Tor-Persistence

Introduction

Let R be a commutative noetherian ring. Recall that a finitely generated R-module M has finite projective dimension if $\mathrm{Tor}_i^R(M,N)=0$ for $i \gg 0$ for each finitely generated R-module N. Indeed, first note that $\mathrm{Tor}_i^R(M,N)=0$ if and only if

$$\mathrm{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) \simeq \mathrm{Tor}_i^R(M,N)_\mathfrak{p} = 0$$

for all prime ideals $\mathfrak{p}$ of R. Thus by replacing R, M, and N with $R_\mathfrak{p}$, $M_\mathfrak{p}$, and $N_\mathfrak{p}$ if necessary, we may assume that $R=(R,\mathfrak{m},\Bbbk)$ is local. Now let F be the minimal R-free resolution of M. Thus

$$\mathrm{Tor}_i^R(M,N) = \mathrm{H}_i(F \otimes_R N).$$

We first prove the easy direction: suppose M has finite projective dimension, say $\mathrm{pd}_R M = p$. This means that $F_p \neq 0$ and $F_i = 0$ for all $i > p$. In particular that $(F \otimes_R N)_i = 0$ for all $i > p$, which implies $\mathrm{Tor}_i^R(M,N)=0$ for all $i>p$. Now we prove the harder direction: suppose $\mathrm{Tor}_i^R(M,N)=0$ for $i \gg 0$ for each finitely generated R-module N. In particular, we have $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i \gg 0$. This implies $\mathrm{H}_i(F_\Bbbk)=0$ for $i \gg 0$ where we set $F_\Bbbk := F \otimes_R \Bbbk$. However F is *minimal*, thus $\mathrm{d}_\Bbbk = 0$, where $\mathrm{d}_\Bbbk$ is the differential of $F_\Bbbk$. Thus we have $\mathrm{H}_i(F_\Bbbk) = F_{i,\Bbbk} := F_i \otimes_R \Bbbk$ and this implies $F_i \otimes_R \Bbbk = 0$ for $i \gg 0$ which implies $F_i = 0$ for $i \gg 0$ by Nakayama's lemma (here is where we used the fact that R is noetherian and M is finitely generated).

Now suppose that the only thing we knew was that $\mathrm{Tor}_i^R(M,M)=0$ for $i \gg 0$. Can we still conclude that the projective dimension of M is finite? This is an open question in general, however it is known to be true for various rings R: we call such rings **Tor-persistent**. It is natural to wonder if in fact every commutative noetherian ring is Tor-persistent. Note that

$$\mathrm{Tor}_i^R(M,M) = \mathrm{H}_i(F \otimes_R M) = \mathrm{H}_i(F^{\otimes 2})$$

where we denoted $F^{\otimes 2} = F \otimes_R F$. One of the main reasons why we could conclude that M had finite projective dimension if $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i \gg 0$ was because the homology of $F_\Bbbk$ was extremely simple: $\mathrm{H}(F_\Bbbk)=F_\Bbbk$. The homology of $F^{\otimes 2}$ is more complicated, thus even if we knew that $\mathrm{H}_i(F^{\otimes 2})=0$ for $i \gg 0$, it is not at all clear why this should imply that $F_i = 0$ for $i \gg 0$. In order to prove this, one would presumably need to use the fact that R is noetherian, M is finitely generated, and F is minimal.

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In what follows, we assume $(R,\mathfrak{m},\Bbbk)$ is a local noetherian ring.