Advanced Linear Programming Homework 6

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Remark 1. I did two different versions of problem 5 because I thought the book may have made a typo. I labeled these problem 5 and problem 5′. Problem 5 is straight from the book, whereas problem 5′ is a slight variation.

Problem 1

For this problem, let $f(\boldsymbol{x}) = (x_2^2 - x_1)^2$.

Problem 1.a

Exercise 1. Find all stationary points of this function.

Solution 1. We have

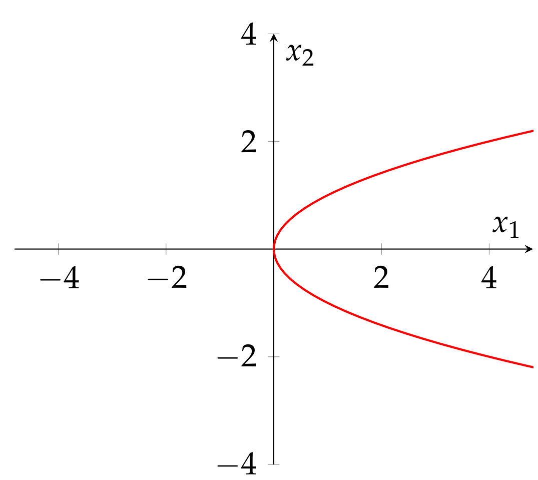
$$\begin{aligned} \boldsymbol{x} \text{ is a stationary point of } f &\iff \nabla f(\boldsymbol{x}) = 0 \\ &\iff 2\begin{pmatrix} x_1 - x_2^2 \\ 2x_2(x_2^2 - x_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff x_1 = x_2^2 \end{aligned}$$

Thus the set of all stationary points of f is given by $\mathcal{S}_f = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 - x_2^2 = 0\}$.

Problem 1.b

Exercise 2. From among the stationary points your found in part (a), identify those that are minima.

Solution 2. Let $\mathcal{Z}_f$ be the zero-set of f, that is $\mathcal{Z}_f = \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}) = 0\}$. Then it's easy to see that $\mathcal{Z}_f = \mathcal{S}_f$. Since $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$, it follows that every stationary point of f is a global minimizer with optimal objective value being 0. The set of all stationary points forms a curve in the plane drawn below:



In particular, none of the stationary points are no *strict* local minimizers.

Problem 2

For this problem, let $f(\boldsymbol{x}) = (x_2 - x_1^2)^2 + (1 - x_1)^2$.

Problem 2.a

Exercise 3. Find all stationary points of this function.

Solution 3. We have

$$\begin{aligned}
\boldsymbol{x} \text{ is a stationary point of } f &\iff \nabla f(\boldsymbol{x}) = 0 \\
&\iff 2\begin{pmatrix} -2(x_2 - x_1^2)x_1 - (1 - x_1) \\ x_2 - x_1^2 \end{pmatrix} = 0 \\
&\iff -2(x_2 - x_1^2)x_1 - (1 - x_1) = 0 \text{ and } x_2 - x_1^2 = 0 \\
&\iff -(1 - x_1) = 0 \text{ and } x_2 - x_1^2 = 0 \\
&\iff x_1 = 1 = x_2.
\end{aligned}$$

Thus there is only one stationary point of f, namely $\mathcal{S}_f = \{\boldsymbol{a}\}$ where we set $\boldsymbol{a} = (1, 1)$.

Problem 2.b

Exercise 4. Classify the points found in part (a) according to whether they are strict minimizing points, minimizing points, maximizing points, or points that are neither minimizing or maximizing points.

Solution 4. Since f is a sum of squares, we have $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Furthermore, we have $f(\boldsymbol{a}) = 0$. It follows that $\boldsymbol{a}$ is a global minimizer of f with optimal objective value being $f(\boldsymbol{a}) = 0$. Furthermore, observe that

$$\begin{aligned}
\boldsymbol{x} \in \mathcal{Z}_f &\iff f(\boldsymbol{x}) = 0 \\
&\iff (x_2 - x_1^2)^2 + (1 - x_1)^2 = 0 \\
&\iff x_2 - x_1^2 = 0 \text{ and } 1 - x_1 = 0 \\
&\iff x_1 = 1 = x_2.
\end{aligned}$$

It follows that $\mathcal{Z}_f = \{\boldsymbol{a}\} = \mathcal{S}_f$. In particular, if $\boldsymbol{x} \neq \boldsymbol{a}$, then $f(\boldsymbol{x}) > 0$. This implies $\boldsymbol{a}$ is a *strict* global minimizer (and the only one at that).

Problem 3

For this problem, let $f(\boldsymbol{x}) = ax_1^2 e^{x_2} + x_2^2 e^{x_3} + x_3^2 e^{x_1}$.

Problem 3.a

Exercise 5. For what values of a is the point $\boldsymbol{x}_0 = [0, 0, 0]^\top$ a stationary point?

Solution 5. We have

$$\begin{aligned}
\boldsymbol{x}_0 \text{ is a stationary point of } f &\iff \nabla f(\boldsymbol{x}_0) = 0 \\
&\iff \begin{pmatrix} 2ax_1 e^{x_2} + x_3^2 e^{x_1} \\ ax_1^2 e^{x_2} + 2x_2 e^{x_3} \\ x_2^2 e^{x_3} + 2x_3 e^{x_1} \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0} = 0 \\
&\iff 0 = 0.
\end{aligned}$$

In particular, for all $a \in \mathbb{R}$, the point $\boldsymbol{x}_0$ is a stationary point.

Problem 3.b

Exercise 6. Classify the points as a strict minimum, minimum, maximum, or strict maximum for the values of a found in part (a).

Solution 6. We break this into three cases:

Case 1: Suppose $a > 0$. Observe that

$$\begin{aligned}
\mathrm{H}_f(\boldsymbol{x}_0) &= \begin{pmatrix} 2ae^{x_2} + x_3^2 e^{x_1} & 2ax_1 e^{x_2} & 2x_3 e^{x_1} \\ 2ax_1 e^{x_2} & ax_1^2 e^{x_2} + 2e^{x_3} & 2x_2 e^{x_3} \\ 2x_3 e^{x_1} & 2x_2 e^{x_3} & 2e^{x_1} + x_2^2 e^{x_3} \end{pmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0} \\
&= \begin{pmatrix} 2a & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\end{aligned}$$

In particular, the Hessian of f at $\boldsymbol{x}_0$ is positive definite. This implies $\boldsymbol{x}_0$ is a strict minimizing point (with objective value being $f(\boldsymbol{x}_0) = 0$). Next, observe that

$$\begin{aligned}
\boldsymbol{x} \in \mathcal{Z}_f &\iff f(\boldsymbol{x}) = 0 \\
&\iff ax_1^2 e^{x_2} + x_2^2 e^{x_3} + x_3^2 e^{x_1} = 0 \\
&\iff (\sqrt{a} x_1 e^{x_2/2})^2 + (x_2 e^{x_3/2})^2 + (x_3 e^{x_1/2})^2 = 0 \\
&\iff \sqrt{a} x_1 e^{x_2/2} = 0 \text{ and } x_2 e^{x_3/2} = 0 \text{ and } x_3 e^{x_1/2} = 0 \\
&\iff x_1 = x_2 = x_3 = 0,
\end{aligned}$$

where we used the fact that $\sqrt{a} > 0$ and where we used the fact that the exponential function is strictly positive on all of $\mathbb{R}$. This implies $f(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathbb{R}^3 \backslash \{0\}$. Thus $\boldsymbol{x}_0$ is a strict global minimizer of f (and in fact the only one).

Case 2: Suppose $a = 0$. Then $f(\boldsymbol{x}) = x_2^2 e^{x_3} + x_3^2 e^{x_1}$. Clearly we have $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and $f(\boldsymbol{x}_0) = 0$. It follows that $\boldsymbol{x}_0$ is a global minimizer with optimal objective value being $f(\boldsymbol{x}_0) = 0$. Next, observe that

$$\begin{aligned}
\boldsymbol{x} \in \mathcal{Z}_f &\iff f(\boldsymbol{x}) = 0 \\
&\iff x_2^2 e^{x_3} + x_3^2 e^{x_1} = 0 \\
&\iff (x_2 e^{x_3/2})^2 + (x_3 e^{x_1/2})^2 = 0 \\
&\iff x_2 e^{x_3/2} = 0 \text{ and } x_3 e^{x_1/2} = 0 \\
&\iff x_2 = 0 = x_3,
\end{aligned}$$

where we used the fact that the exponential function is strictly positive on all of $\mathbb{R}$. Thus

$$\mathcal{Z}_f = \{(t, 0, 0) \mid t \in \mathbb{R}\}.$$

In particular, the value of the objective function neither *increases* or *decreases* if we move along the x_1-axis (in either direction) away from $\boldsymbol{x}_0$ (we could have also determined this by looking at the Hessian and seeing that one of its eigenvalues is zero with $(1, 0, 0)^\top$ being a corresponding eigenvector). It follows that $\boldsymbol{x}_0$ is *not* a strict global minimizer.

Case 3: Suppose $a < 0$. For each $t \in \mathbb{R}$, let $\boldsymbol{x}_t = (t, 0, 0)$ and let $\widehat{\boldsymbol{x}}_t = (0, t, 0)$. If $t \neq 0$, then we have

$$f(\boldsymbol{x}_t) = at^2 < 0 \quad \text{and} \quad f(\widehat{\boldsymbol{x}}_t) = t^2 > 0.$$

In particular, the value of the objective function *decreases* if we move along the x_1-axis (in either direction) away from $\boldsymbol{x}_0$, and the value of the objective function *increases* if we move along the x_2-axis (in either direction) away from $\boldsymbol{x}_t$ (we could have also determined this by looking at the Hessian and seeing that it has mixed positive and negative eigenvalues). It follow that $\boldsymbol{x}_0$ is a saddle point.

Problem 4

For this problem, let $f(\boldsymbol{x}) = x_1 - x_2 + x_3$ and let $g(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 - 1$. We consider the following NLP

$$\begin{aligned}
&\text{minimize} && f(\boldsymbol{x}) \\
&\text{subject to} && g(\boldsymbol{x}) \leq 0
\end{aligned}$$

Problem 4.a

Exercise 7. Write the KKT FONC to this NLP.

Solution 7. A KKT point for this NLP is a pair $(\boldsymbol{x}, \mu) = (x_1, x_2, x_3, \mu) \in \mathbb{R}^4$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &\leq 0 && \text{Primal Feasibility} \\ \mu &\geq 0 && \text{Dual Feasibility} \\ \mu g(\boldsymbol{x}) &= 0 && \text{Complementary Slackness} \end{aligned} \tag{1}$$

Let $\mathcal{K} \subseteq \mathbb{R}^4$ denote the set of all KKT points for this NLP. The KKT FONC says that if $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ is a local minimizer and the NLP satisfies some regularity conditions, then there exists $\alpha \geq 0$ such that $(\boldsymbol{a}, \alpha) \in \mathcal{K}$. The regularity condition that we will use is the LICQ condition, which says the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at $\boldsymbol{a}$. What this means in this case is that if $g(\boldsymbol{a}) = 0$, then $\nabla g(\boldsymbol{a}) \neq 0$. Now observe that

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \nabla g(\boldsymbol{x}) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix}.$$

Thus (1) can be rewritten as:

$$\begin{aligned} 1 + 2\mu x_1 &= 0 \\ -1 + 2\mu x_2 &= 0 \\ 1 + 2\mu x_3 &= 0 \\ x_1^2 + x_2^2 + x_3^2 - 1 &\leq 0 \\ \mu &\geq 0 \\ \mu(x_1^2 + x_2^2 + x_3^2 - 1) &= 0 \end{aligned}$$

We also note that the regulartiy condition is always satisfied since $\nabla g(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = 0$, and $g(0) = -1 \neq 0$.

Problem 4.b

Exercise 8. Find all solutions $(\boldsymbol{x}, \mu)$ to the KKT FONC for this NLP.

Solution 8. Suppose $(\boldsymbol{x}, \mu) = (x_1, x_2, x_3, \mu) \in \mathcal{K}$. From the Stationary equations, we have

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) = 0 &\iff \begin{pmatrix} 1 + 2\mu x_1 \\ -1 + 2\mu x_2 \\ 1 + 2\mu x_3 \end{pmatrix} = 0 \\ &\iff \boldsymbol{x} = \frac{1}{2\mu} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Note in particular $\mu \neq 0$. Thus by the Complementary Slackness equation, we have

$$\begin{aligned} 0 &= g(\boldsymbol{x}) \\ &= x_1^2 + x_2^2 + x_3^2 - 1 \\ &= \frac{1}{4\mu^2} + \frac{1}{4\mu^2} + \frac{1}{4\mu^2} - 1 \\ &= \frac{3}{4\mu^2} - 1. \end{aligned}$$

Equivalently, this says $\mu = \sqrt{3}/2$ (where this is the positive square root since $\mu \geq 0$ by the Dual Feasibility property). From this it follows that $x_1 = x_3 = -\sqrt{3}/3$ and $x_2 = \sqrt{3}/3$. Therefore we see that $\mathcal{K}$ consists of one point, namely $\mathcal{K} = \{(\boldsymbol{a}, \alpha)\}$ where

$$(\boldsymbol{a}, \alpha) = (a_1, a_2, a_3, \alpha) = \left(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{2} \right).$$

Problem 4.c

Exercise 9. Find all optimal solutions to this NLP. Give their properties (choose from local, global, strict local, unique global).

Solution 9. From parts a and b, we see that the only possible local minimizer is the point $\boldsymbol{a} = \frac{\sqrt{3}}{3}(-1, 1, -1)^\top$. In fact, since f is affine and g is convex (as $\mathrm{H}_g(\boldsymbol{x}) = 2 \cdot \mathrm{I}_3$), the KKT SOSC are satisfied, which tells us that $\boldsymbol{a}$ is indeed a local minimizer for this NLP. Furthermore, note that the feasible region $\{g \leq 1\} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \|\boldsymbol{x}\| \leq 1\}$ is compact, and combining this with the fact that f is continuous implies that f has a global minimizer in the feasible region (which must be $\boldsymbol{a}$ since it's the only possible one).

Problem 5

For this problem, let $f(\boldsymbol{x}) = x_1^2 - x_2^2$ and let $g(\boldsymbol{x}) = -(x_1 - 2)^2 - x_2^2 + 4$. We consider the following NLP:

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g(\boldsymbol{x}) \leq 0 \end{array}$$

Problem 5.a

Exercise 10. Write the KKT FONC to this NLP.

Solution 10. A KKT point for this NLP is a pair $(\boldsymbol{x}, \mu) = (x_1, x_2, x_3, \mu) \in \mathbb{R}^3$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &\leq 0 && \text{Primal Feasibility} \\ \mu &\geq 0 && \text{Dual Feasibility} \\ \mu g(\boldsymbol{x}) &= 0 && \text{Complementary Slackness} \end{aligned} \tag{2}$$

Let $\mathcal{K} \subseteq \mathbb{R}^3$ denote the set of all KKT points for this NLP. The KKT FONC says that if $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^3$ is a local minimizer and the NLP satisfies some regularity conditions, then there exists $\alpha \geq 0$ such that $(\boldsymbol{a}, \alpha) \in \mathcal{K}$. The regularity condition that we will use is the LICQ condition, which says the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at $\boldsymbol{a}$. What this means in this case is that if $g(\boldsymbol{a}) = 0$, then $\nabla g(\boldsymbol{a}) \neq 0$. Now observe that

$$\nabla f(\boldsymbol{x}) = 2 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \quad \text{and} \quad \nabla g(\boldsymbol{x}) = 2 \begin{pmatrix} -(x_1 - 2) \\ -x_2 \end{pmatrix}.$$

Thus (2) can be rewritten as:

$$\begin{aligned} x_1 - \mu(x_1 - 2) &= 0 \\ -x_2 - \mu x_2 &= 0 \\ 4 - (x_1 - 2)^2 - x_2^2 &\leq 0 \\ \mu &\geq 0 \\ \mu(4 - (x_1 - 2)^2 - x_2^2) &= 0 \end{aligned}$$

We also note that the regulartiy condition is always satisfied since $\nabla g(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = (2, 0)$ and $g(2, 0) = 4$.

Problem 5.b

Exercise 11. Find all solutions $(\boldsymbol{x}, \mu)$ to the KKT FONC for this NLP.

Solution 11. Suppose $(\boldsymbol{x},\mu) = (x_1, x_2, \mu) \in \mathcal{K}$. From the Stationary property, we have

$$\begin{aligned}
\nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) = 0 &\iff \begin{pmatrix} x_1 - \mu(x_1 - 2) \\ -x_2 - \mu x_2 \end{pmatrix} = 0 \\
&\iff \begin{pmatrix} (1-\mu)x_1 + 2\mu \\ (-1-\mu)x_2 \end{pmatrix} = 0 \\
&\iff x_1 = \frac{2\mu}{\mu - 1} \text{ and } x_2(\mu+1) = 0 \\
&\iff x_1 = \frac{2\mu}{\mu - 1} \text{ and } x_2 = 0 && \text{since } \mu \geq 0.
\end{aligned}$$

By the Complementary Slackness property, we either have $\mu = 0$ or $g(\boldsymbol{x}) = 0$. If $\mu = 0$, then $x_1 = 0 = x_2$. Now assume that $\mu \neq 0$, so that $g(\boldsymbol{x}) = 0$. Then

$$\begin{aligned}
g(\boldsymbol{x}) = 0 &\implies \left(\frac{2\mu}{\mu-1} - 2\right)^2 = 4 \\
&\iff \frac{2\mu}{\mu-1} - 2 = 2 \text{ or } \frac{2\mu}{\mu-1} - 2 = -2 \\
&\iff \frac{2\mu}{\mu-1} = 4 \text{ or } \frac{2\mu}{\mu-1} = 0 \\
&\iff \frac{\mu}{\mu-1} = 2 && \text{since } \mu \neq 0 \\
&\iff \mu = 2. \\
&\implies x_1 = 4.
\end{aligned}$$

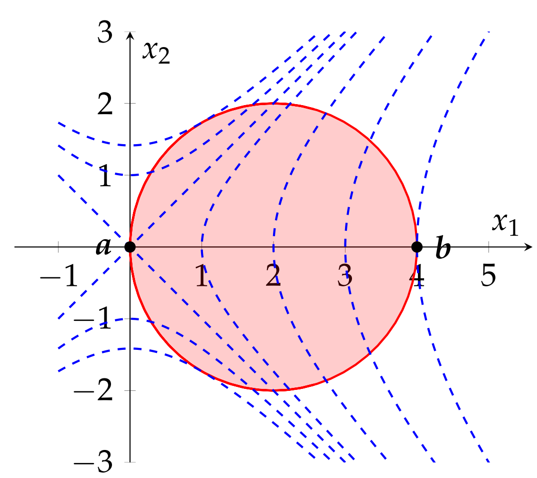
We have found all KKT points, they are given by $\mathcal{K} = \{(\boldsymbol{a},\alpha), (\boldsymbol{b},\beta)\}$, where we set

$$\begin{aligned}
(\boldsymbol{a},\alpha) &= (a_1, a_2, \alpha) = (0,0,0) \\
(\boldsymbol{b},\beta) &= (b_1, b_2, \beta) = (4,0,2).
\end{aligned}$$

Problem 5.c

Exercise 12. Find all optimal solutions to this NLP. Give their properties (choose from local, global, strict local, unique global).

Solution 12. We solve this problem geometrically:



The feasible region (that is $\{g \leq 0\} := \{\boldsymbol{x} \in \mathbb{R}^2 \mid g(\boldsymbol{x}) \leq 0\}$) is everywhere *outside* the region in shaded in red (not including the thick red circle). The blue dashed lines above are level sets $\{f = c\} := \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}) = c\}$ for various values of $c \in \mathbb{R}$ (namely for $c = -2, -1, 0, 1, 4, 9, 16$). We also plotted the points $\boldsymbol{a}$ and $\boldsymbol{b}$ because we know that these are the only possible points that can be local minimizers. Note that the vector $\nabla f(\boldsymbol{b}) = (8,0)^\top$ is normal to the circle $\{g = 0\}$ at the point $\boldsymbol{b}$. This implies that the level set $\{f = 16\}$ is tangent to the circle $\{g = 0\}$, which implies $\boldsymbol{b}$ is a strict local minimizer with optimal objective value $f(\boldsymbol{b}) = 16$ (though $\boldsymbol{b}$ is not a global minimizer since $f(\boldsymbol{a}) = 0 < 16$ for example). On the other hand, suppose $\boldsymbol{a}_t = (0, t)$ and $\widehat{\boldsymbol{a}}_t = (t, 0)$ where $t \in \mathbb{R}\backslash\{0\}$. Then

$$f(\boldsymbol{a}_t) = -t^2 < 0, \quad f(\boldsymbol{a}) = 0, \quad \text{and} \quad f(\widehat{\boldsymbol{a}}_t) = t^2 > 0.$$

Thus if we move away from $\boldsymbol{a}$ in the $(-1, 0)^\top$ direction, then we stay in the feasible region while *decreasing* the objective value, and if we move away from $\boldsymbol{a}$ in the $(0, 1)^\top$ direction, then we stay in the feasible region while *increasing* the objective value (we could have also determined this by looking at the Hessian and seeing that it has mixed positive and negative eigenvalues). It follows that $\boldsymbol{a}$ is a saddle point and hence not an optimal solution.

Problem 5′

For this problem, let $f(\boldsymbol{x}) = x_1^2 - x_2^2$ and let $g(\boldsymbol{x}) = (x_1 - 2)^2 + x_2^2 - 4$. We consider the following NLP:

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g(\boldsymbol{x}) \leq 0 \end{array}$$

Problem 5′.a

Exercise 13. Write the KKT FONC to this NLP.

Solution 13. A KKT point for this NLP is a pair $(\boldsymbol{x}, \mu) = (x_1, x_2, x_3, \mu) \in \mathbb{R}^3$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &\leq 0 && \text{Primal Feasibility} \\ \mu &\geq 0 && \text{Dual Feasibility} \\ \mu g(\boldsymbol{x}) &= 0 && \text{Complementary Slackness} \end{aligned} \tag{3}$$

Let $\mathcal{K} \subseteq \mathbb{R}^3$ denote the set of all KKT points for this NLP. The KKT FONC says that if $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^3$ is a local minimizer and the NLP satisfies some regularity conditions, then there exists $\alpha \geq 0$ such that $(\boldsymbol{a}, \alpha) \in \mathcal{K}$. The regularity condition that we will use is the LICQ condition, which says the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at $\boldsymbol{a}$. What this means in this case is that if $g(\boldsymbol{a}) = 0$, then $\nabla g(\boldsymbol{a}) \neq 0$ Now observe that

$$\nabla f(\boldsymbol{x}) = 2 \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} \quad \text{and} \quad \nabla g(\boldsymbol{x}) = 2 \begin{pmatrix} x_1 - 2 \\ x_2 \end{pmatrix}.$$

Thus (3) can be rewritten as:

$$\begin{aligned} x_1 + \mu(x_1 - 2) &= 0 \\ -x_2 + \mu x_2 &= 0 \\ (x_1 - 2)^2 + x_2^2 - 4 &\leq 0 \\ \mu &\geq 0 \\ \mu(4 - (x_1 - 2)^2 - x_2^2) &= 0 \end{aligned}$$

We also note that the regulartiy condition is always satisfied since $\nabla g(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = (2, 0)$ and $g(2, 0) = -4$.

Problem 5′.b

Exercise 14. Find all solutions $(\boldsymbol{x}, \mu)$ to the KKT FONC for this NLP.

Solution 14. Suppose $(\boldsymbol{x}, \mu) = (x_1, x_2, \mu) \in \mathcal{K}$. From the Stationary equations, we have

$$\begin{aligned}
\nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) = 0 &\iff \begin{pmatrix} x_1 + \mu(x_1 - 2) \\ -x_2 + \mu x_2 \end{pmatrix} = 0 \\
&\iff \begin{pmatrix} (\mu+1)x_1 - 2\mu \\ (\mu - 1)x_2 \end{pmatrix} = 0 \\
&\iff x_1 = \frac{2\mu}{\mu+1} \text{ and } (\mu - 1)x_2 = 0 \\
&\iff x_1 = \frac{2\mu}{\mu+1} \text{ and } (\mu = 1 \text{ or } x_2 = 0)
\end{aligned}$$

If $\mu = 0$, then we must have $x_1 = x_2 = 0$, thus we may assume that $\mu \neq 0$. Then the Complementary Slackness equation implies $g(\boldsymbol{x}) = 0$. If $x_2 = 0$, then this says

$$\begin{aligned}
g(\boldsymbol{x}) = 0 &\iff (x_1 - 2)^2 + x_2^2 = 4 \\
&\iff \left(\frac{2\mu}{\mu+1} - 2\right)^2 = 4 \\
&\iff \frac{2\mu}{\mu+1} - 2 = 2 \text{ or } \frac{2\mu}{\mu+1} - 2 = -2 \\
&\iff \frac{2\mu}{\mu+1} = 4 \text{ or } \frac{2\mu}{\mu+1} = 0 \\
&\iff \frac{\mu}{\mu+1} = 2 && \text{since } \mu \neq 0 \\
&\iff \mu = -2,
\end{aligned}$$

however this contradicts the Dual Feasibility property. Thus we may assume that $x_2 \neq 0$, which implies $\mu = 1$ and hence $x_1 = 1$. Then again by the Complementary Slackness equation, we have

$$\begin{aligned}
g(\boldsymbol{x}) = 0 &\iff (x_1 - 2)^2 + x_2^2 = 4 \\
&\iff 1 + x_2^2 = 4 \\
&\iff x_2 = \pm\sqrt{3}.
\end{aligned}$$

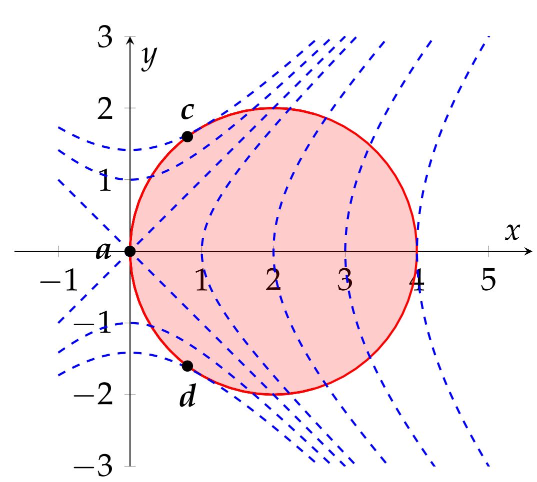
We have found all KKT points, they are given by $\mathcal{K} = \{(\boldsymbol{a}, \alpha), (\boldsymbol{c}, \gamma), (\boldsymbol{d}, \delta)\}$, where we set

$$\begin{aligned}
(\boldsymbol{a}, \alpha) &= (a_1, a_2, 0) = (0, 0, 0) \\
(\boldsymbol{c}, \gamma) &= (c_1, c_2, \gamma) = (1, \sqrt{3}, 1) \\
(\boldsymbol{d}, \delta) &= (d_1, d_2, \delta) = (1, -\sqrt{3}, 1).
\end{aligned}$$

Problem 5'.c

Exercise 15. Find all optimal solutions to this NLP. Give their properties (choose from local, global, strict local, unique global).

Solution 15. We solve this problem geometrically:



The feasible region (that is $\{g \le 0\} := \{\boldsymbol{x} \in \mathbb{R}^2 \mid g(\boldsymbol{x}) \le 0)$ is everywhere *inside* the region shaded in red (including the thick red circle). The blue dashed lines above are level sets $\{f = c\} := \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}) = c\}$ for various values of $c \in \mathbb{R}$ (namely for $c = -2, -1, 0, 1, 4, 9, 16$). We also plotted the points $\boldsymbol{a}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ b ecause we know that these are the only possible points that can be local minimizers. Note that the vector $\nabla f(\boldsymbol{c}) = 2(1, -\sqrt{3})^\top$ is normal to the circle $\{g = 0\}$ at the point $\boldsymbol{c}$. This implies that the level set $\{f = 16\}$ is tangent to the circle $\{g = 0\}$, which implies $\boldsymbol{c}$ is a strict local minimizer with optimal value $f(\boldsymbol{d}) = 16$. A similar argument implies $\boldsymbol{d}$ is a strict local minimizer with optimal value $f(\boldsymbol{c}) = 16$. In fact, both $\boldsymbol{c}$ and $\boldsymbol{d}$ are global minimizers since the feasible region is compact and since f is continuous (so f must have a global minimizer somewhere in the feasible region). Finally, by the same argument as above, $\boldsymbol{a}$ is a saddle point hence not an optimal solution.

Problem 6

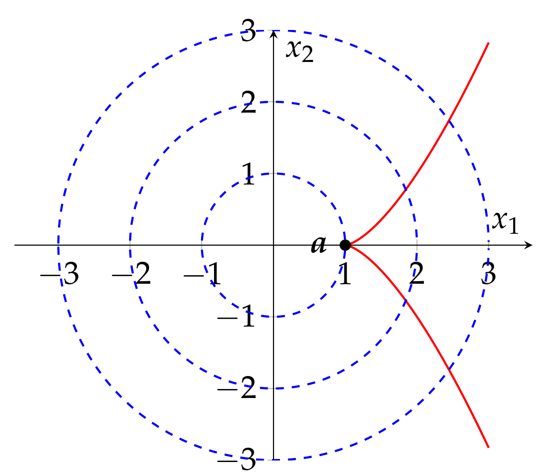
For this problem, let $f(\boldsymbol{x}) = x_1^2 + x_2^2$ and let $g(\boldsymbol{x}) = -(x_1 - 1)^3 + x_2^2$ consider the following NLP

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g(\boldsymbol{x}) \le 0 \end{array}$$

Problem 6.a

Exercise 16. Solve the problem graphically.

Solution 16. We solve this graphically as below:



The feasible region $\{g \le 0\}$ is everything to the *right* of the red curve drawn above. The blue dashed lines above are level sets $\{f = c\}$ for various values of $c \in \mathbb{R}$ (namely for $c = 1, 4, 9$). We also plotted the point $\boldsymbol{a} := (1, 0)$ which is clearly the unique global minimizer with the objective optimal value being $f(\boldsymbol{a}) = 1$. To verify that this is the global minimizer, suppose $\boldsymbol{x} = (x_1, x_2)$ is in the feasible region. If $x_1 = 1$, then $x_2^2 \le 0$ which implies $x_2 = 0$, and hence $\boldsymbol{x} = \boldsymbol{a}$. Now assume that $\boldsymbol{x} \ne \boldsymbol{a}$ so that $x_1 > 1$. Then this implies

$$\begin{aligned} f(\boldsymbol{x}) &= x_1^2 + x_2^2 \\ &> 1 + x_2^2 \\ &\ge 1 \\ &= f(\boldsymbol{a}). \end{aligned}$$

Thus $\boldsymbol{a}$ is the unique global minimizer for this NLP.

Problem 6.b

Exercise 17. Write down the KKT conditions for this problem.

Solution 17. A KKT point for this NLP is a pair $(\boldsymbol{x}, \mu) = (x_1, x_2, \mu) \in \mathbb{R}^3$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &\leq 0 && \text{Primal Feasibility} \\ \mu &\geq 0 && \text{Dual Feasibility} \\ \mu g(\boldsymbol{x}) &= 0 && \text{Complementary Slackness} \end{aligned} \tag{4}$$

Let $\mathcal{K} \subseteq \mathbb{R}^2$ denote the set of all KKT points for this NLP. The KKT FONC says that if $\boldsymbol{x}^* \in \mathbb{R}^3$ is a local minimizer and the NLP satisfies some regularity conditions, then there exists $\mu^* \geq 0$ such that $(\boldsymbol{x}^*, \mu^*) \in \mathcal{K}$. The regularity condition that we will use is the LICQ condition, which says the gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at $\boldsymbol{x}^*$. What this means in this case is that if $g(\boldsymbol{x}^*) = 0$, then $\nabla g(\boldsymbol{x}^*) \neq 0$. Now observe that

$$\nabla f(\boldsymbol{x}) = 2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \nabla g(\boldsymbol{x}) = \begin{pmatrix} -3(x_1 - 1)^2 \\ 2x_2 \end{pmatrix}.$$

Thus (4) can be rewritten as:

$$\begin{aligned} 2x_1 - 3\mu(x_1 - 1)^2 &= 0 \\ x_2 + \mu x_2 &= 0 \\ x_2^2 - (x_1 - 1)^3 &\leq 0 \\ \mu &\geq 0 \\ \mu(x_2^2 - (x_1 - 1)^3) &= 0 \end{aligned}$$

Note that we've already found the optimal solution, namely $\boldsymbol{a} = (1, 0)$, however $\boldsymbol{a}$ does not satisfy the regularity condition since $\nabla g(\boldsymbol{a}) = 0 = g(\boldsymbol{a})$, so it need not be the case that there exists an $\alpha \geq 0$ such that $(\boldsymbol{a}, \alpha) \in \mathcal{K}$ (in fact, the next problem will show that it's not the case).

Problem 6.c

Exercise 18. Do the KKT conditions yield the optimal solution? Explain.

Solution 18. No. Indeed, suppose $(\boldsymbol{x}, \mu) = (x_1, x_2, \mu) \in \mathcal{K}$. From the Stationary equations, we have

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) = 0 &\iff \begin{pmatrix} 2x_1 - 3\mu(x_1 - 1)^2 \\ (\mu + 1)x_2 \end{pmatrix} = 0 \\ &\iff 2x_1 - 3\mu(x_1 - 1)^2 = 0 \text{ and } (\mu + 1)x_2 = 0 \\ &\iff 2x_1 - 3\mu(x_1 - 1)^2 = 0 \text{ and } x_2 = 0 && \text{since } \mu \geq 0 \\ &\iff 2x_1 - 3\mu(x_1 - 1)^2 = 0 \text{ and } x_2 = 0 \end{aligned}$$

If $\mu = 0$, then we must have $x_1 = x_2 = 0$. Now assume for contradiction that $\mu \neq 0$. Then the Complementary Slackness equation implies $g(\boldsymbol{x}) = 0$, thus

$$\begin{aligned} g(\boldsymbol{x}) = 0 &\implies (x_1 - 1)^3 = x_2^2 \\ &\implies (x_1 - 1)^3 = 0 \\ &\implies x_1 = 1. \end{aligned}$$

However if $x_1 = 1$, then plugging this into the equation

$$2x_1 - 3\mu(x_1 - 1)^2 = 0$$

yields $2 = 0$, an obvious contradiction. Thus $\mu = 0$, and hence $x_1 = x_2 = 0$, so the only KKT point is $(0, 0, 0)$, which is not the optimal solution.