Algebraic Topology

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1.1 Simplices

Let $n \in \mathbb{Z}_{\geq 0}$ and let $V = (V, \|\cdot\|)$ be an $(n+1)$-dimensional normed vector space over $\mathbb{R}$. Let $\boldsymbol{v} = v_0, \dots, v_n$ be an ordered list of $n+1$ vectors $v_0, \dots, v_n \in V$ such that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. The ***n*-simplex** $[\boldsymbol{v}] = [v_0, \dots, v_n]$ (or **simplex** if n is understood from context) is the defined to be the convex closure of $\{v_1, \dots, v_n\}$. In other words, $[\boldsymbol{v}]$ is the set of all convex combinations of the v_i:

$$[\boldsymbol{v}] = \left\{ \sum_{i=0}^{n} t_i v_i \mid t_i \in [0,1] \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

In the case where $V = \mathbb{R}^{n+1}$, we call $[\boldsymbol{v}]$ a **Euclidean** n-simplex. Let $\boldsymbol{e} = e_0, \dots, e_n$ be the standard ordered basis of $\mathbb{R}^{n+1}$, where where $e_i = (0, \dots, 1, \dots, 0)^\top$ is the column vector with entry 1 in the ith spot and entry 0 everywhere else, and let $\boldsymbol{t} = t_0, \dots, t_n$ be the corresponding standard coordinates of $\mathbb{R}^{n+1}$. The n-simplex $[\boldsymbol{e}]$ is given a special name: it is called the ***n*-dimensional standard simplex**, and is denoted by $\Delta^n := [\boldsymbol{e}]$. Thus

$$\Delta^n = \left\{ \boldsymbol{t} \in \mathbb{R}^{n+1} \mid t_i \in [0,1] \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

Note that for every $v \in [\boldsymbol{v}]$, there is a unique point $\boldsymbol{t} \in \Delta^n$ such that $v = \sum_{i=0}^{n} t_i v_i$. Indeed, uniqueness follows from the fact that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent: if $\sum_{i=0}^{n} t_i v_i = \sum_{i=0}^{n} t_i' v_i$, then

$$v_0 + \sum_{i=1}^{n} t_i(v_i - v_0) = \sum_{i=0}^{n} t_i v_i = \sum_{i=0}^{n} t_i' v_i = v_0 + \sum_{i=1}^{n} t_i'(v_i - v_0),$$

where we used the fact that $t_0 = 1 - \sum_{i=1}^{n} t_i$ and $t_0' = 1 - \sum_{i=1}^{n} t_i'$. It follows that $\sum_{i=1}^{n} t_i(v_i - v_0) = \sum_{i=1}^{n} t_i'(v_i - v_0)$ and hence $t_i = t_i'$ for all $1 \leq i \leq n$, which further implies $t_0 = t_0'$. The map $\phi = \phi_{\boldsymbol{v}}$ from $[\boldsymbol{v}]$ to Δ^n sending a point $v \in [\boldsymbol{v}]$ to the point $\boldsymbol{t} \in \Delta^n$ (uniquely determined by v) is easily seen to be the unique linear homeomorphism such that $\phi(\boldsymbol{v}) = \boldsymbol{e}$ (that is, such that $\phi(v_i) = e_i$ for all $0 \leq i \leq n$). The coefficients t_i are called the **barycentric coordinates** of v.

Observe that for each $0 \leq i_0 < \cdots < i_k \leq n$, the k-simplex $[v_{i_0}, \dots, v_{i_k}]$ is contained in $[\boldsymbol{v}]$. If $0 \leq i \leq n$, then $[v_i] = v_i$ is called a **vertex** of $[\boldsymbol{v}]$. If $1 \leq i < j \leq n$, then $[v_i, v_j]$ is called an **edge** of $[\boldsymbol{v}]$. If $1 \leq i < j < k \leq n$, then $[v_i, v_j, v_k]$ is called a **face** of $[\boldsymbol{v}]$. More generally, if $0 \leq i_0 < \cdots < i_k \leq n$, then $[v_{i_0}, \dots, v_{i_k}]$ is called a ***k*-face** of $[\boldsymbol{v}]$. The collection of all k-faces of $[\boldsymbol{v}]$ as k ranges from 1 to n has the structure of an abstract simplicial complex. Moreover, the ordering of $\boldsymbol{v}$ induces an orientation on each k-face of $[\boldsymbol{v}]$, and the canonical map $\phi_{\boldsymbol{v}} \colon [\boldsymbol{v}] \to \Delta^n$ preserves this orientation since it preserves the orderings on $\boldsymbol{v}$ and $\boldsymbol{e}$.

1.2 Singular Chain Complex

Let X be a topological space. An ***n*-simplex** of X is a continuous map of the form $\sigma \colon [\boldsymbol{v}] \to X$ where $[\boldsymbol{v}]$ is an n-simplex of an $(n+1)$-dimensional normed vector space over $\mathbb{R}$. In the case where $[\boldsymbol{v}] = \Delta^n$, then we call σ a **singular *n*-simplex** of X. Note that an n-simplex $\sigma \colon [\boldsymbol{v}] \to X$ uniquely determines a singular n-simplex $\widetilde{\sigma} \colon \Delta^n \to X$ given by $\widetilde{\sigma} = \sigma \circ \phi_{\boldsymbol{v}}^{-1}$ where $\phi_{\boldsymbol{v}} \colon [\boldsymbol{v}] \to \Delta^n$ is the unique linear homeomorphism such that which sends $\boldsymbol{v}$ to $\boldsymbol{e}$. We often pass from σ to $\widetilde{\sigma}$ without comment. The set of all singular n-simplices in X is denoted is denoted by $\Sigma_n(X)$ and collection of all simplices of X is denoted $\Sigma(X)$. In particular, note that

$$\Sigma(X) = \bigcup_{n=0}^{\infty} \Sigma_n(X).$$

Let R be a ring. We define an R-complex, called the **singular chain complex of X with coefficients in R**, denoted by $\mathrm{C}(X;R) = (\mathrm{C}(X;R), \partial)$, as follows: the underlying graded R-module of $\mathrm{C}(X;R)$ is given by

$$\mathrm{C}(X;R) = \bigoplus_{n=0}^{\infty} \mathrm{C}_n(X;R) \quad \text{where} \quad \mathrm{C}_n(X;R) = \bigoplus_{\sigma \in \Sigma_n(X)} R\sigma.$$

The elements of $\mathrm{C}_n(X;R)$ are called **singular *n*-chains of** X. The differential of $\mathrm{C}(X;R)$ is defined on singular n-chains $\sigma \in \mathrm{C}_n(X;R)$ by

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[e_0, \dots, \widehat{e}_i, \dots, e_n]} \tag{1}$$

and is extended R-linearly everywhere else. Note that $\sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}$ is technically *not* an element of $\Sigma(X)$ because, even though the map $\sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}$ makes perfect since as a map from the $(n-1)$-simplex $[e_0,\dots,\widehat{e}_i,\dots,e_n]$ to the space X, it is not a singular chain of X since $[e_0,\dots,\widehat{e}_i,\dots,e_n]$ is not the standard $(n-1)$-simplex Δ^{n-1}. So technically speaking, the expression (1) makes no sense (as the differential ∂ needs to be a map from $\mathrm{C}(X;R)$ to itself). The key however is this: let $\boldsymbol{e}_i = e_0,\dots,e_{i-1},e_i,\dots,e_n$ and let $\widetilde{\boldsymbol{e}} = \widetilde{e}_0,\dots,\widetilde{e}_{n-1}$ denote the standard ordered basis of $\mathbb{R}^n$ (so $\Delta^{n-1} = [\widetilde{\boldsymbol{e}}]$), then we know that there is a *unique* linear homeomorphism $\phi_{\boldsymbol{e}_i}\colon [\boldsymbol{e}_i] \to \Delta^{n-1}$ such that $\phi_{\boldsymbol{e}_i}(\boldsymbol{e}_i) = \widetilde{\boldsymbol{e}}$. Then the expression

$$\partial(\sigma) = \sum_{i=0}^{n}(-1)^i(\sigma|_{[\boldsymbol{e}_i]} \circ \phi_{\boldsymbol{e}_i}^{-1}) \tag{2}$$

makes perect sense because $\sigma|_{[\boldsymbol{e}_i]} \circ \phi_{\boldsymbol{e}_i}^{-1}$ is a continuous map from Δ^{n-1} to X, and the maps $\phi_{\boldsymbol{e}_i}$ are uniquely determined by the data $\boldsymbol{e}_i$. So the expression (1) is implicitly understood to be the expression (2). The reason why we can safely ignore $\phi_{\widehat{\boldsymbol{e}}_i}^{-1}$ and use the expression (1), is because $\phi_{\widehat{\boldsymbol{e}}_i}^{-1}$ is completely determined by $\widehat{\boldsymbol{e}}_i$. For instance, let's show that $\partial^2 = 0$ using the expression (1): we have

$$\begin{aligned}
\partial^2(\sigma) &= \sum_{0\le i\le n}(-1)^i\partial(\sigma|_{[\boldsymbol{e}_i]}) \\
&= \sum_{0\le i\le n}(-1)^i\left(\sum_{0\le j<i}(-1)^j\sigma|_{[\boldsymbol{e}_{j,i}]} + \sum_{i<j\le n}(-1)^{j+1}\sigma|_{[\boldsymbol{e}_{i,j}]}\right) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma|_{[\boldsymbol{e}_{j,i}]} + \sum_{0\le i<j\le n}(-1)^{i+j+1}\sigma|_{[\boldsymbol{e}_{i,j}]} \\
&= 0.
\end{aligned}$$

Now we show $\partial^2 = 0$ using the expression (2): we have

$$\begin{aligned}
\partial^2(\sigma) &= \sum_{0\le i\le n}(-1)^i\partial(\sigma|_{[\boldsymbol{e}_i]} \circ \phi_{\boldsymbol{e}_i}^{-1}) \\
&= \sum_{0\le i\le n}(-1)^i\left(\sum_{0\le j<i}(-1)^j(\sigma|_{[\boldsymbol{e}_{j,i}]} \circ \phi_{[\boldsymbol{e}_{j,i}]}^{-1}) + \sum_{i<j\le n}(-1)^{j+1}(\sigma|_{[\boldsymbol{e}_{i,j}]} \circ \phi_{[\boldsymbol{e}_{i,j}]}^{-1})\right) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}(\sigma|_{[\boldsymbol{e}_{j,i}]} \circ \phi_{[\boldsymbol{e}_{j,i}]}^{-1}) + \sum_{0\le i<j\le n}(-1)^{i+j+1}(\sigma|_{[\boldsymbol{e}_{i,j}]} \circ \phi_{[\boldsymbol{e}_{i,j}]}^{-1}) \\
&= 0.
\end{aligned}$$

As you can see, it makes essentially no difference to the computation of $\partial^2 = 0$ whether we included the ϕ^{-1} or not.

1.2.1 Singular Homology

The homology of $\mathrm{C}(X;R)$ is called the **singular homology of X with coefficients in R**, and is denoted by $\mathrm{H}^{\mathrm{sing}}(X;R)$, or more simply by $\mathrm{H}(X;R)$. In the case where $R = \mathbb{Z}$, then we simplify our notation as follows: we write $\mathrm{C}(X;\mathbb{Z}) = \mathrm{C}(X)$ and $\mathrm{H}(X;\mathbb{Z}) = \mathrm{H}(X)$. Note that if $\varphi\colon R \to S$ is a ring homomorphism, then we have a canonical isomorphism of R-complexes

$$\mathrm{C}(X;R) \otimes_R S \simeq \mathrm{C}(X;S)$$

defined on elementary tensors by $\sigma \otimes s \mapsto s\sigma$ for all $s \in S$ and $\sigma \in \Sigma(X)$. In particular, we see that

$$\mathrm{H}(\mathrm{C}(X;R) \otimes_R S) \simeq \mathrm{H}(X;S).$$

Keep in mind though that the lefthand side is *not* isomorphic to $\mathrm{H}(X;R) \otimes_R S$ (taking homology and tensoring with S does not commute). However if R is a principal ideal domain (such as $R = \mathbb{Z}$), then there is a theorem from homological algebra, called the **universal coefficient theorem**, which says that we have a short exact sequence of S-complexes

$$0 \longrightarrow \mathrm{H}_n(X;R) \otimes_R S \longrightarrow \mathrm{H}_n(X;S) \longrightarrow \mathrm{Tor}_1(\mathrm{H}_{n-1}(X;R), S) \longrightarrow 0 \tag{3}$$

which splits (though not naturally).

The Homology of a Point Let x be a point. Let's calculate the singular homology of a x. First, we calculate the singular chain complex $\mathrm{C}(x)$. Note there is exactly one continuous map from Δ^i to x, namely the map $\sigma_{i,x}\colon \Delta^i \to x$ defined by $\sigma_{i,x}(\boldsymbol{t}) = x$ for all $\boldsymbol{t} \in \Delta^n$. In particular, we see that

$$\mathrm{C}_i(x) = \begin{cases} \mathbb{Z}\sigma_{i,x} & \text{if } i \geq 0 \\ 0 & \text{else} \end{cases}$$

Furthermore, it is straightforward to check that the differential is defined by

$$\partial(\sigma_{i,x}) = \begin{cases} \sigma_{i-1,x} & \text{if } i \geq 2 \text{ is even} \\ 0 & \text{else} \end{cases}$$

It follows that

$$\mathrm{H}_i(x) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

More generally, if $X = \{x_1, \dots, x_n\}$ is a finite set of points, then we have

$$\mathrm{H}_i(X) = \begin{cases} \mathbb{Z}^n & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

1.2.2 Singular Cohomology

Let X be a topological space, let R be a ring, and let $C = C(X;R)$ be the singular chain complex of X with coefficients in R. We denote by $C^\star = \mathrm{Hom}_R^\star(C,R)$ to be the hom-complex. Thus the homogeneous component of degree n of the underlying graded R-module $C^\star$ is given by

$$C_n^\star = \{\text{graded } R\text{-module homomorphisms from } C \text{ to } R \text{ of degree } n\}.$$

In particular, we have

$$\begin{aligned} C_0^\star &= \{R\text{-module homomorphisms from } C_0 \text{ to } R\} \\ C^{\star,1} = C_{-1}^\star &= \{R\text{-module homomorphisms from } C_1 \text{ to } R\} \\ &\vdots \\ C^{\star,n} = C_{-n}^\star &= \{R\text{-module homomorphisms from } C_n \text{ to } R\} \\ &\vdots \end{aligned}$$

and we have $C_i^\star = 0$ for all $i > 0$ and where we make the notational convention $C^{\star,n} := C_{-n}^\star$ for all $n \in \mathbb{Z}$. Next, the codifferential $\partial^\star\colon C^\star \to C^\star$ is defined as follows: given an R-module homomorphism $\varphi\colon C_n \to R$, we set $\partial^\star\varphi\colon C_{n+1} \to R$ to be the map defined by

$$(\partial^\star\varphi)(\sigma) = \varphi(\partial\sigma)$$

for all $\sigma \in \Sigma(X)$. We call $C^\star = C(X;R)^\star$ the **singular cochain complex of X with coefficients in R**. The cohomology of $C^\star$ is denoted $\mathrm{H}_{\mathrm{sing}}(X;R)$ and is called the **singular cohomology of X with coefficients in R**. If R is a PID, then we can relate the singular homology with singular cohomology with the Ext version of the universal coefficient theorem. This says that we have a short exact sequence

$$0 \to \mathrm{Ext}_R^1(\mathrm{H}_{n-1}(X;R),S) \to \mathrm{H}^n(X;S) \to \mathrm{Hom}_R(\mathrm{H}_n(X;R),S) \to 0,$$

which splits (though not naturally).

1.2.3 Homotopy Invariance

Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon C(X) \to C(Y)$ to be the unique graded homomorphism of R-modules such $f_*(\sigma) = f \circ \sigma$ for all $\sigma \in \Sigma(X)$. We claim that f_* is more than just a graded homomorphism: it is a chain map. Indeed, we have

$$\begin{aligned} \partial f_*(\sigma) &= \partial(f \circ \sigma) \\ &= \sum_{0 \leq i \leq n} (-1)^i (f \circ \sigma)|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]} \\ &= \sum_{0 \leq i \leq n} (-1)^i \left(f \circ \sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]} \right) \\ &= f_* \left(\sum_{0 \leq i \leq n} (-1)^i \sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]} \right) \\ &= f_*\partial(\sigma). \end{aligned}$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then clearly we have $(f \circ g)_* = f_* \circ g_*$. In particular, we obtain a covariant functor $\mathrm{C}\colon \mathbf{Top} \to \mathbf{Chain}_R$ from the category of topological spaces to the category of R-complexes, which sends a topological space X to the chain complex $C(X)$ and which sends a continuous map $f\colon X \to Y$ to the chain map $f_*\colon C(X) \to C(Y)$. We call C the **singular chain functor**. Note that we also have a homology functor $\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$ which sends an R-complex A to its homology $\mathrm{H}(A)$ and which sends a chain map $\varphi\colon A \to B$ to the induced map in homology $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(B)$. The **singular homology functor** is the composition of these two functors.

Recall that two continuous maps $f, g\colon X \to Y$ are said to be homotopic to each other (as continuous functions), denoted $f \sim g$, if there exists a continuous map $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case, we say H is a homotopy form f to g. Similarly, recall that two chain maps $\varphi, \psi\colon A \to B$ are said to be homotopic to each other (as chain maps), denoted $\varphi \sim \psi$, if there exists a graded R-linear map $h\colon X \to Y$ of degree 1 such that $\varphi - \psi = \mathrm{d}_Y h + h \mathrm{d}_X$. In both **Top** and $\mathbf{Comp}_R$, the homotopy relation $\sim$ is easily seen to be an equivalence relation. We want to now show that the singular chain functor preserves these homotopy equivance relations:

Proposition 1.1. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions such that $f \sim g$ as continuous functions. Then $f_* \sim g_*$ as chain maps.*

Proof. Let $H\colon X \times I \to Y$ be a homotopy from f to g (so H is continuous and $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$). In order to show $f_* \sim g_*$, we need to find a graded homomorphism $h\colon C(X) \to C(Y)$ of degree 1 such that $f_* = g_* + \partial h + h\partial$ which is equivalent to showing

$$\partial h(\sigma) = g_*(\sigma) - f_*(\sigma) + h\partial(\sigma) \tag{4}$$

for all $\sigma \in \Sigma(X)$. With that in mind, let $\sigma \in \Sigma_n(X)$; so we want to find a graded homomorphism $h\colon C(X) \to C(Y)$ of degree 1 such that (4) holds. The idea is that h should somehow come from H. Consider the composite function $H \circ (\sigma \times 1) = H(\sigma \times 1)$ which is a map from $\Delta^n \times I$ to Y. Both $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$ are n-simplices (even Euclidean n-simplices), write them as $\Delta^n \times \{0\} = [v_0, \dots, v_n] = [\boldsymbol{v}]$ and $\Delta^n \times \{1\} = [w_0, \dots, w_n] = [\boldsymbol{w}]$. Observe that

$$H(\sigma \times 1)(x,0) = f(\sigma(x)) \quad \text{and} \quad H(\sigma \times 1)(x,1) = g(\sigma(x))$$

for all $x \in X$. It follows that $H(\sigma \times 1)|_{[\boldsymbol{v}]} = f_*(\sigma)$ and $H(\sigma \times 1)_{[\boldsymbol{w}]} = g_*(\sigma)$. If $n = 0$, then $\Delta^0 \times I$ is a simplex (namely $\Delta^0 \times I = [v_0, w_0]$) and thus $H(\sigma \times 1)$ is a singular chain (up to the unique linear homeomorphism determined by (v_0, w_0)), and so if we defined $h(\sigma) = H(\sigma \times 1)$, then (4) is satisfed since $\partial(\sigma) = 0$. If $n > 0$, then $\Delta^n \times I$ is not a simplex, so this idea doesn't work. The key however, is that we can subdivide $\Delta^n \times I$ into simplices in a nice way to make the $n = 0$ case work more generally. The idea is to pass from $[\boldsymbol{v}]$ to $[\boldsymbol{w}]$ by interpolating a sequence of n-simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. Thus the first step is to move $[\boldsymbol{v}] = [v_0, \dots, v_n]$ up to $[\boldsymbol{v}_{n-1}, \boldsymbol{w}_1] = [v_0, \dots, v_{n-1}, w_n]$, then the second step is to move this up to $[\boldsymbol{v}_{n-2}, \boldsymbol{w}_2] = [v_0, \dots, v_{n-2}, w_{n-1}, w_n]$, and so on. In the typical step $[\boldsymbol{v}_i, \boldsymbol{w}_{n-i}] = [v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ moves up to $[\boldsymbol{v}_{i-1}, \boldsymbol{w}_{n-i+1}] = [v_0, \dots, v_{i-1}, w_i, \dots, w_n]$. The region between these two n-simplices is exactly the $(n+1)$-simplex $[\boldsymbol{v}_i, w_i, \boldsymbol{w}_{n-i}]$ which has $[\boldsymbol{v}_i, \boldsymbol{w}_{n-i}]$ as its lower face and $[\boldsymbol{v}_{i-1}, \boldsymbol{w}_{n-i+1}]$ as its upper face. Altogether, $\Delta^n \times I$ is the union of $(n+1)$-simplices $[\boldsymbol{v}_i, w_i, \boldsymbol{w}_{n-i}]$, each intersecting the next in an n-simplex face. With this understood, we define

$$h(\sigma) = \sum_{0 \le i \le n} (-1)^i H(\sigma \times 1)_{[\boldsymbol{v}_i, w_i, \boldsymbol{w}_{n-i}]}.$$

Given a homotopy $H\colon X \times I \to Y$ from f to g and a singular simplex $\sigma\colon \Delta^n \to X$, we can form the composition $H \circ (\sigma \times 1)$, which is a map from $\Delta^n \times I$ to Y. Using this, we can define **prism operators** $P\colon C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$h(\sigma) = \sum_{0 \le i \le n} (-1)^i H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

It is straightforward to check that h satisfies (4), giving us our desired homotopy from f_* to g_*. $\square$

Corollary 1. *If f and g are homotopically equivalent as continuous functions, then $f_\#$ and $g_\#$ induce the same map on homology.*

1.3 Exact Sequences and Excision

Let X be a topological space and let A a subspace of X. Then the inclusion map $\iota\colon A \to X$ induces a chain map $\iota_*\colon \mathrm{C}(A) \to \mathrm{C}(X)$. We set $\mathrm{C}(X, A)$ to be the cokernel of this chain map:

$$\mathrm{C}(X, A) = \mathrm{C}(X)/\mathrm{C}(A).$$

The homology of $\mathrm{C}(X, A)$ is called **relative homology** and is denoted $\mathrm{H}(X, A)$.

Example 1.1. In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n)$, the maps $H_i(D^n, \partial D^n) \to \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the remaining terms $\widetilde{H}_i(D^n)$ are zero for all i. Thus we obtain the calculation

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.2. *Let X be a topological space and let $x_0 \in X$. Then*

$$\mathrm{H}(X, x_0) = \widetilde{\mathrm{H}}(X).$$

Proof. The inclusion $\{x_0\} \hookrightarrow X$ induces a short exact sequence of complexes

$$0 \longrightarrow \mathrm{C}(x_0) \longrightarrow \mathrm{C}(X) \longrightarrow \mathrm{C}(X, x_0) \longrightarrow 0 \tag{5}$$

which in turn induces a long exact sequence in homology:

$$\cdots \longrightarrow \mathrm{H}_{i+1}(X, x_0) \longrightarrow \mathrm{H}_i(x_0) \longrightarrow \mathrm{H}_i(X) \longrightarrow \mathrm{H}_i(X, x_0) \longrightarrow \mathrm{H}_{i-1}(x_0) \longrightarrow \cdots$$

Since $\mathrm{H}_i(x_0) = 0$ for all $i \geq 1$, we see that $\mathrm{H}_i(X) = \mathrm{H}_i(X, x_0)$ for all $i \geq 2$. Furthermore, the map

$$\mathbb{Z} = \mathrm{H}_0(x_0) \to \mathrm{H}_0(X) = \mathbb{Z}^P,$$

is injective, where P denotes the set of all path-connected components of X. Thus we have $\mathrm{H}_1(X) = \mathrm{H}_1(X, x_0)$ and we have a short exact sequence in homological degree 0:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^P \longrightarrow \mathrm{H}(X, x_0) \longrightarrow 0 \tag{6}$$

This short exact sequence splits, thus $\mathrm{H}_0(X) = \mathrm{H}_0(X, x_0) \oplus \mathbb{Z}$. $\square$

1.3.1 Excision

Theorem 1.1. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces an isomorphism*

$$\mathrm{H}(X \backslash Z, A \backslash Z) \cong \mathrm{H}(X, A).$$

Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism

$$\mathrm{H}(B, A \cap B) \cong \mathrm{H}(X, A).$$

Remark 1. The translation between the two versions is obtained by setting $B = X \backslash Z$ and $Z = X \backslash B$. Then $A \cap B = A \backslash Z$ and the condition $\overline{Z} \subset \mathrm{int}(A)$ is equivalent to $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ since $X \backslash \mathrm{int}(B) = \overline{Z}$.

For a space X, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $\mathrm{C}^{\mathcal{U}}(X)$ be the subcomplex of $\mathrm{C}(X)$ consisting of chains $\sum_i m_i \sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$. We denote the homology groups of this chain complex by $H_n^{\mathcal{U}}(X)$.

Proposition 1.3. *The inclusion $\iota \colon \mathrm{C}^{\mathcal{U}}(X) \hookrightarrow \mathrm{C}(X)$ is a homotopy equivalence, that is, there is a chain map $\rho \colon \mathrm{C}(X) \to \mathrm{C}^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence ι induces a graded isomorphism $\mathrm{H}^{\mathcal{U}}(X) \cong \mathrm{H}(X)$.*

Proof. The barycentric subdivision process will be performed at four levels, beginning with the most geometric and becoming increasingly algebraic.

(1) Barycentric Subdivision of Simplices: The points of a simplex $[v_0, \ldots, v_n]$ are the linear combinations $\sum_i t_i v_i$ with $\sum t_i = 1$ and $t_i \in [0, 1]$ for each i. The **barycenter** or 'center of gravity' of the simplex $[v_0, \ldots, v_n]$ is the point $b = \sum t_i v_i$ whose barycentric coordinates t_i are all equal, namely $t_i = 1/(n+1)$ for each i. The

barycentric subdivision of $[v_0,\dots,v_n]$ is the decomposition of $[v_0,\dots,v_n]$ into the n-simplices $[b,w_0,\dots,w_{n-1}]$ where, inductively, $[w_0,\dots,w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0,\dots,\widehat{v}_i,\dots,v_n]$. The induction starts with the case $n=0$ when the barycentric subdivision of $[v_0]$ is defined to be just $[v_0]$ itself. It follows from the inductive definition that the vertices of simplices in the barycentric subdivision of $[v_0,\dots,v_n]$ are exactly the barycenters of all the k-dimensional faces $[v_{i_0},\dots,v_{i_k}]$ of $[v_0,\dots,v_n]$ for $0\le k\le n$. When $k=0$ this gives the original vertices v_i since the barycenter of 0-simplex is itself. The barycenter of $[v_{i_0},\dots,v_{i_k}]$ has barycentric coordinates $t_i=1/(k+1)$ for $i=i_0,\dots,i_k$ and $t_i=0$ otherwise.

The n-simplices of the barycentric subdivision of Δ^n, together with all their faces, do in fact form a Δ-complex structure on Δ^n, indeed a simplicial complex structure, though we shall not need to know this in what follows.

A fact we will need is that the diameter of each simplex of the barycentric subdivision of $[v_0,\dots,v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0,\dots,v_n]$. Here the diameter of a simplex is by definition the maximum distance between any two of its points, and we are using the metric from the ambient Euclidean space $\mathbb{R}^m$ containing $[v_0,\dots,v_n]$. The diameter of a simplex equals the maximum distance between any of its vertices because the distance between the points v and $\sum t_iv_i$ of $[v_0,\dots,v_n]$ satisfies the inequality

$$\begin{aligned}\left|v-\sum_{i=0}^{n}t_iv_i\right| | &= \left|\sum_{i=0}^{n}t_i(v-v_i)\right|\\ &\le \sum_{i=0}^{n}t_i|v-v_i|\\ &\le \sum_{i=0}^{n}t_i\max_{0\le j\le n}|v-v_j|\\ &= \max_{0\le j\le n}|v-v_j|.\end{aligned}$$

The significance of the factor $n/(n+1)$ is that by repeated barycentric subdivision we can produce simplices of arbitrarily small diameter since $(n/(n+1))^r$ approaches 0 as r goes to infinity. It is important that the bound $n/(n+1)$ does not depend on the shape of the simplex since repeated barycentric subdivision produces simplices of many different shapes.

To obtain the bound $n/(n+1)$ on the ratio of diameters, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex $[w_0,\dots,w_n]$ of the barycentric subdivision of $[v_0,\dots,v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0,\dots,v_n]$.

(2) Barycentric Subdivision of Linear Chains. The main part of the proof will be to construct a subdivision operator $\mathcal{S}\colon S_n(X)\to S_n(X)$ and show that this is chain homotopic to the identity map. First we will construct $\mathcal{S}$ and the chain homotopy in a more restricted linear setting.

For a convex set Y in some Euclidean space, the linear maps $\Delta^n\to Y$ generate a subgroup of $S_n(Y)$ that we denote $L_n(Y)$, the **linear chains**. Note that $L(Y)$ is ∂-stable, so the linear chains form a subcomplex of $(S(Y),\partial)$. We can uniquely designate a linear map $\lambda\colon\Delta^n\to Y$ by $[w_0,\dots,w_n]$ where w_i is the image under λ of the ith vertex of Δ^n. Indeed, by linearity we have $\lambda(\sum t_ie_i)=\sum t_i\lambda(e_i)$. To avoid having to make exceptions for 0-simplices, it will be convenient to augment the complex $(L(Y),\partial)$ by setting $L_{-1}(Y)=R$ generated by the empty simplex $[\varnothing]$, with $\partial[w_0]=[\varnothing]$ for all 0-simplices $[w_0]$.

Each point $b\in Y$ determines a graded homomorphism $b\colon L(Y)\to L(Y)$ of degree 1, defined on basis elements by $b([w_0,\dots,w_n])=[b,w_0,\dots,w_n]$. Geometrically, the homomorphism b can be regarded as a cone operator, sending a linear chain to the cone having the linear chain as the base of the cone and the point b as the tip of the cone. Applying the usual formula for ∂, we obtain the relation

$$\begin{aligned}\partial b([w_0,\dots,w_n]) &= \partial[b,w_0,\dots,w_n])\\ &= [w_0,\dots,w_n]-b(\partial[w_0,\dots,w_n]).\end{aligned}$$

By linearity it follows that $\partial b(\alpha)=\alpha-b(\partial\alpha)$ for all $\alpha\in L(Y)$. This expresses algebraically the geometric fact that the boundary of a cone consists of its base together with the cone on the boundary of its base. The relation $\partial b(\alpha)=\alpha-b(\partial\alpha)$ can be rewritten as

$$\partial b+b\partial=1,$$

so b is a chain homotopy between the identity map and the zero map of the augmented chain complex $(L(Y),\partial)$.

Now we define a graded homomorphism $\mathcal{S}\colon L(Y)\to L(Y)$ by induction on n. Let $\lambda\colon\Delta^n\to Y$ be a generator of $L(Y)$ and let b_λ be the image of the barycenter of Δ^n under λ. Then the inductive formula for $\mathcal{S}$ is

$$\mathcal{S}(\lambda)=b_\lambda(\mathcal{S}(\partial\lambda)),$$

where $b_\lambda \colon L(Y) \to L(Y)$ is the cone operator defined in the preceding paragraph. The induction starts with $\mathcal{S}([\varnothing]) = [\varnothing]$, so $\mathcal{S}$ is the identity on $L_{-1}(Y)$. To get a feel for the map $\mathcal{S}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned}\mathcal{S}[w_0] &= w_0\left(\mathcal{S}(\partial[w_0])\right) \\ &= w_0(\mathcal{S}[\varnothing]) \\ &= w_0[\varnothing] \\ &= [w_0].\end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter b_{01}. Then

$$\begin{aligned}\mathcal{S}[w_0, w_1] &= b_{01}\left(\mathcal{S}(\partial[w_0, w_1])\right) \\ &= b_{01}(\mathcal{S}[w_1] - \mathcal{S}[w_0]) \\ &= b_{01}([w_1] - [w_0]) \\ &= [b_{01}, w_1] - [b_{01}, w_0].\end{aligned}$$

Now let $[w_0, w_1, w_2] \in L_2(Y)$ with barycenter b_{012}. Then

$$\begin{aligned}\mathcal{S}[w_0, w_1, w_2] &= b_{012}\left(\mathcal{S}(\partial[w_0, w_1, w_2])\right) \\ &= b_{012}(\mathcal{S}[w_1, w_2] - \mathcal{S}[w_0, w_2] + \mathcal{S}[w_0, w_1]) \\ &= b_{012}([b_{12}, w_2] - [b_{12}, w_1] - [b_{02}, w_2] + [b_{02}, w_0] + [b_{01}, w_1] - [b_{01}, w_0]) \\ &= [b_{012}, b_{12}, w_2] - [b_{012}, b_{12}, w_1] + [b_{012}, b_{02}, w_0] - [b_{012}, b_{02}, w_2] + [b_{012}, b_{01}, w_1] - [b_{012}, b_{01}, w_0],\end{aligned}$$

where b_{12}, b_{02}, and b_{01} are the barycenters for the simplices $[w_1, w_2]$, $[w_0, w_2]$, and $[w_0, w_1]$ respectively. In general, when λ is an embedding, with image a genuine n-simplex $[w_0, \ldots, w_n]$, then $\mathcal{S}(\lambda)$ is the sum of the n-simplices in the barycentric subdivision of $[w_0, \ldots, w_n]$, with certain signs that could be computed explicitly.

Let us check that $\mathcal{S} \colon L(Y) \to L(Y)$ is a chain map, i.e. $\partial\mathcal{S} = \mathcal{S}\partial$. Since $\mathcal{S} = 1$ on $L_0(Y)$ and $L_{-1}(Y)$, we certainly have $\partial\mathcal{S} = \mathcal{S}\partial$ on $L_0(Y)$. The result for larger n is given by the following calculation, in which we omit some parentheses to unclutter the formulas:

$$\begin{aligned}\partial\mathcal{S}\lambda &= \partial b_\lambda(\mathcal{S}\partial\lambda) \\ &= (1 - b_\lambda\partial)(\mathcal{S}\partial\lambda) \\ &= \mathcal{S}\partial\lambda - b_\lambda\partial(\mathcal{S}\partial\lambda) \\ &= \mathcal{S}\partial\lambda - b_\lambda\mathcal{S}(\partial\partial\lambda) \\ &= \mathcal{S}\partial\lambda,\end{aligned}$$

where $\partial S(\partial\lambda) = S\partial(\partial\lambda)$ follows by induction on n.

We next build a chain homotopy $\mathcal{T} \colon L(Y) \to L(Y)$ between $\mathcal{S}$ and the identity. We define $\mathcal{T}$ on $L_n(Y)$ inductively by setting $\mathcal{T} = 0$ for $n = -1$ and let $\mathcal{T}\lambda = b_\lambda(\lambda - \mathcal{T}\partial\lambda)$ for $n \geq 0$. The induction starts with $\mathcal{T}[\varnothing] = 0$. To get a feel for the map $\mathcal{T}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned}\mathcal{T}[w_0] &= w_0\left([w_0] - \mathcal{T}\partial[w_0]\right) \\ &= w_0\left([w_0] - \mathcal{T}[\varnothing]\right) \\ &= [w_0, w_0].\end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter b_{01}. Then

$$\begin{aligned}\mathcal{T}[w_0, w_1] &= b_{01}\left([w_0, w_1] - \mathcal{T}\partial[w_0, w_1]\right) \\ &= b_{01}\left([w_0, w_1] - \mathcal{T}[w_1] + \mathcal{T}[w_0]\right) \\ &= [b_{01}, w_0, w_1] - [b_{01}, w_1, w_1] + [b_{01}, w_0, w_0].\end{aligned}$$

The geometric motivation for this formula is an inductively defined subdivision of $\Delta^n \times I$ obtained by joining all simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$. What $\mathcal{T}$ actually does is take the image of this subdivision under the projection $\Delta^n \times I \to \Delta^n$.

The chain homotopy formula $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ is trivial on $L_{-1}(Y)$ where $\mathcal{T} = 0$ and $\mathcal{S} = 1$. Verifying the formula on $L_n(Y)$ with $n \geq 0$ is done by the calculation

$$\begin{aligned}\partial\mathcal{T}\lambda &= \partial b_\lambda(\lambda - \mathcal{T}\partial\lambda) \\ &= (1 - b_\lambda\partial)(\lambda - \mathcal{T}\partial\lambda) \\ &= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\mathcal{T}\partial\lambda \\ &= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda(1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda \\ &= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\lambda - b_\lambda\mathcal{S}\partial\lambda - b_\lambda\mathcal{T}\partial\partial\lambda \\ &= \lambda - \mathcal{T}\partial\lambda - b_\lambda\mathcal{S}\partial\lambda \\ &= \lambda - \mathcal{T}\partial\lambda - \mathcal{S}\lambda.\end{aligned}$$

where $\partial\mathcal{T}\partial\lambda = (1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda$ follows by induction on n. Now we discard $L_{-1}(Y)$ and the relation $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ still holds since $\mathcal{T}$ was zero on $L_{-1}(Y)$.

(3) Barycentric Subdivision of General Chains. Define $\mathcal{S}\colon S_n(X) \to S_n(X)$ by setting $\mathcal{S}\sigma = \sigma_\#\mathcal{S}\Delta^n$ for a singular n-simplex $\sigma\colon \Delta^n \to X$. Since $\mathcal{S}\Delta^n$ is the sum of the n-simplices in the barycentric subdivision of Δ^n, with certain signs, $\mathcal{S}\sigma$ is the corresponding signed sum of the restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. For example, if $\sigma \in S_1(X)$, then

$$\begin{aligned}\mathcal{S}\sigma &= \sigma_\#\mathcal{S}[e_0, e_1] \\ &= \sigma \circ ([b, e_1] - [e_0, b]) \\ &= \sigma|_{[b,e_1]} - \sigma|_{[e_0,b]},\end{aligned}$$

where $b = (e_0 + e_1)/2$ is the barycenter of $[e_0, e_1]$.

The operator $\mathcal{S}$ is a chain map since

$$\begin{aligned}\partial\mathcal{S}\sigma &= \partial\sigma_\#\mathcal{S}\Delta^n \\ &= \sigma_\#\partial\mathcal{S}\Delta^n \\ &= \sigma_\#\mathcal{S}\partial\Delta^n \\ &= \sigma_\# S\Big(\sum_i (-1)^i\Delta_i^n\Big) \\ &= \sum_i (-1)^i\sigma_\# S\Delta_i^n \\ &= \sum_i (-1)^i S(\sigma|_{\Delta_i^n}) \\ &= S\left(\sum_i (-1)^i\sigma|_{\Delta_i^n}\right) \\ &= S(\partial\sigma).\end{aligned}$$

where Δ_i is the ith face of Δ^n.

In similar fashion we define $T\colon S_n(X) \to S_n(X)$ by $T\sigma = \sigma_\# T\Delta^n$, and this gives a chain homotopy between S and the identity, since the formula $\partial T + T\partial = 1 - S$ holds by the calculation

$$\begin{aligned}\partial T\sigma &= \partial\sigma_\# T\Delta^n \\ &= \sigma_\#\partial T\Delta^n \\ &= \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n) \\ &= \sigma - S\sigma - \sigma_\# T\partial\Delta^n \\ &= \sigma - S\sigma - T(\partial\sigma)\end{aligned}$$

where the last equality follows just as in the previous displayed calculation, with S replaced by T.

(4) Iterated Barycentric Subdivision. A chain homotopy between 1 and the iterate S^m is given by the operator $D_m = \sum_{0 \le i < m} TS^i$ since

$$\begin{aligned}\partial D_m + D_m\partial &= \sum_{0\le i<m} (\partial TS^i + TS^i\partial) \\ &= \sum_{0\le i<m} (\partial TS^i + T\partial S^i) \\ &= \sum_{0\le i<m} (\partial T + T\partial)S^i \\ &= \sum_{0\le i<m} (1 - S)S^i \\ &= \sum_{0\le i<m} (S^i - S^{i+1}) \\ &= 1 - S^m.\end{aligned}$$

For each singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an m such that $S^m(\sigma)$ lies in $S_n^{\mathcal{U}}(X)$ since the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}(\text{int}(U_j))$ if m

is large enough. (Recall that a Lebesgue number for an open cover of a compact metric space is a number $\varepsilon > 0$ such that every set of diameter less than ε lies in some set of the cover; such a number exists by an elementary compactness argument). We cannot expect the same number m to work for all σ's, so let us define $m(\sigma)$ to be the smallest m such that $S^m(\sigma)$ is in $S_n^{\mathcal{U}}(X)$.

We now define $D\colon S_n(X) \to S_{n+1}(X)$ by setting $D\sigma = D_{m(\sigma)}\sigma$ for each singular n-simplex $\sigma\colon \Delta^n \to X$. For this D we would like to find a chain map $\rho\colon S_n(X) \to S_n(X)$ with image in $S_n^{\mathcal{U}}(X)$ satisfying the chain homotopy equation

$$\partial D + D\partial = 1 - \rho. \tag{7}$$

A quick way to do this is to simply regard this equation as defining ρ, so we let $\rho = 1 - \partial D - D\partial$. It follows easily that ρ is a chain map since

$$\begin{aligned} \partial\rho(\sigma) &= \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma \\ &= \partial\sigma - \partial D\partial\sigma \\ &= \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma \\ &= \rho(\partial\sigma). \end{aligned}$$

To check that ρ takes $S_n(X)$ to $S_n^{\mathcal{U}}(X)$, we compute $\rho(\sigma)$ more explicitly:

$$\begin{aligned} \rho(\sigma) &= \sigma - \partial D\sigma - D(\partial\sigma) \\ &= \sigma - \partial D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \\ &= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma). \end{aligned}$$

The term $S^{m(\sigma)}\sigma$ lies in $S_n^{\mathcal{U}(X)}$ by the definition of $m(\sigma)$. The remaining terms $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ for σ_j the restriction of σ to a face of Δ^n, so $m(\sigma_j) \leq m(\sigma)$ and hence the difference $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ consists of terms $TS^i(\sigma_j)$ with $i \geq m(\sigma_j)$, and these terms lie in $S_n^{\mathcal{U}}(X)$ since T takes $S_{n-1}^{\mathcal{U}}(X)$ to $S_n^{\mathcal{U}}(X)$.

View ρ as a chain map $S_n(X) \to S_n^{\mathcal{U}}(X)$, the equation (7) says that $\partial D + D\partial = 1 - \iota\rho$ for $\iota\colon S_n^{\mathcal{U}}(X) \hookrightarrow S_n(X)$ the inclusion. Furthermore, $\rho\iota = 1$ since D is identically zero on $S_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ if σ is in $S_n^{\mathcal{U}}(X)$, hence the summation defining $D\sigma$ is empty. Thus we have shown that ρ is a chain homotopy inverse for ι. □

1.4 Mayer-Vietoris

Let $A, B \subseteq X$ be subspaces such that X is the union of the interiors of A and B. We set $\mathrm{C}(A+B)$ to be the subgroup of $\mathrm{C}(X)$ consisting of chains that are sums of chains in A and chains in B. In particular, since $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, the inclusion $\mathrm{C}(A+B) \hookrightarrow \mathrm{C}(X)$ is a homotopy equivalence, so $\mathrm{H}(X) = \mathrm{H}(\mathrm{C}(A+B))$. Now, observe that we obtain a short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{C}(A\cap B) \xrightarrow{\varphi} \mathrm{C}(A) \oplus \mathrm{C}(B) \xrightarrow{\psi} \mathrm{C}(A+B) \longrightarrow 0 \tag{8}$$

where $\varphi(\sigma) = (\sigma, -\sigma)$ and $\psi(\sigma, \tau) = \sigma + \tau$. This short exact sequence induces a long exact sequence in homology

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}(X) \arrow[dlll, rounded corners] \\ \mathrm{H}_i(A\cap B) \arrow[r] & \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \arrow[r] & \mathrm{H}_i(X) \arrow[dll, rounded corners] & \\ \mathrm{H}_{i-1}(A\cap B) \arrow[r] & \cdots & & \end{tikzcd}$$

This long exact sequence is called the **Mayer-Vietoris sequence**.

1.5 Singular Cohomology

Let R be a ring and N and R-module. If M is a graded R-module, then we set $\mathrm{Hom}_R(M,N)_{\mathrm{gr}}$ to be the graded R-module whose homogeneous component in degree n is $M_n := \mathrm{Hom}_R(M_n, N)$. If (M, d) is a chain complex over R, where M is considered a graded R-module and d is considered a graded endomorphism $d\colon M \to M$ of degree

-1, then we obtain a cochain complex over R given by $(\mathrm{Hom}_R(M,N)_{\mathrm{gr}}, d_*)$, where if $\psi \in \mathrm{Hom}_R(M_{n-1},N)$ then $d_*(\psi) = \psi \circ d \in \mathrm{Hom}_R(M_n,N)$.

In particular, we obtain a cochain complex $(\mathrm{Hom}_R(S(X),N)_{\mathrm{gr}}, \partial_*)$ called the **singular cochain complex of X over R with values in N**. Elements in $S_n(X)^\vee$ are called **singular n-cochains** and the nth cohomology, called the **singular cohomology of X over R**, is denoted $H^n_{\mathrm{sing}}(X,R)$. For notational purposes, we denote $\delta := \partial^\vee_{\mathrm{gr}}$ and $S^n(X,R) := S_n(X,R)^\vee$. We can work out δ explicitly as follows: if $\psi \in S^n(X)$, then $\delta(\psi) \in S^{n+1}(X)$ is given by

$$\delta(\psi)(\sigma) = \psi(\partial(\sigma)) = \sum_i (-1)^i \psi(\sigma_i)$$

for all $\sigma \in S_{n+1}(X)$.

1.5.1 Delta Complex

A **Δ-complex** structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha|_{\Delta^n \setminus \partial\Delta^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\Delta^n \setminus \partial\Delta^n}$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism betweem them that preserves the ordering of the vertices.
3. A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Among other things, this last condition rules out trivialities like regarding all the points of X are individual vertices.

2 $C(X,Y)$

2.1 Compact-Open Topology

Let X and Y be topological spaces. We define $C(X,Y)$ to be the set of all continuous maps from X to Y. We endow $C(X,Y)$ with a topology, called the **compact-open topology**, where a subbase of $C(X,Y)$ is given by the collection of all sets of the form

$$\mathrm{B}(K,V) = \{f \in C(X,Y) \mid f(K) \subseteq V\}$$

where $K \subseteq X$ is compact and where $V \subseteq Y$ is open. In particular, if $\Omega \subseteq C(X,Y)$ is open and $f \in \Omega$, then we can find a $\mathrm{B}(K,V)$ such $f \in \mathrm{B}(K,V) \subseteq \Omega$. In other words, we can find $K \subseteq X$ compact and $V \subseteq Y$ open such that $f(K) \subseteq V$ and for any $g \in C(X,Y)$ with the property that $g(K) \subseteq V$, we have $g \in \Omega$. Such a g can be thought of as being "close" to f. Whenever we write "let $\mathrm{B}(K,V) \subseteq C(X,Y)$ be a basic open", then as long as context is clear, it will be understood that $K \subseteq X$ is compact and $V \subseteq Y$ is open. We will try to be as consistent with our notation as possible. For instance, we typically use Δ, K, L, M to denote compact sets, Ω, U, V, W to denote open sets, and Γ, E, F, G to denote closed sets. Furthermore, we will try to be as lexicographically consistent with our notation as possible (for instance, $K \subseteq X$, $L \subseteq Y$, $M \subseteq Z$ is lexicographically consistent, whereas $V \subseteq X$, $W \subseteq Y$, and $U \subseteq Z$ is not). Even though there's no guarantee that we will adhere to this principal at all times, we will always try to be as clear as possible from context.

Now we assume that X and Y are locally compact Hausdorff spaces and let $f \in \mathrm{B}(K,V)$. Since f is continuous, we see that $L := f(K)$ is compact, and since Y is Hausdorff, this implies $L \subseteq Y$ is closed. For each $y \in L$, choose a compact neighborhood of y, say $y \in V_y \subseteq L_y$ where we may also assume that $V_y \subseteq V$ (if we need to, we can always replace V_y with $V_y \cap V$ which is again open). Note that $\{V_y\}_{y \in L}$ is a covering of L, so we can extract a finite subcovering, say

$$L \subseteq \bigcup_{i=1}^n V_{y_i} \subseteq \bigcup_{i=1}^n L_{y_i}$$

Set $V' = \bigcup_{i=1}^n V_{y_i}$ and set $L' = \bigcup_{i=1}^n L_{y_i}$, so $L \subseteq V' \subseteq L'$ (the open V' is smushed inbetween two compacts!). Note that since Y is Hausdorff, we have $\overline{V'} \subseteq L'$, and in pariticular this implies $\overline{V'}$ is compact. In particular, we have shown that we can replace $\mathrm{B}(K,V)$ with a smaller open neighborhood of f, namely $\mathrm{B}(K,V')$, which has the property that V' has compact closure.
Note that since Y is Hausdorff, we have $\overline{V'} \subseteq L'$, and in pariticular this implies $\overline{V'}$ is compact.

We claim that $\{\mathrm{B}(K,V)\}$ forms a basis in this case. Indeed, it suffices to show that if $f \in \mathrm{B}(K_1,V_1) \cap \mathrm{B}(K_2,V_2)$, then we can find a $\mathrm{B}(K,V)$ such that $f \in \mathrm{B}(K,V) \subseteq \mathrm{B}(K_1,V_1) \cap \mathrm{B}(K_2,V_2)$.

A natural space to consider is the iterated space $C(X, C(Y, Z))$. The basic opens in this space have the form $\mathrm{B}(K, \Omega)$ (where again it is implicitly understood that $K \subseteq X$ is compact and $\Omega \subseteq C(Y, Z)$ is open). Write Ω as a union of the basic opens of $C(Y, Z)$, say $\Omega = \bigcup_{i \in I} \mathrm{B}(L_i, W_i)$. We claim that

$$\mathrm{B}(K, \Omega) = \bigcup_{i \in I} \mathrm{B}(K, \mathrm{B}(L_i, W_i)).$$

Indeed, given $f \in C(X, C(Y, Z))$, we have

$$\begin{aligned} f \in \bigcup_{i \in I} \mathrm{B}(K, \mathrm{B}(L_i, W_i)) &\iff f \in \mathrm{B}(K, \mathrm{B}(L_i, W_i)) \text{ for some } i \\ &\iff f(K) \subseteq \mathrm{B}(L_i, W_i) \text{ for some } i \\ &\iff f(K) \subseteq \bigcup_{i \in I} \mathrm{B}(L_i, W_i) \\ &\iff f(K) \subseteq \Omega \\ &\iff f \in \mathrm{B}(K, \Omega) \end{aligned}$$

Thus the sets of the form $\mathrm{B}(K, \mathrm{B}(L, W))$ serve as a basis for $C(X, C(Y, Z))$. Another natural space to consider is $C(X \times Y, Z)$. The basic opens of this space have the form $\mathrm{B}(\Delta, W)$ where $\Delta \subseteq X \times Y$ is compact. Note that Δ has the form $\Delta = K \times L$ where $K \subseteq X$ and $L \subseteq Y$ are compact (namely $K = \pi_1(X \times Y)$ and $L = \pi_2(X \times Y)$). So the sets of the form $\mathrm{B}(K \times L, W)$ serve as a basis for $C(X \times Y, Z)$.

Proposition 2.1. *Define a map $(-)^\diamond \colon C(X, C(Y, Z)) \to C(X \times Y, Z)$ by*

$$f^\diamond(x, y) := (f(x))(y) \tag{9}$$

*where $f \in C(X, C(Y, Z))$, where $x \in X$, and $y \in Y$. If X, Y, and Z are Hausdorff and locally compact, then Φ is a homeomorphism. In this case, we can write $(Z^Y)^X = Z^{X \times Y}$; this formula provides a justification for the notation Y^X and is called the **exponential law**.*

Note that in (9), f is a continuous function from X to $C(Y, Z)$, so it takes an element x and spits out another continuous function $f(x) \colon Y \to Z$, which takes an element y and spits out and element $(f(x))(y)$. We can express this whole process by simply writing $(f(x))(y)$ where the parethenthesis gives us context of what is what; for instance, the fact that we have a parenthesis surrounding $f(x)$ in $(f(x))(y)$ should tell you that $f(x)$ is a function of y. Before we give a proof of this proposition, we want to point out that $(-)_\diamond$ is already a bijection as a set-theoretic function, with inverse $(-)_\diamond \colon C(X \times Y, Z) \to C(X, C(Y, Z))$ defined by

$$(g_\diamond(x))(y) := g(x, y) \tag{10}$$

where $g \in C(X \times Y, Z)$, where $x \in X$, and where $y \in Y$. Again, the parenthesis in (10) should tell us how to interpret this equation. One should think of the maps $(-)^\diamond$ and $(-)_\diamond$ as applying some sort of associative law. To see this, we first simplify our notation and write (9) and (10) as

$$f^\diamond(x, y) := (fx)y \quad \text{and} \quad (g_\diamond x)y := g(x, y) \tag{11}$$

instead (this is similar to the notation used in linear analysis where we typically write Tx instead of $T(x)$ for a linear map T and a vector x). Intuitively, one thinks of $f^\diamond(x, y) = (fx)y$ as applying the "associative law" where the diamond in the superscript tells us that we can "pull back" the parenthesis. Similarly, one thinks of $(g_\diamond x)y = g(x, y)$ as applying the "associative law" where the diamond in the subscript tells us that we can "push forward" the parenthesis. With this notational simplification in mind, it is very easy to see why $(-)^\diamond$ and $(-)_\diamond$ are inverse to each other as set-theoretic functions: we are just applying the "associative law": we have

$$((f^\diamond)_\diamond x)y = f^\diamond(x, y) = f(x)y \quad \text{and} \quad (g_\diamond)^\diamond(x, y) = (g_\diamond x)y = g(x, y) \tag{12}$$

where context makes it clear how to interpret all of the symbols in (12) In particular, one should note that the reason why $(-)_\diamond$ and $(-)^\diamond$ are inverse to each other is precisely due to the way we defined them in the first place. Another added benefit that we get when using this notation is that when we write an interpretable string using the symbols $\{\diamond, (,), f, g, h, x, y, z\}$, then it becomes visibly clear how we could interpret this string, where we consider a string interpretable if we can obtain a new string without any diamond symbols by applying the associative law a finite number of times to the original string. For instance, the string $f_\diamond(x, y)$ is uninterpretable in our language since we can't "pullback" the parenethesis and remove the diamond in the subscript. On the other had, the string $h^\diamond(gx, (f_\diamond x)y)$ is interpretable: if we apply the "associative law" one time, we can remove the subscript diamond and obtain $h^\diamond(gx, f(x, y))$. If we apply the associative law again, we can remove the superscript diamond and obtain $(h(gx))f(x, y)$. Since this string doesn't contain any diamonds, we can give a reasonable interpretation to it. For instance, h can be thought of as a function in $C(A_1, C(A_2, A_3))$, with maps the element $gx \in A_1$ to the function $h(gx) \in C(A_2, A_3)$ whose value at $f(x, y) \in A_2$ is $(h(gx))f(x, y)$. Let us now prove Proposition (2.1).

Proof. It suffices to show that both $(-)^\diamond$ and $(-)_\diamond$ are continuous. Observe that

$$\begin{aligned} g_\diamond \in \mathrm{B}(K, \mathrm{B}(L,W)) &\iff g_\diamond K \subseteq \mathrm{B}(L,W) \\ &\iff g_\diamond x \in \mathrm{B}(L,W) \text{ for all } x \in K \\ &\iff (g_\diamond x)y \in W \text{ for all } y \in L \text{ for all } x \in K \\ &\iff g(x,y) \in W \text{ for all } y \in L \text{ for all } x \in K \\ &\iff g(K \times L) \in W \\ &\iff g \in \mathrm{B}(K \times L, W). \end{aligned}$$

□

3 Homotopy

Let X and Y be topological spaces and let $f, g \colon X \to Y$ be continuous maps. We say that f and g are **homotopic**, written $f \sim g$, if there exists a continuous function $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. The map H is called a **homotopy** joining f and g. We say X and Y are **homotopic** if there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. The equivalence classes for the homotopy relation in $C(X,Y)$ are called **homotopy classes**. The set of homotopy classes in $C(X,Y)$ is denoted by $\pi(X,Y)$.

Lemma 3.1. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be continuous functions which are homotopic to $f' \colon X \to Y$ and $g' \colon Y \to Z$ respectively (denoted $f \sim f'$ and $g \sim g'$). Then $gf \sim g'f'$ (where $gf = g \circ f$ and $g'f' = g' \circ f'$ denotes composition).*

Proof. Let $F \colon X \times I \to Y$ be a homotopy from f to f' and let $G \colon Y \times I \to Z$ be a homotopy from g to g'. Thus

$$\begin{aligned} F(x,0) &= f(x) \\ F(x,1) &= f'(x) \\ G(y,0) &= g(y) \\ G(y,1) &= g'(y) \end{aligned}$$

Define $H \colon X \times I \to Z$ by $H(x,t) = G(F(x,t),t)$. We can think of H as the composite map $X \times I \to Y \times I \to Z$ where the map $X \times I \to Y \times I$ sending (x,t) to $(F(x,t),t)$ is continuous since each component function is continuous and where the map $Y \times I \to Z$ sending (y,t) to $G(y,t)$ is continuous since G is a homotopy. Therefore, H is a continuous map. Furthermore it is straightforward to check that $H(-,0) = gf$ and $H(-,1) = g'f'$. Thus H is a homotopy from gf to $g'f'$, that is, $gf \sim g'f'$. □

Remark 2. Let $f_1, f_1' \colon X_1 \to X_2$, and $f_2, f_2' \colon X_2 \to X_3$, and $f_3, f_3' \colon X_3 \to X_4$ be continuous functions such that $f_1 \sim f_1'$, and $f_2 \sim f_2'$, and $f_3 \sim f_3'$. Write $f = f_3 f_2$ and $f' = f_3' f_2'$. By the lemma above, we have $f \sim f'$, which implies

$$\begin{aligned} f_3 f_2 f_1 &= (f_3 f_2) f_1 \\ &= f f_1 \\ &\sim f' f_1' \\ &= (f_3' f_2') f_1' \\ &= f_3' f_2' f_1'. \end{aligned}$$

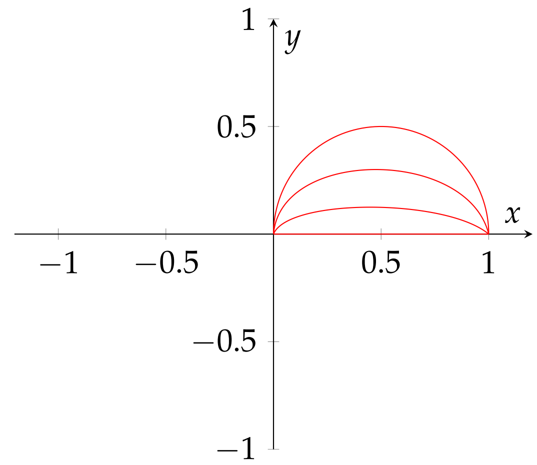
This shows that we may replace a function in a composite with a homotopic map without having to worry about associativy.

Example 3.1. Let $*$ denote the one-point space and let X be a topological space. Then $\pi(*, X) = \pi_0(X)$. Indeed, a continuous map from $*$ to X corresponds to a point $x \in X$. Then there is a homotopy between two points if and only if they are path-connected. More generally if Y is another topological space, then $\pi_0(C(X,Y)) = \pi(X,Y)$. Indeed, let $f, g : X \to Y$ and let $H : X \times I \to Y$ be a homotopy from f to g. Then the function $\widetilde{H} : I \to C(X,Y)$, given by $\widetilde{H}(s) = H(-,s)$ for all $s \in I$, is a continuous map, and thus represents a path from f to g. Similarly, if $G : I \to C(X,Y)$ is a path from f to g, then $\widehat{G} : X \times I \to Y$, given by $\widehat{G}(x,s) = G(s)(x)$ for all $(x,s) \in X \times I$, is a homotopy from f to g.

Example 3.2. Let $\gamma_1 : I \to \mathbb{R}^2$ and $\gamma_2 : I \to \mathbb{R}^2$ be continuous maps given by $\gamma_1(t) = (t,0)$ and $\gamma_2(t) = (1/2)(-\cos \pi t + 1, \sin \pi t)$ respectively for all $t \in I$. We claim that γ_1 and γ_2 are homotopic. Indeed, define $H : I \times I \to \mathbb{R}^2$ by

$$H(t,s) = (1-s)(t,0) + (s/2)(-\cos \pi t + 1, \sin \pi t)$$

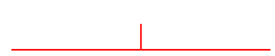
for all $(t,s) \in I \times I$. Then H is a continuous function since its component functions H_1 and H_2 are continuous[1]. Moreover, $H(t,0) = \gamma_1(t)$ and $H(t,1) = \gamma_2(t)$ for all $t \in I$. Therefore H is a homotopy joining γ_1 and γ_2. We can visualize it likeso:



On the other hand, if we replace $\mathbb{R}^2$ with $\mathbb{R}^2 \backslash \{(1/8, 1/4)\}$, then this homotopy no longer works since $H(1/2, 1/2) = (1/8, 1/4)$ (there is a "hole" preventing H from being defined at $(1/2, 1/2)$).

Remark 3. More generally, all continuous maps from an abitrary space X to any convex subset A of $\mathbb{R}^n$ are homotopic to each other: A homotopy $H : X \times I \to A$ joining continuous maps $f, g : X \to A$ is defined by the formula $H(x,s) = (1-s)f(x) + sg(x)$ for all $(x,s) \in X \times I$. Since A is convex, H is defined everywhere in $X \times I$. Moreover, it is continuous with $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Example 3.3. Let $X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times I) \subseteq \mathbb{R}^2$ and $Y = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. We illustarte X and Y below respectively:



We claim that X and Y are not homeomorphic. Indeed, assume (to obtain a contradiction) that $f : X \to Y$ is a homeomorphism. Then f induces a homeomorphism $f|_{X \backslash \{(0,0)\}} : X \backslash \{(0,0)\} \to Y \backslash \{f(0,0)\}$. However, $\pi_0(f|_{X \backslash \{(0,0)\}})$ *cannot* be a bijection since $|\pi_0(X \backslash \{(0,0)\})| = 3$ and $|\pi_0(Y \backslash \{f(0,0)\})| = 2$. This contradicts the fact that π_0 is a topological invariant, and hence our claim is proved.

On the other hand, X and Y are homotopic: Let $\iota : Y \to X$ be the inclusion map, given by $\iota(x,y) = (x,y)$ for all $(x,y) \in Y$, and let $\rho : X \to Y$ be the projection map, given by $\rho(x,y) = (x,0)$ for all $(x,y) \in X$. Then $\rho \circ \iota \sim \mathrm{id}_Y$ because $\rho \circ \iota = \mathrm{id}_Y$, and $\iota \circ \rho \sim \mathrm{id}_Y$ because the function $H : X \times I \to Y$, given by $H((x,y),s) = (x, sy)$ for all $((x,y),s) \in X \times I$, is a homotopy from $f \circ g$ to id_Y.

Theorem 3.2. *If X and Y are homotopic, then $\pi_0(X)$ and $\pi_0(Y)$ are bijective.*

Remark 4. This theorem tells us that π_0 is a homotopy invariant.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be continuous functions such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. Let H_1 be a homotopy joining $f \circ g$ and id_Y and let H_2 be a homotopy joining $g \circ f$ and id_X. We claim that the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection. To prove this claim, we first show that it is surjective. Let $[y] \in Y$. We need to find an $[x] \in X$ such that $[f(x)] = [y]$, or in other words, we need to find an $x \in X$ such that $f(x) \sim y$. We claim that $x = g(y)$ works. Indeed, we just need to show that $f(g(y)) \sim y$. But this is clear, since $H_1(y, -)$ is a path from $f(g(y))$ to y. By the same argument, $\pi_0(g) : \pi_0(Y) \to \pi_0(X)$ is surjective as well. This implies that $\pi_0(X)$ and $\pi_0(Y)$ are bijective. $\square$

Remark 5. Suppose that $f : X \to Y$ is a homeomorphism. Then it's very easy to see that $\pi_0(f)$ is surjective. Indeed, if $[y] \in \pi_0(Y)$, then we use the fact that f is surjective and choose an x such that $f(x) = y$. Then $\pi_0(f)$ maps $[x]$ to $[f(x)] = [y]$. The point here is that we did not use path-connectedness in this argument. We only used the fact that $f : X \to Y$ is surjective. In the proof above, we did use path-connectedness: we found an $x \in X$ such that $f(x)$ is homotopic to y, and not necessarily equal to y.

[1] Let X be a topological space and let $p_i : \mathbb{R}^n \to \mathbb{R}$ be the map given by projecting to the ith coordinate. A function $f : X \to \mathbb{R}^n$ is continuous if and only if its component functions $f_i := f \circ p_i$ are continuous.

3.1 Fundamental Group

Let X be a topological space. A **path** on X is a continuous function $\alpha : I \to X$. We call $\alpha(0)$ the **source** of α and $\alpha(1)$ the **target** of α. The **constant path** $c_x \colon I \to X$ at a point x in X is given by $c_x(t) = x$ for all $t \in [0,1]$. If $\alpha \colon I \to X$ is a path on X, then its **inverse** α^- is a path $\alpha^- \colon I \to X$ given by $\alpha^-(t) = \alpha(1-t)$ for all $t \in I$. A **loop** is a path such that the source equals the target. If $\alpha \colon I \to X$ is a path such that $\alpha(0) = x = \alpha(1)$, then we will say that α is a **loop based at** x.

Let $x \in X$. Define $\mathcal{L}(X,x)$ to be the set of loops based at x. There is a natural binary operation $\star$ on $\mathcal{L}(X,x)$ given as follows: For α and β in $\mathcal{L}(X,x)$, we define the loops $\beta \star \alpha \colon I \to X$ based at x by setting

$$(\beta \star \alpha)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta\left(2\left(t - \frac{1}{2}\right)\right) & \frac{1}{2} < t \leq 1. \end{cases}$$

for all $t \in I$. The idea is that we traverse around α and then β all in one day. The $2t$ comes from the fact that we need to traverse α twice as fast. The $t - \frac{1}{2}$ comes from the fact that we need to wait half a day before we start traversing β. Finally, the $2\left(t - \frac{1}{2}\right)$ comes from the fact that we need to traverse β twice as fast.

The binary operation we defined on $\mathcal{L}(X,x)$ is not a very nice one (it isn't associative, has no identity, no inverses, etc...). To remedy this situation, we consider paths up to homotopy. Let $\mathcal{L}(X,x)/\sim$ denote the set of homotopy classes of loops based at x. We claim that $(\mathcal{L}(X,x)/\sim, \star)$ is a group.

First we need to show that the binary operation $\star$ passes to the quotient $\mathcal{L}(X,x)$: suppose $\alpha \sim \alpha'$ and $\beta \sim \beta'$, with H_α and H_β being their homotopies respectively. Let $H \colon I \times I \to X$ be given by

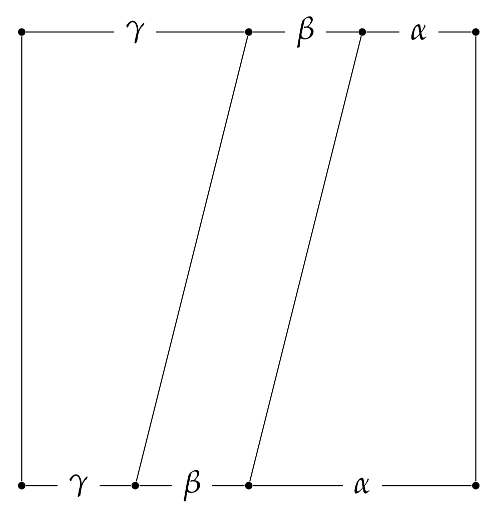
$$H(t,s) = \begin{cases} H_\alpha(2t,s) & 0 \leq t \leq \frac{1}{2} \\ H_\beta(2t-1,s) & \frac{1}{2} < t \leq 1. \end{cases}$$

Then H is easily seen to be a homotopy from $\beta \star \alpha$ to $\beta' \star \alpha'$.

Next we show that $(\mathcal{L}(X,x)/\sim, \star)$ is associative. Suppose α, β, and γ are loops based at x. Define $H \colon I \times I \to X$ by

$$H(t,s) = \begin{cases} \gamma\left(\frac{4}{s+1} \cdot t\right) & 0 \leq t \leq \frac{s+1}{4} \\ \beta\left(4 \cdot \left(t - \left(\frac{s+1}{4}\right)\right)\right) & \frac{s+1}{4} < t \leq \frac{s+2}{4} \\ \alpha\left(\frac{4}{2-s} \cdot \left(t - \left(\frac{s+2}{4}\right)\right)\right) & \frac{s+2}{4} < t \leq 1 \end{cases}$$

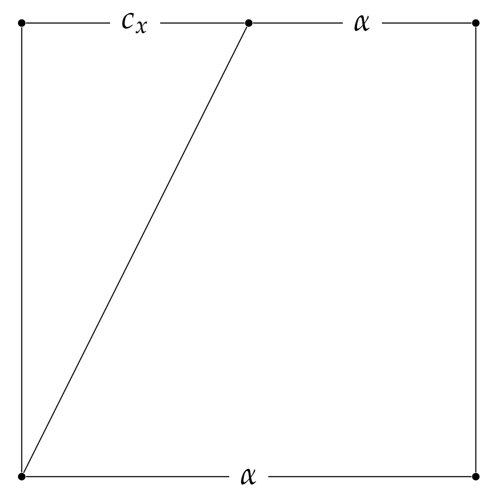
for all $(t,s) \in I \times I$. Then H is easily seen to be a homotopy from $\alpha \star (\beta \star \gamma)$ to $(\alpha \star \beta) \star \gamma$. One may visualize this homotopy as below:



Next, we want to show that c_x represents the identity element in $(\mathcal{L}(X,x)/\sim, \star)$. Suppose α is a loop based at x. Define $H \colon I \times I \to X$ by

$$H(t,s) = \begin{cases} c_x\left(\frac{2}{s} \cdot t\right) & 0 \leq t < \frac{s}{2} \\ \alpha\left(\frac{2}{2-s} \cdot \left(t - \frac{s}{2}\right)\right) & \frac{s}{2} \leq t \leq 1 \end{cases}$$

for all $(t,s) \in I \times I$. Then H is easily seen to be a homotopy from α to $\alpha \star c_x$. One may visualize this homotopy as below:

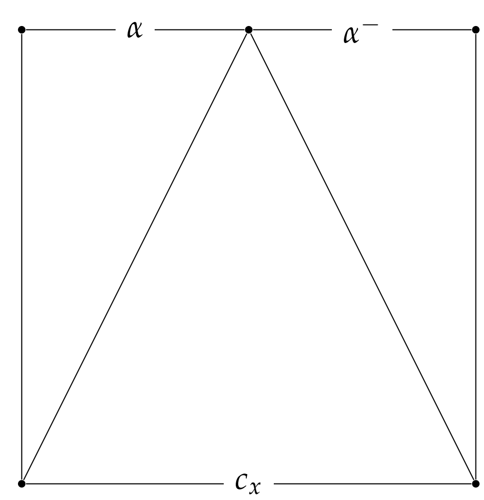


A similary argument gives a homotopy from α to $c_x \star \alpha$.

Finally, we want to show that $(\mathcal{L}(X,x)/\sim, \star)$ has inverses. Suppose α is a loop based at x. Define $H\colon I \times I \to X$ by

$$H(t,s) = \begin{cases} \alpha\left(\frac{2}{s} \cdot t\right) & 0 \le t < \frac{s}{2} \\ c_x\left(\frac{1}{1-s} \cdot \left(t - \frac{s}{2}\right)\right) & \frac{s}{2} \le t \le \frac{2-s}{2} \\ \alpha^-\left(\frac{2}{s} \cdot \left(t - \frac{2-s}{2}\right)\right) & \frac{2-s}{2} < t \le 1 \end{cases}$$

for all $(t,s) \in I \times I$. Then H is easily seen to be a homotopy from c_x to $\alpha \star \alpha^-$. One may visualize this homotopy as below:



A similary argument gives a homotopy from c_x to $\alpha^- \star \alpha$.

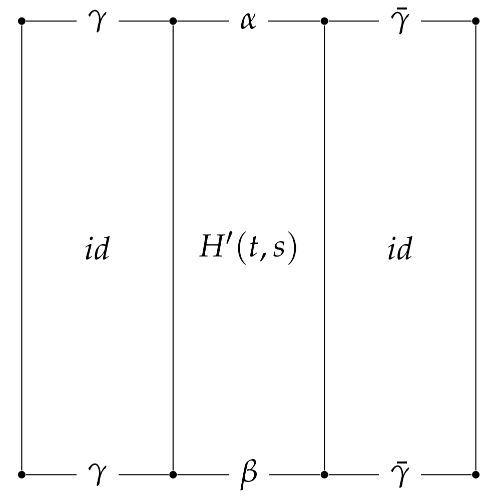
We have shown that $(\mathcal{L}(X,x)/\sim, \star)$ forms a group. It is called the **fundamental group** of X based at x, and usually denoted as $\pi_1(X,x)$. What happens if you change the base point x to another base point y?

Proposition 3.1. *If X is path connected, then $\pi_1(X,x) \cong \pi_1(X,y)$.*

Proof. Pick a path γ whose source is x and whose target is y. Define $\Phi : \pi_1(X,y) \to \pi_1(X,x)$ by $\Phi[\alpha] = [\bar{\gamma} \star \alpha \star \gamma]$. We need to show this is well-defined. So choose another representative $\beta \sim \alpha$. Then there is a continuous map $H' : [0,1] \times [0,1] \to X$ such that $H'(t,0) = \beta(t)$ and $H'(t,1) = \alpha(t)$. We define a homotopy H from $\bar{\gamma} \star \alpha \star \gamma$ to $\bar{\gamma} \star \beta \star \gamma$ as follows

$$H(t,s) = \begin{cases} \gamma(3t) & 0 \le t < \frac{1}{3} \\ H(t,s) & \frac{1}{3} \le t < \frac{2}{3} \\ \bar{\gamma}\left(3\left(t - \frac{2}{3}\right)\right) & \frac{2}{3} \le t \le 1 \end{cases}$$

Illustration:



So $\bar{\gamma} \star \alpha \star \gamma \sim \bar{\gamma} \star \beta \star \gamma$, hence Φ is well-defined. Next we show Φ is a group homomorphism. Suppose $\alpha, \beta \in \pi_1(X, y)$. Then

$$\begin{aligned}\Phi[\alpha] \star \Phi[\beta] &= (\bar{\gamma} \star \alpha \star \gamma) \star (\bar{\gamma} \star \beta \star \gamma)\\ &\sim \bar{\gamma} \star (\alpha \star \beta) \star \gamma\\ &= \Phi[\alpha \star \beta].\end{aligned}$$

To show that Φ is an isomorphism, we describe an inverse $\bar{\Phi}$ to Φ: Define $\bar{\Phi} : \pi_1(X, x) \to \pi_1(X, y)$ by $\Phi[\alpha] = [\gamma \star \alpha \star \bar{\gamma}]$. This is a group homomorphism by the same reasoning as above. It is also a right inverse to Φ, since

$$\begin{aligned}\Phi[\bar{\Phi}[\alpha]] &= \Phi[\gamma \star \alpha \star \bar{\gamma}]\\ &= \bar{\gamma} \star (\gamma \star \alpha \star \bar{\gamma}) \star \gamma\\ &\sim \alpha.\end{aligned}$$

for all $\alpha \in \pi_1(X, x)$. For similar reasons, $\bar{\Phi}$ is also a left inverse to Φ. Therefore $\bar{\Phi}$ and Φ are inverses, hence $\pi_1(X, x) \cong \pi_1(X, y)$. □

Proposition (3.1) tells us that whenever X is path-connected, we can simplify the notation $\pi_1(X, x)$ to $\pi_1(X)$.

4 CW-Complexes

Definition 4.1. An n**-cell** is a space homeomorphic to the open n-disk $\operatorname{int} D^n$. A **cell** is a space which is an n-cell for some $n \geq 0$. Note that $\operatorname{int}(D^m)$ and $\operatorname{int}(D^n)$ are homeomorphic if and only if $m = n$. Thus it makes sense to talk about the **dimension** of a cell. An n-cell is said to have dimension n.

Definition 4.2. A **cell-decomposition** of X is a collection $\mathcal{E}$ of cells of X whose disjoint union is X:

$$X = \coprod_{e \in \mathcal{E}} e.$$

We write $X = (X, \mathcal{E})$ when we want to think of X being equipped with a cell-decomposition $\mathcal{E}$ of X. In this case, we define the n**-skeleton** of X to be the subspace

$$X^n = \coprod_{e \in \mathcal{E}^n} e$$

where $\mathcal{E}^n$ is the set of all n-cells in $\mathcal{E}$.

Note that if $\mathcal{E}$ is a cell-decomposition of a space X, then the cells of $\mathcal{E}$ can have many different dimensions. For example, one cell-decomposition of S^1 is given by $\mathcal{E} = \{e_a, e_b\}$ where e_a is an arbitrary point $p \in S^1$ and $e_b = S^1 \backslash \{p\}$. Here e_a is a 0-cell and e_b is a 1-cell. There are no restrictions on the number of cells in a cell-decomposition. Thus we can have uncountable many cells in such a decomposition. For instance, any space X has a cell-decomposition where each point of X is a 0-cell. A finite cell-decomposition is a cell decomposition consisting of finitely many cells.

Definition 4.3. A pair $(X, \mathcal{E})$ consisting of a Hausdorff space X and a cell-decomposition $\mathcal{E}$ of X is called a **CW-complex** if the following three axioms hold:

1. (characteristic maps) For each n-cell $e \in \mathcal{E}$ there is a map $\Phi_e \colon D^n \to X$ such that $\Phi_e|_{\operatorname{int} D^n} \colon \operatorname{int} D^n \to e$ is a homeomorphism and such that $\Phi_e|_{S^{n-1}}$ lands in X^{n-1}.
2. (closure finiteness) For any cell $e \in \mathcal{E}$ the closure $\bar{e}$ intersects only a finite number of other cells in $\mathcal{E}$.
3. (weak topology) A subset $A \subseteq X$ is closed if and only if $A \cap \bar{e}$ is closed in X for each $e \in \mathcal{E}$.

Remark 6. Note that axioms 2 and 3 are only needed in case $\mathcal{E}$ is infinite.

Lemma 4.1. *Let $(X, \mathcal{E})$ be a Hausdorff space X together with a cell-decomposition $\mathcal{E}$. If $(X, \mathcal{E})$ satisfies axiom 1 in Definition (4.3), then we have $\bar{e} = \Phi_e(D^n)$ for any cell $e \in \mathcal{E}$. In particular, $\bar{e}$ is a compact subspace of X and the cell boundary $\bar{e} \backslash e = \Phi_e(S^{n-1})$ lies in X^{n-1}.*

Proof. Observe that $\Phi_e(D^n) \subseteq \overline{\Phi_e(\operatorname{int} D^n)} = \bar{e}$. Since $\Phi_e(D^n)$ is compact, it is closed in X (since X is Hausdorff), thus $\Phi_e(D^n) = \bar{e}$. By axiom 1, we have $\Phi_e(\operatorname{int} D^n) = e$ and $\Phi_e(S^{n-1}) \cap e = \emptyset$, thus $\Phi_e(S^{n-1}) = \bar{e} \backslash e$. $\square$

Example 4.1. (A CW decompositions of S^n) For a finite n, there are two canonical CW decompositions of the sphere S^n. The first consists of two cells: a point e_1 (for example $(1, 0, \ldots, 0)$) and the set $e_2 = S^n \backslash e^0$. A characteristic map $f \colon D^n \to S^n$ for e_2 can be chosen like the usual making a sphere from a ball by gluing all points of the boundary sphere into one point:

$$f(\boldsymbol{a}) = \left(-\cos(\|\boldsymbol{a}\|\pi), \frac{a_1}{\|\boldsymbol{a}\|} \sin(\|\boldsymbol{a}\|\pi), \ldots, \frac{a_n}{\|\boldsymbol{a}\|} \sin(\|\boldsymbol{a}\|\pi), \right)$$

where $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\|\boldsymbol{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$. Notice that

$$\begin{aligned} \|f(\boldsymbol{a})\|^2 &= \cos^2(\|\boldsymbol{a}\|\pi) + \left(\frac{a_1^2 + \cdots + a_n^2}{\|\boldsymbol{a}\|^2} \right) \sin^2(\|\boldsymbol{a}\|\pi) \\ &= \cos^2(\|\boldsymbol{a}\|\pi) + \sin^2(\|\boldsymbol{a}\|\pi) \\ &= 1. \end{aligned}$$

Also notice that if $\|\boldsymbol{a}\| = 1$, then $f(\boldsymbol{a}) = (1, 0, \ldots, 0) = e_1$. The other classical CW decomposition of S^n consists of $2n + 2$ cells $e^0_\pm, \ldots, e^n_\pm$ where

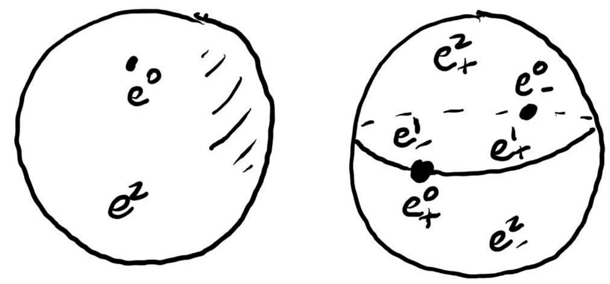
$$e^k_\pm = \{\boldsymbol{a} \in S^n \mid a_{k+2} = \cdots = a_{k+1} = 0 \text{ and } \pm a_{k+1} > 0\}.$$

Here we do not need to care about characteristic maps: Closures of all cells are obviously homeomorphic to balls. Nonetheless, we summarize the characteristic maps in the table below

Characteristic Map	Description
$f^2_+ : D^2 \to S^2$	$(a_1, a_2) \mapsto \left(a_1, a_2, \sqrt{1 - a_1^2 - a_2^2}\right)$
$f^2_- : D^2 \to S^2$	$(a_1, a_2) \mapsto \left(a_1, a_2, -\sqrt{1 - a_1^2 - a_2^2}\right)$
$f^1_+ : D^1 \to S^2$	$a \mapsto \left(\sqrt{1 - a^2}, a, 0\right)$
$f^1_- : D^1 \to S^2$	$a \mapsto \left(-\sqrt{1 - a^2}, a, 0\right)$
$f^0_+ : D^0 \to S^2$	$1 \mapsto (0, 1, 0)$
$f^0_- : D^0 \to S^2$	$1 \mapsto (0, -1, 0)$

Observe that the restriction of f^2_+ to S^1 given by the union of e^1_+ and e^1_-.

Notice that both CW decompositions described above are obtained from the only possible cellular decomposition of S^0 (the two-point space) by the canonical cellular version of suspension. In the first case, we use the base point version of suspension, and in the second case we take the usual suspension.



Proposition 4.1. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Then $\overline{f(U)} \supset f(\overline{U})$ for all open sets $U \subset X$. Moreover, if f maps closed sets to closed sets, then we have equality $\overline{f(U)} = f(\overline{U})$.*

Proof. Let U be an open set in X. First we show that $\overline{f(U)} \supset f(\overline{U})$. To show this, we just need to show that if E is any closed set containing $f(U)$, then E also contains $f(\overline{U})$. So let E be a closed set which contains $f(U)$. Then $f^{-1}(E)$ is a closed set which contains U. Hence $f^{-1}(E)$ contains $\overline{U}$, and thus E contains $\overline{U}$. This is what we wanted to show.

Now assume that f maps closed sets to closed set. We want to show that $\overline{f(U)} \subset f(\overline{U})$. Since $\overline{U}$ is closed, $f(\overline{U})$ is closed. Therefore, since $f(\overline{U})$ is closed and contains $f(U)$, it must also contain $\overline{f(U)}$. □

Summary Table
$e_i^q = f_i^q\left(\mathrm{Int}\left(D^q\right)\right)$
$\cup_{j \in J} e_j^{q-1} = \dot{e}_i^q = f_i^q\left(S^{q-1}\right)$
$\overline{e}_i^q \supset f_i^q\left(D^q\right)$

Example 4.2. (Projective Spaces) The identification of the antipodal points of the sphere S^n glues together the cells e^q_+, e^q_- of the above described CW decomposition of S^n into $2n+2$ cells This gives a decomposition of $\mathbb{RP}^n$ into $n+1$ cells e^q, one in every dimension from 0 to n. The other way of describing this CW decomposition of $\mathbb{RP}^n$ is provided by the formula

$$e^q = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{RP}^n \mid a_q \neq 0 \text{ and } a_{q+1} = \cdots = a_n = 0\}.$$

One more description is provided by the chain of inclusions

$$\emptyset = \mathbb{RP}^{-1} \subset \mathbb{RP}^0 \subset \mathbb{RP}^1 \subset \cdots \subset \mathbb{RP}^n :$$

We set $e^q = \mathbb{RP}^q \backslash \mathbb{R}^{q-1}$. A characteristic map for e^q may be chosen as the composition of the canonical projection $D^q \to \mathbb{RP}^q$ and the inclusion $\mathbb{RP}^q \hookrightarrow \mathbb{RP}^n$. For $n = \infty$, this construction provides a CW decomposition of $\mathbb{RP}^\infty$ with one cell in every dimension. For example, in $\mathbb{RP}^2$,

$$e^2 = \{(a_0 : a_1 : 1) \in \mathbb{RP}^2\} \qquad e^1 = \{(a_0 : 1 : 0) \in \mathbb{RP}^2\} \qquad \text{and} \qquad e^0 = \{(1 : 0 : 0) \in \mathbb{RP}^2\}$$

The construction also has complex, quaternionic, and Cayley analogs. In the complex case, we get a CW decomposition of $\mathbb{CP}^n$ into $n+1$ cells $e^0, e^2, e^4, \ldots, e^{2n}$ and also a CW decomposition of $\mathbb{CP}^\infty$ with one cell of every even dimension. For example, in $\mathbb{CP}^2$,

$$e^4 = \{(x_0 + iy_0 : x_1 + iy_1 : 1) \in \mathbb{CP}^2\} \qquad e^2 = \{(x_0 + iy_0 : 1 : 0) \in \mathbb{CP}^2\} \qquad \text{and} \qquad e^0 = \{(1 : 0 : 0) \in \mathbb{CP}^2\}$$

In the quaternionic case, we get a CW decomposition of $\mathbb{HP}^n$ into $n+1$ cells $e^0, e^4, e^8, \ldots, e^{4n}$ and also a CW decomposition of $\mathbb{HP}^\infty$ with one cell of every dimension divisible by 4. For the Cayley projective plane $\mathbb{C}a\mathbb{P}^2$, we get a CW decomposition into cells of dimensions $0, 8$, and 16.

4.1 Cellular Homology

Let $X = (X, \mathcal{E})$ be a CW-complex. The short exact sequence of complexes

$$0 \longrightarrow C(X^n) \longrightarrow C(X^{n+1}) \longrightarrow C(X^{n+1})/C(X^n) \longrightarrow 0 \tag{13}$$

induces a long exact sequence in homology modules:

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{i+1}(X^n, X^{n-1}) \arrow[dll, rounded corners] \\ \mathrm{H}_i(X^{n-1}) \arrow[r] & \mathrm{H}_i(X^n) \arrow[r] & \mathrm{H}_i(X^n, X^{n-1}) \arrow[dll, rounded corners] \\ \mathrm{H}_{i-1}(X^{n-1}) \arrow[r] & \cdots & \end{tikzcd} \tag{14}$$

Recall that $\mathrm{H}_i(X^n, X^{n-1}) = 0$ for all $i \neq n$ and $\mathrm{H}_n(X^n, X^{n-1})$ is free with basis in one-to-one correspondence with the set of n-cells $\mathcal{E}^n$ of X.

$$0 \to \mathrm{H}_n(X^n) \xrightarrow{j} \mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\iota} \mathrm{H}_{n-1}(X) \to 0$$

Therefore

$$0 \to \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{j} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\partial} \mathrm{H}_{n-2}(X^{n-2})$$

We set $\mathrm{d} = \partial \circ \pi$ and we set $E(X)$ to be the graded module whose graded component in degree n is

$$S_n(X) = \mathrm{H}_n(X^n, X^{n-1}).$$

We claim that $(S(X), \mathrm{d})$ is a complex. Indeed, it suffices to show that $\mathrm{d}^2 = 0$. Let $e_i^n \in S_n(X)$ where e_i^n is an n-cell of X. Then we have

$$\begin{aligned} \mathrm{d}^2(e_i^n) &= \pi\partial\pi\partial \\ &= \end{aligned}$$

$$0 \to \mathrm{H}_n(X^{n-1}) \to \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial} \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^n) \to 0$$

$$0 \to \mathrm{H}_{n-1}(X^{n-2}) \to \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\partial} \mathrm{H}_{n-2}(X^{n-2}) \to \mathrm{H}_{n-2}(X^{n-1}) \to 0$$

$$0 \to \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial} \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^n) \to 0$$

$$0 \to \mathrm{H}_{n-1}(X^{n-2}) \to \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{\partial} \mathrm{H}_{n-2}(X^{n-2}) \to \mathrm{H}_{n-2}(X^{n-1}) \to 0$$

We can
The map $\partial \colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1})$ is given by $[e^n] \mapsto [\partial(e^n)]$ for all n-cells e^n of X.

4.2 Cellular Boundary Formula

Let X be a CW complex. We write $\mathrm{H}_n(X^n, X^{n-1}) = \langle$

5 Mod Two Homology and Cohomology

5.1 Simplicial Complexes

Definition 5.1. A **simplicial complex** K consists of

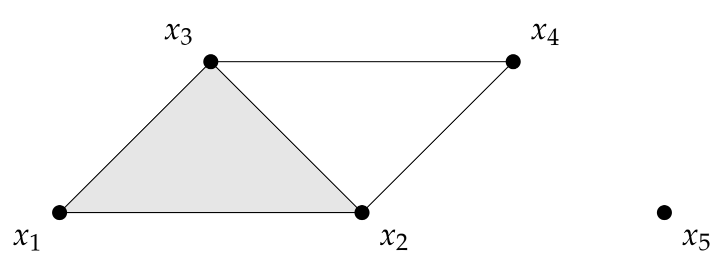
- A set $V(K)$, the set of **vertices** of K.
- A set $\mathcal{S}(K)$ of finite nonempty subsets of $V(K)$ which is closed under containment: if $\sigma \in S(K)$ and $\sigma \supset \tau$, then $\tau \in S(K)$. We require that $\{v\} \in S(K)$ for all $v \in V(K)$.

An element σ of $S(K)$ is called a **simplex** of K. If $|\sigma| = m+1$, we say that σ is of **dimension** m or that σ is an m-simplex. The set of m-simplexes of K is denoted $S_m(K)$. The set $S_0(K)$ of 0-simplexes is in bijection with $V(K)$, and we usually identify $v \in V(K)$ with $\{v\} \in S_0(K)$. We say that K is of **dimension** $\leq n$ if $S_m(K) = \emptyset$ for $m > n$, and that K is of **dimension** n or (n-dimensional) if it is of dimension $\leq n$ but not of dimension $\leq n-1$. A simplicial complex of dimension ≤ 1 is called a **simplicial graph**. A simplicial complex K is called **finite** if $V(K)$ is a finite set.

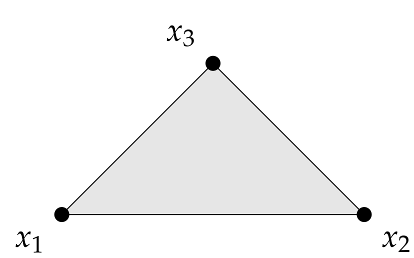
If $\sigma \in S(K)$ and $\tau \subset \sigma$, we say that τ is a **face** of σ. As $S(K)$ is closed under inclusion, it is determined by its subset $S(K)_{\max}$ of **maximal** simplexes (if K is finite dimensional). A **subcomplex** L of K is a simplicial complex such that $V(L) \subset V(K)$ and $S(L) \subset S(K)$. If $U \subset S(K)$, we denote by $\overline{U}$ the subcomplex generated by U, i.e. the smallest subcomplex of K such that $U \subset S(\overline{U})$. The m**-skeleton** K^m of K is the subcomplex of K generated by the union of $\mathcal{S}_k(K)$ for $k \leq m$.

Let $\sigma \in \mathcal{S}(K)$. We denote by $\overline{\sigma}$ (or $\mathcal{K}_\sigma$) the subcomplex of $\mathcal{K}$ formed by σ and all its faces. The subcomplex $\dot{\sigma}$ of $\overline{\sigma}$ generated by the proper faces of σ is called the **boundary** of σ.

Example 5.1. Let $\mathcal{K}$ be the simplical complex with $V(\mathcal{K}) = \{x_1, x_2, x_3, x_4, x_5\}$ and $S(\mathcal{K})_{\max} = \{x_1x_2x_3, x_2x_4, x_3x_4, x_5\}$, where we use the monomial notation $x_{i_1}x_{i_2}\cdots x_{i_k}$ to mean $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. We may visualize $\mathcal{K}$ as



The subcomplex $\mathcal{K}_{x_1x_2x_3}$ of $\mathcal{K}$ can be visualized as



5.2 Geometric Realization

Let $\mathcal{K}$ be a simplicial complex. The **geometric realization** $|\mathcal{K}|$ of $\mathcal{K}$ is, as a set, defined by

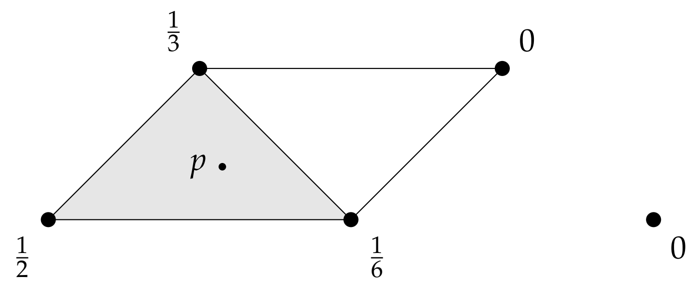
$$|\mathcal{K}| := \left\{ p : V(\mathcal{K}) \to [0,1] \mid \sum_{x \in V(\mathcal{K})} p(x) = 1 \text{ and } p^{-1}((0,1]) \in S(\mathcal{K}) \right\}$$

The condition $p^{-1}((0,1]) \in S(\mathcal{K})$ says that the set of all $x \in \mathcal{V}(K)$ such that $p(x) \neq 0$ must form a simplex of K. There is a distance on $|K|$ defined by

$$d(p,q) = \sqrt{\sum_{x \in V(\mathcal{K})} (p(x) - q(x))^2},$$

which defined the metric topology on $|\mathcal{K}|$. The set $|\mathcal{K}|$ with the metric topology is denoted by $|\mathcal{K}|_d$. For instance, if $\sigma \in S_m(\mathcal{K})$, then $|\mathcal{K}_\sigma|_d$ is isometric to the standard Euclidean simplex $\Delta^m = \{(a_0, \ldots, a_m) \in \mathbb{R}^{m+1} \mid a_i \geq 0 \text{ and } \sum a_i = 1\}$.

Example 5.2. Let $\mathcal{K}$ be the simplical complex as in Example (5.4). We can visualize a function $p \in |\mathcal{K}|$ by attaching a number in $(0,1]$ to each vertex likeso:



We can actually think of p here as the vector $v = \frac{1}{2}e_1 + \frac{1}{6}e_2 + \frac{1}{3}e_3 \in \mathbb{R}^3$, where e_i denote the standard basis. The distance function then is just the normal euclidean distance function ($d(v,w) = \|v - w\|$).

A more used topology for $|\mathcal{K}|$ is the **weak topology**, for which $A \subset |\mathcal{K}|$ is closed if and only if $A \cap |\mathcal{K}_\sigma|_d$ is closed in $|\mathcal{K}_\sigma|_d$ for all $\sigma \in S(\mathcal{K})$. The notation $|\mathcal{K}|$ stands for the set $|\mathcal{K}|$ endowed with the weak topology. A map f from $|\mathcal{K}|$ to a topological space X is then continuous if and only if its restriction to $|\mathcal{K}_\sigma|_d$ is continuous for each $\sigma \in S(\mathcal{K})$. In particular, the identity $|\mathcal{K}| \to |\mathcal{K}|_d$ is continuous, which implies that $|\mathcal{K}|$ is Hausdorff. The weak and the metric topology coincide if and only if $\mathcal{K}$ is locally finite, that is, each vertex is contained in a finite number of simplexes. When $\mathcal{K}$ is not locally finite, $|\mathcal{K}|$ is not metrizable.

5.3 Simplicial Join, Stars, and Links

5.3.1 Simplicial Join

Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes. Their **join** is the simplicial complex $\mathcal{K} \star \mathcal{L}$ defined by

$$\begin{aligned} V(\mathcal{K} \star \mathcal{L}) &= V(\mathcal{K}) \uplus V(\mathcal{L}) \\ S(\mathcal{K} \star \mathcal{L}) &= S(\mathcal{K}) \cup S(\mathcal{L}) \cup \{\sigma \cup \tau \mid \sigma \in S(\mathcal{K}) \text{ and } \tau \in S(\mathcal{L})\}. \end{aligned}$$

5.3.2 Stars and Links

Let $\mathcal{K}$ be a simplicial complex and $\sigma \in S(\mathcal{K})$. The **star** $\mathbf{St}(\sigma)$ of σ is the subcomplex of $\mathcal{K}$ generated by all the simplexes containing σ. The **link** $\mathbf{Lk}(\sigma)$ of σ is the subcomplex of $\mathcal{K}$ formed by the simplexes $\tau \in S(\mathcal{K})$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in S(\mathcal{K})$. Thus, $\mathrm{Lk}(\sigma)$ is a subcomplex of $\mathrm{St}(\sigma)$ and

$$\mathrm{St}(\sigma) = \mathcal{K}_\sigma \star \mathrm{Lk}(\sigma).$$

Example 5.3. Let $\mathcal{K}$ be the simplical complex as in Example (5.4). Then

$$\begin{aligned} \mathrm{Lk}(x_1x_3)_{\max} &= \{x_2\} & \mathrm{St}(x_1x_3)_{\max} &= \{x_1x_2x_3\} \\ \mathrm{Lk}(x_1)_{\max} &= \{x_2x_3\} & \mathrm{St}(x_1)_{\max} &= \{x_1x_2x_3\} \\ \mathrm{Lk}(x_2)_{\max} &= \{x_1x_3, x_4\} & \mathrm{St}(x_2)_{\max} &= \{x_1x_2x_3, x_2x_4\} \\ \mathrm{Lk}(x_4)_{\max} &= \{x_2, x_3\} & \mathrm{St}(x_4)_{\max} &= \{x_3x_4, x_2x_4\} \\ \mathrm{Lk}(x_5)_{\max} &= \emptyset & \mathrm{St}(x_5)_{\max} &= \emptyset \end{aligned}$$

5.4 Simplicial Maps

Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes. A **simplicial map** $f : \mathcal{K} \to \mathcal{L}$ is a map $f : V(\mathcal{K}) \to V(\mathcal{L})$ such that the image of a simplex is a simplex:$\sigma \in S(\mathcal{K})$ implies $f(\sigma) \in S(\mathcal{L})$. Simplicial complexes and simplicial maps form a category, the **simplicial category**, denoted by **Simp**.

A simplicial map $f : \mathcal{K} \to \mathcal{L}$ induces a continuous map $|f| : |\mathcal{K}| \to |\mathcal{L}|$ defined, for $x \in V(\mathcal{L})$, by

$$|f|(p)(y) = \sum_{x \in f^{-1}(y)} p(x).$$

Example 5.4. Let $\mathcal{K}$ be the simplical complex as in Example (5.4), $\mathcal{L}$ be the simplical complex with $V(\mathcal{L}) = \{y_1, y_2, y_3\}$ and $S(\mathcal{L})_{\max} = \{y_1y_3, y_2\}$, and $\mathcal{M}$ be the simplicial complex with $V(\mathcal{M}) = \{z_1, z_2, z_3\}$ and $S(\mathcal{M})_{\max} = \{z_1z_2, z_1z_3, z_2z_3\}$. Then the maps $f : \mathcal{K} \to \mathcal{L}$ and $g : \mathcal{K} \to \mathcal{M}$ induced by

$$\begin{aligned} f(x_1) &= y_1 \\ f(x_2) &= y_3 \\ f(x_3) &= y_1 \\ f(x_4) &= y_3 \\ f(x_5) &= y_1 \end{aligned} \quad \text{and} \quad \begin{aligned} g(x_1) &= z_1 \\ g(x_2) &= z_2 \\ g(x_3) &= z_2 \\ g(x_4) &= z_3 \\ g(x_5) &= z_1 \end{aligned}$$

are simplicial maps.

- Triangulations

A **triangulation** of a topological space X is a homeomorphism $h : |\mathcal{K}| \to X$, where $\mathcal{K}$ is a simplicial complex. A topological space is **triangulable** if it admits a triangulation. A compact subspace A of $\mathbb{R}^n$ is a **convex cell** if it is the set of solutions of families of affine equations and inequalities

$$f_i = 0, \quad i = 1, \dots, r \quad \text{and} \quad g_j \geq 0, \quad j = 1, \dots s$$

A face B of A is a convex cell obtained by repacing some of the inequalities $g_j \geq 0$ by the set equations $g_j = 0$. For example, the standard Euclidean simplex $\Delta^2 \subset \mathbb{R}^3$ is a convex cell with

$$f_1 = x + y + z - 1, \quad g_1 = x, \quad g_2 = y, \quad \text{and} \quad g_3 = z$$

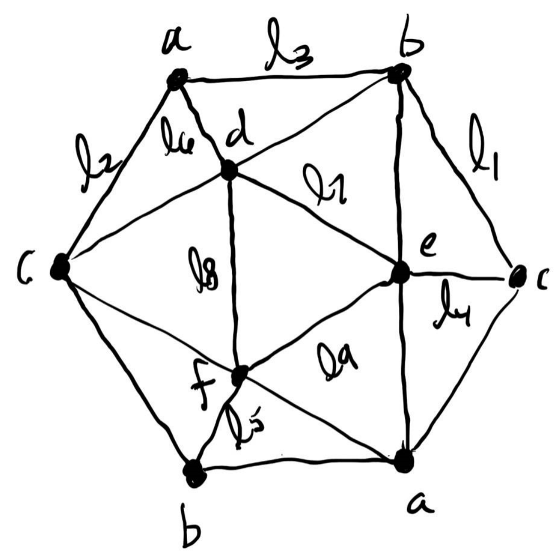
One face of Δ^2 is given by

$$f_1 = x + y + z - 1, \quad f_2 = x, \quad g_1 = y, \quad \text{and} \quad g_2 = z$$

Example 5.5. The real projective plane $\mathbb{RP}^2$ admits the following triangulation: Let

$$\begin{array}{llllll} \ell_1 = x & \ell_4 = x - y & \ell_7 = x - y + z & a = [1:0:0] & d = [0:1:1] \\ \ell_2 = y & \ell_5 = x - z & \ell_8 = x + y - z & b = [0:1:0] & e = [1:1:0] \\ \ell_3 = z & \ell_6 = y - z & \ell_9 = -x + y + z & c = [0:0:1] & f = [1:0:1] \end{array}$$

This gives us the following triangulation of $\mathbb{RP}^2$.



6 Deformation Retract

Definition 6.1. Let A be a subspace of a topological space X. Write $\iota\colon A \hookrightarrow X$ for the inclusion map. A **retraction** from X to A is a is a continuous function $r\colon X \twoheadrightarrow A$ such that $r\iota = 1_A$ (here we use the notation $\iota r := \iota \circ r$). If $A \neq X$, then we obviously don't have $\iota r = 1_X$, but we still might have $\iota r \sim 1_X$. When this happens, we call A a **deformation retract** of X. If $F\colon X \times \mathbb{I} \to X$ is a homotopy from 1_X to ιr, then we call F a **deformation retraction**. In other words, F is a deformation rectraction if it is continuous and satisfies

$$\begin{aligned} F(x, 0) &= x \\ F(x, 1) &\in A \\ F(a, 1) &= a \end{aligned}$$

for all $a \in A$ and $x \in X$. If in addition of a deformation retraction, we add the requirement that

$$F(a, t) = a$$

for all $a \in A$ and $t \in \mathbb{I}$, then we say F is a **strong deformation retraction**.

Remark 7. In other words, A is a deformation retract of X if the inclusion map $A \hookrightarrow X$ is a homotopy equivalence. It follows that

Example 6.1. In this example, we construct an explicit strong deformation retraction of $X = \mathbb{R}^n \backslash \{0\}$ onto S^{n-1}. Define $F\colon X \times I \to X$ by

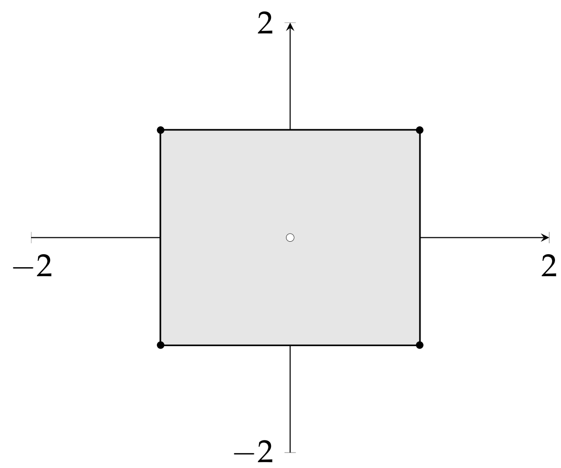
$$F(x,t) = (1-t)x + t(x/\|x\|)$$

where $\|\cdot\|$ is the usual Euclidean norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$. Note that $f_0 = 1_X$ and f_1 is a retraction map. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in S^n$, then $\|x\| = 1$, and thus $F(x,t) = x$ for all $t \in I$.

Example 6.2. In this example, we construct an explicit strong deformation retraction of the torus T with one point deleted onto a graph G consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus. Let $\|\cdot\|_\infty$ denote the sup norm on $\mathbb{R}^2$ defined by $\|x\|_\infty = \max\{x_1, x_2\}$ for all $x \in \mathbb{R}^2$. Note that the sup norm induces the same topology as the usual Euclidean norm does (in particular, $\|\cdot\|_\infty \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous). Now set

$$X = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\} \backslash \{0\} \quad \text{and} \quad A = \{x \in \mathbb{R}^2 \mid \|x\|_\infty = 1\}.$$

We illustrate X and A below: X is the grey shaded region (including the borders) whereas A is the black shaded borders of the square.



We define $F\colon X \times I \to X$ by

$$F(x,t) = (1-t)x + t(x/\|x\|_\infty).$$

Note that $f_0(x) := F(x,0) = x$ and $f_1(x) = F(x,1) = x/\|x\|_\infty$. In particular, $f_0 = 1_X$ and f_1 is a retraction. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in A$, then $\|z\|_\infty = 1$, and thus $F(z,t) = z$ for all $t \in I$.

Next we identity T with the quotient space $[X] := X/\sim$ where $\sim$ is defined by

$$(-1,b) \sim (1,b) \text{ and } (a,-1) \sim (a,1)$$

for all $a, b \in [-1,1]$. Similarly we identify G with the quotient space $[A] := A/\sim$. Note that if $x \sim y$, then $F(x,t) \sim F(y,t)$ for all $t \in I$. Thus F induces a continuous map $[F]\colon [X] \times I \to [X]$. It is easy to see that $[F]$ is a strong deformation retract of $[X]$ onto $[A]$ since it inherits these properties from F.

Lemma 6.1. *Let B be a deformation retact of C and let A be a deformation retract of B. Then A is a deformation retract of C.*

Proof. Choose deformation retractions $G\colon C \times I \to C$ and $F\colon B \times I \to B$; so F and G are continuous maps such that $G(-,0) = 1_C$, $F(-,0) = 1_B$, and $G(-,1) = s$ and $F(-,1) = r$ are both retracts: we view s as map $s\colon C \to B$ such that $s(b) = b$ for all $b \in B$ and we view r as a map $r\colon B \to A$ such that $r(a) = a$ for all $a \in A$. Note that rs is a map from C to A such that $rs(a) = a$ for all $a \in A$, i.e. rs is a retraction of C onto A. Let ι denote the inclusion map $\iota\colon B \to C$ and define $\widetilde{F}\colon C \times I \to C$ be defined by $\widetilde{F}(c,t) = \iota F(s(c),t)$. Finally, to get a deformation retraction with respect to $A \subseteq C$, we glue $\widetilde{F}$ and and G together as follows: define $H\colon C \times I \to C$ by

$$H(c,t) = \begin{cases} G(c,2t) & 0 \leq t \leq 1/2 \\ \widetilde{F}(c,2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Then H is continuous and satisfies $H(-,0) = 1_C$ and $H(-,1) = rs$. Thus A is a deformation retract of C with H being a deformation retraction. $\square$

Example 6.3. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segmnet $[0,1] \times \{0\}$ together with all the verticle segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Let us show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. Let $A = [0,1] \times \{0\}$ and let $\boldsymbol{a} = (a, 0)$ be a point in A. We show X deformation retracts to A then we show A deformation retracts to $\boldsymbol{a}$. First we show X deformation rectracts to A. Define $F\colon X \times I \to X$ as follows: let $\boldsymbol{x} = (x_1, x_2)$ be a point in X. If $x_2 = 0$, then we set $F(\boldsymbol{x}, t) = 0$ for all $t \in I$. Otherwise, x_1 is rational and $0 < x_2 \leq 1 - x_1$. In this case, we set

$$F(\boldsymbol{x}, t) = (1-t)(x_1, x_2) + t(x_1, 0)$$

for all $t \in I$. Then F is a homotopy from 1_X to a retraction map $r\colon X \to A$. In fact, it's easy to see that F is a *strong* deformation retraction. This shows that X deformation retracts to A. Now we show that A deformation retracts to $\boldsymbol{a}$. Define $G\colon A \times I \to A$ as follows: let $\boldsymbol{x} = (x, 0)$ be a point in A. We set

$$G(\boldsymbol{x}, t) = (1-t)\boldsymbol{x} + t\boldsymbol{a}$$

for all $t \in I$. Then it's easy to see that G is a deformation rectraction which shows that A deformation retracts to $\boldsymbol{a}$.

Proposition 6.1. *The retract of a contractible space is contractible.*

Proof. Let A be a retract of a contractible space X. Thus we have a continuous map $r\colon X \to A$ such that $r \circ \iota = 1_A$ where $\iota\colon A \to X$ is the inclusion map. Since X is contractible, there exists $z \in X$ such that $1_X \sim \mathrm{c}_z$ where $\mathrm{c}_z\colon X \to X$ is the constant map defined by $\mathrm{c}_z(x) = z$ for all $x \in X$. We claim that A is contractible with $1_A \sim \mathrm{c}_{r(z)}$. Indeed, let $F\colon X \times I \to I$ be a homotopy from 1_X to c_z; so F is continuous and $F(-,0) = 1_X$ and $F(-,1) = \mathrm{c}_z$. Let G be the composite map

$$A \times I \xrightarrow{\iota \times 1} X \times I \xrightarrow{F} X \xrightarrow{r} A.$$

Concretely $G(a,t) = r(F(a,t))$ for all $a \in A$ and $t \in I$. Then G is continuous (being a composite of continuous functions) and we have $G(-,0) = 1_A$ and $G(-,1) = \mathrm{c}_{r(z)}$. □

Proposition 6.2. *Let $X = (X, x_0)$ and $Y = (Y, y_0)$ be based topological spaces. Suppose there exists an open neighborhoods U of x_0 and V of y_0 such that U has a strong deformation retraction to x_0 and V has a strong deformation retraction to y_0. Then we have*

$$\widetilde{\mathrm{H}}(X \vee Y) = \widetilde{\mathrm{H}}(X) \oplus \widetilde{\mathrm{H}}(Y).$$

Proof. Let $F\colon U \times I \to U$ be a strong deformation retraction of U to x_0. Thus F is a continuous map such that

$$\begin{aligned} F(u,0) &= u \\ F(u,1) &= x_0 \\ F(x_0,t) &= x_0 \end{aligned}$$

for all $u \in U$ and $t \in I$. Similarly let $G\colon V \times I \to V$ be a strong deformation retraction of V to y_0. □

Exercise 1. Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$.

1. Compute the homology of X and Y and confirm that the homology is the same in every dimension.
2. Describe the universal covering spaces of X and Y.
3. Show that the universal covering spaces of X and Y do not have the same homology.

Solution 1. 1. We use Kunneth theorem which tells us that $\mathrm{H}(X) \simeq \mathrm{H}(S^1) \otimes \mathrm{H}(S^1)$ as graded modules. In particular, this implies

$$\mathrm{H}_i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

Next, note that the identified basepoint in the wedge sum $S^1 \vee S^1 \vee S^2$ is a deformation retract of open neighborhoods in S^1 and S^2. Thus one can use the Mayer-Vietoris sequence to deduce that $\widetilde{\mathrm{H}}(Y) \simeq \widetilde{\mathrm{H}}(S^1) \oplus \widetilde{\mathrm{H}}(S^1) \oplus \widetilde{\mathrm{H}}(S^2)$ as graded modules, where the tilde denoted "reduced homology". In particular, this implies

$$\mathrm{H}_i(Y) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

where we use the fact that Y is connected so get $H_0(Y) = \mathbb{Z}$.

2. First we describe the universal covering space of X. Recall we have a homeomorphism $\mathbb{R}/\mathbb{Z} \simeq S^1$ defined by

$$\overline{x} \mapsto e^{2\pi i x}$$

for all $\overline{x} \in \mathbb{R}/\mathbb{Z}$. Thus it suffices to describe the universal covering space of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. The universal covering space of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is given by $\pi\colon \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ where π is canonical projection map defined by

$$\pi(\boldsymbol{x}) = \pi(x_1, x_2) = (\overline{x}_1, \overline{x}_2),$$

for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Since $\mathbb{R}^2$ is simply-connected, this is the univeral covering space.

Next we describe the universal covering space of Y. First, we want to describe the universal covering space of $S^1 \vee S^1$. We describe a cell complex $T \subseteq \mathbb{Q}^2$ as follows:

1) We first describe the vertices of T. We partition the vertices of T in terms of their **length** $\ell \in \mathbb{N} \cup \{\infty\}$. First, we have one vertex of length $\ell = 0$, namely the origin $(0,0) \in \mathbb{Q}^2$. Next, we have four vertices of length $\ell = 1$, namely the points $(\pm 1/2, 0), (0, \pm 1/2) \in \mathbb{Q}^2$. More generally, suppose $\ell > 1$. There are $3^{\ell-1}4$ vertices of length ℓ, namely the points of the form

$$v = v_{(\boldsymbol{r},\boldsymbol{u})} = \left(\sum_{i=1}^{\ell} r_i 2^{-i}, \sum_{i=1}^{\ell} u_i 2^{-i} \right)$$

where $\boldsymbol{r}, \boldsymbol{u} \in \{-1, 0, 1\}^\ell$ such that $|a_i| + |b_i| = 1$ for all $i = 1, \dots, \ell$, where we use the notation $\boldsymbol{r} = (r_1, \dots, r_\ell)$ and $\boldsymbol{u} = (u_1, \dots, u_\ell)$. In particular, this condition says for each $i = 1, \dots \ell$, we have

1. if $r_i = 0$, then $u_i = \pm 1$,
2. if $b_i = 0$, then $a_i = \pm 1$,
3. if $a_i =$
4. or $b_i = 0$ and $a_i = \pm 1$.

For instance, the vertices of length $\ell = 2$ are given in the table below

$v \in \mathbb{R}^2$	$\boldsymbol{a} \in \{-1,0,1\}^2$	$\boldsymbol{b} \in \{-1,0,1\}^2$
$(1/2 + 1/4, 0)$	$(1,1)$	$(0,0)$
$(-1/2 - 1/4, 0)$	$(-1,-1)$	$(0,0)$
$(0, 1/2 + 1/4)$	$(0,0)$	$(1,1)$
$(0, -1/2 - 1/4)$	$(0,0)$	$(-1,-1)$
$(1/2, 1/4)$	$(1,0)$	$(0,1)$
$(1/2, -1/4)$	$(1,0)$	$(0,-1)$
$(-1/2, 1/4)$	$(-1,0)$	$(0,1)$
$(-1/2, -1/4)$	$(-1,0)$	$(0,-1)$
$(1/4, 1/2)$	$(0,1)$	$(1,0)$
$(1/4, -1/2)$	$(0,1)$	$(-1,0)$
$(-1/4, 1/2)$	$(0,-1)$	$(1,0)$
$(-1/4, -1/2)$	$(0,-1)$	$(-1,0)$

We have a norm N on $\mathbb{Q}^2$ defined by

$$\mathrm{N}(\boldsymbol{x}) = \max\{|x_1|_2, |x_2|_2\}$$

where $|\cdot|_2$ is the 2-adic absolute value on $\mathbb{Q}^2$. For instance

$$\begin{aligned} N\left(\left(\frac{1}{2}+\frac{1}{4}, -\frac{1}{8}\right)\right) &= 8 \\ N\left(\left(\frac{1}{8}, \frac{1}{2}-\frac{1}{4}\right)\right) &= 8 \\ N\left(\left(\frac{1}{2}, 0\right)\right) &= 2, \end{aligned}$$

and so on.
Observe that

$$v_{(\boldsymbol{a},\boldsymbol{b})} + v_{(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})} = \left(\sum_{i=1}^{\ell}(a_i+\widetilde{a}_i)2^{-i}, \sum_{i=1}^{\ell}(b_i+\widetilde{b}_i)2^{-i}\right) + \left(\sum_{i=1}^{\ell} a_i 2^{-i}, \sum_{i=1}^{\ell} b_i 2^{-i}\right)$$

$$\begin{aligned} \|v_{(\boldsymbol{a},\boldsymbol{b})} - v_{(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})}\|^2 &= \left\|\left(\sum_{i=1}^{\ell}(a_i-\widetilde{a}_i)2^{-i}, \sum_{i=1}^{\ell}(b_i-\widetilde{b}_i)2^{-i}\right)\right\|^2 \\ &= \left(\sum_{i=1}^{\ell}(a_i-\widetilde{a}_i)2^{-i}\right)^2 + \left(\sum_{i=1}^{\ell}(a_i-\widetilde{a}_i)2^{-i}\right)^2 \\ &= \sum_{i=1}^{\ell}(a_i^2 - 2a_i\widetilde{a}_i + \widetilde{a}_i^2 + b_i^2 - 2b_i\widetilde{b}_i + \widetilde{b}_i^2)2^{-2i} \\ &= \sum_{i=1}^{\ell}(a_i^2 - 2a_i\widetilde{a}_i + \widetilde{a}_i^2 + b_i^2 - 2b_i\widetilde{b}_i + \widetilde{b}_i^2)2^{-2i} \end{aligned}$$

$$\begin{aligned} \|v_{(\boldsymbol{a},\boldsymbol{b})} - v_{(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})}\| = 0 &\iff \left\|\left(\sum_{i=1}^{\ell}(a_i-\widetilde{a}_i)2^{-i}, \sum_{i=1}^{\ell}(b_i-\widetilde{b}_i)2^{-i}\right)\right\| = 0 \\ &\iff \sum_{i=1}^{\ell}(a_i-\widetilde{a}_i)2^{-i} = 0 \text{ and } \sum_{i=1}^{\ell}(b_i-\widetilde{b}_i)2^{-i} = 0 \\ &\iff \sum_{i=1}^{\ell}(\widetilde{b}_i-b_i)2^{-i} = 0 \text{ and } \sum_{i=1}^{\ell}(b_i-\widetilde{b}_i)2^{-i} = 0 \qquad \text{since } a_i = 1-b_i \text{ and } \widetilde{a}_i \\ &= \sum_{i=1}^{\ell}(a_i^2 - 2a_i\widetilde{a}_i + \widetilde{a}_i^2 + b_i^2 - 2b_i\widetilde{b}_i + \widetilde{b}_i^2)2^{-2i} \end{aligned}$$

Given a vertex $v = v_{(\boldsymbol{a},\boldsymbol{b})}$ in T, we assign a number $Q = Q_{(\boldsymbol{a},\boldsymbol{b})}$ given by

$$Q = \sum_{i=1}^{\ell}(a_i+b_i)2^{-i} = \sum_{i=1}^{\ell} c_i 2^{-i} = 2^{-\ell}\sum_{i=1}^{\ell} c_i 2^{\ell-i}$$

where we set $c_i = a_i + b_i$ for $i = 1, \ldots, \ell$. Note that

$$\begin{aligned} Q_{(\boldsymbol{a},\boldsymbol{b})} = Q_{(\boldsymbol{a}',\boldsymbol{b}')} &\iff Q_{(\boldsymbol{a},\boldsymbol{b})} - Q_{(\boldsymbol{a}',\boldsymbol{b}')} \\ &\iff \sum_{i=1}^{\ell}(c_i - c_i')2^{-i} = 0 \\ &\iff \sum_{i=1}^{\ell}(c_i - c_i')\overline{2^{\ell}} = 0 \end{aligned}$$

$Q_{(\boldsymbol{a},\boldsymbol{b})} = Q_{(\boldsymbol{a}',\boldsymbol{b}')}$ if and only if $\boldsymbol{a}' = \boldsymbol{b}$ and $\boldsymbol{b}' = \boldsymbol{a}$.

We draw below

Problem 2

7 Fundamental Group of S^1

Theorem 7.1. *Let $\omega_n \colon I \to S^1$ be given by*

$$\omega_n(t) = (\cos(2n\pi t), \sin(2n\pi t))$$

The map $\Phi \colon \mathbb{Z} \to \pi_1(S^1)$ given by $n \mapsto [\omega_n]$ is an isomorphism.

8 Mapping Cones and Mapping Cylinders

Let $f \colon X \to Y$ be a continuous map of topological spaces. The **mapping cylinder** M_f of f is defined to be the quotient space

$$M_f := (X \times \mathbb{I}) \coprod Y/\sim,$$

where $(x, 1) \sim f(x)$ for all $x \in X$. The **mapping cone** C_f of f is defined to be the quotient space

$$C_f := (X \times \mathbb{I}) \coprod Y/\sim$$

where $(x, 1) \sim f(x)$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Now let $\varphi \colon A \to B$ be the chain map induced by $f \colon X \to Y$ where $A = \mathrm{C}(X)$ and $B = \mathrm{C}(Y)$. The mapping cone of φ is the chain complex C_φ whose underlying graded module is $C_\varphi = B + eA$, where e is viewed as an exterior variable of degree 1, and whose differential is defined by

$$\mathrm{d}(b + ea) = \mathrm{d}b + \varphi(a) - e\mathrm{d}(a),$$

for all homogeneous $a \in A$ and $b \in B$. One can show that $\mathrm{C}(C_f)$ is homotopy equivalent to C_φ.

Part I
Homework

9 Homework 1

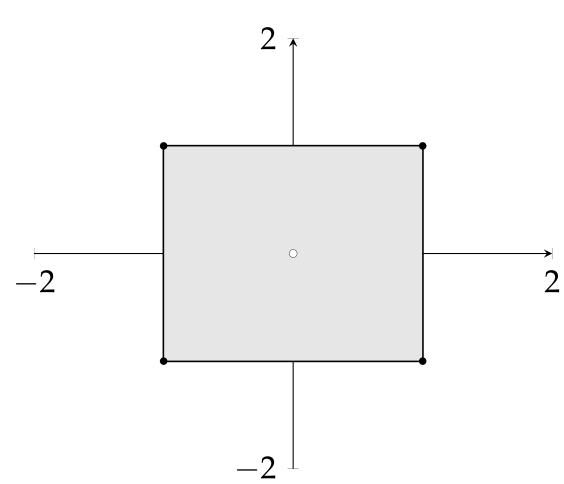
9.1 Constructing an Explicit Deformation Retraction

Exercise 2. Construct an explicit deformation retraction of the torus T with one point deleted onto a graph G consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution 2. Let $\|\cdot\|_\infty$ denote the sup norm on $\mathbb{R}^2$ defined by $\|x\|_\infty = \max\{x_1, x_2\}$ for all $x \in \mathbb{R}^2$. Note that the sup norm induces the same topology as the usual Euclidean norm does (in particular, $\|\cdot\|_\infty \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous). Now set

$$X = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\} \backslash \{0\} \quad \text{and} \quad A = \{x \in \mathbb{R}^2 \mid \|x\|_\infty = 1\}.$$

We illustrate X and A below: X is the grey shaded region (including the borders) whereas A is the black shaded borders of the square.



We define $F\colon X \times I \to X$ by
$$F(x,t) = (1-t)x + t(x/\|x\|_\infty).$$
Note that $f_0(x) := F(x,0) = x$ and $f_1(x) = F(x,1) = x/\|x\|_\infty$. In particular, $f_0 = 1_X$ and f_1 is a retraction. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in A$, then $\|z\|_\infty = 1$, and thus $F(z,t) = z$ for all $t \in I$.

Next we identity T with the quotient space $[X] := X/\sim$ where $\sim$ is defined by
$$(-1,b) \sim (1,b) \text{ and } (a,-1) \sim (a,1)$$
for all $a, b \in [-1,1]$. Similarly we identify G with the quotient space $[A] := A/\sim$. Note that if $x \sim y$, then $F(x,t) \sim F(y,t)$ for all $t \in I$. Thus F induces a continuous map $[F]\colon [X] \times I \to [X]$. It is easy to see that $[F]$ is a deformation retract of $[X]$ onto $[A]$ since it inherits these properties from F.

Exercise 3. Construct an explicit deformation retraction of $X = \mathbb{R}^n \backslash \{0\}$ onto S^{n-1}.

Solution 3. Define $F\colon X \times I \to X$ by
$$F(x,t) = (1-t)x + t(x/\|x\|)$$
where $\|\cdot\|$ is the usual Euclidean norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$. Note that $f_0 = 1_X$ and f_1 is a retraction map. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in S^n$, then $\|x\| = 1$, and thus $F(x,t) = x$ for all $t \in I$.

9.2 Homotopies Respect Composition

To solve this problem (as well as the next problem), we will make use of the following lemma which says homotopies pass through the composition operation:

Lemma 9.1. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions which are homotopic to $f'\colon X \to Y$ and $g'\colon Y \to Z$ respectively (denoted $f \sim f'$ and $g \sim g'$). Then $gf \sim g'f'$ (where $gf = g \circ f$ and $g'f' = g' \circ f'$ denotes composition).*

Proof. Let $F\colon X \times I \to Y$ be a homotopy from f to f' and let $G\colon Y \times I \to Z$ be a homotopy from g to g'. Thus
$$\begin{aligned} F(x,0) &= f(x) \\ F(x,1) &= f'(x) \\ G(y,0) &= g(y) \\ G(y,1) &= g'(y) \end{aligned}$$
Define $H\colon X \times I \to Z$ by $H(x,t) = G(F(x,t),t)$. We can think of H as the composite map $X \times I \to Y \times I \to Z$ where the map $X \times I \to Y \times I$ sending (x,t) to $(F(x,t),t)$ is continuous since each component function is continuous and where the map $Y \times I \to Z$ sending (y,t) to $G(y,t)$ is continuous since G is a homotopy. Therefore, H is a continuous map. Furthermore it is straightforward to check that $H(-,0) = gf$ and $H(-,1) = g'f'$. Thus H is a homotopy from gf to $g'f'$, that is, $gf \sim g'f'$. □

Remark 8. Let $f_1, f_1'\colon X_1 \to X_2$, and $f_2, f_2'\colon X_2 \to X_3$, and $f_3, f_3'\colon X_3 \to X_4$ be continuous functions such that $f_1 \sim f_1'$, and $f_2 \sim f_2'$, and $f_3 \sim f_3'$. Write $f = f_3 f_2$ and $f' = f_3' f_2'$. By the lemma above, we have $f \sim f'$, which implies
$$\begin{aligned} f_3 f_2 f_1 &= (f_3 f_2) f_1 \\ &= f f_1 \\ &\sim f' f_1' \\ &= (f_3' f_2') f_1' \\ &= f_3' f_2' f_1'. \end{aligned}$$
This shows that we may replace a function in a composite with a homotopic map without having to worry about associativy.

Now we state and solve problem 3:

Exercise 4. Prove the following:

1. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

2. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

3. Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution 4. 1. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be homotopy equivalences with homotopy inverses $\widetilde{f}\colon Y \to X$ and $\widetilde{g}\colon Z \to Y$ respectively. Thus we have $\widetilde{f}f \sim 1_X$, $f\widetilde{f} \sim 1_Y$, $\widetilde{g}g \sim 1_Y$, and $g\widetilde{g} \sim 1_Z$. In particular, this implies

$$\begin{aligned}(gf)(\widetilde{f}\widetilde{g}) &= g(f\widetilde{f})\widetilde{g} \\ &\sim g1_Y\widetilde{g} \\ &= g\widetilde{g} \\ &\sim 1_Z\end{aligned}$$

A similar computation gives us $(\widetilde{f}\widetilde{g})(gf) \sim 1_X$. It follows that $gf\colon X \to Z$ is a homotopy equivalence. In particular, this says that if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$ (this shows that $\sim$ is transitive; that $\sim$ is reflexive and symmetric is obvious).

2. Let $f, g, h\colon X \to Y$ be continuous functions such that $f \sim g$ and $g \sim h$, say $F\colon X \times I \to Y$ is a homotopy from f to g and $G\colon X \times I \to Y$ is a homotopy from g to h. Define $H\colon X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}$$

Clearly H is continuous. Furthermore, we have $H(-,0) = f$, $H(-,1/2) = g$, and $H(-,1) = h$. In particular, H is a homotopy from f to h. It follows that $\sim$ is transitive (that $\sim$ is reflexive and symmetric is obvious).

3. Let $f\colon X \to Y$ be a homotopy equivalence with $\widetilde{f}\colon Y \to X$ being its homotopy inverse and suppose $f'\colon X \to Y$ is a map which is homotopic to f. Then by the lemma above, we have $1_Y \sim f\widetilde{f} \sim f'\widetilde{f}$ and $1_X \sim \widetilde{f}f \sim \widetilde{f}f'$. This shows that f' is a homotopy equivalence as well.

9.3 Weak Deformation Retraction

Exercise 5. A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t\colon X \to X$ such that $f_0 = 1_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $\iota\colon A \to X$ is a homotopy equivalence.

Solution 5. Define $r\colon X \to A$ by $r(x) = f_1(x)$ (thus $\iota r = f_1$). We claim that r is the homotopy inverse to ι. Indeed, we have $r\iota \sim 1_A$ since the map $R\colon A \times I \to A$ given by $R(a,t) = f_t(a)$ is a homotopy from 1_A to $r\iota$ (notice we needed the fact that $f_t(A) \subseteq A$ in order for this map to make sense). On the other hand, we have $\iota r \sim 1_X$ since $F\colon X \times I \to X$ is a homotopy from 1_X to ιr.

10 Homework 2

10.1 Transitivity of Deformation Retractions

Lemma 10.1. *Let B be a deformation retact of C and let A be a deformation retract of B. Then A is a deformation retract of C.*

Proof. Choose deformation retractions $G\colon C \times I \to C$ and $F\colon B \times I \to B$; so F and G are continuous maps such that $G(-,0) = 1_C$, $F(-,0) = 1_B$, and $G(-,1) = s$ and $F(-,1) = r$ are both retracts: we view s as map $s\colon C \to B$ such that $s(b) = b$ for all $b \in B$ and we view r as a map $r\colon B \to A$ such that $r(a) = a$ for all $a \in A$. Note that rs is a map from C to A such that $rs(a) = a$ for all $a \in A$, i.e. rs is a retraction of C onto A. Let ι denote the inclusion map $\iota\colon B \to C$ and define $\widetilde{F}\colon C \times I \to C$ be defined by $\widetilde{F}(c,t) = \iota F(s(c),t)$. Finally, to get a deformation retraction with respect to $A \subseteq C$, we glue $\widetilde{F}$ and and G together as follows: define $H\colon C \times I \to C$ by

$$H(c,t) = \begin{cases} G(c,2t) & 0 \le t \le 1/2 \\ \widetilde{F}(c,2t-1) & 1/2 \le t \le 1 \end{cases}$$

Then H is continuous and satisfies $H(-,0) = 1_C$ and $H(-,1) = rs$. Thus A is a deformation retract of C with H being a deformation retraction. $\square$

Exercise 6. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segmnet $[0,1] \times \{0\}$ together with all the verticle segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Solution 6. Let $A = [0,1] \times \{0\}$ and let $\boldsymbol{a} = (a, 0)$ be a point in A. We show X deformation retracts to A then we show A deformation retracts to $\boldsymbol{a}$. First we show X deformation rectracts to A. Define $F\colon X \times I \to X$ as follows: let $\boldsymbol{x} = (x_1, x_2)$ be a point in X. If $x_2 = 0$, then we set $F(\boldsymbol{x}, t) = 0$ for all $t \in I$. Otherwise, x_1 is rational and $0 < x_2 \leq 1 - x_1$. In this case, we set

$$F(\boldsymbol{x}, t) = (1-t)(x_1, x_2) + t(x_1, 0)$$

for all $t \in I$. Then F is a homotopy from 1_X to a retraction map $r\colon X \to A$. In fact, it's easy to see that F is a *strong* deformation retraction. This shows that X deformation retracts to A. Now we show that A deformation retracts to $\boldsymbol{a}$. Define $G\colon A \times I \to A$ as follows: let $\boldsymbol{x} = (x, 0)$ be a point in A. We set

$$G(\boldsymbol{x}, t) = (1-t)\boldsymbol{x} + t\boldsymbol{a}$$

for all $t \in I$. Then it's easy to see that G is a deformation rectraction which shows that A deformation retracts to $\boldsymbol{a}$.

10.2 Contractible Spaces

Definition 10.1. Let X be a topological space. We say X is **contractible** if it is homotopy equivalent to a point. Equivalently, the identitiy map $1_X\colon X \to X$ is a homotopic to the constant map $\mathrm{c}_x\colon X \to X$ for some $x \in X$.

Exercise 7. Show that a retract of a contractible space is contractible.

Solution 7. Let A be a retract of a contractible space X. Thus we have a continuous map $r\colon X \to A$ such that $r \circ \iota = 1_A$ where $\iota\colon A \to X$ is the inclusion map. Since X is contractible, there exists $z \in X$ such that $1_X \sim \mathrm{c}_z$ where $\mathrm{c}_z\colon X \to X$ is the constant map defined by $\mathrm{c}_z(x) = z$ for all $x \in X$. We claim that A is contractible with $1_A \sim \mathrm{c}_{r(z)}$. Indeed, let $F\colon X \times I \to I$ be a homotopy from 1_X to c_z; so F is continuous and $F(-,0) = 1_X$ and $F(-,1) = \mathrm{c}_z$. Let G be the composite map

$$A \times I \xrightarrow{\iota \times 1} X \times I \xrightarrow{F} X \xrightarrow{r} A.$$

Concretely $G(a,t) = r(F(a,t))$ for all $a \in A$ and $t \in I$. Then G is continuous (being a composite of continuous functions) and we have $G(-,0) = 1_A$ and $G(-,1) = \mathrm{c}_{r(z)}$.

Exercise 8. Show that a space X is contractible iff every map $f\colon X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f\colon Y \to X$ is nullhomotopic.

Solution 8. It suffices to show the first part of the exercise since the proof of the second part is almost identical to the proof of the first part. Suppose X is contractible and let $f\colon X \to Y$ be an arbitrary continuous map. Since X is contractible, there exists $z \in X$ such that $1_X \sim \mathrm{c}_z$. Choose a homotopy from 1_X to c_z, say $F\colon X \times I \to X$. Let G be the composite map

$$X \times I \xrightarrow{F} X \xrightarrow{f} Y.$$

Concretely, $G(x,t) = f(F(x,t))$ for all $(x,t) \in X \times I$. Then G is continuous (being a composite of continuous functions) and we have $G(-,0) = f$ and $G(-,1) = \mathrm{c}_{f(z)}$. Thus f is nullhomotopic. Conversely, suppose every continuous map $f\colon X \to Y$ is nullhomotopic. Then in particular, $1_X\colon X \to X$ is nullhomotopic. However this implies X is contractible, by definition.

10.3 $\pi_0(X)$

Before we solve this exercise, we introduce some terminology as well as state and prove a lemma. Let X be a topological space. We denote by $\pi_0(X)$ to be the set of path-connected components of X. We write $[x] \in \pi_0(X)$ for the equivalence class with $x \in X$ as a particular choice of representative. Thus if $x' \in [x]$, then there exists a path $\gamma\colon I \to X$ form x to x', i.e. such that $\gamma(0) = x$ and $\gamma(1) = x'$. Next let $f\colon X \to Y$ be a continuous map. Define a map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ as follows: given $[x] \in \pi_0(X)$, we set

$$\pi_0(f)[x] := [f(x)].$$

To see that this is well-defined, let $x' \in [x]$ be another representative. Choose a path $\gamma\colon I \to X$ from x to x'. Then $f\gamma$ is path from $f(x)$ to $f(x')$. Indeed, it is continuous since it is a composite of two continuous functions and we have $f\gamma(0) = f(x)$ and $f\gamma(1) = f(x')$ (note we are using the notation $f\gamma$ to mean $f \circ \gamma$). It is straightforward to check that we obtain a functor $\pi_0 \colon \mathbf{Top} \to \mathbf{Set}$ which takes a topological space X to the set $\pi_0(X)$ and which takes a continuous map $f \colon X \to Y$ to the function $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$. In particular, this means that π_0 preserves compositions: if $g \colon Y \to Z$ is another continuous map, then we have $\pi_0(gf) = \pi_0(g)\pi_0(f)$. Similarly, this means π_0 preserves identities: we have $\pi_0(1_X) = 1_{\pi_0(X)}$. The functor π_0 turns out to be invariant under homotopy:

Lemma 10.2. *Let $f, g \colon X \to Y$ be two continuous maps such that $f \sim g$. Then $\pi_0(f) = \pi_0(g)$.*

Proof. Choose a homotopy from f to g, say $F \colon X \times I \to Y$; so F is continuous and $F(-,0) = f$ and $F(-,1) = g$. Let $[x_0]$ be a path-connected component in X. Then observe that $F(x_0, -) \colon I \to Y$ is a path from $f(x_0)$ to $g(x_0)$ by our assumption of F (the map $F(x_0, -) \colon I \to Y$ is deifned by sending $t \in I$ to $F(x_0, t)$; so x_0 is fixed and t varies). In particular, it follows that $[f(x_0)] = [g(x_0)]$. Since $[x_0]$ was arbitrary, it follows that $\pi_0(f) = \pi_0(g)$. □

Now we solve the exercise:

Exercise 9. Show that a homotopy equivalence $f \colon X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X .

Solution 9. Let $f \colon X \to Y$ and $g \colon Y \to X$ be continuous functions such that $fg \sim 1_Y$ and $gf \sim 1_X$. It follows from the fact that π_0 is a homotopy invariant functor that $\pi_0(f)\pi_0(g) = 1_Y$ and $\pi_0(g)\pi_0(f) = 1_X$. In other words, $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$ is a bijection with $\pi_0(g)$ being its inverse. For the second part of the exercise, let $P \subseteq X$ be a connected component of X. Then $f(P)$ is contained in a connected component of Y since f is continuous, say $f(P) \subseteq Q$. Similarly, $g(Q)$ is contained in a connected component of X. Since $gf \sim 1_X$, we must have $g(Q) \subseteq P$.

11 Homework 3

11.1 Homotopy Equivalences

Exercise 10. Show that $f \colon X \to Y$ is a homotopy equivalence if there exist maps $g, h \colon Y \to X$ such that $fg \sim 1_Y$ and $hf \sim 1_X$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Solution 10. labelsol Observe that $h \sim hfg \sim g$. In particular, we have $fg \sim 1_Y$ and $gf \sim hf \sim 1_X$, thus g is a homotopic inverse of f. It follows that $f \colon X \to Y$ is a homotopy equivalence. For the generalization, suppose $u \colon X \to X$ is a homotopic inverse of $hf \colon X \to X$ and suppose $v \colon Y \to Y$ is a homotopic inverse of $fg \colon Y \to Y$. Then observe that $1_Y \sim (fg)v = f(gv)$ and $1_X \sim u(hf) = (uh)f$. It follows from the previous case that f is a homotopy equivalence.

11.2 S^∞ is Contractible

Exercise 11. Show that S^∞ is contractible.

Solution 11. labelsol Recall that S^∞ is the unit circle
Let $T \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the shift operator defined as follows: given a sequence $\boldsymbol{x} = (x_1, x_2, x_3, \dots,)$ in $\mathbb{R}^\infty$, we set $T(\boldsymbol{x}) = (0, x_1, x_2, \dots)$. Define $F \colon S^\infty \times I \to S^\infty$ by

$$F(\boldsymbol{x}, \lambda) = \frac{(1 - \lambda + \lambda T)\boldsymbol{x}}{\|(1 - \lambda + \lambda T)\boldsymbol{x}\|} \tag{15}$$

where $\boldsymbol{x} \in S^\infty$ and where $\lambda \in I$. Observe that F is continuous since $1 - \lambda + \lambda T$ is a bounded linear operator and since the denominator (15) is never zero. Furthermore observe that $F(-,0) = 1_{S^\infty}$ and $F(-,1) = T$. Thus F is a homotopy from 1_{S^∞} to T. Next let $\boldsymbol{e} = (1, 0, 0, \dots)$ and define $G \colon S^\infty \times I \to S^\infty$ by

$$G(\boldsymbol{x}, \lambda) = (1 - \lambda)T\boldsymbol{x} + \lambda \boldsymbol{e}.$$

Clearly G is a homotopy from T to the constant map $c_{\boldsymbol{e}}$. Thus $1_{S^\infty} \sim T \sim c_{\boldsymbol{e}}$ which means S^∞ is contractible.

11.3 Criterion for when CW Complex is Contractible

Exercise 12. Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Solution 12. labelsol Let X be a CW complex and suppose $X = A \cup B$ with A and B two contractible subcomplexes of X such that $Z = A \cap B$ is also contractible. Since Z is a contractible subspace of X, we have $X \sim X/Z$. If we can show X/Z is contractible, then it will follow that X is contractible since contractibility is preserved under homotopy equivalences. Thus by replacing X with X/Z if necessary, we may assume that $A \cap B = \{z\}$ is a singleton set. Since A is a contractible subspace of X, we have $X \sim X/A = B$. Finally since B is contractible, it follows that X is contractible.

Remark 9. Note that we are appealing to propositions 0.16 and 0.17 in Hatcher here which says if (X, A) is a CW pair, then (X, A) has the homotopy extension property and thus $X \to X/A$ is a homotopy equivalence. This need not hold for arbitrary topological spaces $A \subseteq X$.

Exercise 13. Read the proof of Proposition 0.19 in Hatcher's book and explain the square figure that's part of the proof on Page 17.

Solution 13. labelsol The square represents the homotopy of homotopies $K\colon X \times I \times I \to X$ denoted $(x, t, u) \mapsto K(x, t, u)$. The square is basically the parameter domain $I \times I$ for the pairs (t, u) with t-axis horizontal and u-axis vertical. Each point (t, u) in the square represents a continuous function $K(-, t, u)\colon X \to X$. For instance, the bottom left corner of the square represents the continuous function $K(-, 0, 0) = g_1 f$ and the bottom right corner of the square represents the continuous function $K(-, 1, 0) = 1_X$. The bottom edge of the square represents the homotopy $K(-, -, 0) = F$ where $F\colon X \times I \to X$ is the homotopy from $g_1 f$ to 1_X defined by

$$F(-, t) = \begin{cases} g_{1-2t} f & 0 \leq t \leq 1/2 \\ h_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$$

More generally for each $u \in I$, the line segment $\{u\} \times I$ represents a homotopy $K(-, -, u)$ from $g_1 f$ to 1_X. This is why we call K a homotopy of homotopies.

12 Homework 4

12.1 Composition of Paths Satisfies Cancellation Property

Exercise 14. Show that composition of paths satisfies the following cancellation property: if $f_0 \cdot g_0 \sim f_1 \cdot g_1$ and $g_0 \sim g_1$, then $f_0 \sim f_1$.

Solution 14. Let $\widetilde{g}_0$ be the inverse path of g_0, so $\widetilde{g}_0(t) = g_0(1 - t)$. Then we have

$$\begin{aligned} f_0 &\sim f_0 \cdot (g_0 \cdot \widetilde{g}_0) \\ &\sim (f_0 \cdot g_0) \cdot \widetilde{g}_0 \\ &\sim (f_1 \cdot g_1) \cdot \widetilde{g}_0 \\ &\sim f_1 \cdot (g_1 \cdot \widetilde{g}_0) \\ &\sim f_1 \cdot (g_0 \cdot \widetilde{g}_0) \\ &\sim f_1 \end{aligned}$$

12.2 X is Simply-Connected if All Maps $S^1 \to X$ are Homotopic

Exercise 15. Show that for a space X, the following three conditions are equivalent:

1. Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
2. Every map $S^1 \to X$ extends to a map $D^2 \to X$.
3. $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Solution 15. In this problem, we identify S^1 with the unit circle in the complex plane. Similarly, we identify D^2 with the unit disc in the complex plane.

We first show 1 implies 2. Let $f\colon S^1 \to X$ be a continuous map. Denote $x = f(1)$ and let $F\colon S^1 \times I \to X$ be a homotopy from c_x to f (so $F(-,0) = \mathrm{c}_x$ and $F(-,1) = f$). Define $\widetilde{f}\colon D^2 \to X$ by

$$\widetilde{f}(w) = \begin{cases} x & \text{if } w = 0 \\ F(w/|w|, |w|) & \text{else} \end{cases}$$

for all $w \in D^2$. Then $\widetilde{f}$ is easily seen to be an extension of f.

Next we show 2 implies 3. Pick $x_0 \in X$ and let $f\colon S^1 \to X$ be a loop in X based at x_0 (so $f(1) = x_0$). Let $F\colon D^2 \to X$ be a continuous extension of f, so $F|_{S^1} = f$. Next define $H\colon S^1 \times I \to X$ by

$$H(e^{2\pi is}, t) = \begin{cases} F(te^{2\pi i(s/t)} + (1-t)) & \text{if } 0 \le s \le t \\ x_0 & \text{if } t \le s \le 1 \end{cases}$$

Notice that when $t = 1$, we have $H(e^{2\pi is}, 1) = F(e^{2\pi is}) = f(e^{2\pi is})$. Thus $H(-,1) = f$. Similarly, when $t = 0$, we have $H(e^{2\pi is}, 0) = x_0$, thus $H(-,0) = \mathrm{c}_{x_0}$. Finally, it is easy to see that H is a homotopy from c_{x_0} to f with fixed endpoints. It follows that $\pi_1(X, x_0) = 0$.

3 implies 1 follows by definition.

12.3 $\pi_1(X,x)$ Forms a Group

Exercise 16. Find the explicit reparametrization that shows for paths γ_1, γ_2, and γ_3 with $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$ such that $[(\gamma_1 \cdot \gamma_2) \cdot \gamma_3] = [\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)]$.

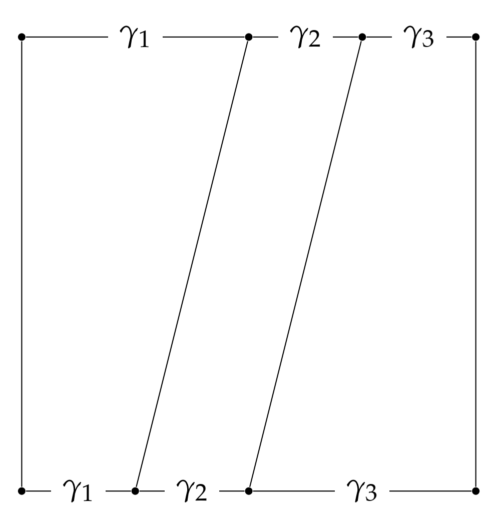
Solution 16. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \gamma_1\left(\frac{4s}{t+1}\right) & 0 \le s \le \frac{t+1}{4} \\ \gamma_2\left(4\left(s - \left(\frac{t+1}{4}\right)\right)\right) & \frac{t+1}{4} < s \le \frac{t+2}{4} \\ \gamma_3\left(\frac{4}{2-t}\left(s - \left(\frac{t+2}{4}\right)\right)\right) & \frac{t+2}{4} < s \le 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy from $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ to $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$. Let's explain what's happening in

$$\gamma_3\left(\frac{4}{2-t}\left(s - \left(\frac{t+2}{4}\right)\right)\right)$$

in order to get a better idea of how H is defined. Here, the $(t+2)/4$ part is telling us to delay γ_3 by $(t+2)/4$ seconds. Thus when $t = 0$, we wait half a second before follow the γ_3 path. The $4/(2-t)$ part is telling us to speed up the γ_3 path by $4/(2-t)$ seconds. Thus when $t = 0$, we follow the γ_3 twice as fast. All of the other parts of H can be understood in an analagous way. One may visualize this homotopy as below:



with horizontal axis being the s-axis (s is the path variable) and with vertical axis being the t-axis (t is the homotopy variable). The diagonal lines partition the square into three regions. Note that the left-most diagonal line above is given by the equation $t = 4s - 1$. Thus when $t \ge 4s - 1$ (or equivalently when $0 \le s \le (t+1)/4$), we are in the left-most region of the square above.

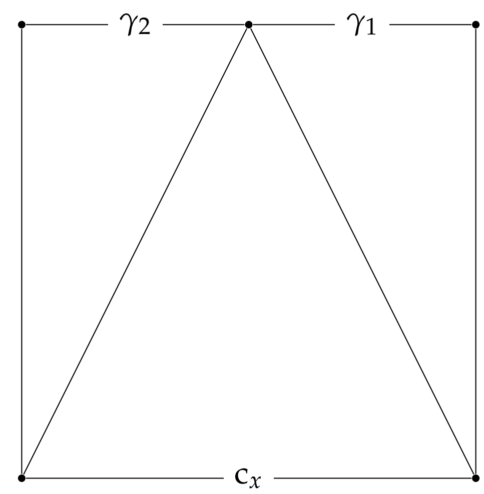
Remark 10. Usually one uses the horizontal axis as the homotopy axis (so the t-axis), however I drew these diagrams in the past and didn't think it was necessary to re-draw them for this problem. The same applies to problem 4 as well.

Exercise 17. Find the explicit homotopy that shows that for a loops γ_1 and γ_2 where $\gamma_2(t) = \gamma_1(1-t)$ the composite $\gamma_1 \cdot \gamma_2$ is homotopic to a constant loop.

Solution 17. Let x be the point at which both γ_1 and γ_2 are based at. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \gamma_1\left(\frac{2s}{t}\right) & 0 \leq s < \frac{t}{2} \\ \mathrm{c}_x\left(\frac{1}{1-t}\left(s - \frac{t}{2}\right)\right) & \frac{t}{2} \leq s \leq \frac{2-t}{2} \\ \gamma_2\left(\frac{2}{t}\left(s - \frac{2-t}{2}\right)\right) & \frac{2-t}{2} < s \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy from c_x to $\gamma_1 \cdot \gamma_2$. One may visualize this homotopy as below:



13 Homework 5

13.1 Borsuk-Ulam Theorem

Remark 11. In this problem, we are identifying S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Thus an element in S^1 has the form $\overline{\theta}$ where $\theta \in \mathbb{R}$.

Exercise 18. Does the Borsuk–Ulam theorem hold for the torus? In other words, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(\overline{\theta}, \overline{\vartheta}) \in S^1 \times S^1$ such that $f(\overline{\theta}, \overline{\vartheta}) = f(\overline{\theta + \pi}, \overline{\vartheta + \pi})$?

Solution 18. No: let $\iota_{r,R}\colon S^1 \times S^1 \to \mathbb{R}^3$ be the embedding of the torus in $\mathbb{R}^3$ given parametrically by

$$\begin{aligned} x(\overline{\theta}, \overline{\vartheta}) &= (R + r\cos\overline{\theta})\cos\overline{\vartheta} \\ y(\overline{\theta}, \overline{\vartheta}) &= (R + r\cos\overline{\theta})\sin\overline{\vartheta} \\ z(\overline{\theta}, \overline{\vartheta}) &= r\sin\overline{\theta} \end{aligned}$$

Here R is the distance from the center of the tube to the center of the torus and r is the radius of the tube. For this problem it doesn't matter what r and R are; we can set them both equal to 1 and denote $\iota = \iota_{1,1}$. Note that this map is well-defined since the cosine and sin functions are 2π-periodic. Next let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map given by $\pi(x,y,z) = (x,y)$. Clearly both ι and π are continuous, so the composite $f := \pi \circ \iota$ is also continuous. Furthermore, it is straightforward to check that $f(\overline{\theta}, \overline{\vartheta}) = f(\overline{\theta + \pi}, \overline{\vartheta + \pi})$ for any $(\overline{\theta}, \overline{\vartheta}) \in S^1 \times S^1$.

Exercise 19. Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk–Ulam theorem to show that there is one plane $\mathcal{P} \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Solution 19. Step 1: Fix $\boldsymbol{s} \in S^2$ and let A be an arbitrary compact set in $\mathbb{R}^3$. We will find a plane with normal vector $\boldsymbol{s}$ which divides A into two pieces of equal measure. Let $t \in \mathbb{R}$, and let $P(\boldsymbol{s}, t)$ be the plane in $\mathbb{R}^3$ which passes through the point $t\boldsymbol{s}$ and with normal vector $\boldsymbol{s}$. Thus $P(\boldsymbol{s}, t)$ is given by

$$P(\boldsymbol{s}, t) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \ell(\boldsymbol{s}, t) = 0\}$$

where $\ell(\boldsymbol{s}, t) = s_1x_1 + s_2x_2 + s_3x_3 - t$. The plane $P(\boldsymbol{s}, t)$ partitions the compact set A into two pieces, namely $A = A^+(\boldsymbol{s}, t) \cup A^-(\boldsymbol{s}, t)$ where

$$A^+(\boldsymbol{s}, t) = \{\boldsymbol{a} \in A \mid \boldsymbol{a} \geq \ell(\boldsymbol{s}, t)\} \quad \text{and} \quad A^-(\boldsymbol{s}, t) = \{\boldsymbol{a} \in A \mid \boldsymbol{a} \leq \ell(\boldsymbol{s},t)\}.$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = \mathrm{m}(A^+(\boldsymbol{s}, t))$. It is easy to show that since A is bounded, the function f is continuous in t, and that there exists $T \in \mathbb{R}$ such that $f(-T) = 0$ and $f(T) = 1$. By the intermediate value theorem, there exists $t_0 \in [-T, T]$ such that $f(t_0) = 1/2$. Let $a = \inf\{t \in \mathbb{R} \mid f(t) = 1/2\}$ and let $b = \sup\{t \in \mathbb{R} \mid f(t) = 1/2\}$. We set $t_A(\boldsymbol{s}) = (a+b)/2$. Thus any plane of the form $P(\boldsymbol{s}, t)$, where $a \leq t \leq b$, divides A into two pieces, and the plane $P(\boldsymbol{s}, t_A(\boldsymbol{s}))$ is the one in the "middle" which divides A into two pieces of equal measure.

Step 2: For each $\boldsymbol{s} \in S^2$, let $P_i(\boldsymbol{s}, t_i(\boldsymbol{s}))$ be the "middle" plane which divides A_i into two pieces of equal measure where $t_i(\boldsymbol{s}) = t_{A_i}(\boldsymbol{s})$ for each $i = 1, 2, 3$. Define $\varphi\colon S^2 \to \mathbb{R}^2$ by

$$\varphi(\boldsymbol{s}) = (t_3(\boldsymbol{s}) - t_1(\boldsymbol{s}), t_3(\boldsymbol{s}) - t_2(\boldsymbol{s})).$$

This is a continuous map such that $\varphi(-\boldsymbol{s}) = -\varphi(\boldsymbol{s})$, so by Borsuk-Ulam, there exists $\boldsymbol{s}_0 \in S^2$ such that $\varphi(\boldsymbol{s}_0) = \varphi(-\boldsymbol{s}_0)$, which is equivalent to saying

$$t_1(\boldsymbol{s}_0) = t_2(\boldsymbol{s}_0) = t_3(\boldsymbol{s}_0).$$

In other words, $P_i(\boldsymbol{s}_0, t_i(\boldsymbol{s}_0))$ is the same plane for each $i = 1, 2, 3$.

Remark 12. I used https://math.stackexchange.com/questions/1166179/hatcher-exercise-9-chapter-1-using-borsuk-ulams-theorem as a reference for this solution.

Exercise 20. Show that there are no retractions $r\colon X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
3. $X = S^1 \times D^2$ and A the circle shown in the figure.
4. $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
5. X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
6. X the Möbius band and A its boundary circle.

Solution 20. First consider the most general case where X is an arbitrary topological space and where A is an arbitary subspace of X with $\iota\colon A \to X$ denoting the inclusion map. Suppose a retraction $r\colon X \to A$ exists. Since $\pi_1\colon \mathbf{Top} \to \mathbf{Gp}$ is a functor, we have

$$\begin{aligned} 1_{\pi_1(A)} &= \pi_1(1_A) \\ &= \pi_1(r \circ \iota) \\ &= \pi_1(r) \circ \pi_1(\iota). \end{aligned}$$

Thus we have the identity

$$1_{\pi_1(A)} = \pi_1(r) \circ \pi_1(\iota). \tag{16}$$

There are at least two ways we can obtain a contradiction from (16):

- If $\pi_1(A) \neq 0$ and $\pi_1(r)$ is not surjective, then $\pi_1(r) \circ \pi_1(\iota)$ is not surjective which contradicts (16).
- If $\pi_1(A) \neq 0$ and $\pi_1(\iota) = 0$, then $1_{\pi_1(A)} \neq 0 = \pi_1(r) \circ \pi_1(\iota)$ which contradicts (16).

We now consider the special cases:

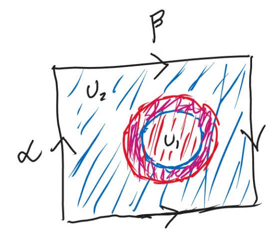
1. In this case, we have $\pi_1(A) = \mathbb{Z}$ and $\pi_1(\iota) = 0$ (since $\pi_1(X) = 0$), which contradicts (16).
2. In this case, we have $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(r)$ is not surjective (since $\pi_1(X) = \mathbb{Z}$), which contradicts (16).
3. In this case, we have $\pi_1(A) = \mathbb{Z} = \pi_1(X)$ and $\pi_1(\iota) = 2$. In particular, $\mathrm{im}(\pi_1(r) \circ \pi_1(\iota)) \subseteq 2\mathbb{Z}$ which contradicts (16).
4. In this case, we have $\pi_1(A) = \mathbb{Z} \star \mathbb{Z}$ and $\pi_1(\iota) = 0$ (since $\pi_1(X) = 0$), which contradicts (16).
5. In this case, we have $\pi_1(A) = \mathbb{Z} \star \mathbb{Z}$ and $\pi_1(r)$ is not surjective (since $\pi_1(X) = \mathbb{Z}$), which contradicts (16).
6. In this case, we have $\pi_1(A) = \mathbb{Z} = \pi_1(X)$ and $\pi_1(\iota) = 2$. In particular, $\mathrm{im}(\pi_1(r) \circ \pi_1(\iota)) \subseteq 2\mathbb{Z}$ which contradicts (16).

Problem 4

Exercise 21. Use van Kampen's theorem to compute the fundamental group of the Klein bottle and projective plane.

Solution 21. I wrote this solution down by hand:

Klein Bottle K



$U_1 \sim \bullet \qquad \pi_1(U_1) = 1$

$U_2 \sim \infty \qquad \pi_1(U_2) = \mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta \rangle$

$U_1 \cap U_2 \sim S^1 \qquad \pi_1(U_1 \cap U_2) = \mathbb{Z}$

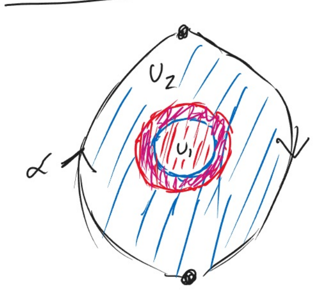
$$\begin{aligned} \pi_1(K) &= \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \\ &= 1 *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z}) \\ &= \langle \alpha, \beta \mid 1 = \alpha\beta\alpha^{-1}\beta \rangle \end{aligned}$$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1(K) \\ & \mathbb{Z} * \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

$1 \mapsto 1$

$1 \mapsto \alpha\beta\alpha^{-1}\beta$

Projective Plane $\mathbb{RP}^2$



$U_1 \sim \bullet \qquad \pi_1(U_1) = 1$

$U_2 \sim S^1 \qquad \pi_1(U_2) = \mathbb{Z} = \langle \alpha \rangle$

$U_1 \cap U_2 \sim S^1 \qquad \pi_1(U_1 \cap U_2) = \mathbb{Z}$

$$\begin{aligned} \pi_1(K) &= \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \\ &= 1 *_{\mathbb{Z}} \mathbb{Z} \\ &= \langle \alpha \mid \alpha^2 = 1 \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1(\mathbb{RP}^2) \\ & \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

$1 \mapsto 1$

$1 \mapsto \alpha^2$

14 Fundamental Group Homework

Exercise 22. On Page 14 of the online version of Hatcher, there is a diagram of a genus three surface as a quotient of a 12-gon. Compute the fundamental group of this surface in the following two ways:

1. As a quotient of a free group on 6 elements via the attached disk.
2. Using the Seifert-van Kampen theorem by splitting the genus three surface into a punctured genus two and a punctured genus one surface (please let me know if you need a sketch of this setup).

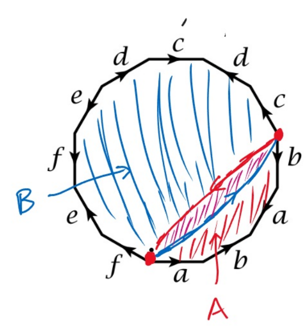
Solution 22. Let X be the genus three surface and let x be the point in X corresponding to any one of the vertices of the 12-gon. Then $\pi_1(X)$ is generated by the loops a, b, c, d, e, f subject to the the relation

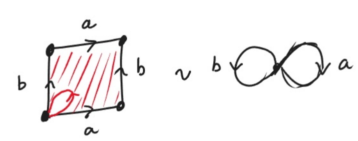
$$[a, b][c, d][e, f] = 1$$

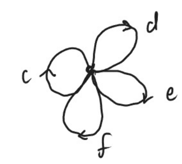
where $[\cdot, \cdot]$ denotes the commutator, given by $[x, y] = xyx^{-1}y^{-1}$. Thus

$$\pi_1(X) = \langle a, b, c, d, e, f \mid [a, b][c, d][e, f] = 1\rangle$$

2. We work this out below:



$A =$ 

$B \sim$ 

$$\begin{tikzcd} & \mathbb{Z}*\mathbb{Z} \arrow[dr] & \\ \mathbb{Z} \arrow[ur, "1 \mapsto {[a,b]^{-1}}"] \arrow[dr, "1 \mapsto {[c,d][e,f]}"'] & & \pi_1(X) \\ & \mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z} \arrow[ur] & \end{tikzcd}$$

$A \cap B \sim S^1$

$$\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

$$= (\mathbb{Z}*\mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z})$$

$$= \langle a, b, c, d, e, f \mid [a,b]^{-1} = [c,d][e,f] \rangle$$

Problem 2

Exercise 23. Let $X = S^1 \vee S^1$ and let a and b be the generators of $\pi_1(X)$ corresponding to the two summands.

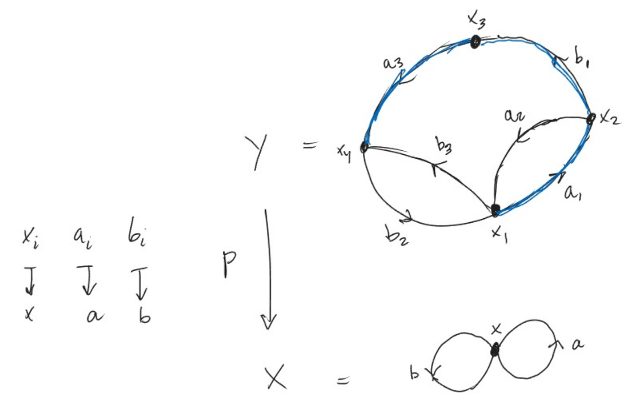
1. Draw a picture of the covering space of X with fundamental group $\langle a^2, b^2, (ab)^2\rangle$ and explain why this covering space corresponds to the given group. Does this covering space have any deck transformations?

2. Draw a picture of the covering space of X with fundamental group the normal group generated by a^2, b^2, and $(ab)^2$ and explain why this covering space corresponds to the given group. Find all deck transformations of this covering space.

Solution 23. 1. We denote this covering space by Y and work out the details below:

$$(abab^{-1})^{-1}(abab) = b^2$$

$$(abab)(b^2)^{-1} = abab^{-1}$$



$$H = p_* \pi_1(Y) = \langle a^2, abab, abab^{-1} \rangle = \langle a^2, abab, b^2 \rangle$$

$$G = \pi_1(X) = \langle a, b \rangle$$

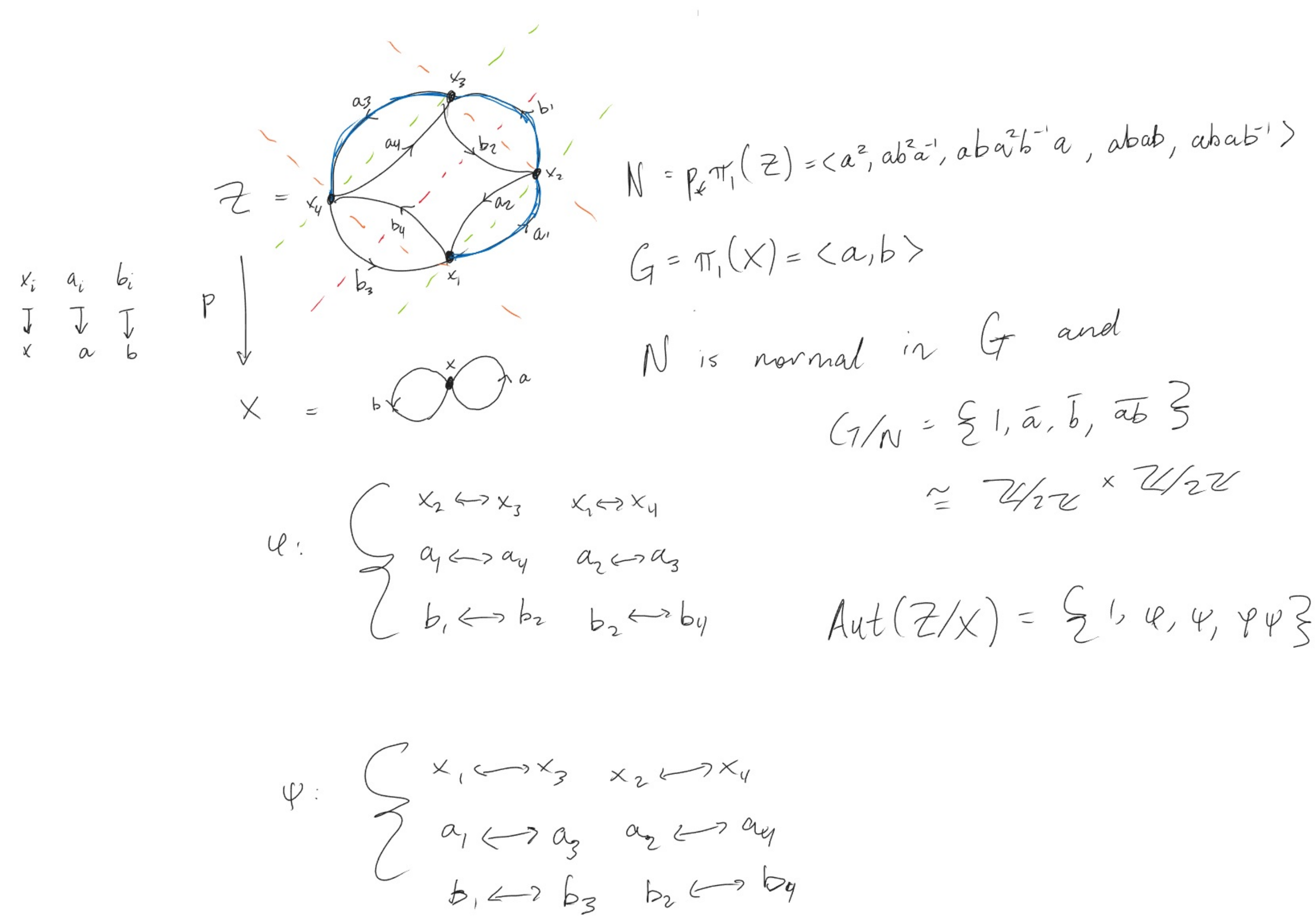
$N_G(H) = H$, thus $\mathrm{Aut}(Y/X) = N_G(H)/H = 1$

We calculate $\pi_1(Y)$ using Proposition 1A.2 in Hatcher where the maximal tree we use is colored in blue. This tells us that $\pi_1(Y) = \langle a^2, abab, abab^{-1} \rangle$, and since

$$\begin{aligned} (abab^{-1})^{-1}(abab) &= b^2 \\ b^2(abab) &= abab^{-1}, \end{aligned}$$

it follows that $\pi_1(Y) = \langle a^2, abab, b^2 \rangle$. Finally, note that there are no deck transformations here. The reason is that if $\varphi \colon Y \to Y$ is a homeomorphism such that $p \circ \varphi = \varphi$, then we are forced to have $\varphi(x_i) = x_i$ for $i = 1, 2, 3, 4$. This further implies that $\varphi(a_i) = a_i$ and $\varphi(b_i) = b_i$. Thus φ is the identity map.

2. We denote this covering space by Z and work out the details below:



We again use Proposition 1A.2 in Hatcher to calculate the fundamental group. We have two deck transformations φ and ψ which generate all of $\mathrm{Aut}(Z/X)$. We can think of φ as acting on Z via reflections across the dashed lines in the image above, and we can think of ψ as acting on Z via a 180 degree counterclockwise rotation. Altogether we have $\mathrm{Aut}(Z/X) = \{1, \varphi, \psi, \varphi\psi\}$, and we know that this is all of them since

$$\mathrm{Aut}(Z/X) \cong \mathrm{N}_G(N)/N = G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

15 Cohomology Homework

In this homework, we will make use of the universal coefficient theorem for cohomology involving the Ext functor says that if G is an abelian group, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X), G) \longrightarrow \mathrm{H}^n(X; G) \xrightarrow{[[\cdot]]} \mathrm{Hom}(\mathrm{H}_n(X), G) \longrightarrow 0 \tag{17}$$

where $[[\cdot]]$ is defined as follows: if $[\varphi] \in \mathrm{H}^n(X; G)$ where $\varphi\colon C_n(X) \to G$ satisfies $\varphi\partial = 0$, and if $[a] \in \mathrm{H}_n(X)$ where $a \in C_n(X)$ satisfes $\partial(a) = 0$, then we set $[[\varphi]]$ to be the map from $\mathrm{H}_n(X) \to G$ given by

$$[[\varphi]][a] = \varphi(a).$$

This is well-defined since if $[\varphi] = [\varphi + \psi\partial]$ where $\psi\colon C_{n-1}(X) \to G$ and $[a] = [a + \partial(b)]$ where $b \in C_{n+1}(X)$, then we have

$$\begin{aligned} [[\varphi + \psi\partial]][a + \partial b] &= (\varphi + \psi\partial)(a + \partial b) \\ &= \varphi(a) + \varphi\partial(b) + \psi\partial(a) + \psi\partial\partial(b) \\ &= \varphi(a). \end{aligned}$$

Moreover, the map $[[\cdot]] = [[\cdot]]_X$ is *natural* in X. This means that if $f\colon X \to Y$ is a continuous map, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^n(Y; G) \arrow[r, "{[[\cdot]]_Y}"] \arrow[d, "\mathrm{H}(f^*)"'] & \mathrm{Hom}(\mathrm{H}_n(Y), G) \arrow[d, "(\mathrm{H}(f_*))^*"] \\ \mathrm{H}^n(X; G) \arrow[r, "{[[\cdot]]_X}"'] & \mathrm{Hom}(\mathrm{H}_n(X), G) \end{tikzcd}$$

Indeed, if $[a] \in \mathrm{H}_n(X)$ and $[\psi] \in \mathrm{H}^n(Y;G)$, then we have

$$\begin{aligned}((f_\star)^*([[\psi]]_Y)[a] &= [[\psi]]_Y[f_\star(a)] \\ &= \psi(f_\star(a)) \\ &= (f^\star\psi)(a) \\ &= [[f^\star\psi]]_X[a].\end{aligned}$$

It follows that $[[\cdot]]_X \circ \mathrm{H}(f^\star) = (\mathrm{H}(f_\star))^* \circ [[\cdot]]_Y$.

Remark 13. Note that in our notation, we use the $\star$ symbol to denote chain maps. For instance, a continuous map $f\colon X \to Y$ induces a chain map $f_\star\colon C_\star(X) \to C_\star(Y)$ which is defined on singular chains $a = \sum r_i\sigma_i \in C_\star(X)$ by

$$f_\star(a) = \sum r_i(f \circ \sigma).$$

This in turn induces a cochain map $f^\star\colon C^\star(Y) \to C^\star(X)$ which is defined by mapping the singular cochain $\varphi \in C^\star(Y)$ to the singular cochain $f^\star(\varphi) \in C^\star(X)$ which is defined on chains $a \in C_\star(X)$ by

$$f^\star(\varphi)(a) = \varphi(f_\star(a)).$$

Problem 1

Exercise 24. Let T be the torus, let K be the Klein bottle, and let P be the real projective plane.

1. Use the universal coefficient theorem to compute the cohomology of T, K, and P over $\mathbb{Z}$.
2. Use the definition to compute the simplicial cohomology of T, K, and K over $\mathbb{Z}$ using the Δ-complex structure on a square formed from two triangles.

Solution 24. When we set $G = \mathbb{Z}$ in (17), then the universal coefficient theorem takes the form:

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X),\mathbb{Z}) \longrightarrow \mathrm{H}^n(X) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(X),\mathbb{Z}) \longrightarrow 0 \tag{18}$$

We will use this short exact sequence to compute the cohomologies of T, K, and P over $\mathbb{Z}$. We first consider T. Recall that

$$\mathrm{H}_i(T) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

In each case, we have $\mathrm{Ext}^1(\mathrm{H}_{i-1}(T),\mathbb{Z}) = 0$ since $\mathrm{H}_{i-1}(T)$ is a free $\mathbb{Z}$-module for all i (note that 0 is the free module with empty set as basis). Therefore (18) gives us

$$\mathrm{H}^i(T) \simeq \mathrm{Hom}(\mathrm{H}_i(T),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

Now we first consider K. Recall that

$$\mathrm{H}_i(K) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

This time $\mathrm{H}_{i-1}(K)$ is free for all i *expect* $i = 2$. Therefore (18) gives us

$$\mathrm{H}^i(K) \simeq \mathrm{Hom}(\mathrm{H}_i(K),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = 1 \\ 0 & i \neq 0,1,2 \end{cases}$$

where we used the fact that

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \\ &= \mathbb{Z} \oplus 0 \\ &= \mathbb{Z}.\end{aligned}$$

It remains to calculate $\mathrm{H}^2(K)$. In this case, the short exact sequence (18) gives us

$$0 \to \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{H}^2(K) \to 0 \to 0$$

where we used the fact that $\mathrm{H}_2(X) = 0$. Since Ext takes finite direct sums in the first variable to direct sum (more generally it takes direct sums in the first variable to products), we have

$$\begin{aligned}\mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) &= \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &= 0 \oplus \mathbb{Z}/2\mathbb{Z} \\ &= \mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

Thus $\mathrm{H}^2(K) = \mathbb{Z}/2\mathbb{Z}$. Finally, we consider P. Recall that

$$\mathrm{H}_i(P) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

Again, $\mathrm{H}_{i-1}(P)$ is free for all i *expect* $i = 2$. Therefore (18) gives us

$$\mathrm{H}^i(P) \simeq \mathrm{Hom}(\mathrm{H}_i(P), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & i \neq 0, 2 \end{cases}$$

where we used the fact that $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$. It remains to calculate $\mathrm{H}^2(P)$. In this case, the short exact sequence (18) gives us

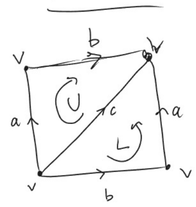
$$0 \to \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{H}^2(P) \to 0 \to 0$$

where we used the fact that $\mathrm{H}_2(P) = 0$. In particular, this imlpies $\mathrm{H}^2(P) = \mathbb{Z}/2\mathbb{Z}$.

2. First we calculate the cohomology of the Torus below:

smith normal form of $\begin{pmatrix}1&1\\1&1\\-1&-1\end{pmatrix}$ is $\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}$

Torus



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{c|cc} & U & L \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ c & -1 & -1 \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{c|ccc} & a & b & c \\ \hline v & 0 & 0 & 0 \end{array}} \underset{0}{\mathbb{Z}} \to 0$$

$U \mapsto a+b-c$

$L \mapsto a+b-c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}} \xrightarrow{\begin{pmatrix}0\\0\\0\end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix}1&1&-1\\1&1&-1\end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$$H_0(F^*) = \mathbb{Z}$$

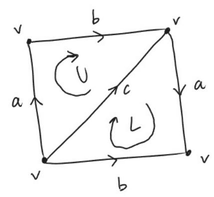
$$\begin{aligned} H_{-1}(F^*) &= \ker\begin{pmatrix}1&1&-1\\1&1&-1\end{pmatrix} \\ &= \ker\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix} \\ &= \mathbb{Z}^2 \end{aligned}$$

$$\begin{aligned} H_{-2}(F^*) &= \mathbb{Z}^2/\mathrm{im}\begin{pmatrix}1&1&-1\\1&1&-1\end{pmatrix} \\ &\cong \mathbb{Z}^2/\mathrm{im}\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix} \\ &= \mathbb{Z} \end{aligned}$$

Next we calculate the cohomology of the Klein bottle below:

smith normal form of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$

Klein Bottle



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{c|cc} & U & L \\ \hline a & 1 & 1 \\ b & 1 & -1 \\ c & -1 & 1 \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{c|ccc} & a & b & c \\ \hline v & 0 & 0 & 0 \end{array}} \underset{0}{\mathbb{Z}} \to 0$$

$U \mapsto a + b - c$

$L \mapsto a - b + c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$H_0(F^*) = \mathbb{Z}$

$H_{-1}(F^*) = \ker \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \ker \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \mathbb{Z}$

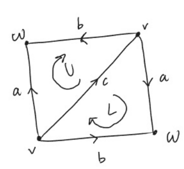
$H_{-2}(F^*) = \mathbb{Z}^2 / \operatorname{im} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \mathbb{Z}^2 / \operatorname{im} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \mathbb{Z}/2\mathbb{Z}$

Finally we calculate the cohomology of the real projective plane below:

smith normal form of $\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$

smith normal form of $\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Real Projective Plane



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{c|cc} & U & L \\ \hline a & 1 & 1 \\ b & -1 & -1 \\ c & -1 & 1 \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{c|ccc} & a & b & c \\ \hline v & -1 & -1 & 0 \\ w & 1 & 1 & 0 \end{array}} \underset{0}{\mathbb{Z}^2} \to 0$$

$U \mapsto a - b - c$

$L \mapsto a - b + c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}^2} \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$H_0(F^*) = \ker \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{Z}$

$H_{-1}(F^*) = \ker \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \Big/ \operatorname{im} \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \cong \ker \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \Big/ \operatorname{im} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$H_{-2}(F^*) = \mathbb{Z}^2 / \operatorname{im} \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \mathbb{Z}^2 / \operatorname{im} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \mathbb{Z}/2\mathbb{Z}$

Problem 2

Exercise 25. Show that if $f\colon S^n \to S^n$ has degree d, then $f^*\colon \mathrm{H}^n(S^n; G) \to \mathrm{H}^n(S^n; G)$ is multiplication by d map.

We prove this in a more general situation:

Exercise 26. Let $f\colon X \to X$ be a continuous map such that $\mathrm{H}(f_\star) = d$ where $d \in \mathbb{Z}$ (that is, $\mathrm{H}(f_\star)\colon \mathrm{H}_\star(X) \to \mathrm{H}_\star(X)$ is the multiplication by d map). Furthermore, assume that $\mathrm{H}_i(X)$ is free for all $i \in \mathbb{Z}$. Then $\mathrm{H}(f^\star) = d$ (that is, $\mathrm{H}(f^\star)\colon \mathrm{H}^\star(X;G) \to \mathrm{H}^\star(X;G)$ is the multiplication by d map).

Solution 25. Since each $\mathrm{H}_i(X)$ is free, the universal coefficient theorem gives us the following commutative diagram

$$\begin{tikzcd} \mathrm{H}^n(X;G) \arrow[r, "{[[\cdot]]}"] \arrow[d, "\mathrm{H}(f^\star)"'] & \mathrm{Hom}(\mathrm{H}_n(X), G) \arrow[d, "d^*"] \\ \mathrm{H}^n(X;G) \arrow[r, "{[[\cdot]]}"'] & \mathrm{Hom}(\mathrm{H}_n(X), G) \end{tikzcd}$$

where $[[\,\cdot\,]]$ is an isomorphism. In particular, we have

$$\mathrm{H}(f^\star) = [[\cdot]]^{-1} \circ d^* \circ [[\cdot]].$$

Note that $d^* = d$ since all maps are $\mathbb{Z}$-linear and d is an integer. Next note that $d \circ [[\cdot]] = [[\cdot]] \circ d$ since d is an integer and $[[\cdot]]$ is a $\mathbb{Z}$-linear isomorphism. Thus we have

$$\begin{aligned} \mathrm{H}(f^\star) &= [[\cdot]]^{-1} \circ d^* \circ [[\cdot]] \\ &= [[\cdot]]^{-1} \circ d \circ [[\cdot]] \\ &= [[\cdot]]^{-1} \circ [[\cdot]] \circ d \\ &= d. \end{aligned}$$

Problem 3

Exercise 27. Use cup products over $\mathbb{Z}/2\mathbb{Z}$ to show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Solution 26. On the one hand, we have

$$\begin{aligned} \mathrm{H}^\star(\mathbb{RP}^2 \vee S^3; \mathbb{Z}/2\mathbb{Z}) &= \mathrm{H}^\star(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^\star(S^3; \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{F}_2[x]/\langle x^3 \rangle \times \mathbb{F}_2[y]/\langle y^2 \rangle, \end{aligned}$$

where $|x| = 1$ and $|y| = 3$. On the other hand, we have

$$\mathrm{H}^\star(\mathbb{RP}^3; \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[z]/\langle z^4 \rangle$$

where $|z| = 1$. These rings are not isomorphic. For instance,

$$\mathrm{Spec}(\mathbb{F}_2[x]/\langle x^3 \rangle \times \mathbb{F}_2[y]/\langle y^2 \rangle) = \{\langle \overline{x} \rangle, \langle \overline{y} \rangle, \langle \overline{x}, \overline{y} \rangle\}$$

consists of three points, however

$$\mathrm{Spec}(\mathbb{F}_2[z]/\langle z^4 \rangle) = \{\langle \overline{z} \rangle\}$$

only has one point.

Appendix

We calculate $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ as follows: let F be the free $\mathbb{Z}$-complex below

$$F = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0,$$

where $F_0 = \mathbb{Z} = F_1$ and $F_i = 0$ for all $i \neq 0, 1$. Then F is a free resolution of $\mathbb{Z}/2\mathbb{Z}$. Next we set $F^\star := \mathrm{Hom}^\star(F, \mathbb{Z})$ (this is the hom-complex where

$$F_i^\star = \{\text{graded homomorphisms of degree } i \text{ from } F \text{ to } \mathbb{Z}\}.$$

In particular,

$$\begin{aligned} F_0^\star &= \{\text{homomorphisms from } F_0 \text{ to } \mathbb{Z}\} = \mathbb{Z} \\ F_{-1}^\star &= \{\text{homomorphisms from } F_1 \text{ to } \mathbb{Z}\} = \mathbb{Z} \end{aligned}$$

and $F_{-1}^\star = 0$ for all $i \neq 0, -1$. The differential $\mathrm{d}_0^\star\colon F_0 \to F_{-1}$ is easily seen to be the multiplication by 2 map, so

$$F^\star = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0.$$

Finally we have

$$\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathrm{H}_{-1}(F^\star) = \mathbb{Z}/2\mathbb{Z}.$$