ADDITIONAL NOTE ON HOUSEHOLDER

Instructor: Dr. Fei Xue, Martin O-203, fxue@clemson.edu.

Let x be a nonzero vector. Define vector $v = x - \|x\|_2 e_1$. Algebraically, we have

$$v^T x = (x - \|x\|_2 e_1)^T x = x^T x - \|x\|_2 x(1) = \|x\|_2^2 - \|x\|_2 x(1), \quad \text{and}$$
$$v^T v = (x - \|x\|_2 e_1)^T (x - \|x\|_2 e_1) = x^T x - 2\|x\|_2 x(1) + \|x\|_2^2 = 2\left(\|x\|_2^2 - \|x\|_2 x(1)\right),$$

where $x(1)$ refers to the first element of x. Recall that the orthogonal projection of x onto v is $v\frac{v^T x}{v^T v}$. As a result, the reflection of the terminal point of x with respect to the hyperplane with normal direction v is

$$Hx = \left(I - 2\frac{vv^T}{v^T v}\right) x = x - 2v\frac{v^T x}{v^T v} = x - 2v\frac{\|x\|_2^2 - \|x\|_2 x(1)}{2\left(\|x\|_2^2 - \|x\|_2 x(1)\right)} = x - v = \|x\|_2 e_1.$$

Geometrically, since the terminal point of x and Hx are reflections of each other with respect to the hyperplane mentioned above, the vector connecting the two terminal points should be the normal direction of this hyperplane. In fact,

$$x - Hx = x - (x - v) = v$$

is indeed the normal direction here. From another perspective, given a nonzero vector x and its target reflection point $Hx = \|x\|_2 e_1$, this normal direction vector connecting the two terminal points of x and Hx should be

$$v = x - Hx = x - \|x\|_2 e_1$$

If we sketch x, the hyperplane, and its reflection point Hx, then connect the two terminal points with v (originating from Hx toward the terminal x), it should be easy to see that the orthogonal projection of x onto the direction of v should be exactly a half of v in length, which has been shown above algebraically as

$$v\frac{v^T x}{v^T v} = \frac{1}{2} v.$$

Please sketch these vectors and the hyperplane, follow the above explanation carefully to make sure that you see these algebraic and geometric relationships. This should help you fully understand how a Household reflector is constructed and why $Hx = \|x\|_2 e_1$ has only a single nonzero element (the first element) whereas all subsequent elements are zeros.

In practice, we may let $v = x + \mathrm{sign}(x(1))\|x\|_2 e_1$, where $\mathrm{sign}(x(1))$ is the sign of $x(1)$. If $x(1) = 0$, we can artificially define $\mathrm{sign}(x(1)) = 1$; if $x(1)$ is a complex number in the form of $\rho e^{i\theta}$, then $\mathrm{sign}(x(1)) = e^{i\theta}$. We can replace the regular vector transpose operation above with complex conjugate transpose, then Householder reflector H satisfies $Hx = -e^{i\theta}\|x\|_2 e_1$. The vector v defined above gives the most generic form of Householder reflectors, which also includes the case of real vectors x. If $x(1) \geq 0$, $\theta = 0$, and if $x(1) < 0$, $\theta = \pi$.