Derivatives

1 Introduction

In this document, I will show you how to compute derivatives in an algebraic and algorithmic manner. Let us recall the definition of the derivative of a function. Let $f(x)$ be a function. We say

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1.1 Product Rule

Recall that if $f(x)$ and $g(x)$ are two functions, then the derivative of their product is given by the **product rule:**

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \cdot g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \cdot g(x) + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(g(x)).$$

Let's calculate some derivatives of product functions.

Example 1.1. Suppose $f(x) = x^{2/3}(x^3 - 5x^2)$. We write $f(x)$ as a product of two functions,

$$g(x) = x^{2/3} \text{ and } h(x) = x^3 - 5x^2.$$

So $f(x) = g(x)h(x)$. We calculate

$$g'(x) = \frac{2}{3}x^{-1/3} \text{ and } h'(x) = 3x^2 - 10x.$$

Therefore

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) \\ &= \left(\frac{2}{3}x^{-1/3}\right)\left(x^3 - 5x^2\right) + x^{2/3}(3x^2 - 10x). \end{aligned}$$

This is the way the book may want you to do it. The way I calculate the derative of $f(x)$ is as follow

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{2/3}(x^3 - 5x^2)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{2/3})(x^3 - 5x^2) + x^{2/3}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x^2) \\ &= \frac{2}{3}x^{-1/3}(x^3 - 5x^2) + x^{2/3}(3x^2 - 10x). \end{aligned}$$

Example 1.2. Let's find the derivative of $e^{2x}\sqrt{x^3 - 5x^2}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\sqrt{x^3 - 5x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{2x})\sqrt{x^3 - 5x^2} + e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^3 - 5x^2\right)^{1/2}\right) \\ &= e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}(2x)\sqrt{x^3 - 5x^2} + e^{2x} \cdot \frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x^2) \\ &= e^{2x} \cdot 2\sqrt{x^3 - 5x^2} + e^{2x} \cdot \frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}(3x^2 - 10x). \end{aligned}$$

Example 1.3. Let's find the derivative of $(4x^2 - x + 1.5)(2(5^x))$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((4x^2 - x + 1.5)(2(5^x))\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(4x^2 - x + 1.5) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2(5^x)) \\
&= (8x - 1) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot 2\frac{\mathrm{d}}{\mathrm{d}x}(5^x) \\
&= (8x - 1) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot 2\ln(5)5^x.
\end{aligned}$$

Example 1.4. Let's find the derivative of $\frac{-2(3^x)}{\sqrt{x}}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-2(3^x)}{\sqrt{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-2(3^x) \cdot x^{-1/2}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(-2(3^x))) \cdot x^{-1/2} + -2(3^x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1/2}\right) \\
&= -2\ln(3)3^x x^{-1/2} + -2(3^x) \cdot \frac{-1}{2}x^{-3/2}.
\end{aligned}$$

Example 1.5. Let's find the derivative of $2.5x\sqrt{x^3 - x}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(2.5x\sqrt{x^3 - x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(2.5x) \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^3 - x}\right) \\
&= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left((x^3 - x)^{1/2}\right) \\
&= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{1}{2} \cdot (x^3 - x)^{-1/2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^3 - x) \\
&= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{1}{2} \cdot (x^3 - x)^{-1/2} \cdot (3x^2 - 1).
\end{aligned}$$

Example 1.6. Let's find the derivative of $(6x - 4)^5(2x + 1)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((6x - 4)^5(2x + 1)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}((6x - 4)^5) \cdot (2x + 1) + (6x - 4)^5 \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2x + 1) \\
&= 5 \cdot (6x - 4)^4 \cdot \frac{\mathrm{d}}{\mathrm{d}x}(6x - 4) \cdot (2x + 1) + (6x - 4)^5 \cdot 2 \\
&= 5 \cdot (6x - 4)^4 \cdot 6 \cdot (2x + 1) + (6x - 4)^5 \cdot 2.
\end{aligned}$$

Example 1.7. Let's find the derivative of $\frac{2x^3 + 7x}{3x - 5}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^3 + 7x}{3x - 5}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((2x^3 + 7x)(3x - 5)^{-1}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(2x^3 + 7x) \cdot (3x - 5)^{-1} + (2x^3 + 7x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left((3x - 5)^{-1}\right) \\
&= (6x^2 + 7)(3x - 5)^{-1} + (2x^3 + 7x) \cdot (-1) \cdot (3x - 5)^{-2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(3x - 5) \\
&= (6x^2 + 7)(3x - 5)^{-1} - (2x^3 + 7x)(3x - 5)^{-2} \cdot 3.
\end{aligned}$$

Example 1.8. Let's find the derivative of $2(5^x)\ln(x)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(2(5^x)\ln(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}(2(5^x)) \cdot \ln(x) + 2(5^x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}(5^x) \cdot \ln(x) + 2(5^x) \cdot \frac{1}{x} \\
&= 2 \cdot \ln(5) \cdot 5^x \cdot \ln(x) + 2(5^x) \cdot \frac{1}{x}.
\end{aligned}$$

1.2 The Chain Rule

So far we know how to compute the derivative of simple functions like x^5 or e^x. We also know how to compute the derivative of composite functions, like e^{x^5}: indeed we learned the chain rule last section:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^5}\right) &= e^{x^5} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^5) \\ &= e^{x^5} \cdot 5x^4.\end{aligned}$$

Now we want to know how to calculate the derivative of a product of two functions like $x^5 e^x$.

The way that we will do it is via the product rule: The product rule says that if you have two functions $f(x)$ and $g(x)$, then the derivative of their product is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \cdot g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \cdot g(x) + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(g(x)).$$

So for example

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^5 e^x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^5) e^x + x^5 \frac{\mathrm{d}}{\mathrm{d}x}(e^x) \\ &= 5x^4 e^x + x^5 e^x.\end{aligned}$$

If we use the prime notation for the derative, then the product rule looks like this:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Let's looks at some examples.

Example 1.9. Suppose $g(x) = 5x^6$ and $h(x) = \ln(x)$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 5x^6 \ln(x).$$

The derivative of their product is given by

$$\begin{aligned}(g \cdot h)'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}((g \cdot h)(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(5x^6 \ln(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(5x^6) \ln(x) + 5x^6 \frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) \\ &= 30x^5 \ln(x) + 5x^6 \cdot \frac{1}{x} \\ &= 30x^5 \ln(x) + 5x^5 \\ &= 5x^5(6\ln(x) + 1).\end{aligned}$$

Example 1.10. Suppose $g(x) = 2(3^x)$ and $h(x) = 3x^2 - 2x + 1$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 2(3^x)(3x^2 - 2x + 1).$$

The derivative of their product is given by

$$\begin{aligned}(g \cdot h)'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}((g \cdot h)(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2(3^x)(3x^2 - 2x + 1)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2(3^x))(3x^2 - 2x + 1) + 2(3^x)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 2x + 1) \\ &= 2\frac{\mathrm{d}}{\mathrm{d}x}(3^x)(3x^2 - 2x + 1) + 2(3^x)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 2x + 1) \\ &= 2\ln(3)3^x(3x^2 - 2x + 1) + 2(3^x)(6x - 2).\end{aligned}$$