Challenge Problems

September 21, 2023

1. Compute $\int_0^{\arcsin x} \sin t \mathrm{d}t$.

Solution: Let $F(x) = \int_0^{\arcsin x} \sin t \mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\arcsin x} \sin t \mathrm{d}t\right) \\ &= \sin(\arcsin x)\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \\ &= \frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

The function $G(x) = -\sqrt{1-x^2}$ is another antiderivative of $F'(x)$. It follows that

$$F(x) = c + G(x) \tag{1}$$

where c is some constant to be determined. To figure out what c is, substitute $x = 0$ to both sides of (**??**) to get $c = 1$. Therefore

$$\int_0^{\arcsin x} \sin t \mathrm{d}t = 1 - \sqrt{1-x^2}.$$

2. Compute $\int_0^{\log x} e^t \mathrm{d}t$.

Solution: Let $F(x) = \int_0^{\log x} e^t \mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\log x} e^t \mathrm{d}t\right) \\ &= e^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}(\log x) \\ &= \frac{x}{x} \\ &= 1. \end{aligned}$$

The function $G(x) = x$ is another antiderivative of $F'(x)$. It follows that

$$F(x) = c + G(x) \tag{2}$$

where c is some constant to be determined. To figure out what c is, substitute $x = 0$ to both sides of (**??**) to get $c = -1$. Therefore

$$\int_0^{\log x} e^t \mathrm{d}t = -1 + x.$$

(**) 3. Compute $\int_0^{\sin x} \arcsin t \mathrm{d}t$

Solution: Let $F(x) = \int_0^{\sin x} \arcsin t \mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\sin x} \arcsin t \mathrm{d}t\right) \\ &= \arcsin(\sin x)\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\ &= x\cos x. \end{aligned}$$

The function $G(x) = x\sin x + \cos x$ is another antiderivative of $F'(x)$. It follows that

$$F(x) = c + G(x) \tag{3}$$

where c is some constant to be determined. To figure out what c is, substitute $x = 0$ to both sides of (**??**) to get $c = -1$. Therefore

$$\int_0^{\sin x} \arcsin t \mathrm{d}t = -1 + x\sin x + \cos x.$$

(***) 4. Compute $\int_2^{\int_2^x \frac{\mathrm{d}t}{\ln t}} \frac{\mathrm{d}t}{\ln t}$

Solution: Let $F(x) = \int_2^{\int_2^x \frac{\mathrm{d}t}{\ln t}} \frac{\mathrm{d}t}{\ln t}$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_2^{\int_2^x \frac{\mathrm{d}t}{\ln t}} \frac{\mathrm{d}t}{\ln t}\right) \\ &= \frac{1}{\ln\left(\int_2^x \frac{\mathrm{d}t}{\ln t}\right)\ln x} \end{aligned}$$

(***) 4. Compute $\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t$

Solution: Let $F(x) = \int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right) \\ &= \varphi\left(\int_0^x \varphi(t)\mathrm{d}t\right)\varphi(x) \end{aligned}$$

(***) 4. Compute $\int_0^{\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t$

Solution: Let $F(x) = \int_0^{\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned} F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right) \\ &= \varphi\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right)\int_0^x \varphi(t)\mathrm{d}t \\ &= \varphi\left(\int_0^x \varphi(t)\mathrm{d}t\right)\varphi(x) \end{aligned}$$

Solution: Let $F(x) = \int_0^{\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t$. Combining the Chain Rule with the First Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right) \\
&= \varphi\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right) \\
&= \varphi\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right)\varphi\left(\int_0^x \varphi(t)\mathrm{d}t\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \varphi(t)\mathrm{d}t\right) \\
&= \varphi\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t\right)\varphi\left(\int_0^x \varphi(t)\mathrm{d}t\right)\varphi(x) \\
&= \varphi\left(\int_0^{\int_0^x \varphi(t)\mathrm{d}t} \varphi(t)\mathrm{d}t + \int_0^x \varphi(t)\mathrm{d}t + x\right).
\end{aligned}$$

In general let $\varphi\colon \mathbb{R}^\times \to \mathbb{R}$ be a group homomorphism. Define

$$\begin{aligned}
F_0(x) &= \varphi_0(x) \\
F_1(x) &= \int_0^x \varphi_1(t)\mathrm{d}t \\
F_2(x) &= \int_0^{F_1(x)} \varphi_2(t)\mathrm{d}t \\
&\vdots \\
F_n(x) &= \int_0^{F_{n-1}(x)} \varphi_n(t)\mathrm{d}t \\
&\vdots
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(F_n(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{F_{n-1}(x)} \varphi(t)\mathrm{d}t\right) \\
&= \varphi(F_{n-1}(x))\frac{\mathrm{d}}{\mathrm{d}x}(F_{n-1}(x)) \\
&= \varphi(F_{n-1}(x))\varphi(F_{n-2}(x))\frac{\mathrm{d}}{\mathrm{d}x}(F_{n-2}(x)) \\
&\vdots \\
&= \varphi_n(F_{n-1}(x))\varphi_{n-1}(F_{n-2}(x))\cdots\varphi_1(F_0(x))\varphi_0(x)
\end{aligned}$$

Let $F_0\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuously differentiable function and assume that $F_0'(x) \neq 0$ for all $x \in \mathbb{R}_{>0}$. For each $n \in \mathbb{N}$, define $F_n\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ recursively by

$$F_n(x) = \int_1^{F_{n-1}(x)} F_{n-1}^{-1}(t)\mathrm{d}t. \tag{4}$$

We prove by induction on $n \geq 1$ that the recursive formula (4) makes sense. For the base case $n = 1$, first note that F_0^{-1} exists since $F_0'(x) \neq 0$ for all $x \in \mathbb{R}_{>0}$. In particular, the formula

$$F_1(x) = \int_1^{F_0(x)} F_0^{-1}(t)\mathrm{d}t$$

makes sense. Now assume that for some $n \geq 1$, we have defined $F_n \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ recursively using the recursive formula (4). Observe that

$$\begin{aligned}
F_n'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{F_{n-1}(x)} F_{n-1}^{-1}(t)\mathrm{d}t\right) \\
&= F_{n-1}^{-1}(F_{n-1}(x))\frac{\mathrm{d}}{\mathrm{d}x}(F_{n-1}(x)) \\
&= xF_{n-1}'(x) \\
&= x^2F_{n-2}'(x) \\
&\vdots \\
&= x^nF_0'(x).
\end{aligned}$$

for all $x \in \mathbb{R}_{>0}$. Therefore $F_n'(x) \neq 0$ for all $x \in \mathbb{R}_{>0}$, and hence F_n^{-1} exists. Therefore we may define

$$F_{n+1}(x) = \int_1^{F_n(x)} F_n^{-1}(t)\mathrm{d}t.$$

This justifies our claim.

Now since $F_n'(x) = x^nF_0'(x)$, we have

$$F_n$$