A Generalized Associator

0.1 A Generalized Associator

Let F be an R-module and let $\mu, \nu \colon F^{\otimes 2} \to F$ and let $\lambda \colon F \to F$ be R-linear maps (where we denote $F^{\otimes 2} := F \otimes_R F$). We set $[\cdot]_{\mu,\nu,\lambda} \colon F^{\otimes 3} \to F$ to be the R-linear map given by

$$[\cdot]_{\mu,\nu,\lambda} := \mu(\nu \otimes \lambda - \lambda \otimes \nu).$$

We denote by $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda} \colon F^3 \to F$ to be the unique R-trilinear map which corresponds to $[\cdot]_{\mu,\nu,\lambda}$. Thus if we denote $a_1 a_2 = \mu(a_1 \otimes a_2)$ and $a_1 \cdot a_2 = \nu(a_1 \otimes a_2)$ for $a_1 \otimes a_2 \in F^{\otimes 2}$, then we have

$$[a_1 \otimes a_2 \otimes a_3]_{\mu,\nu,\lambda} = (a_1 \cdot a_2)\lambda(a_3) - \lambda(a_1)(a_2 \cdot a_3) = [a_1, a_2, a_3]_{\mu,\nu,\lambda}.$$

We often pass back in forth between $[\cdot]_{\mu,\nu,\nu}$ and $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda}$ without explicitily saying so (mostly we will only talk about $[\cdot]_{\mu,\nu,\nu}$ since it is notationally simpler to write). For istance, we call $[\cdot]_{\mu,\nu,\lambda}$ the **associator** with respect to the triple (μ,ν,λ) (or more simply just **associator** if (μ,ν,λ) is understood from context), and thus we also call $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda}$ the **associato**r. If $\mu = \nu$, then we simplify our notation and write $[\cdot]_{\mu,\lambda} := [\cdot]_{\mu,\mu,\lambda}$. Similarly, if $\mu = \nu$ and $\lambda = 1$, then we simplify our notation further and write $[\cdot]_\mu := [\cdot]_{\mu,\mu,1}$.

Observe that $[\cdot]_{\mu,\nu,\lambda}$ is R-trilinear in μ, ν, and λ. In particular, this means that if $\mu', \nu' \colon F^{\otimes 2} \to F$ and $\lambda' \colon F \to F$ are another triple of R-linear maps, and $r \in R$, then we have

$$\begin{aligned}
[\cdot]_{\mu+\mu',\nu,\lambda} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu',\nu,\lambda} \\
[\cdot]_{\mu,\nu+\nu',\lambda} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu,\nu',\lambda} \\
[\cdot]_{\mu,\nu,\lambda+\lambda'} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu,\nu,\lambda'} \\
r[\cdot]_{\mu,\nu,\lambda} &= [\cdot]_{r\mu,\nu,\lambda} = [\cdot]_{\mu,r\nu,\lambda} = [\cdot]_{\mu,\nu,r\lambda}.
\end{aligned}$$

Thus we have an R-linear map

$$[\cdot]_{(-,-,-)} \colon \mathrm{Hom}(F^{\otimes 2}, F)^{\otimes 2} \otimes \mathrm{Hom}(F, F) \to \mathrm{Hom}(F^{\otimes 3}, F)$$

which takes an elementary tensor $\mu \otimes \nu \otimes \lambda$ in $\mathrm{Hom}(F^{\otimes 2}, F)^{\otimes 2} \otimes \mathrm{Hom}(F, F)$ and maps it to $[\cdot]_{\mu,\nu,\lambda}$ in $\mathrm{Hom}(F^{\otimes 3}, F)$. In particular, note that

$$\begin{aligned}
[\cdot]_{\mu+\mu'} &= [\cdot]_{\mu+\mu',\mu+\mu'} & [\cdot]_{r\mu} &= [\cdot]_{r\mu,r\mu} \\
&= [\cdot]_{\mu,\mu} + [\cdot]_{\mu,\mu'} + [\cdot]_{\mu',\mu} + [\cdot]_{\mu',\mu'} & &= r^2[\cdot]_{\mu,\mu} \\
&= [\cdot]_{\mu} + [\cdot]_{\mu'} + [\cdot]_{\mu,\mu'} + [\cdot]_{\mu',\mu} & &= r^2[\cdot]_{\mu}
\end{aligned}$$

Proposition 0.1. *Let $t \in R$ and let $\mu_0, \mu_1 \in \mathrm{Mult}(F)$. Furthermore we set $\mu_t = t\mu_1 + (1-t)\mu_0$. Then we have*

$$[\cdot]_{\mu_t} = t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t(1-t)([\cdot]_{\mu_1,\mu_0} + [\cdot]_{\mu_0,\mu_1}).$$

Proof. We have

$$\begin{aligned}
[\cdot]_{\mu_t} &= [\cdot]_{t\mu_1+(1-t)\mu_0} \\
&= [\cdot]_{t\mu_1} + [\cdot]_{(1-t)\mu_0} + [\cdot]_{t\mu_1,(1-t)\mu_0} + [\cdot]_{(1-t)\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + [\cdot]_{t\mu_1,\mu_0-t\mu_0} + [\cdot]_{\mu_0-t\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + [\cdot]_{t\mu_1,\mu_0} + [\cdot]_{t\mu_1,-t\mu_0} + [\cdot]_{\mu_0,t\mu_1} + [\cdot]_{-t\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t[\cdot]_{\mu_1,\mu_0} - t^2[\cdot]_{\mu_1,\mu_0} + t[\cdot]_{\mu_0,\mu_1} - t^2[\cdot]_{\mu_0,\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t(1-t)([\cdot]_{\mu_1,\mu_0} + [\cdot]_{\mu_0,\mu_1}).
\end{aligned}$$

□

Now suppose $F = (F, \mathrm{d})$ is an R-complex. We view F is a graded R-module and we view $\mathrm{d}\colon F \to F$ as a graded R-linear map of degree -1 which satisfies $\mathrm{d}^2 = 0$. We further assume that μ is a chain map, i.e. it commtues with the differential. To clean notation in what follows, we denote the differentials of $F^{\otimes 2}$ and $F^{\otimes 3}$ by d again, where context will make clear which differential the symbol "d" refers to. For instance, we if $a_1, a_2 \in F$ with a_1 homogeneous, then we have

$$\mathrm{d}(a_1 \otimes a_2) = \mathrm{d}a_1 \otimes a_2 + (-1)^{|a_1|} a_1 \otimes \mathrm{d}a_2. \tag{1}$$

It is clear here that the d on the lefthand side of (1) is the differential for $F^{\otimes 2}$, whereas the d' on the righthand side are the differentials for F. If we wanted to be more formal, then our notation becomes more clunky-looking:

$$\mathrm{d}_{F^{\otimes 2}}(a_1 \otimes a_2) = \mathrm{d}_F(a_1) \otimes a_2 + (-1)^{|a_1|} a_1 \otimes \mathrm{d}_F(a_2).$$

Thus we will avoid this and use the simpler notation instead (where context makes everything clear). Note that since μ is a chain map, we have

$$\mathrm{d}[\cdot]_{\mu,\nu,\lambda} = [\cdot]_{\mathrm{d}\mu,\nu,\lambda} = [\cdot]_{\mu\mathrm{d},\nu,\lambda}.$$

Furthermore, we claim that (up to some minor sign issues) we have

$$\mathrm{d}[\cdot]_{\mu,\nu,\lambda} = [\cdot]_{\mu,\mathrm{d}\nu,\lambda} + [\cdot]_{\mu,\nu,\mathrm{d}\lambda} \quad \text{and} \quad [\cdot]_{\mu,\nu,\lambda}\mathrm{d} = [\cdot]_{\mu,\nu\mathrm{d},\lambda} + [\cdot]_{\mu,\nu,\lambda\mathrm{d}} \tag{2}$$

Indeed the identities follow from the identities

$$\begin{aligned} \mathrm{d}(\nu \otimes \lambda) &= \mathrm{d}\nu \otimes \lambda + \overline{\nu} \otimes \mathrm{d}\lambda & \qquad (\nu \otimes \lambda)\mathrm{d} &= \nu\mathrm{d} \otimes \lambda + (-1)^{|\nu|}\overline{\nu} \otimes \lambda\mathrm{d} \\ \mathrm{d}(\lambda \otimes \nu) &= \mathrm{d}\lambda \otimes \nu + \overline{\lambda} \otimes \mathrm{d}\nu & \qquad (\lambda \otimes \nu)\mathrm{d} &= \lambda\mathrm{d} \otimes \nu + (-1)^{|\lambda|}\overline{\lambda} \otimes \nu\mathrm{d} \end{aligned}$$

where $\overline{\nu}\colon F^{\otimes 2} \to F$ and $\overline{\lambda}\colon F \to F$ are defined by

$$\overline{\nu}(a_1 \otimes a_2) = (-1)^{|a_1|+|a_2|+|\nu|}\nu(a_1 \otimes a_2) \qquad \overline{\lambda}(a) = (-1)^{|a|+|\lambda|}\lambda(a).$$

The identity (2) holds exactly in characteristic 2, however in general one should interpret with (2) with appropriate signs. For instance, we have

$$\begin{aligned} \mathrm{d}[\cdot]_{\mu,\nu,\lambda} &= [\cdot]_{\mathrm{d}\mu,\nu,\lambda} \\ &= [\cdot]_{\mu\mathrm{d},\nu,\lambda} \\ &= \mu\mathrm{d}(\nu \otimes \lambda - \lambda \otimes \nu) \\ &= \mu(\mathrm{d}\nu \otimes \lambda + \overline{\nu} \otimes \mathrm{d}\lambda - \mathrm{d}\lambda \otimes \nu - \overline{\lambda} \otimes \mathrm{d}\nu) \\ &= \mu(\mathrm{d}\nu \otimes \lambda - \overline{\lambda} \otimes \mathrm{d}\nu) + \mu(\overline{\nu} \otimes \mathrm{d}\lambda - \mathrm{d}\lambda \otimes \nu) \\ &= [\cdot]^{(3)}_{\mu,\mathrm{d}\nu,\lambda} + [\cdot]^{(2)}_{\mu,\overline{\nu},\mathrm{d}\lambda} \end{aligned}$$

Proposition 0.2. *Let $\mu \in \mathrm{Mult}(F)$, let $h\colon F^{\otimes 2} \to F$, and set $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$. Then we have*

$$[\cdot]_{\mu_h} = [\cdot]_{\mu} + \mathrm{d}H + H\mathrm{d}$$

where $H = [\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}$.

Proof. We have

$$\begin{aligned} [\cdot]_{\mu_h} &= [\cdot]_{\mu+\mathrm{d}h+h\mathrm{d}} \\ &= [\cdot]_{\mu} + [\cdot]_{\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mathrm{d}h+h\mathrm{d},\mu} \\ &= [\cdot]_{\mu} + [\cdot]_{\mathrm{d}h} + [\cdot]_{h\mathrm{d}} + [\cdot]_{\mathrm{d}h,h\mathrm{d}} + [\cdot]_{h\mathrm{d},\mathrm{d}h} + [\cdot]_{\mu,\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mathrm{d}h+h\mathrm{d},\mu} \\ &= [\cdot]_{\mu} + \mathrm{d}[\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + \mathrm{d}[\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + [\cdot]_{\mathrm{d}h,\mu} + [\cdot]_{h\mathrm{d},\mu} \\ &= [\cdot]_{\mu} + \mathrm{d}[\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + \mathrm{d}[\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}[\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}\mu} + [\cdot]_{h,\mu,\mathrm{d}} \\ &= [\cdot]_{\mu} + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}\mu} + [\cdot]_{h,\mu,\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) \\ &= [\cdot]_{\mu} + [\cdot]_{h,h\mathrm{d}}\mathrm{d} + [\cdot]_{h,\mathrm{d}h}\mathrm{d} + [\cdot]_{h,\mu}\mathrm{d} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) \\ &= [\cdot]_{\mu} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu})\mathrm{d} \\ &= [\cdot]_{\mu} + \mathrm{d}[\cdot]_{\mu,h} + [\cdot]_{\mu,h,\mathrm{d}} + [\cdot]_{\mu,h}\mathrm{d} + [\cdot]_{\mu,h,\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu})\mathrm{d} \\ &= [\cdot]_{\mu} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h})\mathrm{d} \\ &= [\cdot]_{\mu} + \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}) + ([\cdot]_{\mu,h} + [\cdot]_{h,\mu_h})\mathrm{d} \\ &= [\cdot]_{\mu} + \mathrm{d}H + H\mathrm{d}. \end{aligned}$$

Note that

$$\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}})+([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}})\mathrm{d}=[\cdot]_{\mathrm{d}h,h\mathrm{d}}+[\cdot]_{h,h\mathrm{d},\mathrm{d}}+[\cdot]_{h\mathrm{d},h,\mathrm{d}}$$

$$\begin{aligned}
[\cdot]_{\mu_h} &= [\cdot]_{\mu+\mathrm{d}h+h\mathrm{d}} \\
&= [\cdot]_{\mu}+[\cdot]_{\mathrm{d}h+h\mathrm{d}}+[\cdot]_{\mu,\mathrm{d}h+h\mathrm{d}}+[\cdot]_{\mathrm{d}h+h\mathrm{d},\mu} \\
&= [\cdot]_{\mu}+[\cdot]_{\mathrm{d}h}+[\cdot]_{h\mathrm{d}}+[\cdot]_{\mathrm{d}h,h\mathrm{d}}+[\cdot]_{h\mathrm{d},\mathrm{d}h}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+[\cdot]_{\mathrm{d}h,\mu}+[\cdot]_{h\mathrm{d},\mu} \\
&= [\cdot]_{\mu}+\mathrm{d}[\cdot]_{h,\mathrm{d}h}+[\cdot]^{(3)}_{h,\mathrm{d}h\mathrm{d}}+[\cdot]^{(2)}_{h,\overline{h\mathrm{d}},\mathrm{d}}+\mathrm{d}[\cdot]_{h,h\mathrm{d}}+[\cdot]^{(2)}_{h,\overline{\mathrm{d}h},\mathrm{d}}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+\mathrm{d}[\cdot]_{h,\mu}+[\cdot]^{(3)}_{h,\mathrm{d}\mu}+[\cdot]^{(2)}_{h,\overline{\mu},\mathrm{d}} \\
&= [\cdot]_{\mu}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu})+[\cdot]^{(3)}_{h,\mathrm{d}h\mathrm{d}}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+[\cdot]^{(3)}_{h,\mathrm{d}\mu}+([\cdot]^{(2)}_{h,\overline{\mu},}+[\cdot]^{(2)}_{h,\overline{h\mathrm{d}}}+[\cdot]^{(2)}_{h,\overline{\mathrm{d}h}})\mathrm{d}
\end{aligned}$$

$$\begin{aligned}
&= [\cdot]_{\mu}+\mathrm{d}[\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,\mathrm{d}h\mathrm{d}}+[\cdot]_{h,h\mathrm{d},\mathrm{d}}+\mathrm{d}[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mathrm{d}h,\mathrm{d}}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+\mathrm{d}[\cdot]_{h,\mu}+[\cdot]_{h,\mathrm{d}\mu}+[\cdot]_{h,\mu,\mathrm{d}} \\
&= [\cdot]_{\mu}+[\cdot]_{h,\mathrm{d}h\mathrm{d}}+[\cdot]_{h,h\mathrm{d},\mathrm{d}}+[\cdot]_{h,\mathrm{d}h,\mathrm{d}}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+[\cdot]_{h,\mathrm{d}\mu}+[\cdot]_{h,\mu,\mathrm{d}}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu}) \\
&= [\cdot]_{\mu}+[\cdot]_{h,h\mathrm{d}}\mathrm{d}+[\cdot]_{h,\mathrm{d}h}\mathrm{d}+[\cdot]_{h,\mu}\mathrm{d}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu}) \\
&= [\cdot]_{\mu}+[\cdot]_{\mu,\mathrm{d}h}+[\cdot]_{\mu,h\mathrm{d}}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu})+([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu})\mathrm{d} \\
&= [\cdot]_{\mu}+\mathrm{d}[\cdot]_{\mu,h}+[\cdot]_{\mu,h,\mathrm{d}}+[\cdot]_{\mu,h}\mathrm{d}+[\cdot]_{\mu,h,\mathrm{d}}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu})+([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu})\mathrm{d} \\
&= [\cdot]_{\mu}+\mathrm{d}([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu}+[\cdot]_{\mu,h})+([\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu}+[\cdot]_{\mu,h})\mathrm{d} \\
&= [\cdot]_{\mu}+\mathrm{d}([\cdot]_{\mu,h}+[\cdot]_{h,\mu_h})+([\cdot]_{\mu,h}+[\cdot]_{h,\mu_h})\mathrm{d} \\
&= [\cdot]_{\mu}+\mathrm{d}H+H\mathrm{d}.
\end{aligned}$$

□

Theorem 0.1. *Let $R=\Bbbk[x,y,z,w]$, let $\boldsymbol{m}=x^2,w^2,zw,xy,yz$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure. In particular, any multiplication on F will be non-associative at the triple $(\varepsilon_1,\varepsilon_{45},\varepsilon_2)$.*

Proof. Let μ be the usual multiplication on F and let μ' be any other multiplication on F. Then μ' has the form $\mu'=\mu+\mathrm{d}h+h\mathrm{d}$ for some graded R-linear map $h\colon F^{\otimes 2}\to F$ of degree 1. Furthermore, the associator of μ' is given by

$$[\cdot]_{\mu'}=[\cdot]_{\mu}+\mathrm{d}H+H\mathrm{d}$$

where $H=[\cdot]_{h,\mathrm{d}h}+[\cdot]_{h,h\mathrm{d}}+[\cdot]_{h,\mu}+[\cdot]_{\mu,h}$. We claim that $[\varepsilon_1,\varepsilon_{45},\varepsilon_5]_{\mu'}\neq 0$. Indeed, the idea is that

$$[\varepsilon_1,\varepsilon_{45},\varepsilon_2]_{\mu}=-x\varepsilon_{12345}\quad\text{and}\quad(\mathrm{d}H+H\mathrm{d})(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2)\in IF_4$$

where $I=\langle x^2,y,z,w\rangle$, and thus no term in $(\mathrm{d}H+H\mathrm{d})(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2)$ will be able to cancel out $x\varepsilon_{12345}$. To see this, first note that $\mathrm{d}H(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2)=0$, so we only need to focus on the terms in $H\mathrm{d}(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2)$. Now clearly we have

$$\operatorname{im}([\cdot]_{h,\mathrm{d}h})\mathrm{d})\in\mathfrak{m}^2F\subseteq IF\quad\text{and}\quad\operatorname{im}([\cdot]_{h,h\mathrm{d}})\mathrm{d})\in\mathfrak{m}^2F\subseteq IF,$$

since the differential shows up twice in each case. Next note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2) &= x^2[1\otimes\varepsilon_{45}\otimes\varepsilon_2]_{h,\mu}-x[\varepsilon_1\otimes\varepsilon_5\otimes\varepsilon_2]_{h,\mu}+z[\varepsilon_1\otimes\varepsilon_4\otimes\varepsilon_2]_{h,\mu}+w^2[\varepsilon_1\otimes\varepsilon_{45}\otimes 1]_{h,\mu} \\
&= -x[\varepsilon_1\otimes\varepsilon_5\otimes\varepsilon_2]_{h,\mu} \\
&= -xh((z\varepsilon_{14}+x\varepsilon_{45})\otimes\varepsilon_2-\varepsilon_1\otimes(z\varepsilon_{23}+y\varepsilon_{35})) \\
&= 0.
\end{aligned}$$

Similarly in F/IF we have

$$\begin{aligned}
[\cdot]_{\mu,h}\mathrm{d}(\varepsilon_1\otimes\varepsilon_{45}\otimes\varepsilon_2) &= x^2[1\otimes\varepsilon_{45}\otimes\varepsilon_2]_{\mu,h}-x[\varepsilon_1\otimes\varepsilon_5\otimes\varepsilon_2]_{\mu,h}+z[\varepsilon_1\otimes\varepsilon_4\otimes\varepsilon_2]_{\mu,h}+w^2[\varepsilon_1\otimes\varepsilon_{45}\otimes 1]_{\mu,h} \\
&= -x[\varepsilon_1\otimes\varepsilon_5\otimes\varepsilon_2]_{\mu,h} \\
&= 0
\end{aligned}$$

where we used the fact that $\varepsilon_1F_3\in\mathfrak{m}F_4$ and $\varepsilon_2F_3\in\mathfrak{m}F_4$. □

Theorem 0.2. *Let $R = \Bbbk[x,y,z,w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure.*

Proof. Let μ be the usual multiplication on F and let μ' be any other multiplication on F. Then μ' has the form $\mu' = \mu + \mathrm{d}h + h\mathrm{d}$ for some graded R-linear map $h\colon F^{\otimes 2} \to F$ of degree 1. Furthermore, the associator of μ' is given by

$$[\cdot]_{\mu'} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d}$$

where $H = [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h}$. We claim that $[e_{12}, e_5, e_2]_{\mu'} \neq 0$. Indeed, the idea is that

$$[e_{12}, e_5, e_2]_\mu = x^2yze_{1234} \quad \text{and} \quad (\mathrm{d}H + H\mathrm{d})(e_{12} \otimes e_5 \otimes e_2) \in IF_4$$

where $I = \langle x^3, y^2, z^2, w\rangle$, and thus no term in $(\mathrm{d}H + H\mathrm{d})(e_{12} \otimes e_5 \otimes e_2)$ will be able to cancel out x^2yze_{1234}. To see this, first note that $\mathrm{d}H(e_{12} \otimes e_5 \otimes e_2) = 0$, so we only need to focus on the terms in $H\mathrm{d}(e_{12} \otimes e_5 \otimes e_2)$. Note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &= x^2[e_2, e_5, e_2]_{h,\mu} + w^2[e_1, e_5, e_2]_{h,\mu} + y^2z^2[e_{12}, 1, e_2]_{h,\mu} + w^2[e_{12}, e_5, 1]_{h,\mu} \\
&= x^2[e_2, e_5, e_2]_{h,\mu} \\
&= x^2h((y^2ze_{23} + we_{35}) \otimes e_2 - e_2 \otimes (y^2ze_{23} + we_{35})) \\
&= 0.
\end{aligned}$$

Similarly in F/IF we have

$$\begin{aligned}
[\cdot]_{\mu,h}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &= x^2[e_2, e_5, e_2]_{\mu,h} + w^2[e_1, e_5, e_2]_{\mu,h} + y^2z^2[e_{12}, 1, e_2]_{\mu,h} + w^2[e_{12}, e_5, 1]_{\mu,h} \\
&= x^2[e_2, e_5, e_2]_{\mu,h} \\
&= x^2(e_2h(e_5 \otimes e_2) - h(e_2 \otimes e_5)e_2) \\
&= 0.
\end{aligned}$$

where we used the fact that $e_2F_3 \in \langle w\rangle F_4$. Next note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,h\mathrm{d}}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &= x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} + w^2[e_1, e_5, e_2]_{h,h\mathrm{d}} + y^2z^2[e_{12}, 1, e_2]_{h,h\mathrm{d}} + w^2[e_{12}, e_5, 1]_{h,h\mathrm{d}} \\
&= x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} \\
&= x^2h(h\mathrm{d}(e_2 \otimes e_5) \otimes e_2 - e_2 \otimes h\mathrm{d}(e_2 \otimes e_5)) \\
&= x^2h(w^2h(1 \otimes e_5) \otimes e_2 - y^2z^2h(e_2 \otimes 1) \otimes e_2 - w^2e_2 \otimes h(1 \otimes e_5) + y^2z^2e_2 \otimes h(e_2 \otimes 1))) \\
&= 0.
\end{aligned}$$

□

Proof. Let μ be the usual multiplication on F. Any other multiplication on F must be of the form $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ where $h\colon F^{\otimes 2} \to F$ is a graded R-linear map of degree 1 such that $h|_{F\otimes 1}$, $h|_{1\otimes F}$, and $h\sigma$ are all chain maps where $\sigma\colon F^{\otimes 2} \to F^{\otimes 2}$ is defined by

$$\sigma(a_1 \otimes a_2) = a_1 \otimes a_2 - (-1)^{|a_1||a_2|}a_2 \otimes a_1$$

for all homogeneous $a_1, a_2 \in F$. By Proposition (0.2), we have

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d}$$

where $H = [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h}$. We claim that $[e_{12}, e_5, e_2]_{\mu_h} \neq 0$. The idea is that $[e_{12}, e_5, e_2] = x^2yze_{1234}$ but term in $(\mathrm{d}H + H\mathrm{d})(e_{12} \otimes e_5 \otimes e_2)$ will be able to cancel out x^2yze_{1234}. Note that $\mathrm{d}H(e_{12} \otimes e_5 \otimes e_2) = 0$, so we only need to focus on the terms in

$$H\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) = x^2H(e_2 \otimes e_5 \otimes e_2) - w^2H(e_1 \otimes e_5 \otimes e_2) + y^2z^2H(e_{12} \otimes 1 \otimes e_2) - w^2H(e_{12} \otimes e_5 \otimes 1).$$

Clearly only the terms in $x^2H(e_2 \otimes e_5 \otimes e_2)$ can possibly cancel out x^2yze_{1234}, so we focus on that. Now observe that

$$\begin{aligned}
x^2[e_2 \otimes e_5 \otimes e_2]_{h,h\mathrm{d}} &\in \langle x^2w^2, x^2y^2z^2\rangle F_4 \\
x^2[e_2 \otimes e_5 \otimes e_2]_{h,\mu} &\in \langle x^2y^2z, x^2w\rangle F_4 \\
x^2[e_2 \otimes e_5 \otimes_2]_{\mu,h} &\in \langle x^2w\rangle F_4,
\end{aligned}$$

so only the terms in $x^2[e_2 \otimes e_5 \otimes e_2]_{h,\mathrm{d}h}$ can possibly cancel out x^2yze_{1234}. Now observe that since $h\sigma$ is a chain map, we have

$$\begin{aligned}
\mathrm{d}[e_2 \otimes e_5 \otimes e_2]_{h,\mathrm{d}h} \bmod \langle y^2z^2, w^2\rangle F_4 &\equiv \mathrm{d}h(\mathrm{d}h(e_2 \otimes e_5) \otimes e_2 - e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \bmod \langle y^2z^2, w^2\rangle F_4 \\
&\equiv \mathrm{d}h(\mathrm{d}h(e_5 \otimes e_2) \otimes e_2 - e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \bmod \langle y^2z^2, w^2\rangle F_4 \\
&\equiv \mathrm{d}h\sigma(\mathrm{d}h(e_5 \otimes e_2) \otimes e_2) \bmod \langle y^2z^2, w^2\rangle F_4 \\
&\equiv h\sigma\mathrm{d}(\mathrm{d}h(e_5 \otimes e_2) \otimes e_2) \bmod \langle y^2z^2, w^2\rangle F_4 \\
&\equiv 0 \bmod \langle y^2z^2, w^2\rangle F_4.
\end{aligned}$$

It follows that $\mathrm{d}(x^2[e_2 \otimes e_5 \otimes e_2]_{h,\mathrm{d}h}) \in \langle x^2y^2z^2, x^2w^2\rangle F_4$ which implies $[e_2 \otimes e_5 \otimes e_2]_{h,\mathrm{d}h} \in \langle xy^2z^2, x^2yz^2, x^2y^2z, xw^2, x^2w\rangle F_4$

$\square$