boldsymbol{w} = w_1,\dots,w_n$ is an ordered basis for W and let $\boldsymbol{y} = y_1,\dots,y_n$ denote the corresponding dual basis. Then a vector $w = \sum y_i w_i$ in W is identified with the point $\boldsymbol{y} = (y_1,\dots,y_n)$ in $\mathbb{R}^n$ via the canonical linear isomorphism $[\cdot]_{\boldsymbol{w}}\colon W \simeq \mathbb{R}^n$ which sends the ith basis element w_i in W to the ith standard basis element $\boldsymbol{e}_i = (0,\dots,1,\dots 0)$ in $\mathbb{R}^n$. The open subset $U \subseteq V$ is identified with the open subset $[U]_{\boldsymbol{v}} \subseteq [V]_{\boldsymbol{v}} = \mathbb{R}^m$ and the map $f\colon U \to W$ then is identified with the map $[f]_{\boldsymbol{v}}^{\boldsymbol{w}} := [\cdot]_{\boldsymbol{w}} \circ f \circ [\cdot]_{\boldsymbol{v}}^{-1}$ from $[U]_{\boldsymbol{v}}$ to $[W]_{\boldsymbol{w}} = \mathbb{R}^n$. Under these identifications, the map $f\colon U \to W$ has the form

$$\begin{aligned} f\colon U &\longrightarrow W \\ \boldsymbol{x} = (x_1,\dots,x_n)^\top &\mapsto (f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))^\top = f(\boldsymbol{x}) = \boldsymbol{y}, \end{aligned}$$

where the $f_j\colon U \to \mathbb{R}$ for $1 \le j \le n$ are called the **component functions** of f. We claim that the linear map $\mathrm{D}f(\boldsymbol{x})$ is identified the **Jacobian matrix** of f at $\boldsymbol{x}$:

$$\mathrm{D}f(\boldsymbol{x}) = \mathrm{J}_f(\boldsymbol{x}) := \begin{pmatrix} \partial_{x_1} f_1(\boldsymbol{x}) & \cdots & \partial_{x_m} f_1(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_n(\boldsymbol{x}) & \cdots & \partial_{x_m} f_n(\boldsymbol{x}) \end{pmatrix} = (\partial_{x_i} f_j(\boldsymbol{x})).$$

Indeed, suppose $\boldsymbol{h} = (0,\dots,h_i,\dots,0)$ and let $\mathrm{D}f(\boldsymbol{x})^i$ (respectively $\mathrm{D}f(\boldsymbol{x})^i_j$) denote the ith column vector (respectively the (j,i) entry) of the matrix $\mathrm{D}f(\boldsymbol{x})$. Then as $h_i \to 0$, we see that

$$\frac{\|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - \mathrm{D}f(\boldsymbol{x})\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = \left\| \frac{f(x_1,\dots,x_i+h_i,\dots,x_m) - f(x_1,\dots,x_m) - \mathrm{D}f(\boldsymbol{x})^i h_i}{h_i} \right\| \to 0.$$

In particular, this implies the jth component of the vector

$$\frac{f(x_1,\dots,x_i+h_i,\dots,x_m)^\top - f(x_1,\dots,x_m)^\top - \mathrm{D}f(\boldsymbol{x})^i h_i}{h_i} \in \mathbb{R}^n$$

tends to zero as $h_i \to 0$. The jth component of this vector is given by

$$\frac{f_j(x_1,\dots,x_i+h_i,\dots,x_n) - f_j(x_1,\dots,x_n) - h_i \mathrm{D}f(\boldsymbol{x})^i_j}{h_i}.$$

In particular, this means $\mathrm{D}f(\boldsymbol{x})^i_j = \partial_{x_i} f_j(\boldsymbol{x})$.

Example 6.1. Let V be a two dimensional $\mathbb{R}$-vector space. Let $\boldsymbol{e} = (e_1, e_2)$ be an ordered basis for V and let $\boldsymbol{x} = (x_1, x_2)^\top$ be the corresponding dual basis. Thus every $v \in V$ can expressed as $v = a_1 v_1 + a_2 v_2$ for unique $a_1, a_2 \in \mathbb{R}$ where $a_1 = x_1(v)$ and $a_2 = x_2(v)$. We often get lazy and simply write $v = x_1 v_1 + x_2 v_2$ where we think of x_1 and x_2 as the coordinates of v. Let $f\colon V \to \mathbb{R}$ be given by $f = x_1^2 + x_1 x_2 + x_2$, so

$$\begin{aligned} f(v) &= (x_1^2 + x_1 x_2 + x_2)(v) \\ &= x_1(v)^2 + x_1(v)x_2(v) + x_2(v) \\ &= x_1^2 + x_1 x_2 + x_2 \end{aligned}$$

where we got lazy at the end simply wrote $x_1 = x_1(v)$ and $x_2 = x_2(v)$. Notice that we are using the x_i's in two different (and admittedly contradictory) ways here. When we write $f = x_1^2 + x_1 x_2 + x_2$, we are thinking of the x_i as linear functions $x_i\colon V \to \mathbb{R}$. When we write $f(v) = x_1^2 + x_1 x_2 + x_2$, we are thinking of the x_i as the coordinates of $v = x_1 e_1 + x_2 e_2$. At the end of the day however, context will always makes clear how we are thinking of the x_i's. The matrix representation of the differential of f at a point $v = x_1 e_1 + x_2 e_2$ is given by

$$\begin{aligned} [\mathrm{D}f(v)]_{\boldsymbol{e}} &= \nabla_{\boldsymbol{x}} f(\boldsymbol{x})^\top \\ &= (\partial_{x_1} f(\boldsymbol{x}), \partial_{x_2} f(\boldsymbol{x})) \\ &= (2x_1 + x_2, x_1 + 1). \end{aligned}$$

Now suppose $\widetilde{\boldsymbol{e}} = (\widetilde{e}_1, \widetilde{e}_2)$ is another ordered basis of V. Let $C = \left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ be a change of basis matrix from $\boldsymbol{e}$ to $\widetilde{\boldsymbol{e}}$, so $\boldsymbol{e}C = \widetilde{\boldsymbol{e}}$ and $\boldsymbol{x} = C\widetilde{\boldsymbol{x}}$ where $\widetilde{\boldsymbol{x}} = (\widetilde{x}_1, \widetilde{x}_2)^\top$ is the corresponding dual basis of $\widetilde{\boldsymbol{e}}$. Then f expressed in this new basis is given by

$$\begin{aligned} f &= x_1^2 + x_1 x_2 + x_2 \\ &= (c_{11}\widetilde{x}_1 + c_{12}\widetilde{x}_2)^2 + (c_{11}\widetilde{x}_1 + c_{12}\widetilde{x}_2)(c_{21}\widetilde{x}_1 + c_{22}\widetilde{x}_2) + c_{21}\widetilde{x}_1 + c_{22}\widetilde{x}_2 \\ &= c_{11}^2\widetilde{x}_1^2 + 2c_{11}c_{12}\widetilde{x}_1\widetilde{x}_2 + c_{12}^2\widetilde{x}_2^2 + c_{11}c_{21}\widetilde{x}_1^2 + c_{12}c_{21}\widetilde{x}_1\widetilde{x}_2 + c_{11}c_{22}\widetilde{x}_1\widetilde{x}_2 + c_{12}c_{22}\widetilde{x}_2^2 + c_{21}\widetilde{x}_1 + c_{22}\widetilde{x}_2 \\ &= (c_{11}^2 + c_{11}c_{21})\widetilde{x}_1^2 + (2c_{11}c_{12} + c_{12}c_{21} + c_{11}c_{22})\widetilde{x}_1\widetilde{x}_2 + (c_{12}^2 + c_{12}c_{22})\widetilde{x}_2^2 + c_{21}\widetilde{x}_1 + c_{22}\widetilde{x}_2. \end{aligned}$$

Let us consider the special case where $C = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -2 \end{smallmatrix}\right)$. Then $f = \widetilde{x}_1^2 - \widetilde{x}_2^2 - 2\widetilde{x}_2$. and

$$\begin{aligned} v &= \boldsymbol{e}\boldsymbol{x} \\ &= \boldsymbol{e}CC^{-1}\boldsymbol{x} \\ &= \widetilde{\boldsymbol{e}}\widetilde{\boldsymbol{x}} \end{aligned}$$

We can calculate the matrix representation of $\mathrm{D}f(v)$ with respect to the $\widetilde{\boldsymbol{e}}$ basis in two ways: the first way is

$$\begin{aligned} [\mathrm{D}f(v)]_{\widetilde{\boldsymbol{e}}} &= \nabla_{\widetilde{\boldsymbol{x}}} f(\widetilde{\boldsymbol{x}})^\top \\ &= (\partial_{\widetilde{x}_1} f(\widetilde{\boldsymbol{x}}), \partial_{\widetilde{x}_2} f(\widetilde{\boldsymbol{x}})) \\ &= (2\widetilde{x}_1, -2\widetilde{x}_2 - 2) \end{aligned}$$

The second way is

$$\begin{aligned} [\mathrm{D}f(v)]_{\widetilde{\boldsymbol{e}}} &= [\mathrm{D}f(v)]_{\boldsymbol{e}} C \\ &= (2x_1 + x_2, x_1 + 1) \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \\ &= (2x_1 + x_2, x_2 - 2) \\ &= (2\widetilde{x}_1, -2\widetilde{x}_2 - 2). \end{aligned}$$

Example 6.2. Let U be the open subset of $\mathbb{R}^2$ given by

$$U = \{\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2 \mid t_1 t_2 \in \mathbb{R} \backslash \{\pi/2 + \pi\mathbb{Z}\} \text{ and } t_1 + t_2 \in (0, \infty)\}$$

and let $f \colon U \to \mathbb{R}^2$ be defined by

$$f(\boldsymbol{t}) = (\tan(t_1 t_2), \ln(t_1 + t_2)) = (f_1(\boldsymbol{t}), f_2(\boldsymbol{t})).$$

Then f is differentiable with its derivative at a point $\boldsymbol{t} = (t_1, t_2)$ in U defined by

$$\mathrm{D}f(\boldsymbol{t}) = \begin{pmatrix} t_2 \sec^2(t_1 t_2) & t_1 \sec^2(t_1 t_2) \\ \frac{1}{t_1 + t_2} & \frac{1}{t_1 + t_2} \end{pmatrix}.$$

Notice that $\mathrm{D}f(\boldsymbol{t})$ is literally a matrix in this case (and not an abstract linear map between abstract vector spaces). This is because we are working specifically in $\mathbb{R}^2$.

Example 6.3. Let V and W be two dimensional $\mathbb{R}$-vector spaces. Let $\boldsymbol{v} = (v_1, v_2)$ be an ordered basis for V with corresponding dual basis $\boldsymbol{x} = (x_1, x_2)^\top$ and let $\boldsymbol{w} = (w_1, w_2)$ be an ordered basis for W with corresponding dual basis $\boldsymbol{y} = (y_1, y_2)^\top$. Define $f \colon V \to W$ by

$$f(v) = f(x_1 v_1 + x_2 v_2) = \tan(x_1 x_2) w_1 + \ln(x_1 + x_2) w_2 = f_1(v) w_1 + f_2(v) w_2$$

where $f_1 = f \circ y_1$ and $f_2 = f \circ y_2$. The matrix representation of the differential of f at a point $v = x_1 e_1 + x_2 e_2$ with respect to the ordered bases $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\begin{aligned} [\mathrm{D}f(v)]_{\boldsymbol{v}}^{\boldsymbol{w}} &= \begin{pmatrix} \partial_{x_1} f_1(\boldsymbol{x}) & \partial_{x_2} f_1(\boldsymbol{x}) \\ \partial_{x_1} f_2(\boldsymbol{x}) & \partial_{x_2} f_2(\boldsymbol{x}) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \sec^2(x_1 x_2) & x_1 \sec^2(x_1 x_2) \\ \frac{1}{x_1 + x_2} & \frac{1}{x_1 + x_2} \end{pmatrix}, \end{aligned}$$

which is the same matrix that we calculated in Example (6.2), except now we are expressing it using $\boldsymbol{x}$ coordinates. If $\widetilde{\boldsymbol{v}} = (\widetilde{v}_1, \widetilde{v}_2)$ is another ordered basis of V with corresponding change of basis matrix $C \in \mathrm{GL}_2(\mathbb{R})$, and $\widetilde{\boldsymbol{w}} = (w_1, w_2)$ is another ordered basis of W with corresponding change of basis matrix $D \in \mathrm{GL}_2(\mathbb{R})$, then it is straightforward to check that the matrix representation of the differential of f at the point $v = \boldsymbol{v}\widetilde{\boldsymbol{x}}$ is given by

$$[\mathrm{D}f(v)]_{\widetilde{\boldsymbol{v}}}^{\widetilde{\boldsymbol{w}}} = D[\mathrm{D}f(v)]_{\boldsymbol{v}}^{\boldsymbol{w}} C.$$

6.1.1 Derivative of a Linear Map

Let $\boldsymbol{e} = (e_1, \dots, e_n)$ be an ordered basis for V and let $\boldsymbol{x} = (x_1, \dots, x_n)^\top$ be the corresponding dual basis. Let $T \colon V \to V$ be a linear isomorphism and let $u \in U$. Identify $\boldsymbol{e}$ with the standard ordered basis of $\mathbb{R}^n$. Then T gets identified to a matrix $T = (T_j^i)$, the point u gets identified to a vector $\boldsymbol{u} = (u_1, \dots, u_n)^\top$, and the derivative of T at u gets identified to the Jacobian of T at u. Viewing T as a map from $\mathbb{R}^n \to \mathbb{R}^n$, we see that it has the form

$$T(\boldsymbol{u}) = \begin{pmatrix} T_1^1 & \cdots & T_1^n \\ \vdots & \ddots & \vdots \\ T_n^1 & \cdots & T_n^n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} T_1^1 u_1 + \cdots + T_1^n u_n \\ \vdots \\ T_n^1 u_1 + \cdots + T_n^n u_n \end{pmatrix} = \begin{pmatrix} T_1(\boldsymbol{u}) \\ \vdots \\ T_n(\boldsymbol{u}) \end{pmatrix}.$$

Thus the component functions of T correspond to its rows as matrix. These component functions are given by

$$T_j = T_j^1 x_1 + \cdots + T_j^n x_n$$

for all $1 \leq i \leq n$. With this in mind, we have

$$\begin{aligned} (\mathrm{D}T)(u) &= \mathrm{J}_{\boldsymbol{u}}(T) \\ &= \begin{pmatrix} (\partial_{x_1} T_1)(\boldsymbol{u}) & \cdots & (\partial_{x_n} T_1)(\boldsymbol{u}) \\ \vdots & \ddots & \vdots \\ (\partial_{x_1} T_n)(\boldsymbol{u}) & \cdots & (\partial_{x_n} T_n)(\boldsymbol{u}) \end{pmatrix} \\ &= \begin{pmatrix} T_1^1 & \cdots & T_1^n \\ \vdots & \ddots & \vdots \\ T_n^1 & \cdots & T_n^n \end{pmatrix} \\ &= T. \end{aligned}$$

So the derivative of a linear map at a point u is just the linear map itself. Notice that we proved this by choosing an ordered basis, but we could have also proven this directly using the coordinate-free definition of differentiability (13). Indeed, if we set $\mathrm{D}T(u) = T$, then for any h we have

$$\frac{\|T(u+h) - T(u) - \mathrm{D}T(u)(h)\|}{\|h\|} = 0,$$

so clearly $\mathrm{D}T(u)$ exists, and it must be equal to T itself.

6.1.2 Chain Rule

Proposition 6.1. *Let V, V',and V'' be finite-dimensional $\mathbb{R}$-vector spaces of dimensions n, n', and n'' respectively, let U be an open subset of V, let U' be an open subset of V', and let $f \colon U \to V'$ and $g \colon U' \to V''$ be differentiable functions. Then the map $g \circ f \colon U \cap f^{-1}(U') \to V''$ is differentiable and for all $u \in U \cap f^{-1}(U')$ we have*

$$\mathrm{D}(g \circ f)(u) = \mathrm{D}g(f(u)) \circ \mathrm{D}f(u). \tag{14}$$

Proof. If $U \cap f^{-1}(U') = \emptyset$, then there is nothing to prove, so we may assume that it is nonempty. After choosing bases for V, V', and V'', we may identify $\mathrm{D}(g \circ f)(u)$ with the $n'' \times n$ matrix whose (i'', i) entry is $\partial_{x_i}(g \circ f)_{i''}(u)$, we may identify $\mathrm{D}g(f(u))$ with the $n'' \times n'$ matrix whose (i'', i') entry is $\partial_{f_{i'}} g_{i''}(f(u))$, and we may identify $\mathrm{D}f(u)$ with the $n' \times n$ matrix whose (i', i) entry is $\partial_{x_i} f_{i'}(u)$. Then (14) turns into the matrix equation

$$(\partial_{x_i}(g \circ f)_{i''}(u)) = (\partial_{f_{i'}} g_{i''}(f(u))) \cdot (\partial_{x_i} f_{i'}(u))$$

which gives us a system of equations

$$\partial_{x_i}(g \circ f)_{i''}(u) = \sum_{i'=1}^{n'} \partial_{f_{i'}} g_{i''}(f(u)) \partial_{x_i} f_{i'}(u) \tag{15}$$

for each $1 \leq i \leq n$ and $1 \leq i'' \leq n''$. Therefore (14) holds if and only if (15) holds for all i, i'', and (15) holds since this is just the chain rule in the classical case. □

Here's a coordinate-free proof of the Chain Rule: again we may assume $U \cap f^{-1}(U') \neq \emptyset$. Furthermore, by replacing U with $U \cap f^{-1}(U')$ if necessary, we may assume that $U = f^{-1}(U')$. Fix $u \in U$ (we show $g \circ f$ is differentiable at u with derivative given by (14)). There exists open balls $\mathrm{B}_\varepsilon(0) \subseteq V$ and $\mathrm{B}_{\varepsilon'}(0) \subseteq V'$ together with functions $R_1 \colon \mathrm{B}_\delta(0) \to V'$ and $R_2 \colon \mathrm{B}_{\delta'}(0) \to V''$ such that

$$f(u+h) = f(u) + \mathrm{D}f(u)h + R_1(h)$$
$$g(f(u)+h') = g(f(u)) + \mathrm{D}g(f(u))h' + R_2(h')$$

for all $h \in \mathrm{B}_\delta(0)$ and $h' \in \mathrm{B}_{\delta'}(0)$, where R_1 and R_2 have the additional property that

$$\|R_1(h)\| \leq A_u(h)\|h\| \quad \text{and} \quad \|R_2(h')\| \leq B_{f(u)}(h')\|h'\|,$$

where $A_u \colon \mathrm{B}_\delta(0) \to \mathbb{R}_{>0}$ and $B_{f(u)} \colon \mathrm{B}_{\delta'}(0) \to \mathbb{R}_{>0}$ satisfy

$$\lim_{h \to 0} A_u(h) = 0 \quad \text{and} \quad \lim_{h' \to 0} B_{f(u)}(h') = 0.$$

Moreover, the convergence $A_u(h) \to 0$ as $h \to 0$ is uniform on a compact subspace $K \subseteq U$ where $u \in K$. Similarly, the convergence $B_{f(u)}(h) \to 0$ as $h \to 0$ is uniform on a compact subspace $K' \subseteq U'$ with $f(u) \in K'$. In particular, given $\varepsilon, \varepsilon' > 0$, by replacing δ and δ' with smaller values if necessary, we have

$$\sup_{\substack{\|h\|<\delta \\ v \in K}} \|A_v(h)\| < \varepsilon \quad \text{and} \quad \sup_{\substack{\|h'\|<\delta' \\ v' \in K'}} \|B_{v'}(h')\| < \varepsilon'.$$

We don't need the full strength of this fact; mainly we just need the fact that $\|h\| < \delta$ implies $\|A_u(h)\| < \varepsilon$ and similarly $\|h'\| < \delta'$ implies $\|B_{f(u)}(h)\| < \varepsilon'$. Now, in a moment, we are going to want to replace δ with something smaller (if necessary), but before we do this, let's see how the proof ought to work: to show $g \circ f$ is differentiable at u, it suffices to show that it has a first order approximation of the form:

$$(g \circ f)(u+h) = g(f(u)) + (\mathrm{D}g(f(u)) \circ \mathrm{D}f(u))h + R_3(h),$$

where $h \in \mathrm{B}_r(0)$ where $r > 0$ is sufficiently small and where $R_3 \colon \mathrm{B}_r(0) \to V''$ is a function with the property that

$$\|R_3(h)\| \leq C_u(h)\|h\|$$

where $C_u \colon \mathrm{B}_r(0) \to \mathbb{R}$ satisfies $C_u(h) \to 0$ as $h \to 0$. To determine this first order approximation, write

$$\begin{aligned}
(g \circ f)(u+h) &= g(f(u+h)) \\
&= g(f(u) + \mathrm{D}f(u)h + R_1(h)) \\
&= g(f(u) + h') \\
&= g(f(u)) + \mathrm{D}g(f(u))h' + R_2(h') \\
&= g(f(u)) + (\mathrm{D}g(f(u)) \circ \mathrm{D}f(u))h + (\mathrm{D}g(f(u)) \circ R_1(h) + R_2(h')) \\
&= g(f(u)) + (\mathrm{D}g(f(u)) \circ \mathrm{D}f(u))h + R_3(h)
\end{aligned}$$

where we set $h' = \mathrm{D}f(u)h + R_1(h)$ and where $R_3(h) = \mathrm{D}g(f(u)) \circ R_1(h) + R_2(h')$. In order for this to work, we need to justify the jump from the third line to the fourth line above. We show that we can do this by replacing δ with something smaller if necessary. Note that

$$\begin{aligned}
\|h'\| &= \|\mathrm{D}f(u)h + R_1(h)\| \\
&\leq \|\mathrm{D}f(u)\|\|h\| + A_u(h)\|h\| \\
&= (\|\mathrm{D}f(u)\| + A_u(h))\|h\| \\
&< (\|\mathrm{D}f(u)\| + \varepsilon)\|h\| \\
&< (\|\mathrm{D}f(u)\| + \varepsilon)\delta.
\end{aligned}$$

So replacing δ with $\delta'/(\|\mathrm{D}f(u)\| + \varepsilon)$ if necessary, we can ensure that $\|h\| < \delta$ implies $\|h'\| < \delta'$, so that it makes sense to go from the third line to the fourth line in our "initial proof". Finally, observe that

$$\begin{aligned}
\|R_3(h)\| &= \|\mathrm{D}g(f(u)) \circ R_1(h) + R_2(h')\| \\
&\leq \|\mathrm{D}g(f(u))\|\|R_1(h)\| + \|R_2(h')\| \\
&\leq \|\mathrm{D}g(f(u))\|A_u(h)\|h\| + B_{f(u)}(h')\|h'\| \\
&< \|\mathrm{D}g(f(u))\|A_u(h)\|h\| + B_{f(u)}(h')(\|\mathrm{D}f(u)\| + \varepsilon)\|h\| \\
&= C_u(h)\|h\|,
\end{aligned}$$

where

$$C_u(h) = \|\mathrm{D}g(f(u))\|A_u(h) + B_{f(u)}(h')(\|\mathrm{D}f(u)\| + \varepsilon).$$

Then note that $h \to 0$ implies $h' \to 0$ which implies $A_u(h) \to 0$ and $B_{f(u)}(h') \to 0$. So clearly $h \to 0$ implies $C_u(h) \to 0$, and we are done. It follows that $g \circ f$ is differentiable at u with its derivative given by (14).

6.1.3 Derivative of a Chart

Let V be a finite dimensional vector space and let (U, φ) be a smooth chart of V centered at $u \in U$. This means U is an open subset of V and $\varphi\colon U \to \mathbb{R}^n$ is a diffeomorphism onto its image. Then by the chain rule, we have

$$\begin{aligned} 1 &= \mathrm{D}(1_{\varphi(U)})(\varphi(u)) \\ &= \mathrm{D}(\varphi \circ \varphi^{-1})(\varphi(u)) \\ &= \mathrm{D}\varphi(u) \circ \mathrm{D}\varphi^{-1}(\varphi(u)). \end{aligned}$$

In particular, $\mathrm{D}\varphi(u)$ is invertible (and hence nonzero) with its inverse given by $\mathrm{D}\varphi^{-1}(\varphi(u))$.

6.2 C^p maps

Definition 6.2. Let V and W be finite-dimensional spaces, let $U \subseteq V$ be open, and let $f\colon U \to W$. We say f is C^1 on U (or more simply C^1 if U is understood from context) if it is differentiable and its total derivative $\mathrm{D}f\colon U \to \mathrm{Hom}(V, W)$ is continuous. We say f is C^2 if $\mathrm{D}f$ is C^1, meaning $\mathrm{D}f\colon U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W))$ is differentiable (with total derivative denoted $\mathrm{D}^2 f = \mathrm{D}(\mathrm{D}f)$). Note that

Thus we have

$$\begin{aligned} f(u) &\in W & f &\in \mathrm{Map}(U, W) \\ (\mathrm{D}f)(u) &\in \mathrm{Hom}(V, W) & \mathrm{D}f &\in \mathrm{Map}(U, \mathrm{Hom}(V, W)) \\ (\mathrm{D}(\mathrm{D}f))(u) &\in \mathrm{Hom}(V, \mathrm{Hom}(V, W)) & \mathrm{D}(\mathrm{D}f) &\in \mathrm{Map}(U, \mathrm{Hom}(V, \mathrm{Hom}(V, W))) \\ &\vdots & &\vdots \end{aligned}$$

Note that there are unique isomorphisms

$$(-)^\diamond\colon \mathrm{Hom}(V, \mathrm{Hom}(V, W)) \to \mathrm{Hom}(V \otimes V, W) \quad \text{and} \quad (-)_\diamond\colon \mathrm{Hom}(V \otimes V, W) \to \mathrm{Hom}(V, \mathrm{Hom}(V, W))$$

of $\mathbb{R}$-vector spaces (both natural in V and W) such that

$$\varphi^\diamond(v_1 \otimes v_2) = (\varphi v_1)v_2 \quad \text{and} \quad (\psi_\diamond v_1)v_2 = \psi(v_1 \otimes v_2)$$

for all $v_1, v_2 \in V$, $\varphi \in \mathrm{Hom}(V, \mathrm{Hom}(V, W))$ and $\psi \in \mathrm{Hom}(V \otimes V, W)$. We denote

$$\mathrm{D}^2 f(u) = ((\mathrm{D}(\mathrm{D}f))(u))^\diamond$$

and think of $\mathrm{D}^2 f(u)$ as being a map bilinear map $\mathrm{D}^2 f(u)\colon V \times V \to W$. Similarly, we think of $\mathrm{D}^2 f$ as being a map from U to $\mathrm{Mult}(V^2, W)$. Now let us fix a basis $\boldsymbol{e} = e_1, \dots, e_m$ of V with $\boldsymbol{x} = x_1, \dots, x_n$ being the corresponding dual basis. Let $f_1, \dots, f_n$ be the component functions of $f\colon U \to W$ with respect to a basis of W (say $\widetilde{\boldsymbol{e}} = \widetilde{e}_1, \dots, \widetilde{e}_n$ with corresponding dual basis $\widetilde{\boldsymbol{x}} = \widetilde{x}_1, \dots, \widetilde{x}_n$ (so $f_j = \widetilde{x}_j \circ f$)). Then the jth component function of $\mathrm{D}f(u)\colon V \to W$ correspond to the jth row vector of the matrix representation of $\mathrm{D}f(u)$ with respect to $\boldsymbol{e}$ and $\widetilde{\boldsymbol{e}}$. In particular, the jth component vector of $\mathrm{D}f(u)$ is given by

$$\widetilde{x}_j \circ \mathrm{D}f(u) = (\mathrm{D}f)_j(u) = \sum_{i=1}^m \partial_{x_i} f_j(u) x_i.$$

We have

Theorem 6.1. *For $1 \leq i_1, \dots, i_p \leq m$, we have*

$$\mathrm{D}^p f(u)(e_{i_1}, \dots, e_{i_p}) = ((\partial_{x_{i_p}} \cdots \partial_{x_{i_1}} f_1)(u), \dots, (\partial_{x_{i_p}} \cdots \partial_{x_{i_1}} f_m)(u)) \in \mathbb{R}^n.$$

Proof. We induct on p, the base case $p = 1$ being the old theorem on the determination of the matrix for the derivative map $Df(u) : V \to W$ in terms of first-order partials of the component functions for f (using linear coordinates on W to define these component functions, and using linear coordinates on V to define the relevant partial derivative operators on these functions). Now we assume $p \geq 2$.

By definition of the isomorphism in Corollary (**??**), we have

$$D^p f(u)(v_1, \dots, v_p) = (\cdots((D^p f(u)(v_1))(v_2) \cdots)(v_p) \in W$$

for any ordered p-tuple $v_1, \dots, v_p \in V$. Let $F = Df : U \to \mathrm{Hom}(V, W)$. Using the given linear coordinates on V and W, the associated "matrix entries" are taken as the linear coordinates on $\mathrm{Hom}(V, W)$ to get component functions F_{ij} for F (with $1 \leq i \leq m$ and $1 \leq j \leq n$). Considering $v_2, \dots, v_p$ as fixed but v_1 as varying, we have

$$D^p f(u)(\cdot, v_2, \dots, v_p) = (\cdots((D^{p-1}F)(u)(v_2)) \cdots)(v_p) = D^{p-1}F(u)(v_2, \dots, v_p) \in \mathrm{Hom}(V, W)$$

where $\mathrm{Hom}(V, W)$ is the target vector space for F. Setting $v_k = e_{j_k}$ for $2 \leq k \leq p$, the inductive hypothesis applied to $F : U \to \mathrm{Hom}(V, W) = \mathrm{Mat}_{m \times n}(\mathbb{R})$ gives

$$D^{p-1}F(u)(e_{j_2}, \ldots, e_{j_p}) = (\partial_{j_p} \cdots \partial_{j_2} F_{ij}(u)) \in \mathrm{Mat}_{m \times n}(\mathbb{R}).$$

In view of how the matrix coordinatization of $\mathrm{Hom}(V, W)$ was *defined* using the chosen ordered bases on V and W, evaluating e_{j_1} in $\mathrm{Hom}(V, W) \simeq \mathrm{Mat}_{m \times n}(\mathbb{R})$ corresponds to pass to the j_1th column of a matrix. Hence taking $v_1 = e_{j_1}$ gives

$$D^p f(u)(e_{j_1}, e_{j_2}, \ldots, e_{j_p}) = (\partial_{j_p} \cdots \partial_{j_2} F_{1j_1}(u), \ldots, \partial_{j_p} \cdots \partial_{j_2} F_{mj_1}(u)) \in \mathbb{R}^m = W.$$

By the C^1 case, $F = Df : U \to \mathrm{Hom}(V, W) = \mathrm{Mat}_{m \times n}(\mathbb{R})$ has ij-component function $F_{ij} = \partial_j f_i$, so $F_{ij_1} = \partial_{j_1} f_i$. Thus, we get the desired formula. □

Therefore

$$\partial_{x_i} (\mathrm{D}f)_j(u) = \sum_{i=1}^{m} \partial_{x_i}^2 f_j(u)$$

It follows that for any $1 \leq i < i' \leq m$, we have

$$\mathrm{D}^2 f(u)(e_{i_1}, e_{i_2}) = \mathrm{D}((\mathrm{D}f)(u)e_{i_1})e_{i_2}$$

$$\mathrm{D}^2 f(u)(e_{i_1}, e_{i_2}) = (\mathrm{D}(\mathrm{D}f)(u))^{\diamond}(e_{i_1}, e_{i_2}) = (\mathrm{D}(\mathrm{D}f(u)e_{i_1})e_{i_2}$$

$$\begin{aligned}
\mathrm{D}^2 f(u)(e_{i_1}, e_{i_2}) &= ((\mathrm{D}(\mathrm{D}f))(u))^{\diamond}(e_{i_1}, e_{i_2}) \\
&= (((\mathrm{D}(\mathrm{D}f))(u))e_{i_1})e_{i_2} \\
&= \\
&= \left(\mathrm{D}\left(\sum_{j=1}^{n} \partial_{x_i} f_j\right)(u)\right) e_{i'} \\
&= \left(\mathrm{D}\left(\sum_{j=1}^{n} \partial_{x_i} f_j\right)(u)\right) e_{i'} \\
&= \sum
\end{aligned}$$

$$\mathrm{D}(\mathrm{D}f)$$

$$\mathrm{D}^p f(u)(e_{i_1}, \ldots, e_{i_p}) = ((\partial_{x_{i_p}} \cdots \partial_{x_{i_1}} f_1)(u), \ldots, (\partial_{x_{i_p}} \cdots \partial_{x_{i_1}} f_m)(u)) \in \mathbb{R}^n.$$

$$\mathrm{D}f(u) = (\partial_{x_i} f_j(u))_j^i$$

$$\mathrm{D}(\mathrm{D}f)(u) = (\partial_{x_i}(\partial_{x_{i'}} f_j(u))_{i',j}^i$$

$$\mathrm{D}\left(\sum_{j=1}^{n} \partial_{x_i} f_j\right)(u) = (\partial_{x_i} \left(\sum_{j=1}^{n} \partial_{x_i} f_j(u)\right)_j (u))_j^i$$

In particular, the jth component vector of $\mathrm{D}(\mathrm{D}f)(u)$ is given by

$$\begin{aligned}
\mathrm{D}(\mathrm{D}f)(u)_j &= \sum_{i=1}^{m} \partial_{x_i} (\mathrm{D}f)_j(u) x_i \\
&= \sum_{i=1}^{m} \partial_{x_i} \left(\sum_{i'=1}^{m} \partial_{x_{i'}} f_j(u) x_{i'}\right) x_i \\
&= \sum_{i=1}^{m} \sum_{i'=1}^{m} \partial_{x_i} f_j(u) x_i \\
&= \sum_{1 \leq i, i' \leq m} \partial_{x_i} \partial_{x_{i'}} f_j(u) x_{i'} x_i
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}(\mathrm{D}f)(u)_j &= \sum_{i=1}^{m} \partial_{x_i}(\mathrm{D}f)_j(u)x_i \\
&= \sum_{i=1}^{m} \partial_{x_i}\left(\sum_{i'=1}^{m} \partial_{x_{i'}} f_j(u)x_{i'}\right) x_i \\
&= \sum_{i=1}^{m}\sum_{i'=1}^{m} \partial_{x_i} f_j(u)x_i \\
&= \sum_{1\le i,i'\le m} \partial_{x_i}\partial_{x_{i'}} f_j(u)x_{i'}x_i
\end{aligned}$$

$$\mathrm{D}(\mathrm{D}f)(u)_j = \sum_{i=1}^{m} \partial_{x_i}(\mathrm{D}f)_j(u)x_i =$$

It follows that for any $1 \le i < i' \le m$, we have

$$\begin{aligned}
\mathrm{D}^2 f(u)(e_i, e_{i'}) &= (\mathrm{D}(\mathrm{D}f)(u))^{\diamond}(e_i, e_{i'}) \\
&= (\mathrm{D}(\mathrm{D}f)(u)e_i)e_{i'} \\
&= \\
&= \mathrm{D}f(u)^{i}_{i'}
\end{aligned}$$

Where we used the fact that

$$\mathrm{D}(\mathrm{D}f)(u)e_i = \mathrm{D}(\mathrm{D}f)(u)^i$$

So we just need to figure out $(\mathrm{D}(\mathrm{D}f)(u))^{\diamond}(e_i, e_{i'})$

$$\mathrm{D}f(u) = (\partial_{x_i} f_j(u))$$

$$\mathrm{D}(\mathrm{D}f)(u) = (\partial_{x_i}(\mathrm{D}f)_j(u))$$

$$\mathrm{D}^2 f(u)(e_i, e_{i'}) = (\mathrm{D}(\mathrm{D}f)(u))^{\diamond}(e_i, e_{i'}) =$$

the matrix representation of $\mathrm{D}((\mathrm{D}f(u))(u')$ is given by

$$\mathrm{D}^2 f(u) = \mathrm{D}^2 f(u \otimes u) = (\mathrm{D}(\mathrm{D}f))^{\diamond}(u \otimes u) = \mathrm{D}((\mathrm{D}f(u))(u) = \mathrm{D}(\mathrm{D}f(u))(u) = \mathrm{D}(\mathrm{D}f)(u),$$

Now let us fix a basis $\boldsymbol{e} = e_1, \dots, e_m$ of V with $\boldsymbol{x} = x_1, \dots, x_n$ being the corresponding dual basis. Let ∂_i denote ∂_{x_i} and let $f_1, \dots, f_n$ be the component functions of $f\colon U \to W$ with respect to a basis of W (say $\widetilde{\boldsymbol{e}} = \widetilde{e}_1, \dots, \widetilde{e}_n$ with corresponding dual basis $\widetilde{\boldsymbol{x}} = \widetilde{x}_1, \dots, \widetilde{x}_n$ (so $f_j = \widetilde{x}_j \circ f$)). Then the jth component function of $\mathrm{D}f(u)\colon V \to W$ correspond to the jth row vector of the matrix representation of $\mathrm{D}f(u)$ with respect to $\boldsymbol{e}$ and $\widetilde{\boldsymbol{e}}$. In particular, the jth component vector of $\mathrm{D}f(u)$ is given by

$$\widetilde{x}_j \circ \mathrm{D}f(u) = \mathrm{D}f(u)_j = \sum_{i=1}^{m} \partial_{x_i} f_j(u)x_i.$$

It follows that the matrix representation of $\mathrm{D}((\mathrm{D}f(u))(u')$ is given by

$$\mathrm{D}((\mathrm{D}f(u))(u') = \mathrm{J}_{\mathrm{D}f(u)}(u') := \begin{pmatrix} (\partial_{x_1}\partial_{x_1} f_1)(u) & \cdots & (\partial_{x_m} f_1)(u) \\ \vdots & \ddots & \vdots \\ (\partial_{x_1} f_n)(u) & \cdots & (\partial_{x_m} f_n)(u) \end{pmatrix}.$$

(Thus we can think of .= when we identity $\mathrm{D}f(u)$ to a ts rows when we view it as a matrix:as given by

$$\mathrm{D}f(u)_j =$$

$$\widetilde{x}_j \circ \mathrm{D}f(u) =$$

Then for $1 \le i_1, i_2 \le m$, we have

Now let us fix linear coordinates on V and W.

$$\begin{aligned} D^2f(u_1 \otimes u_2) &= (D(Df))^\diamond(u_1 \otimes u_2) \\ &= D((Df(u_1))(u_2). \end{aligned}$$

Thus if we fix linear coordinates on V and W, we see that it has the form

$$Df(u_1)^j_i = \partial_{x_i} f_j(u_1)$$

$$D^2f(u_1 \otimes u_2) = (D(Df))^\diamond(u_1 \otimes u_2) = (D(Df)u$$
$$D^2f(u)(u_1 \otimes u_2) = (D(Df)(u))^\diamond(u_1 \otimes u_2) = (D(Df))u$$
$$D^2f(u) = (D(Df)(u))^\diamond$$

What does it mean to say that $Df\colon U \to \mathrm{Hom}(V, W)$ is continuous? Upon fixing linear coordinates on V and W, such continuity amounts to continuity for each of the component functions $\partial_{x_i} f_j\colon U \to \mathbb{R}$ of the matrix-valued Df, and so the concrete definition of f being C^1 is equivalent to the coordinate-free property that $f\colon U \to W$ is differentiable and that the associated total derivative map $Df\colon U \to \mathrm{Hom}(V, W)$ from U to a new vector space $\mathrm{Hom}(V, W)$ is continuous. With this latter point of view, wherein Df is a map from the open set $U \subseteq V$ into a finite-dimensional vector space $\mathrm{Hom}(V, W)$, a very natural question is: what does it mean to say that Df is differentiable, or even continuously so?

Lemma 6.2. *Suppose $f\colon U \to W$ is a C^1 map, and let $Df\colon U \to \mathrm{Hom}(V, W)$ be the associated total derivative map. As a map from an open set in V to a finite-dimensional vector space, Df is C^1 if and only if (relative to a choice of linear coordinates on V and W) all second-order partials $\partial_{x_{i_1}} \partial_{x_{i_2}} f_j\colon U \to \mathbb{R}$ exist and are continuous.*

Proof. Fixing linear coordinates identifies Df with a map from an open set $U \subseteq \mathbb{R}^m$ to a Euclidean space of $n \times m$ matrices, with component functions $\partial_{x_i} f_j$ for $1 \le i \le m$ and $1 \le j \le n$. Hence, this map is C^1 if and only if these components admit all first-order partials that are moreover continuous, and this is exactly the statement that the f_j's admit all second-order partials and that such partials are continuous. □

Let us say that $f\colon U \to W$ is C^2 when it is differentiable and $Df\colon U \to \mathrm{Hom}(V, W)$ is C^1. By the lemma, this is just a fancy way to encode the concrete definition that all component functions of f (relative to linear coordinizations of V and W) admit continuous second-order partials. Next let us consider the total derivative of Df, that is,

$$D^2f = D(Df)\colon U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W)) \cong \mathrm{Hom}(V \otimes V, W)$$

More to the point, how do we work with the vector space $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$? I claim that it is not nearly as complicated as it may seem, and that once we understand how to think about this iterated Hom-space we will see that the theory of higher-order partials admits a very pleasing reformulation in the language of multilinear mappings. The underlying mechanism is a certain isomorphism in linear algebra, so we now digress to discuss the algebraic preliminaries in a purely algebraic setting over any field.

Definition 6.3. In general, for an integer $p \ge 1$ we say that $f\colon U \to W$ is a C^p **map**, or is p **times continuously differentiable**, if it is differentiable and $Df\colon U \to \mathrm{Hom}(V, W)$ is a C^{p-1} map. If f is a C^p map for every p, we shall say that f is a C^∞ **map**, or is **infinitely differentiable**.

6.3 Higher Derivatives as Symmetric Multilinear Maps

Let V and W be finite-dimensional vector spaces over $\mathbb{R}$, and let U be open in V. Let $f : U \to W$ be a map of sets. We say f is a C^0 **map** if it is continuous. We have seen above that f is differentiable with

$$Df : U \to \mathrm{Hom}(V, W)$$

continuous if and only if, with respect to a choice of linear coordinates, the components f_j of f admit continuous first-order partial derivatives across all of U with respect to the coordinates on V. This property of f is called being a C^1 **map**, and we may rephrase it as the property that f is differentiable and Df is continuous. We now make a recursive definition:

Definition 6.4. In general, for an integer $p \ge 1$ we say that $f : U \to W$ is a C^p **map**, or is p **times continuously differentiable**, if it is differentiable and

$$Df : U \to \mathrm{Hom}(V, W)$$

is a C^{p-1} map. If f is a C^p map for every p, we shall say that f is a C^∞ **map**, or is **infinitely differentiable**.

Assuming f is C^2, we write $D^2f(u)$ to denote $D(Df)(u)$, and by definition since $Df : U \to \mathrm{Hom}(V, W)$ is a differentiable map from an open in V to the vector space $\mathrm{Hom}(V, W)$, we see that $D^2f(u)$ is a linear map from V to $\mathrm{Hom}(V, W)$. That is, we have

$$D^2f : U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W))$$

and this is continuous (as f is C^2). More generally, if f is C^p, then for $i \le p$ we write $D^i f = D(D^{i-1}f)$, and arguing recursively we see that $D^p f(u)$ is a linear map from V to $\mathrm{Hom}(V, \mathrm{Hom}(V, \ldots, \mathrm{Hom}(V, W)\ldots))$ where there are $p-1$ iterated Hom's. That is, we have

$$D^p f : U \to \mathrm{Hom}(V, \mathrm{Hom}(V, \ldots, \mathrm{Hom}(V, W)\ldots)) \simeq \mathrm{Mult}(V^p, W).$$

Theorem 6.3. *Suppose $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. Let $U \subseteq V$ be open an let $f_i : U \to \mathbb{R}$ denote the ith component of f, so f is described as a map $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m = W$. Let $p \ge 0$ be a non-negative integer. Then f is a C^p map if and only if all p-fold iterated partial derivatives of the f_i's exist and are continuous on U. Likewise, f is C^∞ if and only if all f_i's admit all iterated partials of all orders.*

Proof. We induct on p, the case $p = 0$ being the old result that a map f to a product space is continuous if and only if its component maps f_i are continuous. For $p = 1$, the theorem is our earlier observation that f is differentiable with $Df : U \to \mathrm{Hom}(V, W)$ continuous if and only if the component functions f_i of f admit continuous first-order partials.

Now we assume $p > 1$, so in either direction of implication in the theorem we know (from the C^1 case which has been established) that f admits a continuous derivative map Df and that all partials $\partial_{x_j} f_i$ exist as continous functions on U. Also, we know that the map

$$Df : U \to \mathrm{Hom}(V, W) \simeq \mathrm{Mat}_{m\times n}(\mathbb{R})$$

to the vector space of $m \times n$ matrices has as its component functions (i.e. "matrix entries") precisely the first-order partials $\partial_{x_j} f_i : U \to \mathbb{R}$.

By definition, f is C^p if and only if Df is C^{p-1}, but since this latter map has the $\partial_{x_j} f_i$'s as its component functions, by the inductive hypothesis applied to Df (with the target vector space now $\mathrm{Hom}(V, W)$ rather than W, and linear coordinates given by matrix entries), it follows that Df is C^{p-1} if and only if all $\partial_{x_j} f_i$'s admit all $(p-1)$-fold iterated partial derivatives in the linear coordinates on V and that these are continuous. Since an arbitrary $(p-1)$-fold partial of an arbitrary first order partial $\partial_{x_j} f_i$ is nothing more or less than an arbitrary p-fold partial of f_i with respect to the linear coordinates on V, we conclude that f is C^p if and only if all p-fold partials of all f_i's with respect to the linear coordinates on V exist and are continuous. □

Let $f : U \to W$ be a C^p mapping with $p \ge 1$, and consider the continuous pth derivative mapping

$$D^p f : U \to \mathrm{Mult}(V^p, W).$$

We want to describe this in terms of partial derivatives using linear coordinates on V and W. That is, we fix ordered bases $\{e_1, \ldots, e_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W, so for each $u \in U$ the multilinear mapping

$$D^p f(u) : V^p \to W = \mathbb{R}^m$$

is uniquely determined by the m-tuples

$$D^p f(u)(e_{j_1}, \ldots, e_{j_p}) \in W = \mathbb{R}^m$$

for $1 \le j_1, \ldots, j_p \le n$. What are the m components of this vector in $\mathbb{R}^m$? The answer is very nice:

Theorem 6.4. *With notation as above, let $x_1, \ldots, x_n \in V^\vee$ be the dual basis to the basis $\{e_1, \ldots, e_n\}$ of V. Let ∂_j denote ∂_{x_j}, and let $f_1, \ldots, f_m$ be the component functions of $f : U \to W$ with respect to the basis of w_i's of W. For $1 \le j_1, \ldots, j_p \le n$,*

$$D^p f(u)(e_{j_1}, \ldots, e_{j_p}) = ((\partial_{j_p} \cdots \partial_{j_1} f_1)(u), \ldots, (\partial_{j_p} \cdots \partial_{j_1} f_m)(u)) \in \mathbb{R}^m.$$

Proof. We induct on p, the base case $p = 1$ being the old theorem on the determination of the matrix for the derivative map $Df(u) : V \to W$ in terms of first-order partials of the component functions for f (using linear coordinates on W to define these component functions, and using linear coordinates on V to define the relevant partial derivative operators on these functions). Now we assume $p \ge 2$.

By definition of the isomorphism in Corollary (**??**), we have

$$D^p f(u)(v_1, \ldots, v_p) = (\cdots((D^p f(u)(v_1))(v_2)\cdots)(v_p) \in W$$

for any ordered p-tuple $v_1, \ldots, v_p \in V$. Let $F = Df : U \to \mathrm{Hom}(V, W)$. Using the given linear coordinates on V and W, the associated "matrix entries" are taken as the linear coordinates on $\mathrm{Hom}(V, W)$ to get component functions F_{ij} for F (with $1 \le i \le m$ and $1 \le j \le n$). Considering $v_2, \ldots, v_p$ as fixed but v_1 as varying, we have

$$D^p f(u)(\cdot, v_2, \ldots, v_p) = (\cdots((D^{p-1}F)(u)(v_2))\cdots)(v_p) = D^{p-1}F(u)(v_2, \ldots, v_p) \in \mathrm{Hom}(V, W)$$

where $\mathrm{Hom}(V,W)$ is the target vector space for F. Setting $v_k = e_{j_k}$ for $2 \leq k \leq p$, the inductive hypothesis applied to $F : U \to \mathrm{Hom}(V,W) = \mathrm{Mat}_{m\times n}(\mathbb{R})$ gives

$$D^{p-1}F(u)(e_{j_2},\dots,e_{j_p}) = (\partial_{j_p}\cdots\partial_{j_2}F_{ij}(u)) \in \mathrm{Mat}_{m\times n}(\mathbb{R}).$$

In view of how the matrix coordinatization of $\mathrm{Hom}(V,W)$ was *defined* using the chosen ordered bases on V and W, evaluating e_{j_1} in $\mathrm{Hom}(V,W) \simeq \mathrm{Mat}_{m\times n}(\mathbb{R})$ corresponds to pass to the j_1th column of a matrix. Hence taking $v_1 = e_{j_1}$ gives

$$D^p f(u)(e_{j_1}, e_{j_2},\dots,e_{j_p}) = (\partial_{j_p}\cdots\partial_{j_2}F_{1j_1}(u),\dots,\partial_{j_p}\cdots\partial_{j_2}F_{mj_1}(u)) \in \mathbb{R}^m = W.$$

By the C^1 case, $F = Df : U \to \mathrm{Hom}(V,W) = \mathrm{Mat}_{m\times n}(\mathbb{R})$ has ij-component function $F_{ij} = \partial_j f_i$, so $F_{ij_1} = \partial_{j_1} f_i$. Thus, we get the desired formula. □

Example 6.4. Suppose $W = \mathbb{R}$ and let $x_1,\dots,x_m$ be linear coordinates on V relative to some ordered basis $\boldsymbol{e} = (e_1,\dots,e_m)$ on V. Then $\mathrm{D}^2 f(u)$ is identified with the **Hessian** of f at u:

$$\mathrm{D}^2 f(u) = \mathrm{H}_f(u) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(u)\right).$$

Hence, the Hessian that appears in the second derivative test in several variables is *not* a linear map (as might be suggested by its traditional presentation as a matrix) but rather is intrinsically seen to be a symmetric bilinear form.

6.4 Higher-Dimensional Taylor's Formula: Motivation and Preparations

As an application of the formalism of higher derivatives as multilinear mappings, we wish to state and prove Taylor's formula (with an integral remainder term) for C^α maps $f : U \to W$ on any open $U \subseteq V$. In the special case $V = W = \mathbb{R}$ and U a non-empty open interval, this will recover the usual Taylor formula from calculus. There is also a more traditional version of the multivariable Taylor formula given with loads of mixed partials and factorials, and we will show that this traditional version is equivalent to the version we will prove in the language of higher derivatives as multilinear mappings. The power of our point of view is that it permits one to give a proof of Taylor's formula that is virtually identical in appearance to the proof in the classical case (with $V = W = \mathbb{R}$); proofs of Taylor's formula in the classical language of mixed partials tend to become a big mess with factorials, and the integral formula and error bound for the remainder term are unpleasant to formulate in the classical language.

Before we state the general case, let us first recall the 1-variable Taylor formula for a C^p function $f\colon I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ with $a \in I$ an interior point: for $|h|$ sufficiently small so that $(a-h, a+h) \in I$ we have

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(p)}(a)}{p!}h^p + R_{p,a}(h)$$

with error term $R_{p,a}$ is given by

$$R_{p,a}(h) = \int_0^1 \frac{f^{(p)}(a+th) - f^{(p)}(a)}{(p-1)!}\cdot(1-t)^{p-1}h^p\mathrm{d}t = h^p\psi_{p,a}(h),$$

where $|\psi_{p,a}(h)|$ can be made below any desired ε for h near 0 (uniformly for a in a compact subinterval of I) since the continuous $f^{(p)}$ is uniformly continuous on compacts in I. In particular, as $h \to 0$ we have $|R_{p,a}(h)|/|h|^p \to 0$ *uniformly* for a in a compact subinterval of I.
We calculate

$$\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}f(a+th)\right)\mathrm{d}t$$

We want to show Observe that

$$\begin{aligned} f(x+h) - f(x) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}f(x+th)\mathrm{d}t \\ &= \int_0^1 hf'(x+th)\mathrm{d}t \\ &= h\int_0^1 f'(x+th)\mathrm{d}t \\ &= h\psi(h), \end{aligned}$$

where we set $\psi(h) = \int_0^1 f'(x+th)\mathrm{d}t$. Notice that

$$\begin{aligned}
\lim_{h\to 0} \int_0^1 f'(x+th)\mathrm{d}t &= \int_0^1 \lim_{h\to 0} f'(x+th)\mathrm{d}t \\
&= \int_0^1 \lim_{h\to 0} f'(x+th)\mathrm{d}t \\
&= \int_0^1 f'(x)\mathrm{d}t \\
&= f'(x)
\end{aligned}$$

We claim that

$$\lim_{h\to 0} f(h,x) = \lim_{n\to\infty} f(1/n,x).$$

Indeed, suppose $\lim_{h\to 0} f(h,x) = f(x)$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|h| < \delta$ implies $|f(h,x) - f(x)| < \varepsilon$. Then $1/n < \delta$ implies $|f(1/n,x) - f(x)| < \varepsilon$.

6.4.1 Taylor's Formula: Statement and Proof

Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, U and open subset in V, and $f : U \to W$ a C^p map with $p \geq 1$. We choose $a \in U$ and $r > 0$ such that $B_r(a) \subseteq U$ (relative to an arbitrary but fixed choice of norm on V). Thus, $f(a+h)$ makes sense for $h \in V$ satisfying $\|h\| < r$. Now we can state and prove Taylor's formula by essentially just copying the proof from calculus!

Theorem 6.5. *With the notation as above,*

$$f(a+h) = \sum_{j=0}^{p} \frac{(\mathrm{D}^j f)(a)}{j!}(h^{(j)}) + R_{p,a}(h) \tag{16}$$

in W, where

$$R_{p,a}(h) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\right)(h^{(p)})\mathrm{d}t$$

satisfies

$$\|R_{p,a}(h)\| \leq C_{p,a}(h)\|h\|^p \text{ and } \lim_{h\to 0} C_{p,a}(h) = 0 \tag{17}$$

with

$$C_{p,a}(h) = \sup_{t\in[0,1]} \frac{\|(\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\|}{p!}.$$

The convergence $C_{p,a}(h) \to 0$ as $h \to 0$ is uniform for a supported in a compact subset of U.

Remark 9. The norm on $\mathrm{Mult}(V^p, W)$ that is implicit in the numerator defining $C_{p,h,a}$ is defined in terms of arbitrary but fixed choices of norms on V and W: for any multilinear $\mu\colon V^p \to W$ there exists a constant $B \geq 0$ such that $\|\mu(v_1, \ldots, v_p)\| \leq B \prod_{j=1}^{p} \|v_j\|$, by elementary arguments exactly as in the simplest case $p = 1$, and the infimum of all such B's also works and is called $\|\mu\|$. More concretely, $\|\mu\|$ is the minimum of $\|\mu(v_1, \ldots, v_n)\|$ for the compact set of points $(v_1, \ldots, v_p) \in V^p$ satisfying $\|v_j\| = 1$ for all j. It is easy to check that $\mu \mapsto \|\mu\|$ is a norm on the finite-dimensional vector space $\mathrm{Mult}^p(V, W)$, and in particular is a continuous $\mathbb{R}$-valued function on this space of multilinear mappings. Thus

$$\begin{aligned}
\|R_{p,a}(h)\| &= \left\| \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a) \right) (h^{(p)}) \mathrm{d}t \right\| \\
&\leq \int_0^1 \left\| \frac{(1-t)^{p-1}}{(p-1)!} \left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a) \right) (h^{(p)}) \right\| \mathrm{d}t \\
&= \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \left\| \left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a) \right) (h^{(p)}) \right\| \mathrm{d}t \\
&\leq \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \|(\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\| \|h\|^p \mathrm{d}t \\
&\leq \left(\sup_{t \in [0,1]} \|(\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\| \right) \|h\|^p \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \mathrm{d}t \\
&= \frac{1}{p!} \left(\sup_{t \in [0,1]} \|(\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\| \right) \|h\|^p \\
&= C_{p,a}(h) \|h\|^p.
\end{aligned}$$

One important consequence of the error estimate (17) is that it shows the error $R_{p,a}(h)$ in the "degree p" expansion (16) of $f(a+h)$ about a dies off more rapidly than $\|h\|^p$ as $h \to 0$, that is $\|R_{p,a}(h)\| / \|h\|^p \to 0$ as $h \to 0$ with the rate of such decay actually *uniform* for a supported in a fixed compact subset of U. This is tremendously important for some applications.

A particular important case is $p = 2$: the approximation

$$f(a+h) = f(a) + (\mathrm{D}f(a))(h) + (\mathrm{D}^2 f(a))(h, h) + (\ldots)$$

has an error which dies more rapidly than $\|h\|^2$. This is what underlies the reason why the symmetric bilinear Hessian $\mathrm{H}_f(a) = (\mathrm{D}^2 f)(a)$ governs the structure of f near critical points (that is those with $\mathrm{D}f(a) = 0$, such as local extrema) in the case when $W = \mathbb{R}$. That is, the signature of the quadratic form associated to $\mathrm{H}_f(a)$ encodes much of the local geometry for f near a when $\mathrm{D}f(a) = 0$.

Theorem 6.6. *Let $U \subseteq V$ be an open set and let $f\colon U \to \mathbb{R}$ be a C^1 function. If a is a local minimizer for f, then $\nabla f(a) = 0$. In this case, we say a is a **critical point** of f.*

Proof. Replacing f with $f - f(a)$, we may assume that $f(a) = 0$. By Taylor's formula, for small h we have

$$f(a+h) = \nabla f(a)^\top h + R_a(h)$$

where $\|R_a(h)\| \leq C_{a,h} \|h\|$ and $C_{a,h} \to 0$ as $h \to 0$. Setting $h_t = -t \nabla f(a)$ where $t \in (0, 1)$ gives us

$$f(a+h_t) = -t \|\nabla f(a)\|^2 + R_a(h_t).$$

Choose $\delta > 0$ such that $t < \delta$ implies $\|R_a(h_t)\| \leq \|h_t\|/2$. Then $t < \delta$ implies

$$\begin{aligned}
f(a+h_t) &= -t \|\nabla f(a)\|^2 + R_a(h_t) \\
&\leq -t \|\nabla f(a)\|^2 + \frac{1}{2} \|h_t\| \\
&= -t \|\nabla f(a)\|^2 + \frac{t}{2} \|\nabla f(a)\|^2 \\
&= -\frac{t}{2} \|\nabla f(a)\|^2 \\
&< 0.
\end{aligned}$$

Since f has a local minimum at a, we must have $\|\nabla f(a)\| = 0$, which implies $\nabla f(a) = 0$. $\square$

Theorem 6.7. *Let $U \subseteq V$ be an open set and let $f\colon U \to \mathbb{R}$ be a C^2 function. Suppose that a is a critical point for f in the sense that $\mathrm{D}f(a) = 0$ for some $a \in U$. Let $\mathrm{H}_f(a)\colon V \times V \to \mathbb{R}$ be the symmetric bilinear Hessian $\mathrm{D}^2 f(a)$, and let $q_{f,a}\colon V \to \mathbb{R}$ be the associated quadratic form. If $\mathrm{H}_f(a)$ is non-degenerate, then f has an isolated local minimum at a when $q_{f,a}$ is positive-definite, an isolated local maximum at a when $q_{f,a}$ is negative-definite, and neither a local minimum nor a maximum in the indefinite case.*

Proof. Replacing f with $f - f(a)$, we may assume that $f(a) = 0$. By Taylor's formula, for small h we have

$$\frac{f(a+h)}{\|h\|^2} = \frac{1}{2}\mathrm{H}_f(a)(\widehat{h}, \widehat{h}) + R_a(h) = q_{f,a}(\widehat{h}) + R_a(h)$$

where $R_a(h) \to 0$ as $h \to 0$ and $\widehat{h} = h/\|h\|$ is a unit vector pointing in the same direction as h. Thus, $f(a+h)/\|h\|^2$ is approximated by $q_{f,a}(\widehat{h})$ up to an error that ends to 0 locally uniformly in a as $h \to 0$. Provided that $q_{f,a}$ is non-degenerate, in the positive-definite case it is bounded below by some $c > 0$ on the unit sphere, and hence (depending on c) by taking h sufficiently small we get $f(a+h)/\|h\|^2 \geq c/2 > 0$. This shows that f has an isolated local minimum at a, and a similar argument gives an isolated local maximum at a if $q_{f,a}$ is negative-definite.

Now suppose that $q_{f,a}$ is indefinite. By the spectral theorem, if we choose the norm on V to come from an inner product, then the pairing $\mathrm{H}_f(a)$ is given by the inner product against an orthogonal linear map. Hence, in such cases we can find an orthonormal basis with respect to which $q_{f,a}$ is diagonalized, and so in the indefinite case there are lines on which the restriction of $q_{f,a}$ is positive-definite and there are lines on which the restriction of $q_{f,a}$ is negative-definite. Approaching a along such directions gives different types of behavior for f at a (isolated local minimum when approaching through the positive light cone for $q_{f,a}$, and an isolated local maximum when approaching through the negative light cone for $q_{f,a}$, provided the approach is not tangential to the null cone of vectors $v \in V$ for which $q_{f,a}(v) = 0$). This gives the familiar "saddle point" picture for the behavior of f, with the shape of the saddle governed by the eigenspace decomposition for the orthogonal map arising from the Hessian $\mathrm{H}_f(a)$ and the choice of inner product on V. □

Now we prove Taylor's Theorem.

Proof. By the second Fundamental Theorem of Calculus (applied componentwise using a basis of W, say), we have

$$\begin{aligned}
f(a+h) &= f(a) + (f(a+h) - f(a)) \\
&= f(a) + \int_0^1 \mathrm{D}f(a+th)h\mathrm{d}t \\
&= f(a) + \int_0^1 \mathrm{D}f(a+th)h\mathrm{d}t + \mathrm{D}f(a)h - \mathrm{D}f(a)h \\
&= f(a) + \int_0^1 \mathrm{D}f(a+th)h\mathrm{d}t + \mathrm{D}f(a)h - \int_0^1 \mathrm{D}f(a)h\mathrm{d}t \\
&= f(a) + \mathrm{D}f(a)h + \int_0^1 (\mathrm{D}f(a+th) - \mathrm{D}f(a))h\mathrm{d}t
\end{aligned}$$

in W. This takes care of the case $p = 1$. Now we assume $p > 1$ and we use induction. Since f is also of class C^{p-1}, we have

$$f(a+h) = \sum_{j=0}^{p-1} \frac{\mathrm{D}^j f(a)}{j!}(h^{(j)}) + R_{p-1,a}(h)$$

in W, where

$$R_{p-1,a}(h) = \int_0^1 \frac{(1-t)^{p-2}}{(p-2)!}\left((\mathrm{D}^{p-1}f)(a+th) - (\mathrm{D}^{p-1}f)(a)\right)(h^{(p-1)})\mathrm{d}t.$$

in W. Thus we just have to show that

$$R_{p-1,a}(h) = \frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) + R_{p,a}(h)$$

in W, where

$$R_{p,a}(h) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\right)(h^{(p)})\mathrm{d}t.$$

Note that

$$\begin{aligned}
\frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) + R_{p,a}(h) &= \frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}\left((\mathrm{D}^p f)(a+th) - (\mathrm{D}^p f)(a)\right)(h^{(p)})\mathrm{d}t \\
&= \frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}(\mathrm{D}^p f)(a+th)(h^{(p)})\mathrm{d}t - \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}(\mathrm{D}^p f)(a)(h^{(p)})\mathrm{d}t \\
&= \frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}(\mathrm{D}^p f)(a+th)(h^{(p)})\mathrm{d}t - \frac{1}{p!}\mathrm{D}^p f(a)(h^{(p)}) \\
&= \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}(\mathrm{D}^p f)(a+th)(h^{(p)})\mathrm{d}t,
\end{aligned}$$

so we really just need to show that

$$R_{p-1,a}(h) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \mathrm{D}^p f(a+th)(h^{(p)}) \mathrm{d}t$$

in W. By using the identification $\mathrm{Mult}(V^p, W) \simeq \mathrm{Hom}(V, \mathrm{Mult}(V^{p-1}, W))$ it suffices to prove that in $\mathrm{Mult}(V^{p-1}, W)$ we have an equality

$$\int_0^1 (p-1)(1-t)^{p-2}(\mathrm{D}^{p-1}f(a+th) - \mathrm{D}^{p-1}f(a))\mathrm{d}t = \int_0^1 (1-t)^{p-1}\mathrm{D}^p f(a+th)h\mathrm{d}t,$$

where the evaluation at $h \in V$ on the right is really evaluation in the first slot of a symmetric multilinear mapping on V^p, since then evaluation on $h^{(p-1)} = (h, \cdots, h) \in V^{p-1}$ (which can be moved inside a definite integral) and division by $(p-1)!$ will yield what we want. Let $g = \mathrm{D}^{p-1}f \colon U \to \mathrm{Mult}(V^{p-1}, W)$, so g is a C^1 map and we want to show

$$\int_0^1 (p-1)(1-t)^{p-2}(g(a+th) - g(a))\mathrm{d}t = \int_0^1 (1-t)^{p-1}\mathrm{D}g(a+th)(h)\mathrm{d}t.$$

This essentially comes down to integration by parts. We can rewrite our desired equation as

$$g(a) = \int_0^1 ((p-1)(1-t)^{p-2}g(a+th) - (1-t)^{p-1}\mathrm{D}g(a+th)h)\mathrm{d}t.$$

Consider the map

$$\phi \colon (-1, 1+\varepsilon) \to \mathrm{Mult}(V^{p-1}, W)$$

defined by

$$\phi(t) = -(1-t)^{p-1}g(a+th)$$

where $\varepsilon > 0$ is small enough so that $(1+\varepsilon)\|h\| < r$ and hence $a + th \in \mathrm{B}_r(a)$ for $t \in (-1, 1+\varepsilon)$. Since g is C^1, a straightforward application of the Chain Rule yields that ϕ is C^1 with

$$\phi'(t) = \mathrm{D}\phi(t)(1) = (p-1)(1-t)^{p-2}g(a+th) - (1-t)^{p-1}\mathrm{D}g(a+th)h$$

in $\mathrm{Mult}(V^{p-1}, W)$. This is exactly the integrand we need, so we are reduced to proving $g(a) = \int_0^1 \phi'$. But by the second Fundamental Theorem of Calculus (applied componentwise with respect to a basis of the vector space $\mathrm{Mult}(V^{p-1}, W)$, say), this latter integral is equal to $\phi(1) - \phi(0)$, and from the definition of ϕ this is exactly $g(a)$ as desired. □

7 Morse Lemma

Let V be a finite-dimensional nonzero $\mathbb{R}$-vector space and let $f \colon U \to \mathbb{R}$ be a C^p-function with $2 \le p \le \infty$. Suppose for $u_0 \in U$ we have $\mathrm{d}f(u_0) = 0$; that is, u_0 is a critical point of f. We seek a convenient coordinate system on a neighborhood of u_0 in U that will help us to see how f behaves near u_0. The Morse Lemma in the C^∞ case is this:

Theorem 7.1. *Let V be a finite-dimensional vector space and $U \subseteq V$ an open set. Let $f \colon U \to \mathbb{R}$ be a C^∞-function and suppose f has a non-degenerate critical point at $u_0 \in U$. For a suitable C^∞-coordinate system*

$$\varphi = (x_1, \ldots, x_n) : U_0 \to \mathbb{R}^n$$

on an open $U_0 \subseteq U$ around u_0 with $\varphi(u_0) = 0$, the mapping $[f] = f \circ \varphi^{-1} \colon \varphi(U_0) \to \mathbb{R}$ that is "f in the x_i-coordinates" is given by

$$[f](\boldsymbol{a}) = \sum_{i=1}^{r} a_i^2 - \sum_{j=1}^{s} a_{r+j}^2 \tag{18}$$

for all $\boldsymbol{a} \in \varphi(U_0)$, with $(r, s) = (r, n-r)$ the signature of the quadratic form $\mathrm{q}_f(u_0) \colon V \to \mathbb{R}$ associated to the symmetric bilinear form $\mathrm{H}_f(u_0)$ on V.

Remark 10. We could rewrite (18) as

$$[f] = \sum_{i=1}^{r} x_i^2 - \sum_{j=1}^{s} x_{r+j}^2$$

where $x_1, \ldots, x_n$ are the standard coordinate functions on $\mathbb{R}^n$.

7.1 Separation of Variables

We shall deduce the Morse lemma from a more general result that is called "separation of variables"

Theorem 7.2. *Let V be a finite-dimensional vector space and $U \subseteq V$ an open set. Let $f: U \to \mathbb{R}$ be a C^∞-function and suppose f has a non-degenerate critical point at $u_0 \in U$. There exists a C^∞-coordinate system $\varphi = (x_1, \dots, x_n): U_0 \to \mathbb{R}^n$ on an open neighborhood of u_0 in U with $\varphi(u_0) = 0$ such that $[f] = f \circ \varphi^{-1}$ is given by $\varepsilon x_1^2 + F$ on $\varphi(U_0) \subseteq \mathbb{R}^n$ with F a C^∞-function of $x_2, \dots, x_n$.*

Remark 11. For $n = 1$, this theorem just says that if f is a smooth function near the origin in $\mathbb{R}$ with $f(0) = f'(0) = 0$ but $f''(0) \neq 0$ (i.e. 0 is a non-degenerate critical point of f), then $f = \varepsilon k^2$ for $\varepsilon = \pm 1$ and some smooth function k near the origin with $k(0) = 0$ but $k'(0) \neq 0$ (as such a k provides a local C^∞-coordinate near the origin on the real line).

Step 1: We show that since f is smooth and $f(0) = 0$, we have $f(t) = tg(t)$ for a smooth function g near the origin. Indeed, we define

$$g(t) := \int_0^1 f'(ty)\mathrm{d}y.$$

By the theorem on differentiation through the integral sign, we have

$$g^{(n)}(t) = \int_0^1 y^n f^{(n+1)}(ty)\mathrm{d}y. \tag{19}$$

where $g^{(n)}(t)$ is continuous in t since the integrand on the righthand side of (19) is continuous in t with y fixed and since the integrand on the righthand side of (19) is dominated by some constant (we are secretly applying the dominated convergence theoerem here). It follows that g is a smooth function, and by the Fundamental Theorem of Calculus, we have

$$g(t) = \frac{f(ty)}{t}\Big|_{y=0}^{y=1} = \frac{f(t)}{t},$$

whenever $t \neq 0$. In other words, we have $f(t) = tg(t)$ for all t (including $t = 0$ since $f(0) = 0 = g(0)$).

Step 2: Since g is smooth and $g(0) = f'(0) = 0$, we can repeat the process and obtain $f(t) = t^2 G(t)$ with G smooth near the origin. Thus $G(0) = f''(0) \neq 0$, so if this has the same sign as $\varepsilon = \pm 1$ then $f(t) = \varepsilon t^2(\varepsilon G)(t)$ with εG a smooth function that is positive at the origin. Hence, it admits a smooth positive square root (possibly on a smaller open neighborhood of 0), so we get the result for f by setting $k = t\sqrt{\varepsilon G}$. This establishes the case where $n = 1$.

Proof. We may assume that $n = \dim V > 1$. Additive translation has no effect on derivative maps, nor on Hessians. Thus by replacing U with $U - u_0 = \{u - u_0 \mid u \in U\}$ and replacing $f: U \to \mathbb{R}$ with $f_{u_0}: U - u_0 \to \mathbb{R}$ (where $f_{u_0}(u - u_0) = f(u)$) if necessary, we may suppose $u_0 = 0$ in V (thus $f(0) = 0$ and $\mathrm{D}f(0) = 0$). Since the symmetric bilinear form $\mathrm{H}_f(0)$ is nonzero, its associated quadratic form $\mathrm{q}_f(0): V \to \mathbb{R}$ is nonzero. By the structure theorem for quadratic spaces over $\mathbb{R}$, we may choose linear coordinates $\{y_1, \dots, y_n\}$ on V such that $\mathrm{q}_f(0)$ is in standard diagonal form, say $\varepsilon y_1^2 + \cdots$ with $\varepsilon = \pm 1$. By replacing f with $-f$ if necessary, we may assume that $\varepsilon = 1$.

Observe that $\partial_{y_1} f(0, \mathbf{0}) = 0$ and $\partial_{y_1}^2 f(0, \mathbf{0}) = 1 \neq 0$. Thus the implicit function theorem implies for each $\boldsymbol{y} = (y_2, \dots, y_n)$ near the origin $\mathbf{0}$ there exists a unique $g(\boldsymbol{y})$ near 0 satisfying

$$\partial_{y_1} f(g(\boldsymbol{y}), \boldsymbol{y}) = 0,$$

(so $g(\mathbf{0}) = 0$) and g is a C^∞ function. Thus if we fix $c > 0$ then by continuity of g we conclude that for $|a_2|, \dots, |a_n|$ sufficiently small (depending on c) the function $f(y_1, \boldsymbol{a})$ has a unique critical point at $y_1 = g(\boldsymbol{a})$ in the interval $(-c, c)$ and the second derivative at this critical point is 1. By taking c possibly smaller, we can assume that $|a_2|, \dots, |a_n| < c$ is "sufficiently small". So $f(y_1, \boldsymbol{a})$ on $(-c, c)$ has a unique minimum at $y_1 = g(\boldsymbol{a})$ with positive second derivative there. Define the function

$$k(y_1, \boldsymbol{y}) = f(y_1, \boldsymbol{y}) - f(g(\boldsymbol{y}), \boldsymbol{y}).$$

Thus for fixed $a_2, \dots, a_n \in (-c, c)$, the function $k(y_1, \boldsymbol{a})$ is non-negative with a unique zero at $y_1 = g(\boldsymbol{a})$ and a positive second derivative at this minimum point.

Suppose that $k(y_1, \boldsymbol{y})$ is the square of a C^∞ function h near the origin. By defining the C^∞ function

$$F(\boldsymbol{y}) := f(g(\boldsymbol{y}), \boldsymbol{y})$$

near the origin we get $f(y_1, \boldsymbol{y}) = h^2 + F(\boldsymbol{y})$, so we would be done as long as $\{h, \boldsymbol{y}\}$ is a C^∞ coordinate system near the origin. By the inverse function theorem, this amounts to the condition that $(\partial_{y_1} h)(0, \mathbf{0}) \neq 0$. But such non-vanishing is clear because for y_1 near 0 we see that

$$h(y_1, \mathbf{0})^2 = f(y_1, \mathbf{0}) - f(g(\mathbf{0}), \mathbf{0}) = f(y_1, \mathbf{0})$$

has Taylor expansion $y_1^2 + \cdots$ at the origin (as $f(0, \mathbf{0}) = 0$, $\partial_{y_1} f(0, \mathbf{0}) = 0$, and $\partial_{y_1}^2 f(0, \mathbf{0}) = \varepsilon = 1$), so the Taylor expansion of $h(y_1, \mathbf{0})$ at the origin must be $\pm y_1 + \cdots$.

It remains to show that k is the square of a C^∞ function near the origin. Indeed, let $y_1' = y_1 - g(\boldsymbol{y})$ and note that by the inverse function theorem, $\{y_1', \boldsymbol{y}\}$ is a C^∞ coordinate system near the origin. Thus if we let K denote k expressed in these coordinates, then $K(y_1', \boldsymbol{y})$ is a C^∞ function near the origin that vanishes for $y_1' = 0$. By applying the fundamental theorem of calculus to $u(t) = K(ty_1', \boldsymbol{y})$ with $y_1', \boldsymbol{y}$ all fixed, we see that

$$\begin{aligned} k(y_1, \boldsymbol{y}) &= K(y_1', \boldsymbol{y}) \\ &= \int_0^1 \partial_t K(ty_1', \boldsymbol{y}) \mathrm{d}t \\ &= y_1' \int_0^1 (\partial_1 K)(ty_1', \boldsymbol{y}) \mathrm{d}t \\ &= y_1' I(y_1, \boldsymbol{y}) \\ &= (y_1 - g(\boldsymbol{y})) I(y_1, \boldsymbol{y}) \end{aligned}$$

where we set

$$I(y_1, \boldsymbol{y}) = \int_0^1 (\partial_1 K)(ty_1', \boldsymbol{y}) \mathrm{d}t,$$

where the integrand is C^∞ in $y_1', \boldsymbol{y}$ (by differentiation through the integral sign and the C^∞ property of K). Thus, we have made a factorization

$$k(y_1, \boldsymbol{y}) = (y_1 - g(\boldsymbol{y})) I(y_1, \boldsymbol{y}) \tag{20}$$

with I a C^∞ function near the origin. Fix $\boldsymbol{y} = \boldsymbol{a}$ with $|a_i| < c$. As we have seen above, $k(y_1, \boldsymbol{a}) \geq 0$ has a critical point with positive second derivative at its unique minimum $y_1 = g(\boldsymbol{a})$ on $(-c, c)$ with $k(y_1, \boldsymbol{a})$ vanishing at this point, so the Taylor expansion for $k(y_1, \boldsymbol{a})$ at $y_1 = g(\boldsymbol{a})$ begins in degree 2 with positive coefficient. In particular, by considering Taylor expansions it follows from (20) that $I(y_1, \boldsymbol{a})$ vanishes at $y_1 = g(\boldsymbol{a})$ and has positive derivative at this point. Running through the same integration trick with the fundamental theorem of calculus again, we get

$$I(y_1, \boldsymbol{y}) = (y_1 - g(\boldsymbol{y})) J(y_1, \boldsymbol{y})$$

with $J(g(\boldsymbol{y}), \boldsymbol{y}) > 0$ for $y_1, \boldsymbol{y}$ near the origin. Feeding this into (20) and working with $y_1', \boldsymbol{y}$ as C^∞ coordinates near the origin we have

$$\begin{aligned} K(y_1', \boldsymbol{y}) &= k(y_1, \boldsymbol{y}) \\ &= (y_1 - g(\boldsymbol{y})) I(y_1, \boldsymbol{y}) \\ &= (y_1 - g(\boldsymbol{y})^2) J(y_1, \boldsymbol{y}) \\ &= y_1'^2 \widetilde{J}(y_1', \boldsymbol{y}) \end{aligned}$$

with $\widetilde{J}(0, \mathbf{0}) > 0$. We may therefore extract a C^∞ positive square root of $\widetilde{J}$ near the origin, so indeed K (and thus k) is a square of a C^∞ function near the origin. □

8 Manifold with Corners

Let W be an m-dimensional $\mathbb{R}$-vector space where $m \geq 1$. For $1 \leq k \leq m$, a k-**sector** in W is a non-empty subset of the form

$$\Sigma = \{w \in W \mid \ell_1(w) \geq c_1, \ldots, \ell_k(w) \geq c_k\} \tag{21}$$

with $c_1, \ldots, c_k \in \mathbb{R}$ and linearly independent $\ell_1, \ldots \ell_k \in W^\vee$. We often use the shorthand notation $\Sigma = \{\ell_1 \geq c_1, \ldots, \ell_k \geq c_k\}$ to denote (21). Observe that if $w \in W$ is a point, then the translation $w + \Sigma$ is also a k-sector since

$$w + \Sigma = \{\ell_1 \geq c_1 + \ell_1(w), \ldots, \ell_k \geq c_k + \ell_k(w)\}.$$

A 0-**sector** is $\Sigma = W$. A **sector** $\Sigma \subseteq W$ is a k-sector for some $0 \leq k \leq m$.

Lemma 8.1. *Let $\Sigma = \{\ell_1 \geq c_1, \ldots, \ell_k \geq c_k\}$ be a k-sector. There are exactly k translated hyperplanes H in W such that $H \cap \partial\Sigma$ contains a non-empty open set in H. In particular, these H's are of the form $\{\ell_i = c_i\}$ for all $1 \leq i \leq k$. In particular, the subset $\Sigma \subseteq W$ uniquely determines k and the pairs (ℓ_i, c_i) up to positive scaling.*

Proof. Extend $\{\ell_1, \ldots, \ell_k\}$ to an ordered basis $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_k, \ldots, \ell_m)$ of $W^\vee$ and let $\boldsymbol{e} = (e_1, \ldots, e_m)$ be the ordered basis of W whose dual basis is given by $\boldsymbol{\ell}$. Thus we have

$$\ell_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

for all $1 \le i, j \le m$. Let $w = \sum_{i=1}^k c_i e_i$ and let Σ_w denote the translated k-sector $\Sigma - w$. Then after identifying W with $\mathbb{R}^m$ using the ordered basis $\boldsymbol{e}$, we see that

$$\Sigma_w = [0, \infty)^k \times \mathbb{R}^{m-k}.$$

It suffices to show that there are exactly k hyperplanes H in W such that $H \cap \partial\Sigma_w$ contains a non-empty open set in H. First let us calculate $\partial\Sigma_w$. Observe that

$$\begin{aligned} \operatorname{int}\Sigma_w &= \operatorname{int}([0,\infty)^k \times \mathbb{R}^{m-k}) \\ &= \operatorname{int}([0,\infty))^k \times \operatorname{int}(\mathbb{R})^{m-k} \\ &= (0,\infty)^k \times \mathbb{R}^{m-k}. \end{aligned}$$

It follows that

$$\partial\Sigma_w = \bigcup_{i=1}^k [0,\infty)^{i-1} \times \{0\} \times [0,\infty)^{k-i} \times \mathbb{R}^{m-k},$$

where we make the convention that $[0,\infty)^0 = \{0\}$ in the union above. Thus $\partial\Sigma_w$ is the union of k sets $\Sigma_w \cap H_i = \partial\Sigma_w \cap H_i$ where $H_i = \{\ell_i = 0\}$ for $1 \le i \le k$, each of which contains a non-empty open in the hyperplane H_i, namely

$$U_i = \{\ell_i = 0\} \cap \bigcap_{j \ne i} \{\ell_j > 0\}.$$

Assume for a contradiction that $H \subseteq W$ is some other hyperplane such that $H \cap \partial\Sigma_w$ contains a non-empty open subset in H. Since $H \ne H_i$, the intersection $H \cap H_i$ is a proper subspace of H for all i. Hence, $H \cap \partial\Sigma_w$ is contained in the union of the $H \cap H_i$'s, but this implies that a finite union of proper subspaces of H contains a non-empty open subset in H which is a contradiction. Since the subset $\{\ell_i = c_i\}$ in W determines the pair (ℓ_i, c_i) up to a nonzero scaling factor, it remains to prove that if we switch the order of any of the initial defining inequalities then the sector changes. But this is obvious. $\square$

The lemma above makes the following definition well-posed.

Definition 8.1. Let $\Sigma = \{\ell_1 \ge c_1, \ldots, \ell_k \ge c_k\}$ be a k-sector in W. For each $1 \le i \le k$ let $H_i = \{\ell_i = c_i\}$ be the k translated hyperplanes which are uniquely determined by the subset $\Sigma \subseteq W$. We also set

$$U_i = \{\ell_i = 0\} \cap \bigcap_{j \ne i} \{\ell_j > 0\}.$$

Notice that the topological boundary of Σ in W is given by

$$\partial\Sigma = \bigcup_{i=1}^k \{\ell_1 \ge c_1, \ldots, \ell_i = c_i, \ldots, \ell_k \ge c_k\}.$$

A point $x \in \Sigma$ has **index** r if $\ell_i(x) = c_i$ for exactly r indices i (with $0 \le r \le k$). We define Σ_r to be the set of points $x \in \Sigma$ with index r, or equivalently $x \in H_j$ for exactly r values of j.

Example 8.1. Let $\Sigma = [0,\infty)^3 \subseteq \mathbb{R}^3$. Then we have

$$\begin{aligned} \Sigma_0 &= \{x_1 > 0, x_2 > 0, x_3 > 0\} \\ \Sigma_1 &= \{x_1 = 0, x_2 > 0, x_3 > 0\} \cup \{x_1 > 0, x_2 = 0, x_3 > 0\} \cup \{x_1 > 0, x_2 > 0, x_3 = 0\} \\ \Sigma_2 &= \{x_1 = 0, x_2 = 0, x_3 > 0\} \cup \{x_1 = 0, x_2 > 0, x_3 = 0\} \cup \{x_1 > 0, x_2 = 0, x_3 = 0\} \\ \Sigma_3 &= \{x_1 = 0, x_2 = 0, x_3 = 0\} = \{(0,0,0)^\top\} \end{aligned}$$

The following result summarizes some nice topological relations (easily visualized by picturing the non-negative orthant $\Sigma = [0,\infty)^3 \subseteq \mathbb{R}^3 = W$ and the 2-sector $\Sigma = [0,\infty)^2 \times \mathbb{R}$ in $\mathbb{R}^3$.

Theorem 8.2. *Let $\Sigma = \{\ell_1 \geq c_1, \ldots, \ell_k \geq c_k\}$ be a k-sector in W.*

1. *We have* $\operatorname{int}\Sigma = \Sigma_0$ *and* $\Sigma = \overline{\Sigma_0}$.
2. *For $1 \leq r \leq k$, we have $\Sigma_r \neq \emptyset$ and the connected components of Σ_r are open in Σ_r and are given by the intersections of Σ_r with $H_{i_1} \cap \cdots \cap H_{i_r}$ for each $1 \leq i_1 < \cdots < i_r \leq k$, with this intersection also open in $H_{i_1} \cap \cdots \cap H_{i_r}$.*
3. *For $0 \leq r \leq k$, we have $\overline{\Sigma_r} = \bigcup_{r' \geq r} \Sigma_{r'}$.*
4. *For $r \geq 1$, Σ_r is the set of $x \in \Sigma$ that lie in the closure of exactly r connected components of Σ_1.*

Remark 12. In particular, using just Σ and Σ_1 we can locally topologically encode the property of having index $r \geq 0$: $x \in \Sigma$ has index r if and ony if x admits arbitrarily small open neighborhoods U in Σ that meet the closures of exactly r connected components of $U_1 = U \cap \Sigma_1$. This is tremendously important for globalization to manifolds with corners.

8.1 Calculus on Sectors

Let V and V' be two finite-dimensional vector spaces over $\mathbb{R}$, and let $\Sigma \subseteq V$ and $\Sigma' \subseteq V'$ be two sectors. Fix $1 \leq p \leq \infty$. Suppose that we are given non-empty open sets $U \subseteq \Sigma$ and $U' \subseteq \Sigma'$ and let $f\colon U \to U'$ be a C^p-morphism. Then for each $x \in U$ there is a derivative $\mathrm{D}f(x)$ that is a linear map $V \to V'$, so by the Chain Rule if f is a C^p isomorphism then $\mathrm{D}f(x)$ is a linear isomorphism and hence $\dim V = \dim V'$. In general, if f is a C^p map then it is impossible to say anything about the index of $f(x) \in U' \subseteq \Sigma'$ in terms of the index of $x \in U \subseteq \Sigma$. For example, the index could go up or down; consider putting $[0,1)$ into $\mathbb{R}$ or along the edge of a square in the plane. However, to get the theory of C^p-premanifolds with corners off of the ground we just need to build a consistent theory of local C^p-charts, and so rather than studying general C^p maps what we need to study are C^p isomorphisms. That is, we need to prove:

Theorem 8.3. *If $f\colon U \to U'$ is a C^p-isomorphism then $f(x)$ has the same index in Σ' as x in Σ for all $x \in U$. In other words, we have $f(U \cap \Sigma_r) = U' \cap \Sigma'_r$.*

Proof. To prove the theorem, let $g\colon U' \to U$ be the C^p-inverse of f. Since U and U' are non-empty, the Chain Rule ensures $\dim V = \dim V'$; let n be this common dimension. Let $U_r = U \cap \Sigma_r$ and let $U'_r = U' \cap \Sigma'_r$. We first show that f must carry U_0 into U'_0 and g must carry U'_0 into U_0, so $U'_0 = f(U_0)$. By symmetry, we consider f.

First note that U_0 is an open set in V since it is the intersection of two open subsets of V:

$$U_0 = U \cap \Sigma_0 = U \cap \operatorname{int}\Sigma.$$

The map $f|_{U_0}$ is therefore a C^p mapping in the usual sense, with $(\mathrm{D}f|_{U_0})(u_0) = \mathrm{D}f(u_0)$ as linear maps from V to V'. Since $\mathrm{D}f(u_0)$ is a linear isomorphism (by the Chain Rule for f and g) the mapping $f|_{U_0}\colon U_0 \to V'$ between open sets in vector spaces satisfies the hypotheses for the usual inverse function theorem at u_0 (that is, its total derivative map at u_0 is a linear isomorphism). Thus, by the usual inverse function theorem, $f|_{U_0}$ gives a C^p isomorphism between small opens around u_0 and $f(u_0)$ in U_0 and V' respectively. In particular, $f(U_0) \subseteq U' \subseteq V'$ contains an open set around $f(u_0)$ in V'. Hence,

$$f(u_0) \in U' \cap \operatorname{int}_{V'}(\Sigma') = U' \cap \Sigma'_0 = U'_0$$

as desired.

For $r > 1$, note that U_r is topologically determined in U by U_1 and U_0. More precisely, U_r is the set of points $x \in U \backslash U_0$ admitting arbitrarily small open neighborhoods meeting the closures of exactly r connected components of U_1. The same holds for U'_r in terms of U'_0 and U'_1, so since f and g are inverse homeomorphisms and we have already proved that they identify U_0 and U'_0 we are reduced to the case of index 1. If $x \in U_1$ then $f(x) \notin f(U_0) = U'_0$, so $f(x)$ has index at least 1 in $U' \subseteq \Sigma'$. The problem is to prove that $f(x)$ has index exactly 1. One this is setttled, it makes sense to define the notion of a C^p-premanifold with corners (in the sense of being a structured $\mathbb{R}$-space locally isomorphic to an open in a sector in a vector space equipped with its natural $\mathbb{R}$-space structure given by C^p-functions on its open subsets), but we will need to show more, namely that the locally closed set of points with a given index has a natural structure of C^p-premanifold. We take up these issues and more in what follows.

Using the notation as in the preceding discussion, we have $x \in U_1$ and we seek a contradiction if $f(x) \in U'_r$ with $r \geq 2$, which is to say (after relabelling) that we seek a contradiction if $f(x) \in H'_1 \cap H'_2$ for two of the translated hyperplanes that give "faces" of Σ' (this possibility can only occur if $n \geq 2$, so we now assume this to be the case). By translation, we may and do assume (for simplicity of language) that x and $f(x)$ are the origin in their respective vector spaces. In particular, any translated hyperplane through these points is a genuine hyperplane.

We claim that in fact if $f(x) \in H'$ for a hyperplane H' that gives a "face" of Σ' then the map $\mathrm{D}f(x)\colon V \to V'$ carries H into H', where H is the unique hyperplane in V that is a "face" of Σ and contains x (here we use that x has index 1, so $x \in \Sigma_1$). Granting this, it follows that $\mathrm{D}f(x)$ sends H into $H'_1 \cap H'_2$, but this is impossible for dimension reasons because $\mathrm{D}f(x)\colon V \to V'$ is an isomorphism and $H'_1 \cap H'_2$ has codimension 2 in V'. This contradiction settles the problem for points with index 1, granting the above claim that must now be proved.

By suitable choice of linear coordinates on V and V', we can assume $V = \mathbb{R}^n$, and $\{t_n = 0\}$ is the unique hyperplane H in V through the origin x giving a face of Σ, and that associated to this hyperplane the inequality "$t_n \geq 0$" (rather than "$-t_n \geq 0$") arises in the definition of the sector Σ. We can likewise suppose $V' = \mathbb{R}^n$ with $H' = \{t'_n = 0\}$, and that "$t'_n \geq 0$" is the corresponding inequality that arises in the definition of Σ'. Since x is a point of index 1, near x an open set in Σ is open in the half-space $\{t_n \geq 0\}$. Thus, since our problems are local near x, we may replace Σ with $\mathbf{H} = \{t_n \geq 0\}$ and Σ' with $\mathbf{H}' = \{t'_n \geq 0\}$ to reduce to the setup in the following result. $\square$

Theorem 8.4. *Let V and V' be finite-dimensional nonzero vector spaces over $\mathbb{R}$, and let $\mathbf{H} = \{\ell \geq 0\}$ and $\mathbf{H}' = \{\ell' \geq 0\}$ be closed half-spaces defined by nonzero linear functionals $\ell \in V^\vee$ and $\ell' \in V'^\vee$. Let $U \subseteq \mathbf{H}$ be an open subset around a point $x \in \partial\mathbf{H} = \{\ell = 0\}$ and let $f\colon U \to \mathbf{H}'$ be a C^1-map such that $f(x) \in \partial\mathbf{H}' = \{\ell' = 0\}$. The map $\mathrm{D}f(x)\colon V \to V'$ sends the hyperplane $\partial\mathbf{H}$ into the hyperplane $\partial\mathbf{H}'$.*

8.2 C^p-Structure on Singular Strata

Definition 8.2. For $0 \leq p \leq \infty$, a C^p **premanifold with corners** is a structured R-space $(X, \mathcal{O})$ that is locally isomorphic (in the sense of structured $\mathbb{R}$-spaces) to an open subset of a sector in a finite-dimensional vector space (equipped with its natural $\mathbb{R}$-space structure given by C^p-functions on open subsets of itself). If the underlying topological space is Hausdorff and second-countable, then we call it a C^p**-manifold with corners**. We usually write X rather than $(X, \mathcal{O})$.

In view of the local results on sectors, we may use any C^p-chart to determine the property of $x \in X$ having index $r \geq 0$, and the subset $X_r \subseteq X$ of points with index r is locally closed in X. The subsets $X_{\geq r} = \cup_{i \geq r} X_i$ are closed in X, and $X_{\geq 1}$ is called the **boundary** of X and is denoted ∂X; the intrinsic notion (that makes no reference to an ambient topological space containing X) must not be confused with the (extrinsic) notion of topological boundary for a subset of a topological space.

8.3 Whitney's Extension Theorem

Theorem 8.5. *(Whitney). Let V and V' be finite-dimensional nonzero vector spaces over $\mathbb{R}$ and let $\Sigma \subseteq V$ be a sector. Let $U \subseteq \Sigma$ be an open subset and $x_0 \in U$ a point. Fix $0 \leq p \leq \infty$.*

Any C^p map $f\colon U \to V'$ locally extends to a C^p map on an open neighborhood of x_0 in V. That is, there exists an open set $\widetilde{U} \subseteq V$ around x_0 and a C^p map $\widetilde{f}\colon \widetilde{U} \to V'$ such that $\widetilde{f}|_{U \cap \widetilde{U}} = f$.

9 The Derivative of a C^p-Map

Let X and Y be two C^p-premanifolds and let $F\colon X \to Y$ be a C^p-mapping. Let $x \in X$ be a point and let $y = F(x)$. We can define a linear mapping $\mathrm{d}F(x)\colon \mathrm{T}_x(X) \to \mathrm{T}_y(Y)$ called the **derivative** of F at x as follows: if $\vec{v} \in \mathrm{T}_x(X)$ is a tangent vector at x (so it is a point-derivation $\vec{v} : \mathcal{O}_{X,x} \to \mathbb{R}$), then we define

$$\mathrm{d}F(x)(\vec{v}) = \vec{v} \circ F^*$$

with $F^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ the "pullback map" defined on germs via $f \mapsto f \circ F$. That $\vec{v} \circ F^*\colon \mathcal{O}_{Y,y} \to \mathbb{R}$ is a point-derivation at y and that the resulting map of sets $\mathrm{d}F(x)\colon \mathrm{T}_x(X) \to \mathrm{T}_y(Y)$ sending $\vec{v}$ to $\vec{v} \circ F^*$ is $\mathbb{R}$-linear follows from the fact that $F^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is an $\mathbb{R}$-algebra map. Indeed, let $[V, f]_y$ and $[V, g]_y$ be two germs at y (with V being a sufficiently small neighborhood of y). Then working with representatives, we have

$$\begin{aligned}
(\vec{v} \circ F^*)(f \cdot g) &= \vec{v}(F^*(f \cdot g)) \\
&= \vec{v}(F^*(f) \cdot F^*(g)) \\
&= \vec{v}(F^*(f)) \cdot F^*(g)(x) + F^*(f)(x) \cdot \vec{v}(F^*(g)) \\
&= (\vec{v} \circ F^*)(f) \cdot g(y) + f(y) \cdot (\vec{v} \circ F^*)(g).
\end{aligned}$$

This establishes Leibnitz Rule. Similarly, for $r \in \mathbb{R}$ we have

$$\begin{aligned}
(\vec{v} \circ F^*)(f + rg) &= \vec{v}(F^*(f + rg)) \\
&= \vec{v}'(F^*(f) + rF^*(g)) \\
&= \vec{v}(F^*(f)) + r\vec{v}(F^*(g)) \\
&= (\vec{v} \circ F^*)(f) + r(\vec{v} \circ F^*)(g).
\end{aligned}$$

This establishes $\mathbb{R}$-linearity.

9.0.1 Matrix Representation of Derivative is a Jacobian Matrix

Let (U, φ) and (V, ψ) be respective C^p charts centered at x and y in X and Y respectively, with $\varphi \colon U \simeq \varphi(U) \subseteq \mathbb{R}^m$ and $\psi \colon V \simeq \varphi(V) \subseteq \mathbb{R}^n$ having respective component functions $\varphi = (\varphi_1, \ldots, \varphi_m)$ and $\psi = (\psi_1, \ldots, \psi_n)$ on the source and target. Thus, $\mathrm{T}_x(X)$ has the ordered basis $\{\partial_{\varphi_i}|_x\}$ and $\mathrm{T}_y(Y)$ has the ordered basis $\{\partial_{\psi_j}|_y\}$. It is natural to ask for the matrix of the linear map $\mathrm{d}F(x) : \mathrm{T}_x(X) \to \mathrm{T}_y(Y)$ with respect to these ordered bases.

Theorem 9.1. *Write $F_j = \psi_j \circ F$ for each $1 \leq j \leq n$. The matrix of $\mathrm{d}F(x)$ with respect to the ordered bases $\{\partial_{\varphi_i}|_x\}$ and $\{\partial_{\psi_j}|_y\}$ has (j,i)-entry given by $(\partial_{\varphi_i} F_j)(x)$.*

Proof. By replacing U with $U \cap F^{-1}(V)$ if necessary, we may assume that $F|_U$ lands in V. Observe that for each $1 \leq j \leq n$, we have

$$\begin{aligned} \mathrm{d}F(x)(\partial_{\varphi_i}|_x)(\psi_j) &= (\partial_{\varphi_i} \circ F^*)(\psi_j)|_x \\ &= \partial_{\varphi_i}(\psi_j \circ F)(x) \\ &= (\partial_{\varphi_i} F_j)(x) \end{aligned}$$

It follows that

$$\mathrm{d}F(x)(\partial_{\varphi_i}|_x) = \sum_{j=1}^{n} ((\partial_{\varphi_i} F_j)(x)) \partial_{\psi_j}|_y.$$

□

Remark 13. Recall that $\varphi \colon U \xrightarrow{\simeq} \varphi(U) \subseteq \mathbb{R}^m$ is a C^p-mapping between two C^p-premanifolds. Let's calculate the derivative of φ at x: if $\varphi(x) = (t_1, \ldots, t_m) = \boldsymbol{t}$, where the t_j denote the standard coordinates of $\mathbb{R}^m$, then by definition $\varphi_j = t_j \circ \varphi$, so we have

$$\begin{aligned} \mathrm{d}\varphi(x)(\partial_{\varphi_i}|_x)(t_j) &= (\partial_{\varphi_i} \circ \varphi^*)(t_j)|_x \\ &= \partial_{\varphi_i}(t_j \circ \varphi)(x) \\ &= (\partial_{\varphi_i} \varphi_j)(x). \end{aligned}$$

It follows that

$$\mathrm{d}\varphi(x)(\partial_{\varphi_i}|_x) = \partial_{t_i}|_{\boldsymbol{t}}.$$

9.0.2 The Chain Rule

Theorem 9.2. *Let $G \colon X'' \to X$ and $F \colon X' \to X$ be C^p mappings between C^p premanifolds. For any $x'' \in X''$ with $G(x'') = x' \in X'$ and $F(x') = x \in X$, the composite linear mapping*

$$(\mathrm{d}F)(G(x'')) \circ \mathrm{d}G(x'') \colon \mathrm{T}_{x''}(X'') \to \mathrm{T}_{x'}(X') \to \mathrm{T}_x(X)$$

is equal to $\mathrm{d}(F \circ G)(x'')$.

Proof. We choose a tangent vector $\vec{v}'' \in \mathrm{T}_{x''}(X'')$, so we want to prove

$$\mathrm{d}(F \circ G)(x'')(\vec{v}'') = (\mathrm{d}F)(G(x''))(\mathrm{d}G(x'')(\vec{v}''))$$

in $\mathrm{T}_x(X)$. This is an equality of point derivations $\mathcal{O}_x \to \mathbb{R}$, and by the definitions of the derivative mappings the left side is $\vec{v}'' \circ (F \circ G)^*$ and the right side is $(\vec{v}'' \circ G^*) \circ F^* = \vec{v}'' \circ (G^* \circ F^*)$. Hence, it suffices to show that the composite of the mappings $F^* \colon \mathcal{O}_x \to \mathcal{O}'_{x'}$ and $G^* \colon \mathcal{O}'_{x'} \to \mathcal{O}''_{x''}$ is equal to $(F \circ G)^*$; that is $(F \circ G)^* = G^* \circ F^*$, however this is clear. □

9.1 Properties of Derivative Mappings

Let (U, φ) and (V, ψ) be respective C^p charts centered at x and y in X and Y respectively, with $\varphi \colon U \simeq \varphi(U) \subseteq \mathbb{R}^m$ and $\psi \colon V \simeq \varphi(V) \subseteq \mathbb{R}^n$ having respective component functions $\varphi = (x_1, \ldots, x_m)$ and $\psi = (y_1, \ldots, y_n)$ on the source and target. Thus, $\mathrm{T}_x(X)$ has the ordered basis $\{\partial_{x_i}|_x\}$ and $\mathrm{T}_y(Y)$ has the ordered basis $\{\partial_{y_j}|_y\}$. It is natural to ask for the matrix of the linear map $\mathrm{d}F(x) : T_x(X) \to T_y(Y)$ with respect to these ordered bases.

Theorem 9.3. *Write $F_j = y_j \circ F$ for each $1 \leq j \leq n$. The matrix of $\mathrm{d}F(x)$ with respect to the ordered bases $\{\partial_{x_i}|_x\}$ and $\{\partial_{y_j}|_y\}$ has (j,i)-entry given by $(\partial_{x_i} F_j)(x)$.*

Proof. By replacing U with $U \cap F^{-1}(V)$ if necessary, we may assume that $F|_U$ lands in V. Observe that for each $1 \leq j \leq n$, we have

$$\begin{aligned} \mathrm{d}F(x)(\partial_{x_i}|_x)(y_j) &= (\partial_{x_i} \circ F^*)(y_j)|_x \\ &= \partial_{x_i}(y_j \circ F)(x) \\ &= (\partial_{x_i} F_j)(x) \end{aligned}$$

It follows that

$$\mathrm{d}F(x)(\partial_{x_i}|_x) = \sum_{j=1}^{n} ((\partial_{x_i} F_j)(x)) \partial_{y_j}|_y.$$

In particular, if $t = \varphi(x)$, then we have

$$\begin{aligned} \mathrm{d}\varphi(x)(\partial_{x_i}|_x) &= \sum_{j=1}^{n} ((\partial_{x_i} \varphi_j)(x)) \partial_{t_j}|_t. \\ &= \sum_{j=1}^{n} ((\partial_{x_i} x_j)(x)) \partial_{t_j}|_t \\ &= \partial_{t_i}|_t. \end{aligned}$$

□

9.2 Parametric Curves and Velocity Vectors

Let X be a C^p premanifold with corners. A **parametrized C^p curve** at $x \in X$ is a C^p map $c \colon I \to X$ with $I \subseteq \mathbb{R}$ a nontrivial (not a point nor the empty set) interval, $0 \in I$, and $c(0) = p$. If $x \notin \partial X$, then we also require 0 to be in the interior of I. Note that a parametirze curve is the data of the mapping c and it may not be injective or have smooth image. For instance, the map $c \colon (-1,1) \to \mathbb{R}^2$ defined by $c(t) = (t^2, t^3)$ has image contained in the locus $x_2^2 - x_1^3$ in the plane that has a cusp at the origin. The map $c' \colon (-1/2, 1/2) \to \mathbb{R}^2$ defined by $c'(t) = c(2t)$ has the same image as c, but we consider c' different than c. Intuitively, c' moves twice as quickly as c.

Definition 9.1. Let $I \subseteq \mathbb{R}$ be a nontrivial interval and let $\gamma \colon I \to X$ be a C^p mapping. For each $a \in I$, the **velocity vector** to γ at a is

$$\gamma'(a) := \mathrm{d}\gamma(a)(\partial_t|_a) \in \mathrm{T}_{\gamma(a)}(X)$$

where $\partial_t|_a \in \mathrm{T}_a(I)$ is the canonical basis vector (sending a C^p germ f at a to the old-fashioned derivative $f'(a)$ in the sense of calculus). In particular, suppose (U, φ) is a chart at $x \in X$. Then we have

$$\gamma'(a) = \sum \gamma_i'(a) \partial_{\varphi_i}|_{\gamma(a)}.$$

10 Charts

Definition 10.1. Let X be a topological space. A **real n-chart** of X (or more simply **n-chart** or just **chart**) consists of a pair (U, φ) where $U \subseteq X$ is open and nonempty and where $\varphi \colon U \to \mathbb{R}^n$ is a homeomorphism onto its image $\varphi(U) \subseteq \mathbb{R}^n$ which is open in $\mathbb{R}^n$. Given a point $x \in X$, we say the chart (U, φ) is **contains** x if $x \in U$, and we say (U, φ) is **centered** at x if it contains x and $\varphi(x) = 0$. We say X is a **topological n-premanifold** (or more simply an **n-premanifold**) if every point $x \in X$ is contained in an n-chart of X. We say X is a **topological n-manifold** (or more simply **n-manifold** or just **manifold**) if it is an n-premanifold and is Hausdorff and second countable.

Let (U, φ) be a chart of X where $\varphi \colon U \simeq \varphi(U) \subseteq \mathbb{R}^n$. For any set S, let $\mathrm{Map}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Note that $\mathrm{Map}(S, \mathbb{R})$ has the structure of an $\mathbb{R}$-algebra, where addition and multiplication are defined pointwise. Let

$$\varphi^* \colon \mathrm{Map}(\varphi(U), \mathbb{R}) \to \mathrm{Map}(U, \mathbb{R})$$

be defined by $\varphi^* g = g \circ \varphi$ for all functions $g \colon \varphi(U) \to \mathbb{R}$. We call $\varphi^* g$ the **pullback** of g with respect to φ. It is straightforward to check that φ^* is an $\mathbb{R}$-algebra homomorphism. Similarly, if $f \colon U \to \mathbb{R}$ is a function, we define it **pushforward** with respect to φ to be the map $(\varphi^{-1})^* f = f \circ \varphi^{-1}$. Let $\{t_1, \ldots, t_n\}$ denote the standard linear coordinates on $\varphi(U) \subseteq \mathbb{R}^n$: thus $t_i(\boldsymbol{a}) = a_i$ for all $\boldsymbol{a} \in \mathbb{R}^n$. We often denote by φ_i to be the pullback of t_i with respect to φ: thus $\varphi_i = t_i \circ \varphi$. We can think of the φ_i as being coordinate functions on U: we call them **local**

coordinates of X. For instance, the function $f = \varphi_1^3 + \cdots + \varphi_n^3$ is defined by

$$\begin{aligned} f(x) &= (\varphi_1^3 + \cdots + \varphi_n^3)(x) \\ &= (t_1^3 + \cdots + t_n^3)\varphi(x) \\ &= (t_1^3 + \cdots + t_n^3)(\boldsymbol{a}) \\ &= a_1^3 + \cdots + a_n^3, \end{aligned}$$

where we set $\varphi(x) = \boldsymbol{a} = (a_1, \ldots, a_n)$. Functions are not the only thing we can pullback (or pushforward). Indeed, we can pullback the partial derivative ∂_{t_i}: we denote by $\partial_{\varphi_i} := \partial_{t_i} \circ (\varphi^{-1})^*$. In particular, observe that

$$\partial_{\varphi_i}(\varphi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Furthermore, note that ∂_{t_i} is the pushforward of ∂_{φ_i} in this case: $\partial_{t_i} = \partial_{\varphi_i} \circ \varphi^*$. Finally, note that ∂_{φ_i} is an $\mathbb{R}$-linear map which satisfies Leibniz law precisely because $(\varphi^{-1})^*$ is an $\mathbb{R}$-algebra homomorphism.

Example 10.1. Recall that the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is defined by

$$S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| = 1\}.$$

Here we write $\boldsymbol{x} = (x_1, \ldots, x_{n+1})$. This description S^n comes equipped with *global* coordinates: every point in S^n has the form $\boldsymbol{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that $\|\boldsymbol{x}\| = 1$. Let $\boldsymbol{x}_N = (0, \ldots, 0, 1)$ be the north pole and let $U_N = S^n \backslash \{\boldsymbol{x}_N\}$. Let (U_N, φ_N) be a chart where $\varphi_N = (t_1, \ldots, t_n)$ is defined by

$$\varphi_N(\boldsymbol{x}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right) = (t_1, \ldots, t_n) = \boldsymbol{t}.$$

Then φ_N is a homeomorphism onto its image $\varphi_N(U_N) = \mathbb{R}^n$ whose inverse $\varphi_N^{-1} \colon \varphi(U_N) \to U_N$ is defined by

$$\varphi_N^{-1}(\boldsymbol{t}) = \left(\frac{2t_1}{\|\boldsymbol{t}\|^2 + 1}, \ldots, \frac{2t_n}{\|\boldsymbol{t}\|^2 + 1}, \frac{\|\boldsymbol{t}\|^2 - 1}{\|\boldsymbol{t}\|^2 + 1} \right) = \boldsymbol{x}.$$

To see why this is the case, first note that φ_N is continuous since each of its component functions $\varphi_{N,i} = t_i \circ \varphi_N$ (given by $\varphi_{N,i} = x_i/(1 - x_{n+1})$) is continuous as $x_{n+1} \neq -1$. A similar argument shows φ_N^{-1} is continuous as well. Futhermore, observe that

$$\begin{aligned} (\varphi_{N,i} \circ \varphi_N^{-1})(\boldsymbol{t}) &= \varphi_{N,i}(\boldsymbol{x}) \\ &= \frac{x_i}{1 - x_{n+1}} \\ &= \left(\frac{1}{1 - \frac{\|\boldsymbol{t}\|^2 - 1}{1 + \|\boldsymbol{t}\|^2}} \right) \left(\frac{2t_i}{1 + \|\boldsymbol{t}\|^2} \right) \\ &= \left(\frac{1 + \|\boldsymbol{t}\|^2}{2} \right) \left(\frac{2t_i}{1 + \|\boldsymbol{t}\|^2} \right) \\ &= t_i. \end{aligned}$$

It follows that $\varphi_N(\varphi_N^{-1}(\boldsymbol{t})) = \boldsymbol{t}$. A similar calculation shows $\varphi_N^{-1}(\varphi_N(\boldsymbol{x})) = \boldsymbol{t}$.

Now let $\boldsymbol{x}_S = (0, \ldots, 0, -1)$ be the south pole and let $U_S = S^n \backslash \{\boldsymbol{x}_S\}$. Let (U_N, φ_N) be a chart where $\varphi_N = (\widetilde{t}_1, \ldots, \widetilde{t}_n)$ defined by

$$\varphi_S(\widetilde{\boldsymbol{x}}) = \left(\frac{\widetilde{x}_1}{1 + \widetilde{x}_{n+1}}, \ldots, \frac{\widetilde{x}_n}{1 + \widetilde{x}_{n+1}} \right) = (\widetilde{t}_1, \ldots, \widetilde{t}_n) = \widetilde{\boldsymbol{t}}.$$

Then φ_S is a homeomorphism onto its image $\varphi_S(U_S) = \mathbb{R}^n$ whose inverse $\varphi_S^{-1} \colon \varphi(U_S) \to U_S$ is defined by

$$\varphi_S^{-1}(\widetilde{\boldsymbol{t}}) = \left(\frac{2\widetilde{t}_1}{1 + \|\widetilde{\boldsymbol{t}}\|^2}, \ldots, \frac{2\widetilde{t}_n}{1 + \|\widetilde{\boldsymbol{t}}\|^2}, \frac{1 - \|\widetilde{\boldsymbol{t}}\|^2}{1 + \|\widetilde{\boldsymbol{t}}\|^2} \right) = \widetilde{\boldsymbol{x}}.$$

Now observe that for each $1 \leq i \leq n$, we have

$$
\begin{aligned}
(\varphi_{N,i} \circ \varphi_S^{-1})(\widetilde{\boldsymbol{t}}) &= \varphi_{N,i}(\widetilde{\boldsymbol{x}}) \\
&= \frac{\widetilde{x}_i}{1 - \widetilde{x}_{n+1}} \\
&= \left(\frac{2\widetilde{t}_i}{1 + \|\widetilde{\boldsymbol{t}}\|^2} \right) \left(\frac{1}{1 - \left(\frac{1 - \|\widetilde{\boldsymbol{t}}\|^2}{1 + \|\widetilde{\boldsymbol{t}}\|^2} \right)} \right) \\
&= \left(\frac{2\widetilde{t}_i}{1 + \|\widetilde{\boldsymbol{t}}\|^2} \right) \left(\frac{1 + \|\widetilde{\boldsymbol{t}}\|^2}{2\|\widetilde{\boldsymbol{t}}\|^2} \right) \\
&= \frac{\widetilde{t}_i}{\|\widetilde{\boldsymbol{t}}\|^2}.
\end{aligned}
$$

In particular, $\varphi_N \circ \varphi_S^{-1}$ is C^∞ on $\varphi_S(U_N \cap U_S) = \mathbb{R}^n \backslash \{0\}$. A similar calculation shows $\varphi_{S,i} \circ \varphi_N^{-1} = t_i / \|\boldsymbol{t}\|^2$ and hence is C^∞ on $\varphi_N(U_N \cap U_S) = \mathbb{R}^n \backslash \{0\}$.

In Example (10.1), we noticed that the sphere S^n came equipped with global coordinates. This makes it easy, for example, to define functions out of S^n. In general however, a manifold X will not come equipped with global coordinates (but when they do, we should definitely make use of them!). Here's another manifold which admits nice global coordinates:

Example 10.2. (Real Projective Plane) The real projective space $\mathbb{RP}^n$ is defined to be the set of all lines in $\mathbb{R}^{n+1}$ which pass through the origin. We wish to describe every point in $\mathbb{RP}^n$ using "global coordinates". For each $\boldsymbol{a} = (a_0, a_1, \dots, a_n)$ in $\mathbb{R}^{n+1} \backslash \{0\}$, we let $\ell_{\boldsymbol{a}}$ denote the linear form $\ell_{\boldsymbol{a}} = \sum_{i=0}^n a_i x_i$. With this notation in mind, observe that if L is a line which passes through the origin, then there exists an $\boldsymbol{a} \in \mathbb{R}^{n+1} \backslash \{0\}$ such that L has the form:

$$
L = L_{\boldsymbol{a}} = \mathrm{V}(\ell_{\boldsymbol{a}}) := \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1} \mid \ell_{\boldsymbol{a}}(\boldsymbol{x}) = 0 \right\}.
$$

Moreover, $\boldsymbol{a}$ is unique up to scaling by a nonzero constant. This means that if $L_{\boldsymbol{a}} = L_{\boldsymbol{a}'}$, then there exists $\lambda \in \mathbb{R} \backslash \{0\}$ such that $\lambda \boldsymbol{a} = \boldsymbol{a}'$ (that is, such that $\lambda a_i = a_i'$ for all $0 \leq i \leq n$). Therefore we have a map $\mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ given by $\boldsymbol{a} \mapsto L_{\boldsymbol{a}}$ which is surjective but not one-to-one. However if we define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} \backslash \{0\}$ by declaring $\boldsymbol{a} \sim \boldsymbol{a}'$ if and only if there exists $\lambda \in \mathbb{R} \backslash \{0\}$ such that $\lambda \boldsymbol{a} = \boldsymbol{a}'$, then the map $\mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ induces a bijection from $(\mathbb{R}^{n+1} \backslash \{0\})/\sim$ to $\mathbb{RP}^n$ which sends the equivalence class $[\boldsymbol{a}] := [a_0 : a_1 \cdots : a_n]$ (represented by $\boldsymbol{a} = (a_0, a_1, \dots, a_n)$) to the line $L_{\boldsymbol{a}}$. Thus our global coordinates for $\mathbb{RP}^n$ look like $[\boldsymbol{a}] = [a_0 : a_1 \cdots : a_n]$, where we keep in mind that $[\boldsymbol{a}] = [\lambda \boldsymbol{a}]$ for all $\lambda \in \mathbb{R} \backslash \{0\}$.

Now that we've described what the global coordinates on $\mathbb{RP}^n$ look like, let's give $\mathbb{RP}^n$ the structure of a topological space. Let $\pi \colon \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{RP}^n$ denote the projection map. Then $\mathbb{RP}^n$ inherits the sturcture a topological space via the quotient topology. This is the weakest topology on $\mathbb{RP}^n$ which makes π continuous. In particular, a set $\widetilde{U} \subseteq \mathbb{RP}^n$ is said open if and only if $\pi^{-1}(U) \subseteq \mathbb{R}^{n+1} \backslash \{0\}$ is open. Note that in this case, π is an open map. Indeed, let $\mathrm{B}_r(\boldsymbol{a})$ be the open ball in $\mathbb{R}^{n+1} \backslash \{0\}$ centered at the point $\boldsymbol{a}$ and with radius $r > 0$. Then observe that

$$
\begin{aligned}
\pi^{-1}\pi(\mathrm{B}_r(\boldsymbol{a})) &= \{\boldsymbol{b} \in \mathbb{R}^{n+1} \mid \pi(\boldsymbol{b}) \in \pi(\mathrm{B}_r(\boldsymbol{a}))\} \\
&= \{\boldsymbol{b} \in \mathbb{R}^{n+1} \mid [\boldsymbol{b}] = [\boldsymbol{a}'] \text{ where } \|\boldsymbol{a} - \boldsymbol{a}'\| < r\} \\
&= \{\boldsymbol{b} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{a} - \boldsymbol{b}/\lambda\| < r \text{ for some } \lambda \neq 0\} \\
&= \{\boldsymbol{b} \in \mathbb{R}^{n+1} \mid \|\lambda \boldsymbol{a} - \boldsymbol{b}\| < \lambda r \text{ for some } \lambda \neq 0\} \\
&= \bigcup_{\lambda \in \mathbb{R} \backslash \{0\}} \mathrm{B}_{\lambda r}(\lambda \boldsymbol{a}).
\end{aligned}
$$

It follows that π is open. Now in general, if X is a topological space $\sim$ is an equivalence relation on X, then we call $\sim$ **open** if the corresponding quotient map $\rho \colon X \to X/\sim$ is open. In this case, a lot of nice things happen. First of all, $X/\sim$ is automatically second-countable. Secondly, if $\{B_i\}$ is a basis for X, then its images $\{\rho(B_i)\}$ form a basis for $X/\sim$. Thirdly, the space $X/\sim$ is Hausdorff if and only if the graph $\Gamma_\sim = \{(x, y) \in X \times X \mid x \sim y\}$ of $\sim$ is closed in $X \times X$. In particular, $\mathbb{RP}^n$ is second countable since $\mathbb{R}^{n+1} \backslash \{0\}$ is and to see that $\mathbb{RP}^n$ is Hausdorff, we just need to check that the graph $\Gamma = \Gamma_\sim$ is closed in $\mathbb{R}^{n+1} \backslash \{0\} \times \mathbb{R}^{n+1} \backslash \{0\}$. In our case, the graph looks like

$$
\begin{aligned}
\Gamma &= \{(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{n+1} \backslash \{0\} \times \mathbb{R}^{n+1} \backslash \{0\} \mid [\boldsymbol{a}] = [\boldsymbol{b}]\} \\
&= \{(\boldsymbol{a}, \lambda \boldsymbol{a}) \in \mathbb{R}^{n+1} \backslash \{0\} \times \mathbb{R}^{n+1} \backslash \{0\} \mid \text{for some } \lambda \neq 0\}.
\end{aligned}
$$

We can show that Γ is closed in $\mathbb{R}^{n+1}\backslash\{0\} \times \mathbb{R}^{n+1}\backslash\{0\}$ by using the sequential criterion for subspaces of a metric space to be closed: if $((\boldsymbol{a}_n, \lambda_n \boldsymbol{a}_n))$ is a sequence in Γ such that $(\boldsymbol{a}_n, \lambda_n \boldsymbol{a}_n) \to (\boldsymbol{a}, \lambda \boldsymbol{a})$ in $\mathbb{R}^{n+1}\backslash\{0\} \times \mathbb{R}^{n+1}\backslash\{0\}$, then $\boldsymbol{a}_n \to \boldsymbol{a}$ in $\mathbb{R}^{n+1}\backslash\{0\}$ and $\lambda_n \to \lambda$ in $\mathbb{R}\backslash\{0\}$, and therefore $(\boldsymbol{a}, \lambda\boldsymbol{a}) \in \Gamma$. It follows that Γ is closed in $\mathbb{R}^{n+1}\backslash\{0\} \times \mathbb{R}^{n+1}\backslash\{0\}$, hence $\mathbb{RP}^n$ is Hausdorff.

Finally, we give $\mathbb{RP}^n$ the structure of a real manifold. Observe that $\mathbb{RP}^n$ is covered by $n+1$ open sets

$$\widetilde{U}_i := \mathrm{D}(x_i) := \{[\boldsymbol{x}] \in \mathbb{RP}^n \mid x_i \neq 0\}.$$

Notice that $\widetilde{U}_i$ is well-defined since if $[\boldsymbol{x}] \in \widetilde{U}_i$, then $[\lambda \boldsymbol{x}] \in \widetilde{U}_i$ for all $\lambda \in \widetilde{\mathbb{R}}\backslash\{0\}$ (since $x_i \neq 0$ if and only if $\lambda x_i \neq 0$). This also implies that it is open since

$$\pi^{-1}(\widetilde{U}_i) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}\backslash\{0\} \mid x_i \neq 0\}.$$

Notice that every element in $\widetilde{U}_i$ has a nice representative: if $[\boldsymbol{x}] \in \widetilde{U}_i$, then we have

$$[\boldsymbol{x}] = [x_0 : \cdots : x_i : \cdots : x_n] = \left[\frac{x_0}{x_i} : \cdots : 1 : \cdots : \frac{x_n}{x_i}\right] = [\boldsymbol{x}/x_i].$$

This gives us an idea of how to define our chart: let $(\widetilde{U}_i, \phi_i)$ be the chart, where $\phi_i = (x_{0,i}, \dots, x_{j,i}, \dots x_{n,i}) = \boldsymbol{x}_i$ with $0 \leq j \leq n$ and $j \neq i$, and where ϕ_i is defined by

$$\phi_i([\boldsymbol{x}]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_j}{x_i}, \dots, \frac{x_n}{x_i}\right) = \boldsymbol{x}_i.$$

In particular, if $\phi_{j,i} := t_j \circ \phi_i$ denotes the jth coordinate function of ϕ_i, then $\phi_{j,i} \colon \mathbb{RP}^n \to \mathbb{R}$ is defined by $\phi_{j,i}([\boldsymbol{x}]) = x_j/x_i$. Then ϕ_i is a homeomorphism onto its image $\phi_i(\widetilde{U}_i) = \mathbb{R}^n$ whose inverse $\phi_i^{-1} \colon \phi_i(\widetilde{U}_i) \to U_i$ is defined by

$$\phi_i^{-1}(\boldsymbol{x}_i) = \left[x_{0,i} : \cdots : x_{j,i} : \cdots : x_{n,i}\right] = [\boldsymbol{x}].$$

Observe that if $i \neq i'$, then

$$\begin{aligned}\phi_{j,i} \circ \phi_{i'}^{-1}(\boldsymbol{x}_{i'}) &= \phi_{j,i}(\left[x_{0,i'} : \cdots : x_{j,i'} : \cdots : x_{n,i'}\right]) \\ &= x_{j,i'}/x_{i,i'}.\end{aligned}$$

10.1 Construction of Products

Let $X_1, \dots, X_n$ be C^p premanifolds. The topological product $\prod X_i$ ought to admit a natural C^p premanifold structure, using as local C^p charts the maps

Definition 10.2.

11 Manifolds

We first recall a few definitions from point-set topology. A topological space is **second countable** if it has a countable basis. A **neighborhood** of a point p in a topological space M is any open set containing p. A topological space M is **Hausdorff** if for every pair of points $x, y \in M$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. An **open cover** of M is a collection $\{U_i\}_{i \in I}$ of open sets in M whose union $\bigcup_{i \in I} U_i$ is M.

The Hausdorff condition and second countability are "hereditary properties"; they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff.

Proposition 11.1. *Let M' be a subspace of a topological space M.*

1. *If M is Hausdorff, then M' is Hausdorff.*
2. *If M is second countable, then M' is second countable.*

Proof. (1) : Suppose $x, y \in M'$. Since $x, y \in M$ and M is Hausdorff, choose a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. Then $U' = U \cap M'$ is a neighboorhood of x in the subspace topology and $V' = V \cap M'$ is a neighborhood of y in the subspace topology and $U' \cap V' = \emptyset$. (2) : If $\{B_i\}_{i \in \mathbb{N}}$ is a countable basis for M, then $\{B_i'\}_{i \in \mathbb{N}}$ is a countable basis for M', where $B_i' = B_i \cap M'$. □

Definition 11.1. A topological space M is **locally Euclidean of dimension** n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system on** U. We say that a chart (U, ϕ) is **centered** at $p \in U$ if $\phi(p) = 0$.

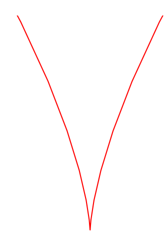
Proposition 11.2. *Let (U, ϕ) be a chart on the topological space M. If V is an open subset U, then $(V, \phi|_V)$ is a chart on M.*

Proof. This follows from the fact that if $\phi : U \to \phi(U)$ is a homeomorphism, then $\phi|_V : V \to \phi(V)$ is a homeomorphism. □

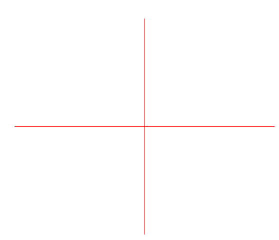
Definition 11.2. A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Example 11.1. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, 1_U)$.

Example 11.2. (A cusp). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold. By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean because it is homeomorphic to $\mathbb{R}$ via the projection $(x, x^{2/3}) \mapsto x$.



Example 11.3. (A cross). The cross can be described as $\{(r, 0) \mid r \in \mathbb{R}\} \cup \{(0, r) \mid r \in \mathbb{R}\}$. We show that the cross in $\mathbb{R}^2$ with the subspace topology is not locally Euclidean at the intersection $p = (0, 0)$, and so cannot be a manifold. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B_\varepsilon(0) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism $U \to B$ restricts to a homemorphism $U \setminus \{p\} \to B \setminus \{0\}$. Now $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components of $n = 1$. Since $U \setminus \{p\}$ has four connected components, there can be no homeomorphism from $U \setminus \{p\}$ to $B \setminus \{p\}$. This contradiction proves that the cross is not locally Euclidean at p.



11.1 Compatible Charts

Suppose $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 11.3. Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞**-compatible** if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called the **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞ compatible. To simplify this notation, we will sometimes write U_{ij} for $U_i \cap U_j$ and U_{ijk} for $U_i \cap U_j \cap U_k$. We will also sometimes write $\phi_{ij} = \phi_i \circ \phi_j^{-1}$. Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of compatible charts.

C^∞ compatibility is clearly reflexive and symmetric, but not necessarily transitive. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2)^{-1} \circ (\phi_2 \circ \phi_1^{-1})$$

is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$. A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} \setminus U_{123})$.

Definition 11.4. A C^∞ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_i, \phi_i)\}_{i \in I}$ of pairwise C^∞-compatible charts that cover M, i.e. such that $M = \bigcup_{i \in I} U_i$.

Example 11.4. (A C^∞ atlas on a circle). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{2\pi it} \in \mathbb{C} \mid 0 \le t \le 1\}$. Let U_1 and U_2 be the two open subsets of S^1

$$U_1 = \{e^{2\pi it} \in \mathbb{C} \mid -\frac{1}{2} < t < \frac{1}{2}\} \qquad U_2 = \{e^{2\pi it} \in \mathbb{C} \mid 0 < t < 1\}$$

and define $\phi_i : U_i \to \mathbb{R}$ for $i = 1, 2$ by

$$\phi_1(e^{2\pi it}) = t \qquad \phi_2(e^{2\pi it}) = t$$

Both ϕ_1 and ϕ_2 are branches of the complex log funciton $(1/i)\log z$ and are homeomorphisms onto their respective images. Thus (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection U_{12} consists of two connected components, the lower half A and the upper half B:

$$A = \{e^{2\pi it} \mid -\frac{1}{2} < t < 0\} \qquad B = \{e^{2\pi it} \mid 0 < t < \frac{1}{2}\}$$

with

$$\phi_1(U_{12}) = \phi_1(A \cup B) = \phi_1(A) \cup \phi_1(B) = \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$

$$\phi_2(U_{12}) = \phi_2(A \cup B) = \phi_2(A) \cup \phi_2(B) = \left(\frac{1}{2}, 1\right) \cup \left(0, \frac{1}{2}\right)$$

The transisition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_{12}) \to \phi_2(U_{12})$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 1 & \text{for } t \in \left(-\frac{1}{2}, 0\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Similarly,

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t - 1 & \text{for } t \in \left(\frac{1}{2}, 1\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1.

We say that a chart (V, ψ) is **compatible with an atlas** $\{(U_i, \phi_i)\}_{i \in I}$ if it is compatible with all the charts (U_i, ϕ_i) of the atlas.

Lemma 11.1. *Let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_i, \phi_i)\}_{i \in I}$, then they are compatible with each other.*

Proof. We want to show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. For all $i \in I$, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is C^∞ on $\psi(V \cap W \cap U_i)$. Therefore $\sigma \circ \psi^{-1}$ is C^∞ on $\bigcup_{i \in I} \psi(V \cap W \cap U_i) = \psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. □

Remark 14. The domain of $\sigma \circ \psi^{-1}$ is $\psi(V \cap W)$ and the domain of $(\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is $\psi(U \cap V \cap W)$. What the equality means in the proof above is that the two maps are equal on their common domain.

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas; in other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 11.5. A **smooth** or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a **differentiable structure** on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a **curve**. A 2-dimensional manifold is a **surface**, and an n-dimensional manifold an n-manifold.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, because of the following proposition.

Proposition 11.3. *Any atlas $\mathfrak{U} = \{(U_i, \phi_i)\}_{i \in I}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Lemma (11.1), the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves existence. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{U}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{U}$ and so by construction must belong to $\mathfrak{M}$. This proves $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. This proves uniqueness. □

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that

1. M is Hausdorff and second countable
2. M has a C^∞ atlas.

From now on, a "manifold" will mean a C^∞ manifold. We use the terms "smooth" and "C^∞" interchangeably. In the context of manifolds, we denote the standard coordinates of $\mathbb{R}^n$ by $r^1, \ldots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. Thus for $p \in U$, $(x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \ldots, x^n$ are called **coordinates** or **local coordinates** on U. By abuse of notation, we sometimes omit the p. So the notations $(x^1, \ldots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$.

Remark 15. A topological manifold can be endowed with different (non-compatible) differentiable structures. For instance, consider $X = \mathbb{R}$. We can give the space the structure of a C^∞-manifold using the chart $(\mathbb{R}, \varphi_1)$, where φ_1 maps $x \to x$. We can also give the space the structure of a C^∞ manifold using the chart $(\mathbb{R}, \varphi_2)$, where φ_2 maps $x \mapsto x^3$. These two charts are not C^∞-compatible since $\varphi_1 \circ \varphi_2^{-1}$ maps $x \mapsto x^{\frac{1}{3}}$, and this is *not* C^∞ on $\mathbb{R}$: $\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ is not continuous at $x = 0$.

11.1.1 An Atlas For a Product

Proposition 11.4. *If $\mathfrak{U} = \{(U_i, \phi_i) \mid i \in I\}$ and $\mathfrak{V} = \{(V_j, \psi_j) \mid j \in J\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection*

$$\mathfrak{U} \times \mathfrak{V} = \{(U_i \times V_j, \phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n) \mid (i,j) \in I \times J\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. Clearly the set $\{U_i \times V_j \mid (i,j) \in I \times J\}$ covers $M \times N$, so we just need to show that any two charts in $\mathfrak{U} \times \mathfrak{V}$ are pairwise compatible. Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two charts in $\mathfrak{U} \times \mathfrak{V}$. Then $(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1}$ is C^∞, since

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} = \left(\phi_1 \circ \phi_2^{-1}\right) \times \left(\psi_2 \times \psi_2^{-1}\right),$$

and both $\phi_1 \circ \phi_2^{-1}$ and $\psi_2 \times \psi_2^{-1}$ are C^∞ on their respective domains. The same proof shows that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1}$ is C^∞. Thus $\mathfrak{U} \times \mathfrak{V}$ is a collection of pairwise C^∞ compatible charts that cover $M \times N$. □

Example 11.5. It follows from Proposition (11.4) that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds.

11.2 Examples of Smooth Manifolds

11.2.1 Euclidean Space

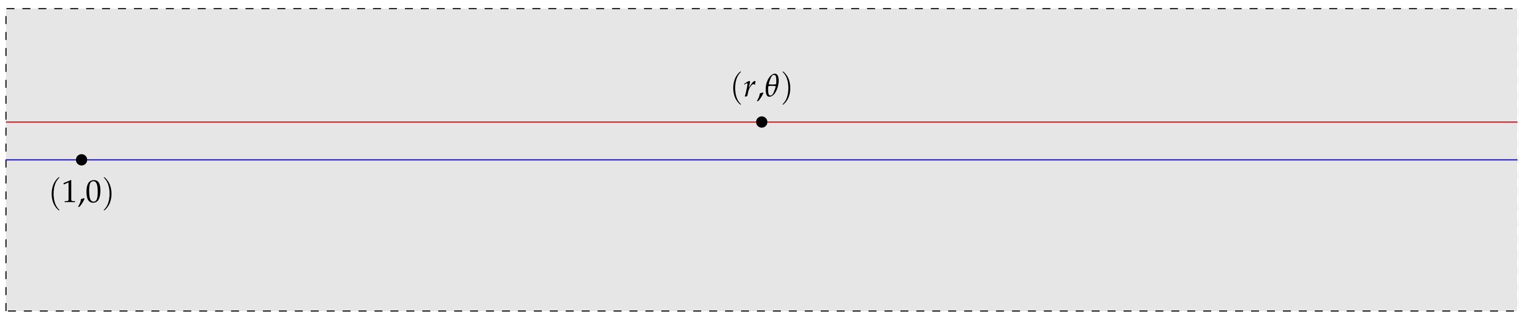
Example 11.6. (Euclidean space). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, \mathrm{id})$. We use $x_1, \ldots, x_n$ to denote coordinates functions and $a_1, \ldots, a_n$ to denote real numbers. Thus, if $p = (a_1, \ldots, a_n)$ is a point in $\mathbb{R}^n$, we have $x_1(p) = a_1$, $x_2(p) = a_2$, and etc...

Example 11.7. The real half line $\mathbb{R}_{>0} : \{a \in \mathbb{R} \mid a > 0\}$ is also a smooth manifold, with a single chart $(\mathbb{R}_{>0}, \mathrm{id})$. In fact, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$. A homeomorphism from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is given by $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

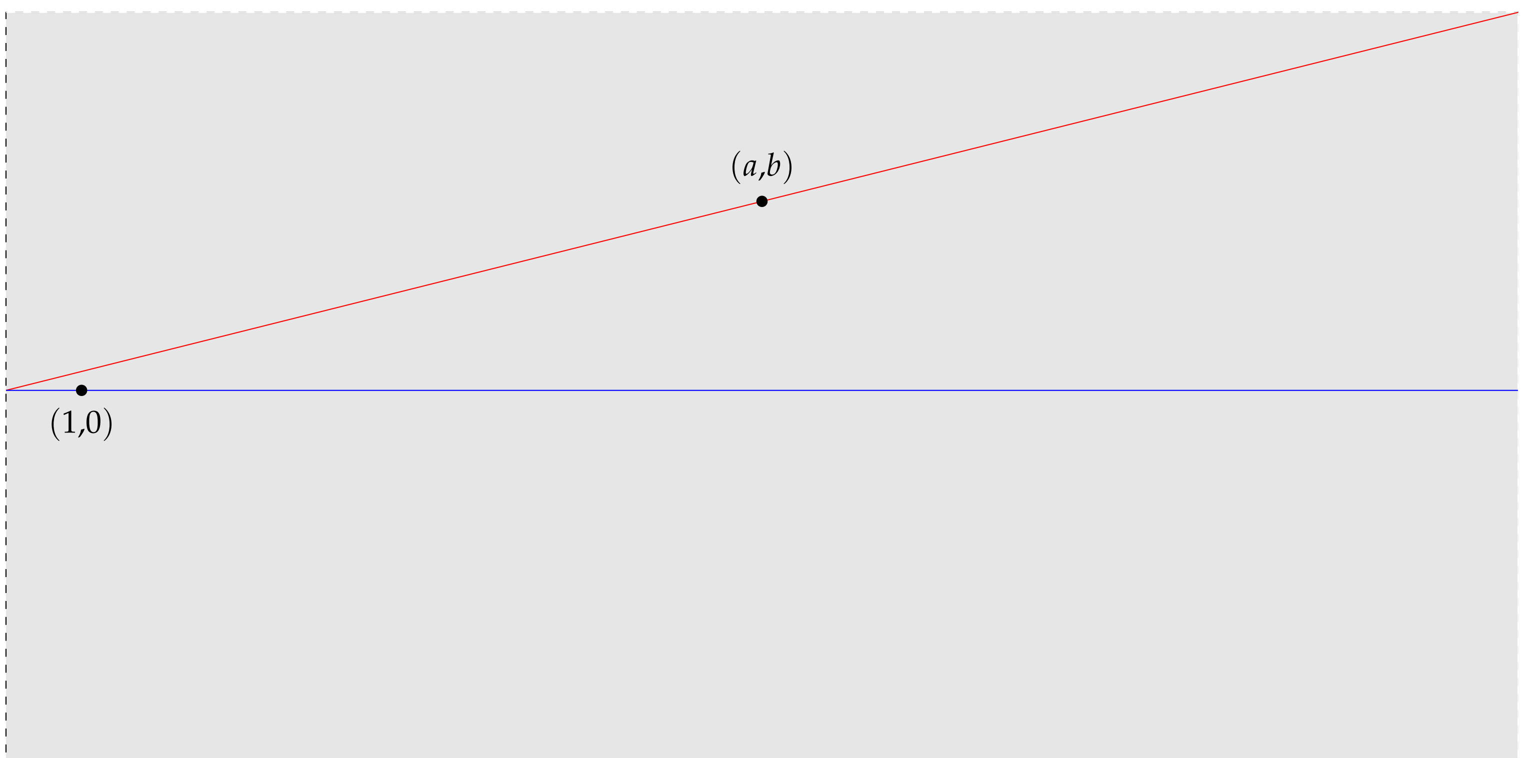
Now consider the half-open interval $(0, 2\pi)$. Open sets of the form (a, b) where and $0 \leq a < b < 2\pi$ form a basis for this topological space.

11.2.2 Right-Half Infinite Strip and the Right-Half Plane

Let $M = \mathbb{R}_{>0} \times \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. We illustrate this space below:



Now let $N = \mathbb{R}_{>0} \times \mathbb{R}$ be the right-half plane. We illustrate this space below:



We can give both M and N the structure of a smooth manifold by simply using the identity charts.

Let $\varphi : M \to N$ be given by $\varphi(r, \theta) = (\varphi_1(r, \theta), \varphi_2(r, \theta))$, where

$$\varphi_1(r, \theta) = r \sin\theta$$
$$\varphi_2(r, \theta) = r \cos\theta$$

Then φ is a diffeomorphism from M to N. The Jacobian of φ at a point $(r, \theta) \in M$:

$$J_{(r,\theta)}(\varphi) = \begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$$

The inverse to $\varphi : M \to N$ is $\psi : N \to M$, given by $\psi(a, b) = (\psi_1(a, b), \psi_2(a, b))$ where

$$\psi_1(a, b) = \sqrt{a^2 + b^2}$$
$$\psi_2(a, b) = \arctan\left(\frac{a}{b}\right)$$

The Jacobian of ψ at a point $(a, b) \in N$:

$$J_{(a,b)}(\psi) = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

11.2.3 Manifolds of Dimension Zero

Example 11.8. (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

11.2.4 Graph of a Smooth Function

Example 11.9. (Graph of a smooth function). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$, the **graph** of f is defined to be the subset

$$\Gamma(f) = \{(p, f(p)) \in A \times \mathbb{R}^n\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^∞, then the two maps

$$\phi : \Gamma(f) \to U \qquad (p, f(p)) \mapsto p$$

and

$$(1, f) : U \to \Gamma(f) \qquad p \mapsto (p, f(p))$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^n$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

11.2.5 Circle S^1

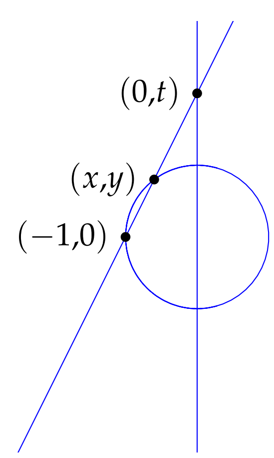
Example 11.10. (Circle) Let S^1 be the unit circle centered at the origin in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We shall describe an atlas on S^1 using stereographic projection. Let $U_1 = S^1 \backslash \{(-1, 0)\}$. Consider the line L which passes through the points $(-1, 0)$ and $(0, t)$ where $t \in \mathbb{R}$. The equation of this line is given by

$$Y = t(X + 1).$$

Since L passes through $(-1, 0)$ and is not tangent to $(-1, 0)$, it must pass through a unique point (x, y) in S^1. This is illustrated in the image below:



Since (x, y) lies on the line L and the unit circle, we get the relations

$$x^2 + y^2 - 1 = 0,$$
$$y - t(x + 1) = 0.$$

Using the second relation, we have $y = t(x + 1)$. Plugging in $t(x + 1)$ for y in the first relation, we get

$$t^2 = \frac{(1 - x)^2}{(1 + x)^2} = \frac{1 - x}{1 + x}.$$

Now we solve for x in terms of t, to get:

$$x = \frac{1 - t^2}{1 + t^2},$$
$$y = \frac{2t}{1 + t^2}.$$

Now, let $\phi_1 : U_1 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1+x}.$$

This map is clearly C^∞ in its domain U_1, since $x \neq -1$, and the inverse $\phi_1^{-1} : \mathbb{R} \to U_1$ is given by

$$t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

Next, let $U_2 = S^1 \backslash \{(1, 0)\}$. Following the same line of reasoning as the paragraph above, let $\phi_2 : U_2 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1-x}.$$

Again, this map is clearly C^∞ in its domain U_2, since $x \neq 1$, and the inverse $\phi_2^{-1} : \mathbb{R} \to U_2$ is given by

$$t \mapsto \left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2} \right).$$

Let us calculate the transition map $\phi_{12} := \phi_1 \circ \phi_2^{-1}$:

$$\begin{aligned} \phi_{12}(t) &= (\phi_1 \circ \phi_2^{-1})(t) \\ &= \phi_1 \left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2} \right) \\ &= \frac{1}{t}. \end{aligned}$$

Remark 16. We think of t as a local coordinate of S^1 and x, y as global coordinates of S^1.

11.2.6 Projective Line

Example 11.11. Let $\mathbb{P}^1(\mathbb{R})$ be the projective line over $\mathbb{R}$. Define in $\mathbb{P}^1(\mathbb{R})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0} \in \mathbb{R},$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1} \in \mathbb{R}.$$

These maps are clearly C^∞ in their domains. The inverse maps are given by

$$\phi_0^{-1}(t) = (1 : t) \in U_0 \qquad \phi_1^{-1}(t) = (t : 1) \in U_1.$$

Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1}$:

$$\begin{aligned} \phi_0 \circ \phi_1^{-1}(t) &= \phi_0 \circ \phi_1^{-1}(t) \\ &= \phi_0 (t : 1) \\ &= \frac{1}{t}. \end{aligned}$$

Recall that this is the same transition map we calculated in Example (11.10).

11.2.7 Sphere S^2

Example 11.12. (Sphere) Let S^2 be the unit sphere

$$S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres - the front, rear, right , left, upper, and lower hemispheres

$$\begin{aligned} U_1 &= \{(a, b, c) \in S^2 \mid a > 0\} \qquad &\phi_1(a, b, c) = (b, c) \\ U_2 &= \{(a, b, c) \in S^2 \mid a < 0\} \qquad &\phi_2(a, b, c) = (b, c) \\ U_3 &= \{(a, b, c) \in S^2 \mid b > 0\} \qquad &\phi_3(a, b, c) = (a, c) \end{aligned}$$

$$\begin{aligned}
U_4 &= \{(a,b,c) \in S^2 \mid b < 0\} & \phi_4(a,b,c) &= (a,c) \\
U_5 &= \{(a,b,c) \in S^2 \mid c > 0\} & \phi_5(a,b,c) &= (a,b) \\
U_6 &= \{(a,b,c) \in S^2 \mid c < 0\} & \phi_6(a,b,c) &= (a,b)
\end{aligned}$$

The open set U_{14} is $\{(a,b,c) \in S^2 \mid b < 0 < a\}$ and $\phi_4(U_{14}) = \{(a,c) \in \mathbb{R}^2 \mid a^2 + c^2 < 1 \text{ and } a > 0\}$.
Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned}
\phi_{14}(a,c) &= \phi_1 \circ \phi_4^{-1}(a,c) \\
&= \phi_1\left(a, \sqrt{1-c^2-a^2}, c\right) \\
&= \left(\sqrt{1-c^2-a^2}, c\right).
\end{aligned}$$

It is easy to see that this is indeed a smooth map in its domain (since $1 - c^2 - a^2 \neq 0$). The Jacobian of ϕ_{14} at the point (a,c) is

$$J_{(a,c)}(\phi_{14}) = \begin{pmatrix} \frac{a}{\sqrt{1-c^2-a^2}} & \frac{c}{\sqrt{1-c^2-a^2}} \\ 0 & 1 \end{pmatrix}$$

Now let's compute the transition map ϕ_{45}:

$$\begin{aligned}
\phi_{45}(a,b) &= \phi_4 \circ \phi_5^{-1}(a,b) \\
&= \phi_4\left(a, b, \sqrt{1-a^2-b^2}\right) \\
&= \left(a, \sqrt{1-a^2-b^2}\right).
\end{aligned}$$

11.2.8 The Sphere S^n

Example 11.13. Recall that the sphere $S^n \subseteq \mathbb{R}^{n+1}$ is defined by

$$S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| = 1\}.$$

Here we write $\boldsymbol{x} = (x_1, \dots, x_n, x_{n+1})$. In particular, this description S^n comes equipped with *global* coordinates: every point in S^n has the form $\boldsymbol{x} = (x_1, \dots, x_{n+1})$ where $\|\boldsymbol{x}\| = 1$. Let $\boldsymbol{x}_N = (0, \dots, 0, 1)$ be the north pole and let $U_N = S^n \backslash \{\boldsymbol{x}_N\}$. Define $\varphi_N \colon U_N \to \mathbb{R}^n$ by

$$\varphi_N(\boldsymbol{x}) = \varphi_N(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}}\right) = (y_1(\boldsymbol{x}), \dots, y_n(\boldsymbol{x})),$$

where $y_i = t_i \circ \varphi_N$ for each $1 \leq i \leq n$ where the t_i denote the standard coordinates of $\mathbb{R}^n$. We denote this map by $\varphi_N = (y_1, \dots, y_n)$. We often get lazy and write $y_i(\boldsymbol{x}) = y_i$ where we think of y_i in this case as a real number (and not a function) which gives the ith coordinate of $\boldsymbol{x}$. For instance, the function $f \colon U_N \to \mathbb{R}$ given by $f = y_1^3 + \cdots + y_n^3$ is defined by

$$\begin{aligned}
f(\boldsymbol{x}) &= (y_1^3 + \cdots + y_n^3)(\boldsymbol{x}) \\
&= y_1(\boldsymbol{x})^3 + \cdots + y_n(\boldsymbol{x})^3 \\
&= y_1^3 + \cdots + y_n^3
\end{aligned}$$

for all $\boldsymbol{x} \in U_N$, where we got lazy at the end and simply wrote $y_i(\boldsymbol{x}) = y_i$. When we write $f = y_1^3 + \cdots + y_n^3$, we are thinking of the y_i as functions $y_i \colon U_N \to \mathbb{R}$When we write $f(\boldsymbol{x}) = y_1^3 + \cdots + y_n^3$, we are thinking of the y_i as the coordinates of $\boldsymbol{x} = \varphi_N^{-1}(y_1, \dots, y_n) = \varphi_N^{-1}(\boldsymbol{y})$.

$$f(\boldsymbol{x}) = y_1(\boldsymbol{x}) + \cdots$$

we have a function $f \colon U_N \to \mathbb{R}$ defined by $f =$

$$f = y_1 + y_2 + \cdots + y_n.$$

Thus the y_i are used in to different (and admittedly contradictory) ways: either as a function $y_i \colon U_N \to \mathbb{R}$ or as the ith coordinate of x.
Here, the y_i gives us *local* coordinates of S^n in the open set U_N: each $\boldsymbol{x} \in U_N$ can be identified with the there is a unique point $(y_1(\boldsymbol{x}), \dots, y_n(\boldsymbol{x})) \in \mathbb{R}^n$We denote this map by $\varphi_N = (y_1, \dots, y_n)$. Note that we often get lazy and

write $(y_1, \ldots, y_n) = (y_1(\boldsymbol{x}), \ldots, y_n(\boldsymbol{x}))$ where now we we think of the y_i in this case as It is straightforward to check that this map is a homeomorphism. Here, the y_i gives us *local* coordinates of S^n in the open set U_N: every point $\boldsymbol{x}$ in U_N can be identitified uniquely the form $\boldsymbol{x} = (y_1(\boldsymbol{x}), \ldots, y_n(\boldsymbol{x}))$ Here, we view the Then $\boldsymbol{y}$ gives us local coordinates of S^n on U_N, and we can

Using stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $\boldsymbol{x}_N = (0, \ldots, 0, 1)$ be the north pole $p_N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the north pole and $p_S = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the south pole. Define the maps $\phi_N : S^n \backslash \{p_N\} \to \mathbb{R}^n$ and $\phi_S : S^n \backslash \{p_S\}$, called **stereographic projection** from the north pole (resp. south pole), by

$$\phi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \phi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\phi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, -1 + \sum_{i=1}^n x_i^2\right)$$

and

$$\phi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, 1 - \sum_{i=1}^n x_i^2\right).$$

Thus, if we let $U_N = S^n \backslash \{p_N\}$ and $U_S = S^n \backslash \{p_S\}$, we see that U_N and U_S are two open subsets convering S^n, both homeomorphic to $\mathbb{R}^n$. Furthermore, it is easily checked that on the overlap, $U_N \cap U_S$, the transition maps

$$\phi_S \circ \phi_N^{-1} = \phi_N \circ \phi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $p_O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n \backslash \{O\}$, so we conclude that (U_N, ϕ_N) and (U_S, ϕ_S) form a smooth atlas for S^n.

11.2.9 Real Projective Plane

Example 11.14. (Projective Plane) Let $\mathbb{P}^2(\mathbb{R})$ be the projective plane over $\mathbb{R}$. Define in $\mathbb{P}^2(\mathbb{R})$ the three charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1 : x_2) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) =: (a, b)$$

$$U_1 = D(X_1) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) =: (c, d)$$

$$U_2 = D(X_2) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_2 \neq 0\} \qquad \phi_2(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) =: (e, f)$$

The reason the map ϕ_1 is a homeomorphism is because given that $x_1 \neq 0$, we use the equivalence relation to write the point $p = (x_0 : x_1 : x_2)$ as $p = \left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right)$. Now $\frac{x_0}{x_1}$ and $\frac{x_2}{x_1}$ are two real rumbers which uniquely determine the point (a, b). We think of a and b as the local coordinates in the (U_0, ϕ_0) chart.

Let U_{01} be the intersection of U_0 and U_1, that is, $U_{01} := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$. Then $\phi_0(U_{01}) = \left\{\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$ and $\phi_1(U_{01}) = \left\{\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$. We can also write this in terms of local coordinates as $\phi_0(U_{01}) = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ and $\phi_1(U_{01}) = \{(c, d) \in \mathbb{R}^2 \mid c \neq 0\}$. Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1} : \phi_1(U_{01}) \to \phi_0(U_{01})$ using the local coordinates. We have

$$\begin{aligned}
\phi_{01}(c,d) &= \phi_0 \circ \phi_1^{-1}(c,d) \\
&= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right) \\
&= \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \\
&= \left(\frac{1}{c}, \frac{d}{c}\right).
\end{aligned}$$

It's easy to see that ϕ_{01} is C^∞. Indeed, writing ϕ_{01}^1 and ϕ_{01}^2 for the components of ϕ_{01} (so $\phi_{01}^1(c,d) = \frac{1}{c}$ and $\phi_{01}^2(c,d) = \frac{d}{c}$), the partial derivatives $\partial_c^m \partial_d^n \phi_{01}^i$ exist and are continuous everywhere in $\phi_1(U_{01})$ for all $m,n \in \mathbb{N}$ and $i = 1,2$. This is because ϕ_{01}^1 and ϕ_{01}^2 are rational functions (i.e. ratio of two polynomials) and are they are defined everywhere since $c \neq 0$ in$\phi_1(U_{01})$.

Similarly, one can easily show that

$$\begin{aligned}
\phi_{10}(a,b) &= \left(\frac{1}{a}, \frac{b}{a}\right) \\
\phi_{20}(a,b) &= \left(\frac{1}{b}, \frac{a}{b}\right) \\
\phi_{02}(e,f) &= \left(\frac{f}{e}, \frac{1}{e}\right) \\
\phi_{12}(e,f) &= \left(\frac{e}{f}, \frac{1}{f}\right) \\
\phi_{21}(c,d) &= \left(\frac{c}{d}, \frac{1}{d}\right)
\end{aligned}$$

It is instructive to check that $\phi_{ij} \circ \phi_{ji} = 1$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$.

11.2.10 Riemann Sphere

Example 11.15. (Riemann sphere) In this example we describe a **complex manifold**. A complex manifold is the complex analogue of a manifold, however in the complex manifold case, we require the transition maps to be holomorphic, and not just C^∞. Let $\mathbb{P}^1(\mathbb{C})$ be the projective line over $\mathbb{C}$ (also known as the Riemann sphere). Define in $\mathbb{P}^1(\mathbb{C})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0}$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1}$$

This time, let $z = \frac{x_0}{x_1}$. The open set U_{01} is $\{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$ and $\phi_1(U_{01}) = \mathbb{C}^\times$. Now

$$\begin{aligned}
\phi_0 \circ \phi_1^{-1}(z) &= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0}\right) \\
&= \frac{x_1}{x_0} \\
&= \frac{1}{z}.
\end{aligned}$$

One can show that the map $z \mapsto \frac{1}{z}$ is holomorphic in the domain $\mathbb{C}^\times$.

11.2.11 Mobius Strip

Example 11.16. Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold. First we consider the set of all nonvertical lines in $\mathbb{R}^2$, which we denote by U_v. A nonvertial is of the form $\ell^v_{a,b} = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell^v_{a,b} \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism. Next we consider the set of all nonhorizontal lines in $\mathbb{R}^2$, which we denote by U_h. A nonhorizontal is of the form $\ell^h_{c,d} = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell^h_{c,d} \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism. Now we have $U_v \cup U_h = \mathcal{L}$. To get a topology on $\mathcal{L}$, we glue the topologies from U_v and U_h: a set $U \subset \mathcal{L}$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}
\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d) \\
&= \varphi_v\left(\ell^h_{c,d}\right) \\
&= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right) \\
&= \left(\frac{1}{c}, -\frac{d}{c}\right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b) \\
&= \varphi_h\left(\ell^v_{a,b}\right) \\
&= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right) \\
&= \left(\frac{1}{a}, -\frac{b}{a}\right).
\end{aligned}$$

These maps are clearly C^∞. In fact, they look very similar to the transition maps for the projective plane, except they are twisted by a negative sign.

Remark 17. We can also describe $\mathcal{L}$ as $\mathbb{RP}^2 \backslash \{[0:0:1]\}$: Any line in the euclidean plane is of the form $ax + by + c = 0$, for some $a,b,c \in \mathbb{R}$. First note that these coefficients uniquely determine the line and they are homogeneous. Hence there is a well defined map $\phi : \mathcal{L} \to \mathbb{RP}^2$, given by mapping the line $\mathbf{V}(ax+by+c)$ to the point $[a:b:c]$. Now ϕ is injective, but not surjective. However if we remove the point $[0:0:1]$, then the induced map $\phi : \mathcal{L} \to \mathbb{RP}^2 \backslash \{[0:0:1]\}$ is a bijection.

11.2.12 Grassmannians

The **Grassmannian** $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the **rank** of a matrix A, denoted by rk$A$, is defined to be the number of linearly independent columns of A. This matrix is called a **matrix representative** of the k-plane.

Two bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k,\mathbb{R})$ such that

$$b_j = \sum_{i=1}^{k} a_i g_{ij}$$

for all $1 \le k \le n$. In matrix notation, this says $B = Ag$. Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \text{ if and only if there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

A **real Grassmann manifold** $G(n,k)$ is defined as the space of all k-dimensional subspaces of the space $\mathbb{R}^n$. The topology in $G(n,k)$ may be described as induced by the embedding $G(n,k) \to \mathrm{End}(\mathbb{R}^n)$ which assigns to a $P \in G(n,k)$, the orthogonal projection $\mathbb{R}^n \to P$ combined with the inclusion map $P \to \mathbb{R}^n$.

In $G(4,2)$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} \sim \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = \begin{pmatrix} c_{11}a_{11}+c_{12}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24} \\ c_{21}a_{11}+c_{22}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24} \end{pmatrix}$$

where $c_{11}c_{22} - c_{21}c_{12} \neq 0$.

11.2.13 Grassmanians: Algebraic Theory

Let F be a field and let V be a vector space over F of dimension $n+1$ where $n \geq 1$. We let $\mathbb{G}(V)$ denote the set of all subspaces of V. Also, for each $0 \leq d \leq n+1$, we let $\mathbb{G}_d(V)$ denote the set of codimension-d subspaces of V. Thus we have a decomposition

$$\mathbb{G}(V) = \bigcup_{d=0}^{n+1} \mathbb{G}_d(V)$$

where

$$\begin{aligned} \mathbb{G}_0(V) &= V \\ \mathbb{G}_1(V) &= \{\text{hyperplanes in } V\} \\ &\vdots \\ \mathbb{G}_n(V) &= \{\text{lines that pass through the origin in } V\} \\ \mathbb{G}_{n+1}(V) &= 0 \end{aligned}$$

When $V = F^{n+1}$, then this set is called the **Grassmanian of codimension-d subspaces in $(n+1)$-space** (over F) and is denoted $\mathbb{G}(d,n)(F)$ (or more simply by $\mathbb{G}(d,n)$ if F is understood. For $d = 1$ it is called the **projective space** and is denoted $\mathbb{P}(V)$. By the dual relationship between subspaces of V and of $V^\vee$ (whereby a subspace $W \subseteq V$ "corresponds" to the subspace $(V/W)^\vee$ in $V^\vee$), $\mathbb{G}_d(V)$ can also be viewed as $\mathbb{G}_{n+1-d}(V^\vee)$.

Fix an ordered basis $\boldsymbol{e} = e_0, \ldots, e_n$ of V.

Lemma 11.2. *Let $I = \{i_1, \ldots, i_d\}$ be a set of d distinct indices where $0 \leq i_1 < \cdots < i_d \leq n$. Let $U_I \subseteq \mathbb{G}_d(V)$ denote the subset of codimension-d linear subspaces $W \subseteq V$ for which the e_i's with $i \in I$ projective to a basis of the d-dimensional V/W. The U_I's cover $\mathbb{G}_d(V)$ as a set.*

Proposition 11.5. *We have a bijection*

$$\begin{aligned} \mathbb{G}_{n+1-d}(\mathbb{R}^{n+1}) \quad &\longleftrightarrow \quad \{A \in \mathrm{M}_{n+1,d}(\mathbb{R}) \mid A \text{ has full rank}\}/\sim \\ W \quad &\rightarrow \\ \operatorname{im} A \quad &\leftarrow \quad A \end{aligned}$$

where the equivalence relation is given by

$$A \sim B \text{ if and only if } AC = B \text{ for some } C \in \mathrm{GL}_d(\mathbb{R}).$$

Example 11.17. Let us consider the case where $V = \mathbb{R}^3$. We want to describe the codimension-1 subspaces of $\mathbb{R}^3$. In other words, we want to describe the subspaces of $\mathbb{R}^3$ which have dimension 2. A subspace of dimension m can be described using a

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We w

11.2.14 Grassmanians: Topological Theory

Theorem 11.3. *Let V be a finite-dimensional vector space over $\mathbb{R}$*

12 Smooth Maps on a Manifold

Now that we've defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined.

12.1 Smooth Functions

Definition 12.1. Let M be a smooth manifold of dimension n. A function $f\colon M \to \mathbb{R}$ is said to be C^∞ or **smooth at a point** p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

Observe that the definition of the smoothness of a function f at a point p is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) is any other chart about p in M, then on $\psi(U \cap V)$, we have

$$f \circ \psi^{-1} |_{\psi(U\cap V)} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$

which is C^∞ at $\psi(p)$. Thus $f \circ \psi^{-1}$ must be C^∞ at $\psi(p)$. Also observe that in the definition above, $f\colon M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are only interested in functions that are smooth on an open set, there is no loss of generality in assuming at the onset that f is continuous.

Proposition 12.1. *Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

1. *The function $f : M \to \mathbb{R}$ is C^∞.*
2. *The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
3. *For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

Proof. We will prove the proposition as a cyclic chain of implications.

(2 $\implies$ 1): This follows directly from the definition of a C^∞ function, since by (2) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

(1 $\implies$ 3): Let (V, ψ) be an arbitrary chart on M and let $p \in V$. By the remark above, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

(3 $\implies$ 2): Obvious.

□

The smoothness conditions of Proposition (12.1) will be a recurrent motif: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on *every* chart.

Definition 12.2. Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

Remark 18. In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

Example 12.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by an angle θ and let x, y denote the standard coordinate functions on $\mathbb{R}^2$. Then

$$\begin{aligned}\phi^* x &= (\cos\theta)x - (\sin\theta)y \\ \phi^* y &= (\sin\theta)x + (\cos\theta)y.\end{aligned}$$

Indeed, let e_1, e_2 denote the standard coordinates on $\mathbb{R}^2$; so $x(e_1) =$

$$\begin{aligned}(\phi^* x)(a, b) &= x\left(\phi(a, b)\right) \\ &= x\left(\cos\theta a - \sin\theta b, \sin\theta a + \cos\theta b\right) \\ &= \cos\theta a - \sin\theta b \\ &= ((\cos\theta)x - (\sin\theta)y)(a, b).\end{aligned}$$

12.2 Smooth Maps Between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms "C^∞" and "smooth" interchangeably.

Definition 12.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ **at a point** p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

Remark 19.

1. In the definition, we needed $F^{-1}(V)$ to be open so that $\phi(F^{-1}(V) \cap U)$ is open. Thus, C^∞ maps between manifolds are by definition continuous.

2. In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ as a chart about $F(p)$ in $\mathbb{R}^m$. According to the definition above, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p in N such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

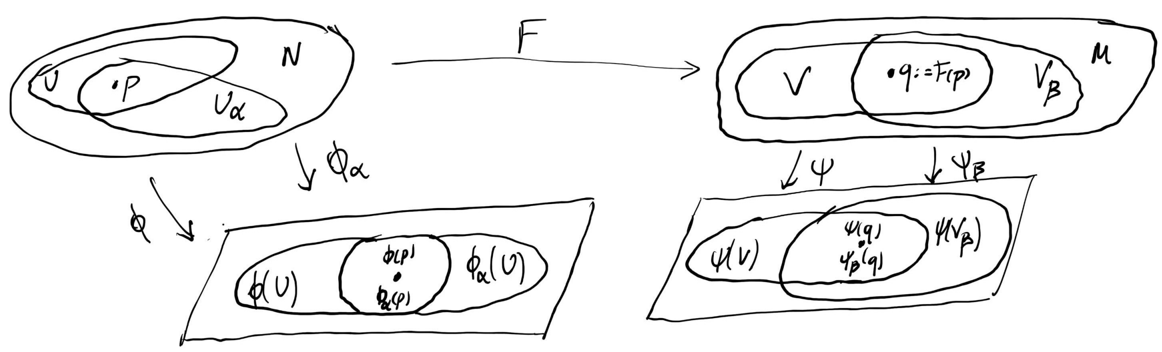
We show now that the definition of the smoothness of a map $F : N \to M$ at a point is independent of the choice of charts.

Proposition 12.2. *Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.*

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1} \,|_{\phi\left(F^{-1}(V \cap V_\beta) \cap U \cap U_\alpha\right)} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1}),$$

is C^∞ at $\phi(p)$. Therefore $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.



□

Proposition 12.3. *(Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:*

1. *The map $F : N \to M$ is C^∞.*

2. *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map*

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

   is C^∞.

3. *For every chart (U, ϕ) on N and (V, ψ) on M, the map*

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

   is C^∞.

Proof.
($2 \implies 1$): Let $p \in N$. Suppose (U, ϕ) is a chart about p in $\mathfrak{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathfrak{V}$. By (2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

($1 \implies 3$): Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \emptyset$. Let $p \in U \cap F^{-1}(V)$. Then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition (12.2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitary point of $\phi(U \cap F^{-1}(V))$, the map $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞.

($3 \implies 2$): Obvious. $\square$

Proposition 12.4. *(Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.*

Proof. Let (U, ϕ), (V, ψ), and (W, σ) be charts on N, M, and P respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}).$$

Since F and G are C^∞, the maps $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are also C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞, and thus $G \circ F$ is C^∞. $\square$

12.2.1 Diffeomorphisms

A **diffeomorpism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 12.5. *If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\left\{(\phi(U), \mathrm{id}_{\phi(U)})\right\}$ with a single chart on $\phi(U)$. Since

$$\mathrm{id}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$$

is the identity map, it is C^∞. By Proposition (12.3), ϕ is C^∞.

To test smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since

$$\phi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} : \phi(U) \to \phi(U)$$

is the identity map, the map ϕ^{-1} is also C^∞. $\square$

Proposition 12.6. *Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition (12.5). As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U, F) is in the atlas. $\square$

12.2.2 Smoothness in Terms of Components

In this subsection, we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 12.7. *(Smoothness of a vector-valued function) Let N be a manifold and let $F : N \to \mathbb{R}^m$ be a continuous map. The following are equivalent:*

1. *The map $F : N \to \mathbb{R}^m$ is C^∞.*
2. *The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*
3. *For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*

Proposition 12.8. *(Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F_1, \ldots, F_m : N \to \mathbb{R}$ are all C^∞.*

Proof. The $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the functions $F_i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ if and only if the functions $F_i : N \to \mathbb{R}$ are all C^∞. $\square$

12.3 Germs of C^∞ functions

Let M be an n-dimensional manifold and let p be a point in M. Consider the set of all pairs (f, U), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. Just as in the $\mathbb{R}^n$ case, we introduce an equivalence relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(M)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Let (f, U) be represent a germ in $C_p^\infty(M)$ and suppose (U_0, ϕ) is a chart centered at p. Then $(U_0 \cap U, \phi_{|U})$ is a chart centered at p and clearly we have $(f, U) \sim (f\,|_{U_0 \cap U}, U_0 \cap U)$. Thus we may always assume that a germ can be represented by (f, U) where (U, ϕ) is a chart centered at p. In particular, we obtain an isomorphism

$$\widehat{\phi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \phi^{-1}, \phi(U))$. Of course this map depends on our choice of chart. If (V, φ) was another chart, then we'd obtain another isomorphism

$$\widehat{\varphi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \varphi^{-1}, \varphi(U))$. We can relate these two isomorphisms via the transition function $\phi \circ \varphi^{-1}$.

Let M be a manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on M. We describe the structure of a premanifold as follows: if U is an open subset of M, then we set

$$\mathcal{O}_M(U) := \{f : U \to \mathbb{R} \mid f|_{U \cap U_i} \circ \phi_i^{-1} : \phi_i(U \cap U_i) \to \mathbb{R} \text{ is } C^\infty\} = \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty\}.$$

To see that this is a premanifold, fix $i_0 \in I$. For $U \subseteq U_{i_0}$ open let $f : U \to \mathbb{R}$ be a map such that $f \circ \phi_{i_0}^{-1} : \phi_{i_0}(U_{i_0} \cap U) \to \mathbb{R}$ is a C^∞ function. Then $f \in \mathcal{O}_M(U)$ because the change of charts between i and i_0 are C^∞-diffeomorphisms. Indeed, we have

$$f|_{U \cap U_i} \circ \phi_i^{-1} = (f \circ \phi_{i_0}^{-1}) \circ (\phi_{i_0} \circ \phi_i^{-1}).$$

Therefore ϕ_{i_0} yields an isomorphism $(U_{i_0}, \mathcal{O}_{M|U_{i_0}}) \cong (Y_{i_0}, \mathcal{O}_{i_0})$, where $\mathcal{O}_{Y_0}$ is the sheaf of C^∞ functions on Y_{i_0}. Hence, $(M, \mathcal{O}_M)$ is a ringed space that is locally isomorphic to a manifold. Hence it is a premanifold.

12.4 Examples of Smooth Maps

Example 12.2. We show that the map $F : \mathbb{R} \to S^1$ given by $F(t) = (\cos t, \sin t)$ is C^∞. For $\mathbb{R}$, we use the atlas which consists of a single chart $(\mathbb{R}, \mathrm{id})$. For S^1 we use the atlas which consists of the charts (U_1, ϕ_1), (U_2, ϕ_2), (U_3, ϕ_3) and (U_4, ϕ_4) where

$$\begin{aligned}
U_1 &= \{(a, b) \in S^1 \mid a > 0\} & \phi_1(a, b) &= b \\
U_2 &= \{(a, b) \in S^2 \mid a < 0\} & \phi_2(a, b) &= b \\
U_3 &= \{(a, b) \in S^2 \mid b > 0\} & \phi_3(a, b) &= a \\
U_4 &= \{(a, b) \in S^2 \mid b < 0\} & \phi_4(a, b) &= a
\end{aligned}$$

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned}
\phi_{14}(a) &= \phi_1 \circ \phi_4^{-1}(a) \\
&= \phi_1\left(a, \sqrt{1 - a^2}\right) \\
&= \sqrt{1 - a^2}.
\end{aligned}$$

Similar computations shows that

$$\begin{aligned}
\phi_{13}(a) &= \sqrt{1 - a^2} \\
\phi_{24}(a) &= \sqrt{1 - a^2} \\
\phi_{23}(a) &= \sqrt{1 - a^2}.
\end{aligned}$$

Now, we need to show that $\phi_i \circ F \circ \mathrm{id}$ is C^∞ for $i = 1, 2, 3, 4$. Let's compute $\phi_1 \circ F \circ \mathrm{id}$:

$$\begin{aligned}
(\phi_1 \circ F \circ \mathrm{id})(t) &= \phi_1(F(t)) \\
&= (\phi_1((\cos t, \sin t)) \\
&= \sin t.
\end{aligned}$$

Similar computations shows that

$$\begin{aligned}(\phi_2 \circ F \circ \mathrm{id})(t) &= \sin t\\ (\phi_3 \circ F \circ \mathrm{id})(t) &= \cos t\\ (\phi_4 \circ F \circ \mathrm{id})(t) &= \cos t.\end{aligned}$$

These maps are all C^∞.

Example 12.3. Consider $N = \mathbb{R}$ and $M = \mathbb{R}^2$ and let $f : N \to M$ be given by $f(t) = (t^2, t^3)$.

Example 12.4. Let S^2 be the unit sphere with its smooth structure given in Example (11.12). Let $f : S^2 \to \mathbb{R}$ be given by

$$f(a,b,c) = c^2.$$

We claim that f is C^∞. To see this, we need to show that f is C^∞ at every point $p = (a,b,c)$ in S^2. First assume that $p \in U_6$. Using the chart (U_6, ϕ_6), we find that

$$\begin{aligned}(f \circ \phi_6^{-1})(a,b) &= f\left(\phi_6^{-1}(a,b)\right)\\ &= f\left(a, b, \sqrt{1-a^2-b^2}\right)\\ &= 1 - a^2 - b^2,\end{aligned}$$

which is clearly C^∞.

Example 12.5. Let us show that a C^∞ function $f(x,y)$ on $\mathbb{R}^2$ restricts to a C^∞-function on S^1. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a,b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a,b) = a$ and $y(a,b) = b$. Suppose that we can show that x and y restrict to C^∞-functions on S^1. Then the inclusion map $i : S^1 \to \mathbb{R}^2$, given by $i(p) = (x(p), y(p))$ is C^∞ on S^1, and so the composition $f\,|_{S^1} = f \circ i$ with be C^∞ on S^1 too.

Consider first the function x. We use the following atlas (U_i, ϕ_i) for S^1, where

$$\begin{aligned}U_1 &= \{(a,b) \in S^1 \mid b > 0\} & \phi_1(a,b) &= a\\ U_2 &= \{(a,b) \in S^1 \mid b < 0\} & \phi_2(a,b) &= a\\ U_3 &= \{(a,b) \in S^1 \mid a > 0\} & \phi_3(a,b) &= b\\ U_4 &= \{(a,b) \in S^2 \mid a < 0\} & \phi_4(a,b) &= b\end{aligned}$$

Since x is a coordinate function on U_1 and U_2, it is a coordinate function on $U_1 \cup U_2$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1} : \phi_3(U_3) \to \mathbb{R}$.

$$(x \circ \phi_3^{-1})(b) = x\left(\sqrt{1-b^2}, b\right) = \sqrt{1-b^2}.$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1-b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3. On U_4, we have

$$(x \circ \phi_4^{-1})(b) = x\left(-\sqrt{1-b^2}, b\right) = -\sqrt{1-b^2}.$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1. The proof that y is C^∞ on S^1 is similar.

Example 12.6. Let S^2 be the unit sphere with its smooth structure given in Example (11.12). Let's construct a smooth function on S^2. First note that

$$\phi_1(U_{16}) = \{(b,c) \in \mathbb{R}^2 \mid b^2 + c^2 < 1 \text{ and } c < 0\} \qquad \text{and} \qquad \phi_6(U_{16}) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \text{ and } 0 < a\}.$$

Let $f : \phi_1(U_{16}) \to \mathbb{R}^2$ be given by

$$f(b,c) = b^2 + c^2.$$

Let's pullback $f : \phi_1(U_{16}) \to \mathbb{R}^2$ to $\phi_{16}^*(f) : \phi_6(U_{16}) \to \mathbb{R}^2$ using the transition function ϕ_{16}, where

$$\begin{aligned}\phi_{16}(a,b) &= \phi_1 \circ \phi_6^{-1}(a,b)\\ &= \phi_1\left(a, b, \sqrt{1-b^2-a^2}\right)\\ &= \left(b, \sqrt{1-b^2-a^2}\right).\end{aligned}$$

We have,

$$\begin{aligned}\phi_{16}^*(f)(a,b) &= (f \circ \phi_{16})(a,b)\\ &= f\left(b, \sqrt{1-b^2-a^2}\right)\\ &= 1 - a^2.\end{aligned}$$

12.4.1 Diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B_1(0)$

Let $\beta : \mathbb{R}^n \to B_1(0)$ be given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right) := \beta(x)$$

for all $x \in \mathbb{R}^n$. Then β is a diffeomorphism from $\mathbb{R}^n$ to $B_1(0)$ with inverse given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) := \beta^{-1}(x)$$

for all $x \in B_1(0)$. Indeed, let us first check that $\beta(x) \in B_1(0)$:

$$\begin{aligned} \|\beta(x)\| &= \sqrt{\left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2 + \cdots + \left(\frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2} \\ &= \sqrt{\frac{\sum_{i=1}^n x_i^2}{1 + \sum_{i=1}^n x_i^2}} \\ &< \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2}} \\ &= 1. \end{aligned}$$

Thus $\beta(x) \in B_1(0)$. Next we check that β is smooth. This comes down to checking the component functions β_i are smooth:

$$x := (x_1, \dots, x_n) \mapsto \frac{x_i}{\sqrt{1 + \sum_{i=1}^n x_i^2}} := \beta_i(x).$$

This follows from the fact that $1 + \sum_{i=1}^n x_i^2 > 0$. That β^{-1} is smooth follows by the same reasoning. Finally, checking that $\beta(\beta^{-1}(x)) = x$ is tedious but trivial:

$$\begin{aligned} \beta(\beta^{-1}(x)) &= \frac{1}{\sqrt{1 + \sum_{i=1}^n \left(\frac{x_i}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right)^2}} \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \cdots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) \\ &= \frac{1}{\sqrt{1 - \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i^2}} (x_1, \dots, x_n) \\ &= (x_1, \dots, x_n). \end{aligned}$$

12.5 Inverse Function Theorem

We say that a C^∞ map $F : N \to M$ is **locally invertible** or a **local diffeomorphism** at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism. Given n smooth functions $F_1, \dots, F_n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F_1, \dots, F_n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 12.1. *(Inverse function theorem for $\mathbb{R}^n$). Ler $F : W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant $det(J(F)_p)$ is not zero.*

Theorem 12.2. *(Inverse function theorem for manifolds). Let $F : N \to M$ be a C^∞*

13 Tangent Spaces

By definition, the **tangent space** to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map, called its **differential**, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map. In this sense, the differential of a map between manifolds is a generalization of the derivative of a map between Euclidean spaces.

13.1 The Tangent Space at a Point

Just as for $\mathbb{R}^n$, we define a **germ** of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; which scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a **derivation at a point** in a manifold M, or a **point-derivation** of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition 13.1. A **tangent vector** at a point p in a manifold M is a derivation at p.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by mapping (x, y) to $x^3 + y^3 + x + 1 := t$. Let $p = (x_0, y_0)$ be a point in $\mathbb{R}^2$. Then f induces a map $T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ by taking a derivation D in $T_p\mathbb{R}^2$ to the derivation f_*D in $T_{f(p)}\mathbb{R}$, where $(f_*D)(g) = D(g \circ f)$.

Example 13.1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^3 - x := t$. Then $\partial_x \mapsto (3x_0^2 - 1)\partial_t$.

Example 13.2. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \mapsto (f_1(x, y), f_2(x, y))$, where

$$f_1(x, y) = x$$
$$f_2(x, y) = \frac{xy^2}{y^2 + 1}$$

Then

$$J_{(x_0,y_0)}(f_1, f_2) = \begin{pmatrix} 1 & 0 \\ \frac{y_0^2}{y_0^2+1} & \frac{2x_0y_0}{(y_0^2+1)^2} \end{pmatrix}$$
$$\frac{2x_0y_0}{(y_0^2 + 1)^2} = 0$$

Let M be an n-dimensional manifold and let p be a point in M. We describe T_pM in another way. Let $\mathcal{P}_p$ be the set of paths through p:

$$\mathcal{P}_p := \{\gamma : (-a, a) \to M \mid \gamma \text{ is } C^\infty \text{ and } \gamma(0) = p\}.$$

We define an equivalence relation on $\mathcal{P}_p$ as follows: we say $\gamma_1 \sim \gamma_2$ if there exist a chart (U, ϕ) centered at p such that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Here, $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are paths in $\mathbb{R}^n$.

Remark 20. This is independent of the choice of chart. If (V, ψ) is another chart centered at p, then

$$\begin{aligned}
(\psi \circ \gamma_1)' &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_1)' \\
&= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_1)' \\
&= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_2)' \\
&= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\
&= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\
&= (\psi \circ \gamma_2)'
\end{aligned}$$

Here, $(\psi \circ \varphi^{-1})'$ is the Jacobian.

Definition 13.2. The tangent space at p in M is

$$T_pM := \mathcal{P}_p/\sim.$$

Example 13.3. Let $M = \{(\cos\alpha, \sin\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$ be the cylinder. Define the two charts (U, φ) and (M, ψ) where $U = M \cap \left\{(\frac{-\pi}{,2}, \frac{\pi}{2}) \times (\frac{-\pi}{,2}, \frac{\pi}{2}) \times \mathbb{R}\right\}$ and

$$\varphi(\cos\alpha, \sin\alpha, \beta) = (\alpha, \beta) \qquad \text{and} \qquad \psi(\cos\alpha, \sin\alpha, \beta) = (\cos\alpha, \beta).$$

Now let γ_1 and γ_2 be two paths in M given by

$$\gamma_1(t) = (\cos(t^2), \sin(t^2), t) \qquad \text{and} \qquad \gamma_2(t) = (0, 1, t).$$

Using the chart (U, φ), we have

$$(\varphi \circ \gamma_1)(t) = (t^2, t) \qquad \text{and} \qquad (\varphi \circ \gamma_2)(t) = \left(\frac{\pi}{2}, t\right).$$

Therefore

$$(\varphi \circ \gamma_1)'(0) = (0, 1) = (\varphi \circ \gamma_2)'(0)$$

and so $\gamma_1 \sim \gamma_2$.

13.2 Partial Derivatives

On a manifold M of dimension n, let (U, ϕ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, ϕ has n components $\phi_1, \ldots, \phi_n$. This means that if $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$, then $\phi_i = x_i \circ \phi$. For $p \in U$, we define the **partial derivative of f with respect to ϕ_i**, denoted $\partial_{\phi_i} f$, to be

$$\partial_{\phi_i} |_p \, f := (\partial_{\phi_i} f)(p) := \partial_{x_i}(f \circ \phi^{-1})(\phi(p)) := \partial_{x_i} |_{\phi(p)} \, (f \circ \phi^{-1}).$$

Example 13.4. Consider the projective plane $\mathbb{P}^2(\mathbb{R})$. Use the chart (U, ϕ) where

$$U = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2(\mathbb{R}) \mid a_0 \neq 0\} \qquad \phi(a_0 : a_1 : a_2) = \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right).$$

Then

$$\begin{aligned}
\phi_1(a_0 : a_1 : a_2) &= (x_1 \circ \phi)(a_0 : a_1 : a_2) \\
&= x_1(\phi(a_0 : a_1 : a_2)) \\
&= x_1\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right) \\
&= \frac{a_1}{a_0}.
\end{aligned}$$

Similarly, $\phi_2(a_0 : a_1 : a_2) = a_2/a_0$.

13.2.1 Polar Coordinates

Consider the following smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\left(\frac{y}{x}\right)
\end{aligned}$$

Then

$$\begin{aligned}
dr &= \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy \\
d\theta &= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy
\end{aligned}$$

Then

$$\begin{aligned}
r dr d\theta &= \sqrt{x^2 + y^2} \left(\frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy\right) \left(\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy\right) \\
&= (x dx + y dy) \left(\frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy\right) \\
&= \frac{1}{x^2 + y^2} (x dx + y dy)(-y dx + x dy) \\
&= \frac{1}{x^2 + y^2} \left(-xy dx dx + x^2 dx dy - y^2 dy dx + xy dy dy\right) \\
&= \frac{1}{x^2 + y^2} \left(x^2 dx dy - y^2 dy dx\right) \\
&= \frac{1}{x^2 + y^2} \left(x^2 dx dy + y^2 dx dy\right) \\
&= \frac{x^2 + y^2}{x^2 + y^2} dx dy \\
&= dx dy.
\end{aligned}$$

Therefore, we can integrate the Gaussian as follows:

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdx = \int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta.$$

The inverse map is given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

13.3 Immersion, Embedding, Submersion

Let $F : N \to M$ be a C^∞ map and let p be a point in N. Then

1. F is called an **immersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is injective.
2. F is called an **immersion** if it is an immersion at every point in N.
3. F is called a **submersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective.
4. F is called an **submersion** if it is an submersion at every point in N.

Example 13.5. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$i(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi(a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) = (a_1, \ldots, a_m).$$

Example 13.6. If U is an open subset of a manifold M, then the inclusion $i : U \to M$ is both an immersion and submersion. This example shows in particular that a submersion need not be onto.

13.3.1 Critical Point

Definition 13.3. Let $F : N \to M$ be a C^∞ map, p a point in N, and q a point in M. Then

1. We say p is a **critical point** of F if $F_{*,p}$ is not surjective.
2. We say q is a **critical value** of F if the set $F^{-1}(q) := \{p \in N \mid F(p) = q\}$ contains a critical point.

Theorem 13.1. *Let M be a manifold and let f be a C^∞ function. Then the set of critical values of f has measure zero.*

Measure Theory on $\mathbb{R}$

Let μ be the Lebesgue measure on $\mathbb{R}$. Recall that

- If $I = (a, b)$, then $\mu(I) = b - a$.
- If I and J are disjoint intervals, then $\mu(I \cup J) = \mu(I) + \mu(J)$.
- A set $E \subset \mathbb{R}$ has measure 0 if for all $\varepsilon > 0$, you can cover E by a union of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\mu\left(\bigcup_n I_n\right) < \varepsilon$.

Example 13.7. Let $E = \{0, 1\}$. For each $n \in \mathbb{N}$, define the set

$$A_n := \left(\frac{-1}{4n}, \frac{1}{4n}\right) \cup \left(1 - \frac{1}{4n}, 1 + \frac{1}{4n}\right)$$

Then for each $n \in \mathbb{N}$, A_n is a disjoint union of intervals which covers E and

$$\begin{aligned} \mu(A_n) &= \frac{1}{4n} - \frac{-1}{4n} + 1 + \frac{1}{4n} - \left(1 - \frac{1}{4n}\right) \\ &= \frac{1}{n}. \end{aligned}$$

As $n \to \infty$, $\mu(A_n) \to 0$. Thus, E has measure 0.

Example 13.8. Let $M = \{(x, x + \sin(x)) \mid x \in \mathbb{R}\}$ and let $\pi : M \to \mathbb{R}$ be the projection onto the y-axis map, given by $(x, x + \sin(x)) \mapsto x + \sin(x)$.

Definition 13.4. A critical point is **degenerate** if the associated Hessian matrix is **singular** (i.e. has determinant equal to 0).

Example 13.9. Let $M = \{(\cos\theta, \theta, \sin\theta + 2) \mid \theta \in \mathbb{R}\}$ and $N = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Note that M is homeomorphic to $\mathbb{R}$ and N is homeomorphic to $\mathbb{R}^2$. Let $\varphi : M \to N$ be the projection map, given by $(\cos\theta, \theta, \sin\theta + 2) \mapsto (\cos\theta, \theta, 0)$. Let $\{(M, \psi_M)\}$ be an atlas on M and $\{(N, \psi_N)\}$ be an atlas on N where

$$\psi_M(\cos\theta, \theta, \sin\theta + 2) = \theta \qquad \text{and} \qquad \psi_N(x, y, 0) = (x, y)$$

What are the coordinates of $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, 2 + \frac{\sqrt{2}}{2}\right) \in M$? Then answer is $\frac{\pi}{4}$.

Theorem 13.2. *Let f be a continuous function*

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$$

A critical point is **degenerate** if the associated Hessian matrix is **singular**

13.4 Tangent Bundle

Let M be an n-dimensional manifold. The **Tangent Bundle** of M is

$$TM := \bigcup_{p\in M} T_pM.$$

Let $\mathcal{A} := \{(U_i, \phi_i)\}$ be an atlas for M. Then an atlas for TM is given by

$$\mathcal{A}_T := \{(U_i \times \mathbb{R}^n, \phi_i \times \mathrm{id})\}$$

Thus, if we denote $\Phi_i := \phi_i \times \mathrm{id}$. Then $\Phi_i : U_i \times \mathbb{R}^n \to \mathbb{R}^{2n}$ and we think of $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ as the local coordinates of TM, where $(x_1, \ldots, x_n)$ is a point in M and $(y_1, \ldots, y_n)$ is a vector.

Example 13.10. The tangent bundle of the circle S^1 is diffeomorphic to the cylinder.

Remark 21. There exist a canonical map $\pi : TM \to M$ given by $(p, v) \mapsto v$.

Definition 13.5. A **vector field** is a smooth function ω from M to TM such that $\pi \circ \omega = \mathrm{id}$.

Remark 22. Intuitively, a vector field is the data of a vector at every point in M.

A vector field ω comes with two gadgets. The first gadget is called a 1-**parameter** flow and is denoted ω^t. The second gadget is called a **differential operator** and is denoted L_ω.

13.5 Vector Bundles

On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi : TM \to M$, given by $\pi(p, v) = p$, makes TM into a C^∞ **vector bundle** over M, which we now define.

Given any map $\pi : E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the **fiber** at p. The fiber at p is often written as E_p. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be **fiber-preserving** if $\phi(E_p) \subset E'_p$ for all $p \in M$. Equivalently, this says that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank** r if

1. Each fiber E_p has the structure of a vector space of dimension r.

2. For each $p \in M$, there are an open neighborhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for every $q \in U$ the restriction

$$\phi\,|_{E_q} \colon E_q \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. Such an open set U is called a **trivializing open set** for E, and ϕ is called a **trivialization** of E over U.

The collection $\{(U, \phi)\}$, with $\{U\}$ and open cover of M, is called a **local trivialization** for E, and $\{U\}$ is called a **trivializing open cover** of M for E. A C^∞**vector bundle of rank** r is a triple (E, M, π) consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the **total space** of the vector bundle and M the **base space**. By abuse of language, we say that E is a **vector bundle over** M. Properly speaking, the tangent bundle of a manifold M is a triple (TM, M, π), and TM is the total space of the tangent bundle. In common usage, TM is often referred to as the tangent bundle.

13.5.1 Gluing

Given two local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ and $\phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\phi_j \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R}).$$

These are called **transition functions** of the bundle, and since they come from $\phi_j \circ \phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$, as well as the cocycle condition

$$g_{ij} g_{jk} g_{ki} = \mathrm{id}\,|_{U_i \cap U_j \cap U_k}$$

Example 13.11. To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$. Then glue these together on the double overlaps $\{U_i \cap U_j\}$ via functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.

Example 13.12. Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection to the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r, called the **product bundle** of rank r over M. The vector space structure on the fiber $\pi^{-1}(p) = \{(p, v) \mid v \in \mathbb{R}^r\}$ is the obvious one:

$$(p, u) + (p, v) = (p, u + v) \text{ and } b \cdot (p, v) = (p, bv) \text{ for } b \in \mathbb{R}.$$

A local trivialization on $M \times \mathbb{R}$ is given by the identity map $1_{M \times \mathbb{R}} : M \times \mathbb{R} \to M \times \mathbb{R}$. For example, the infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle.

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles, possibly of different ranks. A **bundle map** from E to F is a pair of maps $(f, \widetilde{f})$, where $f : M \to N$ and $\widetilde{f} : E \to F$ such that

1. The diagram

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

   is commutative.

2. $\widetilde{f}$ is linear one each fiber; i.e. $\widetilde{f} : E_p \to F_{f(p)}$ is a linear map of vector spaces for each $p \in M$.

The collection of all vector bundles together with bundle maps between them forms a category.

Example 13.13. A smooth map $f : N \to M$ of manifolds induces a bundle map $(f, \widetilde{f})$, where $\widetilde{f} : TN \to TM$ is given by

$$\widetilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subset TM$$

for all $v \in T_pN$. This gives rise to a covariant functor T from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps: to each manifold M, we associate its tangent bundle TM, and to each C^∞ map $f : N \to M$ of manifolds, we associate the bundle map $T(f) = (f, \widetilde{f})$.

If E and F are two vector bundles over the same manifold M, then a bundle map from E to F over M is a bundle map in which the base map is the identity 1_M. For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundles maps over M. In this category it makes sense to speak of an isomorphism of vector bundles over M. Any vector bundle over M is isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a **trivial bundle**.

Example 13.14. Let

$$M = \left\{ (x, y) \in \mathbb{R}^2 \mid \det \begin{pmatrix} x & 1-y \\ y & x \end{pmatrix} = 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4} \right\}.$$

This can be realized as the circle of radius $\frac{1}{2}$ centered at the point $(0, 1/2)$ in the plane. There is natural vector bundle associated to M. Indeed, to each point $(x, y) \in M$, let $E_p := \operatorname{Ker} \left(\begin{smallmatrix} x & 1-y \\ y & x \end{smallmatrix} \right)$. Note that E_p is nonzero since $\det \left(\begin{smallmatrix} x & 1-y \\ y & x \end{smallmatrix} \right) = 0$.

13.5.2 Smooth Sections

A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = 1_M$. This condition means precisely that for each p in M, s maps p into the fiber E_p. We say that a section is **smooth** if it is smooth as a map from M to E. A **vector field** X on a manifold M is a function that assigns a tangent vector $X_p \in T_p M$ to each point p in M. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is **smooth** if it is smooth as a map from M to TM.

Example 13.15. The formula

$$X_{(x,y)} = -y\partial_x + x\partial_y$$

defines a smooth vector field on $\mathbb{R}^2$.

13.5.3 Whitney Sum

Let (E, M, π) and (E', M, π') be two vector bundles. We can construct a new vector bundle called the **Whitney sum**, given by $(\pi, \pi') : E \oplus E' \to M$.

Example 13.16. Suppose $E = L \oplus L'$ where L and L' are line bundles. Then we can make a new bundle called $\det(E)$.

Throughout this section, let R be a commutative ring.

Definition 13.6. An R**-ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally ringed** R**-space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ to be the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Remark 23. As every ring has a unique structure as a $\mathbb{Z}$-algebra, we simply say **(locally) ringed space** instead of **(locally)** $\mathbb{Z}$**-ringed space**. Usually we will denote a (locally) R-ringed space by $(X, \mathcal{O}_X)$ simply by X.

Example 13.17. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by C_X^∞ the sheaf of C^∞-functions, i.e.

$$C_X^\infty(U) := \{ f : U \to \mathbb{R} \mid f \text{ is } C^\infty \text{ function} \}.$$

Then C_X^∞ is a sheaf of $\mathbb{R}$-algebras.

14 Differential Forms

14.1 Differential 1-Forms

Let M be a smooth manifold and p a point in M. The **cotangent space** of M at p, denoted by $T_p^* M$, is defined to be the dual space of the tangent space $T_p M$:

$$T_p^* M = (T_p M)^\vee = \operatorname{Hom}_{\mathbb{R}}(T_p M, \mathbb{R}).$$

An element of the cotangent space $T_p^* M$ is called a **covector** at p. Thus, a covector ω_p at p is a linear function

$$\omega_p : T_p M \to \mathbb{R}.$$

A **covector field**, also called a **differential** 1-**form** or more simply a 1-**form**, on M is a function ω that assigns to each point p in M a covector ω_p at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p. There are many reasons for the great utility of differential forms in manifold theory, among which is the fact that they can be pulled back under a map. This is in contrast to vector fields, which in general cannot be pushed forward under a map.

14.1.1 The Differential of a Function

If f is a C^∞ real-valued function on a manifold M, its **differential** is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$, we have

$$(df)_p(X_p) = X_p f.$$

Instead of $(df)_p$ we also write $df\,|_p$ for the value of the 1-form df at p.

15 Bump Functions and Partitions of Unity

A partition of unity on a manifold is a collection of nonnegative functions that sum to 1. Usually one demands in addition that the partition of unity be **subordinate** to an open cover $\{U_i\}_{i\in I}$. What this means is that the partition of unity $\{\rho_i\}_{i\in I}$ is indexed over the same set as the open cover $\{U_i\}_{i\in I}$, and for each i in the index I, the support of ρ_i is contained in U_i.

The existence of a C^∞ partition of unity is one of the most technical tools in the theory of C^∞ manifolds. It is the single feature that makes the behavior of C^∞ manifolds so different from that of real-analytic or complex manifolds. In this section we construct C^∞ bump functions on any manifold and prove the existence of a C^∞ partition of unity on a compact manifold. The proof of the existence of a C^∞ partition of unity of a general manifold is more technical and is postponed.

A partition of unity is used in two ways:

1. to decompose a global object on a manifold into a locally finite sum of local objects on the open sets U_i of an open cover.

2. to patch together local objects on the open sets U_i into a global object on the manifold.

Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local coordinates on a manifold, there may be no global coordinates.

15.1 C^∞ Bump Functions

The **support** of a real-valued function f on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$:

$$\operatorname{supp} f := \overline{f^{-1}(\mathbb{R}\backslash\{0\})}.$$

Let q be a point in M, and U a neighborhood of q. By a **bump function at** q **supported in** U we mean any continuous nonnegative function ρ on M that is 1 in a neighborhood of q with $\operatorname{supp}\rho \subset U$.

Example 15.1. The support of the function $f : (-1, 1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is the open interval $(-1, 1)$, and not the closed interval $[-1, 1]$, because the closure of $f^{-1}(\mathbb{R}\backslash\{0\})$ is taken in the domain $(-1, 1)$ and not in $\mathbb{R}$.

Recall from Example (5.2) the smooth function f defined on $\mathbb{R}$ by the formula

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

We wish to build a smooth bump function function from f. The main challenge in building a smooth bump function from f is to construct a smooth version of a step function. We seek $g(t)$ by dividing $f(t)$ by a positive function $\ell(t)$, for the quotient $f(t)/\ell(t)$ will be zero for $t \leq 0$. The denominator $\ell(t)$ should be a positive function that agrees with $f(t)$ for $t \geq 1$, for then $f(t)/\ell(t)$ will be identically 1 for $t \geq 1$. The simplest way to construct such an $\ell(t)$ is to add to $f(t)$ a nonnegative function that vanishes for $t \geq 1$. One such nonnegative function is $f(1-t)$. This suggests that we take $\ell(t) = f(t) + f(1-t)$ and consider

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}.$$

Given two positive real numbers $a < b$, we make a linear change of variables to map $[a^2, b^2]$ to $[0, 1]$:

$$x \mapsto \left(\frac{x - a^2}{b^2 - a^2} \right):$$

Let $h : \mathbb{R} \to [0, 1]$ be given by

$$h(x) = g\left(\frac{x - a^2}{b^2 - a^2} \right).$$

Then h is a C^∞ step function such that

$$h(x) = \begin{cases} 0 & \text{if } x \le a^2 \\ 1 & \text{if } x \ge b^2. \end{cases}$$

Now replace x by x^2 to make the function symmetric in x: $k(x) = h(x^2)$. Finally, set

$$\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2} \right).$$

This $\rho(x)$ is a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. For any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

It is easy to extend the construction of a bump function from $\mathbb{R}$ to $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0}$ in $\mathbb{R}^n$ that is 1 on the closed balled $\overline{B_a(\mathbf{0})}$ and has support in the closed ball $\overline{B_b(\mathbf{0})}$, set

$$\sigma(x) = \rho\left(\|x\|\right) = 1 - g\left(\frac{x_1^2 + \cdots + x_n^2 - a^2}{b^2 - a^2} \right).$$

As a composition of C^∞ functions, σ is C^∞. To get a C^∞ bump function at q in $\mathbb{R}^n$, take $\sigma(x - q)$.

15.1.1 Extending C^∞ Bump Functions to M

Now suppose we have manifold M, and open subset U of M, and a point q in U. Choose a chart (ϕ_i, U_i) such that $U_i \subseteq U$ and $\phi_i(U_i) \cong B_b(\phi(q))$, for some $b > 0$, and choose an open neighborhood V_i of p such that $V_i \subseteq U_i$ and $\phi_i(V_i) \cong B_a(\phi(q))$ for some $a < b$. We've shown now to construct a bump function ρ at $\phi_i(q)$ such that $\rho(x) = 1$ for all $x \in B_a(\phi_i(q))$ and such that $\rho(x) = 0$ outside $B_b(\phi_i(q))$. Now we pull back ρ by ϕ to get a bump function on U_i:

$$(\phi_i^* \rho)(q') = \rho(\phi_i(q')) \text{ for all } q' \in U_i.$$

Finally we extend this function a bump function $\widetilde{\rho}$ on M by setting

$$\widetilde{\rho}(q') = \begin{cases} (\phi_i^* \rho)(q') & \text{if } q' \in U_i \\ 0 & \text{if } q' \notin U_i \end{cases}$$

Let us show that this function is C^∞. For $q' \in U_i$, we simply choose the chart (ϕ_i, U_i). Then

$$(\widetilde{\rho} \circ \phi_i^{-1})(x) = (\phi^* \rho)(\phi_i^{-1}(x)) = \rho(x),$$

shows that $\widetilde{\rho}$ is C^∞ in (ϕ_i, U_i). For $q' \notin U_i$, we choose a chart (ϕ_j, U_j) such that $U_i \cap U_j = \emptyset$. Then

$$(\widetilde{\rho} \circ \phi_j^{-1})(x) = \widetilde{\rho}(\phi_j^{-1}(x)) = 0.$$

Thus, the function $\widetilde{\rho}$ we constructed is C^∞ everywhere.

15.1.2 C^∞ Extension of a Function

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on M; an example is the function $\sec x$ on the open interval $(-\pi/2, \pi/2)$ in $\mathbb{R}$. However, if we require that the global function on M agree with the given function only on some neighborhood of a point in U, then a C^∞ extension is possible.

Proposition 15.1. *(C^∞ extension of a function) Suppose f is a C^∞ function defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ function $\widetilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p.*

Proof. Choose a C^∞ bump map $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p. Define

$$\widetilde{f}(q) = \begin{cases} \rho(q)f(q) & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

As the product of two C^∞ functions on U, $\widetilde{f}$ is C^∞ on U. If $q \notin U$, then $q \notin \operatorname{supp}\rho$, and so there is an open set containing q on which $\widetilde{f}$ is 0, since $\operatorname{supp}\rho$ is closed. Therefore $\widetilde{f}$ is also C^∞ at every point $q \notin U$. Finally, since $\rho = 1$ on V, the function $\widetilde{f}$ agrees with f on V. □

Remark 24. This proposition says that the natural map $C^\infty(M) \to C_p^\infty(M)$ is surjective. Thus, every germ in $C_p^\infty(M)$ can be represented by (f, M), where f is a C^∞ function on M.

15.2 Partitions of Unity

If $\{U_i\}_{i\in I}$ is a finite open cover of M, a C^∞ **partition of unity subordinate to** $\{U_i\}_{i\in I}$ is a collection of nonnegative functions $\{\rho_i : M \to \mathbb{R}\}$ such that $\operatorname{supp}\rho_i \subset U_i$ and

$$\sum_{i\in I} \rho_i = 1. \tag{22}$$

When I is an infinite set, for the sum in (22) to make sense, we will impose a **local finiteness** condition. A collection $\{A_\alpha\}$ of subsets of a topological space X is said to be **locally finite** if every point x in X has a neighborhood that meets only finitely many of the sets A_α. In particular, every point $x \in X$ is contained in only finitely many of the A_α's.

Example 15.2. Let $U_{r,n}$ be the open interval $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ on the real line $\mathbb{R}$. Then the open cover $\{U_{r,n} \mid r \in \mathbb{Q},\ n \in \mathbb{N}\}$ of $\mathbb{R}$ is not locally finite.

Definition 15.1. A C^∞ **partition of unity** on a manifold is a collection of nonnegative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i\in I}$ such that

1. The collection of supports, $\{\operatorname{supp}\rho\}_{i\in I}$, is locally finite,
2. $\sum_{i\in I} \rho_i = 1$.

Given an open cover $\{U_i\}_{i\in I}$ of M, we say that a partition of unity $\{\rho_i\}_{i\in I}$ is **subordinate to the open cover** $\{U_i\}_{i\in I}$ if $\operatorname{supp}\rho_i \subset U_i$ for every $i \in I$.

Remark 25. Since the collection of supports, $\{\operatorname{supp}\rho_i\}_{i\in I}$, is locally finite, every point q lies in only finitely many of the sets $\operatorname{supp}\rho_i$. Hence $\rho_i(q) \neq 0$ for only finitely many i. It follows that the sum $\sum_{i\in I}\rho_i(q)$ is finite.

Example 15.3. Let U and V be the open intervals $(-\infty, 2)$ and $(-1, \infty)$ in $\mathbb{R}$ respectively, and let ρ_V be a smooth step function which is equal to 0 on $(-\infty, 0)$ and equal to 1 on $(1, \infty)$. Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp}\rho_V \subset V$ and $\operatorname{supp}\rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

15.3 Existence of a Partition of Unity

In this subsection we begin a proof of the existence of a C^∞ partition of unity on a manifold. Because the case of a compact manifold is somewhat easier and already has some of the features of the general case, for pedagogical reasons we give a separate proof for the compact case.

Lemma 15.1. *If $\rho_1, \ldots, \rho_m$ are real-valued funcitons on a manifold M, then*

$$supp\left(\sum_{i=1}^m \rho_i\right) \subset \bigcup_{i=1}^m supp\rho_i.$$

Proof. Suppose $q \in \operatorname{supp}\left(\sum_{i=1}^m \rho_i\right)$. Thus $\sum_{i=1}^m \rho_i(q) \neq 0$. In particular, we must have $\rho_i(q) \neq 0$ for some $i = 1, \ldots, m$. Thus, $q \in \operatorname{supp}\rho_i \subset \bigcup_{i=1}^m \operatorname{supp}\rho_i$. □

Proposition 15.2. *Let M be a compact manifold and $\{U_i\}_{i\in I}$ an open cover of M. There exists a C^∞ partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$.*

Proof. For each $q \in M$, find an open set U_i containing q from the given cover and let ψ_q be a C^∞ bump function at q supported in U_i. Because $\psi_q(q) > 0$, there is a neighborhood W_q of q on which $\psi_q > 0$. By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \ldots, W_{q_m}\}$. Let $\psi_{q_1}, \ldots, \psi_{q_m}$ be the corresponding bump functions. Then $\psi := \sum_{j=1}^m \psi_{q_j}$ is positive at every point q in M because $q \in W_{q_j}$ for some i. Define

$$\varphi_j = \frac{\psi_{q_j}}{\psi}, \quad j = 1, \ldots, m.$$

Clearly $\sum_{j=1}^m \varphi_j = 1$. Moreover, since $\psi > 0$, $\varphi_j(q) \neq 0$ if and only if $\psi_{q_j}(q) \neq 0$, so

$$\mathrm{supp}\varphi_j = \mathrm{supp}\psi_{q_j} \subset U_i$$

for some $i \in I$. This shows that $\{\varphi_i\}$ is a partition of unity such that for every j, $\mathrm{supp}\varphi_j \subset U_i$ for some $i \in I$.

The next step is to make the index set of the partition of unity the same as that of the open cover. For each $j = 1, \ldots, m$, choose $\tau(j) \in I$ to be an index such that

$$\mathrm{supp}\varphi_j \subset U_{\tau(j)}.$$

We group the collection of functions $\{\varphi_j\}$ into subcollections according to $\tau(j)$ and define for each $i \in I$,

$$\rho_i = \sum_{\tau(j)=i} \varphi_j;$$

if there is no j for which $\tau(j) = i$, the sum is empty and we define $\rho_i = 0$. Then

$$\sum_{i \in I} \rho_i = \sum_{i \in I} \sum_{\tau(j)=i} \varphi_j = \sum_{j=1}^m \varphi_j = 1.$$

Moreover by Lemma (15.1),

$$\mathrm{supp}\rho_i \subset \bigcup_{\tau(j)=i} \mathrm{supp}\varphi_j \subset U_i.$$

So $\{\rho_i\}$ is a partition of unity subordinate to $\{U_i\}$. □

16 Integration on Manifolds

16.1 Riemann Integral of a Function on $\mathbb{R}^n$

A **closed rectangle** in $\mathbb{R}^n$ is a Cartesian product $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ of closed intervals in $\mathbb{R}$, where $a_i, b_i \in \mathbb{R}$. Let $f : R \to \mathbb{R}$ be a bounded function defined on a closed rectangle R. The **volume** $\mathbf{vol}(R)$ of the closed rectangle R is defined to be

$$\mathrm{vol}(R) := \prod_{i=1}^n (b_i - a_i).$$

A **partition** of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A **partition** of the rectangle R is a collection $P = \{P_1, \ldots, P_n\}$, where each P_i is a partition of $[a_i, b_i]$. The partition P divides the rectangle R into closed subrectangles, which we denote by R_j.

We define the **lower sum** and the **upper sum** of f with respect to the partition P to be

$$L(f, P) := \sum_{R_j} \left(\inf_{R_j} f \right) \mathrm{vol}(R_j), \qquad U(f, P) := \sum_{R_j} \left(\sup_{R_j} f \right) \mathrm{vol}(R_j),$$

where each sum runs over all subrectangles of the partition P. For any partition P, clearly $L(f, P) \leq U(f, P)$. In fact, more is true: for any two partitions P and P' of the rectangle R,

$$L(f, P) \leq U(f, P'),$$

which we show next.

A partition $P' = \{P'_1, \dots, P'_n\}$ is a **refinement** of the partition $P = \{P_1, \dots, P_n\}$ if $P_i \subset P'_i$ for all $i = 1, \dots, n$. If P' is a refinement of P, then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of P', and it is easily seen that

$$L(f, P) \le L(f, P'),$$

because if $R'_{jk} \subset R_j$, then $\inf_{R_j} f \le \inf_{R'_{jk}} f$. Similarly, if P' is a refinement of P, then

$$U(f, P') \le U(f, P).$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \dots, Q_n\}$ with $Q_i = P_i \cup P'_i$, and thus

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P').$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions P of R is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions of R. We define these two numbers to be the **lower integral** $\underline{\int}_R f$ and the **upper integral** $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f := \sup_P L(f, P), \qquad \overline{\int}_R f := \inf_P L(f, P).$$

Definition 16.1. Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be **Riemann integrable** if $\underline{\int}_R f = \overline{\int}_R f$; in this case, the Riemann integral of f is this common value, denoted by $\int_R f(x) dx_1 \cdots dx_n$, where $x_1, \dots, x_n$ are the standard coordinates on $\mathbb{R}^n$.

Example 16.1. Let f be a bounded monotone increasing function on $[-1, 1]$. Then f is Riemann integrable. Indeed, consider the partition $P_n = \{p_0 < p_1 < \cdots < p_{2n-1} < p_{2n}\}$ where $p_i = -1 + i/3$. Then

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \frac{1}{n} \sum_{i=1}^{2n} f(-1 + i/3) - \frac{1}{n} \sum_{i=0}^{2n-1} f(-1 + i/3) \\
&= \frac{1}{n} \left(\sum_{i=1}^{2n} f(-1 + i/3) - \sum_{i=0}^{2n-1} f(-1 + i/3) \right) \\
&= \frac{1}{n} \left(f(1) - f(-1) \right),
\end{aligned}$$

which tends to 0 as $n \to \infty$.

If $f : A \subset \mathbb{R}^n \to \mathbb{R}$, then the **extension of** f **by zero** is the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Now suppose $f : A \to \mathbb{R}$ is a bounded function on a bounded set A in $\mathbb{R}^n$. Enclose A in a closed rectangle R and define the Riemann integral of f over A to be

$$\int_A f(x) dx_1 \cdots dx_n = \int_R \tilde{f}(x) dx_1 \cdots dx_n$$

if the right-hand side exists. In this way we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$. The **volume** $\mathrm{vol}(A)$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1 dx_1 \cdots dx_n$ if the integral exists.

16.2 Integrability Conditions

In this section we describe some conditions under which a function defined on an open subset of $\mathbb{R}^n$ is Riemann integrable.

Definition 16.2. A set $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a countable cover $\{R_i\}_{i=1}^{\infty}$ of A by closed rectangles R_i such that $\sum_{i=1}^{\infty} \mathrm{vol}(R_i) < \varepsilon$.

Theorem 16.1. *(Lebesgue's theorem) A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set $\mathrm{Disc}(\tilde{f})$ of discontinuities of the extended function $\tilde{f}$ has measure zero.*

Proposition 16.1. *If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.*

Proof. Being continuous on a compact set, the function f is bounded. Being compact, the set $\text{supp}(f)$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\widetilde{f}$ is continuous.

Since $\widetilde{f}$ agrees with f on U, the extended function $\widetilde{f}$ is continuous on U. It remains to show that $\widetilde{f}$ is continuous on the complement of U in $\mathbb{R}^n$ as well. If $p \notin U$, then $p \notin \text{supp}(f)$. Since $\text{supp}(f)$ is a closed subset of $\mathbb{R}^n$, there is an open ball B containing p and disjoint from $\text{supp}(f)$. On this open ball, $\widetilde{f} = 0$, which implies that $\widetilde{f}$ is continuous at $p \notin U$. Thus, $\widetilde{f}$ is continuous on $\mathbb{R}^n$. By Lebesgue's theorem, f is Riemann integrable on U. □

Example 16.2. The continuous function $f : (-1,1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is defined on an open subset of finite length in $\mathbb{R}$, but it not bounded. The support of f is the open interval $(-1,1)$, which is not compact. Thus, the function f does not satisfy the hypotheses of either Lebesgue's theorem or Proposition (16.1). Note that it is not Riemann integrable.

Definition 16.3. A subset $A \subset \mathbb{R}^n$ is called a **domain of integration** if it is bounded and its topological boundary $\text{bd}(A)$ is a set of measure zero.

Proposition 16.2. *Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.*

Proof. Let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero. Since f is continuous on A the extension $\widetilde{f}$ is necessarily continuous at all interior points of A. Clearly, $\widetilde{f}$ is continuous at all exterior points of A also, because every exterior point has a neighborhood contained entirely in $\mathbb{R}^n \backslash A$, on which $\widetilde{f}$ is identically zero. Therefore, the set $\text{Disc}(\widetilde{f})$ of discontinuities of $\widetilde{f}$ is a subset of $\partial(A)$, a set of measure zero. By Lebesgue's theorem, f is Riemann integrable. □

16.3 The Integral of an n-Form on $\mathbb{R}^n$

Once a set of coordinates $x_1, \ldots, x_n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 16.4. Let $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x_1, \ldots, x_n$. Its **integral** over a subset $A \subset U$ is defined to be the Riemann integral of $f(x)$:

$$\int_A \omega = \int_A f(x) dx_1 \wedge \cdots \wedge dx_n := \int_A f(x) dx_1 \cdots dx_n,$$

if the Riemann integral exists.

Example 16.3. If f is a bounded continuous function defined on a domain of integration A in $\mathbb{R}^n$, then the integral $\int_A f(x) dx_1 \wedge \cdots \wedge dx_n$ exists.

Let us see how the integral of an n-form $\omega = f dx_1 \wedge \cdots \wedge dx_n$ on an open subset $U \subset \mathbb{R}^n$ transforms under a change of variables. A change of variables on U is given by a diffeomorphism $T : \mathbb{R}^n \supset V \to U \subset \mathbb{R}^n$. Let $x_1, \ldots, x_n$ be the standard coordinates on U and $y_1, \ldots, y_n$ be the standard coordinates on V. Then $T_i := x_i \circ T$ is the ith component of T. We will assume that U and V are connected, and write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$dT_1 \wedge \cdots \wedge dT_n = \det(J(T))\, dy_1 \wedge \cdots \wedge dy_n.$$

Hence,

$$\begin{aligned}
\int_V T^*\omega &= \int_V (T^*f) T^*dx_1 \wedge \cdots \wedge T^*dx_n \\
&= \int_V (f \circ T) dT_1 \wedge \cdots \wedge dT_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \wedge \cdots \wedge dy_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \cdots dy_n.
\end{aligned}$$

On the other hand, the change-of-variables formula from advanced calculus gives

$$\int_U \omega = \int_U f dx_1 \cdots dx_n = \int_V (f \circ T) |\det(J(T))| dy_1 \cdots dy_n,$$

with an absolute-value sign around the Jacobian determinant. Hence,

$$\int_V T^*\omega = \pm \int_U \omega,$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative. In particular, the integral of a differential form is not invariant under all diffeomorphisms of V with U, but only under orientation-preserving diffeomorphisms.

16.4 Integral of a Differential Form over a Manifold

Integration of an n-form on $\mathbb{R}^n$ is not so different from integration of a function. Our approach to integration over a general manifold has several distinguishing features:

1. The manifold must be oriented.
2. On a manifold of dimension n, one can integrate only n-forms, not functions.
3. The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of M. Denote by $\Omega_c^k(M)$ the vector space of C^∞ k-forms with compact support on M. Suppose (U, ϕ) is a chart in this atlas. If $\omega \in \Omega_c^n(U)$ is an n-form with compact support on U, then because $\phi : U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega = \int_{\phi(U)} (\phi^{-1})^*\omega.$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\phi \circ \psi^{-1}\psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^*(\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega.$$

Thus, the integral $\int_U \omega$ on a chart U of the atlas is well defined, independent of the choice of coordinates on U. By linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a *finite* sum. Also since $\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}(\rho_\alpha) \cap \operatorname{supp}(\omega)$, $\operatorname{supp}(\rho_\alpha\omega)$ is a closed subset of the compact set $\operatorname{supp}(\omega)$. Hence, $\operatorname{supp}(\rho_\alpha\omega)$ is compact. Since $\rho_\alpha\omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha\omega. \tag{23}$$

For this integral to be well defined, we must show that it is independent of the choices of oriented atlas and partition of unity. Let $\{V_\beta, \psi_\beta\}$ be another oriented atlas of M specifying the orientation of M, and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\{(U_\alpha \cap V_\beta, \phi_\alpha|_{U_\alpha \cap V_\beta})\}$ and $\{(U_\alpha \cap V_\beta, \psi_\beta|_{U_\alpha \cap V_\beta})\}$ are two new atlases of M specifying the orientation of M, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta\omega && \text{(because } \sum_\beta \chi_\beta = 1\text{)} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha\chi_\beta\omega && \text{(these are finite sums)} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha\chi_\beta\omega,
\end{aligned}$$

where the last line follows from the fact that the support of $\rho_\alpha\chi_\beta$ is contained in $U_\alpha \cap V_\beta$. By symmetry, $\sum_\beta \int_{V_\beta} \chi_\beta\omega$ is equal to the same sum. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha\omega = \sum_\beta \int_{V_\beta} \chi_\beta\omega,$$

proving that the integral (23) is well defined.

17 Quotients and Gluing

There are many important topological spaces (and manifolds) that are constructed by "identifying" pieces of spaces. This typically takes the form of gluing along open sets or passing to quotients by (reasonable) equivalence relations.

17.1 The Quotient Topology

Recall that an equivalence relation on a set X is a reflexive, symmetric, and transitive relation. The **equivalence class** $[x]$ of $x \in X$ is the set of all elements in X equivalent to x. An equivalence relation on X partitions X into disjoint equivalence classes. We denote the set of equivalence classes by $X/\sim$ and call this set the **quotient** of X by the equivalence relation $\sim$. There is a natural **projection map** $\pi : X \to X/\sim$ that sends $x \in X$ to its equivalence class $[x]$.

Assume now that X is a topological space. We define a topology on $X/\sim$ by declaring a set U in $X/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in X. Clearly, both the empty set $\emptyset$ and the entire quotient $X/\sim$ are open. Further, since

$$\pi^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i) \text{ and } \pi^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} \pi^{-1}(U_i),$$

the collection of open sets in $X/\sim$ is closed under arbitrary unions and finite intersections, and is therefore a topology. It is called the **quotient topology** on $X/\sim$. With this topology, $X/\sim$ is called the **quotient space** of X by the equivalence relation $\sim$. The way we defined the topology on $X/\sim$ makes the projection map π continuous.

17.1.1 Continuity of a Map on a Quotient

Suppose f is a map from X to Y and is constant on each equivalence class. Then it induces a map $\overline{f} : X/\sim \to Y$, given by $\overline{f}([x]) = f(x)$ where $x \in X$.

Proposition 17.1. *The induced map $\overline{f} : X/\sim \to Y$ is continuous if and only if the map $f : X \to Y$ is continuous.*

Proof. If $\overline{f}$ is continuous, then f is continuous since $f = \overline{f} \circ \pi$ is a composition of two continuous functions. Conversely, suppose f is continuous. Let V be an open set in Y. Then $f^{-1}(V) = \pi^{-1}\left(\overline{f}^{-1}(V)\right)$ is open in X. By the definition of quotient topology, $\overline{f}^{-1}(V)$ is open in $X/\sim$. Thus $\overline{f}$ is continuous since V was arbitary. $\square$

17.1.2 Identification of a Subset to a Point

If A is a subspace of a topological space X, we can define a relation $\sim$ on X by declaring

$$x \sim x \text{ for all } x \in X \text{ and } x \sim y \text{ for all } x, y \in A.$$

This is an equivalence relation on X. We say that the quotient space $X/\sim$ is obtained from X by **identifying** A **to a point**.

Example 17.1. Let I be the unit interval $[0,1]$ and $I/\sim$ be the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1$, given by $f(x) = e^{2\pi i x}$, assumes the same value at 0 and 1, and so induces a function $\overline{f} : I/\sim \to S^1$. Since f is continuous, $\overline{f}$ is continuous. As the continuous image of a compact set I, the quotient $I/\sim$ is compact. Thus $\overline{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. Hence it is a homeomorphism.

17.2 Open Equivalence Relations

An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection map $\pi : X \to X/\sim$ is open. In other words, the equivalence relation $\sim$ on X is open if and only if for every open set U in X, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U} [x]$$

of all points equivalent to some point of U is open.

Example 17.2. Let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 and let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map. Then π is not an open map. Indeed, let V be the open interval $(-2, 0)$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\},$$

which is not open in $\mathbb{R}$.

Given an equivalence relation $\sim$ on X, let R be the subset of $X \times X$ that defines the relation

$$R = \{(x, y) \in X \times X \mid x \sim y\}.$$

We call R the **graph** of the equivalence relation $\sim$.

Theorem 17.1. *Suppose $\sim$ is an open equivalence relation on a topological space X. Then the quotient space $X/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $X \times X$.*

Proof. There is a sequence of equivalent statements: R is closed in $X \times X$ iff $(X \times X)\backslash R$ is open in $X \times X$ iff for every $(x, y) \in (X \times X)\backslash R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \emptyset$ iff for every pair $x \not\sim y$ in X, there exist neighborhoods U of x and V of y in X such that no element of U is equivalent to an element of V iff for any two points $[x] \neq [y]$ in $X/\sim$, there exist neighborhoods U of x and V of y in X such that $\pi(U) \cap \pi(V) = \emptyset$ in $X/\sim$.

We now show that this last statement is equivalent to $X/\sim$ being Hausdorff. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $X/\sim$ containing $[x]$ and $[y]$ respectively, so $X/\sim$ is Hausdorff. Conversely, suppose $X/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $X/\sim$. Then there exist disjoint open sets A and B in $X/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$. Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $X/\sim$. □

Theorem 17.2. *Let $\sim$ be an open equivalence relation on a topological space X. If $\mathcal{B} = \{B_\alpha\}$ is a basis for X, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $X/\sim$.*

Proof. Since π is an open map, $\{\pi(B(\alpha)\}$ is a collection of open sets in $X/\sim$. Let W be an open set in $X/\sim$ and $[x] \in W$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that $x \in B \subset \pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B) \subset W$, which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary 1. *If $\sim$ is an open equivalence relation on a second-countable space X, then the quotient space is second-countable.*

17.3 Quotients by Group Actions

Many important manifolds are constructed as quotients by actions of groups on other manifolds, and this often provides a useful way to understand spaces that may have been constructed by other means. As a basic example, the Klein bottle will be defined as a quotient of $S^1 \times S^1$ by the action of a group of order 2. The circle as defined concretely in $\mathbb{R}^2$ is isomorphic to the quotient of $\mathbb{R}$ by additive translation by $\mathbb{Z}$.

Definition 17.1. Let X be a topological space, let G be a discrete group, and let μ be a right action of G on X.

1. We say μ is **continuous** if it is continuous as a map $\mu \colon X \times G \to G$. In other words, μ is continuous if for each $g \in G$ the map $\mu_g \colon X \to X$ defined by $\mu_g(x) = xg$ is continuous (and hence a homeomorphism with inverse being $\mu_{g^{-1}}$).

2. We say μ is **free** if for each $x \in X$ the stabilizer subgroup

$$\mathrm{Stab}_G(x) = \{g \in G \mid xg = x\}$$

   is the trivial subgroup (in other words, $xg = x$ implies $g = 1$).

3. We say μ is **properly discontinuous** when it is continuous for the discrete topology on G and each $x \in X$ admits an open neighborhood U_x such that the G-translate $U_x g$ meets U_x for only finitely many $g \in G$. In particular, $\mathrm{Stab}_G(x)$ is necessarily finite.

What does it mean for μ to be continuous at a point $(x, g) \in X \times G$? It means that for all open neighborhoods $V \subseteq X$ of xg, there exists an open neighborhood $U \subseteq X$ of x such that if $y \in U$, then $gy \in V$. Alternatively, we can characterize continuity of μ using the sequential criterion: let (x_n, g_n) be a sequence in $X \times G$ which converges to (x, g) in $X \times G$. Then the sequence $(x_n g_n)$ in X converges to xg in X. Note that since G has the discrete topology, eventually we must have $g_n = g$. Thus $(x_n, g_n) \to (x, g)$ is equivalent to saying $x_n \to x$ and $g \in G$. Thus we can restate the sequential criterion as: if $x_n \to x$ in X and $g \in G$, then $x_n g \to xg$ in X.

Example 17.3. Suppose that X is a locally Hausdorff space, and that G acts on X on the right via a properly discontinuous action. For each $x \in X$, we get an open subset U_x such that U_x meets $U_x g$ for only finitely many $g \in G$. This property is unaffected by replacing U_x with a smaller open subset around x, so by the locally Hausdorff property we can assume that U_x is Hausdorff. The key is that we can do better: there exists an open set $U'_x \subseteq U_x$ such that U'_x meets $U'_x g$ if and only if $x = xg$. Thus, if the action is also free then U'_x is disjoint from $U'_x g$ for all $g \in G$ with $g \neq 1$.

Indeed, observe that if $xg \neq x$ and $U_x \cap U_x g \neq \emptyset$, then by the Hausdorff property of U_x, there exists an open neighborhood $U_x^g \subseteq U_x$ of x and an open neighborhood $V_x^g \subseteq U_x$ of xg such that $U_x^g \cap V_x^g = \emptyset$. By continuity of μ_g at the point x, there exists an open neighborhood $\widetilde{U}_x^g \subseteq U_x^g$ of x such that $\widetilde{U}_x^g g \subseteq V_x^g$. By replacing U_x^g with $\widetilde{U}_x^g$ if necessary, we may assume that $U_x^g g \subseteq V_x^g$ so that $U_x^g \cap U_x^g g = \emptyset$. We now set

$$U'_x = \bigcap_{\substack{g \in G \setminus \mathrm{Stab}(x) \\ U_x \cap U_x g \neq \emptyset}} U_x^g$$

The intersection is finite, so U'_x is open (and contains x since each U_x^g contains x). Furthermore we have $U'_x \cap U'_x g \neq \emptyset$ if and only if $xg = g$.

The interest in free and properly discontinuous actions is that for such actions in the locally Hausdorff case we may find an open U_x around each $x \in X$ such that U_x is disjoint from $U_x g$ whenever $g \neq 1$. Thus, for such actions we may say that in X/G we are identifying points in the same G-orbit with this identification process not "crushing" the space X by identifying points in X that are arbitrarily close to each other. An example where things go horribly wrong is the action of $G = \mathbb{Q}$ on $\mathbb{R}$ via additive translations. This is a continuous action, but the quotient $\mathbb{R}/\mathbb{Q}$ is very bad: any two $\mathbb{Q}$-orbits in $\mathbb{R}$ contain arbitrarily close points! Here are some examples of free and properly discontinuous actions.

Example 17.4. The antipodal map on S^n, given by

$$\boldsymbol{x} = (x_1, \ldots, x_{n+1}) \mapsto (-x_1, \ldots, -x_{n+1}) = -\boldsymbol{x},$$

viewed as an action of C_2 on S^n (where C_2 is the cyclic group of order 2) is free and properly discontinuous: freeness is clear, as is continuity, and for any $\boldsymbol{x} \in S^n$ the points near $\boldsymbol{x}$ all have their antipodes far away! For instance, consider the small open ball centered at $\boldsymbol{x}$ with radius $\varepsilon > 0$

$$\mathrm{B}_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^{n+1} \mid \|\boldsymbol{y} - \boldsymbol{x}\| < \varepsilon\},$$

and set $\mathrm{B}_\varepsilon^{S^n}(\boldsymbol{x}) := \mathrm{B}_\varepsilon(\boldsymbol{x}) \cap S^n$. Then $\mathrm{B}_\varepsilon^{S^n}(\boldsymbol{x}) \subseteq S^n$ is open and a neighborhood of $\boldsymbol{x}$, and choosing $\varepsilon > 0$ small enough (say $\varepsilon \leq 1/2$), we can ensure $\mathrm{B}_\varepsilon^{S^n}(\boldsymbol{x}) \cap \mathrm{B}_\varepsilon^{S^n}(-\boldsymbol{x}) = \emptyset$.

Example 17.5. Consider the curve $X := \mathrm{V}_{\mathbb{C}}(x^3 + y^3 + z^3 - 1) \subseteq \mathbb{C}^3$. The map $(x, y, z) \mapsto (\zeta_3 x, \zeta_3 y, \zeta_3 z)$, viewed as an action of C_3 on X is free and properly discontinuous.

Example 17.6. Let $X = S^1 \times S^1$ be a product of two circles, where the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is viewed as a topological group (using multiplication in $\mathbb{C}$, so both the group law and inversion $z \mapsto 1/z = \overline{z}$ on S^1 are continuous). The visibly continuous map $(z, w) \mapsto (1/z, -w) = (\overline{z}, -w)$ reflects through the x-axis in the first circle and rotates 180-degree in the second circle, and it is its own inverse. Thus, this give an action by C_2 on X which is free and properly discontinuous. The associated quotient X/G will be called the (set-theoretic) **Klein bottle**.

Theorem 17.3. *Let X be a locally Hausdorff topological space equipped with a free and properly discontinuous action by a group G. There is a unique topology on X/G such that the quotient map $\pi\colon X \to X/G$ is a continuous map that is a local homeomorphism (i.e. each $x \in X$ admits a neighborhood mapping homeomorphically onto an open subset of X/G). Moreover, the quotient map is open.*

A subset $S \subseteq X/G$ is open if and only if its preimage in X is open, and if $U \subseteq X$ is an open set that is disjoint from Ug for all nontrivial $g \in G$ then the map $U \to X/G$ is a homeomorphism onto its open image $\overline{U}$ and the natural map $U \times G \to \pi^{-1}(\overline{U})$ over $\overline{U}$ given by $(u, g) \mapsto ug$ is a homeomorphism when G is given the discrete topology.

Remark 26. The topology in this theorem is called the **quotient topology**, and it is locally Hausdorff since $X \to X/G$ is a local homeomorphism.

Proof. Sketch: we show that π is an open map. Let $x \in X$ and pick U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$. We first show that $\pi(U_x)$ is open. The inverse image of $\pi(U_x)$ under π is a disjoint union of open sets $\bigcup_{g \in G} U_x.g$. Therefore $\pi(U_x)$ is open. Now let U be any open subset of X. For each $x \in X$, choose U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$ and $U_x \subset U$. Then

$$\pi(U) = \pi\left(\bigcup_{x \in U} U_x\right) = \bigcup_{x \in U} \pi(U_x)$$

implies $\pi(U)$ is open. □

Example 17.7. (Möbius Strip) Choose $a > 0$. Let $X = (-a, a) \times S^1$, and let the group of order 2 act on it with the non-trivial element acting by $(t, w) \mapsto (-t, -w)$. This is easily checked to be a continuous action for the discrete topology of the group of order 2, and it is free and properly discontinuous. The quotient M_a is the **Möbius strip** of height $2a$.

To check that the Möbius strip M_a is Hausdorff, we use the quotient criterion: the set of points in $X \times X$ with the form $((t, w), (t', w'))$ with $(t', w') = (t, w)$ or $(t', w') = (-t, -w)$ is checked to be closed by using the sequential criterion in $X \times X$: suppose $(t_n, w_n) \sim (t'_n, w'_n)$ are sequences in $X \times X$ which converge (t, w) and (t', w') respectively. Then we need to show that $(t, w) \sim (t', w')$. Assume that $(t, w) \neq (t', w')$. Choose open neighborhoods U of (t, w) and U' of (t', w') respectively such that $U \cap U' = \emptyset$ and such that eventually $(t_n, w_n) \neq (t'_n, w'_n)$ (We can do this because they converge to different limits and our space $X \times X$ is Hausdorff). Thus, eventually we have $(t'_n, w'_n) = (-t_n, -w_n) \to (-t, -w)$.

Example 17.8. We have a right action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by

$$\boldsymbol{x} \cdot \boldsymbol{a} = (x_1 + a_1, x_2 + a_2) \tag{24}$$

for all $\boldsymbol{a} = (a_1, a_2) \in \mathbb{Z}^2$ and $\boldsymbol{x} \in (x_1, x_2) \in \mathbb{R}^2$. The action (24) is free since if $\boldsymbol{x} \cdot \boldsymbol{a} = \boldsymbol{x}$ implies $\boldsymbol{a} = 0$. Next observe that the action (24) is properly discontinuous. Indeed, it is continuous as a map $\mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{R}^2$ since for fixed $\boldsymbol{a} \in \mathbb{Z}^2$, the map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x_1, x_2) = \boldsymbol{x} \mapsto \boldsymbol{x} \cdot \boldsymbol{a} = (x_1 + a_1, x_2 + a_2)$$

is continuous (as the component functions are continuous). Furthermore, given $\boldsymbol{x} \in \mathbb{R}^2$, choose

$$U_{\boldsymbol{x}} = \{\boldsymbol{y} \in \mathbb{R}^2 \mid \|\boldsymbol{y} - \boldsymbol{x}\|_\infty < 1/2\} = (x_1 - 1/2, x_1 + 1/2) \times (x_2 - 1/2, x_2 + 1/2),$$

that is, $U_{\boldsymbol{x}}$ is the open square centered at $\boldsymbol{x}$ whose sides have length 1. Then clearly $U_{\boldsymbol{x}} \cdot \boldsymbol{a}$ is disjoint from $U_{\boldsymbol{x}}$ for all $\boldsymbol{a} \in \mathbb{Z}^2 \backslash \{0\}$.

17.4 Möbius Strip in $\mathbb{R}^3$

Recall that the Möbius strip M_a (with height $2a$) was defined as an abstract smooth manifold made as a quotient of $(-a, a) \times S^1$ by a free and properly discontinuous action by the group of order 2. Using the C^∞ isomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and the circle $S^1 \subseteq \mathbb{R}^2$ via $\theta \mapsto (\cos\theta, \sin\theta)$, we consider the standard parameter $\theta \in \mathbb{R}$ as a local coordinate on S^1. For finite $a > 0$, consider the C^∞ map $f : (-a, a) \times S^1 \to \mathbb{R}^3$ defined by

$$(t, \theta) \mapsto (2a\cos 2\theta + t\cos\theta\cos 2\theta, 2a\sin 2\theta + t\cos\theta\sin 2\theta, t\sin\theta).$$

Since $f(-t, \pi + \theta) = f(t, \theta)$ by inspection, it follows from the universal property of the quotient map $(-a, a) \times S^1 \to M_a$ that f uniquely factors through this via a C^∞ map $\overline{f} : M_a \to \mathbb{R}^3$. Our goal is to prove that $\overline{f}$ is an embedding and to use this viewpoint to understand some basic properties of the Möbius strip.

17.4.1 Embedding

Theorem 17.4. *The map $\overline{f}$ is an immersion.*

Proof. We first reduce the problem to working with f, as f is given by a simple explicit formula across its entire domain (M_a does not have global coordinates. Of course, working locally for $\overline{f}$ is "the same" as working locally for f, so the reduction step to working with f isn't really necessary if one says things a little differently. However, it seems a bit cleaner to just make the reduction step right away and so to thereby work with the map f that feel a bit more concrete than the map $\overline{f}$ at the global level.)

Let $p : (-a, a) \times S^1 \to M_a$ be the natural quotient map. Each point in M_a has the form $p(\xi_0)$ for some ξ_0 and the Chain Rule gives that the injection $df(\xi_0)$ factors as $d\overline{f}(p(\xi_0)) \circ dp(\xi_0)$ with $dp(\xi_0)$ an isomorphism (as p is a local C^∞ isomorphism, via the theory of quotients by free and properly discontinuous group actions). Hence, the tangent map for $\overline{f}$ is injective at $p(\xi_0)$ if and only if the tangent map for f is injective at ξ_0. It is therefore enough (even equivalent!) to prove that f is an immersion.

□

17.5 Construction of Manifolds From Gluing Data

The definition of a manifold assumes that the underlying set, M, is already known. However, there are situations where we only have some indirect information about the overlap of the domains U_i, of the local charts defining our manifold, M, in terms of the transition functions

$$\phi_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

but where M itself is not known. Our goal in this subsection is to try and reconstruct a manifold M by gluing open subsets of $\mathbb{R}^n$ using the transition functions ϕ_{ij}.

Definition 17.2. Let n be an integer with $n \geq 1$ and let k be either an integer with $k \geq 1$ or $k = \infty$. A set of **gluing data** is a triple

$$\mathcal{G} = ((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\phi_{ji})_{(i,j) \in K}),$$

satisfying the following properties, where I is a (nonempty) countable set and $K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\}$:

1. For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{R}^n$ called a **parametrization domain**, for short, p-**domain**, and the Ω_i are pairwise disjoint (i.e. $\Omega_i \cap \Omega_j) = \emptyset$ for all $i \neq j$).

2. For every pair $(i,j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ij} \neq \emptyset$ if and only if $\Omega_{ji} \neq \emptyset$. Each nonempty Ω_{ij} (with $i \neq j$) is called a **gluing domain**.

3. The maps $\phi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i,j) \in K$ called a **transition function** (or **gluing function**) and the following condition holds:

   (a) The **cocycle condition** holds: for all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ and

   $$\phi_{ki}(x) = (\phi_{kj} \circ \phi_{ji})(x)$$

   for all $x \in \phi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

4. For every pair $(i,j) \in K$ with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and every $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}.

Remark 27.

1. In practical applications, the index set, I, is of course finite and the open subsets, Ω_i, may have special properties (for example, connected; open simplicies, etc.).

2. Observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, as Ω_i and Ω_j are disjoint, so are Ω_{ij} and Ω_{ji}.

3. The cocycle condition may seem overly complicated but it is actually needed to guarantee the transitivity of the relation, $\sim$, which we will define shortly. Since the ϕ_{ji} are bijections, the cocycle condition implies the following conditions

   (a) $\phi_{ii} = \mathrm{id}_{\Omega_i}$ for all $i \in I$. This follows by setting $i = j = k$.

   (b) $\phi_{ij} = \phi_{ji}^{-1}$ for all $(i,j) \in K$. This follows from (a) and by setting $k = i$.

4. Let M be a C^k manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on it. Then set $\Omega_i = \phi_i(U_i)$, $\Omega_{ij} = \phi_i(U_i \cap U_j)$, and let $\phi_{ij} : \Omega_{ji} \to \Omega_{ij}$ be the corresponding transition maps. Then it's easy to check that the open sets Ω_i, Ω_{ij}, and the gluing functions ϕ_{ij}, satisfy the conditions of Definition (17.2). Indeed,

$$\begin{aligned}
\phi_{ji}^{-1}(\Omega_{jk}) &= (\phi_i \circ \phi_j^{-1})(\phi_j(U_j \cap U_k)) \\
&= \phi_i(U_j \cap U_k) \\
&= \phi_i(U_i \cap U_j \cap U_k) \\
&\subseteq \phi_i(U_i \cap U_k) \\
&= \Omega_{ik}.
\end{aligned}$$

Let us show that a set of gluing data defines a C^k manifold in a natural way.

Proposition 17.2. *For every set of gluing data $\mathcal{G} = (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\phi_{ji})_{(i,j) \in K})$, there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the ϕ_{ji}'s.*

Proof. Define the binary relation, $\sim$, on the disjoint union, $\Omega := \coprod_{i \in I} \Omega_i$, of the open sets, Ω_i, as follows: For all

$x, y \in \Omega$,

$$x \sim y \text{ if and only if there exists } (i,j) \in K \text{ such that } x \in \Omega_{ij},\ y \in \Omega_{ji}, \text{ and } y = \phi_{ji}(x).$$

The cocycle condition ensures that this is an equivalence relation. Indeed, (a) implies reflexivity and (b) implies symmetry. The crucial step is to check transitivity. Assume that $x \sim y$ and $y \sim z$. Then there are some i, j, k such that $\phi_{ji}(x) = y$ and $\phi_{kj}(y) = z$. But then $(\phi_{kj} \circ \phi_{ji})(x) = \phi_{ki}(x) = z$. That is, $x \sim z$, as desired.

Since $\sim$ is an equivalence relation, let

$$M_{\mathcal{G}} := \Omega / \sim$$

be the quotient space by the equivalence relation $\sim$. We claim that $\sim$ is an open equivalence relation. Indeed, let $U := \coprod_{i \in I} U_i$ be an open subset of Ω, where U_i is an open subset of Ω_i for each i. Then

$$\pi^{-1}(\pi(U)) = \coprod_{i \in I} \left(\bigcup_{j \in I} \phi_{ij}(U_j \cap \Omega_{ji}) \cup U_i \right),$$

which is open in Ω since $\phi_{ij}(U_j \cap \Omega_{ji})$ is open in Ω_i for all $i \in I$. Therefore, $M_{\mathcal{G}}$ is second-countable since Ω is second-countable.

Since $\sim$ is an open equivalence relation, we can use Theorem (17.1) to show that $M_{\mathcal{G}}$ is Hausdorff by showing that the graph

$$R = \{(x,y) \in \Omega \times \Omega \mid x \sim y\}$$

is closed in $\Omega \times \Omega$. We do this by showing that if (x_n, y_n) is a sequence in R that converges to $(x,y) \in \Omega \times \Omega$, then $(x,y) \in R$. That is to say, if $x_n \sim y_n$, then $x \sim y$. Since $(x,y) \in \Omega_i \times \Omega_j$, we may assume that $(x_n, y_n) \in \Omega_i \times \Omega_j$ (since it will eventually be in there anyways). If $i = j$, then $x_n = y_n$, and hence $x = y$, so assume $i \neq j$.

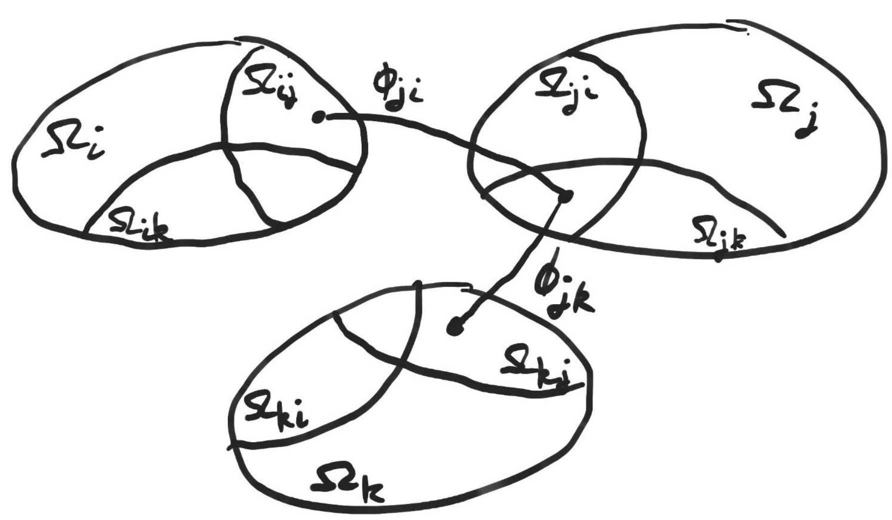
In order for us to have $x_n \sim y_n$, we must have $x_n \in \Omega_{ij}$ and $y_n \in \Omega_{ji}$. If $x \in \Omega_{ij}$, then it is easy to see that $y \in \Omega_{ji}$ and that $x \sim y$, since $x_n \sim y_n$ in arbitrarily small neighborhoods of x and y. Thus we need to show that either $x \in \Omega_{ij}$ or $y \in \Omega_{ji}$. Assume for a contradiction, that $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$. Choose open balls V_x and V_y centered at x and y so that no point in $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}. But this implies that no point in V_x is equivalent to some point in V_y. This contradicts the fact that $x_n \to x$ and $y_n \to y$, as the sequence (x_n, y_n) must eventually be in the neighborhoods V_x and V_y. Therefore $M_{\mathcal{G}}$ is Hausdorff.

Finally, for every $i \in I$, let $\mathrm{in}_i : \Omega_i \to \coprod_{i \in I} \Omega_i$ be the natural injection and let

$$\tau_i := \pi \circ \mathrm{in}_i : \Omega_i \to M_{\mathcal{G}}.$$

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we conclude that every τ_i is injective. If we let $U_i = \tau_i(\Omega_i)$ and $\phi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, ϕ_i) are charts and this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. □

Remark 28. Note that the condition $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ is needed in order for $\sim$ to be transitive. The picture below illustrates how things could go wrong:



17.5.1 Mobius Strip

Example 17.9. Let X be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold.

Let U_v be the set of all nonvertical lines in $\mathbb{R}^2$. A nonvertical is of the form $\ell^v_{a,b} = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by

$\ell^v_{a,b} \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism.

Next let U_h be the set of all nonhorizontal lines in $\mathbb{R}^2$. A nonhorizontal is of the form $\ell^h_{c,d} = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell^h_{c,d} \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism.

Now we have $U_v \cup U_h = X$. To get a topology on X, we glue the topologies from U_v and U_h: a set $U \subset X$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}
\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d) \\
&= \varphi_v\left(\ell^h_{c,d}\right) \\
&= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right) \\
&= \left(\frac{1}{c}, -\frac{d}{c}\right),
\end{aligned}$$

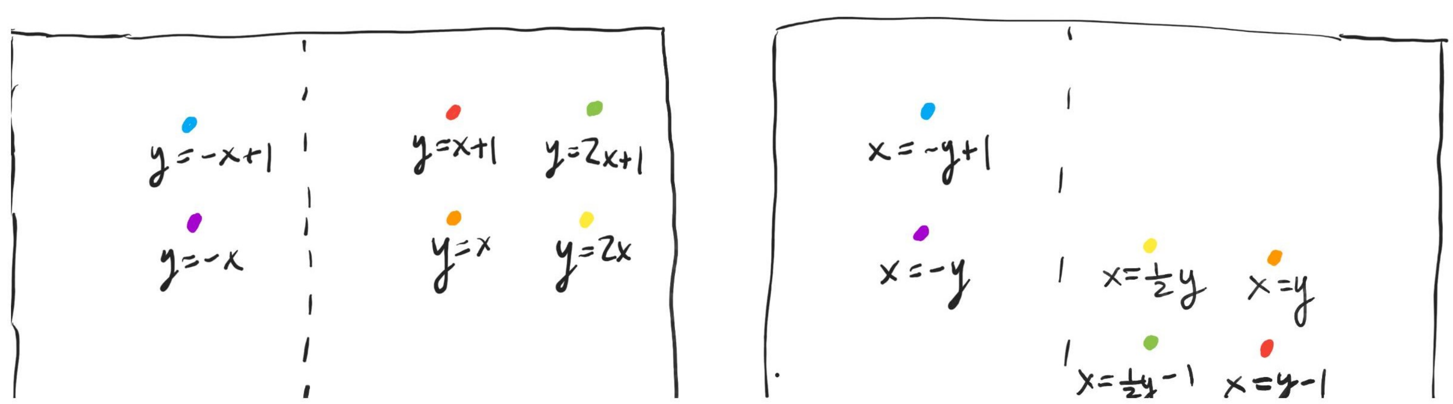
which is C^∞ whenever $c \neq 0$. Similarly,

$$\begin{aligned}
\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b) \\
&= \varphi_h\left(\ell^v_{a,b}\right) \\
&= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right) \\
&= \left(\frac{1}{a}, -\frac{b}{a}\right),
\end{aligned}$$

which is C^∞ whenever $a \neq 0$. Altogether, our gluing data consists of

$$\Omega_1 = \Omega_2 = \mathbb{R}^2, \qquad \Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}, \qquad \phi_{12} : (a,b) \mapsto \left(\frac{1}{a}, -\frac{b}{a}\right).$$

This manifold is called the **Mobius strip**. We can visualize it as below:



Remark 29. We can describe this manifold in another way as follows: let G be the group given by

$$G := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a,b \in \mathbb{R} \right\}.$$

The group G has a natural open subgroup

$$\mathrm{Aff}(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a,b \in \mathbb{R} \text{ and } a \neq 0 \right\}.$$

Clearly G can be identified with $\Omega_1 = \Omega_2 = \mathbb{R}^2$ and $\mathrm{Aff}(\mathbb{R})$ can be identified with $\Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}$. Using these identifications, the transition map ϕ_{12} is identified with the inverse map! Indeed, the inverse of $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1/a & -b/a \\ 0 & 1 \end{smallmatrix}\right)$.

Given a set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, it is natural to consider the collection of manifolds, M, parametrized by maps, $\theta_i : \Omega_i \to M$, whose domains are the Ω_i's and whose transition functions are given by the ϕ_{ji}'s, that is, such that

$$\phi_{ji} = \theta_j^{-1} \circ \theta_i.$$

We will say that such manifolds are **induced** by the set of gluing data $\mathcal{G}$.

The parametrization maps τ_i satisfy the property: $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ if and only if $(i,j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Furthermore, they also satisfy the consistency condition:

$$\tau_i = \tau_j \circ \phi_{ji},$$

for all $(i,j) \in K$. If M is a manifold induced by the set of gluing data $\mathcal{G}$, then because the θ_i's are injective and $\phi_{ji} = \theta_j^{-1} \circ \theta_i$, the two properties stated above for the τ_i's also hold for the θ_i's. We will see that the manifold $M_{\mathcal{G}}$ is a "universal" manifold induced by $\mathcal{G}$ in the sense that every manifold induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by some C^k map.

Interestingly, it is possible to characterize when two manifolds induced by the same set of gluing data are isomorphic in terms of a condition on their transition functions.

Proposition 17.3. *Given any set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, for any two manifolds M and M' induced by $\mathcal{G}$ given by families of parametrizations $(\Omega_i, \theta_i)_{i\in I}$ and $(\Omega_i, \theta'_i)_{i\in I}$, respectively, if $f : M \to M'$ is a C^k isomorphism, then there are C^k bijections, $\rho_i : W_{ij} \to W'_{ij}$, for some open subsets $W_{ij}, W'_{ij} \subseteq \Omega_i$, such that*

$$\phi'_{ji}(x) = \rho_j \circ \phi_{ji} \circ \rho_i^{-1}(x),$$

for all $x \in W_{ij}$ with $\phi_{ji} = \theta_j^{-1} \circ \theta_i$ and $\phi'_{ji} = \theta'^{-1}_j \circ \theta'_i$. Furthermore, $\rho_i = (\theta'^{-1}_i \circ f \circ \theta_i) \mid_{W_{ij}}$ and if $\theta'^{-1}_i \circ f \circ \theta_i$ is a bijection from Ω_i to itself and $\theta'^{-1}_i \circ f \circ \theta_i(\Omega_{ij}) = \Omega_{ij}$ for all i, j, then $W_{ij} = W'_{ij} = \Omega_i$.

18 Ringed Spaces

Definition 18.1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally** R**-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

19 Equivalence between C^p-structures and maximal C^p-atlases

Fix $0 \leq p \leq \infty$, and let X be a topological premanifold. Let $\mathscr{A} = \{(\phi_i, U_i)\}$ and $\mathscr{A}' = \{(\phi'_{i'}, U_{i'})\}$ be two C^p-atlases on X, so $\phi_i \colon U_i \to V_i$ and $\phi'_{i'} \colon U'_{i'} \to V'_{i'}$ are homeomorphisms onto nonempty open subsets of finite-dimensional $\mathbb{R}$-vector spaces, and the resulting homeomorphisms

$$\phi_{i_1} \circ \phi_{i_2}^{-1} \colon \phi_{i_2}(U_{i_1} \cap U_{i_2}) \to \phi_{i_1}(U_{i_1} \cap U_{i_2}) \quad \text{and} \quad \phi'_{i'_1} \circ \phi - 1 \colon \phi'_{i'_2}(U'_{i'_1} \cap U'_{i'_2}) \to \phi'_{i'_1}(U'_{i'_1} \cap U'_{i'_2})$$

between open domains in the vector spaces are C^p isomorphisms in the usual sense. Let us say that a C^p-atlas $\mathscr{A} - \{(\phi_i, U_i)\}_{i\in I}$ is **standardized** if the following two conditions hold:

1. for each (ϕ_i, U_i) in $\mathscr{A}$ the target vector space for $\phi_i \colon U_i \to V_i$ is a Euclidean space $\mathbb{R}^{n_i}$ (with n_i uniquely determined by U_i, as U_i is nonempty), and

2. $\mathscr{A}$ has no repititions in the sense that whenever $i \neq j$ we have either $U_i \neq U_j$, or $U_i = U_j$ (so $n_i = n_j$, and $U_i = U_j$ is nonempty) the maps $\phi_i, \phi_j \colon U_i \rightrightarrows \mathbb{R}^{n_i}$ do not coincide.

Since we are insisting on the lack of repetitions in $\mathscr{A}$, we may and do drop the indexing set for such atlases: a standardized C^p atlas is a certain kind o fsubset of the set of pairs (ϕ, U) where $U \subseteq X$ is a nonempty open set and $\phi \colon U \to \mathbb{R}$ is a homeomorphism onto an open subset of a Euclidean space (note that n is permitted to vary, though it is determined by (ϕ, U) since $U \neq \emptyset$).

If $\mathscr{A}$ and $\mathscr{A}'$ are standarized C^p-atlases on X, then it makes sense to ask if $\mathscr{A} \subseteq \mathscr{A}'$. This means that each $(\phi, U) \in \mathscr{A}$ is equal to some $(\phi', U') \in \mathscr{A}'$. We say that a standarized atlas $\mathscr{A}'$ **dominates** a standarized atlas $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{A}'$ in the sense just defined. It is clear that if two standardized atlases dominate each other then they are literally equal. A standardized C^p-atlas $\mathscr{A}$ on X is **maximal** if it is not strictly contained inside of another standarized C^p atlas of X.

19.1 From C^p-Structures to Maximal C^p-Atlases

Let $\mathcal{O}$ be a C^p-structure on X. Let $\mathscr{A}$ be the set of all pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi\colon (U, \mathcal{O}\mid_U) \to \mathbb{R}^n$ is a C^p-isomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$ (with $\mathbb{R}^n$ given its usual C^p-structure). The collection $\mathscr{A}$ is a C^p atlas because of two facts: a composite of C^p maps is C^p, and for maps between opens in finite-dimensional $\mathbb{R}$-vector spaces the "old" notion of C^p is the same as the "new" notion (in terms of structured $\mathbb{R}$-spaces). It is obvious that $\mathscr{A}$ is standardized. We want to prove that the standardized C^p atlas $\mathscr{A}$ is maximal.

Let us see that we can recover $\mathcal{O}$ from $\mathscr{A}$:

Theorem 19.1. *For any nonempty open $U_0 \subseteq X$, $\mathcal{O}(U_0)$ is the set of functions $f\colon U_0 \to \mathbb{R}$ such that for each $(\phi, U) \in \mathscr{A}$, the function $f \circ \phi^{-1}\colon \phi(U \cap U_0) \to \mathbb{R}$ is C^p on the open subset $\phi(U \cap U_0)$ in the target Euclidean space $\mathbb{R}^n$ for ϕ.*

Proof. The condition that $f \circ \phi^{-1}$ be C^p on the open set $\phi(U \cap U_0)$ says exactly that $f \circ \phi^{-1} \in \mathcal{O}_{\mathbb{R}^n}(\phi(U \cap U_0))$, with $\mathcal{O}_{\mathbb{R}^n}$ denoting the usual C^p-structure on $\mathbb{R}^n$. Thus, since ϕ defines a C^p-isomorphism between $(U, \mathcal{O}|_U)$ and $(\phi(U), \mathcal{O}_{\mathbb{R}^n}|_{\phi(U)})$, by definition of $\mathscr{A}$ in terms of $\mathcal{O}$, it follows that composition with ϕ^{-1} carries $\mathcal{O}_{\mathbb{R}^n}(\phi(U'))$ bijectively over to $\mathcal{O}(U')$ for any open subset $U' \subseteq U$. Taking $U' = U \cap U_0$, we conclude that the condition on f with respect to (ϕ, U) in the theorem says exactly that $f \in \mathcal{O}(U \cap U_0)$. Since $\mathscr{A}$ is an atlas, so as we vary $(\phi, U) \in \mathscr{A}$ the opens U cover X, it follows that as we vary $(\phi, U) \in \mathscr{A}$ the opens $U \cap U_0$ cover U_0. By the locality axiom for the $\mathbb{R}$-space structure $\mathcal{O}$, it follows that $f\colon U_0 \to \mathbb{R}$ lies in $\mathcal{O}(U_0)$ if and only if its restriction to each $U \cap U_0$ lies in $\mathcal{O}(U \cap U_0)$, and hence if and only if $f \circ \phi^{-1}\colon \phi(U \cap U_0) \to \mathbb{R}$ is a C^p function on the open set $\phi(U \cap U_0)$ in $\mathbb{R}^n$. □

19.2 From Maximal C^p-Atlases to C^p-Structures

Let $\mathscr{A}$ be a maximal standarized C^p-atlas on X. For any non-empty open set $U_0 \subseteq X$, we define $\mathcal{O}(U_0)$ to be the set of functions $f\colon U_0 \to \mathbb{R}$ such that for all $(U, \phi) \in \mathscr{A}$, the composite map

$$f \circ \phi^{-1}\colon \phi(U \cap U_0) \to \mathbb{R}$$

is a C^p function on the open subset $\phi(U \cap U_0)$ in the Euclidean space $\mathbb{R}^n$ that is the target of ϕ. Also define $\mathcal{O}(\emptyset) = \{0\}$.

Lemma 19.2. *The correspondence $U_0 \mapsto \mathcal{O}(U_0)$ is an $\mathbb{R}$-space structure on X. For any $(U, \phi) \in \mathcal{A}$ and open $U_0 \subseteq U$, $\mathcal{O}(U_0)$ is the set of $f : U_0 \to \mathbb{R}$ such that $f \circ \phi^{-1} : \phi(U_0) \to \mathbb{R}$ is a C^p function on the open domain $\phi(U_0)$ in a Euclidean space.*

Proof. The usual notion of C^p function on an open set in a Euclidean space is preserved under restirction to smaller opens and can be checked by working on an open covering. Thus, the first claim in the lemma follows easily from the definition of $\mathcal{O}$. □

20 deRham Cohomology

Suppose $F(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a smooth vector field representing a force on an open subset U of $\mathbb{R}^2$, and C is a parametrized curve $c(t) = (x(t), y(t))$ in U from a point p to a point q with $a \le t \le b$. Then the work done by the force in moving a particle from p to q along C is given by the line integral $\int_C Pdx + Qdy$.

Such a line integral is easy to compute if the vector field F is the gradient of a scalar function $f(x, y)$:

$$F = \mathrm{grad} f = \langle \partial_x f, \partial_y f\rangle.$$

By Stoke's theorem, the line integral is simply

$$\int_C \partial_x f dx + \partial_y f dy = \int_C df = f(q) - f(p).$$

A necessary condition for the vector field $F = \langle P, Q\rangle$ to be a gradient is that

$$P_y = \partial_y \partial_x f = \partial_x \partial_y f = Q_x.$$

The question is now the following: if $P_y - Q_x = 0$, is the vector field $F = \langle P, Q\rangle$ on U the gradient of some scalar function $f(x, y)$ on U? In terms of differential forms, the question becomes the following: if the 1-form $\omega = Pdx + Qdy$ is closed on U, is it exact? The answer to this question is sometimes yes and sometimes no, depending on the topology of U.

20.1 de Rham Complex

Let M be a manifold and let R denote the ring $\Omega^0(M) := C^\infty(M)$. Then we have the following cochain complex over R

$$(\Omega(M), d) := \qquad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \tag{25}$$

where d denotes the exterior derivative. We denote $H_{\mathrm{dR}}(M)$ to be the cohomology of $(\Omega(M), d)$ and call it the **deRham cohomology** of M. We denote by $Z(M)$ to be the cycles of $(\Omega(M), d)$ and $B(M)$ to be the boundaries of $(\Omega(M), d)$.

Proposition 20.1. *If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered- r-tuple of real numbers, each real number representing a constant function on a connected component of M.*

Proof. Since there are no nonzero exact 0-forms,

$$H^0(M) = Z^0(M).$$

Suppose f is a closed 0-form on M, i.e. f is a C^∞ function on M such that $df = 0$. On any chart $(U, x_1, \ldots, x_n)$, we have

$$df = \sum_{\lambda=1}^{n} (\partial_{x_\lambda} f) dx_\lambda.$$

Thus $df = 0$ on U if and only if all the partial derivatives $\partial_{x_\lambda} f$ vanish identically on U. This in turn is equivalent to f being locally constant on U. Hence, the closed 0-forms on M are precisely the locally constant functions on M. Such a function must be constant on each connected component on M. If M has r connected components, then a locally constant function on M can be specified by an ordered set of r real numbers. Thus, $Z^0(M) = \mathbb{R}^r$. □

Proposition 20.2. *On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.*

Proof. At any point $p \in M$, then tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, the space of alternating k-linear functions on T_pM. If $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form on M is the zero form. □

20.1.1 Examples of de Rham Cohomology

Example 20.1. (De Rham cohomology of the real line) Since the real line $\mathbb{R}^1$ is connected, we have

$$H^0(\mathbb{R}^1) = \mathbb{R}.$$

For dimensional reasons, there are no nonzero 2-forms on $\mathbb{R}^1$. This implies that every 1-form on $\mathbb{R}^1$ is closed. A 1-form $f(x)dx$ on $\mathbb{R}^1$ is exact if and only if three is a C^∞ function $g(x)$ on $\mathbb{R}^1$ such that

$$f(x)dx = dg = g'(x)dx,$$

where $g'(x)$ is the calculus derivative of g with respect to x. Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example

$$g(x) = \int_0^x f(t)dt.$$

This proves that every 1-form on $\mathbb{R}^1$ is exact. Therefore, $H^1(\mathbb{R}^1) = 0$.

Example 20.2. (De Rham cohomology of the circle) Let S^1 be the unit circle in the xy-plane. Since S^1 is connected, we have $H^0(S^1) = \mathbb{R}$, and since S^1 is one-dimensional, we have $H^k(S^1) = 0$ for all $k \geq 2$. It remains to compute $H^1(S^1)$.

Let $h : \mathbb{R} \to S^1$ be given by $h(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$ and let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restricting the domain of h to $[0, 2\pi]$ gives a parametrization $F := h \circ i : [0, 2\pi] \to S^1$ of the circle. A nowhere-vanishing 1-form on S^1 is given by $\omega = -ydx + xdy$. Note that

$$\begin{aligned} h^*\omega &= -\sin t d(\cos t) + \cos t d(\sin t) \\ &= (\sin^2 t + \cos^2 t)dt \\ &= dt. \end{aligned}$$

Thus

$$\begin{aligned} F^*\omega &= i^*h^*\omega \\ &= i^*dt \\ &= dt, \end{aligned}$$

and so

$$\begin{aligned} \int_{S^1} \omega &= \int_{F([0,2\pi])} \omega \\ &= \int_{[0,2\pi]} F^*\omega \\ &= \int_0^{2\pi} dt \\ &= 2\pi. \end{aligned}$$

Since the cirle has dimension 1, all 1-forms on S^1 are closed, so $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1-forms on S^1 defines a linear map

$$\varphi : Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}, \quad \varphi(\alpha) = \int_{S^1} \alpha.$$

Because $\varphi(\omega) = 2\pi \neq 0$, the linear map $\varphi : \Omega^1(S^1) \to \mathbb{R}$ is onto.

By Stokes's theorem, the exact 1-forms on S^1 are in $\mathrm{Ker}(\varphi)$. Conversely, we will show that all 1-forms in $\mathrm{Ker}(\varphi)$ are exact. Suppose $\alpha = f\omega$ is a smooth 1-form on S^1 such that $\varphi(\alpha) = 0$. Let $\overline{f} = h^*f = f \circ h \in \Omega^0(\mathbb{R})$. Then $\overline{f}$ is periodic of period 2π and

$$\begin{aligned} 0 &= \int_{S^1} \alpha \\ &= \int_{F([0,2\pi])} F^*\alpha \\ &= \int_{[0,2\pi]} (i^*h^*f)(t) \cdot F^*\omega \\ &= \int_0^{2\pi} \overline{f}(t)dt. \end{aligned}$$

20.2 The C^∞ Hairy Ball Theorem

Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ with $n > 0$. FIf n is odd then there exists a nowhere-vanishing smooth vector field on S^n. Indeed, if $n = 2k+1$ the consider the vector field $\vec{v}$ on $\mathbb{R}^{n+1} = \mathbb{R}^{2k+2}$ given by

$$\vec{v} = (-x_2\partial_{x_1} + x_1\partial_{x_2}) + \cdots + (-x_{2k+2}\partial_{x_{2k+1}} + x_{2k+1}\partial_{x_{2k+2}}) = \sum_{j=0}^{k} (-x_{2j+2}\partial_{x_{2j+1}} + x_{2j+1}\partial_{x_{2j+2}}).$$

For any point $p \in S^n$ it is easy to see that $\vec{v}(p) \in T_p(\mathbb{R}^{2k+2})$ is perpendicular to the line spanned by $\sum_i x_i(p)\partial_{x_i}|_p$, so it lies in the hyperplane $T_p(S^n)$ orthogonal to this line. In other words, the smooth section $\vec{v}|_{S^n}$ of the pullback bundle $(T(\mathbb{R}^{n+1}))|_{S^n}$ over S^n takes values in the subbundle $T(S^n)$, which is to say that $\vec{v}|_{S^n}$ is a smooth vector field on the manifold S^n. This is a visibly nowhere-vanishing vector field.

The above construction does not work if n is even, so there arises the question of whether there exists a nowhere-vanishing smooth vector field on S^n for even n. The answer is negative, and is called the **hairy ball** theorem.

Theorem 20.1. *A smooth vector field on S^n must vanish somewhere if n is even.*

Proof. Let $\vec{v}$ be a smooth vector field on S^n, and assume that it is nowhere-vanishing. For each $p \in S^n$, let $\gamma_p : [0, \pi/\|\vec{v}(p)\|] \to S^n$ be the smooth parametric great circle (with constant speed) going from p to $-p$ with velocity vector $\gamma_p'(0) = \vec{v}(p) \neq 0$ at $t = 0$ (This would not make sense if $\vec{v}(p) = 0$). Working in the plane spanned by $p \in \mathbb{R}^{n+1}$ and $\vec{v}(p) \in T_p(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$, we get the formula

$$\gamma_p(t) = \cos\left(t\|\vec{v}(p)\|\right) p + \sin\left(t\|\vec{v}(p)\right) \frac{\vec{v}(p)}{\|\vec{v}(p)\|} \in S^n \subseteq \mathbb{R}^{n+1}.$$

(These algebraic formulas would not make sense if $\vec{v}$ vanishes somewhere on S^n). Consider the "flow" mapping

$$F : S^n \times [0,1] \to S^n,$$

defined by $(p,t) \mapsto \gamma_p(\pi t/\|\vec{v}(p)\|)$. The formula for $\gamma_p(t)$ makes it clear that F is a smooth map (and is continuous if $\vec{v}$ is merely continuous and nowhere-vanishing). Now obviously $F(p,0) = p$ for all $p \in S^n$ and $F(p,1) = -p$ for all $p \in S^n$. Hence, F defines a smooth homotopy from the identity map on S^n to the antipodal map $p \mapsto -p$ on S^n (and is a continuous homotopy if $\vec{v}$ is merely continuous and nowhere-vanishing). Thus, to prove the hairy ball theorem we just have to prove that if n is even then the identity and antipodal maps $S^n \to S^n$ are not smoothly homotopic to each other; likewise to get the continuous version we just need to prove that there is no continuous homotopy deforming one of these maps into the other.

To prove the *non-existence* of such a homotopy, we shall use the (smooth) homotopy invariance of deRham cohomology. Indeed, by this homotopy-invariance we get that under the existence of such a $\vec{v}$ the antipodal map $A : S^n \to S^n$ induces the identity map $A^* : \mathrm{H}^k_{\mathrm{dR}}(S^n) \to \mathrm{H}^k_{\mathrm{dR}}(S^n)$ on the kth deRham cohomology of S^n for all $k \geq 0$. Let us focus on the case $k = n$. To get a contradiction, we just have to prove that if n is even then A^* as a self-map of $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is *not* the identity map.

Consider the n-form on $\mathbb{R}^{n+1}$ defined by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Clearly $d\omega = (n+1)dx_1 \wedge \cdots \wedge dx_{n+1}$, so for the unit ball $B^{n+1} \subseteq \mathbb{R}^{n+1}$ with its standard orientation we have

$$\int_{B^{n+1}} d\omega = (n+1)\mathrm{vol}(B^{n+1}) \neq 0.$$

By Stokes' theorem for B^{n+1}, if we let $\eta = \omega|_{S^n}$ and we give $S^n = \partial B^{n+1}$ the induced boundary orientation, then

$$\int_{S^n} \eta = \int_{B^{n+1}} d\omega \neq 0.$$

Hence, by Stokes' theorem for the boundaryless smooth compact oriented manifold S^n we conclude that the top-degree differential form η on S^n is not exact. That is, its deRham cohomology class $[\eta] \in H^n_{\mathrm{dR}}(S^n)$ is non-zero. (Note that ω is not closed as an n-form on $\mathbb{R}^{n+1}$, but its pullback η on S^n is necessarily closed on S^n purely for elementary reasons, as S^n is n-dimensional.)

By the existence of the smooth homotopy between A and the identity map, it follows that A^* on $H^n_{\mathrm{dR}}(S^n)$ is the identity map, so $[A^*(\eta)] = A^*([\eta])$ is equal to $[\eta]$. That is, the top-degree differential forms $A^*(\eta)$ and η on S^n differ by an exact form. But the antipodal map $A : S^n \to S^n$ is induced by the negation map $N : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, and by inspection of the definition of $\omega \in \Omega^n_{\mathbb{R}^{n+1}}(\mathbb{R}^{n+1})$ we have $N^*(\omega) = (-1)^{n+1}\omega$. Hence, pulling back this equality to the sphere gives $A^*(\eta) = (-1)^{n+1}\eta$ in $\Omega^n_{S^n}(S^n)$. Thus, in $H^n_{\mathrm{dR}}(S^n)$ we have

$$[\eta] = A^*([\eta]) = [A^*(\eta)] = [(-1)^{n+1}\eta] = (-1)^{n+1}[\eta].$$

If n is even we therefore have $[\eta] = -[\eta]$, so $[\eta] = 0$. But we have already seen via Stokes' theorem for the boundaryless manifold S^n and for the manifold with boundary B^{n+1} that $[\eta]$ is nonzero. This completes the proof. □

21 Exercises

21.1 $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \right\} \subset \mathbb{R}^4$. Let $\gamma : [0,1] \to \mathrm{SL}_2(\mathbb{R})$ be a path in $\mathrm{SL}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

such that $\gamma(0) = I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then by differentiating the identity $a(t)d(t) - b(t)c(t) = 1$ and evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= \dot{a}(0)d(0) + a(0)\dot{d}(0) - \dot{b}(0)c(0) - b(0)\dot{c}(0) \\ &= \dot{a}(0) + \dot{d}(0). \end{aligned}$$

Or $\dot{a}(0) = -\dot{d}(0)$. In particular, this means that $\mathrm{Tr}\,(\dot{\gamma}(0)) = 0$.

Conversely, suppose we have a matrix A such that $\mathrm{Tr}(A) = 0$. Can we find a path γ in $\mathrm{SL}_2(\mathbb{R})$ such that $\dot{\gamma}(0) = A$? Indeed, we can. The matrix exponential works:

$$e^{tA} := I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots.$$

This is because

$$\begin{aligned}\frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left(I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \cdots\right)\\ &= \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \frac{1}{2}\frac{d}{dt}(t^2A^2) + \frac{1}{6}\frac{d}{dt}(t^3A^3) + \cdots\\ &= A + tA^2 + \frac{1}{2}t^2A^3 + \cdots\end{aligned}$$

Thus, $\frac{d}{dt}\left(e^{tA}\right)_{|t=0} = A$. Also we have $e^{tA} \in \mathrm{SL}_2(\mathbb{R})$ since

$$\begin{aligned}\det(e^{tA}) &= e^{\mathrm{Tr}(tA)}\\ &= e^0\\ &= 1.\end{aligned}$$

21.2 $\mathrm{SO}_2(\mathbb{R})$

Let $\mathrm{SO}_2(\mathbb{R}) := \left\{A \in \mathrm{SL}_2(\mathbb{R}) \mid AA^t = I\right\}$. Let $\gamma : [0,1] \to \mathrm{SO}_2(\mathbb{R})$ be a path in $\mathrm{SO}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix}.$$

such that $\gamma(0) = I := \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Then by differentiating the identity $I = \gamma(t)\gamma(t)^t$ and evaluating at $t = 0$, we get

$$\begin{aligned}0 &= \dot\gamma(0)\gamma(0)^t + \gamma(0)\dot\gamma(0)^t\\ &= \dot\gamma(0) + \dot\gamma(0)^t.\end{aligned}$$

In particular, this means that $\dot\gamma(0)$ is a skew-symmetric matrix.

21.3 Vector Field in $\mathbb{R}^3$

Let ω be a vector field in $\mathbb{R}^3$ given by $\omega := (0, y, 0) := y\partial_y$. Let's find a path γ in $\mathbb{R}^3$ such that $\dot\gamma = \omega(\gamma)$. A general path γ in $\mathbb{R}^3$ has the form

$$\gamma(t) := (a(t), b(t), c(t)) \qquad \text{and} \qquad \dot\gamma(t) = (\dot a(t), \dot b(t), \dot c(t)).$$

Therefore ω(So we need

$$\begin{aligned}\dot a(t) &= 0\\ \dot b(t) &= b(t)\\ \dot c(t) &= 0.\end{aligned}$$

In particular, $\gamma(t) = (a(0), b(0)e^t, c(0))$ works.

21.4 Lie Groups

Definition 21.1. A **Lie group** is a C^∞ manifold G that is also a group such that the two group operations, multiplication

$$G \times G \to G, \qquad (a, b) \mapsto ab,$$

and inverse

$$G \to G, \qquad a \mapsto a^{-1},$$

are C^∞.

For $a \in G$, denote by $\ell_a : G \to G$, where $\ell_a(x) = ax$, the operation of **left multiplication by** a, and by $r_a : G \to G$, where $r_a(x) = xa$, the operation of **right multiplication** by a. We also call left and right multiplications **left** and **right translations**.

Actually smoothness of invsersion can be dropped from the definition of a Lie Group.

Theorem 21.1. *Let G be a C^∞ manifold and suppose it is equipped with a group structure such that the composition law $m : G \times G \to G$ is C^∞. Then the inversion $G \to G$ is C^∞.*

Proof. Consider the "shearing transformation"

$$\Sigma : G \times G \to G \times G,$$

defined by $\Sigma(g,h) = (g,gh)$. This is bijective since we are using a group law, and it is C^∞ since the composition law m is assumed to be C^∞. (Recall that if M, M', M'' are C^∞ manifolds, a map $M \to M' \times M''$ is C^∞ if and only if its component maps $M \to M'$ and $M \to M''$ are C^∞, due to the nature of product manifold structures.)

We claim that Σ is a diffeomorphism. Granting this,

$$G = \{e\} \times G \longrightarrow G \times G \xrightarrow{\Sigma^{-1}} G \times G$$

is C^∞, but explicitely this composite map is $g \mapsto (g, g^{-1})$, so its second component $g \mapsto g^{-1}$ is C^∞ as desired. Since Σ is a C^∞ bijection, the C^∞ property for its inverse is equivalent to Σ being a **local isomorphism** (i.e. each point in its source has an open neighborhood carried diffeomorphically onto an open neighborhood in the target). By the Inverse Function Theorem, this is equivalent to the isomorphism property for the tangent map

$$d\Sigma(g,h) : T_g(G) \oplus T_h(G) = T_{(g,h)}(G \times G) \to T_{(g,gh)}(G \times G) = T_g(G) \oplus T_{gh}(G)$$

for all $g, h \in G$.

We shall now use left and right translations to reduce this latter "linear" problem to the special case $g = h = e$, and in that special case we will be able to compute the tangent map explicitely and see the isomorphism property by inspection.

□

Part III
Algebraic Curves and Riemann Surfaces

Example 21.1. Let S^2 denote the unit 2-sphere

Part IV
Algebraic Geometry

Throughout these notes, let K be a field and let $\overline{K}$ be an algebraic closure of K Unless otherwise specified, we let n be a positive integer. In this case, we often write $\boldsymbol{x} = (x_1, \dots, x_n)$ to denote a point in K^n whenever context is clear. Similarly we often write $K[\boldsymbol{T}] = K[T_1, \dots, T_n]$ to denote a polynomial ring in the variables $\boldsymbol{T} = T_1, \dots, T_n$ with coefficients in K whenever context is clear.

22 Affine Algebraic Sets

In this section, we will define **affine algebraic sets**. Before we do this, we first introduce the following notation: Let $\mathcal{P}$ be a set of polynomials in $K[\boldsymbol{T}]$. We denote by $\mathrm{V}_K(\mathcal{P})$ to be the set of common zeros of the polynomials in $\mathcal{P}$:

$$\mathrm{V}_K(\mathcal{P}) = \{\boldsymbol{x} \in K^n \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in \mathcal{P}\}.$$

If the underlying field K is understood from context, then we will simplify our notation and write $\mathrm{V}(\mathcal{P})$ instead of $\mathrm{V}_K(\mathcal{P})$. If $\mathcal{Q}$ is another set of polynomials in $K[\boldsymbol{T}]$ such that $\mathcal{P} \subseteq \mathcal{Q}$, then we have $\mathrm{V}(\mathcal{P}) \supseteq \mathrm{V}(\mathcal{Q})$. In other words, V is **inclusion-reversing**. Now let $\mathfrak{a}$ be the ideal generated by $\mathcal{P}$. Recall that $K[\boldsymbol{T}]$ is a Noetherian ring, and thus $\mathfrak{a}$ is finitely generated as an ideal, say $\mathfrak{a} = \langle f_1, \dots, f_m \rangle$. Observe that

$$\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\mathcal{P}) = \mathrm{V}(f_1, \dots, f_m),$$

where we denote $\mathrm{V}(f_1, \dots, f_m) = \mathrm{V}(\{f_1, \dots, f_m\})$. Indeed, it suffices to show that $\mathrm{V}(\mathfrak{a}) \supseteq \mathrm{V}(f_1, \dots, f_m)$ since the reverse inclusion follows from the fact that V is inclusion-reversing. Given $\boldsymbol{x} \in \mathrm{V}(f_1, \dots, f_m)$, then $f_i(\boldsymbol{x}) = 0$ for all $1 \leq i \leq m$. This implies that

$$\left(\sum_{i=1}^m g_i f_i\right)(\boldsymbol{x}) = \sum_{i=1}^m g_i(\boldsymbol{x}) f_i(\boldsymbol{x})$$
$$= \sum_{i=1}^m g_i(\boldsymbol{x}) \cdot 0$$
$$= 0$$

for all $\sum_{i=1}^m g_i f_i \in \mathfrak{a}$. Thus we have $\mathrm{V}(\mathfrak{a}) \supseteq \mathrm{V}(f_1, \ldots, f_m)$.

22.0.1 Maximal ideals defined by points

Let $\boldsymbol{x} \in K^n$ and let $\mathrm{ev}_{\boldsymbol{x}} \colon K[\boldsymbol{T}] \to K$ be the unique K-algebra homomorphism given by $\mathrm{ev}_{\boldsymbol{x}}(T_i) = x_i$ for all $i = 1, \ldots, n$. Denote by $\mathfrak{m}_{\boldsymbol{x}}$ to be the kernel of $\mathrm{ev}_{\boldsymbol{x}}$:

$$\mathfrak{m}_{\boldsymbol{x}} = \{f \in K[\boldsymbol{T}] \mid f(\boldsymbol{x}) = 0\}.$$

Then $\mathfrak{m}_{\boldsymbol{x}}$ is a maximal ideal of $K[\boldsymbol{T}]$ since $K[\boldsymbol{T}]/\mathfrak{m}_x \cong K$. Now let $\mathfrak{a}$ be another ideal of $K[\boldsymbol{T}]$. Then observe that $\boldsymbol{x} \in \mathrm{V}(\mathfrak{a})$ if and only if $\mathfrak{m}_{\boldsymbol{x}} \supseteq \mathfrak{a}$. In particular, we can express $\mathrm{V}(\mathfrak{a})$ in terms of the maximal ideals $\mathfrak{m}_{\boldsymbol{x}}$ as follows:

$$\mathrm{V}(\mathfrak{a}) = \{\boldsymbol{x} \in K^n \mid \mathfrak{m}_{\boldsymbol{x}} \supseteq \mathfrak{a}\}$$

We will use this reformulation many times throughout this article.

22.1 The Zariski Topology

We are almost ready to define algebraic sets, but first we need to prove the following lemma:

Lemma 22.1. *The following relations hold:*

1. $\mathrm{V}(0) = K^n$ *and* $\mathrm{V}(1) = \varnothing$.
2. *For two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$, *we have*
$$\mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{V}(\mathfrak{a}\mathfrak{b}) = \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b}).$$
3. *For every family* $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ *of ideals, we have*
$$\mathrm{V}\left(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(\mathfrak{a}_\lambda).$$

Proof. 1. We have $\mathrm{V}(0) = K^n$ since $\mathfrak{m}_x \supseteq \langle 0 \rangle$ for all $x \in K^n$. Similarly we have $\mathrm{V}(1) = \varnothing$ since $\mathfrak{m}_x \not\supseteq \langle 1 \rangle$ for all $x \in K^n$.

2. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{b}$, it follows that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$ from the inclusion-reversing property of V. It remains to show that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$. To do this, we just need to show that $\mathfrak{m}_x \supseteq \mathfrak{a}\mathfrak{b}$ implies either $\mathfrak{m}_x \supseteq \mathfrak{a}$ or $\mathfrak{m}_x \supseteq \mathfrak{b}$ for all $x \in K^n$. But this follows from the fact that $\mathfrak{m}_x$ is a prime ideal.

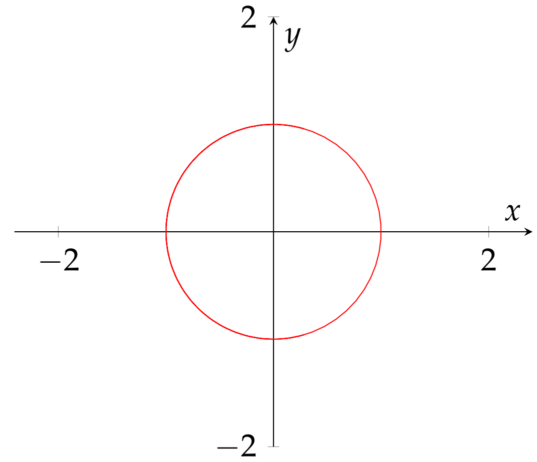
3. That $\mathrm{V}\left(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right)$ follows from the fact that $\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is the ideal generated by $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$. That $\mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(\mathfrak{a}_\lambda)$ follows from the fact that $\mathfrak{m}_x \supseteq \sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ if and only if $\mathfrak{m}_x \supseteq \mathfrak{a}_\lambda$ for all $\lambda \in \Lambda$ and for all $x \in K^n$.

□

Remark 30. It is very important to pay close attention to what is actually used in proofs. For example, in the proof of the second statement of this lemma, we only used the fact that $\mathfrak{m}_{\boldsymbol{x}}$ is a prime ideal (even though it is a maximal ideal). This gives us an idea for how we can generalize things. In particular, we will be replacing maximal ideals of the form $\mathfrak{m}_{\boldsymbol{x}}$ with arbitrary prime ideals. Keep this in mind!

This lemma implies that there is a unique topology on K^n for which the closed subsets are exactly those of the form $\mathrm{V}(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of $K[\boldsymbol{T}]$. We call this topology the **Zariski topology** and write $\mathbb{A}^n(K)$ to mean the set K^n equipped with the Zariski topology. We call $\mathbb{A}^n(K)$ an n-**dimensional affine space**. Closed subspaces of $\mathbb{A}^n(K)$ are called **affine algebraic sets**. In particular, note that singletons are closed since $\{\boldsymbol{x}\} = \mathrm{V}(\mathfrak{m}_{\boldsymbol{x}})$.

Example 22.1. The unit circle in $\mathbb{R}^2$ can be described as the variety $\mathrm{V}(x^2 + y^2 - 1)$ and can be pictured below:



There is a nice parametrization of the unit circle which we now describe: Suppose L is a line which passes through the point $(-1,0)$ and such that L is not the tangent line to the unit circle at the point $(-1,0)$. Then $L = \mathrm{V}(y - m(x+1))$, where m is the slope of the line, and L passes through a point $(x,y) \neq (-1,0)$ on the unit circle. Since (x,y) lies on the line L and the unit circle, we get the relations

$$\begin{aligned} x^2 + y^2 - 1 &= 0, \\ y - m(x+1) &= 0. \end{aligned}$$

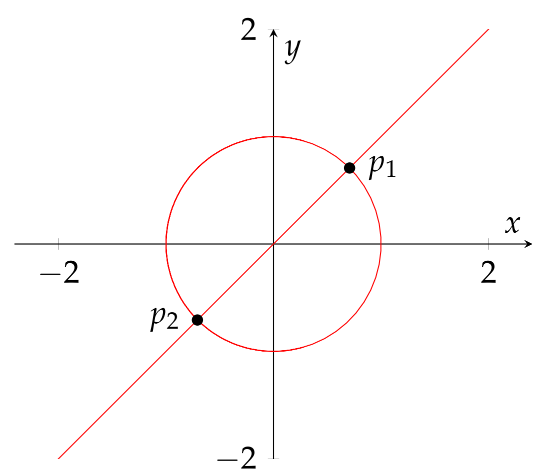
Using the second relation, we have $y = m(x+1)$. Plugging in $m(x+1)$ for y in the first relation, we get

$$m^2 = \frac{(1-x)^2}{(1+x)^2} = \frac{1-x}{1+x}.$$

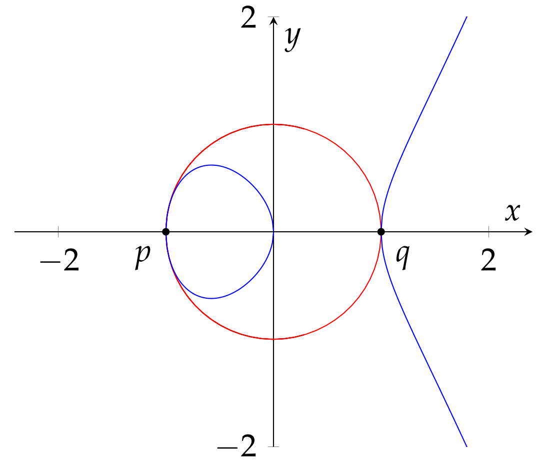
Now we solve for x in terms of m, to get the following parametrization:

$$\begin{aligned} x &= \frac{1-m^2}{1+m^2}, \\ y &= \frac{2m}{1+m^2}. \end{aligned}$$

Now suppose we throw in the polynomial $y - x$. What does the affine variety $\mathrm{V}(x^2+y^2-1, y-x)$ look like? The affine variety $\mathrm{V}(x^2+y^2-1, y-x)$ consists of the points (r_1, r_2) in $\mathbb{R}^2$ such that $r_1^2 + r_2^2 - 1 = 0$ and $r_2 - r_1 = 0$. There are two such points p_1 and p_2, and they correspond to two intersection points of the unit circle with the line $y = x$ as pictured below:



Example 22.2. Let $X = \mathrm{V}_K(x^2+y^2-1) = \mathrm{V}(f)$ and let $Y = \mathrm{V}_K(y^2 - x^3 + x) = \mathrm{V}(g)$. If $K = \mathbb{R}$, then we can see that X intersects Y at the points $p = (-1,0)$ and $q = (1,0)$ as pictured below



Let $A = K[x,y]/\langle f\rangle$ be the coordinate ring for X and let $B = K[x,y]/\langle g\rangle$ be the coordinate ring for Y. Then the coordinate ring of $X \cap Y$ is given by $A \otimes_K B = K[x,y]/\langle f, g\rangle$. The point $p = (-1, 0)$ corresponds to the maximal ideal $\mathfrak{m} = \langle x+1, y\rangle$ of $K[x,y]$, thus the local ring of A at p is given by

$$A_p = K[x,y]_{\mathfrak{m}}/\langle y^2 - u(x+1)\rangle_{\mathfrak{m}}$$

where $u = 1-x$ (this is a unit in A_p). Similarly, the local ring of B at p is given by

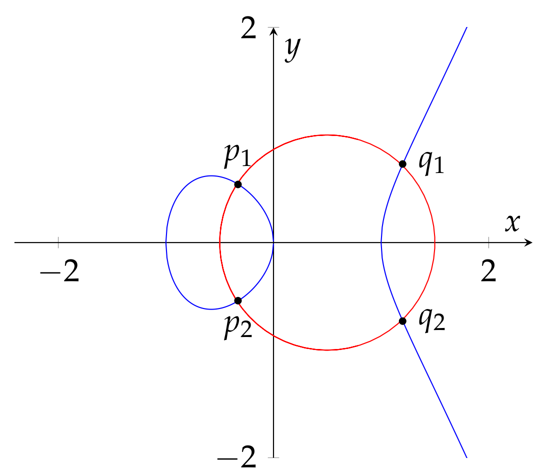
$$B_p = K[x,y]_{\mathfrak{m}}/\langle y^2 - uv(x+1)\rangle_{\mathfrak{m}}$$

where $u = 1-x$ and $v = x$ (these are both units in B_p). Thus

$$A_p \otimes_K B_p = K[x,y]_{\mathfrak{m}}/\langle y^2 - u(x+1), y^2 - uv(x+1)\rangle_{\mathfrak{m}} = K[x,y]_{\mathfrak{m}}/\langle y^2, x+1\rangle.$$

Clearly we have $\dim_K(A_p \otimes_K B_p) = 2$. Thus X and Y intersect at the point p with multiplicity 2.

Now suppose we perturn f a bit; say $X_t = \mathrm{V}_K((x+t)^2 + y^2 - 1) = \mathrm{V}(f_t)$ where $t \in K$. For instance, if $K = \mathbb{R}$, then we can picture $X_{1/2}$ intersecting Y as below:



Thus the point p splits into two points p_1 and p_2.

22.2 Hilbert's Nullstellensatz

Let R be a ring. It is often advantageous to characterize the ideals of R in terms of some underlying geometric spaces. For instance, if R is a ring a functions on some space, then perhaps the ideals of R can be described in terms of certain subspaces of that space. Let's us consider two examples of when we can/cannot do this:

Example 22.3. Let $C_0(\mathbb{R})$ be the ring of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ which vanish at $\pm\infty$. We want to understand what the ideals of $C_0(\mathbb{R})$ look like. First let's start with the maximal ideals. For each $x \in \mathbb{R}$, we define $\mathfrak{m}_x = \{f \in C_0(\mathbb{R}) \mid f(x) = 0\}$. It is easy to check that $\mathfrak{m}_x$ is a maximal ideal of $C_0(\mathbb{R})$. Thus we can describe a bunch of maximal ideals of $C_0(\mathbb{R})$ in terms points of $\mathbb{R}$. This gives us a nice geometric way of describing some of the maximal ideals of R, but the question remains: does this describe all of the maximal ideals of $C_0(\mathbb{R})$? The answer is no. Indeed, let I be the set of all $f \in C_0(\mathbb{R})$ such that f vanishes outside some compact set. It is easy to check that I is an ideal of $C_0(\mathbb{R})$, and hence can be extended to a maximal ideal $\mathfrak{m}$ of $C_0(\mathbb{R})$, but $\mathfrak{m}$ is not of the form $\mathfrak{m}_x$ for any $x \in \mathbb{R}$ since we can always find an $f \in \mathfrak{m}$ such that $f(x) \neq 0$. The takeaway here is that the ideals of $C_0(\mathbb{R})$ are complicated, and the space $\mathbb{R}$ doesn't do a good enough job at characterizing these ideals. One would need to extend $\mathbb{R}$ to a much larger space (like the hyperreals ${}^*\mathbb{R}$) in order to characterize the ideals of $C_0(\mathbb{R})$

Example 22.4. Let X be a compact Hausdorff space and let $C(X)$ be the ring of continuous functions $f\colon X \to \mathbb{R}$. Again, we can characterize many of the maximal ideals of $C(X)$ in terms of the points of X, namely for each $x \in X$ we set $\mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}$. In this case, it turns out that all of the maximals ideals of $C(X)$ are of the form $\mathfrak{m}_x$ for some $x \in X$. Indeed, let $\mathfrak{m}$ be a maximal ideal of $C(X)$ and assume for a contradiction that $\mathfrak{m} \neq \mathfrak{m}_x$ for any $x \in X$. Then for each $x \in X$, we can find an $f_x \in C(X)$ such that $f_x(x) \neq 0$. For each $x \in X$, choose an open neighborhood U_x of x such that $f_x(y) \neq 0$ for all $y \in U_x$. The collection $\{U_x \mid x \in X\}$ is an open cover of X. Since X is compact, there exists a finite subcollection of $\{U_x\}$ which covers X, say $\{U_{x_1}, \dots, U_{x_n}\}$. Now define

$$f = \sum_{i=1}^{n} f_{x_i}.$$

Clearly $f \in \mathfrak{m}$ since $\mathfrak{m}$ is closed under addition. However note that $f(y) \neq 0$ for all $y \in X$, thus $1/f$ exists, or in other words, f is a unit. This is a contradiction. We've just shown that the maximal ideals of $C(X)$ are in one-to-one correspondence with the points of X. Thus the space X does a nice job at describing the maximal ideals of $C(X)$. On the other hand, there is still much more work to be done if we want to characterize *all* ideals of $C(X)$ in terms of subspaces of X.

Now we consider the ring $\overline{K}[\boldsymbol{T}]$. We can interpret $\overline{K}[\boldsymbol{T}]$ as the ring of functions on $\mathbb{A}^n(\overline{K})$. We also know some nice subspaces of $\mathbb{A}^n(\overline{K})$, namely the affine algebraic subspaces of $\mathbb{A}^n(\overline{K})$ (which also happen to be the *closed* subspaces of $\mathbb{A}^n(\overline{K})$ in the Zariski topology). Every such affine algebraic subspace has the form $\mathrm{V}(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of $\overline{K}[T]$. Thus we can characterize the affine algebraic subspaces of $\mathbb{A}^n(\overline{K})$ in terms of the ideals of $\overline{K}[T]$. However we really want to go backwards: can we characterize the ideals of $\overline{K}[T]$ in terms of the affine algebraic subspaces of $\mathbb{A}^n(\overline{K})$? Hilbert Nullstellensatz tells us that we can almost do this; in particular we can characterize the *radical* ideals of $\overline{K}[T]$ in terms of the affine algebraic subspaces of $\mathbb{A}^n(\overline{K})$. In other words, there is a one-to-one correspondence:

$$\{\text{radical ideals of } \overline{K}[T]\} \leftrightarrow \{\text{affine algebraic subspaces of } \mathbb{A}^n(\overline{K})\}.$$

Thus the space $\mathbb{A}^n(\overline{K})$ gives a nice characterization of the radical ideals of $\overline{K}[T]$. With this motivation out of the way, we now are ready to state Hilbert's Nullstellensatz in a more general setting:

Theorem 22.2. *(Hilbert's Nullstellensatz) Let A be a finitely generated K-algebra. Then A is* ***Jacobson****, that is, for every prime ideal $\mathfrak{p}$ of A, we have*

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \supset \mathfrak{p} \\ \mathfrak{m} \text{ is maximal}}} \mathfrak{m}.$$

Moroever, if $\mathfrak{m}$ is a maximal ideal of A, then the field extension $K \subseteq A/\mathfrak{m}$ is finite.

We shall not prove this result here since a proof is best left for a course in Commutative Algebra (see my Algebra notes for such a proof). Instead, we will focus on the consequences of this theorem in regards to algebraic geometry. First let us consider some consequences when working over the algebraically closed field $\overline{K}$:

Proposition 22.1. *Let $\mathfrak{m}$ be a maximal ideal of $\overline{K}[\boldsymbol{T}]$. Then there exists an $\boldsymbol{x} \in \mathbb{A}^n(\overline{K})$ such that $\mathfrak{m} = \mathfrak{m}_x$.*

Proof. Observe that $\overline{K}[\boldsymbol{T}]$ is a finitely generated $\overline{K}$-algebra, thus from the Nullstellenatz, we see that $\overline{K} \hookrightarrow \overline{K}[\boldsymbol{T}]/\mathfrak{m}$ is a finite extension of fields; hence $\overline{K}[\boldsymbol{T}]/\mathfrak{m} \cong \overline{K}$ since $\overline{K}$ is algebraically closed. Now let x_i be the image of T_i by the homomorphism $\overline{K}[\boldsymbol{T}] \to \overline{K}[\boldsymbol{T}]/\mathfrak{m} \cong \overline{K}$. Then $\mathfrak{m}$ is a maximal ideal which contains the maximal ideal $\mathfrak{m}_{\boldsymbol{x}} = \langle T_1 - x_1, \dots, T_n - x_n \rangle$. Therefore both are equal. □

The proposition above tells us that the set of all points $\boldsymbol{x}$ in $\mathbb{A}^n(\overline{K})$ are in one-to-one correspondence with the set of all maximal ideals $\mathfrak{m}$ of $\overline{K}[\boldsymbol{T}]$. Note that we really do need $\overline{K}$ to be algebraically closed for this to hold. For instance, $\langle T^2 + 1 \rangle$ is a maximal ideal of $\mathbb{R}[T]$ since $\mathbb{R}[T]/\langle T^2 + 1 \rangle \cong \mathbb{C}$, however $\langle T^2 + 1 \rangle$ does not come from a point in $\mathbb{R}$ since $T^2 + 1$ doesn't even vanish on $\mathbb{R}$. Thus there are more maximal ideals in $\mathbb{R}[T]$ than just the ones which correspond to points in $\mathbb{R}$ (there are more maximals ideals in $\mathbb{R}[T]$ than just those of the form $\mathfrak{m}_x = \langle T - x \rangle$ where $x \in \mathbb{R}$). On the other hand, the Nullstellensatz guarentees that for all maximal ideals $\mathfrak{m}$ of $\mathbb{R}[T]$, we will have either $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{C}$ or $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{R}$. In the second case, we will have $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in \mathbb{R}$, and in the first case, we will have $\mathfrak{m} = \mathfrak{m}_{z,\overline{z}} = \langle (T - z)(T - \overline{z}) \rangle$ where z is a complex number in the upper-half plane ($\mathrm{Im}(z) > 0$).

22.3 The Correspondence Between Radical Ideals and Affine Algebraic Sets

Now we shall focus on the consequences of Hilbert's Nullstellensatz when working over K (not necessarily algebraically closed). To do this, we introduce the following notation: let Z be a subset of $\mathbb{A}^n(K)$. We denote by $\mathrm{I}_K(Z)$ to be the set of all functions which vanish on Z:

$$\mathrm{I}_K(Z) = \{f \in K[\boldsymbol{T}] \mid f(\boldsymbol{x}) = 0 \text{ for all } x \in Z\}.$$

If the underlying field K is understood from context, then we will simplify our notation and write $\mathrm{I}(Z)$ instead of $\mathrm{I}_K(Z)$. It is easy to see that $\mathrm{I}(Z)$ is an ideal of $K[\boldsymbol{T}]$, thus it is also called the ideal of functions which vanish on Z. Furthemore, since $f(\boldsymbol{x}) = 0$ if and only if $f \in \mathfrak{m}_{\boldsymbol{x}}$, we have

$$\mathrm{I}(Z) = \bigcap_{\boldsymbol{x} \in Z} \mathfrak{m}_{\boldsymbol{x}}.$$

Now let A be a finitely generated K-algebra and let $\mathfrak{a}$ be an ideal of A. Then we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{a} \subseteq \mathfrak{p} \subset A \\ \mathfrak{p} \text{ prime ideal}}} \mathfrak{p} = \bigcap_{\substack{\mathfrak{a} \subseteq \mathfrak{m} \subset A \\ \mathfrak{m} \text{ maximal ideal}}} \mathfrak{m}.$$

Indeed, the first equality holds in arbitrary commutative rings and the second equality follows from Hilbert's Nullstellensatz. We shall use this fact to prove the following proposition:

Proposition 22.2. *Let $\mathfrak{a}$ be an ideal of $\overline{K}[\boldsymbol{T}]$ and let $Z \subseteq \mathbb{A}^n(\overline{K})$ be a subset. Then*

1. $\mathrm{IV}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$
2. $\mathrm{VI}(Z) = \overline{Z}$ *where $\overline{Z}$ is the closure of Z in $\mathbb{A}^n(\overline{K})$.*

Proof. 1 We have

$$\begin{aligned} \mathrm{IV}(\mathfrak{a}) &= \bigcap_{\boldsymbol{x} \in \mathrm{V}(\mathfrak{a})} \mathfrak{m}_{\boldsymbol{x}} \\ &= \bigcap_{\substack{\mathfrak{m} \supseteq \mathfrak{a} \\ \text{maximal ideal}}} \mathfrak{m} \\ &= \sqrt{\mathfrak{a}}. \end{aligned}$$

2. This is a simple assertion for which we do not need the Nullstellensatz. On the one hand we have $Z \subseteq \mathrm{VI}(Z)$ and $\mathrm{VI}(Z)$ is closed. This shows $\mathrm{VI}(Z) \supseteq \overline{Z}$. On the other hand let $V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ be a closed subset that contains Z. Then we have $f(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in Z$ and $f \in \mathfrak{a}$. This shows $\mathfrak{a} \subseteq \mathrm{I}(Z)$ and hence $\mathrm{VI}(Z) \subseteq \mathrm{V}(\mathfrak{a})$.

□

22.4 Changing the Underlying Field

Let L/K be an extension of fields and let V be an affine algebraic subset of $\mathbb{A}^n(L)$. We say V is **defined over** K if $\mathrm{I}_L(V)$ can be generated by polynomials in $K[\boldsymbol{T}]$, or equivalently, if $\mathrm{I}_L(V) = \mathrm{I}_K(V)L[\boldsymbol{T}]$. The set of K**-rational points** of V is the set

$$V(K) = V \cap \mathbb{A}^n(K).$$

Now assume that L/K is a Galois extension and let $G = \mathrm{Gal}(L/K)$. There is a natural action of G on $\mathbb{A}^n(L)$ by

$$\sigma(\boldsymbol{x}) = \sigma(x_1, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n))$$

for all $\sigma \in G$ and $\boldsymbol{x} \in \mathbb{A}^n(L)$. If $f \in K[\boldsymbol{T}]$ and $\boldsymbol{x} \in \mathbb{A}^n(L)$, then for any $\sigma \in G$, we have

$$\sigma(f(\boldsymbol{x})) = f(\sigma(\boldsymbol{x})).$$

In particular, if V is defined over K, then the action of G on $\mathbb{A}^n(L)$ induces an action on V, and clearly

$$V(K) = \{\boldsymbol{x} \in V \mid \sigma(\boldsymbol{x}) = \boldsymbol{x} \text{ for all } \sigma \in G\}.$$

Example 22.5. Consider $L = \overline{\mathbb{Q}}$ and $K = \mathbb{Q}$. Let $n \geq 3$ and let $V = \mathrm{V}_{\overline{\mathbb{Q}}}(T_1^n + T_2^n - 1)$. Clearly V is defined over $\mathbb{Q}$. Fermat's last theorem, proven by Andrew Wiles in 1995, states that

$$V(\mathbb{Q}) = \begin{cases} \{(1,0),(0,1)\} & \text{if } n \text{ is odd.} \\ \{(\pm 1,0),(0,\pm 1)\} & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, $\mathrm{V}(\overline{\mathbb{Q}})$ has infinitely many points. Let us try to describe these points using a fixed embedding $\overline{\mathbb{Q}} \subseteq \mathbb{C}$. For each $z \in \mathbb{C}$, let $z = re^{i\theta}$ be its polar representation where $r > 0$ and $\theta \in (-\pi, \pi]$. We set $z^{1/n} = r^{1/n}e^{i\theta/n}$. We also set $\zeta_n = e^{2\pi i/n}$. In particular, given $\alpha \in \overline{\mathbb{Q}} \subseteq \mathbb{C}$, we have

$$\mathrm{V}(\overline{\mathbb{Q}}) = \begin{cases} (\alpha, \zeta_n^k(1-\alpha^n)^{1/n}) & 1 \leq k \leq n \text{ if } 1 - \alpha^n \neq 0 \\ (\alpha, 0) & \text{else} \end{cases}$$

22.5 Morphisms of Affine Algebraic Sets

Having defined affine algebraic sets, we now wish to define morphisms between them.

Definition 22.1. Let $X = \mathrm{V}(I)$ be an affine algebraic subset of $\mathbb{A}^m(K)$, let $Y = \mathrm{V}(J)$ be an affine algebraic subset of $\mathbb{A}^n(K)$, and let $f\colon X \to Y$ be a function. Then there exists functions $f_1, \dots, f_n\colon \mathbb{A}^m(K) \to \mathbb{A}^1(K)$ such that for each $\boldsymbol{x} = (x_1, \dots, x_m)$ in X, we have

$$f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_n(\boldsymbol{x})) = (y_1, \dots, y_n) = \boldsymbol{y}$$

where $\boldsymbol{y} = (y_1, \dots, y_n) \in Y$. We call the n-tuple $(f_1, \dots, f_n)$ a **representation** of f. The f_j are called the **components** of this representation. Note that this representation need not be unique: there may exist a different n-tuple of functions $(\widetilde{f}_1, \dots, \widetilde{f}_n)$ which represents f as well. We say $(f_1, \dots, f_n)$ is a **polynomial representation** of f if each component function is a polynomial in $K[\boldsymbol{x}] = K[x_1, \dots, x_m]$. We say f is a **morphism** if there exists a polynomial representation of f. We say $f\colon X \to Y$ is an **isomorphism** if there exists a morphim $g\colon Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. The set of all morphisms from X to Y is denoted $\mathrm{Hom}(X, Y)$.

Remark 31. To say that f is a morphism from $X \subseteq \mathbb{A}^m_K$ to $Y \subseteq \mathbb{A}^n_K$ represented by $(f_1, \dots, f_n)$ means that $(f_1(x), \dots, f_n(x))$ must satisfy the defining equations of Y for all points $x \in X$.

22.5.1 Examples of morphisms

Example 22.6. Let $X = \mathbb{A}^1(K)$, let $Y = \mathrm{V}(y_2^2 - y_1)$, and let $f\colon X \to Y$ be defined by $f(x) = (x^2, x) = \boldsymbol{y}$ for all $x \in X$. Then f is a morphism since it is represented by the polynomials $f_1 = x^2$ and $f_2 = x$ in $K[x]$ and since it lands in Y: for all $x \in X$ we have $f(x) \in Y$ since $(x)^2 - x^2 = 0$. Now define $g\colon Y \to X$ by $g(\boldsymbol{y}) = y_2$ for all $\boldsymbol{y} = (y_1, y_2)$ in Y. Then g is a morphism since it is represented by the polynomial $g = y_2$ and since it clearly lands in X. Moreover, observe that for all $\boldsymbol{y} = (y_1, y_2)$ in Y, we have

$$\begin{aligned}(f \circ g)(\boldsymbol{y}) &= f(y_2) \\ &= (y_2^2, y_2) \\ &= (y_1, y_2) \\ &= \boldsymbol{y},\end{aligned}$$

where we used the fact that $\boldsymbol{y} \in Y$ to get from the second line to the third line. A similar calculation shows $(g \circ f)(x) = x$ for all $x \in X$. It follows that $f\colon X \to Y$ is an isomorphism with $g\colon Y \to X$ being its inverse.

Example 22.7. Let $X = \mathrm{V}(x_2 - x_1^2, x_3 - x_1^3) = \mathrm{V}(p_1, p_2)$, let $Y = \mathrm{V}(y_2 - y_1 - y_1^2) = \mathrm{V}(q)$, and let $f\colon X \to Y$ be defined by $f(\boldsymbol{x}) = (x_1x_2, x_1^2x_2^2 + x_3) = \boldsymbol{y}$ for all $\boldsymbol{x} = (x_1, x_2, x_3) \in X$. Then f is represented by the polynomials $f_1 = x_1x_2$ and $f_2 = x_1^2x_2^2 + x_3$, thus to see if f is a morphism, we just need to check that f lands in Y. To see this, we must check that the polynomial $q = y_2 - y_1 - y_1^2$ vanishes at $\boldsymbol{y} = f(\boldsymbol{x})$: we have

$$\begin{aligned}q(\boldsymbol{y}) &= y_2 - y_1 - y_1^2 \\ &= x_1^2x_2^2 + x_3 - x_1x_2 - (x_1x_2)^2 \\ &= x_3 - x_1x_2 \\ &= x_3 - x_1^3 \\ &= 0\end{aligned}$$

where we used the fact that $\boldsymbol{x} \in X$ to get from the third line to the fifth line. Thus f is in fact a morphism of affine algebraic sets. Note that the morphism f induces a homomorphism of K-algebras $f^*\colon K[\boldsymbol{y}]/\langle q \rangle \to K[\boldsymbol{x}]/\langle p_1, p_2 \rangle$ where f^* is the unique K-algebra homomorphism such that $f^*(y_1) = f_1$ and $f^*(y_2) = f_2$, that is,

$$\begin{aligned}f^*(y_1) &= x_1x_2 \\ f^*(y_2) &= x_1^2x_2^2 + x_3.\end{aligned}$$

Example 22.8. Let $X = \mathbb{A}^1(K)$, let $Y = \mathrm{V}(T_2'^2 - T_1')$, and let $f\colon X \to Y$ be defined by $f(x) = (x^2, x)$ for all $x \in X$. Then f is a morphism since it is represented by the polynomials $f_1 = T$ and $f_2 = T^2$ and since it lands in Y: for all $x \in X$ we have $f(x) \in Y$ since $(x)^2 - x^2 = 0$. Now define $g\colon Y \to X$ by $g(y) = g(y_1, y_2) = y_2$ for all $y = (y_1, y_2)$ in Y. Then g is a morphism since it is represented by the polynomial $g = T_2'$ and since it clearly lands in X. Moreover, observe that for all $y = (y_1, y_2)$ in Y, we have

$$\begin{aligned}(f \circ g)(y) &= f(g(y)) \\ &= f(y_2) \\ &= (y_2^2, y_2) \\ &= (y_1, y_2) \\ &= y \\ &= 1_Y(y).\end{aligned}$$

where we used the fact that $y \in Y$ to get from the third line to the fourth line. Similarly, for all $x \in X$, we have

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2, x) \\ &= x \\ &= 1_X(x).\end{aligned}$$

It follows that $f\colon X \to Y$ is an isomorphism of affine algebraic sets.

Example 22.9. Let $X = \mathrm{V}(T_2 - T_1^2, T_3 - T_1^3) = \mathrm{V}(p_1, p_2)$, let $Y = \mathrm{V}(T_2' - T_1' - T_1'^2) = \mathrm{V}(q)$, and let $f\colon X \to Y$ be defined by

$$f(x) = (x_1x_2, x_1^2x_2^2 + x_3) = (y_1, y_2)$$

for all $x = (x_1, x_2, x_3)$ in X. Then f is represented by the polynomials $f_1 = T_1T_2$ and $f_2 = T_1^2T_2^2 + T_3$, thus to see if f is a morphism, we only need to check that f lands in Y. To see this, we must check that the polynomial $q = T_2' - T_1' - T_1'^2$ vanishes at $y = (y_1, y_2)$: we have

$$\begin{aligned}q(y) &= y_2 - y_1 - y_1^2 \\ &= x_1^2x_2^2 + x_3 - x_1x_2 - (x_1x_2)^2 \\ &= x_3 - x_1x_2 \\ &= x_3 - x_1^3 \\ &= 0\end{aligned}$$

where we used $p_1(x) = 0 = p_2(x)$ to get from the third line to the fifth line. Thus f is in fact a morphism of affine algebraic sets. Note that the morphism f induces a homomorphism

$$f^*\colon K[T_1', T_2']/\langle T_2' - T_1' - T_1'^2\rangle \to K[T_1, T_2, T_3]/\langle T_2 - T_1^2, T_3 - T_1^3\rangle$$

of K-algebras, where f^* is the unique K-algebra homomorphism such that

$$\begin{aligned}f^*(T_1') &= T_1T_2 \\ f^*(T_2') &= T_1^2T_2^2 + T_3.\end{aligned}$$

Example 22.10. The map $\mathbb{A}^1(K) \to V(T_2^2 - T_1^2(T_1 + 1))$, given by $x \mapsto (x^2 - 1, x(x^2 - 1))$, is a morphism of affine algebraic sets. For $\mathrm{char}(K) \neq 2$, it is not bijective: 1 and -1 are both mapped to the origin $(0, 0)$. In $\mathrm{char}(K) = 2$, it is bijective but not an isomorphism.

Example 22.11. Let $X = V(1 - T_1T_2)$, let $Y = \mathbb{A}^1(K)$, and let $f\colon X \to Y$ be defined by $f(x) = x_1$ for all $x = (x_1, x_2)$ in X. Then $f(X) = \mathbb{A}^1(K)\backslash\{0\}$ is not an algebraic set. This shows that the image of an algebraic set is not necessarily an algebraic set.

Example 22.12. Let $f = x^2 + 4x + 1$. By completing the square, we see that

$$\begin{aligned}f &= x^2 + 4x + 1 \\ &= (x + 2)^2 - 3 \\ &= \widetilde{x}^2 - 3 \\ &= \widetilde{f},\end{aligned}$$

where we set $\widetilde{x} = x + 2$ and $\widetilde{f} = \widetilde{x}^2 - 3$. Now set $V = \mathrm{V}_{\mathbb{R}}(f)$ and set $\widetilde{V} = \mathrm{V}_{\mathbb{R}}(\widetilde{f})$. Thus we have

$$V = \{-2 - \sqrt{3}, -2 + \sqrt{3}\} \quad \text{and} \quad \widetilde{V} = \{-\sqrt{3}, \sqrt{3}\}.$$

Clearly we have a morphism $V \to \widetilde{V}$ given by $x \mapsto x + 2 := \widetilde{x}$ and similarly a morphism $\widetilde{V} \to V$ given by $\widetilde{x} \mapsto \widetilde{x} - 2 := x$. These maps are inverse to each other, so $V \cong \widetilde{V}$.

The notion of an affine algebraic set is still not satisfactory. We list three problems:

- Open subsets of affine algebraic sets do not carry the structure of an affine algebraic set in a natural way. In particular, we cannot glue affine algebraic sets along open subsets (although this is a "natural operation" for geometric objects.
- Intersections of affine algebraic sets in $\mathbb{A}^n(k)$ are closed and hence again affine algebraic sets. But we cannot distinguish between $\mathrm{V}(X) \cap \mathrm{V}(Y) \subset \mathbb{A}^2(k)$ and $\mathrm{V}(Y) \cap \mathrm{V}(X^2 - Y) \subset \mathbb{A}^2(k)$ although the geometric situation seems to be different.
- Affine algebraic sets seem not to help in studying solutions of polynomial equations in more general rings than algebraically closed fields.

22.5.2 Morphisms are continuous with respect to the Zariski topology

We now want to show that morphisms are continuous with respect to the Zariski topology. We first need a lemma:

Lemma 22.3. *Let Y be an affine algebraic subset of $\mathbb{A}^n(K)$ and let $f\colon \mathbb{A}^m(K) \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Suppose $Y = \mathrm{V}(p_1, \dots, p_r)$ and let Z be a closed subset of Y. Then Z has the form

$$Z = \mathrm{V}(p_1, \dots, p_r) \cap \mathrm{V}(q_1, \dots, q_s) = \mathrm{V}(p_1, \dots, p_r, q_1, \dots, q_s).$$

where $\mathrm{V}(q_1, \dots, q_s)$ is another closed subset of $\mathbb{A}^n(K)$. In particular,

$$f^{-1}(Z) = \mathrm{V}(p_1 \circ f, \dots, p_r \circ f, q_1 \circ f, \dots, q_s \circ f)$$

is a closed subset of $\mathbb{A}^m(K)$. □

Proposition 22.3. *Let X be an affine algebraic subset of $\mathbb{A}^m(K)$, let Y be an affine algebraic subset of $\mathbb{A}^n(K)$, and let $f\colon X \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Lift f to a morphism $\widetilde{f}\colon \mathbb{A}^m(K) \to Y$ such that $\widetilde{f}|_X = f$ (choosing a lift of f is equivalent to choosing a polynomial representation $(f_1, \dots, f_n)$ of f). By Lemma (22.3), $\widetilde{f}$ is continuous. Therefore its restriction $\widetilde{f}|_X = f$ must be continuous also. □

Example 22.13. Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic sets and let $\pi_i : \mathbb{A}^n(k) \to \mathbb{A}^1(k)$ be the projection to the ith coordinate map (i.e. $\pi_i(y_1, \dots, y_i, \dots, y_n) = y_i$). Then $\pi_i|_X$ is continuous with respect to the Zariski topology. Let us show this directly: let $\{z_1, \dots, z_n\}$ be a closed subset of $\mathbb{A}^1(k)$ where the z_j are distinct points in $\mathbb{A}^1(k)$ (every closed subset in $\mathbb{A}^1(k)$ is just a finite set of points). Then

$$\pi_i|_X^{-1}(\{z_1, \dots, z_n\}) = X \cap \left(\bigcap_{j=1}^{n} V(\pi_i - z_j) \right).$$

Thus the inverse image a closed subset in $\mathbb{A}^1(k)$ is a closed subset in X.

22.5.3 Maps which are continuous with respect to the Zariski topology are not necessarily morphisms

A continuous map with respect to the Zariski topology does not have to be a morphism. Indeed, consider the complex conjugation map $\bar{\cdot} : \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$. This map is continuous with respect to the Zariski topology. To see why, let $V(p_1, \dots, p_r)$ be a closed subset of $\mathbb{A}^1(\mathbb{C})$. Then the inverse image of $V(p_1, \dots, p_r)$ under $\bar{\cdot}$ is the closed subset $V(\overline{p}_1, \dots, \overline{p}_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $\overline{p}_i = \sum_{j=1}^{n_i} \overline{a}_{n_j} z^{n_j}$. On the other hand, $\bar{\cdot}$ does not have a polynomial representation: the only root of $\bar{\cdot}$ is $z = 0$, but $\bar{\cdot} \neq T^m$ for any $m \in \mathbb{N}$.

More generally, if L/K galois extension with galois group $G = \mathrm{Gal}(L/K)$. Then for all $g \in G$ the map $g\cdot : \mathbb{A}^1(L) \to \mathbb{A}^1(L)$, given by $x \mapsto g \cdot x$, is Zariski continuous because the inverse image of a closed subset $V(p_1, \dots, p_r)$ of $\mathbb{A}^1(L)$ under $g\cdot$ is the closed subset $V(g \cdot p_1, \dots, g \cdot p_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $g \cdot p_i = \sum_{j=1}^{n_i} (g \cdot a_{n_j}) z^{n_j}$. On the other hand, $g\cdot$ does not have a polynomial representation: the only root of $g\cdot$ is $z = 0$, but $g\cdot \neq T^m$ for any $m \in \mathbb{N}$. Note that $g\cdot$ gives rise to a K-algebra homomorphism $\Gamma(g\cdot) : L[T] \to L[T]$ (and not an L-algebra homomorphism).

Remark 32. One may wonder why we restrict our morphisms between affine algebraic sets in the first place. Why do we not consider $\mathrm{Hom}(X, Y)$ to be the set of all Zariski-continuous maps? The point is that the category of affine algebraic sets are naturally thought of as being objects in the category of locally ringed spaces (we will define what these are later on) rather than just in the category of topological spaces.

22.6 Affine Algebraic Sets as Reduced Finitely-Generated K-Algebras

We make the following definitions.

Definition 22.2. Let $X \subseteq \mathbb{A}^n(K)$ be an affine algebraic set.

1. The **affine coordinate ring** of X, denoted $\Gamma(X)$, is the K-algebra
$$\Gamma(X) := K[\boldsymbol{T}]/\mathrm{I}(X).$$
Notice that $\mathrm{Hom}(X, \mathbb{A}^1(K))$ has the structure of a K-algebra in the natural way, where addition and multiplication are defined pointwise, and that $\Gamma(X) \cong \mathrm{Hom}(X, \mathbb{A}^1(K))$ as K-algebras. Thus $\mathrm{Hom}(X, \mathbb{A}^1(K))$ is an equivalent description of $\Gamma(X)$ which we shall often use.

2. Let $\boldsymbol{x} \in X$. We denote by $\mathfrak{m}_{X,\boldsymbol{x}}$ to be the maximal ideal of $\Gamma(X)$ given by
$$\mathfrak{m}_{X,\boldsymbol{x}} = \{f \in \Gamma(X) \mid f(\boldsymbol{x}) = 0\}.$$
Often we simplify our notation (when context is clear) and write $\mathfrak{m}_{\boldsymbol{x}}$ instead of $\mathfrak{m}_{X,\boldsymbol{x}}$.

3. Let $\mathfrak{a}$ be an ideal of $\Gamma(X)$. We denote by $\mathrm{V}_X(\mathfrak{a})$ to be the subset of X given by
$$\mathrm{V}_X(\mathfrak{a}) = \{\boldsymbol{x} \in \mathrm{X} \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in \mathfrak{a}\}.$$
Let $\widetilde{\mathfrak{a}}$ be an ideal of $K[\boldsymbol{T}]$ which lifts the ideal $\mathfrak{a}$ with respect to the surjective map $\pi \colon K[\boldsymbol{T}] \to \Gamma(X)$. Then observe that $\mathrm{V}_X(\mathfrak{a}) = X \cap \mathrm{V}(\widetilde{\mathfrak{a}})$. In particular, the $\mathrm{V}_X(\mathfrak{a})$ (as $\mathfrak{a}$ ranges) are the closed sets in the subspace topology X in $\mathbb{A}^n(K)$. We again call this subspace topology the **Zariski topology** of X. Often it is clear from context that we are working in X, so we will often simplify our notation and write $\mathrm{V}(\mathfrak{a})$ instead of $\mathrm{V}_X(\mathfrak{a})$.

4. For each $f \in \Gamma(X)$, we denote by $\mathrm{D}_X(f)$ to be the subset of X given by
$$\mathrm{D}_X(f) = \{\boldsymbol{x} \in \mathrm{X} \mid f(\boldsymbol{x}) \neq 0\}.$$
These are principal open subsets of X, which we call the **principal open subsets** of X. Again we often simplify our notation by writing $\mathrm{D}(f)$ instead of $\mathrm{D}_X(f)$.

Lemma 22.4. *Let $X \subseteq \mathbb{A}^n(K)$ be an affine algebraic set. The open sets $\mathrm{D}(f)$, for $f \in \Gamma(X)$, form a basis of the topology (i.e. finite intersections of principal open subsets are again principal open subsets and for every open subset $U \subseteq X$ there exist $f_i \in \Gamma(X)$ with $U = \bigcup_i D(f_i)$.*

Proof. Clearly we have $D(f) \cap D(g) = D(fg)$ for $f, g \in \Gamma(X)$. It remains to show that every open subset U is a union of principal open subsets. We write $U = X \backslash V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. For generators $f_1, \ldots, f_n$ of this ideal we find $V(\mathfrak{a}) = \bigcap_{i=1}^n V(f_i)$, and hence $U = \bigcup_{i=1}^n D(f_i)$. □

Remark 33. Let $f : X \to Y$ be a morphism of affine algebraic sets. Then f is continuous with respect to the Zariski topology. Indeed, if $D(g)$ is a basic open set in Y, then $f^{-1}(D(g)) = D(f^*g)$.

Proposition 22.4. *Let X be an affine algebraic set. The affine coordinate ring $\Gamma(X)$ is a reduced finitely generated k-alebra. Moreover, X is irreducible if and only if $\Gamma(X)$ is an integral domain.*

Proof. As $\Gamma(X) = k[T_1, \ldots, T_n]/I(X)$, it is a finitely generated k-algebra. As $I(X) = \sqrt{I(X)}$, we find that $\Gamma(X)$ is reduced. Also, X is irreducible if and only if $I(X)$ is prime if and only if $\Gamma(X)$ is an integral domain. □

Proposition 22.5. *Let $f : X \to Y$ be a morphism between affine algebraic sets. Then*

1. *$f(X)$ is dense in Y if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is injective.*

2. *$f(X) \subset Y$ is a closed subvariety and $f : X \to f(X)$ is an isomorphism if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is surjective.*

Proof.

1. First assume that $f(X)$ is dense in Y. Suppose that $f^*h = f^*g$ where $g, h \in \Gamma(Y)$. Then for all $x \in X$, we have $h(f(x)) = g(f(x))$. Or in other words, we have $(h - g)(y) = 0$ for all $y \in f(X)$. Since $f(X)$ is dense in Y and $h - g$ is continuous, we must therefore have $(h - g)(y) = 0$ for all $y \in Y$. Thus, $h = g$, which shows that f^* is injective. Conversely, assume that f^* is injective. Suppose $f(X)$ is not dense in Y. Denote $Z := \overline{f(X)}$ and pick $y \in Y$ such that $y \notin Z$. Then $Z \subset Y$ implies $I(Z) \supset I(Y)$. Thus, we can find an $g \in I(Z)$ such that $g \notin I(Y)$. This means $g(z) = 0$ for all $z \in Z$ and there exists $y \in Y$ such that $g(y) \neq 0$. But f^* is injective, $f^*g = f^*0$ implies $g = 0$, which is a contradiction.

$\square$

Remark 34. We say $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y.

22.6.1 Equivalence of Categories Between Affine Algebraic Sets and Reduced Finitely Generated k-Algebras

Let $f : X \to Y$ be a morphism of affine algebraic sets. The map

$$\Gamma(f) : \mathrm{Hom}(Y, \mathbb{A}^1(k)) \to \mathrm{Hom}(X, \mathbb{A}^1(k)),$$

given by $g \mapsto f^*g := g \circ f$, defines a homomorphism of k-algebras. We obtain a functor

$$\Gamma : (\text{affine algebraic sets})^{\mathrm{opp}} \to (\text{reduced finitely generated } k\text{-algebras}).$$

Proposition 22.6. *The functor Γ induces an equivalence of categories. By restriction one obtains an equivalence of categories*

$$\Gamma : \textit{(irreducible affine algebraic sets)}^{opp} \to \textit{(integral finitely generated k-algebras)}.$$

Proof. A functor induces an equivalence of categories if and only if it is fully faithful and essentially surjective. We first show that Γ is fully faithful, i.e. that for affine algebraic sets $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$, the map $\Gamma : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\Gamma(Y), \Gamma(X))$ is bijective. We define an inverse map. If $\varphi : \Gamma(Y) \to \Gamma(X)$ is given, there exists a k-algebra homomorphism $\widetilde{\varphi}$ that makes the following diagram commutative

$$\begin{tikzcd} k[T_1', \dots, T_m'] \arrow[r, "\widetilde{\varphi}"] \arrow[d] & k[T_1, \dots, T_n] \arrow[d] \\ \Gamma(Y) \arrow[r, "\varphi"] & \Gamma(X) \end{tikzcd}$$

We define $f : X \to Y$ by

$$f(x) := (\widetilde{\varphi}(T_1')(x), \dots, \widetilde{\varphi}(T_n')(x))$$

and obtain the desired inverse homomorphism.

It remains to show that the functor is essentially surjective, i.e. that for every reduced finitely generated k-algebra A there exists an affine algebraic set X such that $A \cong \Gamma(X)$. By hypothesis, A is isomorphic to $k[T_1, \dots, T_n]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $k[T_1, \dots, T_n]$ with $\mathfrak{a} = \sqrt{\mathfrak{a}}$. If we set $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$, we have

$$\Gamma(X) = k[T_1, \dots, T_n]/I(V(\mathfrak{a})) = k[T_1, \dots, T_n]/\mathfrak{a}.$$

$\square$

Remark 35. Let $X \subseteq \mathbb{A}^m(K)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic sets and let $f : X \to Y$ whose components are f_i for $i = 1, \dots, m$. Write the affine coordinate rings of X and Y as $\Gamma(X) = K[T_1, \dots, T_m]/I(X)$ and $\Gamma(Y) = K[T_1', \dots, T_n']/I(Y)$. Then $\Gamma(f)(T_i) := T_i \circ f = f_i$ for all $i = 1, \dots, m$. Indeed, for all points $x \in X$, we have

$$\begin{aligned} \Gamma(f)(T_i)(x) &= T_i(f(x)) \\ &= T_i(f_1(x), \dots, f_i(x), \dots, f_n(x)) \\ &= f_i(x). \end{aligned}$$

Using the bijective correspondence between points of affine algebraic sets X and maximal ideals of $\Gamma(X)$, we also have the following description of morphisms.

Proposition 22.7. *Let $f : X \to Y$ be a morphism of affine algebraic sets and let $\Gamma(f) : \Gamma(Y) \to \Gamma(X)$ be the corresponding homomorphism of the affine coordinate rings. Then $\Gamma(f)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ for all $x \in X$.*

Proof. This follows from $g(f(x)) = \Gamma(f)(g)(x)$ for all $g \in \Gamma(Y) = \mathrm{Hom}(Y, \mathbb{A}^1(k))$. $\square$

22.7 Affine Algebraic Sets as Spaces with Functions

We will now define the notion of a **space with functions**. For us this will be the prototype of a "geometric object". It is a special case of a so-called ringed space on which the notion of a scheme will be based on.

Definition 22.3.

1. A **space with functions over** K is a topological space X together with a family $\mathcal{O}_X$ of K-subalgebras $\mathcal{O}_X(U) \subseteq \operatorname{Map}(U, K)$ for every open subset $U \subseteq X$ that satisfy the following properties:
   (a) If $U' \subseteq U \subseteq X$ are open and $f \in \mathcal{O}_X(U)$, then the restriction $f|_{U'} \in \operatorname{Map}(U', K)$ is an element of $\mathcal{O}_X(U')$.
   (b) Given an open covering $\{U_i\}_{i \in I}$ of an open subset U of X and elements $f_i \in \mathcal{O}_X(U_i)$ such that
   $$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$
   for all $i, j \in I$, then there exists a unique function $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.
2. A **morphism** $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of spaces with functions is a continuous map $g : X \to Y$ such that for all open subsets V of Y and functions $f \in \mathcal{O}_Y(V)$, the function $g^* f := f \circ g|_{g^{-1}(V)} : g^{-1}(V) \to K$ lies in $\mathcal{O}_X(g^{-1}(V))$.

Clearly spaces with functions over K form a category.

Definition 22.4. Let X be a space with functions and let U be an open subset of X. We denote by $(U, \mathcal{O}_{X|U})$ the space U with functions
$$\mathcal{O}_{X|U}(V) = \mathcal{O}_X(V)$$
for $V \subseteq U$ open.

22.7.1 The Space with Functions of an Irreducible Affine Algebraic Set

Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. It is endowed with the Zariski topology and we want to define for every open subset $U \subseteq X$ a k-algebra of functions $\mathcal{O}_X(U)$ such that $(X, \mathcal{O}_X)$ is a space with functions.

As X is irreducible, the k-algebra $\Gamma(X)$ is a domain, and by definition all the sets $\mathcal{O}_X(U)$ will be k-subalgebras of its field of fractions.

Definition 22.5. The field of fractions $K(X) := \operatorname{Frac}(\Gamma(X))$ is called the **function field** of X.

If we consider $\Gamma(X)$ as the set of morphisms $X \to \mathbb{A}^1(k)$, elements of the function field f/g, where $f, g \in \Gamma(X)$ and $g \neq 0$, usually do not define functions on X because the denominator may have zeros on X, but certainly f/g defines a function $D(g) \to \mathbb{A}^1(k)$[1] We will use functions of this kind to make X into a space with functions.

Lemma 22.5. *Let X be an irreducible affine algebraic set and let f_1/g_1 and f_2/g_2 be elements of $K(X)$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$ if and only if there exists a non-empty open subset $U \subseteq D(g_1 g_2)$ with*
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in U$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$.

Proof. First suppose $f_1/g_1 = f_2/g_2$ in $K(X)$. This means $f_1 g_2 = f_2 g_1$ in $\Gamma(X)$. In particular,
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in D(g_1 g_2)$. Conversely, let $U \subseteq D(g_1 g_2)$ be a non-empty open subset such that
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in U$. Then the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. □

Proof. We have $(f_1 g_2 - f_2 g_1)(x) = 0$ for all $x \in U$. Therefore the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. □

[1] It might be even defined on a bigger open subset of X as there exist representations of the fraction with different denominators.

Definition 22.6. Let X be an irreducible affine algebraic set and let $U \subseteq X$ be open. We denote by $\mathfrak{m}_x$ the maximal ideal of $\Gamma(X)$ corresponding to $x \in X$ and by $\Gamma(X)_{\mathfrak{m}_x}$ the localization of the affine coordinate ring with respect to $\mathfrak{m}_x$. We define

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \Gamma(X)_{\mathfrak{m}_x} \subset K(X).$$

The localization $\Gamma(X)_{\mathfrak{m}_x}$ can be described in this situation as the union

$$\Gamma(X)_{\mathfrak{m}_x} = \bigcup_{f \in \Gamma(X) \setminus \mathfrak{m}_x} \Gamma(X)_f \subset K(X).$$

Remark 36. Note that

$$\Gamma(X)_{\mathfrak{m}_x} = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \right\}.$$

Indeed, $g(x) \neq 0$ is equivalent to $g \notin \mathfrak{m}_x$. It may be tempting to think that

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \text{ for all } x \in U \right\},$$

but this is not necessarily the case. For instance, let $X \subset \mathbb{A}^4$ be the variety defined by the equation $T_1 T_4 = T_2 T_3$. Then $T_1/T_2 \in \mathcal{O}_X(D(T_2))$ and $T_3/T_4 \in \mathcal{O}_X(D(T_4))$ and by the equation of X, these two functions coincide where they are both defined;

$$\frac{T_1}{T_2}\Big|_{\mathrm{D}(T_2 T_4)} = \frac{T_3}{T_4}\Big|_{\mathrm{D}(T_2 T_4)}$$

So this gives rise to a regular function on $\mathrm{D}(T_2) \cup \mathrm{D}(T_4)$, but there is no representation of this function as a quotient of two polynomials in $K[T_1, T_2, T_3, T_4]$ that works on all of $\mathrm{D}(T_2) \cup \mathrm{D}(T_4)$; we have to use different representations at different points. On the other hand, it is true that

$$\mathcal{O}_{\mathbb{A}^n(K)}(U) = \left\{ \frac{f}{g} \mid f, g \in K[\boldsymbol{T}] \text{ and } g(\boldsymbol{x}) \neq 0 \text{ for all } \boldsymbol{x} \in U \right\}.$$

For instance, let $X = \mathbb{A}^2(k)$ and $U = \mathbb{A}^2(k) \setminus \{0\}$. Suppose $f \in \mathcal{O}_X(U)$ and $x \in U$. Since $f \in O_{X,p}$, we can write

$$f|_{D(g_1)} = \frac{f_1}{g_1},$$

where $g_1(x) \neq 0$. We may assume f_1 and g_1 share no common factors. If g_1 is not a constant, then there exists another point $y \in U$ such that $g_1(y) = 0$. Since $f \in \mathcal{O}_{X,y}$, we must be able to write

$$f|_{D(g_2)} = \frac{f_2}{g_2},$$

where $g_2(y) \neq 0$. This implies

$$\frac{f_1}{f_2}|_{D(g_1 g_2)} = f = \frac{f_2}{g_2}|_{D(g_1 g_2)}.$$

Thus, $f_1/g_1 = f_2/g_2$ in $K(X)$. But the only way we can have $f_1/g_1 = f_2/g_2$ is if $g_1 = hf_1$ and $g_2 = hf_2$, where $h \in k[T_1, T_2]$.[a] But this implies $g_2(y) = h(y) f_2(y) = 0$, which is a contradiction.

[a]This is related to the fact that $\langle g_1, g_2 \rangle$ has depth 2.

To consider $(X, \mathcal{O}_X)$ as a space with functions, we first have to explain how to identify elements $f \in \mathcal{O}_X(U)$ with functions $U \to k$. Given $x \in U$, the element f is by definition in $\Gamma(X)_{\mathfrak{m}_x}$ and we may write $f = g/h$ where $g, h \in \Gamma(X)$ and $h \notin \mathfrak{m}_x$. But then $h(x) \neq 0$ and we may set $f(x) := g(x)/h(x) \in k$. The value of $f(x)$ is well defined and Lemma (22.5) implies that this construction defines an injective map $\mathcal{O}_X(U) \to \mathrm{Map}(U, k)$.

If $V \subseteq U \subseteq X$ are open subsets we have $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V)$ by definition and this inclusion corresponds via the identification with maps $U \to k$ resp. $V \to k$ to the restriction of functions.

To show that $(X, \mathcal{O}_X)$ is a space with functions, we still have to show that we may glue functions together. But this follows immediately from the definition of $\mathcal{O}_X(U)$ as subsets of the function field $K(X)$. We call $(X, \mathcal{O}_X)$ the **space of functions associated with** X. Functions on principal open subsets $D(f)$ can be explicitely described as follows.

Proposition 22.8. *Let $(X, \mathcal{O}_X)$ be the space with functions associated to the irreducible affine algebraic set X and let $f \in \Gamma(X)$. Then there is an equality*
$$\mathcal{O}_X(D(f)) = \Gamma(X)_f$$
(as subsets of $K(X)$). In particular $\mathcal{O}_X(X) = \Gamma(X)$ (taking $f = 1$).

Proof. Clearly we have $\Gamma(X)_f \subset \mathcal{O}_X(D(f))$. Let $g \in \mathcal{O}_X(D(f))$. If we can show that $f^n g = h$, for some $n \in \mathbb{N}$ and $h \in \Gamma(X)$, then $g = h/f^n$ would show that $g \in \Gamma(X)_f$. To do this, we will work with ideals, because our argument will use Nullstellensatz which is a theorem about ideals. So set

$$\mathfrak{a} = \{q \in \Gamma(X) \mid qg \in \Gamma(X)\}.$$

Obviously $\mathfrak{a}$ is an ideal of $\Gamma(X)$ and we have to show that $f \in \operatorname{rad}\mathfrak{a}$. By Hilbert's Nullstellensatz we have $\operatorname{rad}\mathfrak{a} = I(V(\mathfrak{a}))$. Therefore it suffices to show $f(x) = 0$ for all $x \in V(\mathfrak{a})$. Let $x \in X$ be a point with $f(x) \neq 0$, i.e. $x \in D(f)$. As $g \in \mathcal{O}_X(D(f))$, we find $g_1, g_2 \in \Gamma(X)$ with $g_2 \notin \mathfrak{m}_x$ and $g = g_1/g_2$. Thus $g_2 \in \mathfrak{a}$ and as $g_2(x) \neq 0$ we have $x \notin V(\mathfrak{a})$. □

Remark 37.

1. Note that we needed to use Nullstellensatz here. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1(\mathbb{R})$, but not polynomial.

2. The proposition shows that we could have defined $(X, \mathcal{O}_X)$ also in another way, namely by setting

$$\mathcal{O}_X(D(f)) = \Gamma(X)_f \text{ for } f \in \Gamma(X).$$

   As the $D(f)$ for $f \in \Gamma(X)$ form a basis of the topology, the axiom of gluing implies that at most one such space with functions can exist. It would remain to show the existence of such a space (i.e. that for $f, g \in \Gamma(X)$ with $D(f) = D(g)$ we have $\Gamma(X)_f = \Gamma(X)_g$ and that gluing of functions is possible). This is more or less the same as the proof of Proposition (22.8). The way we chose is more comfortable in our situation. For affine schemes we will use the other approach.

Remark 38. If A is an integral finitely generated k-algebra we may construct the space with functions $(X, \mathcal{O}_X)$ of "the" corresponding irreducible affine algebraic set directly without choosing generators of A. Namely, we obtain X as the set of maximal ideals in A. Closed subsets of X are sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{m} \subset A \text{ maximal} \mid \mathfrak{m} \supseteq \mathfrak{a}\},$$

where $\mathfrak{a}$ is an ideal in A. For an open subset $U \subseteq X$ we finally define

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{m} \in U} A_{\mathfrak{m}} \subset \operatorname{Frac}(A).$$

This defines a space with functions $(X, \mathcal{O}_X)$ which coincides the space with functions of the irreducible affine algebraic set X corresponding in A. This approach is the point of departure for the definition of schemes.

22.7.2 The Functor from the Category of Irreducible Affine Algebraic Sets to the Category of Spaces with Functions

Proposition 22.9. *Let X, Y be irreducible affine algebraic sets and $f : X \to Y$ a map. The following assertions are equivalent.*

1. *The map f is a morphism of affine algebraic sets.*

2. *If $g \in \Gamma(Y)$, then $g \circ f \in \Gamma(X)$.*

3. *The map f is a morphism of spaces with functions, i.e. f is continuous and if $U \subseteq Y$ open and $g \in \mathcal{O}_Y(U)$, then $g \circ f \in \mathcal{O}_X(f^{-1}(U))$.*

Proof. The equivalence of (1) and (2) has already been proved in Proposition (22.6). Moreover, it is clear that (2) is implied by (3) by taking $U = Y$. Let us show that (2) implies (3). Let $f^* : \Gamma(Y) \to \Gamma(X)$ be the homomorphism $h \mapsto h \circ f$. For $g \in \Gamma(Y)$ we have

$$\begin{aligned} f^{-1}(D(g)) &= \{x \in X \mid f(x) \in D(g)\} \\ &= \{x \in X \mid g(f(x)) \neq 0\} \\ &= D(f^*(g)). \end{aligned}$$

As the principal open subsets form a basis of the topology, this shows that f is continuous. The homomorphism f^* induces a homomorphism of the localizations $\Gamma(Y)_g \to \Gamma(X)_{f^*(g)}$. By definition of f^* this is the map $\mathcal{O}_Y(D(g)) \to \mathcal{O}_X(D(f^*(g)))$, given by $h \mapsto h \circ f$. This shows the claim if U is principal open. As we can obtain functions on arbitrary open subsets of Y by gluing functions on principal open subsets, this proves (3). □

Altogether we obtain

Theorem 22.6. *The above construction $X \mapsto (X, \mathcal{O}_X)$ defines a fully faithful functor*

$$\text{(Irreducible affine algebraic sets)} \mapsto \text{(Spaces with functions over } k\text{)}.$$

23 Prevarieties

We have seen that we can embed the category of irreducible affine algebraic sets into the category of spaces with functions. Of course we do not obtain all spaces with functions in this way. We will now define prevarieties as those connected spaces with functions that can be glued together from finitely many spaces with functions attached to irreducible affine algebraic sets.

23.1 Definition of Prevarieties

We call a space with functions $(X, \mathcal{O}_X)$ **connected**, if the underlying topological space X is connected.

Definition 23.1.

1. An **affine variety** is a space with functions that is isomorphic to a space with functions associated to an irreducible affine algebraic set.
2. A **prevariety** is a connected space with functions $(X, \mathcal{O}_X)$ with the property that there exists a finite covering $X = \bigcup_{i=1}^n U_i$ such that the space with functions $(U_i, \mathcal{O}_{X|U_i})$ is an affine variety for all $i = 1, \dots, n$.
3. A **morphism** of prevarieties is a morphism of spaces with functions.

Corollary 2. *The following categories are equivalent.*

1. *The opposed category of finitely generated k-algebras without zero divisors.*
2. *The category of irreducible affine algebraic sets.*
3. *The category of affine varieties.*

We define an **open affine covering of a prevariety** X to be a family of open subspaces with functions $U_i \subseteq X$ that are affine varieties such that $X = \bigcup_i U_i$.

Proposition 23.1. *Let $(X, \mathcal{O}_X)$ be a prevariety. The topological space X is Noetherian (in particular quasi-compact) and irreducible.*

Proof. The first assertion follows from the fact that X has a finite covering of Noetherian spaces, which implies that X is Noetherian. The second assertion follows from the fact that X is connected and has a finite covering of irreducible spaces, which implies X is irreducible. □

23.1.1 Open Subprevarieties

We are now able to endow open subsets of affine varieties, and more general of prevarieties with the structure of a prevariety. Note that in general open subprevarieties of affine varieties are not affine.

Lemma 23.1. *Let X be an affine variety and let $f \in \Gamma(X)$. and let $\mathrm{D}(f) \subseteq X$ be the corresponding principal open subset. Let $\Gamma(X)_f$ be the localization of $\Gamma(X)$ by f and let $(Y, \mathcal{O}_Y)$ be the affine variety corresponding to this integral finitely generated k-algebra. Then $(D(f), \mathcal{O}_{X|D(f)})$ and $(Y, \mathcal{O}_Y)$ are isomorphic spaces with functions. In particular, $(D(f), \mathcal{O}_{X|D(f)})$ is an affine variety.*

Proof. By Proposition (22.8) we have $\mathcal{O}_X(\mathrm{D}(f)) = \Gamma(X)_f$. As two affine varieties are isomorphic if and only if their coordinate rings are isomorphic, it suffices to show that $(\mathrm{D}(f), \mathcal{O}_X|_{\mathrm{D}(f)})$ is an affine variety.

Let $X \subseteq \mathbb{A}^n(k)$ and $\mathfrak{a} = I(X) \subseteq k[T_1, \dots, T_n]$ be the corresponding radical ideal. We consider $k[T_1, \dots, T_n]$ as a subring of $k[T_1, \dots, T_n, T_{n+1}]$ and denote by $\mathfrak{a}' \subseteq k[T_1, \dots, T_n, T_{n+1}]$ the ideal generated by $\mathfrak{a}$ and the polynomial $fT_{n+1} - 1$. Then the affine coordinate ring of Y is $\Gamma(Y) = \Gamma(X)_f \cong k[T_1, \dots, T_n, T_{n+1}]/\mathfrak{a}'$, and we can identify Y with $V(\mathfrak{a}') \subseteq \mathbb{A}^{n+1}(k)$.

The projection $\mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ to the first n coordinates induces a bijective map

$$j : Y = \{(x, x_{n+1}) \in X \times \mathbb{A}^1(k) \mid x_{n+1}f(x) = 1\} \to D(f) = \{x \in X \mid f(x) \neq 0\}.$$

We will show that j is an isomorphism of spaces with functions. As a restriction of a continuous map, j is continuous. It is also open, because for $\frac{g}{f^N} \in \Gamma(Y)$, with $g \in \Gamma(X)$, we have

$$j\left(D\left(\frac{g}{f^N}\right)\right) = j(D(gf)) = D(gf).$$

Thus j is a homeomorphism.

It remains to show that for all $g \in \Gamma(X)$ the map $\mathcal{O}_X(D(fg)) \to \Gamma(Y)_g$, given by $s \mapsto s \circ j$, is an isomorphism. But we have

$$\mathcal{O}_X(D(fg)) = \Gamma(X)_{fg} = \Gamma(Y)_g$$

and this identification corresponds to the composition with j. □

Proposition 23.2. *Let $(X, \mathcal{O}_X)$ be a prevariety and let $U \subseteq X$ be a non-empty open subset. Then $(U, \mathcal{O}_{X|U})$ is prevariety and the inclusion $U \hookrightarrow X$ is a morphism of prevarieties.*

Proof. As X is irreducible, U is connected. The previous lemma shows that U can be covered by open affine subsets of X. As X is Noetherian, U is quasi-compact. Thus a finite covering suffices. □

23.1.2 Function Field of a Prevariety

Let X be a prevariety. If $U, V \subseteq X$ are non-empty open affine subvarieties, then $U \cap V$ is open in U and non-empty. We have

$$\mathcal{O}_X(U) \subseteq \mathcal{O}_X(U \cap V) \subseteq K(U)$$

by the definition of functions on U, and therefore $\mathrm{Frac}(\mathcal{O}_X(U \cap V)) = K(U)$. The same argument for V shows $K(U) = K(V)$. Thus the function field of a non-empty open affine subvariety U of X does not depend on U and we denote it by $K(X)$.

Definition 23.2. The field $K(X)$ is called the **function field** of X.

Remark 39. Let $f : X \to Y$ be a morphism of affine varieties. As the corresponding homomorphism $\Gamma(Y) \to \Gamma(X)$ between the affine coordinate rings is not injective in general, it does not induce a homomorphism of function fields $K(Y) \to K(X)$. Thus $K(X)$ is not functorial in X. But if $f : X \to Y$ is a morphism of prevarieties whose image contains a non-empty open (and hence dense) subset, f induces a homomorphism $K(Y) \to K(X)$. Such morphisms will be called **dominant**.

Proposition 23.3. *Let X be a prevariety and $U \subseteq X$ a non-empty open subset. Then $\mathcal{O}_X(U)$ is a k-subalgebra of the function field $K(X)$. If $U' \subseteq U$ is another open subset, the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ is the inclusion of subalgebras of $K(X)$. If $U, V \subseteq X$ are arbitrary open subsets, then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$.*

Proof. Let $f : U \to \mathbb{A}^1(k)$ be an element of $\mathcal{O}_X(U)$. Then its vanishing set $f^{-1}(0) \subseteq U$ is closed because f is continuous and $\{0\} \subseteq \mathbb{A}^1(k)$ is closed. Therefore if the restriction of f to U' is zero, then f is zero because U' is dense in U. This shows that restriction maps are injective. The axiom of gluing implies therefore $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$ for all open subsets $U, V \subseteq X$. □

23.1.3 Closed Subprevarieties

Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. We want to define on Z the structure of a prevariety. For this we have to define functions on open subsets of Z. We define:

$$\mathcal{O}'_Z(U) = \{f \in \mathrm{Map}(U, k) \mid \text{for all } x \in U, \text{ there exists } V \subseteq U \text{ open and } g \in \mathcal{O}_X(V) \text{ such that } f|_{U \cap V} = g|_{U \cap V}\}.$$

The definition shows that $(Z, \mathcal{O}'_Z)$ is a space with functions and that $\mathcal{O}'_X = \mathcal{O}_X$. Once we have shown the following lemma, we will always write $\mathcal{O}_Z$ (instead of $\mathcal{O}'_Z$).

Remark 40. $\mathcal{O}'_Z$ is the sheafification of the sheaf $\mathcal{O}_{X|Z}$.

Lemma 23.2. *Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set and let $Z \subseteq X$ be an irreducible closed subset. Then the space with functions $(Z, \mathcal{O}_Z)$ associated to the affine algebraic set Z and the above defined space with functions $(Z, \mathcal{O}'_Z)$ coincide.*

Proof. In both case Z is endowed with the topology induced by X. As the inclusion $Z \to X$ is a morphism of affine algebraic sets it induces a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$. The definition of $\mathcal{O}'_Z$ shows that $\mathcal{O}'_Z(U) \subseteq \mathcal{O}_Z(U)$ for all open subsets $U \subseteq Z$.

Conversely, let $f \in \mathcal{O}_Z(U)$. For $x \in U$ there exists $h \in \Gamma(Z)$ with $x \in D(h) \subseteq U$. The restriction $f\mid_{D(h)} \in \mathcal{O}_Z(D(h)) = \Gamma(Z)_h$ has the form $f = g/h^n$ where $n \geq 0$ and $g \in \Gamma(Z)$. We lift g and h to elements in $\widetilde{g}, \widetilde{h} \in \Gamma(X)$, set $V := D(\widetilde{h}) \subseteq X$, and obtain $x \in V$, $\widetilde{g}/\widetilde{h}^n \in \mathcal{O}_X(D(\widetilde{h}))$ and $f\mid_{U \cap V} = \frac{\widetilde{g}}{\widetilde{h}^n}\mid_{U \cap V}$. □

As a corollary of the lemme we obtain:

Proposition 23.4. *Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. Let $\mathcal{O}_Z$ be the system of functions defined above. Then $(Z, \mathcal{O}_Z)$ is a prevariety.*

23.2 Gluing Prevarieties

The most general way to construct prevarieties is to take some affine varieties and patch them together:

Example 23.1. Let X_1 and X_2 be prevarieties, $U_1 \subset X_1$ and $U_2 \subset X_2$ be non-empty open subsets, and let $f : (U_1, \mathcal{O}_{X_1}\mid_{U_1}) \to (U_2, \mathcal{O}_{X_2}\mid_{U_2})$ be an isomorphism. Then we can define a prevariety X, obtained by **gluing** X_1 and X_2 along U_1 and U_2 via the isomorphism f:

- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation $x \sim f(x)$ for all $x \in U_1$.
- As a topological space, we endow X with the so-called **quotient topology** induced by the above equivalence relation, i.e. we say that a subset $U \subset X$ is open if $U \cap X_1 \subset X_1$ is open in X_1 and $U \cap X_2 \subset X_2$ is open in X_2.
- As a ringed space, we define the structure sheaf $\mathcal{O}_X$ by

  $$\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2), \text{ and } s_1 = s_2 \text{ on the overlap (i.e. } f^*(s_2\mid_{U \cap U_2}) = s_1\mid_{U \cap U_1})\}$$

Example 23.2. Let $X_1 = X_2 = \mathbb{A}^1(k)$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $t \mapsto \frac{1}{t} := t'$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $k = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$. Let us show that $\mathcal{O}_X(X) \cong k$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2\mid_{U_2}) = b_m T^{-m} + b_{m-1} T^{1-m} + \cdots + b_0\mid_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0\mid_{U_2}.$$

  The only way this happens is if $a_0 = b_0$ and $a_i = b_j = 0$ for all $i, j > 0$. Thus, $(s_1, s_2) = (a_0, a_0)$.

- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X. Let us show that $\mathcal{O}_X(X) \cong k[T]$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2\mid_{U_2}) = b_m T^m + b_{m-1} T^{m-1} + \cdots + b_0\mid_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0\mid_{U_2}.$$

  The only way this happens is if $m = n$ and $a_i = b_i$ for all $i = 0, \ldots, n$.

Example 23.3. Let X be the complex affine curve

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2\widetilde{x}^4 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x},\widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x,y) \mapsto (\widetilde{x},\widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^4}\right)$$
$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x},\widetilde{y}) \mapsto (x,y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^4}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

24 Projective Varieties

By far the most important example of prevarieties are projective space $\mathbb{P}^n(K)$ and subvarieties of $\mathbb{P}^n(K)$, called (quasi-)projective varieties.

24.1 Homogeneous Polynomials

To describe the functions on projective space we start with some remarks on homogeneous polynomials. Throughout this subsection, let R be a ring. To clean our notation in what follows, we often write $R[\boldsymbol{X}]$ to denote $R[X_0, \dots, X_n]$. A monomial in $R[\boldsymbol{X}]$ is denoted by $\boldsymbol{X}^{\boldsymbol{\alpha}} = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$ where $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$. We also denote $|\boldsymbol{\alpha}| = \sum_{i=0}^n \alpha_i$. The vector $(1, \cdots, 1)$ in $\mathbb{Z}^n_{\geq 0}$ is denoted $\mathbf{1}$, thus $X_0 \cdots X_n = \boldsymbol{X}^{\mathbf{1}}$. A point in R^{n+1} is denoted by $\boldsymbol{x} = (x_0, \dots, x_n)$. We will frequently use this notation whenever context is clear.

Definition 24.1. A polynomial $f \in R[X_0, \dots, X_n]$ is called **homogeneous** of degree $d \in \mathbb{Z}_{\geq 0}$ if f is the sum of monomials of degree d.

Lemma 24.1. *Assume R is an integral domain with infinitely many elements and let $f \in R[\boldsymbol{X}]$ be a nonzero polynomial. Then f is homogeneous of degree d if and only if*

$$f(\lambda \boldsymbol{x}) = \lambda^d f(\boldsymbol{x}) \tag{26}$$

for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R \backslash \{0\}$.

Proof. One direction is obvious, so we will only prove the other direction. We will prove the other direction by induction on the number of terms of a polynomial. For the base case, let f be a monomial in $R[\boldsymbol{X}]$, say $f = c\boldsymbol{X}^{\boldsymbol{\alpha}}$ where $c \neq 0$ and assume that f satisfies (26) for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$. Clearly f is homogeneous, but we still need to show that it has degree d.

Let K be the fraction field of R. Since K has infinitely many elements and since $f \neq 0$, there exists a point $\boldsymbol{a} \in \mathrm{D}(\boldsymbol{X}^{\mathbf{1}}) \cap \mathrm{D}(f)$. By clearing the denominators of $\boldsymbol{a}$ if necessary, we may assume that $\boldsymbol{a} \in R$. Then for all $\lambda \in R\backslash\{0\}$, we have

$$\lambda^d \boldsymbol{a}^{\boldsymbol{\alpha}} = \lambda^d f(\boldsymbol{a}) = f(\lambda \boldsymbol{a}) = \lambda^{|\boldsymbol{\alpha}|} \boldsymbol{a}^{\boldsymbol{\alpha}}.$$

Since R is a domain and $\boldsymbol{a}^{\boldsymbol{\alpha}} \neq 0$, it follows that $\lambda^d = \lambda^{|\boldsymbol{\alpha}|}$ for all $\lambda \in R\backslash\{0\}$. Assume without loss of generality that $d \geq |\boldsymbol{\alpha}|$ and set $r = d - |\boldsymbol{\alpha}|$. If $r > 0$, then $T^r - 1$ has infinitely many solutions in R (in fact every nonzero element of R is a solution). This is a contradiction since $T^r - 1$ can have at most r solutions in R. Thus $r = 0$, which implies $|\boldsymbol{\alpha}| = d$; hence f has degree d.

For the induction step, assume that we have proven the statement for all polynomials with k terms, where $k \geq 1$. Let f be a polynomial with $k+1$ terms such that f satisfies (26) for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$. Write f as

$$f = c\boldsymbol{X}^{\boldsymbol{\alpha}} + g,$$

where $\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^{n+1}$, where $c \in R\backslash\{0\}$, and where g is a nonzero polynomial in $R[\boldsymbol{X}]$ such that $\boldsymbol{X}^{\boldsymbol{\alpha}} \nmid g$. Let K be the fraction field of R and let $\boldsymbol{a} \in K^{n+1}$ be a point such that $\boldsymbol{a}^{\boldsymbol{\alpha}} \neq 0$ and $g(\boldsymbol{a}) = 0$. Note that such a point exists since $\mathrm{V}(g) \cap \mathrm{V}(\boldsymbol{X}^{\boldsymbol{\alpha}}) \neq \emptyset$. Indeed, otherwise we'd have $\mathrm{V}(g) \subseteq \mathrm{V}(\boldsymbol{X}^{\boldsymbol{\alpha}})$ which would imply $\boldsymbol{X}^{\boldsymbol{\alpha}} \mid g$, a contradiction. By clearing the denominators if necessary, we may assume that $\boldsymbol{a} \in R^{n+1}$. In particular, for all $\lambda \in R\backslash\{0\}$, we have

$$\lambda^d \boldsymbol{a}^{\boldsymbol{\alpha}} = \lambda^d f(\boldsymbol{a}) = f(\lambda \boldsymbol{a}) = \lambda^{\boldsymbol{\alpha}} \boldsymbol{a}^{\boldsymbol{\alpha}}.$$

Arguing as before, this implies $|\boldsymbol{\alpha}| = d$. Now observe that for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$, we have

$$\begin{aligned} g(\lambda \boldsymbol{x}) &= f(\lambda \boldsymbol{x}) - c(\lambda \boldsymbol{X})^{\boldsymbol{\alpha}} \\ &= \lambda^d f(\boldsymbol{x}) - c\lambda^{|\boldsymbol{\alpha}|} \boldsymbol{x}^{\boldsymbol{\alpha}} \\ &= \lambda^d f(\boldsymbol{x}) - c\lambda^d \boldsymbol{x}^{\boldsymbol{\alpha}} \\ &= \lambda^d (f(\boldsymbol{x}) - c\boldsymbol{x}^{\boldsymbol{\alpha}}) \\ &= \lambda^d g(\boldsymbol{x}). \end{aligned}$$

Thus by induction, we see that g must be homogeneous of degree d; hence f is homogeneous of degree d. □

24.1.1 Dehomogenization and Homgenization

To simplify notation in what follows, we write $\boldsymbol{X} = X_0, \dots, X_n$ and $\boldsymbol{X}_i = X_{i,0}, \dots, \widehat{X}_{i,i}, \dots, X_{i,n}$ for each $0 \leq i \leq n$. Intuitively, we think of the variable $X_{j,i}$ as being the fraction X_j/X_i. For each $0 \leq i \leq n$ and $d \geq 0$, we define $\mathrm{D}_i^d \colon R[\boldsymbol{X}]_d \to R[\boldsymbol{X}_i]_{\leq d}$, called **dehomogenization**, by

$$\mathrm{D}_i^d(f)(\boldsymbol{X}_i) = \mathrm{D}_i^d(f(\boldsymbol{X})) = f(X_{i,0}, \dots, X_{i,i-1}, 1, X_{i,i+1}, \dots, X_{i,n})$$

for all $f \in R[\boldsymbol{X}]_d$. Esentially D_i^d takes a polynomial $f(\boldsymbol{X})$, divides it by X_i^d, and then makes the substitution $X_{j,i} = X_i/X_j$ for each $j \neq i$. For instance, we have

$$\begin{aligned} X_0^2 X_2 + X_1^3 + X_1 X_2^2 &\mapsto \frac{X_0^2 X_2 + X_1^3 + X_1 X_2^2}{X_1^3} \\ &\mapsto \left(\frac{X_0}{X_1}\right)^2 \left(\frac{X_2}{X_1}\right) + 1 + \left(\frac{X_2}{X_1}\right)^2 \\ &\mapsto X_{0,1}^2 X_{2,1} + 1 + X_{2,1}^2 \\ &= \mathrm{D}_1^3(X_0^2 X_2 + X_1^3 + X_1 X_2^2) \end{aligned}$$

We also define $\mathrm{H}_i^d \colon R[\boldsymbol{X}_i]_{\leq d} \to R[\boldsymbol{X}]_d$, called **homogenization**, by

$$\mathrm{H}_i^d(g)(\boldsymbol{X}) = \mathrm{H}_i^d(g(\boldsymbol{X}_i)) = \sum_{k=0}^{d} X_i^{d-k} g_k(X_0, \dots, \widehat{X}_i, \dots, X_n)$$

for all $g \in R[\boldsymbol{X}_i]_{\leq d}$ where g_k is the homogoneous component of g of degree k. Esentially H_i^d takes a polynomial $g(\boldsymbol{X}_i)$, multiplies it homogeneous component in degree k by X_i^{d-k}, and then makes the substitution $X_{j,i} = X_j$ for each $j \neq i$. For instance, we have

$$\begin{aligned} X_{0,1}^2 X_{2,1} + X_{2,1}^2 + 1 &\mapsto X_{0,1}^2 X_{2,1} + X_1 X_{2,1}^2 + X_1^3 \\ &\mapsto X_0^2 X_2 + X_1 X_2^2 + X_1^3 \\ &= \mathrm{H}_1^3(X_{0,1}^2 X_{2,1} + X_{2,1}^2 + 1) \end{aligned}$$

Lemma 24.2. *D_i^d is an R-linear isomorphism with H_i^d being its inverse.*

Proof. Clearly D_i^d is R-linear (it is just an evaluation map). Furthermore we have, if $f \in R[\boldsymbol{X}]$, then we have

$$\begin{aligned} \mathrm{H}_i^d(\mathrm{D}_i^d(f))(\boldsymbol{X}) &= \mathrm{H}_i^d(\mathrm{D}_i^d(f)(\boldsymbol{X}_i)) \\ &= \mathrm{H}_i^d(f(X_{i,0}, \dots, X_{i,i-1}, 1, X_{i,i+1}, \dots, X_{i,n})) \\ &= \sum_{k=0}^{d} X_i^{d-k} f_k(X_0, \dots, \widehat{X}_i, \dots, X_n) \\ &= f(\boldsymbol{X}). \end{aligned}$$

□

For $f \in R[\boldsymbol{X}]_d$ and $g \in R[\boldsymbol{X}]_e$, the product fg is homogeneous of degree $d+e$ and we have

$$\mathrm{D}_i^d(f)\mathrm{D}_i^e(g) = \mathrm{D}_i^{d+e}(fg). \tag{27}$$

If $R = K$ is a field, we will extend homogenization and dehomogenization to fields of fractions as follows: let $\mathcal{F}$ be the subset of $K(\boldsymbol{X})$ that consists of those elements f/g, where $f, g \in K[\boldsymbol{X}]$ are homogeneous polynomials of the same degree. It is easy to check that $\mathcal{F}$ is a subfield of $K(\boldsymbol{X})$. By (27), we have a well defined isomorphism of K-extensions

$$\mathrm{D}_i : \mathcal{F} \to K(\boldsymbol{X}_i), \tag{28}$$

given by $f/g \mapsto \mathrm{D}_i(f)/\mathrm{D}_i(g)$. For instance, we have

$$\begin{aligned}\mathrm{D}_1\left(\frac{X_1^3}{X_2^3 + X_1X_0^2}\right) &= \frac{\mathrm{D}_1(X_1^3)}{\mathrm{D}_1(X_2^3 + X_1X_0^2)} \\ &= \frac{1}{X_{2,1}^3 + X_{0,1}^2}.\end{aligned}$$

Often, we will identify $K(\boldsymbol{X}_i)$ with the subring $K(\boldsymbol{X}/X_i) = K(X_0/X_i, \ldots, X_n/X_i)$ of the field $K(\boldsymbol{X})$.

24.2 Definition of the Projective Space $\mathbb{P}^n(K)$

The projective space $\mathbb{P}^n(K)$ is an extremely important prevariety within algebraic geometry. Many prevarieties of interest are subprevarieties of the projective space. Moreover, the projective space is the correct environment for projective geometry which remedies the "defect" of affine geometry of missing points at infinity. As a set, we define

$$\mathbb{P}^n(K) := \{\text{lines through the origin in } K^{n+1}\} = (K^{n+1}\backslash\{0\})/K^\times.$$

Here a line through the origin is per definition a 1-dimensional k-subspace and we denote by $(K^{n+1}\backslash\{0\})/K^\times$ the set of equivalence classes in $K^{n+1}\backslash\{0\}$ with respect to the equivalence relation:

$$(x_0, \ldots, x_n) \sim (x_0', \ldots, x_n') \text{ if and only if there exists } \lambda \in K^\times \text{ such that } x_i = \lambda x_i' \text{ for all } 0 \le i \le n.$$

The equivalence class of a point $\boldsymbol{x} = (x_0, \ldots, x_n)$ is denoted by $[\boldsymbol{x}] = [x_0 : \cdots : x_n]$. We call the x_i the **homogeneous coordinates** of $\mathbb{P}^n(K)$. For each $0 \le i \le n$ we set

$$U_i := \{[\boldsymbol{x}] \in \mathbb{P}^n(K) \mid x_i \neq 0\}$$

This subset is well-defined and the union of the U_i is all of $\mathbb{P}^n(K)$. Note that for each i we have a bijection $\varphi_i \colon U_i \to \mathbb{A}^n(K)$ which is defined by

$$\varphi_i([\boldsymbol{x}]) = \varphi_i([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_i}, \cdots \frac{\widehat{x_i}}{x_i}, \cdots, \frac{x_n}{x_i}\right) = \widehat{\boldsymbol{x}}_i/x_i$$

for all $[\boldsymbol{x}] \in U_i$. F Using this bijection, we endow U_i with the structure of a space with functions, isomorphic to $(\mathbb{A}^n(K), \mathcal{O}_{\mathbb{A}^n(K)})$, which we denote by $(U_i, \mathcal{O}_{U_i})$. We want to define on $\mathbb{P}^n(k)$ the structure of a space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ such that U_i becomes an open subset of $\mathbb{P}^n(k)$ and such that $\mathcal{O}_{\mathbb{P}^n(k)}|_{U_i} = \mathcal{O}_{U_i}$ for all $i = 0, \ldots, n$. As $\bigcup_i U_i = \mathbb{P}^n(k)$, there's at most one way to do this:

We define the topology on $\mathbb{P}^n(k)$ by calling a subset $U \subseteq \mathbb{P}^n(k)$ open if $U \cap U_i$ is open in U_i for all i. This defines a topology on $\mathbb{P}^n(k)$ as for all $i \neq j$ the set $U_i \cap U_j = \mathrm{D}(T_j) \subseteq U_i$ is open (we use here on $U_i \cong \mathbb{A}^n(k)$ the coordinates $T_0, \ldots, \widehat{T_i}, \ldots, T_n$). With this definition, $\{U_i\}_{i \in \{0,\ldots,n\}}$ becomes an open covering of $\mathbb{P}^n(k)$.

We still have to define functions on open subsets $U \subseteq \mathbb{P}^n(k)$. For this, we set

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f \in \mathrm{Map}(U, k) \mid f|_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i) \text{ for all } i = 0, \ldots, n\}.$$

It is clear that this defines the structure of a space with functions on $\mathbb{P}^n(k)$, although we still have to see that $\mathcal{O}_{\mathbb{P}^n(k)}|_{U_i} = \mathcal{O}_{U_i}$ for all i. This follows from the following description of the k-algebras $\mathcal{O}_{\mathbb{P}^n(k)}(U)$ using the inverse isomorphism of the function field $k(T_0, \ldots, \widehat{T_i}, \ldots, T_n)$ of U_i with the subfield $\mathcal{F}$ of $k(X_0, \ldots, X_n)$.

Proposition 24.1. *Let $U \subseteq \mathbb{P}^n(K)$ be open. Then*

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f : U \to k \mid \forall x \in U,\ \exists x \in V \subseteq U \text{ open and } g, h \in k[X_0, \ldots, X_n] \\ \text{homogeneous of same degree such that } h(v) \neq 0 \text{ and } f(v) = g(v)/h(v) \text{ for all } v \in V\}.\end{aligned}$$

Proof. Let $f \in \mathcal{O}_{\mathbb{P}^n(k)}(U)$. As $f|_{U\cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$, the function f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g}, \widetilde{h} \in k[T_0, \dots, \widehat{T}_i, \dots, T_n]$. Applying the inverse of (28) yields the desired form of f.

Conversely, let f be an element of the right hand side. We fix $i \in \{0, \dots, n\}$. Thus locally on $U \cap U_i$ the function f has the form g/h where $g, h \in k[X_0, \dots, X_n]_d$ for some d. Once more applying the isomorphism (28) we obtain that f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g}, \widetilde{h} \in k[T_0, \dots, \widehat{T}_i, \dots, T_n]$. This shows $f|_{U\cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$. □

Example 24.1. Consider $\mathbb{P}^2(k)$ and

$$f|_{U\cap U_1} = \frac{T_2^2+1}{T_0+1}.$$

Then the inverse of (28) yields

$$\frac{X_2^2+X_1^2}{X_0^2+X_1^2}$$

Corollary 3. *Let $i \in \{0, \dots, n\}$. The bijection $U_i \cong \mathbb{A}^n(k)$ induces an isomorphism*

$$(U_i, \mathcal{O}_{\mathbb{P}^n(k)}|_{U_i}) \cong \mathbb{A}^n(k).$$

of spaces with functions. The space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a prevariety.

Proof. The first assertion follows from the proof of Proposition (24.1). This shows that $\mathbb{P}^n(k)$ is a space with functions that has a finite open covering by affine varieties. Moreover, $\mathbb{P}^n(k)$ is irreducible since it is connected and is covered by finitely many irreducible open subsets. □

The function field $K(\mathbb{P}^n(k))$ of $\mathbb{P}^n(k)$ is by its very definition the function field $K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right)$ of U_i. Using the isomorphism Φ_i, we usually describe $K(\mathbb{P}^n(k))$ as the field

$$K(\mathbb{P}^n(k)) = \{f/g \mid f, g \in k[X_0, \dots, X_n] \text{ homogeneous of the same degree}\}.$$

For $0 \le i, j \le n$ the identification of $K(U_i) \cong K(U_j)$ is then given by $\Phi_j \circ \Phi_i^{-1}$. This can be described explicitely

$$K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \mapsto k\left(\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right) = K(U_j), \qquad \frac{X_\ell}{X_i} \mapsto \frac{X_\ell}{X_i}\frac{X_i}{X_j} = \frac{X_\ell}{X_j}.$$

We use these explicit descriptions to prove the following result.

Proposition 24.2. *The only global functions on $\mathbb{P}^n(k)$ are the constant functions, i.e. $\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = k$. In particular, $\mathbb{P}^n(k)$ is not an affine variety for $n \ge 1$.*

Proof. By Proposition (23.3) we have

$$\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = \bigcap_{0\le i\le n} \mathcal{O}_{\mathbb{P}^n(k)}(U_i) = \bigcap_{0\le i\le n} k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] = k,$$

where the intersection is taken in $K(\mathbb{P}^n(k))$. The last assertion follows because if $\mathbb{P}^n(k)$ were affine, its set of points would be in bijection to the set of maximal ideals in the ring $k = \mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k))$. This implies that $\mathbb{P}^n(k)$ consists of only one point, so $n = 0$. □

24.2.1 Gluing $\mathbb{A}^1(k)$ With $\mathbb{A}^1(k)$ to Make $\mathbb{P}^1(k)$

We now want to describe how we can glue $\mathbb{A}^1(k)$ with $\mathbb{A}^1(k)$ to make $\mathbb{P}^1(k)$ in explicit detail. First we start with the rings $k[S]$ and $k[T]$

Let X_0 and X_1 be the homogeneous coordinates of $\mathbb{P}^1(k)$ and denote $T := \frac{X_1}{X_0}$ and $S := \frac{X_0}{X_1}$.

24.3 Projective Varieties

Definition 24.2. A prevariety is called a **projective variety** if it is isomorphic to a closed subprevariety of a projective space $\mathbb{P}^n(k)$.

As in the affine case, we speak of projective varieties rather than prevarieties. Similarly, we will talk about subvarieties of projective space, instead of subprevarieties. For $[\boldsymbol{x}] \in \mathbb{P}^n(k)$ and $f \in k[\boldsymbol{X}]$ the value $f([\boldsymbol{x}])$ obviously depends on the choice of the representative of $[\boldsymbol{x}]$ and we cannot consider f as a function on $\mathbb{P}^n(k)$. But if f is homogeneous, at least the question whether the value is zero or nonzero is independent of the choice of a representative.

Let $\boldsymbol{f} = f_1, \dots, f_m$ be a finite collection of homogeneous polynomials in $k[\boldsymbol{X}]$. We define

$$\mathrm{V}_+(\boldsymbol{f}) = \{[\boldsymbol{x}] \in \mathbb{P}^n(k) \mid f_i(\boldsymbol{x}) = 0 \text{ for all } i = 1, \dots, m\}.$$

$$V_+(f_1, \dots, f_m) = \{(x_0 : \dots : x_n) \in \mathbb{P}^n(k) \mid f_i(x_0 : \dots : x_n) = 0 \text{ for all } i = 1, \dots, m\}.$$

Subsets of the form $V_+(f_1, \dots, f_m)$ are closed. More precisely we have $i = 0, \dots, n$:

$$V_+(f_1, \dots, f_m) \cap U_i = V(\Phi_i(f_1), \dots, \Phi_i(f_m)),$$

where Φ_i denotes as usual dehomogenization with respect to X_i. We will see that all closed subsets of the projective space are of this form. To do this we consider the map

$$f : \mathbb{A}^{n+1}(k) \backslash \{0\} \to \mathbb{P}^n(k), \qquad (x_0, \cdots, x_n) \mapsto (x_0 : \cdots : x_n).$$

As for all i its restriction $f\mid_{f^{-1}(U_i)}: f^{-1}(U_i) \to U_i$ is a morphism of prevarieties, this holds for f. If $Z \subseteq \mathbb{P}^n(k)$ is a closed subset, $f^{-1}(Z)$ is a closed subset of $\mathbb{A}^{n+1}(k) \backslash \{0\}$ and we denote by $C(Z)$ its closure in $\mathbb{A}^{n+1}(k)$. Affine algebraic sets $X \subseteq \mathbb{A}^{n+1}(k)$ are called **affine cones** if for all $x \in X$ we have $\lambda x \in X$ for all $\lambda \in k^\times$. Clearly $C(Z)$ is an affine cone in $\mathbb{A}^{n+1}(k)$. It is called the **affine cone of** Z.

Proposition 24.3. *Let $X \subseteq \mathbb{A}^{n+1}(k)$ be an affine algebraic set such that $X \neq \{0\}$. Then the following assertions are equivalent.*

1. *X is an affine cone.*
2. *$I(X)$ is generated by homogeneous polynomials.*
3. *There exists a closed subset $Z \subset \mathbb{P}^n(k)$ such that $X = C(Z)$.*

If in this case $I(X)$ is generated by homogeneous polynomials $f_1, \dots, f_m \in k[X_0, \dots, X_n]$, then $Z = V_+(f_1, \dots, f_m)$.

24.3.1 Segre Embedding

Consider $\mathbb{P}^n(k)$ with homogeneous coordinates $X_0, \dots, X_n$ and $\mathbb{P}^m(k)$ with homogeneous coordinates $Y_0, \dots, Y_m$. We want to find an easy description of the product $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$.

Let $\mathbb{P}^N(k) = \mathbb{P}^{(n+1)(m+1)-1}$ be projective space with homogeneous coordinates $Z_{i,j}$ where $0 \leq i \leq n$ and $0 \leq j \leq m$. There is an obviously well-defined set-theoretic map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ given by $z_{i,j} = x_i y_j$.

Lemma 24.3. *Let $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ be the set-theoretic map as above. Then:*

1. *The image $X = f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ is a projective variety in $\mathbb{P}^N(k)$, with ideal generated by the homogeneous polynomials $Z_{i,j}Z_{i',j'} - Z_{i,j'}Z_{i',j}$ for all $0 \leq i, i' \leq n$ and $0 \leq j, j' \leq m$.*
2. *The map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to X$ is an isomorphism. In particular, $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is a projective variety.*
3. *The closed subsets of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ are exactly those subsets that can be written as the zero locus of polynomials in $k[X_0, \dots, X_n, Y_0, \dots, Y_m]$ that are bihomogeneous in the X_i and Y_i.*

Proof.

1. It is obvious that the points of $f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ satisfy the given equations. Conversely, let z be a point in $\mathbb{P}^N(k)$ with coordinates $z_{i,j}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to affine coordinates by setting $z_{0,0} = 1$. Then we have $z_{i,j} = z_{i,0}z_{0,j}$; so by setting $x_i = z_{i,0}$ and $y_j = z_{0,j}$ we obtain a point (x, y) in $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ that is mapped to z by f.
2. Continuing the above notation, let $z \in f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ be a point with $z_{0,0} = 1$. If $f(x, y) = z$, it follows that $x_0 \neq 0$ and $y_0 \neq 0$, so we can assume $x_0 = 1$ and $y_0 = 1$ as the x_i and y_j are only determined up to a common scalar. But then it follows that $x_i = z_{i,0}$ and $y_j = z_{0,j}$, i.e. f is bijective. The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps; so f is an isomorphism.
3. It follows by the isomorphism of (2) that any closed subset of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is the zero locus of homogeneous polynomials in the $Z_{i,j}$, i.e. of bihomogeneous polynomials in the X_i and Y_j (of the same degree). Conversely, a zero locus of bihomogeneous polynomials can always be rewritten as a zero locus of bihomogeneous polynomials of the same degree in the X_i and Y_j. But such a polynomial is obviously a polynomial in the $Z_{i,j}$, so it determines an algebraic set in $X \cong \mathbb{P}^n \times \mathbb{P}^m$.

□

Example 24.2. Consider the case where $n = 1$ and $m = 2$. Then Segre embedding $f : \mathbb{P}^1(k) \times \mathbb{P}^2(k) \to \mathbb{P}^5(k)$ is given by

$$([x_0 : x_1], [y_0 : y_1 : y_2]) \mapsto [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2] := [z_{00} : z_{01} : z_{02} : z_{10} : z_{11} : z_{12}].$$

By Lemma (24.3), the vanishing ideal of $f(\mathbb{P}^1(k) \times \mathbb{P}^2(k))$ is given by

$$\langle Z_{00}Z_{11} - Z_{01}Z_{10}, Z_{00}Z_{12} - Z_{02}Z_{10}, Z_{01}Z_{12} - Z_{02}Z_{11} \rangle.$$

We can view this as the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \end{pmatrix}.$$

This is an example of a **determinantal variety**.

24.4 A Quartic Curve

In this subsection, we go over an example of a quartic curve which will demonstrate many of the concepts introduced above. Let $A = \mathbb{Z}[x,y]/\langle f(x,y)\rangle$ where

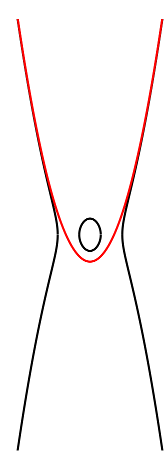
$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 \tag{29}$$

Note that from the expression of f in (29) we see that $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$ are units in A. Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g(x)}]$ where

$$f = y^2 - (x-1)(x-2)(x-3)(x-4) = y^2 - g(x). \tag{30}$$

The expression of f in (30) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (30) we can read off useful information of A viewed as a finite module extension, whereas from (29) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day.

Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we first look at its $\mathbb{R}$-valued points: $X(\mathbb{R}) = \operatorname{Spec} \mathbb{R} \otimes_{\mathbb{Z}} A = \operatorname{Spec} \mathbb{R}[x,y]/f$. We can visualize the $\mathbb{R}$-valued points of X in the picture below:



The thick black curve is $X(\mathbb{R}) = V_{\mathbb{R}}(f)$ whereas the thick red curve is $V_{\mathbb{R}}(u)$. Notice that $V_{\mathbb{R}}(u)$ and $X(\mathbb{R})$ do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X$. Note that the closed points of $X(\mathbb{R})$ have the form $\mathfrak{p}_{a,b} = \langle x - a, y - b \rangle$ where $(a,b) \in \mathbb{R}^2$ such that $f(a,b) = 0$. There's also the generic point $\eta \in X(\mathbb{R})$ corresponding to the 0 ideal.

Now let $p(x) = x^2 - 5x + 5$, so $u = y - p$ and $v = y + p$. The existence of u and v tells us that A is not antilocal (if you look at the curves $V_{\mathbb{R}}(u)$ and $V_{\mathbb{R}}(f)$ in $\mathbb{R}^2$, you'll see that they just barely miss each other), however we can still ask: how far away is A from being antilocal? If we add u and v together, we obtain $u + v = 2y$, which is not a unit in A since the line $V_{\mathbb{R}}(y)$ intersects the curve $V_{\mathbb{R}}(f)$ at four points (you could also see this by plugging in $y = 0$ in (29) above). More generally, we have

$$y^2 - p^2 - 1$$

$$mu + nv = (m+n)y + (n-m)(x^2 - 5x + 5).$$

$$K[u,v,y]/\langle uv - 1\rangle$$

$$K[u,v,y]/\langle uv-1, mu+nv\rangle$$

In particular, we have $n(v-u) = 2n(x^2-5x+5) = 2np$.

$$\mathbb{Z}u = \mathbb{Z}(y-p) \quad \text{and} \quad \mathbb{Z}v$$
$$\mathbb{Z}(u+v) = 2\mathbb{Z}y \quad \text{and} \quad \mathbb{Z}(v-u) = 2\mathbb{Z}p.$$

If particular, all combinations of the form

$$m(u+v) + n(v-u) = 2my + 2np$$

where $m, n \neq 0$ gives us a new unit. Indeed, in this case if both u and v vanish, then both y and p vanishes too, and the function f takes value -1 here (as can be seen in the expression (29)). For instance, the function

$$2(u+v) + (v-u) = 4y + 2p = 4y + 2x^2 - 10x + 10$$

$$\begin{aligned}
f &= (y-p)(y+p) - 1 \\
f &= uv - 1 \\
p &= x^2 - 5x + 5 \\
u + v &= 2y \\
v - u &= 2p \\
y &= \frac{u+v}{2} \\
p &= \frac{v-u}{2} \\
u &= y - p \\
v &= y + p
\end{aligned}$$

$$y^2 = p^2 + 1$$
$$2u + v = 3y - p = 3y - x^2 + 5x - 5$$

$$5u + v = 6y - 4p = 6y - 4x^2 + 20x - 20$$

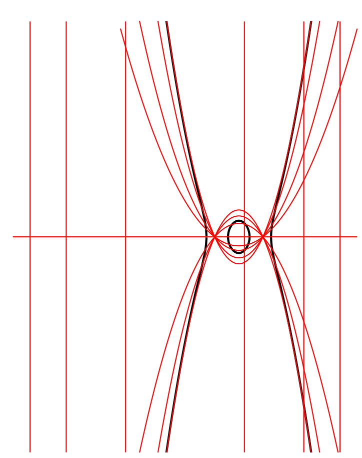
$$2(y-p) + (y+p) = 3y - p$$

$$2(y-p) + y + p = 3y$$
$$y = \frac{m-n}{m+n}(x^2 - 5x + 5)$$

On the other hand, we have $y^2 = (x-1)(x-2)(x-3)(x-4)$. In particular, $y = \sqrt{24} \approx 4.89$.

$$2n(x^2 - 5x + 5)$$

is another unit of A. If you graph the zero set of this function, you'll see that it gets closer to the curve $\mathrm{V}(f)$, but it still doesn't quite intersect it.



$mu + v = (m+1)y + (1-m)(x^2 - 5x + 5)$

$$y = \frac{m-1}{m+1}(x^2 - 5x + 5)$$

25 Irreducible Spaces

25.1 Connected Spaces

Let X be a topological space. We say X is **connected** if it is impossible to write X as a union of two non-empty disjoint open subsets of X: if $X = U \cup V$ where U and V are open subsets of X and $U \cap V = \emptyset$, then one of U or V is empty. If X is not connected, then we say it is **disconnected**. A subspace $C \subseteq X$ is said to be connected if it is connected in the subspace topology. Set $\mathcal{C} = (\mathcal{C}, \subseteq)$ to be the poset whose underlying set is

$$\mathcal{C} = \{C \subseteq X \mid C \text{ is connected}\},$$

and whose partial order is the inclusion map. The maximal elements of $\mathcal{C}$ are called **connected components** of X. Note that the emptyset $\emptyset$ and singletons $\{x\}$ belong to $\mathcal{C}$. Also note that if C_1, C_2 are two distinct maximal elements in $\mathcal{C}$, then they are necessarily closed (since if C is connected, then $\overline{C}$ is connected), and they must be disjoint (since if $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is connected). In particular, if (C_i) is a chain in $\mathcal{C}$, then their union $\bigcup_i C_i$ is an upper bound of this chain in $\mathcal{C}$. Thus Zorn's lemma implies that every connected subset is contained in a connected component of X. In particular, every point of X is contained in a connected component. This shows that X is the union of its connected components.

Note that X being connected is equivalent to saying that it is impossible to write X as a union of two proper disjoint closed subsets of X. Indeed, if $X = U \cup V$ where U and V are two proper open subsets of X which are disjoint from one another, then $U = V^c$ and $V = U^c$ are also two proper closed subsets of X which are disjoint from one another. If we relax this condition a bit, then we get the concept of irreducible spaces: we say X is **irreducible** if cannot be expressed as the union of two proper closed subsets of X: if $X = E \cup F$ where E and F are closed subsets of X, the either $E = X$ or $F = X$. If X is not irreducible, then we say X is **reducible**. A subspace $D \subseteq X$ is said to be irreducible if it is irreducible in the subspace topology. Set $\mathcal{D} = (\mathcal{D}, \subseteq)$ to be the poset whose underlying set is

$$\mathcal{D} = \{D \subseteq X \mid D \text{ is irreducible}\},$$

and whose partial order is the inclusion map. The maximal elements of $\mathcal{D}$ are called **irreducible components** of X. Note that the emptyset $\emptyset$ and singletons $\{x\}$ belong to $\mathcal{D}$. Also note that if D_1, D_2 are two distinct maximal elements in $\mathcal{D}$, then they are necessarily closed (since if D is irreducible, then $\overline{D}$ is irreducible), and they must be disjoint (since if $D_1 \cap D_2 \neq \emptyset$, then $D_1 \cup D_2$ is irreducible). In particular, if (D_i) is a chain in $\mathcal{D}$, then their union $\bigcup_i D_i$ is an upper bound of this chain in $\mathcal{D}$. Thus Zorn's lemma implies that every irreducible subset is contained in a irreducible component of X. In particular, every point of X is contained in an irreducible component. This shows that X is the union of its irreducible components.

Clearly $\mathcal{D} \subseteq \mathcal{C}$ (every irreducible subset of X is connected), however for many spaces this is a strict inclusion. For instance, if X is Hausdorff and contains at least two distinct points, then it is *always* reducible. Indeed, pick two points $x, y \in X$ together with two neighborhoods U_x, U_y of x and y respectively such that $U_x \cap U_y = \emptyset$. Then $X = U_x^c \cup U_y^c$ expresses X as a union of two proper closed subsets of X. Irreducible spaces show up a lot in algebraic geometry (for instance, the Zariski topology of an affine algebraic variety is irreducible). The open subsets of irreducible property are *very* large in the sense of the following proposition:

Proposition 25.1. *Assume X is not empty. The following assertions are equivalent.*

1. *X is irreducible.*
2. *Any two non-empty open subsets of X have a non-empty intersection.*
3. *Every non-empty open subset is dense in X.*
4. *Every non-empty open subset is connected.*
5. *Every non-empty open subset is irreducible.*

Proof.

($1 \implies 2$): Let U and V be open subsets of X such that $U \cap V = \emptyset$. Then $X = U^c \cup V^c$ implies either $U^c = X$ or $V^c = X$ which implies either $U = \emptyset$ ro $V = \emptyset$.

($2 \implies 3$): Let U be a non-empty open subset of X. Then U and $\overline{U}^c$ are disjoint open subsets of X. Since U is non-empty, we must have $\overline{U}^c = \emptyset$, which implies $\overline{U} = X$.

($2 \implies 4$): Let U be a non-empty open subset of X. Assume that U is not connected: write $U = (U_1 \cap U) \cup (U_2 \cap U)$ where U_1 and U_2 are open subsets in X and $U_1 \cap U \neq \emptyset$ and $U_2 \cap U \neq \emptyset$. This is a contradiction since $U_1 \cap U$ and $U_2 \cap U$ are non-empty open subset of X which have non-empty intersection.

($3 \implies 5$): Let U be a non-empty open subset of X. We show that every non-empty open subset V of U is dense in U (this shows that U is irreducible). Now V is also open in X and therefore dense in X. But then V is certainly dense in U.

($5 \implies 1$): Obvious.

□

Corollary 4. *Let $f : X \to Y$ be a continuous map of topological spaces. If $Z \subseteq X$ is an irreducible subspace, its image $f(Z)$ is irreducible.*

Proof. If U_1 and U_2 are non-empty open subsets of $f(Z)$, their preimages in Z have a non-empty intersection. This shows that $U_1 \cap U_2 \neq \emptyset$. □

Lemma 25.1. *Let X be a topological space. A subspace $Y \subseteq X$ is irreducible if and only if its closure $\overline{Y}$ is irreducible.*

Proof. A subset Z of X is irreducible if and only if for any two open subsets U and V of X with $Z \cap U \neq \emptyset$ and $Z \cap V \neq \emptyset$ we have $Z \cap (U \cap V) \neq \emptyset$. This implies the lemma because an open subset meets Y if and only if it meets $\overline{Y}$. Indeed, one direction is trivial. For the other direction, we prove the contrapositive: $U \cap Y = \emptyset$ implies $U \cap \overline{Y} = \emptyset$. If $U \cap Y = \emptyset$, then $X \backslash U$ is a closed subset of X which contains Y. Therefore, $X \backslash U$ must contain $\overline{Y}$, as $\overline{Y}$ is the smallest closed subset of X which contains Y. This implies that $U \cap \overline{Y} = \emptyset$. □

If U is and open subset of X and if Z is an irreducible closed subset of X, then $Z \cap U$ is open in Z and hence an irreducible closed subset of U whose closure in X is Z. Together with Lemma (25.1), this shows that there are mutually inverse bijective maps

$$\{Y \subseteq U \mid Y \text{ is irreducible and closed}\} \quad \leftrightarrow \quad \{Z \subseteq X \mid Z \text{ is irreducible and closed with } Z \cap U \neq \emptyset\}$$

where $Y \mapsto \overline{Y}$ and $Z \mapsto Z \cap U$.

Definition 25.1. A maximal irreducible subset of a topological space X is called an **irreducible component** of X.

Let X be a topological space. Lemma (25.1) shows that every irreducible component is closed. The set of irreducible subsets of X is ordered inductively, as for every chain of irreducible subsets their union is again irreducible. Thus Zorn's lemma implies that every irreducible subset is contained in an irreducible component of X. In particular, every point of X is contained in an irreducible component. This shows that X is the union of its irreducible components.

25.2 Irreducible Affine Algebraic Sets

Proposition 25.2. *Let $Z \subseteq \mathbb{A}^n(k)$ be a closed subset. Then Z is irreducible if and only if $I(Z)$ is a prime ideal. In particular $\mathbb{A}^n(k)$ is irreducible.*

Proof. Suppose $I(Z)$ is a prime ideal and suppose $Z = Z_1 \cup Z_2$ where Z_1, Z_2 are closed subsets of Z. Then $I(Z) = I(Z_1) \cap I(Z_2)$ and since $I(Z)$ is prime, we must either have $I(Z) \supset I(Z_1)$ or $I(Z) \supset I(Z_2)$. Without loss of generality, assume $I(Z) \supset I(Z_1)$. Now we apply V to both sides to get $Z \subset Z_1$. Thus Z is irreducible.

Conversely, suppose Z is irreducible and suppose $fg \in I(Z)$. Then $\langle fg \rangle \subset I(Z)$, and after applying V to both sides, we obtain

$$V\langle fg \rangle = V(f) \cup V(g) \supset Z.$$

Since Z is irreducible, either $V(f) \supset Z$ or $V(g) \supset Z$. Without loss of generality, say $V(f) \supset Z$. Applying I to both sides, we obtain

$$f \in I(V(f)) \subset I(Z),$$

so $I(Z)$ is prime. □

Remark 41. Note that the Nullstellensatz was not used in this proof.

26 Quasi-Compact and Noetherian Topological Spaces

Definition 26.1. A topological space X is called **quasi-compact** if every open covering of X has a finite subcovering.

Definition 26.2. A topological space X is called **Noetherian** if every descending chain

$$X \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$$

of closed subsets of X becomes stationary, i.e. we have $Z_\lambda = Z_{\lambda+1} = Z_{\lambda+2} = \cdots$ for some $\lambda \geq 1$.

Proposition 26.1. *Let X be a topological space. The following are equivalent:*

1. *X is Noetherian.*
2. *Every non-empty set of closed subsets of X has a minimal element.*
3. *Every non-empty set of open subsets of X has a maximal element.*

Proof. The equivalence of (2) and (3) is trivial. Let us show that (1) is equivalent to (2). First assume that X is Noetherian and let $\mathscr{F}$ be a non-empty family closed subsets of X. Assume that $\mathscr{F}$ has no minimal element. Since $\mathscr{F}$ is nonempty, there exists $Z_0 \in \mathscr{F}$. Since $\mathscr{F}$ has no minimal element, there exists $Z_1 \in \mathscr{F}$ such that $Z_0 \supset Z_1$. Again since $\mathscr{F}$ has no minimal element, there exists $Z_2 \in \mathscr{F}$ such that $Z_0 \supset Z_1 \supset Z_2$. Continuing in this way, we obtain a descending chain of closed subsets of X

$$Z_0 \supset Z_1 \supset Z_2 \supset \cdots,$$

which does not become stationary.

Conversely, suppose that every non-empty set of closed subsets of X has a minimal element. Let

$$X \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \tag{31}$$

be a descending chain of closed subsets. Then the $\{Z_i\}_{i\in\mathbb{N}}$ is a non-empty family of closed subsets of X, and hence must have a minimal element. This implies that the chain (31) becomes stationary. □

Lemma 26.1. *Let X be a topological space that has a finite covering $X = \bigcup_{i=1}^r X_i$ by Noetherian subspaces. Then X is itself Noetherian.*

Proof. Let

$$X \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$$

be a descending chain of closed subsets of X. Then for each i, we obtain a descending chain of closed subsets of X_i :

$$X_i \supseteq Z_1 \cap X_i \supseteq Z_2 \cap X_i \supseteq \cdots.$$

Since X_i is Noetherian, this chain must terminate, say at $Z_{\lambda_i} \cap X_i$. Let $\lambda = \max_i(\lambda_i)$. Then for any $\mu \geq \lambda$, we have

$$\begin{aligned} Z_\mu &= X \cap Z_\mu \\ &= \left(\bigcup_{i=1}^r X_i\right) \cap Z_\mu \\ &= \bigcup_{i=1}^r (X_i \cap Z_\mu) \\ &= \bigcup_{i=1}^r (X_i \cap Z_{\mu+1}) \\ &= \left(\bigcup_{i=1}^r X_i\right) \cap Z_{\mu+1} \\ &= Z_{\mu+1}. \end{aligned}$$

□

Lemma 26.2. *Let X be a Noetherian topological space. Then*

1. *Every subspace of X is Noetherian.*
2. *Every open subset of X is quasi-compact (in particular, X is quasi-compact).*
3. *Every closed subset $Z \subseteq X$ has only finitely many irreducible components.*

Proof.

1. Let Z be a subspace of X and suppose

$$Z \supseteq Z \cap Z_1 \supseteq Z \cap Z_2 \supseteq \cdots$$

is a descending chain of closed subsets of Z. Then

$$X \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$$

is a descending chain of closed subsets of X. Since X is Noetherian, we must have $Z_\mu = Z_{\mu+1}$ for all $\mu \geq \lambda$ for some $\lambda \geq 1$. In particular, this implies $Z \cap Z_\mu = Z \cap Z_{\mu+1}$ for all $\mu \geq \lambda$ for some $\lambda \geq 1$.

2. By (1), it suffices to show that X is quasi-compact. Let $\{U_i\}_{i \in I}$ be an open cover of X and let $\mathcal{U}$ be the set of those open subsets of X that are finite unions of the subsets of U_i. As X is Noetherian, $\mathcal{U}$ has a maximal element V. Clearly $V = X$, otherwise there existed an U_i such that V is properly contained in $V \cup U_i \in \mathcal{U}$. This shows that $\{U_i\}_{i \in I}$ has a finite subcovering.

3. It suffices to show that every Noetherian space X can be written as a finite union of irreducible subsets. If the set $\mathcal{M}$ of closed subsets of X that cannot be written as a finite union of irreducible subsets were non-empty, there existed a minimal element $Z \in \mathcal{M}$. The set Z is not irreducible and thus is the union of two proper closed subsets which do not lie in $\mathcal{M}$. This leads to a contradiction.

□

Proposition 26.2. *Let $X \subseteq \mathbb{A}^n(k)$ be any subspace. Then X is Noetherian.*

Proof. By Lemma (26.2) it suffices to show that $\mathbb{A}^n(k)$ is Noetherian. But descending chains of closed subsets of $\mathbb{A}^n(k)$ correspond to ascending chains of radical ideals of $k[T_1, \dots, T_n]$. As $k[T_1, \dots, T_n]$ is Noetherian by Hilbert's basis theorem, this proves the proposition. □

27 Dimension

Definition 27.1. Let X be a (non-empty) irreducible topological space.

1. Let $\mathrm{Chain}(X)$ denote the set of all **chains of irreducible closed subsets** of X, that is,

$$\mathrm{Chain}(X) := \{\wp = (\emptyset \neq X_0 \subset X_1 \subset \cdots \subset X_n \subset X) \mid X_i \text{ is irreducible closed subset of } X\}.$$

2. If $\wp = (\emptyset \neq X_0 \subset X_1 \subset \cdots \subset X_n \subset X) \in \mathrm{Chain}(X)$, then $\mathbf{length}(\wp) := n$.

3. The **dimension** of X is defined as $\dim(X) = \sup\{\mathrm{length}(\wp) \mid \wp \in \mathrm{Chain}(X)\}$.

4. If X is any Noetherian topological space, then the dimension of X is defined to be the supremum of the dimensions of its irreducible components.

5. A space of dimension 1 is called a **curve** and a space of dimension 2 is called a **surface**.

Example 27.1. Let $X = V(T_1T_3, T_2T_3)$, $X_1 = V(T_3)$, and $X_2 = V(T_1, T_2)$. Then X_1 and X_2 are the irreducible components of X. One can show that a maximal chain of irreducible closed subsets of X_1 is given by

$$\wp_1 = (V(T_1, T_2, T_3) \subset V(T_1, T_3) \subset V(T_3)),$$

and thus $\dim(X_1) = 2$. Similary one can show that $\dim(X_2) = 1$. Therefore $\dim(X) = 2$.

28 Spec A as a topological space

We start with the following basic definition. Let A be a ring. The **prime spectrum** of A is the set

$$\operatorname{Spec} A := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal of } A\}.$$

We will now endow $\operatorname{Spec} A$ with the structure of a topological space. For every subset S of A, we denote by $\mathrm{V}(S)$ to be the set of prime ideals of A which contain S. Similarly we denote by $\mathrm{D}(S)$ to be the complement of $\mathrm{V}(S)$ in $\operatorname{Spec} A$. In other words, $\mathrm{D}(S)$ consists of the set of all prime ideals of A which do not contain S. Clearly, if $\mathfrak{a}$

is the ideal generated by S, then $\mathrm{V}(S) = \mathrm{V}(\mathfrak{a})$ and $\mathrm{D}(S) = \mathrm{D}(\mathfrak{a})$. For any $f \in A$, we write $\mathrm{V}(f)$ and $\mathrm{D}(f)$ instead of $\mathrm{V}(\{f\})$ and $\mathrm{D}(\{f\})$ in order to simplify notation. Let

$$\tau_A = \{\mathrm{D}(\mathfrak{a}) \mid \mathfrak{a} \text{ is an ideal of } A\}.$$

It will follow from the following proposition that τ_A gives $\operatorname{Spec} A$ the structure of a topological space.

Proposition 28.1. *The following statements holds.*

1. *We have* $\mathrm{V}(0) = \operatorname{Spec} A$ *and* $\mathrm{V}(1) = \emptyset$. *Equivalently, we have* $\mathrm{D}(0) = \emptyset$ *and* $\mathrm{D}(1) = \operatorname{Spec} A$

2. *For two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$, *we have*
$$\mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{V}(\mathfrak{a}\mathfrak{b}) = \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b}).$$
Equivalently, we have
$$\mathrm{D}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{D}(\mathfrak{a}\mathfrak{b}) = \mathrm{D}(\mathfrak{a}) \cap \mathrm{D}(\mathfrak{b}).$$

3. *For every family* $\{\mathfrak{a}_i\}_{i \in I}$ *of ideals of* A, *we have*
$$\mathrm{V}\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = \mathrm{V}\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} \mathrm{V}(\mathfrak{a}_i).$$
Equivalently, we have
$$\mathrm{D}\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = \mathrm{D}\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcup_{i \in I} \mathrm{D}(\mathfrak{a}_i).$$

Proof. 1. This is trivial.

2. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{b}$, it follows that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$ since V is inclusion-reversing. It remains to show that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$. Assume for a contradiction that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \not\subseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$. Then there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \in \mathrm{V}(\mathfrak{a}\mathfrak{b})$ but $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a})$ and $\mathfrak{p} \notin \mathrm{V}(\mathfrak{b})$. In other words, there exists a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \supseteq \mathfrak{a}\mathfrak{b}$ but $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $\mathfrak{p} \not\supseteq \mathfrak{b}$. Hence there exists $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x, y \notin \mathfrak{p}$. However $xy \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, which contradicts the fact that $\mathfrak{p}$ is prime.

3. We have $\mathrm{V}\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = \mathrm{V}\left(\sum_{i \in I} \mathfrak{a}_i\right)$ since $\sum_{i \in I} \mathfrak{a}_i$ is the ideal generated by $\bigcup_{i \in I} \mathfrak{a}_i$. We have $\mathrm{V}\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} \mathrm{V}(\mathfrak{a}_i)$ since $\mathfrak{p} \supseteq \sum_{i \in I} \mathfrak{a}_i$ if and only if $\mathfrak{p} \supseteq \mathfrak{a}_i$ for all $i \in I$ and for all prime ideals $\mathfrak{p}$ of A. □

Thus τ_A gives $\operatorname{Spec} A$ the structure of a topological space. The topology τ_A is called the **Zariski topology**. The closed subsets of $\operatorname{Spec} A$ are of the form $\mathrm{V}(\mathfrak{a})$ and the open subsets of $\operatorname{Spec} A$ are of the form $\mathrm{D}(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of A. Open sets of the form $\mathrm{D}(a)$ where $a \in A$ are given a special name: they are called **principal open sets**.

Proposition 28.2. *The following two statements hold:*

1. *The collection* $\{\mathrm{D}(a) \mid a \in A\}$ *is a basis for* $\operatorname{Spec} A$.

2. *The principal open sets are quasi-compact. In particular,* $\operatorname{Spec} A$ *is quasi-compact.*

Proof. 1. First note that $\{\mathrm{D}(a) \mid a \in A\}$ covers $\operatorname{Spec} A$ since $\mathrm{D}(1) = \operatorname{Spec} A$. Next, let $\mathrm{D}(\mathfrak{a})$ and $\mathrm{D}(\mathfrak{b})$ be two open subsets of $\operatorname{Spec} A$ and let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \in \mathrm{D}(\mathfrak{a}) \cap \mathrm{D}(\mathfrak{b}) = \mathrm{D}(\mathfrak{a} \cap \mathfrak{b})$. Then $\mathfrak{p} \not\supseteq \mathfrak{a} \cap \mathfrak{b}$, so there exists an $a \in \mathfrak{a} \cap \mathfrak{b}$ such that $a \notin \mathfrak{p}$, or equivalently $\mathfrak{p} \in \mathrm{D}(a)$. Since D is inclusion-preserving, we see that $\mathfrak{p} \in \mathrm{D}(a) \subseteq \mathrm{D}(\mathfrak{a} \cap \mathfrak{b})$. It follows that $\{\mathrm{D}(a) \mid a \in A\}$ is a basis for $\operatorname{Spec} A$.

2. Let $x \in A$ and let $\{y_i\}_{i \in I}$ be a collection of elements of A such that $\mathrm{D}(x) \subseteq \bigcup_{i \in I} \mathrm{D}(y_i)$, or equivalently, such that$\mathrm{V}(x) \supseteq \mathrm{V}(\langle y_i \mid i \in I \rangle)$. Applying I to both sides gives us $x \in \sqrt{\langle x \rangle} \subseteq \sqrt{\langle y_i \mid i \in I \rangle}$. Hence there exists $n \in \mathbb{N}$ such that $x^n \in \langle y_i \mid i \in I \rangle$ which implies there exists $y_{i_1}, \dots, y_{i_k} \in \{y_i\}_{i \in I}$, and $a_{i_1}, \dots, a_{i_k} \in A$ such that

$$x^n = a_{i_1} y_{i_1} + \cdots + a_{i_k} y_{i_k}.$$

In particular, we see that $\mathrm{V}(x) \supseteq \mathrm{V}(y_{i_1}, \dots, y_{i_k})$ or equivalently that $\mathrm{D}(x) \subseteq \mathrm{D}(y_{i_1}) \cup \cdots \cup \mathrm{D}(y_{i_k})$. It follows that $\mathrm{D}(x)$ is quasi-compact. □

Proposition 28.3. *Let* U *be an open subset of* $\operatorname{Spec} A$. *Then* U *is quasi-compact if and only if it can be expressed in the form* $U = \mathrm{D}(\mathfrak{a})$ *where* $\mathfrak{a}$ *is a finitely generated ideal of* A.

Proof. Suppose $U = \mathrm{D}(\mathfrak{a})$ where $\mathfrak{a}$ is a finitely generated ideal of A, say $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$. Then since $\mathrm{D}(x_1, \dots, x_n) = \mathrm{D}(x_1) \cup \cdots \mathrm{D}(x_n)$, we see that U ios a finite union of quasi-compact spaces. It follows that U is quasi-compact. .

Conversely, suppose that U is quasi-compact. Write $U = \mathrm{D}(\mathfrak{b})$ where $\mathfrak{b} = \langle b_i \mid i \in I \rangle$. Then $\{\mathrm{D}(b_i)\}_{i \in I}$ covers U since $U = \mathrm{D}(\mathfrak{b}) = \bigcup_{i \in I} \mathrm{D}(b_i)$. Since U is quasi-compact, there exists a finite subcovering of $\{\mathrm{D}(b_i)\}_{i \in I}$ which covers U, say $\{\mathrm{D}(b_{i_1}), \dots, \mathrm{D}(b_{i_n})\}$. In particular, if we set $\mathfrak{a} = \langle b_{i_1}, \dots, b_{i_n} \rangle$, then we see that

$$U = \mathrm{D}(b_{i_1}) \cup \cdots \cup \mathrm{D}(b_{i_n}) = \mathrm{D}(\mathfrak{a})$$

where $\mathfrak{a}$ is finitely generated. □

Remark 42. If x is a point in Spec A, we will often write $\mathfrak{p}_x$ instead of x when we think of x as a prime ideal of A.

Proposition 28.4. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Then $V(\mathfrak{a}) = V\left(\sqrt{\mathfrak{a}}\right)$.*

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, we have $V(\mathfrak{a}) \supset V(\sqrt{\mathfrak{a}})$. For the reverse inclusion, let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \supset \mathfrak{a}$. Assume, for a contradiction, that $\mathfrak{p} \not\supset \sqrt{\mathfrak{a}}$. Choose $a \in \sqrt{\mathfrak{a}}$ such that $a \notin \mathfrak{p}$. Then $a^n \in \mathfrak{a} \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. But this contradicts the fact that $\mathfrak{p}$ is a prime ideal. □

For every subset Y of Spec A, we set

$$\mathrm{I}(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

We obtain an inclusion-reversing map $Y \mapsto \mathrm{I}(Y)$ from the set of subsets of Spec A to the set of ideals of A. Note that $\mathrm{I}(\emptyset) = A$. The maps V, D, and I are all related as follows.

Proposition 28.5. *Let $\mathfrak{a}$ an ideal in A and let Y a subset of* Spec A. *We have*

1. $\sqrt{\mathrm{I}(Y)} = \mathrm{I}(Y)$.
2. $\mathrm{IV}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$.
3. $\mathrm{VI}(Y) = \overline{Y}$.
4. $\mathrm{ID}(\mathfrak{a}) = \sqrt{0} : \mathfrak{a}$.
5. $\mathrm{DI}(Y) = \bigcap_{\mathfrak{p} \in Y} \mathrm{D}(\mathfrak{p})$.
6. *The maps*

$$\{\textit{ideals } \mathfrak{a} \textit{ of } A \textit{ with } \mathfrak{a} = \textit{rad}(\mathfrak{a})\} \underset{I}{\overset{V}{\rightleftarrows}} \{\textit{closed subsets } Y \textit{ of Spec } A\}$$

are mutually inverse bijections.

Proof. 1. The relation $\mathfrak{a} = \sqrt{\mathfrak{a}}$ means that $f^n \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ for all $f \in A$. This certainly holds for prime ideals and therefore for arbitrary intersections of prime ideals as well.

2. This follows from the fact that the radical of an ideal equals the intersection of all prime ideals containing it.

3. Let $\mathfrak{b}$ be an ideal of A. Observe that

$$\begin{aligned} \mathrm{V}(\mathfrak{b}) \supseteq Y &\iff \mathrm{IV}(\mathfrak{b}) \subseteq \mathrm{I}(Y) \\ &\iff \sqrt{\mathfrak{b}} \subseteq \mathrm{I}(Y) \\ &\iff \mathrm{V}(\sqrt{\mathfrak{b}}) \supseteq \mathrm{VI}(Y) \\ &\iff \mathrm{V}(\mathfrak{b}) \supseteq \mathrm{VI}(Y). \end{aligned}$$

Therefore $\mathrm{VI}(Y)$ is the smallest closed subset of Spec A which contains Y.

4. We first show that $\sqrt{0} : \mathfrak{a} \subseteq \mathrm{ID}(\mathfrak{a})$. Let $x \in \sqrt{0} : \mathfrak{a}$ and assume (to obtain a contradiction) that $x \notin \mathrm{ID}(\mathfrak{a})$. Since $x \notin \mathrm{ID}(\mathfrak{a})$, there exists a prime $\mathfrak{p}$ of A such that $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $x \notin \mathfrak{p}$. Since $x \in \sqrt{0} : \mathfrak{a}$, we have $x\mathfrak{a} \subseteq \sqrt{0} \subseteq \mathfrak{p}$. In particular, either $\mathfrak{p} \supseteq \mathfrak{a}$ or $x \in \mathfrak{p}$. This is a contradiction. Thus we have $\sqrt{0} : \mathfrak{a} \subseteq \mathrm{ID}(\mathfrak{a})$.

Now we will show that $\sqrt{0} : \mathfrak{a} \supseteq \mathrm{ID}(\mathfrak{a})$. Let $x \in \mathrm{ID}(\mathfrak{a})$ (so x belongs to every prime ideal which does not contain $\mathfrak{a}$) and assume (to obtain a contradiction) that $x \notin \sqrt{0} : \mathfrak{a}$. Since $x \notin \sqrt{0} : \mathfrak{a}$, there exists $a \in \mathfrak{a}$ such that $ax \notin \sqrt{0}$. In particular, $\{(ax)^n\}_{n \in \mathbb{N}}$ forms a multiplicative set, and so we can localize at ax. Let $\mathfrak{q}$ be a prime ideal in A_{ax} and let $\mathfrak{p} := \iota_{ax}^{-1}(\mathfrak{q})$, where $\iota_{ax} \colon A \to A_{ax}$ is the canonical ring homomorphism. Then $\mathfrak{p}$ is a prime ideal in A which does not contain ax. This implies that $\mathfrak{p}$ does not contain $\mathfrak{a}$ nor x (if it did, then it'd certainly contain ax). This is a contradiction. Thus we have $\sqrt{0} : \mathfrak{a} \supseteq \mathrm{ID}(\mathfrak{a})$.

5. We have

$$\begin{aligned} \mathrm{DI}(Y) &= \mathrm{D}\left(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\right) \\ &= \bigcap_{\mathfrak{p}\in Y} \mathrm{D}(\mathfrak{p}) \end{aligned}$$

6. This follows from part 2.

□

28.1 Properties of Spec A

Example 28.1. Let $A = K[x,y]$, $\mathfrak{a} = \langle x^2, y^2\rangle$, and $\mathfrak{b} = \langle x^2, xy, y^2\rangle$. Even though $\mathfrak{a} \subset \mathfrak{b}$ (where the inclusion is strict), we have $V(\mathfrak{a}) = V(\mathfrak{b})$, since $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Proposition 28.6. *Let A be a ring. A subset Y of Spec A is irreducible if and only if $\mathfrak{p} := I(Y)$ is a prime ideal. In this case $\{\mathfrak{p}\}$ is dense in $\overline{Y}$.*

Proof. Assume that Y is irreducible. Let $f, g \in A$ with $fg \in \mathfrak{p}$. Then

$$Y \subseteq V(fg) = V(f) \cup V(g).$$

As Y is irreducbile, either $Y \subseteq V(f)$ or $Y \subseteq V(g)$ which implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.
Conversely let $\mathfrak{p}$ be a prime. Then by Proposition (28.5),

$$\overline{Y} = V(\mathfrak{p}) = V(I(\{\mathfrak{p}\}) = \overline{\{\mathfrak{p}\}}.$$

Therefore $\overline{Y}$ is the closure of the irreducible set $\{\mathfrak{p}\}$ and therefore irreducible. This implies that the dense subset Y is also irreducible. □

Note that for arbitrary irreducible subsets Y the prime ideal $I(Y)$ is not necessarily a point in Y. But this is clearly true if Y is closed, or more generally, if Y is locally closed.

Corollary 5. *The map $\mathfrak{p} \mapsto V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is a bijection from Spec A onto the set of closed irreducible subsets of Spec A. Via this bijection, the minimal prime ideals of A correspond to the irreducible components of Spec A.*

Definition 28.1. Let X be an arbitrary topological space.

1. A point $x \in X$ is called **closed** if the set $\{x\}$ is closed,
2. We say that a point $\eta \in X$ is a **generic point** if $\overline{\{\eta\}} = X$.
3. We say x and x' be two points of X. We say that x is a **generization** or that x' is a **specialization** of x if $x' \in \overline{\{x\}}$.
4. A point $x \in X$ is called a **maximal point** if its closure $\overline{\{x\}}$ is an irreducible component of X.

Thus a point $\eta \in X$ is generic if and only if it is a generization of every point of X. As the closure of an irreducible set is again irreducible, the existence of a generic point implies that X is irreducible.

Example 28.2. If $X =$ Spec A is the spectrum of a ring, the notions introduced in Definition (28.1) have the following algebraic meaning.

1. A point $x \in X$ is closed if and only if $\mathfrak{p}_x$ is a maximal ideal.
2. A point $\eta \in X$ is a generic point of X if and only if $\mathfrak{p}_\eta$ is the unique minimal prime ideal. This exists if and only if the nilradical of A is a prime ideal.
3. A point x is a generization of a point x' (in other words, x' is a specialization of x) if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_{x'}$.
4. A point $x \in X$ is a maximal point if and only if $\mathfrak{p}_x$ is a minimal prime ideal.

28.2 The Functor $A \mapsto \operatorname{Spec} A$

We will now show that $A \mapsto \operatorname{Spec} A$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\varphi\colon A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a prime ideal of B, then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of. Therefore we obtain a map ${}^{\mathrm{a}}\varphi = \operatorname{Spec}\varphi$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ given by

$$
{}^{\mathrm{a}}\varphi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})
$$

for all $\mathfrak{q} \in \operatorname{Spec} B$. The following proposition will show that ${}^{\mathrm{a}}\varphi$ is a continuous map.

Proposition 28.7. *Let S be a subset of A and let $\mathfrak{b}$ be an ideal of B. Then*

1. *We have $({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{V}(S)) = \mathrm{V}(\varphi(S))$ and $({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{D}(S)) = \mathrm{D}(\varphi(S))$. In particular, ${}^{\mathrm{a}}\varphi$ is continuous.*
2. *We have $\overline{{}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))} = \mathrm{V}(\varphi^{-1}(\mathfrak{b}))$.*
3. *Assume that $\varphi\colon A \to B$ is an integral extension. Then ${}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b})) = \mathrm{V}(\varphi^{-1}(\mathfrak{b}))$. In particular, ${}^{\mathrm{a}}\varphi$ is a closed map in this case.*

Proof. 1. Let $\mathfrak{q}$ be a prime ideal of B. Then

$$
\begin{aligned}
\mathfrak{q} \in ({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{V}(S)) &\iff ({}^{\mathrm{a}}\varphi)(\mathfrak{q}) \in \mathrm{V}(S) \\
&\iff \varphi^{-1}(\mathfrak{q}) \in \mathrm{V}(S) \\
&\iff \varphi^{-1}(\mathfrak{q}) \supseteq S \\
&\iff \mathfrak{q} \supseteq \varphi(S) \\
&\iff \mathfrak{q} \in \mathrm{V}(\varphi(S)).
\end{aligned}
$$

It follows that $({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{V}(S)) = \mathrm{V}(\varphi(S))$. Similarly, we have

$$
\begin{aligned}
({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{D}(S)) &= ({}^{\mathrm{a}}\varphi)^{-1}((\mathrm{V}(S))^c) \\
&= (({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{V}(S)))^c \\
&= (\mathrm{V}(\varphi(S))^c \\
&= \mathrm{D}(\varphi(S)).
\end{aligned}
$$

2. By Proposition (28.5), we have

$$
\begin{aligned}
\overline{{}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))} &= \mathrm{VI}({}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))) \\
&= \mathrm{VI}(\{{}^{\mathrm{a}}\varphi(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{b})\}) \\
&= \mathrm{VI}(\{\varphi^{-1}(\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{b})\}) \\
&= \mathrm{V}\left(\bigcap_{\mathfrak{q} \in \mathrm{V}(\mathfrak{b})} \varphi^{-1}(\mathfrak{q})\right) \\
&= \mathrm{V}\left(\varphi^{-1}\left(\bigcap_{\mathfrak{q} \in \mathrm{V}(\mathfrak{b})} \mathfrak{q}\right)\right) \\
&= \mathrm{V}(\varphi^{-1}(\sqrt{\mathfrak{b}})) \\
&= \mathrm{V}\left(\sqrt{\varphi^{-1}(\mathfrak{b})}\right) \\
&= \mathrm{V}(\varphi^{-1}(\mathfrak{b}))
\end{aligned}
$$

3. We want to show ${}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b})) = \mathrm{V}(\varphi^{-1}(\mathfrak{b}))$. Let

□

The proposition shows in particular that ${}^{\mathrm{a}}\varphi$ is continuous. As ${}^{\mathrm{a}}(\psi \circ \varphi) ={}^{\mathrm{a}} \varphi \circ{}^{\mathrm{a}} \psi$ for any ring homomorphism $\psi\colon B \to C$, we obtain a contravariant functor $A \mapsto \operatorname{Spec} A$ from the category of rings to the category of topological spaces.

Corollary 6. *The map ${}^{a}\varphi$ is dominant (i.e. its image is dense in* $\operatorname{Spec} A$*) if and only if every element of* $\ker\varphi$ *is nilpotent.*

Proof. We apply (**??**) to (2) $\mathfrak{b} = 0$. □

Proposition 28.8. *Let A be a ring.*

1. *Let $\varphi\colon A \to B$ be a surjective homomorphism of rings with kernel $\mathfrak{a}$. Then ${}^{a}\varphi$ induces a homeomorphism of ${}^{a}\varphi\colon \operatorname{Spec} B \to \mathrm{V}(\mathfrak{a})$ from $\operatorname{Spec} B$ onto the closed subset $\mathrm{V}(\mathfrak{a})$ of $\operatorname{Spec} A$.*

2. *Let S be a multiplicative subset of A and let $\varphi\colon A \to S^{-1}A =: B$ be the canonical homomorphism. Then ${}^{a}\varphi$ induces a homeomorphism ${}^{a}\varphi\colon \operatorname{Spec} A_S \to \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\}$ from $\operatorname{Spec} A_S$ onto the subspace of $\operatorname{Spec} A$ consisting of prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \emptyset$.*

Proof. 1. We first check that ${}^{a}\varphi$ lands in $\mathrm{V}(\mathfrak{a})$. Let $\mathfrak{q}$ be a prime ideal of B. The $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of A which contains $\mathfrak{a}$. It follows that ${}^{a}\varphi$ lands in $\mathrm{V}(\mathfrak{a})$. Let us now show that ${}^{a}\varphi$ is a bijection. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two distinct prime ideals of B. Then $\varphi^{-1}(\mathfrak{q}') \neq \varphi^{-1}(\mathfrak{q})$ since φ is surjective. Thus ${}^{a}\varphi$ is injective. To see that ${}^{a}\varphi$ is surjective, let $\mathfrak{p}$ be a prime ideal of A which contains $\mathfrak{a}$. We claim that $\varphi(\mathfrak{p})$ is a prime ideal of B and that $\mathfrak{p} = \varphi^{-1}(\varphi(\mathfrak{p}))$ which will establish ${}^{a}\varphi$ being surjective.

First, to see that $\varphi(\mathfrak{p})$ is a prime ideal of B, let $\varphi(a), \varphi(a') \in B$ such that $\varphi(a)\varphi(a') \in {}'(\mathfrak{p})$ and $\varphi(a') \notin \varphi(\mathfrak{p})$ where $a, a' \in A$ (every element in B can be expressed in the form $\varphi(a)$ for some $a \in A$ since φ is surjective). Then $a' \notin \mathfrak{p}$ and there exists $x \in \mathfrak{p}$ such that $\varphi(a)\varphi(a') = \varphi(x)$, or in other words, such that $aa' - x \in \mathfrak{a}$. Since $\mathfrak{p} \supseteq \mathfrak{a}$, this implies $aa' \in \mathfrak{p}$, and since $\mathfrak{p}$ is prime and $a' \notin \mathfrak{p}$, this implies $a \in \mathfrak{p}$. Thus $\varphi(a) \in \varphi(\mathfrak{p})$. It follows that $\varphi(\mathfrak{p})$ is prime. Next we will show that $\varphi^{-1}(\varphi(\mathfrak{p})) = \mathfrak{p}$. Clearly we have $\varphi^{-1}(\varphi(\mathfrak{p})) \supseteq \mathfrak{p}$. For the reverse inclusion, let $a \in \varphi^{-1}(\varphi(\mathfrak{p}))$, so $\varphi(a) \in \varphi(\mathfrak{p})$, which means $\varphi(a) = \varphi(x)$ for some $x \in \mathfrak{p}$. It follows that $a - x \in \mathfrak{a}$. Since $\mathfrak{p} \supseteq \mathfrak{a}$ and $x \in \mathfrak{p}$, it follows that $a \in \mathfrak{p}$. Thus we have the reverse inclusion $\varphi^{-1}(\varphi(\mathfrak{p})) \subseteq \mathfrak{p}$.

Thus ${}^{a}\varphi$ is a continuous bijection. To see that it is a homeomorphism, we need to show that ${}^{a}\varphi$ maps closed sets to closed sets. To see this, note that a prime ideal $\mathfrak{q}$ of B contains an ideal $\mathfrak{b}$ of B if and only if the prime ideal $\varphi^{-1}(\mathfrak{q})$ of A contains the ideal $\varphi^{-1}(\mathfrak{b})$ of A. In particular, we have

$$ {}^{a}\varphi(\mathrm{V}(\mathfrak{b})) = \mathrm{V}(\varphi^{-1}(\mathfrak{b})) \cap \mathrm{V}(\mathfrak{a}). $$

Therefore ${}^{a}\varphi$ is a homeomorphism onto its image.

2. Left as an exercise.

□

Corollary 7. *Let A_{red} be the reduced ring of A obtained by quotienting out all nilpotent elements in A, and let $\pi\colon A \to A_{\mathrm{red}}$ be the quotient homomorphism. The ${}^{a}\pi$ induces a homeomorphism $\operatorname{Spec} A \cong \operatorname{Spec} A_{\mathrm{red}}$.*

Proof. Recall that the set of all nilpotent elements of A is given by $\sqrt{0}$. In particular, since $\mathrm{V}(\sqrt{0}) = \mathrm{V}(0) = \operatorname{Spec} A$, the corrollary follows immediately from Proposition (28.8). □

Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A. Proposition (28.8) shows that the passage from A to $A_{\mathfrak{p}}$ cuts out all prime ideals except those contained in $\mathfrak{p}$. The passage from A to $A/\mathfrak{q}$ cuts out all prime ideals except those containing $\mathfrak{q}$. Hence, if $\mathfrak{q} \subseteq \mathfrak{p}$, then by localizing with respect to $\mathfrak{p}$ and then taking the quotient modulo $\mathfrak{q}$ (in either order as these operations commute) we obtain a ring whose prime ideals are those prime ideals of A that lie between $\mathfrak{q}$ and $\mathfrak{p}$. For $\mathfrak{q} = \mathfrak{p}$, we obtain the field

$$ \kappa(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Frac}(A/\mathfrak{p}), $$

which is called the **residue field** at $\mathfrak{p}$.

29 Spectrum of a Ring as a Locally Ringed Space

Let A be a ring. We will now endow the topological space Spec A with the structure of a locally ringed space and obtain a functor $A \mapsto \operatorname{Spec} A$ from the category of rings to the category of locally ringed spaces which we will show to be fully faithful.

29.1 Structure Sheaf on Spec A

We set $X = \operatorname{Spec} A$. Recall that the principal open sets $\mathrm{D}(f)$ for $f \in A$ form a basis of the topology of X. We will define a presheaf $\mathcal{O}_X$ on this basis and then prove that the sheaf axioms are satisfied. The basic idea is this: looking back at the analogy with prevarieties, we certainly want to have $\mathcal{O}_X(X) = A$. More generally, for $f \in A$, we consider the localization A_f of A. Denote by $\iota_f\colon A \to A_f$ the canonical ring homomorphism $a \mapsto a/1$. By Proposition (28.8), ${}^{a}\iota_f$ is a homeomorphism of $\operatorname{Spec} A_f$ onto $\mathrm{D}(f)$. So it seems reasonable to set $\mathcal{O}_X(\mathrm{D}(f)) = A_f$. Let us check that this is a sensible definition: we must check that $A_f = A_g$ whenever $\mathrm{D}(f) = \mathrm{D}(g)$, define restriction maps, and check that the sheaf axioms are satisfied.

For $f, g \in A$, we have $\mathrm{D}(f) \subseteq \mathrm{D}(g)$ if and only if there exists an integer $n \geq 1$ such that $f^n \in \langle g \rangle$ or, equivalently, $g/1 \in (A_f)^{\times}$. In this case we obtain a unique ring homomorphism $\rho_{f,g}\colon A_g \to A_f$ such that $\rho_{f,g} \circ \iota_g = \iota_f$ by the universal mapping property of localization. Whenever $\mathrm{D}(f) \subseteq \mathrm{D}(g) \subseteq \mathrm{D}(h)$, we have $\rho_{f,g} \circ \rho_{g,h} = \rho_{f,h}$. In particular, if $\mathrm{D}(f) = \mathrm{D}(g)$, then $\rho_{f,g}$ is an isomorphism, which we use to identify A_g and A_f. Therefore we can define

$$\mathcal{O}_X(\mathrm{D}(f)) := A_f$$

and obtain a presheaf of rings on the basis $\mathcal{B} = \{\mathrm{D}(f) \mid f \in A\}$ for the topological space Spec A. The restriction maps are the ring homomorphism $\rho_{f,g}$.

Theorem 29.1. *The presheaf $\mathcal{O}_X$ is a sheaf.*

We denote the sheaf of rings on X associated to $\mathcal{O}_X$ again by $\mathcal{O}_X$. For all points $x \in X = \operatorname{Spec} A$, we have

$$\mathcal{O}_{X,x} = \operatorname{colim}_{\mathrm{D}(f) \ni x} \mathcal{O}_X(\mathrm{D}(f)) = \operatorname{colim}_{f \notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x}.$$

In particular, $(X, \mathcal{O}_X)$ is a locally ringed space. We will often simply write Spec A instaed of $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Proof. Let $\mathrm{D}(f)$ be a principal open set and let $\{\mathrm{D}(f_i)\}_{i \in I}$ be an open covering over $\mathrm{D}(f)$. We have to show the following two properties:

1. Let $s \in \mathcal{O}_X(\mathrm{D}(f))$ be such that $s|_{\mathrm{D}(f_i)} = 0$ for all $i \in I$. Then $s = 0$.
2. For $i \in I$, let $s_i \in \mathcal{O}_X(\mathrm{D}(f_i))$ be such that $s_i|_{\mathrm{D}(f_i f_j)} = s_j|_{\mathrm{D}(f_i f_j)}$ for all $i, j \in I$. Then there exists $s \in \mathcal{O}_X(\mathrm{D}(f))$ such that $s|_{\mathrm{D}(f_i)} = s_i$ for all $i \in I$.

As $\mathrm{D}(f)$ is quasi-compact, we can assume that I is finite. Restricting the presheaf $\mathcal{O}_X$ to $\mathrm{D}(f)$ and replacing A by A_f if necessary, we may assume that $f = 1$ and hence $\mathrm{D}(f) = X$ to ease the notation. The relation $X = \bigcup_{i \in I} \mathrm{D}(f_i)$ is equivalent to $\langle f_i \mid i \in I \rangle = A$ (indeed $\sqrt{\mathfrak{a}} = A$ implies $\mathfrak{a} = A$). As $\mathrm{D}(f_i) = \mathrm{D}(f_i^n)$ for all integers $n \geq 1$ there exists elements $b_i \in A$ (depending on n) such that

$$\sum_{i \in I} b_i f_i^n = 1. \tag{32}$$

Proof of 1: let $s = a \in A$ be such that the image of a in A_{f_i} is zero for all $i \in I$. As I is finite, there exists an integer $n \geq 1$, independent of i, such that $f_i^n a = 0$. By (32),

$$a = \left(\sum_{i \in I} b_i f_i^n \right) a = 0.$$

Proof of 2: as I is finite, we can write $s_i = a_i / f_i^n$ for some n independent of i. By hypothesis, the images of a_i / f_i^n and of a_j / f_j^n in $A_{f_i f_j}$ are equal for all $i, j \in I$. Therefore there exists an integer $m \geq 1$ (which again we can choose independent of i and j) such that

$$(f_i f_j)^m (f_j^n a_i - f_i^n a_j) = 0.$$

Replacing a_i by $f_i^m a_i$ and n by $n + m$ (which does not change s_i), we see that $f_j^n a_i = f_i^n a_j$ for all $i, j \in I$. We set

$$s := \sum_{j \in I} b_j a_j \in A,$$

where the b_j are the elements in (32). Then

$$\begin{aligned} f_i^n s &= f_i^n \left(\sum_{j \in I} b_j a_j \right) \\ &= \sum_{j \in I} b_j (f_i^n a_j) \\ &= \sum_{j \in I} b_j (f_j^n a_i) \\ &= \left(\sum_{j \in I} b_j f_j^n \right) a_i \\ &= a_i. \end{aligned}$$

This means that the image of s in A_{f_i} is s_i. □

Remark 43. We have just proved that the sequence

$$0 \longrightarrow A \longrightarrow \bigoplus_{i \in I} A_{f_i} \longrightarrow \bigoplus_{i,j \in I} A_{f_i f_j}$$

is exact.

29.2 The Functor $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

Definition 29.1. A locally ringed space $(X, \mathcal{O}_X)$ is called an **affine scheme**, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$. A **morphism** of affine schemes is a morphism of locally ringed spaces. We obtain the category of affine schemes which we denote by **Aff**.

Recall that **Ring** denotes the category of commutative rings and ring homomorphisms. We can view Spec as a contravariant functor from **Ring** to **Aff** which takes a ring A to the affine scheme $\operatorname{Spec} A = X$ and which takes a ring homomorphism $\phi\colon A \to B$ to the morphism $f\colon Y \to X$ of locally ringed spaces where $Y = \operatorname{Spec} B$, where the underlying continuous map $f\colon Y \to X$ is defined by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for all primes $\mathfrak{q}$ of B, and where the morphism of sheaves $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is defined as follows: given a principal open subset $\mathrm{D}(s)$ of X, we define $f^\flat_{\mathrm{D}(s)}\colon \mathcal{O}_X(\mathrm{D}(s)) \to \mathcal{O}_Y(\mathrm{D}(\phi(s)))$ to be the map $\phi_s\colon A_s \to B_{\phi(s)}$ where ϕ_s is the localization of ϕ with respect to the multiplicative set $\{s^n\}_{n\in\mathbb{N}}$. In other words, we have

$$f^\flat_{\mathrm{D}(s)}(a/s^n) = \phi(a)/\phi(s)^n$$

for all $a/s^n \in A_s$. As the principal open subsets form a basis of the topology, this defines a homomorphism $f^\flat\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves of rings. For instance, if $U = \mathrm{D}(s_1, s_2) = \mathrm{D}(s_1) \cup \mathrm{D}(s_2)$, then $f^{-1}(U) = \mathrm{D}(\phi(s_1), \phi(s_2))$ and $f^\flat_U$ is the map

$$A_{s_1} \times_{A_{s_1 s_2}} A_{s_2} \to B_{\phi(s_1)} \times_{B_{\phi(s_1 s_2)}} B_{\phi(s_2)}$$

given by $(\alpha_1, \alpha_2) \mapsto$

For each $y \in Y$, the homomorphism $f^\#_y\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is the local ring homomorphism $\phi_{\mathfrak{q}_y}\colon A_{\phi^{-1}(\mathfrak{q}_y)} \to B_{\mathfrak{q}_y}$. Conversely, we can view Γ as a contravariant functor from **Aff** to **Ring** which takes the affine scheme X to the commutative ring $\mathcal{O}_X(X)$ and which takes a morphism of affine schemes $(f, f^\flat)\colon X \to Y$ to the homomorphism of rings $\phi\colon B \to A$ where $B = \mathcal{O}_Y(Y)$, $A = \mathcal{O}_X(X)$, and where $\phi = f^\flat_Y$.

Theorem 29.2. *The functors* Spec *and* Γ *define an anti-equivalence between the category of rings and the category of affine schemes.*

Proof. The functor Spec is by definition essentially surjective. Moreover, $\Gamma \circ \operatorname{Spec}$ is clearly isomorphic to $1_{\mathbf{Ring}}$. Therefore it suffices to show that for any two rings A and B, the maps

$$\operatorname{Hom}_{\mathbf{Ring}}(A, B) \underset{\Gamma}{\overset{\operatorname{Spec}}{\rightleftarrows}} \operatorname{Hom}_{\mathbf{Aff}}(\operatorname{Spec} B, \operatorname{Spec} A)$$

are mutually inverse bijections. Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes and set $\phi = \Gamma(f)$. We have to show that ${}^{\mathrm{a}}\phi = f$. If $\mathfrak{q}_y$ is a prime ideal of B corresponding to a point $y \in Y = \operatorname{Spec} B$, then $f^\#_y$ is the unique ring homomorphism which makes the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A_{\mathfrak{p}_{f(y)}} \arrow[r, "f^\#_y"'] & B_{\mathfrak{q}_y} \end{tikzcd}$$

commutative. This shows that $\phi^{-1}(\mathfrak{q}_y) \subseteq \mathfrak{p}_{f(y)}$. As $f^\#_y$ is local, we have equality. This shows that ${}^{\mathrm{a}}\phi = f$ as continuous maps. Now the definitions of ${}^{\mathrm{a}}\phi^\#$ shows that ${}^{\mathrm{a}}\phi^\#_y$ makes the diagram above commutative as well and hence ${}^{\mathrm{a}}\phi^\#_y = f^\#_y$ for all $y \in Y$. This proves ${}^{\mathrm{a}}\phi^\# = f^\#$. □

Proposition 29.1. *Let A_{red} be the reduced ring of A obtained by quotienting out the nilpotent elements of A and let $\pi\colon A \to A_{\mathrm{red}}$ be the quotient map. Set $X = \operatorname{Spec} A$, set $Y = \operatorname{Spec} A_{\mathrm{red}}$, and set $f =^{\mathrm{a}} \pi$. Then We have* $\operatorname{Spec} A \cong \operatorname{Spec} A_{\mathrm{red}}$.

30 Schemes

30.1 Definition of Schemes

Definition 30.1. A **scheme** is a locally ringed spaced $X = (X, \mathcal{O}_X)$ which admits an open covering $X = \bigcup_{i\in I} U_i$ such that all locally ringed spaces $U_i = (U_i, \mathcal{O}_{X|U_i})$ are affine schemes. A **morphism** of schemes is a morphism of locally ringed spaces. We obtain a category of schemes which we will denote by **Sch**.

Remark 44. To simplify notation in what follows, we denote a scheme $(X, \mathcal{O}_X)$ simply by X. If we write "let X be a scheme", then it is understood that the corresponding structure sheaf of X is denoted $\mathcal{O}_X$.

Definition 30.2. Fix a scheme S. The category of of **schemes over** S (or S**-schemes**), denoted $\mathbf{Sch}_S$ is the category whose objects are morphisms $X \to S$ of schemes, and whose morphisms from $X \to S$ to $Y \to S$ are the morphisms $X \to Y$ of schemes with the property that

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The morphism $X \to S$ is called the **structural morphism** of the S-scheme X (and often is silently omitted from the notation). The scheme S is also sometimes called the **base scheme**. In the case that $S = \operatorname{Spec} R$ is an affine scheme, one also speaks about R**-schemes** or **schemes over** R instead. For S-schemes X and Y we denote the set of morphisms $X \to Y$ in the category of S-schemes by $\operatorname{Hom}_S(X, Y)$ (or by $\operatorname{Hom}_R(X, Y)$ if $S = \operatorname{Spec} R$ is affine).

30.2 Open subschemes

Let $f\colon X \to Y$ be a continuous map and let $\mathcal{G}$ be a sheaf on Y. Recall that there is a morphism $f^\diamond\colon \mathcal{G} \to f_*(f^{-1}\mathcal{G})$ which is defined as follows: for all $V \subseteq Y$ open, set $U = f^{-1}(V)$ and define

$$f^\diamond_V(t) = [V, t]_{f(U)}$$

for all $t \in \mathcal{G}(V)$. Notice that this map makes sense because $V \subseteq f(U)$. Furthermore it is easy to check that the $f^\diamond_V$ ranging over all $V \subseteq Y$ open constitute a morphism $f^\diamond\colon \mathcal{G} \to f_*(f^{-1}\mathcal{G})$ of sheaves on Y. Let us see what this morphism looks like in the case where X is a scheme and where $\iota\colon U \to X$ is the inclusion map from an open set $U \subseteq X$ to X. First of all, recall that we define a sheaf $\mathcal{O}_{X|U}$ on U by $\mathcal{O}_{X|U} := \iota^{-1}\mathcal{O}_X$ where $\iota^{-1}\mathcal{O}_X$ is the sheafification of $\iota^+\mathcal{O}_X$. Since ι is an open map, we have

$$\iota^+\mathcal{O}_X(U') = \mathcal{O}_X(U')$$

for all $U' \subseteq U$ open. In particular, $\iota^+\mathcal{O}_X$ is already a sheaf. Thus $\mathcal{O}_{X|U} = \iota^{-1}\mathcal{O}_X = \iota^+\mathcal{O}_X$, and it is very easy to describe what the sections in $\mathcal{O}_{X|U}(U')$ look like (namely $\mathcal{O}_{X|U}(U') = \mathcal{O}_X(U')$). Also note that for every $V \subseteq X$ open, we have

$$\iota_*\mathcal{O}_{X|U}(V) = \mathcal{O}_{X|U}(U \cap V) = \mathcal{O}_X(U \cap V).$$

Taking this altogether, we see that

$$\iota_*(\iota^{-1}\mathcal{O}_X)(V) = \iota_*\mathcal{O}_{X|U}(V) = \mathcal{O}_X(U \cap V)$$

for all $V \subseteq X$ open, and the corresponding morphism $\iota^\diamond\colon \mathcal{O}_X \to \iota_*\mathcal{O}_{X|U}$ has a simple description: it is given by

$$\iota^\diamond_V(t) = t|_{U \cap V}$$

for all $V \subseteq X$ open and for all $t \in \mathcal{O}_X(V)$.

Proposition 30.1. *Let X be a scheme and let U be an open subset of X. The locally ringed space $U = (U, \mathcal{O}_{X|U})$ is a scheme. We call U an **open subscheme** of X. If U is an affine scheme, then U is called an **affine open subscheme**. The affine open subschemes of X form a basis of the topology. If $X = \operatorname{Spec} A$ is an affine scheme, then $\mathrm{D}(f) = (\mathrm{D}(f), \mathcal{O}_{X|\mathrm{D}(f)})$ is an affine scheme with coordinate ring A_f. Subschemes of this form are called **principally open**.*

Proof. By definition the locally ringed space X can be covered by affine schemes, and each of these affine schemes has a basis of its topology consisting of affine schemes. □

Remark 45. If $U \subseteq X$ is open, then we think of it as an open subscheme of X. Whenever we speak about a morphism of schemes from U to X, then we are just talking about the morphism $(\iota, \iota^\diamond)$ where ι is the inclusion map and where $\iota^\diamond$ was defined above.

30.3 Morphisms into Affine Schemes

Morphisms of an arbitrary scheme (or even an arbitrary locally ringed space) into an affine scheme are easy to understand, as the following proposition shows:

Proposition 30.2. *Let X be a locally ringed space and let $Y = \operatorname{Spec} A$ be an affine scheme. Then the natural map*

$$\operatorname{Hom}_{\mathbf{Sch}}(X, Y) \to \operatorname{Hom}_{\mathbf{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

given by $(f, f^\flat) \mapsto f^\flat_Y$ is a bijection.

Proof. We will only prove this in the case where X is a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. For all U_i the natural map

$$\mathrm{Hom}_{\mathbf{Sch}}(U_i, Y) \to \mathrm{Hom}_{\mathbf{Ring}}(A, \Gamma(U_i, \mathcal{O}_X))$$

given by $(f, f^\flat) \mapsto f^\flat_{U_i}$ is a bijection by Theorem (29.2). For an affine open $V \subseteq U_i \cap U_j$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}(U_i, Y) \arrow[r] \arrow[d] & \mathrm{Hom}(A, \Gamma(U_i, \mathcal{O}_X)) \arrow[d] \\
\mathrm{Hom}(V, Y) \arrow[r] & \mathrm{Hom}(A, \Gamma(V, \mathcal{O}_X))
\end{tikzcd}$$

is commutative, since $\Gamma(-)$ is functorial. The assertion now follows from the following general proposition about gluing of morphisms. □

Proposition 30.3. *(Gluing of morphisms) Let X and Y be locally ringed spaces. For every $U \subseteq X$ open, let* $\mathrm{Hom}(U, Y)$ *be the set of morphisms $U \to Y$ of locally ringed spaces. Then $U \mapsto$* $\mathrm{Hom}(U, Y)$ *is a sheaf of sets on X.*

Proof. $\mathrm{Hom}(-, Y)$ is clearly a presheaf. Let $\{U_i\}_{i \in I}$ be an open cover of U and let $\{f_i\}_{i \in I}$ be a compatible collection local sections: so $f_i \in \mathrm{Hom}(U_i, Y)$ for each $i \in I$ and compatibility means

$$f_i|_{U_{ij}} = f_j|_{U_{ij}} \quad \text{and} \quad f_i^\flat|$$

and $f_i^\flat$ for each $i, j \in I$. Recall that the morphisms $f_i = (f_i, f_i^\flat)$ consists of two parts: the continuo where f_i: □

30.4 Gluing of Schemes

Definition 30.3. A gluing datum of schemes consists of the following data:

- an index set I,
- for all $i \in I$ a scheme U_i,
- for all $i, j \in I$ and open subset $U_{i,j} \subseteq U_i$ (viewed as an open subscheme) where $U_{i,i} = U_i$ for all $i \in I$.
- for all $i, j \in I$ an isomorphism $\varphi_{ij} \colon U_{j,i} \to U_{i,j}$ of schemes such that the following cocycle condition

  $$\varphi_{ij}\varphi_{jk} = \varphi_{ik}$$

  holds on $U_{i,j} \cap U_{i,k}$ for all $i, j, k \in I$.

We denote such a gluing datum by $\mathbf{G} = (I, \{U_i\}, \{U_{i,j}\}, \{\varphi_{i,j}\})$.

In the cocycle condition we implicitly assume that in particular $\varphi_{jk}(U_{k,i} \cap U_{k,j}) \subseteq U_{j,i}$ (otherwise the composition is meaningless). For $i = j = k$, the cocycle condition implies that $\varphi_{i,i} = 1_{U_i}$. For $i = k$, the cocycle condition implies that $\varphi_{i,j}^{-1} = \varphi_{j,i}$ and that φ_{ij} restricts to an isomorphism from $U_{j,i} \cap U_{j,k}$ to $U_{i,j} \cap U_{i,k}$.

Proposition 30.4. *Let $\mathbf{G} = (I, \{U_i\}, \{U_{i,j}\}, \{\varphi_{i,j}\})$ be a gluing datum of schemes. Then there exists a scheme $X_{\mathbf{G}}$ together with morphisms $\psi_i \colon U_i \to X$ such that*

- *for all i the map ψ_i yields an isomorphism from U_i onto an open subscheme of $X_{\mathbf{G}}$,*
- *$\psi_j\varphi_{ij} = \psi_i$ on $U_{i,j}$ for all i, j*
- *$X_{\mathbf{G}} = \bigcup_{i \in I} \psi_i(U_i)$,*
- *$\psi_i(U_i) \cap \psi_j(U_j) = \psi_i(U_{i,j}) = \psi_j(U_{j,i})$ for all $i, j \in I$.*

Furthermore, $X_{\mathbf{G}}$ together with the ψ_i is uniquely determined up to unique isomorphism. If $\mathbf{G}$ is understood from context, then we often simplify our ntoation and simply write X instead of $X_{\mathbf{G}}$, and call X (together with the morphisms ψ_i) the scheme obtained from the gluing datum $\mathbf{G}$.

Proof. The first step is to define the underlying topological space of X. We start with the disjoint union $\coprod_{i \in I} U_i$ of the (underlying topological spaces of the) U_i and define an equivalence relation $\sim$ on it as follows: points $x_i \in U_i$ and $x_j \in U_j$ are said to be equivalent, denoted $x_i \sim x_j$, if and only if $x_i = \varphi_{ji}(x_i)$. In particular, if $x_i \sim x_j$, then it is necessary that $x_i \in U_{i,j}$ and $x_j \in U_{j,i}$. We denote the equivalence class of x_i by $[x_i]$. As a set, we define X to be the set of equivalence classes,

$$X = \coprod_{i \in I} U_i / \sim .$$

The natural maps $\psi_i\colon U_i \to X$, given by $\psi_i(x_i) = [x_i]$, are injective and we have $\psi_i(U_{i,j}) = \psi_i(U_i) \cap \psi_j(U_j)$ for all $i, j \in I$. We equip X with the quotient topology, that is, with the coarsest topology such that all ψ_i are continuous. That means that a subset $V \subseteq X$ is open if and only if for all i the preimage $\psi_i^{-1}(V)$ is open in U_i. In particular, the $\psi_i(U_i)$ and the $\psi_i(U_{i,j}) = \psi_i(U_i) \cap \psi_j(U_j)$ are open in X.

To clean our notation in what follows, set $V_i = \psi_i(U_i)$ and note that $V_{ij} = V_i \cap V_j = \psi_i(U_{i,j})$ for each $i, j \in I$. To obtain a locally ringed space, we have to "glue" the structure sheaves on the U_i so as to define a sheaf $\mathcal{O}_X$ of rings on X. Observe that $\{V_i\}$ forms an open cover of X. Also for each $i \in I$, we have a sheaf $\mathcal{O}_i := (\psi_i)_*\mathcal{O}_{U_i}$ on V_i, defined by $\mathcal{O}_i(V) = \mathcal{O}_{U_i}(\psi_i^{-1}(V))$ for all open $V \subseteq X$. We also have isomorphisms $\psi_{ij}\colon \mathcal{O}_i|_{V_{ij}} \to \mathcal{O}_j|_{V_{ij}}$ where $\psi_{ij} := (\psi_i)_*(\varphi_{ij})$ is defined by $(\psi_{ij})_V = (\varphi_{ij})_{\psi_i^{-1}(V)}$ for all open $V \subseteq X$. In particular $\{\psi_{ij}\}$ satisfy the cocycle condition $\psi_{jk}\psi_{ij} = \psi_{ik}$ for all i, j, k. It follows from (1.2) that there exists a sheaf $\mathcal{O}_X$ on X and for all $i \in I$ together with isomorphisms $\phi_i\colon \mathcal{O}_i \to \mathcal{O}_{X|U_i}$ such that $\phi_j\psi_{ij} = \phi_j$ on V_{ij} for all $i, j \in I$. Moreover, $\mathcal{O}_X$ and ϕ_i are uniquely detetrmined up to unique isomorphism by these conditions. An explicit description of $\mathcal{O}_X$ is given by

$$\mathcal{O}_X(V) = \left\{ (s_i)_{i\in I} \in \prod_{i \in I} \mathcal{O}_i(V_i \cap V) \mid s_j|_{V_{ij}\cap V} = \phi_{ij}(s_i)|_{V_{ij}\cap V} \text{ for all } i, j \in I. \right\}$$

for all open $V \subseteq X$.
where $\psi_{ij}\colon \mathcal{O}_i|_{\psi_i(U_{i,j})} \to \mathcal{O}_j|_{\psi_i(U_{i,j})}$ is defined by from $= (\psi_i)_*(\varphi_{ij})\colon \mathcal{O}_i \to \mathcal{O}_j$
The sheaf $\mathcal{O}_X$ is uniquely determined by its sections (and corresponding restriction maps) on a basis of the topology. It is thus sufficient to define it on these open subsets $U \subseteq X$ which are contained in one of the $\psi_i(U_i)$, and to check that this is well-defined and satisfies the sheaf axioms. For each such U, we fix once and for all an i with $U \subseteq \psi_i(U_i)$, and we set $\mathcal{O}_X(U) = \mathcal{O}_{U_i}(\psi^{-1}(U))$. If $U \subseteq U_i \cap U_j$, then we identify the rings $\mathcal{O}_{U_i}(\psi_i^{-1}(U))$ and $\mathcal{O}_{U_j}(\psi_j^{-1}(U))$ with $\mathcal{O}_{U_{i,j}}(U)$ via φ_{ji}. This allows us to define restriction maps. We obtain a sheaf $\mathcal{O}_X$ of rings on X which is independent of our choices. Since all the U_i are locally ringed spaces, the same is true for X.

Furthermore, with this definition the ψ_i are morphisms of locally ringed spaces; they identify U_i with $(\psi_i(U_i), \mathcal{O}_{X|\psi_i(U_i)})$. Finally all the U_i are schemes by assumption, that is, they are covered, as locally ringed spaces, by affine schemes, and therefore X is a scheme as well. By construction, we have $X = \bigcup_{i\in I} U_i$. □

30.4.1 Construction of $\mathbb{P}^n$

Let $\mathbf{G} = (I, \{U_i\}, \{U_{i,j}\}, \{\varphi_{ij}\})$ be the gluing datum where

$$\begin{aligned} I &= \{0, 1, \dots, n\} \\ U_i &= \operatorname{Spec} \mathbb{Z}[\boldsymbol{X}_i] \\ U_{i,j} &= \operatorname{Spec} \mathbb{Z}[\boldsymbol{X}_i]_{X_{i,j}} \\ {}^{\mathrm{a}}\varphi_{ij} &= (\text{given by } X_{i,k} \mapsto X_{j,k}X_{j,i}^{-1} \text{ for each } k \neq j) \end{aligned}$$

The $X_{\mathbf{G}} = \mathbb{P}^n$.

31 Local Properties of Schemes

31.1 The Tangent Space

Let $X = (X, \mathcal{O})$ be a scheme and let $x \in X$. Recall that we set $\mathfrak{m}_x$ to be the maximal ideal of the local ring $\mathcal{O}_x$, and we set $\kappa(x)$ to be the residue field $\kappa(x) := \mathcal{O}_x/\mathfrak{m}_x$. Note that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a $\kappa(x)$-vector space.

Definition 31.1. The **tangent space** of X at x is defined to be the dual space:

$$\mathrm{T}_x(X) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^* := \operatorname{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)).$$

Remark 46. This notion is best behaved if X is a κ-scheme and $x \in X$ is a point with residue field κ. On the other hand, if η is a generic point of any integral scheme X, then we have $\mathfrak{m}_\eta = 0$, so $\mathrm{T}_\eta(X)$ does not contain any information about X.

The tangent space is functorial in (X, x) in the following sense: let $f\colon X \to Y$ be a morphism of schemes and let $x \in X$ be a point such that $\dim_{\kappa(y)} \mathrm{T}_y(Y)$ is finite, where we set $y = f(x)$. Then the local homomorphism $f_x^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a $\kappa(x)$-linear map

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) \to \mathfrak{m}_x/\mathfrak{m}_x^2.$$

It the extension $\kappa(x)/\kappa(y)$ is finite or $\mathrm{T}_y(Y)$ is a finite-dimensional $\kappa(y)$-vector space, then dualizing we obtain an induced map on tangent space

$$\mathrm{d}f_x\colon \mathrm{T}_x(X) \to \mathrm{T}_y(Y) \otimes_{\kappa(y)} \kappa(x).$$

This construction is compatible with composition of morhpisms in the obvious way.

32 Proj

Let A be a graded ring. Recall that the **irrelevant ideal** of A is the ideal of elements of positive degree

$$A_+ = \bigoplus_{n \geq 1} A_n$$

We define $\operatorname{Proj} A$ to be the set of all homogeneous prime ideals of A which do not contain the irrelevant ideal:

$$\operatorname{Proj} A := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a homogeneous prime ideal of } A \text{ such that } \mathfrak{p} \not\supseteq A_+\}.$$

We will now endow $\operatorname{Proj} A$ with the structure of a topological space. For every subset S of A, we denote by $\mathrm{V}(S)$ to be the set of all $\mathfrak{p} \in \operatorname{Proj} A$ such that $\mathfrak{p} \supseteq S$. Similarly, we denote by $\mathrm{D}(S)$ to be the complement of $\mathrm{V}(\mathrm{S})$ in $\operatorname{Proj} A$. Clearly, if $\mathfrak{a}$ is a homogeneous ideal generated by S (meaning $\mathfrak{a}$ is the smallest homogeneous ideal of A which contains S), then $\mathrm{V}(S) = \mathrm{V}(\mathfrak{a})$ and similarly $\mathrm{D}(S) = \mathrm{D}(\mathfrak{a})$. Let

$$\tau_A = \{\mathrm{D}(\mathfrak{a}) \mid \mathfrak{a} \text{ is a homogeneous ideal of } A\}.$$

It is straightforward to check that τ_A is a topology on $\operatorname{Proj} A$. We call this topology the **Zariski topology**.

We also construct a sheaf on $\operatorname{Proj} S$, called the **structure sheaf** which gives it the structure of a scheme. For any open set U of $\operatorname{Proj} S$, we define the ring

$$\mathcal{O}_X(U) = \{f$$

33 Functor of Points

Let R be a ring, let $\boldsymbol{f} = f_1, \ldots, f_m$ be polynomials in $R[\boldsymbol{t}] = R[t_1, \ldots, t_n]$, and let A be an R-algebra. We have a bijection which is natural in A:

$$\{\boldsymbol{x} \in A^n \mid \boldsymbol{f}(\boldsymbol{x}) = 0\} \simeq \operatorname{Hom}_R(R[\boldsymbol{t}]/\boldsymbol{f}, A),$$

where the righthand side is understood to be the set of all R-algebra homomorphisms from $R[\boldsymbol{t}]/\boldsymbol{f}$ to A, and the lefthand side can be thought of as the set of all **A-valued points of** $R[\boldsymbol{t}]/\boldsymbol{f}$ (over R). Indeed, if $\boldsymbol{x} \in A^n$ such that $\boldsymbol{f}(\boldsymbol{x}) = 0$ (meaning $f_i(\boldsymbol{x}) = 0$ for all i) , then we define $\varphi_{\boldsymbol{x}}\colon R[\boldsymbol{t}]/\boldsymbol{f} \to A$ by $\varphi_{\boldsymbol{x}}(t_j) = x_j$ for all j. That $\varphi_{\boldsymbol{x}}$ is well-defined follows from the fact that $\boldsymbol{f}(\boldsymbol{x}) = 0$. Conversely, if $\varphi\colon R[\boldsymbol{t}]/\boldsymbol{f} \to A$ is an R-algebra homomorphism, then we get a point $\boldsymbol{x}^\varphi \in A^n$ where $x_j^\varphi = \varphi(t_j)$. In particular, if we set $X = \operatorname{Spec} R[\boldsymbol{t}]/\boldsymbol{f}$ and $T = \operatorname{Spec} A$, then we see that

$$\{\boldsymbol{x} \in A^n \mid \boldsymbol{f}(\boldsymbol{x}) = 0\} \simeq \operatorname{Hom}_R(T, X),$$

where the righthand side is understood to be the set of all R-scheme homomorphisms from T to X. Again, this bijection is natural in A (and hence in T). Thus it is natural to attach to a scheme X the functor $h_X\colon \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ which takes a scheme T and and sends it to the set of all scheme homomorphisms $h_X(T) := \operatorname{Hom}(T, X)$, and which takes a morphism of schemes $f\colon T' \to T$ and sends it to the function $h_X(f)\colon h_X(T) \to h_X(T')$ given by $h_X(f)(g) = gf$. We often denote $h_X(f) = f^*$ and $h_X(T) = X(T)$ and we $X(T)$ call this set the **T-valued points of** X. If $T = \operatorname{Spec} A$ is affine, then we often set $X(A) := X(\operatorname{Spec} A)$ and call this set the **A-valued points of** X. More generally, we might consider an arbitrary functor $F\colon \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ as a "geometric object" and we call $F(T)$ the set of **T-valued points of** F.

More generally, let S be a fixed scheme. Indead fo the category **Sch**, we consider $\mathbf{Sch}/S$: the category of S-schemes. Again, every S-scheme X provides a functor from $\mathbf{Sch}/S$ to **Set**, which is given by $T \mapsto \operatorname{Hom}_S(T, X) := X_S(T)$ on objects. If it is understood that all schemes are considered S-schemes, then we simplify our notation further by writing $X_S(T) = X(T)$. If $S = \operatorname{Spec} R$ or $T = \operatorname{Spec} A$ is affine, we also write $X_R(T)$ or $X_S(A)$ (or even $X_R(A)$ if both $S = \operatorname{Spec} R$ and $T = \operatorname{Spec} A$ are affine).

Example 33.1. Let $A =$

Example 33.2. Let K be an algebraically closed field and let X be a K-scheme locally of finite type. Then $X(K)$ is givven by the set of all pairs (x, ι_x) where $x \in X$ and $\iota_x\colon \kappa_x \to K$ is a field extension. Recall that the scheme homorphisms

$$X(K) := \operatorname{Hom}(\operatorname{Spec} K, X) = \{(x, \iota_x) \mid x \in X \text{ and } \iota$$

34 Fibre Products

Let X be a scheme. We would like to gain some intuition as to what X is. To this end, let k be a field. The **k-valued points** of X, denoted $X(k)$, is defined to be the set of all scheme homomoprhisms from $\operatorname{Spec} k$ to X. For instance, if $X = \operatorname{Spec} \mathbb{Z}[\boldsymbol{T}]/\langle f_1, \dots, f_m \rangle$, then

$$X(k) = \{x = (x_1, \dots, x_n) \in k^n \mid f_1(x) = \cdots = f_m(x) = 0\} = \mathrm{V}_k(f_1, \dots, f_m)$$

35 Separated Morphisms

Proposition 35.1. *Let X be a topological space. The following conditions are equivalent.*

1. *X is Hausdorff.*
2. *The diagonal $\Delta_X := \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.*
3. *For every topological space Y and every continuous map $f \colon Y \to X$, its graph $\Gamma_f : \{(y, f(y) \mid y \in Y\}$ is closed in $Y \times X$.*
4. *For every topological space Y and any two continuous maps $f, g \colon Y \to X$, then kernel $K_{f,g} := \{y \in Y \mid f(y) = g(y)\}$ is closed in Y.*

Proof. First we show 1 is equivalent to 2. Observe that Δ_X is closed in $X \times X$ if and only if Δ_X^c is open in $X \times X$ if and only if for every $(x, x') \in X \times X$ with $x \neq x'$, there exists an open neighborhood $U \times U' \subseteq X \times X$ of (x, x') such that $\Delta_X \cap (U \times U') = \emptyset$ if and only if X is Hausdorff.

Next we show 2 is equivalent to 3. Clearly 2 is a special case of 3 with f being the identity function on X, so one direction is clear. Conversely, suppose $f \colon Y \to X$ is a continuous map. Consider the continuous map $F \colon Y \times X \to X \times X$ defined by $F(y, x) = (f(y), x)$ for all $(y, x) \in Y \times X$. Observe that $F(y, x) \in \Delta_X$ if and only if $x = f(y)$. If follows that $\Gamma_f = F^{-1}(\Delta_X)$. Thus if $\Delta_X \subseteq X \times X$ is closed, then $\Gamma_f \subseteq Y \times X$ is closed.

Finally we show 2 is equivalent to 4. First note that 2 is a special case of 4 (take $Y = X \times X$, $f = \pi_1$, and $g = \pi_2$, then $K_{f,g} = \Delta_X$). Conversely, suppose $f, g \in Y \to X$ are continuous maps. Consider the continuous map $F \colon Y \to X \times X$ defined by $F(y) = (f(y), g(y))$ for all $y \in Y$. Observe that $F(y) \in \Delta_X$ if and only if $f(y) = g(y)$. It follows that $K_{f,g} = F^{-1}(\Delta_X)$. In particular, if $\Delta_X \subseteq X \times X$ is closed, then $K_{f,g} \subseteq Y$ is closed. □

The underlying topological spaces of schemes are rarely Hausdorff (for instance, every irreducible topological space is not Hausdorff), but the analagous properties (2-4) in Proposition (35.1) can be used to define an analogue of the Hausdorff property for schemes. Since the topology on fiber products of schemes is (usually) not the product topology, this gives rise to the different (and in fact very useful) notion of a separated scheme. Z