MATH 8610 (SPRING 2023) HOMEWORK 10

Assigned 04/26/23, due 05/06/23 (Saturday) by 11:59pm.

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1. **[Q1]** (10 pts). Show that SOR fails to converges for any matrix if $\omega \le 0$ or $\omega \ge 2$. (Hint: first, show that SOR iteration can be written as

$$(I - \omega D^{-1}E)x_{k+1} = \left[(1-\omega)I + \omega D^{-1}F\right] x_k + \omega D^{-1}b$$

so that the iteration matrix of SOR is $(I - \omega D^{-1}E)^{-1}\left[(1-\omega)I + \omega D^{-1}F\right]$. Now let $\{\lambda_j\}$ denote the eigenvalues of this iteration matrix. We can show that $\left|\Pi_{j=1}^{n}\lambda_j\right| = \det(I-\omega D^{-1}E)^{-1} \cdot \det\left[(1-\omega)I + \omega D^{-1}F\right] = \det\left[(1-\omega)I + \omega D^{-1}F\right] = |1-\omega|^n$, so that at least one eigenvalue λ_j satisfies $|\lambda_j| \ge |1-\omega|$)

2. **[Q2]** (25 pts) Consider an $n \times n$ symmetric tridiagonal matrix of the form

$$T(\alpha) = \begin{bmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & -1 \\ & & & -1 & \alpha \end{bmatrix}$$

where α is a real parameter.

(a) Verify that the eigenvalues of $T(\alpha)$ are given by $\lambda_k = \alpha - 2\cos\frac{k\pi}{n+1}$ $(k = 1, \dots, n)$, and that an eigenvector associated with λ_k is $v_k = \left[\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \dots, \sin\frac{n\pi}{n+1}\right]^T$. Under what condition on α is $T(\alpha)$ positive definite?

(b) Let $\alpha = 2$. Show that $T(2)$ is obtained by setting up a uniform mesh on $[a, b]$, namely, $x_k = a + k\frac{b-a}{n+1}$ $(0 \le k \le n+1)$ and applying the 2nd order centered finite difference approximation for the 1D Poisson equation $-u''(x) = f(x)$ with Dirichlet boundary condition $u(a) = u_0$ and $u(b) = u_{n+1}$ (both values given).

(c) Does the Jacobi iteration converge for $T(2)$? If so,what is the convergence factor?

(d) Does Gauss-Seidel converge for $T(2)$? If so,what is the convergence factor?

(e) For which values of ω does the SOR iteration converge for $T(2)$?