Mathematical Programming Homework 7

Michael Nelson

Problem 1

Exercise 1. Consider the linear program (LP) in standard form. Suppose at the end of Phase 1, a basic feasible solution to this LP has been found with only artificial variables basic. Prove that at the optimality of the Phase 1 linear program, all reduced costs are equal to zero for the original variables and are equal to one for the artificial variables.

Solution 1. Recall that the reduced cost $\widehat{c}_i$ of x_i is equal to 0 if and only if the objective function does not change if went enter x_i into the basis. The objective function in question is $z' = -\sum a_j$ which clearly doesn't change as we increase/decrease x_i. Similarly, the objective function changes by 1 since $\partial_{a_j} z' = -1$, so the reduced cost for a_j must be $\widehat{c}_j = 1$.

Problem 2

Exercise 2. Find a dual problem to the following linear program. For full credit present the dual in the most compact form (i.e., with as few variables and constraints as possible, and with no "minus" signs or subtractions).

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A_1 \boldsymbol{x} \leq \boldsymbol{b}^1 \\ & -A_2 \boldsymbol{x} = \boldsymbol{b}^2 \\ & \boldsymbol{x} \leq \boldsymbol{u} \end{aligned}$$

Solution 2. First we make the substitutions

$$\begin{aligned} \widetilde{\boldsymbol{x}} &= \boldsymbol{x} - \boldsymbol{u} \\ \widetilde{\boldsymbol{b}}^1 &= \boldsymbol{b}^1 - A_1 \boldsymbol{u} \\ \widetilde{\boldsymbol{b}}^2 &= \boldsymbol{b}^2 - A_2 \boldsymbol{u} \\ \widetilde{z} &= z + \boldsymbol{c}^\top \boldsymbol{u}. \end{aligned}$$

Then the primal problem has the form

$$\begin{aligned} \text{minimize} \quad & \widetilde{z} = \boldsymbol{c}^\top \widetilde{\boldsymbol{x}} \\ \text{s.t.} \quad & A_1 \widetilde{\boldsymbol{x}} \leq \widetilde{\boldsymbol{b}}^1 \\ & -A_2 \widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}^2 \\ & \widetilde{\boldsymbol{x}} \leq 0 \end{aligned}$$

The dual to this problem then has the form

$$\begin{aligned} \text{maximize} \quad & \widetilde{w} = (\widetilde{\boldsymbol{b}}^1)^\top \widetilde{\boldsymbol{y}}^1 + (\widetilde{\boldsymbol{b}}^2)^\top (\widetilde{\boldsymbol{y}})^2 \\ \text{s.t.} \quad & A_1^\top \widetilde{\boldsymbol{y}}^1 + A_2^\top \widetilde{\boldsymbol{y}}^2 \geq \boldsymbol{c} \\ & \widetilde{\boldsymbol{y}}^1 \leq 0, \widetilde{\boldsymbol{y}}^2 \text{ free} \end{aligned}$$

Problem 3

Exercise 3. Prove the following: Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to the minimization and maximization problems, respectively, in the canonical dual pair. Then they are respectively optimal if and only if $(\boldsymbol{c} - A^\top \boldsymbol{y})^\top \boldsymbol{x} = 0$ and $\boldsymbol{y}^\top (A\boldsymbol{x} - \boldsymbol{b}) = 0$.

Solution 3. It suffices to show $\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{y}^\top A\boldsymbol{x}$ since by the strong duality theorem we already have $\boldsymbol{y}^\top \boldsymbol{b} = \boldsymbol{c}^\top \boldsymbol{x}$ (which would give us the other equality) By the strong duality theorem, we have

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &= \boldsymbol{y}^\top \boldsymbol{b} \\ &\leq \boldsymbol{y}^\top A\boldsymbol{x} \end{aligned}$$

where we used the fact that $A\boldsymbol{x} \geq \boldsymbol{b}$. Conversely, since $\boldsymbol{c}^\top \geq \boldsymbol{y}^\top A$, we also have

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x} &\geq (\boldsymbol{y}^\top A)\boldsymbol{x} \\ &= \boldsymbol{y}^\top A\boldsymbol{x}. \end{aligned}$$

Problem 4

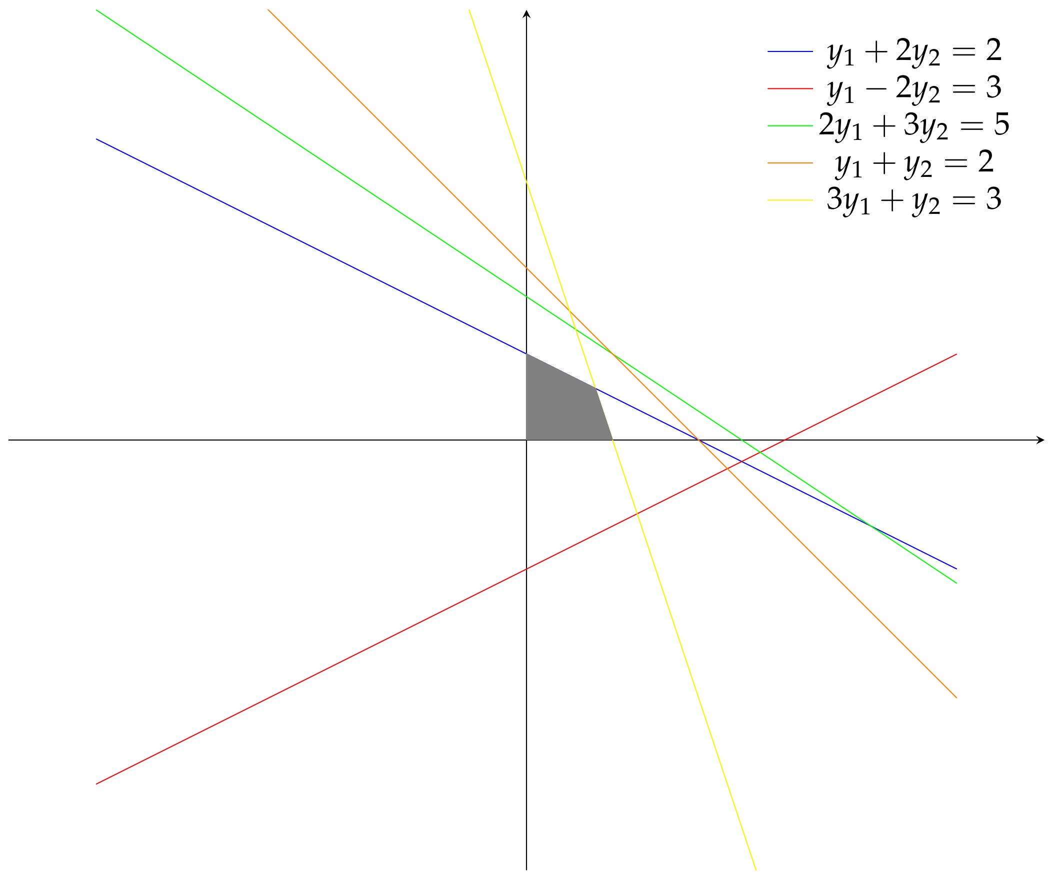
Exercise 4. Without using any Simplex method (or any computer software) solve the following linear program (LP):

$$\begin{aligned} \text{minimize} \quad & z = 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \\ & 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Solution 4. The dual to this LP is given by

$$\begin{aligned} \text{maximize} \quad & w = 4y_1 + 3y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \leq 2 \\ & y_1 - 2y_2 \leq 3 \\ & 2y_1 + 3y_2 \leq 5 \\ & y_1 + y_2 \leq 2 \\ & 3y_1 + y_2 \leq 3 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

The dual proglem is much easier to solve since we can visualize the feasible region shaded in grey as below:



This region is bounded and its extreme points are given by $(0,0)^\top$, $(0,1)^\top$, $(4/5,3/5)^\top$, and $(1,0)^\top$. The function w is maximized at one of these extreme points, and a quick calculation shows that it takes maximum value at $\boldsymbol{y}_* = (4/5,3/5)^\top$ with optimal objective value given by $w = 5$ (the objective value at the other extreme points is < 5, so $\boldsymbol{y}_*$ is *the* optimal solution). To find the dual optimal solution, we find the kernel of $(\boldsymbol{c} - A^\top \boldsymbol{y})^\top$:

$$
\begin{aligned}
(\boldsymbol{c} - A^\top \boldsymbol{y}_*)^\top \boldsymbol{x}_* = 0 &\iff \left(\begin{pmatrix} 2 \\ 3 \\ 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 1 & -2 \\ 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix} \right)^\top \boldsymbol{x}_* = 0 \\
&\iff \left(\begin{pmatrix} 2 \\ 3 \\ 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -2/5 \\ 17/5 \\ 7/5 \\ 3 \end{pmatrix} \right)^\top \boldsymbol{x}_* = 0 \\
&\iff \begin{pmatrix} 0 \\ 17/5 \\ 8/5 \\ 3/5 \\ 0 \end{pmatrix}^\top \boldsymbol{x}_* = 0 \\
&\iff (0, 17/5, 8/5, 3/5, 0) \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{pmatrix} = 0.
\end{aligned}
$$

We must have $x_2^* = x_3^* = x_4^* = 0$ and $z = 2x_1^* + 3x_5^* = 5$. A quick calculation shows that setting $x_1^* = 1 = x_5^*$ gives us a feasible point $\boldsymbol{x}^*$.

Problem 5

Exercise 5. Use the Dual Simplex method and solve the following linear program

$$\begin{aligned} \text{maximize} \quad & z = -x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 + x_3 \geq 4 \\ & 2x_1 + x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

1. Report an optimal primal solution and its value;
2. Report an optimal dual solution and its value.

Solution 5. First we set $z' = -z$ to turn this into a minimization problem; so the linear program now has the form:

$$\begin{aligned} \text{minimize} \quad & z' = x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 + x_3 \geq 4 \\ & 2x_1 + x_2 - x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

If excess variables but not artificial variables are added, then the tableau for this problem becomes

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z'$	1	2	0	0	0	0
	1	-2	1	-1	0	4
	2	1	-1	0	-1	6

Consider the initial basis $x_B = (x_4, x_5)^\top$. We express the tableau in the current basis by multiplying the constraints by -1:

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z'$	1	2	0	0	0	0
x_4	-1	2	-1	1	0	-4
x_5	-2	-1	1	0	1	-6

This basis is not primal feasible (since both x_4 and x_5 are negative), however it is primal optimal (the reduced costs are positive). The most negative variable in the basis is x_5, so we choose it to be the leaving variable. The ratio test indicates that x_1 needs to be our entering variable. We now apply elementary operations to the tableau to express it in the new basis:

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z'$	0	$3/2$	$1/2$	0	$1/2$	-3
x_4	0	$5/2$	$-3/2$	1	$-1/2$	-1
x_1	1	$1/2$	$-1/2$	0	$-1/2$	3

This basis is still primal feasible since x_4 is negative, however it is primal optimal (the reduced costs are positive). Since x_4 is the only negative variable, we use it as the leaving variable. The ratio test indicates that x_3 needs to be our entering variable. We now apply elementary operations to the tableau to obtain a new basic solution:

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z'$	0	$7/3$	0	$1/3$	$1/3$	$-10/3$
x_3	0	$-5/3$	1	$-2/3$	$1/3$	$2/3$
x_1	1	$-1/3$	0	$-1/3$	$-1/3$	$10/3$

This basis is optimal and feasible, so we stop.

1. So an optimal primal solution is given by $x_1 = 10/3$, $x_2 = 0$, and $x_3 = 2/3$, and the corresponding objective value is $z = -z' = -10/3$.

2. The corresponding dual solution is give by $y_1 = 1/5$, $y_2 = 2/5$, and the corresponding objective value is $w = -w' = -16/5$.

Problem 6

Exercise 6. Consider the following linear program (LP):

$$\begin{aligned} \text{maximize} \quad & z = 101x_1 - 87x_2 - 23x_3 \\ \text{s.t.} \quad & 6x_1 - 13x_2 - 3x_3 \leq 11 \\ & 6x_1 + 11x_2 + 2x_3 \leq 45 \\ & x_1 + 5x_2 + x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

with the following optimal basic feasible solution:

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z$	0	0	0	12	4	5	372
x_1	1	0	0	1	-2	7	5
x_2	0	1	0	-4	9	-30	1
x_3	0	0	1	19	-43	144	2

Apply sensitivity analysis to answer the following questions:

1. What is an optimal solution when the value of $b_2 = 45$ decreases to 30?
2. What is the maximum decrease and increase of b_2 without changing the optimal basis?
3. What is an optimal solution when the value of $c_1 = -101$ increases by 25?
4. What is the maximum decrease and increase of $c_3 = 23$ without changing the optimal basis?

Solution 6. Just as in the previous problem, we set $z' = -z$ to convert this into a minimization problem:

$$\begin{aligned} \text{minimize} \quad & z' = -101x_1 + 87x_2 + 23x_3 \\ \text{s.t.} \quad & 6x_1 - 13x_2 - 3x_3 \leq 11 \\ & 6x_1 + 11x_2 + 2x_3 \leq 45 \\ & x_1 + 5x_2 + x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

with the following optimal basic feasible solution:

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z'$	0	0	0	12	4	5	-372
x_1	1	0	0	1	-2	7	5
x_2	0	1	0	-4	9	-30	1
x_3	0	0	1	19	-43	144	2

The current basis is $x_B = (x_1, x_2, x_3)^\top$ and we have

$$B = \begin{pmatrix} 6 & -13 & -3 \\ 6 & 11 & 2 \\ 1 & 5 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & -2 & 7 \\ -4 & 9 & -30 \\ 19 & -43 & 144 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B^{-1}N = \begin{pmatrix} 1 & -2 & 7 \\ -4 & 9 & -30 \\ 19 & -43 & 144 \end{pmatrix}$$

$$\boldsymbol{c}_B = \begin{pmatrix} -101 \\ 87 \\ 23 \end{pmatrix} \qquad \boldsymbol{c}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{y} = B^{-\top}\boldsymbol{c}_B = \begin{pmatrix} -12 \\ -4 \\ -5 \end{pmatrix}$$

$$B^{-1}\boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} \qquad \widehat{\boldsymbol{c}}_N = \boldsymbol{c}_N - N^\top \boldsymbol{y} = \begin{pmatrix} 12 \\ 4 \\ 5 \end{pmatrix}$$

We answer part 2 of this question first, then part 1, then part 4, then part 3:

2. Set $\widetilde{b}_2 = b_2 + \delta$ and set $\widetilde{\boldsymbol{b}} = \boldsymbol{b} + \Delta$ where $\Delta = (0, \delta, 0)^\top$. This change has no effect on the optimality conditions ($\widehat{\boldsymbol{c}}_N \geq 0$), but it does affect the feasibility conditions ($B^{-1}\widetilde{\boldsymbol{b}} \geq 0$). The feasibility condition will remain satisfied as long as $B^{-1}\widetilde{\boldsymbol{b}} \geq 0$, or equivalently, as long as

$$
\begin{aligned}
0 &\leq B^{-1}\widetilde{\boldsymbol{b}} \\
&= B^{-1}\boldsymbol{b} + B^{-1}\Delta \\
&= \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 7 \\ -4 & 9 & -30 \\ 19 & -43 & 144 \end{pmatrix} \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 5 - 2\delta \\ 1 + 9\delta \\ 2 - 43\delta \end{pmatrix}.
\end{aligned}
$$

In particular, the feasibility conditions remains satisfied if and only if $\delta \not< -1/9$, $\delta \not> 5/2$, and $\delta \ngeq 2/43$, or in other words if and only if $\delta \in [-1/9, 2/43]$ with the new objective value given by

$$
\begin{aligned}
\widetilde{z} &= \boldsymbol{c}_B^\top B^{-1}\widetilde{\boldsymbol{b}} \\
&= -4\delta - 372
\end{aligned}
$$

This answers part 2.

1. For part 1, we set $\delta = -15$. Then since $\delta < -1/9$, we see that the basis changes since $\widetilde{\boldsymbol{b}}$ is now infeasible. In terms of the current basis, the perturbed problem is

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z'$	0	0	0	12	4	5	-312
x_1	1	0	0	1	-2	7	35
x_2	0	1	0	-4	9	-30	-134
x_3	0	0	1	19	-43	144	647

The reduced cost are unchanged, thus optimality conditions remain satisfied. We now use the dual simplex method to find an optimal solution: we use x_2 as a leaving variable and we use x_6 as an entering variable, and we obtain

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z'$	0	$1/6$	0	$34/3$	$11/2$	0	$-1003/3$
x_1	1	$7/30$	0	$1/15$	$1/10$	0	$56/15$
x_6	0	$-1/30$	0	$2/15$	$-3/10$	1	$67/15$
x_3	0	$24/5$	1	$-1/5$	$1/5$	0	$19/5$

This basis is optimal and feasible, so we've found an optimal solution, namely $x_1 = 56/15$, $x_2 = 67/15$, and $x_3 = 19/5$. This answers part 1.

4. Set $\widetilde{c}_3 = c_3 + \delta$ and set $\widetilde{\boldsymbol{c}}_B = \boldsymbol{c}_B + \Delta$ where $\Delta = (0, 0, \delta)^\top$. This change has no effect on the feasibility conditions ($B^{-1}\boldsymbol{b} \geq 0$), but it does affect the optimality condition ($\boldsymbol{c}_N - \widetilde{\boldsymbol{c}}_B^\top B^{-1}N \geq 0$). The optimality condition will remain satisfied as long as $\boldsymbol{c}_N - \widetilde{\boldsymbol{c}}_B^\top B^{-1}N \geq 0$, or equivalently, as long as

$$
\begin{aligned}
0 &\leq \boldsymbol{c}_N - \widetilde{\boldsymbol{c}}_B^\top B^{-1}N \\
&= (101, -87, -23 + \delta) \begin{pmatrix} 1 & -2 & 7 \\ -4 & 9 & -30 \\ 19 & -43 & 144 \end{pmatrix} \\
&= (19\delta + 12, 4 - 43\delta, 144\delta + 5)
\end{aligned}
$$

In particular, the feasibility conditions remains satisfied if and only if $\delta \not< -12/19$, $\delta \not> 4/43$, and $\delta \not< 5/144$, or in other words if and only if $\delta \in [5/144, 4/43]$.

3. For part 3, we set $\delta = 25$. Then since $\delta > 4/43$, we see that the current basis is no longer optimal. We apply the primal simplex method to the perturbed problem:

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z'$	0	0	0	12	4	5	-312
x_1	1	0	0	1	-2	7	35
x_2	0	1	0	-4	9	-30	-134
x_3	0	0	1	19	-43	144	647

The reduced cost are unchanged, thus optimality conditions remain satisfied. We now use the dual simplex method to find an optimal solution: we use x_2 as a leaving variable and we use x_6 as an entering variable, and we obtain

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z'$	0	$1/6$	0	$34/3$	$11/2$	0	$-1003/3$
x_1	1	$7/30$	0	$1/15$	$1/10$	0	$56/15$
x_6	0	$-1/30$	0	$2/15$	$-3/10$	1	$67/15$
x_3	0	$24/5$	1	$-1/5$	$1/5$	0	$19/5$

This basis is optimal and feasible, so we've found an optimal solution, namely $x_1 = 56/15$, $x_2 = 67/15$, and $x_3 = 19/5$. This answers part 1.