Tangent Space of Local Ring

Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherain ring. Recall the tangent space of R is defined to be the $\mathbb{k}$-vector space:

$$\mathrm{T}_{\mathfrak{m}}(R) = \mathrm{Hom}_{\mathbb{k}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{k}).$$

Recall that a **point-derivation** $\partial \colon R \to \mathbb{k}$ is a $\mathbb{k}$-linear map which satisfies Leibniz law, meaning

$$\partial(r_1 r_2) = \partial(r_1)\overline{r}_2 + \overline{r}_1 \partial(r_2)$$

for all $r_1, r_2 \in R$ where $\overline{r} \in \mathbb{k}$ denotes the image of $r \in R$ under the canonical quotient map $R \twoheadrightarrow \mathbb{k}$. The set of all point-derivations $\partial \colon R \to \mathbb{k}$ forms an R-module and is given by

$$\mathrm{Der}_{\mathbb{k}}(R, \mathbb{k}) = \mathrm{Hom}_R(\Omega_{R/\mathbb{k}}, \mathbb{k}),$$

where $\Omega_{R/\mathbb{k}}$ is the module of Kahler differentials of R over $\mathbb{k}$.

Definition 0.1. A map $\theta \colon R \to R$ is called a **derivation** if θ satisfies Leibniz law, meaning

$$\theta(r_1 r_2) = \theta(r_1) r_2 + r_1 \theta(r_2)$$

for all $r_1, r_2 \in R$, and if the map $\vartheta \colon \mathfrak{m}^2 \to R$ defined by

$$\vartheta(x_1, x_2) := \theta(x_1 + x_2) - \theta(x_1) - \theta(x_2),$$

lands in $\mathfrak{m}$.

Remark. If θ is a derivation, then observe that

1. $\theta(\mathfrak{m}^2) \subseteq \mathfrak{m}$
2. $[r, x]_\theta := \theta(rx) - r\theta(x) = \theta(r)x \in \mathfrak{m}$.

In particular, we get a well-defined $\mathbb{k}$-linear map $\overline{\theta} \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{k}$. Conversely, suppose $\tau \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{k}$ is any $\mathbb{k}$-linear map. Let $\overline{x}_1, \dots, \overline{x}_m$ be a basis for $\mathfrak{m}/\mathfrak{m}^2$ as a $\mathbb{k}$-vector space, so $x_1, \dots, x_m$ is a minimal generating set for $\mathfrak{m}$. Furthermore, set $\tau(\overline{x}_i) = c_i$ for each i and let

$$\partial := c_1 \partial_{x_1} + \cdots + c_m \partial_{x_m}.$$