Trees

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1 Basic Definitions

1.1 Trees

A **tree** is an undirected graph in which any two vertices are connected by exactly one path. It is common to refer to the vertices of a tree as **nodes**. This is the convention we will take. If T is a tree, then we denote the set of all nodes of T as $\mathrm{N}(T)$. A **rooted tree** is a tree in which a special vertex is singled out. This singled out vertex is called the **root**.

Let T be a rooted tree. We say $\ell \in \mathrm{N}(T)$ is a **leaf** of T if ℓ is not the root and $\deg(\ell) = 1$. We denote the set of leaves of T as $\mathrm{L}(T)$. An **internal node** of a rooted tree is a node which isn't a leaf. We say a tree is **finite** if it has finitely many nodes. From here on out, we will assume that all trees discussed in this article are finite.

1.1.1 Height

Let T be a rooted tree and let $v \in \mathrm{N}(T)$. The **height** of v in T, denoted $h_T(v)$, is the number of edges along the unique path between v and the root. The **height** of T, denoted h_T (or simply h if context is clear), is

$$h := \max_{\ell \in \mathrm{L}(T)} h_T(\ell).$$

For all $i \in \mathbb{N}$, we define

$$\mathrm{N}_i(T) := \{v \in \mathrm{N}(T) \mid h_T(v) = i\}.$$

Since T is assumed to be finite, it is clear that h_T is finite and that $\mathrm{N}_i(T) = \emptyset$ for all $i > h$. Moreover, we claim that $\mathrm{N}_i(T) \neq \emptyset$ for all $i = 0, 1, \dots, h$. Indeed, if ℓ is a leaf of T such that $h_T(\ell) = h$, then there is a unique path $r \to v_1 \to v_2 \to \cdots\cdots \to \ell$ from r to ℓ. Then setting $v_0 = r$ and $v_{h_T} = \ell$, we see that $v_i \in \mathrm{N}_i(T)$ for all $i = 0, 1, \dots, h$.

1.1.2 Ancestors and Descendants

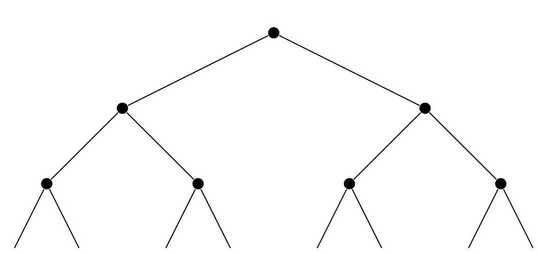
Let T be a rooted tree and let $v_1, v_2 \in \mathrm{N}(T)$ such that $h_T(v_2) \geq h_T(v_1)$. If the unique path from v_1 to v_2 does not contain the root, then we say v_1 is an **ancestor** of v_2 and v_2 is a **descendant** of v_1. If $h_T(v_2) = h_T(v_1) + 1$, then we say v_1 is a **parent** of v_2 and v_2 is a **child** of v_1. We say a node v in T is **prolific** if it has more than one child.

Ancestory gives rise to a partial order on the set of nodes of a tree. If T is a rooted tree with $v_1, v_2 \in \mathrm{N}(T)$, then we say $v_1 \leq v_2$ if v_1 is an ancestor of v_2. The pair $(\mathrm{N}(T), \leq)$ forms a partially ordered set. We say a node v is the **greatest common ancestor** of v_1 and v_2, denoted $v = \gcd(v_1, v_2)$, if$v \leq v_1, v_2$ and for all other nodes w such that $w \leq v_1, v_2$, we have $w \leq v$. Since the root of a tree is an ancestor to all other nodes, a common ancestor always exists. Moreover, a greatest common ancestor always exists since $\mathbb{N}$ is well-ordered.

1.1.3 Ordered Trees

An **ordered tree** is a rooted tree in which a total ordering is specified for the children of each vertex. This is sometimes called a "plane tree" because an ordering of the children is equivalent to an embedding of the tree in the plane, with the root at the top and the children of each vertex lower than that vertex.

Example 1.1. Here is a visual representation of an ordered tree:



Given an ordered tree T, we get an induced total ordering $\mathrm{L}(T)$: Let $\ell, \ell' \in \mathrm{L}(T)$, $v = \gcd(\ell, \ell')$, and let $\ell \to v_1 \cdots \to v_k \to v$ and $\ell' \to v'_1 \cdots \to v'_{k'} \to v$ be the unique paths from ℓ to v and ℓ' to v respectively (so v_k and $v'_{k'}$ are children of v). We say $\ell \leq \ell'$ if $v_k \leq v'_{k'}$. If $|\mathrm{L}(T)| = n$, then often denote the leaves of T as $\ell_1, \dots, \ell_n$, where it is understood that $\ell_i \leq \ell_j$ for all $1 \leq i \leq j \leq n$.

1.2 Permutohedra and Associahedra

1.2.1 The Permutohedron

The **permutohedron** $\mathbf{P}_n$ is the $(n-1)$-dimensional polytope in $\mathbb{R}^n$ defined as the convex hull of all vectors obtained from all permutations of the vector $(1, 2 \dots, n)$:

$$\mathbf{P}_n := \mathrm{ConvexHull}((\sigma(1), \dots, \sigma(n)) \mid \sigma \in S_n),$$

where S_n is the symmetric group. For each $0 \leq k \leq n-1$, we denote by $\Delta_k(\mathbf{P}_n)$ to be the set of k-cells of $\mathbf{P}_n$.

1.2.2 The Associahedron

An **associahedron** K_n is an $(n-2)$-dimensional polytope whose $(n-k)$-cells correspond to A-trees on n leaves and with $k-1$ internal nodes.

Jean-Louis Loday gave a simple formula for realizing the Stasheff polytopes as a convex hull of integer coordinates in Euclidean space (Loday 2004). Let Y_n denote the set of (rooted planar) binary trees with $n+1$ leaves (and hence n internal vertices). For any binary tree $T \in Y_n$, enumerate the leaves by left-to-right order, denoted $\ell_1, \dots, \ell_{n+1}$ and enumerate the internal vertices as $v_1, \dots, v_n$ where v_i is the greatest common ancestor of ℓ_i and ℓ_{i+1}. Define a vector $M(T) \in \mathbb{R}^n$ whose ith coordinate is the product $a_i b_i$ of the number a_i of leaves that are left descendants of v_i and the number b_i of leaves that are right descendants of v_i

Theorem 1.1. . *(Loday). The convex hull of the points $\{M(T) \in \mathbb{R}^n \mid T \in Y_n\}$ is a realization of the Stasheff polytope of dimension $n-1$, and is included in the affine hyperplane $\{x_1, \dots, x_n \mid x_1 + \cdots + x_n = n^2\}$.*

2 P-trees and A-trees

We now discuss two special trees which we call P-trees and A-trees. We will see that there is a bijection between the set of P-trees on n leaves and the set of cells of $\mathbf{P}_n$. We will also see that there is a bijection between the set of A-trees on n leaves and the set of cells of the associahedron $\mathbf{K}_n$.

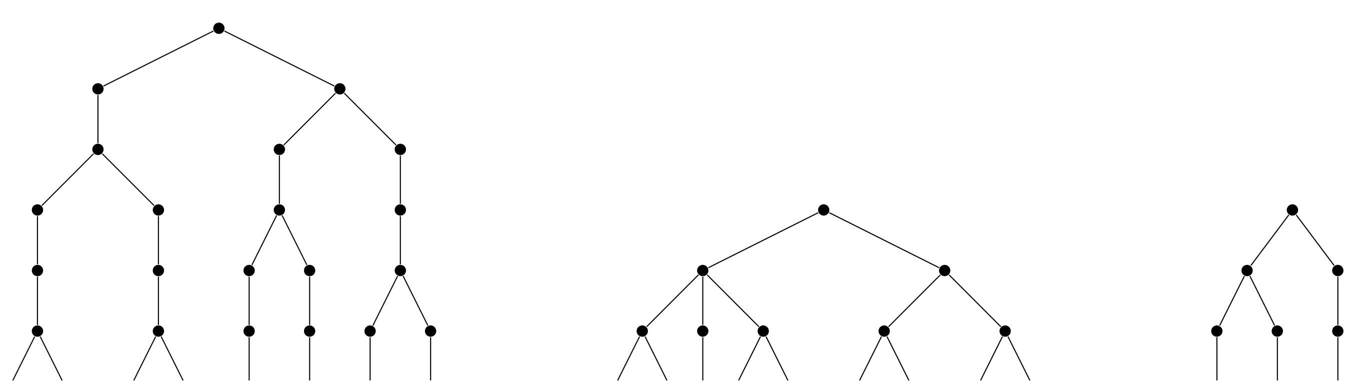
2.1 P-trees

Definition 2.1. Let T be an ordered tree. We say T is a P-**tree** if

1. Every leaf of T has the same height.

2. For all $i = 0, 1, \dots, h_T - 1$, there exists at least one prolific node in $\mathrm{N}_i(T)$.

Example 2.1. Here we illustrate a P-tree of height 6 and with 8 leaves, a P-tree of height 3 and with 9 leaves, and an ordered tree which is not a P-tree.



2.1.1 P-Trees and Total Preorders

Let X be a set. A **total preorder** on X is a binary relation $\leq$ which satisfies the following properties

1. (reflexive) $x \leq x$ for all $x \in X$.
2. (transitive) If $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in X$.
3. (total) Either $x \leq y$ or $y \leq x$ for all $x, y \in X$.

Remark. Note that $x \leq y$ and $y \leq x$ does not imply $x = y$.

Given a total preorder $\leq$ on X, we say x is a **maximal element** if $y \leq x$ for every $y \in X$. Let X_1 be the set of all maximal elements in $(X, \leq)$. Intuitively, these are the elements which "come in first place". The total preorder $\leq$ on X induces a total preorder, which we again denote by $\leq$, on $X \backslash X_1$. Let X_2 be the set of all maximal elements in $(X \backslash X_1, \leq)$. Intuitively, these are the elements which "come in second place". We proceed inductively on $k \in \mathbb{N}$ and obtain

$$X_k := \left\{ \text{maximal elements in } X \backslash \left(\bigcup_{i=1}^{k-1} X_i \right) \right\}.$$

Intuitively, these are the elements which "come in kth place".

Now assume X is finite. We define the **height** of $(X, \leq)$, denoted $h_{(X,\leq)}$ (or simply h if context is clear), to be the least $k \in \mathbb{N}$ such that $X_k \neq \emptyset$. The collection $\{X_1, \ldots, X_h\}$ forms an ordered set partition of X. Conversely, if we are given an ordered set partition $\{X_1, \ldots, X_h\}$ of X. Then there is a unique total preorder $\leq$ on X such that X_k is the set of all elements in $(X, \leq)$ which "come in kth place", for all $1 \leq k \leq h$. Thus, we arrive at:

Theorem 2.1. *Let $\mathcal{T}_X$ be the set of all total preorders on X and let $\mathcal{S}_X$ be the set of all ordered set partitions of X. Then we have a bijection $\mathcal{T}_X \cong \mathcal{S}_X$.*

Now consider the case where $X = \{1, \ldots, n\}$. For each $k = 1, \ldots, h$, let $\lambda_k = |X_k|$ and list the elements in X_k in increasing order as $x_{k,1}, \ldots, x_{k,\lambda_k}$, where by increasing order we means $x_{k,1} < \cdots < x_{k,\lambda_k}$ where $<$ denotes the usual order on the set $\{1, \ldots, n\}$. We use the shorthand notation $[x_{1,1} \cdots x_{1,\lambda_1}][x_{2,1} \cdots x_{2,\lambda_2}] \cdots [x_{h,1} \cdots x_{h,\lambda_h}]$ to refer to this total preorder.

Theorem 2.2. *Let $\mathcal{P}_n$ be the set of all P-trees with $n+1$ leaves and let $\mathcal{T}_n$ be the set of all total preorders on the set $X = \{1, 2, \ldots, n\}$. Then we have a bijection $\mathcal{P}_n \cong \mathcal{T}_n$.*

Proof. Let T be a P-tree with leaves $\ell_1, \ldots, \ell_{n+1}$ and let $v_i = \gcd(\ell_i, \ell_{i+1})$ for all $i = 1, \ldots, n$. We define a total preorder $\leq_T$ on X as follows: for all $i, j \in [n]$ we say

$$i \leq_T j \text{ if and only if } h_T(v_i) \leq h_T(v_j).$$

Conversely, let $\leq$ be a total preorder on X with $\{X_1, \ldots, X_h\}$ being the corresponding ordered set partition. Furthermore, for each $k = 1, \ldots, h$, let $\{X_{k,1}, \ldots, X_{k,\mu_k}\}$ be a partition of X_k into its $\leq$-consecutive parts.
Then we can construct a P-tree on $n+1$ leaves whose gaps v_i are appropriately placed using the total preorder.

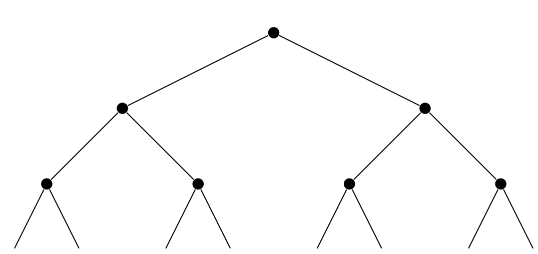
Let $[x_{1,1} \cdots x_{1,\lambda_1}][x_{2,1} \cdots x_{2,\lambda_2}] \cdots [x_{h,1} \cdots x_{h,\lambda_h}]$ represent this preorder. To build a P-tree, we start with the root r. The root r should have $\lambda_h + 1$ children.

Let us see how we can build a P-tree. We start with leaves $\ell_1, \ldots, \ell_{n+1}$. Now we focus our attention on $X_1 = \{x_{1,1}, \ldots, x_{1,\lambda_1}\}$. Let $\{X_{k,1}, \ldots, X_{k,\mu_k}\}$ be a partition of X_1 into its $\leq$-consecutive parts. We partition X_1 into its $\leq$-consecutive parts: X

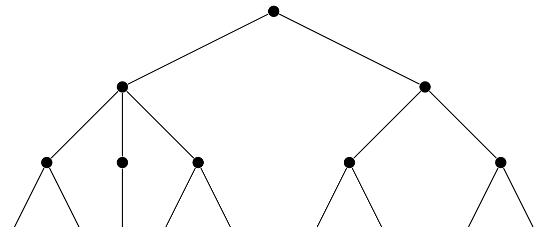
□

Let us go over several examples:

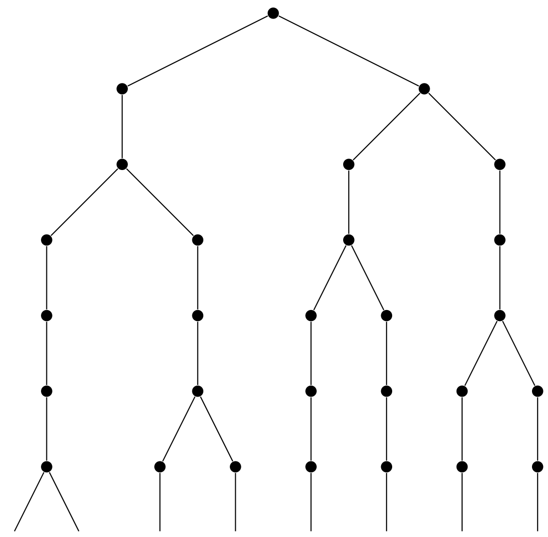
Example 2.2. The P-tree on 8 leaves below corresponds to the total preorder given by $[1357][26][5]$.



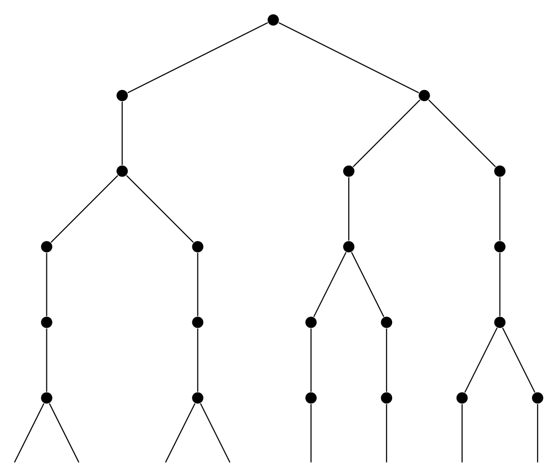
Example 2.3. The P-tree on 9 leaves below corresponds to the total preorder given by $[1468][237][5]$.



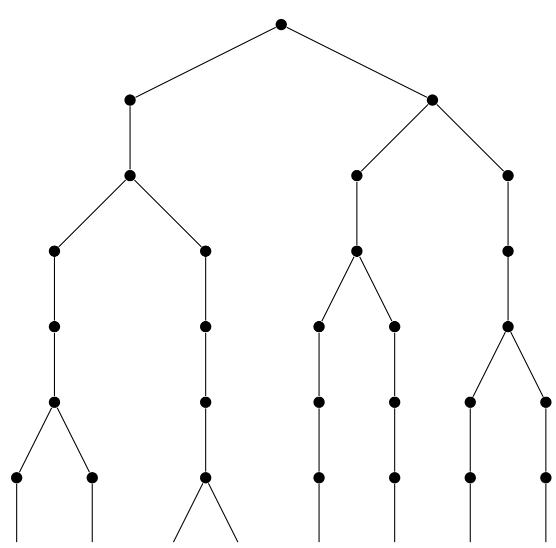
Example 2.4. The non-degenerate P-tree on 8 leaves below corresponds to the total preorder given by $[1][3][7][5][2][6][4]$.



Example 2.5. The non-degenerate P-tree on 8 leaves below corresponds to the total preorder given by $[13][7][5][2][6][4]$.



Example 2.6. The non-degenerate P-tree on 8 leaves below corresponds to the total preorder given by $[3][1][7][5][2][6][4]$.

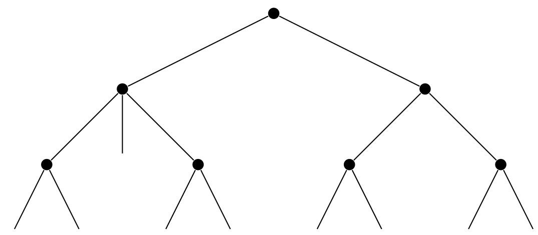


2.2 A-trees

Definition 2.2. Let T be an ordered tree. We say T is an A**-tree** if every internal node is prolific.

There is a natural way to obtain an A-tree from a P-tree; simply delete the internal nodes which are not prolific.

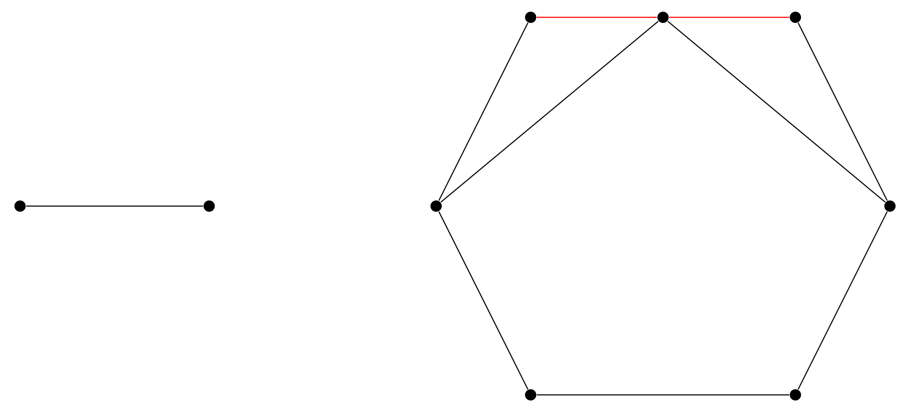
Example 2.7. Here's the A-tree obtained from the P-tree [1468][237][5] by deleting a single node of degree 2:



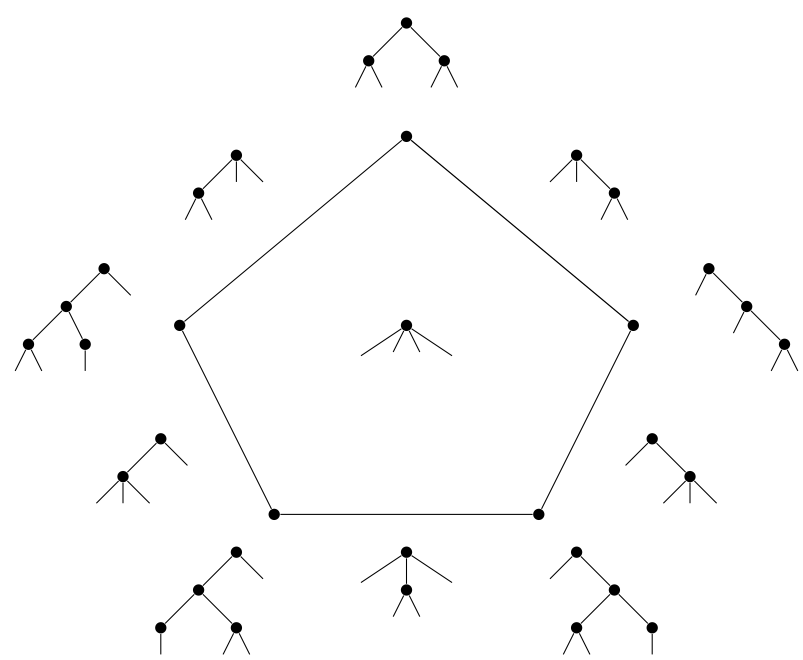
There is a unique way to turn a P-tree into an A-tree: simply delete the nodes of degree 2. However there are many ways to turn an A-tree into a P-tree.

3 Number of P-trees which collapse to an A-tree.

The permutohedron $\mathbf{P}_3$ with P-trees on 4 leaves attached to each k-cell of $\mathbf{P}_3$:



The associahedron $\mathbf{K}_2$ with A-trees on 4 leaves attached to each k-cell of $\mathbf{K}_2$:



In this section, we count the number of k-cells on the n-dimensional permutohedron $\mathbf{P}_n$ which collapse to a 0-cell on an associahedron K_{n-1}.

Lemma 3.1. *Let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be elements of some set. Then there are $\binom{m}{k}\binom{m+n-k}{m}$ ways of ordering these elements such that $x_i < x_j$ and $y_i < y_j$ whenever $i < j$, and k of the y_i's are tied with k of the x_j's.*

Proof. There are $\binom{m}{k}$ ways to select k of the x_i's to be tied with and $\binom{m+n-k}{m}$ ways of rearranging everything else. □

Example 3.1. Here is a way to order the list $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4$ such that $x_i < x_j$ and $y_i < y_j$ whenever $i < j$, and 2 of the y_i's are tied with 2 of the x_j's:

$$x_1 < x_2 < y_1 < x_3 = y_2 < y_3 < x_4 = y_4 < x_5$$

Let T be a binary A-tree, L be the subtree whose root is the left child of the root of T, and R be the subtree whose root is the right child of the root of T. Let n_T, n_L, and n_R be the number of internal nodes of each tree respectively. Finally, let $P_k(T)$, $P_k(L)$, and $P_k(R)$ be the number of k-cells on the permutohedron which collapse to the 0-cell on the associahedron which corresponds to each binary tree respectively. Then

$$\sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{n_L - i}{k - i - j} \binom{n_L + n_R - k}{n_L - i} P_i(L) P_j(R) = P_k(T)$$