Koszul Complexes

Let R be a ring and let $\boldsymbol{r} = r_1, \ldots, r_m$ be a sequence of elements in R.

1. The Koszul algebra $\mathbb{K} = \mathcal{K}(\boldsymbol{r})$ is defined to be the R-complex whose underlying graded R-module is given by
$$\mathbb{K} = \bigoplus_{\sigma \subseteq \{1\ldots,m\}} e_\sigma R,$$
where we use the notation $e_\sigma = \prod_{i\in\sigma} e_i$ and where e_σ is homogeneous with $|e_\sigma| = \#\sigma$. The differential d of E is defined on the homogeneous basis by $\mathrm{d}e_i = r_i$ and extended everywhere else using the Leibniz law. In particular, we have
$$\mathrm{d}e_\sigma = \sum_{i\in\sigma} (-1)^{\mathrm{pos}(i,\sigma)} r_i e_{\sigma\backslash i}.$$
For example, we haveFor example, if $m = 3$ then we have
$$\mathrm{d}(1) = 0 \qquad \begin{aligned} \mathrm{d}e_1 &= r_1 \\ \mathrm{d}e_2 &= r_2 \\ \mathrm{d}e_3 &= r_3 \end{aligned} \qquad \begin{aligned} \mathrm{d}e_{23} &= e_3 r_2 - e_2 r_3 \\ \mathrm{d}e_{13} &= e_3 r_1 - e_1 r_3 \\ \mathrm{d}e_{12} &= e_2 r_1 - e_1 r_2 \end{aligned} \qquad \mathrm{d}e_{123} = e_{23} r_1 - e_{13} r_2 + e_{12} r_3$$
An alternative description of $\mathbb{K}$ is the the iterated tensor product of complexes:
$$\mathbb{K}(\boldsymbol{r}) \simeq \mathbb{K}(r_1) \otimes_R \mathbb{K}(r_2) \otimes_R \cdots \otimes_R \mathbb{K}(r_m).$$
If M is an R-module, then we set $\mathbb{K}(\boldsymbol{r}, M) := \mathbb{K} \otimes_R M$ and we denote its homology by $\mathrm{H}(\boldsymbol{x}, M)$.

2. Another Koszul complex we are interested in is called the **dual Koszul complex**: it is given by $\mathbb{K}^\star := \mathrm{Hom}_R^\star(\mathbb{K}, R)$. The underlying graded R-module is given by
$$\mathbb{K}^\star = \bigoplus_{\sigma \subseteq \{1,\ldots,m\}} R e_\sigma^\star.$$
Here $e_\sigma^\star \colon E \to R$ is an R-linear map, graded of degree $-(\#\sigma)$, which is defined by
$$e_\sigma^\star(e_\tau) = \begin{cases} 1 & \text{if } \sigma = \tau \\ 0 & \text{else} \end{cases}$$
The differential $\mathrm{d}^\star$ of $E^\star$ is defined by $\mathrm{d}^\star e_\sigma^\star = e_\sigma^\star \mathrm{d}$. In particular, we have
$$\mathrm{d}^\star e_\sigma^\star = (-1)^{\#\sigma+1} \sum_{i\in\sigma^\star} (-1)^{\mathrm{pos}(i,\sigma^\star)} r_i e_{\sigma\cup i}^\star,$$
where $\sigma^\star := \{1,\ldots,m\}\backslash\sigma$. For example, if $m = 3$ then we have
$$\mathrm{d}^\star(1) = -r_1 e_1^\star - r_2 e_2^\star - r_3 e_3^\star \qquad \begin{aligned} \mathrm{d}^\star e_1^\star &= r_3 e_{13}^\star + r_2 e_{12}^\star \\ \mathrm{d}^\star e_2^\star &= r_3 e_{23}^\star - r_1 e_{12}^\star \\ \mathrm{d}^\star e_3^\star &= -r_2 e_{23}^\star - r_1 e_{13}^\star \end{aligned} \qquad \begin{aligned} \mathrm{d}^\star e_{23}^\star &= r_1 e_{123}^\star \\ \mathrm{d}^\star e_{13}^\star &= -r_2 e_{123}^\star \\ \mathrm{d}^\star e_{12}^\star &= r_3 e_{123}^\star \end{aligned} \qquad \mathrm{d}^\star e_{123}^\star = 0$$
Note that the nonzero components of $E^\star$ live in negative homological degree, that is, if $0 < k < m$, then $E_k^\star = 0$ and $E_{-k}^\star \neq 0$. We often think of $E^\star$ as a cochain complex using the upper sign convention $E_{-k}^\star = E^{\star,k}$ and $\mathrm{d}_{-k}^\star = \mathrm{d}^{\star,k}$. Note that the map $\varphi \colon \Sigma^n \mathbb{K}^\star \to \mathbb{K}$ defined by
$$\varphi(e_\sigma^\star) = \mathrm{sign}(\sigma^\star, \sigma) e_{\sigma^\star}$$
is an isomorphism of R-complexes. In particular we obtain $\mathrm{H}_i(\mathbb{K}) \simeq \mathrm{H}_{i-m}(\mathbb{K}^\star)$.

3. The **stable Koszul complex** $\widetilde{\mathbb{K}}$ is complex whose underlying graded R-module is given by

$$\widetilde{\mathbb{K}} = \bigoplus_{\sigma \subseteq \{1,\dots,m\}} \widetilde{e}_\sigma R_{r_\sigma}$$

For example, if $m = 3$ then we have

$$\widetilde{\mathrm{d}}(1) = \widetilde{e}_1 + \widetilde{e}_2 + \widetilde{e}_3 \qquad \begin{aligned} \widetilde{\mathrm{d}}\widetilde{e}_1 &= \widetilde{e}_{13} - \widetilde{e}_{12} \\ \widetilde{\mathrm{d}}\widetilde{e}_2 &= \widetilde{e}_{23} - \widetilde{e}_{12} \\ \widetilde{\mathrm{d}}\widetilde{e}_3 &= \widetilde{e}_{23} - \widetilde{e}_{13} \end{aligned} \qquad \begin{aligned} \widetilde{\mathrm{d}}\widetilde{e}_{23} &= \widetilde{e}_{123} \\ \widetilde{\mathrm{d}}e_{13} &= -\widetilde{e}_{123} \\ \widetilde{\mathrm{d}}\widetilde{e}_{12} &= \widetilde{e}_{123} \end{aligned} \qquad \widetilde{\mathrm{d}}\widetilde{e}_{123} = 0$$

Observe that

$$\widetilde{\mathbb{K}} = \varinjlim \mathbb{K}^\star(\boldsymbol{r}^n),$$

where $\boldsymbol{r}^n = r_1^n, \dots, r_m^n$. In particular, it follows that

$$\mathrm{H}(\boldsymbol{r}^\infty, M) = \bigcup_{n \geq 0} \mathrm{H}(\boldsymbol{r}^n, M) = \varinjlim \mathrm{H}(\boldsymbol{r}^n, M).$$