# Contractibility

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## Introduction

Let $\varphi\colon (A,\mathfrak{m}) \to (B,\mathfrak{n})$ be a local ring homomorphism and assume that $\mathfrak{m} \neq 0$ (so $A$ is not a field hence $B$ is not a field hence $\mathfrak{n} \neq 0$). Being a local ring homomorphism means $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. Since $\varphi^{-1}(\mathfrak{n})$ is necessarily a prime ideal of $A$, the condition $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ is equivalent to the condition $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$. Now equip $A$ with the $\mathfrak{m}$-adic filtration, so $A = (A_n)$ where $A_n = \mathfrak{m}^n$ and let $A' = (A'_n)$ be the filtered $A$-module where $A'_n = \varphi^{-1}(\mathfrak{n}^n)$ (so in particular we have $A_0 = A = A'_0$ and $A_1 = \mathfrak{m} = A'_1$). Note that $(A'_n)$ really is an $\mathfrak{m}$-filtration since if $x \in A_m = \mathfrak{m}^m$ and $y \in A'_n = \varphi^{-1}(\mathfrak{n}^n)$, then

$$\varphi(xy) = \varphi(x)\varphi(y) \in \varphi(\mathfrak{m}^m)\mathfrak{n}^n \subseteq \mathfrak{n}^{m+n},$$

implies $xy \in A'_{m+n}$. Now, let $S = \mathrm{S}_{B,A}$ denote the standard stabilizing function of $(A'_n)$ with respect to to $(A_n)$, that is, $S\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ is given by

$$S(k) = \inf\{m \mid A'_m \subseteq A_k\} = \inf\{m \mid \varphi^{-1}(\mathfrak{n}^m) \subseteq \mathfrak{m}^k\}.$$

Thus $n \geq S(k)$ implies

$$A'_n \subseteq A'_{S(k)} \subseteq A_k,$$

and if $S(k) \neq 1$, then $A'_{S(k)-1} \not\subseteq A_k$. Note that if $k_2 \geq k_1$, then

$$A'_{S(k_2)} \subseteq A_{k_2} \subseteq A_{k_1}$$

implies $S(k_2) \geq S(k_1)$. Thus the sequence $(S(k)/k)_{k\in\mathbb{N}}$ is monotone increasing, so it makes sense to define the limit

$$c = \mathrm{c}_{B,A} = \lim_{k\to\infty} \frac{S(k)}{k} \in [0,\infty].$$

We call $c$ the **contractibility** of $B$ with respect to $A$. Note that since $\varphi$ is a local ring homomorphism, we have $A'_k \supseteq A_k$ for all $k$. In particular, if $A$ is not Artinian (so $(A_n)$ is strictly descending), then we must have $S \geq \mathbf{1}_k$ (we write $\mathbf{1}_k$ for the function $\mathbb{N} \to \mathbb{N}$ defined by $\mathbf{1}_k(k) = k$). In this case we have $\mathrm{c}_{B,A} \in [1,\infty]$.

**Example 0.1.** Consider the case where $A = K[y]_{\langle y \rangle}$, $B = K[x,y]_{\langle x,y \rangle}/\langle y^2 - x^3 \rangle$, and $\varphi\colon A \to B$ is the inclusion map. We calculate $A'_n := \varphi^{-1}(\mathfrak{n}^n)$ for various $n \in \mathbb{N}$. We have

$$\begin{aligned}
A'_1 &= \varphi^{-1}(\mathfrak{n}) = \mathfrak{m} && \\
A'_2 &= \varphi^{-1}(\mathfrak{n}^2) = \mathfrak{m}^2 && \\
A'_3 &= \varphi^{-1}(\mathfrak{n}^3) = \mathfrak{m}^2 && \text{since } y^2 = x^3 \text{ in } B \\
A'_4 &= \varphi^{-1}(\mathfrak{n}^4) = \mathfrak{m}^3 && \text{since } y^3 = x^3 y \text{ in } B \\
A'_5 &= \varphi^{-1}(\mathfrak{n}^5) = \mathfrak{m}^4 && \text{since } y^4 = x^6 \text{ in } B \\
A'_6 &= \varphi^{-1}(\mathfrak{n}^6) = \mathfrak{m}^4 && \text{since } y^4 = x^6 \text{ in } B \\
&\vdots
\end{aligned} \tag{1}$$

If $S$ denotes the standard stabilizing function of $(A'_n)$ with respect to $(\mathfrak{m}^n)$, then the calculations (1) tells us that the sequence $(S(k))_{k\geq 1}$ starts out as

$$(S(k))_{k\geq 1} = (1,2,4,5,7,8,\dots)$$

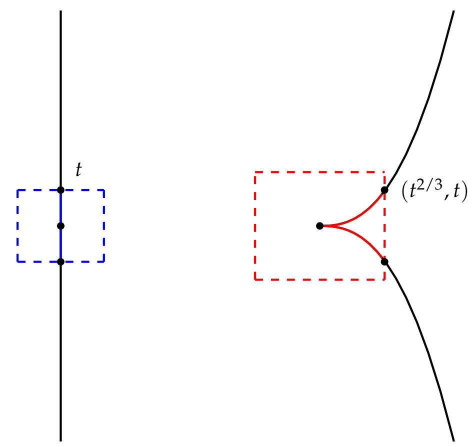
More generally, we have

$$S(n) = \begin{cases} 3m-2 & \text{if } n = 2m-1 \text{ where } m \geq 1 \\ 3m-1 & \text{if } n = 2m \text{ where } m \geq 1 \end{cases}$$

It follows that the contractibility of $B$ with respect to $A$ is given by

$$c = \mathrm{c}_{B,A} = \lim_{k\to\infty} \frac{S(k)}{k} = \frac{3}{2}.$$

To see what's going on geometrically, consider the image below:




The red square represents the open box neighborhood of $\mathfrak{n}$ given by $\{x \in \mathbb{R}^2 \mid \|x\|_\infty < t^{2/3}\}$ (where $t < 1$) and the blue square represents the open box neighborhood of $\mathfrak{m}$ given by $\{x \in \mathbb{R}^2 \mid \|x\|_\infty < t\}$. Intuitively, we think of the ring homomorphism $\varphi\colon A \to B$ as inducing a map $f\colon Y \to X$ given by $f(\mathfrak{n}) = \mathfrak{m}$ where we set $Y = \operatorname{Spec} B = \{0, \mathfrak{n}\}$ and $X = \operatorname{Spec} A = \{0, \mathfrak{m}\}$. The map $f\colon Y \to X$ is thought of as a contraction map with contractibility factor being $3/2$ (the red box whose side length is $2t^{2/3}$ is contracted to the blue box whose side length is $2t$).

**Example 0.2.** Consider the case where $A = K[y]_{\langle y\rangle}$ and $B = K[y, \boldsymbol{x}]_{\langle y, \boldsymbol{x}\rangle}$ where $\boldsymbol{x} = (x_1, x_2, \dots, x_n, \dots)$. Since

$$A'_k = \varphi^{-1}(\mathfrak{n}^k) = \mathfrak{m}^k = A_k$$

for all $k \in \mathbb{N}$, it follows that $\mathrm{S}_{B,A} = \mathbf{1}_k$ and hence $\mathrm{c}_{B,A} = 1$.

**Example 0.3.** Consider the case where $A = K[y]_{\langle y\rangle}$ and $B = K[y, \boldsymbol{x}]_{\langle y, \boldsymbol{x}\rangle} / \langle y^2 - x_1^3, y^2 - x_2^4, \dots, y^2 - x_n^{n+2}, \dots\rangle$. Then observe that for each $n > 2$, we have

$$A'_n = \varphi^{-1}(\mathfrak{n}^n) = \mathfrak{m}^2 = A_2$$

since $y^2 = x_{n-2}^n$ in $B$. In particular, there does not exist an $m$ such that $A'_m \subseteq \mathfrak{m}^3$. It follows that $\mathrm{S}_{B,A}(k) = \infty$ for $k \geq 2$ and hence $\mathrm{c}_{B,A} = \infty$.

**Example 0.4.** Consider the case where $A = K[y]_{\langle y\rangle}$ and $B = K[y, \boldsymbol{x}]_{\langle y, \boldsymbol{x}\rangle} / \langle y^3 - x_1^2, y^4 - x_2^2, \dots, y^{n+2} - x_n^2, \dots\rangle$. Then observe that for each $n > 2$, we have

$$A'_2 = \varphi^{-1}(\mathfrak{n}^2) \subseteq \mathfrak{m}^n = A_n$$

since $y^n = x_{n-2}^2$ in $B$. In particular, we have $\mathrm{S}_{B,A}(k) = 2$ for $k \geq 2$ and hence $\mathrm{c}_{B,A} = 0$.

## Questions

For "nice" local ring homorphisms $A \to B$, the following properties should hold:

1. we have $\mathrm{c}_{B,A} \in \mathbb{Q} \cap [0, \infty]$,
2. if $B \to C$ is another local ring homomorphism, then $\mathrm{c}_{C,B}\mathrm{c}_{B,A} \geq \mathrm{c}_{C,A}$ (where equality holds when something nice happens).

The question we ask now is, what are the "nice" local ring homomorphisms which give rise to those properties? For instance, here's how property (1) could be proved: suppose there exists $k_0 \in \mathbb{N}$ such that

$$\mathrm{c}_{B,A} := \lim_{k\to\infty} \mathrm{S}_{B,A}(k)/k = \mathrm{S}_{B,A}(k_0)/k_0.$$

Then clearly $\mathrm{c}_{B,A} \in \mathbb{Q} \cap [0, \infty]$. Next, suppose that

$$\mathrm{c}_{C,A} = \frac{\mathrm{S}_{C,A}(k_0)}{k_0} \quad \text{and} \quad \mathrm{c}_{B,A} = \frac{\mathrm{S}_{B,A}(k_0)}{k_0}$$

Then if $A$ is not Artinian, we have

$$\begin{aligned} \mathrm{c}_{C,B} &\geq \frac{\mathrm{S}_{C,A}(\mathrm{S}_{B,A}(k_0))}{\mathrm{S}_{B,A}(k_0)} \\ &\geq \frac{\mathrm{S}_{C,A}(k_0)}{\mathrm{S}_{B,A}(k_0)} \\ &= \frac{\mathrm{c}_{C,A}}{\mathrm{c}_{B,A}}, \end{aligned}$$

so this gives us the inequality $\mathrm{c}_{C,B}\mathrm{c}_{B,A} \geq \mathrm{c}_{C,A}$.