Spectrum of a Ring

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1 Spec A as a Topological Space

We start with the following basic definition. Let A be a ring. We set

$$\operatorname{Spec} A := \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

We will now endow Spec A with the structure of a topological space. For every subset S of A, we denote by $V(S)$ the set of prime ideals of A containing S. Clearly, if $\mathfrak{a}$ is the ideal generated by S, then $V(S) = V(\mathfrak{a})$. For any $f \in A$, we write $V(f)$ instead of $V(\{f\})$.

Lemma 1.1. *The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is an inclusion reversing map from the set of ideals of A to the set of subsets of Spec A. Moreover, the following relations hold:*

1. *$V(0) = \operatorname{Spec} A$ and $V(1) = \emptyset$.*
2. *For two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have*
$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$
3. *For every family $\{\mathfrak{a}_i\}_{i \in I}$ of ideals, we have*
$$V\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

Proof.

1. Trivial.
2. Since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{b}$, it follows that $V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a} \cap \mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b})$. It remains to show that $V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. Assume that $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$ but $\mathfrak{p} \not\supset \mathfrak{a}$ and $\mathfrak{p} \not\supset \mathfrak{b}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. Then there exists $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x, y \notin \mathfrak{p}$. But $xy \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ contradicts the fact that $\mathfrak{p}$ is prime.
3. That $V\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$ follows from the fact that $\sum_{i \in I} \mathfrak{a}_i$ is the ideal generated by $\bigcup_{i \in I} \mathfrak{a}_i$. That $V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$ follows from the fact that $\mathfrak{p} \supset \sum_{i \in I} \mathfrak{a}_i$ if and only if $\mathfrak{p} \supset \mathfrak{a}_i$ for all $i \in I$ and for all primes $\mathfrak{p} \in \operatorname{Spec} A$.

□

The lemma shows that the subsets $V(\mathfrak{a})$ of Spec A form the closed sets of a topology on Spec A. This leads us to the following definition.

Definition 1.1. Let A be a ring. The set Spec A of all prime ideals of A with the topology whose closed sets are the sets $V(\mathfrak{a})$, where $\mathfrak{a}$ runs through the set of ideals of A, is called the **prime spectrum** of A or simply the **spectrum** of A. The topology thus defined is called the **Zariski topology** on Spec A.

Remark. If x is a point in Spec A, we will often write $\mathfrak{p}_x$ instead of x when we think of x as a prime ideal of A.

Proposition 1.1. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Then $V(\mathfrak{a}) = V\left(\sqrt{\mathfrak{a}}\right)$.*

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, we have $V(\mathfrak{a}) \supset V(\sqrt{\mathfrak{a}})$. For the reverse inclusion, let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \supset \mathfrak{a}$. Assume, for a contradiction, that $\mathfrak{p} \not\supset \sqrt{\mathfrak{a}}$. Choose $a \in \sqrt{\mathfrak{a}}$ such that $a \notin \mathfrak{p}$. Then $a^n \in \mathfrak{a} \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. But this contradicts the fact that $\mathfrak{p}$ is a prime ideal. □

For every subset Y of Spec$A$, we set

$$I(Y) := \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}.$$

We obtain an inclusion reversing map $Y \mapsto I(Y)$ from the set of subsets of Spec A to the set of ideals of A. Note that $I(\emptyset) = A$. The maps V and I are related as follows.

Proposition 1.2. *Let A be a ring, $\mathfrak{a}$ an ideal in A, and Y a subset of Spec A.*

1. $\sqrt{I(Y)} = I(Y)$.
2. $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
3. $V(I(Y)) = \overline{Y}$.
4. $I(D(\mathfrak{a})) = \sqrt{0} : \mathfrak{a}$.
5. $D(I(Y)) = \bigcap_{\mathfrak{p}\in Y} D(\mathfrak{p})$.
6. *The maps*

$$\{\textit{ideals } \mathfrak{a} \textit{ of } A \textit{ with } \mathfrak{a} = rad(\mathfrak{a})\} \underset{I}{\overset{V}{\rightleftarrows}} \{\textit{closed subsets } Y \textit{ of Spec } A\}$$

are mutually inverse bijections.

Proof.

1. The relation $\mathfrak{a} = \sqrt{\mathfrak{a}}$ means that for $f \in A$, $f^n \in \mathfrak{a}$ implies already $f \in \mathfrak{a}$. This certainly holds for prime ideals and therefore for arbitrary intersections of prime ideals as well.

2. This follows from the fact that the radical of an ideal equals the intersection of all prime ideals containing it.

3. Observe that $V(\mathfrak{b}) \supset Y$ if and only if $\sqrt{\mathfrak{b}} \subset I(Y)$ if and only if $V(\mathfrak{b}) = V\left(\sqrt{\mathfrak{b}}\right) \supset V(I(Y))$. Therefore $V(I(Y))$ is the smallest closed subset of Spec A containing Y.

4. We first show that $\sqrt{0} : \mathfrak{a} \subseteq I(D(\mathfrak{a}))$. Let $x \in \sqrt{0} : \mathfrak{a}$ and assume (to obtain a contradiction) that $x \notin I(D(\mathfrak{a}))$. Since $x \notin I(D(\mathfrak{a}))$, there exists a prime $\mathfrak{p} \subseteq A$ such that $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $x \notin \mathfrak{p}$. Since $x \in \sqrt{0} : \mathfrak{a}$, we have $x\mathfrak{a} \subseteq \sqrt{0} \subseteq \mathfrak{p}$. In particular, either $\mathfrak{p} \supseteq \mathfrak{a}$ or $\mathfrak{p} \supseteq x$. Contradiction.

Now we will show that $\sqrt{0} : \mathfrak{a} \supseteq I(D(\mathfrak{a}))$. Let $x \in I(D(\mathfrak{a}))$ (so x belongs to every prime ideal which does not contain $\mathfrak{a}$) and assume (to obtain a contradiction) that $x \notin \sqrt{0} : \mathfrak{a}$. Since $x \notin \sqrt{0} : \mathfrak{a}$, there exists $a \in \mathfrak{a}$ such that $ax \notin \sqrt{0}$. In particular, $\{(ax)^n\}_{n\in\mathbb{N}}$ forms a multiplicative set, and so we can localize at ax. Let $\mathfrak{q}$ be a prime ideal in A_{ax} and let $\mathfrak{p} := \iota_{ax}^{-1}(\mathfrak{q})$, where $\iota_{ax} \colon A \to A_{ax}$ is the canonical ring homomorphism. Then $\mathfrak{p}$ is a prime ideal in A which does not contain ax. This implies that $\mathfrak{p}$ does not contain $\mathfrak{a}$ or x (if it did, then it'd certain contain ax). Contradiction.

5. We have

$$\begin{aligned} D(I(Y)) &= D\left(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\right) \\ &= \bigcap_{\mathfrak{p}\in Y} D(\mathfrak{p}) \end{aligned}$$

6. This follows from part 2.

□

1.0.1 Partially Ordered Sets

Definition 1.2.

1.1 Properties of Spec A

Let A be a ring and let $\mathfrak{a}$ be an ideal in A. We set

$$D(\mathfrak{a}) := \operatorname{Spec}(A) \backslash V(\mathfrak{a}).$$

If $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = \langle f_1, \ldots, f_n \rangle$, then we write $D(f_1, \ldots, f_n)$ instead of $D(\langle f_1, \ldots, f_n \rangle)$. Open subsets of Spec A of the form $D(f)$ are called **principal open sets** of Spec A. Clearly, $D(0) = \emptyset$ and $D(u) =$ Spec A for any unit $u \in A$. As for a prime ideal $\mathfrak{p}$ and two elements $f, g \in A$ we have $fg \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, we find

$$D(f) \cap D(g) = D(fg).$$

Lemma 1.2. *Let (f_i) be a family of elements in A and let $g \in A$. Then $D(g) \subseteq \bigcup_i D(f_i)$ if and only if there exists an integer $n > 0$ such that g^n is contained in the ideal $\mathfrak{a}$ generated by the f_i.*

Proof. Indeed, $D(g) \subseteq \bigcup_i D(f_i)$ is equivalent to $V(g) \supseteq V(\mathfrak{a})$ which is equivalent to $g \in \sqrt{\mathfrak{a}}$. □

Remark. Applying this to $g = 1$ it follows that $(D(f_i))_i$ is a covering of Spec$A$ if and only if the ideal generated by the f_i is equal to A.

Proposition 1.3. *Let A be a ring. The principal open subsets $D(f)$ for $f \in A$ form a basis of the topology of Spec A. For all $f \in A$ the open sets $D(f)$ are quasi-compact. In particular, the space Spec A is quasi-compact.*

Proof. Every closed subset of Spec$A$ is the intersection of closed sets of the form $V(f)$. By taking complements we see that the $D(f)$ form a basis for the topology.

Let $(g_i)_{i \in I}$ be a family of elements of A such that $D(f) \subseteq \bigcup_{i \in I} D(g_i)$. Then there exists an integer $n \geq 1$ such that $f^n = \sum_{i \in I} a_i g_i$ where $a_i \in A$ and $a_i = 0$ for all $i \notin J$, $J \subseteq I$ a suitable finite subset. Hence $D(f) \subseteq \bigcup_{j \in J} D(g_j)$. This proves that $D(f)$ is quasi-compact by the first part of the proposition. □

Proposition 1.4. *Let A be a ring and let U be an open subset of Spec A. Then U is quasi-compact if and only if it is the complement of a closed set of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is a finitely generated ideal.*

Proof. Suppose $U = D(\mathfrak{a})$ where $\mathfrak{a}$ is a finitely generated. Then $\mathfrak{a} = \langle f_1, \ldots, f_n \rangle$ for some $f_1, \ldots, f_n \in A$. In particular,

$$U = D(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n).$$

As U is a finite union of compact spaces, it must be compact.

Conversely, suppose that U is quasi-compact. Since $\{D(g)\}_{g \in A}$ forms a basis for the topology, we can write

$$U = \bigcup_{i \in I} D(g_i)$$

for some $g_i \in A$. In particular, $\{D(g_i)\}_{i \in I}$ is an open covering of U. Since U is quasi-compact, there exists a finite subcovering, say $\{D(g_1), \ldots, D(g_n)\}$. Thus,

$$U = \bigcup_{i=1}^{n} D(g_i) = D(g_1, \ldots, g_n).$$

In particular, setting $\mathfrak{a} = \langle g_1, \ldots, g_n \rangle$, we can write $U = D(\mathfrak{a})$ where $\mathfrak{a}$ is finitely generated. □

Example 1.1. Let $A = K[x, y]$, $\mathfrak{a} = \langle x^2, y^2 \rangle$, and $\mathfrak{b} = \langle x^2, xy, y^2 \rangle$. Even though $\mathfrak{a} \subset \mathfrak{b}$ (where the inclusion is strict), we have $V(\mathfrak{a}) = V(\mathfrak{b})$, since $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Proposition 1.5. *Let A be a ring. A subset Y of Spec A is irreducible if and only if $\mathfrak{p} := I(Y)$ is a prime ideal. In this case $\{\mathfrak{p}\}$ is dense in $\overline{Y}$.*

Proof. Assume that Y is irreducible. Let $f, g \in A$ with $fg \in \mathfrak{p}$. Then

$$Y \subseteq V(fg) = V(f) \cup V(g).$$

As Y is irreducbile, either $Y \subseteq V(f)$ or $Y \subseteq V(g)$ which implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Conversely let $\mathfrak{p}$ be a prime. Then by Proposition (1.2),

$$\overline{Y} = V(\mathfrak{p}) = V(I(\{\mathfrak{p}\}) = \overline{\{\mathfrak{p}\}}.$$

Therefore $\overline{Y}$ is the closure of the irreducible set $\{\mathfrak{p}\}$ and therefore irreducible. This implies that the dense subset Y is also irreducible. □

Note that for arbitrary irreducible subsets Y the prime ideal $I(Y)$ is not necessarily a point in Y. But this is clearly true if Y is closed, or more generally, if Y is locally closed.

Corollary. *The map $\mathfrak{p} \mapsto V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is a bijection from Spec A onto the set of closed irreducible subsets of Spec A. Via this bijection, the minimal prime ideals of A correspond to the irreducible components of Spec A.*

Definition 1.3. Let X be an arbitrary topological space.

1. A point $x \in X$ is called **closed** if the set $\{x\}$ is closed,
2. We say that a point $\eta \in X$ is a **generic point** if $\overline{\{\eta\}} = X$.
3. We say x and x' be two points of X. We say that x is a **generization** or that x' is a **specialization** of x if $x' \in \overline{\{x\}}$.
4. A point $x \in X$ is called a **maximal point** if its closure $\overline{\{x\}}$ is an irreducible component of X.

Thus a point $\eta \in X$ is generic if and only if it is a generization of every point of X. As the closure of an irreducible set is again irreducible, the existence of a generic point implies that X is irreducible.

Example 1.2. If $X = \operatorname{Spec} A$ is the spectrum of a ring, the notions introduced in Definition (1.3) have the following algebraic meaning.

1. A point $x \in X$ is closed if and only if $\mathfrak{p}_x$ is a maximal ideal.
2. A point $\eta \in X$ is a generic point of X if and only if $\mathfrak{p}_\eta$ is the unique minimal prime ideal. This exists if and only if the nilradical of A is a prime ideal.
3. A point x is a generization of a point x' (in other words, x' is a specialization of x) if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_{x'}$.
4. A point $x \in X$ is a maximal point if and only if $\mathfrak{p}_x$ is a minimal prime ideal.

1.2 The Functor $A \mapsto$ Spec A

We will now show that $A \mapsto \operatorname{Spec} A$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\varphi : A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a prime ideal of B, then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of A ($A/\varphi^{-1}(\mathfrak{q})$ is a subring of the domain $B/\mathfrak{q}$, hence $A/\varphi^{-1}(\mathfrak{q})$ is a domain) Therefore we obtain a map

$$^a\varphi = \operatorname{Spec} \varphi : \operatorname{Spec} B \to \operatorname{Spec} A, \qquad \mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q}).$$

Proposition 1.6. *Let A be a ring.*

1. *For every subset $S \subseteq A$, the relation*
$$^a\varphi^{-1}(V(S)) = V(\varphi(S))$$
holds. In particular, for $f \in A$,
$$^a\varphi^{-1}(D(f)) = D(\varphi(f)).$$
2. *For every ideal $\mathfrak{b}$ of B,*
$$V(\varphi^{-1}(\mathfrak{b})) = \overline{^a\varphi(V(\mathfrak{b}))}. \tag{1}$$

Proof.

1. A prime ideal $\mathfrak{q}$ of B contains $\varphi(S)$ if and only if $\varphi^{-1}(\mathfrak{q})$ contains S.
2. By Proposition (1.2), we can rewrite the right hand side as $V(I(^a\varphi(V(\mathfrak{b}))))$. But
$$I(^a\varphi(V(\mathfrak{b}))) = \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \varphi^{-1}(\mathfrak{q}) = \varphi^{-1}\left(\sqrt{\mathfrak{b}}\right) = \sqrt{\varphi^{-1}(\mathfrak{b})},$$
and the claim follows after applying $V(-)$.

□

The proposition shows in particular that $^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous. As $^a(\psi \circ \varphi) =^a \varphi \circ^a \psi$ for any ring homomorphism $\psi : B \to C$, we obtain a contravariant functor $A \mapsto \operatorname{Spec} A$ from the category of rings to the category of topological spaces.

Corollary. *The map ${}^a\varphi$ is dominant (i.e. its image is dense in Spec A) if and only if every element of Ker(φ) is nilpotent.*

Proof. We apply (1) to $\mathfrak{b} = 0$. $\square$

Proposition 1.7. *Let A be a ring.*

1. *Let $\varphi : A \to B$ be a surjective homomorphism of rings with kernel $\mathfrak{a}$. Then ${}^a\varphi$ is a homeomorphism of Spec B onto the closed subset $V(\mathfrak{a})$ of Spec A.*

2. *Let S be a multiplicative subset of A and let $\varphi : A \to S^{-1}A =: B$ be the canonical homomorphism. Then ${}^a\varphi$ is a homeomorphism of Spec $S^{-1}A$ onto the subspace of Spec A consisting of prime ideals $\mathfrak{p} \subset A$ with $S \cap \mathfrak{p} = \emptyset$.*

Proof. In both cases it is clear that ${}^a\varphi$ is injective with the stated image. Moreover in both cases a prime ideal $\mathfrak{q}$ of B contains an indeal $\mathfrak{b}$ of B if and only if $\varphi^{-1}(\mathfrak{q})$ contains $\varphi^{-1}(\mathfrak{b})$. This shows that ${}^a\varphi(V(\mathfrak{b})) = V(\varphi^{-1}(\mathfrak{b})) \cap \mathrm{Im}({}^a\varphi)$. Therefore ${}^a\varphi$ is a homeomorphism onto its image.

$\square$

Remark. Let A be a ring and let $\mathfrak{p}, \mathfrak{q} \subset A$ be prime ideals. Proposition (1.7) shows that the passage from A to $A_\mathfrak{p}$ cuts out all prime ideals except those contained in $\mathfrak{p}$. The passage from A to $A/\mathfrak{q}$ cuts out all prime ideals except those containing $\mathfrak{q}$. Hence, if $\mathfrak{q} \subseteq \mathfrak{p}$ localizing with respect to $\mathfrak{p}$ and taking the quotient modulo $\mathfrak{q}$ (in either order as these operations commute) we obtain a ring whose prime ideals are those prime ideals of A that lie between $\mathfrak{q}$ and $\mathfrak{p}$. For $\mathfrak{q} = \mathfrak{p}$, we obtain the field

$$\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Frac}(A/\mathfrak{p}),$$

which is called the **residue field** at $\mathfrak{p}$.

2 Spectrum of a Ring as a Locally Ringed Space

Let A be a ring. We will now endow the topological space Spec A with the structure of a locally ringed space and obtain a functor $A \mapsto$ Spec A from the category of rings to the category of locally ringed spaces which we will show to be fully faithful.

2.1 Structure Sheaf on Spec A

We set $X = \mathrm{Spec}\, A$. Recall that the principal open sets $D(f)$ for $f \in A$ form a basis of the topology of X. We will define a presheaf $\mathcal{O}_X$ on this basis and then prove that the sheaf axioms are satisfied. The basic idea is this: Looking back at the analogy with prevarieties, we certainly want to have $\mathcal{O}_X(X) = A$. More generally, for $f \in A$, we consider the localization A_f of A. Denote by $\iota_f : A \to A_f$ the canonical ring homomorphism $a \mapsto a/1$. By Proposition (1.7), ${}^a\iota_f$ is a homeomorphism of Spec A_f onto $D(f)$. So it seems reasonable to set $\mathcal{O}_X(D(f)) = A_f$. Let us check that this is a sensible definition: we must check that $A_f = A_g$ whenever $D(f) = D(g)$, define restriction maps, and check that the sheaf axioms are satisfied.

For $f, g \in A$, we have $D(f) \subseteq D(g)$ if and only if there exists an integer $n \geq 1$ such that $f^n \in \langle g \rangle$ or, equivalently, $g/1 \in (A_f)^\times$. In this case we obtain a unique ring homomorphism $\rho_{f,g} : A_g \to A_f$ such that $\rho_{f,g} \circ \iota_g = \iota_f$. Whenever $D(f) \subseteq D(g) \subseteq D(h)$, we have $\rho_{f,g} \circ \rho_{g,h} = \rho_{f,h}$. In particular, if $D(f) = D(g)$, then $\rho_{f,g}$ is an isomorphism, which we use to identify A_g and A_f. Therefore we can define

$$\mathcal{O}_X(D(f)) := A_f$$

and obtain a presheaf of rings on the basis $\mathcal{B} = \{D(f) \mid f \in A\}$ for the topological space Spec A. The restriction maps are the ring homomorphism $\rho_{f,g}$.

Theorem 2.1. *The presheaf $\mathcal{O}_X$ is a sheaf on $\mathcal{B}$.*

We denote the sheaf of rings on X associated to $\mathcal{O}_X$ again by $\mathcal{O}_X$. For all points $x \in X = \mathrm{Spec}\, A$, we have

$$\mathcal{O}_{X,x} = \lim_{D(f) \ni x} \mathcal{O}_X(D(f)) = \lim_{f \notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x}.$$

In particular, $(X, \mathcal{O}_X)$ is a locally ringed space. We will often simply write Spec A instaed of (Spec $A, \mathcal{O}_{\mathrm{Spec}\, A}$).

Proof. Let $D(f)$ be a principal open set and let $\{D(f_i)\}_{i \in I}$ be an open covering over $D(f)$. We have to show the following two properties:

1. Let $s \in \mathcal{O}_X(D(f))$ be such that $s|_{D(f_i)} = 0$ for all $i \in I$. Then $s = 0$.

2. For $i \in I$, let $s_i \in \mathcal{O}_X(D(f_i))$ be such that $s_i|_{D(f_i)\cap D(f_j)} = s_j|_{D(f_i)\cap D(f_j)}$ for all $i, j \in I$. Then there exists $s \in \mathcal{O}_X(D(f))$ such that $s|_{D(f_i)} = s_i$ for all $i \in I$.

As $D(f)$ is quasi-compact, we can assume that I is finite. Restricting the presheaf $\mathcal{O}_X$ to $D(f)$ and replacing A by A_f, we may assume that $f = 1$ and hence $D(f) = X$ to ease the notation. The relation $X = \bigcup_{i\in I} D(f_i)$ is equivalent to $\langle f_i \mid i \in I\rangle = A$ (indeed $\sqrt{\mathfrak{a}} = A$ implies $\mathfrak{a} = A$). As $D(f_i) = D(f_i^n)$ for all integers $n \geq 1$ there exists elements $b_i \in A$ (depending on n) such that

$$\sum_{i\in I} b_i f_i^n = 1. \tag{2}$$

1. Proof of (1). Let $s = a \in A$ be such that the image of a in A_{f_i} is zero for all $i \in I$. As I is finite, there exists an integer $n \geq 1$, independent of i, such that $f_i^n a = 0$. By (2),

$$a = \left(\sum_{i\in I} b_i f_i^n\right) a = 0.$$

2. Proof of (2). As I is finite, we can write $s_i = a_i / f_i^n$ for some n independent of i. By hypothesis, the images of a_i / f_i^n and of a_j / f_j^n in $A_{f_i f_j}$ are equal for all $i, j \in I$. Therefore there exists an integer $m \geq 1$ (which again we can choose independent of i and j) such that

$$(f_i f_j)^m (f_j^n a_i - f_i^n a_j) = 0.$$

Replacing a_i by $f_i^m a_i$ and n by $n + m$ (which does not change s_i), we see that $f_j^n a_i = f_i^n a_j$ for all $i, j \in I$. We set

$$s := \sum_{j\in I} b_j a_j \in A,$$

where the b_j are the elements in (2). Then

$$\begin{aligned} f_i^n s &= f_i^n \left(\sum_{j\in I} b_j a_j\right) \\ &= \sum_{j\in I} b_j (f_i^n a_j) \\ &= \sum_{j\in I} b_j (f_j^n a_i) \\ &= \left(\sum_{j\in I} b_j f_j^n\right) a_i \\ &= a_i. \end{aligned}$$

This means that the image of s in A_{f_i} is s_i.

□

Remark. We have just proved that the sequence

$$0 \longrightarrow A \longrightarrow \bigoplus_{i\in I} A_{f_i} \longrightarrow \bigoplus_{i,j\in I} A_{f_i f_j}$$

is exact.

2.2 The Functor $A \mapsto (\textbf{Spec } A, \mathcal{O}_{\textbf{Spec } A})$

Definition 2.1. A locally ringed space $(X, \mathcal{O}_X)$ is called an **affine scheme**, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphism to (Spec $A, \mathcal{O}_{\text{Spec } A}$).