Fundamental Group Questions

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Problem 1

Exercise 1. On Page 14 of the online version of Hatcher, there is a diagram of a genus three surface as a quotient of a 12-gon. Compute the fundamental group of this surface in the following two ways:

1. As a quotient of a free group on 6 elements via the attached disk.
2. Using the Seifert-van Kampen theorem by splitting the genus three surface into a punctured genus two and a punctured genus one surface (please let me know if you need a sketch of this setup).

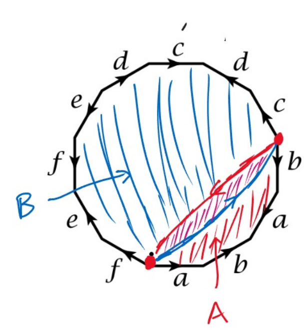
Solution 1. Let X be the genus three surface and let x be the point in X corresponding to any one of the vertices of the 12-gon. Then $\pi_1(X)$ is generated by the loops a, b, c, d, e, f subject to the the relation

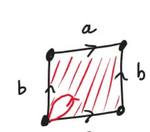
$$[a, b][c, d][e, f] = 1$$

where $[\cdot, \cdot]$ denotes the commutator, given by $[x, y] = xyx^{-1}y^{-1}$. Thus

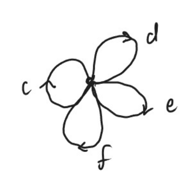
$$\pi_1(X) = \langle a, b, c, d, e, f \mid [a, b][c, d][e, f] = 1\rangle$$

2. We work this out below:



$A =$ 

$\sim$ 

$B \sim$ 

$$\begin{tikzcd} & \mathbb{Z} * \mathbb{Z} \arrow[dr] & \\ \mathbb{Z} \arrow[ur, "{[a,b]^{-1}}"] \arrow[dr, "{[c,d][e,f]}"'] & & \pi_1(X) \\ & \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

$A \cap B \sim S^1$

$$\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

$$= (\mathbb{Z} * \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z})$$

$$= \langle a, b, c, d, e, f \mid [a, b]^{-1} = [c, d][e, f] \rangle$$

Problem 2

Exercise 2. Let $X = S^1 \vee S^1$ and let a and b be the generators of $\pi_1(X)$ corresponding to the two summands.

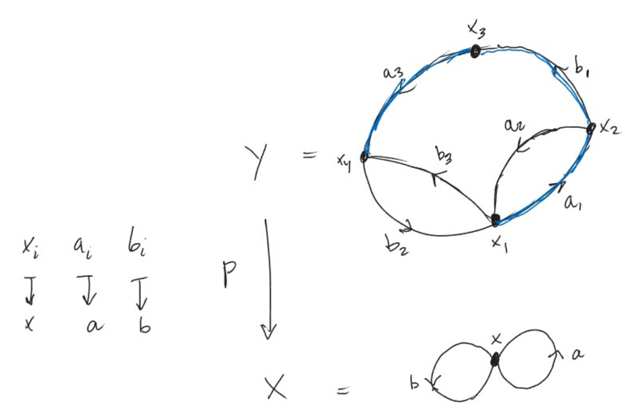
1. Draw a picture of the covering space of X with fundamental group $\langle a^2, b^2, (ab)^2 \rangle$ and explain why this covering space corresponds to the given group. Does this covering space have any deck transformations?

2. Draw a picture of the covering space of X with fundamental group the normal group generated by a^2, b^2, and $(ab)^2$ and explain why this covering space corresponds to the given group. Find all deck transformations of this covering space.

Solution 2. 1. We denote this covering space by Y and work out the details below:

$(abab^{-1})^{-1}(abab) = b^2$

$(abab)(b^2)^{-1} = abab^{-1}$



$H = p_*\pi_1(Y) = \langle a^2, abab, abab^{-1} \rangle = \langle a^2, abab, b^2 \rangle$

$G = \pi_1(X) = \langle a, b \rangle$

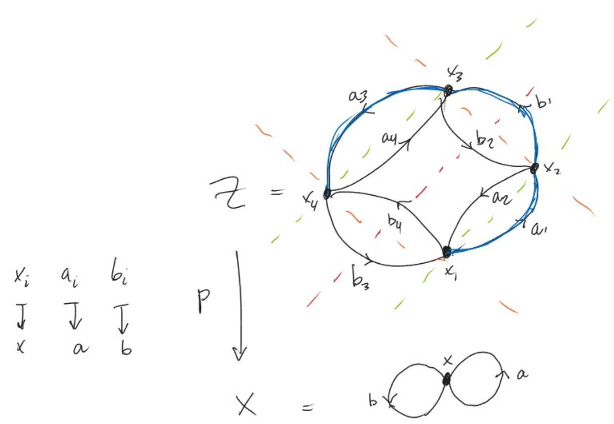
$N_G(H) = H$, thus $\mathrm{Aut}(Y/X) = N_G(H)/H = 1$

We calculate $\pi_1(Y)$ using Proposition 1A.2 in Hatcher where the maximal tree we use is colored in blue. This tells us that $\pi_1(Y) = \langle a^2, abab, abab^{-1} \rangle$, and since

$$(abab^{-1})^{-1}(abab) = b^2$$
$$b^2(abab) = abab^{-1},$$

it follows that $\pi_1(Y) = \langle a^2, abab, b^2 \rangle$. Finally, note that there are no deck transformations here. The reason is that if $\varphi\colon Y \to Y$ is a homeomorphism such that $p \circ \varphi = \varphi$, then we are forced to have $\varphi(x_i) = x_i$ for $i = 1, 2, 3, 4$. This further implies that $\varphi(a_i) = a_i$ and $\varphi(b_i) = b_i$. Thus φ is the identity map.

2. We denote this covering space by Z and work out the details below:



$N = p_*\pi_1(Z) = \langle a^2, ab^2a^{-1}, aba^2b^{-1}a, abab, abab^{-1} \rangle$

$G = \pi_1(X) = \langle a, b \rangle$

N is normal in G and

$G/N = \{1, \bar{a}, \bar{b}, \overline{ab}\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$$\varphi : \begin{cases} x_2 \leftrightarrow x_3 & x_1 \leftrightarrow x_4 \\ a_1 \leftrightarrow a_4 & a_2 \leftrightarrow a_3 \\ b_1 \leftrightarrow b_2 & b_2 \leftrightarrow b_4 \end{cases}$$

$\mathrm{Aut}(Z/X) = \{1, \varphi, \psi, \varphi\psi\}$

$$\psi : \begin{cases} x_1 \leftrightarrow x_3 & x_2 \leftrightarrow x_4 \\ a_1 \leftrightarrow a_3 & a_2 \leftrightarrow a_4 \\ b_1 \leftrightarrow b_3 & b_2 \leftrightarrow b_4 \end{cases}$$

We again use Proposition 1A.2 in Hatcher to calculate the fundamental group. We have two deck transformations φ and ψ which generate all of $\mathrm{Aut}(Z/X)$. We can think of φ as acting on Z via reflections across the dashed lines in the image above, and we can think of ψ as acting on Z via a 180 degree counterclockwise rotation. Altogether we have $\mathrm{Aut}(Z/X) = \{1, \varphi, \psi, \varphi\psi\}$, and we know that this is all of them since

$$\mathrm{Aut}(Z/X) \cong \mathrm{N}_G(N)/N = G/N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$