MLDG Algebras and Modules

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Contents

1 Basic Definitions

Throughout this document, let R be a commutative ring.

1.1 MLDG Algebras

Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ and $\lambda\colon A \to A$ be chain maps. If $\sum_{i=1}^{n} a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^{m} a_i \otimes b_i\right) = \sum_{i=1}^{m} a_i \star_\mu b_i = \sum_{i=1}^{m} a_i b_i,$$

where we denote its image under μ by $\sum_i a_i b_i$ only if μ is understood from context. Since μ is a chain map, the Leibniz law is satisfied, which in this context says

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\mathrm{d}(b)$$

for all $a, b \in A$ with a homogeneous, where $|a|$ denotes the homological degree of a. We call the quadruple $(A, \mathrm{d}, \lambda, \mu)$ a **pre-MLDG R-algebra**. In the case where $R = \mathbb{Z}$ (or when the base ring isn't relevant to the present discussion), then we say $(A, \mathrm{d}, \lambda, \mu)$ is a **pre-MLDG algebra**. We call the triple (A, d, μ) a **pre-MDG algebra** (so this is just an MLDG algebra except we forget the extra data of the chain map "$\lambda\colon A \to A$"). Similarly, we call the triple (A, d, λ) a **pre-LDG algebra**, the pair (A, μ) a **pre-MG algebra**, and so on. In fact, this will be a common theme throughout the document: if we define an "MLDG" object, then will have also (implicitly) define an "MDG/LDG/MG/etc..." object by forgetting various data.

In order to simplify our notation in what follows, we often refer to the quadruple $(A, \mathrm{d}, \lambda, \mu)$ via its underlying graded R-module A, where we think of A as a graded R-module which is equipped with a differential $\mathrm{d}\colon A \to A$, giving it the structure of an R-complex, and which is further equipped with the chain maps $\mu\colon A \otimes_R A \to A$ and $\lambda\colon A \to A$ (we write "let $A = (A, \mathrm{d}, \lambda, \mu)$ be a pre-MLDG algebra" to denote this). For isntance, if μ satisfies a property (such as being commutative), then we also say A satisfies that property. With this notational convention in mind, here are some properties we can impose on A which we will be interested in:

Definition 1.1. Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ and $\lambda\colon A \to A$ be chain maps. If $\sum_{i=1}^{n} a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^{m} a_i \otimes b_i\right) = \sum_{i=1}^{m} a_i \star_\mu b_i = \sum_{i=1}^{m} a_i b_i,$$

where we denote its image under μ by $\sum_i a_i b_i$ only if μ is understood from context. Since μ is a chain map, the Leibniz law is satisfied, which in this context says

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\mathrm{d}(b)$$

for all $a, b \in A$ with a homogeneous, where $|a|$ denotes the homological degree of a. We call the quadruple $(A, \mathrm{d}, \lambda, \mu)$ a **pre-MLDG R-algebra**.

Definition 1.2. Let $A = (A, \mathrm{d}, \lambda, \mu)$ be a pre-MLDG algebra.

1. We say A is **unital** (or μ is **unital**) if there exists $e \in A$ such that
$$ae = a = ea$$
for all $a \in A$. In this case, we can view R as a subset of A sitting in degree 0 via the embedding $R \xrightarrow{e} A$ defined by $r \mapsto re$ for all $r \in R$.

2. We say A is **graded-commutative** (or μ is **graded-commutative**) if
$$ab = (-1)^{|a||b|}ba$$
for all homogeneous $a, b \in A$. We say it is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that
$$a^2 = 0$$
for all homogeneous $a \in A$ whenever $|a|$ is odd.

3. We say A is **multiplicative** (or λ is **μ-multiplicative**) if it satisfies the **multiplicative law**:
$$\lambda(ab) = \lambda(a)\lambda(b)$$
for all $a, b \in A$. Note that we can also write this identity without using elements as
$$\lambda \circ \mu = \mu \circ \lambda^{\otimes 2}.$$

4. We say A is **hom-associative** (or μ is **λ-associative**) if it satisfies the **hom-associative law**:
$$\lambda(a)(bc) = (ab)\lambda(c)$$
for all $a, b, c \in A$. If $\lambda = 1_A$, then we will simply say A is **associative**. Note that we can also write this identity without using elements as
$$\mu \circ (\lambda \otimes \mu) = \mu \circ (\mu \otimes \lambda).$$

5. We say A is **permutative** (or μ is **λ-permutative**) if it satisfies the **permutative law**:
$$\lambda(ab)(\lambda(c)\lambda(d)) = (\lambda(a)\lambda(b))\lambda(cd) \tag{1}$$
for all $a, b, c, d \in A$. Note that we can also write this identity without using elements as
$$\mu \circ (\lambda \otimes \mu) \circ (\mu \otimes \lambda^{\otimes 2}) = \mu \circ (\mu \otimes \lambda) \circ (\lambda^{\otimes 2} \otimes \mu).$$

We say A is an **MLDG algebra** if A is strictly graded-commutative and unital. In this case, we call μ the **multiplication** of A and we call λ the **perturbation** of A. The multiplication of A is sometimes denoted μ_A and the perturbation of A is sometimes denoted λ_A in case context is not clear, just like how the differential of A is sometimes denoted d_A.

Note that
$$a(\lambda(bc)) =$$

So A is an MLDG algebra if it satisfies the first two properties above. In general, A does not need to satisfy the remaining three properties above, however we can use the underlying R-complex (A, d) to measure "how far away" it is from satisfying those properties. To this end, here are some tools which will help us do this:

Definition 1.3. Let $A = (A, \mathrm{d}, \lambda, \mu)$ be an MLDG algebra.

1. The **multiplicator** of A is the chain map

$$[\cdot, \cdot]_A := \lambda \circ \mu - \mu \circ \lambda^{\otimes 2}.$$

If A is understood from context, then we simplify our notation by writing $[\cdot, \cdot] = [\cdot, \cdot]_A$. The multiplicator of the pair (a, b) where $a, b \in A$ is just the image of $a \otimes b$ under $[\cdot, \cdot]$:

$$[a, b] = \lambda(ab) - \lambda(a)\lambda(b).$$

The **multiplicator complex** of A is the subcomplex of A given by $[A, A] := \operatorname{im}[\cdot, \cdot]$, so the underlying graded module of A is

$$[A, A] := \operatorname{span}_R\{[a, b] \mid a, b \in A\}$$

and the differential of $[A, A]$ is simply the restriction of the differential of A to $[A, A]$ (so we denote the differential of $[A, A]$ simply by d again). Note that since d commutes with both μ and λ, we have

$$\mathrm{d}[a, b] = [\mathrm{d}(a), b] + (-1)^{|a|}[a, \mathrm{d}(b)].$$

The **multiplicator homology** of A is just the homology of $[A, A]$. We say A is **homologically multiplicative** if $\mathrm{H}[A, A] = 0$, and we say it is homologically multiplicative in degree i if $\mathrm{H}_i[A, A] = 0$ (of course we also say it is multiplicative in degree i if $[A, A]_i = 0$).

2. The **hom-associator** of A is the chain map

$$[\cdot, \cdot, \cdot]_A := \mu \circ (\lambda \otimes \mu) - \mu \circ (\mu \otimes \lambda).$$

If A is understood from context, then we simplify our notation by writing $[\cdot, \cdot, \cdot] = [\cdot, \cdot, \cdot]_A$. The hom-associator of the triple (a, b, c) where $a, b, c \in A$ is just the image of $a \otimes b \otimes c$ under $[\cdot, \cdot, \cdot]$:

$$[a, b, c] = \lambda(a)(bc) - (ab)\lambda(c).$$

The **hom-associator complex** of A is the subcomplex of A given by $[A, A, A] := \operatorname{im}[\cdot, \cdot, \cdot]$, so the underlying graded module of A is

$$[A, A, A] := \operatorname{span}_R\{[a, b, c] \mid a, b, c \in A\}$$

and the differential of $[A, A, A]$ is simply the restriction of the differential of A to $[A, A, A]$ (so we denote the differential of $[A, A, A]$ simply by d again). Note that since d commutes with both μ and λ, we have

$$\mathrm{d}[a, b, c] = [\mathrm{d}(a), b, c] + (-1)^{|a|}[a, \mathrm{d}(b), c] + (-1)^{|a|+|b|}[a, b, \mathrm{d}(c)].$$

The **hom-associator homology** of A is just the homology of $[A, A, A]$. We say A is **homologically hom-associative** if $\mathrm{H}[A, A, A] = 0$, and we say it is homologically homassociative in degree i if $\mathrm{H}_i[A, A, A] = 0$ (of course we also say it is hom-associative in degree i if $[A, A]_i = 0$).

3. We also have the **permutator / permutator complex / etc...** which is define analagously as 1 and 2 above.

Remark 1. Thus if we have an expression of the form $\lambda(ab)$, then we can apply the multiplicative law but with the cost of a multiplicator $\lambda(ab) = \lambda(a)\lambda(b) + [a,b]$. Similarly, we can apply the hom-associative law to $\lambda(a)(bc)$ but with an additive cost of a hom-associator $\lambda(a)(bc) = (ab)\lambda(c) + [a,b,c]$. Finally, we can apply the permutative law to $\lambda(ab)(\lambda(c)\lambda(d))$ but with an additive cost of a permutator $\lambda(ab)(\lambda(c)\lambda(d)) = (\lambda(a)\lambda(b))(\lambda(cd)) + [a,b,c,d]$. In particular, suppose $\lambda x = [a,b]$ and $\lambda y = [c,d]$. Then we have $[a,1] = \lambda a(1-e)$ where $e = \lambda(1)$.

$$
\begin{aligned}
[a,b,c,d] &= \lambda(ab)(\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))\lambda(cd) \\
&= [a,b](\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))[c,d] \\
&= \lambda(x)(\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))\lambda(y) \\
&= (x\lambda(c))\lambda^2(d) - \lambda^2(a)(\lambda(b)y) + [x,\lambda c,\lambda d] - [\lambda a,\lambda b,y] \\
&= (x\lambda(c))\lambda^2(d) - \lambda^2(a)(\lambda(b)y) + [x,\lambda c,\lambda d] - [\lambda a,\lambda b,y]
\end{aligned}
$$

$$
[a,b,c,d] = \lambda(ab)(\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))\lambda(cd) = [a,b](\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))[c,d].
$$

1.1.1 Basic Relationships Between Multipicativity, Hom-Associativity, and Permutativity

Let $A = (A, \mathrm{d}, \lambda, \mu)$ be an MLDG algebra. By definition, A is strictly graded-commutative and unital. Note that these two conditions are telling us that the multiplication μ of A must satisfy some special properties: being strictly graded-commutative tells us that $\mu(a \otimes b) = (-1)^{|a||b|}\mu(b \otimes a)$ for all $a, b \in A$ and being unital tells us that $\mu(1 \otimes a) = a = \mu(a \otimes 1)$ for all $a \in A$. Meanwhile, the pertubation λ of A isn't required to satisfy any additional properties. Therefore it is natural for us to impose at least one condition on λ; one such natural condition is that λ is unital, meaning $\lambda(1) = 1$ where 1 is the identity element of A. If we impose this condition on λ, then it turns out that permutativity and commutativity are equivalent concepts:

Proposition 1.1. *Let $A = (A, \mathrm{d}, \lambda, \mu)$ be an MLDG algebra.*

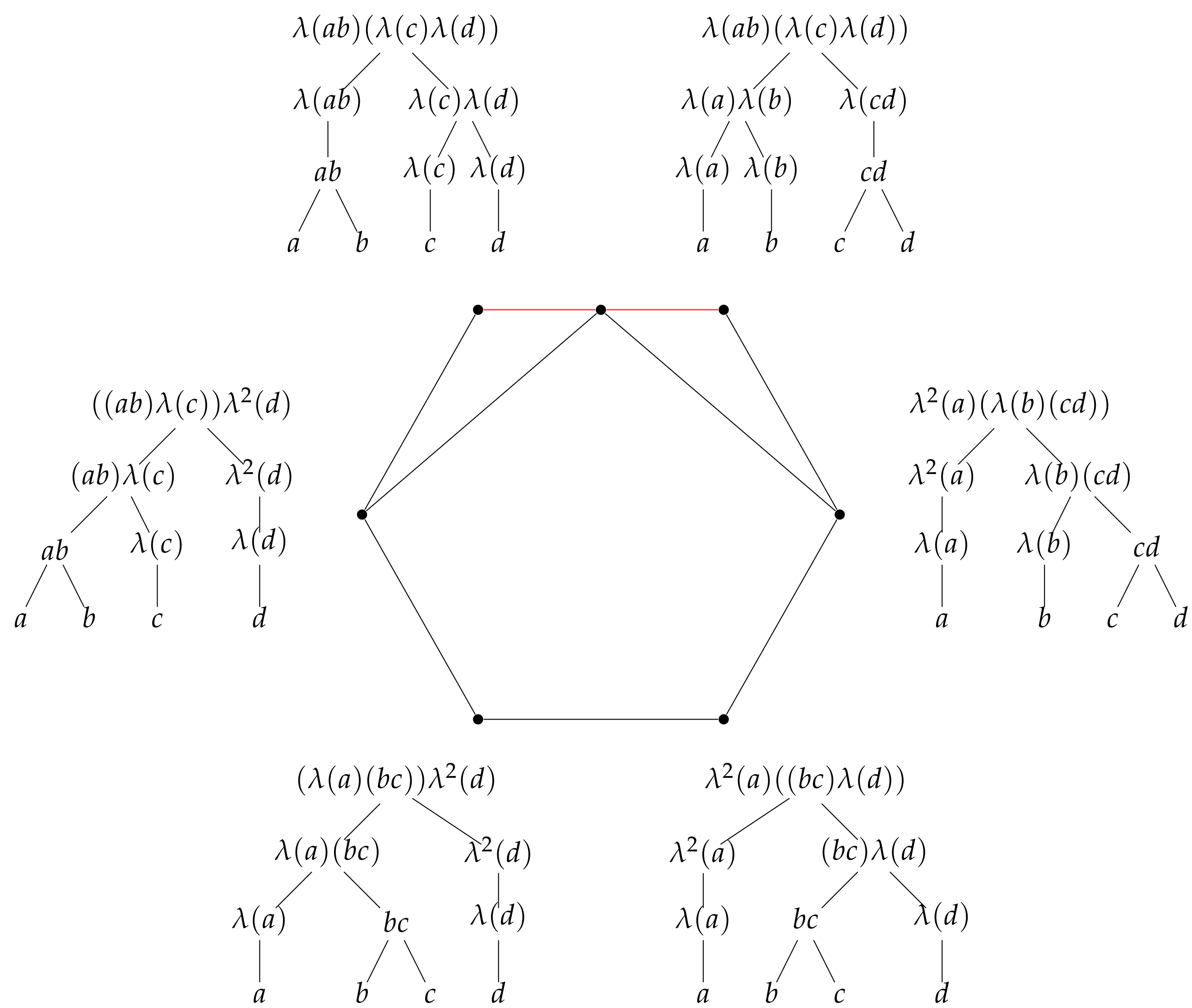
1. *If A is multilpicative, then A is permutative. The converse is true if λ is unital.*
2. *If A is hom-associative, then A is permutative. In particular, if λ is unital, then hom-associativity implies multiplicativity (so hom-associativity is a stronger property in this case).*

Proof. 1. It is clear that if A is multiplicative, then A is permutative. Now suppose that λ fixes the identity element and that A is permutative. Then setting $c = 1 = d$ in (1) shows that A is multiplicative. In the general case where λ is not necessarily unital, we have $\lambda(1) = e$ where $e \in A_0$. In this case, the permutative law would imply that e associates with all of the other elements, and furthermore it would tell us that $e\lambda(ab) = e^2\lambda(a)\lambda(b)$ for all $a, b \in A$ (which is not quite the same as A being multiplicative).

2. Suppose A is hom-associative. Then for all $a, b, c, d \in A$, we have

$$
\begin{aligned}
\lambda(ab)(\lambda(c)\lambda(d)) &= ((ab)\lambda(c))\lambda^2(d) \\
&= (\lambda(a)(bc))\lambda^2(d) \\
&= \lambda^2(a)((bc)\lambda(d)) \\
&= \lambda^2(a)(\lambda(b)(cd)) \\
&= (\lambda(a)\lambda(b))\lambda(cd).
\end{aligned}
$$

There's a cute way to visualize this by tracing the edges of the permutohedron (the hexagon) below:



Note that the red edge "collapses" to the associahedra (the pentagon) if $\lambda = 1$. $\square$

Example 1.1. Let $\lambda \in R$ and let A be an MLDG R-algebra with $\lambda_A = \mathrm{m}_\lambda$ being the multilpication by λ map given by $a \mapsto \lambda a$. Recall that A is R-linear, so in particular the element λ must associative with all pairs of elements of A. It follows that A is permutative since

$$\begin{aligned}\lambda(ab)(\lambda(c)\lambda(d)) &= \lambda^3((ab)(cd)) \\ &= (\lambda(a)\lambda(b))\lambda(cd).\end{aligned}$$

On the other hand, A is not necessarily hom-associative. Indeed, we have

$$\lambda(a)(bc) = (ab)\lambda(c) \iff \lambda(a(bc) - (ab)c)$$

for all $a, b, c \in A$ and the righthand side need not be zero. It is easy to see though that A is hom-associative if and only if λ kills $\mathrm{im}\,[\cdot,\cdot,\cdot]$ where $[\cdot,\cdot,\cdot]$ is the usual associator map defined by $[a,b,c] = a(bc) - (ab)c$ for all $a, b, c \in A$. Similarly, A is not necessarily multiplicative. Indeed, we have

$$\begin{aligned}\lambda(ab) - \lambda(a)\lambda(b) &= \lambda(ab - \lambda ab) \\ &= \lambda(1-\lambda)ab\end{aligned}$$

for all $a, b \in A$. If we assume that R is local and that $\lambda \in \mathfrak{m}$, then $1 - \lambda$ is a unit. Then in this case, it is easy to see that A is multiplicative if and only if λ kills $\mathrm{im}\,\mu$.

Proposition 1.2. *Let A be a hom-associative MLG algebra. Then we have*

1. $\lambda(a)b = a\lambda(b)$ *for all* $a, b \in A$,
2. $a\lambda(1) = \lambda(a)$ *for all* $a \in A$,
3. $ab = a\lambda(b)$ *for all* $a, b \in A$.

Proof. 1. We have $\lambda(a)b = \lambda((a1)b) = (a1)\lambda(b) = a\lambda(b)$. We obtain 2 by setting $b = 1$, and for 3 we have $a\lambda(b) = 1(a\lambda(b)) = (\lambda(1)a)b = ab$. $\square$

Proposition 1.3. *Let $A = (A, \lambda, \mu)$ be a hom-associative MLG algebra. Then $\lambda(A)$ is*

1.1.2 MLDG Algebra Homomorphisms

Having defined MLDG algebras, we now wish to define MLDG algebras homomorphisms.

Definition 1.4. Let A and B be two MLDG algebras and let $f\colon A \to B$ be a chain map. We say f is an MLDG algebra **homomorphism** (or a homomorphism of MLDG algebras) if it satisfies the following four properties:

1. It is **multipicative** (or preserves multiplication), meaning $f(a_1a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in A$.
2. It is **perturbative** (or preserves perturbation), meaning $f(\lambda_A(a)) = \lambda_B(f(a))$ for all $a \in A$.
3. It is **unital** (or preserves the identity element), meaning $f(e_A) = e_B$ where e_A is the identity element of A and e_B is the identity element of B.

The collection of all MLDG algebras together with the collection of all MLDG algebra homomorphisms forms a category, which we denote by **MLDG** (or by $\mathbf{MLDG}_R$ if the base ring R is relevant).

Remark 2. Note that properties 1, 2, and 3 in Definition (1.4) can be interpreted as saying the following diagrams are commutative:

$$\begin{tikzcd} A \otimes_R A \arrow[r, "f\otimes f"] \arrow[d, "\mu_A"'] & B \otimes_R B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\lambda_A"'] & B \arrow[d, "\lambda_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & R \arrow[ul, "e_A"] \arrow[ur, "e_B"'] & \end{tikzcd}$$

We often simplify our notation by letting 1 denote the identity element in (any) MLDG R-algebra. With this notation in mind, property 3 can be replaced with $f(1) = 1$.

1.1.3 Transporting MLDG R-algebra structures

The following example (which we often reference) will help provide motivation for some of the concepts discussed in this paper.

Example 1.2. Assume that $(R, \mathfrak{m})$ is a local Noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, let (F, d) be the minimal free resolutions of R/I over R and let $\sigma\colon F \to R/I$ denote its augmentation map. Let $\varepsilon\colon F \to R/I$ be the minimal free resolution of R/I over R. We view R/I as an MLDG R-algebra in the obvious way, where as an R-complex, R/I is the trivial one who only nonzero homogeneous component is in homological degree 0, where the perturbation is just the identity map $1\colon R/I \to R/I$, and where the multiplication is the usual multiplication $\mathrm{m}\colon R/I \otimes_R R/I \to R$. Since F is a free resolution of R/I, we can lift the multiplication map $\mathrm{m}\colon R/I \otimes_R R/I \to R/I$ to a chain map $\mu\colon F \otimes_R F \to F$ which can be chosen in such a way so that $F = (F, \mathrm{d}, \mu)$ is an MDG R-algebra. In particular, μ is unital and strictly graded-commutative, but not necessarily associative. Furthermore, we can also lift the identity map $1\colon R/I \to R/I$ to a chain map $\kappa\colon F \to F$ which can be chosen in such a way so that $F = (F, \mathrm{d}, \kappa, \mu)$ is an MLDG R-algebra. In particular, note that $\kappa(1) = 1$. Notice that κ is homotopic to the identity map $1\colon F \to F$. It is easy to check that the augmentation map $\varepsilon\colon F \to R/I$ is an MLDG R-algebra homorphism. We say F is an **MLDG R-algebra resolution** of R/I over R. Now let (G, ∂) be another free resolution of R/I over R and let $\tau\colon G \to R/I$ denote its augmentation map. Then just as in the case of F, we can choose chain maps $\lambda\colon G \to G$ and $\nu\colon G \otimes_R G \to G$ such that $G = (G, \partial, \lambda, \nu)$ is an MLDG R-algebra resolution of R/I over R.

Now since F is minimal, we can choose chain maps $\iota\colon F \to G$ and $\pi\colon G \to F$ such that $\tau\iota = \varepsilon$, $\varepsilon\pi = \tau$, and $\pi\iota = 1$. In particular, we view $\iota\colon F \to G$ as the inclusion map and $\pi\colon G \to F$ as a projection map. Using the comparision maps ι and π , we can transport the MLDG R-algebra structure on F to another MLDG R-algebra structure on G, and vice-versa. Let's first explain how to do this from F to G: we set $\widehat{\kappa} = \iota\kappa\pi$ and we set $\widehat{\mu} = \iota\mu\pi^{\otimes 2}$. Then $\widehat{G} = (G, \partial, \widehat{\kappa}, \widehat{\mu})$ is an MLDG R-algebra whose underlying complex is (G, ∂). Similarly, we set $\widehat{\lambda} = \pi\lambda\iota$ and $\widehat{\nu} = \pi\nu\iota^{\otimes 2}$. Then $\widehat{F} = (F, \mathrm{d}, \widehat{\lambda}, \widehat{\nu})$ is an MLDG R-algebra whose underlying complex is (F, d).

Proposition 1.4. *Keeping the notation as in (1.2), we have the following:*

1. *Both $\pi\colon \widehat{G} \to F$ and $\iota\colon F \to \widehat{G}$ are MLDG R-algebra homomorphisms.*
2. *We have $[\cdot,\cdot,\cdot]_{\widehat{G}} = \iota[\cdot,\cdot,\cdot]_F \pi^{\otimes 3}$ and $[\cdot,\cdot,\cdot]_F = \pi[\cdot,\cdot,\cdot]_{\widehat{G}}\iota^{\otimes 3}$.*
3. *We have $[\cdot,\cdot]_{\widehat{G}} = \iota[\cdot,\cdot]_F \pi^{\otimes 2}$ and $[\cdot,\cdot]_F = \pi[\cdot,\cdot]_{\widehat{G}}\iota^{\otimes 3}$.*
4. *The map π restricts to a surjective chain map $\pi|_{[\widehat{G},\widehat{G},\widehat{G}]}\colon [\widehat{G},\widehat{G},\widehat{G}] \to [F,F,F]$ of hom-associator complexes. In particular, if $\widehat{G}$ is hom-associative, then F is hom-associative too. Similarly the map ι restricts to a chain map $\iota|_{[F,F,F]}\colon [F,F,F] \to [\widehat{G},\widehat{G},\widehat{G}]$. Furthermore, $\iota|_{[F,F,F]}$ splits $\pi|_{[\widehat{G},\widehat{G},\widehat{G}]}$ on the right, so the hom-associator complex $[\widehat{G},\widehat{G},\widehat{G}]$ is the direct sum of $[F,F,F]$ and a trivial complex. In particular, F is homologically hom-associative if and only if $\widehat{G}$ is homologically hom-associative (and since F is minimal, then if $\widehat{G}$ is homologically hom-associative, then F hom-associative).*
5. *Replace the hom-associator complexes $[\widehat{G},\widehat{G},\widehat{G}]$ and $[F,F,F]$ in 4 with the multiplicator complexes $[\widehat{G},\widehat{G}]$ and $[F,F]$.*

Proof. 1. Clearly both π and ι are chain maps which preserve the identity element 1. Both π and ι preserve multiplication since $\pi\widehat{\mu} = \pi\iota\mu\pi^{\otimes 2} = \mu\pi^{\otimes 2}$ and similarly $\widehat{\mu}\iota^{\otimes 2} = \iota\mu\pi^{\otimes 2}\iota^{\otimes 2} = \iota\mu$. In terms of the elements, this says that if $b_1, b_2 \in G$, then

$$\pi(b_1 \star_{\widehat{\mu}} b_2) = \pi(b_1) \star_{\mu} \pi(b_2) = \pi(b_1)\pi(b_2).$$

Both π and ι preserve perturbation since $\pi\widehat{\kappa} = \pi\iota\kappa\pi = \kappa\pi$ and similarly $\widehat{\kappa}\iota = \iota\kappa\pi\iota = \iota\kappa$. In terms of the elements, this says that if $b \in G$, then

$$\pi\widehat{\kappa}(b) = \kappa\pi(b).$$

2. Let's prove this in two ways just to get a feel for what this identity is saying. First we do this without elements:

$$\begin{aligned}
[\cdot,\cdot,\cdot]_{\widehat{G}} &= \widehat{\mu}(\widehat{\kappa}\otimes\widehat{\mu} - \widehat{\mu}\otimes\widehat{\kappa}) \\
&= \iota\mu\pi^{\otimes 2}(\iota\kappa\pi\otimes\iota\mu\pi^{\otimes 2} - \iota\mu\pi^{\otimes 2}\otimes\iota\kappa\pi) \\
&= \iota\mu\pi^{\otimes 2}(\iota^{\otimes 2}(\kappa\pi\otimes\mu\pi^{\otimes 2}) - \iota^{\otimes 2}(\mu\pi^{\otimes 2}\otimes\kappa\pi)) \\
&= \iota\mu((\kappa\pi\otimes\mu\pi^{\otimes 2}) - (\mu\pi^{\otimes 2}\otimes\kappa\pi)) \\
&= \iota\mu((\kappa\otimes\mu)\pi^{\otimes 2} - (\mu\otimes\kappa)\pi^{\otimes 2}) \\
&= \iota\mu(\kappa\otimes\mu - \mu\otimes\kappa)\pi^{\otimes 3} \\
&= \iota[\cdot,\cdot,\cdot]_F\pi^{\otimes 3},
\end{aligned}$$

Applying π to the left and $\iota^{\otimes 3}$ to the right also gives us $[\cdot,\cdot,\cdot]_F = \pi[\cdot,\cdot,\cdot]_{\widehat{G}}\iota^{\otimes 3}$. Next we prove this with elements: given $b_1, b_2, b_3 \in \widehat{G}$, we have

$$\begin{aligned}
[\pi(b_1),\pi(b_2),\pi(b_3)]_F &= \kappa\pi(b_1)(\pi(b_2)\pi(b_3)) - (\pi(b_1)\pi(b_2))\kappa\pi(b_3) \\
&= \pi\widehat{\kappa}(b_1)(\pi(b_2)\pi(b_3)) - (\pi(b_1)\pi(b_2))\pi\widehat{\kappa}(b_3) \\
&= \pi\widehat{\kappa}(b_1)\pi(b_2\star_{\widehat{\mu}} b_3) - \pi(b_1\star_{\widehat{\mu}} b_2)\pi\widehat{\kappa}(b_3) \\
&= \pi(\widehat{\kappa}(b_1)\star_{\widehat{\mu}}(b_2\star_{\widehat{\mu}} b_3)) - \pi((b_1\star_{\widehat{\mu}} b_2)\star_{\widehat{\mu}}\widehat{\kappa}(b_3)) \\
&= \pi(\widehat{\kappa}(b_1)\star_{\widehat{\mu}}(b_2\star_{\widehat{\mu}} b_3) - (b_1\star_{\widehat{\mu}} b_2)\star_{\widehat{\mu}}\widehat{\kappa}(b_3)) \\
&= \pi[b_1,b_2,b_3]_{\widehat{G}}.
\end{aligned}$$

since π is an MLDG algebra homomorphism. Applying ι on the left to both sides gives us $\iota[\pi(b_1),\pi(b_2),\pi(b_3)]_F = [b_1,b_2,b_3]_{\widehat{G}}$. The hom-associator complex of $\widehat{G}$ is defined by $[\widehat{G},\widehat{G},\widehat{G}] = \mathrm{im}\,[\cdot,\cdot,\cdot]_{\widehat{G}}$. In other words, it's underlying grading R-module is

$$[\widehat{G},\widehat{G},\widehat{G}] = \mathrm{span}_R\{[b_1,b_2,b_3]_{\widehat{G}} \mid b_1,b_2,b_3 \in \widehat{G}\},$$

and its differential is simply the restriction of ∂ to the graded submodule $[\widehat{G},\widehat{G},\widehat{G}]$. The hom-associator complex of F is defined in a similar fashion. Since π is an MLDG algebra homomorphism, it restricts to a chain map $\pi|_{[\widehat{G},\widehat{G},\widehat{G}]}\colon [\widehat{G},\widehat{G},\widehat{G}] \to [F,F,F]$ given by

$$\pi[b_1,b_2,b_3]_{\widehat{G}} = [\pi(b_1),\pi(b_2),\pi(b_3)]_F,$$

where we write $\pi = \pi|_{[\widehat{G},\widehat{G},\widehat{G}]}$ in order to simplify notation. Similarly, ι restricts to a chain map $\iota|_{[F,F,F]}\colon [F,F,F] \to [\widehat{G},\widehat{G},\widehat{G}]$ given by

$$\iota[a_1,a_2,a_3]_F = [a_1,a_2,a_3]_{\widehat{G}},$$

where we write $\iota(a_i) = a_i$ since ι is just the inclusion map.

3. We have

$$
\begin{aligned}
[\cdot,\cdot]_{\widehat{G}} &= \widehat{\kappa}\widehat{\mu} - \widehat{\mu}\widehat{\kappa}^{\otimes 2} \\
&= \iota\kappa\pi\iota\mu\pi^{\otimes 2} - \iota\mu\pi^{\otimes 2}\iota^{\otimes 2}\kappa^{\otimes 2}\pi^{\otimes 2} \\
&= \iota\kappa\mu\pi^{\otimes 2} - \iota\mu\kappa^{\otimes 2}\pi^{\otimes 2} \\
&= \iota(\kappa\mu - \mu\kappa^{\otimes 2})\pi^{\otimes 2} \\
&= \iota[\cdot,\cdot]_F\pi^{\otimes 2}
\end{aligned}
$$

Applying π to the left and $\iota^{\otimes 3}$ to the right gives us $[\cdot,\cdot]_F = \pi[\cdot,\cdot]_{\widehat{G}}\iota^{\otimes 3}$. □

Corollary 1. *Keeping the notation as in (1.2),*

Corollary 2. *Keeping the notation as in (1.2), suppose $\kappa = 1$ (so $\widehat{\kappa} = \iota\pi$). If F is associative, then $\widehat{G}$ is hom-associative.*

1.2 MLDG Modules

We now want to define MLDG modules. Let $A = (A, \mathrm{d}, \lambda, \mu)$ be an MLDG R-algebra. Let (X, d_X) be an R-complex, let $\mu_{A,X}\colon A \otimes_R X \to X$ be a chain map, let $\mu_{X,A}\colon X \otimes_R A \to X$ be a chain map, and let $\lambda_X\colon X \to X$ be chain map. We call the quintuple $X = (X, \mathrm{d}_X, \lambda_X, \mu_{A,X}, \mu_{X,A})$ a **pre-MLDG A-module**. In this case, we call $\mu_{A,X}$ the **left A-scalar multilpication** of X, we call $\mu_{X,A}$ the **right A-scalar multiplication** of X, and we call λ_X the **perturbation** of X. As usual, we denote the image of $a \otimes x$ under $\mu_{A,X}$ by ax, and similarly we denote the image of $x \otimes a$ under $\mu_{X,A}$ by xa. We often simplify our notation even further by denoting $\mathrm{d} = \mathrm{d}_X$ and $\lambda = \lambda_X$ (context will always make it clear which differential the symbol d stands for, and similarly which perturbation the symbol λ stands for). With this convention in mind, we see that for all $a, b \in A$ homogeneous, $x, y \in X$ homogeneous, and $r \in R$ we have identities of the form

$$
\begin{aligned}
a(x+y) &= ax + ay \\
(a+b)x &= ax + bx \\
r(ax) &= a(rx) = (ax)r \\
(xa)r &= x(ar) = (xr)a \\
\mathrm{d}(ax) &= \mathrm{d}(a)x + (-1)^{|a|}a\mathrm{d}(x) \\
\mathrm{d}(xa) &= \mathrm{d}(x)a + (-1)^{|x|}x\mathrm{d}(a),
\end{aligned}
$$

and so on. For X to be an MLDG A-module, we want it to satisfy two additional properties:

Definition 1.5. Let $X = (X, \mathrm{d}_X, \lambda_X, \mu_{A,X}, \mu_{X,A})$ be a pre-MLDG A-module.

1. We say X is **unital** if $1x = x = x1$ for all $x \in X$.
2. We say X is **graded-commutative** if
$$ax = (-1)^{|a||x|}xa$$
for all $a \in A$ homogeneous and $x \in X$ homogeneous. In this case, $\mu_{X,A}$ is completely determnined by $\mu_{A,X}$, and thus we completely forget about it and write $\mu_X = \mu_{A,X}$. We call μ_X the **A-scalar multiplication** of X.

We say X is an **MLDG A-module** if it is graded-commutative and unital.

The definitions (3-5) in Definition (4.2) also apply to X. For instance, we say X is multiplicative if

$$\lambda_X(ax) = \lambda_A(a)\lambda_X(x) \tag{2}$$

for all $a \in A$ and $x \in X$. If X is multiplicative, then we say λ_X is an A-linear map. In particular, if $\lambda_A = 1$ and λ_X is a perturbation of X which is not necessarily equal to the identity function on X, then the multiplicative law looks like $\lambda_X(ax) = a\lambda_X(x)$ (so calling λ_X an A-linear map here agrees with the usual case). Note that we used the full symbols λ_A and λ_X in (2), but we can also write this in a much cleaner format as $\lambda(ax) = \lambda(a)\lambda(x)$: the point is that the "λ" in $\lambda(ax)$ obviously has to be λ_X, and the "λ" in $\lambda(a)$ obviously has to be λ_A. The reason why we abuse our notation in this way is because it makes formulas look much cleaner. For instance, the permutative law for X is rigorously expressed as

$$\lambda_A(ab)(\lambda_A(c)\lambda_X(x)) = (\lambda_A(a)\lambda_A(b))\lambda_X(cx),$$

but we can also express this law as $\lambda(ab)(\lambda(c)\lambda(x)) = (\lambda(a)\lambda(b))\lambda(cx)$ which is visibly much cleaner.

The definitions in Definition (1.3) also apply to X as well. For instance, the hom-associator of X is the chain map

$$[\cdot,\cdot,\cdot]_{A,X} = \mu_X \circ (\lambda_A \otimes \mu_X) - \mu_X \circ (\mu_A \otimes \lambda_X) = \mu(\lambda \otimes \mu - \mu \otimes \lambda).$$

If A and X are understood from context, then simplify our notation by writing $[\cdot,\cdot,\cdot] = [\cdot,\cdot,\cdot]_X = [\cdot,\cdot,\cdot]_{A,X}$.

1.2.1 Hom-Associator Identities

We want to familiarize ourselves with the hom-associator of X, so in this subsubsection we collect together some identities which the hom-associator of X satisfies:

- For all $a,b \in A$ homogeneous and $x \in X$, we have the Leibniz law:
$$\mathrm{d}[a,b,x] = [\mathrm{d}(a),b,x] + (-1)^{|a|}[a,\mathrm{d}(b),x] + (-1)^{|a|+|b|}[a,b,\mathrm{d}(x)].$$
- For all $a,b \in A$ homogeneous and $x \in X$ homogeneous we have
$$[a,b,x] = -(-1)^{|a||b|+|a||x|+|b||x|}[x,b,a]. \tag{3}$$
- For all $a,b \in A$ homogenous and $x \in X$ homogeneous we have a graded-commutative version of the Jacobi identity:
$$[a,b,x] = -(-1)^{|a||x|+|b||x|}[x,a,b] - (-1)^{|a||b|+|a||x|}[b,x,a] = 0 \tag{4}$$
- For all $a,b \in A$ and $x \in X$ we have
$$(\lambda^2 a)[b,x] - [ab,\lambda x] + [\lambda a, bx] - [a,b](\lambda^2 x) = [\lambda a, \lambda b, \lambda x] - \lambda[a,b,x] \tag{5}$$
- For all $a,b,c \in A$ and $x \in X$ we have
$$\lambda^2(a)[b,c,x] - [ab,\lambda(c),\lambda(x)] + [\lambda(a),bc,\lambda(x)] - [\lambda(a),\lambda(b),cx] + [a,b,c]\lambda^2(x) = [a,b,c,x] \tag{6}$$

The way the signs in (3) show up can be interpreted as follows: in order to go from $[a,b,x]$ to $[x,b,a]$, we have to first swap a with b (this is where the $(-1)^{|a||b|}$ comes from), then swap a with x (this is where the $(-1)^{|a||x|}$ comes from), and then finally swap b with x (this is where the $(-1)^{|b||x|}$ comes from). The proof of (3) doesn't go exactly like that; instead it looks like this:

$$\begin{aligned}
[a,b,x] &= \lambda(a)(bx) - (ab)\lambda(x) \\
&= (-1)^{|b||x|}\lambda(a)(xb) - (-1)^{|a|b|}(ba)\lambda(x) \\
&= (-1)^{|b||x|+|a||x|+|a||b|}(xb)\lambda(a) - (-1)^{|a|b|+|b||x|+|a||x|}\lambda(x)(ba) \\
&= (-1)^{|b||x|+|a||x|+|a||b|}((xb)\lambda(a) - \lambda(x)(ba)) \\
&= -(-1)^{|b||x|+|a||x|+|a||b|}(\lambda(x)(ba) - (xb)\lambda(a)) \\
&= -(-1)^{|b||x|+|a||x|+|a||b|}[x,b,a].
\end{aligned}$$

Nevertheless, the analogy is still helpful when trying to keep track of the signs. A similar interpretation is also given to (4). For instance, in order to get from $[a,b,x]$ to $[x,a,b]$, we have to swap x with b and then swap x with a (this is where the $(-1)^{|a||x|+|b||x|}$ comes from). The signs can become confusion and oftentimes we don't even need to keep track of them. For this reason, we make the convention that we will ignore them from time to time and pretend as if we are working in characteristic 2, with the understanding in the back of our minds that all of our constructions/results should still work if we put the signs back in. When we need to include the signs, then we definitely will, but our notational convention will make things much cleaner in the end. Finally, we can write (5) without using element as

$$\delta^3([\cdot,\cdot]) = [\cdot,\cdot,\cdot] \circ \lambda^{\otimes 3} - \lambda \circ [\cdot,\cdot,\cdot]$$

Similarly, we can write (6) without using elements as

$$\delta^4([\cdot,\cdot,\cdot]) = [\cdot,\cdot,\cdot,\cdot]$$

where δ^3 and δ^4 are thought of as codifferential maps.

Proposition 1.5. *Assume X is a two-sided MLDG A-module and let $a \in A$ and $x \in X$.*

1. *We have $[a,a,x] = 0$ if and only if $[x,a,a] = 0$.*
2. *If $[a,a,x] = 0$, then $[a,x,a] = 0$. The converse holds if we are working in characteristic $\neq 2$.*
3. *In characteristic 2, we have $[a,x,a] = 0$.*

Proof. Let $a \in A$ and let $x \in X$ where both a and x are homogeneous. From (3) we have

$$\begin{aligned}[a,a,x] &= -(-1)^{|a||a|+|a||x|+|a||x|}[x,a,a] \\ &= -(-1)^{|a|}[x,a,a].\end{aligned}$$

The signs make this look hideous, so instead we can simply write this as $[a,a,x] = [x,a,a]$ (and we can think of this as being true "up to a sign"). Thus since $[a,a,x] = [x,a,a]$ "up to a sign", it follows at once that 1 and 3 are equivalent (since $[a,a,x] = 0$ and $[a,x,a] = 0$ are also true up to sign). Now if we combine (3) with (4), then we see that

$$\begin{aligned}0 &= [a,a,x] + [x,a,a] + [a,x,a] \\ &= [x,a,a] + [x,a,a] + [a,x,a] \\ &= \varepsilon[x,a,a] + [a,x,a].\end{aligned}$$

At this point we need to be careful because we are not *actually* working in characteristic 2 (so $[x,a,a] + [x,a,a]$ don't cancel out necessarily). Thus to finish the calculation, we really need to pay attention to the signs. So working with the signs, we see that

$$\begin{aligned}[a,x,a] &= -(-1)^{|a||x|+|a|}[a,a,x] - (-1)^{|a||x|+|a|}[x,a,a] \\ &= (-1)^{|a||x|}[x,a,a] - (-1)^{|a||x|+|a|}[x,a,a] \\ &= (-1)^{|a||x|}(1-(-1)^{|a|})[x,a,a]. \\ &= -(-1)^{|a||x|}((-1)^{|a|}-1)[x,a,a].\end{aligned}$$

So we have the formulas

$$\begin{aligned}[a,a,x] &= -(-1)^{|a|}[x,a,a] \\ [a,x,a] &= -(-1)^{|a||x|}((-1)^{|a|}-1)[x,a,a].\end{aligned}$$

In particular, we see that

$$[a,x,a] = \begin{cases} (-1)^{|x|}2[x,a,a] = (-1)^{|x|}2[a,a,x] & \text{if } |a| \text{ is odd} \\ 0 & \text{if } |a| \text{ is even} \end{cases} \tag{7}$$

for all homogeneous $a \in A$ and homogeneous $x \in X$. So in characteristic 2 we always have $[a,x,a] = 0$, though it may be possible that $[x,a,a] = a(ax) = [a,a,x] \neq 0$. Similarly, we have

$$[a,a,x] = \begin{cases} (-1)^{|x|}\frac{1}{2}[a,x,a] & \text{if } |a| \text{ is odd and } \operatorname{char} R \neq 2 \\ (-1)^{|a|}[x,a,a] & \text{if } |a| \text{ is even} \end{cases} \tag{8}$$

□

1.2.2 Hom-Associator Complex and its Homology

Let X be an MLDG A-module. Clearly if X is hom-associative, then X is homologically hom-associative. The converse is also true under certain conditions.

Proposition 1.6. *Assume that $(R,\mathfrak{m})$ is a local ring and that $[X]$ is minimal. If X is hom-associative in degree i, then X is hom-associative in degree $i+1$ if and only if X is homologically hom-associative in degree $i+1$. In particular, if $[X]$ is bounded below and minimal, then X is hom-associative if and only if X is homologically hom-associative.*

Proof. Clearly if X is hom-associative in degree $i+1$, then it is homologically hom-associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically hom-associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume $\mathrm{H}_{i+1}([A,A,X]) = 0$ and $[A,A,X]_{i+1} \neq 0$. By Nakayama's Lemma, we can find a triple (a,b,x) such that $|a|+|b|+|x| = i+1$ and such that $[a,b,x] \notin \mathfrak{m}[A,A,X]_{i+1}$. Since $[A,A,X]_i = 0$ by assumption, we have $\mathrm{d}_X[a,b,x] = 0$. Also, since X is minimal, we have $\mathrm{d}_X[A,A,X] \subseteq \mathfrak{m}[A,A,X]$. Thus $[a,b,x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. □

Remark 3. Note that if both A and X are both minimal, then the Leibniz law (**??**) implies $[X]$ is minimal too, however the converse need not hold.

The proof of Proposition (1.6) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition:

Definition 1.6.

1. Assume that $[X]$ is bounded below. The **lower associative index** of X, denoted $\mathrm{la}(X)$, is defined to be the smallest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\mathrm{la}(X) = \infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded below by setting $\mathrm{la}(X) = -\infty$.

2. Assume that $[X]$ is bounded above. The **upper associative index** of X, denoted $\mathrm{ua}(X)$, is defined to be the largest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\mathrm{ua}(X) = -\infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded above by setting $\mathrm{ua}(X) = \infty$.

With the lower associative index of X defined, we see, after analyzing the proof of Proposition (1.6), that if R is local and $[X]$ is nonzero and minimal, then

$$\mathrm{la}(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i([X]) \neq 0\}.$$

Thus, in this case, the lower associative index of X can be measured homologically.

1.2.3 MLDG Module Homomorphisms

Having defined MLDG A-modules, we now wish to define MLDG A-module homomorphisms between them.

Definition 1.7. Let X and Y be two MLDG A-modules and let $\varphi\colon X \to Y$ be a chain map of degree i where $i \in \mathbb{Z}$. We say φ is an MLDG A**-module homomorphism of degree** i if it satisfies the following two additional properties:

1. It preserves the scalar-multiplications of X and Y, meaning $\varphi(ax) = (-1)^{|a||\varphi|} a\varphi(x)$ for all $a \in A$ and $x \in X$.

2. It preserves the perturbations of X and Y, meaning $\varphi(\lambda(x)) = \lambda(\varphi(x))$ for all $x \in X$.

In the case where $|\varphi| = 0$, then we will simply call φ an MLDG A-module homomorphism. The collection of all MLDG A-modules together with the colleciton of all MLDG A-module homomorphisms forms a category, which we denote by $\mathbf{MLDGmod}_A$.

Remark 4. Note that properties 1 and 2 in Definition (1.7) can be interpreted as saying the following diagrams are commutative (up to an appropriate sign in the first diagram):

$$\begin{tikzcd} A \otimes_R X \arrow[r, "1\otimes\varphi"] \arrow[d, "\mu_{A,X}"'] & A \otimes_R Y \arrow[d, "\mu_{A,Y}"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\lambda_X"'] & Y \arrow[d, "\lambda_Y"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

1.2.4 MLDG Submodules and MLDG Ideals

Let X and Y be two MLDG A-modules such that $X \subseteq Y$. We say X is an MLDG A-submodule of Y if the inclusion map $\iota\colon X \to Y$ is an MLDG A-module homomorphism. In particular, this means that the differential of Y restricts to the differential of X, the perturbation of Y restricts to the perturbation of X, and the multiplication of Y restricts to the multiplication of X. The MLDG A-submodules of A are called the **MLDG ideals** of A. Recall that everytime we define an "MLDG" object, we also implicitly define corresponding "MLG/MDG/MG/etc..." objects by forgetting various data. For instance, we say $\mathfrak{a}$ is an **MG ideal** of A (or more simply just an **ideal** of A), if it is abelian subgroup of A which is closed under A-scalar multiplicaiton; meaning if $x, y \in \mathfrak{a}$ and $a \in A$, then $ax \in \mathfrak{a}$ and $x + y \in \mathfrak{a}$. Thus we make no reference to differential or perturbation of A. Similarly, we say A is an **MDG ideal** of A if it is an MG ideal of A and the differential of A takes $\mathfrak{a}$ to itself, meaning $\mathrm{d}\mathfrak{a} \subseteq \mathfrak{a}$ (so d gives $\mathfrak{a}$ the structure of an R-complex).

Now suppose $\mathfrak{a}$ is just an MG ideal of A.

1.2.5 The hom-associator complex and homology of an MLDG A-module

1.2.6 The nucleus of an MDG A-module

Let X be an MDG A-module. It will be important to keep track of what elements in X associate with everything else. This leads us to the following definitions:

Definition 1.8. Let X be an MDG A-module.

1. Suppose X is a left MDG A-module. The **left nucleus** of X, denoted $\mathrm{N}_l(X)$, is the subset of X defined by
$$\mathrm{N}_l(X) = \{x \in X \mid [a,b,x] = 0 \text{ for all } a,b \in A\}.$$
2. Suppose X is a right MDG A-module. The **right nucleus** of X, denoted $\mathrm{N}_r(X)$, is the subset of X defined by
$$\mathrm{N}_r(X) = \{x \in X \mid [x,a,b] = 0 \text{ for all } a,b \in A\}.$$
3. Suppose X is two-sided MDG A-module. The **middle nucleus** of X, denoted $\mathrm{N}_m(X)$, is the subset of X defined by
$$\mathrm{N}_m(X) = \{x \in X \mid [a,x,b] = 0 \text{ for all } a,b \in A\},$$
and the **nucleus** of X, denoted $\mathrm{N}(X)$, is subset of X defined by
$$\mathrm{N}(X) = \mathrm{N}_l(X) \cap \mathrm{N}_m(X) \cap \mathrm{N}_r(X),$$
where $\mathrm{N}_l(X)$ is the left nucleus of X where we view X as a left PDG A-module, and where $\mathrm{N}_r(X)$ is the right nucleus of X where we view X as a right PDG A-module.

Remark 5. Suppose X is a two-sided MDG A-module. Notice that the identity (3) implies $\mathrm{N}_l(X) = \mathrm{N}_r(X)$. Also notice that (4) implies $\mathrm{N}_l(X) \cap \mathrm{N}_r(X) \subseteq \mathrm{N}_m(X)$. Combining these facts together, we see that $\mathrm{N}(X) = \mathrm{N}_l(X) = \mathrm{N}_r(X)$. Thus if Y is a left (resp. right) PDG A-module, then we can define the **nucleus** of Y, denoted $\mathrm{N}(Y)$, to be left (resp. right) nucleus of Y without any threat of conflict in our notation. With the remark above in mind, we also wish to consider the subcomplex of A given by
$$\mathrm{N}_A(X) = \{a \in A \mid [a,b,x] = 0 \text{ for all } b \in A \text{ and } x \in X\}.$$
We call $\mathrm{N}_A(X)$ the nucleus of A with respect to X.

Lemma 1.1. *Let (A, d, μ) be an MDG R-algebra such that $A_0 = R$ and set $I = \mathrm{d}(A_1)$. Then I kills $\mathrm{H}(A)$.*

Proof. Let $t \in I$ and let $\mathrm{m}_t \colon A \to A$ be the multiplication by t map, defined by
$$\mathrm{m}_t(a) = ta$$
for all $a \in A$. We claim that m_t is null-homotopic. Indeed, choose $e \in A$ such that $\mathrm{d}(e) = t$, and let $\mathrm{m}_e \colon A \to A$ be the multiplication by e map, defined by
$$\mathrm{m}_e(a) = ea$$
for all $a \in A$. Note that m_e is a graded R-linear map of degree 1. Also note that for all $a \in A$, we have
$$\begin{aligned}
(\mathrm{d}\mathrm{m}_e + \mathrm{m}_e\mathrm{d})(a) &= \mathrm{d}\mathrm{m}_e(a) + \mathrm{m}_e\mathrm{d}(a) \\
&= \mathrm{d}(ea) + e\mathrm{d}(a) \\
&= \mathrm{d}(e)a - e\mathrm{d}(e) + e\mathrm{d}(a) \\
&= \mathrm{d}(e)a \\
&= ta \\
&= \mathrm{m}_t(a).
\end{aligned}$$
Thus m_e is a homotopy from m_t to the zero map; hence m_t is null-homotopic. It follows that t kills $\mathrm{H}(A)$, and since $t \in I$ was arbitrary, we see that I kills $\mathrm{H}(A)$. □

Corollary 3. *Let K be a field and let (A, d, μ) be an MDG K-algebra such that $A_0 = K$. Assume that $\mathrm{d}(A_1) \neq 0$. Then $\mathrm{H}(A) = 0$.*

Corollary 4. *Let (A, d, μ) be an MDG R-algebra such that $A_0 = R$ and set $I = \mathrm{d}(\mathrm{N}(A)_1)$. Then I kills $\mathrm{H}([A])$.*

Proof. Let $t \in I$ and choose $e \in \mathrm{N}(A)$ such that $\mathrm{d}(e) = t$. Note that the restriction of m_e lands in $[A]$ when it is restricted to $[A]$ since (3) implies $\mathrm{m}_e[a,b,c] = [ea,b,c]$ for all $a,b,c \in A$. It follows that $\mathrm{m}_e|_{[A]}$ is a homotopy from $[\mathrm{m}_t]$ to the zero map; hence $[\mathrm{m}_t]$ is null-homotopic. It follows that t kills $\mathrm{H}([A])$, and since $t \in I$ was arbitrary, we see that I kills $\mathrm{H}([A])$. □

Corollary 5. *Let K be a field and let (A, d, μ) be an MDG K-algebra such that $A_0 = K$. Assume that $\mathrm{d}(\mathrm{N}(A)_1) \neq 0$. Then $\mathrm{H}([A]) = 0$.*

1.2.7 The Hom-Associator Functor

Let $\varphi \colon X \to Y$ be an MLDG A-module homomorphism. Observe that for all $a, b \in A$ and $x \in X$, we have

$$\begin{aligned}\varphi[a,b,x] &= \varphi(\lambda(a)(bx) - (ab)\lambda(x)) \\ &= \lambda(a)\varphi(bx) - (ab)\varphi(\lambda(x)) \\ &= \lambda(a)(b\varphi(x)) - (ab)\varphi(\lambda(x)) \\ &= \lambda(a)(b\varphi(x)) - (ab)\lambda(\varphi(x)) \\ &= [a,b,\varphi(x)].\end{aligned}$$

Thus φ restricts to a chain map $\varphi \colon [A,A,X] \to [A,A,Y]$ of R-complexes given by $\varphi[a,a,x] = [a,a,\varphi(x)]$. It is straightforward to check that the assignment $X \mapsto [A,A,X]$ and $\varphi \mapsto \varphi|_{[A,A,X]}$ gives rise to a functor from $\mathbf{MLDGmod}_A$ to $\mathbf{Comp}_R$. We call this functor the **hom-associator functor** of X. The hom-associator functor need not be exact. To see what goes wrong, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{9}$$

be a short exact sequence of MLDG A-modules. We obtain an induced sequence of R-complexes

$$0 \longrightarrow [A,A,X] \xrightarrow{\varphi} [A,A,Y] \xrightarrow{\psi} [A,A,Z] \longrightarrow 0 \tag{10}$$

It's easy to see that we have exactness at $[A,A,X]$ and $[A,A,Z]$ since these are just restriction maps. On the other hand, we may not have exactness at $[A,A,Y]$. To see why, let $\sum_{i=1}^n [a_i,b_i,y_i] \in \ker\psi$. Then by exactness of (9), there exists an $x \in X$ such that $\varphi(x) = \sum_{i=1}^n [a_i,b_i,y_i]$. It is not necessarily the case however that $x \in [A,A,X]$ (we will see a counterexample to this later on). In order to ensure exactness of (10), we need to place a condition on (9). This leads us to consider the following definition:

Definition 1.9. Let X be an MLDG A-submodule of Y. We say X is an **h-stable** MLDG A-submodule of Y if it satisfies

$$[A,A,X] = X \cap [A,A,Y].$$

The short exact sequence (9) is called **h-stable** if $\varphi(X)$ is an h-stable MLDG A-submodule of Y.

It is easy to check that (10) is a short exact sequence of R-complexes if and only if it is h-stable. In this case, the h-stable short exact sequence (10) of R-complexes induces a long exact sequence in homology

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}([A,A,Z]) \arrow[dll, rounded corners] \\ & \mathrm{H}_i([A,A,X]) \arrow[r] & \mathrm{H}_i([A,A,Y]) \arrow[r] & \mathrm{H}_i([A,A,Z]) \arrow[dll, rounded corners] \\ \mathrm{H}_{i-1}([A,A,X]) \arrow[r] & \cdots & & \end{tikzcd} \tag{11}$$

From this long exact sequences we obtain the following theorem:

Theorem 1.2. *Suppose X is an h-stable MLDG A-submodule of Y. Then Y is homologically hom-associative if and only if X and Y/X are homologically hom-associative.*

1.2.8 Application of long exact sequence

We want to discuss another application of (13). Suppose that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be a free MDG R-algebra resolution of R/I, and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $\mathrm{C}(r)$ is a free resolution of $R/\langle I, r\rangle$ over R. We can give $\mathrm{C}(r)$ the structure of a free MDG R-algebra resolution of $R/\langle I, r\rangle$ as follows: first note that the underlying graded R-module of $\mathrm{C}(r)$ has the form $F + eF$ where e is an exterior variable of degree -1 and where $\{1, e\}$ is an F-linearly independent set. In other words, every element in $F + eF$ can be expressed in the form $\alpha + e\beta$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i, then α and β are homogeneous of degrees i and $i-1$ respectively. With this understood, we extend the multplication μ_F on F to a multiplication $\mu_{\mathrm{C}(r)}$ on $\mathrm{C}(r)$ by setting

$$(\alpha + e\beta)(\gamma + e\delta) = \alpha\gamma + e(\beta\gamma + (-1)^{|\alpha|}\alpha\delta + \varphi(\alpha, \beta))$$

for all homogeneous $\alpha, \beta, \gamma, \delta \in F$ and extending this linearly everywhere else. Note that the mapping cone $\mathrm{C}(r)$ inherits a natural *right* MDG F-module structure via restriction of scalars. The reason why it is more naturally viewed as a right MDG F-module rather than a left MDG F-module can be seen in the way the mapping cone differential acts on elements: given $\alpha, \beta \in F$ where α is homogeneous, we have

$$\mathrm{d}_{\mathrm{C}(r)}(\alpha + e\beta) = \mathrm{d}_F(\alpha) + r\beta - e\mathrm{d}_F(\beta).$$

Thus the mapping cone differential behaves as if e is an exterior variable of degree -1 (which is what we want).

Thus there are two associator complexes to consider: the associator complex of $\mathrm{C}(r)$ where we view $\mathrm{C}(r)$ an MDG R-algebra and the associator complex of $\mathrm{C}(r)$ where we view $\mathrm{C}(r)$ as a right MDG F-module. In fact, these associator complexes are the same. To see this, first note that $e \in \mathrm{N}(\mathrm{C}(r))$ by construction of $\mu_{\mathrm{C}(r)}$. In particular, this implies

$$\begin{aligned}
[\alpha, \beta, \gamma + e\delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|+|\beta|} e[\alpha, \beta, \delta] \\
[\alpha, \beta + e\gamma, \delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|} e[\alpha, \gamma, \delta] \\
[\alpha + e\beta, \gamma, \delta] &= [\alpha, \gamma, \delta] + e[\beta, \gamma, \delta]
\end{aligned}$$

Using these identities together with the fact that $e^2 = 0$ and the fact that identity $1 \in \mathrm{N}(\mathrm{C}(r))$, we obtain

$$\begin{aligned}
[\alpha + e\beta, \gamma + e\delta, \varepsilon + e\zeta] &= [\alpha, \gamma, \varepsilon] + e[\beta, \gamma, \varepsilon] + (-1)^{|\alpha||} e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|} e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||} e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|} e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||}[1 + e\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}[1 + e\alpha, \gamma, \zeta]
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F$. It follows that these associator complexes are the same, and thus we may denote this common associator complex by $[\mathrm{C}(r)]$ without there being any source of confusion.

The homothety map $\mathrm{m}_r \colon F \to F$ gives rise to a short exact sequence of right MDG F-modules

$$0 \longrightarrow F \xrightarrow{\iota} \mathrm{C}(r) \xrightarrow{\pi} \Sigma F \longrightarrow 0 \tag{12}$$

where $\iota \colon F \to \mathrm{C}(r)$ is the inclusion map and where $\pi \colon \mathrm{C}(r) \to \Sigma F$ is the projection map given by $\pi(\alpha + e\beta) = \alpha$ for all $\alpha, \beta \in F$.

Proposition 1.7. *The short exact sequence (12) is stable.*

Proof. We must check that $[\mathrm{C}(r)] \cap F \subseteq [F]$ since the reverse inclusion is trivial. Suppose $\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i]$ is an abitrary element in $[\mathrm{C}(r)] \cap F$ where $r_i \in R$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ for each $1 \leq i \leq m$. Observe that

$$\begin{aligned}
\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] &= \sum_{i=1}^m r_i([\alpha_i, \gamma_i, \delta_i] + e[\beta_i, \gamma_i, \delta_i]) \\
&= \sum_{i=1}^m r_i[\alpha_i, \gamma_i, \delta_i] + e\sum_{i=1}^m r_i[\beta_i, \gamma_i, \delta_i].
\end{aligned}$$

Since $\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in F$, it follows that $\sum_{i=1}^m r_i[\beta_i, \gamma_i, \delta_i] = 0$. Thus

$$\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] = \sum_{i=1}^m r_i[\alpha_i, \gamma_i, \delta_i] \in [F].$$

Therefore $[\mathrm{C}(r)] \cap F \subseteq [F]$. ☐

Remark 6. The same exact proof also shows that if $\varphi \colon X \to Y$ is an MDG A-module homomorphism, then Y is a stable MDG A-submodule of $\mathrm{C}(\varphi) = Y + eX$ where $\mathrm{C}(\varphi)$ is the mapping cone with respect to φ. Here we are using the fact that $\mathrm{C}(\varphi)$ can be given the structure of a right MDG A-module by defining the right scalar-multiplication map $\mu_{\mathrm{C}(\varphi),A}$ via the formula

$$(y + ex)a = ya + e(xa)$$

for all $y + ex \in \mathrm{C}(\varphi)$ and $a \in A$.

Since (12) is stable, we obtain a long exact sequence in homology

$$\begin{tikzcd}
& & \cdots \arrow[r] & \mathrm{H}_i([F]) \arrow[dll, "r", rounded corners, to path={[pos=0.25] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\mathrm{H}_i([F]) \arrow[r] & \mathrm{H}_i([\mathrm{C}(r)]) \arrow[r] & \mathrm{H}_{i-1}([F]) \arrow[dll, "r"] & \\
\mathrm{H}_{i-1}([F]) \arrow[r] & \cdots & &
\end{tikzcd} \tag{13}$$

Using this long exact sequence together with Nakayama's lemma, we can deduce the following theorem:

Theorem 1.3. *Keep the notation as above.*

1. *We have*
$$\mathrm{la}(\mathrm{C}(r)) = \mathrm{la}(F).$$

2. *Set $k = \mathrm{ua}(F)$ and assume that $-\infty < k < \infty$. We have*
$$\mathrm{ua}(\mathrm{C}(r)) = \begin{cases} \mathrm{ua}(F) & \textit{if } r \textit{ is } \mathrm{H}_k([F])\textit{-regular} \\ \mathrm{ua}(F)+1 & \textit{else} \end{cases}$$

1.3 Permutativity

Let X be a permutative A-module, so $\delta^4([\cdot,\cdot,\cdot]) = 0$. Then the identity (6) becomes
$$(\lambda_A^2 a)[b,c,x] + [a,b,c](\lambda_X^2 x) = [ab, \lambda_A c, \lambda_X x] - [\lambda_A a, bc, \lambda_X x] + [\lambda_A a, \lambda_A b, cx].$$
If we set $a = \lambda^{n-2}(a)$ and $x = \lambda^{n-2}(x)$ where $2 \geq 0$, then the identity (6) becomes
$$(\lambda_A^n a)[b,c,x] + [a,b,c](\lambda_X^n x) = [ab, \lambda_A c, \lambda_X^{n-1} x] - [\lambda_A^{n-1} a, bc, \lambda_X^{n-1} x] + [\lambda_A^{n-1} a, \lambda_A b, cx]. \tag{14}$$
for all $n \geq 2$. Now suppose we also have $\delta^3([\cdot,\cdot]) = 0$. Then the identity (5) becomes $\lambda[a,b,x] = [\lambda a, \lambda b, \lambda c]$, and thus we can pull all of the λ's out of (14) and obtain
$$(\lambda_A^n a)[b,c,x] + [a,b,c](\lambda_X^n x) = \lambda_A^n([ab,c,x] - [a,bc,x] + [a,b,cx])$$
Finally, suppose that $\lambda_A^n = 1$ and $\lambda_X^n = 0$ for some $n \geq 2$. Then it follows that
$$a[b,c,x] = [ab,c,x] - [a,bc,x] + [a,b,cx].$$
In this case, it follows that $[A,A,X]$ is an MLDG A-submodule of X. Next observe that
$$\begin{aligned}
a[b,c,[d,e,x]] &= [ab,c,[d,e,x]] - [a,bc,[d,e,x]] + [a,b,c[d,e,x]] \\
&= [ab,c,[d,e,x]] - [a,bc,[d,e,x]] + [a,b,[cd,e,x] - [c,de,x] + [c,d,ex]] \\
&= [ab,c,[d,e,x]] - [a,bc,[d,e,x]] + [a,b,[cd,e,x]] - [a,b,[c,de,x]] + [a,b,[c,d,ex]].
\end{aligned}$$
It follows that $[A,A,X]^{[2]} := [A,A,[A,A,X]]$ is an MLDG A-module. More generally, a similar calculation shows
$$[A,A,X]^{[n]} := [A,A,[A,A,[\cdots[A,A,[A,A,X]]\cdots]]]$$
is an MLDG A-module. We obtain a decreasing sequence of MLDG A-submodules of X:
$$X \supseteq [A,A,X] \supseteq [A,A,X]^{[2]} \supseteq \cdots \supseteq [A,A,X]^{[n]} \supseteq \cdots.$$
This gives rise to short exact sequences
$$0 \to [A,A,X]^{[n]} \to A \to A/[A,A,X]^{[n]} \to 0,$$
where $A/[A,A,X]$ is hom-associative.

Definition 1.10. Let X be an MLDG A-module and let $n \geq 2$. We say X is ***n*-permutative** if $\delta^4([\cdot,\cdot,\cdot]) = 0 = \delta^3([\cdot,\cdot])$ and $\lambda_A^n = 1$ and $\lambda_X^n = 0$.

Theorem 1.4. *Let X be a minimal n-permutative MLDG A-module. Then A is hom-associative if and only if A is acyclic.*

Proof. We a short exact sequence of MLDG A-modules
$$0 \to [A,A,X] \to A \to A/[A,A,X] \to 0.$$
Furthermore, $A/[A,A,X]$ is hom-associative. Now, if A is acyclic, then we see that $\mathrm{H}[A,A,X] = \mathrm{H}(A/[A,A,X])$. In other words, the homology of the associator complex of A is the homology of the hom-associative algebra $A/[A,A,X]$. □

1.4 Multiplicativity

Let X be an MLDG A-module. To keep track of the failure for X to be multiplicative, we introduce the following definition:

Definition 1.11. We define the **multiplicator map** of X to be the chain map $[\cdot,\cdot]_{A,X}\colon A\otimes_R X\to X$ defined by

$$[\cdot,\cdot]_{A,X} := \lambda_X\mu_{A,X} - \mu_{A,X}(\lambda_A\otimes\lambda_X).$$

We simplify our notation by writing $[\cdot,\cdot]_X$ (or even just $[\cdot,\cdot]$) instead of $[\cdot,\cdot]_{A,X}$ whenever context is clear. Thus, the multiplicator map of X is the difference of the two maps (both of which begin at the top) in the diagram below:

$$\begin{tikzcd}
 & A\otimes_R X \arrow[dl, "\mu_{A,X}"'] \arrow[dr, "\lambda_A\otimes\lambda_X"] & \\
X \arrow[dr, "\lambda_X"'] & & A\otimes_R X \arrow[dl, "\mu_{A,X}"] \\
 & X &
\end{tikzcd}$$

In particular, $[\cdot,\cdot]$ measures the failure of X to be multilplicative. In terms of elements, we have

$$[a,x] = \lambda(ax) - \lambda(a)\lambda(x)$$

for all $a\in A$ and $x\in X$.

1.4.1 The multiplicator complex and its homology

Let X be an MLDG A-module. Now we want to introduce a complex associated with X which we call the multiplicator complex.

Definition 1.12. The **multiplicator complex** of X is the R-subcomplex of X given by the image of $[\cdot,\cdot]$. We denote this complex by $[A,X]$. Thus the underlying graded R-module of $[A,X]$ is given by

$$[A,X] = \mathrm{span}_R\{[a,x] \mid a\in A \text{ and } x\in X\}, \tag{15}$$

and the differential $\mathrm{d}_{[A,X]}$ is just the restriction of the differential d_X to $[A,X]$. The **multiplicator homology** of X is the homology of the multiplicator complex of X. We say X is **homologically multiplicative** if $\mathrm{H}([A,X])=0$. We say X is **homologically multiplicative in degree** i if $\mathrm{H}_i([A,X])=0$. Similarly, we say X is **multiplicative in degree** i if $[A,X]_i=0$.

1.4.2 The multiplicator complex and the associator complex

Let X be an MLDG A-module. Recall from Proposition (4.1) that if X is hom-associative, then X is multiplicative. In other words, if $[A,A,X]=0$, then $[A,X]=0$. This leads one to wonder if $[A,X]$ is a subcomplex of $[A,A,X]$. The answer is: not quite, however $[A,X]$ is a subcomplex of $\langle[A,A,X]\rangle$. Indeed, if $a\in A$ and $x\in X$, then we have

$$\begin{aligned}
[a,x] &= [1,1,a,x] \\
&= [1,a,x] - [1,\lambda(a),\lambda(x)] + [1,a,\lambda(x)] - [1,1,ax] + [1,1,a]\lambda^2(x) \\
&= [1,a,x] - [1,\lambda(a),\lambda(x)] + [1,a,\lambda(x)] - [1,1,ax] + (a-\lambda(a))\lambda^2(x) \\
&\in \langle[A,A,X]\rangle.
\end{aligned}$$

It follows that $[A,X]\subseteq\langle[A,A,X]\rangle$. In particular, notice that if $\lambda(a)=a$ or if $\lambda^2(x)=0$, then $[a,x]\subseteq[A,A,X]$.

1.5 MLDG Homology

Let $A=(A,\mathrm{d},\lambda,\mu)$ be an MLDG R-algebra and let X be an MLDG A-module such that both A and X are hom-associative. We construct an R-complex, denoted $\mathrm{C}(A,X)$, as follows: the homogeneous component of degree $n\in\mathbb{Z}$ of the underlying graded R-module $\mathrm{C}(A,X)$ is

$$\mathrm{C}^n(A,X) := \begin{cases} \{\text{chain maps } \varphi\colon A^{\otimes n}\to X \text{ such that } \varphi\lambda^{\otimes n}=\lambda\varphi\} & \text{if } n>0 \\ 0 & \text{if } n\le 0 \end{cases}$$

where we write $A^{\otimes n}$ to mean A tensored with itself n times. The codifferential, denoted ffi$_{A,X}$ (or more simply just δ if context is clear) is defined by

$$\delta(\varphi) = \mu(\lambda^{n-1} \otimes \varphi) + \sum_{i=1}^{n-1} (-1)^i \varphi(\lambda^{\otimes i-1} \otimes \mu \otimes \lambda^{n-i-2}) + (-1)^n \mu(\varphi \otimes \lambda^{n-1})$$

for all $\varphi \in \mathrm{C}(A,X)_n$. Thus, if $\varphi_1 \in \mathrm{C}^1(A,X)$, $\varphi_2 \in \mathrm{C}^2(A,X)$, and $\varphi_3 \in \mathrm{C}^3(A,X)$, then we have

$$\begin{aligned}
\delta(\varphi_3)(a_1,a_2,a_3,a_4) &= \lambda^2(a_1)\varphi_3(a_2,a_3,a_4) - \varphi_3(a_1a_2,\lambda(a_3),\lambda(a_4)) + \varphi_3(\lambda(a_1),a_2a_3,\lambda(a_4)) - \varphi_3(\lambda(a_1),\lambda(a_2),a_3a_4) + \varphi_3(a_1,a_2,a_3)\lambda^2(a_4) \\
\delta(\varphi_2)(a_1,a_2,a_3) &= \lambda(a_1)\varphi_2(a_2,a_3) - \varphi_2(a_1a_2,\lambda(a_3)) + \varphi_2(\lambda(a_1),a_2a_3) - \varphi_2(a_1,a_2)\lambda(a_3) \\
\delta(\varphi_1)(a_1,a_2) &= a_1\varphi_1(a_2) - \varphi_1(a_1a_2) + \varphi_1(a_1)a_2
\end{aligned}$$

for all $a_1,a_2,a_3,a_4 \in A$. One checks that hom-associativity of X implies $\delta^2 = 0$, so that $\mathrm{C}(A,X)$ is indeed a complex. We denote its cohomology by $\mathrm{H}(A,X)$. Notice that μ defines a cocycle in $\mathrm{C}(A,A)$: indeed we have

$$\begin{aligned}
\delta(\mu) &= \mu(\lambda \otimes \mu) - \mu(\mu \otimes \lambda) + \mu(\lambda \otimes \mu) - \mu(\mu \otimes \lambda) \\
&= \mu(\lambda \otimes \mu - \mu \otimes \lambda) + \mu(\lambda \otimes \mu - \mu \otimes \lambda) \\
&= 0,
\end{aligned}$$

by the hom-associative law. In fact, μ is a coboundary in $\mathrm{C}(A,A)$ since $\delta(1) = \mu$ where $1\colon A \to A$ is the identity map.

Definition 1.13. Let $A = (A,\mathrm{d},\lambda,\mu)$ be a hom-associative MLDG R-algebra and let X be a hom-associative MLDG algebra with $\lambda_X = 0$. An **extension** of A by L is a hom-associative MLDG R-algebra E, together with an exact sequence

Theorem 1.5. *Let φ be a 2-cocycle in $\mathrm{C}(A,X)$. We define an MLDG R-algebra E as follows: the underlying graded R-complex is $X \oplus A$ with differential $\mathrm{d}_X \oplus \mathrm{d}$ defined pointwise. Multiplication and perturbation are defined by*

$$(x,a)(y,b) = (xb + ay + \varphi(a,b), ab) \quad \text{and} \quad \lambda_E(xa) = \lambda_X(x)\lambda(a)$$

for all $a,b \in A$ and $x,y \in X$. Using the fact that φ is a 2-cocycle, it is easy to verify that E is an MLDG R-algebra.

$$\mathrm{d}(x,a)(y,b) = (\mathrm{d}(x)b + x\mathrm{d}(b) + \mathrm{d}(a)y + a\mathrm{d}(y) + \mathrm{d}\varphi(a,b), \mathrm{d}(a)b + a\mathrm{d}(b))$$

$$\begin{aligned}
&= (\mathrm{d}(x)b + \mathrm{d}(a)y + \varphi\mathrm{d}^{\otimes 2} \\
&= (\mathrm{d}(x),\mathrm{d}(a))(y,b) + (x,a)(\mathrm{d}(y),\mathrm{d}(b)) \\
&= (\mathrm{d}(x,a))(y,b) + (x,a)(\mathrm{d}(y,b))
\end{aligned}$$

$$\delta(1) = \mu$$

1.6 MDG resolutions

Assume that $(R,\mathfrak{m})$ is local and let $I \subseteq \mathfrak{m}$ be an ideal of R. Let $(F,\mathrm{d},\mu) \xrightarrow{\varepsilon} R/I$ and $(G,\delta,\nu) \xrightarrow{\tau} R/I$ be MDG R-algebra resolutions of R/I. Assume that both F and G are degree-wise finite and that F is minimal. Then there exists comparison maps $\iota\colon F \to G$ and $\pi\colon G \to F$ where ι is injective and where π splits ι, that $\pi\iota = 1_F$ (here we are using the shorthand notation $\pi\iota := \pi \circ \iota$ in order to simplify our notation in what follows). Using these comparison maps, we can transport the multiplication μ on F to a multiplicaiton $\mu' = \iota\mu(\pi \otimes \pi)$ on G, and similarly, we can transport the multiplication ν on G to a multiplication $\nu' = \pi\nu(\iota \otimes \iota)$ on F. Note that $\pi\colon (G,\mu') \to (F,\mu)$ is an MDG R-algebra homomorphism since $\pi\iota = 1$. Furthermore, observe that $\mu'' = \mu$ since $\pi\iota = 1$. Indeed,

$$\begin{aligned}
\mu'' &= \pi\mu'(\iota \otimes \iota) \\
&= \pi\iota\mu(\pi \otimes \pi)(\iota \otimes \iota) \\
&= \pi\iota\mu(\pi\iota \otimes \pi\iota) \\
&= \mu.
\end{aligned}$$

In particular, we can always find a multiplication η on G such that $\eta' = \mu$ (namely $\eta = \mu'$).

Proposition 1.8. *π is multiplicative if and only if μ is associative.*

Proof. First assume π is multiplicative. $\square$

1.7 Identities

Let X be an MLDG A-module. For all homogeneous $a, b, c \in A$ and $x \in X$, we have

1. $\mathrm{d}([a,b,x]) = [\mathrm{d}(a), b, x] + (-1)^{|a|}[a, \mathrm{d}(b), x] + (-1)^{|a|+|b|}[a, b, \mathrm{d}(x)]$
2. $\lambda^2(a)[b,c,x] - [ab, \lambda(c), \lambda(x)] + [\lambda(a), bc, \lambda(x)] - [\lambda(a), \lambda(b), cx] + [a,b,c]\lambda^2(x) = [a,b,c,x]$

- For all $a, b \in A$ and $x \in X$, we have
$$\mathrm{d}([a,b,x]) = [\mathrm{d}(a), b, x] + (-1)^{|a|}[a, \mathrm{d}(b), x] + (-1)^{|a|+|b|}[a, b, \mathrm{d}(x)].$$
- For all $a, b, c \in A$ and $x \in X$ we have
$$\lambda^2(a)[b,c,x] - [ab, \lambda(c), \lambda(x)] + [\lambda(a), bc, \lambda(x)] - [\lambda(a), \lambda(b), cx] + [a,b,c]\lambda^2(x) = [a,b,c,x] \tag{16}$$
where
$$[a,b,c,x] = \lambda(ab)(\lambda(c)\lambda(x)) - (\lambda(a)\lambda(b))(\lambda(cx))$$
In particular, if X is multiplicative, then
$$\lambda^2(a)[b,c,x] - [ab, \lambda(c), \lambda(x)] + [\lambda(a), bc, \lambda(x)] - [\lambda(a), \lambda(b), cx] + [a,b,c]\lambda^2(x) = 0 \tag{17}$$
- If X is two-sided, then for all homogeneous $a, b \in A$ and homogeneous $x \in X$ we have a graded-commutative version of the Jacobi identity:
$$(-1)^{(|a|+|x|)|b|}[a,b,x] + (-1)^{(|b|+|x|)|a|}[x,a,b] + (-1)^{(|a|+|b|)|x|}[b,x,a] = 0 \tag{18}$$
- If X is two-sided, then for all homogeneous $a, b \in A$ and homogeneous $x \in X$ we have
$$[a,b,x] + (-1)^{|a||b|+|a||x|+|b||x|}[x,b,a] = 0. \tag{19}$$

We put these identities to use in the next proposition.

2 MDG R-algebras supported on monomial resolutions

Throughout this subsection, let $\boldsymbol{x} = x_1, \dots, x_n$, let $R = K[\boldsymbol{x}]$, and let $\boldsymbol{m} = m_1, \dots, m_r$ be monomials in R. For each nonempty subset $\sigma \subseteq [r]$, we set $m_\sigma := \mathrm{lcm}(m_\lambda \mid \lambda \in \sigma)$ and we set $\boldsymbol{a}_\sigma \in \mathbb{N}^n$ to be the exponent vector of m_σ. For completeness, we set $m_\varnothing = 1$ and $a_\varnothing = (0, \dots, 0)$. Let Re_σ be the free R-module generated by e_σ whose multidegree is $\boldsymbol{a}_\sigma$. Let Δ be a finite simplical complex with r vertices. We label the vertices of Δ by $m_1, \dots, m_r$. More generally, if σ is a face of Δ, then we label it by m_σ. For each $\boldsymbol{a} \in \mathbb{N}^n$, let $\Delta_{\boldsymbol{a}}$ be the subcomplex of Δ defined by $\Delta_{\boldsymbol{a}} = \{\sigma \in \Delta \mid m_\sigma \mid \boldsymbol{x}^{\boldsymbol{a}}\}$. The differential on $\mathcal{S}(\Delta)$ is denoted ∂, and the differential on $\mathcal{S}(\Delta_{\boldsymbol{a}})$ is denoted $\partial_{\boldsymbol{a}}$. Note that $\partial_{\boldsymbol{a}}$ is just the restriction of ∂ to $\mathcal{S}(\Delta_{\boldsymbol{a}})$.

2.1 Monomial resolution induced by a labeled simplicial complex

Definition 2.1. We define an R-complex, denoted F_Δ (or more simply denoted F if Δ is understood from context) and called R-**complex induced by** Δ (or the R**-complex of** Δ **over** R), as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R-module of F is given by

$$F_k := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $\mathrm{d}(e_\varnothing) = 0$ and

$$\mathrm{d}(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \backslash \{\varnothing\}$ where $\mathrm{pos}(i, \sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma \backslash i$ denotes the face obtained from σ by removing i. In the case where Δ is the r-simplex, we call F the **Taylor complex** of $R/\boldsymbol{m}$ over R.

Observe that F also has the structure of an $\mathbb{Z}^n$-graded K-complex. In other words, we have a decomposition of K-complexes

$$F = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} F_{\boldsymbol{a}},$$

where the K-complex $F_{\boldsymbol{a}}$ in multidegree $\boldsymbol{a} \in \mathbb{Z}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded K-vector space of $F_{\boldsymbol{a}}$ is given by

$$F_{k,\boldsymbol{a}} := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} K \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{a}} \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $\mathrm{d}_{\boldsymbol{a}}$ of $F_{\boldsymbol{a}}$ is just the restriction of d to $F_{\boldsymbol{a}}$. In particular, d is homogeneous with respect to that $\mathbb{Z}^n$-grading; hence

$$\begin{aligned} \mathrm{H}(F) &= \ker \mathrm{d} / \operatorname{im} \mathrm{d} \\ &= \left(\bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} \ker \mathrm{d}_{\boldsymbol{a}} \right) / \left(\bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} \operatorname{im} \mathrm{d}_{\boldsymbol{a}} \right) \\ &\cong \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} (\ker \mathrm{d}_{\boldsymbol{a}} / \operatorname{im} \mathrm{d}_{\boldsymbol{a}}) \\ &= \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} \mathrm{H}(F_{\boldsymbol{a}}). \end{aligned}$$

Now assume that $\Delta_{\boldsymbol{a}}$ contains a nonempty face, say $\sigma \in \Delta_{\boldsymbol{a}}$, then we have

$$\begin{aligned} \mathrm{d}_{\boldsymbol{a}} \left(\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma \right) &= \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} \mathrm{d}(e_\sigma) \\ &= \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{a}} m_\sigma}{m_\sigma m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_{\sigma \setminus i}} e_{\sigma \setminus i}. \end{aligned}$$

Thus if we define $\varphi_{\boldsymbol{a}} \colon F_{\Delta,\boldsymbol{a}}(1) \to \mathcal{S}(\Delta_{\boldsymbol{a}})$ to be the unique graded K-linear isomorphism such that $\varphi_{\boldsymbol{a}} \left(\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma \right) = \sigma$. Then from the computation above, we see that $\mathrm{d}_{\boldsymbol{a}} \partial_{\boldsymbol{a}} = \partial_{\boldsymbol{a}} \mathrm{d}_{\boldsymbol{a}}$; hence $\varphi_{\boldsymbol{a}}$ gives an isomorphism of K-complexes

$$\Sigma^{-1} F_{\boldsymbol{a}} \cong \mathcal{S}(\Delta_{\boldsymbol{a}}).$$

In particular, we obtain an isomorphism of homologies

$$\mathrm{H}(F_{\boldsymbol{a}})(1) \cong \widetilde{\mathrm{H}}(\Delta_{\boldsymbol{a}}, K),$$

where $\widetilde{\mathrm{H}}(\Delta_{\boldsymbol{a}}, K)$ is the simplicial homology of the simplicial complex $\Delta_{\boldsymbol{a}}$ over K. In other words, for each $k \in \mathbb{Z}$, we have

$$\mathrm{H}_{k+1}(F_{\boldsymbol{a}}) \cong \widetilde{\mathrm{H}}_k(\Delta_{\boldsymbol{a}}, K).$$

The following theorem follows immediately from the discussion above:

Theorem 2.1. *F is a free resolution of $R/\boldsymbol{m}$ over R if and only if for all $\boldsymbol{a} \in \mathbb{Z}^n$ either $\Delta_{\boldsymbol{a}}$ is the void complex or $\Delta_{\boldsymbol{a}}$ is acyclic. In particular, the Taylor complex of $R/\boldsymbol{m}$ over R is a free resolution. Moreover, F is minimal if and only if $m_\sigma \neq m_{\sigma'}$ for every proper subface σ' of a face σ.*

Example 2.1. For each $1 \leq i \leq r$, set $d_i = \deg m_i$. Order the m_i so that

$$d_1 \leq d_2 \leq \cdots \leq d_r$$

for each $1 \leq i \leq r$ and set $d = d_r$. We "homogenize" the monomial m_i to obtain the new monomial $\widehat{m}_i = x_0^{d-d_i} m_i$. This gives us another monomial ideal $\widehat{\boldsymbol{m}} = \langle \widehat{m}_1, \ldots, \widehat{m}_r \rangle$. For each $\sigma \subseteq \{1, \ldots, r\}$, set $\varepsilon_\sigma = d - d_{\min \sigma}$ and set $\delta_\sigma = d_{\min(\sigma \setminus \min \sigma)} - d_{\min \sigma}$. A quick calculation gives us

$$\begin{aligned} \widehat{m}_\sigma &= \operatorname{lcm}(\widehat{m}_i \mid i \in \sigma) \\ &= x_0^{d - d_{\min \sigma}} m_\sigma \\ &= x_0^{\varepsilon_\sigma} m_\sigma \end{aligned}$$

Let $\widehat{\Delta}$ be the labeled simplicial complex obtained by replacing the labels m_σ in Δ with $\widehat{m}_\sigma$. Let $\widehat{F}$ be the $R[x_0]$-complex induced by $\widehat{\Delta}$. Note that

$$\begin{aligned}\widehat{\mathrm{d}}(e_\sigma) &= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{\widehat{m}_\sigma}{\widehat{m}_{\sigma\setminus i}}e_{\sigma\setminus i}\\ &= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}x_0^{\varepsilon_\sigma-\varepsilon_{\sigma\setminus i}}\frac{m_\sigma}{m_{\sigma\setminus i}}e_{\sigma\setminus i}\\ &= \frac{m_\sigma}{m_{\sigma\setminus\min\sigma}}x_0^{\delta_\sigma}\widehat{e}_{\sigma\setminus\min\sigma} - \sum_{i\in\sigma\setminus\min\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{m_\sigma}{m_{\sigma\setminus i}}e_{\sigma\setminus i}.\end{aligned}$$

In particular, we see that

$$(\widehat{\mathrm{d}}-\mathrm{d})(e_\sigma) = \frac{(x_0^{\delta_\sigma}-1)m_\sigma}{m_{\sigma\setminus\min\sigma}}e_{\sigma\setminus\min\sigma}.$$

For each $\sigma\subseteq[r]$, denote

$$f_\sigma = \frac{(x_0^{\delta_\sigma}-1)m_\sigma}{m_{\sigma\setminus\min\sigma}}$$

Assume that Consider

$$\begin{aligned}m_1 &= x^2\\ m_2 &= y^5\\ m_3 &= xz^7\end{aligned}$$

Then

$$\begin{aligned}\widehat{m}_1 &= t^6x^2\\ \widehat{m}_2 &= t^3y^5\\ \widehat{m}_3 &= xz^7\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathrm{d}}(e_{123}) &= \frac{\widehat{m}_{123}}{\widehat{m}_{23}}e_{23} - \frac{\widehat{m}_{123}}{\widehat{m}_{13}}e_{13} + \frac{\widehat{m}_{123}}{\widehat{m}_{12}}e_{12}\\ &= t^3\frac{m_{123}}{m_{23}}e_{23} - \frac{m_{123}}{m_{13}}e_{13} + \frac{m_{123}}{m_{13}}e_{12}\\ &=\end{aligned}$$

$$\begin{aligned}\widehat{\mathrm{d}}(e_{12}) &= \frac{\widehat{m}_{12}}{\widehat{m}_2}e_2 - \frac{\widehat{m}_{12}}{\widehat{m}_1}e_1\\ &= t^3\frac{m_{12}}{m_2}e_2 - \frac{m_{12}}{m_1}e_1\end{aligned}$$

Therefore we have

$$\widehat{\mathrm{d}}(e_\sigma) = \mathrm{d}(e_\sigma) + \frac{(x_0^{\delta_\sigma}-1)m_\sigma}{m_{\sigma\setminus\min\sigma}}e_{\sigma\setminus\min\sigma}$$

where $p\mid\delta_\sigma$ for each σ. Then we notice that

$$\widehat{R}/\langle\Phi_p(x_0),\widehat{\boldsymbol{m}}\rangle = \mathbb{Q}[x_0,x_1,\dots,x_n]/\langle\Phi_p(x_0),\widehat{\boldsymbol{m}}\rangle \cong \mathbb{Q}(\zeta_p)[x_1,\dots,x_n]/\langle\boldsymbol{m}\rangle = (R/\boldsymbol{m})\otimes_R\mathbb{Q}(\zeta_p)$$

and $\mathrm{d} = \widehat{\mathrm{d}} \bmod \Phi_p(x_0)$. Thus if $\widehat{F}$ were a resolution, then

$$\mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/\widehat{\boldsymbol{m}},\widehat{R}/\Phi_p(x_0)\rangle = \mathrm{H}_i(\widehat{F}\otimes_R\widehat{R}/\Phi_p(x_0)) = 0$$

$$0\to\widehat{R}\to\widehat{R}\to\widehat{R}/\Phi_p(x_0)\to 0$$

If $\widehat{F}$ is a resolution of $\widehat{R}/\widehat{\boldsymbol{m}} = \mathbb{Q}[t,\boldsymbol{x}]/\langle t^6x^2,t^3y^5,xz^7\rangle$ over $\widehat{R}$, then $F = \widehat{F}\otimes_{\mathbb{Q}[t,\boldsymbol{x}]}\mathbb{Q}(\zeta_p)[\boldsymbol{x}]$ is a resolution of $R/\boldsymbol{m} = \mathbb{Q}(\zeta_p)[\boldsymbol{x}]/\langle x^2,y^5,xz^7\rangle$ over R. Indeed, we would then have

$$\begin{aligned}\mathrm{H}_i(F) &= \mathrm{H}_i(\widehat{F}\otimes_{\widehat{R}}R)\\ &= \mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/\boldsymbol{m},R)\\ &= \mathrm{Tor}_i^{\widehat{R}}(\widehat{R}/\boldsymbol{m},\widehat{R}/\Phi_p)\end{aligned}$$

and using the short exact sequence

$$0 \to \widehat{R} \xrightarrow{\Phi_p} \widehat{R} \to \widehat{R}/\Phi_p \to 0,$$

we find that

$$\mathrm{Tor}_1^{\widehat{R}}(\widehat{R}/\boldsymbol{m}, \widehat{R}/\Phi_p) = \ker(\widehat{R}/\widehat{\boldsymbol{m}} \xrightarrow{\Phi_p} \widehat{R}/\widehat{\boldsymbol{m}}) = 0.$$

$$\mathrm{H}_i(\widehat{F} \otimes_R \mathbb{Q}(\zeta_p)[\boldsymbol{x}]) = \mathrm{H}_i(F)$$

$$\mathrm{Tor}_1^{\mathbb{Q}[x_0,\boldsymbol{x}]}(\mathbb{Q}[x_0,\boldsymbol{x}]/\widehat{\boldsymbol{m}}, \mathbb{Q}(\zeta_p)[\boldsymbol{x}]) = \ker(\mathbb{Q}[x_0,\boldsymbol{x}] \xrightarrow{\Phi_p(x_0)} \mathbb{Q}[x_0,\boldsymbol{x}]) = 0$$

$$\mathrm{Tor}_0^{\mathbb{Q}[x_0,\boldsymbol{x}]}(\mathbb{Q}[x_0,\boldsymbol{x}]/\widehat{\boldsymbol{m}}, \mathbb{Q}(\zeta_p)[\boldsymbol{x}]) = L[\boldsymbol{x}]/\boldsymbol{m}$$

for all $i > 0$.

$$X^{pk} - 1 =$$

Now assume that F is a free resolution, and let $\boldsymbol{a} = \mathbb{Z}^{n+1}$. Write $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ and set $\boldsymbol{b} = (a_1, \ldots, a_n)$, so $x^{\boldsymbol{a}} = x_0^{a_0} x^{\boldsymbol{b}}$. Assume that $\Delta_{\widehat{\boldsymbol{a}}}$ has nontrivial homology. Then there exists $\alpha \in \widehat{F}_k$ be an element which represents this nontrivial element in homology, and express α in the form

$$\alpha = \sum_{1 \le s \le t} c_s x_0^{a_s} \frac{x^{\boldsymbol{b}_s}}{m_{\sigma_s}} e_{\sigma_s}$$

where $c_s \in K$, $a_s \geq 1$, each m_{σ_s} divides $x^{\boldsymbol{b}_s}$, and $|e_{\sigma_s}| = k$ for all s. Then observe that

$$\begin{aligned}
\mathrm{d}(\alpha) &= \mathrm{d}(\alpha) - \widehat{\mathrm{d}}(\alpha) \\
&= (\mathrm{d} - \widehat{\mathrm{d}})(\alpha) \\
&= \sum_{1 \le s \le t} c_s x_0^{a_s} \frac{x^{\boldsymbol{b}_s}}{m_{\sigma_s}} (\mathrm{d} - \widehat{\mathrm{d}})(e_{\sigma_s}) \\
&= \sum_{1 \le s \le t} c_s x_0^{a_s} \frac{x^{\boldsymbol{b}_s}}{m_{\sigma_s}} \frac{(x_0^{\delta_{\sigma_s}} - 1) m_{\sigma_s}}{m_{\sigma_s \setminus \min \sigma_s}} e_{\sigma_s \setminus \min \sigma_s} \\
&= \sum_{1 \le s \le t} c_s x_0^{a_s} x^{\boldsymbol{b}_s} (x_0^{\delta_{\sigma_s}} - 1) \frac{e_{\sigma_s \setminus \min \sigma_s}}{m_{\sigma_s \setminus \min \sigma_s}}
\end{aligned}$$

In particular, we can express α in the form

$$\alpha = \sum_{1 \le s \le t} c_s x_0^{a_s} \frac{x^{\boldsymbol{b}_s}}{m_{\sigma_s}} e_{\sigma_s}$$

In other words, we have

$$c_1 \frac{x^{\boldsymbol{a}_1}}{\widehat{m}_{\sigma_1}} e_{\sigma_1} + \cdots + c_k \frac{x^{\boldsymbol{a}_1}}{\widehat{m}_{\sigma_k}} e_{\sigma_1} = c_1 \frac{x_0^{a_0} x^{\boldsymbol{b}}}{x_0^{\varepsilon_\sigma} m_\sigma} e_{\sigma_1} + \cdots + c_k \frac{x^{\boldsymbol{a}_1}}{\widehat{m}_{\sigma_k}} e_{\sigma_1}$$

$$\alpha = \sum_{\sigma} \frac{x^{\boldsymbol{a}}}{\widehat{m}_\sigma} e_\sigma = \sum_{\sigma}$$

$$F_{k,\boldsymbol{a}} := \begin{cases} \bigoplus_{\dim \sigma = k-1} K \frac{x^{\boldsymbol{a}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{a}} \text{ and } 0 \le k \le \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

Thus for all $\boldsymbol{a} \in \mathbb{Z}^n$, either $\Delta_{\boldsymbol{a}}$ is the void complex
In particular, this implies $\Delta_{\boldsymbol{a}} = \{\sigma \in \Delta \mid \boldsymbol{a}_\sigma \le \boldsymbol{a}\}$.

Note that

$$(\widehat{R}/\widehat{\boldsymbol{m}}) \otimes_{\widehat{R}} \widehat{R}/\langle x_0 - 1 \rangle \cong R/\boldsymbol{m}$$

Proposition 2.1. *Suppose $\widehat{F}$ is a free resolution of $\widehat{R}/\widehat{\boldsymbol{m}}$ over $\widehat{R}$. Then F is a free resolution of $R/\boldsymbol{m}$ over R.*

Proof. Since $\widehat{F}$ is a free resolution of $\widehat{R}/\widehat{\boldsymbol{m}}$ over $\widehat{R}$, we have

$$
\begin{aligned}
\mathrm{H}(F) &= \mathrm{H}(\widehat{F} \otimes_{\widehat{R}} R) \\
&= \mathrm{Tor}^{\widehat{R}}(\widehat{R}/\widehat{\boldsymbol{m}}, R) \\
&= \mathrm{H}((\widehat{R}/\widehat{\boldsymbol{m}}) \otimes_{\widehat{R}} \mathcal{K}^{\widehat{R}}(1-t)) \\
&= \mathrm{H}(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m}))
\end{aligned}
$$

where $\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{\boldsymbol{m}})$ is the Koszul complex

$$
0 \to \widehat{R}/\widehat{\boldsymbol{m}} \xrightarrow{1-t} \widehat{R}/\widehat{\boldsymbol{m}} \to 0
$$

In particular, its homology is given by

$$
\mathrm{H}_i(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m})) = \begin{cases} R/\boldsymbol{m} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}
$$

where we used the fact that $1-t$ is $\widehat{R}/\widehat{\boldsymbol{m}}$-regular to conclude that $\mathrm{H}_1(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m})) = 0$. Thus F is a free resolution of $R/\boldsymbol{m}$ over R. □

Proposition 2.2. *Suppose $R = S/\langle \boldsymbol{x} \rangle$ where $\boldsymbol{x} = x_1, \dots, x_r$ is a regular sequence in S. is a free resolution of $\widehat{R}/\widehat{\boldsymbol{m}}$ over $\widehat{R}$. Then F is a free resolution of $R/\boldsymbol{m}$ over R.*

Proof. Since $\widehat{F}$ is a free resolution of $\widehat{R}/\widehat{\boldsymbol{m}}$ over $\widehat{R}$, we have

$$
\begin{aligned}
\mathrm{H}(F) &= \mathrm{H}(\widehat{F} \otimes_{\widehat{R}} R) \\
&= \mathrm{Tor}^{\widehat{R}}(\widehat{R}/\widehat{\boldsymbol{m}}, R) \\
&= \mathrm{H}((\widehat{R}/\widehat{\boldsymbol{m}}) \otimes_{\widehat{R}} \mathcal{K}^{\widehat{R}}(1-t)) \\
&= \mathrm{H}(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m}))
\end{aligned}
$$

where $\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{\boldsymbol{m}})$ is the Koszul complex

$$
0 \to \widehat{R}/\widehat{\boldsymbol{m}} \xrightarrow{1-t} \widehat{R}/\widehat{\boldsymbol{m}} \to 0
$$

In particular, its homology is given by

$$
\mathrm{H}_i(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m})) = \begin{cases} R/\boldsymbol{m} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}
$$

where we used the fact that $1-t$ is $\widehat{R}/\widehat{\boldsymbol{m}}$-regular to conclude that $\mathrm{H}_1(\mathcal{K}^{\widehat{R}}(1-t; \widehat{R}/\widehat{m})) = 0$. Thus F is a free resolution of $R/\boldsymbol{m}$ over R. □

2.2 MDG-algebra structures on the monomial resolution induced by a labeled simplicial complex

Let (F, d) denote the R-complex induced by Δ. Let $\mu \colon F \otimes_R F \to F$ be a chain map such that

1. μ gives F the structure of an MDG R-algebra resolution of $R/\boldsymbol{m}$.
2. μ respects the multigrading: this means that if $\alpha \in F_{\boldsymbol{a}}$ and $\beta \in F_{\boldsymbol{b}}$, then $\alpha\beta \in F_{\boldsymbol{a}+\boldsymbol{b}}$ for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$.

For each $\sigma, \tau \in \Delta$ we have

$$
e_\sigma e_\tau = \sum_{v \in \Delta} f^v_{\sigma,\tau}(\mu) e_v \tag{20}
$$

where $f^v_{\sigma,\tau}(\mu) \in K[\boldsymbol{x}]$ for each $v \in \Delta$. The $f^v_{\sigma,\tau}(\mu)$ uniquely determine μ; they are called the **structured R-coefficients** of μ. If μ is understood from context, then we'll simplify our notation by writing $f^v_{\sigma,\tau} = f^v_{\sigma,\tau}(\mu)$. Since μ is a graded map, we must have $f^v_{\sigma,\tau} = 0$ whenever $|e_\sigma| + |e_\tau| \neq |e_v|$. In fact, since μ respects the multigrading, we must have

$$
f^v_{\sigma,\tau}(\mu) = c^v_{\sigma,\tau}(\mu) \frac{m_\sigma m_\tau}{m_v}
$$

where $c^{v}_{\sigma,\tau}(\mu) \in K$ for all $\sigma, \tau, v \in \Delta$ where $c^{v}_{\sigma,\tau}(\mu) = 0$ whenever $|e_\sigma| + |e_\tau| \neq |e_v|$ or $m_\sigma m_\tau \nmid m_v$. The $c^{v}_{\sigma,\tau}(\mu)$ also uniquely determine μ; they are called the **structured K-coefficients** of μ. Again, if μ is understood from context, then we'll simplify our notation by writing $c^{v}_{\sigma,\tau}(\mu) = c^{v}_{\sigma,\tau}$. It would be nice if we could re-express (20) as

$$\left(\frac{e_\sigma}{m_\sigma}\right)\left(\frac{e_\tau}{m_\tau}\right) = \sum_{v} c^{v}_{\sigma,\tau}\left(\frac{e_v}{m_v}\right), \tag{21}$$

but the problem is that F does not contain terms like e_σ/m_σ. In order to make sense of (21), we need to adjoin inverves to our base ring R.

2.2.1 Base change

Let S be the multiplicatively closed set generated by $\{m_\sigma \mid \sigma \in \Delta\}$ and let T be the multiplicatively closed set generated by $x_1 \cdots x_n \in R$. The localization of R at T has a natural description in terms of the Laurent polynomial ring over K, that is $R_T = K[\boldsymbol{x}, \boldsymbol{x}^{-1}]$, and the localization of R at S has a natural description as an R-submodule of R_T which is generated by Laurent monomials m_σ/m_τ for all $\sigma, \tau \in \Delta$. In general, any R-submodule M of R_T which is generated by Laurent monomials $\boldsymbol{x}^{\boldsymbol{a}}$ where $\boldsymbol{a} \in \mathbb{Z}^n$ is called a **monomial module**.

We want to perform a base change from R to R_S in order to make sense of (21). The localization functor $-_S$ from the category of R-complexes to the category of R_S-complexes is an exact functor which preserves quasiisomorphisms. In particular, it takes the quasiisomorphism $F \to R/\boldsymbol{m}$ to the quasiisomorphim $F_S \to 0$. Note that $F_S \to 0$ being a quasiisomorphism is equivalent to saying F_S is an exact R-complex. We can give a natural description of F_S as an MDG R_S-complex as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R_S-module of F_S is given by

$$F_{S,k} := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} R_S e_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

The differential d_S of F_S is defined via the rule $\mathrm{d}_S(\boldsymbol{x}^{\boldsymbol{a}} e_\sigma) = \boldsymbol{x}^{\boldsymbol{a}} \mathrm{d}(e_\sigma)$ for all homogeneous generators e_σ and for all $\boldsymbol{a} \in \mathbb{Z}^n$. Similarly, the multiplication μ_S of F_S is defined via the rule $(\boldsymbol{x}^{\boldsymbol{a}} e_\sigma)(\boldsymbol{x}^{\boldsymbol{b}} e_\tau) = \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{b}} e_\sigma e_\tau$ for all homogeneous generators e_σ, e_τ and for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$. Note that d_S and μ_S are just the localizations of d and μ with respect to S, so our notation here is consistent. Since $-_S$ is an exact functor, we have $[F]_S \cong [F_S]$, where $[F]_S$ is the localization (with repsect to S) of the associator complex of F and where $[F_S]$ is the associator complex of F_S.

2.2.2 Determining associativity of F using $\widetilde{F}$

To simplifiy our notation in what follows, let us denote $\widetilde{F} = F_{S,\boldsymbol{0}}$, that is, $\widetilde{F}$ is the multidegree $\boldsymbol{0}$ part of F_S. The multiplication (21) makes perfect sense in $\widetilde{F}$. Denoting $\widetilde{e}_\sigma = e_\sigma/m_\sigma$ for each $\sigma \in \Delta$, we re-express (21) as

$$\widetilde{e}_\sigma \widetilde{e}_\tau = \sum_{v} c^{v}_{\sigma,\tau} \widetilde{e}_v.$$

It is straightforward to check that $\widetilde{F}$ inherits the structure of an MDG K-algebra. We shall denote its differential by $\widetilde{\mathrm{d}}$ and we shall denote its multiplication by $\widetilde{\mu}$.

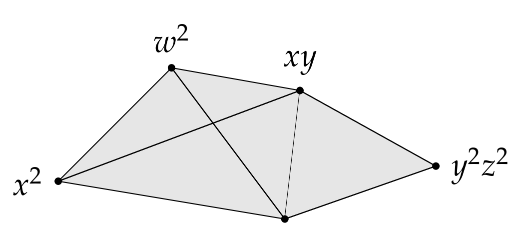
Theorem 2.2. *F is a DG R-algebra if and only if $\widetilde{F}$ is a DG K-algebra.*

Proof. A straightforward calculation gives us

$$[e_\sigma, e_\tau, e_v]_\mu = m_\sigma m_\tau m_v [\widetilde{e}_\sigma, \widetilde{e}_\tau, \widetilde{e}_v]_{\widetilde{\mu}}$$

for all $\sigma, \tau, v \in \Delta$. Thus μ is associative if and only if $\widetilde{\mu}$ is associative. □

Example 2.2. Consider the case where $R = \mathbb{Q}[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and where Δ is the labeled simplicial complex which can be pictured below



We chose to only label the vertices in the picture above in order to keep things clean, but note that every face of the simplicial complex above should be labeled by an appropriate monomial. Note that the R-complex F induced by Δ is in fact the minimal free resolution of $R/\boldsymbol{m}$. Since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have

$$\begin{aligned}
\widetilde{e}_1\widetilde{e}_5 &= c_{1,5}^{14}\widetilde{e}_{14} + c_{1,5}^{45}\widetilde{e}_{45}\\
\widetilde{e}_2\widetilde{e}_5 &= c_{2,5}^{23}\widetilde{e}_{23} + c_{2,5}^{35}\widetilde{e}_{35}\\
\widetilde{e}_2\widetilde{e}_{45} &= c_{2,45}^{234}\widetilde{e}_{234} + c_{2,45}^{345}\widetilde{e}_{345}\\
\widetilde{e}_1\widetilde{e}_{35} &= c_{1,35}^{134}\widetilde{e}_{134} + c_{1,35}^{345}\widetilde{e}_{345}\\
\widetilde{e}_2\widetilde{e}_{14} &= c_{2,14}^{124}\widetilde{e}_{124}\\
\widetilde{e}_1\widetilde{e}_{23} &= c_{1,23}^{123}\widetilde{e}_{123}
\end{aligned}$$

where each $c_{\sigma,\tau}^{v}$ above is nonzero. At this point however, $\widetilde{\mu}$ is already not associative since $[\widetilde{e}_1,\widetilde{e}_5,\widetilde{e}_2] \neq 0$. Indeed, we use Singular to calculate $[\widetilde{e}_1,\widetilde{e}_5,\widetilde{e}_2]$ using the code below:

```
intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);

ring A=(2,c14,c45,c23,c35,c234,c345,c134,c123,c124),
(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+c14*e14+c45*e45;
poly f(2)(5) = e2*e5+c23*e23+c35*e35;
poly f(2)(45) = e2*e45+c234*e234+c345*e345;
poly f(1)(35) = e1*e35+c134*e134+c345*e345;
poly f(1)(23) = e1*e23+c123*e123;
poly f(2)(14) = e2*e14+c124*e124;
poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);
reduce(s(1)(5)(2),I);
```

Plugging this code into Singular gives us

$$[\widetilde{e}_1,\widetilde{e}_5,\widetilde{e}_2] = c_{2,5}^{23}c_{1,23}^{123}e_{123} - c_{1,5}^{14}c_{2,14}^{124}e_{124} + c_{2,5}^{35}c_{1,35}^{134}e_{134} - c_{1,5}^{45}c_{2,45}^{234}e_{134} + (c_{1,5}^{45}c_{2,45}^{345} - c_{2,5}^{35}c_{1,35}^{345})e_{345} \neq 0$$

Now we want to calculate the homology of the associator complex. To do this, we first note that since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have $e_3, e_4 \in \mathrm{N}(F)$. In particular, e_3 and e_4 behave like Taylor variables. For instance, $\widetilde{e}_3\widetilde{e}_5 = c_{3,5}^{35}\widetilde{e}_{35}$ and $\widetilde{e}_4\widetilde{e}_5 = c_{4,5}^{45}\widetilde{e}_{45}$. Using this fact, observe that

$$\begin{aligned}
\mathrm{d}[e_1,e_{45},e_2] &= [\mathrm{d}(e_1),e_{45},e_2] - [e_1,\mathrm{d}(e_{45}),e_2] - [e_1,e_{45},\mathrm{d}(e_2)]\\
&= -[e_1,\mathrm{d}(e_{45}),e_2]\\
&= -[e_1,xe_5 - z^2e_4,e_2]\\
&= -x[e_1,e_5,e_2] + z^2[e_1,e_4,e_2]\\
&= -x[e_1,e_5,e_2].
\end{aligned}$$

Similar calculations gives us

$$\begin{aligned}
\mathrm{d}[e_{14},e_5,e_2] &= y[e_1,e_5,e_2]\\
\mathrm{d}[e_1,e_5,e_{23}] &= z[e_1,e_5,e_2]\\
\mathrm{d}[e_1,e_{35},e_2] &= -w[e_1,e_5,e_2].
\end{aligned}$$

It follows that

$$\mathrm{H}_i[F,F,F] \cong \begin{cases} \mathbb{Q} & \text{if } i=3\\ 0 & \text{else}\end{cases}$$

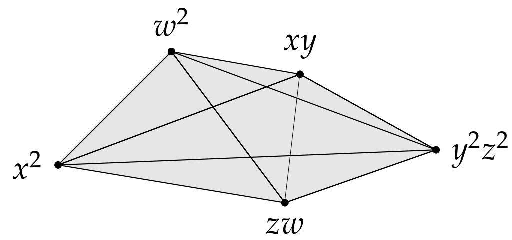
Note that if we are allowed to divide by x, then we can express the associator $[e_1,e_5,e_2]$ as the boundary of another associator, namely

$$[e_1,e_5,e_2] = \mathrm{d}[e_1/x,e_{45},e_2].$$

Thus we clearly have $\mathrm{H}[F_{\{x\}},F_{\{x\}},F_{\{x\}}] = 0$ even though $[F_{\{x\}},F_{\{x\}},F_{\{x\}}] \neq 0$ (the reason we have this is that $F_{\{x\}}$ is not a minimal $R_{\{x\}}$-complex). For multidegree reasons, we must have $[e_1,e_5,e_2] = \mathrm{d}(me_{1234})$ for some monomial m. Now note that the multidegree of $[e_1, e_5, e_2]$ is $m_1 m_5 m_2 = x^2 y^2 z^2 w^2$ and the multidegree of e_{1234} is $m_{1234} = x^2 yzw^2$. It follows that

$$[e_1, e_5, e_2] = yz\mathrm{d}(e_{1234}).$$

Example 2.3. Continuing with (2.2), let T be the Taylor algebra of $R/\boldsymbol{m}$. Note that T is the R-complex induced by the labeled simplicial complex pictured below



Let $\iota\colon F \to T$ and $\pi\colon T \to F$ be comparison maps such that

1. ι is the inclusion map.

2. π respects the multigrading (obviously ι respects the multigrading automatically).

3. π splits ι, meaning $\pi\iota = 1$. In particular, we can view π as a map $\pi\colon T \to T$ such that $\pi^2 = \pi$.

We set $\lambda = \iota\pi$. Observe that

1. λ preserves the multigrading.

2. λ is homotopic to the identity $1\colon T \to T$. In particular, there exists a homotopy $\mathrm{h}\colon T \to T$ such that $\lambda - 1 = \mathrm{dh} + \mathrm{hd}$.

3. λ is a projection, meaning $\lambda^2 = \lambda$.

We extend $\mu\colon F \otimes_R F \to F$ to a multiplication $\nu\colon T \otimes_R T \to T$ on T by setting $\nu = \iota\mu\pi^{\otimes 2}$. We note that $T = (T, \mathrm{d}_T, \lambda, \nu)$ has the structure of an MLDG R-algebra whose underlying R-complex is the Taylor complex.

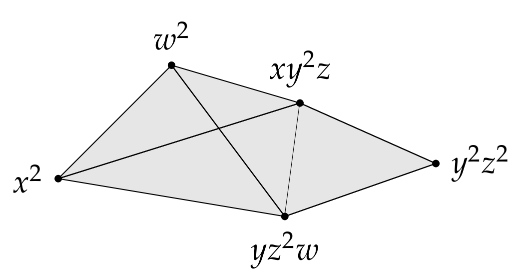
Proposition 2.3. *Let $\nu' = \iota\mu\pi^{\otimes 2}$, let $\kappa' = \iota\mu$, and let $T' = (T, \mathrm{d}_T, \kappa', \nu')$ be the corresponding MLDG R-algebra whose underlying R-complex is the Taylor complex. Then F is associative if and only if T' is hom-associative.*

Proof. Indeed, we have

$$\begin{aligned}
[\alpha, \beta, \gamma]_{\nu'} &= \kappa'(\alpha) \star_{\nu'} (\beta \star_{\nu'} \gamma) - (\alpha \star_{\nu'} \beta) \star_{\nu'} \kappa'(\gamma) \\
&= \pi(\alpha) \star_{\nu'} (\beta \star_{\nu'} \gamma) - (\alpha \star_{\nu'} \beta) \star_{\nu'} \pi(\gamma) \\
&= \pi(\alpha) \star_{\nu'} (\pi(\beta)\pi(\gamma)) - (\pi(\alpha)\pi(\beta)) \star_{\nu'} \pi(\gamma) \\
&= \pi^2(\alpha)\pi(\pi(\beta)\pi(\gamma)) - \pi(\pi(\alpha)\pi(\beta))\pi^2(\gamma) \\
&= \pi(\alpha)\pi(\pi(\beta)\pi(\gamma)) - \pi(\pi(\alpha)\pi(\beta))\pi(\gamma) \\
&= \pi(\alpha)(\pi^2(\beta)\pi^2(\gamma)) - (\pi^2(\alpha)\pi^2(\beta))\pi(\gamma) \\
&= \pi(\alpha)(\pi(\beta)\pi(\gamma)) - (\pi(\alpha)\pi(\beta))\pi(\gamma) \\
&= [\pi(\alpha), \pi(\beta), \pi(\gamma)]_{\mu}
\end{aligned}$$

where we used the fact that π is multiplicative on $\operatorname{im}\pi$. The proposition now easily follows from the fact that π is surjective. □

Example 2.4. Consider the case where $R = K[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, yz^2w, xy^2z, y^2z^2$, and where Δ is the labeled simplicial complex which can be pictured below



Note that the free resolution F supported on this labeled simplicial complex is in fact the minimal free resolution of $R/\boldsymbol{m}$. Since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have

$$\begin{aligned}
\widetilde{e}_1\widetilde{e}_5 &= c^{14}_{1,5}\widetilde{e}_{14} + c^{45}_{1,5}\widetilde{e}_{45}\\
\widetilde{e}_2\widetilde{e}_5 &= c^{23}_{2,5}\widetilde{e}_{23} + c^{35}_{2,5}\widetilde{e}_{35}\\
\widetilde{e}_2\widetilde{e}_{45} &= c^{234}_{2,45}\widetilde{e}_{234} + c^{345}_{2,45}\widetilde{e}_{345}\\
\widetilde{e}_1\widetilde{e}_{35} &= c^{134}_{1,35}\widetilde{e}_{134} + c^{345}_{1,35}\widetilde{e}_{345}\\
\widetilde{e}_2\widetilde{e}_{14} &= c^{124}_{2,14}\widetilde{e}_{124}\\
\widetilde{e}_1\widetilde{e}_{23} &= c^{123}_{1,23}\widetilde{e}_{123}
\end{aligned}$$

where each $c^{v}_{\sigma,\tau}$ above is nonzero. We again have $[e_1, e_5, e_2] \neq 0$ as seen in the example above. A quick calculation gives us

$$\begin{aligned}
\mathrm{d}[e_1, e_{45}, e_2] &= x[e_1, e_5, e_2]\\
\mathrm{d}[e_{14}, e_5, e_2] &= y^2z[e_1, e_5, e_2]\\
\mathrm{d}[e_1, e_5, e_{23}] &= yz^2[e_1, e_5, e_2]\\
\mathrm{d}[e_1, e_{35}, e_2] &= w[e_1, e_5, e_2]\\
\mathrm{d}[e_1, e_5, e_{24}] &= xy^2z[e_1, e_5, e_2]\\
\mathrm{d}[e_{13}, e_5, e_2] &= yz^2w[e_1, e_5, e_2]\\
\mathrm{d}[e_{12}, e_5, e_2] &= w^2[e_1, e_5, e_2]\\
\mathrm{d}[e_1, e_5, e_{12}] &= x^2[e_1, e_5, e_2]
\end{aligned}$$

In particular, the associator complex for F is different from the example above; it is given by

$$\mathrm{H}_i([F]) \cong \begin{cases} K[y,z]/\langle y^2z, yz^2\rangle & \text{if } i = 3\\ 0 & \text{else}\end{cases}$$

2.2.3 $\mathbb{Z}^n$-graded MDG $\widetilde{F}$-algebra structure of F_S

Observe that F_S has the structure of a $\mathbb{Z}^n$-graded MDG $\widetilde{F}$-module. Indeed, we have a decomposition of F_S into MDG $\widetilde{F}$-modules

$$F_S = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n} F_{S,\boldsymbol{a}},$$

where the MDG $\widetilde{F}$-module $F_{S,\boldsymbol{a}}$ in multidegree $\boldsymbol{a} \in \mathbb{Z}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded $\widetilde{F}$-module of $F_{S,\boldsymbol{a}}$ is given by

$$F_{S,\boldsymbol{a}} := \begin{cases} \bigoplus_{\dim\sigma = k-1} K\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma}e_\sigma & \sigma\in\Delta,\ 0\le k\le \dim\Delta+1, \text{ and } \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma}\in R_S\\ 0 & \text{else}\end{cases}$$

The differential $\mathrm{d}_{S,\boldsymbol{a}}$ of $F_{S,\boldsymbol{a}}$ is just the restriction of d_S to $\mathrm{d}_{S,\boldsymbol{a}}$, and the multiplication $\mu_{S,\boldsymbol{a}}$ of $F_{S,\boldsymbol{a}}$ is just the restriction of μ_S to $\mu_{S,\boldsymbol{a}}$. Note that we need μ to respect the multi-grading in order for $F_{S,\boldsymbol{a}}$ to be an MDG $\widetilde{F}$-module so that the scalar-multiplicaton map $\widetilde{F}\times F_{S,\boldsymbol{a}} \to F_{S,\boldsymbol{a}}$ is well-posed.

Since d_S homogeneous with respect to the $\mathbb{Z}^n$-grading, we have

$$0 = \mathrm{H}(F_S) \cong \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n}\mathrm{H}(F_{S,\boldsymbol{a}}).$$

Hence each K-complex $F_{S,\boldsymbol{a}}$ is an exact complex. It is also straightforward to show that

$$[F_S] = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n}[F_{S,\boldsymbol{a}}],$$

where $[F_{S,\boldsymbol{a}}]$ is the associator complex of $F_{S,\boldsymbol{a}}$ as an MDG $\widetilde{F}$-module. In particular,

$$[F_{S,\boldsymbol{a}}] = \mathrm{span}_{R_S}\{[\alpha,\beta,\gamma] \mid \alpha,\beta\in\widetilde{F} \text{ and } \gamma\in F_{S,\boldsymbol{a}}\}.$$

Thus since $-_S$ is an exact functor, we have

$$\begin{aligned}
\mathrm{H}([F])_S &\cong \mathrm{H}([F]_S)\\
&\cong \mathrm{H}([F_S])\\
&\cong \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n}\mathrm{H}([F_{S,\boldsymbol{a}}]).
\end{aligned}$$

3 Associativity test using Gröbner bases

3.1 Setup

Let K be a field and let (F, d, μ) be an MDG K-algebra. Let $n \geq 1$ and assume that $(1, e_1, \dots, e_n)$ is an ordered homogeneous basis of F such that

1. $|e_i| \geq 1$ for all $1 \leq i \leq n$,
2. if $|e_j| > |e_i|$, then $j > i$.

In this section, we will use some tools from the theory of Gröbner bases to determine whether or not F is DG algebra, that is, whether or not F is associative. For simplicity, we will only describe how this works in the case where K has characteristic 2. Let $(c_{i,j}^k)$ be the structured K-coefficients of μ. Thus for each $0 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=0}^{n} c_{i,j}^k e_k,$$

where we use $\star$ to denote multiplication with respect to μ. Let S be the weighted polynomial ring $K[e_1, \dots, e_n]$ where e_i is weighted of degree $|e_i|$ for each $1 \leq i \leq n$. A monomial of S has the form $\boldsymbol{e}^{\boldsymbol{a}} = e_1^{a_1} \cdots e_n^{a_n}$ where $\boldsymbol{a} \in \mathbb{N}^n$ where we identify the monomial $\boldsymbol{e}^{(0,\dots,0)}$ with 1 in this notation. Given a monomial $\boldsymbol{e}^{\boldsymbol{a}}$, we define its **degree**, denoted $\deg(\boldsymbol{e}^{\boldsymbol{a}})$, and its **weighted degree**, denoted $|\boldsymbol{e}^{\boldsymbol{a}}|$, by

$$\deg(\boldsymbol{e}^{\boldsymbol{a}}) = \sum_{i=1}^{n} a_i \quad \text{and} \quad |\boldsymbol{e}^{\boldsymbol{a}}| = \sum_{i=1}^{n} a_i |e_i|.$$

For each $k \in \mathbb{N}$, we shall write

$$S_k = \operatorname{span}_K \{\boldsymbol{e}^{\boldsymbol{a}} \mid \deg(\boldsymbol{e}^{\boldsymbol{a}}) = k\}.$$

We identity F with $S_0 + S_1 = K + \sum_{i=1}^n K e_i$. In order to keep notation consistent, we shall write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to μ, and we shall write $\alpha\beta$ to denote their multiplication with respect to $\cdot$ in S. In particular, note that $\deg(e_i \star e_j) = 1$ and $\deg(e_i e_j) = 2$.

For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Note that since we are working over a field of characteristic 2, we have $f_{i,j} = f_{j,i}$ for all $1 \leq i, j \leq n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let I be the ideal of S generated by $\mathcal{F}$. We equip S with a weighted lexicographic ordering $>_w$ with respect to the weight vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials $\boldsymbol{e}^{\boldsymbol{a}}$ and $\boldsymbol{e}^{\boldsymbol{b}}$ in S, we say $\boldsymbol{e}^{\boldsymbol{a}} >_w \boldsymbol{e}^{\boldsymbol{b}}$ if either

1. $|\boldsymbol{e}^{\boldsymbol{a}}| > |\boldsymbol{e}^{\boldsymbol{a}}|$ or;
2. $|\boldsymbol{e}^{\boldsymbol{a}}| = |\boldsymbol{e}^{\boldsymbol{b}}|$ and there exists $1 \leq i \leq n$ such that $\alpha_i > \beta_i$ and $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\dots$, $\alpha_{i-1} = \beta_{i-1}$.

Observe that for each $1 \leq i \leq j \leq n$, we have $\mathrm{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since μ is graded, we must have $|e_i e_j| = |e_i| + |e_j| = |e_k|$. It follows that $|e_k| > |e_i|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ by our assumption on $(e_1, \dots, e_n)$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

3.2 Main theorem

Before we state and prove the main theorem, let us introduce one more piece of notation. We denote $\mathcal{M} = \{\boldsymbol{e}^{\boldsymbol{a}} \mid \boldsymbol{e}^{\boldsymbol{a}} \notin \mathrm{LT}(I)\}$. Since $\mathrm{LT}(f_{i,j}) = e_i e_j$ for all $1 \leq i, j \leq n$, we see that $\mathcal{M}$ is a subset of $\{e_1, \dots, e_n\}$. Now we are ready to state and prove the main theorem:

Theorem 3.1. *The following statements are equivalent:*

1. *F is associative.*
2. *$\mathcal{F}$ is a Gröbner basis.*
3. *$\mathcal{M} = \{e_1, \dots, e_n\}$.*

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq n$. We have

$$\begin{aligned} S_{i,j,k} &:= S(f_{j,k}, f_{i,j}) \\ &= e_i f_{j,k} - f_{i,j} e_k \\ &= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\ &= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\ &= \left(\sum_l c_{i,j}^l e_l \right) e_k - e_i \left(\sum_l c_{j,k}^l e_l \right) \\ &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l. \end{aligned}$$

Now we divide $S_{i,j,k}$ by $\mathcal{F}$:

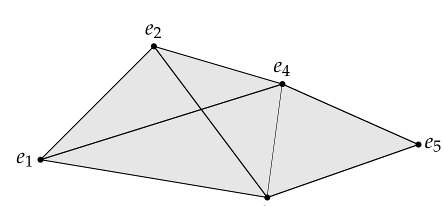
$$\begin{aligned} S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} \\ &= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) + \sum_l c_{j,k}^l (f_{i,l} - e_i e_l) \\ &= \sum_l c_{i,j}^l (e_l e_k - e_l e_k + e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l - e_i e_l) \\ &= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\ &= \left(\sum_l c_{i,j}^l e_l \right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l \right) \\ &= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\ &= [e_i, e_j, e_k]. \end{aligned}$$

Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$. It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. A straightforward comptutation also shows that $S(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \leq i \leq n$. Finally, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq l \leq n$. We have

$$\begin{aligned} S_{i,j,k,l} &:= S(f_{k,l}, f_{i,j}) \\ &= e_i e_j f_{j,k} - f_{i,j} e_k e_l \\ &= (f_{i,j} + e_i \star e_j) f_{j,k} - f_{i,j}(f_{k,l} + e_k \star e_l) \\ &= (e_i \star e_j) f_{j,k} - f_{i,j}(e_k \star e_l). \end{aligned}$$

From this, it's easy to see that $S_{i,j,k,l}^{\mathcal{F}} = 0$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. $\square$

Example 3.1. Let Δ be the simplicial complex below



and let (F, d) be the $\mathbb{F}_2$-complex induced by Δ. Let's write the homogeneous components of F as a graded $\mathbb{F}_2$-vector space

$$\begin{aligned} F_0 &= \mathbb{F}_2 \\ F_1 &= \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5 \\ F_2 &= \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45} \\ F_3 &= \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345} \\ F_4 &= \mathbb{F}_2 e_{1234} \end{aligned}$$

Let μ be a multiplication on F such that

$$\begin{aligned} e_1 \star_\mu e_5 &= e_{14} + e_{45} \\ e_2 \star_\mu e_5 &= e_{23} + e_{35} \\ e_2 \star_\mu e_{45} &= e_{234} + e_{345} \\ e_1 \star_\mu e_{35} &= e_{134} + e_{345} \\ e_1 \star_\mu e_{23} &= e_{123} \\ e_2 \star_\mu e_{14} &= e_{124}. \end{aligned}$$

Then F is not associative with respect to μ since $[e_1, e_5, e_2]_\mu = e_{123} + e_{124} + e_{234} + e_{134} \neq 0$. We use Singular to determine this:

```
intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2),I);

// e123+e124+e234+e134 = [e1,e5,e2]
```

Let us extend μ by setting $e_5 \star_\mu e_{12} = e_{123} + e_{134} + e_{345}$. Extending the code above gives us the following code:

```
intvec w=(1,1,1,2,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e12,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(2) = e1*e2+e12;
poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;
poly f(5)(12) = e5*e12+e123+e134+e345;

ideal I = f(1)(2),f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14),f(5)(12);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2),I);

// e123+e124+e234+e134 = [e1,e5,e2]

poly s(2)(1)(5) = e2*f(1)(5)+e5*f(1)(2);
reduce(s(2)(1)(5),I);

// e123+e124+e234+e134 = [e2,e1,e5]

poly s(5)(2)(1) = e5*f(1)(2)+e1*f(2)(5);
reduce(s(5)(2)(1),I);

// 0 = [e5,e2,e1]
```

From this code, we see that $[e_2, e_1, e_5] \neq 0$ and $[e_5, e_2, e_1] = 0$. In fact, we have $[e_1, e_5, e_2] = [e_2, e_1, e_5]$.

4 Algorithm

Let $R = \mathbb{F}_2$ and let $\boldsymbol{m} = (m_1, \ldots, m_d) = (1, \ldots, 1)$. Let Δ be the full $(d-1)$-simplex and labeled by $\boldsymbol{m}$. In particular, every face of Δ is labeled by 1. Let F be the $\mathbb{F}_2$-complex induced by Δ. We fix the following ordered basis on F: we say $e_\sigma \leq e_\tau$ if either $|e_\sigma| < |e_\tau|$ (meaning $\#\sigma < \#\tau$) or $|e_\sigma| = |e_\tau|$ and $\min(\sigma \backslash \tau) < \min(\tau \backslash \sigma)$. This is just the standard degree lexicographic ordering. Also for notational convenience, we write Δ_i to be set of all i-faces of Δ: so $\Delta_i = \{\sigma \subseteq [d] \mid \#\sigma = i\}$. We will describe an algorithm which gives us multiplication μ on F which gives it the structure of an MDG algebra. In particular, μ needs to be strictly graded-commutative, unital, and satisfy Leibniz law. Clearly $e_\varnothing$ needs to be the identity element, so we must have $e_\varnothing e_\sigma = e_1 = e_\sigma e_\varnothing$ for all $\sigma \in \Delta$. Furthermore, we wish to make μ as associative as possible (though this will inevitably break down). We now break the algorithm into two parts:

Part 1: In the first part of the algorithm, we construct multiplications of the form $e_1 e_\sigma$ for all $\sigma \in \Delta \backslash \Delta_0$. Once we've constructed all multiplications of this form, we then define $e_\sigma e_1 := e_1 e_\sigma$ for all $\sigma \in \Delta \backslash \Delta_0$ in order to ensure commutativity.

Step 1: First we define all multiplications of the form $e_1 e_i$ where $1 \leq i \leq d$. Clearly we need to have $e_1 e_1 = 0$, so we just need to define $e_1 e_i$ where $i \neq 1$. Let G_1^1 be the graph whose vertices are the 0-faces of Δ and whose edges are the 1-faces of Δ, so $G_{1,1}^1 = \Delta$and choose T_1^1 to be a spanning tree of G_1^1. For each $i \neq 1$, let $P_{1,i}^1$ be the unique path from e_1 to e_i in T_1^1. We define

$$e_1 e_i := \sum_{\sigma \in P_{1,i}^1} e_\sigma.$$

In particular, if e_2 is adjacent to e_1 in T_1^1, then $e_1 e_2 = e_{12}$. If e_3 is 2-adjacent to e_1 in T_1^1 and $P_{1,2}^1$ is the path $e_1 - e_2 - e_3$, then $e_1 e_3 = e_{12} + e_{23}$. Note that in this case we have

$$\begin{aligned} \mathrm{d}(e_1 e_3) &= \mathrm{d}(e_{12} + e_{23}) \\ &= \mathrm{d}(e_{12}) + \mathrm{d}(e_{23}) \\ &= e_1 + e_2 + e_2 + e_3 \\ &= e_3 + e_1 \\ &= \mathrm{d}(e_1) e_3 + e_1 \mathrm{d}(e_3). \end{aligned}$$

This shows that Leibniz law is satisfied at (e_1, e_i). An easy induction argument shows that Leibniz law is satisfied at all pairs (e_1, e_i).

Next, in order to ensure that the multiplication is alternative, we define

$$e_1 e_\sigma := 0 \tag{22}$$

for all edges σ in T_1^1. To see why we need this condition, first consider the case where e_2 is adjacent to e_1 in T_1^1. Then since $e_{12} = e_1 e_2$, we see that $e_1 e_{12} = e_1(e_1 e_2)$. If μ was alternative at (e_1, e_1, e_2), then we'd continue with the computation and find that $e_1(e_1 e_2) = (e_1 e_1) e_2 = 0$. So clearly we must have $e_1 e_{12} = 0$ in order to ensure that the multiplication is alternative at (e_1, e_1, e_2). Next we consider the case where e_3 is 2-adjacent to e_1 with $e_1 - e_2 - e_3$ being the unique path from e_1 to e_3 in T_1^1. Again, in order to ensure that the multiplication is alternative at (e_1, e_1, e_3), we must have $e_1(e_1 e_3) = 0$. It would then follow that

$$\begin{aligned} 0 &= e_1(e_1 e_3) \\ &= e_1(e_{12} + e_{23}) \\ &= e_1 e_{12} + e_1 e_{23} \\ &= e_1 e_{23}. \end{aligned}$$

An easy induction argument shows that in order to ensure the multiplication is alternative at the pair (e_1, e_1, e_i) for all i, then we must have (22).

Step 2: In step 1, we defined all multiplications of the form $e_1 e_i$ as well multiplications of the form $e_1 e_\sigma$ for all $\sigma \in T_1^1$. We now construct another graph denoted G_1^2 whose underlying vertex set is $\{e_1, e_\sigma \mid \sigma \in \Delta_1 \backslash T_1^1\}$. Thus the vertices of G_1^2 corresponds to e_1 as well as the $e_\sigma \in F_2$ such that $e_1 e_\sigma$ hasn't been defined yet. Each $\sigma \in \Delta_1 \backslash T_1^1$ gives rise to a cycle in

Definition 4.1. Let be a graded R-module, and let $\mu\colon A \otimes_R A \to A$ and $\lambda\colon A \to A$ be R-linear maps. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^m a_i \otimes b_i\right) = \sum_{i=1}^m a_i \star_\mu b_i = \sum_{i=1}^m a_i b_i,$$

where we denote its image under μ by $\sum_i a_i b_i$ only if μ is understood from context. We call the triple (A, λ, μ) a **pre-MLG R-algebra**.

Definition 4.2. Let $A = (A, \lambda, \mu)$ be a pre-MLDG algebra.

1. We say A is **unital** (or μ is **unital**) if there exists $1 \in A$ such that

$$a1 = a = 1a$$

for all $a \in A$. In this case, we can view R as a subset of A sitting in degree 0 via the embedding $R \xrightarrow{e} A$ defined by $r \mapsto re$ for all $r \in R$.

2. We say A is **graded-commutative** (or μ is **graded-commutative**) if

$$ab = (-1)^{|a||b|} ba$$

for all homogeneous $a, b \in A$. We say it is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that

$$a^2 = 0$$

for all homogeneous $a \in A$ whenever $|a|$ is odd.

3. We say A is **multiplicative** (or λ is **μ-multiplicative**) if it satisfies the **multiplicative law**:

$$\lambda(ab) = \lambda(a)\lambda(b)$$

for all $a, b \in A$. The **multiplicator** if the map $[\cdot,\cdot]\colon A^{\otimes 2} \to A$ defined by

$$[a, b] = \lambda(ab) - \lambda(a)\lambda(b)$$

for all $a, b \in A$. .

4. We say A is **hom-associative** (or μ is **λ-associative**) if it satisfies the **hom-associative law**:

$$\lambda(a)(bc) = (ab)\lambda(c)$$

for all $a, b, c \in A$. If $\lambda = 1_A$, then we will simply say A is **associative**. The **hom-associator** of A is the map $[\cdot,\cdot,\cdot]\colon A^{\otimes 3} \to A$ defined by

$$[a, b, c] = \lambda(a)(bc) - (ab)\lambda(c)$$

for all $a, b, c \in A$.

5. We say A is **permutative** (or μ is **λ-permutative**) if it satisfies the **permutative law**:

$$\lambda(ab)(\lambda(c)\lambda(d)) = (\lambda(a)\lambda(b))\lambda(cd) \tag{23}$$

for all $a, b, c, d \in A$. The **permutator** of A is the map $[\cdot,\cdot,\cdot,\cdot]\colon A^{\otimes 4} \to A$ defined by

$$[a, b, c, d] = \lambda(ab)(\lambda(c)\lambda(d)) - (\lambda(a)\lambda(b))\lambda(cd)$$

for all $a, b, c, d \in A$.

We say A is an **MLG algebra** if A is strictly graded-commutative and unital. In this case, we call μ the **multiplication** of A and we call λ the **perturbation** of A. The multiplication of A is sometimes denoted μ_A and the perturbation of A is sometimes denoted λ_A in case context is not clear.

Proposition 4.1. *Let $A = (A, \lambda, \mu)$ be an MLG algebra and let $e = \lambda(1)$.*

1. *If A is permutative, then we have $e \in [\lambda A, \lambda A, \lambda A]$ and $e^2\lambda(ab) = e\lambda(a)\lambda(b)$ for all $a, b \in A$. Conversely, if $e \in [\lambda A, \lambda A, \lambda A]$ and $e^2\lambda(ab) = e\lambda(a)\lambda(b)$, then $e^2[a, b, c, d] = 0$ for all $a, b, c, d \in A$. In particular, if e is nonzerodivisor, then A is permutative if and only if $e \in [\lambda A, \lambda A, \lambda A]$ and $e\lambda(ab) = \lambda(a)\lambda(b)$ for all $a, b \in A$. In this case, we have $[a, b] = (1 - e)\lambda(ab)$.*

2. *If A is hom-associative, then A is permutative. More generally, we have*

$$[a, b, c, d] = \lambda^2(a)[b, c, d] - [ab, \lambda(c), \lambda(d)] + [\lambda(a), bc, \lambda(d)] - [\lambda(a), \lambda(b), cd] + [a, b, c]\lambda^2(d) := \delta^4([\cdot, \cdot, \cdot])(a, b, c, d)$$

for all $a, b, c, d \in A$.

Proof. Suppose that A is permutative. Then setting $a = 1 = b$ in the permutative law (1) gives us $e(\lambda(c)\lambda(d)) = e^2\lambda(cd)$. Setting $a = 1 = c$ in the permutative law (1) gives us $\lambda(b)(e\lambda(d)) = (e\lambda(b))\lambda(d) = (\lambda(b)e)\lambda(d)$ where the last equality follows from the fact that A is graded-commutative. Conversely, suppose $e \in [\lambda A, \lambda A, \lambda A]$ and that $e^2\lambda(ab) = e\lambda(a)\lambda(b)$ for all $a, b \in A$. Then

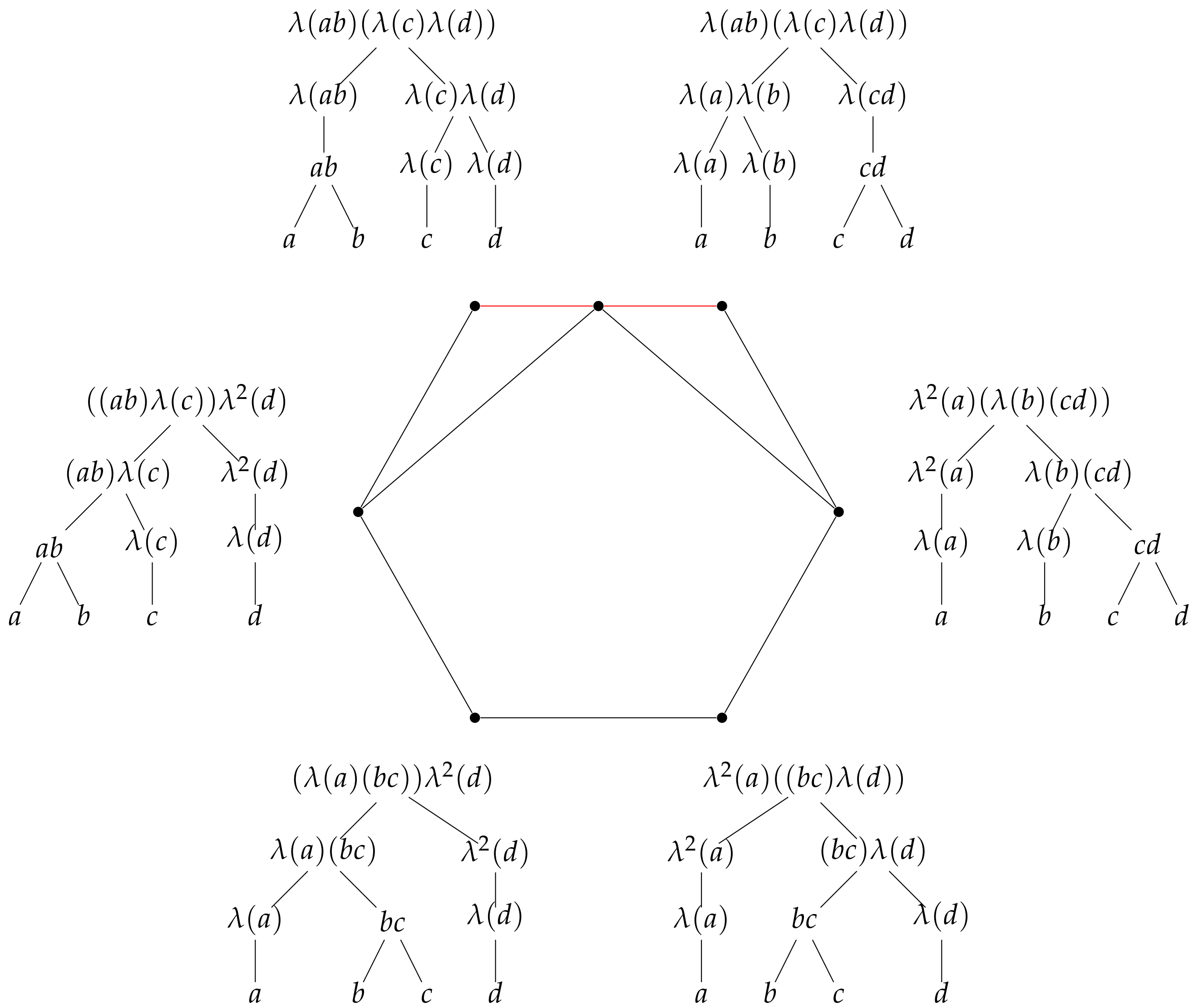
$$\begin{aligned} e^2\lambda(ab)(\lambda(c)\lambda(d)) &= (e\lambda(a)\lambda(b))(\lambda(c)\lambda(d)) \\ &= (\lambda(a)\lambda(b))(e\lambda(c)\lambda(d)) \\ &= (\lambda(a)\lambda(b))(\lambda(cd)) \\ &= e^2(\lambda(a)\lambda(b))\lambda(cd). \end{aligned}$$

It follows that $e^2[a, b, c, d] = 0$.

2. Suppose A is hom-associative. Then for all $a, b, c, d \in A$, we have

$$\begin{aligned} \lambda(ab)(\lambda(c)\lambda(d)) &= ((ab)\lambda(c))\lambda^2(d) \\ &= (\lambda(a)(bc))\lambda^2(d) \\ &= \lambda^2(a)((bc)\lambda(d)) \\ &= \lambda^2(a)(\lambda(b)(cd)) \\ &= (\lambda(a)\lambda(b))\lambda(cd). \end{aligned}$$

There's a cute way to visualize this by tracing the edges of the permutohedron (the hexagon) below:



Note that the red edge "collapses" to the associahedra (the pentagon) if $\lambda = 1$. $\square$

4.0.1 Hom-Associator Identities

We want to familiarize ourselves with the hom-associator of X, so in this subsubsection we collect together some identities which the hom-associator of X satisfies:

- For all $a, b \in A$ homogeneous and $x \in X$, we have the Leibniz law:

$$\mathrm{d}[a,b,x] = [\mathrm{d}(a), b, x] + (-1)^{|a|}[a, \mathrm{d}(b), x] + (-1)^{|a|+|b|}[a, b, \mathrm{d}(x)].$$

- For all $a, b \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a,b,x] = -(-1)^{|a||b|+|a||x|+|b||x|}[x,b,a]. \tag{24}$$

- For all $a, b \in A$ homogenous and $x \in X$ homogeneous we have a graded-commutative version of the Jacobi identity:

$$[a,b,x] = -(-1)^{|a||x|+|b||x|}[x,a,b] - (-1)^{|a||b|+|a||x|}[b,x,a] = 0 \tag{25}$$

- For all $a, b \in A$ and $x \in X$ we have

$$(\lambda^2 a)[b,x] - [ab, \lambda x] + [\lambda a, bx] - [a,b](\lambda^2 x) = [\lambda a, \lambda b, \lambda x] - \lambda[a,b,x] \tag{26}$$

- For all $a, b, c \in A$ and $x \in X$ we have

$$\lambda^2(a)[b,c,x] - [ab, \lambda(c), \lambda(x)] + [\lambda(a), bc, \lambda(x)] - [\lambda(a), \lambda(b), cx] + [a,b,c]\lambda^2(x) = [a,b,c,x] \tag{27}$$

Example 4.1. Let V be a 2-dimensional vector space over a field K and let $\boldsymbol{e} = e_1, e_2$ be a basis of V. Define $\lambda\colon V \to V$ by $\lambda(e_1) = e_2$ and $\lambda(e_2) = 0$.