Tor-Persistence

Introduction

Let R be a commutative noetherian ring. Recall that a finitely generated R-module M has finite projective dimension if $\mathrm{Tor}_i^R(M,N)=0$ for $i \gg 0$ for each finitely generated R-module N. Indeed, first note that $\mathrm{Tor}_i^R(M,N)=0$ if and only if

$$\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) \simeq \mathrm{Tor}_i^R(M,N)_{\mathfrak{p}} = 0$$

for all prime ideals $\mathfrak{p}$ of R. Thus by replacing R, M, and N with $R_{\mathfrak{p}}$, $M_{\mathfrak{p}}$, and $N_{\mathfrak{p}}$ if necessary, we may assume that $R=(R,\mathfrak{m},\Bbbk)$ is local. Now let F be the minimal free resolution of M over R. Thus

$$\mathrm{Tor}_i^R(M,N) = \mathrm{H}_i(F \otimes_R N).$$

We first prove the easy direction: suppose M has finite projective dimension, say $\mathrm{pd}_R M = p$. This means that $F_p \neq 0$ and $F_i = 0$ for all $i > p$. In particular that $(F\otimes_R N)_i = 0$ for all $i > p$, which implies $\mathrm{Tor}_i^R(M,N)=0$ for all $i>p$. Now we prove the harder direction: suppose $\mathrm{Tor}_i^R(M,N)=0$ for $i \gg 0$ for each finitely generated R-module N. In particular, we have $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i \gg 0$. This implies $\mathrm{H}_i(F_{\Bbbk})=0$ for $i \gg 0$ where we set $F_{\Bbbk} := F \otimes_R \Bbbk$. However F is *minimal*, thus $\mathrm{d}_{\Bbbk}=0$, where $\mathrm{d}_{\Bbbk}$ is the differential of $F_{\Bbbk}$. Thus we have $\mathrm{H}_i(F_{\Bbbk}) = F_{\Bbbk,i} := F_i \otimes_R \Bbbk$ and this implies $F_i \otimes_R \Bbbk = 0$ for $i \gg 0$ which implies $F_i = 0$ for $i \gg 0$ by Nakayama's lemma (here is where we used the fact that R is noetherian and M is finitely generated).

Now suppose that the only thing we knew was that $\mathrm{Tor}_i^R(M,M)=0$ for $i \gg 0$. Can we still conclude that the projective dimension of M is finite? This is an open question in general, however it is known to be true for various rings R: we call such rings **Tor-persistent**. It is natural to wonder if in fact every commutative noetherian ring is Tor-persistent. Note that

$$\mathrm{Tor}_i^R(M,M) = \mathrm{H}_i(F\otimes_R M) = \mathrm{H}_i(F^{\otimes 2})$$

where we denoted $F^{\otimes 2} = F \otimes_R F$. One of the main reasons why we could conclude that M had finite projective dimension if $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i \gg 0$ was because the homology of $F_{\Bbbk}$ was extremely simple, namely $\mathrm{H}(F_{\Bbbk}) = F_{\Bbbk}$. The homology of $F^{\otimes 2}$ is more complicated however, thus even if we knew that $\mathrm{H}_i(F^{\otimes 2})=0$ for $i \gg 0$, it is not at all clear why this should imply that $F_i=0$ for $i \gg 0$. In order to prove this, one would presumably need to use the fact that R is noetherian, M is finitely generated, and F is minimal.

Reduction to Complete Local Ring

Let $(R,\mathfrak{m},\Bbbk)$ be a local noetherian ring and let M be a finitely generated R-module. Then

$$\mathrm{Tor}_i^R(M,M)\otimes_R \widehat{R} = \mathrm{Tor}_i^{\widehat{R}}(\widehat{M},\widehat{M}) \quad \text{and} \quad \mathrm{Tor}_i^R(M,\Bbbk)\otimes_R \widehat{R} = \mathrm{Tor}_i^{\widehat{R}}(\widehat{M},\Bbbk),$$

where $\widehat{R}$ and $\widehat{M}$ denote the completions of R and M in the $\mathfrak{m}$-adic topology. In particular, since $R \to \widehat{R}$ is faithfully flat, it follows that $\mathrm{Tor}_i^R(M,M)=0$ for all $i \gg 0$ if and only if $\mathrm{Tor}_i^{\widehat{R}}(\widehat{M},\widehat{M})=0$ for all $i \gg 0$, and $\mathrm{pd}_R(M) = \mathrm{pd}_{\widehat{R}}(\widehat{M})$. Thus we may as well assume that R is complete with respect to the $\mathfrak{m}$-adic topology in what follows.

Reduction to Depth Zero

Lemma 0.1. *Let $(R,\mathfrak{m},\Bbbk)$ be a local noetherian ring and let M be a finitely generated R-module. Suppose that $x \in \mathfrak{m}$ is an R-regular and M-regular element. Then $\mathrm{Tor}_i^R(M,M)=0$ for all $i \gg 0$ if and only if $\mathrm{Tor}_i^{R/x}(M/x,M/x)=0$ for all $i \gg 0$. Furthermore, M has finite projective dimension over R if and only if M/x has finite projective dimension over R/x.*

Proof. Consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/x \longrightarrow 0 \tag{1}$$

After tensoring (1) with M, we obtain a long exact sequence of Tor modules

$$\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(M, M/x) \longrightarrow \mathrm{Tor}_i^R(M,M) \xrightarrow{x} \mathrm{Tor}_i^R(M,M) \longrightarrow \mathrm{Tor}_i^R(M,M/x) \longrightarrow \mathrm{Tor}_{i-1}^R(M,M) \longrightarrow \cdots$$

In particular, if $\mathrm{Tor}_i^R(M,M) = 0$ for all $i \gg 0$, we see that $\mathrm{Tor}_i^R(M,M/x) = 0$ for all $i \gg 0$. Conversely, if $\mathrm{Tor}_i^R(M,M/x) = 0$ for all $i \gg 0$, then Nakayama's lemma implies that $\mathrm{Tor}_i^R(M,M) = 0$ for all $i \gg 0$. Similarly, after tensoring (1) with M/x, we obtain the long exact sequence of Tor modules

$$\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(M/x, M/x) \longrightarrow \mathrm{Tor}_i^R(M,M/x) \xrightarrow{x} \mathrm{Tor}_i^R(M,M/x) \longrightarrow \mathrm{Tor}_i^R(M/x,M/x) \longrightarrow \mathrm{Tor}_{i-1}^R(M,M/x) \longrightarrow \cdots$$

By the same argument as above, we see that $\mathrm{Tor}_i^R(M,M/x) = 0$ for all $i \gg 0$ if and only if $\mathrm{Tor}_i^R(M/x,M/x) = 0$ for all $i \gg 0$. Now let F be the minimal free resolution of M over R. Then F/x is the minimal free resolution of M/x over R/x and $\mathrm{C}(x) = F \oplus eF$ is the minimal free resolution of M over R. In particular, note that

$$\begin{aligned}
\mathrm{Tor}_i^R(M/x,M/x) &= \mathrm{H}_i((F \oplus eF) \otimes_R M/x) \\
&= \mathrm{H}_i((F/x) \otimes_R M \oplus e((F/x) \otimes_R M)) \\
&= \mathrm{H}_i(F/x \otimes_{R/x} M/x) \oplus \mathrm{H}_{i+1}(F/x \otimes_{R/x} M/x) \\
&= \mathrm{Tor}_i^{R/x}(M/x,M/x) \oplus \mathrm{Tor}_{i-1}^{R/x}(M/x,M/x)
\end{aligned}$$

where we used the fact that $\mathrm{d}e = 0$ in F/x. It follows at once that $\mathrm{Tor}_i^R(M/x,M/x) = 0$ for all $i \gg 0$ if and only if $\mathrm{Tor}_i^{R/x}(M/x,M/x) = 0$ for all $i \gg 0$. Finally, note that

$$\mathrm{pd}_R(M) = \mathrm{length}(F) = \mathrm{length}(F/x) = \mathrm{pd}_{R/x}(M/x).$$

$\square$

Remark 1. Let M be a finitely generated R-module and let M_n denote the nth syzygy of M for each $n \geq 0$ with $M_0 = M$. Then we have

$$\mathrm{Tor}_i^R(M_n, M_n) = \mathrm{Tor}_{i+2n}^R(M,M) \quad \text{and} \quad \mathrm{Tor}_i^R(M_n, \Bbbk) = \mathrm{Tor}_{i+n}^R(M, \Bbbk).$$

Thus M satisfies Tor persistence if and only if M_n satisfies Tor persistence. Furthermore, if $\operatorname{depth} M < \operatorname{depth} R$, then $\operatorname{depth} M_1 = \operatorname{depth} M + 1$, so by replacing M with M_n for n large enough, we may reduce to the case where $\operatorname{depth} M \geq \operatorname{depth} R$. Then by Lemma (0.1), we may further reduce to the case where $\operatorname{depth} R = 0$. After this, we can then further reduce to the case where $\operatorname{depth} R = \operatorname{depth} M = 0$.

Note that anytime short exact sequence of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0 \tag{2}$$

, then virtually by the same argument as in the lemma above, if $\operatorname{Tor}_i^R(M,M) = 0$ for all $i \gg 0$, then $\operatorname{Tor}_i^R(E,E) = 0$ for all $i \gg 0$. The R-module E is called an extension of M by M. The isomorphism classes of extensions of M by M is in bijection with $\operatorname{Ext}_R^1(M,M)$.

Lemma 0.2. *Let E be an extension of M by M. Then $\operatorname{pd}_R(E) = \operatorname{pd}_R(M)$.*

Proof. After tensoring (2) by $\Bbbk$, we obtain the long exact sequence of Tor modules

$$\begin{tikzcd}
 & \cdots \arrow[r] & \operatorname{Tor}_{i+1}^R(M,\Bbbk) \arrow[dll, rounded corners] \\
\operatorname{Tor}_i^R(M,\Bbbk) \arrow[r] & \operatorname{Tor}_i^R(E,\Bbbk) \arrow[r] & \operatorname{Tor}_i^R(M,\Bbbk) \arrow[dll, rounded corners] \\
\operatorname{Tor}_{i-1}^R(M,\Bbbk) \arrow[r] & \cdots &
\end{tikzcd}$$

In particular, suppose $p = \operatorname{pd}_R(M)$. we see that

$$\operatorname{pd}_R(M) = \sup\{\operatorname{Tor}_i^R(M,\Bbbk) \neq 0 \mid i \in \mathbb{N}\}$$

$\square$

Reduction to Indecomposable Modules

Lemma 0.3. *Let $(R,\mathfrak{m},\Bbbk)$ be a local noetherian ring and let M be a finitely generated R-module such that $\operatorname{Tor}_i^R(M,M) = 0$ for all $i \gg 0$ and such that $M = M_1 \oplus M_2$ where M_1 and M_2 are R-modules. Then*

$$\operatorname{Tor}_i^R(M_1,M_1) = 0, \quad \operatorname{Tor}_i^R(M_2,M_2) = 0, \quad \textit{and} \quad \operatorname{Tor}_i^R(M_1,M_2) = 0$$

for all $i \gg 0$. Furthermore, we have

$$\operatorname{pd}_R(M) = \max\{\operatorname{pd}_R(M_1), \operatorname{pd}_R(M_2)\}.$$

Then $\operatorname{Tor}_i^R(N,N) = 0$ for all $i \gg 0$ and N has finite projective dimension if and only if M has finite projective dimension.

Proof. For $i \gg 0$, we have

$$\begin{aligned}
0 &= \operatorname{Tor}_i^R(M,M) \\
&= \operatorname{Tor}_i^R(M_1 \oplus M_2, M_1 \oplus M_2) \\
&= \operatorname{Tor}_i^R(M_1,M_1) \oplus \operatorname{Tor}_i^R(M_1,M_2)^2 \oplus \operatorname{Tor}_i^R(M_2,M_2).
\end{aligned}$$

This establishes the first part of the lemma. For the second part, note that

$$\begin{aligned}
\operatorname{Tor}_i^R(M,\Bbbk) &= \operatorname{Tor}_i^R(M_1 \oplus M_2, \Bbbk) \\
&= \operatorname{Tor}_i^R(M_1,\Bbbk) \oplus \operatorname{Tor}_i^R(M_2,\Bbbk).
\end{aligned}$$

It follows that $\operatorname{pd}_R(M) = \max\{\operatorname{pd}_R(M_1), \operatorname{pd}_R(M_2)\}$. $\square$

Finite Length Case

Lemma 0.4. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring and let M be a finitely generated R-module such that $\ell(M) = 2$. Thus there is a short exact sequence*

$$0 \longrightarrow \Bbbk \longrightarrow M \longrightarrow \Bbbk \longrightarrow 0 \tag{3}$$

If $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \gg 0$, then M has finite projective dimension of R.

Proof. Let F be the minimal free resolution of M over R. After tensoring (4) with $- \otimes_R M$ and taking homology, we obtain isomorphisms

$$\mathrm{Tor}_i(M, \Bbbk) := F_{\Bbbk,i} \xrightarrow{\partial_i} F_{\Bbbk,i-1} := \mathrm{Tor}_{i-1}(M, \Bbbk)$$

for all $i \gg 0$ where ∂_i is the connecting map from the long exact sequence in Tor modules. Next let E be the minimal free resolution of $\Bbbk$ over R. Then after tensoring (4) with $- \otimes_R \Bbbk$ and taking homology, we see that $\ell(E_{\Bbbk,i}) = \ell(E_{\Bbbk,i-1})$ for $i \gg 0$. This implies $E_{\Bbbk,i} = 0$ for $i \gg 0$ since E is a DG algebra. It follows that $F_{\Bbbk,i} = 0$ for $i \gg 0$.

we obtain
We claim that $\partial_i = 0$. Indeed, the connecting map is defined as follows: let F be the minimal free resolution of M over R. Given $a \otimes \overline{1} \in F_{\Bbbk}$ in homological degree i, we lift $a \otimes \overline{1}$ to $a \otimes 1 \in F^{\otimes 2}$ and then we apply the differential to get $\mathrm{d}(a \otimes 1) = \mathrm{d}a \otimes 1 \in F^{\otimes 2}$. Note that $\mathrm{d}a \in \mathfrak{m}F$ □

Induction: now suppose we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow \Bbbk \longrightarrow 0 \tag{4}$$

. If $\mathrm{Tor}_i^R(M', M') = 0$ for $i \gg 0$, then by induction on length, we would have $F'_{\Bbbk,i} = 0$ for $i \gg 0$ where F' is the minimal free resolution of M'. Then this would imply $0 = \mathrm{Tor}_i(M, M') = F_{\Bbbk,i+1}$ for $i \gg 0$.

Tor-Persistence

In what follows, we assume $(R, \mathfrak{m}, \Bbbk)$ is a local noetherian ring. Let F be the minimal R-free resolution of the cyclic R-module R/I where $I \subseteq \mathfrak{m}$ is an ideal of R. Choose a multiplication μ on F giving it the structure of an MDG R-algebra. We denote $\mu(a_1 \otimes a_2) = a_1 a_2$ for all $a_1, a_2 \in F$ in order to simplify notation in what follows. Define a chain map $\{\cdot\}_\mu \colon F^{\otimes 3} \to F^{\otimes 2}$ by the formula

$$\{a_1 \otimes a_2 \otimes a_3\} = a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 = \{a_1, a_2, a_3\},$$

where we remove the subscript μ from $\{\cdot\}_\mu$ when context is clear and where we set $\{\cdot, \cdot, \cdot\} \colon F^3 \to F^{\otimes 2}$ to be the unique R-trilinear map corresponding to $\{\cdot\}$ via the universal mapping property of tensor products. Our goal is to determine what $\ker\{\cdot\}$ and $\operatorname{im}\{\cdot\}$ look like. First we consider $\operatorname{im}\{\cdot\}$. For each $a_1, a_2, a_3 \in F$, we have

$$\begin{aligned}
\{a_1, a_2, 1\} &= a_1 a_2 \otimes 1 - a_1 \otimes a_2 \\
\{1, a_2, a_3\} &= a_2 \otimes a_3 - 1 \otimes a_2 a_3 \\
\{a_1, 1, a_3\} &= 0 \\
\{a, a, b\} &= a^2 \otimes b - a \otimes ab
\end{aligned}$$

Thus if $ab = 0$, then $a \otimes b \in \operatorname{im}\{\cdot\}$. Furthermore we have $a \otimes 1 - 1 \otimes a \in \operatorname{im}\{\cdot\}$. Now suppose that

$$\{e_{i_1}, e_{i_2}, e_{i_3}\} = e_{i_1} e_{i_2} \otimes e_{i_3} - e_{i_1} \otimes e_{i_2} e_{i_3} = 0.$$

Then we must have $e_{i_1} = e_{i_1} e_{i_2}$ and $e_{i_3} = e_{i_2} e_{i_3}$. Or in other words, we must have $e_{i_1}(1 - e_{i_2}) = 0$ and $e_{i_3}(1 - e_{i_2}) = 0$. By considering homological degrees as well as using the fact that R is local, one sees that the only solution to these equations is

$$\{(0, e_{i_2}, 0), (0, 1, e_{i_3}), (e_{i_1}, 1, 0), (e_{i_1}, 1, e_{i_3})\}.$$

In particular, this spans $F^{\oplus 3} \oplus F^{\otimes 2}$.

Proposition 0.1. *Suppose $\mathrm{H}_i(F) = 0 = \mathrm{H}_i(F^{\otimes 2})$ for $i \gg 0$. Then $\mathrm{H}_i(F^{\otimes n}) = 0$ for $i \gg 0$ for all $n \geq 1$.*

Proof. Consider the short exact sequence $0 \to F \to F^{\otimes 3} \to F^{\otimes 2} \to 0$. Actually this even shows $\mathrm{Tor}_+^R(S, S) = \mathrm{H}_+(F^{\otimes n})$ for all $n \geq 2$. □