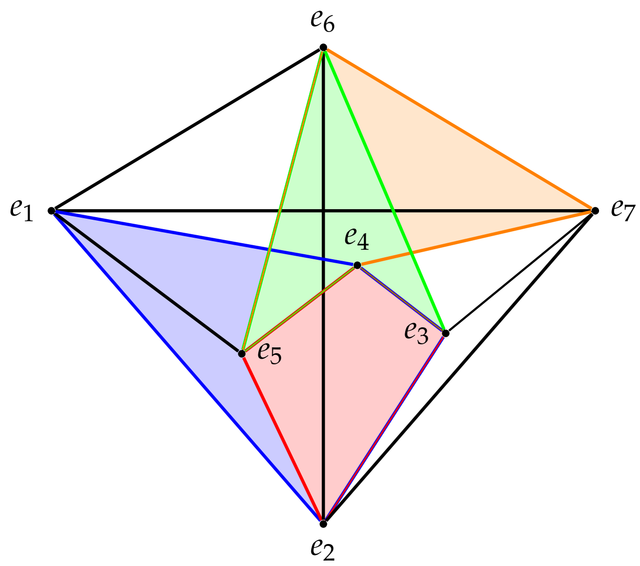
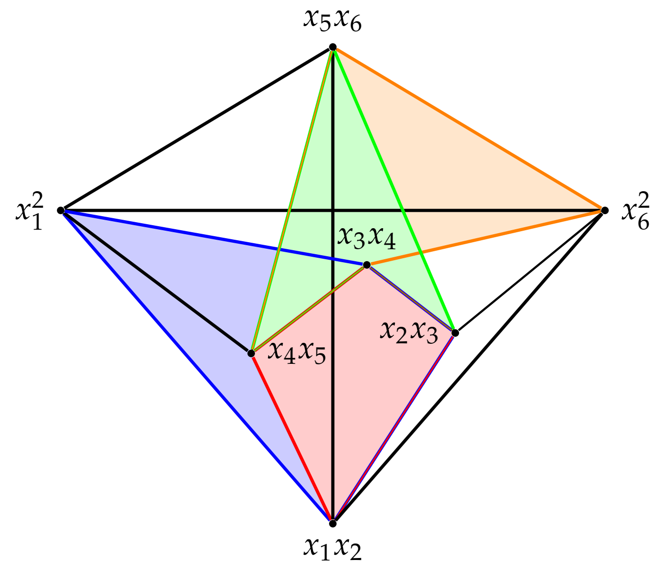
Example 0.1. Let $R = \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$, let $\boldsymbol{m} = x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ of R. Then F is the R-complex supported on the $\boldsymbol{m}$-labeled cellular complex below:





We label the homogeneous generators of F corresponding to the simplicial faces in the usual way. In particular, the complex in homological degree 1 consists of seven 0-simplices corresponding to the seven generators $e_1, \dots, e_7$ of F_1, and the complex in homological degree 2 consists of sixteen 1-simplices corresponding to the sixteen generators $e_{12}, e_{23}, \dots, e_{67}$ of F_2. The differential is defined on the generators corresponding to the simplicial faces via the Taylor rule (for example $\mathrm{d}e_1 = x_1^2$ and $\mathrm{d}e_{12} = x_2e_1 - x_1e_2$). The complex in homological degree 3 consists of thirteen 2-simplices and four squares (which we shaded in blue, red, green, and orange above). The differential on the squares is given by

$$\begin{aligned}
\mathrm{d}e_{1234} &= x_3x_4e_{12} + x_1x_4e_{23} - x_2e_{14} + x_1^2e_{34}\\
\mathrm{d}e_{2345} &= x_4x_5e_{23} + x_1x_5e_{34} - x_3e_{25} + x_1x_2e_{45}\\
\mathrm{d}e_{3456} &= x_5x_6e_{34} + x_2x_6e_{45} - x_4e_{36} + x_2x_3e_{56}\\
\mathrm{d}e_{4567} &= x_6^2e_{45} + x_3x_6e_{56} - x_5e_{47} + x_3x_4e_{67}
\end{aligned}$$

The complex in homological degree 4 consists of three 3-simplices, three Avramov tetrahedra, and two pyramids. The differential on the Avramov tetrahedra and pyramids is given by

$$\begin{aligned}
\mathrm{d}e_{12345} &= x_5e_{1234} - x_3e_{125} + x_2e_{145} - x_1e_{2345}\\
\mathrm{d}e_{23456} &= x_6e_{2345} - x_4e_{236} + x_3e_{256} - x_1e_{3456}\\
\mathrm{d}e_{34567} &= x_6e_{3456} - x_5e_{347} + x_4e_{367} - x_2e_{4567}\\
\mathrm{d}e_{123457} &= x_6^2e_{1234} - x_3x_4e_{127} - x_1x_4e_{237} + x_2e_{147} - x_1^2e_{347}\\
\mathrm{d}e_{134567} &= x_6^2e_{145} + x_3x_6e_{156} - x_5e_{147} + x_3x_4e_{167} - x_1^2e_{4567}
\end{aligned}$$

Finally, the complex in homological degree 5 consists of one 4-cell, and the differential on it is given by

$$\mathrm{d}e_{1234567} = x_6^2e_{12345} + x_3x_6e_{1256} + x_1x_6e_{23456} - x_5e_{123457} + x_3x_4e_{1267} + x_1x_4e_{2367} - x_2e_{134567} + x_1^2e_{34567}$$

Now equip F with a multigraded multiplication μ (i.e. $\mu\colon F^{\otimes 2} \to F$ is a multigraded chain map which is strictly graded-commutative and unital though not necessarily associative). Upon considerations of the Leibniz rule and multigrading, one can show that we already have three non-trivial associators corresponding to the three Avramov tetrahedra:

$$[e_1, e_3, e_5]_\mu = \mathrm{d}e_{12345}, \quad [e_2, e_4, e_6]_\mu = \mathrm{d}e_{23456}, \quad \text{and} \quad [e_3, e_5, e_7]_\mu = \mathrm{d}e_{34567}.$$

We claim that *any* multiplication on F will also be non-associative at these three triples. Indeed, let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F where $h\colon F^{\otimes 2} \to F$ is a homotopy (i.e. a graded R-linear map of degree 1). It suffices to show that μ_h is not associative at the triple (e_1, e_3, e_5) as the argument for non-associativity at the other triples is essentially the same. Observe that the associator for μ_h is given by

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d},$$

where $[\cdot]_\mu$ is the associator for μ and where $H = [\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}$. Here, we set

$$[\cdot]_{\mu,h} = \mu(h \otimes 1 - 1 \otimes h) \quad \text{and} \quad [\cdot]_{h,\mu_h} = h(\mu_h \otimes 1 - 1 \otimes \mu_h)$$

where additional signs will appear in $[\cdot]_{\mu,h}$ when applied to elements due to the Koszul sign rule. We can decompose $[\cdot]_{h,\mu_h}$ further as

$$[\cdot]_{h,\mu_h} = [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}}$$

where we use the same notation as above (for example $[\cdot]_{h,h\mathrm{d}} = h(h\mathrm{d} \otimes 1 - 1 \otimes h\mathrm{d})$). Let M be the R-submodule of F given by

$$M = \mathfrak{m}^2 F \oplus \left(\bigoplus_{m_\sigma \nmid x_1^2 x_2 x_3 x_4 x_5} Re_\sigma \right)$$

The idea behind the proof is that on the one hand we have $[e_1, e_3, e_5]_\mu \notin M$ but on the other hand

$$(\mathrm{d}H + H\mathrm{d})(e_1 \otimes e_3 \otimes e_5) \in M,$$

and in particular it will follow that $[e_1, e_3, e_5]_{\mu_h} \equiv [e_1, e_3, e_5]_\mu \neq 0$ in $\overline{F} := F/M$. Indeed, in $\overline{F}$ we have

$$\begin{aligned}
(\mathrm{d}H + H\mathrm{d})(e_1 \otimes e_3 \otimes e_5) &= \mathrm{d}H(e_1 \otimes e_3 \otimes e_5) + x_1^2 H(1 \otimes e_3 \otimes e_5) - x_2 x_3 H(e_1 \otimes 1 \otimes e_5) + x_1^2 H(1 \otimes e_3 \otimes e_5) \\
&\equiv \mathrm{d}H(e_1 \otimes e_3 \otimes e_5) \\
&= \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})(e_1 \otimes e_3 \otimes e_5) \\
&\equiv \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu})(e_1 \otimes e_3 \otimes e_5), \\
&\equiv \mathrm{d}([\cdot]_{\mu,h})(e_1 \otimes e_3 \otimes e_5) \\
&\equiv 0
\end{aligned}$$

where in the fourth line we used the fact that $\mathrm{d}F \subseteq \mathfrak{m}F$, where in the fifth line we used the fact that the multigraded and Leibniz rule forces us to have $e_1 \star_\mu e_3 \in \mathfrak{m}F$ and $e_3 \star_\mu e_5 \in \mathfrak{m}F$, and where in the sixth line we used that the multigraded and Leibniz rule forces us to have $e_1 \star_\mu \overline{F}_3 \in \mathfrak{m}\overline{F}$ and $e_5 \star_\mu \overline{F}_3 \in \mathfrak{m}\overline{F}$.