Homological Associativity of Differential Graded Algebras and Gröbner Bases

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Abstract

We investigate associativity of multiplications on chian complexes over commutative noetherian rings from two perspectives. First, we introduce a natural associator subcomplex and show how its homology can detect associativity. Second, we use Gröbner bases to compute the associator.

Contents

Introduction

In this paper, we study algebraic structures that we can attach to free resolutions. In particular, we are motivated by the following problem: let $(R, \mathfrak{m}, \Bbbk)$ be a local (or standard graded) commutative noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let $F = (F, \mathrm{d})$ be the minimal free resolution of R/I over R. The usual multiplication map $\mathrm{m}: R/I \otimes_R R/I \to R/I$ can be lifted to a chain map $\mu: F \otimes_R F \to F$, denoted $a_1 \otimes a_2 \mapsto a_1 \star_\mu a_2$ where $a_1, a_2 \in F$ (where we simplify notation to $a_1 \star_\mu a_2 = a_1 a_2$ whenever μ is clear from context). Further, we can choose μ to be unital (with $1 \in F_0 = R$ being the identify element) and strictly graded-commutative; see Definition (1.1). In this case we call μ a **multiplication** on F, and when we equip F with this multiplication, we say F is an **MDG algebra** (M stands for multiplication, D stands for differential, and G stands for graded). See Section 1 below for foundational material on MDG algebras and modules. It was first shown that F always possesses an MDG algebra structure by Buchsbaum and Eisenbud in [BE77], and in that paper they posed the following question:

Question A: Does F possess the structure of a DG algebra? In other words, can μ be chosen such that it is associative?

One reason this question is interesting is that when we know the answer is "yes", then we gain a lot of information about the "shape" of F. For instance, Buchsbaum and Eisenbud proved that if we further assume R is a domain and we know that an associative multiplication on F exists, then one obtains important lower bounds of the Betti numbers $\beta_i = \beta_i^R(R/I)$. It turns out however, that the answer to Question A is that F need not have a DG algebra structure on it (see [Avr81, Luk26] for counterexamples).

With this in mind, we pose the following question:

Question B: Equip F with a multiplication μ giving it the structure of an MDG algebra. How can we measure the failure of F to be associative?

We answer this question in Section 2 below. In short, in Subsection 2.1, we define the **associator** submodule of an MDG module X over an MDG algebra A to be the smallest MDG submodule containing all "associators" of X:

$$\langle X \rangle = \{(a_1 a_2)x - a_1(a_2 x) \mid a_1, a_2 \in A \text{ and } x \in X\} \subseteq X.$$

It is clear that if X is associative, then $\mathrm{H}(\langle X \rangle) = 0$. The first main result of this paper Theorem (2.1) shows that the converse holds under certain conditions. In Section 3.1, we exploit a criterion for exactness. We apply this criterion in our second main result, Theorem (3.1) to demonstrate associativity of exterior extensions. In the final section of this paper, we construct the symmetric DG algebra of an R-complex A which is centered at R (meaning $A_0 = R$ and $A_i = 0$ for all $i < 0$), denoted by $\mathrm{S}_R(A) = S$. This section contains our third result of the paper, namely Theorem (4.3), which says that if we fix a multiplication μ on A, then the quotient $A^{\mathrm{as}} := A/\langle A \rangle$ can be presented as a quotient of S by a DG S-ideal $\mathfrak{s} = \mathfrak{s}(\mu)$ which is constructed from μ in a functorial way. In particular, we can study MDG algebra structures on A by studying certain DG ideals of S. This presentation allows us to use Gröbner bases to help calculate A^{as} when working over an integral domain where we can see how associators naturally arise when performing Buchberger's algorithm to certain set of polynomials using this monomial ordering.

1 MDG Algebras and Modules

We begin by defining MDG algebras. After defining MDG algebras, we then motivate their study by explaining how they arise naturally in the study of minimal free resolutions of cyclic modules.

1.1 MDG Algebras

Let R be a commutative ring and let $A = (A, \mathrm{d})$ be an R-complex:

$$A := \cdots \longrightarrow A_{n+1} \xrightarrow{\mathrm{d}_{n+1}} A_n \xrightarrow{\mathrm{d}_n} A_{n-1} \longrightarrow \cdots.$$

We view A as a graded R-module

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

equipped with an R-linear map $\mathrm{d}: A \to A$ which is graded of degree -1 and satisfies $\mathrm{d}^2 = 0$. We further equip A with a chain map $\mu: A \otimes_R A \to A$. We denote by $\star_\mu: A \times A \to A$ (or more simply by $\cdot$ if context is clear) to be the unique graded R-bilinear map which corresponds to μ via the universal mapping property of tensors products. Thus we have

$$\mu(a_1 \otimes a_2) = a_1 \star_\mu a_2 = a_1 a_2$$

for all $a_1, a_2 \in A$, where we further simplify the notation by writing $a_1 \star_\mu a_2 = a_1 a_2$ when context is clear. Note that since μ is a chain map, $\star_\mu$ satisfies the **Leibniz law** which says

$$\mathrm{d}(a_1 a_2) = \mathrm{d}(a_1)a_2 + (-1)^{|a_1|} a_1 \mathrm{d}(a_2)$$

for all $a_1, a_2 \in A$ with a_1 homogeneous, where $|a_1|$ denotes the homological degree of a_1. Note also that the chain map μ induces a chain map $\overline{\mu} \colon \mathrm{H}(A) \otimes_R \mathrm{H}(A) \to \mathrm{H}(A)$, given by

$$\overline{\mu}(\overline{a}_1 \otimes \overline{a}_2) = \overline{a_1 a_2} \tag{1}$$

for all $\overline{a}_1, \overline{a}_2 \in \mathrm{H}(A)$ (where $a_1, a_2 \in A$ such that $\mathrm{d}a_1 = 0 = \mathrm{d}a_2$ are representatives of $\overline{a}_1$ and $\overline{a}_2$) where the Leibniz law ensures the equation (1) is well-defined. Here, we view $\mathrm{H}(A)$ as a trivial R-complex whose underlying graded R-module is $\mathrm{H}(A)$ and whose differential is the zero map. Thus $\overline{\mu}$ being a chain map is equivalent to it being just a graded R-linear map.

In order to simplify our notation in what follows, we often refer to the triple (A, d, μ) via its underlying graded R-module A, where we think of A as a graded R-module which is equipped with a differential $\mathrm{d} \colon A \to A$, giving it the structure of an R-complex, and which is further equipped with a chain map $\mu \colon A \otimes_R A \to A$. For instance, if μ satisfies a property (such as being associative), then we also say A satisfies that property.

Definition 1.1. With the notation as above, we make the following definitions:

1. We say A is **unital** if there exists $1 \in A$ such that $1a = a = a1$ for all $a \in A$.
2. We say A is **graded-commutative** if $a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_1$ for all homogeneous $a_1, a_2 \in A$.
3. We say A is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that $a^2 = 0$ for all elements $a \in A$ with $|a|$ odd.
4. We say A is **associative** if $(a_1 a_2)a_3 = a_1(a_2 a_3)$ for all for all $a_1, a_2, a_3 \in A$.

We say A is an **MDG R-algebra** if A is strictly graded-commutative, unital, and $\mathrm{H}(A)$ is associative. Thus $\mathrm{H}(A)$ obtains the structure of an associative, strictly graded-commutative R-algebra. We call μ the **multiplication** of A just as we call d the **differential** of A. We say A is **centered** at R if $A_0 = R$ and $A_i = 0$ for all $i < 0$. Suppose B is another MDG R-algebra and let $\varphi \colon A \to B$ be a function.

1. We say φ is **unital** if $\varphi(1) = 1$.
2. We say φ is **multiplicative** if $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$.

We say $\varphi \colon A \to B$ is an **MDG R-algebra homomorphism** if it is a chain map which is both unital and multiplicative. We denote by $\mathbf{MDG}_R$ to be the category of all MDG R-algebras and MDG R-algebra homomorphisms.

Remark 1. Let A be an MDG R-algebra. We also view R as an MDG R-algebra over itself. We have a canonical MDG R-algebra homomorphism $\iota \colon R \to A$ defined by $\iota(r) = r \cdot 1$ where we write $\cdot$ to denote the R-scalar multiplication $R \times A \to A$.

1.2 MDG Algebra Resolutions of a Cyclic Module

In this subsection, we describe the MDG algebras we are mostly interested in. Throughout this subsection, let I be an ideal of R, and let F be a free resolution of R/I over R such that $F_0 = R$. We denote by $\mathcal{C}(F^{\otimes 2}, F)$ to be the set of all chain maps from $F^{\otimes 2} := F \otimes_R F$ to F (more generally, if X and Y are two R-complexes, then we denote by $\mathcal{C}(X, Y)$ to be the set of all chain maps from X to Y).

Definition 1.2. A **multiplication** on F is a chain map $\mu \in \mathcal{C}(F^{\otimes 2}, F)$ which is unital (with $1 \in F$ being the identity element) and strictly graded-commutative (if we decide to equip F with a particular multiplication μ, giving it the structure of an MDG R-algebra, then we write $F = (F, \mathrm{d}, \mu)$ and refer to μ as *the* multiplication of F). We denote by $\mathrm{Mult}(F)$ to be the set of all multiplications on F.

We claim that every multiplication on F is automatically a lift of the usual multiplication m on R/I. Indeed, first note that F comes equipped with a canonical quasi-isomorphism $\tau \colon F \to R/I$. Here we view R/I as a trivial R-complex which sits in homological degree 0. In homological degree 0, we have $\tau_0 \colon R \to R/I$ where τ_0 is the canonical projection map. In homological degree i where $i \neq 0$, we have $\tau_i \colon F_i \to 0$ is the zero map. With this understood, the multiplication μ is a lift of m if the following diagram of R-complexes commutes:

$$\begin{tikzcd} F \otimes_R F \arrow[r, "\mu"] \arrow[d, "\tau^{\otimes 2}"'] & F \arrow[d, "\tau"] \\ R/I \otimes_R R/I \arrow[r, "\mathrm{m}"] & R/I. \end{tikzcd} \tag{2}$$

In homological degree $i \neq 0$, this diagram commutes for trivial reasons, so the only thing that we need to check is that the diagram commutes in homological degree 0. In homological degree 0, the diagram looks like:

$$\begin{tikzcd} R \otimes_R R \arrow[r, "\mu_0"] \arrow[d, "\tau_0^{\otimes 2}"'] & R \arrow[d, "\tau_0"] \\ R/I \otimes_R R/I \arrow[r, "\mathrm{m}"] & R/I. \end{tikzcd} \tag{3}$$

Note that μ_0 is R-linear, so it is completely determined by where it sends $1 \otimes 1$. The diagram (3) will commute if and only if μ_0 sends $1 \otimes 1$ to $1 + x$ for some $x \in I$. In fact, μ_0 is already forced to send $1 \otimes 1$ to 1 since μ is assumed to be unital with identity element 1. Thus if $r_1, r_2 \in R$, then

$$r_1 \star_\mu r_2 = (r_1 r_2)(1 \star_\mu 1) = r_1 r_2.$$

In other words, μ_0 agrees with the usual multiplication on R, and the diagram (3) automatically commutes in this case as well.

Next, let J be an ideal contained in I and let G be an R-free resolution of R/J such that $G_0 = R$. Fix multiplications μ on F and ν on G giving them the structure of MDG R-algebras. Choose $\varphi \colon G \to F$ to be a lift of the map $R/J \to R/I$. We claim that if R is local and φ is multiplicative, then φ is automatically unital. Indeed, suppose φ is multiplicative and write $\varphi(1) = r$ for some $r \in R$. Since φ is a lift of $R/J \to R/I$, we must have $r = 1 + x$ for some $x \in I$. Since R is local, this implies r is a unit. However multiplicativity of φ already implies $r^2 = r$, and thus we must have $r = 1$ since r is a unit. Thus under these assumptions, $\varphi \colon G \to F$ is an MDG algebra homomorphism if and only if it is multiplicative.

1.2.1 Viewing $\mathrm{Mult}(F)$ as a Convex Subset

We now want to explain how $\mathrm{Mult}(F)$ can be viewed as a "convex" subset of $\mathcal{C}(F^{\otimes 2}, F)$. To see what this means, first recall that $\mathcal{C}(F^{\otimes 2}, F)$ has a natural R-module structure on it, but this R-module structure does not induce an R-module structure on $\mathrm{Mult}(F)$ (if $r \in R \backslash \{1\}$ and $\mu \in \mathrm{Mult}(F)$, then $r\mu$ will not be unital). The following lemma shows that it is better to interpret $\mathrm{Mult}(F)$ as some sort of convex subset of $\mathcal{C}(F^{\otimes 2}, F)$:

Lemma 1.1. *Suppose $\mu, \nu \in \mathrm{Mult}(F)$ and $\lambda \in \mathcal{C}(F, F)$. Then $\lambda\mu + (1 - \lambda)\nu \in \mathrm{Mult}(F)$.*

Proof. Clearly $\lambda\mu + (1-\lambda)\nu$ is a chain map. It is also unital since if $a \in F$, then we have

$$\begin{aligned} (\lambda\mu + (1-\lambda)\nu)(1 \otimes a) &= \lambda\mu(1 \otimes a) + (1-\lambda)\nu(1 \otimes a) \\ &= \lambda a + (1-\lambda)a \\ &= a. \end{aligned}$$

A similar computation shows $(\lambda\mu + (1-\lambda)\nu)(a \otimes 1) = a$. Finally, it is graded-commutative since if $a_1, a_2 \in F$ are homogeneous, then we have

$$\begin{aligned} (\lambda\mu + (1-\lambda)\nu)(a_1 \otimes a_2) &= \lambda\mu(a_1 \otimes a_2) + (1-\lambda)\nu(a_1 \otimes a_2) \\ &= (-1)^{|a_1||a_2|}\lambda\mu(a_2 \otimes a_1) + (-1)^{|a_1||a_2|}(1-\lambda)\nu(a_2 \otimes a_1) \\ &= (-1)^{|a_1||a_2|}(\lambda\mu + (1-\lambda)\nu)(a_2 \otimes a_1). \end{aligned}$$

$\square$

1.2.2 Multiplications up to Homotopy

A chain map $\mu \in \mathcal{C}(F^{\otimes 2}, F)$ which lifts the usual multiplication map on R/I is unique up to homotopy. What this means is that if $\mu' \in \mathcal{C}(F^{\otimes 2}, F)$ is another chain map which lifts the multiplication map on R/I, then there exists a graded R-linear map $\nu\colon F^{\otimes 2} \to F$ of degree one, then $\mu' = \mu_\nu$ where

$$\mu_\nu := \mu + \mathrm{d}\nu + \nu\mathrm{d}. \tag{4}$$

If μ is a multiplication, then we want to determine the conditions ν needs to satisfy in order for μ_ν to be a multiplication also. To this end, write $\star\colon F^2 \to F$ for the R-bilinear map associated to μ, write $\star_\nu\colon F^2 \to F$ for the R-bilinear map associated to μ_ν, and write $[\cdot,\cdot]_\nu\colon F^2 \to F$ for the R-bilinear map associated to ν. Thus for each $a_1, a_2 \in F$ homogeneous, we have

$$a_1 \star_\nu a_2 = a_1 \star a_2 + \mathrm{d}[a_1, a_2]_\nu + [\mathrm{d}a_1, a_2]_\nu + (-1)^{|a_1|}[a_1, \mathrm{d}a_2]_\nu. \tag{5}$$

Let $\sigma\colon F^{\otimes 2} \to F^{\otimes 2}$ be the chain map defined on homogeneous elements $a, a_1, a_2 \in F$ by

$$\sigma(a_1 \otimes a_2) = a_1 \otimes a_2 - (-1)^{|a_1||a_2|} a_2 \otimes a_1$$

Finally, write $[\cdot,\cdot]_{\nu\sigma}\colon F^2 \to F$ for the R-bilinear map associated to $\nu\sigma$. Thus we have

$$[a_1, a_2]_{\nu\sigma} = [a_1, a_2]_\nu - (-1)^{|a_1||a_2|}[a_2, a_1]_\nu$$

for all homogeneous $a_1, a_2 \in F$.

Lemma 1.2. *With the notation as above we have the following:*

1. *μ_ν is graded-commutative if and only if the composite $\nu\sigma$ is a chain map of degree one, meaning*

$$\mathrm{d}[a_1, a_2]_{\nu\sigma} = -[\mathrm{d}a_1, a_2]_{\nu\sigma} - (-1)^{|a_1|}[a_1, \mathrm{d}a_2]_{\nu\sigma}$$

for all homogeneous $a_1, a_2 \in F$.

2. *We have $a_1 \star_\nu a_2 = a_1 \star a_2$ if and only if $[\cdot,\cdot]_\nu$ satisfies Leibniz law at the pair (a_1, a_2), meaning*

$$\mathrm{d}[a_1, a_2]_\nu = -[\mathrm{d}a_1, a_2]_\nu - (-1)^{|a_1|}[a_1, \mathrm{d}a_2]_\nu.$$

In particular, if μ_ν is unital if and only if both $\nu|_{F\otimes 1}$ and $\nu|_{1\otimes F}$ are chain maps of degree one for all $a \in F$, meaning

$$\mathrm{d}[a, 1]_\nu = -[\mathrm{d}a, 1]_\nu \quad \textit{and} \quad \mathrm{d}[1, a] = -[1, \mathrm{d}a]_\nu,$$

for all $a \in F$. Similarly, μ_ν is strictly graded-commutative if and only if $\nu\sigma$ is a chain map of degree one and

$$\mathrm{d}[a, a]_\nu = [\mathrm{d}a, a]_{\nu\sigma}$$

for all homogeneous $a \in F$ such that $|a|$ is odd.

Proof. Note that μ_ν is graded-commutative if and only if $\mu_\nu\sigma = 0$. Thus by applying σ to the right on both sides in (4) and using the fact that μ is graded-commutative and σ is a chain map, we see that μ_ν is graded-commutative if and only if $\mathrm{d}(\nu\sigma) = -(\nu\sigma)\mathrm{d}$, that is, if and only if $\nu\sigma$ is a chain map of degree one. The remaining identities are obtained by considering (5). □

1.3 Multigraded MDG Algebras

In this subsection, discuss a combinatorial setting where MDG algebras shows up as well as provide some examples of MDG algebras. Throughout this subsection, we fix the following notation: let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_n]$ where $\Bbbk$ is a field and let $I = \langle \boldsymbol{m} \rangle = \langle m_1, \dots, m_d \rangle$ is a monomial ideal in R. For each subset $\sigma \subseteq \{1, \dots, r\}$, we denote $e_\sigma := \{e_i \mid i \in \sigma\}$ (thus $e_{123} = \{e_1, e_2, e_3\}$). We also set $m_\sigma := \mathrm{lcm}(m_i \mid i \in \sigma)$ and we set $\boldsymbol{\alpha}_\sigma \in \mathbb{Z}^n$ to be the exponent vector of m_σ. Let Δ be a finitely simplicial complex with d-vertices denoted $e_1, \dots, e_d$. The sequence of monomials $\boldsymbol{m}$ induces a labeling of the faces of Δ as follows: we label the vertices $e_1, \dots, e_d$ of Δ by the monomials $m_1, \dots, m_d$ (so e_i is labeled by m_i). More generally, if e_σ a face of Δ, then we label it by m_σ. With the faces labeled this way, we call Δ an **$\boldsymbol{m}$-labeled simplicial complex** (or a labeled simplicial complex if $\boldsymbol{m}$ is understood from context). Also, for each $\boldsymbol{\alpha} \in \mathbb{Z}^n$, let $\Delta_{\boldsymbol{\alpha}}$ be the subcomplex of Δ defined by

$$\Delta_{\boldsymbol{\alpha}} = \{\sigma \in \Delta \mid m_\sigma \text{ divides } x^{\boldsymbol{\alpha}}\}.$$

We often denote the faces of $\Delta_{\boldsymbol{\alpha}}$ by $(x^{\boldsymbol{\alpha}}/m_\sigma)e_\sigma$ instead of σ whenever context is clear. With the notation as above, we obtain the following R-complex (which was first described in [BPS98]):

Definition 1.3. We define an R-complex, denoted F_Δ (or more simply denoted F if Δ is understood from context) and called the R-**complex induced by** Δ as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R-module of F is given by

$$F_k := \begin{cases} \bigoplus_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \le k \le \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $\mathrm{d}(e_\varnothing) = 0$ and

$$\mathrm{d}(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma\setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \setminus \{\varnothing\}$ where $\mathrm{pos}(i,\sigma)$ is the number of elements preceding i in the ordering of σ, and where $\sigma \setminus i$ denotes the face obtained from σ by removing i. In the case where Δ is the d-simplex, we call F the **Taylor complex**.

Observe that F also has the structure of a multigraded $\Bbbk$-complex (or an $\mathbb{N}^n$-graded $\Bbbk$-complex) since the differential d respects the multigrading. In other words, we have a decomposition of $\Bbbk$-complexes

$$F = \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} F_{\boldsymbol{\alpha}},$$

where the $\Bbbk$-complex $F_{\boldsymbol{\alpha}}$ in multidegree $\boldsymbol{\alpha} \in \mathbb{N}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded $\Bbbk$-vector space is given by

$$F_{k,\boldsymbol{\alpha}} := \begin{cases} \bigoplus_{\dim \sigma = k-1} \Bbbk \frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{\alpha}} \text{ and } 0 \le k \le \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $\mathrm{d}_{\boldsymbol{\alpha}}$ of $F_{\boldsymbol{\alpha}}$ is just the restriction of d to $F_{\boldsymbol{\alpha}}$. Notice that the differential behaves exactly like boundary map of $\Delta_{\boldsymbol{\alpha}}$ does:

$$\begin{aligned} \mathrm{d}_{\boldsymbol{\alpha}} \left(\frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma \right) &= \frac{x^{\boldsymbol{\alpha}}}{m_\sigma} \mathrm{d}(e_\sigma) \\ &= \frac{x^{\boldsymbol{\alpha}}}{m_\sigma} \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma\setminus i}} e_{\sigma\setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{x^{\boldsymbol{\alpha}} m_\sigma}{m_\sigma m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{x^{\boldsymbol{\alpha}}}{m_{\sigma\setminus i}} e_{\sigma\setminus i}. \end{aligned}$$

Thus if we define $\varphi_{\boldsymbol{\alpha}} \colon F_{\boldsymbol{\alpha}}(1) \to \mathcal{S}(\Delta_{\boldsymbol{\alpha}})$ to be the unique graded $\Bbbk$-linear isomorphism such that $\frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma \mapsto \sigma$, then from the computation above, we see that $\mathrm{d}_{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} = \partial_{\boldsymbol{\alpha}} \mathrm{d}_{\boldsymbol{\alpha}}$, and hence $\varphi_{\boldsymbol{\alpha}}$ gives an isomorphism of $\Bbbk$-complexes $\varphi \colon \Sigma^{-1} F_{\boldsymbol{\alpha}} \simeq C(\Delta_{\boldsymbol{\alpha}}; \Bbbk)$, where $C(\Delta_{\boldsymbol{\alpha}}, \Bbbk)$ is the reduced chain complex of $\Delta_{\boldsymbol{\alpha}}$ over $\Bbbk$. In particular, this implies

$$\begin{aligned} \mathrm{H}(F) &= \ker \mathrm{d} / \operatorname{im} \mathrm{d} \\ &= \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \ker \mathrm{d}_{\boldsymbol{\alpha}} \right) \Big/ \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \operatorname{im} \mathrm{d}_{\boldsymbol{\alpha}} \right) \\ &\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} (\ker \mathrm{d}_{\boldsymbol{\alpha}} / \operatorname{im} \mathrm{d}_{\boldsymbol{\alpha}}) \\ &= \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \mathrm{H}(F_{\boldsymbol{\alpha}}) \\ &\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \widetilde{\mathrm{H}}(\Delta_{\boldsymbol{\alpha}}, \Bbbk)(-1). \end{aligned}$$

In other words, we have

$$\mathrm{H}_i(F) \cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \mathrm{H}_i(F_{\boldsymbol{\alpha}}) \cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{Z}^n} \widetilde{\mathrm{H}}_{i-1}(\Delta; \Bbbk).$$

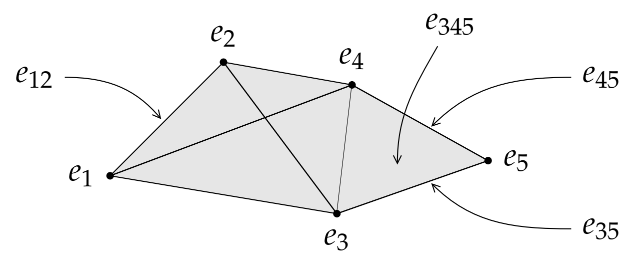
for all $i \in \mathbb{Z}$. From this we easily get the following theorem from [BPS98]:

Theorem 1.3. *F is an R-free resolution of $R/\boldsymbol{m}$ if and only if for all $\boldsymbol{\alpha} \in \mathbb{Z}^n$ either $\Delta_{\boldsymbol{\alpha}}$ is the void complex or $\Delta_{\boldsymbol{\alpha}}$ is acyclic. In particular, the Taylor complex is an R-free resolution of $R/\boldsymbol{m}$. Moreover, F is minimal if and only if $m_\sigma \neq m_{\sigma'}$ for every proper subface σ' of a face σ.*

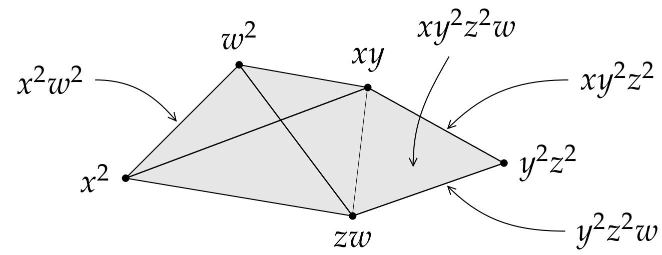
1.3.1 Examples of Multigraded MDG Algebras

Throughout this subsubsection, let $R = \Bbbk[x, y, z, w]$. We consider six examples of multigraded MDG R-algebras. The first two examples were considered in [Luk26] and [Avr81] respectively, and were both shown to be examples of minimal free resolutions which do not admit a DG algebra structure on them.

Example 1.1. ([Luk26]) Let $\Delta_K = \Delta$ be the simplicial complex whose vertex set is $\{e_1, e_2, e_3, e_4, e_5\}$ and whose faces consists of all subsets of $e_{1234} = \{e_1, e_2, e_3, e_4\}$ and $e_{345} = \{e_3, e_4, e_5\}$, pictured below:



Let $\boldsymbol{m}_K = \boldsymbol{m} = x^2, w^2, xy, zw, y^2z^2$. Then we obtain an $\boldsymbol{m}$-labeled simplicial complex $\Delta = (\Delta, \boldsymbol{m})$ which is pictured below:



Let $F_K = F$ be the multigraded R-complex induced by Δ. Thus the homogeneous components of F as a graded R-module look like:

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234}
\end{aligned}$$

The differential $\mathrm{d}_K = \mathrm{d}$ of F behaves just like the usual boundary map of the simplicial complex above except some monomials can show up as coefficients (which makes it so that the differential respects the multidegree). For instance, we have

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

Now equip F with a multiplication $\mu_K = \mu$ which respects the multigrading, giving it the structure of a multigraded MDG algebra. Since μ respects the multigrading and satisfies Leibniz law, we are forced to have:

$$\begin{aligned}
e_1 \star e_5 &= yz^2e_{14} + xe_{45} \\
e_1 \star e_2 &= e_{12} \\
e_2 \star e_5 &= y^2ze_{23} + we_{35} \\
e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= -e_{124}
\end{aligned}$$

At this point however, one can conclude that F is not associative since

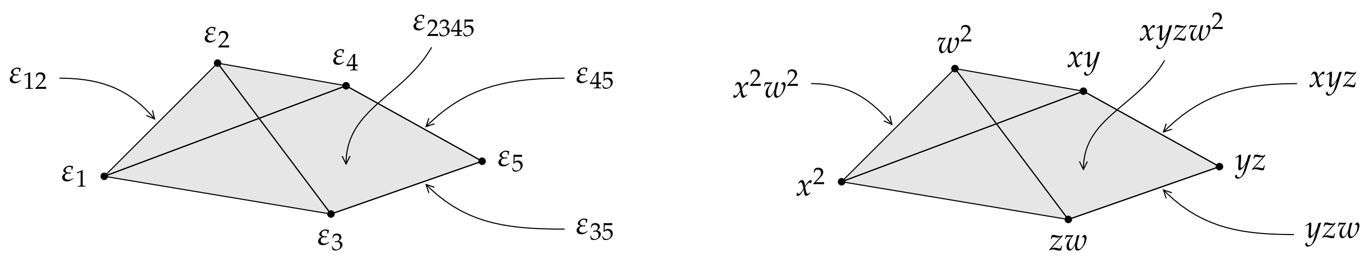
$$[e_1, e_5, e_2] := (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{6}$$

The multiplication is not uniquely determined on all pairs (e_σ, e_τ); for instance there are two possible ways in which μ is defined at the pair (e_5, e_{12}). We assume that μ is defined at (e_5, e_{12}) by

$$e_5 \star e_{12} = yz^2 e_{124} + xyze_{234} + xwe_{345}.$$

Finally, we would still like for μ to be as associative as possible (even though we already know it is not associative at the triple (e_1, e_5, e_2)). In particular, we want μ to be associative on all triples of the form $(e_\sigma, e_\sigma, e_\tau)$. It turns out this can be done and we will assume that μ is associative on all such triples.

Example 1.2. ([Avr81]) Let $\boldsymbol{m}_{\mathrm{A}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz$, and let $F_{\mathrm{A}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F can be realized as the R-complex induced by the $\boldsymbol{m}$-labeled cellular complex pictured below:



We write down the homogeneous components of F as a graded module below:

$$\begin{aligned}
F_0 &= R \\
F_1 &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_2 &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_3 &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_4 &= R\varepsilon_{12345}
\end{aligned}$$

The differential $\mathrm{d}_{\mathrm{A}} = \mathrm{d}$ on the non-simplicial faces as below

$$\begin{aligned}
\mathrm{d}(\varepsilon_{12345}) &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\
\mathrm{d}(\varepsilon_{1345}) &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\
\mathrm{d}(\varepsilon_{2345}) &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}.
\end{aligned}$$

We obtain a multiplication μ_{A} on F_{A} from the one we constructed on F_{K} as follows: first note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{A}}$ induces a multigraded comparison map $\pi\colon F_{\mathrm{K}} \to F_{\mathrm{A}}$ defined by

$$\begin{aligned}
\pi(e_5) &= yz\varepsilon_5 & \pi(e_{345}) &= 0 \\
\pi(e_{35}) &= yz\varepsilon_{35} & \pi(e_{234}) &= \varepsilon_{2345} \\
\pi(e_{45}) &= yz\varepsilon_{45} & \pi(e_{134}) &= \varepsilon_{1345} \\
\pi(e_{34}) &= x\varepsilon_{35} - w\varepsilon_{45} & \pi(e_{1234}) &= \varepsilon_{12345}
\end{aligned}$$

and $\pi(e_\sigma) = \varepsilon_\sigma$ for the remaining homogeneous basis elements. This map is locally invertible. Indeed, by base changing to R_{yz}, we obtain quasi-isomorphisms $F_{\mathrm{A},yz} \to 0 \leftarrow F_{\mathrm{K},yz}$. In particular, there exists a comparison map $\iota\colon F_{\mathrm{A},yz} \to F_{\mathrm{K},yz}$ which splits comparison map $\pi\colon F_{\mathrm{K},yz} \to F_{\mathrm{A},yz}$. By considering the multigrading as well as the Leibniz law, we see that

$$\begin{aligned}
\iota(\varepsilon_5) &= e_5/yz & \iota(\varepsilon_{2345}) &= -e_{234} + e_{345}/yz \\
\iota(\varepsilon_{35}) &= e_{35}/yz & \iota(\varepsilon_{1345}) &= e_{134} - e_{345}/yz \\
\iota(\varepsilon_{45}) &= e_{45}/yz & \iota(\varepsilon_{12345}) &= e_{1234}
\end{aligned}$$

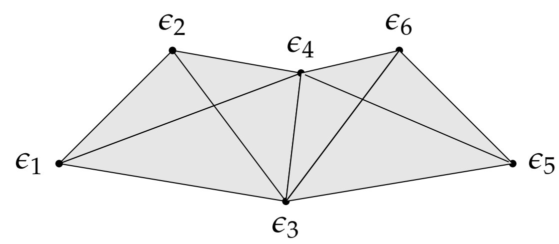
and $\iota(\varepsilon_\sigma) = e_\sigma$ for the remaining homogeneous basis elements. With this in mind, we define a multiplication μ_{A} on F_{A} using the multiplication μ_{K} on $F_{\mathrm{K},yz}$ by setting $\mu_{\mathrm{A}} = \pi\mu_{\mathrm{K}}\iota^{\otimes 2}$. In other words, we have

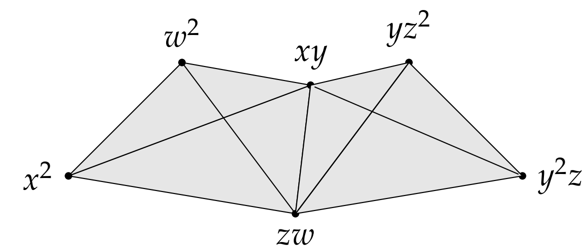
$$\varepsilon_\sigma \star_{\mu_{\mathrm{A}}} \varepsilon_\tau = \pi(\iota(\varepsilon_\sigma) \star_{\mu_{\mathrm{K}}} \iota(\varepsilon_\tau)) \tag{7}$$

for all homogeneous basis elements $\varepsilon_\sigma, \varepsilon_\tau$ of $F_{\mathrm{A},yz}$. It is straightforward to check that μ_{A} restricts to a multiplication on F_{A} (the coefficients in (7) are in R). Note that μ_{A} is not associative since

$$[\varepsilon_1, \varepsilon_5, \varepsilon_2] = -\mathrm{d}(\varepsilon_{1234}) \neq 0.$$

Example 1.3. Let $\boldsymbol{m}_{\mathrm{M}} = \boldsymbol{m} = x^2, w^2, zw, xy, y^2z, yz^2$, and let $F_{\mathrm{M}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ of R. Then F can be realized as the R-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:





The homogeneous components of F as a graded R-module are given below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= R\epsilon_1 + R\epsilon_2 + R\epsilon_3 + R\epsilon_4 + R\epsilon_5 + R\epsilon_6 \\
F_2 &= R\epsilon_{12} + R\epsilon_{13} + R\epsilon_{14} + R\epsilon_{23} + R\epsilon_{24} + R\epsilon_{34} + R\epsilon_{35} + R\epsilon_{36} + R\epsilon_{45} + R\epsilon_{46} + R\epsilon_{56} \\
F_3 &= R\epsilon_{123} + R\epsilon_{124} + R\epsilon_{134} + R\epsilon_{234} + R\epsilon_{345} + R\epsilon_{346} + R\epsilon_{356} + R\epsilon_{456} \\
F_4 &= R\epsilon_{1234} + R\epsilon_{3456}.
\end{aligned}
$$

The canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{M}}$ induces multigraded comparison maps $\pi_\lambda \colon F_{\mathrm{K}} \to F_{\mathrm{M}}$ where $\lambda \in \Bbbk$ and where π_λ is defined by

$$
\begin{aligned}
\pi_\lambda(e_5) &= \lambda z\epsilon_5 + (1-\lambda)y\epsilon_6 \\
\pi_\lambda(e_{35}) &= \lambda z\epsilon_{35} + (1-\lambda)y\epsilon_{36} \\
\pi_\lambda(e_{45}) &= \lambda z\epsilon_{45} + (1-\lambda)y\epsilon_{46} \\
\pi_\lambda(e_{345}) &= \lambda z\epsilon_{345} + (1-\lambda)y\epsilon_{346}
\end{aligned}
$$

and $\pi_\lambda(e_\sigma) = \epsilon_\sigma$ for the remaining homogeneous basis elements. We will choose $\lambda = 1$ and view F_{K} as a subcomplex of F_{M} via $\pi = \pi_1$. We define a multigraded multiplication μ_{M} on F_{M} so that it extends the multiplication μ_{K} on F_{K}. Considerations of the Leibniz and multigrading tells us that we are already forced to have:

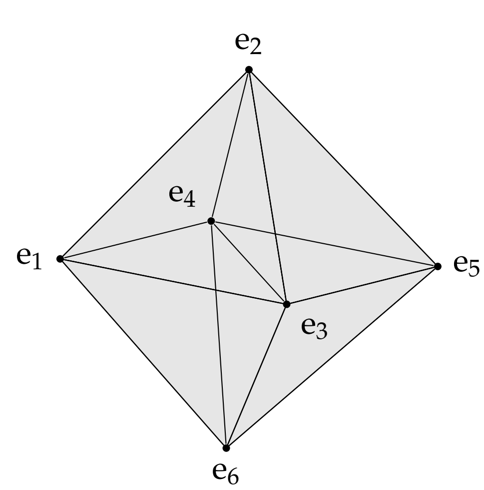
$$
\begin{aligned}
\epsilon_1 \star \epsilon_5 &= yz\epsilon_{14} + x\epsilon_{45} & \qquad \epsilon_1 \star \epsilon_6 &= z^2 e_{14} + xe_{46} \\
\epsilon_2 \star \epsilon_5 &= y^2\epsilon_{23} + w\epsilon_{35} & \qquad \epsilon_2 \star \epsilon_6 &= yz\epsilon_{23} + w\epsilon_{36} \\
\epsilon_2 \star \epsilon_{45} &= -y\epsilon_{234} + w\epsilon_{345} & \qquad \epsilon_2 \star \epsilon_{46} &= -ze_{234} + w\epsilon_{346} \\
\epsilon_1 \star \epsilon_{35} &= y\epsilon_{134} - x\epsilon_{345} & \qquad \epsilon_1 \star \epsilon_{36} &= z\epsilon_{134} - x\epsilon_{346}.
\end{aligned}
$$

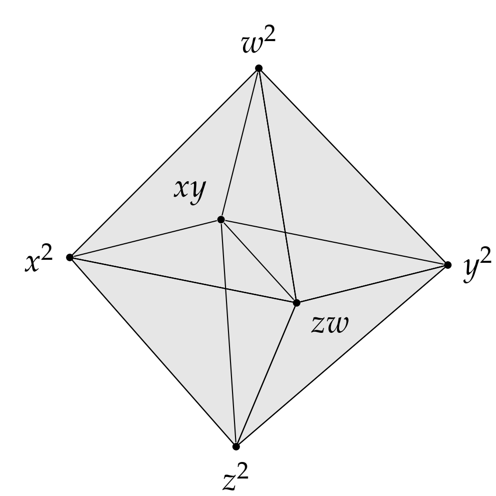
In particular, μ_{K} is not associative (and in fact any multigraded multiplication on F_{M} is not associative) since we will always have:

$$[\epsilon_1, \epsilon_5, \epsilon_2] = -y\mathrm{d}(\epsilon_{1234}) \neq 0 \quad \text{and} \quad [\epsilon_1, \epsilon_6, \epsilon_2] = -z\mathrm{d}(\epsilon_{1234}) \neq 0.$$

On the other hand, since the multiplication of F_{M} extends the multiplication of F_{K}, we see that the comparison map $F_{\mathrm{K}} \to F_{\mathrm{M}}$ is multiplicative, and hence F_{K} is an MDG subalgebra of F_{M}.

Example 1.4. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = \boldsymbol{m}_{\mathrm{O}} = x^2, w^2, zw, xy, y^2, z^2$, and let $F_{\mathrm{O}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F can be realized as the R-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:





The homogeneous components of F as a graded R-module are given below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= R\mathrm{e}_1 + R\mathrm{e}_2 + R\mathrm{e}_3 + R\mathrm{e}_4 + R\mathrm{e}_5 + R\mathrm{e}_6 \\
F_2 &= R\mathrm{e}_{12} + R\mathrm{e}_{13} + R\mathrm{e}_{14} + R\mathrm{e}_{16} + R\mathrm{e}_{23} + R\mathrm{e}_{24} + R\mathrm{e}_{25} + R\mathrm{e}_{34} + R\mathrm{e}_{35} + R\mathrm{e}_{36} + R\mathrm{e}_{45} + R\mathrm{e}_{46} + R\mathrm{e}_{56} \\
F_3 &= R\mathrm{e}_{123} + R\mathrm{e}_{124} + R\mathrm{e}_{134} + R\mathrm{e}_{136} + R\mathrm{e}_{146} + R\mathrm{e}_{234} + R\mathrm{e}_{235} + R\mathrm{e}_{245} + R\mathrm{e}_{345} + R\mathrm{e}_{346} + R\mathrm{e}_{356} + R\mathrm{e}_{456} \\
F_4 &= R\mathrm{e}_{1234} + R\mathrm{e}_{1346} + R\mathrm{e}_{2345} + R\epsilon_{3456}.
\end{aligned}
$$

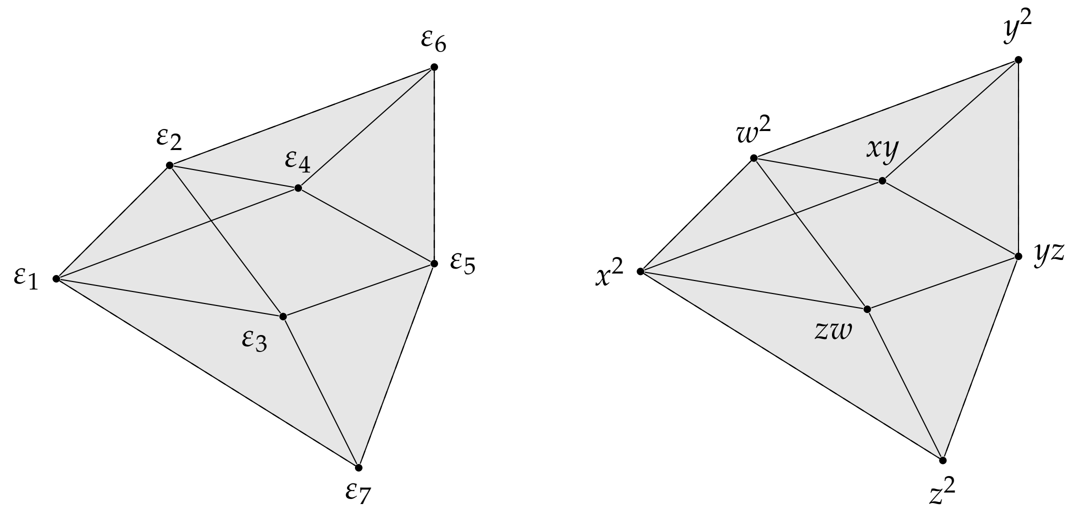
The canonical map $R/\boldsymbol{m}_{\mathrm{M}} \to R/\boldsymbol{m}_{\mathrm{O}}$ induces an injective multigraded comparison map $F_{\mathrm{M}} \to F_{\mathrm{O}}$ and we identify F_{M} with this subcomplex of F_{O}. This time it is impossible extend the multiplication of F_{M} to a multigraded multiplication on F_{O}. Indeed, assuming we could extend the multiplication, then

$$\begin{aligned} z(\mathrm{e}_2 \star \mathrm{e}_5) &= \mathrm{e}_2 \star (z\mathrm{e}_5) \\ &= \epsilon_2 \star \epsilon_5 \\ &= y^2\epsilon_{23} + w\epsilon_{35} \\ &= y^2\mathrm{e}_{23} + w\mathrm{e}_{35}, \end{aligned}$$

which would imply $\mathrm{e}_2 \star \mathrm{e}_5 = (y^2/z)\mathrm{e}_{23} + (w/z)\mathrm{e}_{35}$. However this is obviously not in F_{O} since the coefficients are not in R. On the other hand, it turns out that there is a better choice of multigraded multiplication that we can use on F_{O} anyways; namely namely $\mathrm{e}_2 \star \mathrm{e}_5 = \mathrm{e}_{25}$. In fact, this is the only possible choice we can make if we want the multiplication to be multigraded. Similarly, we are forced to have $\mathrm{e}_1 \star \mathrm{e}_6 = \mathrm{e}_{16}$. One can show that this extends to an *associative* multigraded multiplication on F_{O}. We define it below on some of the homogeneous basis elements:

$$\begin{aligned} \mathrm{e}_1 \star \mathrm{e}_5 &= y\mathrm{e}_{14} + x\mathrm{e}_{45} & \qquad \mathrm{e}_2 \star \mathrm{e}_{46} &= -z\mathrm{e}_{234} + w\mathrm{e}_{346} \\ \mathrm{e}_2 \star \mathrm{e}_6 &= z\mathrm{e}_{23} + w\mathrm{e}_{35} & \mathrm{e}_2 \star \mathrm{e}_{56} &= -z\mathrm{e}_{235} + w\mathrm{e}_{356} \\ \mathrm{e}_1 \star \mathrm{e}_{25} &= y\mathrm{e}_{124} - x\mathrm{e}_{245} & \mathrm{e}_2 \star \mathrm{e}_{146} &= \mathrm{e}_{1234} + \mathrm{e}_{1346} \\ \mathrm{e}_1 \star \mathrm{e}_{35} &= y\mathrm{e}_{134} - x\mathrm{e}_{345} & \mathrm{e}_2 \star \mathrm{e}_{456} &= \mathrm{e}_{2345} + \mathrm{e}_{3456} \\ \mathrm{e}_1 \star \mathrm{e}_{56} &= y\mathrm{e}_{146} + x\mathrm{e}_{456} & \mathrm{e}_1 \star \mathrm{e}_{235} &= \mathrm{e}_{1234} + \mathrm{e}_{2345} \\ \mathrm{e}_2 \star \mathrm{e}_{16} &= -z\mathrm{e}_{123} - w\mathrm{e}_{136} & \mathrm{e}_1 \star \mathrm{e}_{356} &= \mathrm{e}_{1346} + \mathrm{e}_{3456} \end{aligned}$$

Example 1.5. Let $\boldsymbol{m}_{\mathrm{N}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz, y^2, z^2$, and let $F_{\mathrm{N}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F can be realized as the R-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:



It is visibly clear that the map $R/\boldsymbol{m}_{\mathrm{A}} \to R/\boldsymbol{m}_{\mathrm{N}}$ induces a comparison map $\iota\colon F_{\mathrm{A}} \to F_{\mathrm{N}}$ defined by $\iota(\varepsilon_\sigma) = \varepsilon_\sigma$ for all homogeneous basis element ε_σ of F_{A} (in particular, there are no monomials showing up in this comparison map). Thus we run into the same problem as in Example (1.2), and so there is no way to choose a multigraded multiplication on F_{N} which is associative.

Example 1.6. Let $m = xyzw$, let $\boldsymbol{m} = mx, my, mz, mw$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F is just the Taylor resolution with respect to $\boldsymbol{m}$ and is supported on the 3-simplex. Usually F comes equipped with an associative multiplication giving it the structure of a DG algebra, however we wish to consider a different multiplication μ which gives it the structure of a non-associative MDG algebra. In particular, this multiplication will start out as:

$$\begin{aligned} e_1 \star e_2 &= xyzwe_{12} \\ e_1 \star e_3 &= xyz^2e_{14} - x^2yze_{34} \\ e_2 \star e_3 &= xyzwe_{23} \\ e_1 \star e_{23} &= xyzwe_{123} + xy^2ze_{134} \\ e_2 \star e_{14} &= -xyzwe_{124} \\ e_2 \star e_{34} &= xyzwe_{234} \end{aligned}$$

At this point, no matter how we extend this multiplication, it will not be associative since

$$[e_1, e_2, e_3] = x^2y^2z^2w\mathrm{d}(e_{1234}).$$

1.4 MDG Modules

We now want to define MDG A-modules where A is an MDG R-algebra.

Definition 1.4. Let X be an R-complex equipped with chain maps $\mu_{A,X}\colon A \otimes_R X \to X$ and $\mu_{X,A}\colon X \otimes_R A \to X$, denoted $a \otimes x \mapsto ax$ and $x \otimes a \mapsto xa$ respectively.

1. We say X is **unital** if $1x = x = x1$ for all $x \in X$.
2. We say X is **graded-commutative** if $ax = (-1)^{|a||x|}xa$ for all $a \in A$ homogeneous and $x \in X$ homogeneous. In this case, $\mu_{X,A}$ is completely determined by $\mu_{A,X}$, and thus we completely forget about it and write $\mu_X = \mu_{A,X}$.
3. We say X is **associative** if $a_1(a_2x) = (a_1a_2)x$ for all $a_1, a_2 \in A$ and $x \in X$.

We say X is an **MDG A-module** if it is graded-commutative, unital, and the graded R-linear map

$$\overline{\mu}_X\colon \mathrm{H}(A) \otimes_R \mathrm{H}(X) \to \mathrm{H}(X)$$

induced by μ_X gives $\mathrm{H}(X)$ the structure of an associative graded-commutative $\mathrm{H}(A)$-module. In this case, we call μ_X the **A-scalar multiplication** of X. If X is also associative, then we say X is a **DG A-module**.

Definition 1.5. A map $\varphi\colon X \to Y$ between MDG A-modules X and Y is called an **MDG A-module homomorphism** if it is a chain map which is also **multiplicative**, meaning

$$\varphi(ax) = a\varphi(x)$$

for all $a \in A$ and $x \in X$. We obtain a category, denoted $\mathbf{MDGmod}_A$ whose objects are MDG A-modules and whose morphisms are MDG A-module homomorphisms.

Example 1.7. Let A and B be MDG R-algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we give B the structure of an MDG A-module by defining an A-scalar multiplication on B via

$$a \cdot b = \varphi(a)b$$

for all $a \in A$ and $b \in B$. Note that we need $\varphi(1) = 1$ in order for B to be unital as an MDG A-module. Also note that φ is an MDG A-module homomorphism if and only if it is an algebra homomorphism. Indeed, it is an A-module homomorphism if and only if for all $a_1, a_2 \in A$ we have

$$\varphi(a_1a_2) = a_1 \cdot \varphi(a_2) = \varphi(a_1)\varphi(a_2),$$

which is equivalent to saying φ is an algebra homomorphism (since we already have $\varphi(1) = 1$).

1.4.1 The Category of All MDG A-Modules

Let A be an MDG R-algebra. The category of all MDG A-modules forms an abelian category which is enriched over the category of all R-modules. Indeed, if X and Y are MDG A-modules, then the set of all MDG A-module homomorphisms from X to Y, denoted $\mathrm{Hom}_A(X,Y)$, has the structure of an R-module, and moreover, the usual composition operation

$$\circ\colon \mathrm{Hom}_A(Y,Z) \times \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_A(X,Z),$$

denoted $(g, f) \mapsto g \circ f = fg$, is R-bilinear. We also have a zero object, binary biproducts, as well as kernels and cokernels. For instance, if $\varphi\colon X \to Y$ is an MDG A-module homomorphism, then the kernel of φ, denoted $\ker\varphi$, is defined in the usual way as

$$\ker\varphi = \{x \in X \mid \varphi(x) = 0\}$$

together with the canonical inclusion map $\iota\colon \ker\varphi \to X$. The differential and A-scalar multiplication of $\ker\varphi$ are simply the ones obtained from X via restriction to $\ker\varphi$. Similarly the cokernel of φ is defined in the usual way as well. Thus the category of all MDG A-modules shares many of the same properties as the category of all DG B-modules where B is a DG R-algebra. Thus, the language we use in the category of MDG A-modules is often similar to the language used in the category of all DG B-modules. For instance, if X and Y are two MDG A-modules such that $X \subseteq Y$, then we say X is an MDG A-submodule of Y if the inclusion map $\iota\colon X \to Y$ is an MDG A-module homomorphism. In particular, this means that both the differential and A-scalar multiplication of Y restricts to a differential and A-scalar multiplication on X. Similarly, the MDG A-submodules $\mathfrak{a}$ of A are often called MDG ideals of A or MDG A-ideals.

Having said all of this, there are also some notable differences between the category of all DG B-modules and the category of all MDG A-modules. For instance, one must be careful when defining localization, tensor, and hom in the latter. In particular, if X and Y are MDG A-modules, then one can define the tensor complex $X \otimes_A Y$ as well as the hom complex $\mathrm{Hom}^{\star}_A(X, Y)$ in the usual way. Then tensor complex $X \otimes_A Y$ turns out to be an MDG A-module with the obvious A-scalar multiplication, however it need not be true that $A \otimes_A X \simeq X$. On the other hand, it may not be possible to give the hom complex $\mathrm{Hom}^{\star}_A(X, Y)$ the structure of an MDG A-module by defining A-scalar multiplication in the obvious way. Finally, if $S \subseteq A$ is a multiplicatively closed set, then one can make sense of the localization X_S, but only in the case where S satisfies some extra conditions. We include more details on these constructions in the appendix.

Remark 2. Let A be an MDG algebra and let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{8}$$

be a short exact sequence of MDG A-modules. If we just view (8) as a short exact sequence of chain complexes, then we know that we get an induced long exact sequence of abelian groups:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(Y) \arrow[r, "\psi_{i+1}"] & \mathrm{H}_{i+1}(Z) \arrow[dll, rounded corners, "\partial_{i+1}"] \\
\mathrm{H}_i(X) \arrow[r, "\varphi_i"] & \mathrm{H}_i(Y) \arrow[r, "\psi_i"] & \mathrm{H}_i(Z) \arrow[dll, rounded corners, "\partial_i"] \\
\mathrm{H}_{i-1}(X) \arrow[r, "\varphi_{i-1}"] & \mathrm{H}_{i-1}(Y) \arrow[r] & \cdots
\end{tikzcd} \tag{9}$$

where the connecting map $\partial\colon \mathrm{H}(Z) \to \mathrm{H}(X)$ is a graded $\mathrm{H}_0(A)$-module homomorphism of degree -1 which is defined as follows: let $\overline{z} \in \mathrm{H}(Z)$ where $z \in Z$ is homogeneous and $\mathrm{d}z = 0$. Choose $y \in Y$ such that $\psi y = z$. Then there is a unique $x \in X$ such that $\varphi x = \mathrm{d}y$. We set $\partial \overline{z} = \overline{x}$ and verify that this is a well-defined map (i.e. it does not depend on any of the choices we made). However we get more when we obtain a little more when we view (8) as a short exact sequence of MDG A-modules. Indeed, ∂ is not just an $\mathrm{H}_0(A)$-linear: it is $\mathrm{H}(A)$-linear! Thus we obtain a sequence of graded $\mathrm{H}(A)$-modules:

$$\mathrm{H}(X) \xrightarrow{\varphi} \mathrm{H}(Y) \xrightarrow{\psi} \mathrm{H}(Z) \xrightarrow{\partial} \Sigma\mathrm{H}(X) \xrightarrow{\varphi} \Sigma\mathrm{H}(Y)$$

which is exact at $\mathrm{H}(Y)$, $\mathrm{H}(Z)$, and $\Sigma\mathrm{H}(X)$.

2 Associators and Multiplicators

In order to get a better understanding as to how far away MDG objects are from being DG objects, we need to discuss associators and multiplicators. Associators will help us measure how far away an MDG A-module X is from being associative, whereas multiplicators will help up measure how far away a chain map $\varphi\colon X \to Y$ is from being multiplicative.

2.1 Associators

We begin by defining associators. Throughout this subsection, let A be an MDG R-algebra and let X be an MDG A-module.

Definition 2.1. The **associator** of X is the chain map, denoted $[\cdot]_X$ (or more simply by $[\cdot]$ if X is understood from context), from $A \otimes_R A \otimes_R X$ to X defined by

$$[\cdot] := \mu(\mu \otimes 1 - 1 \otimes \mu).$$

Note that we use μ to denote both the multiplication μ_A on A and the A-scalar multiplication μ_X on X where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot]\colon A \times A \times X \to X$ to be the unique R-trilinear map which corresponds to $[\cdot]$ via the universal mapping property of tensor products. Thus we have

$$[a_1 \otimes a_2 \otimes x] = (a_1 a_2)x - a_1(a_2 x) = [a_1, a_2, x]$$

for all $a_1, a_2 \in A$ and $x \in X$.

2.1.1 Associator Identities

In order to familiarize ourselves with the associator we collect together some useful identities that the associator satisfies in this subsubsection:

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ we have the Leibniz law

$$\mathrm{d}[a_1, a_2, x] = [\mathrm{d}a_1, a_2, x] + (-1)^{|a_1|}[a_1, \mathrm{d}a_2, x] + (-1)^{|a_1|+|a_2|}[a_1, a_2, \mathrm{d}x]. \tag{10}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1]. \tag{11}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||x|+|a_2||x|}[x, a_1, a_2] - (-1)^{|a_1||a_2|+|a_1||x|}[a_2, x, a_1] \tag{12}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = (-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] \tag{13}$$

- For all $a_1, a_2, a_3 \in A$ and $x \in X$ we have

$$a_1[a_2, a_3, x] = [a_1a_2, a_3, x] - [a_1, a_2a_3, x] + [a_1, a_2, a_3x] - [a_1, a_2, a_3]x \tag{14}$$

The way the signs in (11) show up can be interpreted as follows: in order to go from $[a_1, a_2, x]$ to $[x, a_2, a_1]$, we have to first swap a_1 with a_2 (this is where the $(-1)^{|a_1|a_2|}$ comes from), then swap a_1 with x (this is where the $(-1)^{|a_1||x|}$ comes from), and then finally swap a_2 with x (this is where the $(-1)^{|a_2||x|}$ comes from). We then obtain one extra minus sign by swapping terms in the associator at the final step:

$$\begin{aligned}
[a_1, a_2, x] &= (a_1a_2)x - a_1(a_2x) \\
&= (-1)^{|a_1|a_2|}(a_2a_1)x - (-1)^{|a_2|||x|}a_1(xa_2) \\
&= (-1)^{|a_1||a_2|+|a_2||x|+|a_1||x|}x(a_2a_1) - (-1)^{|a_2||x|+|a_1||x|+|a_1||a_2|}(xa_2)a_1 \\
&= (-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}(x(a_2a_1) - (xa_2)a_1) \\
&= -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1].
\end{aligned}$$

A similar interpretation is also given to (12) and (13). For instance, in order to get from $[a_1, a_2, x]$ to $[x, a_1, a_2]$, we have to swap x with a_2 and then swap x with a_1 (this is where the $(-1)^{|a_1||x|+|a_2||x|}$ comes from). We do add an extra minus sign in (13) however since we never swap terms in the associator:

$$\begin{aligned}
(-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] &= (a_1a_2)x - (-1)^{|a_1||a_2|}a_2(a_1x) + (-1)^{|a_2||x|}(a_1x)a_2 - a_1(a_2x) \\
&= (a_1a_2)x - (-1)^{|a_1||a_2|}a_2(a_1x) + (-1)^{|a_1|a_2|}a_2(a_1x) - a_1(a_2x) \\
&= (a_1a_2)x - a_1(a_2x) \\
&= [a_1, a_2, x].
\end{aligned}$$

2.1.2 Alternative MDG Modules

If X is not associative, then one is often interested in knowing whether or not X satisfies the following weaker property:

Definition 2.2. We say X is **alternative** if $[a, a, x] = 0$ for all $a \in A$ and $x \in X$.

In other words, X is alternative if for each $a \in A$ and $x \in X$, we have $a^2x = a(ax)$. The reason behind the name "alternative" comes from the fact that in the case where $X = A$, then A is alternative if and only if the associator $[\cdot, \cdot, \cdot]$ is alternating.

Proposition 2.1. *Let $a \in A$ and $x \in X$ be homogeneous.*

1. *We have $[a, a, x] = 0$ if and only if $[x, a, a] = 0$.*
2. *If $[a, a, x] = 0$, then $[a, x, a] = 0$. The converse holds if $|a|$ is odd and* $\operatorname{char} R \neq 2$.
3. *If $|a|$ is even, we have $[a, x, a] = 0$, and if $|a|$ is odd, we have $[a, x, a] = (-1)^{|x|}2[a, a, x]$. In particular, if* $\operatorname{char} R = 2$, *we always have $[a, x, a] = 0$.*

Proof. From identities (11) and (13) we obtain

$$\begin{aligned}
[a, a, x] &= -(-1)^{|a|}[x, a, a] \\
[a, x, a] &= (-1)^{|x||a|}(1 - (-1)^{|a|})[a, a, x].
\end{aligned}$$

In particular, we see that

$$[a, x, a] = \begin{cases} = (-1)^{|x|}2[a, a, x] = -(-1)^{|x|}2a(ax) & \text{if } a \text{ is odd} \\ 0 & \text{if } a \text{ is even} \end{cases} \tag{15}$$

Similarly we have

$$[a, a, x] = \begin{cases} (-1)^{|x|}\frac{1}{2}[a, x, a] & \text{if } a \text{ is odd and char } R \neq 2 \\ (-1)^{|a|}[x, a, a] & \text{if } a \text{ is even} \end{cases} \tag{16}$$

□

Remark 3. Suppose F is an MDG R-algebra whose underlying graded R-module is finite and free with $e_1, \dots, e_n$ being a homogeneous basis. In order to show F is alternative, it is *not* enough to check $[e_i, e_i, e_j] = 0$ for all e_i, e_j in the homogeneous basis. Indeed, even in this case, observe that if e_i and e_j are odd, then

$$\begin{aligned}
[e_i + e_j, e_i + e_j, e_k] &= [e_i, e_i, e_k] + [e_i, e_j, e_k] + [e_j, e_i, e_k] + [e_j, e_j, e_k] \\
&= [e_i, e_j, e_k] + [e_j, e_i, e_k] \\
&= [e_i, e_j, e_k] - [e_j, e_i, e_k] + (-1)^{|e_k|}[e_j, e_k, e_i] \\
&= (-1)^{|e_k|}[e_j, e_k, e_i].
\end{aligned}$$

Thus in order for F to be alternative, we certainly need $[a_1, a_2, a_3] = 0$ for all $a_1, a_2, a_3 \in F$ whenever both $|a_1|$ and $|a_3|$ are odd. For instance, consider the MDG R-algebra F_K given in Example (1.1). Then we have $[e_\sigma, e_\sigma, e_\tau] = 0$ for all $\sigma, \tau \in \Delta$, however F is not alternative since $[e_1, e_5, e_2] \neq 0$.

2.1.3 The Maximal Associative Quotient

Definition 2.3. The **associator R-subcomplex** of X, denoted $[X]$, is the R-subcomplex of X given by the image of the associator of X. Thus the underlying graded R-module of $[X]$ is

$$[X] = \operatorname{span}_R\{[a_1, a_2, x] \mid a_1, a_2 \in A \text{ and } x \in X\},$$

and the differential of $[X]$ is simply the restriction of the differential of X to $[X]$. The **associator A-submodule** of X, denoted $\langle X \rangle$, is defined to be the smallest A-submodule of X which contains $[X]$. The underlying graded R-module of $\langle X \rangle$ also has a simple description. Indeed, observe that

$$a_1(a_2[a_3, a_4, x]) = (a_1a_2)[a_3, a_4, x] - [a_1, a_2, [a_3, a_4, x]] \tag{17}$$

for all $a_1, a_2, a_3, a_4 \in A$ and $x \in X$. Using identities like (17) together with graded-commutativity, one can show that the underlying graded R-module of $\langle X \rangle$ is given by

$$\langle X \rangle = \operatorname{span}_R\{a_1[a_2, a_3, x] \mid a_1, a_2, a_3 \in A \text{ and } x \in X\}$$

The quotient $X^{\mathrm{as}} := X/\langle X \rangle$ is a DG A-module (i.e. an associative MDG A-module). We call X^{as} (together with its canonical quotient map $X \twoheadrightarrow X^{\mathrm{as}}$) the **maximal associative quotient** of X.

The maximal associative quotient of X satisfies the following universal mapping property:

Proposition 2.2. *Every* MDG *A-module homomorphism $\varphi\colon X \to Y$ in which Y is associative factors through a unique* MDG *A-module homomorphism $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$, meaning $\overline{\varphi}\rho = \varphi$ where $\rho\colon X \twoheadrightarrow X^{\mathrm{as}}$ is the canonical quotient map. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & X^{\mathrm{as}} \arrow[d, dashed, "\overline{\varphi}"] \\ & Y \end{tikzcd} \tag{18}$$

Proof. Indeed, suppose $\varphi\colon X \to Y$ is any MDG A-module homomorphism where Y is associative. In particular, we must have $[X] \subseteq \ker \varphi$, and since $\langle X \rangle$ is the smallest MDG A-submodule of X which contains $[X]$, it follows that $\langle X \rangle \subseteq \ker \varphi$. Thus the map $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$ given by $\overline{\varphi}(\overline{x}) := \varphi(x)$ where $\overline{x} \in X^{\mathrm{as}}$ is well-defined. Furthermore, it is easy to see that $\overline{\varphi}$ is an MDG A-module homomorphism and the unique such one which makes the diagram (18) commute. □

Corollary 1. *Taking the maximal associative quotient extends to a functor*

$$(-)^{\mathrm{as}}\colon \mathbf{MDGmod}_A \to \mathbf{DGmod}_A,$$

and this functor is left adjoint to the forgetful functor. In particular, the functor $(-)^{\mathrm{as}}$ preserves all colimits and the forgetful functor preserves all limits.

2.1.4 Homological Associativity

Definition 2.4. The **associator homology** of X is the homology of the associator A-submodule of X. We often simplify notation and denote the associator homology of X by $\mathrm{H}\langle X \rangle$ instead of $\mathrm{H}(\langle X \rangle)$. We say X is **homologically associative** if $\mathrm{H}\langle X \rangle = 0$ and we say X is **homologically associative in degree** i if $\mathrm{H}_i\langle X \rangle = 0$. Similarly we say X is associative in degree if $\langle X \rangle_i = 0$.

Clearly, if X is associative, then X is homologically associative. The converse holds under certain conditions. This is the first main theorem given in the introduction.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a local ring, let A be an* MDG *R-algebra, and let X be an* MDG *A-module such that $\langle X \rangle$ is minimal (meaning $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$), and such that each $\langle X \rangle_i$ is a finitely generated R-module. If X is associative in degree i, then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $\langle X \rangle$ is also bounded below (meaning $\langle X \rangle_i = 0$ for $i \ll 0$), then X is associative if and only if X is homologically associative.*

Proof. Assume that X is associative in degree i. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume

$$\mathrm{H}_{i+1}\langle X \rangle = 0 \quad \text{and} \quad \langle X \rangle_{i+1} \neq 0.$$

Then by Nakayama's Lemma, we can find homogeneous $a_1, a_2, a_3 \in A$ and homogeneous $x \in X$ such that such that $a_1[a_2, a_3, x] \notin \mathfrak{m}\langle X \rangle_{i+1}$. Since $\langle X \rangle_i = 0$ by assumption, we have $\mathrm{d}(a_1[a_1, a_2, x]) = 0$. Also, since $\langle X \rangle$ is minimal, we have $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$. Thus $a_1[a_2, a_3, x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. □

We are often also interested in the homology of the maximal associative quotient of X as well. To this end, observe that the short exact sequence of MDG A-modules

$$0 \longrightarrow \langle X \rangle \longrightarrow X \longrightarrow X^{\mathrm{as}} \longrightarrow 0$$

induces a sequence of graded $\mathrm{H}(A)$-modules

$$\mathrm{H}\langle X \rangle \longrightarrow \mathrm{H}(X) \longrightarrow \mathrm{H}(X^{\mathrm{as}}) \xrightarrow{\overline{\mathrm{d}}} \Sigma\mathrm{H}\langle X \rangle \longrightarrow \Sigma\mathrm{H}(X)$$

which is exact at $\mathrm{H}\langle X \rangle$, $\mathrm{H}(X)$, and $\mathrm{H}(X^{\mathrm{as}})$ and where the connecting map $\overline{\mathrm{d}}\colon \mathrm{H}(X^{\mathrm{as}}) \to \Sigma\mathrm{H}\langle X \rangle$ is essentially defined in terms of the differential d of X, namely given $\overline{x} \in \mathrm{H}(X^{\mathrm{as}})$, we set $\overline{\mathrm{d}}\overline{x} = \overline{\mathrm{d}x}$.

Example 2.1. Let X be an MDG A-module. Assume that $(R, \mathfrak{m})$ is a local noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let F be the minimal R-free resolution of R/I. Equip F with a multiplication μ giving it the structure of an MDG R-algebra. Then

$$\mathrm{H}_i(F^{\mathrm{as}}) \cong \begin{cases} R/I & \text{if } i = 0 \\ \mathrm{H}_{i-1}\langle F \rangle & \text{else} \end{cases}$$

2.1.5 Computing Annihilators of the Associator Homology

In this subsubsection, we assume that A is centered at R. Set I to be the image of $\mathrm{d}_1\colon A_1 \to R$. In particular, we have $\mathrm{H}_0(A) = R/I$.

Proposition 2.3. *I annihilates $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(X^{\mathrm{as}})$.*

Proof. Let $t \in I$. Thus $t = \mathrm{d}(a)$ where $|a| = 1$. Let $\mathrm{m}_a \colon X \to X$ be the multiplication by a map given by $\mathrm{m}_a(x) = ax$. In particular, m_a restricts to an R-linear map $\mathrm{m}_a \colon \langle X \rangle \to \langle X \rangle$ and thus induces an R-linear map $\overline{\mathrm{m}}_a \colon \mathrm{X}^{\mathrm{as}} \to \mathrm{X}^{\mathrm{as}}$. Observe that if $x \in X$, then

$$\begin{aligned}(\mathrm{d}\mathrm{m}_a + \mathrm{m}_a\mathrm{d})(x) &= \mathrm{d}(ax) + a\mathrm{d}(x)\\ &= \mathrm{d}(a)x - a\mathrm{d}(x) + a\mathrm{d}(x)\\ &= tx\\ &= \mathrm{m}_t(x).\end{aligned}$$

In particular, we see that m_a is a homotopy from m_t to the zero map, which restricts to a homotopy $\mathrm{m}_a \colon \langle X \rangle \to \langle X \rangle$ from $\mathrm{m}_t \colon \langle X \rangle \to \langle X \rangle$ to the zero map. A similar argument shows that $\overline{\mathrm{m}}_a$ is a homotopy from $\overline{\mathrm{m}}_t \colon X^{\mathrm{as}} \to X^{\mathrm{as}}$ to the zero map. It follows that t annihilates both $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(\mathrm{X}^{\mathrm{as}})$. □

We now assume that R is an integral domain with quotient field K. Furthermore we assume both A and X are free as graded R-modules. In this case, we set

$$A_K = \{a/r \mid a \in A \text{ and } r \in R\backslash\{0\}\} \quad \text{and} \quad X_K = \{x/r \mid x \in X \text{ and } r \in R\backslash\{0\}\}.$$

Note that A_K is an MDG K-algebra centered at K. Next we consider the conductor:

$$\mathfrak{c} = \{c \in A_K \mid c\langle X \rangle \subseteq \langle X \rangle\}.$$

The Leibniz law implies $\mathfrak{c}$ is an R-complex. We set $Q = \mathrm{d}(\mathfrak{c}_1) \cap R$. Then by the same argument as in the proposition above, we see that Q annihilates $\mathrm{H}(X)$, $\mathrm{H}\langle X \rangle$, and $\mathrm{H}(\mathrm{X}^{\mathrm{as}})$.

Example 2.2. Let us revisit example (1.1) where we keep the same notation. Observe that

$$\begin{aligned}\frac{e_1}{x}[e_1, e_5, e_2] &= \frac{1}{x}\left([e_1^2, e_5, e_2] - [e_1, e_1e_5, e_2] + [e_1, e_1, e_5e_2] - [e_1, e_1, e_5]e_2\right)\\ &= -\frac{1}{x}[e_1, e_1e_5, e_2]\\ &= -\frac{1}{x}[e_1, yz^2e_{14} + xe_{45}, e_2]\\ &= -\frac{yz^2}{x}[e_1, e_{14}, e_2] - [e_1, e_{45}, e_2]\\ &= -[e_1, e_{45}, e_2].\end{aligned}$$

It follows that $\mathrm{d}(e_1/x) = x$ annihilates $\mathrm{H}\langle F \rangle$. Similar calculations likes this shows that $\mathfrak{m} = \langle x, y, z, w \rangle$ annihilates $\mathrm{H}\langle F \rangle$. It follows that

$$\mathrm{H}_i\langle F \rangle \cong \begin{cases}\Bbbk & \text{if } i = 3\\ 0 & \text{else}\end{cases}$$

One can interpret this as saying that the multiplication μ is very close to being associative (the failure for μ to being associative is reflected in the fact that $\dim_{\Bbbk}(\mathrm{H}\langle F \rangle) = 1$). Note that μ is not associative in homological degree 4 since

$$[e_1, e_{45}, e_2] = xyze_{1234} \neq 0.$$

In some sense however, the nonzero associator $[e_1, e_{45}, e_2]$ is not really anything *new*. Indeed, one could argue that $[e_1, e_{45}, e_2]$ being nonzero is simply a direct consequence of $[e_1, e_5, e_2]$ being nonzero. More generally, an element $\gamma \in \langle F \rangle$ should only be thought of as contributing something new towards the failure for μ to being associative if $\mathrm{d}\gamma = 0$ (otherwise one could argue that γ being nonzero is simply a consequence of the associators in $\mathrm{d}\gamma$ being nonzero). Similarly, if $\gamma = \mathrm{d}\gamma'$ for some $\gamma' \in \langle F \rangle$, then again γ is not contributing anything new towards the failure for μ to being associative since one could argue that γ being nonzero is a direct consequence of γ' being nonzero. Thus the associators which really do contribute something new towards the failure for μ to being associative should be the ones which represent nonzero elements in homology. This is how we interpret the associator homology of F. In this case, we have precisely one nontrivial associator $[e_1, e_5, e_2]$ which represents a nonzero element in homology (all of the other nonzero associators are derived from the fact that $[e_1, e_5, e_2] \neq 0$). Finally, let $U \colon R^4 \to R$ be the map given by $U = (xyz, y^2z, yz^2, yzw)$. One can show that

$$F_i^{\mathrm{as}} = \begin{cases}\mathrm{coker}(U^\top) & \text{if } i = 4\\ \mathrm{coker}(U) & \text{if } i = 3\\ F_i & \text{else}\end{cases}$$

Example 2.3. Let us revisit example (1.3) where we keep the same notation. One can check that

$$\mathrm{H}_i\langle F\rangle \cong \begin{cases} \Bbbk \oplus \Bbbk & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

Example 2.4. Let us revisit example (1.4) where we keep the same notation. We extend the multiplication initially defined in that example so that $e_4 \in \mathrm{N}(F)$ and $[e_\sigma, e_\sigma, e_\tau] = 0$ for all e_σ, e_τ in the homogeneous basis. Then we have

$$\begin{aligned} \mathrm{d}[e_{12}, e_3, e_2] &= -y[e_1, e_3, e_2] \\ \mathrm{d}[e_{13}, e_3, e_2] &= -z[e_1, e_3, e_2] \\ \mathrm{d}[e_{14}, e_3, e_2] &= -w[e_1, e_3, e_2] \\ \mathrm{d}[e_1, e_3, e_{12}] &= x[e_1, e_3, e_2]. \end{aligned}$$

It follows that

$$\mathrm{H}_i\langle F\rangle \cong \begin{cases} \Bbbk & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

2.1.6 The Nucleus

Let A be an MDG R-algebra and let X be an MDG A-module. The **nuclear subcomplex** of X, denoted $\mathrm{N}(X)$, is the R-subcomplex of X given by

$$\mathrm{N}(X) := \{x \in \mathrm{X} \mid [a_1, a_2, x] = 0 \text{ for all } a_1, a_2 \in A\}.$$

Indeed, the Leibniz law implies $\mathrm{d}(\mathrm{N}(X)) \subseteq \mathrm{N}(X)$, so the differential of $\mathrm{N}(X)$ is simply the differential of X restricted to $\mathrm{N}(X)$. The **nucleus** of X, denoted $\mathrm{N}\langle X\rangle$, is defined to be the smallest MDG A-submodule of X which contains $\mathrm{N}(X)$. The nucleus of X plays a role that is similar to the center of a group G. In particular, every associative A-submodule of X is contained in $\mathrm{N}\langle X\rangle$. We will also be interested in studying the **nuclear complex of X in A**, denoted $\mathrm{N}_A(X)$. This is the R-subcomplex of A given by

$$\mathrm{N}_A(\mathrm{X}) := \{a \in A \mid [a, a', x] = 0 \text{ for all } a \in A \text{ and } x \in X\}.$$

Note that if $a_1, a_2 \in \mathrm{N}_A(X)$, then $a_1 a_2 \in \mathrm{N}_A(X)$. However in general, if $a \in \mathrm{N}_A(X)$ and $b \in A$, then $[ab, c, x] = a[b, c, x]$ The **nucleus of X in A**, denoted $\mathrm{N}_A\langle X\rangle$, is defined to be the smallest MDG A-ideal which contains $\mathrm{N}_A(X)$. There is also the following weaker notion we may consider: we define the **middle nuclear complex of** X, denoted $\mathrm{M}(X)$, to be the R-subcomplex of X given by

$$\mathrm{M}(X) := \{x \in X \mid [a_1, x, a_2] = 0 \text{ for all } a_1, a_2 \in A\},$$

By combining (11) with (12), one can check that $\mathrm{N}(X) \subseteq \mathrm{M}(X)$, however this inclusion may be strict. Indeed, by combining the identities (11) with (12) we obtain the identity

$$[a_1, x, a_2] = (-1)^{|a_1||a_2| + |a_2||x|}((-1)^{|a_1||a_2|}[a_2, a_1, x] - [a_1, a_2, x]) \tag{19}$$

In particular, we have $x \in \mathrm{M}(X)$ if and only if $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a_1, a_2 \in A$. However just because we have $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a, b \in A$ does not necessarily mean $[a_1, a_2, x] = 0$ for all $a_1, a_2 \in A$.

Proposition 2.4. *Let A be an MDG algebra. Then $\mathrm{N}(A)$ is a DG subalgebra of A.*

Proof. Clearly we have $1 \in A$. Let $a, a' \in \mathrm{N}(A)$. Then for each $a_1, a_2 \in A$, we have

$$[aa', a_1, a_2] = a[a', a_1, a_2] + [a, a'a_1, a_2] - [a, a', a_1 a_2] + [a, a', a_1]a_2 = 0.$$

It follows that $aa' \in \mathrm{N}(A)$. Similarly, we have

$$[\mathrm{d}a, a_1, a_2] = \mathrm{d}[a, a_1, a_2] - (-1)^{|a|}[a, \mathrm{d}a_1, a_2] - (-1)^{|a| + |a_1|}[a, a_1, \mathrm{d}a_2] = 0.$$

It follows that $\mathrm{d}a \in \mathrm{N}(A)$. □

By using the identities (12), (13), and (14), one can show that every element in $\langle A\rangle$ can be expressed as the R-span of all elements of the form $a_1[a_2, a_3, a_4]$ where $|a_1| \leq |a_2|, |a_3|, |a_4|$. In fact, we can often do better than even this. Indeed, suppose $a_1 = az \neq 0$ for some homogeneous $a \in A$ with $|a| < |a_1|$ and homogeneous $z \in \mathrm{N}(A)$. Then we have $a_1[a_2, a_3, a_4] = a[za_2, a_3, a_4]$. It follows that we can express every element in $\langle A\rangle$ as an R-linear combination of elements of the form $a_1[a_2, a_3, a_4]$ where

$$|a_1| \leq \min\{|a_1|, |a_2|, |a_3|\} \quad \text{and} \quad a_1 \notin \mathrm{N}\langle A\rangle.$$

2.1.7 Multigraded Associativity Test

Suppose $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_d]$ and $\langle \boldsymbol{m} \rangle = \langle m_1, \dots, m_\ell \rangle$ be a monomial ideal in R, and let F be the minimal R-free resolution of R/I. Choose a multiplication μ on F which respects the multigrading giving it the structure of a multigraded MDG R-algebra. We denote by $\star = \star_\mu$ to be the R-bilinear map corresponding to μ in what follows. Let $e_1, \dots, e_\ell, e_{\ell+1}, \dots e_n$ be an ordered homogeneous basis of F where each e_i is multigraded with $\mathrm{multideg}(e_i) = m_i$. Recall that for each $1 \leq i, j \leq n$, there exists unique $r_{i,j}^k \in R$ such that

$$e_i \star e_j = \sum_{k=0}^{n} r_{i,j}^k e_k, \tag{20}$$

Since μ also respects the multigrading, we must have

$$r_{i,j}^k = c_{i,j}^k \frac{m_i m_j}{m_k},$$

where m_i, m_j, m_k are the monomials corresponding to the multidegrees of e_i, e_j, e_k, and where $c_{i,j}^k \in \Bbbk$ are called the **structured** $\Bbbk$**-coefficients** of μ. It would be nice if we could re-express (20) as

$$\left(\frac{e_i}{m_i}\right)\left(\frac{e_j}{m_j}\right) = \sum_k c_{i,j}^k \left(\frac{e_k}{m_k}\right), \tag{21}$$

but the problem is that F does not contain terms like e_i/m_i. In order to make sense of (20), we perform a base change. Namely let S be the multiplicatively closed set generated by $\{m_1, \dots, m_n\}$. We set $\widetilde{F} = F_{S,\mathbf{0}}$ to be the multidegree $\mathbf{0}$ component of F_S. The $\mathbb{N}^n$-graded MDG R-algebra structure on F induces an MDG $\Bbbk$-algebra structure on $\widetilde{F}$. The multiplication (21) makes perfect sense in the MDG $\Bbbk$-algebra $\widetilde{F}$. Denoting $\widetilde{e}_i = e_i/m_i$ for each i, we can re-express (21) as

$$\widetilde{e}_i \widetilde{e}_j = \sum_k c_{i,j}^k \widetilde{e}_k.$$

Theorem 2.2. *F is a DG R-algebra if and only if $\widetilde{F}$ is a DG $\Bbbk$-algebra.*

Proof. A straightforward calculation gives us

$$[e_i, e_j, e_k]_\mu = m_i m_j m_k [\widetilde{e}_i, \widetilde{e}_j, \widetilde{e}_k]_{\widetilde{\mu}}$$

for all i, j, k. Thus μ is associative if and only if $\widetilde{\mu}$ is associative. □

2.2 Multiplicators

Having discussed associators, we now wish to discuss multiplicators. Throughout this subsection, let A be an MDG R-algebra, let X be and Y be MDG A-modules, and let $\varphi \colon X \to Y$ be a chain map.

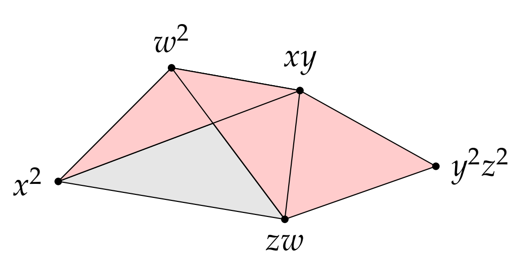
Definition 2.5. The are two types of multiplicators were are interested in:

1. The **multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi$, from $A \otimes_R X$ to Y defined by
$$[\cdot]_\varphi := \varphi\mu - \mu(1 \otimes \varphi).$$
Note that we use μ to denote both A-scalar multiplications μ_X and μ_Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot]_\varphi \colon A \times X \to Y$ (or more simply by $[\cdot,\cdot]$ if context is clear) to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot]_\varphi$). Thus we have
$$[a \otimes x]_\varphi = \varphi(ax) - a\varphi(x) = [a, x]$$
for all $a \in A$ and $x \in X$. We say φ is **multiplicative** if $[\cdot]_\varphi = 0$.

2. The 2-**multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi^{(2)}$, from $A \otimes_R A \otimes_R X$ to Y defined by
$$[\cdot]_\varphi^{(2)} := \varphi[\cdot]_\mu - [\cdot]_\mu(1 \otimes 1 \otimes \varphi)$$
where we write $[\cdot]_\mu$ to denote both the associator of X and the associator Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot]_\varphi \colon A \times X \to Y$ to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi^{(2)}$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot,\cdot,\cdot]_\varphi$). Thus we have
$$[a_1 \otimes a_2 \otimes x]_\varphi^{(2)} = \varphi([a_1, a_2, x]) - [a_1, a_2, \varphi(x)] = [a_1, a_2, x]_\varphi$$
for all $a_1, a_2 \in A$ and $x \in X$. We say φ is 2-**multiplicative** if $[\cdot]_\varphi^{(2)} = 0$.

Remark 4. If A and B are MDG R-algebras and $\varphi \colon A \to B$ is a chain map such that $\varphi(1) = 1$, then we view B as an MDG A-module with the A-scalar multiplication defined by $a \cdot b = \varphi(a)b$. In this case, the multiplicator of φ has the form
$$[a_1, a_2]_\varphi = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$
for all $a_1, a_2 \in A$.

Example 2.5. Let us continue with Example (1.1) where we keep the same notation except we write $F = F_{\mathrm{K}}$ and $\boldsymbol{m} = \boldsymbol{m}_{\mathrm{K}}$. Let $\boldsymbol{m}' = x^2, w^2, y^2z^2$ and let $E' = \mathcal{K}(\boldsymbol{m}')$ be the Koszul R-algebra which resolves $R/\boldsymbol{m}'$. The standard homogeneous basis of E' is denoted by e'_σ. Choose a comparison map $\iota' \colon E' \to F$ which lifts the projection $R/\boldsymbol{m}' \to R/\boldsymbol{m}$ such that ι' is unital and respects the multigrading. Then ι' being a chain map together with the fact that it is unital and respects the multigrading forces us to have
$$\begin{aligned} \iota'(e'_1) &= e_1 & \iota'(e'_{12}) &= e_{12} \\ \iota'(e'_2) &= e_2 & \iota'(e'_{13}) &= yz^2 e_{14} + xe_{45} \\ \iota'(e'_3) &= e_5 & \iota'(e'_{23}) &= y^2 z e_{23} + we_{35}. \end{aligned}$$
Moreover, ι' can be defined at e'_{123} in two possible ways. Assume that it is defined by
$$\iota'(e'_{123}) = yz^2 e_{124} + xyze_{234} - xwe_{345}.$$
We can picture $\iota'(E')$ inside of F as being supported on the red-shaded subcomplex below:

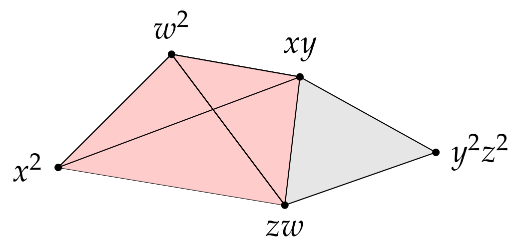


We now ask: is ι' an MDG algebra homomorphism? The answer is no. Indeed, clearly this map is a chain map which fixes the identity element, however it is not multiplicative. In fact, it is not even 2-multiplicative. To see this, assume for a contradiction that it was 2-multiplicative. Then we would have

$$\begin{aligned}0 &= \iota'(0)\\ &= \iota'([e'_1, e'_2, e'_3])\\ &= [\iota'(e'_1), \iota'(e'_2), \iota'(e'_3)]\\ &= [e_1, e_2, e_5]\\ &\neq 0,\end{aligned}$$

which is an obvious contradiction.

Next let $\boldsymbol{m}'' = x^2, w^2, zw, xy$ and let $T'' = \mathcal{T}(\boldsymbol{m}'')$ be the Taylor algebra which resolves $R/\boldsymbol{m}''$. The standard homogeneous basis of T'' is denoted by e''_σ. Choose a comparison map $\iota''\colon T'' \to F$ which lifts the projection $R/\boldsymbol{m}'' \to R/\boldsymbol{m}$ such that ι'' is unital and respects the multigrading. Then ι'' being a chain map together with the fact that it is multigraded forces us to have $\iota''(e''_\sigma) = e_\sigma$ for all σ. We can picture $\iota''(T'')$ inside of F as being supported on the red-shaded subcomplex below:



This time it is easy to check that ι'' is an MDG algebra homomorphism. We give F the structure of an MDG T''-module using ι'' in the usual way. Notice that F is *not* associative as a T''-module, that is F is not a DG T''-module. Indeed, we have $[e_1, e_2, e_5] \neq 0$.

Finally let $\boldsymbol{t} = x^2 + w^2, w^2 + xy, x^2 + zw$. One can check that $\boldsymbol{t}$ is an R-regular sequence contained in $\langle \boldsymbol{m} \rangle$. Let $E = \mathcal{K}(\boldsymbol{t})$ be the Koszul R-algebra which resolve $R/\boldsymbol{t}$. The standard homogeneous basis of E is denoted by ϵ_σ. We begin to construct a comparison map $\iota\colon E \to F$ which lifts the projection $R/\boldsymbol{t} \to R/\boldsymbol{m}$ by setting

$$\begin{aligned}\iota(\epsilon_1) &= e_1 + e_2\\ \iota(\epsilon_2) &= e_2 + e_3\\ \iota(\epsilon_3) &= e_3 + e_4\end{aligned}$$

It is straightforward to check that this extends to a unique MDG algebra homomorphism by setting

$$\iota(\epsilon_\sigma) = \prod_{i \in \sigma} \iota(\epsilon_i).$$

We give F the structure of an MDG E-module using ι in the usual way. Again, note that F is not a DG E-module, however $\iota\colon E \to F$ *is* an MDG algebra homomorphism.

2.2.1 Multiplicator Identities

We want to familiarize ourselves with the multiplicator of $\varphi\colon X \to Y$, so in this subsubsection we collect together some identities which the multiplicator satisfies:

- For all $a \in A$ homogeneous and $x \in X$, we have the Leibniz law:

$$\mathrm{d}[a, x] = [\mathrm{d}a, x] + (-1)^{|a|}[a, \mathrm{d}x].$$

- For all $a \in A$ homogeneous and $x \in X$ homogeneous, we have

$$[a, x] = (-1)^{|a||x|}[x, a]. \tag{22}$$

- For all $a_1, a_2 \in A$ and $x \in X$, we have

$$a_1[a_2, x] - [a_1 a_2, x] + [a_1, a_2 x] = [a_1, a_2, x]_\varphi \tag{23}$$

Furthermore, if Z is another MDG A-module and $\psi\colon Y \to Z$ is another chain map, then for all $a \in A$ and $x \in X$, we have

$$[a, x]_{\psi\varphi} = \psi([a, x]_\varphi) + [a, \varphi(x)]_\psi \tag{24}$$

In particular, if ψ is multiplicative, then $\psi([Y]_\varphi) \subseteq [Z]_{\psi\varphi}$.

Remark 5. Let A and B be MDG R-algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we can rewrite (23) as follows: for all $a_1, a_2, a_3 \in A$, we have

$$\varphi(a_1)[a_2, a_3] - [a_1a_2, a_3] + [a_1, a_2a_3] - [a_1, a_2]\varphi(a_3) = [\varphi(a_1), \varphi(a_2), \varphi(a_3)] - \varphi([a_1, a_2, a_3]) \tag{25}$$

Indeed, this follows from the fact that

$$[\varphi(a_1), \varphi(a_2), \varphi(a_3)] = [a_1, a_2, \varphi(a_3)] - [a_1, a_2]\varphi(a_3).$$

In this case, we also have $[a, a]_\varphi = 0$ for all $a \in A$ where $|a|$ is odd.

2.2.2 The Maximal Multiplicative Quotient

The **multiplicator complex** of φ, denoted $[Y]_\varphi$, is the R-subcomplex of Y given by $[Y]_\varphi := \operatorname{im}[\cdot]_\varphi$, so the underlying graded module of $[Y]_\varphi$

$$[Y]_\varphi := \operatorname{span}_R\{[a, x]_\varphi \mid a \in A \text{ and } x \in X\},$$

and the differential of $[Y]_\varphi$ is simply the restriction of the differential of Y to $[Y]_\varphi$. In order to avoid confusion with the associator complex, we will always write φ in the subscript of $[Y]_\varphi$. Even though the multiplicator complex of φ is closed under the differential, it need not be closed under A-scalar multiplication. In other words, if $a_1, a_2 \in A$ and $x \in X$, then it need not be the case that $a_1[a_2, x]_\varphi \in [Y]_\varphi$. We denote by $\langle Y\rangle_\varphi$ to be the MDG A-submodule of Y generated by $[Y]_\varphi$. In other words, $\langle Y\rangle_\varphi$ is the smallest MDG A-submodule of Y which contains $[Y]_\varphi$. Unlike the associator submodule, the multiplicator submodule is difficult to describe in terms of an R-span of elements. Indeed, as a first guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{[a, x]_\varphi \mid a \in A \text{ and } x \in X\}. \tag{26}$$

However this is clearly incorrect in general as we may need to adjoin elements of the form $a_1[a_2, x]$ to (26). As a second guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi \mid a_1, a_2 \in A \text{ and } x \in X\}. \tag{27}$$

However this is not correct in general either since the identity

$$a_1(a_2[a_3, x]_\varphi) = (a_1a_2)[a_3, x]_\varphi - [a_1, a_2, [a_3, x]_\varphi]$$

tells us that should really adjoin elements of the form $a_1[a_2, a_3, [a_4, x]]$ to (27) as well. As a third guess, one might think that $\langle Y\rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi,\ a_1[a_2, a_3, [a_4, x]_\varphi] \mid a_1, a_2, a_3, a_4 \in A \text{ and } x \in X\}. \tag{28}$$

Again this is not correct in general since the identity

$$a_1(a_2[a_3, a_4, [a_5, x]_\varphi]) = (a_1a_2)[a_3, a_4, [a_5, x]] - [a_1, a_2, [a_3, a_4, [a_5, x]_\varphi]].$$

tells us that we should really adjoin elements of the form $a_1[a_2, a_3, [a_4, a_5, [a_6, x]_\varphi]]$ to (28) as well. The problem continues getting worse with no end in sight. It turns out however, that if φ is 2-multiplicative, then $\langle Y\rangle_\varphi$ given by (26).

Proposition 2.5. *If φ is 2-multiplicative, then for all $a_1, a_2, a_3 \in A$ and $x \in X$ we have*

$$a_1[a_2, x]_\varphi = [a_1a_2, x]_\varphi - [a_1, a_2x]_\varphi \quad \textit{and} \quad [a_1, a_2, [a_3, x]_\varphi] = [[a_1, a_2, a_3], x]_\varphi - [a_1, [a_2, a_3, x]]_\varphi. \tag{29}$$

In particular, $\langle Y\rangle_\varphi$ is given by (26).

Proof. A straightforward calculation yields

$$a_1[a_2, a_3, x]_\varphi = [a_1a_2, a_3, x]_\varphi - [a_1, a_2a_3, x]_\varphi + [a_1, a_2, a_3x]_\varphi - [[a_1, a_2, a_3], x]_\varphi + [a_1, [a_2, a_3, x]]_\varphi - [a_1, a_2, [a_3, x]_\varphi].$$

Using this identity together with the identity (23), we see that if φ is 2-multiplicative, then we obtain (29). This implies all elements of the form $a_1[a_2, x]$ and $a_1[a_2, a_3, [a_4, x]]$ belong to (26). An easy induction argument shows that $\langle Y\rangle_\varphi$ is given by (26). □

The quotient $Y/\langle Y\rangle_\varphi$ is an MDG A-module. We denote by $\pi\colon Y\to Y/\langle Y\rangle_\varphi$ to be the canonical quotient map. Note that both π and $\pi\varphi$ are multiplicative. Therefore (24) implies $[Y]_\varphi\subseteq\ker\pi$ which implies $\langle Y\rangle_\varphi\subseteq\ker\pi$. We call $Y/\langle Y\rangle_\varphi$ (together with its canonical quotient map π) the **maximal multiplicative quotient** of $\varphi\colon X\to Y$; it satisfies the following universal mapping property:

Proposition 2.6. *For all MDG A-modules Z and for all chain maps $\psi\colon Y\to Z$ where both ψ and $\psi\varphi$ are MDG A-module homomorphisms, there exists a unique MDG A-module homomorphism $\overline{\psi}\colon Y/\langle Y\rangle_\varphi\to Z$ such that $\overline{\psi}\pi=\psi$. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \arrow[dl, "\psi"'] \arrow[d, "\pi"] \\ Z & Y/\langle Y\rangle_\varphi \arrow[l, dashed, "\overline{\psi}"] \end{tikzcd} \tag{30}$$

Proof. Suppose $\psi\colon Y\to Z$ is such a map. Then (24) implies $[Y]_\varphi\subseteq\ker\psi$ which implies $\langle Y\rangle_\varphi\subseteq\ker\psi$. Thus the map $\overline{\psi}\colon Y/\langle Y\rangle_\varphi\to Z$ given by

$$\overline{\psi}(\overline{y}):=\psi(y),$$

where $\overline{y}\in Y/\langle Y\rangle_\varphi$ and where $y\in Y$ is a choice of an element in Y such that $\pi(y)=\overline{y}$, is well-defined. Furthermore, it is easy to check that $\overline{\psi}$ is an MDG A-module homomorphism and the unique such map which makes the diagram (45) commute. □

3 The Associator Functor

Let X and Y be MDG A-modules and let $\varphi\colon X\to Y$ be a chain map. If φ is multiplicative, then observe that for all $a_1,a_2,a_3\in A$ and $x\in X$, we have

$$\varphi(a_1[a_2,a_3,x])=a_1[a_2,a_3,\varphi(x)]. \tag{31}$$

Thus φ restricts to an MDG A-module homomorphism $\varphi\colon\langle X\rangle\to\langle Y\rangle$. In particular, the assignment $X\mapsto\langle X\rangle$ induces a functor from category of MDG A-modules to itself. We call this the **associator functor.**

3.1 Failure of Exactness

The associator functor need not be exact. Indeed, let

$$0\longrightarrow X\xrightarrow{\varphi}Y\xrightarrow{\psi}Z\longrightarrow 0 \tag{32}$$

be a short exact sequence of MDG A-modules. We obtain an induced sequence of MDG A-modules

$$0\longrightarrow\langle X\rangle\xrightarrow{\varphi}\langle Y\rangle\xrightarrow{\psi}\langle Z\rangle\longrightarrow 0 \tag{33}$$

which is exact at $\langle X\rangle$ and $\langle Z\rangle$ but not necessarily exact at $\langle Y\rangle$. In order to ensure exactness of (33), we need to place a condition on (32). This leads us to consider the following definition:

Definition 3.1. Let X be an MDG A-submodule of Y. We say Y is an **associative extension** of X if it satisfies

$$\langle X\rangle=X\cap\langle Y\rangle.$$

It is easy to see that (33) is a short exact sequence of MDG A-modules if and only if Y is an associative extension of $\varphi(X)$. In this case, we obtain a long exact sequence in homology:

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}\langle Z\rangle \arrow[dll, rounded corners] \\ & \mathrm{H}_i\langle X\rangle \arrow[r] & \mathrm{H}_i\langle Y\rangle \arrow[r] & \mathrm{H}_i\langle Z\rangle \arrow[dll, rounded corners] \\ & \mathrm{H}_{i-1}\langle X\rangle \arrow[r] & \cdots & \end{tikzcd} \tag{34}$$

We can use this long exact sequence to deduce interesting theorems like:

Theorem 3.1. *Let X be an MDG A-module and suppose Y is an associative extension of X. Then Y is homologically associative if and only if X and Y/X are homologically associative.*

3.2 An Application of the Long Exact Sequence

Assume that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be the minimal R-free resolution of R/I, which is equipped with a multiplication μ giving it the structure of an MDG R-algebra, and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $F + eF$ is the minimal R-free resolution of $R/\langle I, r\rangle$. Here, e is thought of as an exterior variable of degree 1. The differential of the mapping cone is given by

$$\mathrm{d}(a + eb) = \mathrm{d}(a) + rb - e\mathrm{d}(b)$$

for all $a, b \in F$. We give $F + eF$ the structure of an MDG R-algebra by extending the multiplication on F to a multiplication on $F + eF$ by setting

$$(a + eb)(c + ed) = ac + e(bc + (-1)^{|a|}ad)$$

for all $a, b, c, d \in F$. In particular, note that $(eb)c = e(bc)$ for all $b, c \in F$, so e belongs to the nucleus of $F + eF$. We denote by $\iota\colon F \to F + eF$ to be the inclusion map. We can view $F + eF$ either as an MDG F-module or as an MDG R-algebra, thus we potentially have two different associator complexes to consider. It turns out that however these give rise to the same R-complex since e is in the nucleus of $F + eF$. This is the second main theorem from the introduction.

Theorem 3.2. *Let $\langle F + eF\rangle_F$ be the associator F-submodule of $F + eF$ and let $\langle F + eF\rangle$ be the associator $(F + eF)$-ideal of $F + eF$. Then*

$$\langle F + eF\rangle_F = \langle F\rangle + e\langle F\rangle = \langle F + eF\rangle. \tag{35}$$

In particular, $F + eF$ is an associative extension of F. More generally, suppose $\boldsymbol{r} = r_1, \dots, r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$. We set

$$F + \boldsymbol{e}F = F + \sum_{i=1}^{m} e_i F$$

to be minimal R-free resolution of $R/\langle I, \boldsymbol{r}\rangle$ obtained by iterating the mapping cone construction as above, where e_i is an exterior variable of degree 1 which satisfies $\mathrm{d}e_i = r_i$, and where we extend the multiplication of F to a multiplication on $F + \boldsymbol{e}F$ by extending it from $F + \sum_{i=1}^{k} e_i F$ to $F + \sum_{i=1}^{k+1} e_i F$ for each $1 \leq k < m$ as above. Then

$$\langle F + \boldsymbol{e}F\rangle_F = \langle F\rangle + \boldsymbol{e}\langle F\rangle = \langle F + \boldsymbol{e}F\rangle \tag{36}$$

where we set $\boldsymbol{e}\langle F\rangle := \sum_{i=1}^{m} e_i\langle F\rangle$. In particular, $F + \boldsymbol{e}F$ is an associative extension of F.

Proof. Since e is in the nucleus, we have $e[a, b, c] = [ea, b, c]$ for all $a, b, c \in F$. Similarly we have

$$\begin{aligned}
[a, b, ec] &= -(-1)^{|a||b|+|a||ec|+|ec||b|}[ec, b, a] \\
&= -(-1)^{|a||b|+|a||c|+|b||c|}[ec, b, a] \\
&= -(-1)^{|a||b|+|a||c|+|b||c|}e[c, b, a] \\
&= e[a, b, c]
\end{aligned}$$

for all $a, b, c \in F$. Similarly we have

$$\begin{aligned}
[a, eb, c] &= -(-1)^{|a||eb|+|a||c|}[eb, c, a] - (-1)^{|eb||c|+|a||c|}[c, a, eb] \\
&= e(-(-1)^{|a||eb|+|a||c|}[b, c, a] - (-1)^{|eb||c|+|a||c|}[c, a, b]) \\
&= e[a, b, c]
\end{aligned}$$

for all $a, b, c \in F$. Thus we have

$$\begin{aligned}
(a + ea')[b + eb', c + ec', d + ed'] &= (a + ea')[b, c, d] + (a + ea')(e[b', c', d']) \\
&= a[b, c, d] + ea'[b, c, d] + (-1)^{|a|}ea[b', c', d'] \\
&= a[b, c, d] + e(a'[b, c, d] + (-1)^{|a|}a[b', c', d'])
\end{aligned}$$

for all $a, b, c, d, a', b', c', d' \in F$. Thus we obtain (35). To see why (35) implies $F + eF$ is an associative extension of F, note that

$$F \cap \langle F + eF\rangle = F \cap (\langle F\rangle + e\langle F\rangle) = \langle F\rangle.$$

The last part of the theorem follows from induction. □

Theorem 3.3. *Let $\varepsilon = \mathrm{lha}(F)$ and let $\delta = \mathrm{uha}(F)$. Then $\mathrm{lha}(F + eF) = \varepsilon$ and*

$$\mathrm{uha}(F + eF) = \begin{cases} \delta & \text{if } r \text{ is } \mathrm{H}_\delta\langle F\rangle\text{-regular} \\ \delta + 1 & \text{otherwise} \end{cases} \tag{37}$$

Moreover, we have a short exact sequence of $R/\langle I, r\rangle$-modules

$$0 \longrightarrow \mathrm{H}_i\langle F\rangle / r\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + eF\rangle \longrightarrow 0 :_{\mathrm{H}_{i-1}\langle F\rangle} r \longrightarrow 0 \tag{38}$$

for each $i \in \mathbb{Z}$. In particular, we have an isomorphism of $R/\langle I, r\rangle$-modules

$$\mathrm{H}_\varepsilon\langle F\rangle / r\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F + eF\rangle.$$

Proof. Since $F + eF$ is an associative extension of F, we obtain a long exact sequence in homology:

$$\cdots \longrightarrow \mathrm{H}_i\langle F\rangle \xrightarrow{r} \mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + eF\rangle \longrightarrow \mathrm{H}_{i-1}\langle F\rangle \xrightarrow{r} \mathrm{H}_{i-1}\langle F\rangle \longrightarrow \cdots \tag{39}$$

We obtain (40) as well as (39) from this long exact sequence. We obtain $\mathrm{lha}(F + eF) = \varepsilon$ from the long exact sequence together with an application of Nakayama's lemma. □

Corollary 2. *Suppose $\boldsymbol{r} = r_1, \dots, r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$ and let $F + \boldsymbol{e}F$ be the corresponding R-free resolution of $R/\langle I, \boldsymbol{r}\rangle$ obtained by iterating the mapping cone construction. Then we obtain a short exact sequence of $R/\langle I, \boldsymbol{r}\rangle$-modules*

$$0 \longrightarrow \mathrm{H}_i\langle F\rangle / \boldsymbol{r}\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + \boldsymbol{e}F\rangle \longrightarrow 0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r} \longrightarrow 0 \tag{40}$$

In particular, have an isomorphism of $R/\langle I, \boldsymbol{r}\rangle$-modules:

$$\mathrm{H}_\varepsilon\langle F\rangle / \boldsymbol{r}\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F + \boldsymbol{e}F\rangle.$$

We also have the length formula:

$$\ell(\mathrm{H}_i\langle F + \boldsymbol{e}F\rangle) = \ell(\mathrm{H}_i\langle F\rangle / \boldsymbol{r}\mathrm{H}_i\langle F\rangle) + \ell(0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r}),$$

here $\ell(-)$ is the length function.

4 The Symmetric DG Algebra

Let R be a commutative ring, let A be a $\mathbb{Z}$-graded R-module such that $A_0 = R$ which is also equipped with a $\mathbb{Z}$-linear differential $\mathrm{d}\colon A \to A$ giving it the structure of a chain complex. Note that the differential need not be R-linear and note that A may be nonzero in negative homological degree. In this section, we will construct the symmetric DG algebra of A, which we denote by $\mathrm{S}(A)$. After constructing the symmetric DG algebra in this general setting, we then specialize to the case we are mostly interesting in, namely that A is an R-complex centered at R meaning the differential of A is R-linear with $A_0 = R$ and $A_{<0} = 0$. In this case, we sometimes denote the symmetric DG algebra of A by $\mathrm{S}_R(A)$ with R in the subscript in order to emphasize that A is centered at R.

Before we give a rigorous construction of the symmetric DG algebra, we wish to help motivate the reader by giving an informal description of it in this special case where A is an R-complex centered at R. In this case, the underlying graded algebra of $S = \mathrm{S}_R(A)$ is the usual symmetric R-algebra $\mathrm{Sym}(A_+)$ where we view A_+ as just an R-module. However S obtains a bi-graded structure using homological degree and total degree: we have a decomposition of S into R-modules:

$$S = \bigoplus_{i \geq 0} S_i = \bigoplus_{m \geq 0} S^m = \bigoplus_{i,m \geq 0} S_i^m.$$

We refer to the i in the subscript as homological degree and we refer to the m in the superscript as total degree. We have $S_0 = S^0 = S_0^0 = R$ and $S^1 = A_+$. More generally, for $i, m \geq 1$, the R-module S_i^m is the R-span of all homogeneous elementary products of the form $\boldsymbol{a} = a_1 \cdots a_m$ where $a_1, \dots, a_m \in A_+$ are homogeneous (with respect to homological degree of course) such that

$$|\boldsymbol{a}| = |a_1| + \cdots + |a_m| = i.$$

In particular, note that $A = S^{\leq 1} = R + A_+$, thus we view A as being the total degree ≤ 1 part of S. The differential of A extends the differential of S in a natural way and is defined on homogeneous elementary products $\boldsymbol{a} = a_1 \cdots a_m$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \cdots \mathrm{d}(a_j) \cdots a_m. \tag{41}$$

If each of the a_j in (41) live in homological degree ≥ 2, then $\mathrm{d}\boldsymbol{a}$ and $\boldsymbol{a}$ has the same total degree, namely $\deg(\mathrm{d}\boldsymbol{a}) = m = \deg \boldsymbol{a}$. However if one of the a_j in (41) lives in homological degree 1, then $\deg(\mathrm{d}\boldsymbol{a}) = m - 1$. The diagram below illustrates how the differential acts on the bi-graded components:

$$\begin{tikzcd}
A_i \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & S_i^{m-1} \arrow[dl] \arrow[dr] & & S_i^m \arrow[dl] \arrow[dr] & & S_i^{m+1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathbb{K}_i \arrow[dl] \\
& A_{i-1} & & S_{i-1}^{m-2} & & S_{i-1}^{m-1} & & S_{i-1}^m & & S_{i-1}^{m+1} & & \mathbb{K}_{i-1} &
\end{tikzcd}$$

where we set $\mathbb{K}$ to be the koszul DG algebra induced by $\mathrm{d}\colon A_1 \to A_0$. Thus the differential of S connects the usual differential of A on the far left to a koszul differential on the far right. In order to keep track of how the differential operates on the bi-graded components, we express d as

$$\mathrm{d} = \eth + \partial,$$

where $\eth$ is the component of d which respects total degree and where ∂ is the component of d which drops total degree by 1. In the next example, we consider a free resolution of a cyclic module and work out what the symmetric DG algebra looks like in this case.

Example 4.1. Let $R = \mathbb{k}[x, y]$, let $I = \langle x^2, xy \rangle$, and let F be Taylor resolution of $\overline{R} = R/I$. We write down the homogeneous components of F as a graded R-module as well as how the differential acts on the homogeneous basis below:

$$\begin{aligned}
F_0 &= R & \qquad\qquad \mathrm{d}e_1 &= x^2 \\
F_1 &= Re_1 + Re_2 & \mathrm{d}e_2 &= xy \\
F_2 &= Re_{12}, & \mathrm{d}e_{12} &= xe_2 - ye_1,
\end{aligned}$$

Note that the Taylor resolution usually comes equipped with a multiplication called the Taylor multiplication. Let us denote this by $\star$ so as not to confuse it with the multiplication $\cdot$ of $S = \mathrm{S}_R(F)$. Now we write down the homogeneous components of S as a graded R-module (with respect to homological degree):

$$\begin{aligned}
S_0 &= R \\
S_1 &= Re_1 + Re_2 \\
S_2 &= Re_{12} + Re_1e_2 \\
S_3 &= Re_1e_{12} + Re_2e_{12} \\
S_4 &= Re_{12}^2 + Re_1e_2e_{12} \\
&\vdots \\
S_{2k-1} &= Re_1e_{12}^{k-1} + Re_2^{k-1} \\
S_{2k} &= Re_{12}^k + Re_1e_2e_{12}^{k-1} \\
S_{2k+1} &= Re_1e_{12}^k + Re_2e_{12}^k \\
&\vdots
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{d}(e_1e_2 - xe_{12}) &= \mathrm{d}(e_1e_2) - x\mathrm{d}(e_{12}) \\
&= \mathrm{d}(e_1)e_2 - e_1\mathrm{d}(e_2) - x(xe_2 - ye_1) \\
&= x^2e_2 - xye_1 - x^2e_2 + xye_1 \\
&= 0.
\end{aligned}$$

4.1 Construction of the Symmetric DG Algebra of A

We now provide a rigorous construction of $\mathrm{S}(A)$ in the general case where the differential of A need not be R-linear and where $A_{<0}$ is not necessarily zero. Our construction will occur in three steps:

Step 1: We define the **non-unital tensor DG algebra** of A to be

$$\mathrm{U}_{\mathbb{Z}}(A) := \bigoplus_{n=1}^{\infty} A^{\otimes n},$$

where the tensor product is taken as $\mathbb{Z}$-complexes. An elementary tensor in $U = \mathrm{U}_{\mathbb{Z}}(A)$ is denoted $\boldsymbol{a} = a_1 \otimes \cdots \otimes a_n$ where $a_1, \ldots, a_n \in A$ and $n \geq 1$. The differential of U is denoted by d again to simplify notation and is defined on $\boldsymbol{a}$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{n} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots \otimes a_n.$$

We say $\boldsymbol{a}$ is a homogeneous elementary tensors if each a_i is a homogeneous element in A. In this case, we set

$$|\boldsymbol{a}| = \sum_{i=1}^{n} |a_i| \quad \text{and} \quad \deg \boldsymbol{a} = \sum_{i=1}^{n} \deg a_i,$$

where deg is defined on elements $a \in A$ by

$$\deg a = \begin{cases} 1 & \text{if } a \in A_{>0} \\ 0 & \text{if } a \in R \\ -1 & \text{if } a \in A_{<0} \end{cases}$$

We call $|\boldsymbol{a}|$ the **homological degree** of $\boldsymbol{a}$ and we call $\deg \boldsymbol{a}$ the **total degree** of $\boldsymbol{a}$. With $|\cdot|$ and deg defined, we observe that U admits a bi-graded decomposition:

$$U = \bigoplus_{i \in \mathbb{Z}} U_i = \bigoplus_{m \in \mathbb{Z}} U^m = \bigoplus_{i,m \in \mathbb{Z}} U_i^m,$$

where the component U_i^m consists of all finite $\mathbb{Z}$-linear combinations of homogeneous elementary tensors $\boldsymbol{a} \in U$ such that $|\boldsymbol{a}| = i$ and $\deg \boldsymbol{a} = m$. We equip U with an associative (but not commutative nor unital) bi-graded $\mathbb{Z}$-bilinear multiplication which is defined on homogeneous elementary tensors by $(\boldsymbol{a}, \boldsymbol{a}') \mapsto \boldsymbol{a} \otimes \boldsymbol{a}'$ and is extended $\mathbb{Z}$-bilinearly everywhere else. This multiplication is easily seen to satisfy Leibniz law, however note that U is not unital under this multiplication since $(1,1) \mapsto 1 \otimes 1 \neq 1$ (hence why we call this the *non-unital* tensor DG algebra). Also note that U already comes equipped with an R–scalar multiplication (from the R-module structure on A), denoted $(r, \boldsymbol{a}) \mapsto r\boldsymbol{a}$, however the multiplication of U only agrees with the R-scalar multiplication wherever they are both defined and vanish. To rectify this, let $\mathfrak{u} = \mathfrak{u}(A)$ be the U-ideal by all elements of the form

$$[r,a]_\mu = r \otimes a - ra \qquad [a,r]_\mu = a \otimes r - ar$$
$$[r,a]_{\mathrm{d}} = \mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a) \qquad [a,r]_{\mathrm{d}} = (-1)^{|a|} a \otimes \mathrm{d}r - \mathrm{d}(ar) + (\mathrm{d}a)r$$

where $r \in R$ and $a \in A$.

Lemma 4.1. *The differential maps $\mathfrak{u}$ to itself.*

Proof. Indeed, given $r \in R$ and $a \in A$, we have

$$\begin{aligned} \mathrm{d}[r,a]_\mu &= \mathrm{d}(r \otimes a) - \mathrm{d}(ra) \\ &= \mathrm{d}r \otimes a + r \otimes \mathrm{d}a - \mathrm{d}r \otimes a + r(\mathrm{d}a) + [r,a]_{\mathrm{d}} \\ &= r \otimes \mathrm{d}a + r(\mathrm{d}a) + [r,a]_{\mathrm{d}} \\ &= [r, \mathrm{d}a]_\mu + [r,a]_{\mathrm{d}} \\ &\in \mathfrak{u}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathrm{d}[r,a]_{\mathrm{d}} &= \mathrm{d}(\mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r \otimes \mathrm{d}a - [r, \mathrm{d}a]_\mu) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}r \otimes \mathrm{d}a - \mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -\mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -[r, \mathrm{d}a]_{\mathrm{d}} \\ &\in \mathfrak{u}. \end{aligned}$$

Similar calculations show $\mathrm{d}[a,r]_\mu \in \mathfrak{u}$ and $\mathrm{d}[a,r]_{\mathrm{d}} \in \mathfrak{u}$. $\square$

Step 2: We define the **tensor DG algebra** of A to be the quotient

$$\mathrm{T}(A) := \mathrm{U}(A)/\mathfrak{u}(A).$$

The multiplication of $U = \mathrm{U}(A)$ induces a multiplication on $T = \mathrm{T}(A)$ which not only becomes unital but also agrees with the R-scalar multiplication on T where they are both defined. Since $\mathfrak{u} = \mathfrak{u}(A)$ is generated by elements which are homogeneous with respect to homological degree and since the differential of U maps $\mathfrak{u}$ to itself, it follows that the differential of U induces a differential on T, which we again denote by d again. This gives T the structure of a non-commutative (but unital) DG $\Bbbk$-algebra, where

$$\Bbbk = \{r \in R \mid \mathrm{d}r \otimes a = 0 \text{ for all } a \in A\}.$$

In other words, the differential of T satisfies Leibniz law and is $\Bbbk$-linear. Note that the generator $[r,a]_\mu$ of $\mathfrak{u}$ is also homogeneous with respect to total degree, however the generators $[r,a]_{\mathrm{d}}$ is homogeneous with respect to total degree if and only if either $\mathrm{d}r \otimes a = 0$, or $\mathrm{d}(ra) = r\mathrm{d}a$, or $|a| \in \{0,1\}$. In particular, $\mathfrak{u}$ will be homogeneous with respect to total degree if A is an R-complex centered at R (which is a case we are interested in). In this case, T inherits from U a bi-graded R-algebra structure:

$$T = \bigoplus_{i\in\mathbb{Z}} T_i = \bigoplus_{m\in\mathbb{Z}} T^m = \bigoplus_{i,m\in\mathbb{Z}} T_i^m.$$

Example 4.2. Let us describe what the total degree m component of $T = \mathrm{T}_R(A)$ in the case where A is an R-complex centered at R. We have

$$\begin{aligned} T^0 &= R \\ T^1 &= \bigoplus_{1\le i} A_i \\ T^2 &= \bigoplus_{1\le i<j} ((A_i \otimes A_j) \oplus (A_j \otimes A_i)) \oplus \bigoplus_{1\le i} A_i^{\otimes 2} \end{aligned}$$

The component T^3 is slightly more complicated:

$$\bigoplus_{\substack{1\le i<j<k \\ \pi\in S_3}} (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(k)}) \oplus \bigoplus_{\substack{1\le i<j \\ \pi\in S_2}} ((A_{\pi(i)}^{\otimes 2} \otimes A_{\pi(j)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(i)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)}^{\otimes 2})) \oplus \bigoplus_{1\le i} A_i^{\otimes 3}.$$

More generally, there is an interpretation of T^m in terms of certain rooted trees.

Now let $\mathfrak{t} = \mathfrak{t}(A)$ be the T-ideal generated by all elements of the form

$$[a_1,a_2]_\sigma \colon = (-1)^{|a_1||a_2|} a_2 \otimes a_1 - a_1 \otimes a_2 \quad \text{and} \quad [a]_\tau := a \otimes a,$$

where $a, a_1, a_2 \in A$ are homogeneous and $|a|$ is odd.

Lemma 4.2. *The differential of T maps $\mathfrak{t}$ to itself.*

Proof. Indeed, if $a, a_1, a_2 \in A$ are homogeneous with $|a|$ odd, then we have

$$\mathrm{d}[a_1,a_2]_\sigma = [\mathrm{d}a_1,a_2]_\sigma + (-1)^{|a_1|}[a_1,\mathrm{d}a_2]_\sigma \in \mathfrak{t} \quad \text{and} \quad \mathrm{d}[a]_\tau = [\mathrm{d}a,a]_\sigma \in \mathfrak{t}.$$

$\square$

Step 3: We define the **symmetric DG algebra** of A to be the quotient

$$\mathrm{S}(A) := \mathrm{T}(A)/\mathfrak{t}(A)$$

The image of a homogeneous elementary tensor $a_1 \otimes \cdots \otimes a_m$ in $S = \mathrm{S}(A)$ is often denoted $a_1 \cdots a_n$ and is called a homogeneous elementary product. Since $\mathfrak{t} = \mathfrak{t}(A)$ is generated by elements which are homogeneous with respect to both homological degree and since the differential of $T = \mathrm{T}(A)$ maps $\mathfrak{t}$ to itself, we see that the differential of T induces a differential on S, which we again denote by d, giving it the structure of a strictly graded-commutative DG $\Bbbk$-algebra. Furthemore, if T inherits the bi-graded structure from U, then S inherits the bi-graded structure from T since $\mathfrak{t}$ is generated by elements which are homogeneous with respect to total degree.

4.2 Properties of the Symmetric DG Algebra

We now focus our attention to the case where A is an R-complex centered at R and we wish to study $S = \mathrm{S}_R(A)$ the symmetric DG R-algebra of A (note that we sometimes write R in the subscript of $\mathrm{S}_R(A)$ to emphasize that A and $S = \mathrm{S}_R(A)$ are centered at R). In this case, the underlying graded R-algebra of S is the usual symmetric algebra of A_+:

$$\mathrm{Sym}_R(A_+) = \frac{\bigoplus_{m\geq 0} A_+^{\otimes m}}{\langle \{[a_1, a_2]_\sigma, [a]_\tau\}\rangle},$$

where the tensor product is taken over R. Thus the symmetric DG algebra of A inherits all of the properties that are satisfied by the symmetric algebra of A_+ when we forget about the differential. For instance, recall that a bounded below R-complex is semiprojective if and only if its underlying graded R-module is projective as a graded R-module. In particular, if A is semiprojective, then S is semiprojective too. Thus if we assume that A is semiprojective *and* that there exists a chain map $\pi\colon S \to A$ which splits the inclusion map $\iota\colon A \hookrightarrow S$, then we can lift chains maps out of A along surjective quasiisomorphisms, meaning if $\varphi\colon A \to X$ is any chain map and $\tau\colon Y \to X$ is any surjective quasiisomorphism, then there exists a chain map $\widetilde{\varphi}\colon S \to Y$ such that $\tau\widetilde{\varphi} = \varphi$, moreover such a lift is unique up to homotopy. The assumption that A is semiprojective is mild whereas the assumption that there exists a chain map $S \to A$ which splits the inclusion map $A \hookrightarrow S$ is rather subtle. We will see that if A has a DG R-algebra structure on it, then there will be such a map $S \to A$.

Proposition 4.1. *Let R be a commutative ring and let A be an R-complex centered at R.*

1. *(Base Change) Let R' be an R-algebra. Then*
$$\mathrm{S}_R(A) \otimes_R R' = \mathrm{S}_{R'}(A \otimes_R R'). \tag{42}$$

2. *(Exact Sequences) Let*
$$B \longrightarrow A \longrightarrow A' \longrightarrow 0 \tag{43}$$
be an exact sequence of R-complexes where A' is centered at a cyclic R-algebra, say $R' = R/I$ for some ideal I of R. Then we obtain an exact sequence
$$\mathrm{S}_R(A) \otimes_R B \longrightarrow \mathrm{S}_R(A) \longrightarrow \mathrm{S}_{R'}(A') \longrightarrow 0 \tag{44}$$
.

3. *(Universal Mapping Property) For every chain map of the form $\varphi\colon A \to A'$, where A' is a DG algebra centered at a ring R' and where φ restricts to a ring homomorphism $\varphi_0\colon R \to R'$, there exists a unique DG algebra homomorphism $\widetilde{\varphi}\colon \mathrm{S}_R(A) \to A'$ which extends $\varphi\colon A \to A'$, that is, such that $\widetilde{\varphi} \circ \iota = \varphi$ where $\iota\colon A \hookrightarrow \mathrm{S}_R(A)$ is the inclusion map. We express this in terms of a commutative diagram as below:*
$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{S}_R(A) \arrow[d, "\widetilde{\varphi}"] \\ & A' \end{tikzcd} \tag{45}$$

Remark 6. Strictly speaking, one should write $R \otimes_R R'$ in the subscript on the righthand side of Equation (42). However we may view R' as being the homological degree 0 part by identifying R' with $R \otimes_R R'$ via the canonical isomorphism $R' \simeq R \otimes_R R'$.

Proof. We only prove the third property since the first two properties are straightforward to show. Let $\varphi\colon A \to A'$ be such a chain map and denote $S = \mathrm{S}_R(A)$. We define $\widetilde{\varphi}\colon S \to A'$ by setting $\widetilde{\varphi}|_A = \varphi$ and

$$\widetilde{\varphi}(a_1 \cdots a_m) = \varphi(a_1) \cdots \varphi(a_m) \tag{46}$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$ and then extending it R-linearly everywhere else. By construction, $\widetilde{\varphi}$ is multiplicative and extends $\varphi\colon A \to A'$. Furthermore, $\widetilde{\varphi}$ is a chain map since it is a graded R-linear map which commutes with the differential. Indeed, we clearly have $\widetilde{\varphi}\mathrm{d}(1) = 0 = \mathrm{d}\widetilde{\varphi}(1)$, and for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, we have

$$\begin{aligned}
\widetilde{\varphi}\mathrm{d}(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \widetilde{\varphi}(a_1 \cdots \mathrm{d}(a_j) \cdots a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \varphi\mathrm{d}(a_j) \cdots \varphi(a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \mathrm{d}\varphi(a_j) \cdots \varphi(a_m) \\
&= \mathrm{d}(\varphi(a_1) \cdots \varphi(a_m)) \\
&= \mathrm{d}\widetilde{\varphi}(a_1 \cdots a_m).
\end{aligned}$$

Finally, if $\widehat{\varphi} \colon S \to A'$ were another DG algebra homomorphism which extended $\varphi \colon A \to B$, then we would have

$$\widetilde{\varphi}(a_1 \cdots a_m) = \widehat{\varphi}(a_1) \cdots \widehat{\varphi}(a_m) = \varphi(a_1) \cdots \varphi(a_m) = \widehat{\varphi}(a_1 \cdots a_m)$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, which implies $\widehat{\varphi} = \widetilde{\varphi}$. □

Definition 4.1. Let A and B be two R-complexes centered at R. We define their **wedge sum** $A \vee B$ to be the R-complex centered at R whose underlying graded R-module is given by

$$(A \vee B)_i = \begin{cases} A_i \oplus B_i & \text{if } i \geq 1 \\ R & \text{if } i = 0 \end{cases}$$

and whose differential is defined by

$$\mathrm{d}(a, b) = \begin{cases} (\mathrm{d}a, \mathrm{d}b) & \text{if } |a| = |b| \geq 2 \\ \mathrm{d}a - \mathrm{d}b & \text{if } |a| = |b| = 1 \end{cases}$$

Observe that

$$\mathrm{H}_i(A \vee B) = \begin{cases} R/(\mathrm{d}A_1 + \mathrm{d}B_1) & \text{if } i = 0 \\ (A_1 \times_R B_1)/(\mathrm{d}A_2 \oplus \mathrm{d}B_2) & \text{if } i = 1 \\ \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) & \text{if } i \geq 2 \end{cases}$$

Proposition 4.2. *Let A and B be two R-complexes centered at R. Then we have*

$$\mathrm{S}_R(A \vee B) = \mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B).$$

Proof. In terms of the underlying graded R-algebras, we have

$$\begin{aligned}
\mathrm{S}_R(A \vee B) &= \mathrm{Sym}_R(A_+ \oplus B_+) \\
&= \mathrm{Sym}_R(A_+) \otimes_R \mathrm{Sym}_R(B) \\
&= \mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B).
\end{aligned}$$

It is easy to check that the differential of $\mathrm{S}_R(A \vee B)$ is carried over to the differential of $\mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B)$ under this isomorphism (we write equality here because $\mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B)$ satisfies the universal mapping property of the symmetric DG R-algebra of $A \vee B$. □

Proposition 4.3. *Let $\varphi \colon A \to B$ be a chain map of R-complexes centered at R. Let $B + eA$ be the mapping cone of φ. Then we have*

$$\mathrm{S}_R(B + eA) = \mathrm{S}_R(B) + e\mathrm{S}_R(A).$$

In other words, the symmetric DG R-algebra commutes with the mapping cone.

Proof. This follows from the formula

$$(b_1 + ea_1)(b_2 + ea_2) = b_1 b_2 + e(a_1 a_2 + (-1)^{|b_1|} b_1 a_2 + (-1)^{|b_2|} b_2 a_1) \tag{47}$$

for all homogeneous $a_1, a_2 \in A$ and homogeneous $b_1, b_2 \in B$. More generally, a homogeneous elementary product in $\mathrm{S}_R(B + eA)$ can be expressed in terms of a sum of two homogeneous elementary products in $\mathrm{S}_R(B) + e\mathrm{S}_R(A)$ using (47). □

4.3 Presentation of the Maximal Associative Quotient

Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Equip A with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra. In particular, note that if $a_1, a_2 \in A_1$, then

$$a_1 a_2 \in S_2^2, \quad a_1 \star a_2 \in S_2^1, \quad \text{and} \quad [a_1, a_2] \in S_2,$$

where $[a_1, a_2] = a_1 \star a_2 - a_1 a_2$ is the multiplicator of the inclusion map $\iota \colon A \hookrightarrow S$ evaluated at $(a_1, a_2) \in A^2$. Let $\mathfrak{s} = \mathfrak{s}(\mu)$ be the S-ideal generated by all such multiplicators, so

$$\mathfrak{s} = \operatorname{span}_S\{[a_1, a_2] \mid a_1, a_2 \in A\}.$$

Also let $\pi \colon S \to S/\mathfrak{s}$ and $\pi^{\mathrm{as}} \colon A \twoheadrightarrow A^{\mathrm{as}}$ denote the canonical quotient maps. The universal mapping property of the symmetric DG algebra of A implies $\pi^{\mathrm{as}} \colon A \twoheadrightarrow A^{\mathrm{as}}$ extends uniquely to a DG algebra homomorphism $S \twoheadrightarrow A^{\mathrm{as}}$ which we again denote by π^{as}. We let $S^{\geq 2} = S/A$ be the R-complex whose underlying graded R-module is $S^{\geq 2}$ and whose differential $\mathrm{d}^{\geq 2}$ is defined by

$$\mathrm{d}^{\geq 2}|_{S^m} = \begin{cases} \eth|_{S^2} & \text{if } m = 2 \\ \mathrm{d}|_{S^m} & \text{if } m > 2. \end{cases}$$

We also let $\rho \colon S \twoheadrightarrow S/A = S^{\geq 2}$ be the canonical quotient map. We now present the third main theorem from the introduction.

Theorem 4.3. *With the notation as above, we have*

$$A^{\mathrm{as}} = \operatorname{coker}(\mathfrak{s} \hookrightarrow S) = S/\mathfrak{s}$$

More specifically, there is a unique isomorphism $A^{\mathrm{as}} \to S/\mathfrak{s}$ of DG *S-algebras (thus we are justified in writing $\pi \colon S \to A^{\mathrm{as}}$ to denote both $\pi^{\mathrm{as}} \colon S \to A^{\mathrm{as}}$ and $\pi \colon S \to S/\mathfrak{s}$ in order to simplify notation) In particular, this implies*

$$\langle A \rangle = A \cap \mathfrak{s} = \mathfrak{s}^{\leq 1} = \ker(\mathfrak{s} \to S^{\geq 2})$$

Thus we have the following canonically defined hexagonal-shaped diagram of R-complexes which is exact everywhere (in every direction) and which is natural in $A = (A, \mathrm{d}, \mu)$:

$$\begin{tikzcd}
& S^{\geq 2} \arrow[r] & 0 \\
\mathfrak{s} \arrow[ur] \arrow[r, hook, "i"] & S \arrow[u, two heads, "\rho"'] \arrow[r, two heads, "\pi"] & A^{\mathrm{as}} \arrow[u] \\
\mathfrak{s}^{\leq 1} \arrow[u, hook] \arrow[r, hook] & A \arrow[u, hook, "\iota"'] \arrow[ur] &
\end{tikzcd} \tag{48}$$

where the blue arrows are DG *S-module homorphisms, where the green arrows are chain maps as R-complexes, and where the red arrows are* MDG *A-module homomorphisms. In particular, if $\mathrm{H}_+(A) = 0$, then $\mathrm{H}_+(S) = \mathrm{H}(S^{\geq 2})$ and we obtain a canonically defined sequence of graded $\mathrm{H}(S)$-modules:*

$$\mathrm{H}_+(\mathfrak{s}) \longrightarrow \mathrm{H}_+(S) \longrightarrow \mathrm{H}_+(A^{\mathrm{as}}) \longrightarrow \Sigma\mathrm{H}(\mathfrak{s}) \longrightarrow \Sigma\mathrm{H}(S) \tag{49}$$

which is natural in $A = (A, \mathrm{d}, \mu)$.

Remark 7. By "natural in $A = (A, \mathrm{d}, \mu)$" we mean that if R' is an R-algebra and $\varphi \colon A \to A'$ is an MDG R-algebra homomorphism where $A' = (A', \mathrm{d}', \mu')$ is an MDG R'-algebra centered at R', then we obtain canonically defined maps $S \to S'$ and $\mathfrak{s} \to \mathfrak{s}'$, where we set $S' = \mathrm{S}_{R'}(A')$ and $\mathfrak{s}' = \mathfrak{s}(\mu')$, which induces a map of hexagonal-shaped diagrams in which everything commutes. For instance, if $\mathrm{H}_+(A) = 0 = \mathrm{H}_+(A')$, then then we have a commutative diagram of graded $\mathrm{H}(\mathrm{S}')$-modules of the form:

$$\begin{tikzcd}
\mathrm{H}_+(\mathfrak{s}) \arrow[r] \arrow[d] & \mathrm{H}_+(S) \arrow[r] \arrow[d] & \mathrm{H}_+(A^{\mathrm{as}}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(\mathfrak{s}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(S) \arrow[d] \\
\mathrm{H}_+(\mathfrak{s}') \arrow[r] & \mathrm{H}_+(S') \arrow[r] & \mathrm{H}_+((A')^{\mathrm{as}}) \arrow[r] & \Sigma\mathrm{H}(\mathfrak{s}') \arrow[r] & \Sigma\mathrm{H}(S')
\end{tikzcd} \tag{50}$$

We are especially interested in the case where $A = A'$ but allow $\mu \neq \mu'$. In that case, we are basically studying the DG ideals $\mathfrak{s} = \mathfrak{s}(\mu)$ and $\mathfrak{s}' = \mathrm{s}(\mu')$ in $S = S'$.

Proof. Observe that $\pi^{\mathrm{as}}\colon S \twoheadrightarrow A^{\mathrm{as}}$ satifies

$$\begin{aligned}\pi^{\mathrm{as}}[a_1,a_2] &= \pi^{\mathrm{as}}(a_1 \star a_2 - a_1a_2)\\ &= \pi^{\mathrm{as}}(a_1 \star a_2) - \pi^{\mathrm{as}}(a_1a_2)\\ &= \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2) - \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2)\\ &= 0.\end{aligned}$$

Thus the universal mapping property of the quotient $S/\mathfrak{s} = \mathrm{coker}(\mathfrak{s} \hookrightarrow S)$ implies there is a unique DG algebra homomorphism $\overline{\pi}^{\mathrm{as}}\colon S/\mathfrak{s} \to A^{\mathrm{as}}$ such that

$$\overline{\pi}^{\mathrm{as}} \circ \pi = \pi^{\mathrm{as}}.$$

Similarly, note that the composite $\pi \circ \iota\colon A \to S/\mathfrak{s}$ is an MDG algebra homomorphism which is surjective. Indeed, if $a_1 \cdots a_m$ is a homogeneous elementary tensor in S^m, then we have

$$a_1a_2a_3 \cdots a_m = ((\cdots(a_1 \star a_2) \star a_3) \star \cdots) \star a_m$$

in $S/\mathfrak{s}$. Thus every element in $S/\mathfrak{s}$ can be represented by an element in $A = S^1$ which implies $\pi\iota\colon A \twoheadrightarrow S/\mathfrak{s}$ is surjective as claimed. In particular, since $S/\mathfrak{s}$ is associative, it follows from the universal mapping property of the maximal associative quotient of A that there is a unique DG algebra homomorphism $\overline{\pi}\colon A^{\mathrm{as}} \to S/\mathfrak{s}$ such that

$$\pi \circ \iota = \overline{\pi} \circ \pi^{\mathrm{as}}.$$

Combining all of this together, we have a commutative diagram of MDG S-modules:

$$\begin{tikzcd} S \arrow[r, two heads, "\pi"] \arrow[dr, dashed, "\pi^{\mathrm{as}}"] & S/\mathfrak{s} \arrow[d, dashed, shift left, "\overline{\pi}^{\mathrm{as}}"] \\ A \arrow[u, hook, "\iota"] \arrow[r, two heads, "\pi^{\mathrm{as}}"'] & A^{\mathrm{as}} \arrow[u, dashed, shift left, "\overline{\pi}"] \end{tikzcd}$$

where the dashed arrows indicates uniqueness. □

Corollary 3. *Continuing with the notation as above, assume further that A is associative, so $A = A^{\mathrm{as}}$. Then the canonical map $\mathfrak{s} \to S^{\geq 2}$ defined on multiplicators by*

$$[a_1, a_2] \mapsto a_1a_2$$

is an isomorphism of R-complexes. Let $\theta\colon S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s} \hookrightarrow S$ be the composite map where $S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s}$ is the inverse isomorphism of the canonical map $\mathfrak{s} \to S^{\geq 2}$. We obtain a short exact sequence of R-complexes

$$0 \longrightarrow S^{\geq 2} \xrightarrow{\theta} S \xrightarrow{\pi} A \longrightarrow 0 \tag{51}$$

which is split by the inclusion map $\iota\colon A \to S$. Similarly, the short exact sequence of R-complexes

$$0 \longrightarrow A \xrightarrow{\iota} S \xrightarrow{\rho} S^{\geq 2} \longrightarrow 0 \tag{52}$$

is split by $\theta\colon S^{\geq 2} \to S$.

Corollary 4. *Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Then a necessary condition for A to have a DG algebra structure is that the canonical short exact sequence of R-complexes*

$$0 \longrightarrow A \xrightarrow{\iota} S \xrightarrow{\rho} S^{\geq 2} \longrightarrow 0 \tag{53}$$

is split.

Corollary 5. *Continuing with the notation as above, assume that $A = F$ is the minimal free resolution of a cyclic R-module R/I and let J be an ideal of R. If F/JF has a DG algebra structure on it, then $\mathrm{Tor}^R(R/I, R/J)$ is a direct summand of $\mathrm{H}(S/JS)$.*

Proof. Since the symmetric DG algebra construction commutes with base change, we have $S/JS = \mathrm{S}_{R/J}(F/JF)$. Since F/JF has a DG algebra structure on it, the canonical map $F/JF \to S/JF$ is split. Thus $\mathrm{Tor}^R(R/I, R/J) = \mathrm{H}(F/JF)$ is a direct summand of $\mathrm{H}(S/JS)$. □

Example 4.3. One can check that the multiplication defined on F in Example (1.1) becomes associative when we tensor with R/yz. It follows that $\mathrm{Tor}^R(R/I, R/yz)$ is a direct summand of $\mathrm{H}(\mathrm{S}_R(F))$.

Proposition 4.4. *Let R be a commutative ring, let A be an R-complex centered at R, and let $I = \mathrm{d}(A_1)$ (so $\mathrm{H}_0(A) = R/I$). Set $S = \mathrm{S}_R(A)$ to be the symmetric DG algebra of A. Assume further that $\mathrm{d}A \subseteq IA$. Then the canonical quotient map $\rho \colon S \to S^{\geq 2}$ induces an isomorphism*

$$S/IS \simeq A/IA \oplus S^{\geq 2}/IS^{\geq 2}$$

as R-complexes.

Proof. Note S and $S^{\geq 2}$ are the exact same complex in total degree ≥ 3, so the only difference between them is how they behave in total degree ≤ 2. In in particular, we obtain $S^{\geq 2}$ from S by replacing $S^{\leq 1} = A$ with 0 and replacing the labeled arrows in the diagram below with zero maps

$$\begin{tikzcd}
A_{i+1} \arrow[dr, "\eth^1_{i+1}"'] & & S^2_{i+1} \arrow[dl, "\partial^2_{i+1}"'] \arrow[dr] & & S^3_{i+1} \arrow[dl] & \\
& A_i \arrow[dr, "\eth^1_i"'] & & S^2_i \arrow[dl, "\partial^2_i"'] \arrow[dr] & & S^3_i \arrow[dl] \\
& & A_{i-1} & & S^2_{i-1} &
\end{tikzcd}$$

Note that $\mathrm{im}(\eth^1_i) = \mathrm{d}A_i \subseteq IA_i$ and $\mathrm{im}(\partial_i) = IA_i$. Thus we obtain $S/IS = S \otimes_R R/I$ by replacing the labeled arrows above with zero maps.

□

4.4 Homology of the Symmetric DG Algebra

Example 4.4. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2 \rangle$, and let F be the minimal free resolution of R/I over R as in Example (1.1). The homology of the symmetric DG algebra $S = \mathrm{S}_R(F)$ is complicated to describe, but it "knows" about multiplications on F. For instance, the polynomials below each represent a distinct elements which are linearly independent in $\mathrm{H}_2(S)$:

$$\begin{aligned}
f_{12} &= e_1e_2 - e_{12} \\
f_{13} &= e_1e_3 - e_{13} \\
f_{14} &= e_1e_4 - xe_{14} \\
f_{15} &= e_1e_5 - yz^2e_{14} - xe_{45} \\
f_{23} &= e_2e_3 - we_{23}.
\end{aligned}$$

More generally, for each $1 \leq i < j \leq 5$, the polynomial $f_{ij} = e_ie_j - e_i \star e_j$ represents another distinct element in homology and the collection $\{f_{ij}\}$ are all linearly independent in $\mathrm{H}_2(S)$. Note that $\mathrm{d}(e_1e_{14}) = yf_{14}$, so $y \in \mathrm{Ann}(\overline{f}_{14})$. Similar arguments show that $\mathrm{Ann}(\overline{f}_{14}) = \langle x, y, zw, w^2 \rangle = I : x$. On the other hand, one can show that $\mathrm{Ann}(\overline{f}_{12}) = I$. Furthermore, if we set $f_{1,23} = e_1e_{23} - e_{123}$, then we have $\mathrm{d}(f_{1,23}) = zf_{12} - wf_{13}$, so $z\overline{f}_{12} = w\overline{f}_{13}$. Finally, note that

$$f_{12}^2 = x^2e_{12}^2 - 2xe_1e_2e_{12} = \mathrm{d}(e_1e_{12}^2) \quad \text{and} \quad f_{13}^2 = e_{13}^2 - 2e_1e_3e_{13}.$$

In particular, $\overline{f}_{12}^2 = 0$ but $\overline{f}_{13}^2 \neq 0$ since the coefficient for e_{13}^2 is not in $\mathfrak{m} = \langle x, y, z, w \rangle$. More generally one can show that $\overline{f}_{13}^n \neq 0$ for all $n \geq 1$.

Lemma 4.4. *Set $f_{ij} = e_i \star e_j - e_ie_j$ where $|e_i|$ is odd. Then we have*

$$f_{ij}^n = (e_i \star e_j)^{n-1}(e_i \star e_j - ne_ie_j), \quad \textit{and} \quad e_if_{ij}^n = e_i(e_i \star e_j)^n.$$

In particular, if $e_i \star e_j \in \mathfrak{m}F$, then $f_{ij}^n \in \mathfrak{m}^nF$.

Example 4.5. Let us revisit Example (4.6) where $R = \Bbbk[x, y]$, $I = \langle x^2, xy \rangle$, F is the Taylor resolution of $\overline{R} = R/I$, and S is the symmetric DG R-algebra of F. One can show that the homology of S is given by

$$\mathrm{H}_i(S) = \begin{cases} R/\langle k, x \rangle & \text{if } i = 2k+1 \text{ where } k \geq 1 \\ R/\langle x, y \rangle & \text{if } i = 2k \text{ where } k \geq 1 \\ R/\langle x^2, xy \rangle & \text{if } i = 0 \end{cases}$$

Furthermore, one can show that the underlying graded $\overline{R}$-algebra structure of $\mathrm{H}(S)$ looks like

$$\mathrm{H}(S) = \overline{R}[\{f_{2k}, g_{2k+1} \mid k \geq 1\}]/\langle \{xf_{2k}, yf_{2k}, xg_{2k+1}, kg_{2k+1}, f_{2k}f_{2m}, f_{2k}g_{2m+1}, g_{2k+1}g_{2m+1} \mid k, m \geq 1\}\rangle,$$

where $f_{2k} = (e_1e_2 - xe_{12})^k/x^{k-1}$ and where $g_{2k+1} = \mathrm{d}(e_{12}^k)$ for each $k \geq 1$. On the other hand, let us treat e_{12} as a divided variable. Then with respect to the ordered bases $e_{12}^{(k)}, e_1e_2e_{12}^{(k-1)}$ for D_{2k} and $e_1e_{12}^{(k-1)}, e_2e_{12}^{(k-1)}$ for D_{2k+1}, the matrix representation of the differential looks like:

$$[\mathrm{d}_{2k}] = \begin{pmatrix} -y & -xy \\ x & x^2 \end{pmatrix} \quad \text{and} \quad [\mathrm{d}_{2k+1}] = \begin{pmatrix} x^2 & xy \\ -x & -y \end{pmatrix}.$$

In this case, one has $p_{2k} = xe_{12}^{(k)} - e_1e_2e_{12}^{(k-1)}$ and $q_{2k-1} = ye_1e_{12}^{(k-1)} - xe_2e_{12}^{(k-1)} = \mathrm{d}(e_{12}^{(k)})$ generating their respective kernels.

$$k!p_{2k} = f_{2k} \quad \text{and} \quad (k-1)!q_{2k-1} = g_{2k-1}.$$

In particular, for the divided algebra we have

$$\mathrm{H}_i(D) = \begin{cases} 0 & \text{if } i = 2k+1 \text{ where } k \geq 1 \\ R/\langle x, y\rangle & \text{if } i = 2k \text{ where } k \geq 1 \\ R/\langle x^2, xy\rangle & \text{if } i = 0 \end{cases}$$

and the underlying graded $\overline{R}$-algebra of $\mathrm{H}(D)$ looks like:

$$\mathrm{H}(D) = \overline{R}[\{p_{2k} \mid k \geq 1\}]/\langle \{xp_{2k}, yp_{2k}, p_{2k}p_{2m} \mid k, m \geq 1\}$$

Example 4.6. Let $R = \Bbbk[x, y]$, let $I = \langle x, y\rangle$, and let F be Koszul resolution of $\Bbbk = R/I$. We write down the homogeneous components of F as a graded R-module as well as how the differential acts on the homogeneous basis below:

$$\begin{aligned} F_0 &= R & \mathrm{d}e_1 &= x \\ F_1 &= Re_1 + Re_2 & \mathrm{d}e_2 &= y \\ F_2 &= Re_{12}, & \mathrm{d}e_{12} &= xe_2 - ye_1, \end{aligned}$$

Let $S = \mathrm{S}_R(F)$ denote the symmetric DG R-algebra of F. The homogeneous components of S as a graded R-module (with respect to homological degree) looks the same as the previous example:

$$\begin{aligned} S_0 &= R \\ S_1 &= Re_1 + Re_2 \\ S_2 &= Re_{12} + Re_1e_2 \\ S_3 &= Re_1e_{12} + Re_2e_{12} \\ S_4 &= Re_{12}^2 + Re_1e_2e_{12} \\ &\vdots \\ S_{2k-1} &= Re_1e_{12}^{k-1} + Re_2^{k-1} \\ S_{2k} &= Re_{12}^k + Re_1e_2e_{12}^{k-1} \\ S_{2k+1} &= Re_1e_{12}^k + Re_2e_{12}^k \\ &\vdots \end{aligned}$$

where $2k \geq 1$. One can show that the homology of S is given by:

$$\mathrm{H}_i(S) = \begin{cases} 0 & \text{if } i = 2k+1 \text{ where } k \geq 0 \\ R/\langle x, y\rangle & \text{if } i = 2k \text{ where } k \geq 0 \end{cases}$$

Furthermore, one can show that the underlying graded $\Bbbk$-algebra structure of $\mathrm{H}(S)$ is just $\Bbbk[f_2]$ where $f_2 = e_{12} - e_1e_2$.

Proposition 4.5. *Let $R = (R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $F = (F, \mathrm{d})$ be the minimal free resolution of R/I over R where $I \subseteq \mathfrak{m}$. Equip F with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra and let $S = \mathrm{S}_R(F)$ be the symmetric DG R-algebra of F. Finally let*

$$f := [a_1, a_2] = a_1a_2 - a_1 \star a_2,$$

where $a_1, a_2 \in F_1 \backslash \mathfrak{m}F_1$. Then f represents a nonzero element in $\mathrm{H}_2(S)$.

Proof. Clearly we have $\mathrm{d}f = 0$. Suppose that $\mathrm{d}g = f$ where $g \in S_3$. Let g^2 and g^3 be the components of g that lie in S_3^2 and S_3^3 respectively. Then in particular, we must have

$$a_1 a_2 = \partial g^3 + \eth g^2. \tag{54}$$

However this is a contradiction as minimality of F implies that the RHS of (54) lies in $\mathfrak{m}S$ however the LHS of (54) does not lie in $\mathfrak{m}S$ as $a_1, a_2 \notin \mathfrak{m}F$. □

4.5 The Symmetric DG Algebra viewed as a Generalization of the Koszul Algebra

In this subsection, we want to explain how the symmetric DG algebra generalizes the Koszul algebra. Let A be an R-complex centered at R and let X be an R-complex. We set

$$\mathrm{S}_R(A, X) := \mathrm{S}_R(A) \otimes_R X \quad \text{and} \quad \mathrm{H}(A, X) := \mathrm{H}(\mathrm{S}_R(A, X)).$$

We also set

$$\delta(A, X) := \sup\{i \mid \mathrm{H}_i(A, X) \neq 0\}.$$

Note that if A is the R-complex

$$\cdots \longrightarrow 0 \longrightarrow R^n \xrightarrow{\varphi} R \longrightarrow 0 \longrightarrow \cdots \tag{55}$$

where $\varphi\colon R^n \to R$ is an R-linear map with R sitting in homological degree 0, and if X is an R-module M viewed as R-complex with M sitting in homological degree 0, then $\mathrm{S}_R(A)$ is the Koszul algebra $\mathcal{K}^R(\varphi)$ and $\mathrm{S}_R(A, M)$ is the Koszul module $\mathcal{K}^R(\varphi, M)$. Thus the symmetric DG algebra construction we described generalizes the usual Koszul algebra construction. Finally, if X is positive, then we set

$$\chi(A, X) := \sum_{i \in \mathbb{Z}} \ell(\mathrm{H}_i(A, X)),$$

whenever this is defined.

Lemma 4.5. *Let A be an R-complex centered at R and let X be a DG S-module where $S = \mathrm{S}_R(A)$ is the symmetric DG algebra of A. Set I to be the image of $\mathrm{d}_1\colon A_1 \to R$. Then I annihilates $\mathrm{H}(X)$.*

Proof. Let $r \in I$, thus $r = \mathrm{d}a$ where $a \in A_1 = S_1$. If $x \in X$ such that $\mathrm{d}x = 0$, then $\mathrm{d}(ax) = rx$. It follows that r annihilates $\mathrm{H}(X)$, and since $r \in I$ was arbitrary, it follows that I annihilates $\mathrm{H}(X)$. □

Lemma 4.6. *Let F be an R-complex centered at R such that each F_i is a free R-module, let X be an R-complex, and let $r \in I = \mathrm{d}_1(F_1)$ be X-regular (meaning the multiplication by r map $X \to X$ is injective). Then we have a short exact sequence of graded modules:*

$$0 \longrightarrow \mathrm{H}(F, X) \longrightarrow \mathrm{H}(F, X/rX) \longrightarrow \Sigma\mathrm{H}(F, X) \longrightarrow 0. \tag{56}$$

In particular, if $\delta = \delta(F, X)$, then we have

$$\delta(F, X/rX) = \delta + 1 \quad \textit{and} \quad \mathrm{H}_\delta(F, X) \cong \mathrm{H}_{\delta+1}(F, X/rX).$$

Furthermore, if X is positive, then we have

$$\sum_{i=0}^{n} (-1)^n \ell(\mathrm{H}_i(F, X/rX)) = \ell(\mathrm{H}_n(F, X)).$$

whenever this is defined. Thus

$$\chi(F, X/rX) = \lim_{n \to \infty} \ell(\mathrm{H}_n(F, X))$$

Proof. The multiplication by r map from X to itself induces a short exact sequence of R-complexes

$$0 \longrightarrow X \xrightarrow{r} X \longrightarrow X/r \longrightarrow 0 \tag{57}$$

Since F is semiprojective as an R-complex, we see that $\mathrm{S}_R(F)$ is also semiprojective (hence semiflat), thus tensoring (61) with $\mathrm{S}_R(F)$ yields the short exact sequence of R-complexess

$$0 \longrightarrow \mathrm{S}_R(F, X) \xrightarrow{r} \mathrm{S}_R(F, X) \longrightarrow \mathrm{S}_R(F, X/r) \longrightarrow 0 \tag{58}$$

Then after taking homology and using the fact that r annihilates $\mathrm{H}(F, X)$, we obtain the short exact sequence of graded R-modules (56). □

Lemma 4.7. *Assume that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be the minimal free resolution of R/I over R, and let $r \in \mathfrak{m}$ be an R/I-regular element. Then we obtain a long exact sequence in homology:*

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}_i(F,X) \arrow[dll, rounded corners, "r"] \\
\mathrm{H}_i(F,X) \arrow[r] & \mathrm{H}_i(F+eF,X) \arrow[r] & \mathrm{H}_{i-1}(F,X) \arrow[dll, rounded corners, "r"] \\
\mathrm{H}_{i-1}(F,X) \arrow[r] & \cdots &
\end{tikzcd} \tag{59}$$

where $F + eF$ is the mapping cone of the multiplication by r.

Proof. The multiplication by r map from R/I to itself induces a short exact sequence of R-complexes

$$0 \longrightarrow F \longrightarrow F + eF \longrightarrow \Sigma F \longrightarrow 0 \tag{60}$$

The short exact sequence (60) in turn induces the short exact sequence

$$0 \longrightarrow S \longrightarrow S + eS \longrightarrow \Sigma S \longrightarrow 0 \tag{61}$$

where we set $S = \mathrm{S}_R(F)$ and where we used the fact that the symmetric DG algebra functor commutes with mapping cones and shifts of complexes. Then after tensoring (61) with X and taking homology, we obtain the long exact sequence (59). $\square$

Corollary 6. *With the notation as above, we have*

$$\sum_{i=0}^{n} (-1)^n \ell(\mathrm{H}_i(F + eF, X)) = \ell(\mathrm{H}_n(F, X)).$$

4.6 Symmetric Powers of Chain Complexes

In this subsection, we describe a construction given by Tchernev (in [T95]) and explain how it is related to our construction. In particular, let X be an R-complex. We construct a *non-unital* symmetric DG algebra of X over R, denoted $\mathrm{C}_R(X)$ as follows: we begin with the non-unital tensor DG algebra of X over R, given by

$$\mathrm{U}_R(X) = \bigoplus_{n=1}^{\infty} X^{\otimes n}$$

where the tensor product is taken as R-complexes. Just as before, an elementary tensor in $U = \mathrm{U}_R(A)$ is denoted $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_n$ where $x_1, \ldots, x_n \in X$ and $n \geq 1$, and the differential of U is denoted by d again to simplify notation and is defined on $\boldsymbol{x}$ by

$$\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{n} (-1)^{|a_1| + \cdots + |a_{j-1}|} x_1 \otimes \cdots \otimes \mathrm{d}x_j \otimes \cdots \otimes x_n.$$

We say $\boldsymbol{x}$ is a homogeneous elementary tensor if each x_i is a homogeneous element in X. What is different this time is that we equip $U = \mathrm{U}_R(X)$ with a different bi-graded structure; namely we set

$$|\boldsymbol{x}| = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \deg \boldsymbol{x} = n.$$

Thus we make no distinction on whether or not $x_i \in X_0$ or $x_i \in X_{<0}$. With $|\cdot|$ and deg defined, we observe that U admits a bi-graded decomposition:

$$U = \bigoplus_{i \in \mathbb{Z}} U_i = \bigoplus_{n \geq 1} U^n = \bigoplus_{i,n} U_i^n,$$

where the component U_i^n consists of all finite R-linear combinations of homogeneous elementary tensors $\boldsymbol{x} \in U$ such that $|\boldsymbol{x}| = i$ and $\deg \boldsymbol{x} = n$. We equip U with an associative (but not commutative nor unital) bi-graded R-bilinear multiplication which is defined on homogeneous elementary tensors by $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \boldsymbol{x} \otimes \boldsymbol{x}'$ and is extended

R-bilinearly everywhere else. This multiplication is easily seen to satisfy Leibniz law, however note that U is not unital under this multiplication since $(1,1) \mapsto 1 \otimes 1 \neq 1$ (hence why we call this the *non-unital* tensor DG algebra).

Next let $\mathfrak{c} = \mathfrak{c}(X)$ be the U-ideal generated by all elements of the form

$$[x_1, x_2]_\sigma := (-1)^{|x_1||x_2|} x_2 \otimes x_1 - x_1 \otimes x_2 \quad \text{and} \quad [x]_\tau := x \otimes x,$$

where $x, x_1, x_2 \in X$ are homogeneous and $|x|$ is odd. We then define the **non-unital symmetric DG algebra** of X over R to be the quotient

$$\mathrm{C}_R(X) := U/\mathfrak{c}.$$

Since the generators of $\mathfrak{c}$ are homogeneous with respect to both homological and total degree, we see that $C = \mathrm{C}_R(X)$ inherits a bi-graded structure from U. In particular, if X is a positive R-complex (meaning $X_i = 0$ for all $i < 0$), then one has $C_0^n = \mathrm{Sym}_R^n(X_0)$. In general, we call C^n the n**th symmetric power** of X. The second symmetric power and its properties were studied in [FST08]. The next proposition helps clarify how our construction is related to Tchernev's construction:

Proposition 4.6. *Let A be an R-complex centered at R. Denote $S = \mathrm{S}_R(A)$ and $C = \mathrm{C}_R(A)$. We have $S^{\leq n} \cong C^n$ as R-complexes.*

Proof. Define $\varphi_h \colon S^{\leq n} \to C^n$, called **homogenization**, as follows: let $f \in S^{\leq n}$ and express it as $f = \sum_{k=0}^n f^k$ where f^k is the total degree k component of f. We set

$$\varphi_h(f) = 1^{n-1} \otimes f^0 + \sum_{k=1}^{n} 1^{\otimes(n-k)} \otimes f^k.$$

Conversely, define $\varphi_d \colon C^n \to S^{\leq n}$, called **dehomogenization**, as follows: we set

$$\varphi_d(1^{\otimes k} \otimes \boldsymbol{a}) = \boldsymbol{a}$$

where $\boldsymbol{a} \in A_+^{\otimes(n-k)}$ is a homogeneous elementary tensor. We extend φ_d everywhere else R-linearly. It is straightforward to check that both φ_h and φ_d are chain maps and are inverse to each other. $\square$

Let X be an R-complex. Denote $C = \mathrm{C}_R(X)$, $\mathfrak{c} = \mathfrak{c}(X)$, and $U = \mathrm{U}_R(X)$. There's an alternative description of C^n which is often useful. Let $\sigma = (ij)$ be a transposition in the symmetric group Σ_n and let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_n$ be a homogeneous elementary tensor in U. We set

$$\sigma \boldsymbol{x} = \begin{cases} 0 & \text{if } x_i = x_j \text{ and } |x_i| \text{ is odd} \\ (-1)^{|x_i||x_j|} x_1 \otimes \cdots x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n & \text{else.} \end{cases} \tag{62}$$

Then (62) extends to an action of the symmetric group Σ_n on U^n. In other words, U^n has the structure of an $R[\Sigma_n]$-module. With this understood, we have $C^n = (U^n)_{\Sigma_n}$. If R contains $\mathbb{Q}$ (or more generally a characteristic zero field), then the short exact sequence of R-complexes

$$0 \longrightarrow \mathfrak{c} \longrightarrow U \longrightarrow C \longrightarrow 0 \tag{63}$$

is split exact with splitting map $C \to U$ defined on homogeneous elementary products by

$$x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(x_1 \otimes \cdots \otimes x_n).$$

In particular, we may identify C^n with the R-subcomplex of U^n which is fixed by Σ_n in this case.

Proposition 4.7. *Assume that $\mathbb{Q} \subseteq R$. Let $\varphi, \psi \colon X \to X'$ be chain maps of R-complexes. Denote $C = \mathrm{C}_R(X)$, $C' = \mathrm{C}_R(X')$, $U = \mathrm{U}_R(X)$, and $U' = \mathrm{U}_R(X')$, and identify C and C' with the R-subcomplexes of U and U' fixed by the symmetric groups. If φ is homotopic to ψ, then $\varphi^{\otimes n}$ is homotopic to $\psi^{\otimes n}$ for each n. Moreover, we can choose a homotopy $h^n \colon U^n \to U'^n$ from $\varphi^{\otimes n}$ to $\psi^{\otimes n}$ which restricts to a homotopy $h^n|_C \colon C^n \to C'^n$ from $\varphi^{\otimes n}|_C$ to $\psi^{\otimes}|_C$.*

Proof. Let h be a homotopy from φ to ψ. For $n = 1$, we set $h^1 = h$. The case where $n = 2$ was shown in [FST08]. More generally, we set

$$h^n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma \left(\sum_{k=1}^{n-1} (\varphi^{\otimes(n-k)} \otimes h \otimes \psi^{\otimes k}) \right).$$

One checks that h^n is a homotopy from $\varphi^{\otimes n}$ to $\psi^{\otimes n}$ and by construction is restricts to a map from C^n to C'^n. $\square$

Proposition 4.8. *Assume that $\mathbb{Q} \subseteq R$. Let $\varphi, \psi \colon A \to A'$ be chain maps of R-complexes centered at R. Denote $S = \mathrm{S}_R(A)$ and $S' = \mathrm{S}_R(A')$, and let $\widetilde{\varphi}, \widetilde{\psi} \colon S \to S'$ be the lifts of φ and ψ from the universal mapping property. If φ is homotopic to ψ, then $\widetilde{\varphi}$ is homotopic to $\widetilde{\psi}$.*

4.7 The Symmetric DG Algebra of a Finite Free Complex over an Integral Domain

Throughout this subsection, we assume that R is an integral domain with quotient field K. Let F be an R-complex centered at R such that the underlying graded R-module of F is a finite and free as an R-module. Let $e_1, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $|e_j| > |e_i|$, then $j > i$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $R[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $R[\boldsymbol{e}]$. The reason we do not allow elements of odd degree to square to zero is because we will want to calculate the Gröbner basis of an ideal in $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is simpler when we do not have any zero-divisors. In any case, one recovers the symmetric DG R-algebra of F as below:

$$R[\boldsymbol{e}]/\langle \{ e_i^2 \mid |e_i| \text{ is odd} \} \rangle \simeq \mathrm{S}_R(F).$$

Finally, let $(\mu, \star)$ be a multiplication of F. Our goal is to compute the maximal associative quotient of F using the presentation given in Theorem (4.3) as well as the theory of Gröbner bases in $K[\boldsymbol{e}]$. Before we can do this, we need to introduce some notation for Gröbner basis applications in $K[\boldsymbol{e}]$. Our notation mostly follows [BE77] however we introduce some of our own notation as well.

4.7.1 Monomials and Monomial Orderings in $K[\boldsymbol{e}]$

A **monomial** in $K[\boldsymbol{e}]$ is an element of the form

$$e^{\boldsymbol{\alpha}} = e_1^{\alpha_1} \cdots e_n^{\alpha_n} \tag{64}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called the **multidegree** of $e^{\boldsymbol{\alpha}}$ and is denoted $\mathrm{multideg}(e^{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$. Similarly we define its **total degree**, denoted $\deg(e^{\boldsymbol{\alpha}})$, and its **homological degree** denoted $|e^{\boldsymbol{\alpha}}|$, by

$$\deg(e^{\boldsymbol{\alpha}}) = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad |e^{\boldsymbol{\alpha}}| = \sum_{i=1}^{n} \alpha_i |e_i|.$$

By convention we set $e^{\mathbf{0}} = 1$ where $\mathbf{0} = (0, \dots, 0)$ is the zero vector in $\mathbb{N}^n$. We define the **support** of $e^{\boldsymbol{\alpha}}$, denoted $\mathrm{supp}(e^{\boldsymbol{\alpha}})$, to be the set

$$\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{ e_i \mid e_i \text{ divides } e^{\boldsymbol{\alpha}} \} = \{ e_i \mid \alpha_i \neq 0 \}.$$

Note that if the support of $e^{\boldsymbol{\alpha}}$ is empty if and only if $e^{\boldsymbol{\alpha}} = 1$. If $e^{\boldsymbol{\alpha}}$ has non-empty support, then we define its **initial variable** and **terminal variable** to be the variables e_i and e_k respectively where

$$i = \inf\{ j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}}) \} \quad \text{and} \quad k = \max\{ j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}}) \}.$$

For instance, suppose that $\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{ e_{i_1}, \dots, e_{i_k} \}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then we can express (64) as

$$e^{\boldsymbol{\alpha}} = e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{k}},$$

and in this case, e_{i_1} is the initial variable of $e^{\boldsymbol{\alpha}}$ and e_{i_k} is the terminal variable of $e^{\boldsymbol{\alpha}}$.

Remark 8. Note how the ordering matters. In particular, if $i < j$ and both $|e_i|$ and $|e_j|$ are odd, then $e_j e_i$ is not a monomial in $K[\boldsymbol{e}]$ since it can be expressed as a non-trivial coefficient times a monomial:

$$e_j e_i = -e_i e_j.$$

On the other hand, if one of the e_i or e_j is even, then $e_j e_i$ is a monomial in $K[\boldsymbol{e}]$ since $e_j e_i = e_i e_j$.

We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ (the notation for this monomial ordering in Singular is Wp(w)). More specifically, given two monomials $e^{\boldsymbol{\alpha}}$ and $e^{\boldsymbol{\beta}}$ in $K[\boldsymbol{e}]$, we say $e^{\boldsymbol{\beta}} > e^{\boldsymbol{\alpha}}$ if either

1. $|e^{\boldsymbol{\beta}}| > |e^{\boldsymbol{\alpha}}|$ or;
2. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and $\beta_1 > \alpha_1$ or;
3. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Given a nonzero polynoimal $f \in K[\boldsymbol{e}]$, there exists unique $c_1, \dots, c_m \in K \backslash \{0\}$ and unique $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ where $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ for all $1 \leq i < j \leq m$ such that

$$f = c_1 e^{\boldsymbol{\alpha}_1} + \cdots + c_m e^{\boldsymbol{\alpha}_m} = \sum c_i e^{\boldsymbol{\alpha}_i} \tag{65}$$

The $c_i e^{\boldsymbol{\alpha}_i}$ in (65) are called the **terms** of f, and the $e^{\boldsymbol{\alpha}_i}$ in (65) are called the **monomials** of f. By reindexing the $\boldsymbol{\alpha}_i$ if necessary, we may assume that $e^{\boldsymbol{\alpha}_1} > \cdots > e^{\boldsymbol{\alpha}_m}$. In this case, we call $c_1 e^{\boldsymbol{\alpha}_1}$ the **lead term** of f, we call $e^{\boldsymbol{\alpha}_1}$ the **lead monomial** of f, and we call c_1 the **lead coefficient** of f. We denote these, respectively, by

$$\mathrm{LT}(f) = c_1 e^{\boldsymbol{\alpha}_1}, \quad \mathrm{LM}(f) = e^{\boldsymbol{\alpha}_1}, \quad \text{and} \quad \mathrm{LC}(f) = c_1.$$

The **multidegree** of f is defined to be the multidegree of its lead monomial $e^{\boldsymbol{\alpha}_1}$ and is denoted $\mathrm{multideg}(f) = \boldsymbol{\alpha}_1$. The **total degree** of f is defined to be the maximum of the total degrees of its monomials and is denoted

$$\deg(f) = \max_{1 \leq i \leq m} \{\deg(e^{\boldsymbol{\alpha}_i})\}.$$

We say f is **homogeneous** of homological degree i if each of its monomials is homogeneous of homological degree i. In this case, we say f has **homological degree** i and we denote this by $|f| = i$.

Proposition 4.9. *For each $1 \leq i \leq j \leq n$, let $f_{ij} = -[e_i, e_j] = e_i e_j - e_i \star e_j$. We have*

$$\mathrm{LT}(f_{ij}) = e_i e_j.$$

Proof. If $e_i \star e_j = 0$, then this is clear, otherwise term of $e_i \star e_j$ has the form $r_{i,j}^k e_k$ for some k where $r_{i,j}^k \neq 0$. Since $\star$ respects homological degree, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > |e_i|$ and $|e_k| > |e_j|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ and $k > j$ by our assumption on the ordering of $e_1, \dots, e_n$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j > e_k$. □

4.7.2 Gröbner Basis Calculations

Our goal is to use the theory of Gröbner bases to help us calculate

$$F^{\mathrm{as}} = \mathrm{S}_R(F)/\mathfrak{s}(\mu) \simeq R[\boldsymbol{e}]/\langle \{f_{i,j}\} \rangle,$$

where $f_{i,j} \in R[\boldsymbol{e}]$ are defined by

$$f_{i,j} = e_i e_j - e_i \star e_j = e_i e_j - \sum_k r_{i,j}^k e_k,$$

where the $r_{i,j}^k \in R$ are the entries of the matrix representation of μ with respect to the ordered homogeneous basis $e_1, \dots, e_n$. In order to do this though, we first need to base change to K because that is where the theory of Gröbner basis works best. Thus we wish to calculate:

$$F_K^{\mathrm{as}} := F^{\mathrm{as}} \otimes_R K \simeq K[\boldsymbol{e}]/\langle \{f_{i,j}\} \rangle.$$

To this end, let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let $\mathfrak{a}$ be the $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. We wish to construct a left Gröbner basis for $\mathfrak{a}$ (which will turn out to be a two-sided Gröbner basis) via Buchberger's algorithm (as described in [GP02]) using the monomial ordering described above. Suppose f, g are two nonzero polynomials in $K[\boldsymbol{e}]$ with $\mathrm{LT}(f) = re^{\boldsymbol{\alpha}}$ and $\mathrm{LT}(g) = se^{\boldsymbol{\beta}}$. Set $\boldsymbol{\gamma} = \mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and the left S-**polynomial** of f and g to be

$$\mathrm{S}(f, g) = e^{\boldsymbol{\gamma} - \boldsymbol{\alpha}} f \pm (r/s) e^{\boldsymbol{\gamma} - \boldsymbol{\beta}} g \tag{66}$$

where the $\pm$ in (66) is chosen to be $+$ or $-$, depending on which sign will cancel out the lead terms. We begin Buchberger's algorithm by calculating the S-polynomials of all pairs of polynomials in $\mathcal{F}$. In other words, we calculate all S-polynomials of the form $\mathrm{S}(f_{k,l}, f_{i,j})$ where $1 \leq i, j, k, l \leq n$. Note that if $k > l$, then

$$f_{l,k} = (-1)^{|e_k||e_l|} f_{k,l},$$

which implies

$$\mathrm{S}(f_{l,k}, f_{i,j}) = (-1)^{|e_k||e_l|} \mathrm{S}(f_{k,l}, f_{i,j}) = \pm \mathrm{S}(f_{i,j}, f_{k,l}).$$

Similarly, if $i \geq k$, then

$$\mathrm{S}(f_{i,j}, f_{l,k}) = \pm \mathrm{S}(f_{k,l}, f_{i,j}).$$

Thus we may assume that $j \geq i$ and $l \geq k \geq i$. Obviously we have $\mathrm{S}(f_{i,j}, f_{i,j}) = 0$ for each i, j, however something interesting happens when we calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $j > i$ and then divide this by $\mathcal{F}$

(where division by $\mathcal{F}$ means taking the left normal form of $\mathrm{S}(f_{j,k}, f_{i,j})$ with respect to $\mathcal{F}$ using the left normal form described in [GP02]). We have

$$\begin{aligned}\mathrm{S}(f_{j,k}, f_{i,j}) &= e_i(e_je_k - e_j \star e_k) - (e_ie_j - e_i \star e_j)e_k \\ &= (e_i \star e_j)e_k - e_i(e_j \star e_k) \\ &= \sum_l r^l_{i,j}e_le_k - \sum_l r^l_{j,k}e_ie_l \\ &\to \sum_l r^l_{i,j}e_l \star e_k - \sum_l r^l_{j,k}e_i \star e_l \\ &= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\ &= [e_i, e_j, e_k],\end{aligned}$$

where in the fourth line we did division by $\mathcal{F}$ (note that if $[e_i, e_j, e_k] \neq 0$, then $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$). Finally if $j > i$, $l > k$, and $j \neq k$, then we have

$$\begin{aligned}\mathrm{S}(f_{k,l}, f_{i,j}) &= e_ie_jf_{k,l} - f_{i,j}e_ke_l \\ &= (e_i \star e_j)e_ke_l - e_ie_j(e_k \star e_l) \\ &\to (e_i \star e_j) \star (e_k \star e_l) - (e_i \star e_l) \star (e_k \star e_l) \\ &= 0\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Next, suppose that

$$f = re_k + r'e_{k'} + \cdots + r''e_{k''} \in \langle F \rangle$$

where $r, r', r'' \in R$ with $r \neq 0$ and where $\mathrm{LM}(f) = e_k$. Then we have

$$\begin{aligned}\mathrm{S}(f, f_{j,k}) &= e_jf - rf_{j,k} \\ &= r'e_je_{k'} + \cdots + r''e_je_{k''} + re_j \star e_k \\ &\to r'e_j \star e_{k'} + \cdots + r''e_j \star e_{k''} + re_j \star e_k \\ &= e_j \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\ &= e_j \star f \\ &\in \langle F \rangle\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Similarly, we have if $i \neq k \neq j$, then we have

$$\begin{aligned}\mathrm{S}(f, f_{i,j}) &= e_ie_jf - rf_{i,j}e_k \\ &= r'(e_ie_j)e_{k'} + \cdots + r''(e_ie_j)e_{k''} + r(e_i \star e_j)e_k \\ &\to r'(e_i \star e_j) \star e_{k'} + \cdots + r''(e_i \star e_j) \star e_{k''} + r(e_i \star e_j) \star e_k \\ &= (e_i \star e_j) \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\ &= (e_i \star e_j) \star f \\ &\in \langle F \rangle.\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Finally suppose that

$$g = se_m + s'e_{m'} + \cdots + s''e_{m''} \in \langle F \rangle$$

where $s, s', s'' \in R$ with $s \neq 0$ and where $\mathrm{LM}(g) = e_m$. If $k = m$, then we have

$$s\mathrm{S}(f, g) = sf - rg \in \langle F \rangle.$$

On the other hand, if $k \neq m$, then we have

$$\begin{aligned}s\mathrm{S}(f, g) &= se_mf - rge_k \\ &= sr'e_me_{k'} + \cdots + sr''e_me_{k''} - rs'e_{m'}e_k - \cdots - rs''e_{m''}e_k \\ &\to sr'e_m \star e_{k'} + \cdots + sr''e_m \star e_{k''} - rs'e_{m'} \star e_k - \cdots - rs''e_{m''} \star e_k \\ &= se_m \star (r'e_{k'} + \cdots + r''e_{k''}) - r(s'e_{m'} + \cdots + s''e_{m''}) \star e_k \\ &= se_m \star (f - re_k) - r(g - se_m) \star e_k \\ &= se_m \star f + rg \star e_k - sre_m \star e_k + rse_m \star e_k \\ &= se_m \star f + rg \star e_k \\ &\in \langle F \rangle.\end{aligned}$$

It follows that we can construct a Gröbner basis

$$\mathcal{G} := \mathcal{F} \cup \{g_1, \ldots, g_m\}$$

of $\mathfrak{a}$ such that the g_i all belong to $\langle F \rangle$.

Example 4.7. In Example (1.1) we calculate the associator $[e_1, e_5, e_2]$ using the following Singular code:

```
LIB "ncalg.lib";

intvec v= 1:3, 2:5, 3:5;
ring A=(0,x,y,z,w),(e1,e2,e5,e12,e14,e23,e35,e45,e123,e124,e134,e234,e345),Wp(v);

matrix C[13][13]; matrix D[13][13]; int i; int j;
for (i=1; i<=13; i++) {for (j=1; j<=13; j++) {C[i,j]=(-1)^(v[i]*v[j]);}}
ncalgebra(C,D);

poly f(1)(5) = e1*e5-yz2*e14-x*e45;
poly f(1)(2) = e1*e2-e12;
poly f(2)(5) = e2*e5-y2z*e23-w*e35;
poly f(2)(45) = e2*e45+yz*e234-w*e345;
poly f(1)(35) = e1*e35-yz*e134+x*e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14+e124;
poly S(1)(5)(2) = f(1)(5)*e2+e1*f(2)(5);

ideal I = f(2)(14),f(2)(45),f(1)(23),f(1)(35),f(2)(5),f(1)(5);
reduce(S(1)(5)(2),b);

// [e1,e5,e2] = (y^2*z)*e123-(y*z^2)*e124+(y*z*w)*e134-(x*y*z)*e234
```

Appendix

5 Localization, Tensor, and Hom

Let A be an MDG R-algebra and let X and Y be MDG A-modules. In this subsection we define the tensor complex $X \otimes_A Y$ (which turns out to be an MDG A-module with the obvious A-scalar multiplication) as well as the hom complex $\mathrm{Hom}^\star_A(X, Y)$ (which need not be an MDG A-module using the naive A-scalar multiplication since this map need not be well-defined). Before defining these complexes however, we first discuss localization.

5.1 Localization

A subset $S \subseteq A$ is called **multiplicatively closed** if it satisfies the following conditions:

1. We have $1 \in S$ and if $s_1, s_2 \in S$ we have $s_1 s_2 \in S$.
2. Each $s \in S$ must be homogeneous of even degree.
3. We have $S \subseteq \mathrm{N}(A)$.

Given a multiplicatively closed subset $S \subseteq A$, we define an MDG R-algebra A_S, called the **localization of** A **at** S, as follows: as a set, A_S is given by

$$A_S := \{a/s \mid a \in A \text{ and } s \in S\}$$

where a/s denotes the equivalence class of $(a, s) \in A \times S$ with respect to the following equivalence relation:

$$(a, s) \sim (a', s') \text{ if and only if there exists } s'' \in S \text{ such that } s''s'a = s''sa'. \tag{67}$$

Notice how we are not bothering to put in parenthesis in (67) since each $s \in S$ belongs to the nucleus of A and thus associates with everything else. One can check that (67) is indeed an equivalence relation because every

$s \in S$ associates and commutes with everything else. We give A_S the structure of an R-module by defining addition and R-scalar multiplication on A_S by

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2 a_1 + s_1 a_2}{s_1 s_2} \quad \text{and} \quad r \cdot \frac{a}{s} = \frac{ra}{s}, \tag{68}$$

for all a/s, a_1/s_1, and a_2/s_2 in A_S, and for all $r \in R$. Again, (68) is well-defined since $S \subseteq \mathrm{N}(A) \cap \mathrm{Z}(A)$ where $\mathrm{Z}(A)$ is the center of A (the set of all elements which commutes with everything else). In fact, A_S is a graded R-module where the homogeneous component in degree $i \in \mathbb{Z}$, denoted $A_{S,i}$, is the R-span of all fractions of the form a/s where a is homogeneous and where $|a/s| := i = |a| - |s|$. We give A_S the structure of an R-complex by attaching to it the differential $\mathrm{d}_S \colon A_S \to A_S$ which is defined by

$$\mathrm{d}_S \left(\frac{a}{s} \right) = \frac{\mathrm{d}(a)s - (-1)^{|a|} a \mathrm{d}(s)}{s^2}$$

for all $a/s \in A_S$. A straightforward computation shows that $\mathrm{d}_S \colon A_S \to A_S$ is a graded R-linear map of degree -1 which satisfies $\mathrm{d}_S^2 = 0$, so d_S really is a differential. As usual, we denote d_S more simply by d if context is understood. Finally we give A_S the structure of an MDG R-algebra by defining the multiplication μ_S of A_S via the formula

$$\frac{a_1}{s_1} \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$$

for all a_1/s_1 and a_2/s_2 in A_S.

If X is an MDG A-module and $S \subseteq A$ is a multiplicatively closed set such that $S \subseteq \mathrm{N}_A(X)$, then we can also define an MDG A_S-module X_S, called **localization of X with respect to S**. The construction of X_S is almost identical to the construction of A_S, however we really do need to have $S \subseteq \mathrm{N}_A(X)$ (and not just $S \subseteq \mathrm{N}(A)$) in order for this construction to be well-defined). In particular, we cannot view localization as a functor

$$-_S \colon \mathbf{MDGmod}_A \to \mathbf{MDGmod}_{A_S}.$$

However if we consider the subcategory $\mathbf{MDGmod}_A^\star$ of $\mathbf{MDGmod}_A$, where the objects of $\mathbf{MDGmod}_A^\star$ are the MDG A-modules X such that $\mathrm{N}(A) \subseteq \mathrm{N}_A(X)$, then we do obtain a functor

$$-_S \colon \mathbf{MDGmod}_A^\star \to \mathbf{MDGmod}_{A_S}^\star.$$

5.2 Tensor

We now discuss the tensor complex $X \otimes_A Y$. The underlying graded R-module of $X \otimes_A Y$ in degree i is the R-span of homogeneous elementary tensors $x \otimes y$ where $|x| + |y| = i$ subject to the relations

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

for all $x_1, x_2, x \in X$ and $y_1, y_2, y \in Y$ as well as the relations

$$a(x \otimes y) = ax \otimes y = (-1)^{|a||x|} x \otimes ay \tag{69}$$

for all homogeneous $a \in A$, $x \in X$, and $y \in Y$. The differential of the tensor complex $X \otimes_A Y$ is defined on homogeneous elementary tensors $x \otimes y$ by

$$\mathrm{d}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{|x|} x \otimes \mathrm{d}(y).$$

The tensor complex $X \otimes_A Y$ inherits the structure of an MDG A-module where the A-scalar multiplication is defined via (69), thus $X \otimes_A Y$ is in fact an MDG A-module. A calculation shows that

$$[a_1, a_2, x \otimes y] = [a_1, a_2, x] \otimes y = (-1)^{|a_1 + a_2||x|} x \otimes [a_1, a_2, y]$$

for all homogeneous $a_1, a_2 \in A$ and for all homogeneous elementary tensors $x \otimes y \in X \otimes_A Y$. In particular, if either X or Y is associative, then $X \otimes_A Y$ is associative. Here is an important warning to keep in mind when dealing with tensor complexes however: the map $\varphi \colon A \otimes_A X \to X$ defined by $\varphi(a \otimes x) = ax$ is *not* well-defined if X is not associative. Indeed, suppose $[a_1, a_2, x] \neq 0$. Then

$$\begin{aligned} 0 &= \varphi(0) \\ &= \varphi(a_1 a_2 \otimes x - a_1 \otimes a_2 x) \\ &= [a_1, a_2, x] \\ &\neq 0 \end{aligned}$$

shows that φ is not well-defined. More generally, given an MDG A-ideal $\mathfrak{a}$, the map $A/\mathfrak{a} \otimes_A X \to X/\mathfrak{a}X$, defined on elementary tensors by $\overline{a} \otimes x \mapsto \overline{ax}$, is only well-defined if $[X] \subseteq \mathfrak{a}X$. Similarly, given a multiplicative subset $S \subseteq \mathrm{N}(A) \cap \mathrm{N}(X)$, the map $A_S \otimes_A X \to X_S$, defined on elementary tensors by $(a/1) \otimes x \mapsto ax/1$, is only well-defined if $[X]_S = 0$.

5.3 Hom

Next we discuss the hom complex $\mathrm{Hom}_A^\star(X,Y)$. The hom complex $\mathrm{Hom}_A^\star(X,Y)$ is the R-complex whose underlying graded module in degree $i \in \mathbb{Z}$ is

$$\mathrm{Hom}_A^\star(X,Y)_i := \{\varphi\colon X \to Y \mid \varphi \text{ is a graded } A\text{-module homomorphism of degree } i\}.$$

A graded A-module homomorphism of degree $i := |\varphi|$ is a graded linear map $\varphi\colon X \to Y$ of degree $|\varphi|$ which satisfies $\varphi(ax) = (-1)^{|a||\varphi|}a\varphi(x)$ for all homogeneous $a \in A$ and $x \in X$. The differential of $\mathrm{Hom}_A^\star(X,Y)$ is denoted $\mathrm{d}^\star$ and is defined on homogeneous $\varphi \in \mathrm{Hom}_A^\star(X,Y)$ by

$$\mathrm{d}^\star(\varphi) = \mathrm{d}\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}.$$

Note that $\mathrm{d}^\star(\varphi)$ really is a graded A-module homomorhism of degree $|\varphi| - 1$! Indeed, for all homogeneous $a \in A$ and $x \in X$, we have

$$\begin{aligned}
\mathrm{d}^\star(\varphi)(ax) &= (\mathrm{d}\varphi)(ax) - (-1)^{|\varphi|}(\varphi\mathrm{d})(ax) \\
&= (-1)^{|a||\varphi|}\mathrm{d}(a\varphi(x)) - (-1)^{|\varphi|}\varphi(\mathrm{d}(a)x) - (-1)^{|\varphi|+|a|}\varphi(a\mathrm{d}(x)) \\
&= (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) + (-1)^{|a||\varphi|+|a|}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|\varphi|(|a|+1)}\mathrm{d}(a)\varphi(x) - (-1)^{|\varphi|+|a|+|a||\varphi|}a\varphi(\mathrm{d}(x)) \\
&= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|a|(|\varphi|+1)}a\varphi(\mathrm{d}(x)) + (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) - (-1)^{|a||\varphi|}\mathrm{d}(a)\varphi(x) \\
&= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x)) - (-1)^{|\varphi|+|a|(|\varphi|+1)}a(\varphi\mathrm{d}(x)) \\
&= (-1)^{|a|(|\varphi|+1)}a(\mathrm{d}\varphi(x) - (-1)^{|\varphi|}\varphi\mathrm{d}(x)) \\
&= (-1)^{|a|(|\varphi|-1)}a\mathrm{d}^\star(\varphi)(x).
\end{aligned}$$

The hom complex $\mathrm{Hom}_A^\star(X,Y)$ does not necessarily inherit the structure of an MDG A-module where the A-scalar multiplication is defined by $\varphi \mapsto a\varphi$ where $a\varphi\colon X \to Y$ is defined by

$$(a\varphi)(x) = (-1)^{|a||\varphi|}\varphi(ax) = a\varphi(x)$$

for all $x \in X$. Indeed, given homogeneous $a_1, a_2 \in A$ we have

$$\begin{aligned}
(a_1\varphi)(a_2x) &= a_1\varphi(a_2x) \\
&= (-1)^{|a_2||\varphi|}a_1(a_2\varphi(x)) \\
&= (-1)^{|a_2||\varphi|}(a_1a_2)\varphi(x) - (-1)^{|a_2||\varphi|}[a_1,a_2,\varphi(x)] \\
&= (-1)^{|a_2||\varphi|+|a_1||a_2|}(a_2a_1)\varphi(x) - (-1)^{|a_2||\varphi|}[a_1,a_2,\varphi(x)] \\
&= (-1)^{|a_2||\varphi|+|a_1||a_2|}a_2(a_1\varphi(x)) + (-1)^{|a_2||\varphi|+|a_1||a_2|}[a_2,a_1,\varphi(x)] - (-1)^{|a_2||\varphi|}[a_1,a_2,\varphi(x)]
\end{aligned}$$

for all $x \in X$. If we knew that

$$[a_1,a_2,\varphi(x)] = (-1)^{|a_1||a_2|}[a_2,a_1,\varphi(x)], \tag{70}$$

then we could continue the calculation and conclude that $a_1\varphi$ is A-linear, however we need not have the identity (70) in general. However recall that the identity (70) holding for all $a_1, a_2 \in A$ is equivalent to the condition that $\varphi(x) \in \mathrm{M}(Y)$. Therefore if we knew that φ landed in $\mathrm{M}(Y)$, then $a_1\varphi$ would be A-linear.

Just as in the case of the tensor product where it need not be true that $A \otimes_A X \simeq X$, it need not be the case that $\mathrm{Hom}_A^\star(A,X) \simeq X$. In fact, we have

$$\mathrm{Hom}_A^\star(A,X) \simeq \mathrm{N}(X).$$

Indeed, suppose $\varphi \in \mathrm{Hom}_A^\star(A,X)$ and suppose $\varphi(1) = x$. Thus by A-linearity of φ, we have $\varphi(a) = (-1)^{|a||\varphi|}ax$ for all $a \in A$. Note that

$$\begin{aligned}
0 &= \varphi([a_1,a_2,1]) \\
&= [a_1,a_2,\varphi(1)] \\
&= [a_1,a_2,x]
\end{aligned}$$

for all $a_1, a_2 \in A$ forces $x \in \mathrm{N}(X)$.

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