Calculus of Ext and Tor

Unless otherwise specified, let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring. Ext and Tor show up all over the place in commutative algebra.
one is often presented with an R-complex A which is homologically bounded above and homologically bounded below, and would like to know *when* does $\mathrm{H}_i(A)$ vanish? In particular, we want to find an $\varepsilon, \delta \in \mathbb{Z}$ such that $\varepsilon \leq \delta$ and

$$\mathrm{H}_\delta(A) \neq 0$$
$$\mathrm{H}_\varepsilon(A) \neq 0$$
$$\mathrm{H}_i(A) = 0 \text{ for all } i < \varepsilon \text{ and } i > \delta.$$

Vanishing in Ext

1 Shifting and Antishifting

In order to get a better understanding of Ext and Tor, the first step is to understand their shifting/antishifting properties. The lesson that we shall learn is that covariant left exact functors typically satisfy the shift property whereas everything else usually satisfies the antishift property.

1.1 Shifting and Antishifting Depth

1.1.1 Antishift Property of Koszul Homology and Depth

Let R be a noetherian ring, let I be an ideal of R such that $I = \sqrt{\langle x_1, \dots, x_n \rangle} = \sqrt{\langle \boldsymbol{x} \rangle}$ where $x_1, \dots, x_n \in I$, and let N a finitely-generated R-module such that $N \neq IN$. Set $\delta = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(\boldsymbol{x}, N) \neq 0\}$ and let y be an N-regular sequence contained in $\langle \boldsymbol{x} \rangle$. Then we have an isomorphism

$$\mathrm{H}_\delta(\boldsymbol{x}, N) \simeq \mathrm{H}_{\delta+1}(\boldsymbol{x}, N/yN) \tag{1}$$

We think of (1) as an **antishift property** of Koszul homology with respect to depth in the sense that δ increases by one when we replace it by $\delta + 1$ whereas the $\langle \boldsymbol{x} \rangle$-depth (and hence I-depth) of N decreases by one when we replace N by N/yN (slogan: homological degree goes up, depth goes down). This antishift property is derived by considering the short exact sequence of R-modules

$$0 \to N \xrightarrow{y} N \to N/yN \to 0.$$

Then applying the right exact Koszul functor $\mathrm{H}(\mathcal{K}(\boldsymbol{x}, -))$ to this short exact sequence and using the fact that y kills $\mathrm{H}(\boldsymbol{x}, N)$, we obtain the short exact sequence of Koszul homologies

$$0 \to \mathrm{H}_i(\boldsymbol{x}, N) \to \mathrm{H}_i(\boldsymbol{x}, N/yN) \to \mathrm{H}_{i-1}(\boldsymbol{x}, N) \to 0.$$

1.1.2 Shift Property of Ext and Depth

Let R be a noetherian local ring, let I be an ideal of R, let N be a finitely-generated R-module such that $IN \neq N$, and let M be a finitely generated R-module such that $\mathrm{V}(\operatorname{Ann} M) = \mathrm{V}(I)$ (for instance one may take $M = R/I$ or $M = R/\sqrt{I}$). Set $\varepsilon = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}_R(M, N) \neq 0\}$ and let y be an N-regular element contained in $\operatorname{Ann} M$. Then we have an isomorphism

$$\mathrm{Ext}_R^\varepsilon(M, N) \simeq \mathrm{Ext}_R^{\varepsilon-1}(M, N/yN) \tag{2}$$

We think of (2) as a **shift property** of Ext with respect to depth in the sense that ε decreases by one when we replace it by $\varepsilon - 1$ whereas the I-depth of the second component also decreases by one when replace N by N/yN. This shift property is derived by considering the short exact sequence of R-modules

$$0 \to N \xrightarrow{y} N \to N/yN \to 0$$

Then applying the left exact covariant functor $\mathrm{Ext}_R(M, -)$ to this short exact sequence and using the fact that y kills $\mathrm{Ext}_R(M, N)$, we obtain the short exact sequence of Ext modules:

$$0 \to \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N/yN) \to \mathrm{Ext}^{i+1}_R(M, N) \to 0$$

1.2 Shifting and Antishifting Syzygies

Let us explain what we mean: let M be a finitely generated R-module and let F be the minimal R-free resolution of M. Thus we have an exact sequence:

$$\cdots \longrightarrow F_2 \xrightarrow{\mathrm{d}_2} F_1 \xrightarrow{\mathrm{d}_1} F_0 \xrightarrow{\tau} M \longrightarrow 0 \tag{3}$$

For $i \geq 0$, we define the ith **syzygy** of M, denoted M_i, to be the image of $\mathrm{d}_i \colon F_i \to F_{i-1}$. If R is Gorenstein and M is a maximal Cohen-Macaulay R-module, then we can extend this definition to all $i \in \mathbb{Z}$. Indeed, let F be the minimal R-free resolution of $M^\star := \mathrm{Hom}_R(M, R)$. Thus we have an exact sequence:

$$\cdots \longrightarrow F'_2 \xrightarrow{\mathrm{d}'_2} F'_1 \xrightarrow{\mathrm{d}'_1} F'_0 \xrightarrow{\tau'} M^\star \longrightarrow 0 \tag{4}$$

Since $M^\star$ is maximal Cohen-Macaulay, the dual sequence is exact:

$$0 \longrightarrow M^{\star\star} \xrightarrow{(\tau')^\star} F_{-1} \xrightarrow{\mathrm{d}_{-1}} F_{-2} \xrightarrow{\mathrm{d}_{-2}} F_{-3} \longrightarrow \cdots \tag{5}$$

where we set $F_{-i} := (F'_{i-1})^\star$ and $\mathrm{d}_{-i} := (\mathrm{d}'_i)^\star$. Using the fact that M is reflexive, we can splice together (3) and (5) to get the doubly long infinite long exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[rr, "\mathrm{d}_2"] \arrow[dr] & & F_1 \arrow[rr, "\mathrm{d}_1"] \arrow[dr] & & F_0 \arrow[rr, "\mathrm{d}_0"] \arrow[dr] & & F_{-1} \arrow[rr, "\mathrm{d}_{-1}"] \arrow[dr] & & F_{-2} \arrow[r] & \cdots \\
& & M_2 \arrow[ur] & & M_1 \arrow[ur] & & M \arrow[ur] & & M_{-1} \arrow[ur] & &
\end{tikzcd}$$

where we set $\mathrm{d}_0 = (\tau')^\star \tau$. We call this the **completed** R-free resolution of M, and we abuse notation slighly and call this F again. With this understood, we define the ***i*th syzygy** of M, denoted M_i, to be the image of $\mathrm{d}_i \colon F_i \to F_{i-1}$ for all $i \in \mathbb{Z}$.

Proposition 1.1. *Let M and N finitely generated R-modules, and for $i \geq 0$, let M_i and N_i denote their respective syzygies. For $n \geq 1$, we have*

$$\begin{aligned}
\mathrm{Ext}^{n+1}_R(M_i, N) &\cong \mathrm{Ext}^n_R(M_{i+1}, N) \\
\mathrm{Tor}^R_{n+1}(M_i, N) &\cong \mathrm{Tor}^R_n(M_{i+1}, N) \\
\mathrm{Tor}^R_{n+1}(M, N_i) &\cong \mathrm{Tor}^R_n(M, N_{i+1})
\end{aligned}$$

Moreover, assume R is Gorenstein and M and N are maximal Cohen-Macaulay. Then the isomorphisms above continue to make sense for all $i \in \mathbb{Z}$ and we also get

$$\mathrm{Ext}^n_R(M, N_i) \cong \mathrm{Ext}^{n+1}_R(M, N_{i+1}).$$

Proof. For each i we have a short exact sequence of R-modules:

$$0 \longrightarrow M_{i+1} \longrightarrow F_i \longrightarrow M_i \longrightarrow 0 \tag{6}$$

After applying $\mathrm{Hom}_R(-, N)$ to this short exact sequence, we obtain a long exact sequence in homology:

$$
\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}_R^{n-1}(M_{i+1}, N) \arrow[dll, rounded corners] \\
\mathrm{Ext}_R^{n}(M_i, N) \arrow[r] & \mathrm{Ext}_R^{n}(F_i, N) \arrow[r] & \mathrm{Ext}_R^{n}(M_{i+1}, N) \arrow[dll, rounded corners] \\
\mathrm{Ext}_R^{n+1}(M_i, N) \arrow[r] & \cdots &
\end{tikzcd}
$$

Since $\mathrm{Ext}_R^n(F_i, N) = 0$ for all $n \geq 1$, we obtain isomorphisms

$$\mathrm{Ext}_R^{n+1}(M_i, N) \cong \mathrm{Ext}_R^{n}(M_{i+1}, N)$$

for all $n \geq 1$. The proof of the other isomorphisms follows a similar line of logic. $\square$