OBSTRUCTIONS TO THE EXISTENCE OF MULTIPLICATIVE STRUCTURES ON MINIMAL FREE RESOLUTIONS

By LUCHEZAR L. AVRAMOV

Introduction. Let $f: R \to S$ be a finite local homomorphism of noetherian (unitary) local rings, inducing the identity map on their common residue field k. Let M be a finitely-generated S-module, and denote by F and P minimal R-free resolutions of R and M respectively. Then $F \otimes k = \mathrm{Tor}^R(S, k)$ has a natural structure of associative (skew-) commutative graded k-algebra, and $P \otimes k = \mathrm{Tor}^R(M, k)$ becomes a left $\mathrm{Tor}^R(S, k)$-module: both pairings are given by the homological $\pitchfork$-product of Cartan-Eilenberg [4, Chapter XI, Section 4]. Since the $\pitchfork$-product is constructed via comparison of resolutions, it is uniquely defined. In particular, if F comes equipped with a structure of DG (= differential graded) algebra, and P has a structure of DG F-module, then the induced pairings in homology coincide with the $\pitchfork$-product.

A very interesting case, when F carries a particularly simple DG algebra structure, is given by the Koszul resolution of the residue class ring of R modulo a regular sequence. In this situation Buchsbaum and Eisenbud have formulated the following:

Conjecture 1.2′ [3, p. 450]. Suppose $S = R/J$, where J is generated by an R-sequence, and F carries the standard exterior algebra structure. If I is an ideal containing J, and $M = R/I$, then P admits a structure of DG F-module.

The interest of this statement comes in particular from a beautiful remark [3, Proposition 1.4], that for ideals I which satisfy it, an important lower bound holds for the ranks of the syzygies of R/I (R being supposed a domain). The argument is extendable to non-cyclic modules (cf. Proposition 6.4.1 below) and a proof of the conjecture, stretched to finitely-generated modules, would give a solution to a problem of Horrocks [11, Problem 24] on the ranks of syzygies of modules of finite length over regular local rings.

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In a preprint version of their paper, Buchsbaum and Eisenbud conjectured the following much stronger statement holds:

Conjecture 1.2 [3, p. 453]. The minimal R-free resolution P of R/I admits a structure of associative skew-commutative DG algebra.

To see that 1.2′ is implied by 1.2 simply note that using the universal property of the exterior algebra, the map $R/J \to R/I$ can be lifted to a homomorphism of DG algebras $F \to P$.

Except for the complete intersection case, discussed above, and the case when I is the maximal ideal $\mathbf{m}$ of R, the last conjecture was proved [3, Proposition 1.3] to hold when $pd_R R/I \leq 3$. However, as the present author remarked [3, Note added in proof, p. 450], an example of Khinich of an artinian local ring of embedding dimension 4, whose Koszul complex contains an indecomposable Massey triple product (cf. [1, Appendix]), can be used to refute 1.2 in general: cf. Example 5.2.2 below.

Most recently, Kustin and Miller [12] proved that 1.2 does hold for height 4 *Gorenstein* ideals (in Gorenstein rings containing ½), and conjectured its validity for all Gorenstein ideals.

Using the results of this paper, we show the following:

Let R be an arbitrary local ring, and let m and n be integers subject to the conditions depth $R \geq n \geq m \geq 2$. If $n \geq 4$, there exist a perfect ideal I in R of grade n, and an R-regular sequence in I of length m, for which Conjecture 1.2′ *does not hold. If $n \geq 6$, then the ideal I can be chosen to be Gorenstein.*

A similar statement holds in the graded case.

In particular, there exist ideals I for which the minimal R-free resolution of R/I cannot support any associative DG algebra structure (even one which is not supposed to be skew-commutative). Thus the Kustin-Miller conjecture remains undecided only for codimension 5 Gorenstein ideals.

The examples are constructed in Section 2, first under generic circumstances (R is a graded polynomial ring over a field), and then shifted to arbitrary rings by appropriate specializations. The test that they do possess the required properties depends on showing the non-triviality of an obstruction $o^f(M)$, introduced in Section 1 in a general setup: S need not be a complete intersection in R, and M is not supposed cyclic. *The definition of $o^f(M)$ and the investigation of its properties constitute the* *main object of this paper.* The obstruction is studied using a very general change of rings spectral sequence of Eilenberg-Moore type, which is also of independent interest. It is constructed in Section 3, some relevant properties of the Eilenberg-Moore differential derived functors being summarized in a short Appendix. Section 4 contains the proofs of the theorems from the first section, while Section 5 gives a reinterpretation of the obstruction in terms of Massey products, and briefly treats a related (but weaker) obstruction to the existence of *DG* algebra structures on minimal resolutions of finite ring extensions. In the final Section 6 we consider some open questions, in particular the relevance of the obstruction theory to an attack of Horrocks' problem, and to a problem of Eisenbud [5, 6], concerning the structure of minimal resolutions of modules over local complete intersections.

Conventions and notations. Throughout this paper, R will denote either a local ring with maximal ideal $\mathbf{m}$ and residue field $k = R/\mathbf{m}$, or a graded noetherian ring $R = \oplus_{i\geq 0} R_i$, with $R_0 = k$ and irrelevant maximal homogeneous ideal $\mathbf{m} = \oplus_{i\geq 1} R_i$. Either of these situations will be referred to by the abbreviations $(R, \mathbf{m})$ or $(R, \mathbf{m}, k)$. Modules will be (usually) finitely-generated. In the graded case they will moreover be supposed graded, and all homomorphisms between them will respect the degree.

The results will normally be stated only for the local case, and it will be understood, that the precise graded analogue does hold. Occasionally, when additional information can be obtained in the graded case, it will be given in a separate statement with index g.

In objects graded by homological degree (e.g. $\mathrm{Tor}^R(M, N)$, resolutions, etc.), sums of elements of different degrees will never be taken. This means that (e.g.) $\mathrm{Tor}^R(M, N)$ denotes the collection of modules $\{\mathrm{Tor}_i{}^R(M, N)\}_{i\in \mathbf{Z}}$ rather than their direct sum; for more about this point of view cf. [15, Chapter VI]. Consequently, with the exception of zero, all elements of such a graded object have a well-defined degree. Commutativity with respect to the homological degree is always understood in the skew sense: $ab = (-1)^{\deg a \cdot \deg b} ba$, and $a^2 = 0$, when $\deg a$ is odd.

Finally all algebras—commutative, skew-commutative or neither—are always associative.

1. The obstruction and its properties. Denote by $\mathrm{Tor}_+{}^R(S, k)$ the kernel of the composition: $\mathrm{Tor}^R(S, k) \to \mathrm{Tor}_0{}^R(S, k) = S \otimes_R k \to k$.

The naturality of the $\pitchfork$-product, which we shall henceforth denote by a dot or simply by juxtaposition, yields a commutative diagram, in which the vertical maps are induced by the ring homomorphism $f: R \to S$:

$$\begin{array}{ccc} \mathrm{Tor}_+{}^R(S, k) \otimes \mathrm{Tor}^R(M, k) & \xrightarrow{\pitchfork} & \mathrm{Tor}^R(M, k) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_+{}^S(S, k) \otimes \mathrm{Tor}^S(M, k) & \xrightarrow{\pitchfork = 0} & \mathrm{Tor}^S(M, k) \end{array}$$

Hence there exists a canonical map of graded k-spaces:

$$\frac{\mathrm{Tor}^R(M, k)}{\mathrm{Tor}_+{}^R(S, k) \cdot \mathrm{Tor}^R(M, k)} \to \mathrm{Tor}^S(M, k).$$

Definition 1.1. The kernel of this map is denoted $o^f(M)$ and is called the obstruction to the existence of multiplicative structure (on the minimal R-free resolution of M).

Here is the result responsible for the denomination:

THEOREM 1.2. *Suppose the minimal R-free resolution F of S has a structure of DG algebra. If $o^f(M) \neq 0$, then no DG F-module structure exists on the minimal R-free resolution P of M.*

The obstruction possesses useful functorial properties. For the fixed field k denote by $\mathcal{C}_k$ the category, whose objects are pairs $(f: R \to S, M)$, with f a finite (hence local) homomorphism of local rings, inducing the identity on their common residue field k, and M a finitely-generated S-module. The maps of $\mathcal{C}_k$ are triples (g_1, g_2, g_3), where $g_1: R \to R'$ and $g: S \to S'$ are local ring homomorphisms, fitting in the commutative square:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S' \end{array}$$

while $g_3: M \to M'$ is a g_2-linear homomorphism of abelian groups.

Of the following two propositions, the first one is immediate from the naturality of the homological products, and the second one is proved in 3.3.

PROPOSITION 1.3. (Functoriality). *The obstruction $o^{-}(-)$ is a covariant functor from $\mathcal{C}_k$ to the category of graded k-vector spaces and k-linear maps of degree zero.*

Remark 1.3.g. In the graded case for each i the vector space $\operatorname{Tor}_i^R(M, k)$ is graded, as a subquotient of the i-th graded free module in some graded resolution of M. Thus $o^f(M)$ is in fact a bigraded vector space in this case. However, we shall have no occasion to use this second "interior" degree on the elements of the obstruction, so we suppress it from our notations.

PROPOSITION 1.4. (Flat base change). *Let $g: R \to R'$ be a local ring homomorphism, inducing the identity on the residue fields, and such that for $i > 0$:*

$$\operatorname{Tor}_i^R(R', S) = 0, \qquad \textit{and} \quad \operatorname{Tor}_i^R(R', M) = 0.$$

The triple $(g, g \otimes S, g \otimes M)$ induces an isomorphism:

$$o^f(M) \xrightarrow{\sim} o^{R' \otimes f}(R' \otimes M).$$

The next result shows, in particular, that if nontrivial obstructions exist, they already appear for modules of finite length.

THEOREM 1.5. *Let N be a submodule of M, contained in $\mathbf{m}^r M$. There exists a positive integer $r(M)$, depending only on M, such that for $r > r(M)$, the map $o^f(M) \to o^f(M/N)$ is injective.*

In the graded case this result admits an important sharpening. Let $X_1, \ldots, X_n$ be a set of indeterminates over k, which are in 1-1 correspondence with a basis of the graded vector space $\mathbf{m}/\mathbf{m}^2$, and have correspondingly assigned degrees. The R-module M becomes a graded $\tilde{R} = k[X_1, \ldots, X_n]$-module via the surjective map $\tilde{R} \to R$. Let $i(M)$ (resp. $s(M)$) denote the minimal (resp. maximal) among the degrees of generators of the modules in a minimal $\tilde{R}$-free resolution of M. It is easily seen that these integers are invariants of M, and that $i(M)$ equals the minimal degree of a non-zero homogeneous element of M (cf. also 6.1).

THEOREM 1.5.g. *Let N be a graded submodule of M. If $i(N) > s(M)$, the map $o^f(M) \to o^f(M/N)$ is injective.*

To conclude this section we list a few vanishing theorems for the obstruction. Possible implications for the structure of free resolutions are mentioned in 6.5.

PROPOSITION 1.6. *Let* $S = R/J$.

(1) *For any finitely-generated module* M, $o_i^f(M) = 0$ *if* $i \leq 2$.

(2) *Suppose that* $M = R/I$ *and* $J \cap \mathbf{m}I = \mathbf{m}J$, *i.e.* J *is generated by a subset of a minimal system of generators of* I. *Then* $o_i^f(M) = 0$ *for* $i \leq 3$.

THEOREM 1.7. *Let* M *be a finitely-generated* R*-module. There exists an integer* $t(M)$, *such that for any* $J \subset \mathbf{m}^{t(M)} \cap \operatorname{ann} M$, $o^{R\to R/J}(M) = 0$.

In the case of complete intersections the previous result can be considerably improved.

THEOREM 1.8. *Let* J *be an ideal generated by an* R*-regular sequence, and let* M *be a finitely-generated* R/J*-module. If* $J \subset \mathbf{m} \cdot \operatorname{ann} M$, *then* $o^{R\to R/J}(M) = 0$.

Remark 1.8.1. When J is generated by a regular sequence, $\operatorname{Tor}^R(S, k)$ is the exterior algebra on $\operatorname{Tor}_1{}^R(S, k)$ $(S = R/J)$, so that in this case, for any $i > 0$:

$$o_i{}^f(M) = \operatorname{Ker}\left(\frac{\operatorname{Tor}_i{}^R(M, k)}{\operatorname{Tor}_1{}^R(S, k)\cdot \operatorname{Tor}_{i-1}^R(M, k)} \to \operatorname{Tor}_i{}^S(M, k)\right).$$

The results of this section are proved in Section 4.

2. The examples.

2.1. *Rappel on Taylor resolutions* (cf. [7]). Let $R = k[X_1, \ldots, X_n]$, and let E denote the exterior algebra of a rank m free R-module. The elements of the standard basis of E are written $e_{i_1\ldots i_p}$ $(1 \leq i_1 < \cdots < i_p \leq m)$. If $M_1, \ldots, M_m$ are monomials in the indeterminates, then E becomes a free algebra resolution of $S = R/(M_1, \ldots, M_m)$, after introducing a differential by the formula:

$$de_{i_1\cdots i_p} = \sum_{j=1}^{p} (-1)^{j-1} \frac{M_{i_1\cdots i_p}}{M_{i_1\cdots \hat{i}_j\cdots i_p}}\, e_{i_1\cdots \hat{i}_j\cdots i_p}$$

and a new product by the formula:

$$e_{i_1 \cdots i_p} e_{j_1 \cdots j_q} = \frac{M_{i_1 \cdots i_p} M_{j_1 \cdots j_q}}{M_{i_1 \cdots i_p j_1 \cdots j_q}} e_{i_1 \cdots i_p} \wedge e_{j_1 \cdots j_q}$$

(here $M_{i_1 \cdots i_p}$ denotes the least common multiple of the monomials $M_{i_1}, \ldots, M_{i_p}$).

2.2. *Example* I. Let $R' = k[X_1, X_2, X_3, X_4]$, $M' = R'/I'$, $S' = R'/J$, $I' = (X_1{}^2, X_1X_2, X_2X_3, X_3X_4, X_4{}^2)$, $J = (X_1{}^2, X_4{}^2)$. We shall show the minimal R'-free resolution P' of R'/I' admits no DG module structure over the Koszul complex F resolving S'. According to Theorem 1.2, it is sufficient to establish the following claim:

$$\dim_k o_i{}^f(M') = \begin{cases} 0, & \text{if } i \neq 4; \\ 1, & \text{if } i = 4. \end{cases}$$

The vanishing of the obstruction in dimensions > 4 is a consequence of the fact that $pd_{R'} M' \leq 4$, while for $i < 4$ it follows from Proposition 1.6.2. In order to study the remaining dimension we use the isomorphism of k-algebras:

$$H(E \otimes k) \simeq \mathrm{Tor}^{R'}(R'/I', k) \simeq H(K),$$

where E is the Taylor resolution from 2.1, and K is the Koszul complex on x_1, x_2, x_3, x_4, x_j being the image of X_j in R'/I'. Using the explicit formulas of 2.1, one easily sees that

$$\mathrm{Tor}_4{}^{R'}(R'/I', k) \simeq H_4(E \otimes k)$$

is generated by the class of $e_{1245} \otimes 1$, and that

$$\mathrm{Tor}_1{}^{R'}(S', k) \cdot \mathrm{Tor}_3{}^{R'}(R'/I', k)$$
$$\subset \mathrm{Tor}_1{}^{R'}(R'/I', k) \cdot \mathrm{Tor}_3{}^{R'}(R'/I', k) = 0.$$

It follows from Remark 1.8.1 that our claim is equivalent to the nontriviality of $\mathrm{Ker}(\mathrm{Tor}_4{}^f(R'/I', k))$.

Now note that this map is induced by the inclusion of complexes

$$K \subset L = K\langle S_1, S_4; dS_1 = x_1T_1, dS_4 = x_4T_4\rangle$$

where the T_i's form a basis of K_1 such that $dT_i = x_i$, and the angular brackets denote adjunction of variables in the sense of Tate [19]. Indeed, L is the Tate-Zariski minimal algebra resolution of k over the complete intersection S'. Since $x_1x_4T_1T_2T_3T_4$ is a non-zero element of

$$Z_4(K) = H_4(K) = \mathrm{Tor}_4^{R'}(R'/I', k),$$

the formula:

$$x_1x_4T_1T_2T_3T_4 = d(x_4S_1T_2T_3T_4 + x_2S_1S_4T_3) \in L$$

proves our claim.

The idea of this example is derived from a construction of Jorgen Backelin, reproduced in Section 5 below.

2.3. *Example* II. In the first example the ideal is not perfect. We proceed to produce a perfect one. Keeping the notations of 2.2, we set $I_r' = I' + (X_2^r, X_3^r, X_1X_3^{r-1} - X_2^{r-1}X_4)$ i.e. I_r' is the homogeneous ideal of $k[X_1, \ldots, X_4]$ generated by the 8 elements:

$$X_1^2, X_2^r, X_3^r, X_4^2, X_1X_2, X_2X_3, X_3X_4, X_1X_3^{r-1} - X_2^{r-1}X_4. \quad (2.3.1)$$

It is immediately seen from the explicit form of the minimal R'-free resolution of R'/I', obtained by factoring out a homotopically trivial subcomplex of E, that $s(R'/I') = 6$. Hence from Proposition 1.5.g. and (2.2.1) we conclude:

PROPOSITION 2.3.2. *Let $R' = k[X_1, X_2, X_3, X_4]$, and let I_r' be the ideal, generated by the elements* (2.3.1). *Then I_r' is a $(X_1, \ldots, X_4)$-primary ideal, such that $o_4^{R' \to R'/(X_1^2, X_4^2)}(R'/I') \neq 0$ for $r \geq 7$.*

2.4. *Example* III. Building up from the previous example, we now construct a Gorenstein one. Some preparation is required.

Let T be a homomorphic image of S, and let N be a finitely-generated T-module. To a free presentation

$$T^q \xrightarrow{A} T^p \to N \to 0$$

with $p \times q$ matrix $A = (a_{ij})$ we associate the pair $(\tilde{R} \to \tilde{S}, \tilde{T})$ with $\tilde{R} = R[Y_1, \ldots, Y_p]$, $\tilde{S} = S[Y_1, \ldots, Y_p]$, and $\tilde{T} = T[Y_1, \ldots, Y_p]/U$ where U is the ideal generated by the q linear forms $\Sigma_{i=1}^{p} a_{ij}X_i$ and by all the monomials of degree two in the Y's. Note that $\tilde{T} = T \ltimes N$, the trivial extension of T by N. Since the pair $(f:R \to S, T)$ is a retract in $\mathcal{C}_k$ of the pair $(\tilde{f}:\tilde{R} \to \tilde{S}, \tilde{T})$ for the obvious maps, it follows from Proposition 1.3 that $o^f(T)$ is a direct summand of $o^{\tilde{f}}(\tilde{T})$.

We now apply these considerations to the rings R' and S' of 2.2 and $T = R'/I_r'$ where I_r' is generated by the elements (2.3.1). Set $N = \mathrm{Hom}_k(T, k)$; N is a graded R-module for the induced action and the grading

$$N_i = \{u:T \to k \mid u(T_j) = 0 \quad \text{for} \quad j \neq r - i + 1\}.$$

Denoting by $\mathbf{n}$ the maximal ideal of T, it is easily checked that $0:n = (x_1x_4, x_1x_3{}^{r-1})$, and that a minimal free presentation of N is given by the 2×5 matrix:

$$A = \begin{pmatrix} x_1x_4 & x_3{}^{r-1} & x_2{}^{r-1} & 0 & 0 \\ 0 & -x_4 & -x_1 & x_2 & x_3 \end{pmatrix}.$$

Let $\tilde{R} = k[X_1, X_2, X_3, X_4, Y_1, Y_2]$, $\deg X_i = \deg Y_1 = 1$, $\deg Y_2 = r - 1$, and let $\tilde{I}$ be the homogeneous ideal of $\tilde{R}$, generated by the 8 elements (2.3.1) and by 8 additional elements:

$$X_1X_4Y_1, X_2Y_2, X_3Y_2, X_2{}^{r-1}Y_1 - X_1Y_2,$$

$$X_3{}^{r-1}Y_2 - X_4Y_1, Y_1{}^2, Y_1Y_2, Y_2{}^2.$$

Then $\tilde{R}/\tilde{I} \simeq T \ltimes N$ as graded rings, hence for $r \geq 7$ one has $o^{\tilde{R} \to \tilde{R}/(X_1{}^2, X_4{}^2)}(\tilde{R}/\tilde{I}) \neq 0$, by the previous remarks and Proposition 2.3.2. Since the pairing $(T \ltimes N)_i \times (T \ltimes N)_{r-i+1} \to (T \ltimes N)_{r+1} \simeq k$ given by the multiplication in $T \ltimes N$ coincides with the non-degenerate k-bilinear map $(t, u) \times (t, u') \mapsto u(t') \cdot u'(t)$, $T \ltimes N$ is a graded Gorenstein ring, hence $\tilde{I}$ is a Gorenstein ideal.

For aesthetical reasons, one might prefer to have such an ideal in a graded ring, generated by elements of degree 1. To this end it suffices to consider the ring $R'' = k[X_1, \ldots, X_6]$, $\deg X_i = 1$, and the map of degree zero $g:\tilde{R} \to R''$, defined by the requirements $g(X_i) = X_i$

$(1 \leq i \leq 4)$, $g(Y_1) = X_5$, $g(Y_2) = X_6{}^{r-1}$. The extension of the ideal $\tilde{I}$ in R'' is generated by the elements (2.3.1) and the following 8 elements:

$$X_5{}^2, X_6{}^{2r-2}, X_1X_4X_5, X_2X_6{}^{r-1}, X_3X_6{}^{r-1}, X_5X_6{}^{r-1},$$

$$X_2{}^{r-1}X_5 - X_1X_6{}^{r-1}, X_3{}^{r-1}X_5 - X_4X_6{}^{r-1}. \tag{2.4.1}$$

Since R'' becomes a free R-module via g, Proposition 1.4 shows that

$$o^{R \to R/(X_1{}^2, X_4{}^2)}(R/I) \simeq o^{R'' \to R''/(X_1{}^2, X_4{}^2)}(R''/g(I)R'').$$

On the other hand, $R''/g(I)R''$ is a free extension of the Gorenstein ring R/I, with fibre $k[X_6]/(X_6{}^{r-1})$, hence is a Gorenstein ring itself.

Summing up, we have proved

PROPOSITION 2.4.2. *Let $R'' = k[X_1, \ldots, X_6]$, and let I_r'' be the ideal, generated by the elements* (2.3.1) *and* (2.4.1). *Then I_r'' is a Gorenstein* (*i.e. irreducible*) $(X_1, \ldots, X_6)$*-primary ideal, such that* $o_4{}^{R'' \to R''/(X_1{}^2, X_4{}^2)}(R''/I_r'') \neq 0$ *for* $r \geq 7$.

2.5. *Construction of the examples in an arbitrary local ring.* Let $(R, \mathbf{m})$ be a local ring of depth $d \geq l$ and denote by Q the localization of $R[X_1, \ldots, X_l]$ at the maximal ideal $m + (X_1, \ldots, X_l)$. For the moment we suppose that either $l = 4$ or $l = 6$, and depending on the case we write $\bar{R}$ for the localization at the irrelevant maximal ideal either of the ring R' of Proposition 2.3.2 or of the ring R'' of Proposition 2.4.2, writing $\bar{I}_r$ for the corresponding ideal. Consider the ideal I_Q of Q, generated by the elements (2.3.1) for $l = 4$, and by the elements (2.3.1) and (2.4.1) for $l = 6$. For a regular sequence $a_1, \ldots, a_l$ in R, denote by I_r the ideal generated by the appropriate expressions in the a_i's. There is a commutative diagram of local ring homomorphisms:

$$\begin{tikzcd}
\bar{R} \arrow[d] & Q \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\
\bar{R}/\bar{I}_r & Q/I \arrow[l] \arrow[r] & R/I_r
\end{tikzcd}$$

in which the squares are tensor product diagrams, $\bar{R} \leftarrow Q$ is factorization of $\mathbf{m}Q$, while $Q \to R$ is the unique map of local R-algebras sending X_i to a_i.

LEMMA 2.5.1. *For $i > 0$ one has:*

$$\mathrm{Tor}_i^Q(\bar{R}, Q/I_Q) = 0, \textit{ and } \mathrm{Tor}_i^Q(R, Q/I_Q) = 0.$$

Proof. It is easily seen that in case $l = 4$ (resp. $l = 6$) the homomorphic image of $R[X_1, \ldots, X_l]$ modulo the ideal generated by the elements (2.3.1) (resp. (2.3.1) and (2.4.1)) is a free R-module. The argument used in the proof of [2, Theorem 6.2] establishes the claim of the Lemma.

Since X_1^2, X_4^2 form a regular sequence both on $\bar{R}$ and on R, one sees that for $i > 0$:

$$\mathrm{Tor}_i^Q(\bar{R}, Q/(X_1^2, X_4^2)) = 0, \qquad \text{and} \quad \mathrm{Tor}_i^Q(R, Q/(X_1^2, X_4^2)) = 0.$$

Proposition 1.4 now gives the isomorphisms:

$$o^{\bar{R}\to\bar{R}/(X_1^2,X_4^2)}(R/I_r) \simeq o^{Q\to Q/(X_1^2,X_4^2)}(Q/I_Q)$$
$$\simeq o^{R\to R/(a_1^2,a_4^2)}(R/I_r).$$

Writing $\bar{\bar{R}}$ for $k[X_1, \ldots, X_l]$ and $\bar{\bar{I}}_r$ for the corresponding ideal, Proposition 1.4 applied to the localization map implies:

$$o^{\bar{\bar{R}}\to\bar{\bar{R}}/(X_1^2,X_4^2)}(\bar{\bar{R}}/\bar{\bar{I}}_r) \simeq o^{\bar{R}\to\bar{R}/(X_1^2,X_4^2)}(\bar{R}/\bar{I}_r).$$

In view of the above isomorphisms and Propositions 2.3.2 and 2.4.2, we can state

PROPOSITION 2.5.2. *For $l = 4$ (resp. $l = 6$) the ideal I_r of R is a perfect of grade 4 (resp. Gorenstein of grade 6), such that*

$$o_4^{R\to R/(a_1^2,a_4^2)}(R/I_r) \neq 0 \quad \textit{for} \quad r \geq 7.$$

Now take an integer $n : d \geq n \geq l$, and a regular R-sequence $a_1, \ldots, a_n$. Set $I = I_r + (a_{l+1}, \ldots, a_n)$, and $f : R \to R/(a_1^2, a_4^2) = S$. It is easily seen that $a_{l+1}, \ldots, a_n$ form a regular R/I_r-sequence. It follows that I is perfect or Gorenstein together with I_r, and also that:

$$\mathrm{Tor}^R(R/I, k) \simeq \mathrm{Tor}^R(R/I_r, k) \otimes_k D$$

$$\mathrm{Tor}^S(R/I, k) \simeq \mathrm{Tor}^S(R/I_r, k) \otimes_k D$$

where $D = \mathrm{Tor}^S(S/(a_{l+1}, \ldots, a_n), k)$ is the exterior algebra on the $(n - l)$-dimensional vector space D_1. By naturality we have $o^f(R/I) \simeq o^f(R/I_r) \otimes_k D$, hence Proposition 2.5.2 shows that $o_4{}^f(R/I) \neq 0$ for $r \geq 7$.

Finally, with R and I as above, let m be an integer between 2 and n, and extend $a_1{}^2$, $a_4{}^2$ to an R-regular sequence of length m in I. Denote by J the ideal generated by this sequence. If the minimal R-free resolution P of R/I admits a DG module structure over the Koszul complex F resolving R/J, such a structure will be transported to the Koszul complex F' resolving S, since there is a natural inclusion of DG algebras $F' \subset F$. However, this is impossible for $r \geq 7$, in view of Theorem 1.2 and the inequality $o_4{}^f(R/I) \neq 0$ established above.

We have finished the proof of

THEOREM 2.6. *Let R be a local ring and $a_1, \ldots, a_n$, $n \geq 4$ (resp. $n \geq 6$) be an R-regular sequence. Let I_r be the ideal generated by the elements* (2.3.1), *setting $X_i = a_i$, and by $a_5, \ldots, a_n$ (resp. by the elements* (2.3.1) *and* (2.4.1), *and by $a_7, \ldots, a_n$). Let J be the ideal, generated by an R-regular sequence in I of length m* $(2 \leq m \leq n)$ *starting with $a_1{}^2$, $a_4{}^2$.*

Then I_r is a perfect (resp. Gorenstein) ideal of grade n in R, such that the minimal R-free resolution of R/I_r does not admit for $r \geq 7$ any DG module structure over the Koszul complex resolving R/J.

The corresponding statement holds for a graded ring R and an R-sequence of homogeneous elements, whose degrees are subject to the conditions $(d_5 - d_1)/(d_6 - d_2) = (d_5 - d_4)/(d_6 - d_3) = r - 1$, $d_i = \deg a_i$ *(a zero in the denominator implying one in the numerator).*

3. Change of rings. In this section and the next one we shall freely use the properties of Eilenberg-Moore spectral sequences: cf. [17, 8], and the Appendix to this paper.

THEOREM 3.1. *Suppose given an augmented tensor product diagram of (not necessarily noetherian) commutative ring homomorphisms:*

$$\begin{tikzcd} & B \arrow[dr, "g'"] & & \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & & B \otimes_A C \arrow[r, "h"] & D \\ & C \arrow[ur, "f'"'] & & \end{tikzcd}$$

For an (arbitrary) B-module M, considering $\mathrm{Tor}^A(M, C)$ *as a* $\mathrm{Tor}^A(B, C)$*-module via the* $\frown$*-product, and D as a* $\mathrm{Tor}^A(B, C)$*-module via the A-algebra map* $\mathrm{Tor}^A(B, C) \to \mathrm{Tor}_0{}^A(B, C) \xrightarrow{h} D$*, there exists a first quadrant spectral sequence:*

$$E^2_{p,q} = \mathrm{Tor}_{p,q}^{\mathrm{Tor}^A(B,C)}(\mathrm{Tor}^A(M, C), D) \Rightarrow \mathrm{Tor}^B_{p+q}(M, D),$$

whose edge homomorphism e fits into a commutative diagram:

$$\begin{tikzcd}
\mathrm{Tor}^A(M, C) \arrow[d] \arrow[r] & \mathrm{Tor}^f(M, hf') \arrow[d] \\
\mathrm{Tor}^A(M, C) \otimes_{\mathrm{Tor}^A(B,C)} D \arrow[r, "e"'] & \mathrm{Tor}^B(M, D)
\end{tikzcd}$$

Proof. Let V be an A-free algebra resolution of C; denote by i the structure map $A \to V$, and by ϵ_V the augmentation $V \to C$. The homology of the B-free algebra $B \otimes_A V$ is equal to $\mathrm{Tor}^A(B, C)$. In particular, $B \otimes V$ can be extended, by adjunction of new variables [19], to a B-free resolution W of D; call j the inclusion $B \otimes V \to W$, which gives W the structure of a (left) DG V-module, and let $\epsilon_W : W \to D$ be the augmentation. We are now in presence of the commutative diagram of triples:

$$\begin{tikzcd}
(M, A, C) \arrow[r, "{(M,f,hf')}"] & (M, B, D) \\
(M, A, V) \arrow[u, "{(M,A,\epsilon_V)}"] \arrow[r, "{(M,f,j(f\otimes V))}"] \arrow[d, "{(M\otimes i,i,V)}"'] & (M, B, W) \arrow[u, "{(M,B,\epsilon_W)}"'] \arrow[d, "{(M\otimes i,B\otimes i,W)}"] \\
(M \otimes V, V, V) \arrow[r, "{(M\otimes V,f\otimes V,j(f\otimes V))}"] & (M \otimes V, B \otimes V, W)
\end{tikzcd}$$

Since ϵ_V and ϵ_W induce isomorphisms in homology, the vertical arrows of the upper square induce isomorphisms of the corresponding DG torsion functors: cf. Theorem A.3 of the Appendix. For the triples in the lower square, we consider the induced maps of "second" Eilenberg-Moore spectral sequences: cf. Theorem A.2. At the E^1 level, we obtain the commutative diagram:

$$\begin{array}{ccc} \operatorname{Tor}^{A}(M, V^{\#}) & \longrightarrow & \operatorname{Tor}^{B}(M, W^{\#}) \\ \downarrow & & \downarrow \\ \operatorname{Tor}^{V^{\#}}(M \otimes V^{\#}, V^{\#}) & \longrightarrow & \operatorname{Tor}^{B \otimes V^{\#}}(M \otimes V^{\#}, W^{\#}) \end{array}$$

where the torsion products are taken in the classical sense and # denotes the functor, forgetting the differentials. By construction, in each vertex of the diagram, the second argument is free as a module over the corresponding "ring argument," hence $\operatorname{Tor}_{q,*} = 0$ for $q \neq 0$; in case $q = 0$, both vertical maps are obviously bijective. From the convergence of the spectral sequences to the corresponding *DG* torsion functors, we conclude these also are pairwise isomorphic. Summing up the preceding discussion, we arrive to a commutative diagram, in which the torsion products of the lower line are taken in the sense of Eilenberg and Moore:

$$\begin{array}{ccc} \operatorname{Tor}^{A}(M, C) & \xrightarrow{\operatorname{Tor}^{f}(M, hf')} & \operatorname{Tor}^{B}(M, D) \\ \wr\| & & \wr\| \\ \operatorname{Tor}^{V}(M \otimes V, V) & \longrightarrow & \operatorname{Tor}^{B \otimes V}(M \otimes V, W) \end{array}$$

In order to establish the first claim of the Theorem, it is now sufficient to remark that by Theorem A.1 the triple $(M \otimes V, B \otimes V, W)$ defines a spectral sequence, converging to $\operatorname{Tor}^{B \otimes V}(M \otimes V, W)$, and with second term

$$E^2_{p,q} = \operatorname{Tor}_{p,q}^{H(B \otimes V)}(H(M \otimes V), H(W)) = \operatorname{Tor}_{p,q}^{\operatorname{Tor}^{A}(B,C)}(\operatorname{Tor}^{A}(M, C), D).$$

On the other hand, in view of the naturality of the edge homomorphisms, the last diagram above yields a commutative square:

$$\begin{array}{ccc} \operatorname{Tor}^{A}(M, C) & \xrightarrow{\operatorname{Tor}^{f}(M, hf')} & \operatorname{Tor}^{B}(M, D) \\ e \| & & \uparrow e \\ E^2_{0,*}(M \otimes V, V, V) & \longrightarrow & E^2_{0,*}(M \otimes V, B \otimes V, W) \\ \| & & \| \\ \operatorname{Tor}^{A}(M, C) & & \operatorname{Tor}^{A}(M, C) \otimes_{\operatorname{Tor}^{A}(B,C)} D. \end{array}$$

The Theorem is now completely proved.

In the sequel we shall use only a very special case of the previous result:

THEOREM 3.1.1. *Let* $(f\colon R \to S, M)$ *be an object of* $\mathcal{C}_k$ (cf. Proposition 1.3). *There exists a first quadrant spectral sequence:*

$$E^2_{p,q} = \mathrm{Tor}_{p,q}^{\mathrm{Tor}^R(S,k)}(M, k) \Rightarrow \mathrm{Tor}^S_{p+q}(M, k),$$

whose edge homomorphism e fits into the commutative diagram:

$$\begin{tikzcd}
\mathrm{Tor}^R(M, k) \arrow[d] \arrow[dr, "\mathrm{Tor}\, f(M,k)"] & \\
\dfrac{\mathrm{Tor}^R(M, k)}{\mathrm{Tor}_+{}^R(S, k)\cdot \mathrm{Tor}^R(M, k)} \arrow[r] & \mathrm{Tor}^S(M, k)
\end{tikzcd}$$

As a consequence of this result we give a new proof of a theorem of Shamash, which will be used in Sections 4 and 6.

PROPOSITION 3.2. (Shamash [18, Section 3, Theorem 1]). *Let M be a finitely-generated R-module over the local ring* (R, m, k), *and let* $a_1, \ldots, a_n$ *be an R-regular sequence in* $\mathbf{m}\cdot \mathrm{ann}\, M$. *Writing* $P_R{}^M(t) = \sum_{i\geq 0} \dim_k \mathrm{Tor}_i{}^R(M, k)t^i$ *for the Poincaré series of M over R, the following relation holds with* $S = R/(a_1, \ldots, a_n)$:

$$P_S{}^M(t) = P_R{}^M(t)(1 - t^2)^{-n}.$$

Proof. By induction, it is sufficient to treat the case $n = 1$, and we drop the subscript from a_1. Let X be an R-free algebra resolution of k, let $T_1, \ldots, T_m$ be a basis of X_1, and choose $z = \sum b_iT_i \in X_1$ such that $dz = a$ and $b_i \in \mathrm{ann}\, M$ $(1 \leq i \leq m)$. Then $\bar{z} = \sum \bar{b}_iT_i \in S \otimes X_1$ represents a basis element h of $\mathrm{Tor}_1{}^R(S, k) = k[h](h^2 = 0)$. By the choice of $\bar{z}$, $\bar{z}\cdot(M \otimes X) = 0$. Hence for any class $g \in \mathrm{Tor}^R(M, k) = H(M \otimes X)$, the Massey product [16] $\langle h, h, \ldots, h, g\rangle$ is defined and contains zero for any number of occurrences of h: if y is any representative cycle for g, then a defining system of this product consists of zeros of appropriate degrees, since $\bar{z}^2 = 0$ and $\bar{z}y = 0$. It follows from the Gugenheim-May description of the differentials in the Eilenberg-Moore spectral sequence [8, Theorem 5.6 and Remarks after Corollary 5.9], that the spectral sequence of Theorem 3.1.1 degenerates with $E^2 = E^\infty$. In order to compute $E^2 = \mathrm{Tor}^{k[h]}(\mathrm{Tor}^R(M, k), k)$ note that a minimal $k[h]$-free resolution of k is given by $Y = k[h]\langle U;\ dU = h,\ \mathrm{bideg}\ U = (1, 1)\rangle$. Multiplication by h annihilating $\mathrm{Tor}^R(M, k)$,

$$H(\mathrm{Tor}^R(M, k) \otimes_{k[h]} Y) = \mathrm{Tor}^R(M, k) \otimes_k k\langle U\rangle,$$

which immediately implies the required formula for the Poincaré series.

3.3. *Proof of Proposition* 1.4. This is elementary homological algebra.

Let P and F denote R-free resolutions of M and S respectively. Then by our assumptions $R' \otimes_R P$ and $R' \otimes_R F$ are R'-free resolutions of M' and S' respectively, hence

$$\mathrm{Tor}^{R'}(M', k) = H((R' \otimes_R P) \otimes_{R'} k) \simeq H(P \otimes_R k) = \mathrm{Tor}^R(M, k)$$

$$\mathrm{Tor}^{R'}(S', k) = H((R' \otimes_R F) \otimes_{R'} k) \simeq H(F \otimes_R k) = \mathrm{Tor}^R(S, k)$$

and it is easily checked that the first line is an isomorphism of k-vector spaces which is linear over the isomorphism of algebras from the second line.

Let now V be an R-free resolution of R'. Then $S \otimes_R V$ is an S-free resolution of S', hence

$$\mathrm{Tor}^S(S', M) = H((S \otimes_R V) \otimes_S M) \simeq H(V \otimes_R M) = \mathrm{Tor}^R(R', M) = M'.$$

Consequently, if Q is an S-free resolution of M, then $S' \otimes_S Q$ is an S'-free resolution of M' and

$$\mathrm{Tor}^{S'}(M', k) = H((S' \otimes_S Q) \otimes_{S'} k) \simeq H(Q \otimes_S k) = \mathrm{Tor}^S(M, k).$$

Now the claim of the Proposition follows from the naturality of all the isomorphisms involved.

4. Proofs of the results of Section 1

4.1. *Proof of Theorem* 1.2. Let $\{E^i\}_{i \geq 2}$ denote the spectral sequence of Theorem 3.1.1. The result obtained there for the edge homomorphism $e: E^2_{0,*} \to E^\infty_{0,*} \subset \mathrm{Tor}^S(M, k)$ compared to Definition 1.1 shows that $o^f(M) = \mathrm{Ker}\, e$. Hence the Theorem is an immediate consequence of the following

PROPOSITION 4.1.1. *Suppose the minimal R- free resolution F of S has a structure of DG algebra, and the minimal R-free resolution P of M has a structure of DG F-module. Then the spectral sequence of Theorem* 3.1.1 *degenerates with* $E^2 = E^\infty$.

Proof. Consider the maps of triples induced by the augmentations of F, of P, and of an R-free algebra resolution X of k:

$$(M \otimes X, S \otimes X, k) \leftarrow (P \otimes X, F \otimes X, k) \to (P \otimes k, F \otimes k, k).$$

Since all the maps involved induce isomorphisms in homology, the first Eilenberg-Moore spectral sequences (cf. Theorem A.1) are isomorphic from the second term on. But the spectral sequence defined by the triple on the left is the one of Theorem 3.1.1, while the spectral sequence defined by the triple on the right trivially satisfies $E^2 = E^\infty$, the minimality of both P and F implying that both $P \otimes k$ and $F \otimes k$ have zero differentials.

4.2. *Proof of Theorem* 1.5. The map $o^f(M) \to o^f(M/N)$ is the vector space homomorphism induced by the commutative diagram:

$$\begin{array}{ccccccc}
0 & \to & o^f(M) & \to & \dfrac{\mathrm{Tor}^R(M,k)}{\mathrm{Tor}_+{}^R(S,k)\cdot \mathrm{Tor}^R(M,k)} & \to & \mathrm{Tor}^S(M,k) \\
 & & & & \downarrow & & \downarrow \\
0 & \to & o^f(M/N) & \to & \dfrac{\mathrm{Tor}^R(M/N,k)}{\mathrm{Tor}_+{}^R(S,k)\cdot \mathrm{Tor}^R(M/N,k)} & \to & \mathrm{Tor}^S(M/N,k)
\end{array}$$

in which the vertical map is obtained by tensoring over $\mathrm{Tor}^R(S, k)$ with $k = \mathrm{Tor}^R(S, k)/\mathrm{Tor}_+{}^R(S, k)$ the natural homomorphism $\mathrm{Tor}^R(M, k) \to \mathrm{Tor}^R(M/N, k)$. Now apply:

LEMMA 4.2.1. *Let N be a submodule of M, contained in* $\mathbf{m}^r M$. *There exists a positive integer* $r(M)$, *depending only on M, such that for* $r > r(M)$ *the natural map* $\mathrm{Tor}^R(M, k) \to \mathrm{Tor}^R(M/N, k)$ *is a split monomorphism of* $\mathrm{Tor}^R(S, k)$*-modules.*

Proof. Let K be the Koszul complex on a minimal set of generators of $\mathbf{m}$. By the Artin-Rees lemma there is an integer $r(M)$ such that for $r > r(M)$ the equalities

$$\mathbf{m}^{r-i}(M \otimes K_i) \cap Z_i(M \otimes K) = \mathbf{m}(\mathbf{m}^{r-i-1}(M \otimes K_i) \cap Z_i(M \otimes K))$$

hold for all values of i between zero and rank K_1. We suppose from now on that r satisfies the inequality $r > r(M)$.

Let Q denote the subcomplex of $M \otimes K$, defined by setting $Q_i = \mathbf{m}^{r-i}(M \otimes K_i)$. Since the homology of the complexes $\mathbf{m}^j(M \otimes K)$ is annihilated by $\mathbf{m}$ for every $j \geq 0$, one has, by the choice of r, the relations:

$$\begin{aligned} Z_i(Q) &= Z_i(\mathbf{m}^{r-i}(M \otimes K)) = \mathbf{m}^{r-i}(M \otimes K) \cap Z_i(M \otimes K) \\ &= \mathbf{m}(\mathbf{m}^{r-i-1}(M \otimes K) \cap Z_i(M \otimes K)) \\ &= \mathbf{m}(Z_i(\mathbf{m}^{r-i-1}(M \otimes K))) \subset B_i(\mathbf{m}^{r-i-1}(M \otimes K)) \\ &= B_i(Q). \end{aligned}$$

It follows that the natural projection

$$p : M \otimes K \to (M \otimes K)/Q$$

induces an isomorphism in homology.

By definition, we have the inclusions of *DG* K-modules $N \otimes K \subset Q \subset M \otimes K$, hence, viewing k as a trivial *DG* K-module via the augmentation $K \to k$, there are maps of triples:

$$(M \otimes K, K, k) \to ((M \otimes K)/(N \otimes K), K, k) \to ((M \otimes K)/Q, K, k)$$

It follows from Theorems A.3 and A.4 that the induced map

$$\operatorname{Tor}^K(M \otimes K, k) \to \operatorname{Tor}^K((M/N) \otimes K, k)$$

is a split monomorphism of $\operatorname{Tor}^K(S \otimes K, k)$-modules.

On the other hand, consider the map of triples $(—, R, k) \to (— \otimes_R K, K, k)$, which is functorial in the R-module argument. It induces a map of the second Eilenberg-Moore spectral sequences (cf. Theorem A.2):

$$E^1_{p,q}(-, R, k) = \mathrm{Tor}^R_{p,q}(-, k) \to \mathrm{Tor}^{K^\#}_{p,q}(- \otimes_R K^\#, k)$$

$$= E^1_{p,q}(- \otimes_R K, K, k).$$

Since $K^\#$ is free as an R-module, this map is an isomorphism by elementary homological algebra, and in particular we see that, for reasons of convergence, there is a functorial isomorphism:

$$\mathrm{Tor}^R(-, k) \xrightarrow{\sim} \mathrm{Tor}^K(- \otimes_R K, k).$$

This finishes the proof of the lemma.

Remark 4.2.2. The injectivity of the map

$$\mathrm{Tor}^R(M, k) \to \mathrm{Tor}^R(M/N, k)$$

for $N \subset \mathbf{m}^r M$ and for large values of r has been established in [2, Theorem A.4], and this result has been generalized by Levin [14, Lemma 2]. However, neither proof gives the $\mathrm{Tor}^R(S, k)$-splitting of this map, which is essential in the proof of Theorem 1.5.

4.2.g. *Proof of Theorem* 1.5.g. By the remarks made in the proof of the local version, it is sufficient to establish

LEMMA 4.2.2.g. *Let N and M be as in Theorem* 1.5.g. *Then the natural map* $\mathrm{Tor}^R(M, k) \to \mathrm{Tor}^R(M/N, k)$ *is a split monomorphism of* (*bigraded*) $\mathrm{Tor}^R(S, k)$*-modules.*

Proof. Let $t_1, \ldots, t_m$ be a minimal system of homogeneous generators of the graded k-algebra R. Denote by K the Koszul complex on $t_1, \ldots, t_m$, bigraded by the requirement that K_1 has a basis $T_1, \ldots, T_m$ with bideg $T_i = (1, \deg t_i)$. Then the differential d of K, defined by setting $dT_i = t_i$, turns K into a complex of graded R-modules (and maps of degree zero). In particular, with the $\tilde{R}$ introduced in Section 1 (cf. the paragraph preceding the statement of Theorem 1.5.g.), there is an isomorphism of bigraded k-vector spaces $\mathrm{Tor}^{\tilde{R}}(M, k) \simeq H(M \otimes_R K)$. It follows that $H_{ij}(M \otimes K) = 0$ for all i when $j > s(M)$.

Let Q_i be the graded submodule of $(M \otimes_R K_i)$, generated by the homogeneous elements of degree $>s(M)$. The Q_i's form a graded subcomplex of $M \otimes K$. By the first paragraph of the proof, $H_i(Q) = 0$ for all i, hence the natural projection

$$M \otimes_R K \to (M \otimes_R K)/Q$$

induces an isomorphism in homology. Remarking that our assumption on N implies $N \otimes_R K \subset Q$, we can finish the proof as in the local case.

4.3. *Proof of Proposition* 1.6. As in 4.1, identify $o_i^f(M)$ with the kernel of the edge homomorphism $e: E^2_{0,i} \to E^\infty_{0,i}$ of a spectral sequence having

$$E^2_{p,q} = \operatorname{Tor}^{\operatorname{Tor}^R(S,k)}_{p,q}(\operatorname{Tor}^R(M, k), k).$$

Since $\operatorname{Tor}^R(R/J, k)$ is a connected k-algebra, $E^2_{p,q} = 0$ for $p > q$. It follows that $E^2_{0,i} = E^\infty_{0,i}$ for $i \leq 2$, hence the first assertion.

For the proof of the second statement, note that the hypothesis is equivalent to the injectivity in degrees zero and one of the k-algebra homomorphism $\operatorname{Tor}^R(S, k) \to \operatorname{Tor}^R(R/I, k)$. Consequently the induced map $\operatorname{Tor}^{\operatorname{Tor}^R(S,k)}_{p,q}(\operatorname{Tor}^R(S, k), k) \to E^2_{p,q}$ is bijective for $p \geq q$, hence $E^2_{p,q} = 0$ for $p \geq q$, $(p, q) \neq (0, 0)$, and this implies the equalities $E^2_{0,i} = E^\infty_{0,i}$ for $i \leq 3$.

4.4. *Proof of Theorem* 1.7. Let X be a minimal R-algebra resolution of k, and denote by $T(\ )$ the tensor algebra functor. In [14, Proof of Theorem 2], Levin has shown the existence of an integer $t'(M)$, such that for each $t > t'(M)$ there exists a graded free R-module F (depending on t) and a degree -1 R-linear map $\eta: F \to \mathbf{m}^{t-1}X$, for which the graded free R-module $Y = X/\mathbf{m}^t X \otimes_R T(F)$ equipped with a differential defined by:

$$d(x \otimes f_1 \otimes \cdots \otimes f_r) = dx \otimes f_1 \otimes \cdots \otimes f_r$$
$$+ (-1)^{\deg x} x\eta(f_1) \otimes f_2 \otimes \cdots \otimes f_r$$

becomes a minimal $R/\mathbf{m}^t$-free resolution of k. It follows that the obvious map $M/\mathbf{m}^{t-1}M \otimes_R X \to M/\mathbf{m}^{t-1}M \otimes_{R/\mathbf{m}^t} Y$ induces an injective homomorphism

$$\operatorname{Tor}^R(M/\mathbf{m}^{t-1}M, k) \to \operatorname{Tor}^{R/\mathbf{m}^t}(M/\mathbf{m}^{t-1}M, k).$$

Let $r(M)$ be the integer of Lemma 4.2.1 and set

$$t(M) = \max(r(M), t'(M)) + 1.$$

Consider the commutative diagram, defined for an ideal $J \subset \mathbf{m}^t \cap \operatorname{ann} M$, with $t > t(M)$:

$$\begin{tikzcd}
\operatorname{Tor}^R(M, k) \arrow[r] \arrow[d] & \operatorname{Tor}^R(M/\mathbf{m}^{t-1}M, k) \arrow[r] \arrow[d] & \operatorname{Tor}^{R/\mathbf{m}^t}(M/\mathbf{m}^{t-1}M, k) \\
\operatorname{Tor}^{R/J}(M, k) \arrow[r] & \operatorname{Tor}^{R/J}(M/\mathbf{m}^{t-1}M, k) \arrow[ur] &
\end{tikzcd}$$

In the upper line, the left-hand map is injective by Lemma 4.2.1, and the right-hand one is injective by the previous discussion. Hence the left vertical map is injective, which immediately implies $o^{R\to R/J}(M) = 0$ by the definition of the obstruction.

4.5. *Proof of Theorem* 1.8. Denote by D the free divided powers algebra on the S-free module J/J^2, whose elements are assigned degree 2. It is shown in [18, Section 3] and [5, Theorem 7.2] that the differential of $P \otimes_R S$ can be extended to a differential of $Q = P \otimes_R D$, transforming Q into an S-free resolution of M.

In particular, $\operatorname{Tor}^f(M, k)\colon \operatorname{Tor}^R(M, k) \to \operatorname{Tor}^S(M, k)$ is induced by the inclusion of complexes $P \otimes_R k \hookrightarrow Q \otimes_S k$. Noticing that

$$\sum_{i\geq 0} \dim_k(Q \otimes k)_i t^i = P_R{}^M(t)(1 - t^2)^{-1},$$

it follows from the theorem of Shamash (cf. Proposition 3.2 above) that the assumption $J \subset \mathbf{m} \cdot \operatorname{ann} M$ implies the minimality of Q, hence $\operatorname{Tor}^f(M, k)$ is injective, hence $o^{R\to R/J}(M) = 0$.

5. Massey products and obstructions.

5.1. *Interpretation of $o^f(M)$ in terms of matric Massey products.* Let X be an algebra resolution of k over R. Consider the subset of all elements of $H(M \otimes_R X) = \operatorname{Tor}^R(M, k)$, which can be expressed as a value of some Massey product $\langle A_1, \ldots, A_n\rangle$, where A_1 is a row matrix with values in $H(M \otimes X)$, A_i are matrices with entries in $H(S \otimes X)$ for $2 \leq i \leq n - 1$, and A_n is a column matrix with entries in $H_+(S \otimes X)$.

This is a vector subspace of $H(M \otimes X)$: cf. May's paper [16], to which we refer for the definition and basic properties of matric Massey products. Using some results of May, it is easily shown that as a subspace of $\mathrm{Tor}^R(M, k)$, this set can be obtained from any algebra resolution of k (compare [1, Proposition 2.1]). This remark justifies the notation $M(\mathrm{Tor}^R(S, k), \mathrm{Tor}^R(M, k))$, which we shall use to denote the subspace of Massey decomposables of $\mathrm{Tor}^R(M, k)$.

We now apply Theorem 3.1.1 to give a quick proof of an important result of Levin; for the definition of the category $\mathcal{C}_k$ cf. Proposition 1.3.

PROPOSITION 5.1.1. (Levin, [13, Theorem 3.8]). *Let $(f: R \to S, M)$ be an object of $\mathcal{C}_k$. Then*

$$\mathrm{Ker}(\mathrm{Tor}^f(M, k): \mathrm{Tor}^R(M, k) \to \mathrm{Tor}^S(M, k)) = M(\mathrm{Tor}^R(S, k), \mathrm{Tor}^R(M, k)).$$

Proof. Take an R-free DG algebra F', resolving S, and a free F'-module P', resolving M (such objects always exist). Consider the map of triples:

$$(M \otimes X, S \otimes X, k) \leftarrow (P' \otimes X, F' \otimes X, k) \to (P' \otimes k, F' \otimes k, k)$$

As in the proof of Proposition 4.1.1, it is easily seen that these maps induce isomorphisms of the corresponding Eilenberg-Moore functors. The triple on the right satisfies the hypotheses of Gugenheim and May's theorem (cf. Theorem A.5), hence the subspace of Massey decomposables of the $H(F' \otimes k)$-module $H(P' \otimes k)$ coincides with Ker π, where π is the composition of the natural projection

$$H(P' \otimes k) \to \frac{H(P' \otimes k)}{H_+(F' \otimes k) \cdot H(P' \otimes k)} = E^2_{0,*}(P' \otimes k, F' \otimes k, k)$$

with the edge homomorphism $e: E^2_{0,*} \to \mathrm{Tor}^{F' \otimes k}(P' \otimes k, k)$. Theorem 3.1.1 shows that $\pi = \mathrm{Tor}^f(M, k)$, which establishes our claim.

COROLLARY 5.1.2.

$$o^f(M) = \frac{M(\mathrm{Tor}^R(S, k), \mathrm{Tor}^R(M, k))}{\mathrm{Tor}_+{}^R(S, k) \cdot \mathrm{Tor}^R(M, k)}.$$

In view of the last formula Theorem 1.2 admits the following reformulation:

THEOREM 5.1.3. *Suppose the minimal R-free resolution F of S has a structure of DG algebra. If an element of some minimal set of generators of the graded* $\operatorname{Tor}^R(S, k)$*-module* $\operatorname{Tor}^R(M, k)$ *is decomposable in terms of matric Massey products, then the minimal R-free resolution F of M admits no DG F-module structure.*

Example 5.1.4. Let K be the Koszul complex on $x_1, \ldots, x_4 \in R'/I'$, with the ring and ideal of Example 2.2. Then $\operatorname{Tor}^{R'}(R'/I', k) = H(K)$, and the triple Massey product $\langle [x_1T_1], [x_3T_2], [x_4T_4] \rangle$ contains the element $z = [x_1x_4T_1T_2T_3T_4]$, for the defining system

$$a_{12} = x_1T_1T_2T_3,\, a_{23} = 0$$

(square brackets denote homology classes). Since $[x_1T_1] \cdot H_3(K) = 0 = [x_4T_4] \cdot H_3(K)$, z participates in a minimal set of generators of $H(K)$ as a $\operatorname{Tor}^{R'}(R'/(X_1{}^2, X_4{}^2), k)$-module, the last algebra being identified to the subalgebra of $H(K)$ generated by $[x_1T_1]$ and $[x_4T_4]$.

5.2. *An obstruction to the existence of DG algebra structures on the minimal resolution of a finite algebra.*

Let $f: R \to S$ be a finite homomorphism, which is supposed small in the sense of [2]: $\operatorname{Tor}^f(k, k)$ is injective, or, equivalently, the *DG* algebra $S \otimes X$ can be extended to a minimal S-free resolution Y of k (X denotes, as usual, the minimal R-algebra resolution of k). Note that when R is regular f is small if and only if $\operatorname{Ker} f \subset \mathbf{m}^2$. Since any cycle z of $\operatorname{Ker}(S \otimes X \to k)$ becomes a boundary in Y, say $z = -dy$, we can define a degree 1 k-linear map:

$$\sigma: \operatorname{Tor}_+{}^R(S, k) \to \operatorname{Tor}^S(k, k) \otimes_{\operatorname{Tor}^R(k,k)} k$$

by sending z to $1 \otimes y$; σ is called the suspension, and it is shown in [2, Proposition 2.6], that the Gugenheim-May theory can be applied to establish the equality:

$$\operatorname{Ker} \sigma = M(\operatorname{Tor}^R(S, k)),$$

where on the right we have the set of all elements of $\operatorname{Tor}_+{}^R(S, k)$, which decompose in terms of matric Massey products.

On the other hand, σ factors through the edge homomorphism $E^2_{1,*} \to \mathrm{Tor}^S(k, k) \otimes_{\mathrm{Tor}^R(k,k)} k$ of a spectral sequence with second term $E^2_{p,q} = \mathrm{Tor}_{p,q}^{\mathrm{Tor}^R(S,k)}(k, k)$, constructed in [2, Theorem 3.2]. It follows that the equality $\mathrm{Ker}\ \sigma = \mathrm{Tor}_+{}^R(S, k)^2$ holds in case the spectral sequence degenerates with $E^2 = E^\infty$. That this does happen when F admits a *DG R*-algebra structure is shown in [2, Corollary 3.3]. We have proved:

PROPOSITION 5.2.1. *Assume $f: R \to S$ is a finite small homomorphism. If some matric Massey product has a value, not contained in* $\mathrm{Tor}_+{}^R(S, k)^2$, *then the minimal R-free resolution of S admits no structure of DG R-algebra.*

Example 5.2.2. The first example of an indecomposable Massey product was given by Khinich in the Appendix to [1], where he shows that with $R = k[\![X_1, X_2, X_3, X_4]\!]$ and

$$I = (X_1{}^3, X_2{}^3, X_3{}^3 - X_1X_2{}^2, X_1{}^2X_3{}^2, X_1X_2X_3{}^2, X_2{}^2X_4, X_4{}^2)$$

the triple product $\langle [x_1{}^2T_1], [x_2{}^2T_2], [x_4T_4] \rangle$ is not contained in $H_1(K) \cdot H_3(K) + H_2(K)^2$, K being the Koszul complex of R/I. In particular, Proposition 5.2.1 shows that this is a counter-example to the Buchsbaum-Eisenbud Conjecture 1.2 from the Introduction.

Example 5.2.3. Backelin has constructed a simpler example with $M(\mathrm{Tor}^R(S, k)) \neq \mathrm{Tor}_+{}^R(S, k)^2$: take R as above,

$$I = (X_1{}^2, X_1X_2{}^2X_3, X_2{}^2X_3{}^2, X_2X_3{}^2X_4, X_4{}^2),$$

$S = R/I$. Then

$$\langle [x_1T_1], [x_2x_3{}^2T_2], [x_4T_4] \rangle \not\subset \mathrm{Tor}_1{}^R(S, k) \cdot \mathrm{Tor}_3{}^R(S, k) + \mathrm{Tor}_2{}^R(S, k)^2,$$

as can be seen by using Taylor resolutions: cf. 2.1.

I am grateful to Jorgen Backelin for the permission to reproduce his example here.

Remark 5.2.4. The obstruction given by Proposition 5.2.1 is considerably weaker than the one furnished by the appearance of non-vanishing $o^{\bar{f}}(S)$, for some $\bar{f}:R \to \bar{S}$, $g:\bar{S} \to S$ with $g\bar{f} = f$. For instance, in Example 2.2 we have (in the notations used there), the equality $e_{1245} \otimes 1 = (e_{12} \otimes 1)(e_{45} \otimes 1)$, hence $M(\operatorname{Tor}^{R'}(M', k)) = \operatorname{Tor}_+{}^{R'}(M', k)^2$, but the minimal resolution of M' over R' admits no DG algebra structure, because of the non-vanishing of $o_4{}^{R' \to R'/(X_1{}^2, X_4{}^2)}(M')$.

6. Concluding remarks and some open questions. In this section we have assembled some questions stemming from the preceding theory, and some indications for possible further applications.

6.1. *An invariant of graded modules.* Let M be a finitely-generated graded module over the polynomial ring R (which is generated over k by elements of positive degree). Let F_i, $0 \leq i \leq n$ be the free modules in a (graded) minimal R-free resolution of M. As in Proposition 1.5.g., denote by $i(M)$ (resp. $s(M)$) the minimal degree of a homogeneous element of M (resp. the maximal degree of a homogeneous generator of some F_i). From the condition $R_0 = k$ follow the relations: $i(M) = i(F_0) < i(F_1) < \cdots < i(F_n) \leq s(M)$.

What can be said about the relations between the $s(F_i)$? In particular, when does for $s(M) = \max s(F_i)$ hold the equality $s(M) = s(F_n)$?

We note that $s(F_i) < s(F_{i+1})$ always holds for perfect modules, since the $F_i{}^* = \operatorname{Hom}_R(F_i, R)$ are, with the usual grading, free modules of a minimal R-free resolution of $\operatorname{Ext}_R{}^n(M, R)$, and for any i between 0 and n there is equality $s(F_i) = -i(F^*_{n-i})$. The ring M' in Example 2.2 is an instance of a non-perfect module with the same property.

6.2. *Three homological conditions on a local ring.* Denote by K the Koszul complex on a minimal set of generators of the maximal ideal of the local ring $(S, \mathbf{m}, k)$. In [1, Theorem 5.1] a spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^{H(K)}_{p,q}(k, k) \Rightarrow \operatorname{Tor}^S(k, k) \otimes_{K \otimes k} k \qquad (6.2.1)$$

was constructed (cf. also [2, Remarks 2.3]). In particular, there is a coefficientwise inequality $\ll$ of formal power series:

$$P_S{}^k(t) \ll P^k_{H(K)}(t)(1 + t)^n,$$

where n is the embedding dimension $\dim_k \mathbf{m}/\mathbf{m}^2$ of S and

$$P^k_{H(K)} = \sum_{m \geq 0} \sum_{p+q=m} \dim_k(\operatorname{Tor}^{H(K)}_{p,q}(k, k))t^m.$$

Completing S if necessary, we can assume it is a homomorphic image of a regular local ring R of the same embedding dimension. Consider the following conditions on S:

(a) the minimal R-free resolution of S admits a DG algebra structure;

(b) the spectral sequence (6.2.1) degenerates with $E^2 = E^\infty$;

(b′) $P_S{}^k(t) = P^k_{H(K)}(t)(1 + t)^n$;

(c) $\operatorname{Ker} \sigma = (H_+(K))^2$.

The implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (b′) $\Rightarrow$ (c) hold. Indeed, (a) $\Rightarrow$ (b) follows from [2, Corollary 3.3]; (b) $\Leftrightarrow$ (b′) is clear; (b) $\Rightarrow$ (c) is noted in 5.2.

On the other hand (a) $\Rightarrow$ (c) cannot be reversed, as is shown in Remark 5.2.4. I do not know if (b) holds in this case; to show it does, it is sufficient, according to Fröberg's formula for the Poincaré series of a regular local ring modulo an ideal generated by quadratic monomials, to prove that for $S = R'/I'$ of Example 2.2 there is equality $P^k_{H(K)}(t) = (1 - 3t + t^2 + t^3)^{-1}(1 + t)^{-3}$.

Problem 6.2.2. Find examples where one of the implications (a) $\Leftarrow$ (b) and (b) $\Leftarrow$ (c) holds and the other does not.

In view of the topological examples of [10, 8.13], such local rings very probably do exist.

6.3. *Notes on a conjecture of Eisenbud.* Let J be an ideal of R, which is generated by an R-regular sequence of length m. Gulliksen [9] has shown that $\operatorname{Tor}^S(—, —)$ has a structure of graded module over the ring $S[Y_1, \ldots, Y_m]$, functorial in both arguments, where the Y_i's are algebraically independent over S and are assigned degree -2. Eisenbud [5] (cf. also [6]) has produced a new proof of this result, by showing that every complex of free S-modules can be given a structure of "module up to homotopy" over $S[Y_1, \ldots, Y_m]$. Furthermore, he has proved that every S-module M has a resolution Q of the form described in 4.5, which is a DG module over $S[Y_1, \ldots, Y_m]$ [5, Theorem 7.2], and has conjectured, that such a structure is always carried by the minimal S-free resolution [5, p. 4]. As we have noted in the proof of Theorem 1.8, the condition $J \subset \mathbf{m} \cdot \operatorname{ann} M$ is sufficient, by Shamash's theorem, to ensure the minimality of Q, hence Eisenbud's conjecture holds in this case. However, in view of the results of this paper, we are led to make the following

Conjecture 6.3.1. There is an obstruction to the existence of an $S[Y_1, \ldots, Y_m]$-module structure on the minimal resolution of a finitely-generated S-module M, which can be defined in terms of the induced $k[Y_1, \ldots, Y_m]$-module structure on $\mathrm{Ext}_S(M, k) = (\mathrm{Tor}^R(M, k))^*$, and which is in general different from zero, when $J \not\subset \mathbf{m} \cdot \mathrm{ann}\, M$.

Of course, the case of interest is when the regular sequence generating J lies in $\mathbf{m}^2$.

6.4. *Multiplicative structures and ranks of syzygies.* We prove an extension from cyclic to finitely-generated modules of a result of Buchsbaum and Eisenbud [4, Proposition 1.4]. Only a slight rearrangement of their proof is required.

PROPOSITION 6.4.1. *Let M be a finitely-generated module over the integral domain R, and let $x_1, \ldots, x_n$ be a maximal R-regular sequence in* ann M. *Suppose the minimal R-free resolution P of M admits a DG module structure over the Koszul complex F resolving*

$$S = R/(x_1, \ldots, x_n).$$

Then there exists an injective map of complexes $F \to P$.

COROLLARY 6.4.2. *For the i-th syzygy $M_i = \mathrm{Im}(d_i : P_i \to P_{i-1})$ the inequality*

$$\mathrm{rank}\, M_i \geq \binom{n-1}{i-1}$$

holds for $1 \leq i \leq n$.

Proof of the Proposition. Consider an exact sequence

$$0 \to N \to S^m \xrightarrow{p} M \to 0$$

and the induced cohomology exact sequence:

$$\mathrm{Ext}_R{}^{n-1}(N, R) \to \mathrm{Ext}_R{}^n(M, R) \xrightarrow{p^n} \mathrm{Ext}_R{}^n(S^m, R)$$

Since $\mathrm{ann}\, N \supset \mathrm{ann}\, S^m = (x_1, \ldots, x_m)$, the leftmost term vanishes. Since $\mathrm{grade}(\mathrm{ann}\, M) = n$, $\mathrm{Ext}_R{}^n(M, R) \neq 0$, hence $p^n \neq 0$.

Consider the free DG F-module $F \otimes_R R^m = F^m$. Any lifting $\tilde{p}_0 : F_0{}^m \to P_0$ of the map p extends to a homomorphism of DG modules $\tilde{p} : F^m \to P$, which induces in n-dimensional cohomology the map p^n. Hence $p_n \neq 0$.

Let f be a basis element of the R-free module F_n and let $e_1, \ldots, e_m$ be the standard basis of R^m. Choose $a_i \in R$ such that $a = f \otimes \Sigma\, a_i e_i$ is not in the kernel of $\tilde{p}_n$. Define a map of DG F-modules $q : F \to F^m$ by sending the unit of F to $1 \otimes \Sigma\, a_i e_i$. Then $g = \tilde{p} q : F \to P$ is a DG F-module map, such that $g_n \neq 0$. By the integrity assumption on R, g_n is injective, which implies that g is injective, since every ideal of F has a non-zero intersection with F_n.

6.5. *Higher obstructions.* Horrocks' problem [11, Problem 24] asks to find the minimal possible rank of an i-th syzygy of a module of finite length over a regular local ring R of dimension n. According to the last corollary, if the minimal resolution of every module of this type admits a DG module structure over a Koszul complex resolving a quotient of R modulo some regular sequence of length n, chosen in an appropriate way in its annihilator, then the answer to the problem will be

$$\binom{n-1}{i-1}.$$

Theorem 1.8 shows that the first step in the construction of such a sequence must be to take it in $\mathbf{m} \cdot \operatorname{ann} M$: e.g. starting with an arbitrary regular sequence $x_1, \ldots, x_n$, pass to $x_1{}^2, \ldots, x_n{}^2$.

Problem 6.5.1. What are the obstructions to the existence of DG F-module structures on P, in case J is generated by a regular sequence contained in $\mathbf{m} \cdot \operatorname{ann} M$?

More generally, is it possible to construct a sequence of obstructions ${}^{(n)}o^f(M)$, $n = 1, 2, \ldots$, such that the vanishing of ${}^{(n)}o^f(M)$ for all natural values of n constitutes a (necessary and) sufficient condition for the existence of a DG F-module structure on P?

In [10] Halperin and Stasheff have constructed a complete obstruction theory in a different, but at least philosophically and technically deeply related situation.

Our reason for stating the last problem is the hope that a positive answer will permit to construct in the annihilator of every finitely resolved module (say of finite length) a regular sequence with the required property, starting with an arbitrary sequence and modifying it successively so as to kill an extra obstruction at each step. Theorems 1.7 and 1.8 seem to imply that one should look for unobstructed sequences by iterating the operation of replacing a regular sequence by a sequence of powers of its elements.

Appendix: A review of some differential homological algebra. Let (U, V, W) be a triple consisting of a DG ring V, a right DG U-module U, and a left DG U-module W. Using relative homological algebra, Eilenberg and Moore [17] have constructed functors $\mathrm{Tor}^V(U, W)$ to the category of graded abelian groups and maps of degree zero from the category of triples (U, V, W) and maps (a, b, c), where b is a DG ring homomorphism, a is a right b-linear map of DG abelian groups, and c is a left b-linear one. A new approach to these functors is given by Gugenheim and May [8], who in particular established the relevance of May's matric Massey products to the study of the differentials in the spectral sequences. We list below the most important to our purposes properties of the DG torsion functors. It is useful to note that when U, V, and W are trivially graded (i.e. their components of non-zero degree are reduced to 0), the DG $\mathrm{Tor}^V(U, W)$ coincides with the classical one. When the arguments of a classical Tor are graded (with trivial differentials), the torsion functor is bigraded, the first—homological—degree referring to the place of the complex of graded modules with degree zero differentials, where the homology is taken, while the second—internal—one comes from the grading of the modules of this complex. Finally, all spectral sequences live in the first quadrant, and the r-th differentials maps $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$.

THEOREM A.1. (First Eilenberg-Moore spectral sequence). *There exists a functorial spectral sequence:*

$$E^2_{p,q} = \mathrm{Tor}^{H(V)}_{p,q}(H(U), H(W)) \Rightarrow \mathrm{Tor}^V_{p+q}(U, W).$$

THEOREM A.2. (Second Eilenberg-Moore spectral sequence). *There exists a functorial spectral sequence:*

$$E^1_{p,q} = \mathrm{Tor}^{V^\#}_{q,p}(U^\#, W^\#) \Rightarrow \mathrm{Tor}^V_{p+q}(U, W),$$

where # denotes the functor, forgetting differentials.

THEOREM A.3. *If* $(a, b, c):(U, V, W,) \to (U', V', W')$ *is such that* $H(a)$, $H(b)$, *and* $H(c)$ *are isomorphisms, then the induced map*

$$\mathrm{Tor}^b(a, c):\mathrm{Tor}^V(U, W) \to \mathrm{Tor}^{V'}(U', W')$$

is an isomorphism.

In the *DG* theory of Eilenberg and Moore a homological product:

$$\mathrm{Tor}^V(U, W) \otimes \mathrm{Tor}^{V'}(U', W') \to \mathrm{Tor}^{V\otimes V'}(U \otimes U', W \otimes W')$$

can be introduced mimicking the classical construction of comparison of resolutions (cf. also [1, Section 1]). In particular, the analogues of the $\pitchfork$-product exist in the *DG* context, and give the same pairing when taken on functors of arguments with trivial differentials.

THEOREM A.4. *Suppose that there exist homomorphisms of (skew-) commutative DG rings* $U \leftarrow V \to W$. *Then* $\mathrm{Tor}^V(U, W)$ *is a (skew-) commutative graded ring, and the spectral sequences of Theorems* A.1 *and* A.2 *are spectral sequences of rings.*

If moreover U', W' *are graded DG modules over* U, W *respectively, then* $\mathrm{Tor}^V(U', W')$ *is a graded* $\mathrm{Tor}^V(U, W)$*-module, and the spectral sequences of Theorems* A.1 *and* A.2 *are sequences of modules over the corresponding sequences of rings, natural with respect to appropriate maps of triples.*

The last result quoted is [8, Corollary 5.12]:

THEOREM A.5. *Let* V *be an augmented DG algebra over the Noetherian ring* R *and let* U *be a DG* V*-module, such that* $H(V)$ *and* $H(U)$ *are finitely-generated over* R *in each dimension. Then the kernel of the composition of the natural projection* $H(U) \to H(U) \otimes_{H(V)} R = E^2_{0,*}$ *with the edge homomorphism* $E^2_{0,*} \to \mathrm{Tor}^V(U, R)$ *of the first Eilenberg-Moore spectral sequence, consists of all the elements of the* $H(V)$*-module* $H(U)$, *decomposable in terms of matric Massey products.*

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