Impossibility Theorems for Elementary Integration

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1 Analytic Functions

Let U be an open subset of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$. We say f is **analytic** if it is locally expressible as a convergent Taylor series. This means that for each $a \in U$, there exists an open neighborhood U_a of a such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

for all $x \in U_a$. For each open subset U of $\mathbb{R}$, let

$$C^{\omega}_{\mathbb{R}}(U) := \{f\colon U \to \mathbb{R} \mid f \text{ is analytic}\}.$$

If $f, g \in C^{\omega}_{\mathbb{R}}$, then $fg \in C^{\omega}_{\mathbb{R}}$: indeed, let $a \in U$. Then we can express f and g as a power series in the neighborhoods U_a and V_a of a respetively as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

for all $x \in U_a \cap V_a$. Then

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \sum_{n=0}^{\infty} a_n (x-a)^n \sum_{n=0}^{\infty} b_n (x-a)^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) (x-a)^n \\
&= \sum_{n=0}^{\infty} c_n (x-a)^n,
\end{aligned}$$

where $c_n := \sum_{i=0}^{n} a_i b_{n-i}$.

Proposition 1.1.

2 Calculus with $\mathbb{C}$-valued functions

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a $\mathbb{C}$-valued function on $\mathbb{R}$. Then f can be written as $f = u + iv$, where u and v are $\mathbb{R}$-valued functions. We will say f is **continuous** (respectively if u and v are continuous.

2.0.1 Logarithm

If a $\mathbb{C}$-valued function $f(x)$ is analytic and non-vanishing, then f'/f is analytic as well, so upon choosing a point x_0 the integral

$$(\log f)(x) := \int_{x_0}^{x} \frac{f'(t)}{f(t)} dt$$

is analytic funciton called a **logarithm** of f. Such a function depends on the choice of x_0 up to an additive constant, but such ambiguity is irrelevant for our purposes so we will ignore it. Thus, we can equivalently consider a logarithm of f to be a solution to the differential equation $y' = f'/f$. In the special case $x_0 = 1$ and $f(t) = t$ (on the interval $(0, \infty)$) this recovers the traditional logarithm function. If we add a suitable constant to a logarithm of f then we can arrange that $e^{\log(f)} = f$, so the terminology is reasonable.

3 Elementary Fields and Elementary Functions

If $f_1, \dots, f_n$ are meromorphic functions, then $\mathbb{C}(f_1, \dots, f_n)$ denotes the set of meromorphic functions h of the form

$$h = \frac{p(f_1, \dots, f_n)}{q(f_1, \dots, f_n)} = \frac{\sum a_{e_1,\dots,e_n} f_1^{e_1} \cdots f_n^{e_n}}{\sum b_{j_1,\dots,j_n} f_1^{j_1} \cdots f_n^{j_n}}$$

for n-variable polynomials

$$p(X_1, \dots, X_n) = \sum a_{e_1,\dots,e_n} X_1^{e_1} \cdots X_n^{e_n}, \quad \text{and} \quad q(X_1, \dots, X_n) = \sum b_{j_1,\dots,j_n} X_1^{j_1} \cdots X_n^{j_n}$$

in $\mathbb{C}[X_1, \dots, X_n]$ with $q(f_1, \dots, f_n) \neq 0$.

Example 3.1. The field $K = \mathbb{C}(x, \sin(x), \cos(x))$ is the set of ratios

$$\frac{p(x, \sin(x), \cos(x))}{q(x, \sin(x), \cos(x))}$$

for polynomials $p, q \in \mathbb{C}[X, Y, Z]$ such that $q(x, \sin(x), \cos(x)) \neq 0$. For example, we cannot use $q = Y^2 + Z^2 - 1$ since $\sin(x)^2 + \cos(x)^2 - 1 = 0$. Since

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \text{and} \quad e^{ix} = \cos(x) + i\sin(x),$$

we have $K = \mathbb{C}(x, e^{ix})$. So elements in K can also be written in the form $g(x, e^{ix})/h(x, e^{ix})$ with $g, h \in \mathbb{C}[X, Y]$ and $h \neq 0$.

Definition 3.1. A field K of meromorphic functions is an **elementary field** if $K = \mathbb{C}(x, f_1, \dots, f_n)$ with each f_j either an exponential or logarithm of an element of $K_{j-1} = \mathbb{C}(x, f_1, \dots, f_{j-1})$ or else algebraic over K_{j-1} in the sense that $P(f_j) = 0$ for some $P(T) = T^m + a_{m-1}T^{m-1} + \cdots + a_0 \in K_{j-1}[T]$ with all $a_k \in K_{j-1}$. A meromorphic function f is an **elementary function** if it lies in an elementary field of meromorphic functions.

Let K be an elementary field with transcendence degree n. Then K is isomorphic to a finite algebraic extension of the abstract field $\mathbb{C}(T_1, \dots, T_n)$. For example, consider the field $K = \mathbb{C}(x, \sin x, \cos x)$. Then $K \cong \mathbb{C}\left(T_1, T_2, \sqrt{1 - T_2^2}\right)$, where the isomorphism is uniquely determined by $T_1 \mapsto x$ and $T_2 \mapsto \sin x$. Since

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \text{and} \quad e^{ix} = \cos(x) + i\sin(x),$$

we also have $K = \mathbb{C}(x, e^{ix}) \cong \mathbb{C}(T_1', T_2')$, where the isomorphism is uniquely determined by $T_1' \mapsto x$ and $T_2' \mapsto e^{ix}$. Note that $\mathbb{C}(T_1', T_2') \cong \mathbb{C}\left(T_1, T_2, \sqrt{1 - T_2^2}\right)$ where the isomorphism is uniquely determined by $T_1' \mapsto T_1$ and $T_2' \mapsto \sqrt{1 - T_2^2} + iT_1$.

Example 3.2. Consider the elementary field

$$L = \mathbb{C}\left(x, \sqrt{x^3 - 1}\right).$$

This field arises as the function field of the irreducible algebraic set $V(T_2^2 - T_1^3 + 1) \subseteq \mathbb{A}^2(\mathbb{C})$. It is a finite extension of the field $K = \mathbb{C}(x)$. On the other hand, the elementary field

$$E = \mathbb{C}\left(x, e^{ix}\right)$$

is a transcendental extension of the field K. Is $E \cong \mathbb{C}(X, Y)$? We claim that this is indeed the case. Let $\varphi\colon \mathbb{C}(X, Y) \to \mathbb{C}(x, e^{ix})$ be given by

$$\varphi\left(\frac{p(X, Y)}{q(X, Y)}\right) = \frac{p\left(x, e^{ix}\right)}{q\left(x, e^{ix}\right)}.$$

We just need to check that $q\left(x, e^{ix}\right) \neq 0$. If $q\left(x, e^{ix}\right) = 0$, then

Example 3.3. Consider the function

$$f = \frac{\pi x^2 - 3x\log x}{\sqrt{e^x - \sin\left(x/(x^3-7)\right)}} \in C^\omega_{\mathbb{R}}(\sqrt[3]{7}, \infty).$$

Then f is contained in the elementary field

$$\mathbb{C}\left(x, \log x, e^x, e^{ix/(x^3-7)}, \sqrt{e^x - \sin\left(x/(x^3-7)\right)}\right).$$

Note that

$$\mathbb{C}\left(x, \log x, e^x, \sin\left(x/(x^3-7)\right), \sqrt{e^x - \sin\left(x/(x^3-7)\right)}\right)$$

is *not* an elementary field.

Theorem 3.1. *If K is an elementary field, then it is closed under the operation of differentiation.*

Proof. We write $K = \mathbb{C}(x, f_1, \ldots, f_n)$ and we induct on n. The case $n = 0$ is the case $K = \mathbb{C}(x)$. It follows from the usual formulas for derivatives of sums, products, and ratios that $\mathbb{C}(x)$ is closed under differentiation. For the general case, by induction $K_0 = \mathbb{C}(x, f_1, \ldots, f_{n-1})$ is closed under differentiation, and we have $K = K_0(f_n)$ with f_n either algebraic over K_0 or a logarithm or exponential of an element of K_0. Let us now check that it suffices to prove $f_n' \in K_0(f_n)$. Under this assumption, for any polynomial $P(T) = \sum_{j\geq 0} a_j T^j \in K_0[T]$, we have

$$P(f_n)' = a_0' + \sum_{j\geq 1}(a_j' f_n^j + j a_{j-1} f_n^{j-1} f_n') \in K_0(f_n)$$

since $a_j' \in K_0$ for all j. Thus, if $P, Q \in K_0[T]$ are polynomials over K_0 and $Q(f_n) \neq 0$, then

$$\left(\frac{P(f_n)}{Q(f_n)}\right)' = \frac{Q(f_n)P(f_n)' - P(f_n)Q(f_n)'}{Q(f_n)^2} \in K_0(f_n) = K$$

since the numerator and denominator lie in $K_0(f_n)$.

It remains to check that the function f_n that is either algebraic over K_0 or is an exponential or logarithm of an element of K_0 has derivative f_n' that lies in $K_0(f_n)$. If $f_n = e^g$ for some $g \in K_0$, then $f_n' = g' f_n \in K_0(f_n)$. If $f_n = \log(g)$ for some $g \in K_0$, then $f_n' = g'/g \in K_0(f_n)$. Finally, we treat the algebraic case. Suppose $P(f_n) = 0$ for a polynomial

$$P = T^m + a_{m-1}(x)T^{m-1} + \cdots + a_0(x) \in K_0[T].$$

Take P with minimal degree, so

$$P'(T) := mT^{m-1} + (m-1)a_{m-1}(x)T^{m-2} + \cdots + 2a_2(x)T + a_1(x)$$

with degree $m-1$ satisfies $P'(f_n) \neq 0$. But

$$\begin{aligned}
0 &= P(f_n)' \\
&= \left(f_n^m + a_{m-1}(x)f_n^{m-1} + \cdots + a_0(x)\right)' \\
&= (f_n^m)' + (a_{m-1}(x)f_n^{m-1})' + \cdots + a_0'(x) \\
&= mf_n^{m-1}f_n' + (a_{m-1}'(x)f_n^{m-1} + (m-1)a_{m-1}f_n^{m-2}f_n') + \cdots + a_0'(x) \\
&= \sum_{j>0} ja_j(x)f_n^{j-1}f_n' + \sum_{j<m} a_j'(x)f_n^j \\
&= P'(f_n)f_n' + \sum_{j<m} a_j'(x)f_n^j,
\end{aligned}$$

so $P'(f_n)f_n' = -\sum_{j<m} a_j'(x)f_n^j \in K_0[f_n]$. Since $P'(f_n) \neq 0$ and $P'(f_n) \in K_0(f_n)$, we have $f_n' \in K_0(f_n)$ by division. $\square$

Remark. A field K of meromorphic functions that is closed under differentiation si called a **differential field**. The preceding theorem says that elementary fields are examples of differential fields, but the method of proof shows more: if $K = K_0(f)$ with K_0 any differential field and f either algebraic over K_0 or an exponential or logarithm of an element of K_0, then K is a differential field. The field $\mathbb{C}(x, \sin(x), \cos(x))$ is a differential field since $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$. In contrast, $\mathbb{C}(x, \sin(x))$ is *not* a differential field. More specifically, $\sin'(x) = \cos(x)$ but $\cos(x)$ is not an element of $\mathbb{C}(x, \sin(x))$.

Definition 3.2. A meromorphic function f can be **integrated in elementary terms** if $f = g'$ for an elementary function g (and so f is necessarily elementary, by Theorem (3.1)).

Remark. If we only considered $\mathbb{R}$-valued functions (and not $\mathbb{C}$-valued functions), then the definition above would give the *wrong* concept of elementary integration. For example, we would want to say that $1/(1+x^2)$ admits an elementary integral (such as $\arctan(x)$). This is the case in the $\mathbb{C}$-valued setting, since we know that $\arctan(x) = \frac{i}{2}\log\left(\frac{1-ix}{1+ix}\right)$. However, if we work in the $\mathbb{R}$-valued setting and permit only the operations of eponentiation, logarithm, and solving of algebraic equations, then it can be proved that $1/(1+x^2)$ is *not* integrable (over $\mathbb{R}$) in such elementary terms. A way around this technical glitch in the $\mathbb{R}$-valued case is to incorporate all of the usual trigonometric functions and their inverses (and not merely exponentials and logarithms) in an $\mathbb{R}$-valued definition of "integration in elementary terms". Unfortunately, this change in definitions is disastrous for the attempt to push through an $\mathbb{R}$-valued analogue of Liouville's results because such trigonometric functions and their inverses are not solutions to simple first-order differential equations. Since our main interest is in impossibility results, Liouville's work in the $\mathbb{C}$-valued setting will give what we require.

4 Integrability Criterion and Applications

Theorem 4.1. *(Liouville) Let f be an elementary function and let K be an elementary field containing f. The function f can be integrated in elementary terms if and only if there exist nonzero $c_1, \ldots, c_n \in \mathbb{C}$, nonzero $g_1, \ldots, g_n \in K$, and an element $h \in K$ such that*

$$f = \sum c_j \frac{g_j'}{g_j} + h'.$$

The key point is that the g_j's and h can be found in any *elementary field K containing f; $\sum c_j \log(g_j) + h$ is then an elementary integral of f.*

Example 4.1. Consider $f = e^{-x^2}$. This lies in the elementary field $K = \mathbb{C}(x, e^{-x^2})$. Hence, Liouville's theorem says that an elementary anti-derivative of f *must* have the special form $\sum c_j \log(g_j) + h$ for some $h \in \mathbb{C}(x, e^{-x^2})$ and nonzero $c_j \in \mathbb{C}$ and $g_j \in \mathbb{C}(x, e^{-x^2})$. It is not obvious how to prove the non-existence of such h and g_j's, but this still represents a significant advance over the problem of contemplating all elementary functions as candidates for elementary anti-derivatives of e^{-x^2} . We will soon see that the possible form of such an elementary anti-derivative of e^{-x^2} can be made even more special, and so it becomes a problem that we can solve without too much difficulty.

Example 4.2. There is a very interesting class of integrals for which Liouville's result in the above form is immediately applicable without an extra simplification: elliptic integrals. Just as trignometric functions may be introduced through inversion of integral functions of the form $\int dx/\sqrt{x^2-1}$ that arise from calculation of arc length along a unit circle, the theory of elliptic functions grew out of a study of inversion of integral functions of the form $\int dx/\sqrt{P(x)}$ for certain cubic and quartic polynomials $P(X) \in \mathbb{R}[X]$ without repeated roots; such integrals arise in the calculation of arc length along an ellipse. In general, if $P(X) \in \mathbb{R}[X]$ is any monic polynomial with degree ≥ 3 and no repeated roots then we claim that $\int dx/\sqrt{P(x)}$ is not an elementary function. Since $K = \mathbb{C}(x, \sqrt{P})$ is an elementary field, by the criterion in Liouville's theorem, it suffices to prove that there does not exist an identity of the form

$$\frac{1}{\sqrt{P(x)}} = \sum c_j \frac{g_j'}{g_j} + h'$$

with nonzero $c_1, \ldots, c_n \in \mathbb{C}$, nonzero $g_1, \ldots, g_n \in K$, and an element $h \in K$. Such impossibility is a consequence of general facts from the theory of compact Riemann surfaces. More specifically, the above identity is equivalent to the equality of meromorphic 1-forms

$$\frac{dx}{y} = \sum c_j \frac{dg_j}{g_j} + dh$$

on the compact Riemann surface C associated to the equation $y^2 = P(x)$, and for $\deg(P) > 2$ the left side is a nonzero holomorphic 1-form on C. But a nonzero holomorphic 1-form on a compact Riemann surface never admits an expression as a linear combination of logarithmic meromorphic differentials dg/g and exact meromorphic differentials dh.

Theorem 4.2. *Choose $f, g \in \mathbb{C}(X)$ with $f \neq 0$ and g nonconstant. The function $f(x)e^{g(x)}$ can be integrated in elementary terms if and only if there exists a rational function $R \in \mathbb{C}(X)$ such that $R'(X) + g'(X)R(X) = f(X)$ in $\mathbb{C}(X)$.*

The content of the criterion in this theorem is not that the differential equation $R'(x) + g'(x)R(x) = f(x)$ has a solution as a $\mathbb{C}$-valued differentiable function of x (indeed, one such solution is given by $e^{g'(x)x}\left(\int_0^x f(t)e^{-g'(t)t}dt\right)$.