Algebraic Topology Homework 4

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Problem 1

Exercise 1. Show that composition of paths satisfies the following cancellation property: if $f_0 \cdot g_0 \sim f_1 \cdot g_1$ and $g_0 \sim g_1$, then $f_0 \sim f_1$.

Solution 1. Let $\widetilde{g}_0$ be the inverse path of g_0, so $\widetilde{g}_0(t) = g_0(1-t)$. Then we have

$$\begin{aligned} f_0 &\sim f_0 \cdot (g_0 \cdot \widetilde{g}_0) \\ &\sim (f_0 \cdot g_0) \cdot \widetilde{g}_0 \\ &\sim (f_1 \cdot g_1) \cdot \widetilde{g}_0 \\ &\sim f_1 \cdot (g_1 \cdot \widetilde{g}_0) \\ &\sim f_1 \cdot (g_0 \cdot \widetilde{g}_0) \\ &\sim f_1 \end{aligned}$$

Problem 2

Exercise 2. Show that for a space X, the following three conditions are equivalent:

1. Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
2. Every map $S^1 \to X$ extends to a map $D^2 \to X$.
3. $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

Solution 2. In this problem, we identify S^1 with the unit circle in the complex plane. Similarly, we identify D^2 with the unit disc in the complex plane.

We first show 1 implies 2. Let $f\colon S^1 \to X$ be a continuous map. Denote $x = f(1)$ and let $F\colon S^1 \times I \to X$ be a homotopy from c_x to f (so $F(-,0) = c_x$ and $F(-,1) = f$). Define $\widetilde{f}\colon D^2 \to X$ by

$$\widetilde{f}(w) = \begin{cases} x & \text{if } w = 0 \\ F(w/|w|, |w|) & \text{else} \end{cases}$$

for all $w \in D^2$. Then $\widetilde{f}$ is easily seen to be an extension of f.

Next we show 2 implies 3. Pick $x_0 \in X$ and let $f\colon S^1 \to X$ be a loop in X based at x_0 (so $f(1) = x_0$). Let $F\colon D^2 \to X$ be a continuous extension of f, so $F|_{S^1} = f$. Next define $H\colon S^1 \times I \to X$ by

$$H(e^{2\pi i s}, t) = \begin{cases} F(te^{2\pi i(s/t)} + (1-t)) & \text{if } 0 \le s \le t \\ x_0 & \text{if } t \le s \le 1 \end{cases}$$

Notice that when $t = 1$, we have $H(e^{2\pi i s}, 1) = F(e^{2\pi i s}) = f(e^{2\pi i s})$. Thus $H(-,1) = f$. Similarly, when $t = 0$, we have $H(e^{2\pi i s}, 0) = x_0$, thus $H(-,0) = c_{x_0}$. Finally, it is easy to see that H is a homotopy from c_{x_0} to f with fixed endpoints. It follows that $\pi_1(X, x_0) = 0$.

3 implies 1 follows by definition.

Problem 3

Exercise 3. Find the explicit reparametrization that shows for paths γ_1, γ_2, and γ_3 with $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$ such that $[(\gamma_1 \cdot \gamma_2) \cdot \gamma_3] = [\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)]$.

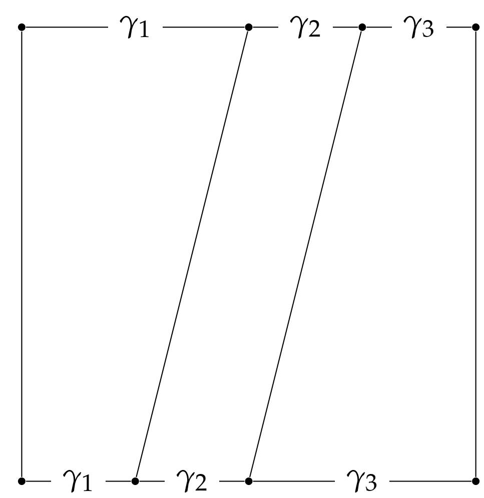
Solution 3. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \gamma_1\left(\frac{4s}{t+1}\right) & 0 \leq s \leq \frac{t+1}{4} \\ \gamma_2\left(4\left(s - \left(\frac{t+1}{4}\right)\right)\right) & \frac{t+1}{4} < s \leq \frac{t+2}{4} \\ \gamma_3\left(\frac{4}{2-t}\left(s - \left(\frac{t+2}{4}\right)\right)\right) & \frac{t+2}{4} < s \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy from $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ to $\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$. Let's explain what's happening in

$$\gamma_3\left(\frac{4}{2-t}\left(s - \left(\frac{t+2}{4}\right)\right)\right)$$

in order to get a better idea of how H is defined. Here, the $(t+2)/4$ part is telling us to delay γ_3 by $(t+2)/4$ seconds. Thus when $t = 0$, we wait half a second before follow the γ_3 path. The $4/(2-t)$ part is telling us to speed up the γ_3 path by $4/(2-t)$ seconds. Thus when $t = 0$, we follow the γ_3 twice as fast. All of the other parts of H can be understood in an analagous way. One may visualize this homotopy as below:



with horizontal axis being the s-axis (s is the path variable) and with vertical axis being the t-axis (t is the homotopy variable). The diagonal lines partition the square into three regions. Note that the left-most diagonal line above is given by the equation $t = 4s - 1$. Thus when $t \geq 4s - 1$ (or equivalently when $0 \leq s \leq (t+1)/4$), we are in the left-most region of the square above.

Remark 1. Usually one uses the horizontal axis as the homotopy axis (so the t-axis), however I drew these diagrams in the past and didn't think it was necessary to re-draw them for this problem. The same applies to problem 4 as well.

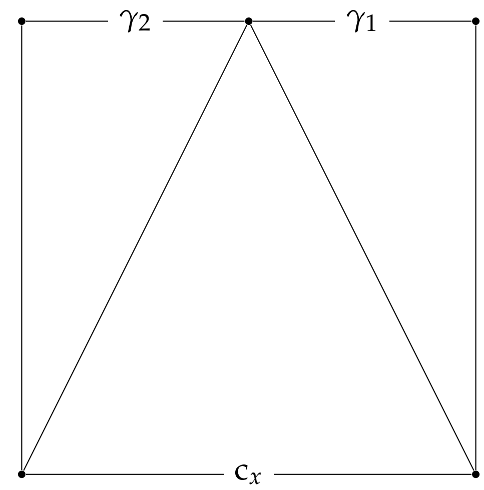
Problem 4

Exercise 4. Find the explicit homotopy that shows that for a loops γ_1 and γ_2 where $\gamma_2(t) = \gamma_1(1-t)$ the composite $\gamma_1 \cdot \gamma_2$ is homotopic to a constant loop.

Solution 4. Let x be the point at which both γ_1 and γ_2 are based at. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \gamma_1\left(\frac{2s}{t}\right) & 0 \leq s < \frac{t}{2} \\ \mathrm{c}_x\left(\frac{1}{1-t}\left(s - \frac{t}{2}\right)\right) & \frac{t}{2} \leq s \leq \frac{2-t}{2} \\ \gamma_2\left(\frac{2}{t}\left(s - \frac{2-t}{2}\right)\right) & \frac{2-t}{2} < s \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy from c_x to $\gamma_1 \cdot \gamma_2$. One may visualize this homotopy as below:


γ_2
γ_1
c_x