δ-rings

Fix a prime p. We want to discuss some aspects of the theory of δ-rings. This theory provides a good language to talk about rings with a lift of Frobenius modulo p.
note

$$\delta(a^p) = \delta(a)a^{(p-1)p} + (a + p\delta(a))\delta(a^{p-1}) = \delta(a)a^{(p-1)p} + (a + p\delta(a))(\delta(a)a^{(p-2)p} + (a + p\delta(a))\delta(a^{p-2})$$

$$\delta(a^p) = 2\delta(a)a^{(p-1)p} + p\delta(a)^2a^{(p-2)p} + (a + p\delta(a))^2\delta(a^{p-2}) + \delta(a^{p-2})$$

$$\delta(a^2) = 2\delta(a)(a^2 + \delta(a))$$

1 Definition and Examples

Definition 1.1. A **δ-ring** is a pair (A, δ) where A is a commutative ring and $\delta\colon A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$ satisfying the following two identities:

1. for all $a, b \in A$ we have
$$\delta(ab) = \delta(a)b^p + a^p\delta(b) + p\delta(a)\delta(b).$$
2. for all $a, b \in A$ we have
$$\delta(a+b) = \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p} = \delta(a) + \delta(b) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a^i b^{p-i}.$$

There is an evident category of δ-rings. If the δ-stucture on A is clear from context, we often supress it from the notation and simply call A as δ-ring.

Suppose A is a commutative ring equipped with a map $\phi\colon A \to A$ that lifts the Frobenius on A/p. Then for each $a \in A$, we have an equation of the form

$$\phi(a) = a^p + p\delta(a),$$

where $\delta\colon A \to A$. In fact, we claim that equipping A with δ gives it the structure of a δ-ring. Indeed, we clearly have $\delta(0) = \delta(1) = 0$. Also, since

$$\begin{aligned} a^pb^p + p\delta(ab) &= \phi(ab) \\ &= \phi(a)\phi(b) \\ &= (a^p + p\delta(a))(b^p + p\delta(b)) \\ &= a^pb^p + p(a^p\delta(b) + b^p\delta(a) + p\delta(a)\delta(b)). \end{aligned}$$

Thus we must have $\delta(ab) = a^p\delta(b) + b^p\delta(a) + p\delta(a)\delta(b)$. Similarly we have

$$\begin{aligned} \delta(a+b) &= \frac{\phi(a+b) - (a+b)^p}{p} \\ &= \frac{\phi(a) + \phi(b) - (a+b)^p}{p} \\ &= \frac{a^p + b^p + p(\delta(a) + \delta(b)) - (a+b)^p}{p} \\ &= \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p}. \end{aligned}$$

We can also go backwards:

Lemma 1.1. *Let A be a commutative ring.*

1. *If $\delta\colon A \to A$ provides a δ-structure on A, then the map $\phi\colon A \to A$ defined by*

$$\phi(a) = a^p + p\delta(a)$$

for all $a \in A$, is an endomorphism of A which lifts the Frobenius on A/p.

2. *When A is p-torsionfree, the construction (1) gives a bijective correspondence between δ-structures on A and Frobenius lifts on A.*

Remark. If A is not necessarily p-torsionfree, it is better to record δ instead of ϕ.