Mathematics Diary

Contents

1 2023

1.1 12/20/2022

Lemma 1.1. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R. Let E be the minimal free resolution of R/J over R, let F be the minimal free resolution of R/I over R, and let $\varphi \colon E \to F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_{\Bbbk} := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_{\Bbbk}$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R.*

Proof. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_{\Bbbk} := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_{\Bbbk}$ are injective. Since $\varphi \colon E \to F$ is injective, we have a short exact sequence of R-complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \tag{1}$$

taking homology gives us a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{i+1}(F/E) \longrightarrow \mathrm{H}_i(E) \longrightarrow \mathrm{H}_i(F) \longrightarrow \mathrm{H}_i(F/E) \longrightarrow \mathrm{H}_{i-1}(E) \longrightarrow \cdots$$

Since E and F are resolutions we conclude that $\mathrm{H}_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $\mathrm{H}_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R-modules (1) with $\Bbbk$ over R gives us the long exact sequence in homology

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \operatorname{Tor}^R_{i+1}(E,\Bbbk) \arrow[dlll, rounded corners] \\
\operatorname{Tor}^R_i(E,\Bbbk) \arrow[r] & \operatorname{Tor}^R_i(F,\Bbbk) \arrow[r] & \operatorname{Tor}^R_i(F/E,\Bbbk) \arrow[dll, rounded corners] & \\
\operatorname{Tor}^R_{i-1}(E,\Bbbk) \arrow[r] & \cdots & &
\end{tikzcd}$$

Since E and F are free R-modules we conclude that $\operatorname{Tor}_i(F/E,\Bbbk) = 0$ for all $i \geq 1$. Since $\overline{\varphi}\colon E \otimes_R \Bbbk \to F \otimes_R \Bbbk$ is injective we conclude that $\operatorname{Tor}_1(F/E,\Bbbk) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $\mathrm{d}(F/E) \subseteq \mathfrak{m}(F/E)$). $\square$

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R-module M connecting maps

$$\operatorname{Tor}^R_{i+1}(R/I, M) \to \operatorname{Tor}^R_i(I/J, M) \quad \text{and} \quad \operatorname{Ext}^i_R(I/J, M) \to \operatorname{Ext}^{i+1}_R(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \to F/E \otimes_R M \quad \text{and} \quad \operatorname{Hom}^\star_R(F/E, M) \to \operatorname{Hom}^\star_R(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\overline{\varphi}\colon E_\Bbbk \to F_\Bbbk$ is injective, then already $\varphi\colon E \to F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of Ideals

Let R be a commutative ring and let $\mathfrak{p}$ be an ideal of R. Recall the **height** of $\mathfrak{p}$ is defined to be the supremum of lengths of chains of primes which descend from $\mathfrak{p}$:

$$\operatorname{ht}\mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

When R is Noetherian, then Krull's prinicipal ideal theorem states that there exists an ideal $\langle \boldsymbol{x} \rangle = \langle x_1, \dots, x_c \rangle \subseteq \mathfrak{p}$ where $c = \operatorname{ht}\mathfrak{p}$ such that $\sqrt{\langle \boldsymbol{x} \rangle} = \mathfrak{p}$, and that if $\langle \boldsymbol{y} \rangle = \langle y_1, \dots, y_m \rangle$ is another ideal such that $\sqrt{\langle \boldsymbol{y} \rangle} = \mathfrak{p}$, then we must have $c \leq m$. If I is an ideal of R, then the **height** of I is defined to be the infimum of the heights of all primes which contain I:

$$\operatorname{ht} I = \inf\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. *Let I_1 and I_2 be ideals of R. Set $c = \operatorname{ht}(I_1 \cap I_2)$, set $c_1 = \operatorname{ht} I_1$, and set $c_2 = \operatorname{ht} I_2$.*

1. *If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.*
2. *We have $c = \min\{c_1, c_2\}$.*

Proof. 1. Let $\mathfrak{p}$ be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2. Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let $\mathfrak{p}$ be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since $\mathfrak{p}$ is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. $\square$

2 2024

1/20/2024 - $\mathrm{V}(\operatorname{Ann} M) = \mathrm{V}(\operatorname{Ann}(0 :_M x))$

Lemma 2.1. *Let R be a commutative ring, let M be an R-module, and let $x \in R$. Then*

$$\mathrm{V}(\operatorname{Ann}(0 :_M x)) = \mathrm{V}(\operatorname{Ann}(0 :_M x^2)).$$

Proof. Note that $0 :_M x \subseteq 0 :_M x^2$ implies $\mathrm{Ann}(0 :_M x^2) \supseteq \mathrm{Ann}(0 :_M x)$ which implies $\mathrm{V}(\mathrm{Ann}(0 :_M x^2)) \subseteq \mathrm{V}(\mathrm{Ann}(0 :_M x))$. For the reverse inclusion, suppose $\mathfrak{p}$ is a prime ideal of R which contains $\mathrm{Ann}(0 :_M x^2)$ and let $r \in \mathrm{Ann}(0 :_M x)$. We claim that $r^2 \in \mathrm{Ann}(0 :_M x^2)$. Indeed, if $u \in 0 :_M x^2$, then

$$\begin{aligned} x^2u = 0 &\implies xu \in 0 :_M x \\ &\implies rxu = 0 \\ &\implies ru \in 0 :_M x \\ &\implies r^2u = 0. \end{aligned}$$

Since u was arbitrary, we see that $r^2 \in \mathrm{Ann}(0 :_M x^2) \subseteq \mathfrak{p}$. However this implies $r \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime. Since r was arbitary, we see that $\mathrm{Ann}(0 :_M x) \subseteq \mathfrak{p}$. □

Corollary 1. *Let R be a commutative ring and let M be a finitely generated R-module. Assume that $x \in R$ acts nilpotently on M. Then*

$$\mathrm{V}(\mathrm{Ann}(M)) = \mathrm{V}(\mathrm{Ann}(0 :_M x)).$$

Proof. Since M is finitely generated, there exists an $n \in \mathbb{N}$ such that $M = 0 :_M x^n$. A straightforward induction on (**??**) gives us

$$\mathrm{V}(\mathrm{Ann}(M)) = \mathrm{V}(\mathrm{Ann}(0 :_M x^n)) = \mathrm{V}(\mathrm{Ann}(0 :_M x)).$$

□

1/21/2024 Some subschemes of $\mathbb{P}^3$

Let $R = \Bbbk[x, y, z, w]$. We consider three cyclic R-algebras, namely $A = R/\boldsymbol{f} = R/\langle f_1, f_2, f_3\rangle$, $B = R/\boldsymbol{g} = R/\langle g_1, g_2, g_3\rangle$, and $C = R/\boldsymbol{h} = R/\langle h_1, h_2, h_3\rangle$ where

$$\begin{aligned} f_1 &= xy - zw & g_1 &= xz - y^2 & h_1 &= xz - y^2 \\ f_2 &= xz - yw & g_2 &= yw - z^2 & h_2 &= x^3 - yzw \\ f_3 &= xw - yz & g_3 &= xw - yz & h_3 &= x^2y - z^2w \end{aligned}$$

We want a geometric picture in mind when thinking of these rings, so let $X = \mathrm{Proj}\, A$, $Y = \mathrm{Proj}\, B$, and $Z = \mathrm{Proj}\, C$. First let us consider X. We can see that X is 8 distinct points in $\mathbb{P}^3(\Bbbk)$ by calculating an irreducible primary decomposition for $I = \langle \boldsymbol{f}\rangle$. Indeed, an irredundant primary decomposition for $\langle \boldsymbol{f}\rangle$ is given by $\langle \boldsymbol{f}\rangle = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_8$ where

$$\begin{aligned} \mathfrak{p}_1 &= \langle y, z, w\rangle & \mathfrak{p}_5 &= \langle x + y, y + z, z + w\rangle \\ \mathfrak{p}_2 &= \langle x, z, w\rangle & \mathfrak{p}_6 &= \langle x + y, y - z, z + w\rangle \\ \mathfrak{p}_3 &= \langle x, y, w\rangle & \mathfrak{p}_7 &= \langle x + y, y - z, z - w\rangle \\ \mathfrak{p}_4 &= \langle x, y, z\rangle & \mathfrak{p}_8 &= \langle x - y, y - z, z - w\rangle. \end{aligned}$$

These primes correspond to the points

$$\begin{aligned} \boldsymbol{p}_1 &= [1 : 0 : 0 : 0] & \boldsymbol{p}_5 &= [-1 : 1 : -1 : 1] \\ \boldsymbol{p}_2 &= [0 : 1 : 0 : 0] & \boldsymbol{p}_6 &= [1 : -1 : -1 : 1] \\ \boldsymbol{p}_3 &= [0 : 0 : 1 : 0] & \boldsymbol{p}_7 &= [-1 : 1 : 1 : 1] \\ \boldsymbol{p}_4 &= [0 : 0 : 0 : 1] & \boldsymbol{p}_8 &= [1 : 1 : 1 : 1] \end{aligned}$$

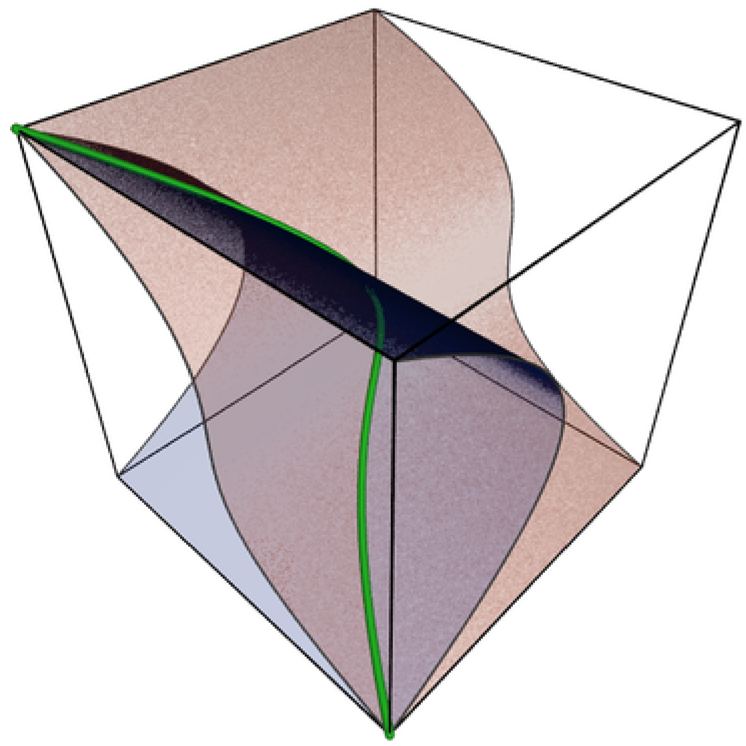
in $\mathbb{P}^3(\Bbbk)$. Note that $\boldsymbol{p}_1, \dots, \boldsymbol{p}_8$ are in linearly general position since the size k minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

are all nonzero for all $1 \leq k \leq 4$. The Betti diagram of A over R is given by

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	-	3	-
2	-	-	-	1

Next we consider Y. In fact, Y is the twisted cubic:



In particular, Y is the image of the map $\mathbb{P}^1(\Bbbk) \to \mathbb{P}^3(\Bbbk)$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Note that $\langle g \rangle$ is a prime of height 2 and so $\langle g \rangle$ can be generated up to radical by two homogeneous polynomials. In particular, we have $\langle g \rangle = \sqrt{\langle g_1, g_4 \rangle}$ where $g_4 = zg_2 - wg_3$. However $\langle g \rangle$ itself cannot be generated by only two polynomials; a minimum of three polynomials are needed. We can see this in Betti diagram of B over R:

	0	1	2
0	1	-	-
1	-	3	2

In particular, the Hilbert-Poincare series of B over R is given by

$$P(t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^4} = \frac{1+2t}{(1-t)^2} = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + \cdots .$$

Thus Y is the set-theoretic complete intersection of $\mathrm{V}(g_1)$ and $\mathrm{V}(g_4)$ however it is not a scheme-theoretic or ideal-theoretic complete intersection. Note also that $\langle g \rangle$ corresponds to the ideal of size 2 minors of the matrix $\left(\begin{smallmatrix} x & y & z \\ y & z & w \end{smallmatrix}\right)$. Up to linear automorphism, the twisted cubic is the only irreducible curve of degree 3 not contained in a plane. Furthermore, any 6 points in linearly general position in $\mathbb{P}^3(\Bbbk)$ lie on a unique twisted cubic. However for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Consequently one can show that if W is a set of 7 points in linearly general position in $\mathbb{P}^3(\Bbbk)$, then there are only two distinct Betti diagrams possible for the homogeneous coordinate ring of W, namely

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	1	6	3

and

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

In the first case, the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the quadrics containing W is the ideal of the unique curve of degree 3 containing W, which is irreducible. Finally, let us write down the minimal free resolution of B over R:

$$R(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} R \longrightarrow 0$$

Now we consider Z. The Betti diagram of C over R is given by

	0	1	2
0	1	-	-
1	-	1	-
2	-	2	2

In particular, the Hilbert-Poincare series of C over R is given by

$$P(t) = \frac{1 - t^2 - 2t^3 + 2t^4}{(1-t)^4} = \frac{1 + 2t + 2t^2}{(1-t)^2} = 1 + 4t + 9t^2 + 14t^3 + 19t^4 + \cdots .$$

In particular, Z is an irreducible curve of degree 5 in $\mathbb{P}^3(\Bbbk)$.

2.1 4/22/2024

Let A be a commutative ring and let B be an A-algebra which is finite as an A-module. Then there exists a surjection $F \twoheadrightarrow B$ of A-modules where $F = A^{n+1}$ where we assume $n \geq 0$ is minimal. We are interested in the question as to whether one can lift the multiplication on B to a multiplication on F. Let K be the kernel of the map $F \twoheadrightarrow B$. In what follows, all tensors products are taken over A.

Lemma 2.2. *The kernel of the map $F^{\otimes 2} \to B^{\otimes 2}$ is given by $K \otimes F + F \otimes K$.*

Proof. This is easily checked via a diagram chase in the diagram below which is exact everywhere and in all directions:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & K^{\otimes 2} \arrow[r] \arrow[d] & K \otimes F \arrow[r] \arrow[d] & K \otimes B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes K \arrow[r] \arrow[d] & F^{\otimes 2} \arrow[r] \arrow[d] & F \otimes B \arrow[r] \arrow[d] & 0 \\
 & B \otimes K \arrow[r] \arrow[d] & B \otimes F \arrow[r] \arrow[d] & B^{\otimes 2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Since $F^{\otimes 2}$ is free (hence projective), we can lift the composite map $F^{\otimes 2} \to B^{\otimes 2} \twoheadrightarrow B$ with respect to the map $F \twoheadrightarrow B$ to obtain an A-linear map $\mu \colon F^{\otimes 2} \to F$. Assume that A is a local noetherian ring. In this case, there exists a minimal generating set of B as an A-module of the form $\{b_0, b_1, \dots, b_n\}$ where $b_0 = 1$. Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ be a basis for F as a free A-module and let $F \twoheadrightarrow B$ be the A-linear map defined by $\varepsilon_i \mapsto b_i$ for all i. For each i, j, we have

$$b_i b_j = \sum_k a_{ij}^k b_k$$

where the $a_{ij}^k \in A$ need not be unique. Since the multiplication on B is unital, we can choose the a_{ij}^k such that

$$a_{j0}^k = a_{0j}^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Furthermore, since the multiplication on B is commutative, we can also choose the a_{ij}^k such $a_{ij}^k = a_{ji}^k$. With these choices of a_{ij}^k in mind, we can define a commutative and unital multiplication μ on F which lifts the multiplication on B by

$$\varepsilon_i \varepsilon_j := \sum_k a_{ij}^k \varepsilon_k.$$

Note that this multiplication need not be associative. Indeed, since the multiplication on B is associative, we have

$$\begin{aligned}[b_i, b_j, b_k] &= (b_i b_j) b_k - b_i (b_j b_k) \\ &= \sum_l (a^l_{ij} b_l b_k - a^l_{jk} b_i b_l) \\ &= \sum_{l,m} (a^l_{ij} a^m_{lk} - a^l_{jk} a^m_{il}) b_m.\end{aligned}$$

However this need not imply that $a^l_{ij} a^m_{lk} - a^l_{jk} a^m_{ik} = 0$ for all i, j, k, l, m (which is what we'd need in order for $[\varepsilon_i, \varepsilon_j, \varepsilon_k] = 0$).

2.2 5/2/2024

Let R be a noetherian ring, let I be an ideal of R, and let $r, r' \in R$. We have an R-linear map

$$\varphi \colon \langle I, r\rangle : r' \twoheadrightarrow (\langle I, r'\rangle : r)/(I : r)$$

defined as follows: if $a \in \langle I, r\rangle : r'$, then we have $ar' = br + x$ for some $b \in R$ and $x \in I$. The map is defined by sending a to the class of b in the quotient. It is straightforward to check that this is well-defined and surjective. Note if $b \in I : r$, then $ar' \in I : r'$. In particular, the kernel of φ is $I : r'$. Thus we've established an isomorphism

$$(\langle I, r\rangle : r'))/(I : r') \cong (\langle I, r'\rangle : r))/(I : r). \tag{2}$$

In particular, if $I : r' = I : r$, then we must have $\langle I, r\rangle : r' = \langle I, r'\rangle : r$. Now assume that $I : r = \mathfrak{p} = \langle I, r\rangle : r'$. Then (2) implies

$$\mathfrak{p}/(I : r') \cong (\langle I, r'\rangle : r)/\mathfrak{p}.$$

Example 2.1. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz\rangle$, let $r = yw$, and let $r' = y$. Then we have

$$\begin{aligned} I : r &= \langle x, z, w\rangle & \langle I, r'\rangle : r &= R \\ I : r' &= \langle x, z, w^2\rangle & \langle I, r\rangle : r' &= \langle x, z, w\rangle. \end{aligned}$$

Now observe that $\langle I : r, r'\rangle \subseteq \langle I, r'\rangle : r$. Indeed, if $a \in \langle I : r, r'\rangle$, then we can express it as $a = b + cr'$ where $b \in I : r$ and $c \in R$. In particular, this means that $ar = br + cr'r \in \langle I, r'\rangle$, and hence $a \in \langle I, r'\rangle : r$.

2.3 5/20/2024

Let $A = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \ldots, x_n]$, let $B = \Bbbk[\boldsymbol{y}] = \Bbbk[y_1, \ldots, y_m]$, and let $\varphi \colon A \to B$ be a $\Bbbk$-algebra homomorphism. Next let $Y = \operatorname{Spec} B$, let $X = \operatorname{Spec} A$, and let $f \colon Y \to X$ be given by $f(\mathfrak{q}) := \varphi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in Y$. We want to describe how f acts all maximal ideals of B of the form $\mathfrak{n}_{\boldsymbol{q}} = \langle y_1 - q_1, \ldots, y_m - q_m\rangle$ where $\boldsymbol{q} \in Y(\Bbbk)$. To this end, for each $1 \le j \le n$ let $f_i = \varphi(x_i)$. Then we have

$$\varphi^{-1}(\mathfrak{n}_{\boldsymbol{q}}) = \mathfrak{m}_{\boldsymbol{p}}$$

where $\boldsymbol{p} = (f_1(\boldsymbol{q}), \ldots, f_n(\boldsymbol{q}))$ and where $\mathfrak{m}_{\boldsymbol{p}} = \langle x_1 - p_1, \ldots, x_n - p_n\rangle$. Indeed, observe that

$$\begin{aligned}\varphi(\mathfrak{m}_{\boldsymbol{p}}) &= \langle \varphi(x_1) - p_1, \ldots, \varphi(x_n) - p_n\rangle \\ &= \langle f_1 - f_1(\boldsymbol{q}), \ldots, f_n - f_n(\boldsymbol{q})\rangle \\ &\subseteq \mathfrak{n}_{\boldsymbol{q}}.\end{aligned}$$

2.4 5/21/2024

Let R be a commutative ring, let M_1 and M_2 be R-modules, and set $T = \operatorname{Tor}^R(M_1, M_2)$. We can turn T into an R-complex as follows: choose projective resolutions F^1 of M_1 and F^2 of M_2 over R. Then $\mathrm{d} \otimes 1 \colon F^1 \otimes_R F^2 \to F^1 \otimes_R F^2$ is a chain map of degree -1, thus it induces a map in homology $\mathrm{d} \otimes 1 \colon T \to T$. Furthermore $(\mathrm{d} \otimes 1)^2 = 0$ and so $\mathrm{d} \otimes 1$ gives T an R-complex structure.