Goldbach Rings

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Abstract

Let $\mathbb{k}$ be a field. We introduce and study an interesting infinite-dimensional $\mathbb{k}$-algebra G which we call the Goldbach ring. As the name suggests, the Goldbach ring is closely related to Goldbach's conjecture. Properties that G satisfies as a ring (such as whether or not it is an integral domain) may give us clues about Goldbach's conjecture itself.

1 Introduction

Let $\mathbb{k}$ be a field. We introduce and study an interesting infinite-dimensional $\mathbb{k}$-algebra which we call the Goldbach ring, which, as the name suggests, is closely related to Goldbach's conjecture. The Goldbach ring G is defined to be the quotient $G = R/I$ where

$$\begin{aligned} R &= \mathbb{k}[\{x_p, x_{p+q} \mid p, q \text{ odd primes}\}] \\ I &= \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\}\rangle \end{aligned}$$

The Goldbach ring has the structure of a bi-graded $\mathbb{k}$-algebra meaning it can be decomposed as

$$G = \bigoplus_{n,d \geq 0} G_{n,d},$$

where the component $G_{n,d}$ in bi-degree $(n, d) \in \mathbb{N}^2$ is a finite-dimensional $\mathbb{k}$-vector space whose dimension we are interested in counting. For instance, Goldbach's conjecture is equivalent to the statement that $\dim_{\mathbb{k}} G_{2k,2} = 1$ for all $k \geq 3$. However this is really just a restatment of Goldbach's conjecture; what's more interesting and new in our view is the following conjecture that we propose:

Conjecture 1. *We have*

$$\dim_{\mathbb{k}} G_{n,d} \leq 1$$

for all $n, d \in \mathbb{N}$.

A counter-example to Conjecture (1) would be the existence of odd primes $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$ such that

$$p_1 + \cdots + p_d = n = q_1 + \cdots + q_d$$

but $x_{p_1} \cdots x_{p_d} \neq x_{q_1} \cdots x_{q_d}$ in G. However we do not believe such a counter-example exists since there are usually many ways to go from $x_{p_1} \cdots x_{p_d}$ to $x_{q_1} \cdots x_{q_d}$ by applying elementary Goldbach relations of the form $x_p x_q = x_{p+q}$. For instance, in $G_{36,4}$ we have $x_3^2 x_{11} x_{19} = x_5^2 x_{13}^2$ since

$$\begin{aligned} x_3^2 x_{11} x_{19} &= x_3 x_{11} x_{22} \\ &= x_3 x_5 x_{11} x_{17} \\ &= x_5 x_{11} x_{20} \\ &= x_5 x_7 x_{11} x_{13} \\ &= x_5 x_{13} x_{18} \\ &= x_5^2 x_{13}^2. \end{aligned}$$

There are many other paths we can take from $x_3^2 x_{11} x_{19}$ to $x_5^2 x_{13}^2$ however it turns out that this is the shortest path. Ultimately any attempt towards a solution to Conjecture (1) will involve tools and techniques from analytic number theory. What we find interesting is that Conjecture (1) also seems to involve a lot of commutative algebra as well. For example, if Conjecture (1) is true, then it woud imply that G is an integral domain. Conversely, one can show that if G is an integral domain and Conjecture (1) holds for n, d sufficiently large, then Conjecture (1) is true.

A deeper relationship between Conjecture (1), analytic number theory, and commutative algebra is realized when one studies G as a direct limit

$$G = \varinjlim G^m$$

of bi-graded noetherian $\Bbbk$-algebras $G^m = R^m/I^m$, where

$$\begin{aligned} R^m &= \Bbbk[x_1, \dots, x_m] \cap R \\ I^m &= \Bbbk[x_1, \dots, x_m] \cap I. \end{aligned}$$

Indeed, for each m we denote by $\delta(m)$ and $\rho(m)$ to be the R^m-depth and R^m-projective dimension of G^m respectively. Then the Auslander-Buchsbaum formula implies

$$\delta(2m) + \rho(2m) = \pi(2m) + m - \kappa(2m) - 3, \tag{1}$$

where $\pi(2m)$ is the usual prime-counting function which counts the number of primes $\leq 2m$ and where $\kappa(2m)$ counts then number of positive even numbers $\leq 2m$ that are counter-examples to Goldbach's conjecture.

2 $\mathcal{A}$-Supported Goldbach Rings

Let $\mathcal{A}$ be a subset of the positive odd integers and set $\mathcal{B} := \mathcal{A} + \mathcal{A} = \{a_1 + a_2 \mid a_1, a_2 \in \mathcal{A}\}$. We further assume that $\mathcal{A} \cap \mathcal{B} = \emptyset$. We set

$$\begin{aligned} R_{\mathcal{A}} &= \Bbbk[\{x_a, x_b \mid a \in \mathcal{A}, b \in \mathcal{B}\}] \\ I_{\mathcal{A}} &= \langle \{x_{a_1}x_{a_2} - x_{a_1+a_2} \mid a_1, a_2 \in \mathcal{A}\} \rangle \\ G_{\mathcal{A}} &= R_{\mathcal{A}}/I_{\mathcal{A}}. \end{aligned}$$

We will refer to $G_{\mathcal{A}}$ as the **$\mathcal{A}$-supported Goldbach ring**. We simplify our notation by writing $\{x_a, x_b\}$ to denote the set $\{x_a, x_b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$. Similarly we write $\{x_{a_1}x_{a_2} - x_{a_1+a_2}\}$ to denote the set $\{x_{a_1}x_{a_2} - x_{a_1+a_2} \mid a_1, a_2 \in \mathcal{A}\}$. We often simplify our notation even further by dropping $\mathcal{A}$ from our notation whenever it is clear from context. For instance, we write "G" instead of "$G_{\mathcal{A}}$" when it's understood that G is the $\mathcal{A}$-supported Goldbach ring. Similarly, if we write "let G be the $\mathcal{A}$-supported Golbach ring", then it's understood that $\mathcal{A}$ is a subset of the positive odd integers and that $\mathcal{B} = \mathcal{A} + \mathcal{A}$.

2.1 Representing Monomials

We will denote by $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$ to be the set of all monomials in $R = R_{\mathcal{A}}$. There are two ways we can represent monomials in R. The first way is as a finite product of the indeterminates $\{x_a, x_b\}$, namely, a monomial can be expressed in the form

$$x_{\boldsymbol{a}} x_{\boldsymbol{b}} := x_{a_1} \cdots x_{a_r} x_{b_1} \cdots x_{b_s}$$

where $\boldsymbol{a} = a_1, \dots, a_r$ is a sequence of elements in $\mathcal{A}$ (not necessarily distinct but often we assume $a_1 \leq \cdots \leq a_r$) and $\boldsymbol{b} = b_1, \dots, b_s$ is a sequence of elements in $\mathcal{B}$ (again not necessarily distinct, but often we assume $b_1 \leq \cdots \leq b_s$). We will use this way of representing monomials to give R a nice bi-graded structure. The second way of representing monomials is described as follows: given a function $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, we define its **support**, denoted $\operatorname{supp} \alpha$, to be the set

$$\operatorname{supp} \alpha = \{m \in \mathbb{N} \mid \alpha(m) \neq 0\}.$$

We denote by $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$ to be the set

$$\mathcal{F} = \{\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0} \mid \operatorname{supp} \alpha \text{ is finite and contained in } \mathcal{A} \cup \mathcal{B}\}.$$

Thus if $\alpha \in \mathcal{F}$, then α takes value 0 zero almost everywhere, and the only places where it may be nonzero is at an element in $\mathcal{A} \cup \mathcal{B}$. There is a bijection from $\mathcal{F}$ to $\mathcal{M}$ given by assigning $\alpha \in \mathcal{F}$ to the monomial

$$x^\alpha := \prod_{m \in \mathbb{N}} x_m^{\alpha(m)} = \prod_{m \in \operatorname{supp} \alpha} x_m^{\alpha(m)}.$$

For instance, suppose $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is defined by

$$\alpha(m) = \begin{cases} 2 & \text{if } m = 3 \\ 2 & \text{if } m = 6 \\ 4 & \text{if } m = 11 \\ 0 & \text{if } m \in \mathbb{N} \backslash \{3, 6, 11\} \end{cases}$$

Then $x^\alpha = x_3^2 x_6^2 x_{11}^4$ and $\operatorname{supp} x^\alpha = \{2, 6, 11\}$. This second way of expressing monimals gives us a cleaner way of expressing nonzero polynomials in R, namely, every nonzero polynomial $f \in R$ can be expressed in the form

$$f = c_1 x^{\alpha_1} + \cdots + c_n x^{\alpha_n}$$

for unique $c_1, \dots c_n \in \Bbbk$ and for unique $\alpha_1, \dots, \alpha_n \in \mathcal{F}$. We often pass back and forth between functions $\alpha \in \mathcal{F}$ and monomials $x^\alpha \in \mathcal{M}$. For instance, given a monomial $x^\alpha \in \mathcal{M}$, we define its **support**, denoted $\operatorname{supp} x^\alpha$, to be $\operatorname{supp} x^\alpha = \operatorname{supp} \alpha$, and etc...

2.2 The Bi-Graded $\Bbbk$-Structure on R and G

We give R and G bi-graded $\Bbbk$-structures as follows: we define $\deg_1 \colon \mathcal{M} \to \mathbb{N}$ and $\deg_2 \colon \mathcal{M} \to \mathbb{N}$ by

$$\deg_1(x_{\boldsymbol{a}} x_{\boldsymbol{b}}) = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j \quad \text{and} \quad \deg_2(x_{\boldsymbol{a}} x_{\boldsymbol{b}}) = r + 2s.$$

In particular, we have For each $n, d \in \mathbb{N}$, we set

$$R_{n,d} = \operatorname{span}_{\Bbbk}\{x^\alpha \in \mathcal{M} \mid \deg_1(x^\alpha) = n \text{ and } \deg_2(x^\alpha) = d\}.$$

Then we have a decomposition of R into $\Bbbk$-vector spaces:

$$R = \bigoplus_{n,d \in \mathbb{N}} R_{n,d},$$

which gives R a bi-graded $\Bbbk$-structure. Since I is homogeneous with respect to this bi-grading, G inherits a bi-graded $\Bbbk$-structure induced by the one on R:

$$G = \bigoplus_{n,d \in \mathbb{N}} G_{n,d}.$$

We set $\Delta_{n,d} = \dim_{\Bbbk} R_{n,d}$ and $\delta_{n,d} = \dim_{\Bbbk} G_{n,d}$. Thus $\Delta_{n,d}$ counts the number of ways we can express n as a sum

$$n = a_1 + \cdots + a_r + c_1 + \cdots + c_s$$

where $a_1, \dots, a_r \in \mathcal{A}$, $b_1, \dots, b_s \in \mathcal{B}$, and $d = r + s$. Whenever we have $\Delta_{n,d} \geq 1$, then we say (n, d) is a **good pair**. In this case, we are interested in determining whether or not $\delta_{n,d} = 1$ for $\delta_{n,d} > 1$. Intuitively, we have $\delta_{n,d} = 1$ when $\mathcal{A}$ is sufficiently "dense" in $\mathbb{N}$ and we have $\delta_{n,d} > 1$ whenever $\mathcal{A}$ is very "sparse" in $\mathbb{N}$.

2.3 Constructing the Minimal Free Resolution of G over R

We now wish to construct the minimal free resolution of G over R. For each $m \geq 1$, set

$$\begin{aligned} R^m &= \Bbbk[\{x_a, x_b \mid a, b \leq m\}] \\ I^m &= \langle\{x_{a_1} x_{a_2} - x_{a_1+a_2} \mid a_1 + a_2 \leq m\}\rangle \\ G^m &= R^m / I^m. \end{aligned}$$

Note that R^m and G^m have bi-graded $\Bbbk$-structures:

$$R^m = \bigoplus_{n,d} R^m_{n,d} \quad \text{and} \quad G^m = \bigoplus_{n,d} G^m_{n,d}.$$

Recall that if $x_a x_b \in R^m_n$, then we must have $a + b = n$ and $a, b \leq m$. In particular, if $m \geq n$ then we have

$$R^m_n = R^n_n = R_n \quad \text{and} \quad G^m_n = G^n_n = G_n.$$

Thus we have directed systems

$$(R^m)_{m \geq 1} \quad \text{and} \quad (G^m)_{m \geq 1}$$

of bi-graded $\Bbbk$-algebras where the bi-graded components $R^m_{n,d}$ and $G^m_{n,d}$ in bi-degree (n, d) stabilizes to $R_{n,d}$ and $G_{n,d}$ respectively for m sufficiently large (for example $m \geq n$). It follows that

$$R = \varinjlim R^m \quad \text{and} \quad G = \varinjlim G^m$$

as bi-graded direct limits.

Next we let $F^m = F^m_{\mathcal{A}}$ be the minimal bi-graded free resolution of G^m over R^m. We set

$$\varepsilon(m) = \varepsilon_{\mathcal{A}}(m) := \operatorname{depth}_{R^m} G^m \quad \text{and} \quad \rho(m) = \rho_{\mathcal{A}}(m) := \operatorname{pd}_{R^m} G^m = \operatorname{length} F^m.$$

Note that these quantities are intrinsic to R^m and G^m (and not R and G). By the Auslander-Buchsbaum formula we have

$$\rho(m) + \varepsilon(m) = \pi_{\mathcal{A}\cup\mathcal{B}}(m) := \#\{z \in \mathcal{A} \cup \mathcal{B} \mid z \leq m\}. \tag{2}$$

Note that F^m has the structure of a bi-graded $\Bbbk$-complex meaning we have a decomposition of $\Bbbk$-complexes:

$$F^m = \bigoplus_{n,d} F^m_{n,d},$$

where $F^m_{n,d}$ is a finite $\Bbbk$-subcomplex of F^m which minimially resolves $G^m_{n,d}$ in the sense that the augmented complex

$$\widetilde{F}^m_{n,d} := \cdots \to F^m_{i,n,d} \to F^m_{i-1,n,d} \to \cdots \to F^m_{1,n,d} \to R^m_{n,d} \to G^m_{n,d} \to 0$$

is exact and where the ith bi-graded Betti number of G^m in bi-degree (n,d) is given by

$$\beta^m_{i,n,d} := \dim_{\Bbbk} \operatorname{Tor}_i^{R^m}(G^m, \Bbbk)_{n,d} = \dim_{\Bbbk}(F^m_{i,n,d}).$$

The canonical maps $G^m \to G^{m+1}$ induce bi-graded comparison maps $F^m \to F^{m+1}$ which we may take to be inclusion maps after identifying F^m with a subcomplex of F^{m+1}. Furthermore, since $R^m_n = R^n_n$ and $G^m_n = G^n_n$ whenever $m \geq n$, we see that $F^m_n = F^n_n$ whenever $m \geq n$. Thus if we define $F = F_{\mathcal{A}}$ to be the direct limit of bi-graded $\Bbbk$-complexes

$$F := \varinjlim F^m,$$

then F is a free resolution of G over R which has the following bi-graded $\Bbbk$-complex structure:

$$F = \bigoplus_{n,d} F_{n,d} = \bigoplus_{n,d} F^n_{n,d}.$$

In particular, if m is sufficiently large, then we see that $\beta^m_{i,n,d} = \beta_{i,n,d}$ where

$$\beta_{i,n,d} := \dim_{\Bbbk} \operatorname{Tor}_i^{R}(G, \Bbbk)_{n,d} = \dim_{\Bbbk}(F_{i,n,d})$$

is the ith bi-graded Betti number of G in bi-degree (n,d). Unlike the quantities $\varepsilon(m)$ and $\rho(m)$, the quantitiy $\beta^m_{i,n,d}$ is actually intrinsic to R and G (and not just R^m and G^m) when m is sufficiently large.

Theorem 2.1. *We have*

$$\delta_{n,d} = \Delta_{n,d} - \sum_{i=1}^{\infty}(-1)^i \beta_{i,n,d} = \sum_{i=0}^{\infty}(-1)^i \dim_{\Bbbk} \operatorname{Tor}_i^R(G, \Bbbk)_{n,d},$$

3 The Goldbach Ring

We now consider the case where $\mathcal{A} = \{\text{positive odd primes}\}$. In this case, we have

$$\begin{aligned} R &= \Bbbk[\{x_p, x_{2k} \mid p \text{ odd prime and } 2k \text{ is a sum of two odd primes}\}] \\ I &= \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\} \rangle \\ G &= R/I. \end{aligned}$$

The homogeneous components of the form $R_{18,d}$ looks like:

$$\begin{aligned} \vdots &= \vdots \\ R_{18,7} &= 0 \\ R_{18,6} &= \Bbbk x_3^6 + \Bbbk x_3^4 x_6 + \Bbbk x_3^2 x_6^2 + \Bbbk x_6^3 \\ R_{18,5} &= 0 \\ R_{18,4} &= \Bbbk x_3^2 x_5 x_7 + \Bbbk x_3 x_5^3 + \Bbbk x_3^2 x_{12} + \cdots + \Bbbk x_5 x_6 x_7 + \Bbbk x_6 x_{12} + \Bbbk x_8 x_{10} \\ R_{18,3} &= 0 \\ R_{18,2} &= \Bbbk x_5 x_{13} + \Bbbk x_7 x_{11} + \Bbbk x_{18} \\ R_{18,1} &= 0 \\ \vdots &= \vdots \end{aligned}$$

Similarly, the homogeneous components of the form $R_{17,d}$ looks like:

$$
\begin{aligned}
\vdots &= \vdots \\
R_{17,6} &= 0 \\
R_{17,5} &= \Bbbk x_3^4 x_5 + \Bbbk x_3^3 x_8 + \Bbbk x_3^2 x_5 x_6 + \Bbbk x_3 x_6 x_8 + \Bbbk x_5 x_6^2 \\
R_{17,4} &= 0 \\
R_{17,3} &= \Bbbk x_3^2 x_{11} + \Bbbk x_3 x_7^2 + \Bbbk x_5^2 x_7 + \Bbbk x_6 x_{11} + \Bbbk x_3 x_{14} + \Bbbk x_7 x_{10} + \Bbbk x_5 x_{12} \\
R_{17,2} &= 0 \\
R_{17,1} &= \Bbbk x_{17} \\
R_{17,0} &= 0 \\
\vdots &= \vdots
\end{aligned}
$$

Staring at the homogeneous components above, we see that $\Delta_{18,4} = 9$ and $\Delta_{17,3} = 7$. More generally, $\Delta_{n,d}$ counts the number of ways we can express n as a sum:

$$
n = p_1 + \cdots + p_r + 2(k_1 + \cdots + k_s), \tag{3}
$$

where $p_1, \dots, , p_r$ are odd primes, $2k_1, \dots, 2k_s$ can be expressed as a sum of two odd primes, and $d = r + 2s$. Next, the homogeneous components of the form $G_{17,d}$ and $G_{18,d}$ looks like:

$$
\begin{aligned}
\vdots &= \vdots & \vdots &= \vdots \\
G_{17,6} &= 0 & G_{18,6} &= \Bbbk \overline{x}_3^6 \\
G_{17,5} &= \Bbbk \overline{x}_3^4 \overline{x}_5 & G_{18,5} &= 0 \\
G_{17,4} &= 0 & G_{18,4} &= \Bbbk \overline{x}_3^2 \overline{x}_5 \overline{x}_7 \\
G_{17,3} &= \Bbbk \overline{x}_3^2 \overline{x}_{11} & G_{18,3} &= 0 \\
G_{17,2} &= 0 & G_{18,2} &= \Bbbk \overline{x}_5 \overline{x}_{13} \\
G_{17,1} &= \Bbbk \overline{x}_{17} & G_{18,1} &= 0 \\
\vdots &= \vdots & \vdots &= \vdots
\end{aligned}
$$

From what we've seen above, it is *very* tempting to consider the following conjecture:

Conjecture 2. *If n is even, then*

$$
\text{we have } \begin{cases} \delta_{n,d} = 1 & \text{if } d \text{ is even and } 2 \leq d \leq \lfloor n/3 \rfloor \\ \delta_{0,0} = 1 & \\ \delta_{n,d} = 0 & \text{else} \end{cases}
$$

If n is odd, then

$$
\text{we have } \begin{cases} \delta_{n,d} = 1 & \text{if } d \text{ is odd and } 3 \leq d \leq \lfloor n/3 \rfloor \\ \delta_{n,d} = 1 & \text{if } p \text{ is an odd prime} \\ \delta_{n,d} = 0 & \text{else} \end{cases}
$$

To get an idea of what this conjecture entails, there are two issues we need to consider:

1. First of all, we'd like to represent each basis element in $G_{n,d}$ by a monomial of the form $x_{\boldsymbol{p}} = x_{p_1} \cdots x_{p_d}$ where $\boldsymbol{p} = p_1, \dots, p_d$ are d odd primes such that $n = p_1 + \cdots + p_d$. However being able to do this is essentially equivalent to Goldbach's Conjecture being true. In particular, Goldbach's Conjecture is already baked in to Conjecture (2).

2. Assume that $\overline{x}^\alpha = \overline{x}_{\boldsymbol{p}}$ and $\overline{x}^\beta = \overline{x}_{\boldsymbol{q}}$ where $\boldsymbol{p} = p_1, \dots, p_d$ and $\boldsymbol{q} = q_1, \dots, q_d$ are d odd primes such that

$$
p_1 + \cdots + p_d = n = q_1 + \cdots + q_d.
$$

Even in this case, it is not entirely clear why $\overline{x}_{\boldsymbol{p}} = \overline{x}_{\boldsymbol{q}}$. Indeed, in $G_{27,3}$, we have $\overline{x}_3 \overline{x}_{11} \overline{x}_{13} = \overline{x}_5^2 \overline{x}_{17}$, however it takes some work to show this:

$$
\begin{aligned}
\overline{x}_3 \overline{x}_{11} \overline{x}_{13} &= \overline{x}_{11} \overline{x}_{16} \\
&= \overline{x}_5 \overline{x}_{11} \overline{x}_{11} \\
&= \overline{x}_5 \overline{x}_{22} \\
&= \overline{x}_5^2 \overline{x}_{17}.
\end{aligned}
$$

Note that at each step in the computation above, we are only allowed to use a relation of the form $\overline{x}_p\overline{x}_q = \overline{x}_{p+q}$. For another example, in $G_{36,4}$ we have $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19} = \overline{x}_5^2\overline{x}_{13}^2$ since

$$\begin{aligned}
\overline{x}_3^2\overline{x}_{11}\overline{x}_{19} &= \overline{x}_3\overline{x}_{11}\overline{x}_{22} \\
&= \overline{x}_3\overline{x}_5\overline{x}_{11}\overline{x}_{17} \\
&= \overline{x}_5\overline{x}_{11}\overline{x}_{20} \\
&= \overline{x}_5\overline{x}_7\overline{x}_{11}\overline{x}_{13} \\
&= \overline{x}_5\overline{x}_{13}\overline{x}_{18} \\
&= \overline{x}_5^2\overline{x}_{13}^2.
\end{aligned}$$

The path we took to get from $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19}$ to $\overline{x}_5^2\overline{x}_{13}^2$ was longer than the path we took to get from $\overline{x}_3\overline{x}_{11}\overline{x}_{13}$ to $\overline{x}_5^2\overline{x}_{17}$, so one can imagine that for n and d large, the path from $x_{\boldsymbol{p}}$ to $x_{\boldsymbol{q}}$ may be even longer. At the same time however, there are *more* ways to get from $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19}$ to $\overline{x}_5^2\overline{x}_{13}^2$ than there are to get from $\overline{x}_3\overline{x}_{11}\overline{x}_{13}$ to $\overline{x}_5^2\overline{x}_{17}$, and hence for n and d large, our intution tells us that there should be many ways to get from $x_{\boldsymbol{p}}$ to $x_{\boldsymbol{q}}$ (as there are many such relations of the form $\overline{x}_p\overline{x}_q = \overline{x}_{p+q}$). In order to prove Conjecture (2), we only need to find *one* path from $x_{\boldsymbol{p}}$ to $x_{\boldsymbol{q}}$.

If Conjecture (2) is true, then G has a nice property as a ring:

Proposition 3.1. *Assume Conjecture (2) is true. Then G is an integral domain.*

Proof. Let $f \in G_{n,d} = \Bbbk\overline{x}^{\alpha}$ and $f' \in G_{n',d'} = \Bbbk\overline{x}^{\alpha'}$ such that $ff' = 0$. Express f and f' as

$$f = c\overline{x}^{\alpha} \quad \text{and} \quad f' = c'\overline{x}^{\alpha'}.$$

Then clearly since $\overline{x}^{\alpha+\alpha'} \neq 0$, so we must have $cc' = 0$, which implies either $c = 0$ or $c' = 0$ which implies either $f = 0$ or $f' = 0$. □

Remark 1. Note that for m sufficiently large, G^m tends to have lots of zerodivisors. For instance, in G^{16} we have $\overline{x}_3\overline{x}_5\overline{x}_{13} = \overline{x}_5^2\overline{x}_{11} = \overline{x}_3\overline{x}_7\overline{x}_{11}$ which implies

$$\overline{x}_3(\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}) = 0.$$

Since $\overline{x}_3 \neq 0$ and $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11} \neq 0$, we see that $\overline{x}_3$ and $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}$ form a zerodivisor pair. The ring homomorphism $G^{16} \to G^{18}$ kills this zerodivisor pair by sending $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}$ to 0, however we pick up another zero-divisor pair in G^{20}: namely $\overline{x}_3$ and $\overline{x}_{11}\overline{x}_{11} - \overline{x}_5\overline{x}_{17}$. Indeed, in G^{20} we have

$$\begin{aligned}
\overline{x}_3\overline{x}_{11}\overline{x}_{11} &= \overline{x}_7\overline{x}_7\overline{x}_{11} \\
&= \overline{x}_5\overline{x}_7\overline{x}_{13} \\
&= \overline{x}_3\overline{x}_5\overline{x}_{17},
\end{aligned}$$

but $\overline{x}_{11}\overline{x}_{11} - \overline{x}_5\overline{x}_{17} \neq 0$ in G^{20}.

3.1 Expressing the Prime Counting Function in terms of Projective Dimension

In this subsection, we assume that Goldbach's conjecture is true. Let us recall some notation we developed in this case and explain what they look like here. For each $m \geq 1$, we have

$$\begin{aligned}
R^m &= \Bbbk[\{x_p, x_{2k} \mid p, 2k \leq m\}] \\
I^m &= \langle \{x_p x_q - x_{p+q} \mid p, q \leq m\} \rangle \\
G^m &= R^m / I^m \\
F^m &\text{ is the minimal resolution of } G^m \text{ over } R \\
F &\text{ is the minimal resolution of } G \text{ over } R \\
\varepsilon(m) &= \operatorname{depth}_{R^m}(G^m) \\
\rho(m) &= \operatorname{pd}_{R^m}(G^m) = \operatorname{length}(F^m)
\end{aligned}$$

Note that (2) has a very nice interpretation in this case. Indeed, we have

$$\rho(2m) + \varepsilon(2m) = \pi(2m) + m - 3, \tag{4}$$

where π is the usual prime-counting function. For m sufficiently large, we should have

$$\varepsilon(2m) = \#\{p \mid m \leq p \leq 2m\}.$$

The idea is that if $p_1, \dots, p_d$ are all of the primes between m and $2m$, then it is easy to check that $\boldsymbol{x} = x_{p_1}, \dots, x_{p_d}$ is a G^{2m}-regular sequence (the hard part is showing that this is in fact a maximal G^{2m}-regular sequence, however we will assume this is true for the moment). Thus we have $\pi(m) = \pi(2m) - \varepsilon(2m)$, and so we should be able to re-express (4) as

$$\pi(m) = \rho(2m) - m + 3. \tag{5}$$

For example, a calculation using a computer algebra program such as Singular shows $\rho(18) = 10$. Then since $\pi(9) = 4$, we have $4 = 10 - 9 + 3$, and thus (5) holds on the nose in this case.

3.1.1 Explicit Calculations of the $\Bbbk$-Complex $F_{n,d}$

Example 3.1. Let's describe $\widetilde{F}_{18,2}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$\begin{aligned}
\widetilde{F}_{1,18,2} &= \Bbbk e_{5,13} + \Bbbk e_{7,11} \\
\widetilde{F}_{0,18,2} &= R_{18,2} = \Bbbk x_5 x_{13} + \Bbbk x_7 x_{11} + \Bbbk x_{18} \\
\widetilde{F}_{-1,18,2} &= G_{18,2} = \Bbbk \overline{x}_5 \overline{x}_{13},
\end{aligned}$$

and $\widetilde{F}_{i,18,2} = 0$ for all $i \neq -1, 0, 1$. The differential is the unique R-linear map defined by $\mathrm{d}(e_{5,13}) = x_5 x_{13} - x_{18}$ and $\mathrm{d}(e_{7,11}) = x_7 x_{13} - x_{18}$. After choosing ordered bases, we can express $\widetilde{F}_{18,2}$ in the form

$$0 \longrightarrow \Bbbk^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} G_{18,2} \longrightarrow 0$$

As expected, we have

$$\delta_{18,2} = \chi(F_{18,2}) = 3 - 2 = 1.$$

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Next, let's describe $\widetilde{F}_{23,3}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$\begin{aligned}
\widetilde{F}_{2,23,3} &= \Bbbk e_{5,7,11} \\
\widetilde{F}_{1,23,3} &= \Bbbk x_{13} e_{3,7} + \Bbbk x_{13} e_{5,5} + \Bbbk x_7 e_{3,13} + \Bbbk x_7 e_{5,11} + \Bbbk x_5 e_{7,11} + \Bbbk x_5 e_{5,13} \\
\widetilde{F}_{0,23,3} &= R_{23,3} = \Bbbk x_{13} x_{10} + \Bbbk x_7 x_{16} + \Bbbk x_5 x_{18} + \Bbbk x_5 x_7 x_{11} + \Bbbk x_3 x_7 x_{13} + \Bbbk x_5^2 x_{13} \\
\widetilde{F}_{-1,23,3} &= G_{23,2} = \Bbbk \overline{x}_5 \overline{x}_7 \overline{x}_{11}
\end{aligned}$$

and $\widetilde{F}_{i,23,3} = 0$ for all $i \neq -1, 0, 1, 2$. The differential is the unique R-linear map defined by

$$\begin{aligned}
\mathrm{d}(e_{5,7,11}) &= x_5 e_{7,11} - x_5 e_{5,13} + x_{13} e_{5,5} - x_{13} e_{3,7} + x_7 e_{3,13} - x_7 e_{5,11} \\
\mathrm{d}(e_{3,7}) &= x_3 x_7 - x_{10} \\
\mathrm{d}(e_{5,5}) &= x_5 x_5 - x_{10} \\
\mathrm{d}(e_{3,13}) &= x_3 x_{13} - x_{16} \\
\mathrm{d}(e_{5,11}) &= x_5 x_{11} - x_{16} \\
\mathrm{d}(e_{7,11}) &= x_7 x_{11} - x_{18} \\
\mathrm{d}(e_{5,13}) &= x_5 x_{13} - x_{18}.
\end{aligned}$$

After choosing ordered basis, we can express $\widetilde{F}_{23,3}$ in the form

$$0 \longrightarrow \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}} \Bbbk^6 \xrightarrow{M} \Bbbk^6 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}} G_{23,3} \longrightarrow 0$$

where M is a matrix whose entries are either -1, 0, or 1.