Algebro-Geometric Classification

Let $\mathbb{k}$ be a commutative ring and let F be a finite free graded $\mathbb{k}$-module such that $F_0 = \mathbb{k}$, $F_i = 0$ for all $i < 0$, and $F_+ \neq 0$. In this note, we give an algebro-geometric classification of various structures we can attach to F. We begin by classifying all $\mathbb{k}$-complex structures on F which fixed the identity element $1 \in \mathbb{k} = F_0$.

Classifying $\mathbb{k}$-Complex Structures on F

Let us state up front what we wish to prove:

Theorem 0.1. *We have the following bijection of sets:*

$$\left\{ \mathrm{GL}_n(\mathbb{k})\text{-orbits of } h_{\mathrm{A}^{\mathrm{d}}_{\mathbb{k}}(F)}(\mathbb{k}) \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of } \mathbb{k}\text{-complex} \\ \text{structures on } F \text{ with fixed identity} \end{array} \right\}$$

where $\mathrm{A}^{\mathrm{d}}_{\mathbb{k}}(F)$ is a $\mathbb{k}$-algebra (to be constructed below) and where

$$h_{\mathrm{A}^{\mathrm{d}}_{\mathbb{k}}(F)}(\mathbb{k}) := \mathrm{Hom}_{\mathbb{k}\text{-}alg}(\mathrm{A}^{\mathrm{d}}_{\mathbb{k}}(F), \mathbb{k})$$

is the $\mathbb{k}$-valued points of $\mathrm{A}^{\mathrm{d}}_{\mathbb{k}}(F)$. Two $\mathbb{k}$-complex structures (F, d) and (F, d') on F are said to be isomorphic with fixed identity if there exists a chain map $\varphi \colon F \to F$ such that $\varphi(1) = 1$.

The proof of this theorem mostly involves setting up our notation which will be used later on when we wish to classify other algebraic structures on F.

Proof. Let d be a $\mathbb{k}$-linear differential on F, meaning $\mathrm{d}\colon F \to F$ is a graded $\mathbb{k}$-linear map of degree -1 which satisfies $\mathrm{d}^2 = 0$. Choose an ordered homogeneous basis $\boldsymbol{e} = (e_0, e_1, \dots, e_n)$ of F where we set $e_0 = 1$ and let $\boldsymbol{d} = (d^i_j)$ be the matrix representation of the differential d with respect to the ordered homogeneous basis $\boldsymbol{e}$. Thus we have $\mathrm{d}\boldsymbol{e} = \boldsymbol{e}\boldsymbol{d}$ where $\mathrm{d}\boldsymbol{e} = (0, \mathrm{d}e_1, \dots, \mathrm{d}e_n)$ and $\boldsymbol{e}\boldsymbol{d}$ is the product of the row vector $\boldsymbol{e}$ on the left with the matrix $\boldsymbol{d}$ on the right. Alternatively we could express this in terms of the matrix entries of $\boldsymbol{d}$: for each $0 \leq j \leq n$ we have

$$\mathrm{d}e_j = \sum_{0 \leq i \leq n} d^i_j e_i.$$

Note that since d is graded of degree -1, we necessarily have $d^i_j = 0$ whenever $|e_i| \neq |e_j| - 1$. Also note that since $\mathrm{d}^2 = 0$, we have $\boldsymbol{d}^2 = 0$. Again we can express this in terms of matrix entries of $\boldsymbol{d}$: for each $0 \leq i, j \leq n$ we have

$$\sum_{0 \leq \iota \leq n} d^\iota_j d^i_\iota = 0 \tag{1}$$

Now consider the following polynomial ring following polynomial ring

$$\mathbb{k}[\boldsymbol{D}] = \mathbb{k}[\{D^i_j \mid 0 \leq i, j \leq n\}]$$

where the D^i_j are coordinates which correspond to the matrix entries of $\boldsymbol{d}$. Let $\mathrm{e}_{\boldsymbol{d}} \colon \mathbb{k}[\boldsymbol{D}] \twoheadrightarrow \mathbb{k}$ be the $\mathbb{k}$-algebra homomorphism given by $\mathrm{e}_{\boldsymbol{d}}(\boldsymbol{D}) = \boldsymbol{d}$ and set $\mathfrak{q}_{\boldsymbol{d}} = \langle \boldsymbol{D} - \boldsymbol{d} \rangle$ to be the kernel of this evaluation map: it is the $\mathbb{k}[\boldsymbol{D}]$-ideal generated by $D^i_j - d^i_j$ for all $0 \leq i, j \leq n$. Note that if $\mathbb{k}$ is an integral domain, then $\mathfrak{q}_{\boldsymbol{d}}$ is a prime ideal since $\mathbb{k}[\boldsymbol{D}]/\mathfrak{q}_{\boldsymbol{d}} \cong \mathbb{k}$, and if $\mathbb{k}$ is a field, then $\mathfrak{q}_{\boldsymbol{d}}$ is a maximal ideal of $\mathbb{k}[\boldsymbol{D}]$ and $\mathbb{k} \to \mathbb{k}[\boldsymbol{D}]/\mathfrak{q}_{\boldsymbol{d}}$ is a finite extension of fields. For each $0 \leq i, j \leq n$ we define the quadratic polynomials $\Delta^i_j \in \mathbb{k}[\boldsymbol{D}]$ by:

$$\Delta^i_j := \sum_{0 \leq \iota \leq n} D^\iota_j D^i_\iota.$$

Then we see that the evaluation map $\mathrm{e}_{\boldsymbol{d}}\colon \Bbbk[\boldsymbol{D}] \twoheadrightarrow R$ factors through a unique $\Bbbk$-algebra homomorphism $\overline{\mathrm{e}}_{\boldsymbol{d}}\colon \mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) \twoheadrightarrow \Bbbk$ where we set

$$\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) := \Bbbk[\boldsymbol{D}]/\langle \boldsymbol{\Delta} \rangle \cup \langle \{D^i_j \mid |e_i| \neq |e_j| - 1\} \rangle$$

where we set $\boldsymbol{\Delta} = (\Delta^i_j)$. Conversely, suppose $\mathrm{e}_{\boldsymbol{r}}\colon \Bbbk[\boldsymbol{D}] \twoheadrightarrow \Bbbk$ is another $\Bbbk$-algebra homomorphism where $\mathrm{e}_{\boldsymbol{r}}(\boldsymbol{D}) = \boldsymbol{r}$ where $\boldsymbol{r} = (r^i_j)$. Then we define a differential $\mathrm{d}_{\boldsymbol{r}}$ on F by $\mathrm{d}_{\boldsymbol{r}}\boldsymbol{e} := \boldsymbol{e}\boldsymbol{r}$. Thus if we set $\mathrm{Diff}_{\Bbbk}(F)$ be the set of all $\Bbbk$-linear differentials on F, then we have a bijection of sets:

$$h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) := \mathrm{Hom}_{\Bbbk\text{-alg}}(\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F), \Bbbk) \simeq \mathrm{Diff}_{\Bbbk}(F).$$

Now suppose that $\boldsymbol{e}' = (1, e'_1, \dots, e'_n)$ is another ordered homogeneous basis of F. Thus there is a graded $\Bbbk$-linear isomorphism $\varphi\colon F \to F$ such that $\varphi\boldsymbol{e} = \boldsymbol{e}'$. Let $\widetilde{\gamma}_\varphi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_\varphi \end{smallmatrix}\right)$ be the matrix representation of φ with respect to $\boldsymbol{e}$ where $\gamma_\varphi \in \mathrm{GL}_n(\Bbbk)$. Thus we have $\varphi\boldsymbol{e} = \boldsymbol{e}' = \boldsymbol{e}\widetilde{\gamma}_\varphi$. Then the matrix representation of d in the $\boldsymbol{e}'$ coordinates is given by $\boldsymbol{d}' = \widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi$ since

$$\begin{aligned} \mathrm{d}\boldsymbol{e}' &= \mathrm{d}\boldsymbol{e}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}\boldsymbol{d}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}'\widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}'\boldsymbol{d}'. \end{aligned}$$

Thus we see that $\mathrm{GL}_n(\Bbbk)$ acts on $h_{\mathrm{A}^{\mathrm{d}}_{R}(F)}(\Bbbk)$ by conjugation $\mathrm{e}_{\boldsymbol{d}} \mapsto \mathrm{e}_{\widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi}$. On the other hand, if we define $\mathrm{d}'\colon F \to F$ by $\mathrm{d}' = \varphi^{-1}\mathrm{d}\varphi$, then we obtain $\mathrm{d}'\boldsymbol{e} = \boldsymbol{e}\boldsymbol{d}'$, hence d' is the differential on F whose matrix representation with respect to our original ordered basis $\boldsymbol{e}$ is $\boldsymbol{d}'$. In particular, $\mathrm{e}_{\boldsymbol{d}}$ and $\mathrm{e}_{\boldsymbol{d}'}$ belong to the same $\mathrm{GL}_n(\Bbbk)$-orbit in $h_{\mathrm{A}^{\mathrm{d}}_{R}(F)}(\Bbbk)$ if and only if the corresponding differentials d and d' give isomorphic $\Bbbk$-complex structures on F with fixed identity. □

Base Change

Suppose that R is a $\Bbbk$-algebra. Then $G := F \otimes_{\Bbbk} A$ is a finite free graded R-module with $G_0 \simeq R$, $G_i = 0$ for all $i < 0$, and $G_+ \neq 0$. We set

$$\mathrm{A}^{\mathrm{d}}_{R}(G) := \mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) \otimes_{\Bbbk} R \simeq R[\boldsymbol{D}]/\langle \boldsymbol{\Delta} \rangle \cup \langle \{D^i_j \mid |e_i| \neq |e_j| - 1\} \rangle.$$

It is clear that we have an inclusion of sets $h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(R) \subseteq h_{\mathrm{A}^{\mathrm{d}}_{R}(G)}(R)$.

Proposition 0.1. *Let $G = \mathrm{Aut}(R/\Bbbk)$. Then G acts on $h_{\mathrm{A}^{\mathrm{d}}_{R}(G)}(R)$ and the set of all fixed points is precisely $h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(R)$.*

Classifying Other Algebraic Structures on F

Let $\lambda\colon F \to F$ and $\mu\colon F \otimes_R F \to F$ be graded R-linear maps. With F equipped with λ and μ as above, we make the following definitions:

1. We say F is **unital** if $\lambda(1) = 1$ and $\mu(1 \otimes a) = a = \mu(a \otimes 1)$ for all $a \in F$.
2. We say F is **graded-commutative** (or μ is **graded-commutative**) if

   $$ab = (-1)^{|a||b|}ba$$

   for all homogeneous $a, b \in F$. We say it is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that

   $$a^2 = 0$$

   for all homogeneous $a \in F$ whenever $|a|$ is odd.
3. We say F is **multiplicative** (or λ is **μ-multiplicative**) if it satisfies the **multiplicative law**:

   $$\lambda(ab) = \lambda(a)\lambda(b)$$

   for all $a, b \in F$

4. We say F is **hom-associative** (or μ is λ-**associative**) if it satisfies the **hom-associative law**:

$$(ab)\lambda(c) = \lambda(a)(bc)$$

for all $a, b, c \in F$.

5. We say F is **permutative** (or μ is λ**-permutative**) if it satisfies the **permutative law**:

$$(\lambda(a)\lambda(b))\lambda(cd) = \lambda(ab)(\lambda(c)\lambda(d)) \tag{2}$$

for all $a, b, c, d \in F$.

Why are we interested in these definitions? Basically we view permutativity as a mixture between hom-associativity and multiplicativity.

Proposition 0.2. *Let $F = (F, \mathrm{d}, \lambda, \mu)$ be an MLDG algebra.*

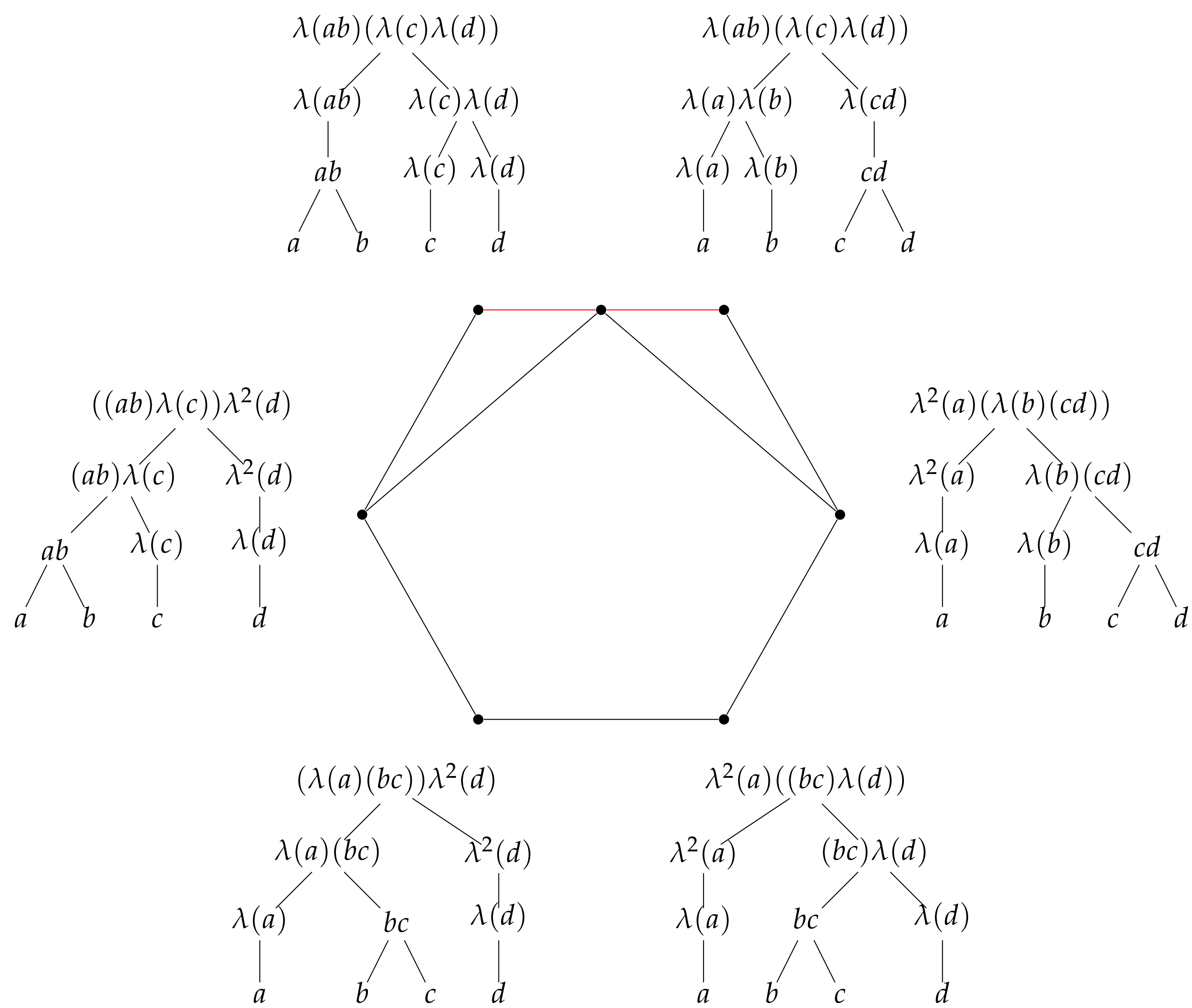
1. *If F is multiplicative, then F is permutative. The converse is true if F is unital.*
2. *If F is hom-associative, then F is permutative. In particular, if F is unital, then hom-associativity implies multiplicativity.*

Proof. 1. It is clear that if F is multiplicative, then F is permutative. Now suppose that F is unital and permutative. Then setting $c = 1 = d$ in (2) shows that F is multiplicative. In the general case where λ is not necessarily unital, we have $\lambda(1) = e$ where $e \in F_0$. In this case, the permutative law would imply that e associates with all of the other elements, and furthermore it would tell us that $e\lambda(ab) = e^2\lambda(a)\lambda(b)$ for all $a, b \in A$ (which is not quite the same as F being multiplicative).

2. Suppose F is hom-associative. Then for all $a, b, c, d \in F$, we have

$$\begin{aligned}
\lambda(ab)(\lambda(c)\lambda(d)) &= ((ab)\lambda(c))\lambda^2(d) \\
&= (\lambda(a)(bc))\lambda^2(d) \\
&= \lambda^2(a)((bc)\lambda(d)) \\
&= \lambda^2(a)(\lambda(b)(cd)) \\
&= (\lambda(a)\lambda(b))\lambda(cd).
\end{aligned}$$

There's a cute way to visualize this by tracing the edges of the permutohedron (the hexagon) below:



Note that the red edge "collapses" to the associahedra (the pentagon) if $\lambda = 1$. $\square$

Example 0.1. Let $\lambda \in R$ and let A be an MLDG R-algebra with $\lambda_A = \mathrm{m}_\lambda$ being the multilpication by λ map given by $a \mapsto \lambda a$. Recall that A is R-linear, so in particular the element λ must associative with all pairs of elements of A. It follows that A is permutative since

$$\begin{aligned}\lambda(ab)(\lambda(c)\lambda(d)) &= \lambda^3((ab)(cd)) \\ &= (\lambda(a)\lambda(b))\lambda(cd).\end{aligned}$$

On the other hand, A is not necessarily hom-associative. Indeed, we have

$$\lambda(a)(bc) = (ab)\lambda(c) \iff \lambda(a(bc) - (ab)c)$$

for all $a, b, c \in A$ and the righthand side need not be zero. It is easy to see though that A is hom-associative if and only if λ kills $\operatorname{im}[\cdot,\cdot,\cdot]$ where $[\cdot,\cdot,\cdot]$ is the usual associator map defined by $[a,b,c] = a(bc) - (ab)c$ for all $a, b, c \in A$. Similarly, A is not necessarily multiplicative. Indeed, we have

$$\begin{aligned}\lambda(ab) - \lambda(a)\lambda(b) &= \lambda(ab - \lambda ab) \\ &= \lambda(1-\lambda)ab\end{aligned}$$

for all $a, b \in A$. If we assume that R is local and that $\lambda \in \mathfrak{m}$, then $1 - \lambda$ is a unit. Then in this case, it is easy to see that A is multiplicative if and only if λ kills $\operatorname{im}\mu$.

We now repeat the same procedure that we did when classifying $\Bbbk$-complex structures on F. Let $\boldsymbol{\lambda} = (\ell_j^i)$ and let $\boldsymbol{m} = (m_{i,j}^k)$ be their matrix representations with respect to $\boldsymbol{e}$ respectively. Thus we have $\lambda\boldsymbol{e} = \boldsymbol{e}\boldsymbol{\lambda}$ we have $\mu(\boldsymbol{e}^\top \otimes \boldsymbol{e}) = \boldsymbol{e}^\top \boldsymbol{m}\boldsymbol{e}$. In terms of the matrix entries, these are given by

$$\lambda(e_j) = \sum_i \ell_j^i e_i \quad \text{and} \quad \mu(e_i \otimes e_j) = \sum_k m_{i,j}^k e_k.$$

for all i, j.

Let $\Bbbk[\boldsymbol{D}, \boldsymbol{L}, \boldsymbol{M}] = \Bbbk[\{D_i^j, L_i^j, M_{i,j}^k\}]$. We express the algebraic laws introduced above in terms of coordinates in the table below:

Algebraic Law	Equation
Graded-Commutative Law	$\Gamma_{i,j}^k = M_{i,j}^k - (-1)^{\lvert e_i \rvert \lvert e_j \rvert} M_{j,i}^k$
Leibniz Law	$\Lambda_{i,j}^k = \sum_\iota (M_{i,j}^\iota D_\iota^k - D_i^\iota M_{\iota,j}^k - (-1)^{\lvert e_i \rvert \lvert e_j \rvert} D_j^\iota M_{i,\iota}^k)$
Multiplicative Law	$\Theta_{i,j}^k = \sum_\iota M_{i,j}^\iota L_\iota^k - \sum_{\iota_1,\iota_2} L_i^{\iota_1} L_j^{\iota_2} M_{\iota_1,\iota_2}^k$
Hom-Associative Law	$H_{i,j,k}^l = \sum_{\iota_1,\iota_2} (M_{i,j}^{\iota_1} L_k^{\iota_2} M_{\iota_1,\iota_2}^l - M_{j,k}^{\iota_1} L_i^{\iota_2} M_{\iota_2,\iota_1}^l)$
Permutative Law	$\Pi_{i,j,k,l}^m = \sum_{\iota_1,\iota_2,\iota_3,\iota_4,\iota_5} (M_{i,j}^{\iota_1} L_k^{\iota_2} L_l^{\iota_3} - M_{k,l}^{\iota_1} L_i^{\iota_2} L_j^{\iota_3}) M_{\iota_2,\iota_3}^{\iota_4} L_{\iota_1}^{\iota_5} M_{\iota_5,\iota_4}^k$

We define

$$
\begin{aligned}
\mathrm{A}_\Bbbk^{\mathrm{p}}(F) &= \Bbbk[\boldsymbol{L}, \boldsymbol{M}] / \langle \boldsymbol{\Pi} \rangle. \\
\mathrm{A}_\Bbbk^{\mathrm{pd}}(F) &= \Bbbk[\boldsymbol{D}, \boldsymbol{L}, \boldsymbol{M}] / \langle \boldsymbol{\Delta}, \boldsymbol{\Pi} \rangle \\
\mathrm{A}_\Bbbk^{\mathrm{h}}(F) &= \Bbbk[\boldsymbol{L}, \boldsymbol{M}] / \langle \boldsymbol{H} \rangle \\
\mathrm{A}_\Bbbk^{`}(F) &= \Bbbk[\boldsymbol{L}, \boldsymbol{M}] / \langle \boldsymbol{\Theta} \rangle \\
\mathrm{A}_\Bbbk^{\mathrm{c}}(F) &= \Bbbk[\boldsymbol{M}] / \langle \boldsymbol{\Gamma} \rangle,
\end{aligned}
$$

and so on.