Non-Existence of DG Algebra Structures

Let R be a noetherian ring, let I be an ideal of R, and let F be the minimal free resolution of R/I over R. A chain map $\mu \in F^{\otimes 2} \to F$ which lifts the multiplication map on R/I is unique up to homotopy. What this means is that if $\mu' \in F^{\otimes 2} \to F$ is another chain map which lifts the multiplication map on R/I, then there exists a graded R-linear map $h\colon F^{\otimes 2} \to F$ of degree one such that $\mu' = \mu_h$ where

$$\mu_h := \mu + \mathrm{d}h + h\mathrm{d}.$$

If both μ and μ_h are graded-commutative, then $h\sigma\colon F^{\otimes 2} \to F$ must be a chain map of degree 1, where $\sigma\colon F^{\otimes 2} \to F^{\otimes 2}$ is defined by

$$\sigma(a_1 \otimes a_2) = a_1 \otimes a_2 - (-1)^{|a_1||a_2|} a_2 \otimes a_1$$

for all homogeneous $a_1, a_2 \in F$. Indeed, since μ_h and μ are graded-commutative, we have

$$\begin{aligned} \mathrm{d}h\sigma + h\sigma\mathrm{d} &= \mathrm{d}h\sigma + h\mathrm{d}\sigma \\ &= (\mathrm{d}h + h\mathrm{d})\sigma \\ &= (\mu_h - \mu)\sigma \\ &= \mu_h\sigma - \mu\sigma \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

Similarly, if both μ and μ_h are unital, then $h|_{F\otimes 1}$ and $h|_{1\otimes F}$ must be chain maps of degree 1. Finally, note that the associator for μ_h is given by

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d} \tag{1}$$

where $H = \overline{[\cdot]}_{\mu,h} + [\cdot]_{h,\mu_h}$. Here, we set

$$\overline{[\cdot]}_{\mu,h} = \mu(h \otimes 1 - \overline{1} \otimes h) \quad \text{and} \quad [\cdot]_{h,\mu_h} = h(\mu_h \otimes 1 - 1 \otimes \mu_h)$$

where $\overline{1}\colon F \to F$ is the map defined by $\overline{1}(a) = (-1)^{|a|}a$ for all homogeneous $a \in A$. Note that we can break $[\cdot]_{h,\mu_h}$ further as

$$[\cdot]_{h,\mu_h} = [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}}$$

where

$$[\cdot]_{h,\mu} = h(\mu \otimes 1 - 1 \otimes \mu), \quad [\cdot]_{h,\mathrm{d}h} = h(\mathrm{d}h \otimes 1 - 1 \otimes \mathrm{d}h), \quad \text{and} \quad [\cdot]_{h,h\mathrm{d}} = h(h\mathrm{d} \otimes 1 - 1 \otimes h\mathrm{d}).$$

Theorem 0.1. *Let $R = \mathbb{k}[x,y,z,w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, yz$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure. In particular, any multiplication on F will be non-associative at the triple $(\varepsilon_1, \varepsilon_{45}, \varepsilon_2)$.*

Proof. Let μ be the usual multiplication and let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F. We claim that $[\varepsilon_1, \varepsilon_{45}, \varepsilon_5]_{\mu_h} \neq 0$. Indeed, the idea is that on the one hand we have $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_\mu = -x\varepsilon_{12345}$ but on the other hand we have

$$(\mathrm{d}H + H\mathrm{d})(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) \in IF$$

where H is the map described in (1) and where $I = \langle x^2, y, z, w \rangle$. In particular, $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_{\mu_h} \not\equiv 0$ modulo IF which implies $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_{\mu_h} \neq 0$. To see this, first note that $\mathrm{d}H(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) = 0$, so we only need to show that

$$H\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) = (\overline{[\cdot]}_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) \in IF.$$

Now clearly we have

$$\operatorname{im}([\cdot]_{h,\mathrm{d}h})\mathrm{d}) \in \mathfrak{m}^2 F \subseteq IF \quad \text{and} \quad \operatorname{im}([\cdot]_{h,h\mathrm{d}})\mathrm{d}) \in \mathfrak{m}^2 F \subseteq IF,$$

where $\mathfrak{m} = \langle x, y, z, w \rangle$, since F is minimal and since the differential shows up twice in each case. Next note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) &\equiv x^2[1 \otimes \varepsilon_{45} \otimes \varepsilon_2]_{h,\mu} - x[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]_{h,\mu} + z[\varepsilon_1 \otimes \varepsilon_4 \otimes \varepsilon_2]_{h,\mu} + w^2[\varepsilon_1 \otimes \varepsilon_{45} \otimes 1]_{h,\mu} \\
&\equiv -x[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]_{h,\mu} \\
&\equiv -xh((z\varepsilon_{14} + x\varepsilon_{45}) \otimes \varepsilon_2 - \varepsilon_1 \otimes (z\varepsilon_{23} + y\varepsilon_{35})) \\
&\equiv 0.
\end{aligned}$$

Similarly in F/IF we have

$$\begin{aligned}
\overline{[\cdot]}_{\mu,h}\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) &\equiv x^2\overline{[1 \otimes \varepsilon_{45} \otimes \varepsilon_2]}_{\mu,h} - x\overline{[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]}_{\mu,h} + z\overline{[\varepsilon_1 \otimes \varepsilon_4 \otimes \varepsilon_2]}_{\mu,h} + w^2\overline{[\varepsilon_1 \otimes \varepsilon_{45} \otimes 1]}_{\mu,h} \\
&\equiv -x\overline{[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]}_{\mu,h} \\
&\equiv 0
\end{aligned}$$

where we used the fact that $\varepsilon_1 F_3 \in \mathfrak{m}F_4$ and $\varepsilon_2 F_3 \in \mathfrak{m}F_4$. $\square$

Theorem 0.2. *Let $R = \Bbbk[x, y, z, w]$ where $\operatorname{char} \Bbbk = 2$, let $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure. In particular, every MDG R-algebra will be non-associative at the triple (e_{12}, e_5, e_2).*

Proof. Let μ be the usual multiplication and let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F. We claim that $[e_{12}, e_5, e_2]_{\mu_h} \neq 0$. Indeed, first note that $[e_{12}, e_5, e_2]_\mu = x^2yze_{1234}$. We will show that

$$(\mathrm{d}H + H\mathrm{d})(e_{12} \otimes e_5 \otimes e_2) \in IF$$

where H is the map described in (1) and where $I = \langle x^3, y^2, z^2, w \rangle$. Again we have $\mathrm{d}H(e_{12} \otimes e_5 \otimes e_2) = 0$, so we only need to show that

$$H\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) = ([\cdot]_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) \in IF$$

First note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,\mu} + w^2[e_1, e_5, e_2]_{h,\mu} + y^2z^2[e_{12}, 1, e_2]_{h,\mu} + w^2[e_{12}, e_5, 1]_{h,\mu} \\
&\equiv x^2[e_2, e_5, e_2]_{h,\mu} \\
&\equiv x^2h((y^2ze_{23} + we_{35}) \otimes e_2 + e_2 \otimes (y^2ze_{23} + we_{35})) \\
&\equiv 0
\end{aligned}$$

Next in F/IF we have

$$\begin{aligned}
[\cdot]_{\mu,h}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{\mu,h} + w^2[e_1, e_5, e_2]_{\mu,h} + y^2z^2[e_{12}, 1, e_2]_{\mu,h} + w^2[e_{12}, e_5, 1]_{\mu,h} \\
&\equiv x^2[e_2, e_5, e_2]_{\mu,h} \\
&\equiv x^2(e_2h(e_5 \otimes e_2) + h(e_2 \otimes e_5)e_2) \\
&\equiv 0,
\end{aligned}$$

where we used the fact that $e_2F_3 \in wF_3$. Next in F/IF we have

$$\begin{aligned}
[\cdot]_{h,h\mathrm{d}}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} + w^2[e_1, e_5, e_2]_{h,h\mathrm{d}} + y^2z^2[e_{12}, 1, e_2]_{h,h\mathrm{d}} + w^2[e_{12}, e_5, 1]_{h,h\mathrm{d}} \\
&\equiv x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} \\
&\equiv x^2h(h\mathrm{d}(e_2 \otimes e_5) \otimes e_2 + e_2 \otimes h\mathrm{d}(e_5 \otimes e_2)) \\
&\equiv 0,
\end{aligned}$$

where we used the fact that $\mathrm{d}e_2 = w^2$ and $\mathrm{d}e_5 = y^2z^2$. Next in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mathrm{d}h}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,\mathrm{d}h} + w^2[e_1, e_5, e_2]_{h,\mathrm{d}h} + y^2z^2[e_{12}, 1, e_2]_{h,\mathrm{d}h} + w^2[e_{12}, e_5, 1]_{h,\mathrm{d}h} \\
&\equiv x^2[e_2, e_5, e_2]_{h,\mathrm{d}h}
\end{aligned}$$

We claim that $[e_2, e_5, e_2]_{h,h\mathrm{d}} \in JF_4$ where $J = \langle w^2, y^2z^2 \rangle$. Once we establish this, the proof will be complete as this implies $[e_2, e_5, e_2]_{h,\mathrm{d}h} \in IF$. Recall that for any $a_1, a_2 \in F$ we have

$$\mathrm{d}h(a_1 \otimes a_2) = \mathrm{d}h(a_2 \otimes a_1) + h\sigma\mathrm{d}(a_1 \otimes a_2).$$

In particular, in F/JF we have

$$\begin{aligned} \mathrm{d}[e_2, e_5, e_2]_{h,\mathrm{d}h} &\equiv \mathrm{d}h(\mathrm{d}h(e_2 \otimes e_5) \otimes e_2 + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv \mathrm{d}h(\mathrm{d}h(e_5 \otimes e_2) \otimes e_2 + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv \mathrm{d}h(e_2 \otimes \mathrm{d}h(e_5 \otimes e_2) + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv 0. \end{aligned}$$

where we used the fact that $\mathrm{d}e_5 = y^2z^2$ and $\mathrm{d}e_2 = w^2$. Now note that

$$\mathrm{H}(F_4/JF_4) = \mathrm{Tor}_4^R(R/I, R/J) = 0$$

Thus we must have $[e_2, e_5, e_2]_{h,\mathrm{d}h} \in JF_4$. $\square$