Some Infinite Minimal Free Resolutions

Example 0.1. Let $S = \Bbbk[x,y]/\langle y^2 - x^3 + x^2 \rangle$, let $\mathfrak{m} = \langle \overline{x}, \overline{y} \rangle$, and let F be the minimal S-free resolution of $S/\mathfrak{m}$. If $\operatorname{char} \Bbbk = 0$, then F is the DG S-algebra $S = R[e_1, e_2, e_3]$ where $|e_1| = 1 = |e_2|$ and $|e_3|$ and where

$$
\begin{aligned}
\mathrm{d}(e_1) &= \overline{x} \\
\mathrm{d}(e_2) &= \overline{y} \\
\mathrm{d}(e_3) &= (\overline{x}^2 - \overline{x})e_1 - \overline{y}e_2.
\end{aligned}
$$

If $\operatorname{char} \Bbbk = p$ where $p > 0$, then this doesn't work since $\mathrm{d}(e_3^p) = p\mathrm{d}(e_3)e_3^{p-1} = 0$. Instead we need to consider divider powers. Thus for each $n \geq 2$, we adjoin a new variable $e_3^{(n)}$ (where intuitively $e_3^{(n)} = e_3^n/n!$) where $|e_3^{(n)}| = n|e_3|$ and where $\mathrm{d}(e_3^{(n)}) = \mathrm{d}(e_3)e_3^{(n-1)}$. The Betti numbers start out as:

$$1, 2, 2, 2, 2, , \dots$$

Therefore we have $\operatorname{cx}_S(\Bbbk) = 1$.

Example 0.2. Let $S = \Bbbk[x,y]/\langle x^2, y^2 \rangle$, let $\mathfrak{m} = \langle \overline{x}, \overline{y} \rangle$, and let F be the minimal S-free resolution of $S/\mathfrak{m}$. If $\operatorname{char} \Bbbk = 0$, then F is the DG S-algebra $S = R[e_1, e_2, e_3, e_4]$ where $|e_1| = 1 = |e_2|$ and $|e_3| = 2 = |e_4|$ and where

$$
\begin{aligned}
\mathrm{d}(e_1) &= \overline{x} \\
\mathrm{d}(e_2) &= \overline{y} \\
\mathrm{d}(e_3) &= \overline{x}e_1. \\
\mathrm{d}(e_4) &= \overline{y}e_2
\end{aligned}
$$

If $\operatorname{char} \Bbbk \neq 0$, we use divided powers again. The Betti numbers start out as:

$$1, 2, 3, 4, 5, \dots$$

Therefore we have $\operatorname{cx}_S(\Bbbk) = 2$.

Example 0.3. Let $S = \Bbbk[x,y]/\langle x^2, xy, y^2 \rangle$, let $\mathfrak{m} = \langle \overline{x}, \overline{y} \rangle$, and let F be the minimal S-free resolution of $S/\mathfrak{m} = \Bbbk$. If $\operatorname{char} \Bbbk = 0$, then F is the DG S-algebra $S = R[e_1, e_2, e_3, e_4, e_5, e_6]$ where $|e_1| = 1 = |e_2|$ and $|e_3| = |e_4| = |e_5| = |e_6|$ and where

$$
\begin{aligned}
\mathrm{d}(e_1) &= \overline{x} \\
\mathrm{d}(e_2) &= \overline{y} \\
\mathrm{d}(e_3) &= \overline{x}e_1. \\
\mathrm{d}(e_4) &= \overline{x}e_2 \\
\mathrm{d}(e_5) &= \overline{y}e_1 \\
\mathrm{d}(e_6) &= \overline{y}e_2.
\end{aligned}
$$

Now consider short exact sequence of S-modules:

$$0 \to \mathfrak{m} \to S \to \Bbbk \to 0.$$

Applying $- \otimes_S \Bbbk$ to this short exact and considering the long exact sequence in Tor, we obtain isomorphisms

$$
\begin{aligned}
\operatorname{Tor}_i^S(\Bbbk, \Bbbk) &\cong \operatorname{Tor}_{i+1}^S(\mathfrak{m}, \Bbbk) \\
&\cong \operatorname{Tor}_{i+1}^S(\Bbbk^2, \Bbbk) \\
&\cong \operatorname{Tor}_i^S(\Bbbk, \Bbbk) \oplus \operatorname{Tor}_i^S(\Bbbk, \Bbbk)
\end{aligned}
$$

for all $i \geq 1$, where we used the fact that $\mathfrak{m} \cong \Bbbk^2$ as S-modules. Therefore, since

$$\beta_i(M) = \beta_i^S(M) = \dim_{\Bbbk}(\operatorname{Tor}_i^S(M, \Bbbk))$$

for all finitely generated S-modules M and for all $i \geq 1$, we see that $\beta_{i+1}(\Bbbk) = 2\beta_i(\Bbbk)$ for all $i \geq 1$, and thus $\beta_i(\Bbbk) = 2^i$ for all $i \geq 1$. The Betti numbers start out as:

$$1, 2, 4, 8, 16, \ldots$$

Therefore we have $\operatorname{cx}_S(\Bbbk) = \infty$. Thus we need to consider the curvature of $\Bbbk$:

$$\begin{aligned} \operatorname{curv}_S(\Bbbk) &= \limsup_{n \to \infty} \beta_n(\Bbbk)^{1/n} \\ &= \limsup_{n \to \infty} (2^n)^{1/n} \\ &= 2. \end{aligned}$$