Massey Triple Product

Definition 0.1. Let A be a DG algebra. The **Massey triple product** of $\overline{a}_1, \overline{a}_2, \overline{a}_3 \in \mathrm{H}A$ is defined by

$$\langle \overline{a}_1, \overline{a}_2, \overline{a}_3 \rangle = \{ \overline{a_{12}a_3 - a_1 a_{23}} \mid \mathrm{d}a_{12} = a_1 a_2 \text{ and } \mathrm{d}a_{23} = (-1)^{|a_1|} a_2 a_3 \}$$

Note that $a_{12}a_3 - a_1 a_{23}$ represents an element in $\mathrm{H}A$ since

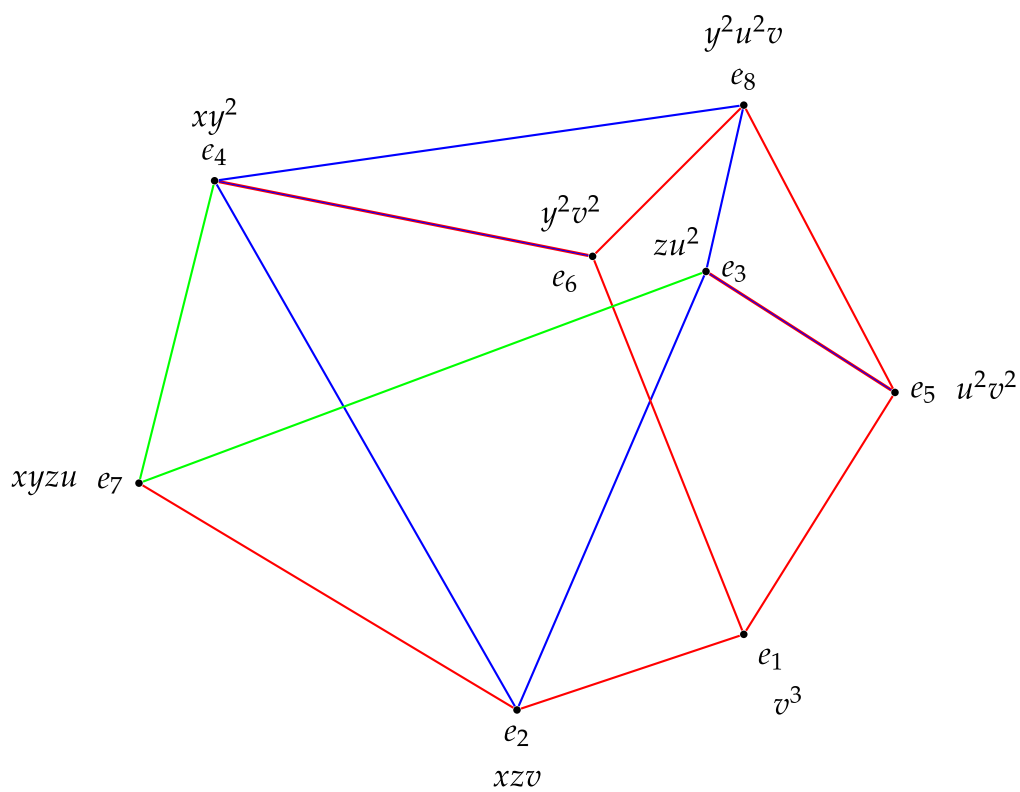
$$\mathrm{d}(a_{12}a_3 - a_1 a_{23}) = [a_1, a_2, a_3] = 0.$$

The Massey product is non-empty if the products $a_1 a_2$ and $a_2 a_3$ are both exact, in which case all of its elements are in the same element of the quotient group

$$\mathrm{H}A / \langle \mathrm{H}A\, \overline{a}_3 + \overline{a}_1\, \mathrm{H}A \rangle.$$

So the Massey product can be regarded as a function defined on triples of classes such that the product of the first or last two is zero, taking values in the above quotient group.

Example 0.1. (Katthän) Let $R = \Bbbk[x, y, z, u, v]$, let $\boldsymbol{m} = v^3, xzv, zu^2, xy^2, u^2v^2, y^2v^2, xyzu, y^2u^2v$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as below:



Let T be the Taylor algebra resolution of $R/\boldsymbol{m}$ over R and let $A = T \otimes_R \Bbbk$. We compute Massey triple products in $\mathrm{H}A = \mathrm{H}(T \otimes_R \Bbbk) = \mathrm{Tor}^R(R/\boldsymbol{m}, \Bbbk) = \mathrm{H}(F \otimes_R \Bbbk)$. We claim that $\langle \overline{e}_1, \overline{e}_3, \overline{e}_4 \rangle$ contains a nonzero element. Indeed, let $e_{1,3} = e_{135}$ and $e_{3,4} = e_{347}$. Note that $\mathrm{d}e_{135} = e_1 e_3$ and $\mathrm{d}e_{347} = e_3 e_4$ so $\overline{e_{1,3}e_4 - e_1 e_{3,4}}$ is an element in $\langle \overline{e}_1, \overline{e}_4, \overline{e}_7 \rangle$. We claim that $\overline{e_{1,3}e_4 - e_1 e_{3,4}} \neq 0$. First we observe that

$$\begin{aligned} e_{1,3}e_4 - e_1 e_{3,4} &= e_{135}e_4 - e_1 e_{347} \\ &= e_{1345} - e_{1347}. \end{aligned}$$

Example 0.2. Avromov says the Massey product $\langle [a_1], [a_2], [a_3] \rangle = \langle [x\varepsilon_1], [z\varepsilon_3], [w\varepsilon_4] \rangle$ contains the element $\gamma = [xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4]$, for the defining system

$$b_{12} = x\varepsilon_1\varepsilon_2\varepsilon_3 \text{ and } b_{23} = 0.$$

In this case, we have

$$[a_1]\mathrm{H}_3(A) + \mathrm{H}_3(A)[a_3] = 0.$$