Permutohedron and Associahedron

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Example 0.1. Let $S = K[x_1, \ldots, x_n]$, let $I_{\mathcal{P}}$ be the permutohedron ideal in S, and let $I_{\mathcal{A}}$ be the associahedron ideal in S. Then there are natural free resolution $F_{\mathcal{P}} \xrightarrow{\tau_{\mathcal{P}}} S/I_{\mathcal{P}}$ and $F_{\mathcal{A}} \xrightarrow{\tau_{\mathcal{A}}} S/I_{\mathcal{A}}$ over S where $F_{\mathcal{P}}$ is supported by the permutohedron and $F_{\mathcal{A}}$ is supported by the associahedron. The inclusion of ideals $I_{\mathcal{P}} \subseteq I_{\mathcal{A}}$ induces a surjective S-linear map $\varphi \colon S/I_{\mathcal{P}} \to S/I_{\mathcal{A}}$ whose kernel is given by $I_{\mathcal{A}}/I_{\mathcal{P}}$. Lift $\varphi\tau_{\mathcal{A}}$ to a chain map $\widetilde{\varphi} \colon F_{\mathcal{P}} \to F_{\mathcal{A}}$ with respect to $\tau_{\mathcal{P}}$, so $\tau_{\mathcal{P}}\widetilde{\varphi} = \varphi\tau_{\mathcal{A}}$. It follows from Theorem (**??**) that $\Sigma C(\widetilde{\varphi})$ is a free resolution of $I_{\mathcal{P}}/I_{\mathcal{A}}$ over S.

Permutohedron

Definition 0.1. Let m be a monomial. The **Permutohedron complex** of m denoted $(\mathcal{P}(m), d^{\mathcal{P}(m)})$ is the R-complex whose graded R-module $\mathcal{P}(\underline{r})$ has

$$\mathcal{P}_i(\underline{r}) := \begin{cases} \bigoplus_{\sigma \in S_i(n)} Re_\sigma & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{K}(\underline{r})}$ is uniquely determined by

$$d^{\mathcal{K}(\underline{r})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle r_\lambda e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq \{1, \ldots, n\}$.

Example 0.2. Let $A = K[x,y,z]$, $I = \langle xy^2z^3, x^2yz^3, x^3yz^2, x^3y^2z, x^2y^3z, xy^3z^2 \rangle$, and $J = \langle x, y \rangle$. We compute $\mathrm{Tor}_i^A(A/I, A/J)$ for all i. A free resolution for A/I comes from the permutohedron of order 3. It is given by

$$0 \longrightarrow A \xrightarrow{\varphi_3} A^6 \xrightarrow{\varphi_2} A^6 \xrightarrow{\varphi_1} A \longrightarrow A/I$$

where

$$\varphi_3 = \begin{pmatrix} xy \\ y^2 \\ yz \\ z^2 \\ xz \\ x^2 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -x & 0 & 0 & 0 & 0 \\ 0 & z & -y & 0 & 0 & 0 \\ 0 & 0 & z & -y & 0 & 0 \\ 0 & 0 & 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 & x & -z \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^3yz^2 & x^3y^2z & x^2y^3z & xy^3z^2 \end{pmatrix}.$$

Now replace the A/I term with 0 and and tensor this new complex with A/J to get:

$$0 \longrightarrow A/J \xrightarrow{\widetilde{\varphi}_3} (A/J)^6 \xrightarrow{\widetilde{\varphi}_2} (A/J)^6 \xrightarrow{\widetilde{\varphi}_1} A/J \longrightarrow 0$$

where $\widetilde{\varphi}_i$ is obtained by setting $x = y = 0$ in the entries of φ_i:

$$\widetilde{\varphi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ z^2 \\ 0 \\ 0 \end{pmatrix}, \quad \widetilde{\varphi}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 0 & 0 & 0 & -z \end{pmatrix}, \quad \widetilde{\varphi}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we see that

$$\begin{aligned}
\operatorname{Tor}_0^A(A/I, A/J) &\cong A/\langle x, y\rangle \\
\operatorname{Tor}_1^A(A/I, A/J) &\cong (A/\langle x, y\rangle)^2 \oplus (A/\langle x, y, z\rangle)^4 \\
\operatorname{Tor}_2^A(A/I, A/J) &\cong (A/\langle x, y\rangle) \oplus \left(A/\langle x, y, z^2\rangle\right)
\end{aligned}$$

and $\operatorname{Tor}_i^A(A/I, A/J) \cong 0$ for all $i \geq 3$.

1 Embedding Permutohedron Resolution Into Taylor Resolution

1.1 Multiplication Rules For Permutohedron

Multiplication rules for $(\mathcal{P}(xy^2z^3), d^{\mathcal{P}(xy^2z^3)})$ are given by

$$\begin{aligned}
e_{xy^2z^3}e_{x^2yz^3} &= xyz^3e_{x^2y^2z^3} \\
e_{xy^2z^3}e_{x^3yz^2} &= x^2yz^2e_{x^2y^2z^3} + xy^2z^2e_{x^3yz^3} \\
e_{xy^2z^3}e_{x^3y^2z} &= x^2y^2ze_{x^2y^2z^3} + xy^3ze_{x^3yz^3} + xy^2z^2e_{x^3y^2z^2} \\
e_{xy^2z^3}e_{x^2y^3z} &= x^2y^2ze_{xy^3z^3} + xy^2z^2e_{x^2y^3z^2} \\
e_{xy^2z^3}e_{xy^3z^2} &= xy^2z^2e_{xy^3z^3} \\
e_{xy^2z^3}e_{x^3y^3z} &= xy^2ze_{x^3y^3z^3}
\end{aligned}$$

Swapping x with y and fixing z gives us

$$\begin{aligned}
e_{x^2yz^3}e_{xy^2z^3} &= xyz^3e_{x^2y^2z^3} \\
e_{x^2yz^3}e_{xy^3z^2} &= xy^2z^2e_{x^2y^2z^3} + x^2yz^2e_{xy^3z^3} \\
e_{x^2yz^3}e_{x^2y^3z} &= x^2y^2ze_{x^2y^2z^3} + x^3yze_{xy^3z^3} + x^2yz^2e_{x^2y^3z^2} \\
e_{x^2yz^3}e_{x^3y^2z} &= x^2y^2ze_{x^3yz^3} + x^2yz^2e_{x^3y^2z^2} \\
e_{x^2yz^3}e_{x^3yz^2} &= x^2yz^2e_{x^3yz^3} \\
e_{x^2yz^3}e_{x^3y^3z} &= x^2yze_{x^3y^3z^3}
\end{aligned}$$

Let us check associativity:

$$(e_{xy^2z^3}e_{x^3yz^2})e_{x^2y^3z} =$$

$$\begin{aligned}
&= e_{xy^2z^3}(e_{x^3yz^2}e_{x^2y^3z}) \\
&= e_{xy^2z^3}(e_{x^3yz^2}e_{x^2y^3z})
\end{aligned}$$

Let us check associativity:

$$\begin{aligned}
(e_me_{\sigma(m)})e_{\sigma\tau(m)} &= (e_me_{s_{i_1j_1}\cdots s_{i_lj_l}(m)})e_{\sigma\tau(m)} \\
&= \left(e_{[m,s_{i_kj_k}(m)]} + e_{[s_{i_kj_k}(m),s_{i_{k-1}j_{k-1}}s_{i_kj_k}(m)]} + \cdots + e_{[s_{i_2j_2}\cdots s_{i_kj_k}(m),s_{i_1j_1}\cdots s_{i_kj_k}(m)]}\right) e_{\sigma\tau(m)} \\
&= \left(\sum_{k=1}^{l} e_{[s_{i_2j_2}\cdots s_{i_lj_l}(m),s_{i_1j_1}\cdots s_{i_kj_k}(m)]}\right) e_{\sigma\tau(m)} \\
\\
&= e_m(\sigma(e_me_{\tau(m)})) \\
&= e_m(\sigma(e_me_{\tau(m)})) \\
&= e_m(e_{\sigma(m)}e_{\sigma\tau(m)})
\end{aligned}$$

Let $\sigma = s_1 \cdots s_l$ and let $\sigma' = s'_1 \cdots s'_{l'}$. Then

$$\begin{aligned}
(e_m e_{\sigma(m)})e_{\sigma'(m)} &= (e_m e_{s_1 \cdots s_l(m)})e_{s'_1 \cdots s'_{l'}(m)} \\
&= \left(e_{[m,s_l(m)]} + e_{[s_l(m),s_{l-1}s_l(m)]} + \cdots + e_{[s_2 \cdots s_l(m),s_1 \cdots s_l(m)]}\right) e_{s'_1 \cdots s'_{l'}(m)} \\
&= e_{[m,s_l(m)]}e_{s'_1 \cdots s'_{l'}(m)} + e_{[s_l(m),s_{l-1}s_l(m)]}e_{s'_1 \cdots s'_{l'}(m)} + \cdots + e_{[s_2 \cdots s_l(m),s_1 \cdots s_l(m)]}e_{s'_1 \cdots s'_{l'}(m)} \\
\\
&= \\
&= e_m(s_1 \cdots s_l(e_{[m,s'_{l'}(m)]} + \cdots + e_{[s'_1 \cdots s'_{l'}(m),s_1 s'_1 \cdots s'_{l'}(m)]} + \cdots + e_{[s_{\ell-1},\ldots,s_1 s'_1 \cdots s'_{l'}(m),s_\ell,\ldots,s_1 s'_1 \cdots s'_{l'}(m)]})) \\
&= e_m(\sigma(e_m e_{s_l \cdots s_1 s'_1 \cdots s'_{l'}(m)})) \\
&= e_m(\sigma(e_m e_{\sigma^{-1}\sigma'(m)})) \\
&= e_m(e_{\sigma(m)}e_{\sigma'(m)})
\end{aligned}$$

Let $\sigma = s_1 \cdots s_l$ and let $\sigma' = s'_1 \cdots s'_{l'}$. Then

$$\begin{aligned}
(e_m e_{\sigma(m)})e_{\sigma'(m)} &= (e_m e_{s_1 \cdots s_l(m)})e_{s'_1 \cdots s'_{l'}(m)} \\
&= \left(e_{[m,s_l(m)]} + \sum_{k=1}^{l} e_{[s_k \cdots s_l(m),s_{k-1} \cdots s_l(m)]}\right) e_{s'_1 \cdots s'_{l'}(m)} \\
&= e_{[m,s_l(m)]}e_{s'_1 \cdots s'_{l'}(m)} + \sum_{k=1}^{l} e_{[s_k \cdots s_l(m),s_{k-1} \cdots s_l(m)]}e_{s'_1 \cdots s'_{l'}(m)} \\
\\
&= \\
&= e_m(s_1 \cdots s_l(e_{[m,s'_{l'}(m)]} + \cdots + e_{[s'_1 \cdots s'_{l'}(m),s_1 s'_1 \cdots s'_{l'}(m)]} + \cdots + e_{[s_{\ell-1},\ldots,s_1 s'_1 \cdots s'_{l'}(m),s_\ell,\ldots,s_1 s'_1 \cdots s'_{l'}(m)]})) \\
&= e_m(\sigma(e_m e_{s_l \cdots s_1 s'_1 \cdots s'_{l'}(m)})) \\
&= e_m(\sigma(e_m e_{\sigma^{-1}\sigma'(m)})) \\
&= e_m(e_{\sigma(m)}e_{\sigma'(m)})
\end{aligned}$$

2 Associativity

$$\begin{aligned}
r(ab)c &= (a(rb))c) \\
&= (a(\sum r_i x_i y_i))c \\
&= \sum r_i(a(x_i y_i))c \\
&= \sum r_i((a x_i) y_i)c \\
&= \sum r_i(a x_i)(y_i c) \\
&= \sum r_i(a(x_i y_i c)) \\
&= \sum (a(r_i x_i y_i c)) \\
&= ra(bc).
\end{aligned}$$

Let $R = K[x,y,z]$ and let $I = \langle xy^2z^3, x^2yz^3, x^3yz^2, x^3y^2z, x^2y^3z, xy^3z^2 \rangle$. We describe two free resolutions of R/I. The first is given by

$$0 \longrightarrow R(-9) \xrightarrow{\varphi_3} R(-7)^6 \xrightarrow{\varphi_2} R(-6)^6 \xrightarrow{\varphi_1} R \longrightarrow 0 \tag{1}$$

where

$$\varphi_3 = \begin{pmatrix} xy \\ y^2 \\ yz \\ z^2 \\ xz \\ x^2 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -x & 0 & 0 & 0 & 0 \\ 0 & z & -y & 0 & 0 & 0 \\ 0 & 0 & z & -y & 0 & 0 \\ 0 & 0 & 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 & x & -z \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^3yz^2 & x^3y^2z & x^2y^3z & xy^3z^2 \end{pmatrix}.$$

This resolution was constructed using the permutohedron $\mathcal{P}(1,2,3)$[1]. In this case, the graded Betti numbers look like

$$\beta_{0,0} = 1$$
$$\beta_{1,6} = 6$$
$$\beta_{2,7} = 6$$
$$\beta_{3,9} = 1$$

The second is given by

$$0 \longrightarrow R(-9) \xrightarrow{\psi_3} R(-7) \oplus R(-8) \oplus R(-7)^2 \oplus R(-8) \oplus R(-7) \xrightarrow{\psi_2} R(-6)^6 \xrightarrow{\psi_1} R \longrightarrow 0 \tag{2}$$

where

$$\psi_3 = \begin{pmatrix} xy \\ x \\ z^2 \\ yz \\ z \\ x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -y^2 & 0 & 0 & 0 & 0 \\ 0 & z^2 & -x & 0 & 0 & 0 \\ 0 & 0 & y & -z & 0 & 0 \\ 0 & 0 & 0 & y & -y^2 & 0 \\ 0 & 0 & 0 & 0 & x^2 & -z \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^2y^3z & x^3y^2z & x^3yz^2 & xy^3z^2 \end{pmatrix},$$

note that ψ_1 differs from φ_1 only by a swap of position of the generators x^3yz^2 and x^2y^3z. This resolution was constructed using the Cayley graph of the symmetric group S_3. In this case, the graded Betti numbers look like

$$\beta_{0,0} = 1$$
$$\beta_{1,6} = 6$$
$$\beta_{2,7} = 4$$
$$\beta_{2,8} = 2$$
$$\beta_{3,9} = 1$$

Swapping gives

$$0 \longrightarrow R(-9) \xrightarrow{\psi_3} R(-7) \oplus R(-8) \oplus R(-7)^2 \oplus R(-8) \oplus R(-7) \xrightarrow{\psi_2} R(-6)^6 \xrightarrow{\psi_1} R \longrightarrow 0 \tag{3}$$

where

$$\psi_3 = \begin{pmatrix} xy \\ x \\ z^2 \\ yz \\ z \\ x^2 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & -y^2 & 0 \\ 0 & 0 & y & -z & 0 & 0 \\ 0 & z^2 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^2 & -z \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^2y^3z & x^3y^2z & x^3yz^2 & xy^3z^2 \end{pmatrix},$$

We have

$$\begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & -y^2 & 0 \\ 0 & 0 & y & -z & 0 & 0 \\ 0 & z^2 & -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^2 & -z \end{pmatrix} \quad \text{and}$$

[1]Recall that $\mathcal{P}(1,2,3)$ is defined to be the convex hull of $\{(\pi(1), \pi(2), \pi(3)) \mid \pi \in S_3\}$ in $\mathbb{R}^3$.