δ-rings

Fix a prime p. We want to discuss some aspects of the theory of δ-rings. This theory provides a good language to talk about rings with a lift of Frobenius modulo p.
note

$$\delta(a^p) = \delta(a)a^{(p-1)p} + (a + p\delta(a))\delta(a^{p-1}) = \delta(a)a^{(p-1)p} + (a + p\delta(a))(\delta(a)a^{(p-2)p} + (a + p\delta(a))\delta(a^{p-2})$$
$$\delta(a^p) = 2\delta(a)a^{(p-1)p} + p\delta(a)^2 a^{(p-2)p} + (a + p\delta(a))^2\delta(a^{p-2}) + \delta(a^{p-2})$$
$$\delta(a^2) = 2\delta(a)(a^2 + \delta(a))$$

1 Definition and Examples

Definition 1.1. A **δ-ring** is a pair (A, δ) where A is a commutative ring and $\delta \colon A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$ satisfying the following two identities:

1. for all $a_1, a_2 \in A$ we have
$$\delta(a_1 a_2) = \delta(a_1)a_2^p + a_1^p\delta(a_2) + p\delta(a_1)\delta(a_2).$$
2. for all $a_1, a_2 \in A$ we have
$$\delta(a_1 + a_2) = \delta(a_1) + \delta(a_2) + \frac{a_1^p + a_2^p - (a_1 + a_2)^p}{p} = \delta(a_1) + \delta(a_2) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_1^i a_2^{p-i}.$$

There is an evident category of δ-rings. If the δ-stucture on A is clear from context, we often supress it from the notation and simply call A as δ-ring.

Suppose A is a commutative ring equipped with a map $\phi \colon A \to A$ that lifts the Frobenius on A/p. Then for each $a \in A$, we have an equation of the form

$$\phi(a) = a^p + p\delta(a),$$

where $\delta \colon A \to A$. In fact, we claim that equipping A with δ gives it the structure of a δ-ring. Indeed, we clearly have $\delta(0) = \delta(1) = 0$. Also, since

$$\begin{aligned} a^p b^p + p\delta(ab) &= \phi(ab) \\ &= \phi(a)\phi(b) \\ &= (a^p + p\delta(a))(b^p + p\delta(b)) \\ &= a^p b^p + p(a^p\delta(b) + b^p\delta(a) + p\delta(a)\delta(b)). \end{aligned}$$

Thus we must have $\delta(ab) = a^p\delta(b) + b^p\delta(a) + p\delta(a)\delta(b)$. Similarly we have

$$\begin{aligned} \delta(a+b) &= \frac{\phi(a+b) - (a+b)^p}{p} \\ &= \frac{\phi(a) + \phi(b) - (a+b)^p}{p} \\ &= \frac{a^p + b^p + p(\delta(a) + \delta(b)) - (a+b)^p}{p} \\ &= \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p}. \end{aligned}$$

We can also go backwards:

Lemma 1.1. *Let A be a commutative ring.*

1. *If $\delta\colon A \to A$ provides a δ-structure on A, then the map $\phi\colon A \to A$ defined by*

$$\phi(a) = a^p + p\delta(a)$$

for all $a \in A$, is an endomorphism of A which lifts the Frobenius on A/p.

2. *When A is p-torsionfree, the construction (1) gives a bijective correspondence between δ-structures on A and Frobenius lifts on A.*

Remark. If A is not necessarily p-torsionfree, it is better to record δ instead of ϕ.

1.1 Perfectoid Fields

Definition 1.2. A **perfectoid field** K is a complete nonarchimedean field (with valuation ring K°) such that:

1. the residue characteristic is p.
2. the associated rank-1 valuation is nondiscrete.
3. the frobenius map $\Phi\colon K^\circ/p \to K^\circ/p$ is surjective.

Example 1.1. Let $K = \mathbb{Q}_p(1/p^{1/\infty})^\wedge$ be the completion of the field obtained by adjoining all p-power roots of p to $\mathbb{Q}_p$.

Lemma 1.2. *Let K be a perfectoid field and let $\mathfrak{m}$ be the maximal ideal of K°. Then $\mathfrak{m}^2 = \mathfrak{m} = \mathfrak{m} \otimes \mathfrak{m}$.*

Proof. Let $x \in \mathfrak{m}$. Then since the frobenius is surjective on K°/p, we have

$$\begin{aligned} x &= x_1^p + px_2 \\ &= x_1^p + p(x_3^p + px_4) \\ &= x_1^p + px_3^p + p^2x_4 \\ &\in \mathfrak{m}^2, \end{aligned}$$

where the last part follows from the fact that $x_1, x_3, x_4 \in \mathfrak{m}$ and $p \geq 2$. It follows that $\mathfrak{m} = \mathfrak{m}^2$. In particular, this implies the map $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, given by $x \otimes y \mapsto xy$, is surjective. That $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ is injective follows from the fact that $\mathfrak{m}$ is flat. $\square$

Remark. Note that since K is complete, the proof gives us the expression

$$x = x_1^p + px_2^p + p^2x_3^p + \cdots = \sum_{n=1}^{\infty} p^n x_n^p$$

which may come in handy.

1.2 Perfect Rings

Definition 1.3. An $\mathbb{F}_p$-algebra A is **perfect** if the Frobenius map $\phi\colon A \to A$ given by $\phi(a) = a^p$ is an isomorphism.

Definition 1.4. Let A be any $\mathbb{F}_p$-algebra. There are two ways we can form a perfect ring out of A:

1. The **direct limit perfection** A_{perf} is the directed limit of the system, $\cdots \to A \to A \to \cdots$ where all the maps are Frobenius. When we just say **perfection**, it will refer to this construction.
2. The **inverse limit perfection** A^{perf} is the inverse limit of the system, $\cdots \to A \to A \to \cdots$ where all the maps are Frobenius.

Example 1.2. The ring $R = \mathbb{F}_p[x^{1/p^\infty}]$ is obtained as the union of polynomial rings $\mathbb{F}_p[x^{1/p^n}]$ as $n \to \infty$.

Let Σ be the category of all $\mathfrak{m}$-torsion submodules of K°. This is a (thick) abelian Serre subcategory, which means it's closed under subobjects, quotients, and (most importantly) extensions.

Definition 1.5. A K°-module is called **almost zero** if it's in Σ. We let $K^{\circ a}$-**mod** be K°-**mod**$/\Sigma$ be the localization of the category K°-**mod** by the Serre subcategory Σ (i.e. objects of the two categories are the same, but we change the hom-sets so that everything in Σ is isomorphic to 0).

Example 1.3. $K^\circ/\mathfrak{m}$ is almost zero, but K°/p is not almost zero.

There is an "almost" functor K°-**mod** $\to K^{\circ a}$-**mod** denoted $M \mapsto M^a$. It has a right adjoint $N \mapsto N_*$ and a left adjoint $N \mapsto N_!$. This means that

$$\mathrm{Hom}_{K^\circ}(M_!, N) = \mathrm{Hom}_{K^{\circ a}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(M, N_*).$$

If $M = T^a$ is an almost module, then

$$(T^a)_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, T) \quad \text{and} \quad (T^a)_! = \mathfrak{m} \otimes T.$$

We call M_* the module of **almost elements** of M. In other words,

$$\mathrm{Hom}_R(M \otimes_R \mathfrak{m}, N) = \mathrm{Hom}_R(M, \mathrm{Hom}_R(\mathfrak{m}, N))$$

Remark. The notation comes from topology: if $j\colon U \to X$ is the inclusion of an open subset, then $j^*\colon \mathbf{Sh}(X) \to \mathbf{Sh}(U)$ has left and right adjoints $j_!$ and j_* respectively.