Advanced Numerical Analysis Homework 2

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Throughout this homework, $\|\cdot\|$ denotes the ℓ_2-norm.

1 Problem 1

Exercise 1. Let $a_0, a_1, \ldots, a_n$ be $n+1$ equispaced points on $[-1,1]$, where $a_0 = -1$ and $a_n = 1$. Assemble these $n+1$ values into a column vector $\boldsymbol{u}$, and use MATLAB's vander to generate Vandermonde matrices A from vector $\boldsymbol{u}$ f $n = 9, 19, 29, 39$. Let $\boldsymbol{x} = (1, 1, \ldots, 1)^\top$ and $\boldsymbol{b} = A\boldsymbol{x}$. Pretend that we do not know $\boldsymbol{x}$ and use numerical algorithms to solve this linear system for $\boldsymbol{x}$. Let $\widehat{\boldsymbol{x}}$ be the computed solution. Compute the relative forward errors $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\| / \|\boldsymbol{x}\|$ and the smallest relative backward errors

$$\frac{\|\boldsymbol{b} - A\widehat{\boldsymbol{x}}\|}{\|A\|\|\widehat{\boldsymbol{x}}\|} = \min\left\{\frac{\|\delta A\|}{\|A\|} \mid (A + \delta A)\widehat{\boldsymbol{x}} = \boldsymbol{b}\right\},$$

where $\|\cdot\|$ denotes the ℓ_2-norm, for the following:

1. GEPP (MATLAB's backlash)
2. QR factorization of A
3. Cramer's rule
4. A^{-1} multiplied by $\boldsymbol{b}$
5. GE without pivoting

Comment on the forward/backward stability of these methods.

Solution 1.

2 Problem 2

Exercise 2. Consider the eigenvalue problem $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Let $(\widehat{\lambda}, \widehat{\boldsymbol{v}})$ be a computed eigenpair, which is assumed to be the exact eigenpair of a perturbed matrix $A + \delta A$. Show that the minimum ℓ_2-norm of all such δA is

$$\frac{\|A\widehat{\boldsymbol{v}} - \widehat{\lambda}\widehat{\boldsymbol{v}}\|}{\|\widehat{\boldsymbol{v}}\|},$$

and find a particular δA whose ℓ_2-norm is the minimum. (Note that this result can help us experimentally determine if an eigenvalue algorithm is backward stable).

Solution 2.

3 Problem 3

Exercise 3. Give a proof that the worst-case growth factor $\rho_n = 2^{n-1}$ for GEPP. Compared to $\rho_n \leq Cn^{\frac{1}{2}+\frac{1}{4}\ln n}$ with complete pivoting and $\rho_n \leq 1.5n^{\frac{3}{4}\ln n}$ with rook pivoting, this is much larger. However, we construct matrices with random elements, each are independent samples from the normal distribution of mean 0 and standard deviation $\frac{1}{\sqrt{n}}$ ($\mathrm{A} = \mathrm{randn(n, n)/sqrt(n)}$). Let $n = 32, 64, \ldots, 512$, and for each n, repeat the experiment 1000 times. Find the percentage of experiments when $\rho_n > \sqrt{n}$. Make brief comments on the chance of having a large ρ_n.

Solution 3.

4 Problem 4

Exercise 4. Though pivoting is needed for factorizing general matrices, it is not needed for symmetric positive definite and diagonally dominant matrices.

1. For a symmetric positive definite A, with the one-step Cholesky factorization

$$A = \begin{pmatrix} a_{11} & w^\top \\ w & K \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{w}{\sqrt{a_{11}}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & K - \frac{ww^\top}{a_{11}} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{w^\top}{\sqrt{a_{11}}} \\ 0 & I \end{pmatrix} = R_1^\top A_1 R_1,$$

   show that the submatrix $K - (ww^\top)/a_{11}$ is symmetric positive definite. Consequently, the factorization can be completed without break-down. Then, show that $\|R\| = \|A\|^{1/2}$, which means the element in R are uniformly bounded by that of $\|A\|$. Explain why this observation leads to the backward stability of Cholesky factorization.

2. Suppose that $A = \begin{pmatrix} \alpha & w^\top \\ v & C \end{pmatrix}$ is column diagonally dominant, with one-step LU factorization

$$A = \begin{pmatrix} 1 & 0 \\ \frac{v}{\alpha} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - \frac{vw^\top}{\alpha} \end{pmatrix} \begin{pmatrix} \alpha & w^\top \\ 0 & I \end{pmatrix}.$$

   Show that the sub-matirx $C - (vw^\top)/\alpha$ is also column diagonally dominant, and no pivoting is needed.

Solution 4.