A Generalized Associator

0.1 A Generalized Associator

Let F be an R-module and let $\mu, \nu \colon F^{\otimes 2} \to F$ and let $\lambda \colon F \to F$ be R-linear maps (where we denote $F^{\otimes 2} := F \otimes_R F$). We set $[\cdot]_{\mu,\nu,\lambda} \colon F^{\otimes 3} \to F$ to be the R-linear map given by

$$[\cdot]_{\mu,\nu,\lambda} := \mu(\nu \otimes \lambda - \lambda \otimes \nu).$$

We denote by $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda} \colon F^3 \to F$ to be the unique R-trilinear map which corresponds to $[\cdot]_{\mu,\nu,\lambda}$. Thus if we denote $a_1 a_2 = \mu(a_1 \otimes a_2)$ and $a_1 \cdot a_2 = \nu(a_1 \otimes a_2)$ for $a_1 \otimes a_2 \in F^{\otimes 2}$, then we have

$$[a_1 \otimes a_2 \otimes a_3]_{\mu,\nu,\lambda} = (a_1 \cdot a_2)\lambda(a_3) - \lambda(a_1)(a_2 \cdot a_3) = [a_1, a_2, a_3]_{\mu,\nu,\lambda}.$$

We often pass back in forth between $[\cdot]_{\mu,\nu,\nu}$ and $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda}$ without explicitily saying so (mostly we will only talk about $[\cdot]_{\mu,\nu,\nu}$ since it is notationally simpler to write). For istance, we call $[\cdot]_{\mu,\nu,\lambda}$ the **associator** with respect to the triple (μ,ν,λ) (or more simply just **associator** if (μ,ν,λ) is understood from context), and thus we also call $[\cdot,\cdot,\cdot]_{\mu,\nu,\lambda}$ the **associator**. If $\mu = \nu$, then we simplify our notation and write $[\cdot]_{\mu,\lambda} := [\cdot]_{\mu,\mu,\lambda}$. Similarly, if $\mu = \nu$ and $\lambda = 1$, then we simplify our notation further and write $[\cdot]_\mu := [\cdot]_{\mu,\mu,1}$.

Observe that $[\cdot]_{\mu,\nu,\lambda}$ is R-trilinear in μ, ν, and λ. In particular, this means that if $\mu', \nu' \colon F^{\otimes 2} \to F$ and $\lambda' \colon F \to F$ are another triple of R-linear maps, and $r \in R$, then we have

$$\begin{aligned}
[\cdot]_{\mu+\mu',\nu,\lambda} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu',\nu,\lambda} \\
[\cdot]_{\mu,\nu+\nu',\lambda} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu,\nu',\lambda} \\
[\cdot]_{\mu,\nu,\lambda+\lambda'} &= [\cdot]_{\mu,\nu,\lambda} + [\cdot]_{\mu,\nu,\lambda'} \\
r[\cdot]_{\mu,\nu,\lambda} &= [\cdot]_{r\mu,\nu,\lambda} = [\cdot]_{\mu,r\nu,\lambda} = [\cdot]_{\mu,\nu,r\lambda}.
\end{aligned}$$

Thus we have an R-linear map

$$[\cdot]_{(-,-,-)} \colon \mathrm{Hom}(F^{\otimes 2}, F)^{\otimes 2} \otimes \mathrm{Hom}(F,F) \to \mathrm{Hom}(F^{\otimes 3}, F)$$

which takes an elementary tensor $\mu \otimes \nu \otimes \lambda$ in $\mathrm{Hom}(F^{\otimes 2}, F)^{\otimes 2} \otimes \mathrm{Hom}(F,F)$ and maps it to $[\cdot]_{\mu,\nu,\lambda}$ in $\mathrm{Hom}(F^{\otimes 3}, F)$. In particular, note that

$$\begin{aligned}
[\cdot]_{\mu+\mu'} &= [\cdot]_{\mu+\mu',\mu+\mu'} \\
&= [\cdot]_{\mu,\mu} + [\cdot]_{\mu,\mu'} + [\cdot]_{\mu',\mu} + [\cdot]_{\mu',\mu'} \\
&= [\cdot]_{\mu} + [\cdot]_{\mu'} + [\cdot]_{\mu,\mu'} + [\cdot]_{\mu',\mu}
\end{aligned}
\qquad\qquad
\begin{aligned}
[\cdot]_{r\mu} &= [\cdot]_{r\mu,r\mu} \\
&= r^2[\cdot]_{\mu,\mu} \\
&= r^2[\cdot]_{\mu}
\end{aligned}$$

Proposition 0.1. *Let $t \in R$ and let $\mu_0, \mu_1 \in \mathrm{Mult}(F)$. Furthermore we set $\mu_t = t\mu_1 + (1-t)\mu_0$. Then we have*

$$[\cdot]_{\mu_t} = t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t(1-t)([\cdot]_{\mu_1,\mu_0} + [\cdot]_{\mu_0,\mu_1}).$$

Proof. We have

$$\begin{aligned}
[\cdot]_{\mu_t} &= [\cdot]_{t\mu_1+(1-t)\mu_0} \\
&= [\cdot]_{t\mu_1} + [\cdot]_{(1-t)\mu_0} + [\cdot]_{t\mu_1,(1-t)\mu_0} + [\cdot]_{(1-t)\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + [\cdot]_{t\mu_1,\mu_0-t\mu_0} + [\cdot]_{\mu_0-t\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + [\cdot]_{t\mu_1,\mu_0} + [\cdot]_{t\mu_1,-t\mu_0} + [\cdot]_{\mu_0,t\mu_1} + [\cdot]_{-t\mu_0,t\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t[\cdot]_{\mu_1,\mu_0} - t^2[\cdot]_{\mu_1,\mu_0} + t[\cdot]_{\mu_0,\mu_1} - t^2[\cdot]_{\mu_0,\mu_1} \\
&= t^2[\cdot]_{\mu_1} + (1-t)^2[\cdot]_{\mu_0} + t(1-t)([\cdot]_{\mu_1,\mu_0} + [\cdot]_{\mu_0,\mu_1}).
\end{aligned}$$

□

Now suppose $F = (F, \mathrm{d})$ is an R-complex. We view F is a graded R-module and we view $\mathrm{d}\colon F \to F$ as a graded R-linear map of degree -1 which satisfies $\mathrm{d}^2 = 0$. We further assume that μ is a chain map, i.e. it commtues with the differential. To clean notation in what follows, we denote the differentials of $F^{\otimes 2}$ and $F^{\otimes 3}$ by d again, where context will make clear which differential the symbol "d" refers to. For instance, we if $a_1, a_2 \in F$ with a_1 homogeneous, then we have

$$\mathrm{d}(a_1 \otimes a_2) = \mathrm{d}a_1 \otimes a_2 + (-1)^{|a_1|} a_1 \otimes \mathrm{d}a_2. \tag{1}$$

It is clear here that the d on the lefthand side of (1) is the differential for $F^{\otimes 2}$, whereas the d' on the righthand side are the differentials for F. If we wanted to be more formal, then our notation becomes more clunky-looking:

$$\mathrm{d}_{F^{\otimes 2}}(a_1 \otimes a_2) = \mathrm{d}_F(a_1) \otimes a_2 + (-1)^{|a_1|} a_1 \otimes \mathrm{d}_F(a_2).$$

Thus we will avoid this and use the simpler notation instead (where context makes everything clear).

Note that since μ is a chain map, we have $\mathrm{d}[\cdot]_{\mu,\nu,\lambda} = [\cdot]_{\mathrm{d}\mu,\nu,\lambda} = [\cdot]_{\mu\mathrm{d},\nu,\lambda}$. We claim that (up to some minor sign issues) we have

$$\mathrm{d}[\cdot]_{\mu,\nu,\lambda} = [\cdot]_{\mu,\mathrm{d}\nu,\lambda} + [\cdot]_{\mu,\nu,\mathrm{d}\lambda} \qquad [\cdot]_{\mu,\nu,\lambda}\mathrm{d} = [\cdot]_{\mu,\nu\mathrm{d},\lambda} + [\cdot]_{\mu,\nu,\lambda\mathrm{d}}. \tag{2}$$

Indeed the identities follow from the identities

$$\begin{aligned} \mathrm{d}(\nu \otimes \lambda) &= \mathrm{d}\nu \otimes \lambda + \overline{\nu} \otimes \mathrm{d}\lambda \qquad & (\nu \otimes \lambda)\mathrm{d} &= \nu \otimes \lambda\mathrm{d} + \nu\mathrm{d} \otimes \overline{\lambda} \\ \mathrm{d}(\lambda \otimes \nu) &= \mathrm{d}\lambda \otimes \nu + \overline{\lambda} \otimes \mathrm{d}\nu \qquad & (\lambda \otimes \nu)\mathrm{d} &= \lambda \otimes \nu\mathrm{d} + \lambda\mathrm{d} \otimes \overline{\nu} \end{aligned}$$

where $\overline{\nu}\colon F^{\otimes 2} \to F$ and $\overline{\lambda}\colon F \to F$ are defined by

$$\overline{\nu}(a_1 \otimes a_2) = (-1)^{|a_1|+|a_2|}\nu(a_1 \otimes a_2) \qquad \overline{\lambda}(a) = (-1)^{|a|}\lambda(a).$$

The identity (2) holds exactly in characteristic 2, however in general one should interpret with (2) with appropriate signs.

Proposition 0.2. *Let $\mu \in \mathrm{Mult}(F)$, let $h\colon F^{\otimes 2} \to F$, and set $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$. Then we have*

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d}$$

where $H = [\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}$.

Proof. We have

$$\begin{aligned} [\cdot]_{\mu_h} &= [\cdot]_{\mu+\mathrm{d}h+h\mathrm{d}} \\ &= [\cdot]_\mu + [\cdot]_{\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mathrm{d}h+h\mathrm{d},\mu} \\ &= [\cdot]_\mu + [\cdot]_{\mathrm{d}h} + [\cdot]_{h\mathrm{d}} + [\cdot]_{\mathrm{d}h,h\mathrm{d}} + [\cdot]_{h\mathrm{d},\mathrm{d}h} + [\cdot]_{\mu,\mathrm{d}h+h\mathrm{d}} + [\cdot]_{\mathrm{d}h+h\mathrm{d},\mu} \\ &= [\cdot]_\mu + \mathrm{d}[\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + \mathrm{d}[\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + [\cdot]_{\mathrm{d}h,\mu} + [\cdot]_{h\mathrm{d},\mu} \\ &= [\cdot]_\mu + \mathrm{d}[\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + \mathrm{d}[\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}[\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}\mu} + [\cdot]_{h,\mu,\mathrm{d}} \\ &= [\cdot]_\mu + [\cdot]_{h,\mathrm{d}h\mathrm{d}} + [\cdot]_{h,h\mathrm{d},\mathrm{d}} + [\cdot]_{h,\mathrm{d}h,\mathrm{d}} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + [\cdot]_{h,\mathrm{d}\mu} + [\cdot]_{h,\mu,\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) \\ &= [\cdot]_\mu + [\cdot]_{h,\mathrm{d}h}\mathrm{d} + [\cdot]_{h,h\mathrm{d}}\mathrm{d} + [\cdot]_{h,\mu}\mathrm{d} + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) \\ &= [\cdot]_\mu + [\cdot]_{\mu,\mathrm{d}h} + [\cdot]_{\mu,h\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu})\mathrm{d} \\ &= [\cdot]_\mu + \mathrm{d}[\cdot]_{\mu,h} + [\cdot]_{\mu,h,\mathrm{d}} + [\cdot]_{\mu,h}\mathrm{d} + [\cdot]_{\mu,h,\mathrm{d}} + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu})\mathrm{d} \\ &= [\cdot]_\mu + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h})\mathrm{d} \\ &= [\cdot]_\mu + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h}) + ([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}} + [\cdot]_{h,\mu} + [\cdot]_{\mu,h})\mathrm{d} \\ &= [\cdot]_\mu + \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}) + ([\cdot]_{\mu,h} + [\cdot]_{h,\mu_h})\mathrm{d} \\ &= [\cdot]_\mu + \mathrm{d}H + H\mathrm{d}. \end{aligned}$$

□

Now let's write

$$[\cdot]_{\varphi,\nu,\mu} = \varphi\nu - \mu\varphi^{\otimes 2}$$

Thus we have

$$\begin{aligned}
[\cdot]_{\varphi+\mathrm{d}h+h\mathrm{d},\nu+\mathrm{d}h'+h'\mathrm{d},\mu} &= (\varphi+\mathrm{d}h+h\mathrm{d})(\nu+\mathrm{d}h'+h'\mathrm{d}) - \mu(\varphi+\mathrm{d}h+h\mathrm{d})^{\otimes 2} \\
&= (\varphi+\mathrm{d}h+h\mathrm{d})(\nu+\mathrm{d}h'+h'\mathrm{d}) - \mu(\varphi+\mathrm{d}h+h\mathrm{d})^{\otimes 2}
\end{aligned}$$

We have

$$\begin{aligned}
[\cdot]_{\mu_h,\mu} &= [\cdot]_{\mu+\mathrm{d}h+h\mathrm{d},\mu} \\
&= [\cdot]_\mu + [\cdot]_{\mathrm{d}h,\mu} + [\cdot]_{h\mathrm{d},\mu} \\
&= [\cdot]_\mu + \mathrm{d}[\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}\mu} + [\cdot]_{h,\mu,\mathrm{d}}
\end{aligned}$$