1. (a) Determine, on paper, a real SVD with the minimal number of minus signs.

**Solution.** Let $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$. The singular values of $A$ are the square roots of the corresponding eigenvalues of $AA^T$ or $A^TA$. Since the former has a nicer structure, we will use it.

$$AA^T = \begin{bmatrix} 125 & 75 \\ 75 & 125 \end{bmatrix} = 25 \begin{bmatrix} 25 & 15 \\ 15 & 25 \end{bmatrix}$$

Thus, we see that $\sigma_{1,2}(A) = 5\sqrt{2}, 10\sqrt{2}$. For the singular vectors, let $A = U\Sigma V^T$ be a reduced SVD of $A$. Then $AA^T = U\Sigma^2 U^T$. Thus, an eigenvalue decomposition of $AA^T$ will give the left singular vectors of $A$. By solving the 4 by 4 system

$$U \begin{bmatrix} 200 & 0 \\ 0 & 50 \end{bmatrix} U^T = A$$

we find that $U$ can be any matrix of the form $U = \begin{bmatrix} x & y \\ x & -y \end{bmatrix}$. Enforcing orthogonality, it follows that $x = \pm\frac{\sqrt{2}}{2}$ and $y = \pm\frac{\sqrt{2}}{2}$. By substituting $U$ into $A = U\Sigma V^T$, we find the right singular vectors by solving for $V$ via

$$A = \frac{\sqrt{2}}{2} \begin{bmatrix} w & z \\ w & -z \end{bmatrix} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix} V^T$$

where $w, z = \pm 1$. This yields $V = \frac{1}{5} \begin{bmatrix} -3w & 4z \\ 4w & 3z \end{bmatrix}$. Thus, to ensure we have the least amount of negative signs, we should pick $w = z = 1$.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a labeled picture of the unit ball in $\mathbb{R}^2$ with the singular vectors marked.

**Solution.** With the above choice of signs, it follows immediately that $U = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $V = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$ and $\sigma_{1,2} = 5\sqrt{2}, 10\sqrt{2}$.

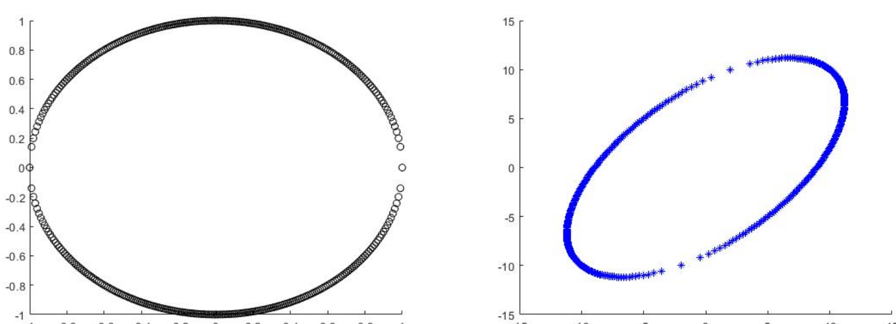


Figure 1: Transformation via Singular values

Not everything is shown in Figure 1, but one can clearly see that the image on the right stretches the diagonals corresponding to the magnitude of the appropriate singular value.

(c)

**Solution.**

$$\begin{aligned}\|A\|_\infty &= 15\\ \|A\|_1 &= 12\\ \|A\|_2 &= 10\sqrt{2}\\ \|A\|_F &= \sqrt{50+200} = 5\sqrt{10}\end{aligned}$$

(d)

**Solution.**

$$A^{-1} = V\Sigma^{-1}U^T = \frac{1}{100}\begin{bmatrix}5 & -11\\ 10 & -2\end{bmatrix}$$

(e)

**Solution.** Not sure what this question was but $\lambda^2 - 3\lambda + 100$ yields solutions $\lambda_{1,2} = \frac{3\pm\sqrt{39i}}{2}$

(f)

**Solution.**

$$\begin{aligned}\det A &= -10 - -110 && \text{direct calculation}\\ &= (\frac{1}{4})(9+391) && \text{product of eigenvalues}\\ &= (5\sqrt{2})(10\sqrt{2}) && \text{product of singular values}\\ &= 100\end{aligned}$$

2. Let $A \in \mathbb{R}^{m\times n}$ have singular values such that $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$.

   (a) If $E \in \mathbb{R}^{m\times n}$, show that

   $$\sigma_1(A+E) \leq \sigma_1(A) + \|E\|_2$$

   and

   $$\sigma_1(A+E) \geq \sigma_1(A) - \|E\|_2$$

   *Proof.* For the first inequality, by triangle inequality of norms, we have that $\|A + E\|_2 \leq \|A\|_2 + \|E\|_2 \implies \sigma_1(A+E) \leq \sigma_1(A) + \|E\|_2$. For the latter, note that $\|A\|_2 = \|A+E-E\|_2 \leq \|A+E\|_2 + \|E\|_2$. Rearranging and definition of the largest singular value gives the desired result. $\square$

   (b) Let $z \in \mathbb{R}^m$. Show that

   $$\sigma_1([A,z]) \geq \sigma_1(A)$$

   and

   $$\sigma_n([A,z]) \leq \sigma_n(A)$$

*Proof.* Since $\sigma_1(A) = \|A\|_2$, by definition of norm, $\|A\|_2 = \sup \frac{\|Ax\|_2}{\|x\|_2}$. Similarly, $\|[A,z]\| = \sup \frac{\|[A,z]y\|_2}{\|y\|_2}$. Letting $y = (x,0)^T$, we see that $\frac{\|[A,z]y\|_2}{\|y\|_2} = \frac{\|Ax\|_2}{\|x\|_2}$. Thus, it follows that $\sup \frac{\|[A,z]y\|_2}{\|y\|_2} \geq \sup \frac{\|[A,z]x\|_2}{\|x\|_2}$. That is, $\sigma_1([A,z]) \geq \sigma_1(A)$.

The same proof technique holds for the latter inequality as well. If $A$ is singular, then the result holds trivially. If $A$ is nonsingular, recall that $\sigma_n(A) = \sigma_1(A^{-1})$. Thus, from the definition of operator norm, it follows that $\sigma_n(A) = \inf \frac{\|Ax\|_2}{\|x\|_2}$. Applying the above proof structure in the same manner gives the second inequality. □

3. (a) Show that if $A \in \mathbb{R}^{m\times n}$, then $\|A\|_F \leq \sqrt{\operatorname{rank} A}\|A\|_2$.

*Proof.* Recall from the text that $\|A\|_F = \sqrt{\sigma_1^2 + \dots \sigma_r^2}$ for a matrix $A$ of rank $r$. Then

$$\begin{aligned}
\|A\|_F &= \sqrt{\sigma_1^2 + \dots \sigma_r^2} \\
&= \|A\|_2\sqrt{1 + \frac{\sigma_2}{\sigma_1}^2 + \dots \frac{\sigma_r}{\sigma_1}^2} \\
&\leq \|A\|_2\sqrt{\operatorname{rank} A}
\end{aligned}$$

which is what we wanted to show. □

(b) Show that if $A \in \mathbb{R}^{m\times n}$ has rank $n$, then $\|A(A^TA)^{-1}A^T\|_2 = 1$.

*Proof.* Let $A = U\begin{bmatrix}\Sigma \\ 0\end{bmatrix}V^T$ be an SVD for $A$.Then

$$\begin{aligned}
A(A^TA)^{-1}A^T &= U\begin{bmatrix}\Sigma \\ 0\end{bmatrix}V^T\left(V\begin{bmatrix}\Sigma \\ 0\end{bmatrix}U^TU\begin{bmatrix}\Sigma \\ 0\end{bmatrix}^T V^T\right)^{-1} V\begin{bmatrix}\Sigma \\ 0\end{bmatrix}^T U \\
&= U\begin{bmatrix}\Sigma \\ 0\end{bmatrix}V^T\left(V\Sigma^2V^T\right)^{-1}V\begin{bmatrix}\Sigma \\ 0\end{bmatrix}^T U \\
&= U\begin{bmatrix}\Sigma \\ 0\end{bmatrix}\Sigma^2\begin{bmatrix}\Sigma \\ 0\end{bmatrix}^T U^T \\
&= U\begin{bmatrix}I_n & 0 \\ 0 & 0\end{bmatrix}U^T
\end{aligned}$$

which is a valid SVD for $A(A^TA)^{-1}A^T$. Thus, $\|A(A^TA)^{-1}A^T)\|_2 = 1$. □

4. (a) Given $A \in \mathbb{R}^{n\times n}$, let $A = U\Sigma V^T$ be an SVD of $A$ where $\det A < 0$. Let $B = [U\operatorname{diag}(1,\dots,1,-1)]\Sigma V^T$ such that $\det(B) = |\det(A)|$ and $\|A - B\|_F = 2\sigma_n$. For any singular values greater than $\sigma_n$, show that there exists $C \in \mathbb{R}^{n\times n}$ such that $\det(C) = \det(B) = |\det(A)|$, and $\|A - C\|_F < \|A - B\|_F = 2\sigma_n$.

(b) Let $P$ be an orthogonal projector. Show that $I - 2P$ is unitary.

*Proof.*

$$\begin{aligned}(I-2P)^T(I-2P) &= I^TI - I^T2P - 2P^TI + 4P^TP \\ &= I - 4P + 4P^2 \\ &= I\end{aligned}$$

This is slightly late, but this is a simple argument that Householder reflectors are unitary. Note that $(I-2P)v = v$ for $v \in \operatorname{null} P$ and that $v - (I-2P)v = 2Pv \in \operatorname{range}(P)$ iff $v \notin \operatorname{null} P$. Thus, for a geometric interpretation, $I - 2P$ is a reflection across the null space of $P$, making it necessarily unitary. □

(c) Show that $\|P\|_2 \geq 1$ and $\|P\|_2 = 1$ iff $P$ is orthogonal.

*Proof.* Since $\|P\| = \|P^2\| \leq \|P\|^2 \implies 1 \leq \|P\|$. If $P$ is orthogonal, then it must have an eigenvalue decomposition, say $P = V\Lambda V^T$ and $\sigma_i = \lambda_i$. By property of projectors, we have that

$$V\Lambda V^T = V\Lambda V^T V\Lambda V^T = V\Lambda^2 V$$

Thus, it's singular values must be either 1 or 0 and $\|P\|_2 = \sigma_1 = 1$. For the other direction, let $P = U\Sigma V^T$ be a reduced SVD of $P$. If $\|P\|_2 = 1$, then $\|\Sigma\|_2 = 1 \implies \Sigma = I_n$. Since $P = P^2$, it follows that $UV^T = UV^TUV^T$, or $UV^T = I$. Thus, $U = V$ since inverses are unique. So $P$ is a symmetric matrix and therefore an orthogonal projector[1] □

5. Read the introduction to the Golub-Kahan-Lanczos (GKL) method and the uploaded code HW3GKLsvds.m.

(a) Give a general description of the functionality of GKL; describe the main difference between the original GKL and the code.

**Solution.** GKL is an algorithm to perform the first phase of SVD approximation. It is used for finding orthogonal matrices $U$ and $V$ such that $UAV^T$ is a bidiagonal matrix. For Dr. Xue's code, there are additional computations used to maintain stability. Lines 31 and 39 start for loops that appear to ensure the orthogonality of $U$ and $V$.

(b) Compare the timing used for computing and the memory used for storing full pictures versus partial SVD approximations.

**Solution.** We begin by noting how nice of a low rank approximation the SVD allows. I would provide pictures, but this won't be printed in color anyways. In any case, the times for a full SVD computation are approximately 3 times as long as the partial SVD with $rk = 160$. This is no doubt impressive, but perhaps the more important statistic is a full SVD uses an order of magnitude more bytes than its approximation. Full results are shown in Table 1 and 2

(c) Are you satisfied with the quality of the image generated by Ahat?

[1] I actually think this method works in both directions but I liked my first argument

| **Color Channel** | **Computation Time (s)** | **Approximation Time (s)** |
|---|---|---|
| **Red** | 1.22 | 0.44 |
| **Green** | 1.23 | 0.42 |
| **Blue** | 1.24 | 0.43 |

Table 1: Computational Time

| | | | |
|---|---|---|---|
| S1 | 1456x1456 | 16959488 | double |
| S2 | 1456x1456 | 16959488 | double |
| S3 | 1456x1456 | 16959488 | double |
| Ss1 | 160x160 | 204800 | double |
| Ss2 | 160x160 | 204800 | double |
| Ss3 | 160x160 | 204800 | double |
| U1 | 1620x1456 | 18869760 | double |
| U2 | 1620x1456 | 18869760 | double |
| U3 | 1620x1456 | 18869760 | double |
| Us1 | 1620x160 | 2073600 | double |
| Us2 | 1620x160 | 2073600 | double |
| Us3 | 1620x160 | 2073600 | double |
| V1 | 1456x1456 | 16959488 | double |
| V2 | 1456x1456 | 16959488 | double |
| V3 | 1456x1456 | 16959488 | double |
| Vs1 | 1456x160 | 1863680 | double |
| Vs2 | 1456x160 | 1863680 | double |
| Vs3 | 1456x160 | 1863680 | double |

Table 2: Terribly Formatted Matrix Storage

**Solution.** Again, it makes little sense to provide pictures here as it will not be in color, but the images are fairly close. I will say that it is not a very high resolution picture to begin with, so any approximation does not look that much different anyways.

6. (a) Implement GK bidiagonalization of a matrix and test it on $F \in \mathbb{R}^{10\times 10}$ generated from $F = randn(10, 10)$.

   **Solution.** Code is provided in the appendix. For the $F$ listed above, I received a normed error of $3.7292 \cdot 10^{-15}$.

   (b) Compare these singular values with those computed by taking the square root of the 5 largest and 5 smallest eigenvalues of $A^T A$. What conclusion do you draw? Is it a good idea to compute the eigenvalues of $\begin{bmatrix} 0 & A^T \\ A^T & 0 \end{bmatrix}$ directly, and why?

   **Solution.** The five largest singular values computed by **both** methods are $(239, 399, 624, 913, 1174)$. However, the smallest singular values are vastly different. They differ by an entire magnitude. As proposed in problem 7, the relative forward error of singular values computed via $A^T A$ is bounded by $\frac{\sigma_1}{\sigma_n}^2 \varepsilon_{\text{mach}}$. On the contrary, computing singular values via $\begin{bmatrix} 0 & A^T \\ A^T & 0 \end{bmatrix}$ directly does not have this square. Thus, for smallest singular values where this leading term is quite large, the forward error is larger and accounts for the discrepancy in the smallest computed singular values.

# 1 Appendix

## 1.1 Script files

### 1.1.1 Question 5

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Spring 2018 Math 8610 w/ Xue
%   Homework 3
%
% Problem
%   5
%
% Function Dependencies
%   HW3_GKLsvds.m
%
% Notes
%   None
%
% Author
%   Trevor Squires
%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear
clc
close all;
load picA

%% A
rk = 160;
tic; [Us1,Ss1,Vs1] = HW3_GKLsvds(pic_A(:,:,1),rk); toc;
tic; [Us2,Ss2,Vs2] = HW3_GKLsvds(pic_A(:,:,2),rk); toc;
tic; [Us3,Ss3,Vs3] = HW3_GKLsvds(pic_A(:,:,3),rk); toc;
tic; [U1,S1,V1] = svd(pic_A(:,:,1),0); toc;
tic; [U2,S2,V2] = svd(pic_A(:,:,2),0); toc;
tic; [U3,S3,V3] = svd(pic_A(:,:,3),0); toc;

%% B
Ahat = zeros(size(pic_A));
Ahat(:,:,1) = Us1*Ss1*Vs1';
Ahat(:,:,2) = Us2*Ss2*Vs2';
Ahat(:,:,3) = Us3*Ss3*Vs3';

disp([norm(Ahat(:,:,1)-pic_A(:,:,1),'fro')/norm(pic_A(:,:,1),'fro') ...
    norm(Ahat(:,:,2)-pic_A(:,:,2),'fro')/norm(pic_A(:,:,2),'fro') ...
    norm(Ahat(:,:,3)-pic_A(:,:,3),'fro')/norm(pic_A(:,:,3),'fro')]);
```

```
figure(1);  image(pic_A);  axis equal;
figure(2);  image(Ahat);  axis equal;
```

#### 1.1.2 Question 6

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Spring 2018 Math 8610 w/ Xue
%   Homework 3
%
% Problem
%   6
%
% Function Dependencies
%   GKNaive.m
%
% Notes
%   None
%
% Author
%   Trevor Squires
%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear
clc
close all;


%% Part A
rng('default');
F = randn(10,10);
B = GKnaive(F);

err = norm(svd(F)-svd(B));

%% Part B
col = linspace(-1,1,1024*1024+1)';
A = col.^(0:31);

[Q,R] = qr(A,0);
B = GKnaive(R);

eigenB = sort(abs(eig([zeros(32) B';B zeros(32)])));
largestB = eigenB(end-4:end);
smallestB = eigenB(1:10);

eigenA = sqrt(abs(eig(A'*A)));
```

```
largestA = eigenA(end-4:end);
smallestA = eigenA(1:5);
```

### 1.2 Accompanying Functions

### 1.3 GK Algorithm

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%
% GKNAIVE.m
%
% DESCRIPTION
%   Performs phase 1 of svd computation by converting a matrix A
%   into a bidiagonal matrix
%
% AUTHOR
%   Trevor Squires
%
% ARGUMENTS
%   A - m x n  matrix
%
% OUTPUT
%   v - matrix of vectors corresponding to Householder transformations
%   B - n x n bidiagonal matrix
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [A] = GKnaive(A)
[m,n] = size(A);


for k = 1:n
    x = A(k:m,k);
    v = sign(x(1))*norm(x,2)*eye(m-k+1,1) + x;
    v = v/norm(v,2);
    A(k:m,k:n) = A(k:m,k:n) - 2*v*(v'*A(k:m,k:n));
    if k < n-1
        x = A(k,k+1:n);
        v = sign(x(1))*norm(x,2)*eye(1,n-k) + x;
        v = v'/norm(v,2);
        A(k:m,k+1:n) = A(k:m,k+1:n) - 2*(A(k:m,k+1:n)*v)*v';
    end

end
end
```