MDG

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Contents

Abstract

We study a class of objects called MDG algebras and MDG modules, which are just DG algebras and DG modules except we don't require the associative law to hold. Many interesting questions regarding DG algebras and DG modules can be studied in the broader class of MDG algebras and MDG modules. Using ideas from homological algebra as well as the theory of Gröbner bases, we develop tools which help us measure how far away MDG objects are from being DG objects.

Introduction

In this paper, we study algebraic structures which are similar to DG algebras, but without the requirement that they be associative. In particular, let R be a local noetherian ring and let F be the minimal R-free resolution of a cyclic R-algebra $S = R/I$. The multiplication map $\mathrm{m}\colon S \otimes_R S \to S$ can be extended to a chain map $\mu\colon F \otimes_R F \to F$, denoted

$$\mu(a_1 \otimes a_2) = a_1 \star_\mu a_2 = a_1 a_2$$

for all $a_1, a_2 \in F$ (where we make the further simplification $a_1 \star_\mu a_2 = a_1 a_2$ whenever context is clear). Up to homotopy, μ is unital, strictly graded-commutative, and associative. It is clear that we can always choose μ to be unital on the nose (with $1 \in F$ being the identity element). Buchsbaum and Eisenbud [BE77] showed that μ can be chosen to be strictly graded-commutative on the nose as well. On the other hand, it is known that μ can't be chosen to be associative on the nose in general (see [Avr81, Luk26]). In any case, we call μ a multiplication on F when it is unital and strictly graded-commutative (though not necessarily associative), and we call $F = (F, \mathrm{d}, \mu)$ an MDG R-algebra.[1] If μ also satisfies the associativity axiom, then we call F a DG R-algebra. Ever since [BE77], a lot of research has been dedicated to the question:

Question: Does there exist a DG R-algebra structure on F? In other words, can we find a multiplication μ on F which is associative?

One reason this question is interesting is that when we know the answer is "yes", then we gain a lot of information about the "shape" of F. For instance, Buchsbaum and Eisenbud [BE77] proved that if we further assume R is a domain and we know that an associative multiplication on F exists, then one obtains important lower bounds of the Betti numbers β_i of R/I. In particular, let $\boldsymbol{t} = t_1, \dots, t_g$ be a maximal R-sequence contained in I and let $E = \mathcal{K}(\boldsymbol{t})$ be the Koszul R-algebra resolution of $R/\boldsymbol{t}$. Any expression of the t_i in terms of the generators for I yields a canonical comparison map $E \to F$. Buchsbaum and Eisenbud showed that under all of these assumptions, this comparison map $E \to F$ is injective, hence we get the lower bound $\binom{m}{i} \leq \beta_i$ for each $i \leq g$. One of the starting points for this paper is based on the observation that one can still obtain these lower bounds even in cases where it is known that F does not possess the structure of a DG R-algebra or even a DG E-module. Indeed, we just need to find a multiplication μ on F together with a comparison map $\varphi\colon E \to F$ such that $\varphi\colon E \to F$ is multiplicative, meaning

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$$

for all $a_1, a_2 \in E$. The proof given in [BE77] which shows $\varphi\colon E \to F$ is injective would still apply to this case. To see that we really do gain something new from this perspective, we will look at an example in Example (2.4) where it is known that we can't find a μ which is associative, nonetheless we can still find a non-associative μ together with a comparison map $\varphi\colon E \to F$ such that φ is multiplicative. Consequently, the lower bounds of the Betti numbers continues to hold even in this case. In their proof, Buchsbaum and Eisenbud used a property that the Koszul algebra E satisfies, namely that every nonzero DG ideal of E intersects the top degree E_g nontrivially. However there are many other MDG algebras which satisfy this property as well (the property being that their nonzero MDG ideals intersect the top degree nontrivially). Thus one may be able to generalize this result even further by replacing $\boldsymbol{t}$ with an ideal J such that $\boldsymbol{t} \subseteq J \subseteq I$ and such that there exists a multiplication on the minimal R-free resolution G of R/J which satisfies this property. It is for this and many other reasons why we believe it will be fruitful to initiate the study of MDG algebras and their modules.

This paper is organized into four sections. In the first section, we work over an arbitrary commutative ring R and define the category of MDG R-algebras. An MDG R-algebra A is essentially just a DG R-algebra except we don't require the associative law to hold. We also define the category of MDG A-modules. An MDG A-module X is essentially just a DG A-module except we don't require the associative law to hold.

[1] The "M" stands for multiplication, the "D" stands for differential, and the "G" stands for grading; this explains our terminology.

In the second section, we introduce tools which help us measure how far away MDG objects are from being DG objects. In particular, we define the associator of X to be the chain map $[\cdot]\colon A\otimes A\otimes X\to X$ defined on elementary tensors by

$$[a_1\otimes a_2\otimes x]=(a_1a_2)x-a_1(a_2x)=[a_1,a_2,x]$$

for all $a_1,a_2\in A$ and $x\in X$, where we denote by $[\cdot,\cdot,\cdot]\colon A\times A\times X\to X$ to be the unique map corresponding to $[\cdot]$ via the universal mapping property of tensor products. We set $\langle X\rangle$ to be the smallest MDG A-submodule of X which contains the image of the associator of X. The quotient $X^{\mathrm{as}}:=X/\langle X\rangle$ is called the maximal associative quotient of X: it plays a role analagous to the role of the maximal abelian quotient of a group. We study the homology of $\langle X\rangle$ as well as the homology of X^{as}. In Theorem (2.2), we show under certain conditions (minimal / local / bounded below / finitely generated) that X is associative if and only if $\mathrm{H}(\langle X\rangle)$ vanishes. We also define the multiplicator of a chain map $\varphi\colon X\to Y$, where X and Y are MDG A-modules, to be the chain map $[\cdot]_\varphi\colon A\otimes X\to Y$ defined on elementary tensors by

$$[a\otimes x]_\varphi=\varphi(ax)-a\varphi(x)=[a,x]$$

for all $a\in A$ and $x\in X$, where we denote by $[\cdot,\cdot]\colon A\times X\to Y$ to be the unique map corresponding to $[\cdot]_\varphi$ via the universal mapping property of tensor products.

In the third section, we turn our attention towards the associator functor which takes an MDG A-module X to the MDG A-module $\langle X\rangle$ and takes an MDG A-module homomorphism $\varphi\colon X\to Y$ to the restriction map $\varphi\colon\langle X\rangle\to\langle Y\rangle$. The associator functor need not be exact. Indeed, let

$$0\longrightarrow X\xrightarrow{\ \varphi\ }Y\xrightarrow{\ \psi\ }Z\longrightarrow 0 \tag{1}$$

be a short exact sequence of MDG A-modules. We obtain an induced sequence of MDG A-modules

$$0\longrightarrow \langle X\rangle\xrightarrow{\ \varphi\ }\langle Y\rangle\xrightarrow{\ \psi\ }\langle Z\rangle\longrightarrow 0 \tag{2}$$

which is exact at $\langle X\rangle$ and $\langle Z\rangle$ but not necessarily exact at $\langle Y\rangle$. In order to ensure exactness of (2), we need to place a condition on (1); the condition being that Y is an **associative extension** of $\varphi(X)$, meaning $X\cap\langle Y\rangle=\langle\varphi(X)\rangle$. When this happens, we obtain a long exact sequence in homology:

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_{i+1}\langle Z\rangle \arrow[dlll, rounded corners] \\
 \mathrm{H}_i\langle X\rangle \arrow[r] & \mathrm{H}_i\langle Y\rangle \arrow[r] & \mathrm{H}_i\langle Z\rangle \arrow[dll, rounded corners] & \\
 \mathrm{H}_{i-1}\langle X\rangle \arrow[r] & \cdots & &
\end{tikzcd} \tag{3}$$

We end this section with a potential application of this long exact sequence: in a future paper, we would like to assign a finite number to a multiplication μ on a minimal R-free resolution F of a cyclic R-module where R is a local noetherian ring. This quantity would measure the failure for μ to being associative. We believe the application of the longest exact sequence that we describe in this section will help us to move closer towards this goal.

In the final section of this paper, we construct the symmetric DG algebra of an R-complex A which is centered at R (meaning $A_0=R$ and $A_i=0$ for all $i<0$), denoted by $\mathrm{S}_R(A)=S$. This section contains our main result of the paper, Theorem (4.1), which says that if we fix a multiplication μ on A, then the maximal associative quotient of A can be presented as a quotient of S by a DG S-ideal $\mathfrak{s}=\mathfrak{s}(\mu)$ which is constructed from μ:

$$A^{\mathrm{as}}=S/\mathfrak{s}.$$

Furthermore this presentation is natural in A in a suitable sense. In particular, we can study MDG algebra structures on A by studying certain DG ideals of S. This presentation also has interesting Gröbner basis applications in the case where $R=K$ is a field and F is an MDG K-algebra centered at K such that the underlying graded K-vector space of F is finite and free as a K-vector space. Indeed, suppose that

$$F_+=Re_1+\cdots+Re_n$$

where $e_1,\dots,e_n$ is an ordered homogeneous basis of F_+ which is ordered in such a way that if $|e_{i'}|>|e_i|$, then $i'>i$, and let $K[\boldsymbol{e}]=K[e_1,\dots,e_n]$ be the free non-strict graded-commutative R-algebra generated by $e_1,\dots,e_n$. We will equip $K[\boldsymbol{e}]$ with a specific monomial ordering and show how associators naturally arise when performing Buchberger's algorithm to certain set of polyhnomials using this monomial ordering.

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1 MDG Algebras and MDG Modules

We begin by defining MDG algebras. After defining MDG algebras, we then motivate their study by explaining how they arise naturally in the study of minimal free resolutions of cyclic modules.

1.1 MDG Algebras

Let R be a commutative ring and let $A = (A, \mathrm{d})$ be an R-complex:

$$A := \cdots \longrightarrow A_{n+1} \xrightarrow{\mathrm{d}_{n+1}} A_n \xrightarrow{\mathrm{d}_n} A_{n-1} \longrightarrow \cdots .$$

We view A as a graded R-module

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

equipped with an R-linear map $\mathrm{d}\colon A \to A$ which is graded of degree -1 and satisfies $\mathrm{d}^2 = 0$. We further equip A with a chain map $\mu\colon A \otimes_R A \to A$. We denote by $\star_\mu\colon A \times A \to A$ (or more simply by $\cdot$ if context is clear) to be the unique graded R-bilinear map which corresponds to μ via the universal mapping property of tensors products. Thus we have

$$\mu(a_1 \otimes a_2) = a_1 \star_\mu a_2 = a_1 a_2$$

for all $a_1, a_2 \in A$, where we make the further simplification in notation $a_1 \star_\mu a_2 = a_1 a_2$ when context is clear. Note that since μ is a chain map, $\star_\mu$ satisfies the **Leibniz law** which says

$$\mathrm{d}(a_1 a_2) = \mathrm{d}(a_1) a_2 + (-1)^{|a_1|} a_1 \mathrm{d}(a_2)$$

for all $a_1, a_2 \in A$ with a_1 homogeneous, where $|a_1|$ denotes the homological degree of a_1. Note also that the chain map μ induces a chain map $\overline{\mu}\colon \mathrm{H}(A) \otimes_R \mathrm{H}(A) \to \mathrm{H}(A)$, given by

$$\overline{\mu}(\overline{a}_1 \otimes \overline{a}_2) = \overline{a_1 a_2} \tag{4}$$

for all $\overline{a}_1, \overline{a}_2 \in \mathrm{H}(A)$ (where $a_1, a_2 \in A$ such that $\mathrm{d}(a_1) = 0 = \mathrm{d}(a_2)$ are representatives of $\overline{a}_1$ and $\overline{a}_2$) where the Leibniz law ensure (4) is well-defined. Here, we view $\mathrm{H}(A)$ as a trivial R-complex whose underlying graded R-module is $\mathrm{H}(A)$ and whose differential is the zero map. Thus $\overline{\mu}$ being a chain map is equivalent to it being just a graded R-linear map.

In order to simplify our notation in what follows, we often refer to the triple (A, d, μ) via its underlying graded R-module A, where we think of A as a graded R-module which is equipped with a differential $\mathrm{d}\colon A \to A$, giving it the structure of an R-complex, and which is further equipped with a chain map $\mu\colon A \otimes_R A \to A$. For instance, if μ satisfies a property (such as being associative), then we also say A satisfies that property.

Definition 1.1. With the notation as above, we make the following definitions:

1. We say A is **unital** if there exists $1 \in A$ such that $1a = a = a1$ for all $a \in A$.

2. We say A is **graded-commutative** if $a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_1$ for all homogeneous $a_1, a_2 \in A$.

3. We say A is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that $a^2 = 0$ for all elements $a \in A$ with $|a|$ odd.

4. We say A is **associative** if $(a_1 a_2) a_3 = a_1 (a_2 a_3)$ for all for all $a_1, a_2, a_3 \in A$.

We say A is an **MDG R-algebra** if A is strictly graded-commutative, unital, and $\mathrm{H}(A)$ is associative. Thus $\mathrm{H}(A)$ obtains the structure of an associative, strictly graded-commutative R-algebra. We call μ the **multiplication** of A just as we call d the differential of A. We say A is **centered** at R if $A_0 = R$ and $A_i = 0$ for all $i < 0$. Suppose B is another MDG R-algebra and let $\varphi \colon A \to B$ be a function.

1. We say φ is **unital** if $\varphi(1) = 1$.

2. We say φ is **multiplicative** if $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$.

We say $\varphi \colon A \to B$ is an **MDG R-algebra homomorphism** if it is a chain map which is both unital and multiplicative. We denote by $\mathbf{MDG}_R$ to be the category of all MDG R-aglebras and MDG R-algebra homomorphisms.

Remark 1. Let A be an MDG R-algebra. We view R itself as an MDG R-algebra itself where R has the trivial R-complex structure (where R sits in homological degree 0 and where the differential of R is the zero map). We have a canonical MDG R-algebra homomorphism $\iota \colon R \to A$ defined by $\iota(r) = r \cdot 1$ where we write $\cdot$ to denote the R-scalar multiplication $R \times A \to A$.

1.1.1 MDG Algebra Resolutions of a Cyclic Module

In this subsubsection, we describe the MDG algebras we are mostly interested in. Let I be an ideal of R, and let F be an R-free resolution of R/I such that $F_0 = R$. We denote by $\mathcal{C}(F \otimes_R F, F)$ to be the set of all chain maps from $F \otimes_R F$ to F (more generally, if X and Y are two R-complexes, then we denote by $\mathcal{C}(X, Y)$ to be the set of all chain maps from X to Y). A **multiplication** on F is a chain map $\mu \in \mathcal{C}(F \otimes_R F, F)$ which is unital (with $1 \in F$ being the identity element) and strictly graded-commutative (if we decide to equip F with a particular multiplication μ, giving it the structure of an MDG R-algebra, then we write $F = (F, \mathrm{d}, \mu)$ and refer to μ as *the* **multiplication** of F). We denote by $\mathrm{Mult}(F)$ to be the set of all multiplications on F.

We claim that every multiplication on F is automatically a lift of the usual multiplication m on R/I. Let us explain what this means: first note that F comes equipped with a canonical quasiisomorphism $\tau \colon F \to R/I$. Here we view R/I as a trivial R-complex which sits in homological degree 0. In homological degree 0, we have $\tau_0 \colon R \to R/I$ where τ_0 is the canonical projection map. In homological degree i where $i \neq 0$, we have $\tau_i \colon F_i \to 0$ is the zero map. With this understood, we say μ is a **lift** of m if the following diagram of R-complexes commutes:

$$\begin{tikzcd} F \otimes_R F \arrow[r, "\mu"] \arrow[d, "\tau^{\otimes 2}"'] & F \arrow[d, "\tau"] \\ R/I \otimes_R R/I \arrow[r, "\mathrm{m}"] & R/I. \end{tikzcd} \tag{5}$$

In homological degree $i \neq 0$, this diagram commutes for trivial reasons, so the only thing that we need to check is that the diagram commutes in homologial degree 0. In homological degree 0, the diagram looks like:

$$\begin{tikzcd} R \otimes_R R \arrow[r, "\mu_0"] \arrow[d, "\tau_0^{\otimes 2}"'] & R \arrow[d, "\tau_0"] \\ R/I \otimes_R R/I \arrow[r, "\mathrm{m}"] & R/I. \end{tikzcd} \tag{6}$$

Note that μ_0 is R-linear, so it's completely determined by where it sends $1 \otimes 1$. The diagram (6) will commute if and only if μ_0 sends $1 \otimes 1$ to $1 + x$ for some $x \in I$. In fact, μ_0 is already forced to send $1 \otimes 1$ to 1 since μ is assumed to be unital with identity element 1. Thus if $r_1, r_2 \in R$, then

$$r_1 \star_\mu r_2 = (r_1 r_2)(1 \star_\mu 1) = r_1 r_2.$$

In other words, μ_0 agrees with the usual multiplication on R, and the diagram (6) automatically commutes in this case as well.

Next, let J be an ideal contained in I and let G be an R-free resolution of R/J such that $G_0 = R$. Fix multiplications μ on F and ν on G giving them the structure of MDG R-algebras. Choose $\varphi\colon G \to F$ to be a lift of the map $R/J \to R/I$. We claim that if R is local and φ is multiplicative, then φ is automatically unital. Indeed, suppose φ is multiplicative and write $\varphi(1) = r$ for some $r \in R$. Since φ is a lift of $R/J \to R/I$, we must have $r = 1 + x$ for some $x \in I$. Since R is local, this implies r is a unit. However multiplicativity of φ already implies $r^2 = r$, and thus we must have $r = 1$ since r is a unit. Thus under these assumptions, $\varphi\colon G \to F$ is an MDG algebra homomorphism if and only if it is multiplicative. Of particular interest is when J is generated by an R-sequence $\boldsymbol{t} = t_1, \dots, t_m$. In this case, we can choose G to be $E = \mathcal{K}(\boldsymbol{t})$: the koszul R-algebra resolution of $R/\boldsymbol{t}$.

1.1.2 Multigraded MDG Algebras

Before we dive into the theory of MDG R-algebras, we provide some motivation for their study by discussing a combinatorial setting where they show up. The following construction was first described in [BPS98]: let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_d]$ where $\Bbbk$ is a field and let $I = \langle \boldsymbol{m} \rangle = \langle m_1, \dots, m_r \rangle$ is a monomial ideal in R. For each subset $\sigma \subseteq \{1, \dots, r\}$, we denote $e_\sigma := \{e_i \mid i \in \sigma\}$ (thus $e_{123} = \{e_1, e_2, e_3\}$). We also set $m_\sigma := \mathrm{lcm}(m_i \mid i \in \sigma)$ and we set $\boldsymbol{\alpha}_\sigma \in \mathbb{Z}^n$ to be the exponent vector of m_σ. Let Δ be a finitely simplicial complex with r-vertices denoted $e_1, \dots, e_r$. The sequence of monomials $\boldsymbol{m}$ induces a labeling of the faces of Δ as follows: we label the vertices $e_1, \dots, e_r$ of Δ by the monomials $m_1, \dots, m_r$ (so e_i is labeled by m_i). More generally, if e_σ a face of Δ, then we label it by m_σ. With the faces labeled this way, we call Δ an ***m*-labeled simplicial complex** (or a labeled simplicial complex if $\boldsymbol{m}$ is understood from context). Also, for each $\boldsymbol{\alpha} \in \mathbb{Z}^n$, let $\Delta_{\boldsymbol{\alpha}}$ be the subcomplex of Δ defined by

$$\Delta_{\boldsymbol{\alpha}} = \{\sigma \in \Delta \mid m_\sigma \text{ divides } x^{\boldsymbol{\alpha}}\}.$$

We often denote the faces of $\Delta_{\boldsymbol{\alpha}}$ by $(x^{\boldsymbol{\alpha}}/m_\sigma)e_\sigma$ instead of σ whenever context is clear.

Definition 1.2. We define an R-complex, denoted F_Δ (or more simply denoted F if Δ is understood from context) and called the R**-complex induced by** Δ as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R-module of F is given by

$$F_k := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $\mathrm{d}(e_\varnothing) = 0$ and

$$\mathrm{d}(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \setminus \{\varnothing\}$ where $\mathrm{pos}(i, \sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma \setminus i$ denotes the face obtained from σ by removing i. In the case where Δ is the r-simplex, we call F the **Taylor complex**.

Observe that F also has the structure of a multigraded $\Bbbk$-complex (or an $\mathbb{N}^n$-graded $\Bbbk$-complex) since the differential d respects the multigrading. In other words, we have a decomposition of $\Bbbk$-complexes

$$F = \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} F_{\boldsymbol{\alpha}},$$

where the $\Bbbk$-complex $F_{\boldsymbol{\alpha}}$ in multidegree $\boldsymbol{\alpha} \in \mathbb{N}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded $\Bbbk$-vector space is given by

$$F_{k,\boldsymbol{\alpha}} := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} \Bbbk \frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{\alpha}} \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $\mathrm{d}_{\boldsymbol{\alpha}}$ of $F_{\boldsymbol{\alpha}}$ is just the restriction of d to $F_{\boldsymbol{\alpha}}$. Notice that the differential behaves exactly like boundary map of $\Delta_{\boldsymbol{\alpha}}$ does:

$$\begin{aligned}
\mathrm{d}_{\boldsymbol{\alpha}}\left(\frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma}e_\sigma\right) &= \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma}\mathrm{d}(e_\sigma) \\
&= \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma}\sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{m_\sigma}{m_{\sigma\setminus i}}e_{\sigma\setminus i} \\
&= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{\boldsymbol{x}^{\boldsymbol{\alpha}}m_\sigma}{m_\sigma m_{\sigma\setminus i}}e_{\sigma\setminus i} \\
&= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_{\sigma\setminus i}}e_{\sigma\setminus i}.
\end{aligned}$$

Thus if we define $\varphi_{\boldsymbol{\alpha}}\colon F_{\boldsymbol{\alpha}}(1)\to\mathcal{S}(\Delta_{\boldsymbol{\alpha}})$ to be the unique graded $\Bbbk$-linear isomorphism such that $\frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma}e_\sigma\mapsto\sigma$, then from the computation above, we see that $\mathrm{d}_{\boldsymbol{\alpha}}\partial_{\boldsymbol{\alpha}}=\partial_{\boldsymbol{\alpha}}\mathrm{d}_{\boldsymbol{\alpha}}$, and hence $\varphi_{\boldsymbol{\alpha}}$ gives an isomorphism of $\Bbbk$-complexes $\varphi\colon\Sigma^{-1}F_{\boldsymbol{\alpha}}\simeq C(\Delta_{\boldsymbol{\alpha}};\Bbbk)$, where $C(\Delta_{\boldsymbol{\alpha}},\Bbbk)$ is the reduced chain complex of $\Delta_{\boldsymbol{\alpha}}$ over $\Bbbk$. In particular, this implies

$$\begin{aligned}
\mathrm{H}(F) &= \ker\mathrm{d}/\operatorname{im}\mathrm{d} \\
&= \left(\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\ker\mathrm{d}_{\boldsymbol{\alpha}}\right)\Big/\left(\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\operatorname{im}\mathrm{d}_{\boldsymbol{\alpha}}\right) \\
&\cong \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}(\ker\mathrm{d}_{\boldsymbol{\alpha}}/\operatorname{im}\mathrm{d}_{\boldsymbol{\alpha}}) \\
&= \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\mathrm{H}(F_{\boldsymbol{\alpha}}) \\
&\cong \bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\widetilde{\mathrm{H}}(\Delta_{\boldsymbol{\alpha}},\Bbbk)(-1).
\end{aligned}$$

In other words, we have

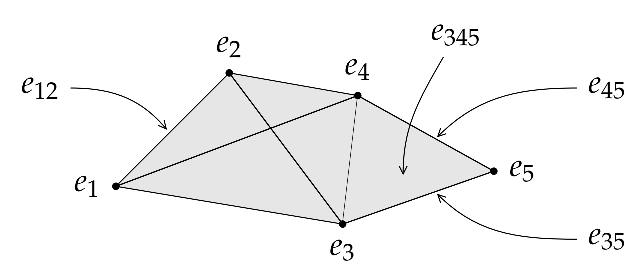
$$\mathrm{H}_i(F)\cong\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\mathrm{H}_i(F_{\boldsymbol{\alpha}})\cong\bigoplus_{\boldsymbol{\alpha}\in\mathbb{Z}^n}\widetilde{\mathrm{H}}_{i-1}(\Delta;\Bbbk).$$

for all $i\in\mathbb{Z}$. From this we easily get the following theorem:

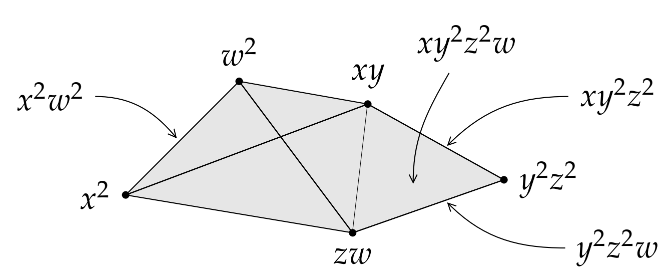
Theorem 1.1. *F is an R-free resolution of $R/\boldsymbol{m}$ if and only if for all $\boldsymbol{\alpha}\in\mathbb{Z}^n$ either $\Delta_{\boldsymbol{\alpha}}$ is the void complex or $\Delta_{\boldsymbol{\alpha}}$ is acyclic. In particular, the Taylor complex is an R-free resolution of $R/\boldsymbol{m}$. Moreover, F is minimal if and only if $m_\sigma\neq m_{\sigma'}$ for every proper subface σ' of a face σ.*

We now assume that Δ satisfies the conditions in Theorem (1.1), so that F is the minimal free R-resolution of $R/\boldsymbol{m}$. One can show that it is always possible choose a multiplication on F which respects the multigrading. The following was shown to be a counterexample first discussed in [Luk26] shows that we cannot choose a multiplication which respects the multigrading and is associative:

Example 1.1. Let Δ be the simplicial complex whose vertex set is $\{e_1,e_2,e_3,e_4,e_5\}$ and whose faces consists of all subsets of $e_{1234}=\{e_1,e_2,e_3,e_4\}$ and $e_{345}=\{e_3,e_4,e_5\}$, pictured below:



Next suppose $R=\Bbbk[x,y,z,w]$ and let $\boldsymbol{m}_{\mathrm{K}}=x^2,w^2,zw,xy,y^2z^2$. Then we obtain an $\boldsymbol{m}_{\mathrm{K}}$-labeled simplicial complex $\Delta=(\Delta,\boldsymbol{m}_{\mathrm{K}})$ which is pictured below:



Let F_{K} be the R-complex induced by Δ. Let's write down the homogeneous components of F_{K} as a graded R-module: we have

$$\begin{aligned}
F_{\mathrm{K},0} &= R \\
F_{\mathrm{K},1} &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_{\mathrm{K},2} &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_{\mathrm{K},3} &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_{\mathrm{K},4} &= Re_{1234}
\end{aligned}$$

The differential $\mathrm{d}\colon F_{\mathrm{K}} \to F_{\mathrm{K}}$ behaves just like the usual simplicial boundary map except some monomials can show up as coefficients. For instance,

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

Now, we begin to construct a multiplication $(\mu, \star)$ on F_{K} as follows: first we want μ to respect the multigrading. Then the multigrading and Leibniz law conditions that we impose on μ forces it to be defined uniquely on many homogeneous basis pairs (e_σ, e_τ). For instance, we are forced to have

$$\begin{aligned}
e_1 \star e_5 &= yz^2 e_{14} + xe_{45} \\
e_1 \star e_2 &= e_{12} \\
e_2 \star e_5 &= y^2 z e_{23} + we_{35} \\
e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= -e_{124}
\end{aligned} \tag{7}$$

At this point however, one can conclude that F_{K} is not associative since

$$[e_1, e_5, e_2] := (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{8}$$

One can work (8) out by hand, however one of the main results of our research is a method for calculating associators like (8) using tools from the theory of Gröbner bases. For instance, we used the following Singular code below to calculate the associator $[e_1, e_5, e_2]$:

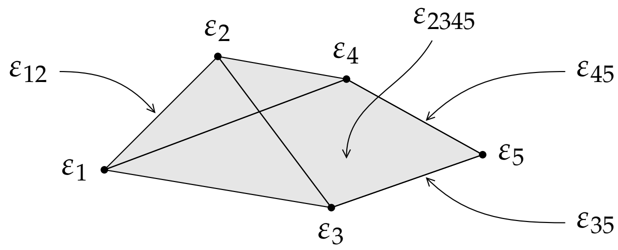
```
LIB "ncalg.lib";

intvec v= 1:3, 2:5, 3:5;
ring A=(0,x,y,z,w),(e1,e2,e5,e12,e14,e23,e35,e45,e123,e124,e134,e234,e345),Wp(v);

matrix C[13][13]; matrix D[13][13]; int i; int j;
for (i=1; i<=13; i++) {for (j=1; j<=13; j++) {C[i,j]=(-1)^(v[i]*v[j]);}}
ncalgebra(C,D);

poly f(1)(5) = e1*e5-yz2*e14-x*e45;
poly f(1)(2) = e1*e2-e12;
poly f(2)(5) = e2*e5-y2z*e23-w*e35;
poly f(2)(45) = e2*e45+yz*e234-w*e345;
poly f(1)(35) = e1*e35-yz*e134+x*e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14+e124;
poly S(1)(5)(2) = f(1)(5)*e2+e1*f(2)(5);

ideal I = f(2)(14),f(2)(45),f(1)(23),f(1)(35),f(2)(5),f(1)(5);
reduce(S(1)(5)(2),b);

// [e1,e5,e2] = (y^2*z)*e123-(y*z^2)*e124+(y*z*w)*e134-(x*y*z)*e234
```

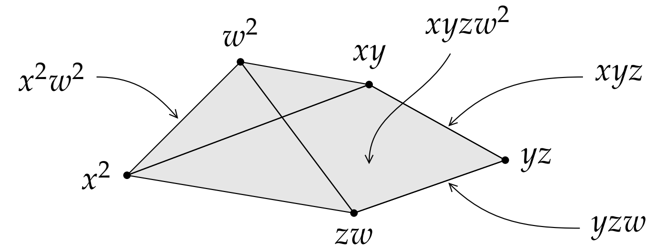
The multiplication isn't uniquely determined on all pairs (e_σ, e_τ), for instance there are two possible ways in which we can define μ at the pair (e_5, e_{12}). We choose to define μ at (e_5, e_{12}) by

$$e_5 \star e_{12} = yz^2 e_{124} + xyze_{234} + xwe_{345}.$$

Finally, we would still like for μ to be as associative as possible (even though we already know it's not associative at the triple (e_1, e_5, e_2)). In particular, we want μ to be associative on all triples of the form $(e_\sigma, e_\sigma, e_\tau)$. It turns out this can be done. In fact, Singular tells us (e.g. by calculating a Gröbner basis of an ideal like the one in the code above) how we should define μ at all other pairs (e_σ, e_τ) in order for this to happen.

Example 1.2. Let $R = \Bbbk[x,y,z,w]$, let $\boldsymbol{m}_{\mathrm{A}} = x^2, w^2, zw, xy, yz$, and let F_{A} be the minimal R-free resolution of $R/\boldsymbol{m}_{\mathrm{A}}$. Then F_{A} can be realized as the R-complex induced by the $\boldsymbol{m}_{\mathrm{A}}$-labeled cellular complex pictured below:





Let's write down the homogeneous components of F_{A} as a graded module: we have

$$
\begin{aligned}
F_{\mathrm{A},0} &= R \\
F_{\mathrm{A},1} &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_{\mathrm{A},2} &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_{\mathrm{A},3} &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_{\mathrm{A},4} &= R\varepsilon_{12345}
\end{aligned}
$$

The differential $\mathrm{d}\colon F_{\mathrm{A}} \to F_{\mathrm{A}}$ on the non-simplicial faces is given below

$$
\begin{aligned}
\mathrm{d}(\varepsilon_{12345}) &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\
\mathrm{d}(\varepsilon_{1345}) &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\
\mathrm{d}(\varepsilon_{2345}) &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}.
\end{aligned}
$$

We obtain a multiplication on F_{A} from the one we constructed on F_{K} as follows: first note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{A}}$ induces a multigraded comparison map $\pi\colon F_{\mathrm{K}} \to F_{\mathrm{A}}$ defined by

$$
\begin{aligned}
\pi(e_5) &= yz\varepsilon_5 \\
\pi(e_{35}) &= yz\varepsilon_{35} \\
\pi(e_{45}) &= yz\varepsilon_{45} \\
\pi(e_{34}) &= x\varepsilon_{35} - w\varepsilon_{45} \\
\pi(e_{345}) &= 0 \\
\pi(e_{234}) &= \varepsilon_{2345} \\
\pi(e_{134}) &= \varepsilon_{1345} \\
\pi(e_{1234}) &= \varepsilon_{12345}
\end{aligned}
$$

and $\pi(e_\sigma) = \varepsilon_\sigma$ for the remaining homogeneous basis elements. This map is locally invertible. Indeed, by base changing to R_{yz}, we obtain quasiisomorphisms $F_{\mathrm{A},yz} \to 0 \leftarrow F_{\mathrm{K},yz}$. In particular, there exists a comparison map $\iota\colon F_{\mathrm{A},yz} \to F_{\mathrm{K},yz}$ which splits comparison map $\pi\colon F_{\mathrm{K},yz} \to F_{\mathrm{A},yz}$. By considering the multigrading as well as the Leibniz law, we see that

$$
\begin{aligned}
\iota(\varepsilon_5) &= e_5/yz \\
\iota(\varepsilon_{35}) &= e_{35}/yz \\
\iota(\varepsilon_{45}) &= e_{45}/yz \\
\iota(\varepsilon_{2345}) &= -e_{234} + e_{345}/yz \\
\iota(\varepsilon_{1345}) &= e_{134} - e_{345}/yz \\
\iota(\varepsilon_{12345}) &= e_{1234}
\end{aligned}
$$

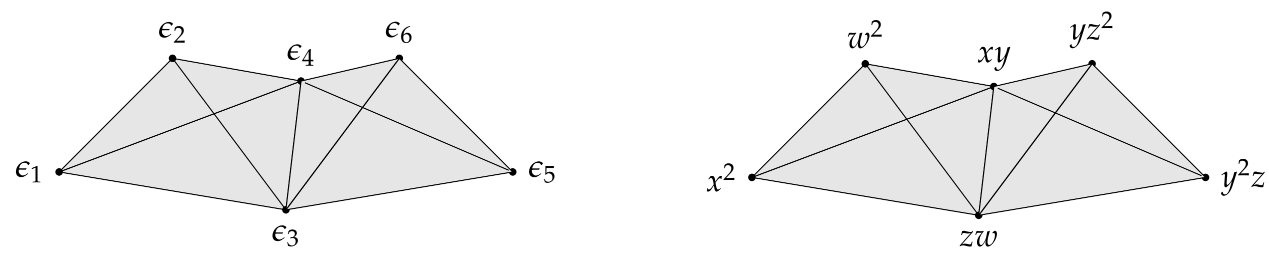
and $\iota(\varepsilon_\sigma) = e_\sigma$ for the remaining homogeneous basis elements. Then we defined a multiplication ν on F using the multiplication μ on $F_{\mathrm{K},yz}$ by setting

$$
\varepsilon_\sigma \star_\nu \varepsilon_\tau = \pi(\iota(\varepsilon_\sigma) \star_\mu \iota(\varepsilon_\tau)) \tag{9}
$$

for all homogeneous basis elements $\varepsilon_\sigma, \varepsilon_\tau$ of $F_{\mathrm{A},yz}$. It is straightforward to check that ν restricts to a multiplication on F_{A} (the coefficients in (9) are all in R). Note that ν is not associative since

$$
[\varepsilon_1, \varepsilon_5, \varepsilon_2] = -\mathrm{d}(\varepsilon_{1234}) \neq 0.
$$

Example 1.3. Let $R = \Bbbk[x,y,z,w]$, let $\boldsymbol{m}_{\mathrm{M}} = x^2, w^2, zw, xy, y^2z, yz^2$, and let F_{M} be the minimal R-free resolution of $R/\boldsymbol{m}_{\mathrm{M}}$. Then F_{M} can be realized as the R-compex induced by the $\boldsymbol{m}_{\mathrm{M}}$-labeled simplicial complex pictured below:



Let's write down the homogeneous components of F_{M} as a graded R-module: we have

$$
\begin{aligned}
F_{\mathrm{M},0} &= R \\
F_{\mathrm{M},1} &= R\epsilon_1 + R\epsilon_2 + R\epsilon_3 + R\epsilon_4 + R\epsilon_5 + R\epsilon_6 \\
F_{\mathrm{M},2} &= R\epsilon_{12} + R\epsilon_{13} + R\epsilon_{14} + R\epsilon_{23} + R\epsilon_{24} + R\epsilon_{34} + R\epsilon_{35} + R\epsilon_{36} + R\epsilon_{45} + R\epsilon_{46} + R\epsilon_{56} \\
F_{\mathrm{M},3} &= R\epsilon_{123} + R\epsilon_{124} + R\epsilon_{134} + R\epsilon_{234} + R\epsilon_{345} + R\epsilon_{346} + R\epsilon_{356} + R\epsilon_{456} \\
F_{\mathrm{M},4} &= R\epsilon_{1234} + R\epsilon_{3456}.
\end{aligned}
$$

Note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{M}}$ induces a multigraded comparison map $\pi_\lambda \colon F_{\mathrm{K}} \to F_{\mathrm{M}}$ where $\lambda \in \Bbbk$ and where π_λ is defined by

$$
\begin{aligned}
\pi_\lambda(e_5) &= \lambda z\epsilon_5 + (1-\lambda)y\epsilon_6 \\
\pi_\lambda(e_{35}) &= \lambda z\epsilon_{35} + (1-\lambda)y\epsilon_{36} \\
\pi_\lambda(e_{45}) &= \lambda z\epsilon_{45} + (1-\lambda)y\epsilon_{46} \\
\pi_\lambda(e_{345}) &= \lambda z\epsilon_{345} + (1-\lambda)y\epsilon_{346}
\end{aligned}
$$

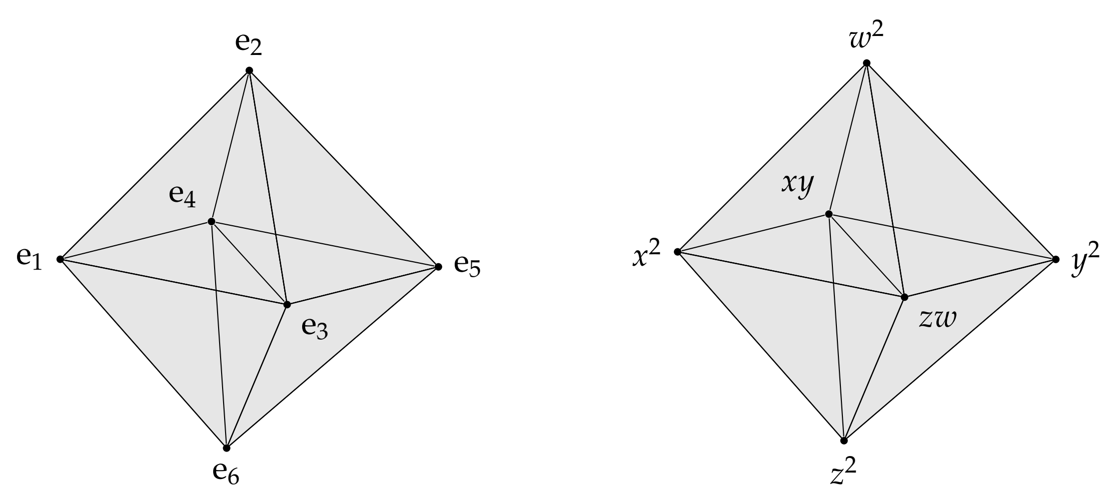
and $\pi_\lambda(e_\sigma) = \epsilon_\sigma$ for the remaining homogeneous basis elements. We define a multiplication on F_{M} as follows: first we take the multiplications given in (7) and we just replace e_1 with ϵ_1, e_5 with $z\epsilon_5$, e_{14} with ϵ_{14}, e_{45} with $z\epsilon_{45}$, and so on. For instance, we have

$$
\begin{aligned}
\epsilon_1 \star \epsilon_5 &= yz\epsilon_{14} + x\epsilon_{45} & \qquad \epsilon_1 \star \epsilon_6 &= z^2 e_{14} + xe_{46} \\
\epsilon_2 \star \epsilon_5 &= y^2\epsilon_{23} + w\epsilon_{35} & \epsilon_2 \star \epsilon_6 &= yz\epsilon_{23} + w\epsilon_{36} \\
\epsilon_2 \star \epsilon_{45} &= -y\epsilon_{234} + w\epsilon_{345} & \epsilon_2 \star \epsilon_{46} &= -ze_{234} + w\epsilon_{345} \\
\epsilon_1 \star \epsilon_{35} &= y\epsilon_{134} - x\epsilon_{345} & \epsilon_1 \star \epsilon_{36} &= z\epsilon_{134} - x\epsilon_{346}.
\end{aligned}
$$

Note that μ is not associative since

$$
[\epsilon_1, \epsilon_5, \epsilon_2] = -y\mathrm{d}(\epsilon_{1234}) \neq 0 \quad \text{and} \quad [\epsilon_1, \epsilon_6, \epsilon_2] = -z\mathrm{d}(\epsilon_{1234}) \neq 0.
$$

Example 1.4. Let $R = \Bbbk[x,y,z,w]$, let $\boldsymbol{m}_{\mathrm{O}} = x^2, w^2, zw, xy, y^2, z^2$, and let F_{O} be the minimal R-free resolution of $R/\boldsymbol{m}_{\mathrm{O}}$. Then F_{O} can be realized as the R-compex induced by the $\boldsymbol{m}_{\mathrm{O}}$-labeled simplicial complex pictured below:



One can show that there is a multigraded multiplication that one can define on F_{O} which turns out to be associative. We define it below on some of the homogeneous basis elements:

$$\begin{aligned}
\mathrm{e}_1 \star \mathrm{e}_5 &= y\mathrm{e}_{14} + x\mathrm{e}_{45} \\
\mathrm{e}_2 \star \mathrm{e}_6 &= z\mathrm{e}_{23} + w\mathrm{e}_{35} \\
\mathrm{e}_1 \star \mathrm{e}_{25} &= y\mathrm{e}_{124} - x\mathrm{e}_{245} \\
\mathrm{e}_1 \star \mathrm{e}_{35} &= y\mathrm{e}_{134} - x\mathrm{e}_{345} \\
\mathrm{e}_1 \star \mathrm{e}_{56} &= y\mathrm{e}_{146} + x\mathrm{e}_{456} \\
\mathrm{e}_2 \star \mathrm{e}_{16} &= -z\mathrm{e}_{123} - w\mathrm{e}_{136} \\
\mathrm{e}_2 \star \mathrm{e}_{46} &= -z\mathrm{e}_{234} + w\mathrm{e}_{346} \\
\mathrm{e}_2 \star \mathrm{e}_{56} &= -z\mathrm{e}_{235} + w\mathrm{e}_{356} \\
\mathrm{e}_2 \star \mathrm{e}_{146} &= \mathrm{e}_{1234} + \mathrm{e}_{1346} \\
\mathrm{e}_2 \star \mathrm{e}_{456} &= \mathrm{e}_{2345} + \mathrm{e}_{3456} \\
\mathrm{e}_1 \star \mathrm{e}_{235} &= \mathrm{e}_{1234} + \mathrm{e}_{2345} \\
\mathrm{e}_1 \star \mathrm{e}_{356} &= \mathrm{e}_{1346} + \mathrm{e}_{3456}.
\end{aligned}$$

1.1.3 Multigraded Multiplications coming from the Taylor Algebra

In this subsubsection, we want to explain how all of the multigraded multiplications that we've considered in the examples above come from a Taylor multiplication in the following sense: let $R = \Bbbk[x_1, \dots, x_d]$, let I be a monomial ideal in R, let F be the minimal R-free resolution of R/I, and let T be the Taylor algebra resolution of R/I. The Taylor multiplication is denoted ν_T. Let ν be a possibly different multiplication on T. We write T_ν to be the MDG R-algebra whose underlying R-complex is the same as the underlying complex of T but whose multiplication is ν. Since F is the minimal R-free resolution of R/I and since T is an R-free resolution of R/I, there exists multigraded chain maps $\iota\colon F \to T$ and $\pi\colon T \to F$ which lift the identity map $R/I \to R/I$ such that $\iota\colon F \to T$ is injective and is split by $\pi\colon T \to F$, meaning $\pi\iota = 1$. By identifying F with $\iota(F)$ if necessary, we may assume that $\iota\colon F \subseteq T$ is inclusion and that $\pi\colon T \to F$ is a **projection**, meaning $\pi\colon T \to F$ is a surjective chain map which satisfies $\pi^2 = \pi$, or alternatively, $\pi\colon T \to T$ is a chain map with $\operatorname{im}\pi = F$. In what follows, we fix $\iota\colon F \subseteq T$ once and for all and we denote by $\mathcal{P}(T, F)$ to be the set of all projections $\pi\colon T \to F$. For each $\mu \in \operatorname{Mult}(F)$, we denote by $\operatorname{Mult}(T/\mu)$ to be the set of all multiplications on T which extends μ:

$$\operatorname{Mult}(T/\mu) = \{\nu \in \operatorname{Mult}(T) \mid \nu|_{F^{\otimes 2}} = \nu\iota^{\otimes 2} = \mu\}.$$

Observe that if $\pi \in \mathcal{P}(T, F)$ and $\nu \in \operatorname{Mult}(T/\mu)$, then $\pi\nu \in \operatorname{Mult}(T/\mu)$. Indeed, $\pi\nu$ is clearly a multiplication on T. Furthermore, since π is a projective and since μ lands in F, we have $\pi\mu = \mu$. Therefore

$$\pi\nu\iota^{\otimes 2} = \pi\mu = \mu,$$

so $\pi\nu$ restricts to μ as well. Next, observe that if $\pi \in \mathcal{P}(T, F)$ and $\mu \in \operatorname{Mult}(F)$, then $\widehat{\mu}_\pi := \mu\pi^{\otimes 2} \in \operatorname{Mult}(T/\mu)$. We call $\widehat{\mu} = \widehat{\mu}_\pi$ the **trivial extension** of μ with respect to π for the following reasons: first note that for each $\nu \in \operatorname{Mult}(T/\mu)$, the inclusion map $\iota\colon F_\mu \subseteq T_\nu$ is multiplicative since $\nu\iota^{\otimes 2} = \mu = \iota\mu$, however $\pi\colon T_\nu \to F_\mu$ need not be multiplicative in general. In the case of the trivial extension $\widehat{\mu}$ however, $\pi\colon T_{\widehat{\mu}} \to F_\mu$ is multiplicative since

$$\pi\widehat{\mu} = \pi\mu\pi^{\otimes 2} = \mu\pi^{\otimes 2}.$$

Next, note that since $\pi\colon T \to F$ splits the inclusion $\iota\colon F \subseteq T$, we obtain isomorphism $\theta_\pi\colon T \simeq F \oplus H$ of R-complexes, where $H = \ker\pi$ is a trivial R-complex with $H_0 = 0 = H_1$, and where $\theta_\pi = (\pi, 1 - \pi)$. There's an obvious multiplication that we can give $F \oplus H$, namely $\mu \oplus 0$, where $0\colon H \otimes H \to H$ is the zero map. Equip $F \oplus H$ with this multiplication. We claim that $\theta_\pi\colon T_{\widehat{\mu}} \to F \oplus H$ is multiplicative, and hence an isomorphism of MDG R-algebras. Indeed, we have

$$\begin{aligned}
\theta_\pi\widehat{\mu} &= (\pi\widehat{\mu}, (1 - \pi)\widehat{\mu}) \\
&= (\pi\widehat{\mu}, \widehat{\mu} - \pi\widehat{\mu}) \\
&= (\widehat{\mu}, \widehat{\mu} - \widehat{\mu}) \\
&= (\widehat{\mu}, 0) \\
&= (\mu\pi^{\otimes 2}, 0) \\
&= (\mu \oplus 0)(\pi^{\otimes 2}, 1 - \pi^{\otimes 2}) \\
&= (\mu \oplus 0)\theta_\pi^{\otimes 2}.
\end{aligned}$$

In particular, every $b \in T$ can expressed in the form $b = a + c$ for unique $a \in F$ and unique $c \in H$. If $b_1, b_2 \in T$ have the unique expressions $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$, then we have $b_1 \star_\nu b_2 = a_1 \star_\mu a_2$.

Example 1.5. The multiplication μ in Example (1.1) is given by $\mu = \pi \nu_T \iota^{\otimes 2}$ where T is the taylor algebra resolution of $R/\mathfrak{m}_M$ and where $\pi\colon T \to F$ is defined by

$$\begin{aligned}
\pi(e_{15}) &= yz^2 e_{14} + xe_{45} \\
\pi(e_{25}) &= y^2 z e_{23} + we_{35} \\
\pi(e_{245}) &= -yze_{234} + we_{35} \\
\pi(e_{235}) &= 0 \\
\pi(e_{2345}) &= 0 \\
&\vdots
\end{aligned}$$

and so on.

1.2 MDG Modules

We now want to define MDG A-modules where A is an MDG R-algebra.

Definition 1.3. Let X be an R-complex equipped with chain maps $\mu_{A,X}\colon A \otimes_R X \to X$ and $\mu_{X,A}\colon X \otimes_R A \to X$, denoted $a \otimes x \mapsto ax$ and $x \otimes a \mapsto xa$ respectively.

1. We say X is **unital** if $1x = x = x1$ for all $x \in X$.
2. We say X is **graded-commutative** if $ax = (-1)^{|a||x|} xa$ for all $a \in A$ homogeneous and $x \in X$ homogeneous. In this case, $\mu_{X,A}$ is completely determined by $\mu_{A,X}$, and thus we completely forget about it and write $\mu_X = \mu_{A,X}$.
3. We say X is **associative** if $a_1(a_2 x) = (a_1 a_2)x$ for all $a_1, a_2 \in A$ and $x \in X$.

We say X is an **MDG A-module** if it is graded-commutative, unital, and the graded R-linear map

$$\overline{\mu}_X\colon \mathrm{H}(A) \otimes_R \mathrm{H}(X) \to \mathrm{H}(X)$$

induced by μ_X gives $\mathrm{H}(X)$ the structure of an associative graded-commutative $\mathrm{H}(A)$-module. We call μ_X the **A-scalar multiplication** of X. If X is also associative, then we say X is a **DG A-module**. A map $\varphi\colon X \to Y$ between MDG A-modules X and Y is called an **MDG A-module homomorphism** if it is a chain map which is also **multiplicative**, meaning $\varphi(ax) = a\varphi(x)$ for all $a \in A$ and $x \in X$. We obtain a category, denoted $\mathbf{Mod}^{\star}_A$, whose objects are MDG A-modules and whose morphisms are MDG A-module homomorphisms.

Example 1.6. Let A and B be MDG R-algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we give B the structure of an MDG A-module by defining an A-scalar multiplication on B via

$$a \cdot b = \varphi(a)b$$

for all $a \in A$ and $b \in B$. Note that we need $\varphi(1) = 1$ in order for B to be unital as an MDG A-module. Also note that φ is an MDG A-module homomorphism if and only if it is an algebra homomorphism. Indeed, it is an A-module homomorphism if and only if for all $a_1, a_2 \in A$ we have

$$\varphi(a_1 a_2) = a_1 \cdot \varphi(a_2) = \varphi(a_1)\varphi(a_2),$$

which is equivalent to saying φ is an algebra homomorphism (since we already have $\varphi(1) = 1$).

1.2.1 The Category of All MDG A-Modules

Let A be an MDG R-algebra. The category of all MDG A-modules forms an abelian category which is enriched over the category of all R-modules. Indeed, if X and Y are MDG A-modules, then the set of all MDG A-module homomorphims from X to Y, denoted $\mathrm{Hom}_A(X, Y)$, has the structure of an R-module, and moreover, the usual composition operation

$$\circ\colon \mathrm{Hom}_A(Y, Z) \times \mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(X, Z),$$

denoted $(g, f) \mapsto g \circ f = fg$, is R-bilinear. We also have a zero object, binary biproducts, as well as kernels and cokernels. For instance, if $\varphi\colon X \to Y$ is an MDG A-module homomorphism, then the kernel of φ, denoted $\ker \varphi$, is defined in the usual way as

$$\ker \varphi = \{x \in X \mid \varphi(x) = 0\}$$

together with the canonical inclusion map $\iota\colon \ker \varphi \to X$. The differential and A-scalar multiplication of $\ker \varphi$ are simply the ones obtained from X via restriction to $\ker \varphi$. Similarly the cokernel of φ is defined in the usual way as well. Thus the category of all MDG A-modules shares many of the same properties as the category of all DG B-modules where B is a DG R-algebra. Thus, the language we use in the category of MDG A-modules is often similar to the language used in the category of all DG B-modules. For instance, if X and Y are two MDG A-modules such that $X \subseteq Y$, then we say X is an **MDG A-submodule** of Y if the inclusion map $\iota\colon X \to Y$ is an MDG A-module homomorphism. In particular, this means that both the differential and A-scalar multiplication of Y restricts to a differential and A-scalar multiplication on X. Similarly, the MDG A-submodules $\mathfrak{a}$ of A are often called **MDG ideals** of A or **MDG A-ideals**. An MDG A-ideal $\mathfrak{p}$ is called a **prime ideal** if it satisfies the following property: if $a_1, a_2 \in A$ such that $a_1 a_2 \in \mathfrak{p}$ and $a_2 \notin \mathfrak{p}$, then $a_1 \in \mathfrak{p}$.

Having said all of this, there are some notable differences between the category of all DG B-modules and the category of all MDG A-modules. In particular, one must be careful when defining localization, tensor, and hom in the latter. In particular, if X and Y are MDG A-modules, then one can define the tensor complex $X \otimes_A Y$ as well as the hom complex $\mathrm{Hom}^\star_A(X, Y)$ in the usual way. Then tensor complex $X \otimes_A Y$ turns out to be an MDG A-module with the obvious A-scalar multiplication, however it need not be true that $A \otimes_A X \simeq X$. On the other hand, it may not be possible to give the hom complex $\mathrm{Hom}^\star_A(X, Y)$ the structure of an MDG A-module by defining A-scalar multiplication in the obvious way. Finally, if $S \subseteq A$ is a multiplicatively closed set, then one can make sense of the localization X_S, but only in the case where S satisfies some extra conditions. We include more details on this in the appendix.

Remark 2. Let A be an MDG algebra and let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{10}$$

be a short exact sequence of MDG A-modules. If we just view (10) as a short exact sequence of chain complexes, then we know that we get an induced long exact sequence of abelian groups:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(Y) \arrow[r, "\psi_{i+1}"] & \mathrm{H}_{i+1}(Z) \arrow[dll, "\partial_{i+1}"] \\
\mathrm{H}_i(X) \arrow[r, "\varphi_i"] & \mathrm{H}_i(Y) \arrow[r, "\psi_i"] & \mathrm{H}_i(Z) \arrow[dll, "\partial_i"] \\
\mathrm{H}_{i-1}(X) \arrow[r, "\varphi_{i-1}"] & \mathrm{H}_{i-1}(Y) \arrow[r] & \cdots
\end{tikzcd} \tag{11}$$

where the connecting map $\partial\colon \mathrm{H}(Z) \to \mathrm{H}(X)$ is a graded $\mathrm{H}_0(A)$-module homomorphism of degree -1 which is defined as follows: let $\overline{z} \in \mathrm{H}(Z)$ where $z \in Z$ is homogeneous and $\mathrm{d}z = 0$. Choose $y \in Y$ such that $\psi y = z$. Then there is a unique $x \in X$ such that $\varphi x = \mathrm{d}y$. We set $\partial \overline{z} = \overline{x}$ and verify that this is a well-defined map (i.e. doesn't depend on any of the choices we made). However we get more when we obtain a little more when we view (10) as a short exact sequence of MDG A-modules. Indeed, ∂ is not just an $\mathrm{H}_0(A)$-linear: it is $\mathrm{H}(A)$-linear! Thus we obtain a sequence of graded $\mathrm{H}(A)$-modules:

$$\mathrm{H}(X) \xrightarrow{\varphi} \mathrm{H}(Y) \xrightarrow{\psi} \mathrm{H}(Z) \xrightarrow{\partial} \Sigma\mathrm{H}(X) \xrightarrow{\varphi} \Sigma\mathrm{H}(Y)$$

which is exact at $\mathrm{H}(Y)$, $\mathrm{H}(Z)$, and $\Sigma\mathrm{H}(X)$.

2 Associators and Multiplicators

In order to get a better understanding as to how far away MDG objects are from being DG objects, we need to discuss associators and multiplicators. Associators will help us measure how far away an MDG A-module X is from being associative, whereas multiplicators will help up measure how far away a chain map $\varphi\colon X \to Y$ is from being multiplicative.

2.1 Associators

We begin by defining associators. Throughout this subsection, let A be an MDG R-algebra and let X be an MDG A-module.

Definition 2.1. The **associator** of X is the chain map, denoted $[\cdot]_X$ (or more simply by $[\cdot]$ if X is understood from context), from $A \otimes_R A \otimes_R X$ to X defined by

$$[\cdot] := \mu(\mu \otimes 1 - 1 \otimes \mu).$$

Note that we use μ to denote both the multiplication μ_A on A and the A-scalar multiplication μ_X on X where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot]\colon A \times A \times X \to X$ to be the unique R-trilinear map which corresponds to $[\cdot]$ via the universal mapping property of tensor products. Thus we have

$$[a_1 \otimes a_2 \otimes x] = (a_1 a_2)x - a_1(a_2 x) = [a_1, a_2, x]$$

for all $a_1, a_2 \in A$ and $x \in X$.

2.1.1 Associator Identities

In order to familiarize ourselves with the associator we collect together some useful identities that the associator satisfies in this subsubsection:

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ we have the Leibniz law

$$\mathrm{d}[a_1, a_2, x] = [\mathrm{d}a_1, a_2, x] + (-1)^{|a_1|}[a_1, \mathrm{d}a_2, x] + (-1)^{|a_1|+|a_2|}[a_1, a_2, \mathrm{d}x]. \tag{12}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1]. \tag{13}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||x|+|a_2||x|}[x, a_1, a_2] - (-1)^{|a_1||a_2|+|a_1||x|}[a_2, x, a_1] \tag{14}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = (-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] \tag{15}$$

- For all $a_1, a_2, a_3 \in A$ and $x \in X$ we have

$$a_1[a_2, a_3, x] = [a_1 a_2, a_3, x] - [a_1, a_2 a_3, x] + [a_1, a_2, a_3 x] - [a_1, a_2, a_3]x \tag{16}$$

The way the signs in (13) show up can be interpreted as follows: in order to go from $[a_1, a_2, x]$ to $[x, a_2, a_1]$, we have to first swap a_1 with a_2 (this is where the $(-1)^{|a_1|a_2|}$ comes from), then swap a_1 with x (this is where the $(-1)^{|a_1||x|}$ comes from), and then finally swap a_2 with x (this is where the $(-1)^{|a_2||x|}$ comes from). We then obtain one extra minus sign by swapping terms in the associator at the final step:

$$\begin{aligned}
[a_1, a_2, x] &= (a_1 a_2)x - a_1(a_2 x) \\
&= (-1)^{|a_1|a_2|}(a_2 a_1)x - (-1)^{|a_2|||x|}a_1(x a_2) \\
&= (-1)^{|a_1||a_2|+|a_2||x|+|a_1||x|}x(a_2 a_1) - (-1)^{|a_2||x|+|a_1||x|+|a_1||a_2|}(x a_2)a_1 \\
&= (-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}(x(a_2 a_1) - (x a_2)a_1) \\
&= -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1].
\end{aligned}$$

A similar interpretation is also given to (14) and (15). For instance, in order to get from $[a_1, a_2, x]$ to $[x, a_1, a_2]$, we have to swap x with a_2 and then swap x with a_1 (this is where the $(-1)^{|a_1||x|+|a_2||x|}$ comes from). We do add an extra minus sign in (15) however since we never swap terms in the associator:

$$\begin{aligned}
(-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] &= (a_1 a_2)x - (-1)^{|a_1||a_2|}a_2(a_1 x) + (-1)^{|a_2||x|}(a_1 x)a_2 - a_1(a_2 x) \\
&= (a_1 a_2)x - (-1)^{|a_1||a_2|}a_2(a_1 x) + (-1)^{|a_1|a_2|}a_2(a_1 x) - a_1(a_2 x) \\
&= (a_1 a_2)x - a_1(a_2 x) \\
&= [a_1, a_2, x].
\end{aligned}$$

2.1.2 Alternative MDG Modules

If X is not associative, then one is often interested in knowing whether or not X satisfies the following weaker property:

Definition 2.2. We say X is **alternative** if $[a,a,x]=0$ for all $a \in A$ and $x \in X$.

In other words, X is alternative if for each $a \in A$ and $x \in X$, we have $a^2x = a(ax)$. The reason behind the name "alternative" comes from the fact that in the case where $X = A$, then A is alternative if and only if the associator $[\cdot,\cdot,\cdot]$ is alternating.

Proposition 2.1. *Let $a \in A$ and $x \in X$ be homogeneous.*

1. *We have $[a,a,x]=0$ if and only if $[x,a,a]=0$.*
2. *If $[a,a,x]=0$, then $[a,x,a]=0$. The converse holds if $|a|$ is odd and* char $R \neq 2$.
3. *If $|a|$ is even, we have $[a,x,a]=0$, and if $|a|$ is odd, we have $[a,x,a]=(-1)^{|x|}2[a,a,x]$. In particular, if* char $R=2$, *we always have $[a,x,a]=0$.*

Proof. From identities (13) and (15) we obtain

$$\begin{aligned}
[a,a,x] &= -(-1)^{|a|}[x,a,a] \\
[a,x,a] &= (-1)^{|x||a|}(1-(-1)^{|a|})[a,a,x].
\end{aligned}$$

In particular, we see that

$$[a,x,a] = \begin{cases} = (-1)^{|x|}2[a,a,x] = -(-1)^{|x|}2a(ax) & \text{if } a \text{ is odd} \\ 0 & \text{if } a \text{ is even} \end{cases} \tag{17}$$

Similarly we have

$$[a,a,x] = \begin{cases} (-1)^{|x|}\frac{1}{2}[a,x,a] & \text{if } a \text{ is odd and char } R \neq 2 \\ (-1)^{|a|}[x,a,a] & \text{if } a \text{ is even} \end{cases} \tag{18}$$

□

Remark 3. Suppose F is an MDG R-algebra whose underlying graded R-module is finite and free with $e_1,\dots,e_n$ being a homogeneous basis. In order to show F is alternative, it is *not* enough to check $[e_i,e_i,e_j]=0$ for all e_i,e_j in the homogeneous basis. Indeed, even in this case, observe that if e_i and e_j are odd, then

$$\begin{aligned}
[e_i+e_j, e_i+e_j, e_k] &= [e_i,e_i,e_k]+[e_i,e_j,e_k]+[e_j,e_i,e_k]+[e_j,e_j,e_k] \\
&= [e_i,e_j,e_k]+[e_j,e_i,e_k] \\
&= [e_i,e_j,e_k]-[e_j,e_i,e_k]+(-1)^{|e_k|}[e_j,e_k,e_i] \\
&= (-1)^{|e_k|}[e_j,e_k,e_i].
\end{aligned}$$

Thus in order for F to be alternative, we certainly need $[a_1,a_2,a_3]=0$ for all $a_1,a_2,a_3 \in F$ whenever both $|a_1|$ and $|a_3|$ are odd. For instance, consider the MDG R-algebra F_{K} given in Example (1.1). Then we have $[e_\sigma,e_\sigma,e_\tau]=0$ for all $\sigma,\tau \in \Delta$, however F is not alternative since $[e_1,e_5,e_2] \neq 0$.

2.1.3 The Maximal Associative Quotient

Definition 2.3. The **associator R-subcomplex** of X, denoted $[X]$, is the R-subcomplex of X given by the image of the associator of X. Thus the underlying graded R-module of $[X]$ is

$$[X] = \mathrm{span}_R\{[a_1, a_2, x] \mid a_1, a_2 \in A \text{ and } x \in X\},$$

and the differential of $[X]$ is simply the restriction of the differential of X to $[X]$. The **associator A-submodule** of X, denoted $\langle X \rangle$, is defined to be the smallest A-submodule of X which contains $[X]$. The underlying graded R-module of $\langle X \rangle$ also has a simple description. Indeed, observe that

$$a_1(a_2[a_3, a_4, x]) = (a_1a_2)[a_3, a_4, x] - [a_1, a_2, [a_3, a_4, x]] \tag{19}$$

for all $a_1, a_2, a_3, a_4 \in A$ and $x \in X$. Using identities like (19) together with graded-commutativity, one can show that the underlying graded R-module of $\langle X \rangle$ is given by

$$\langle X \rangle = \mathrm{span}_R\{a_1[a_2, a_3, x] \mid a_1, a_2, a_3 \in A \text{ and } x \in X\}$$

The quotient $X^{\mathrm{as}} := X/\langle X \rangle$ is a DG A-module (i.e. an associative MDG A-module). We call X^{as} (together with its canonical quotient map $X \twoheadrightarrow X^{\mathrm{as}}$) the **maximal associative quotient** of X.

The maximal associative quotient of X satisfies the following universal mapping property:

Proposition 2.2. *Every* MDG *A-module homomorphism $\varphi\colon X \to Y$ in which Y is associative factors through a unique* MDG *A-module homomorphism $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$, meaning $\overline{\varphi}\rho = \varphi$ where $\rho\colon X \twoheadrightarrow X^{\mathrm{as}}$ is the canonical quotient map. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & X^{\mathrm{as}} \arrow[d, dashed, "\overline{\varphi}"] \\ & Y \end{tikzcd} \tag{20}$$

Proof. Indeed, suppose $\varphi\colon X \to Y$ is any MDG A-module homomorphism where Y is associative. In particular, we must have $[X] \subseteq \ker\varphi$, and since $\langle X \rangle$ is the smallest MDG A-submodule of X which contains $[X]$, it follows that $\langle X \rangle \subseteq \ker\varphi$. Thus the map $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$ given by $\overline{\varphi}(\overline{x}) := \varphi(x)$ where $\overline{x} \in X^{\mathrm{as}}$ is well-defined. Furthermore, it is easy to see that $\overline{\varphi}$ is an MDG A-module homomorphism and the unique such one which makes the diagram (20) commute. □

Lemma 2.1. *We can express $\langle A \rangle$ as the R-span of all elements of the form $a_1[a_2, a_3, a_4]$ where $|a_1| \leq |a_2|, |a_3|, |a_4|$.*

Proof. □

2.1.4 Homological Associativity

Definition 2.4. The **associator homology** of X is the homology of the associator A-submodule of X. We often simplify notation and denote the associator homology of X by $\mathrm{H}\langle X \rangle$ instead of $\mathrm{H}(\langle X \rangle)$. We say X is **homologically associative** if $\mathrm{H}\langle X \rangle = 0$ and we say X is **homologically associative in degree** i if $\mathrm{H}_i\langle X \rangle = 0$. Similarly we say X is associative in degree if $\langle X \rangle_i = 0$.

Clearly, if X is associative, then X is homologically associative. The converse holds under certain conditions.

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a local ring, let A be an* MDG *R-algebra, and let X be an* MDG *A-module such that $\langle X \rangle$ is minimal (meaning $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$), and such that each $\langle X \rangle_i$ is a finitely generated R-module. If X is associative in degree i, then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $\langle X \rangle$ is also bounded below (meaning $\langle X \rangle_i = 0$ for $i \ll 0$), then X is associative if and only if X is homologically associative.*

Proof. Assume that X is associative in degree i. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume

$$\mathrm{H}_{i+1}\langle X \rangle = 0 \quad \text{and} \quad \langle X \rangle_{i+1} \neq 0.$$

Then by Nakayama's Lemma, we can find homogeneous $a_1, a_2, a_3 \in A$ and homogeneous $x \in X$ such that such that $a_1[a_2, a_3, x] \notin \mathfrak{m}\langle X \rangle_{i+1}$. Since $\langle X \rangle_i = 0$ by assumption, we have $\mathrm{d}(a_1[a_1, a_2, x]) = 0$. Also, since $\langle X \rangle$ is minimal, we have $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$. Thus $a_1[a_2, a_3, x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. □

The proof given in Theorem (2.2) tells us something a bit more than what we stated. To see this, we first need a few definitions:

Definition 2.5. Let X be an MDG A-module.

1. Assume that $\langle X\rangle$ is bounded below. The **lower associative index** of X, denoted $\mathrm{la}\langle X\rangle$, is defined to be the smallest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\langle X\rangle_i \neq 0$ where we set $\mathrm{la}\langle X\rangle = \infty$ if X is associative. We extend this definition to case where $\langle X\rangle$ is not bounded below by setting $\mathrm{la}\langle X\rangle = -\infty$.

2. Assume that $\mathrm{H}\langle X\rangle$ is bounded below. The **lower homological associative index** of X, denoted $\mathrm{lha}\langle X\rangle$, is defined to be the smallest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\mathrm{H}_i\langle X\rangle \neq 0$ where we set $\mathrm{lha}\langle X\rangle = \infty$ if X is homologically associative. We extend this definition to case where $\mathrm{H}\langle X\rangle$ is not bounded below by setting $\mathrm{lha}\langle X\rangle = -\infty$.

3. Assume that $\langle X\rangle$ is bounded above. The **upper associative index** of X, denoted $\mathrm{ua}\langle X\rangle$, is defined to be the largest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\langle X\rangle_i \neq 0$ where we set $\mathrm{ua}\langle X\rangle = -\infty$ if X is associative. We extend this definition to case where $\langle X\rangle$ is not bounded above by setting $\mathrm{ua}\langle X\rangle = \infty$.

4. Assume that $\mathrm{H}\langle X\rangle$ is bounded above. The **upper homological associative index** of X, denoted $\mathrm{uha}\langle X\rangle$, is defined to be the largest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\mathrm{H}_i\langle X\rangle \neq 0$ where we set $\mathrm{uha}\langle X\rangle = -\infty$ if X is homologically associative. We extend this definition to case where $\mathrm{H}\langle X\rangle$ is not bounded above by setting $\mathrm{uha}\langle X\rangle = \infty$.

With the lower associative index of X and the lower homological associative index of X defined, we see after analyzing the proof of Theorem (2.2), that if R is local, $\langle X\rangle$ is minimal and bounded below, and each $\langle X\rangle_i$ is finitely generated as an R-module, then we have $\mathrm{la}\langle X\rangle = \mathrm{lha}\langle X\rangle$. On the other hand, even if these conditions are satisfied, we often have $\mathrm{ua}\langle X\rangle > \mathrm{uha}\langle X\rangle$. For instance, we will see in Example (2.3) that $\mathrm{ua}\langle F\rangle = 4$ and $\mathrm{uha}\langle F\rangle = 3$.

Example 2.1. Let A be a positive MDG R-algebra with $A_0 = R$ and $\mathrm{im}\,\mathrm{d}_1 = I$. Let X be an MDG A-module such that the lower associative index $\varepsilon = \mathrm{la}\langle X\rangle$ of X is finite. Then we have

$$\mathrm{H}_\varepsilon\langle X\rangle = \frac{[X]_\varepsilon}{I[X]_\varepsilon + \mathrm{d}[X]_{\varepsilon+1}} \quad \text{and} \quad \mathrm{H}_\varepsilon[X] = \frac{[X]_\varepsilon}{\mathrm{d}[X]_{\varepsilon+1}}.$$

Indeed, the second equality is clear by definition, so let us show the first equality. Since $[X]_{\varepsilon-1} = 0$ by assumption, it suffices to show that

$$\mathrm{im}(\mathrm{d}_{\langle X\rangle,\varepsilon+1}) = I[X]_\varepsilon + \mathrm{d}[X]_{\varepsilon+1}.$$

To see this, note that $\mathrm{im}(\mathrm{d}_{\langle X\rangle,\varepsilon+1})$ is generated (as an R-module) by two types elements: namely $\mathrm{d}(a\gamma)$ or $\mathrm{d}\gamma'$ where $a \in A_1$, where $\gamma \in [X]_\varepsilon$, and where $\gamma' \in [X]_{\varepsilon+1}$. In the first case, we have $\mathrm{d}(a\gamma) = (\mathrm{d}a)\gamma \in I[X]_\varepsilon$ since $\mathrm{d}\gamma = 0$. In the second case, we have $\mathrm{d}\gamma' \in \mathrm{d}[X]_{\varepsilon+1}$. Thus we have

$$\mathrm{im}(\mathrm{d}_{\langle X\rangle,\varepsilon+1}) \subseteq I[X]_\varepsilon + \mathrm{d}[X]_{\varepsilon+1}.$$

The converse direction follows form the fact that $\mathrm{d}(A_1) = I$. A similar calculation shows

$$\mathrm{H}_{\varepsilon+1}\langle X\rangle = \frac{\ker \mathrm{d} \cap \langle X\rangle_{\varepsilon+1}}{I_2[X]_\varepsilon + I_1[X]_{\varepsilon+1} + \mathrm{d}[X]_{\varepsilon+2}},$$

where we set $I_1 = \mathrm{d}(A_1)$ and $I_2 = \mathrm{d}(A_2)$. In particular, calculating $\mathrm{H}\langle X\rangle$ involves the higher syzygies of I. Now let δ be the upper associative index of X and assume that δ is finite. Then we have

$$\mathrm{H}_\delta\langle X\rangle = \frac{\ker \mathrm{d} \cap \langle X\rangle_{\varepsilon+1}}{\sum_{i=\varepsilon}^{\delta-1} I_{(\delta-i)}[X]_i},$$

. suppose that the upper associative index $\delta = \mathrm{ua}$ of X is finite too.

We are often also interested in the homology of the maximal associative quotient of X as well. To this end, observe that the short exact sequence of MDG A-modules

$$0 \longrightarrow \langle X\rangle \longrightarrow X \longrightarrow X^{\mathrm{as}} \longrightarrow 0$$

induces a sequence of graded $\mathrm{H}(A)$-modules

$$\mathrm{H}\langle X\rangle \longrightarrow \mathrm{H}(X) \longrightarrow \mathrm{H}(X^{\mathrm{as}}) \xrightarrow{\overline{\mathrm{d}}} \Sigma\mathrm{H}\langle X\rangle \longrightarrow \Sigma\mathrm{H}(X)$$

which is exact at $\mathrm{H}\langle X\rangle$, $\mathrm{H}(X)$, and $\mathrm{H}(X^{\mathrm{as}})$ and where the connecting map $\overline{\mathrm{d}}\colon \mathrm{H}(X^{\mathrm{as}}) \to \Sigma\mathrm{H}\langle X\rangle$ is essentially defined in terms of the differential d of X, namely given $\overline{x} \in \mathrm{H}(X^{\mathrm{as}})$, we set $\overline{\mathrm{d}}\overline{x} = \overline{\mathrm{d}x}$.

Example 2.2. Let X be an MDG A-module. Assume that $(R, \mathfrak{m})$ is a local noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let F be the minimal R-free resolution of R/I. Equip F with a multiplication μ giving it the structure of an MDG R-algebra. Then

$$\mathrm{H}_i(F^{\mathrm{as}}) \cong \begin{cases} R/I & \text{if } i = 0 \\ \mathrm{H}_{i-1}\langle F \rangle & \text{else} \end{cases}$$

2.1.5 Computing Annihilators of the Associator Homology

In this subsubsection, we assume that A is centered at R. Set I to be the image of $\mathrm{d}_1 \colon A_1 \to R$. In particular, we have $\mathrm{H}_0(A) = R/I$.

Proposition 2.3. *I annihilates both $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(X^{\mathrm{as}})$.*

Proof. Let $t \in I$. Thus $t = \mathrm{d}(a)$ where $|a| = 1$. Let $\mathrm{m}_a \colon X \to X$ be the multiplication by a map given by $\mathrm{m}_a(x) = ax$. In particular, m_a restricts to an R-linear map $\mathrm{m}_a \colon \langle X \rangle \to \langle X \rangle$ and thus induces an R-linear map $\overline{\mathrm{m}}_a \colon X^{\mathrm{as}} \to X^{\mathrm{as}}$. Observe that if $x \in X$, then

$$\begin{aligned}(\mathrm{d}\mathrm{m}_a + \mathrm{m}_a\mathrm{d})(x) &= \mathrm{d}(ax) + a\mathrm{d}(x) \\ &= \mathrm{d}(a)x - a\mathrm{d}(x) + a\mathrm{d}(x) \\ &= tx \\ &= \mathrm{m}_t(x).\end{aligned}$$

In particular, we see that m_a is a homotopy from m_t to the zero map, which restricts to a homotopy $\mathrm{m}_a \colon \langle X \rangle \to \langle X \rangle$ from $\mathrm{m}_t \colon \langle X \rangle \to \langle X \rangle$ to the zero map. A similar argument shows that $\overline{\mathrm{m}}_a$ is a homotopy from $\overline{\mathrm{m}}_t \colon X^{\mathrm{as}} \to X^{\mathrm{as}}$ to the zero map. It follows that t annihilates both $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(X^{\mathrm{as}})$. □

We now assume that R is an integral domain with quotient field K. Furthermore we assume both A and X are free as graded R-modules. In this case, we set

$$A_K = \{a/r \mid a \in A \text{ and } r \in R\backslash\{0\}\} \quad \text{and} \quad X_K = \{x/r \mid x \in X \text{ and } r \in R\backslash\{0\}\}.$$

Note that A_K is an MDG K-algebra centered at K. Next we consider the conductor:

$$\mathfrak{c} = \{c \in A_K \mid c\langle X \rangle \subseteq \langle X \rangle\}.$$

The Leibniz law implies $\mathfrak{c}$ is an R-complex. We set $Q = \mathrm{d}(\mathfrak{c}_1) \cap R$. Then by the same argument as in the proposition above, we see that Q annihilates $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(X^{\mathrm{as}})$.

Example 2.3. Let us revisit example (1.1) where we keep the same notation except we write $F = F_{\mathrm{K}}$. Observe that

$$\begin{aligned}\frac{e_1}{x}[e_1, e_5, e_2] &= \frac{1}{x}\left([e_1^2, e_5, e_2] - [e_1, e_1e_5, e_2] + [e_1, e_1, e_5e_2] - [e_1, e_1, e_5]e_2\right) \\ &= -\frac{1}{x}[e_1, e_1e_5, e_2] \\ &= -\frac{1}{x}[e_1, yz^2e_{14} + xe_{45}, e_2] \\ &= -\frac{yz^2}{x}[e_1, e_{14}, e_2] - [e_1, e_{45}, e_2] \\ &= -[e_1, e_{45}, e_2].\end{aligned}$$

It follows that $\mathrm{d}(e_1/x) = x$ annihilates $\mathrm{H}\langle F \rangle$. Similar calculations likes this shows that $\mathfrak{m} = \langle x, y, z, w \rangle$ annihilates $\mathrm{H}\langle F \rangle$. It follows that

$$\mathrm{H}_i\langle F \rangle \cong \begin{cases} \Bbbk & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

One can interpret this as saying that the multiplication μ is very close to being associative (the failure for μ to being associative is reflected in the fact that $\mathrm{length}(\mathrm{H}\langle F \rangle) = 1$). Note that μ is not associative in homological degree 4 since

$$[e_1, e_{45}, e_2] = xyze_{1234} \neq 0.$$

In particular we have $\mathrm{uha}(F) = \mathrm{lha}(F) = 3$, whereas $\mathrm{ua}(F) = 4$ and $\mathrm{la}(F) = 3$. In some sense however, the nonzero associator $[e_1, e_{45}, e_2]$ isn't really anything *new*. Indeed, we obtained the nonzero associator $[e_1, e_{45}, e_2]$

from the nonzero associator $[e_1, e_5, e_2]$, so one could argue that $[e_1, e_{45}, e_2]$ being nonzero is simply a direct consequence of $[e_1, e_5, e_2]$ being nonzero. More generally, an element $\gamma \in \langle F \rangle$ should only be thought of as contributing something new towards the failure for μ to being associative if $\mathrm{d}\gamma = 0$ (otherwise one could argue that γ being nonzero is simply a consequence of the associators in $\mathrm{d}\gamma$ being nonzero). Similarly, if $\gamma = \mathrm{d}(\gamma')$ for some $\gamma' \in \langle F \rangle$, then again γ isn't contributing anything new towards the failure for μ to being associative since one could argue that γ being nonzero is a direct consequence of γ' being nonzero. Thus the associators which really do contribute something new towards the failure for μ to being associative should be the ones which represent nonzero elements in homology. This is how we interpret the associator homology of F. In this case, we have precisely one nontrivial associator $[e_1, e_5, e_2]$ which represents a nonzero element in homology (all other nonzero associators can be derived from the fact that $[e_1, e_5, e_2] \neq 0$). Finally, let $U\colon R^4 \to R$ be the map given by $U = (xyz, y^2z, yz^2, yzw)$. One can show that

$$F_i^{\mathrm{as}} = \begin{cases} \mathrm{coker}(U^\top) & \text{if } i = 4 \\ \mathrm{coker}(U) & \text{if } i = 3 \\ F_i & \text{else} \end{cases}$$

2.1.6 The Nucleus

Let A be an MDG R-algebra and let X be an MDG A-module. The **nuclear complex of** X, denoted $\mathrm{N}(X)$, is the R-subcomplex of X given by

$$\mathrm{N}(X) := \{x \in X \mid [a_1, a_2, x] = 0 \text{ for all } a_1, a_2 \in A\}.$$

Indeed, the Leibniz law implies $\mathrm{d}(\mathrm{N}(X)) \subseteq \mathrm{N}(X)$, so the differential of $\mathrm{N}(X)$ is simply the differential of X restricted to $\mathrm{N}(X)$. The **nucleus of** X, denoted $\mathrm{N}\langle X \rangle$, is defined to be the smallest MDG A-submodule of X which contains $\mathrm{N}(X)$. The nucleus of X plays a role that's similar to the center of a group G. In particular, every associative A-submodule of X is contianed in $\mathrm{N}\langle X \rangle$. We will also be interested in studying the **nuclear complex of** X **in** A, denoted $\mathrm{N}_A(X)$. This is the R-subcomplex of A given by

$$\mathrm{N}_A(X) := \{a \in A \mid [a, b, x] = 0 \text{ for all } b \in A \text{ and } x \in X\}.$$

Note that if $a_1, a_2 \in \mathrm{N}_A(X)$, then $a_1 a_2 \in \mathrm{N}_A(X)$. However in general, if $a \in \mathrm{N}_A(X)$ and $b \in A$, then $[ab, c, x] = a[b, c, x]$ The **nucleus of** X **in** A, denoted $\mathrm{N}_A\langle X \rangle$, is defined to be the smallest MDG A-ideal which contains $\mathrm{N}_A(X)$. There's also the following weaker notion we may consider: we define the **middle nuclear complex of** X, denoted $\mathrm{M}(X)$, to be the R-subcomplex of X given by

$$\mathrm{M}(X) := \{x \in X \mid [a_1, x, a_2] = 0 \text{ for all } a_1, a_2 \in A\},$$

By combining (13) with (14), one can check that $\mathrm{N}(X) \subseteq \mathrm{M}(X)$, however this inclusion may be strict. Indeed, by combining the identities (13) with (14) we obtain the identity

$$[a_1, x, a_2] = (-1)^{|a_1||a_2| + |a_2||x|}((-1)^{|a_1||a_2|}[a_2, a_1, x] - [a_1, a_2, x]) \tag{21}$$

In particular, we have $x \in \mathrm{M}(X)$ if and only if $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a_1, a_2 \in A$. However just because we have $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a, b \in A$ doesn't necessarily mean $[a_1, a_2, x] = 0$ for all $a_1, a_2 \in A$.

Proposition 2.4. *Let A be an MDG algebra. Then $\mathrm{N}(A)$ is an MDG subalgebra of A.*

Proof. Clearly we have $1 \in A$. Let $a, a' \in \mathrm{N}(A)$. Then for each $a_1, a_2 \in A$, we have

$$[aa', a_1, a_2] = a[a', a_1, a_2] + [a, a'a_1, a_2] - [a, a', a_1 a_2] + [a, a', a_1]a_2 = 0.$$

It follows that $aa' \in \mathrm{N}(A)$. Similarly, we have

$$[\mathrm{d}a, a_1, a_2] = \mathrm{d}[a, a_1, a_2] - (-1)^{|a|}[a, \mathrm{d}a_1, a_2] - (-1)^{|a| + |a_1|}[a, a_1, \mathrm{d}a_2] = 0.$$

It follows that $\mathrm{d}a \in \mathrm{N}(A)$. □

By using the identities (14), (15), and (16), one can show that every element in $\langle A \rangle$ can be expressed as the R-span of all elements of the form $a_1[a_2, a_3, a_4]$ where $|a_1| \leq |a_2|, |a_3|, |a_4|$. In fact, we can often do better than even this. Indeed, suppose $a_1 = az \neq 0$ for some homogeneous $a \in A$ with $|a| < |a_1|$ and homogeneous $z \in \mathrm{N}(A)$. Then we have

$$a_1[a_2, a_3, a_4] = a[za_2, a_3, a_4].$$

2.1.7 Multigraded Associativity Test

Suppose $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_d]$ and $\langle \boldsymbol{m} \rangle = \langle m_1, \dots, m_\ell \rangle$ be a monomial ideal in R, and let F be the minimal R-free resolution of R/I. Choose a multiplication μ on F which respects the multigrading giving it the structure of a multigraded MDG R-algebra. We denote by $\star = \star_\mu$ to be the R-bilinear map corresponding to μ in what follows. Let $e_1, \dots, e_\ell, e_{\ell+1}, \dots e_n$ be an ordered homogeneous basis of F where each e_i is multigraded with $\operatorname{multideg}(e_i) = m_i$. Recall that for each $1 \leq i, j \leq n$, there exists unique $r_{i,j}^k \in R$ such that

$$e_i \star e_j = \sum_{k=0}^{n} r_{i,j}^k e_k, \tag{22}$$

Since μ also respects the multigrading, we must have

$$r_{i,j}^k = c_{i,j}^k \frac{m_i m_j}{m_k},$$

where m_i, m_j, m_k are the monomials corresponding to the multidegrees of e_i, e_j, e_k, and where $c_{i,j}^k \in \Bbbk$ are called the **structured** $\Bbbk$**-coefficients** of μ. It would be nice if we could re-express (22) as

$$\left(\frac{e_i}{m_i}\right)\left(\frac{e_j}{m_j}\right) = \sum_k c_{i,j}^k \left(\frac{e_k}{m_k}\right), \tag{23}$$

but the problem is that F does not contain terms like e_i/m_i. In order to make sense of (22), we perform a base change. Namely let S be the multiplicatively closed set generated by $\{m_1, \dots, m_n\}$. We set $\widetilde{F} = F_{S,\mathbf{0}}$ to be the multidegree $\mathbf{0}$ component of F_S. The $\mathbb{N}^n$-graded MDG R-algebra structure on F induces an MDG $\Bbbk$-algebra structure on $\widetilde{F}$. The multiplication (23) makes perfect sense in the MDG $\Bbbk$-algebra $\widetilde{F}$. Denoting $\widetilde{e}_i = e_i/m_i$ for each i, we can re-express (23) as

$$\widetilde{e}_i \widetilde{e}_j = \sum_k c_{i,j}^k \widetilde{e}_k.$$

Theorem 2.3. *F is a DG R-algebra if and only if $\widetilde{F}$ is a DG $\Bbbk$-algebra.*

Proof. A straightforward calculation gives us

$$[e_i, e_j, e_k]_\mu = m_i m_j m_k [\widetilde{e}_i, \widetilde{e}_j, \widetilde{e}_k]_{\widetilde{\mu}}$$

for all i, j, k. Thus μ is associative if and only if $\widetilde{\mu}$ is associative. □

2.2 Multiplicators

Having discussed associators, we now wish to discuss multiplicators. Throughout this section, let A be an MDG R-algebra, let X be and Y be MDG A-modules, and let $\varphi \colon X \to Y$ be a chain map.

Definition 2.6. The are two types of multiplicators were are interested in:

1. The **multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi$, from $A \otimes_R X$ to Y defined by

$$[\cdot]_\varphi := \varphi\mu - \mu(1 \otimes \varphi).$$

Note that we use μ to denote both A-scalar multiplications μ_X and μ_Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot]_\varphi\colon A \times X \to Y$ (or more simply by $[\cdot,\cdot]$ if context is clear) to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot]_\varphi$). Thus we have

$$[a \otimes x]_\varphi = \varphi(ax) - a\varphi(x) = [a, x]$$

for all $a \in A$ and $x \in X$. We say φ is **multiplicative** if $[\cdot]_\varphi = 0$.

2. The 2-**multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi^{(2)}$, from $A \otimes_R A \otimes_R X$ to Y defined by

$$[\cdot]_\varphi^{(2)} := \varphi[\cdot]_\mu - [\cdot]_\mu(1 \otimes 1 \otimes \varphi)$$

where we write $[\cdot]_\mu$ to denote both the associator of X and the associator Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot]_\varphi\colon A \times X \to Y$ to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi^{(2)}$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot,\cdot,\cdot]_\varphi$). Thus we have

$$[a_1 \otimes a_2 \otimes x]_\varphi^{(2)} = \varphi([a_1, a_2, x]) - [a_1, a_2, \varphi(x)] = [a_1, a_2, x]_\varphi$$

for all $a_1, a_2 \in A$ and $x \in X$. We say φ is 2-**multiplicative** if $[\cdot]_\varphi^{(2)} = 0$.

Remark 4. If A and B are MDG R-algebras and $\varphi\colon A \to B$ is a chain map such that $\varphi(1) = 1$, then we view B as an MDG A-module with the A-scalar multiplication defined by $a \cdot b = \varphi(a)b$. In this case, the multiplicator of φ has the form

$$[a_1, a_2]_\varphi = \varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)$$

for all $a_1, a_2 \in A$.

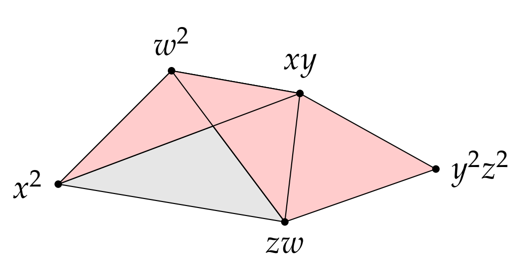
Example 2.4. Let us continue with Example (1.1) where we keep the same notation except we write $F = F_{\mathrm{K}}$ and $\boldsymbol{m} = \boldsymbol{m}_{\mathrm{K}}$. Let $\boldsymbol{m}' = x^2, w^2, y^2z^2$ and let $E' = \mathcal{K}(\boldsymbol{m}')$ be the Koszul R-algebra which resolves $R/\boldsymbol{m}'$. The standard homogeneous basis of E' is denoted by e'_σ. Choose a comparison map $\iota'\colon E' \to F$ which lifts the projection $R/\boldsymbol{m}' \to R/\boldsymbol{m}$ such that ι' is unital and respects the multigrading. Then ι' being a chain map together with the fact that it is unital and respects the multigrading forces us to have

$$\begin{aligned}
\iota'(e'_1) &= e_1 \\
\iota'(e'_2) &= e_2 \\
\iota'(e'_3) &= e_5 \\
\iota'(e'_{12}) &= e_{12} \\
\iota'(e'_{13}) &= yz^2e_{14} + xe_{45} \\
\iota'(e'_{23}) &= y^2ze_{23} + we_{35}.
\end{aligned}$$

Moreover, ι' can be defined at e'_{123} in two possible ways. Assume that it is defined by

$$\iota'(e'_{123}) = yz^2e_{124} + xyze_{234} - xwe_{345}.$$

We can picture $\iota'(E')$ inside of F as being supported on the red-shaded subcomplex below:

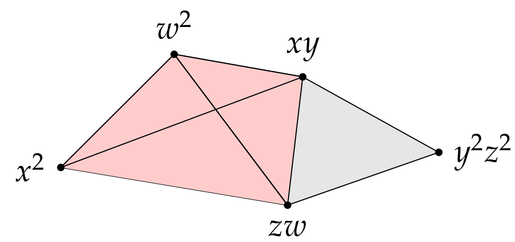


We now ask: is ι' an MDG algebra homomorphism? The answer is no. Indeed, clearly this map is a chain map which fixes the identity element, however it is not multiplicative. In fact, it's not even 2-multiplicative. To see this, assume for a contradiction that it was 2-multiplicative. Then we'd have

$$\begin{aligned}0 &= \iota'(0)\\ &= \iota'([e'_1, e'_2, e'_3])\\ &= [\iota'(e'_1), \iota'(e'_2), \iota'(e'_3)]\\ &= [e_1, e_2, e_5]\\ &\neq 0,\end{aligned}$$

which is an obvious contradiction.

Next let $\boldsymbol{m}'' = x^2, w^2, zw, xy$ and let $T'' = \mathcal{T}(\boldsymbol{m}'')$ be the Taylor algebra which resolves $R/\boldsymbol{m}''$. The standard homogeneous basis of T'' is denoted by e''_σ. Choose a comparison map $\iota''\colon T'' \to F$ which lifts the projection $R/\boldsymbol{m}'' \to R/\boldsymbol{m}$ such that ι'' is unital and respects the multigrading. Then ι'' being a chain map together with the fact that it is multigraded forces us to have $\iota''(e''_\sigma) = e_\sigma$ for all σ. We can picture $\iota''(T'')$ inside of F as being supported on the red-shaded subcomplex below:



This time it is easy to check that ι'' is an MDG algebra homomorphism. We give F the structure of an MDG T''-module using ι'' in the usual way. Notice that F is *not* associative as a T''-module, that is F is not a DG T''-module. Indeed, we have $[e_1, e_2, e_5] \neq 0$.

Finally let $\boldsymbol{t} = x^2 + w^2, w^2 + xy, x^2 + zw$. One can check that $\boldsymbol{t}$ is an R-regular sequence contained in $\langle \boldsymbol{m} \rangle$. Let $E = \mathcal{K}(\boldsymbol{t})$ be the Koszul R-algebra which resolve $R/\boldsymbol{t}$. The standard homogeneous basis of E is denoted by ϵ_σ. We begin to construct a comparison map $\iota\colon E \to F$ which lifts the projection $R/\boldsymbol{t} \to R/\boldsymbol{m}$ by setting

$$\begin{aligned}\iota(\epsilon_1) &= e_1 + e_2\\ \iota(\epsilon_2) &= e_2 + e_3\\ \iota(\epsilon_3) &= e_3 + e_4\end{aligned}$$

It is straightforward to check that this extends to a unique MDG algebra homomorphism by setting

$$\iota(\epsilon_\sigma) = \prod_{i \in \sigma} \iota(\epsilon_i).$$

We give F the structure of an MDG E-module using ι in the usual way. Again, note that F is not a DG E-module, however $\iota\colon E \to F$ *is* an MDG algebra homomorphism.

2.2.1 Multiplicator Identities

We want to familiarize ourselves with the multiplicator of $\varphi\colon X \to Y$, so in this subsubsection we collect together some identities which the multiplicator satisfies:

- For all $a \in A$ homogeneous and $x \in X$, we have the Leibniz law:

$$\mathrm{d}[a, x] = [\mathrm{d}a, x] + (-1)^{|a|}[a, \mathrm{d}x].$$

- For all $a \in A$ homogeneous and $x \in X$ homogeneous, we have

$$[a, x] = (-1)^{|a||x|}[x, a]. \tag{24}$$

- For all $a_1, a_2 \in A$ and $x \in X$, we have

$$a_1[a_2, x] - [a_1 a_2, x] + [a_1, a_2 x] = [a_1, a_2, x]_\varphi \tag{25}$$

Furthermore, if Z is another MDG A-module and $\psi\colon Y \to Z$ is another chain map, then for all $a \in A$ and $x \in X$, we have

$$[a,x]_{\psi\varphi} = \psi([a,x]_\varphi) + [a,\varphi(x)]_\psi \tag{26}$$

In particular, if ψ is multiplicative, then $\psi([Y]_\varphi) \subseteq [Z]_{\psi\varphi}$.

Remark 5. Let A and B be MDG R-algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we can rewrite (25) as follows: for all $a_1, a_2, a_3 \in A$, we have

$$\varphi(a_1)[a_2,a_3] - [a_1a_2,a_3] + [a_1,a_2a_3] - [a_1,a_2]\varphi(a_3) = [\varphi(a_1),\varphi(a_2),\varphi(a_3)] - \varphi([a_1,a_2,a_3]) \tag{27}$$

Indeed, this follows from the fact that

$$[\varphi(a_1),\varphi(a_2),\varphi(a_3)] = [a_1,a_2,\varphi(a_3)] - [a_1,a_2]\varphi(a_3).$$

In this case, we also have $[a,a]_\varphi = 0$ for all $a \in A$ where $|a|$ is odd.

Proposition 2.5. *Let A and B be MDG algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. The multiplicator map $[\cdot]_\varphi\colon A^{\otimes 2} \to B$ defined by*

$$[a_1 \otimes a_2]_\varphi = [a_1,a_2] = a_1a_2 - a_1 \star a_2,$$

where $\cdot$ denotes the multiplication in B and $\star$ denotes the multiplication in A is strictly graded-commutative.

Proof. Graded-commutativity is clear. To see why it is associative, observe that

$$\begin{aligned}
[[a_1,a_2],a_3] - [a_1,[a_2,a_3]] &= [a_1a_2,a_3] - [a_1 \star a_2,a_3] - [a_1,a_2a_3] + [a_1,a_2 \star a_3] \\
&= [a_1 \star a_2,a_3] - [a_1,a_2 \star a_3] + [a_1a_2,a_3] - [a_1,a_2a_3] \\
&= a_1[a_2,a_3] - [a_1,a_2]a_3 + [a_1a_2,a_3] - [a_1,a_2a_3] - [a_1,a_2,a_3]_\mu + [a_1,a_2,a_3]_\nu \\
&= (a_1 \star a_2)a_3 - a_1(a_2 \star a_3) + a_1 \star (a_2a_3) - (a_1a_2) \star a_3 - [a_1,a_2,a_3]_\mu + [a_1,a_2,a_3]_\nu \\
&= [a_1,a_2,a_3]_{\mu,\nu} - [a_1,a_2,a_3]_{\nu,\mu} + [a_1,a_2,a_3]_\nu - [a_1,a_2,a_3]_\mu
\end{aligned}$$

Where we used

$$a_1[a_2,a_3] - [a_1,a_2]a_3 = [a_1 \star a_2,a_3] - [a_1,a_2 \star a_3] + [a_1,a_2,a_3]_\mu - [a_1,a_2,a_3]_\nu$$

Note that

$$\begin{aligned}
a_1[a_2,a_3] &= a_1(a_2a_3) - a_1(a_2 \star a_3) \\
[a_1,a_2]a_3 &= (a_1a_2)a_3 - (a_1 \star a_2)a_3 \\
[a_1a_2,a_3] &= (a_1a_2)a_3 - (a_1a_2) \star a_3 \\
[a_1,a_2a_3] &= a_1(a_2a_3) - a_1 \star (a_2a_3)
\end{aligned}$$

Alternatively we have just shown

$$[\cdot,\cdot,\cdot]_{\nu-\mu} = [\cdot,\cdot,\cdot]_\nu - [\cdot,\cdot,\cdot]_\mu - [\cdot,\cdot,\cdot]_{\nu,\mu} + [\cdot,\cdot,\cdot]_{\mu,\nu}$$

□

2.2.2 The Maximal Multiplicative Quotient

The **multiplicator complex** of φ, denoted $[Y]_\varphi$, is the R-subcomplex of Y given by $[Y]_\varphi := \operatorname{im}[\cdot]_\varphi$, so the underlying graded module of $[Y]_\varphi$

$$[Y]_\varphi := \operatorname{span}_R\{[a,x]_\varphi \mid a \in A \text{ and } x \in X\},$$

and the differential of $[Y]_\varphi$ is simply the restriction of the differential of Y to $[Y]_\varphi$. In order to avoid confusion with the associator complex, we will always write φ in the subscript of $[Y]_\varphi$. Even though the multiplicator complex of φ is closed under the differential, it need not be closed under A-scalar multiplication. In other words, if $a_1, a_2 \in A$ and $x \in X$, then it need not be the case that $a_1[a_2,x]_\varphi \in [Y]_\varphi$. We denote by $\langle Y\rangle_\varphi$ to be the MDG A-submodule of Y generated by $[Y]_\varphi$. In other words, $\langle Y\rangle_\varphi$ is the smallest MDG A-submodule of Y which contains $[Y]_\varphi$. Unlike the associator submodule, the multiplicator submodule is difficult to describe in terms of an R-span of elements. Indeed, as a first guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{[a,x]_\varphi \mid a \in A \text{ and } x \in X\}. \tag{28}$$

However this is clearly incorrect in general as we may need to adjoin elements of the form $a_1[a_2, x]$ to (28). As a second guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi \mid a_1, a_2 \in A \text{ and } x \in X\}. \tag{29}$$

However this isn't correct in general either since the identity

$$a_1(a_2[a_3, x]_\varphi) = (a_1a_2)[a_3, x]_\varphi - [a_1, a_2, [a_3, x]_\varphi]$$

tells us that should really adjoin elements of the form $a_1[a_2, a_3, [a_4, x]]$ to (29) as well. As a third guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi,\ a_1[a_2, a_3, [a_4, x]_\varphi] \mid a_1, a_2, a_3, a_4 \in A \text{ and } x \in X\}. \tag{30}$$

Again this isn't correct in general since the identity

$$a_1(a_2[a_3, a_4, [a_5, x]_\varphi]) = (a_1a_2)[a_3, a_4, [a_5, x]] - [a_1, a_2, [a_3, a_4, [a_5, x]_\varphi]].$$

tells us that we should really adjoin elements of the form $a_1[a_2, a_3, [a_4, a_5, [a_6, x]_\varphi]]$ to (30) as well. The problem continues getting worse with no end in sight. It turns out however, that if φ is 2-multiplicative, then $\langle Y\rangle_\varphi$ given by (28).

Proposition 2.6. *If φ is 2-multiplicative, then for all $a_1, a_2, a_3 \in A$ and $x \in X$ we have*

$$a_1[a_2, x]_\varphi = [a_1a_2, x]_\varphi - [a_1, a_2x]_\varphi \quad \textit{and} \quad [a_1, a_2, [a_3, x]_\varphi] = [[a_1, a_2, a_3], x]_\varphi - [a_1, [a_2, a_3, x]]_\varphi. \tag{31}$$

In particular, $\langle Y\rangle_\varphi$ is given by (28).

Proof. A straightforward calculation yields

$$a_1[a_2, a_3, x]_\varphi = [a_1a_2, a_3, x]_\varphi - [a_1, a_2a_3, x]_\varphi + [a_1, a_2, a_3x]_\varphi - [[a_1, a_2, a_3], x]_\varphi + [a_1, [a_2, a_3, x]]_\varphi - [a_1, a_2, [a_3, x]_\varphi].$$

Using this identity together with the identity (25), we see that if φ is 2-multiplicative, then we obtain (31). This implies all elements of the form $a_1[a_2, x]$ and $a_1[a_2, a_3, [a_4, x]]$ belong to (28). An easy induction argument shows that $\langle Y\rangle_\varphi$ is given by (28). $\square$

The quotient $Y/\langle Y\rangle_\varphi$ is an MDG A-module. We denote by $\pi\colon Y \to Y/\langle Y\rangle_\varphi$ to be the canonical quotient map. Note that both π and $\pi\varphi$ are multiplicative. Therefore (26) implies $[Y]_\varphi \subseteq \ker \pi$ which implies $\langle Y\rangle_\varphi \subseteq \ker \pi$. We call $Y/\langle Y\rangle_\varphi$ (together with its canonical quotient map π) the **maximal multiplicative quotient** of $\varphi\colon X \to Y$; it satisfies the following universal mapping property:

Proposition 2.7. *For all MDG A-modules Z and for all chain maps $\psi\colon Y \to Z$ where both ψ and $\psi\varphi$ are MDG A-module homomorphisms (hence both are multiplicative), there exists a unique MDG A-module homomorphism $\overline{\psi}\colon Y/\langle Y\rangle_\varphi \to Z$ such that $\overline{\psi}\pi = \psi$. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \arrow[dl, "\psi"'] \arrow[d, "\pi"] \\ Z & Y/\langle Y\rangle_\varphi \arrow[l, dashed, "\overline{\psi}"] \end{tikzcd} \tag{32}$$

Proof. Suppose $\psi\colon Y \to Z$ is such a map. Then (26) implies $[Y]_\varphi \subseteq \ker \psi$ which implies $\langle Y\rangle_\varphi \subseteq \ker \psi$. Thus the map $\overline{\psi}\colon Y/\langle Y\rangle_\varphi \to Z$ given by

$$\overline{\psi}(\overline{y}) := \psi(y),$$

where $\overline{y} \in Y/\langle Y\rangle_\varphi$ and where $y \in Y$ is a choice of an element in Y such that $\pi(y) = \overline{y}$, is well-defined. Furthermore, it is easy to check that $\overline{\psi}$ is an MDG A-module homomorphism and the unique such map which makes the diagram (46) commute. $\square$

Remark 6. Let $[a_1, a_2] = a_1a_2 - a_1 \star a_2$. Observe that

3 The Associator Functor

Let X and Y be MDG A-modules and let $\varphi\colon X \to Y$ be a chain map. If φ is multiplicative, then observe that for all $a_1, a_2, a_3 \in A$ and $x \in X$, we have

$$\varphi(a_1[a_2, a_3, x]) = a_1[a_2, a_3, \varphi(x)]. \tag{33}$$

Thus φ restricts to an MDG A-module homomorphism $\varphi\colon \langle X\rangle \to \langle Y\rangle$. In particular, the assignment $X \mapsto \langle X\rangle$ induces a functor from category of MDG A-modules to itself. We call this the **associator functor.**

3.1 Failure of Exactness

The associator functor need not be exact. Indeed, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{34}$$

be a short exact sequence of MDG A-modules. We obtain an induced sequence of MDG A-modules

$$0 \longrightarrow \langle X\rangle \xrightarrow{\varphi} \langle Y\rangle \xrightarrow{\psi} \langle Z\rangle \longrightarrow 0 \tag{35}$$

which is exact at $\langle X\rangle$ and $\langle Z\rangle$ but not necessarily exact at $\langle Y\rangle$. In order to ensure exactness of (35), we need to place a condition on (34). This leads us to consider the following definition:

Definition 3.1. Let X be an MDG A-submodule of Y. We say Y is an **associative extension** of X if it satisfies

$$\langle X\rangle = X \cap \langle Y\rangle.$$

It is easy to see that (35) is a short exact sequence of MDG A-modules if and only if Y is an associative extension of $\varphi(X)$. In this case, we obtain a long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{i+1}\langle Z\rangle \arrow[dll, rounded corners] \\ \mathrm{H}_i\langle X\rangle \arrow[r] & \mathrm{H}_i\langle Y\rangle \arrow[r] & \mathrm{H}_i\langle Z\rangle \arrow[dll, rounded corners] \\ \mathrm{H}_{i-1}\langle X\rangle \arrow[r] & \cdots & \end{tikzcd} \tag{36}$$

We can use this long exact sequence to deduce interesting theorems like:

Theorem 3.1. *Let X be an MDG A-module and suppose Y is an associative extension of X. Then Y is homologically associative if and only if X and Y/X are homologically associative.*

3.2 An Application of the Long Exact Sequence

Assume that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be the minimal R-free resolution of R/I, which is equipped with a multiplication μ giving it the structure of an MDG R-algebra, and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $F + eF$ is the minimal R-free resolution of $R/\langle I, r\rangle$. Here, e is thought of as an exterior variable of degree 1. The differential of the mapping cone is given by

$$\mathrm{d}(a + eb) = \mathrm{d}(a) + rb - e\mathrm{d}(b)$$

for all $a, b \in F$. We give $F + eF$ the structure of an MDG R-algebra by extending the multiplication on F to a multiplication on $F + eF$ by setting

$$(a + eb)(c + ed) = ac + e(bc + (-1)^{|a|}ad)$$

for all $a, b, c, d \in F$. In particular, note that $(eb)c = e(bc)$ for all $b, c \in F$, so e belongs to the nucleus of $F + eF$. We denote by $\iota\colon F \to F + eF$ to be the inclusion map. We can view $F + eF$ either as an MDG F-module or as an MDG R-algebra, thus we potentially have two different associator complexes to consider. It turns out that however these give rise to the same R-complex since e is in the nucleus of $F + eF$:

Theorem 3.2. *Let $\langle F+eF\rangle_F$ be the associator F-submodule of $F+eF$ and let $\langle F+eF\rangle$ be the associator $(F+eF)$-ideal of $F+eF$. Then*

$$\langle F+eF\rangle_F = \langle F\rangle + e\langle F\rangle = \langle F+eF\rangle. \tag{37}$$

In particular, $F+eF$ is an associative extension of F. More generally, suppose $\boldsymbol{r}=r_1,\dots,r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$. We set

$$F+\boldsymbol{e}F = F+\sum_{i=1}^m e_iF$$

to be minimal R-free resolution of $R/\langle I,\boldsymbol{r}\rangle$ obtained by iterating the mapping cone construction as above, where e_i is an exterior variable of degree 1 which satisfies $\mathrm{d}e_i=r_i$, and where we extend the multiplication of F to a multiplication on $F+\boldsymbol{e}F$ by extending it from $F+\sum_{i=1}^k e_iF$ to $F+\sum_{i=1}^{k+1}e_iF$ for each $1\le k<m$ as above. Then

$$\langle F+\boldsymbol{e}F\rangle_F = \langle F\rangle + \boldsymbol{e}\langle F\rangle = \langle F+\boldsymbol{e}F\rangle \tag{38}$$

where we set $\boldsymbol{e}\langle F\rangle := \sum_{i=1}^m e_i\langle F\rangle$. In particular, $F+\boldsymbol{e}F$ is an associative extension of F.

Proof. Since e is in the nucleus, we have $e[a,b,c]=[ea,b,c]$ for all $a,b,c\in F$. Similarly we have

$$\begin{aligned}[a,b,ec] &= -(-1)^{|a||b|+|a||ec|+|ec||b|}[ec,b,a]\\ &= -(-1)^{|a||b|+|a||c|+|b||c|}[ec,b,a]\\ &= -(-1)^{|a||b|+|a||c|+|b||c|}e[c,b,a]\\ &= e[a,b,c]\end{aligned}$$

for all $a,b,c\in F$. Similarly we have

$$\begin{aligned}[a,eb,c] &= -(-1)^{|a||eb|+|a||c|}[eb,c,a]-(-1)^{|eb||c|+|a||c|}[c,a,eb]\\ &= e(-(-1)^{|a||eb|+|a||c|}[b,c,a]-(-1)^{|eb||c|+|a||c|}[c,a,b])\\ &= e[a,b,c]\end{aligned}$$

for all $a,b,c\in F$. Thus we have

$$\begin{aligned}(a+ea')[b+eb',c+ec',d+ed'] &= (a+ea')[b,c,d]+(a+ea')(e[b',c',d'])\\ &= a[b,c,d]+ea'[b,c,d]+(-1)^{|a|}ea[b',c',d']\\ &= a[b,c,d]+e(a'[b,c,d]+(-1)^{|a|}a[b',c',d'])\end{aligned}$$

for all $a,b,c,d,a',b',c',d'\in F$. Thus we obtain (37). To see why (37) implies $F+eF$ is an associative extension of F, note that

$$F\cap\langle F+eF\rangle = F\cap(\langle F\rangle+e\langle F\rangle)=\langle F\rangle.$$

The last part of the theorem follows from induction. □

Theorem 3.3. *Let $\varepsilon=\mathrm{lha}(F)$ and let $\delta=\mathrm{uha}(F)$. Then $\mathrm{lha}(F+eF)=\varepsilon$ and*

$$\mathrm{uha}(F+eF)=\begin{cases}\delta & \text{if } r \text{ is } \mathrm{H}_\delta\langle F\rangle\text{-regular}\\ \delta+1 & \text{otherwise}\end{cases} \tag{39}$$

Moreover, we have a short exact sequence of $R/\langle I,r\rangle$-modules

$$0\longrightarrow \mathrm{H}_i\langle F\rangle/r\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F+eF\rangle \longrightarrow 0:_{\mathrm{H}_{i-1}\langle F\rangle} r\longrightarrow 0 \tag{40}$$

for each $i\in\mathbb{Z}$. In particular, we have an isomorphism of $R/\langle I,r\rangle$-modules

$$\mathrm{H}_\varepsilon\langle F\rangle/r\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F+eF\rangle.$$

Proof. Since $F + eF$ is an associative extension of F, we obtain a long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_i\langle F\rangle \arrow[dll, rounded corners, "r"'] \\ \mathrm{H}_i\langle F\rangle \arrow[r] & \mathrm{H}_i\langle F+eF\rangle \arrow[r] & \mathrm{H}_{i-1}\langle F\rangle \arrow[dll, rounded corners, "r"'] \\ \mathrm{H}_{i-1}\langle F\rangle \arrow[r] & \cdots & \end{tikzcd} \tag{41}$$

We obtain (42) as well as (41) from this long exact sequence. We obtain $\mathrm{lha}(F + eF) = \varepsilon$ from the longe exact sequence together with an application of Nakayama's lemma. $\square$

Corollary 1. *Suppose $\boldsymbol{r} = r_1, \ldots, r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$ and let $F + \boldsymbol{e}F$ be th corresponding R-free resolution of $R/\langle I, \boldsymbol{r}\rangle$ obtained by iterating the mapping cone construction. Then we obtain a short exact sequence of $R/\langle I, \boldsymbol{r}\rangle$-modules*

$$0 \longrightarrow \mathrm{H}_i\langle F\rangle/\boldsymbol{r}\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + \boldsymbol{e}F\rangle \longrightarrow 0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r} \longrightarrow 0 \tag{42}$$

In particular, have an isomorphism of $R/\langle I, \boldsymbol{r}\rangle$-modules:

$$\mathrm{H}_\varepsilon\langle F\rangle/\boldsymbol{r}\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F + \boldsymbol{e}F\rangle.$$

We also have the length formula:

$$\ell(\mathrm{H}_i\langle F + \boldsymbol{e}F\rangle) = \ell(\mathrm{H}_i\langle F\rangle/\boldsymbol{r}\mathrm{H}_i\langle F\rangle) + \ell(0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r}),$$

here $\ell(-)$ is the length function.

4 The Symmetric DG Algebra

Let R be a commutative ring, let A be a $\mathbb{Z}$-graded R-module such that $A_0 = R$ which is also equipped with a $\mathbb{Z}$-linear differential $\mathrm{d}\colon A \to A$ giving it the structure of a chain complex. Note that that the differential need not be R-linear and note that A may be nonzero in negative homological degree. In this section, we will construct the symmetric DG algebra of A, which we denote by $\mathrm{S}(A) = \mathrm{S}$ (where we omit A from the notation when context is clear). After constructing the symmetric DG algebra in this general setting, we then specialize to the case we are mostly interesting in: we say A is **centered** at R if A is an R-complex with $A_0 = R$ and $A_{<0} = 0$ (in particular the differential of A is R-linear). In this case, we sometimes denote the symmetric DG algebra of A by $\mathrm{S}_R(A)$ with R in the subscript in order to emphasize that A is centered at R.

Before we give a rigorous construction of the symmetric DG algebra, we wish to help motivate the reader by giving an informal description of it in this special case where A is centered at R. In this case, the underlying graded algebra of $\mathrm{S} = \mathrm{S}_R(A)$ is the usual symmetric R-algebra $\mathrm{Sym}(A_+)$ where we view A_+ as just an R-module. However S obtains a bi-graded structure using homological degree and total degree: we have a decomposition of S into R-modules:

$$\mathrm{S} = \bigoplus_{i\geq 0} \mathrm{S}_i = \bigoplus_{m\geq 0} \mathrm{S}^m = \bigoplus_{i,m\geq 0} \mathrm{S}_i^m.$$

We refer to the i in the subscript as homological degree and we refer to the m in the superscript as total degree. We have

$$\mathrm{S}_0 = \mathrm{S}^0 = \mathrm{S}_0^0 = R \quad \text{and} \quad \mathrm{S}^1 = A_+.$$

More generally, for $i, m \geq 1$, the R-module S_i^m is the R-span of all homogeneous elementary products of the form $\boldsymbol{a} = a_1 \cdots a_m$ where $a_1, \ldots, a_m \in A_+$ are homogeneous (with respect to homological degree of course) such that

$$|\boldsymbol{a}| = |a_1| + \cdots + |a_m| = i.$$

In particular, note that $A = \mathrm{S}^{\leq 1} = R + A_+$, thus we may (and do) view A a sitting inside S. The differential of S extends the differential of A in a natural way and is defined on homogeneous elementary products $\boldsymbol{a} = a_1 \cdots a_m$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \cdots \mathrm{d}(a_j) \cdots a_m. \tag{43}$$

If each of the a_j in (43) live in homological degree ≥ 2, then $\mathrm{d}\boldsymbol{a}$ and $\boldsymbol{a}$ has the same total degree, namely $\deg(\mathrm{d}\boldsymbol{a}) = m = \deg \boldsymbol{a}$. However if one of the a_j in (43) lives in homological degree 1, then $\deg(\mathrm{d}\boldsymbol{a}) = m - 1$. The diagram below illustrates how the differential acts on the bi-graded components:

$$
\begin{tikzcd}
A_i \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathrm{S}_i^{m-1} \arrow[dl] \arrow[dr] & & \mathrm{S}_i^{m} \arrow[dl] \arrow[dr] & & \mathrm{S}_i^{m+1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathbb{K}_i \arrow[dl] \\
& A_{i-1} & & \mathrm{S}_{i-1}^{m-2} & & \mathrm{S}_{i-1}^{m-1} & & \mathrm{S}_{i-1}^{m} & & \mathrm{S}_{i-1}^{m+1} & & \mathbb{K}_{i-1} &
\end{tikzcd}
$$

where we set $\mathbb{K}$ to be the koszul DG algebra induced by $\mathrm{d}\colon A_1 \to A_0$. Thus the differential of S connects the usual differential of A on the far left to a koszul differential on the far right. In order to keep track of how the differential operates on the bi-graded components, we express d as

$$\mathrm{d} = \eth + \partial,$$

where $\eth$ is the component of d which respects total degree and where ∂ is the component of d which drops total degree by 1. In the next example, we consider a free resolution of a cyclic module and work out what the symmetric DG algebra looks like in this case.

Example 4.1. Let $R = \Bbbk[x, y]$, let $I = \langle x^2, xy \rangle$, and let F be Taylor resolution of R/I. Let's write down the homogeneous components of F as a graded R-module as well as how the differential acts on the homogeneous basis:

$$
\begin{aligned}
F_0 &= R & \qquad\qquad \mathrm{d}e_1 &= x^2 \\
F_1 &= Re_1 + Re_2 & \mathrm{d}e_2 &= xy \\
F_2 &= Re_{12}, & \mathrm{d}e_{12} &= xe_2 - ye_1,
\end{aligned}
$$

Note that the Taylor resolution usually comes equipped with a multiplication called the Taylor multiplication. Let us denote this by $\star$ so as not to confuse it with the multiplication $\cdot$ of $\mathrm{S} = \mathrm{S}_R(F)$. Now let's write down the homogeneous components of S as a graded R-module (with respect to homological degree): we have

$$
\begin{aligned}
\mathrm{S}_0 &= R \\
\mathrm{S}_1 &= Re_1 + Re_2 \\
\mathrm{S}_2 &= Re_{12} + Re_1e_2 \\
\mathrm{S}_3 &= Re_1e_{12} + Re_2e_{12} \\
\mathrm{S}_4 &= Re_{12}^2 + Re_1e_2e_{12} \\
&\vdots
\end{aligned}
$$

Thus we see that S is much larger than F. Note that $\mathrm{S}_4^3 = Re_1e_2e_{12}$ and $\mathrm{S}_4^2 = Re_{12}^2$. Also note that

$$
\begin{aligned}
\mathrm{d}(e_1e_2 - e_1 \star e_2) &= \mathrm{d}(e_1e_2 - xe_{12}) \\
&= \mathrm{d}(e_1)e_2 - e_1\mathrm{d}(e_2) - x\mathrm{d}(e_{12}) \\
&= x^2e_2 - xye_1 - x(xe_2 - ye_1) \\
&= x^2e_2 - xye_1 - x^2e_2 + xye_1 \\
&= 0.
\end{aligned}
$$

In fact, we claim that $f_{12} := e_1e_2 - xe_{12}$ represents a nonzero element in $\mathrm{H}_2(\mathrm{S})$. To see this, note that f_{12} lives in $\mathrm{S}_2^1 \oplus \mathrm{S}_2^2$. Therefore if $\mathrm{d}f = f_{12}$, then f must live in $\mathrm{S}_3^2 = Re_1e_{12} + Re_2e_{12}$ since $\mathrm{S}_3^1 = 0 = \mathrm{S}_3^3$. However a calculation shows

$$
\begin{aligned}
\mathrm{d}(e_1e_{12}) &= x^2e_{12} - e_1(xe_2 - ye_1) = xf_{12} \\
\mathrm{d}(e_2e_{12}) &= xye_{12} - e_2(xe_2 - ye_1) = yf_{12}.
\end{aligned}
$$

In particular we see that $\mathrm{H}_2(\mathrm{S}) = \Bbbk\overline{f}_{12}$.

4.1 Construction of the Symmetric DG Algebra of A

We now provide a rigorous construction of $\mathrm{S}(A) = \mathrm{S}$ in the general case where the differential of A need not be R-linear and where $A_{<0}$ is not necessarily zero. Our construction will occur in three steps:

Step 1: We define the **non-unital tensor DG algebra** of A to be

$$\mathrm{U}(A) = \mathrm{U} := \bigoplus_{n=1}^{\infty} A^{\otimes n},$$

where the tensor product is taken as $\mathbb{Z}$-complexes. An elementary tensor in U is denoted $\boldsymbol{a} = a_1 \otimes \cdots \otimes a_n$ where $a_1, \ldots, a_n \in A$ and $n \geq 1$. The differential of U is denoted by d again to simplify notation and is defined on $\boldsymbol{a}$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{n} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots \otimes a_n.$$

We say $\boldsymbol{a}$ is a homogeneous elementary tensors if each a_i is a homogeneous element in A. In this case, we set

$$|\boldsymbol{a}| = \sum_{i=1}^{n} |a_i| \quad \text{and} \quad \deg \boldsymbol{a} = \sum_{i=1}^{n} \deg a_i,$$

where deg is defined on elements $a \in A$ by

$$\deg a = \begin{cases} 1 & \text{if } a \in A_{>0} \\ 0 & \text{if } a \in R \\ -1 & \text{if } a \in A_{<0} \end{cases}$$

We call $|\boldsymbol{a}|$ the **homological degree** of $\boldsymbol{a}$ and we call $\deg \boldsymbol{a}$ the **total degree** of $\boldsymbol{a}$. With $|\cdot|$ and deg defined, we observe that U admits a bi-graded decomposition:

$$\mathrm{U} = \bigoplus_{i\in\mathbb{Z}} \mathrm{U}_i = \bigoplus_{m\in\mathbb{Z}} \mathrm{U}^m = \bigoplus_{i,m\in\mathbb{Z}} \mathrm{U}_i^m,$$

where the component U_i^m consists of all finite $\mathbb{Z}$-linear combinations of homogeneous elementary tensors $\boldsymbol{a} \in \mathrm{U}$ such that $|\boldsymbol{a}| = i$ and $\deg \boldsymbol{a} = m$. We equip U with an associative (but not commutative nor unital) bi-graded $\mathbb{Z}$-bilinear multiplication which is defined on homogeneous elementary tensors by $(\boldsymbol{a}, \boldsymbol{a}') \mapsto \boldsymbol{a} \otimes \boldsymbol{a}'$ and is extended $\mathbb{Z}$-bilinearly everywhere else. This multiplication is easily seen to satisfy Leibniz law, however note that U is not unital under this multiplication since $(1, 1) \mapsto 1 \otimes 1 \neq 1$ (hence why we call this the *non-unital* tensor DG algebra). Also note that U already comes equipped with an R–scalar multiplication (from the R-module structure on A), denoted $(r, \boldsymbol{a}) \mapsto r\boldsymbol{a}$, however the multiplication of U only agrees with the R-scalar multiplication wherever they are both defined and vanish. Let $\mathfrak{u}$ to be the U-ideal by all elements of the form

$$[r, a]_\mu = r \otimes a - ra \qquad [a, r]_\mu = a \otimes r - ar$$
$$[r, a]_\mathrm{d} = \mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a) \qquad [a, r]_\mathrm{d} = (-1)^{|a|} a \otimes \mathrm{d}r - \mathrm{d}(ar) + (\mathrm{d}a)r$$

where $r \in R$ and $a \in A$. We claim that the differential of U maps $\mathfrak{u}$ to itself. Indeed, given $r \in R$ and $a \in A$, we have

$$\begin{aligned} \mathrm{d}[r, a]_\mu &= \mathrm{d}(r \otimes a) - \mathrm{d}(ra) \\ &= \mathrm{d}r \otimes a + r \otimes \mathrm{d}a - \mathrm{d}r \otimes a + r(\mathrm{d}a) + [r, a]_\mathrm{d} \\ &= r \otimes \mathrm{d}a + r(\mathrm{d}a) + [r, a]_\mathrm{d} \\ &= [r, \mathrm{d}a]_\mu + [r, a]_\mathrm{d} \\ &\in \mathfrak{u}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathrm{d}[r, a]_\mathrm{d} &= \mathrm{d}(\mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r \otimes \mathrm{d}a - [r, \mathrm{d}a]_\mu) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}r \otimes \mathrm{d}a - \mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -\mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -[r, \mathrm{d}a]_\mathrm{d} \\ &\in \mathfrak{u}. \end{aligned}$$

Similar calculations show $\mathrm{d}[a, r]_\mu \in \mathfrak{u}$ and $\mathrm{d}[r, a]_\mathrm{d} \in \mathfrak{u}$.

Step 2: We define the **tensor DG algebra** of A to be the quotient

$$\mathrm{T}(A) = \mathrm{T} := \mathrm{U}/\mathfrak{u}.$$

The multiplication of U induces a multiplication on T which not only becomes unital but also agrees with the R-scalar multiplication on T where they are both defined. Since $\mathfrak{u}$ is generated by elements which are homogeneous with respect to homological degree and since the differential of U maps $\mathfrak{u}$ to itself, it follows that the differential of U induces a differential on T, which we again denote by d again. This gives T the structure of a non-commutative (but unital) DG $\Bbbk$-algebra, where

$$\Bbbk = \{r \in R \mid \mathrm{d}r \otimes a = 0 \text{ for all } a \in A\}.$$

In other words, the differential of T satisfies Leibniz law and is $\Bbbk$-linear. Note that the generators $[r, a]_\mu$ of $\mathfrak{u}$ is also homogeneous with respect to total degree, however the generator $[r, a]_{\mathrm{d}}$ is homogeneous with respect to total degree if and only if either $\mathrm{d}r \otimes a = 0$, or $\mathrm{d}(ra) = r\mathrm{d}a$, or $|a| \in \{0, 1\}$. In particular, $\mathfrak{u}$ will be homogeneous with respect to total degree if A is centered at R (which is a case we are interested in). In this case, T inherits from U a bi-graded R-algebra structure:

$$\mathrm{T} = \bigoplus_{i \in \mathbb{Z}} \mathrm{T}_i = \bigoplus_{m \in \mathbb{Z}} \mathrm{T}^m = \bigoplus_{i, m \in \mathbb{Z}} \mathrm{T}_i^m.$$

If we assume in addition that the differential of A is R-linear to begin with, then we have

$$\mathrm{T}_0 = \mathrm{T}^0 = \mathrm{T}_0^0 = R \quad \text{and} \quad \mathrm{T}^1 = A_+.$$

More generally, for $i, m \geq 1$ the component T_i^m consists of all finite R-linear combinations of homogeneous elementary tensors of the form $\boldsymbol{a} = a_1 \otimes \cdots \otimes a_m$ where $a_1, \dots, a_m \in A_+$ and where $|\boldsymbol{a}| = i$.

Example 4.2. Let us describe what the total degree m component of T looks like in the case where the differential of A is R-linear and where $A_{<0} = 0$. We have

$$\begin{aligned}
\mathrm{T}^0 &= R \\
\mathrm{T}^1 &= \bigoplus_{1 \leq i} A_i \\
\mathrm{T}^2 &= \bigoplus_{1 \leq i < j} ((A_i \otimes A_j) \oplus (A_j \otimes A_i)) \oplus \bigoplus_{1 \leq i} A_i^{\otimes 2}
\end{aligned}$$

The component T^3 is slightly more complicated:

$$\bigoplus_{\substack{1 \leq i < j < k \\ \pi \in S_3}} (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(k)}) \oplus \bigoplus_{\substack{1 \leq i < j \\ \pi \in S_2}} ((A_{\pi(i)}^{\otimes 2} \otimes A_{\pi(j)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(i)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)}^{\otimes 2})) \oplus \bigoplus_{1 \leq i} A_i^{\otimes 3}.$$

We set $\mathfrak{t}$ to be the T-ideal generated by all elements of the form

$$[a_1, a_2]_\sigma \colon = (-1)^{|a_1||a_2|} a_2 \otimes a_1 - a_1 \otimes a_2 \quad \text{and} \quad [a]_\tau := a \otimes a,$$

where $a, a_1, a_2 \in A$ are homogeneous and $|a|$ is odd. Observe that d maps $\mathfrak{t}$ to itself since if $a, a_1, a_2 \in A$ are homogeneous with $|a|$ odd, then we have

$$\mathrm{d}[a_1, a_2]_\sigma = [\mathrm{d}a_1, a_2]_\sigma + (-1)^{|a_1|}[a_1, \mathrm{d}a_2]_\sigma \in \mathfrak{t} \quad \text{and} \quad \mathrm{d}[a]_\tau = [\mathrm{d}a, a]_\sigma \in \mathfrak{t}$$

Step 3: We define the **symmetric DG algebra** of A to be the quotient

$$\mathrm{S}(A) = \mathrm{S} := \mathrm{T}/\mathfrak{t}$$

The image of a homogeneous elementary tensor $a_1 \otimes \cdots \otimes a_m$ in S is often denoted $a_1 \cdots a_n$ and is called a homogeneous elementary product. Since $\mathfrak{t}$ is generated by elements which are homogeneous with respect to both homological degree and since the differential of T maps $\mathfrak{t}$ to itself, we see that the differential of T induces a differential on S, which we again denote by $\mathrm{d} \colon \mathrm{S} \to \mathrm{S}$, giving it the structure of a strictly graded-commutative DG $\Bbbk$-algebra. Furthemore, if T inherits the bi-graded structure from U, then S inherits from T the bi-graded structure from T since $\mathfrak{t}$ is generated by elements which are homogeneous with respect to total degree.

4.2 Properties of the Symmetric DG Algebra

We now focus our attention to the case where A is an R-complex centered at R and we wish to study $S = \mathrm{S}_R(A)$ the symmetric DG R-algebra of A. In this case, the underlying graded R-algebra of S is the usual symmetric algebra of A_+ where:

$$\mathrm{Sym}(A_+) = \mathrm{Sym}_R(A_+) = \frac{\bigoplus_{m\geq 0} A_+^{\otimes m}}{\langle\{[a_1, a_2]_\sigma, [a]_\tau\}\rangle}.$$

Thus the symmetric DG algebra of A inherits all of the properties that are satisfied by the symmetric algebra of A_+ when we forget about the differential. For instance, recall that a bounded below R-complex is semiprojective if and only if its underlying graded R-module is projective as a graded R-module. In particular, if A is semiprojective, then S is semiprojective too. Thus if we assume that A is semiprojective *and* that there exists a chain map $\pi\colon S \to A$ which splits the inclusion map $\iota\colon A \hookrightarrow S$, then we can lift chains maps out of A along surjective quasiisomorphisms, meaning if $\varphi\colon A \to X$ is any chain map and $\tau\colon Y \to X$ is any surjective quasiisomorphism, then there exists a chain map $\widetilde{\varphi}\colon S \to Y$ such that $\tau\widetilde{\varphi} = \varphi$, moreover such a lift is unique up to homotopy. The assumption that A is semiprojective is mild whereas the assumption that there exists a chain map $S \to A$ which splits the inclusion map $A \hookrightarrow S$ is rather subtle. We will see that if A has a DG R-algebra structure on it, then there will be such a map $S \to A$.

Proposition 4.1. *Let R be a commutative ring and let A be an R-complex centered at R.*

1. *(Base Change) Let R' be an R-algebra. Then*

$$\mathrm{S}_R(A) \otimes_R R' = \mathrm{S}_{R'}(A \otimes_R R').$$

2. *(Exact Sequences) Let*

$$B \longrightarrow A \longrightarrow A' \longrightarrow 0 \tag{44}$$

be an exact sequence of R-complexes where A' is centered at a cyclic R-algebra, say $R' = R/I$ for some ideal I of R. Then we obtain an exact sequence

$$\mathrm{S}_R(A) \otimes_R B \longrightarrow \mathrm{S}_R(A) \longrightarrow \mathrm{S}_{R'}(A') \longrightarrow 0 \tag{45}$$

.

3. *(Universal Mapping Property) For every chain map of the form $\varphi\colon A \to A'$, where A' is a DG algebra centered at a ring R' and where φ restricts to a ring homomorphism $\varphi_0\colon R \to R'$, there exists a unique DG algebra homomorphism $\widetilde{\varphi}\colon \mathrm{S}_R(A) \to A'$ which extends $\varphi\colon A \to A'$, that is, such that $\widetilde{\varphi} \circ \iota = \varphi$ where $\iota\colon A \hookrightarrow \mathrm{S}$ is the inclusion map. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{S}_R(A) \arrow[d, "\widetilde{\varphi}"] \\ & A' \end{tikzcd} \tag{46}$$

Proof. We only prove the third property since the first two properties are straightforward to show. Let $\varphi\colon A \to A'$ be such a chain map and denote $S = \mathrm{S}_R(A)$. We define $\widetilde{\varphi}\colon S \to A'$ by setting $\widetilde{\varphi}|_A = \varphi$ and

$$\widetilde{\varphi}(a_1 \cdots a_m) = \varphi(a_1) \cdots \varphi(a_m) \tag{47}$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$ and then extending it R-linearly everywhere else. By construction, $\widetilde{\varphi}$ is multiplicative and extends $\varphi\colon A \to A'$. Furthermore, $\widetilde{\varphi}$ is a chain map since it is a graded R-linear map which commutes with the differential. Indeed, we clearly have $\widetilde{\varphi}\mathrm{d}(1) = 0 = \mathrm{d}\widetilde{\varphi}(1)$, and for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, we have

$$\begin{aligned}
\widetilde{\varphi}\mathrm{d}(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \widetilde{\varphi}(a_1 \cdots \mathrm{d}(a_j) \cdots a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \varphi\mathrm{d}(a_j) \cdots \varphi(a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \mathrm{d}\varphi(a_j) \cdots \varphi(a_m) \\
&= \mathrm{d}(\varphi(a_1) \cdots \varphi(a_m)) \\
&= \mathrm{d}\widetilde{\varphi}(a_1 \cdots a_m).
\end{aligned}$$

Finally, if $\widehat{\varphi}\colon S \to A'$ were another DG algebra homomorphism which extended $\varphi\colon A \to B$, then we'd have

$$\widetilde{\varphi}(a_1 \cdots a_m) = \widehat{\varphi}(a_1) \cdots \widehat{\varphi}(a_m) = \varphi(a_1) \cdots \varphi(a_m) = \widetilde{\varphi}(a_1 \cdots a_m)$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, which implies $\widehat{\varphi} = \widetilde{\varphi}$. □

4.3 Presentation of the Maximal Associative Quotient

Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Equip A with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra. In particular, note that if $a_1, a_2 \in A_1$, then

$$a_1 a_2 \in S_2^2, \quad a_1 \star a_2 \in S_2^1, \quad \text{and} \quad [a_1, a_2] \in S_2,$$

where $[a_1, a_2] = a_1 \star a_2 - a_1 a_2$ is the multiplicator of the inclusion map $\iota\colon A \hookrightarrow S$ evaluated at $(a_1, a_2) \in A^2$. Let $\mathfrak{s} = \mathfrak{s}(\mu)$ be the S-ideal generated by all such multiplicators, so

$$\mathfrak{s} = \mathrm{span}_S\{[a_1, a_2] \mid a_1, a_2 \in A\}.$$

Also let $\pi\colon S \to S/\mathfrak{s}$ and $\pi^{\mathrm{as}}\colon A \twoheadrightarrow A^{\mathrm{as}}$ denote the canonical quotient maps. The universal mapping property of the symmetric DG algebra of A implies $\pi^{\mathrm{as}}\colon A \twoheadrightarrow A^{\mathrm{as}}$ extends uniquely to a DG algebra homomorphism $S \twoheadrightarrow A^{\mathrm{as}}$ which we again denote by π^{as}. We let $S^{\geq 2} = S/A$ be the R-complex whose underlying graded R-module is $S^{\geq 2}$ and whose differential $\mathrm{d}^{\geq 2}$ is defined by

$$\mathrm{d}^{\geq 2}|_{S^m} = \begin{cases} \eth|_{S^2} & \text{if } m = 2 \\ \mathrm{d}|_{S^m} & \text{if } m > 2. \end{cases}$$

We also let $\rho\colon S \twoheadrightarrow S/A = S^{\geq 2}$ be the canonical quotient map.

Theorem 4.1. *With the notation as above, we have*

$$A^{\mathrm{as}} = \operatorname{coker}(\mathfrak{s} \hookrightarrow S) = S/\mathfrak{s}$$

More specifically, there is a unique isomorphism $A^{\mathrm{as}} \to S/\mathfrak{s}$ of DG *S-algebras (thus we are justified in writing $\pi\colon S \to A^{\mathrm{as}}$ to denote both $\pi^{\mathrm{as}}\colon S \to A^{\mathrm{as}}$ and $\pi\colon S \to S/\mathfrak{s}$ in order to simplify notation) In particular, this implies*

$$\langle A \rangle = A \cap \mathfrak{s} = \mathfrak{s}^{\leq 1} = \ker(\mathfrak{s} \to S^{\geq 2})$$

Thus we have the following canonically defined hexagonal-shaped diagram of R-complexes which is exact everywhere (in every direction) and which is natural in $A = (A, \mathrm{d}, \mu)$:

$$\begin{tikzcd}
 & S^{\geq 2} \arrow[r] & 0 \\
\mathfrak{s} \arrow[ur] \arrow[r, hook, "i"] & S \arrow[u, "\rho"] \arrow[r, two heads, "\pi"] & A^{\mathrm{as}} \arrow[u] \\
\mathfrak{s}^{\leq 1} \arrow[u, hook] \arrow[r, hook] & A \arrow[u, hook, "\iota"] \arrow[ur] &
\end{tikzcd} \tag{48}$$

where the blue arrows are DG *S-module homorphisms, where the green arrows are chain maps as R-complexes, and where the red arrows are* MDG *A-module homomorphisms. In particular, if $\mathrm{H}_+(A) = 0$, then $\mathrm{H}_+(S) = \mathrm{H}(S^{\geq 2})$ and we obtain a canonically defined sequence of graded $\mathrm{H}(S)$-modules:*

$$\mathrm{H}_+(\mathfrak{s}) \longrightarrow \mathrm{H}_+(S) \longrightarrow \mathrm{H}_+(A^{\mathrm{as}}) \longrightarrow \Sigma\mathrm{H}(\mathfrak{s}) \longrightarrow \Sigma\mathrm{H}(S) \tag{49}$$

which is natural in $A = (A, \mathrm{d}, \mu)$.

Remark 7. By "natural in $A = (A, \mathrm{d}, \mu)$" we mean that if R' is an R-algebra and $\varphi\colon A \to A'$ is an MDG R-algebra homomorphism where $A' = (A', \mathrm{d}', \mu')$ is an MDG R'-algebra centered at R', then we obtain canonically defined maps $S \to S'$ and $\mathfrak{s} \to \mathfrak{s}'$, where we set $S' = \mathrm{S}_{R'}(A')$ and $\mathfrak{s}' = \mathfrak{s}(\mu')$, which induces a map of hexagonal-shaped diagrams in which everything commutes. For instance, if $\mathrm{H}_+(A) = 0 = \mathrm{H}_+(A')$, then then we have a commutative diagram of graded $\mathrm{H}(\mathrm{S}')$-modules of the form:

$$\begin{tikzcd}
\mathrm{H}_+(\mathfrak{s}) \arrow[r] \arrow[d] & \mathrm{H}_+(S) \arrow[r] \arrow[d] & \mathrm{H}_+(A^{\mathrm{as}}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(\mathfrak{s}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(S) \arrow[d] \\
\mathrm{H}_+(\mathfrak{s}') \arrow[r] & \mathrm{H}_+(S') \arrow[r] & \mathrm{H}_+((A')^{\mathrm{as}}) \arrow[r] & \Sigma\mathrm{H}(\mathfrak{s}') \arrow[r] & \Sigma\mathrm{H}(S')
\end{tikzcd} \tag{50}$$

We are especially interested in the case where $A = A'$ but allow $\mu \neq \mu'$. In that case, we are basically studying the DG ideals $\mathfrak{s} = \mathfrak{s}(\mu)$ and $\mathfrak{s}' = \mathrm{s}(\mu')$ in $S = S'$.

Proof. Observe that $\pi^{\mathrm{as}}\colon S \twoheadrightarrow A^{\mathrm{as}}$ satifies

$$\begin{aligned}
\pi^{\mathrm{as}}[a_1, a_2] &= \pi^{\mathrm{as}}(a_1 \star a_2 - a_1 a_2) \\
&= \pi^{\mathrm{as}}(a_1 \star a_2) - \pi^{\mathrm{as}}(a_1 a_2) \\
&= \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2) - \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2) \\
&= 0.
\end{aligned}$$

Thus the universal mapping property of the quotient $S/\mathfrak{s} = \operatorname{coker}(\mathfrak{s} \hookrightarrow S)$ implies there is a unique DG algebra homomorphism $\overline{\pi}^{\mathrm{as}}\colon S/\mathfrak{s} \to A^{\mathrm{as}}$ such that

$$\overline{\pi}^{\mathrm{as}} \circ \pi = \pi^{\mathrm{as}}.$$

Similarly, note that the composite $\pi \circ \iota\colon A \to S/\mathfrak{s}$ is an MDG algebra homomorphism which is surjective. Indeed, if $a_1 \cdots a_m$ is a homogeneous elementary tensor in S^m, then we have

$$a_1 a_2 a_3 \cdots a_m = ((\cdots (a_1 \star a_2) \star a_3) \star \cdots) \star a_m$$

in $S/\mathfrak{s}$. Thus every element in $S/\mathfrak{s}$ can be represented by an element in $A = S^1$ which implies $\pi\iota\colon A \twoheadrightarrow S/\mathfrak{s}$ is surjective as claimed. In particular, since $S/\mathfrak{s}$ is associative, it follows from the universal mapping property of the maximal associative quotient of A that there is a unique DG algebra homomorphism $\overline{\pi}\colon A^{\mathrm{as}} \to S/\mathfrak{s}$ such that

$$\pi \circ \iota = \overline{\pi} \circ \pi^{\mathrm{as}}.$$

Combining all of this together, we have a commutative diagram of MDG S-modules:

$$\begin{tikzcd}
S \arrow[r, two heads, "\pi"] \arrow[dr, dashed, "\pi^{\mathrm{as}}"] & S/\mathfrak{s} \arrow[d, dashed, shift left, "\overline{\pi}^{\mathrm{as}}"] \\
A \arrow[u, hook, "\iota"] \arrow[r, two heads, "\pi^{\mathrm{as}}"'] & A^{\mathrm{as}} \arrow[u, dashed, shift left, "\overline{\pi}"]
\end{tikzcd}$$

where the dashed arrows indicates uniqueness. □

Corollary 2. *Continuing with the notation as above, assume further that A is associative, so $A = A^{\mathrm{as}}$. Then the canonical map $\mathfrak{s} \to S^{\geq 2}$ defined on multiplicators by*

$$[a_1, a_2] \mapsto a_1 a_2$$

is an isomorphism of R-complexes. Let $\theta\colon S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s} \hookrightarrow S$ be the composite map where $S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s}$ is the inverse isomorphism of the canonical map $\mathfrak{s} \to S^{\geq 2}$. We obtain a short exact sequence of R-complexes

$$0 \longrightarrow S^{\geq 2} \xrightarrow{\ \theta\ } S \xrightarrow{\ \pi\ } A \longrightarrow 0 \tag{51}$$

which is split by the inclusion map $\iota\colon A \to S$. Similarly, the short exact sequence of R-complexes

$$0 \longrightarrow A \xrightarrow{\ \iota\ } S \xrightarrow{\ \rho\ } S^{\geq 2} \longrightarrow 0 \tag{52}$$

is split by $\theta\colon S^{\geq 2} \to S$.

Corollary 3. *Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Then a necessary condition for A to have a DG algebra structure is that the canonical short exact sequence of R-complexes*

$$0 \longrightarrow A \xrightarrow{\ \iota\ } S \xrightarrow{\ \rho\ } S^{\geq 2} \longrightarrow 0 \tag{53}$$

is split.

Proposition 4.2. *Let R be a commutative ring, let A be an R-complex centered at R, and let $I = \mathrm{d}(A_1)$ (so $\mathrm{H}_0(A) = R/I$). Set $S = \mathrm{S}_R(A)$ to be the symmetric DG algebra of A. Assume further that $\mathrm{d}A \subseteq IA$. Then the canonical quotient map $\rho\colon S \to S^{\geq 2}$ induces an isomorphism*

$$S/IS \simeq A/IA \oplus S^{\geq 2}/IS^{\geq 2}$$

as R-complexes.

Proof. Note S and $S^{\geq 2}$ are the exact same complex in total degree ≥ 3, so the only difference between them is how they behave in total degree ≤ 2. In in particular, we obtain $S^{\geq 2}$ from S by replacing $S^{\leq 1} = A$ with 0 and replacing the labeled arrows in the diagram below with zero maps

$$\begin{tikzcd}
A_{i+1} \arrow[dr, "\eth^1_{i+1}"'] & & S^2_{i+1} \arrow[dl, "\partial^2_{i+1}"'] \arrow[dr] & & S^3_{i+1} \arrow[dl] & \\
& A_i \arrow[dr, "\eth^1_i"'] & & S^2_i \arrow[dl, "\partial^2_i"'] \arrow[dr] & & S^3_i \arrow[dl] \\
& & A_{i-1} & & S^2_{i-1} &
\end{tikzcd}$$

Note that $\mathrm{im}(\eth^1_i) = \mathrm{d}A_i \subseteq IA_i$ and $\mathrm{im}(\partial_i) = IA_i$. Thus we obtain $S/IS = S \otimes_R R/I$ by replacing the labeled arrows above with zero maps. □

4.4 Homology of the Symmetric DG Algebra

Proposition 4.3. *Let $R = (R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $F = (F, \mathrm{d})$ be the minimal free resolution of R/I over R where $I \subseteq \mathfrak{m}$. Equip F with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra and let $S = \mathrm{S}_R(F)$ be the symmetric DG R-algebra of F. Finally let*

$$f := [a_1, a_2] = a_1 a_2 - a_1 \star a_2,$$

where $a_1, a_2 \in F_1 \backslash \mathfrak{m}F_1$. Then f represents a nonzero element in $\mathrm{H}_2(S)$.

Proof. Clearly we have $\mathrm{d}f = 0$. Suppose that $\mathrm{d}g = f$ where $g \in S_3$. Let g^2 and g^3 be the components of g that lie in S_3^2 and S_3^3 respectively. Then in particular, we must have

$$a_1 a_2 = \partial g^3 + \eth g^2. \tag{54}$$

However this is a contradiction as minimality of F implies that the RHS of (54) lies in $\mathfrak{m}S$ however the LHS of (54) does not lie in $\mathfrak{m}S$ as $a_1, a_2 \notin \mathfrak{m}F$. □

4.5 The Symmetric DG Algebra of a Finite Free Complex over an Integral Domain

Throughout this subsection, we assume that R is an integral domain with quotient field K. Let F be an R-complex centered at R such that the underlying graded R-module of F is a finite and free as an R-module. Let $e_1, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $|e_j| > |e_i|$, then $j > i$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $R[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $R[\boldsymbol{e}]$. The reason we do not allow elements of odd degree to square to zero is because we will want to calculate the Gröbner basis of an ideal in $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is simpler when we don't have any zero-divisors. In any case, one recovers the symmetric DG R-algebra of F as below:

$$R[\boldsymbol{e}]/\langle \{e_i^2 \mid |e_i| \text{ is odd}\}\rangle \simeq \mathrm{S}_R(F).$$

Finally, let $(\mu, \star)$ be a multiplication of F. Our goal is to compute the maximal associative quotient of F using the presentation given in Theorem (4.1) as well as the theory of Gröbner bases in $K[\boldsymbol{e}]$. Before we can do this, we need to introduce some notation for Gröbner basis applications in $K[\boldsymbol{e}]$. Our notation mostly follows [BE77] however we introduce some of our own notation as well.

4.5.1 Monomials and Monomial Orderings in $K[\boldsymbol{e}]$

A **monomial** in $K[\boldsymbol{e}]$ is an element of the form

$$e^{\boldsymbol{\alpha}} = e_1^{\alpha_1} \cdots e_n^{\alpha_n} \tag{55}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called the **multidegree** of $e^{\boldsymbol{\alpha}}$ and is denoted $\mathrm{multideg}(e^{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$. Similarly we define its **total degree**, denoted $\deg(e^{\boldsymbol{\alpha}})$, and its **homological degree** denoted $|e^{\boldsymbol{\alpha}}|$, by

$$\deg(e^{\boldsymbol{\alpha}}) = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad |e^{\boldsymbol{\alpha}}| = \sum_{i=1}^{n} \alpha_i |e_i|.$$

By convention we set $e^{\boldsymbol{0}} = 1$ where $\boldsymbol{0} = (0, \dots, 0)$ is the zero vector in $\mathbb{N}^n$. We define the **support** of $e^{\boldsymbol{\alpha}}$, denoted $\mathrm{supp}(e^{\boldsymbol{\alpha}})$, to be the set

$$\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_i \mid e_i \text{ divides } e^{\boldsymbol{\alpha}}\} = \{e_i \mid \alpha_i \neq 0\}.$$

Note that if the support of $e^{\boldsymbol{\alpha}}$ is empty if and only if $e^{\boldsymbol{\alpha}} = 1$. If $e^{\boldsymbol{\alpha}}$ has non-empty support, then we define its **initial variable** and **terminal variable** to be the variables e_i and e_k respectively where

$$i = \inf\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\} \quad \text{and} \quad k = \max\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\}.$$

For instance, suppose that $\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_{i_1}, \dots, e_{i_k}\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then we can express (55) as

$$e^{\boldsymbol{\alpha}} = e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_k},$$

and in this case, e_{i_1} is the initial variable of $e^{\boldsymbol{\alpha}}$ and e_{i_k} is the terminal variable of $e^{\boldsymbol{\alpha}}$.

Remark 8. Note how the ordering matters. In particular, if $i < j$ and both $|e_i|$ and $|e_j|$ are odd, then $e_j e_i$ is not a monomial in $K[\boldsymbol{e}]$ since it can be expressed as a non-trivial coefficient times a monomial:

$$e_j e_i = -e_i e_j.$$

On the other hand, if one of the e_i or e_j is even, then $e_j e_i$ is a monomial in $K[\boldsymbol{e}]$ since $e_j e_i = e_i e_j$.

We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ (the notation for this monomial ordering in Singular is Wp(w)). More specifically, given two monomials $e^{\boldsymbol{\alpha}}$ and $e^{\boldsymbol{\beta}}$ in $K[\boldsymbol{e}]$, we say $e^{\boldsymbol{\beta}} > e^{\boldsymbol{\alpha}}$ if either

1. $|e^{\boldsymbol{\beta}}| > |e^{\boldsymbol{\alpha}}|$ or;
2. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and $\beta_1 > \alpha_1$ or;
3. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Given a nonzero polynoimal $f \in K[\boldsymbol{e}]$, there exists unique $c_1, \dots, c_m \in K \backslash \{0\}$ and unique $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ where $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ for all $1 \leq i < j \leq m$ such that

$$f = c_1 e^{\boldsymbol{\alpha}_1} + \cdots + c_m e^{\boldsymbol{\alpha}_m} = \sum c_i e^{\boldsymbol{\alpha}_i} \tag{56}$$

The $c_i e^{\boldsymbol{\alpha}_i}$ in (56) are called the **terms** of f, and the $e^{\boldsymbol{\alpha}_i}$ in (56) are called the **monomials** of f. By reindexing the $\boldsymbol{\alpha}_i$ if necessary, we may assume that $e^{\boldsymbol{\alpha}_1} > \cdots > e^{\boldsymbol{\alpha}_m}$. In this case, we call $c_1 e^{\boldsymbol{\alpha}_1}$ the **lead term** of f, we call $e^{\boldsymbol{\alpha}_1}$ the **lead monomial** of f, and we call c_1 the **lead coefficient** of f. We denote these, respectively, by

$$\mathrm{LT}(f) = c_1 e^{\boldsymbol{\alpha}_1}, \quad \mathrm{LM}(f) = e^{\boldsymbol{\alpha}_1}, \quad \text{and} \quad \mathrm{LC}(f) = c_1.$$

The **multidegree** of f is defined to be the multidegree of its lead monomial $e^{\boldsymbol{\alpha}_1}$ and is denoted multideg$(f) = \boldsymbol{\alpha}_1$. The **total degree** of f is defined to be the maximum of the total degrees of its monomials and is denoted

$$\deg(f) = \max_{1 \leq i \leq m} \{\deg(e^{\boldsymbol{\alpha}_i})\}.$$

We say f is **homogeneous** of homological degree i if each of its monomials is homogeneous of homological degree i. In this case, we say f has **homological degree** i and we denote this by $|f| = i$.

Proposition 4.4. *For each $1 \leq i \leq j \leq n$, let $f_{ij} = -[e_i, e_j] = e_i e_j - e_i \star e_j$. We have*

$$\mathrm{LT}(f_{ij}) = e_i e_j.$$

Proof. If $e_i \star e_j = 0$, then this is clear, otherwise term of $e_i \star e_j$ has the form $r_{i,j}^k e_k$ for some k where $r_{i,j}^k \neq 0$. Since $\star$ respects homological degree, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > |e_i|$ and $|e_k| > |e_j|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ and $k > j$ by our assumption on the ordering of $e_1, \dots, e_n$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j > e_k$. □

4.5.2 Gröbner Basis Calculations

Our goal is to use the theory of Gröbner bases to help us calculate

$$F^{\mathrm{as}} = \mathrm{S}_R(F)/\mathfrak{s}(\mu) \simeq R[\boldsymbol{e}]/\langle \{f_{i,j}\} \rangle,$$

where $f_{i,j} \in R[\boldsymbol{e}]$ are defined by

$$f_{i,j} = e_i e_j - e_i \star e_j = e_i e_j - \sum_k r_{i,j}^k e_k,$$

where the $r_{i,j}^k \in R$ are the entries of the matrix representation of μ with respect to the ordered homogeneous basis $e_1, \dots, e_n$. In order to do this though, we first need to base change to K because that's where the theory of Gröbner basis works best. Thus we wish to calculate:

$$F_K^{\mathrm{as}} := F^{\mathrm{as}} \otimes_R K \simeq K[\boldsymbol{e}]/\langle \{f_{i,j}\} \rangle.$$

To this end, let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let $\mathfrak{a}$ be the $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. We wish to construct a left Gröbner basis for $\mathfrak{a}$ (which will turn out to be a two-sided Gröbner basis) via Buchberger's algorithm (as described in [GP02]) using the monomial ordering described above. Suppose f, g are two nonzero polynomials in $K[\boldsymbol{e}]$ with $\mathrm{LT}(f) = re^{\boldsymbol{\alpha}}$ and $\mathrm{LT}(g) = se^{\boldsymbol{\beta}}$. Set $\boldsymbol{\gamma} = \mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and the left S-**polynomial** of f and g to be

$$\mathrm{S}(f, g) = e^{\gamma - \boldsymbol{\alpha}} f \pm (r/s) e^{\gamma - \boldsymbol{\beta}} g \tag{57}$$

where the $\pm$ in (57) is chosen to be $+$ or $-$, depending on which sign will cancel out the lead terms. We begin Buchberger's algorithm by calculating the S-polynomials of all pairs of polynomials in $\mathcal{F}$. In other words, we calculate all S-polynomials of the form $\mathrm{S}(f_{k,l}, f_{i,j})$ where $1 \leq i, j, k, l \leq n$. Note that if $k > l$, then

$$f_{l,k} = (-1)^{|e_k||e_l|} f_{k,l},$$

which implies

$$\mathrm{S}(f_{l,k}, f_{i,j}) = (-1)^{|e_k||e_l|} \mathrm{S}(f_{k,l}, f_{i,j}) = \pm \mathrm{S}(f_{i,j}, f_{k,l}).$$

Similarly, if $i \geq k$, then

$$\mathrm{S}(f_{i,j}, f_{l,k}) = \pm \mathrm{S}(f_{k,l}, f_{i,j}).$$

Thus we may assume that $j \geq i$ and $l \geq k \geq i$. Obviously we have $\mathrm{S}(f_{i,j}, f_{i,j}) = 0$ for each i, j, however something interesting happens when we calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $j > i$ and then divide this by $\mathcal{F}$ (where division by $\mathcal{F}$ means taking the left normal form of $\mathrm{S}(f_{j,k}, f_{i,j})$ with respect to $\mathcal{F}$ using the left normal form described in [GP02]). We have

$$\begin{aligned}
\mathrm{S}(f_{j,k}, f_{i,j}) &= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
&= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
&= \sum_l r_{i,j}^l e_l e_k - \sum_l r_{j,k}^l e_i e_l \\
&\to \sum_l r_{i,j}^l e_l \star e_k - \sum_l r_{j,k}^l e_i \star e_l \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k],
\end{aligned}$$

where in the fourth line we did division by $\mathcal{F}$ (note that if $[e_i, e_j, e_k] \neq 0$, then $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$). Finally if $j > i$, $l > k$, and $j \neq k$, then we have

$$\begin{aligned}
\mathrm{S}(f_{k,l}, f_{i,j}) &= e_i e_j f_{k,l} - f_{i,j} e_k e_l \\
&= (e_i \star e_j) e_k e_l - e_i e_j (e_k \star e_l) \\
&\to (e_i \star e_j) \star (e_k \star e_l) - (e_i \star e_l) \star (e_k \star e_l) \\
&= 0
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Next, suppose that

$$f = r e_k + r' e_{k'} + \cdots + r'' e_{k''} \in \langle F \rangle$$

where $r, r', r'' \in R$ with $r \neq 0$ and where $\mathrm{LM}(f) = e_k$. Then we have

$$\begin{aligned}
\mathrm{S}(f, f_{j,k}) &= e_j f - r f_{j,k} \\
&= r' e_j e_{k'} + \cdots + r'' e_j e_{k''} + r e_j \star e_k \\
&\to r' e_j \star e_{k'} + \cdots + r'' e_j \star e_{k''} + r e_j \star e_k \\
&= e_j \star (r e_k + r' e_{k'} + \cdots + r'' e_{k''}) \\
&= e_j \star f \\
&\in \langle F \rangle
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Similarly, we have if $i \neq k \neq j$, then we have

$$\begin{aligned}
\mathrm{S}(f, f_{i,j}) &= e_i e_j f - r f_{i,j} e_k \\
&= r'(e_i e_j) e_{k'} + \cdots + r''(e_i e_j) e_{k''} + r(e_i \star e_j) e_k \\
&\to r'(e_i \star e_j) \star e_{k'} + \cdots + r''(e_i \star e_j) \star e_{k''} + r(e_i \star e_j) \star e_k \\
&= (e_i \star e_j) \star (r e_k + r' e_{k'} + \cdots + r'' e_{k''}) \\
&= (e_i \star e_j) \star f \\
&\in \langle F \rangle.
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Finally suppose that

$$g = se_m + s'e_{m'} + \cdots + s''e_{m''} \in \langle F \rangle$$

where $s, s', s'' \in R$ with $s \neq 0$ and where $\mathrm{LM}(g) = e_m$. If $k = m$, then we have

$$s\mathrm{S}(f, g) = sf - rg \in \langle F \rangle.$$

On the other hand, if $k \neq m$, then we have

$$\begin{aligned}
s\mathrm{S}(f, g) &= se_m f - rge_k \\
&= sr'e_m e_{k'} + \cdots + sr''e_m e_{k''} - rs'e_{m'}e_k - \cdots - rs''e_{m''}e_k \\
&\to sr'e_m \star e_{k'} + \cdots + sr''e_m \star e_{k''} - rs'e_{m'} \star e_k - \cdots - rs''e_{m''} \star e_k \\
&= se_m \star (r'e_{k'} + \cdots + r''e_{k''}) - r(s'e_{m'} + \cdots + s''e_{m''}) \star e_k \\
&= se_m \star (f - re_k) - r(g - se_m) \star e_k \\
&= se_m \star f + rg \star e_k - sre_m \star e_k + rse_m \star e_k \\
&= se_m \star f + rg \star e_k \\
&\in \langle F \rangle.
\end{aligned}$$

It follows that we can construct a Gröbner basis

$$\mathcal{G} := \mathcal{F} \cup \{g_1, \dots, g_m\}$$

of $\mathfrak{a}$ such that the g_i all belong to $\langle F \rangle$.

Example 4.3. Consider Example (1.1) with the same notation as in that example and let $K = \Bbbk(x, y, z, w)$ be the fraction field of R. Using Singular, we find that

$$F_K^{\mathrm{as}} := F^{\mathrm{as}} \otimes_R K \simeq K[\boldsymbol{e}]/\mathfrak{s},$$

where $\mathfrak{s}$ is the $K[\boldsymbol{e}]$-ideal which is minimally generated by the following polynomials:

$$\begin{array}{lll}
f_{12} = e_1e_2 - e_{12} & f_{1,23} = e_1e_{23} - e_{123} & \\
f_{13} = e_1e_3 - e_{13} & f_{1,24} = e_1e_{24} - xe_{124} & f_1 = e_1^2 \\
f_{14} = e_1e_4 - xe_{14} & f_{1,34} = e_1e_{34} - xe_{134} & f_2 = e_2^2 \\
f_{23} = e_2e_3 - we_{23} & f_{5,12} = yz^2e_1e_{24} + xe_5e_{12} + x^2yze_{234} + x^2we_{345} & f_3 = e_3^2 \\
f_{24} = e_2e_4 - e_{24} & f_{1,35} = yze_1e_{34} + xe_1e_{35} + x^2e_{345} & f_4 = e_4^2 \\
f_{34} = e_3e_4 - e_{34} & f_{1234} = xe_{1234} & f_5 = e_5^2 \\
f_{25} = y^2ze_2e_3 + we_2e_5 + w^2e_{35} & f_{2,3,5} = y^2z^2e_2e_3 + zwe_2e_5 + w^2e_3e_5 & \\
f_{15} = yz^2e_1e_4 + xe_1e_5 + x^2e_{45} & f_{1,4,5} = y^2z^2e_1e_4 + xye_1e_5 + x^2e_4e_5 &
\end{array}$$

In particular, we have

$$\beta_1^{K[\boldsymbol{e}]}(F_K^{\mathrm{as}}) = \sum_i \beta_i^R(R/I) + 1.$$

Appendix

5 Localization, Tensor, and Hom

Let A be an MDG R-algebra and let X and Y be MDG A-modules. In this subsection we define the tensor complex $X \otimes_A Y$ (which turns out to be an MDG A-module with the obvious A-scalar multiplication) as well as the hom complex $\mathrm{Hom}_A^\star(X, Y)$ (which need not be an MDG A-module using the naive A-scalar multiplication since this map need not be well-defined). Before defining these complexes however, we first discuss localization.

5.1 Localization

A subset $S \subseteq A$ is called **multiplicatively closed** if it satisfies the following conditions:

1. We have $1 \in S$ and if $s_1, s_2 \in S$ we have $s_1s_2 \in S$.

2. Each $s \in S$ must be homogeneous of even degree.

3. We have $S \subseteq \mathrm{N}(A)$.

Given a multiplicatively closed subset $S \subseteq A$, we define an MDG R-algebra A_S, called the **localization of A at S**, as follows: as a set, A_S is given by

$$A_S := \{a/s \mid a \in A \text{ and } s \in S\}$$

where a/s denotes the equivalence class of $(a,s) \in A \times S$ with respect to the following equivalence relation:

$$(a,s) \sim (a',s') \text{ if and only if there exists } s'' \in S \text{ such that } s''s'a = s''sa'. \tag{58}$$

Notice how we are not bothering to put in parenthesis in (58) since each $s \in S$ belongs to the nucleus of A and thus associates with everything else. One can check that (58) is indeed an equivalence relation because every $s \in S$ associates and commutes with everything else. We give A_S the structure of an R-module by defining addition and R-scalar multiplication on A_S by

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1 + s_1a_2}{s_1s_2} \quad \text{and} \quad r \cdot \frac{a}{s} = \frac{ra}{s}, \tag{59}$$

for all a/s, a_1/s_1, and a_2/s_2 in A_S, and for all $r \in R$. Again, (59) is well-defined since $S \subseteq \mathrm{N}(A) \cap \mathrm{Z}(A)$ where $\mathrm{Z}(A)$ is the center of A (the set of all elements which commutes with everything else). In fact, A_S is a graded R-module where the homogeneous component in degree $i \in \mathbb{Z}$, denoted $A_{S,i}$, is the R-span of all fractions of the form a/s where a is homogeneous and where $|a/s| := i = |a| - |s|$. We give A_S the structure of an R-complex by attaching to it the differential $\mathrm{d}_S \colon A_S \to A_S$ which is defined by

$$\mathrm{d}_S\left(\frac{a}{s}\right) = \frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2}$$

for all $a/s \in A_S$. A straightforward computation shows that $\mathrm{d}_S \colon A_S \to A_S$ is a graded R-linear map of degree -1 which satisfies $\mathrm{d}_S^2 = 0$, so d_S really is a differential. As usual, we denote d_S more simply by d if context is understood. Finally we give A_S the structure of an MDG R-algebra by defining the multiplication μ_S of A_S via the formula

$$\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}$$

for all a_1/s_1 and a_2/s_2 in A_S.

If X is an MDG A-module and $S \subseteq A$ is a multiplicatively closed set such that $S \subseteq \mathrm{N}_A(X)$, then we can also define an MDG A_S-module X_S, called **localization of X with respect to S**. The construction of X_S is almost identical to the construction of A_S, however we really do need to have $S \subseteq \mathrm{N}_A(X)$ (and not just $S \subseteq \mathrm{N}(A)$) in order for this construction to be well-defined). In particular, we cannot view localization as a functor

$$-_S \colon \mathbf{MDGmod}_A \to \mathbf{MDGmod}_{A_S}.$$

However if we consider the subcategory $\mathbf{MDGmod}_A^\star$ of $\mathbf{MDGmod}_A$, where the objects of $\mathbf{MDGmod}_A^\star$ are the MDG A-modules X such that $\mathrm{N}(A) \subseteq \mathrm{N}_A(X)$, then we do obtain a functor

$$-_S \colon \mathbf{MDGmod}_A^\star \to \mathbf{MDGmod}_{A_S}^\star.$$

5.2 Tensor

We now discuss the tensor complex $X \otimes_A Y$. The underlying graded R-module of $X \otimes_A Y$ in degree i is the R-span of homogeneous elementary tensors $x \otimes y$ where $|x| + |y| = i$ subject to the relations

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

for all $x_1, x_2, x \in X$ and $y_1, y_2, y \in Y$ as well as the relations

$$a(x \otimes y) = ax \otimes y = (-1)^{|a||x|} x \otimes ay \tag{60}$$

for all homogeneous $a \in A$, $x \in X$, and $y \in Y$. The differential of the tensor complex $X \otimes_A Y$ is defined on homogeneous elementary tensors $x \otimes y$ by

$$\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{|x|} x \otimes \mathrm{d}(y).$$

The tensor complex $X \otimes_A Y$ inherits the structure of an MDG A-module where the A-scalar multiplication is defined via (60), thus $X \otimes_A Y$ is in fact an MDG A-module. A calculation shows that

$$[a_1, a_2, x \otimes y] = [a_1, a_2, x] \otimes y = (-1)^{|a_1+a_2||x|} x \otimes [a_1, a_2, y]$$

for all homogeneous $a_1, a_2 \in A$ and for all homogeneous elementary tensors $x \otimes y \in X \otimes_A Y$. In particular, if either X or Y is associative, then $X \otimes_A Y$ is associative. Here's an important warning to keep in mind when dealing with tensor complexes however: the map $\varphi \colon A \otimes_A X \to X$ defined by $\varphi(a \otimes x) = ax$ is *not* well-defined if X is not associative. Indeed, suppose $[a_1, a_2, x] \neq 0$. Then

$$\begin{aligned} 0 &= \varphi(0) \\ &= \varphi(a_1 a_2 \otimes x - a_1 \otimes a_2 x) \\ &= [a_1, a_2, x] \\ &\neq 0 \end{aligned}$$

shows that φ is not well-defined. More generally, given an MDG A-ideal $\mathfrak{a}$, the map $A/\mathfrak{a} \otimes_A X \to X/\mathfrak{a}X$, defined on elementary tensors by $\overline{a} \otimes x \mapsto \overline{ax}$, is only well-defined if $[X] \subseteq \mathfrak{a}X$. Similarly, given a multiplicative subset $S \subseteq \mathrm{N}(A) \cap \mathrm{N}(X)$, the map $A_S \otimes_A X \to X_S$, defined on elementary tensors by $(a/1) \otimes x \mapsto ax/1$, is only well-defined if $[X]_S = 0$.

5.3 Hom

Next we discuss the hom complex $\mathrm{Hom}_A^\star(X, Y)$. The hom complex $\mathrm{Hom}_A^\star(X, Y)$ is the R-complex whose underlying graded module in degree $i \in \mathbb{Z}$ is

$$\mathrm{Hom}_A^\star(X, Y)_i := \{\varphi \colon X \to Y \mid \varphi \text{ is a graded } A\text{-module homomorphism of degree } i\}.$$

A graded A-module homomorphism of degree $i := |\varphi|$ is a graded linear map $\varphi \colon X \to Y$ of degree $|\varphi|$ which satisfies $\varphi(ax) = (-1)^{|a||\varphi|} a\varphi(x)$ for all homogeneous $a \in A$ and $x \in X$. The differential of $\mathrm{Hom}_A^\star(X, Y)$ is denoted $\mathrm{d}^\star$ and is defined on homogeneous $\varphi \in \mathrm{Hom}_A^\star(X, Y)$ by

$$\mathrm{d}^\star(\varphi) = \mathrm{d}\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}.$$

Note that $\mathrm{d}^\star(\varphi)$ really is a graded A-module homomorhism of degree $|\varphi| - 1$! Indeed, for all homogeneous $a \in A$ and $x \in X$, we have

$$\begin{aligned} \mathrm{d}^\star(\varphi)(ax) &= (\mathrm{d}\varphi)(ax) - (-1)^{|\varphi|}(\varphi\mathrm{d})(ax) \\ &= (-1)^{|a||\varphi|}\mathrm{d}(a\varphi(x)) - (-1)^{|\varphi|}\varphi(\mathrm{d}(a)x) - (-1)^{|\varphi|+|a|}\varphi(a\mathrm{d}(x)) \\ &= (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) + (-1)^{|a||\varphi|+|a|}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|\varphi|(|a|+1)}\mathrm{d}(a)\varphi(x) - (-1)^{|\varphi|+|a|+|a||\varphi|}a\varphi(\mathrm{d}(x)) \\ &= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|a|(|\varphi|+1)}a\varphi(\mathrm{d}(x)) + (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) - (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) \\ &= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|a|(|\varphi|+1)}a(\varphi\mathrm{d}(x)) \\ &= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x) - (-1)^{|\varphi|}\varphi\mathrm{d}(x)) \\ &= (-1)^{|a|(|\varphi|-1)}a\mathrm{d}^\star(\varphi)(x). \end{aligned}$$

The hom complex $\mathrm{Hom}_A^\star(X, Y)$ doesn't necessarily inherit the structure of an MDG A-module where the A-scalar multiplication is defined by $\varphi \mapsto a\varphi$ where $a\varphi \colon X \to Y$ is defined by

$$(a\varphi)(x) = (-1)^{|a||\varphi|}\varphi(ax) = a\varphi(x)$$

for all $x \in X$. Indeed, given homogeneous $a_1, a_2 \in A$ we have

$$\begin{aligned} (a_1\varphi)(a_2 x) &= a_1\varphi(a_2 x) \\ &= (-1)^{|a_2||\varphi|}a_1(a_2\varphi(x)) \\ &= (-1)^{|a_2||\varphi|}(a_1 a_2)\varphi(x) - (-1)^{|a_2||\varphi|}[a_1, a_2, \varphi(x)] \\ &= (-1)^{|a_2||\varphi|+|a_1||a_2|}(a_2 a_1)\varphi(x) - (-1)^{|a_2||\varphi|}[a_1, a_2, \varphi(x)] \\ &= (-1)^{|a_2||\varphi|+|a_1||a_2|}a_2(a_1\varphi(x)) + (-1)^{|a_2||\varphi|+|a_1||a_2|}[a_2, a_1, \varphi(x)] - (-1)^{|a_2||\varphi|}[a_1, a_2, \varphi(x)] \end{aligned}$$

for all $x \in X$. If we knew that

$$[a_1, a_2, \varphi(x)] = (-1)^{|a_1||a_2|}[a_2, a_1, \varphi(x)], \tag{61}$$

then we could continue the calculation and conclude that $a_1\varphi$ is A-linear, however we need not have the identity (61) in general. However recall that the identity (61) holding for all $a_1, a_2 \in A$ is equivalent to the condition that $\varphi(x) \in \mathrm{M}(Y)$. Therefore if we knew that φ landed in $\mathrm{M}(Y)$, then $a_1\varphi$ would be A-linear.

Just as in the case of the tensor product where it need not be true that $A \otimes_A X \simeq X$, it need not be the case that $\mathrm{Hom}^{\star}_A(A, X) \simeq X$. In fact, we have

$$\mathrm{Hom}^{\star}_A(A, X) \simeq \mathrm{N}(X).$$

Indeed, suppose $\varphi \in \mathrm{Hom}^{\star}_A(A, X)$ and suppose $\varphi(1) = x$. Thus by A-linearity of φ, we have $\varphi(a) = (-1)^{|a||\varphi|}ax$ for all $a \in A$. Note that

$$\begin{aligned} 0 &= \varphi([a_1, a_2, 1]) \\ &= [a_1, a_2, \varphi(1)] \\ &= [a_1, a_2, x] \end{aligned}$$

for all $a_1, a_2 \in A$ forces $x \in \mathrm{N}(X)$.

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