MATH 8610 (SPRING 2023) MIDTERM EXAM

Assigned 03/09/23 at 11am, due 03/11/18 by 11:59pm.

Please mark the time you used when submitting it.

In principle, this is **closed-book** exam. You should try best not to refer to your class notes or homework. By doing so, you can have a faithful test of your actual understanding of the course materials and problem-solving skills. These questions would be rather straightforward if you refer to textbooks or class notes.

1. **[Q1]** (a) Let $f(x_1, x_2) = x_1^2 \ln x_2$ where $x_2 > 0$. Find the relative condition number of f. If $x_1 \approx 1$, for what values of x_2 is this evaluation ill-conditioned?

   (b) Let $A \in \mathbb{R}^{n\times n}$ and (λ, v) be an eigenpair such that $Av = \lambda v$. Let $(\hat{\lambda}, \hat{v})$ be a numerically computed approximation to (λ, v). Assume that $\|v\|_2 = \|\hat{v}\|_2 = 1$.
   Find the vector x, such that $(\hat{\lambda}, \hat{v})$ is an eigenpair of $A + \Delta A$, where $\Delta A = x\hat{v}^H$. Give the expression of $\|\Delta A\|_2$ (should not contain x), and explain why $\|A\hat{v} - \hat{\lambda}\hat{v}\|_2 \leq \mathcal{O}(\|A\|_2 \epsilon_{mach})$ means $(\hat{\lambda}, \hat{v})$ is computed by a backward stable algorithm.

2. **[Q2]** (a) Assume that $U, V \in \mathbb{R}^{n\times p}$ $(p < n)$ have full rank, and $V^T U$ is nonsingular. Show that $P = I - U(V^T U)^{-1} V^T$ is a projector, and find the range and null space of P. What can we say about P and $\|P\|_2$, if range(U) = range(V)?

   (b) Let $x \in \mathbb{R}^n$, and consider the vector $z = \begin{bmatrix} 0_{n-1} \\ \|x\|_2 \\ x \end{bmatrix} \in \mathbb{R}^{2n}$. Find the Householder reflector $H = I - 2vv^T$ that reduces z such that Hz is a multiple of e_1 (sufficient to find the expression of v). For $y = \begin{bmatrix} 0_n \\ x \end{bmatrix} \in \mathbb{R}^{2n}$, give the simplfied expression of Hy.

   (c) Given a 6×4 matrix A with all nonzero entries, illustrate the procedure of Golub-Kahan bidiagonalization, and explain how to compute all singular values of A.

3. **[Q3]** (a) For $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$, let $A^\dagger = (A^T A)^{-1} A^T$. Show that $\|A^\dagger\|_2 = \frac{1}{\sigma_n(A)}$. (assume that A has full column rank)

   (b) Let $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{n\times n}$ is nonsingular. Show that $\sigma_n(A) \geq \sigma_n(A_1)$ (explore the relation between $\frac{\|Ax\|_2}{\|x\|_2}$ and $\frac{\|A_1 x\|_2}{\|x\|_2}$), and $\|A^\dagger\|_2 \leq \|A_1^{-1}\|_2$.

   (c) Define the numerical rank of $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ as rank$(A, \epsilon) = \max\{k : \sigma_k \geq \epsilon\}$ $(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n)$. If A has numerical rank $k < n$ for a given ϵ, find a numerically full rank B satisfying $\inf_{\text{rank}(B,\epsilon)=n} \|A - B\|_F$ and show that $\|B - A\|_F \leq \sqrt{n-k}\,\epsilon$.