

Avramov Obstruction Notes

Let $f: R \rightarrow S$ be a finite local ring homomorphism such that the induced map on their common residue field \mathbb{k} is identity and let M be a finitely generated S -module. Let F be an MDG R -algebra resolution of S such that F is minimal. Next let X be any R -free resolution of M . Then the usual S -module structure on M induces an MDG F -module structure on X as follows: we choose a left scalar-multiplication $\mu_X: F \otimes_R X \rightarrow X$, denoted $a \otimes x \mapsto a \star_{\mu_X} x = ax$, which extends the usual left scalar-multiplication $S \otimes_R M \rightarrow M$ such that μ_X is unital, meaning $1x = x$ for all $x \in X$. Note that μ_X is unique up to homotopy. From the left scalar-multiplication, we obtain a right scalar multiplication $X \otimes_R F \rightarrow X$ by setting

$$xa := (-1)^{|a||x|}ax$$

for all $a \in F$ and $x \in X$. Then equipping X with these scalar-multiplication maps gives it the structure of an MDG F -module. We say μ_X gives X the structure of an MDG F -module in this case. If μ_X happens to be associative, then we say μ_X gives X the structure of a DG F -module.

Note that the map μ_X induces a map

$$H(F) \otimes_R H(X) \rightarrow H(X), \quad (1)$$

which is given by $\bar{a} \otimes \bar{x} \mapsto \overline{a \star_{\mu_X} x} = \overline{ax}$, where $\bar{a} \in H(F)$ and $\bar{x} \in H(X)$. Since homotopic chain maps induces the same map in homology, the map (1) does not depend on the choice of μ_X (which is unique up to homotopy). Now let us denote $F_{\mathbb{k}} = F \otimes_R \mathbb{k}$ and $X_{\mathbb{k}} = X \otimes_R \mathbb{k}$. We have $H(F_{\mathbb{k}}) = \text{Tor}^R(S, \mathbb{k})$ and $H(X_{\mathbb{k}}) = \text{Tor}^R(M, \mathbb{k})$, and so the map (1) induces the map in Tor:

$$\text{Tor}^R(S, \mathbb{k}) \otimes \text{Tor}^R(M, \mathbb{k}) \rightarrow \text{Tor}^R(M, \mathbb{k}). \quad (2)$$

Avramov considers the following commutative diagram.

$$\begin{array}{ccc} \text{Tor}_+^R(S, \mathbb{k}) \otimes \text{Tor}^R(M, \mathbb{k}) & \longrightarrow & \text{Tor}^R(M, \mathbb{k}) \\ \downarrow & & \downarrow \\ \text{Tor}_+^S(S, \mathbb{k}) \otimes \text{Tor}^S(M, \mathbb{k}) & \xrightarrow{0} & \text{Tor}^S(M, \mathbb{k}) \end{array}$$

This gives a canonical map of graded \mathbb{k} -vector spaces:

$$\frac{\text{Tor}^R(M, \mathbb{k})}{\text{Tor}_+^R(S, \mathbb{k}) \otimes \text{Tor}^R(M, \mathbb{k})} \rightarrow \text{Tor}^S(M, \mathbb{k}).$$

The kernel of this map is denoted $\text{of}^f(M)$ and is called the **obstruction to the existence of multiplicative structure** (on the minimal R -free resolution of M).

Example 0.1. Let $R = \mathbb{k}[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz \rangle$, and let $t = x^2, w^2$. Here we consider $S = R/t$ and $M = R/I$. Let $K = \mathcal{K}^R(x, y, z, w)$ be the Koszul algebra R -resolution of \mathbb{k} , with Koszul variables denoted $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and let

$$L = K \langle \delta_1, \delta_4 \mid d(\delta_1) = x\varepsilon_1, d(\delta_4) = w\varepsilon_4 \rangle.$$

Then L/tL is the Tate-Zariski minimal algebra S -resolution of \mathbb{k} . Note that $L/tL \otimes_S R/I \simeq L/IL$. In particular, we have

$$H(K/IK) = \text{Tor}^R(R/I, \mathbb{k}) \quad \text{and} \quad H(L/IL) = \text{Tor}^S(R/I, \mathbb{k}),$$

and the inclusion map $\iota: K/IK \hookrightarrow L/IK$ induces the map $\text{Tor}^R(R/I, \mathbb{k}) \rightarrow \text{Tor}^S(R/I, \mathbb{k})$. Now observe that in L/IL , we have

$$xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = d(w\delta_1\varepsilon_2\varepsilon_3\varepsilon_4 + w\delta_1\delta_4\varepsilon_3).$$

On the other hand, there exists no $\gamma \in K/IK$ such that $d(\gamma) = xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ in K/IK . Thus, $xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ represents a nonzero element in $H(K/IK)$, but $xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ represents the zero element in $H(L/IL)$.

0.1 Buchsbaum and Eisenbud Conjecture

Suppose I is an ideal of R and $\mathbf{x} = x_1, \dots, x_g$ is an R -regular sequence contained in I . Then we consider $S = R/\langle \mathbf{x} \rangle$ and $M = R/I$. In this case, we can choose F to be the Koszul algebra $\mathcal{K}(\mathbf{x})$ (in particular F is associative). Any expression of the x_i in terms of the generators for I yields a canonical comparison map $F \rightarrow X$. With this notation in mind, Buchsbaum and Eisenbud made the following conjecture:

Corollary. *X can be given the structure of a DG F -module such that the comparison map $F \rightarrow X$ is a DG F -module homomorphism.*

The reason why this conjecture is interesting is because its validity would imply important lower bounds for the ranks of the syzygies of R/I (where R is assumed to be a domain).

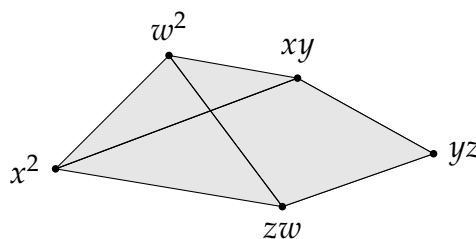
0.2 Avramov's Obstruction

Theorem 0.1. *Suppose the minimal R -free resolution F of S has the structure of a DG algebra. If $\mathfrak{o}^f(M) \neq 0$, then no DG F -module structure exists on the minimal R -free resolution X of M . In particular, in for X to possess the structure of a DG F -module, it is necessary that we have $\mathfrak{o}^f(M) = 0$.*

Example 0.2. Let $R = \mathbb{k}[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz \rangle$, and let $\mathbf{m} = x^2, w^2$. We set $S = R/\mathbf{m}$ and $M = R/I$. There are several complexes we consider: let

F be the Koszul R -algebra resolution of S
 X be the minimal R -free resolution of M
 T be the Taylor R -algebra resolution of M
 K be the Koszul R -algebra resolution of \mathbb{k}

Let $F = \mathcal{K}^R(\mathbf{m})$ be the Koszul algebra resolution of S and let X be the minimal R -free resolution of M . We can visualize X as being supported on the \mathbf{m} -labeled cell complex as below:



Let's write down the homogeneous components of X as a graded R -module: we have

$$\begin{aligned} X_0 &= R \\ X_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\ X_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{35} + Re_{45} \\ X_3 &= Re_{123} + Re_{124} + Re_{1345} + Re_{2345} \\ X_4 &= Re_{12345} \end{aligned}$$

We claim that X does not admit a DG F -module structure. Observe the isomorphism of \mathbb{k} -algebras

$$H(T_{\mathbb{k}}) \simeq \text{Tor}^R(M, \mathbb{k}) \simeq H(K_M).$$

Here T is the Taylor algebra resolution of M over R , and we denote $T_{\mathbb{k}} = T \otimes_R \mathbb{k}$ and $K_M = K \otimes_R M$. Observe that $\text{Tor}_4^R(M, \mathbb{k}) \simeq H_4(T_{\mathbb{k}})$ is generated by the class of $e_{1234} \otimes 1$ and that

$$\text{Tor}_1^R(S, \mathbb{k}) \text{Tor}_3^R(M, \mathbb{k}) \subseteq \text{Tor}_1^R(M, \mathbb{k}) \text{Tor}_3^R(M, \mathbb{k}) = 0.$$

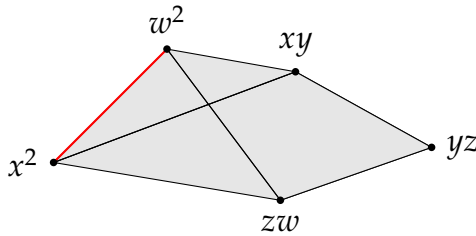
It suffices to show that $\ker(\text{Tor}_4^f(M, \mathbb{k}))$ is nontrivial. Note that this map is induced by the inclusion of complexes $K \subseteq L$, where

$$L = K\langle \delta_1, \delta_4 \mid d(\delta_1) = xe_1 \text{ and } d(\delta_4) = we_4 \rangle.$$

Indeed, L is the Tate-Zariski minimal S -algebra resolution of \mathbb{k} . Since $xw\epsilon_{1234}$ is a non-zero element of $H_4(K \otimes M) \simeq \text{Tor}_4^R(M, \mathbb{k})$, the formula

$$xw\epsilon_{1234} = d(w\delta_1\epsilon_{234} + y\delta_{14}\epsilon_3) \in L$$

proves our claim.



We set $\{a_1, a_2, a_3\} := a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3$. Observe that

$$d\{a_1, a_2, a_3\} = \{da_1, a_2, a_3\} + (-1)^{|a_1|}\{a_1, da_2, a_3\} + (-1)^{|a_1||a_2|}\{a_1, a_2, da_3\}.$$

The short exact sequence $0 \rightarrow V \rightarrow F^{\otimes 2} \xrightarrow{\mu} F \rightarrow 0$ induces isomorphisms $H_+(V) = \text{Tor}_+^R(R/I, R/I)$.

Proposition 0.1. *Assume that $[a_1, a_2, a_3] \neq 0$ represents a nontrivial element in $H[F]$ such that $[da_1, a_2, a_3] = [a_1, da_2, a_3] = [a_1, a_2, da_3] = 0$. Then*

$$d\{a_1, a_2, a_3\} = \{da_1, a_2, a_3\} + (-1)^{|a_1|}\{a_1, da_2, a_3\} + (-1)^{|a_1||a_2|}\{a_1, a_2, da_3\} \quad (3)$$

represents a nontrivial element in $H_+(V)$.

Proof. Each term in the sum (3) belongs to V since $[da_1, a_2, a_3] = [a_1, da_2, a_3] = [a_1, a_2, da_3] = 0$, so $d\{a_1, a_2, a_3\}$ certainly belongs to V . Furthermore, it is easy to see that $d\{a_1, a_2, a_3\} \in \ker d_V$. On the other hand, note that $\{a_1, a_2, a_3\} \notin V$ since $[a_1, a_2, a_3] \neq 0$. If $d\{a_1, a_2, a_3\} = d\tau$ for some $\tau \in V$, then $[a_1, a_2, a_3] = \mu\tau - \mu\{a_1, a_2, a_3\} \in \ker d_F$, implies $[a_1, a_2, a_3] \in \text{im } d_F$ which is a contradiction. \square