Homework 1

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Problem 1

Exercise 1. For this problem, let $f_1(x) = -\sqrt{x+1}$ and let $f_2(x) = x^2 - 4x + 5$. We consider the following biobjective program:

$$\begin{array}{ll} \text{minimize} & [f_1(x), f_2(x)] \\ \text{subject to} & x \geq 0 \end{array}$$

1. Derive the formula representing the outcome set Y in $\mathbb{R}^2$ for this biobjective program.
2. Graph the outcome set Y.
3. Identify and mark the Pareto-nondominated outcomes in Y.
4. Find the Pareto-efficient solutions in X.
5. Find the ideal point.

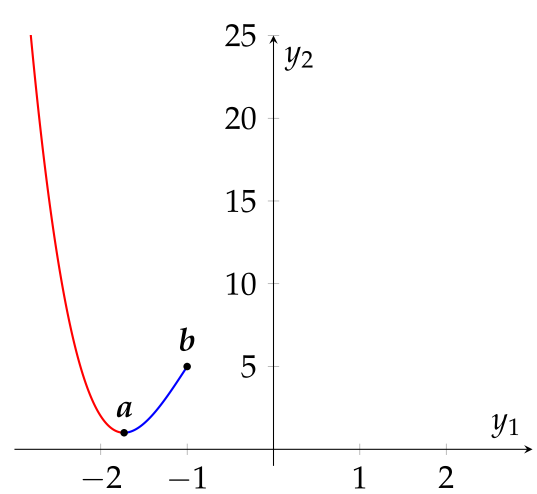
Solution 1. 1. Set $y_1 = f_1(x)$ and set $y_2 = f_2(x)$. We first write y_2 as a function of y_1. Since $x = y_1^2 - 1$, we have

$$\begin{aligned} y_2 &= x^2 - 4x + 5 \\ &= (y_1^2 - 1)^2 - 4(y_1^2 - 1) + 5 \\ &= y_1^4 - 2y_1^2 + 1 - 4y_1^2 + 4 + 5 \\ &= y_1^4 - 6y_1^2 + 10. \end{aligned}$$

Note that if $x \geq 0$, then $y_1 \leq -1$. In particular, the outcome set Y is given by

$$Y = \{(y_1, y_1^4 - 6y_1^2 + 10) \mid y_1 \leq -1\} \subseteq \mathbb{R}^2$$

2. We graph the outcome set Y below:



where $\boldsymbol{a} = (-\sqrt{3}, 1)$ and $\boldsymbol{b} = (-1, 5)$. Note that the outcome set Y consists of both the red and black segments of the curve. Also note that the red segment extends off towards infinity.

3. The red segment of the curve above is the set of all Pareto-nondominated outcomes in Y. Specifically, this is the set of all $\boldsymbol{y} \in Y$ such that $y_1 \leq -\sqrt{3}$.

4. Note that $x = y_1^2 - 1$ and $x \geq 0$. Thus when $y_1 \leq -\sqrt{3}$, we have $x \geq 2$. Thus the efficient solutions in X is given by the interval $[2, \infty)$.

5. The ideal point is given by

$$\boldsymbol{c} = \begin{pmatrix} \inf\limits_{x \geq 2} f_1(x) \\ \inf\limits_{x \geq 2} f_2(x) \end{pmatrix} = \begin{pmatrix} -\infty \\ 1 \end{pmatrix}$$

Problem 2

Exercise 2. For this problem, let $f_1(\boldsymbol{x}) = x_1 - 3x_2$ and let $f_2(\boldsymbol{x}) = -4x_1 + x_2$. Furthermore, let

$$\begin{aligned} g_1(\boldsymbol{x}) &= -x_1 + x_2 - 7/2 \\ g_2(\boldsymbol{x}) &= x_1 + x_2 - 11/2 \\ g_3(\boldsymbol{x}) &= 2x_1 + x_2 - 9 \\ g_4(\boldsymbol{x}) &= x_1 - 4. \end{aligned}$$

Finally let

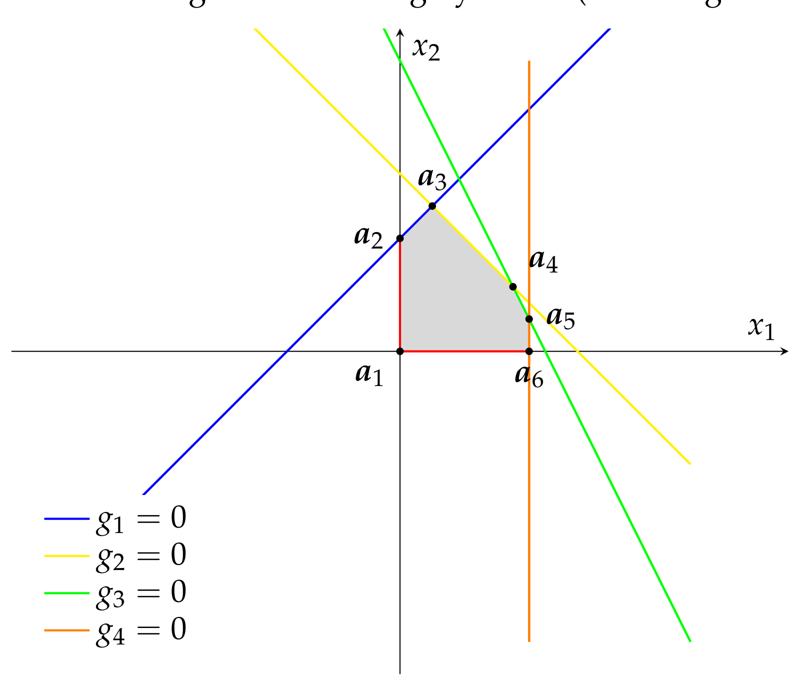
$$X = \{\boldsymbol{x} \in \mathbb{R}^2_{\geq 0} \mid g_j(\boldsymbol{x}) \leq 0 \text{ all } j = 1, 2, 3, 4\}.$$

We consider the following biobjective program:

$$\begin{array}{ll} \text{maximize} & [f_1(\boldsymbol{x}), f_2(\boldsymbol{x})] \\ \text{subject to} & \boldsymbol{x} \in X \end{array}$$

1. Graph the feasible set X in the decision space.
2. Graph the outcome set Y in the objective space $\mathbb{R}^2$. Explain what mathematical property you used to draw Y.
3. Identify and mark the Pareto-nondominated outcomes in Y.
4. Identify and mark the Pareto-efficient solutions in X.
5. Find and graph the ideal point.

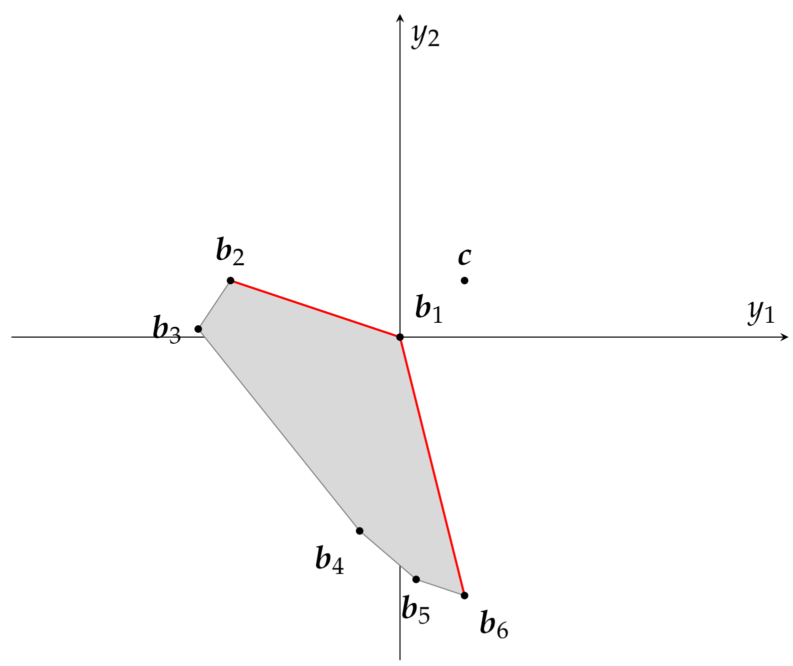
Solution 2. 1. The feasible set X is the region shaded in grey below (including the edges):



where

$$\begin{aligned} \boldsymbol{a}_1 &= (0,0) \\ \boldsymbol{a}_2 &= (0,7/2) \\ \boldsymbol{a}_3 &= (1,9/2) \\ \boldsymbol{a}_4 &= (7/2,2) \\ \boldsymbol{a}_5 &= (4,1) \\ \boldsymbol{a}_6 &= (4,0) \end{aligned}$$

2. The outcome set Y is the region shaded in grey below (including the edges):



where $\boldsymbol{b}_i = \boldsymbol{f}(\boldsymbol{a}_i)$ for all $1 \le i \le 6$. Specifically:

$$\begin{aligned} \boldsymbol{b}_1 &= (0,0) \\ \boldsymbol{b}_2 &= (-21/2,7/2) \\ \boldsymbol{b}_3 &= (-25/2,1/2) \\ \boldsymbol{b}_4 &= (-5/2,-12) \\ \boldsymbol{b}_5 &= (1,-15) \\ \boldsymbol{b}_6 &= (4,-16) \end{aligned}$$

Here we used the fact that $\boldsymbol{f}$ is a linear transformation. Thus it takes the convex closure of the $\boldsymbol{a}_i$ to the convex closure of the $\boldsymbol{b}_i$.

3. The Pareto-nondominated outcomes in Y is the thick red segment in part 2. Specifically, it is given by

$$Y_N = [\boldsymbol{b}_2, \boldsymbol{b}_1] \cup [\boldsymbol{b}_1, \boldsymbol{b}_6],$$

where $[\boldsymbol{b}_2, \boldsymbol{b}_1]$ is the line segment in the plane from $\boldsymbol{b}_2$ to $\boldsymbol{b}_1$, and where $[\boldsymbol{b}_1, \boldsymbol{b}_6]$ is the line segment in the plane from $\boldsymbol{b}_1$ to $\boldsymbol{b}_6$.

4. The Pareto-effecient solutions in X is the thick red segment in part 1. Specifically, it is given by

$$X_E = [\boldsymbol{a}_2, \boldsymbol{a}_1] \cup [\boldsymbol{a}_1, \boldsymbol{a}_6].$$

5. The ideal point is the point

$$\boldsymbol{c} = \begin{pmatrix} \sup\limits_{\boldsymbol{x} \in X_E} f_1(\boldsymbol{x}) \\ \sup\limits_{\boldsymbol{x} \in X_E} f_2(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} 4 \\ 7/2 \end{pmatrix},$$

shown in the graph of part 2.

Problem 3

Exercise 3. Let C_1 and C_2 be finite cones in $\mathbb{R}^p$ and let C_1^* and C_2^* be their dual cones, respectively. Prove the following:

1. If $C_1 \subseteq C_2$, then $C_2^* \subseteq C_1^*$.
2. $(C_1 + C_2)^* = C_1^* \cap C_2^*$.

Solution 3. 1. Let $\boldsymbol{y} \in C_2^*$. Thus $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0$ for all $\boldsymbol{x} \in C_2$ (where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{y}$). In particular, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 0$ for all $\boldsymbol{x} \in C_1$ since $C_1 \subseteq C_2$. It follows that $\boldsymbol{y} \in C_1^*$. Since $\boldsymbol{y} \in C_2^*$ was arbitrary, it follows that $C_2^* \subseteq C_1^*$.

2. First note that if C is a cone, then $0 \in C^*$ and $(C \cup \{0\})^* = C^*$. Thus by replacing C_1 and C_2 with $C_1 \cup \{0\}$ and $C_2 \cup \{0\}$ if necessary, we may assume that both C_1 and C_2 contain 0. In this case, observe that $C_1 \subseteq C_1 + C_2$ and $C_2 \subseteq C_1 + C_2$. Thus by part 1, we have $(C_1 + C_2)^* \subseteq C_1^*$ and $(C_1 + C_2)^* \subseteq C_2^*$. It follows that $(C_1 + C_2)^* = C_1^* \cap C_2^*$.

Problem 4

Exercise 4. Derive the formula representing the polar cone of the cone generated by

1. the vector $\boldsymbol{v} = (2, 3)$ in $\mathbb{R}^2$.
2. the vectors $\boldsymbol{v} = (4, 1)$ and $\boldsymbol{w} = (4, -1)$ in $\mathbb{R}^2$.

Solution 4. Note that if a cone C is generated by vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ in $\mathbb{R}^n$, then we have

$$\begin{aligned} \boldsymbol{x} \in C^+ &\iff \langle \boldsymbol{v}, \boldsymbol{x} \rangle \geq 0 \text{ for all } \boldsymbol{v} \in C \\ &\iff \langle \boldsymbol{v}_i, \boldsymbol{x} \rangle \geq 0 \text{ for all } 1 \leq i \leq m. \\ &\iff A\boldsymbol{x} \geq 0, \end{aligned}$$

where A is the $m \times n$ matrix whose ith row is given by $\boldsymbol{v}_i$. Thus we can express C^+ in inequality form as:

$$C^+ = \{\boldsymbol{x} \mid A\boldsymbol{x} \geq 0\}.$$

In particular, for part 1 we use the 1×2 matrix $A = \begin{pmatrix} 2 & 3 \end{pmatrix}$ and for part 2 we use the 2×2 matrix $A = \begin{pmatrix} 4 & 1 \\ 4 & -1 \end{pmatrix}$.

Problem 5

Exercise 5. Solve the following:

1. Let C be a polyhedral cone defined as

$$C = \{\boldsymbol{x} \in \mathbb{R}^2 \mid A\boldsymbol{x} \geq 0\},$$

   where $A = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$. Derive the generator form for this cone.

2. Let C be a polyhedral cone defined as

$$C = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = B\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\},$$

   where $B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Derive the inequality form for this cone.

Solution 5. 1. We have

$$\begin{aligned} C &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid A\boldsymbol{x} \geq 0\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid A\boldsymbol{x} = \boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\} \\ &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} = A^{-1}\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\} \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} = \frac{1}{5}\begin{pmatrix} -1 & -2 \\ -3 & -1 \end{pmatrix}\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\right\} \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} = \begin{pmatrix} -1 & -2 \\ -3 & -1 \end{pmatrix}\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\right\} \end{aligned}$$

We have

$$
\begin{aligned}
C &= \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = B\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^2 \mid B^{-1}\boldsymbol{x} = \boldsymbol{\lambda},\ \boldsymbol{\lambda} \geq 0\} \\
&= \{\boldsymbol{x} \in \mathbb{R}^2 \mid B^{-1}\boldsymbol{x} \geq 0\} \\
&= \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \frac{1}{2}\left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\boldsymbol{x} \geq 0\right\} \\
&= \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid \left(\begin{smallmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\boldsymbol{x} \geq 0\}\right\}
\end{aligned}
$$

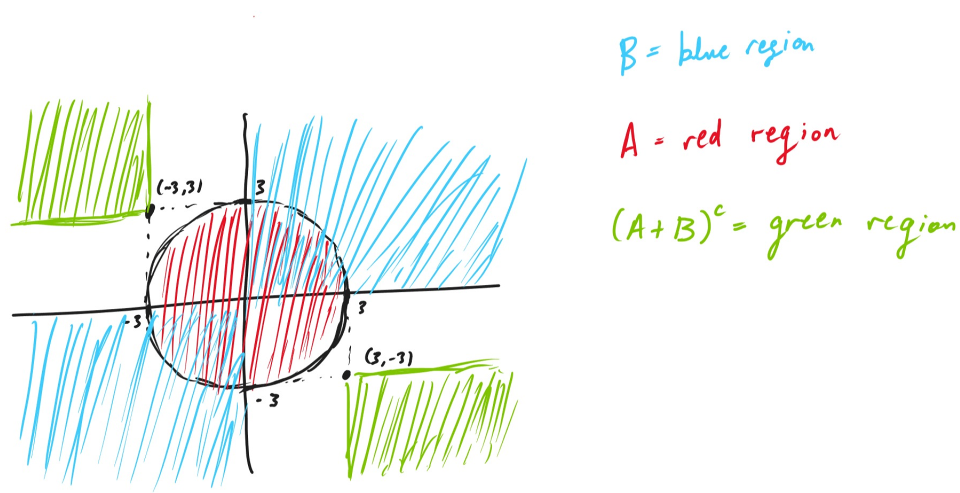
Problem 6

Exercise 6. Graphically find $A - B$, where

1. $A = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 9\}$ and $B = \mathbb{R}^2_{\geq 0} \cup \mathbb{R}^2_{\leq 0}$.
2. A is a set in $\mathbb{R}^2$ and has the shape of a thick letter U rotated 45 degrees to the right and $B = \mathbb{R}^2_{\geq 0}$.

Make sure your pictures are neat and accurate.

Solution 6. 1. First note that $A - B = A + B$ in this case. We find $(A - B)^c = (A + B)^c = \mathbb{R}^2 \backslash (A + B)$ graphically below (we draw the complement $(A + B)^c$ instead $A + B$ since it's easier to visualize).



2. We find $(A - B)^c$ graphically below: