Advanced Mathematical Programming Project

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Introduction

An index is a number that represents the aggregate value of a group of items. In particular, a financial index is composed of a collection of assets, such as stocks or bonds, which captures the value of a specific market or a segment of it. A stock or a bond market index is effectively equivalent to a hypothetical portfolio of assets in the sense that we cannot invest directly on it, i.e., an index is not a financial instrument that we can trade.

Example 0.1. The S&P 500 Index, or Standard & Poor's 500 Index, is a market-capitalization-weighted index of 500 leading publicly traded companies in the U.S. It is not an exact list of the top 500 U.S. companies by market cap because there are other criteria that the index includes. Still, the S&P 500 index is regarded as one of the best gauges of prominent American equities' performance, and by extension, that of the stock market overall.

Index Tracking

It is not possible to trade an index directly. In order to gain access to an index we need to use other financial instruments such as options, futures, and exchange traded funds (ETFs), or create a portfolio of assets that tracks closely a given index.

Example 0.2. The SPDR S&P 500 ETF Trust, also known as the SPY ETF, is one of the most popular funds that aims to track the Standard & Poor's (S&P) 500 index. These stocks are selected by a committee based on market size, liquidity, and industry.

Its top 10 holdings are heavily weighted in technology companies such as Apple, Microsoft, and Amazon. Approximately one-quarter of the SPY ETF is invested in the technology sector.

SPY ETF's Top 10 Holdings (as of January 2022)

Holding (Company)	% SPY Portfolio Weight
Apple Inc. (AAPL)	6.98%
Microsoft Corp. (MSFT)	6.19%
Amazon.com Inc. (AMZN)	3.66%
Tesla Inc. (TSLA)	2.40%
Alphabet Inc. — Class A (GOOGL)	2.15%
Alphabet Inc. — Class C (GOOG)	2.00%
Meta Platforms (Facebook) (FB)	1.97%
NVIDIA Corp. (NVDA)	1.85%
Berkshire Hathaway Inc. — Class B (BRK.B)	1.36%
JPMorgan Chase & Co. (JPM)	1.18%

The Goal of the Paper

With this background material understood, here's the abstract of the paper that my project is based on, which perfectly summarizes the goal of that work:

> "We consider the problem of reproducing the performance of a stock market index, but without purchasing all of the stocks that make up the index, index tracking. We also consider the problem of outperforming the index, enhanced indexation. We present mixed-integer linear programming formulations of these problems. Our formulations include transaction costs, a constraint limiting the number of stocks that can be purchased and a limit on the total transaction cost that can be incurred. As our formulations of these problems are mixed-integer linear programs we can use a standard solver (Cplex). Numeric results are presented for eight data sets drawn from major markets. The largest of these data sets involves over 2000 stocks"

In particular, we want to create our own Tracking Portfolio (denoted TP) which is optimized with respect to some mixed-integer linear program. We use regression analysis in order to determine how successful our TP is. For instance, if X represents the monthly return of the S&P 500 and Y represents the monthly return of our TP, then the least squares regression line (where we are regressing the returns from the portfolio against the returns from the index) has the form

$$Y = \alpha + \beta X.$$

Note here that if our TP perfectly tracks the index, then we would have a regression intercept (alpha) of zero, and a regression slope (beta) of one. Enhanced indexation (sometimes referred to as enhanced index tracking) aims to reproduce the performance of a stock market index, but to generate excess return (return over and above the return achieved by the index). One phrase often encountered with regard to enhanced indexation is "adding alpha".

Problem Formulation

Let us now introduce notation which is used in the paper. We observe over time $0, 1, 2, \ldots, T$ the value of N stocks, as well as the index we are tracking. We are interested in deciding the best set of K stocks to hold (where $K < N$), as well as their appropriate quantities. Let

$$\begin{aligned}
\varepsilon_i &= \text{ be the minimum proportion of the TP that must be held in stock } i \text{ if any of the stock is held} \\
\delta_i &= \text{ be the maximum proportion of the TP that can be held in stock } i \text{ if any of the stock is held} \\
X_i &= \text{ be the number of units of stock } i \text{ in the current TP} \\
V_{i,t} &= \text{ be the value (price) of one unit of stock } i \text{ at time } t \\
I_t &= \text{ be the value of the index at time } t \\
R_t &= \text{ be the single period continuous time return for the index at time } t\text{, i.e. } R_t = \ln(I_t/I_{t-1}) \\
r_{i,t} &= \text{ be the single period continuous time return for stock } i \text{ at time } t\text{, i.e. } r_{i,t} = \ln(V_{i,t}/V_{i,t-1}) \\
C &= \text{ the total value } (\geq 0) \text{ of the current TP at time } T\text{, i.e. } C = \sum_{i=1}^{n} X_i V_{i,T} \text{ plus cash change ;} \\
f_i^s &= \text{ be the fractional cost of selling one unit of stock } i \text{ at time } T \\
f_i^b &= \text{ be the fractional cost of buying one unit of stock } i \text{ at time } T \\
\gamma &= \text{ be the limit } (0 \leq \gamma \leq 1) \text{ on the proportion of } C \text{ that can be consumed by transaction cost}
\end{aligned}$$

Decision Variables

Our decision variables are:

$$\begin{aligned}
x_i &= \text{ the number of units } (\geq 0) \text{ of stock } i \text{ that we choose to hold in the new TP} \\
G_i &= \text{ the transaction cost } (\geq 0) \text{ incurred in selling/buying stock } i \\
z_i &= \; = 1 \text{ if any stock of } i \text{ is held in the new TP, } = 0 \text{ otherwise}
\end{aligned}$$

Without significant loss of generality (since the sums of money involved are large) we allow x_i to take fractional values.

Constraints associated with index tracking problem

The constraints associated with the index tracking problem are:

$$\sum_{i=1}^{N} z_i = K \tag{1}$$

$$\varepsilon_i z_i \leq x_i V_{i,T} \leq \delta_i z_i \qquad 1 \leq i \leq N \tag{2}$$

$$G_i \geq f_i^s (X_i - x_i) V_{i,T} \qquad 1 \leq i \leq N \tag{3}$$

$$G_i \geq f_i^b (x_i - X_i) V_{i,T} \qquad 1 \leq i \leq N \tag{4}$$

$$\sum_{i=1}^{N} G_i \leq \gamma C \tag{5}$$

$$\sum_{i=1}^{N} x_i V_{i,T} = C - \sum_{i=1}^{N} G_i \tag{6}$$

$$x_i, G_i \geq 0 \qquad 1 \leq i \leq N \tag{7}$$

$$z_i \in [0, 1] \qquad 1 \leq i \leq N \tag{8}$$

Equation (1) ensures that there are exactly K stocks in the new TP. Equation (2) ensures that if a stock i is not in the new TP ($z_i = 0$) then x_i is also zero; it also ensures that if the stock is chosen to be in the new TP ($z_i = 1$) then the amount of the stock held satisfies the proportion limits defined. Equations (3) and (4) define the transaction cost and Equation (5) limits the total transaction cost incurred. Equation (6) is a balance constraint such that the total value of the new TP at time T equals the value of the current TP at time T plus the cash change (i.e. C) minus the total transaction cost.

Index Tracking Objective

Adopting the regression viewpoint for index tracking we have that ideally our TP would be chosen such that when we perform a regression of TP returns against index returns we would find that the regression line has an intercept (alpha) of zero, and a slope (beta) of one. However achieving this ideal is not straightforward. There are a number of complications to achieving this ideal within the context of a linear (or linearisable) formulation, as will become apparent below.

The single period continuous time return for the TP in period t is given by

$$\ln\left(\sum_{i=1}^{N} x_i V_{i,t} / \sum_{i=1}^{N} x_i V_{i,t-1}\right).$$

This is a nonlinear function of the decision variables. In the paper, the authors show how to approximate this nonlinear function by a linear one; namely the (linear approximate) return for the TP in period t is given by

$$\sum_{i=1}^{N} w_i r_{i,t}$$

where $r_{i,t}$ represents the return for stock i at time t and is given by the formula $r_{i,t} = \ln(V_{i,t}/V_{i,t-1})$ and where w_i represents the proportion invested in stock i at time T and is given by

$$w_i = v_i x_i \qquad 1 \leq i \leq N \tag{9}$$

where $v_i = V_{i,T}/(C - \gamma C)$ (see the Appendix for more details as to how this is done). Using the log-sum inequality it is possible to prove (assuming we spend γC in transaction cost) that the average return from this approximation

$$\sum_{t=1}^{T} \sum_{i=1}^{N} w_i r_{i.t} / T$$

over-estimates average tracking portfolio

$$\sum_{t=1}^{T} \ln\left(\sum_{i=1}^{N} x_i V_{i,t} / \sum_{i=1}^{N} x_i V_{i,t-1}\right) / T$$

If we regress these TP returns from standard regression theory that the ordinary least-squares estimates, $\widehat{\alpha}$ and $\widehat{\beta}$, for the intercept and slope of the regression line are given by:

$$\alpha = \sum_{i=1}^{N} w_i \alpha_i \tag{10}$$

$$\beta = \sum_{i=1}^{N} w_i \beta_i \tag{11}$$

where α_i and β_i are the ordinary least-squares regression intercept and slope when we regress the returns from stock i ($r_{i,t}$) against the index returns (R_t). Ideally we would like, for index tracking, to choose K stocks and their associated quantities x_i for $1 \leq i \leq N$ such that we achieve $\alpha = 0$ and $\beta = 1$. For real life data this may not, however, be achievable.

Mixed-Integer Linear Program

In order to acheive this, the authors adopted a two stage approach: where the primary objective is to achieve the desired slope of one and the secondary objective is to achieve the desired intercept of zero, i.e.

$$\text{first minimize } |\alpha| \text{ and then minimize } |\beta - 1|. \tag{12}$$

The modulus objectives above are not linear, however there is a standard way to linearize them: introduce variables D and E where

$$D \geq \alpha \tag{13}$$
$$D \geq -\alpha \tag{14}$$
$$E \geq \beta - 1 \tag{15}$$
$$E \geq -(\beta - 1) \tag{16}$$
$$D, E \geq 0 \tag{17}$$

Then our full mixed-integer linear programming formulation for solving the index tracking problem in the first-stage (primary objective to achieve the desired intercept of zero) is

$$\begin{aligned} &\text{minimize } D \\ &\text{subject to Equations 1-11, 13-14, and 17} \end{aligned}$$

In the second-stage, where the emphasis is on achieving the desired slope of one, then when we solve we constrain the intercept α so that it retains the value that it achieved at the first-stage. Formally let α^{opt} be the numeric value for α when our first minimization problem is solved. Then in the second-stage we

$$\begin{aligned} &\text{minimize } E \\ &\text{subject to Equations 1-11, 15-16, and 17} \\ &\qquad\qquad \alpha = \alpha^{\text{opt}} \end{aligned}$$

Our goal is to formulate this MILP using MATLAB. In particuar, we will formulate the first-step.

MATLAB Function

We write a MATLAB function denoted function [R, r, C, a, b, v, alpha, beta] = TrackingPortfolio(I,V, X, cash). Here, $\boldsymbol{v} = (v_1, \ldots, v_N)^\top$ is the vector with $v_i = V_{i,T}/(C - \gamma C)$ (in particular, $\boldsymbol{w} = (w_1, \ldots, w_N)$ is the vector with $w_i = v_i x_i$ where w_i represents the proportion invested in stock i at time T). We also have $\boldsymbol{a} = (a_1, \ldots, a_N)^\top$ and $\boldsymbol{b} = (b_1, \ldots, b_N)^\top$ where $a_i = \alpha_i$ and $b_i = \beta_i$ are the ordinary least-squares regression intercept and slope when we regress the returns $\boldsymbol{r}_i$ from stock i against the returns $\boldsymbol{R}$ of the S&P 500. Thus

$$\alpha = \boldsymbol{a}^\top \boldsymbol{w} = \sum_{i=1}^{N} a_i v_i x_i \quad \text{and} \quad \beta = \boldsymbol{b}^\top \boldsymbol{w} = \sum_{i=1}^{N} b_i v_i x_i,$$

where MATLAB stores these functions as vectors: $\alpha = (v_1 a_1, \ldots, v_N a_N)^\top$ and $\beta = (v_1 b_1, \ldots, v_N b_N)^\top$. The code is given below:

```
function [R,r,C,a,b,Vf,v,alpha,beta] = TrackingPortfolio(I,V,X,cash,gamma)

% I(t) = value of S&P 500 at time t
% V(i,t) = value of stock i at time t.
% X(i) = number of shares of stock i in the current TP.
% cash = either new cash to be invested or cash to be taken out.
% N = number of stocks
% T = final time.
% R(t) = return of S&P at time t.
% r(i,t) = return of stock i at time t.
% C = total value of the current TP at time T.
% regressing r(i,:) against R.
% p has the form p = bx + a.

sz = size(V);
N = sz(1);
T = sz(2);
Vf = V(:,T);
R = zeros(1,T-1);
r = zeros(N,T-1);
a = zeros(N,1);
b = zeros(N,1);
v = zeros(N,1);
p = zeros(N,2);
C = dot(X,Vf) + cash;

for t = 1:T-1
    R(1,t) = log(I(t+1)/I(t));
end;
for i = 1:N
      for t = 1:T-1
         r(i,t) = log(V(i,t+1)/V(i,t));
      end;
end;
for i=1:N
   p(i,:) = polyfit(R,r(i,:),1);
   a(i) = p(i,2);
   b(i) = p(i,1);
   v(i) = Vf(i)/(C-gamma*C);
end;

alpha = v.*a;
beta = v.*b;
```

Example

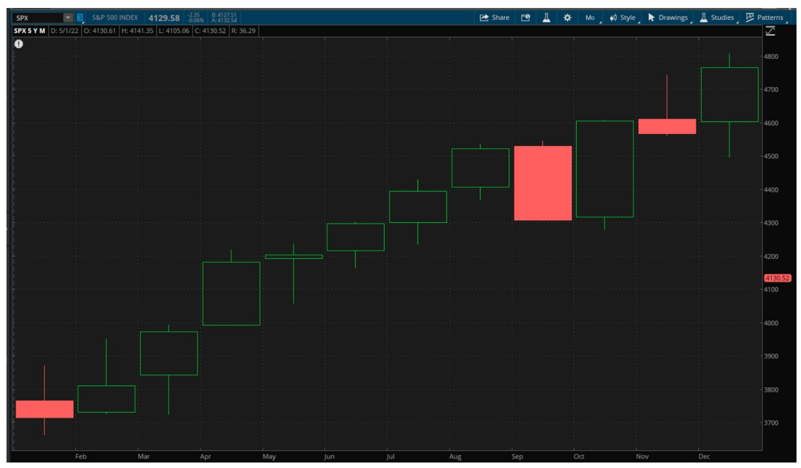
Example 0.3. Let $\boldsymbol{I} = (I_0, I_1, \ldots, I_{12})^\top$ be the vector where the component I_t denotes the value of the S&P 500 at the end of month t (where $t = 0$ corresponds to Dec 2020, $t = 1$ corresponds to Jan 2021, and so on...). Thus the vector $\boldsymbol{I}$ is given (rounding to the nearest integer) by

$$\boldsymbol{I} = (3756, 3714, 3811, 3973, 4181, 4204, 4298, 4395, 4523, 4308, 4605, 4567, 4766)^\top.$$

Next let $\boldsymbol{R} = (R_1, \ldots, R_{12})^\top$ be the vector where the component R_t denotes the single period time return fo the S&P 500 at the end of month t (where $t = 1$ corresponds to Jan 2021, $t = 2$ corresponds to Feb 2021, and so on...). In other words, $R_t = \ln(I_t / I_{t-1})$. Thus the vector $\boldsymbol{R}$ is given (rounding to the fourth decimal place) by

$$\boldsymbol{R} = (-0.0112, 0.0258, 0.0416, 0.0510, 0.0055, 0.0221, 0.0223, 0.0287, -0.0487, 0.0667, -0.0083, 0.0427)^\top.$$

Below is the graph of the S&P500 during 2021:



For instance, we see that the S&P 500 did poorly in the month of September. This corresopnds to the fact that $R_9 = -0.0487$. Next, let $\boldsymbol{V}_i = (V_{i,0}, V_{i,1}, \ldots, V_{i,12})^\top$ be the vector where the component $V_{i,t}$ denotes the value of one share of stock i (where $i = 1$ corresponds to Amazon , $i = 2$ corresponds to Facebook, and $i = 3$ corresponds to Apple) at the end of month t. Similarly, let $\boldsymbol{r}_i = (r_{i,1}, \ldots, r_{i,12})$ be the vector where the the component $r_{i,t}$ denotes the single period time return for stock i at the end of month t (where $t = 1$ corresponds to Jan 2021, $t = 2$ corresponds to Feb 2021, and so on...). In other words, $r_{i,t} = \ln(V_{i,t}/V_{i,t-1})$. Thus the vectors $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, and $\boldsymbol{V}_3$ are given (rounding to the nearest integer) by

$$\boldsymbol{V}_1 = (3257, 3206, 3093, 3094, 3467, 3223, 3440, 3328, 3471, 3285, 3372, 3507, 3304)^\top$$
$$\boldsymbol{V}_2 = (273, 259, 258, 295, 325, 329, 348, 356, 379, 339, 323, 324, 335)^\top.$$
$$\boldsymbol{V}_3 = (133, 132, 121, 122, 131, 125, 137, 146, 152, 141, 150, 165, 178)^\top$$

Let $\boldsymbol{X} = (X_1, X_2, X_3)^\top$ be the vector where X_i denotes number of shares of stock i in the current TP. We will assume that $\boldsymbol{X} = (10, 50, 100)^\top$. Finally, suppose $\gamma = 0.1$ and suppose we have $100,000$ dollars in cash change. In Matlab, we write

```
I = [3756,3714,3811,3973,4181,4204,4298,4395,4523,4308,4605,4567,4766];
V = [3257,3206,3093,3094,3467,3223,3440,3328,3471,3285,3372,3507,3304;
273,259,258,295,325,329,348,356,379,339,323,324,335;
133,132,121,122,131,125,137,146,152,141,150,165,178];
X = [10;50;100];
cash = 100000;
gamma = 0.1;
[R, r, C, a, b, Vf, v, alpha, beta] = TrackingPortfolio(I, V, X, cash, gamma);
```

Matlab gives us

$$C = 167590$$
$$\boldsymbol{v} = (0.022, 0.002, 0.001)^\top$$
$$\boldsymbol{a} = (-0.013, -0.009, 0.005)^\top$$
$$\boldsymbol{b} = (0.718, 1.295, 0.969)^\top.$$

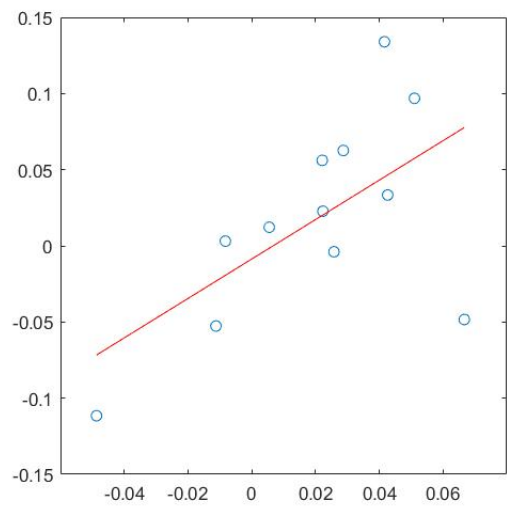
For instance, since $b_2 = 1.295$ and $a_2 = -0.009$, we see that the return of Facebook against the S&P 500 is given by

$$r_2 = -0.009 + 1.295\boldsymbol{R}.$$

We plot this using the code below:

```
t0 = min(R);
t1 = max(R);
t = linspace(t0,t1);
plot(R,r(2,:),'o',t,polyval(p(2,:),t),'r');
```

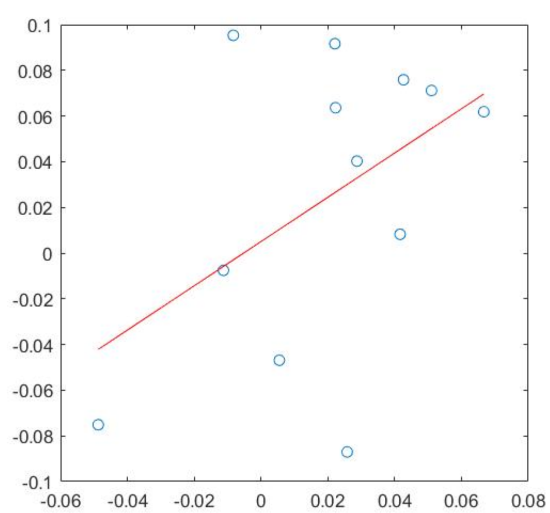
MATLAB gives us the following plot:



For instance, since $b_3 = 0.969$ and $a_3 = 0.005$, we see that the return of Apple against the S&P 500 is given by

$$r_3 = 0.005 + 0.969\boldsymbol{R}.$$

MATLAB gives us the following plot:



Replicating the Results

We want to replicate the results in the paper, however we shall consider a slight variation. Today, one does not need to worry about transaction costs: for instance, if you have an Ameritrade account, it costs precisely 0 dollars to trade any stock on the S&P 500. Thus, we shall ignore the variables f_i^s, f_i^b, and G_i for all $1 \leq i \leq N$, however we still keep γ (where $(1-\gamma)C$ represents the amount we are willing to invest in our new TP). With this in mind, first we solve the following MILP:

$$
\begin{array}{lll}
\text{minimize} & D & \\
\text{subject to} & \displaystyle\sum_{i=1}^{N} a_i v_i x_i \leq D & \\
& \displaystyle -\sum_{i=1}^{N} a_i v_i x_i \leq D & \\
& \varepsilon_i z_i \leq (V_{i,T}/C)x_i \leq \delta_i z_i & 1 \leq i \leq N \\
& \displaystyle\sum_{i=1}^{N} v_i x_i = 1 & \\
& \displaystyle\sum_{i=1}^{N} z_i = K & \\
& x_i, D \geq 0 & 1 \leq i \leq N \\
& z_i \in \{0,1\} & 1 \leq i \leq N
\end{array} \tag{18}
$$

where we recall that $v_i = V_{i,T}/(1-\gamma)C$, or in other words, $V_{i,T} = (1-\gamma)Cv_i$. We will find an optimal solution to this MILP problem using MATLAB, which has a built-in function whose purpose is to solve MILP problems like this. The syntax for this function is [x,cval] = intlinprog(c,intcon,Ain,bin,Aeq,beq,lb,ub), where the MILP solver assumes that the MILP has the form:

$$
\begin{array}{ll}
\text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} & A_{\text{in}}\boldsymbol{x} \leq \boldsymbol{b}_{\text{in}} \\
& A_{\text{eq}}\boldsymbol{x} = \boldsymbol{b}_{eq} \\
& \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \\
& \boldsymbol{x}(\text{intcon}) \text{ are integers}
\end{array}
$$

where $\boldsymbol{c}$, $\boldsymbol{x}$, intcon, $\boldsymbol{b}_{\text{in}}$, $\boldsymbol{b}_{\text{eq}}$, $\boldsymbol{lb}$, $\boldsymbol{ub}$ are vectors. In order to use this function, we need to convert our MILP into this form. First let

$$\boldsymbol{x} = (x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}, x_{2N+1}) = (x_1, \ldots, x_n, z_1, \ldots, z_n, D)^\top.$$

Let intcon $= (N+1, N+2, \ldots, 2N)^\top \in \mathbb{R}^N$ (this tells us that the variables $x_{N+1}, x_{N+2}, \ldots, x_{2N}$ are integers). Let $\boldsymbol{lb} = (0, \ldots, 0)^\top$ where $\boldsymbol{lb} \in \mathbb{R}^{2N+1}$ (this tells us that $\boldsymbol{x} \geq 0$), and let $\boldsymbol{ub} = (\text{Inf}, \ldots, \text{Inf}, 1, \ldots, 1, \text{Inf}) \in \mathbb{R}^{2N+1}$, that is

$$
(ub)_i = \begin{cases} \infty & \text{if } 1 \leq i \leq N \\ 1 & \text{if } N+1 \leq i \leq 2N \\ \infty & \text{if } i = 2N+1 \end{cases}
$$

This tells us that $x_i, x_{2N+1} \leq \infty$ and $x_{N+i} \leq 1$ for all $1 \leq i \leq N$. In particular, since $x_{N+i} \in \mathbb{Z}$, we must have $x_{N+i} \in \{0,1\}$ for all $1 \leq i \leq N$. With this notation, we can rewrite (18) as

$$
\begin{aligned}
\text{minimize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \sum_{i=1}^{N} \alpha_i v_i x_i - x_{2N+1} \leq 0 \\
& -\sum_{i=1}^{N} \alpha_i v_i x_i - x_{2N+1} \leq 0 \\
& (1-\gamma) v_i x_i - \delta_i x_{N+i} \leq 0 && 1 \leq i \leq N \\
& \varepsilon_i x_{N+i} - (1-\gamma) v_i x_i \leq 0 && 1 \leq i \leq N \\
& \sum_{i=1}^{N} v_i x_i = 1 \\
& \sum_{i=1}^{N} x_{N+i} = K \\
& \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \\
& \boldsymbol{x}(\text{intcon}) \text{ are integers}
\end{aligned}
\tag{19}
$$

where $\boldsymbol{c}^\top = (0, \ldots, 0, 1)^\top \in \mathbb{R}^{2N+1}$ is the vector with $c_i = 0$ for all $1 \leq i \leq 2N$ and $c_{2N+1} = 1$. Let

$$
A_{\text{eq}} = \begin{pmatrix} v_1 & \cdots & v_N & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_{\text{eq}} = \begin{pmatrix} 1 \\ K \end{pmatrix}
$$

where $A_{\text{eq}} \in \mathbb{R}^{2 \times (2N+1)}$ and $\boldsymbol{b}_{\text{eq}} \in \mathbb{R}^2$. Next, let

$$
A^1_{\text{in}} = \begin{pmatrix} \alpha_1 v_1 & \cdots & \alpha_N v_N & 0 & \cdots & 0 & -1 \\ -\alpha_1 v_1 & \cdots & -\alpha_N v_N & 0 & \cdots & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^1_{\text{in}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

where $A^1_{\text{in}} \in \mathbb{R}^{2 \times (2N+1)}$ and $\boldsymbol{b}^1_{\text{in}} \in \mathbb{R}^2$. Next, let

$$
A^2_{\text{in}} = \begin{pmatrix} (1-\gamma) v_1 & & 0 & -\delta_1 & & 0 & 0 \\ & \ddots & & & \ddots & & \vdots \\ 0 & & (1-\gamma) v_n & 0 & & -\delta_N & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^2_{\text{in}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

where $A^2_{\text{in}} \in \mathbb{R}^{N \times (2N+1)}$ and $\boldsymbol{b}^2_{\text{in}} \in \mathbb{R}^N$. Next, let

$$
A^3_{\text{in}} = \begin{pmatrix} (\gamma-1) v_1 & & 0 & \varepsilon_1 & & 0 & 0 \\ & \ddots & & & \ddots & & \vdots \\ 0 & & (\gamma-1) v_n & 0 & & \varepsilon_N & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}^3_{\text{in}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

where $A^3_{\text{in}} \in \mathbb{R}^{N \times (2N+1)}$ and $\boldsymbol{b}^3_{\text{in}} \in \mathbb{R}^N$. Finally, let

$$
A_{\text{in}} = \begin{pmatrix} A^1_{\text{in}} \\ A^2_{\text{in}} \\ A^3_{\text{in}} \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_{\text{in}} = \begin{pmatrix} \boldsymbol{b}^1_{\text{in}} \\ \boldsymbol{b}^2_{\text{in}} \\ \boldsymbol{b}^3_{\text{in}} \end{pmatrix},
$$

where $A_{\text{in}} \in \mathbb{R}^{(2N+2) \times (2N+1)}$ and $\boldsymbol{b}_{\text{in}} \in \mathbb{R}^{2N+2}$. Finally, we can rewrite (19) as

$$
\begin{aligned}
\text{minimize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & A_{\text{in}} \boldsymbol{x} \leq \boldsymbol{b}_{\text{in}} \\
& A_{\text{eq}} \boldsymbol{x} = \boldsymbol{b}_{eq} \\
& \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \\
& \boldsymbol{x}(\text{intcon}) \text{ are integers}
\end{aligned}
$$

which is the desired form.

Example

We are now ready to work in MATLAB. Let function [x,cval] = MinimizeAlpha(I,V,X,cash,gamma,epsilon,delta,K) be the MATLAB function which we code below:

```
function [x,cval] = MinimizeAlpha(I,V,X,cash,gamma,epsilon,delta,K);

[R, r, C, a, b, Vf, v, alpha, beta] = TrackingPortfolio(I, V, X, cash , gamma);

N = length(v);

c = [zeros(2*N,1);1];

Aeq = [v' , zeros(1,N+1) ; zeros(1,N), ones(1,N), 0];
beq = [1; K];

Ain1 = [(a.*v)', zeros(1,N), -1; -(a.*v)', zeros(1,N), -1];
Ain2 = [diag((1-gamma)*v), diag(-delta), zeros(N,1)];
Ain3 = [diag((gamma-1)*v), diag(epsilon), zeros(N,1)];
Ain = [Ain1; Ain2; Ain3];
bin = zeros(2*N+2,1);

intcon = N+1:2*N;
intcon = intcon'
lb = [zeros(2*N+1,1)];
ub = [Inf(N,1); ones(N,1) ; Inf];

[x, cval] = intlinprog(c,intcon,Ain,bin,Aeq,beq,lb,ub);
```

Example 0.4. We continue with the notation as in Example (0.3). If we set $K = 1$, $\boldsymbol{\varepsilon} = (0;0;0)^\top$, and $\boldsymbol{\delta} = (1;1;1)^\top$, then we should expect our program to tell us that holding shares in Apple is the optimal solution since $|a_3| = |\alpha_3| = 0.005$ is optimal (we have $|a_1| = 0.013$ and $|a_2| = 0.009$, and clearly $|a_3| < |a_1|$ and $|a_3| < |a_2|$). In MATLAB, we write

```
I = [3756,3714,3811,3973,4181,4204,4298,4395,4523,4308,4605,4567,4766];
V = [3257,3206,3093,3094,3467,3223,3440,3328,3471,3285,3372,3507,3304;
273,259,258,295,325,329,348,356,379,339,323,324,335;
133,132,121,122,131,125,137,146,152,141,150,165,178];
X = [10;50;100];
cash = 100000;
epsilon = [0;0;0];
delta = [1;1;0];
gamma = 0.1;
K = 1;
[x,cval] = MinimizeAlpha(I,V,X,cash,gamma,epsilon,delta,K);
```

MATLAB tells us that the optimal solution is $\boldsymbol{x} = (0, 0, 847.37, 0, 0, 1, 0.005)^\top$ which corresponds to holding ≈ 847 shares of Apple (and no shares of Amazon or Facebook). Notice that

$$847 \text{ shares of Apple} \cdot 178 \text{ dollars} = 150,766 \text{ value of new TP}.$$

Furthermore, we have $(1-\gamma)C = 150,831$, so this also satisfies our constraint. Next, suppose we set $\boldsymbol{\delta} = (1;1;0)^\top$. Then we should expect our program to tell use that holding shares in Facebook is the optimal solution since $|a_2| < |a_1|$. Replacing $\boldsymbol{\delta}$ in the code above and running it through MATLAB again, we see that MATLAB gives us the optimal solution $\boldsymbol{x} = (0, 450.2418, 0, 0, 1, 0, 0.0086)^\top$ which corresponds to holding ≈ 450 shares of Facebook (and no shares of Amazon or Apple).

Conclusion

In this project, we introduced MATLAB code which solves the first-step in the MILP introduced in the paper under consideration. Today, most trading accounts have no transaction costs, thus we do not need to consider the variables f_i^s, f_i^b, and G_i. We still keep the variable $\gamma \in (0, 1)$ since this can be used to represent how much of the total value of our account that we'd like to invest.

Appendix

Adopting the regression viewpoint for index tracking we have that ideally our TP would be chosen such that when we perform a regression of TP returns against index returns we would find that the regression line has an intercept (alpha) of zero, and a slope (beta) of one. However achieving this ideal is not straightforward. There are a number of complications to achieving this ideal within the context of a linear (or linearisable) formulation, as will become apparent below.

The single period continuous time return for the TP in period t is given by

$$\ln\left(\sum_{i=1}^{N} x_i V_{i,t} / \sum_{i=1}^{N} x_i V_{i,t-1}\right).$$

This is a nonlinear function of the decision variables. In order to linearize (in an approximate fashion) this return we shall assume that it can be expressed as a linear weighted sum of individual stock returns, where the weights, summing to one, reflect the proportion invested in each stock at time t. The assumption that portfolio return can be expressed as a linear weighted sum of individual stock returns is a common assumption in finance. Hence the return on the TP at time t is given by

$$\sum_{i=1}^{N} W_{i,t} r_{i,t}$$

where $r_{i,t} = \ln(V_{i,t}/V_{i,t-1})$ and where

$$W_{i,t} = x_i V_{i,t} / \sum_{j=1}^{N} x_j V_{j,t}$$

is the weight associated with investment in stock i at time t and $\sum_{i=1}^{N} W_{i,t} = 1$ for all t. Now the $W_{i,t}$ are nonlinear expressions involving our decision variables x_i, as well as the value $\sum_{j=1}^{N} x_j V_{j,t}$ of the TP at time t. In order to proceed, we shall approximate $W_{i,t}$ by a constant term which is independent of time; namely replace $W_{i,t}$ by w_i where

$$w_i = x_i V_{i,T} / \sum_{j=1}^{N} x_j V_{j,T}$$

represents the proportion invested in stock i at time T. Hence the return on the TP at time t is given by

$$\sum_{i=1}^{N} w_i r_{i,t}.$$

The expresssion for w_i is still nonlinear in the x_i so to linearize it we use the constraint (6) to replace $\sum_{j=1}^{N} x_j V_{j,T}$ by $C - \sum_{j=1}^{N} G_j$. This is also a function of our variables, but we know from (5) that $\sum_{j=1}^{N} G_j$ is bounded above by γC. Hence we approximate w_i using the linear expression:

$$w_i = x_i V_{i,T}/(C - \gamma C) \quad 1 \leq i \leq N.$$

Finally therefore we have a linear expression (approximation) for the returns on the TP as

$$\sum_{i=1}^{N} w_i r_{i,t} \qquad 1 \leq t \leq T$$

Using the log-sum inequality it is possible to prove (assuming we spend γC in transaction cost) that the average return from this approximation

$$\sum_{t=1}^{T} \sum_{i=1}^{N} w_i r_{i.t}/T$$

over-estimates average tracking portfolio

$$\sum_{t=1}^{T} \ln\left(\sum_{i=1}^{N} x_i V_{i,t} / \sum_{i=1}^{N} x_i V_{i,t-1}\right)/T$$