Mathematics Diary

Contents

1 2023

1.1 12/20/2022

Lemma 1.1. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R. Let E be the minimal free resolution of R/J over R, let F be the minimal free resolution of R/I over R, and let $\varphi\colon E \to F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi\colon E \to F$ and $\overline{\varphi}\colon E_\Bbbk := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_\Bbbk$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R.*

Proof. Assume both $\varphi\colon E \to F$ and $\overline{\varphi}\colon E_\Bbbk := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_\Bbbk$ are injective. Since $\varphi\colon E \to F$ is injective, we have a short exact sequence of R-complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \tag{1}$$

taking homology gives us a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(F/E) \arrow[dll, rounded corners] & \\
\mathrm{H}_i(E) \arrow[r] & \mathrm{H}_i(F) \arrow[r] & \mathrm{H}_i(F/E) \arrow[dll, rounded corners] \\
\mathrm{H}_{i-1}(E) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are resolutions we conclude that $\mathrm{H}_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $\mathrm{H}_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R-modules (1) with $\Bbbk$ over R gives us the long exact sequence in homology

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_{i+1}^R(E, \Bbbk) \arrow[dll, rounded corners] & \\
\mathrm{Tor}_i^R(E, \Bbbk) \arrow[r] & \mathrm{Tor}_i^R(F, \Bbbk) \arrow[r] & \mathrm{Tor}_i^R(F/E, \Bbbk) \arrow[dll, rounded corners] \\
\mathrm{Tor}_{i-1}^R(E, \Bbbk) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are free R-modules we conclude that $\mathrm{Tor}_i(F/E, \Bbbk) = 0$ for all $i \geq 1$. Since $\overline{\varphi}\colon E \otimes_R \Bbbk \to F \otimes_R \Bbbk$ is injective we conclude that $\mathrm{Tor}_1(F/E, \Bbbk) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $\mathrm{d}(F/E) \subseteq \mathfrak{m}(F/E)$). □

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R-module M connecting maps

$$\mathrm{Tor}^R_{i+1}(R/I, M) \to \mathrm{Tor}^R_i(I/J, M) \quad \text{and} \quad \mathrm{Ext}^i_R(I/J, M) \to \mathrm{Ext}^{i+1}_R(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \to F/E \otimes_R M \quad \text{and} \quad \mathrm{Hom}^\star_R(F/E, M) \to \mathrm{Hom}^\star_R(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\overline{\varphi}\colon E_\Bbbk \to F_\Bbbk$ is injective, then already $\varphi\colon E \to F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of Ideals

Let R be a commutative ring and let $\mathfrak{p}$ be an ideal of R. Recall the **height** of $\mathfrak{p}$ is defined to be the supremum of lengths of chains of primes which descend from $\mathfrak{p}$:

$$\mathrm{ht}\,\mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

Furthermore, if I is an ideal of R, then the **height** of I is defined to be the infimum of the heights of all primes which contain I:

$$\mathrm{ht}\, I = \inf\{\mathrm{ht}\,\mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. *Let I_1 and I_2 be ideals of R. Set $c = \mathrm{ht}(I_1 \cap I_2)$, set $c_1 = \mathrm{ht}\, I_1$, and set $c_2 = \mathrm{ht}\, I_2$.*

1. *If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.*
2. *We have $c = \min\{c_1, c_2\}$.*

Proof. 1. Let $\mathfrak{p}$ be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2. Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let $\mathfrak{p}$ be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since $\mathfrak{p}$ is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. □