Fourier Analysis

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A **Fourier series** on the real line is the following type of series in sines and cosines:

$$f(x) = \sum_{n\geq 0} a_n \cos(2\pi nx) + \sum_{n\geq 1} b_n \sin(2\pi nx)$$

This is 1-periodic. Since $e^{2\pi inx} = \cos(2\pi nx) + i\sin(2\pi nx)$ and $e^{-2\pi inx} = \cos(2\pi nx) - i\sin(2\pi nx)$, a fourier series can also be written in terms of complex exponentials:

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$$

where the summation runs over all integers ($c_n = \frac{1}{2}(a_n - ib_n)$ for $n > 0$, $c_n = \frac{1}{2}(a_{|n|} + ib_{|n|})$ for $n < 0$, and $c_0 = a_0$). The convenient algebraic property of $e^{2\pi inx}$, which is not shared by sines and cosines, is that it is a group homomorphism from $\mathbb{R}$ to the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$:

$$e^{2\pi in(x+x')} = e^{2\pi inx}e^{2\pi inx'}$$

Given a fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$$

We can get the coeffiecients c_m using the following trick:

$$\begin{aligned}
\int_0^1 f(x)e^{-2\pi imx}dx &= \int_0^1 \left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}\right) e^{-2\pi imx}dx \\
&= \int_0^1 \left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i(n-m)x}\right) dx \\
&= \sum_{n\in\mathbb{Z}} c_n \int_0^1 e^{2\pi i(n-m)x}dx \\
&= c_m
\end{aligned}$$

This works because given $k \in \mathbb{Z}$

$$\int_0^1 e^{2\pi ikx}dx = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$

An important link between a function $f(x)$ and its Fourier coefficients c_n is given by Parseval's formula

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \int_0^1 |f(x)|^2 dx$$

In addition to Fourier series there are Fourier integrals. The **Fourier transform** of a function f that decays rapidly at $\pm\infty$ is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by the integral formula

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx$$

The analogue of the expansion of a periodic function into a Fourier series is the Fourier inversion formula, which expresses f in terms of its Fourier transform $\hat{f}$:

$$f(x) = \int_{\mathbb{R}} \hat{f}(y)e^{2\pi ixy}dy$$

Define a Hermitian inner product of two functions f_1 and f_2 from $\mathbb{R}$ to $\mathbb{C}$ by the integral

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}} f_1(x)\bar{f}_2(x)dx \in \mathbb{C}$$

Plancherel's theorem compares the inner product of two functions and the inner product of their Fourier transforms:

$$\langle \hat{f}_1, \hat{f}_2 \rangle = \langle f_1, f_2 \rangle$$

In particular when $f_1 = f_2 = f$ the result is

$$\int_{\mathbb{R}} |\hat{f}(y)|^2 dx = \int_{\mathbb{R}} |f(y)|^2 dx$$

which is called Parseval's formula. The **convolution** of two functions f_1 and f_2 from $\mathbb{R}$ to $\mathbb{C}$ is a new function from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$(f_1 \star f_2)(x) = \int_{\mathbb{R}} f_1(t) f_2(x-t) dt$$

and the Fourier transform turns this convolution into pointwise multiplication:

$$\widehat{f_1 \star f_2}(y) = \hat{f}_1(y)\hat{f}_2(y)$$

Poisson Summation Formula

A link between Fourier series and Fourier integrals is the **Poisson summation formula**: for a "nice" function $f : \mathbb{R} \to \mathbb{C}$ that decays rapidly enough at $\pm\infty$,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{1}$$

For example, when $f(x) = e^{-bx^2}$ (with $b > 0$), the Poisson summation formula says

$$\sum_{n\in\mathbb{Z}} e^{-bn^2} = \sum_{n\in\mathbb{Z}} \sqrt{\frac{\pi}{b}} e^{-\pi^2 n^2/b}.$$

To prove the Poisson summation formula, we use Fourier series. Periodize $f(x)$ as

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n).$$

Since $F(x+1) = F(x)$, write F as a Fourier series: $F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$. Then

$$\begin{aligned}
c_n &= \int_0^1 F(x) e^{-2\pi i n x} dx \\
&= \int_0^1 \left(\sum_{m\in\mathbb{Z}} f(x+m) \right) e^{-2\pi i n x} dx \\
&= \sum_{m\in\mathbb{Z}} \int_0^1 f(x+m) e^{-2\pi i n x} dx \\
&= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x) e^{-2\pi i n x} dx \\
&= \int_{\mathbb{R}} f(x) e^{-2\pi i n x} dx \\
&= \hat{f}(n).
\end{aligned}$$

Therefore the expansion of $F(x)$ into a Fourier series is equivalent to

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x},$$

which becomes the Poisson summation formula (1) by setting $x = 0$. We can generalize the Poisson summation formula by replacing a sum over $\mathbb{Z}$ with a sum over any one-dimensional lattice $L = a\mathbb{Z}$ in $\mathbb{R}$, where $a \neq 0$, the summation formula becomes

$$\sum_{\lambda\in L} f(\lambda) = \frac{1}{|a|} \sum_{\mu\in L^\top} \hat{f}(\mu),$$

where $L^\top = (1/a)\mathbb{Z}$ is the dual lattice:

$$L^\top = \{\mu \in \mathbb{R} : e^{2\pi i \lambda\mu} = 1 \text{ for all } \lambda \in L\}.$$

Here's how we prove this when $a \in \mathbb{Z}^+$: Periodize $f(x)$ as

$$F(x) = \sum_{\lambda\in L} f(x+\lambda) = \sum_{n\in\mathbb{Z}} f(x+an).$$

Since $F(x+a) = F(x)$, write F as a Fourier series: $F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i(n/a)x}$. Then

$$\begin{aligned}
c_n &= \int_0^a F(x) e^{-2\pi i(n/a)x} dx \\
&= \int_0^a \left(\sum_{m\in\mathbb{Z}} f(x+am) \right) e^{-2\pi i(n/a)x} dx \\
&= \sum_{m\in\mathbb{Z}} \int_0^a f(x+am) e^{-2\pi i(n/a)x} dx \\
&= \sum_{m\in\mathbb{Z}} \int_{am}^{a(m+1)} f(x) e^{-2\pi i(n/a)x} dx \\
&= \int_{\mathbb{R}} f(x) e^{-2\pi i(n/a)x} dx \\
&= \frac{1}{a} \int_{\mathbb{R}} f(x/a) e^{-2\pi i n x} dx \\
&= \frac{1}{a} \hat{f}(n/a)
\end{aligned}$$

So

$$\sum_{\lambda\in L} f(x+\lambda) = \sum_{n\in\mathbb{Z}} f(x+an) = \frac{1}{a} \sum_{n\in\mathbb{Z}} \hat{f}(n/a) e^{2\pi i n x} = \frac{1}{a} \sum_{\mu\in L^{\top}} f(\mu) e^{2\pi i n x},$$

and this becomes the generalized Poisson summation formula when we set $x = 0$.