Mathematical Programming Homework 2

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Problem 1

Problem 1.a

Exercise 1. Find the equation of the plane passing through the points $A = (4, 0, 0)^\top$, $B = (0, 6, 0)^\top$, and $C = (0, 0, 12)^\top$. Write this equation in the form $a^\top x = k$.

Solution 1. Plugging in the points A, B, C into the equation $a^\top x = k$ gives us the three equations

$$\begin{aligned} 4a_1 &= k \\ 6a_2 &= k \\ 12a_3 &= k. \end{aligned}$$

A solution to this system of equations is $k = 12$ and $a = (3, 2, 1)^\top$. Thus the plane defined given by the equation

$$\begin{aligned} 12 &= 3x_1 + 2x_2 + x_3 \\ &= a_1x_1 + a_2x_2 + a_3x_3 \\ &= a^\top x \end{aligned}$$

contains the points A, B, and C.

Problem 1.b

Exercise 2. Is the point $x^0 = (1, 2, 5)^\top$ located in this plane? Explain why.

Solution 2. Yes, because the point x^0 is a solution to the equation $a^\top x = 12$:

$$\begin{aligned} a^\top x^0 &= 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 5 \\ &= 3 + 4 + 5 \\ &= 12. \end{aligned}$$

Problem 1.c

Exercise 3. Write the equation of this plane in the form $a^\top (x - x^0) = 0$ where $x^0 = (1, 2, 5)^\top$.

Solution 3. We again use $a = (3, 2, 1)^\top$. We have

$$\begin{aligned} 0 &= a^\top (x - x^0) \\ &= 3(x_1 - 1) + 2(x_2 - 2) + (x_3 - 5) \\ &= 3x_1 - 3 + 2x_2 - 4 + x_3 - 5 \\ &= 3x_1 + 2x_2 + x_3 - 12. \end{aligned}$$

Problem 2

Exercise 4. Let $f\colon [-2.5, 5.5] \to \mathbb{R}$ be defined by

$$f(x) = 3x^4 - 20x^3 - 24x^2 + 240x + 400.$$

Find all local/global minima/maxima and inflection points of this function on its domain.

Solution 4. First we calculate

$$\begin{aligned} f'(x) &= 12(x^3 - 5x^2 - 4x + 20) \\ f''(x) &= 12(3x^2 - 10x - 4). \end{aligned}$$

Now we calculate the roots of f':

$$\begin{aligned} f'(x) = 0 &\iff x^3 - 5x^2 - 4x + 20 = 0 \\ &\iff (x-5)(x-2)(x+2) = 0 \\ &\iff x \in \{-2, 2, 5\}. \end{aligned}$$

With this information so far, we can determine what the local minima/maxima are:

1. Since $f''(-2) = 336 > 0$, we see that f has a local minimum at $x = -2$.
2. Since $f''(2) = -144 < 0$, we see that f has a local maximum at $x = 2$.
3. Since $f''(5) = 252 > 0$, we see that f has a local minimum at $x = 5$.

Since $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to-\infty} f(x)$, we see that f does not have a global maximum, but does have a global minimum. The only possible places where f can have a global minimum is at the local minima. Since

$$\begin{aligned} f(-2) &= 32 \\ &< 375 \\ &= f(5), \end{aligned}$$

we see that f has a global minimum at $x = -2$ (and only has a local minimum at $x = 5$). Finally, note that

$$\begin{aligned} f''(x) = 0 &\iff 3x^2 - 10x - 4 = 0 \\ &\iff \left(x - \frac{5-\sqrt{37}}{3}\right)\left(x - \frac{5+\sqrt{37}}{3}\right) = 0 \\ &\iff x \in \left\{\frac{5-\sqrt{37}}{3}, \frac{5+\sqrt{37}}{3}\right\}, \end{aligned}$$

since $x = (5 \pm \sqrt{37})/3$ are simple roots of f'' (meaning multiplicity one), they must correspond to the inflection points of f.

Problem 3

Exercise 5. Consider a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with level sets defined as

$$S_\alpha = \{x \in \mathbb{R}^n \mid f(x) \le \alpha\}$$

for $\alpha \in \mathbb{R}$.

1. Prove that if f is a convex function, then the level sets S_α are convex sets.
2. If the level set S_α is a convex set for all $\alpha \in \mathbb{R}$, is the function f necessarily convex? Explain.

Solution 5. 1. Assume f is a convex function. Let $\alpha \in \mathbb{R}$, let $x, y \in S_\alpha$, and let $t \in (0,1)$. Then observe that

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\leq t\alpha + (1-t)\alpha \\ &= (t + 1 - t)\alpha \\ &= \alpha. \end{aligned}$$

It follows that $tx + (1-t)y \in S_\alpha$. Since $\alpha \in \mathbb{R}$ was arbitrary, we see that S_α is convex for all $\alpha \in \mathbb{R}$.

2. No, consider $n = 1$ and $f(x) = -e^x$. Observe that

$$S_\alpha = \begin{cases} \mathbb{R} & \text{if } \alpha \geq 0 \\ (-\infty, \ln \alpha] & \text{if } \alpha < 0 \end{cases}$$

In each case, S_α is convex, even though $-e^x$ is not convex.

Problem 4

Exercise 6. Check that the function $f(x) = 2x_1^2 x_2^{-1}$ is convex or strictly convex on the strictly positive orthant $\{x \in \mathbb{R}^2 \mid x > 0\}$.

Solution 6. Let $x \in \{x \in \mathbb{R}^2 \mid x > 0\}$, We calculate the Hessian matrix of f at x:

$$\begin{aligned} \mathrm{H}_f(x) &= \begin{pmatrix} \partial_{x_1}\partial_{x_1} f(x) & \partial_{x_1}\partial_{x_2} f(x) \\ \partial_{x_2}\partial_{x_1} f(x) & \partial_{x_2}\partial_{x_2} f(x) \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{x_2} & -\frac{4x_1}{x_2^2} \\ -\frac{4x_1}{x_2^2} & \frac{4x_1^2}{x_2^3} \end{pmatrix}. \end{aligned}$$

The leading principal minors of the Hessian are

$$\frac{16x_1^2}{x_2^4} - \frac{16x_1^2}{x_2^4} = 0 \quad \text{and} \quad \frac{4}{x_2} > 0.$$

In particular we see that the Hessian is positive semidefinite but not positive definite, so f is convex but not strictly convex.

Problem 5

Exercise 7. Consider the problem

$$\text{minimize } f(x_1, x_2) = (x_2 - x_1)^2 (x_2 - 2x_1^2)$$

1. Check whether the first- and second-order necessary conditions and the second-order sufficient conditions for optimality are satisfied at $(0,0)^\top$.
2. Show that $(0,0)^\top$ is a local minimizer of f along any line passing through the origin (i.e., consider a line $x_2 = mx_1$).
3. Show that $(0,0)^\top$ is not a local minimizer of f along any curve passing through the origin (i.e., consider a curve $x_2 = mx_1^2$).

Solution 7. 1. First calculate the gradient of f at $(0,0)$:

$$\begin{aligned} \nabla f(x)\Big|_{(0,0)} &= \begin{pmatrix} -2(x_1 - x_2)(4x_1^2 - 2x_1x_2 - x_2) \\ (4x_1^2 + x_1 - 3x_2)(x_1 - x_2) \end{pmatrix}\Bigg|_{(0,0)} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Next we calculate the Hessian of f at $(0,0)$:

$$\begin{aligned} \mathrm{H}_f(x_1,x_2)\Big|_{(0,0)} &= \begin{pmatrix} -2(x_1-x_2)(4x_1^2-2x_1x_2-x_2) \\ (4x_1^2+x_1-3x_2)(x_1-x_2) \end{pmatrix}\Bigg|_{(0,0)} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Problem 6

Exercise 8. Consider the problem

$$\text{minimize } f(x_1,x_2) = x_1^2 + x_1x_2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}$$

1. Write down the first-order necessary conditions for optimality.
2. Check whether the point $(0,0)^\top$ is a local optimal solution. If not, find a direction $d \in \mathbb{R}^2$ along which the function decreases.
3. Minimize the function starting from $(0,0)^\top$ along the direction d you have found above.

Solution 8. 1. The first-order necessary condition states that $\overline{x} = (\overline{x}_1, \overline{x}_2)$ is a minimizer only if $\nabla f(\overline{x}) = 0$, that is, only if

$$\begin{pmatrix} 2\overline{x}_1 + \overline{x}_2 - 2 + e^{\overline{x}_1+\overline{x}_2} \\ \overline{x}_1 + 4\overline{x}_2 + e^{\overline{x}_1+\overline{x}_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

2. Since

$$\begin{aligned} \nabla f(0) &= \begin{pmatrix} 2\cdot 0 + 0 - 2 + e^{0+0} \\ 0 + 4\cdot 0 + e^{0+0} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &\neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

we see that $(0,0)^\top$ is not a local optimal solution. Since $\partial_{x_1} f(0,0) = -1$, we see that the function decreases along the direction $d = (1,0)$.

3. Now we minimize the function starting from $(0,0)^\top$ along the direction $d = (1,0)$. Let $g(t) = f(t,0)$, so

$$g(t) = t^2 - 2t + e^t.$$

Observe that

$$\begin{aligned} g'(t) = 0 &\iff 2t - 2 + e^t = 0 \\ &\iff t \approx 0.314923 \end{aligned}$$

$$\overline{x} \text{ is a minimizer only if } \nabla f(\overline{x}) = 0 \iff \overline{x} \text{ is a minimizer only if } \begin{pmatrix} 2\overline{x}_1 + \overline{x}_2 - 2 + e^{\overline{x}_1+\overline{x}_2} \\ \overline{x}_1 + 4\overline{x}_2 + e^{\overline{x}_1+\overline{x}_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Thus we need

$$\begin{aligned} 2\overline{x}_1 + \overline{x}_2 - 2 + e^{\overline{x}_1+\overline{x}_2} &= 0 \\ \overline{x}_1 + 4\overline{x}_2 + e^{\overline{x}_1+\overline{x}_2} &= 0 \end{aligned}$$

From these two equations we obtain $\overline{x}_1 - 3\overline{x}_2 - 2 = 0$, or in other words, $\overline{x}_1 = 3\overline{x}_2 + 2$.