# Models and Applications Project

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## Introduction

Suppose an investor wishes to select a set of assets wants to maximize (as high as possible) their return on investment and simultaneously minimize (as low as possible) the risk of investment as well as their losses. In order to achieve this, the investor may want to consider this problem from an MCDM/MCDA problem where there are many techniques/tools the investor can use to solve this in the best way possible (according to their preference). In this project, we will focus on the paper "Fuzzy Numbers and MCDM Methods for Portfolio Optimization" by Thi T. Nguyen and Lee N. Gordon-Brown, and will try to discuss how they converted this into an MCDA problem.

## Notation

Let $U(\omega)$ denote the utility of terminal wealth $\omega$. Thus $U(\omega)$ is an analytic function defined on $\mathbb{R}_{>0}$ which is increasing (for instance, one can use $U(\omega) = \ln \omega$). Let $\boldsymbol{R} = (R_1, \dots, R_n)^\top$ be the rates of return of $n$ risky assets (which one may regard as being fixed) and let $\boldsymbol{\mu} = \mathrm{E}(\boldsymbol{R}) = (\mu_1, \dots, \mu_n)^\top$. Let $\boldsymbol{w} = (w_1, \dots, w_n)^\top$ be a weight vector representing the proportiond of wealth allocated to various assets (we do not regard $\boldsymbol{w}$ as being fixed at the moment: the $w_i$ will be our decision variables for the upcoming optimization problem we will set up). In particular, $\boldsymbol{w}$ should satisfy the following constraints:

$$\sum_i w_i = 1 \text{ and } \boldsymbol{w} \geq 0. \tag{1}$$

We shall assume that $\omega$ is normalized in the sense that $\omega = 1 + r_\mathrm{p}$ where $r_\mathrm{p}$ is the rate of return on our entire portfolio (with respect to $\boldsymbol{w}$):

$$r_\mathrm{p} = \boldsymbol{w}^\top \boldsymbol{R} = \sum_i w_i R_i.$$

Thus if $r_\mathrm{p}$ increases, then the utility function $U(\omega)$ increase as well. Since $U(\omega)$ is analytic at $\mathrm{E}(\omega)$, we can express it locally at $\mathrm{E}(\omega)$ in terms of the infinite Taylor series as:

$$U(\omega) = \sum_{k=0}^{\infty} \frac{U^{(k)}(\mathrm{E}(\omega))}{k!} (\omega - \mathrm{E}(\omega))^k \tag{2}$$

Applying the expectation operator to both sides of (2) gives us

$$\mathrm{E}(U(\omega)) = \sum_{k=0}^{\infty} \frac{U^{(k)}(\mathrm{E}(\omega))}{k!} \mathrm{E}((\omega - \mathrm{E}(\omega))^k). \tag{3}$$

The expected utility from an investment in risky assets depends on all central moments, however for numerical purposes, we only consider the first four moments. Thus we approximate $\mathrm{E}(U(\omega))$ using the fourth-order Taylor expansion:

$$\mathrm{E}(U(\omega)) \approx U(\mathrm{E}(\omega)) + \frac{1}{2!} U''(\mathrm{E}(\omega)) \sigma_\mathrm{p}^2 + \frac{1}{3!} U'''(\mathrm{E}(\omega)) s_\mathrm{p}^3 + \frac{1}{4!} U''''(\mathrm{E}(\omega)) \kappa_\mathrm{p}^4.$$

where we set

$$\begin{aligned}
\mu_\mathrm{p} &= \mathrm{E}(r_\mathrm{p}) = \boldsymbol{w}^\top \boldsymbol{\mu} \\
\sigma_\mathrm{p}^2 &= \mathrm{E}((r_\mathrm{p} - \mu_\mathrm{p})^2) = \mathrm{E}((\omega - \mathrm{E}(\omega))^2) \\
s_\mathrm{p}^3 &= \mathrm{E}((r_\mathrm{p} - \mu_\mathrm{p})^3) = \mathrm{E}((\omega - \mathrm{E}(\omega))^3) \\
\kappa_\mathrm{p}^4 &= \mathrm{E}((r_\mathrm{p} - \mu_\mathrm{p})^4) = \mathrm{E}((\omega - \mathrm{E}(\omega))^4)
\end{aligned}$$

These are called the **expected return**, **variance**, **skewness**, and the **kurtosis** of our portfolio respectively. We define the $(n, n)$ covariance matrix $M_2$, the $(n, n^2)$ coskewness matrix $M_3$, and the $(n, n^3)$ cokurtosis matrix by

$$\begin{aligned} M_2 &= \mathrm{E}((\boldsymbol{R}-\boldsymbol{\mu})(\boldsymbol{R}-\boldsymbol{\mu})^\top) = (\sigma_{ij}) \\ M_3 &= \mathrm{E}((\boldsymbol{R}-\boldsymbol{\mu})(\boldsymbol{R}-\boldsymbol{\mu})^\top \otimes (\boldsymbol{R}-\boldsymbol{\mu})^\top) = (s_{ijk}) \\ M_4 &= \mathrm{E}((\boldsymbol{R}-\boldsymbol{\mu})(\boldsymbol{R}-\boldsymbol{\mu})^\top \otimes (\boldsymbol{R}-\boldsymbol{\mu})^\top \otimes (\boldsymbol{R}-\boldsymbol{\mu})^\top) = (\kappa_{ijkl}), \end{aligned}$$

where $\otimes$ dentoes the Kronecker product and where

$$\begin{aligned} \sigma_{ij} &= \mathrm{E}((R_i-\mu_i)(R_j-\mu_j)) \\ s_{ijk} &= \mathrm{E}((R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)) \\ \kappa_{ijkl} &= \mathrm{E}((R_i-\mu_i)(R_j-\mu_j)(R_k-\mu_k)(R_l-\mu_l)) \end{aligned}$$

for all $1 \leq i, j, k, l \leq n$. Given a portfolio weight $\boldsymbol{w}$, we calculate the higher momenta of our portfolio using the matrices $M_1$, $M_2$, and $M_3$ defined above as:

$$\begin{aligned} \mu_{\mathrm{p}} &= \boldsymbol{w}^\top \boldsymbol{\mu} = \sum_i w_i \mu_i \\ \sigma_{\mathrm{p}}^2 &= \boldsymbol{w}^\top M_2 \boldsymbol{w} = \sum_{i,j} w_i w_j \sigma_{ij} \\ s_{\mathrm{p}}^3 &= \boldsymbol{w}^\top M_3 (\boldsymbol{w}^{\otimes 2}) = \sum_{i,j,k} w_i w_j w_k s_{ijk} \\ \kappa_{\mathrm{p}}^4 &= \boldsymbol{w}^\top M_4 (\boldsymbol{w}^{\otimes 3}) = \sum_{i,j,k,l} w_i w_j w_k w_l \kappa_{ijkl}. \end{aligned}$$

The higher momenta of our portfolio gives us a lot of information in regards to how our portfolio is strucutred. For instance, a high value of $\sigma_{ii}$ indicates asset $i$ has high volatility or has high risk, wheras a low value of $\sigma_{ii}$ indicates asset $i$ has low volatility or has low risk. A negative value of $\sigma_{ij}$ where $i \neq j$ indicates that the values of the assets $i$ and $j$ move in opposite directions (which is a desirable feature in a diversified portfolio), and a positive value of $\sigma_{ij}$ where $i \neq j$ indicates that the values of the assets $i$ and $j$ move in the same direction (which often occurs with stocks in companies in the same industry). The skewness and kurtosis have their own interpretations as well. Typically, an investor would prefer to have high portfolio skewness and low variance/kurtosis, however this may vary depending on the investors preferences.

## Marginal Impact

The **marginal impact** of the asset $i$ to the porfolio's return, variance, skewness, and kurtosis is defined to be the $i$th component of the vectors below:

$$\begin{aligned} \nabla_{\boldsymbol{w}} \mu_{\mathrm{p}} &= \boldsymbol{\mu} \\ \nabla_{\boldsymbol{w}} \sigma_{\mathrm{p}}^2 &= 2 M_2 \boldsymbol{w} \\ \nabla_{\boldsymbol{w}} s_{\mathrm{p}}^3 &= 3 M_3 \boldsymbol{w}^{\otimes 2} \\ \nabla_{\boldsymbol{w}} \kappa_{\mathrm{p}}^4 &= 4 M_4 \boldsymbol{w}^{\otimes 3}. \end{aligned}$$

Assets with higher marginal impact will make relatively large changes to the moments of our portfolio with respect to small changes of the weight of that asset; thus they will have *more* influence to our overall portfolio compared to other assets. Note that the marginal impact of asset $i$ to the whole portfolio return is given by the expected return $\mu_i$ which does not depend on the other assets $j$ where $j \neq i$, however the marginal impact of the asset $i$ to the higher moments will typically take into account the other assets $j$ where $j \neq i$. For instance, since the $\sigma_{ij}$ are symmetric in $i$ and $j$, the portfolio's variance can be expressed as

$$\sigma_{\mathrm{p}}^2 = \sum_{1 \leq i \leq n} w_i^2 \sigma_i^2 + 2 \sum_{1 \leq i < j \leq n} w_i w_j \sigma_{ij}.$$

The second term on the right represents the diversification effect of the overall portfolio's variance. The variance marginal impact of asset $i$ to the whole portfolio's variance is given by

$$\partial_{w_i} \sigma_{\mathrm{p}}^2 = 2 \sum_{1 \leq j \leq n} w_j \sigma_{ij},$$

which clearly depends on all assets in the portfolio. Note that the $\sigma_{ij}$ can be obtained from historical/simulation data, however the $w_j$ are still unknown at this stage. Thus estimates of the marginal impacts are required in order to evaluate the performance of different assets. We proceed with the following strategy: for each criteria (variance, skewness, and kurtosis) we calculate two weight vector relateds to minimum and maximum circumstances. We then use them to calculate the *exact* marginal imapcts of the assets to the portfolio momenta using the partial derivative formulas above. Thus we can calculate the marginal impacts of each asset in the extreme cases, however the exact contribution of an asset on the portfolio's higher moments is uncertain before choosing a weight vector $\boldsymbol{w}$. We handle this problem using fuzzy numbers.

### Fuzzy Numbers

A **fuzzy set** $A$ is a pair $(X, f)$ where $X$ is a subset of $\mathbb{R}$ and $f$ is a function from $X$ to $[0,1]$. We denote $f = f_A$ and call it the **membership function** of $A$. For each $x \in X$, the value $f_A(x)$ is called the **grade** of membership of $x$ in $A$. Intuitively, if $f_A(x) = 1$, then we think of $x$ as *fully* belonging to the fuzzy set $A$, and if $f_A(x) = \varepsilon$ where $0 < \varepsilon < 1$ then we think of $x$ as *partially* belonging to $A$. We set

$$\begin{aligned} A^{\geq\alpha} &:= \{f_A \geq \alpha\} \\ A^{>\alpha} &:= \{f_A > \alpha\} \\ \mathrm{Supp}(A) &:= \{f_A > 0\} \\ \mathrm{Core}(A) &:= \{f_A = 1\} \end{aligned}$$

We call these the **$\alpha$-cut** of $A$, the **strict $\alpha$-cut** of $A$, the **support** of $A$, and the **core** of $A$ respectively. For this project, we only consider fuzzy sets of the form $A = (\mathbb{R}, f)$ where the membership function is given by

$$f_A(x) = \begin{cases} (x-a)/(b-a) & a \leq x \leq b \\ 1 & b \leq x \leq c \\ (d-x)/(d-c) & c \leq x \leq d \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

These are called trapezoidal fuzzy numbers since the graph of $f_A$ takes the shape of a trapezoid.

### Weighting Schemes

As we mentioned before, rational investor would typically prefer high portfolio skewness and low variance/kurtosis. Thus they would like to maximize (as high as possible) skewness and minimize (as low as possible) variance and kurtosis. Obviously there are many ways we can do this, depending on the investors preference or utility. To this end, we consider a weighting scheme $\mathrm{s} = (\mathrm{s}_r : \mathrm{s}_v : \mathrm{s}_s : s_\kappa)$ which measures the investors preference. For instance, the scheme $(4:3:2:1)$ indicates that the investor favors return the most, then favors variance the second-most, then favors skewness the third-most, and favors kurtosis the least. In the paper, the authors consider various weighting schemes like this, and to each weighting scheme, they attach a trapezoidal fuzzy number expressed in vector notation. For example, the fuzzy number corresponding to the weighting scheme $(4:3:2:1)$ is given by $(1, 3/4, 1/2, 1)$.

## MCDM Approaches to Portfolio Selection

In the paper, the authors discuss two possible MCDM approaches to portfolio selection: the SAW method and the TOPSIS method. We focus on the SAW method.

### The Decision Variables

The decision matrix is the $n \times 4$ matrix $D = (x_{ij})$. The $i$th row of $D$ is denoted $A_i$ and the $i$th column of $D$ is denoted $C_j$. The $A_i$ correspond to possible assets among which an investor chooses to allocate their initial wealth, the $C_j$ correspond return, variance, skewness, and kurtosis criteria with which asset performance is measured, and the $x_{ij}$ is the centroid of a trapezoidal fuzzy number which represents the maginal impact of asset $A_i$ with respect to criterion $C_j$.

## Benefit Criteria

The SAW method requires a comparable scale for all elements in the decision matrix. This will be a normalized matrix $R = (r_{ij})$ which we define as follows: let $\varepsilon_j$ and $\delta_j$ denote the minimum and maximum values in $C_j$. We set

$$r_{ij} = \frac{x_{ij} - \varepsilon_j}{\delta_j - \varepsilon_j}$$

for all $i, j$. We refer to this as benefit criteria (the larger the rating, the greater the preference). WE weight of each criterion is obtained from the investor's normalized preference vector $\bar{\mathrm{s}} = \mathrm{s}/\|\mathrm{s}\|$. The performance score $p_i$ of the $i$th asset is given by

$$p_i = \sum_j \bar{s}_j r_{ij}.$$