MATH 8100 Fall 2021 Homework 5 due November 1, 2021.

100 points (LPs part 1)

Guidance for writing your assignment:
a) make sure that your writing is clear and legible
b) start your answer to each part of a problem on a new page
c) wherever appropriate, underline or rewrite the final answer
d) show all work for full credit

Please make every effort to follow this guidance and facilitate the reading of your assignment. The assignments that do not follow this guidance will be returned.

1. (20 points) Reformulate the following linear program into Standard Form (SF).

$$\begin{array}{ll} \text{maximize} & x_1 - x_2 + 2x_3 \\ \text{s.t.} & 2x_1 + 3x_2 \leq 4 \\ & x_1 - x_3 \geq 2 \\ & x_1 + 2x_2 = 1 \\ & x_1 \text{ and } x_2 \text{ free, } x_3 \geq 0 \end{array}$$

For full credit, introduce as few new variables as possible and present the data of the final LP in the form:

x =
c =
A =
b =

2. Consider the following polyhedral set

$$X = \{ \mathbf{x} \in R^2 : -2x_1 + x_2 \leq 5,\ ½x_1 - x_2 \leq 2,\ x_1 \geq 0,\ x_2 \geq 0 \}$$

a) (3 points) Use a method of your choice and find the set of all extreme points (EPs) in X.

b) (10 points) Use an appropriate algebraic derivation an find the set of all recession directions in X.

c) (2 points) What recession directions that you found in part b) are extreme? Explain why.

d) (5 points) Apply the Representation Theorem and find a representation for the point $\mathbf{x} = (4, 6)^T \in X$.

e) (5 points) Is your representation unique? Explain why.

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3. Consider the linear program (LP) in R^2:

$$\text{minimize } \mathbf{c}^T\mathbf{x}, \text{ s.t. } \mathbf{x} \in X,$$

where X is defined as in question 2 above. Apply a method for solving LPs in R^2 and find the set of all optimal solutions to this LP for each cost vectors below. Write each set in an appropriate form.

a) (5 points) Let $\mathbf{c} = (1 \quad 1)^T$.

b) (5 points) Let $\mathbf{c} = (2 \quad -1)^T$.

c) (5 points) Let $\mathbf{c} = (0 \quad 1)^T$.

4. (20 points) Apply the Simplex Algorithm and find an optimal solution to the linear program. In every iteration select the most negative reduced cost.

$$\begin{array}{ll} \text{minimize} & -6x_1 - 14x_2 - 13x_3 \\ \text{s.t.} & x_1 + 4x_2 + 2x_3 \le 48 \\ & x_1 + 2x_2 + 4x_3 \le 60 \\ & x_i \ge 0, i = 1, 2, 3 \end{array}$$

5.

a) (10 points) Let $\mathbf{x}$ be feasible but not a basic feasible solution for $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$. Prove that the columns of A corresponding to the nonzero entries of $\mathbf{x}$ are linearly independent.

b) (10 points) Let $\mathbf{x}$ be a feasible point of $X = \{\mathbf{x} \in R^n: A\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ that is not an extreme point. Prove that there exists a vector $\mathbf{p} \in R^n$, $\mathbf{p} \ne \mathbf{0}$ such that

$$A\mathbf{p} = \mathbf{0}$$
$$p_i = 0 \text{ if } x_i = 0.$$

(Hint: Use the result from part a).)