Adic Rings

0.1 Formal Schemes

Definition 0.1. An **adic ring** is a topological ring A carrying the $\mathfrak{a}$-adic topology, called an **ideal of definition**.

Remark 1. Note that the topology of A is part of the data, but the ideal of definition is not (there may be many ideals of definition).

For an adic ring A, we set $\operatorname{Spf} A$ to be the set of open prime ideals of A. If $\mathfrak{a}$ is an ideal of definition, then

$$\operatorname{Spf} A = \mathrm{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

We give $\operatorname{Spf} A$ the structure of a topological ringed space as follow: for each $s \in A$ we define

$$\mathrm{D}(s) = \{\mathfrak{p} \in \operatorname{Spf} A \mid s \notin \mathfrak{p}\},$$

and declare that the $\mathrm{D}(s)$ generate the topology of $\operatorname{Spf} A$. Note that if $s \in \mathfrak{a}$, then clearly $\mathrm{D}(s) = \emptyset$. The structure sheaf $\mathcal{O} = \mathcal{O}_{\operatorname{Spf} A}$ is defined by setting $\mathcal{O}(\mathrm{D}(s))$ to be the $\mathfrak{a}$-adic completion of A_s.

Definition 0.2. A **formal scheme** is a topologically ringed space which is locally for the form $\operatorname{Spf} A$ for an adic ring A.

Remark 2. Let A be a ring, let M be an A-module, and let $\mathfrak{a}$ be a finitely generated ideal of A. Then one has

$$\widehat{M}/\mathfrak{a}\widehat{M} = M/\mathfrak{a}M,$$

where $\widehat{M}$ denotes the $\mathfrak{a}$-adic completion of M. This implies in particular that $\widehat{M}$ is $\mathfrak{a}$-adically complete:

$$\varprojlim \widehat{M}/\mathfrak{a}^n\widehat{M} = \varprojlim M/\mathfrak{a}^n M = \widehat{M}.$$

For this reason we usually only concern ourselves with finitely generated ideals $\mathfrak{a}$.

The category of formal schemes contains the category of schemes as a full subcategory, via the functor which carries $\operatorname{Spec} A$ to $\operatorname{Spf} A$ where A is considered with the discrete topology. A typical example of a formal scheme is $X = \operatorname{Spf} \mathbb{Z}_p$, the formal unit disc over $\mathbb{Z}_p[\![x]\!]$. In this case, if R is any adic $\mathbb{Z}_p$-algebra, one has $X(R) = R^{\circ\circ}$, the ideal of topologically nilpotent elements in R (i.e the set of all r such that $r^n \to 0$ as $n \to \infty$). In particular if $K/\mathbb{Q}_p$ is an extension of nonarchimedean fields, and $K^\circ \subset K$ is its ring of integers, then $X(K^\circ) = K^{\circ\circ}$ is the open unit disc in K.

0.2 Rigid-analytic Spaces

Let K be a nonarchimedean field (a field complete with respect to a nontrivial absolute value $|\cdot|$). For each $n \geq 0$, we have the **Take K-algebra** $K\langle \boldsymbol{x}\rangle = K\langle x_1, \ldots, x_n\rangle$ which is the completion of $K[\boldsymbol{x}]$ under the Gauss norm. Equivalently, $K\langle \boldsymbol{x}\rangle$ is the ring of formal power series in $\boldsymbol{x}$ with coefficients in K tending to 0. A **K-affinoid algebra** is a topological K-algebra A which is isomorphic to a quotient of some $K\langle \boldsymbol{t}\rangle$.

Suppose A is a K-affinoid algebra. For a point