Fibers

Definition 0.1. Let S be an R-algebra and let $\mathfrak{p}$ be a prime ideal of R. We define the **fiber of** S **over** $\mathfrak{p}$ to be the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_R S$ where $\kappa(\mathfrak{p}) = K(R/\mathfrak{p})$ denotes the quotient field of $R/\mathfrak{p}$. In particular, if $\mathfrak{m}$ is a maximal ideal of R, then the fiber of S over $\mathfrak{m}$ is the $R/\mathfrak{m}$-algebra $R/\mathfrak{m} \otimes_R S \simeq S/\mathfrak{m}S$. If R is an integral domain with fraction field K, then **generic fiber of** S is the K-algebra $K \otimes_R S$.

Remark 1. Let $\iota\colon A \to B$ be an inclusion of $\mathbb{k}$-algebras where $\mathbb{k}$ is a field. Geometrically speaking, the inclusion map $\iota\colon A \to B$ of $\mathbb{k}$-algebras corresponds to the morphism $\pi\colon Y \to X$ of affine $\mathbb{k}$-schemes, where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and where π is defined by

$$\pi(\mathfrak{q}) = A \cap \mathfrak{q}$$

for all primes $\mathfrak{q}$ of B. If $\iota\colon A \to B$ is an integral extension, then π is surjective (this is referred to as the **lying over** property for integral extensions). Note that π is continuous with respect to the Zariski topology, for if $U := \mathrm{D}(a)$ is an open subset of X where $a \in A$, then

$$\pi^{-1}(U) = \pi^{-1}(\mathrm{D}(a)) = \mathrm{D}(\iota(a)) := V.$$

In other words, we have $a \notin A \cap \mathfrak{q}$ if and only if $a \notin \mathfrak{q}$ for all primes $\mathfrak{q}$ of B. Now, given a prime $\mathfrak{p}$ of A, the fiber of $\pi\colon Y \to X$ over $\mathfrak{p}$, denoted $Y_{\mathfrak{p}}$, is the pullback of $\pi\colon Y \to X$ with respect to the morphism $\varepsilon_{\mathfrak{p}}\colon \operatorname{Spec}(\kappa(\mathfrak{p})) \to X$ where $\varepsilon_{\mathfrak{p}}$ is the morphism which corresponds to the $\mathbb{k}$-algebra homomorphism $A \to \kappa(\mathfrak{p})$. In particular, $Y_{\mathfrak{p}}$ is an affine $\mathbb{k}$-scheme and the $\mathbb{k}$-algebra which corresponds to $Y_{\mathfrak{p}}$ is $\kappa(\mathfrak{p}) \otimes_A B$, which is precisely how we defined the fiber of B over $\mathfrak{p}$ in the first place.

Example 0.1. Let $R = \mathbb{k}[\boldsymbol{a}] = \mathbb{k}[a_1, a_2, a_3]$ and let $S = R[\boldsymbol{x}]/f = R[x_1, x_2]/f$ where $f = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2$. Also for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{k}^3$, we set

$$\mathfrak{m}_{\boldsymbol{\alpha}} = \langle a_1 - \alpha_1, a_2 - \alpha_2, a_3 - \alpha_3 \rangle \quad \text{and} \quad f_{\boldsymbol{\alpha}} = \alpha_1x_1^2 + \alpha_2x_1x_2 + \alpha_3x_2^2.$$

Then the fiber of S over $\mathfrak{m}_{\boldsymbol{\alpha}}$ is the $\mathbb{k}$-algebra $S_{\boldsymbol{\alpha}} := \mathbb{k}[\boldsymbol{a}, \boldsymbol{x}]/f_{\boldsymbol{\alpha}}$. Geometrically speaking, the inclusion map $\iota\colon R \to S$ of $\mathbb{k}$-algebras corresponds to the projection $\pi\colon Y \to X$ of affine $\mathbb{k}$-schemes, where $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. The fiber of π over $\mathfrak{m}_{\boldsymbol{\alpha}}$ is given by

$$\pi^{-1}(\{\mathfrak{m}_{\boldsymbol{\alpha}}\}) = \mathrm{V}(f_{\boldsymbol{\alpha}}) = \operatorname{Spec}(S_{\boldsymbol{\alpha}}).$$

Example 0.2. Let $R = \mathbb{k}[t]$, let $S = R[x]/\langle x^2 - t \rangle$, and let $\mathfrak{p}_\tau = \langle t - \tau \rangle$ where $\tau \in \mathbb{k}$. Then for $\tau \neq 0$, the fiber of S over $\mathfrak{p}_\tau$ is $\mathbb{k}[x]/\langle x^2 - \tau \rangle \cong \mathbb{k} \times \mathbb{k}$. The fiber over $\mathfrak{p}_0$ is $S_0 := \mathbb{k}[x]/\langle x^2 \rangle$. Finally, the fiber over the zero ideal $\langle 0 \rangle$ is $\mathbb{k}(t)[x]/\langle x^2 - t \rangle$, a field of degree 2 over the residue field $\kappa(\langle 0 \rangle) = \mathbb{k}(t)$. We see that for each prime $\mathfrak{p}$, the fiber over $\mathfrak{p}$ is a vector space of dimension 2 over its residue field $\kappa(\mathfrak{p})$. In fact, S is a free R-module on the generators $(1, x)$. Thus $S \otimes_R N = N \oplus N$ for any R-module N, and it follows that S is flat.

Proposition 0.1. *Let $\varphi\colon A \to B$ be a ring homomorphism and let $\mathfrak{p}$ be a prime ideal of A. Let $f\colon Y \to X$ be the corresponding map of affine schemes where $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. Then $\mathfrak{p}$ is in the image of f if and only if the fiber of B over $\mathfrak{p}$ is nonzero.*

Proof. First note that if $\mathfrak{q}$ is a prime of B that lies over $\mathfrak{p}$, then $\mathfrak{q}_{\mathfrak{q}}$ is a prime of $B_{\mathfrak{q}}$ that lies over $\mathfrak{p}_{\mathfrak{p}}$. Conversely, if $\mathfrak{r}$ is a prime of $B_{\mathfrak{q}}$ that lies over $\mathfrak{p}_{\mathfrak{p}}$, then it must have the form $\mathfrak{r} = \mathfrak{q}_{\mathfrak{p}}$ for some prime $\mathfrak{q}$ of B. Thus, by localizing at $\mathfrak{p}$ if necessary, we may assume that $A = (A, \mathfrak{p}, \mathbb{k})$ is a local ring. Now if $\mathfrak{q}$ if a prime of B that lies over $\mathfrak{p}$, then $\overline{\mathfrak{q}} := \mathfrak{q}/\mathfrak{p}B$ is a prime of $\overline{B} := B/\mathfrak{p}B$ which implies $\overline{B}$ is nonzero. Conversely, if $\overline{B} \neq 0$, then there exists a prime $\mathfrak{r}$ of $\overline{B}$, which must have the form $\mathfrak{r} = \overline{\mathfrak{q}} := \mathfrak{q}/\mathfrak{p}B$ for some prime $\mathfrak{q}$ of B which necessarily lies over $\mathfrak{p}$. $\square$