Advanced Numerical Analysis Homework 10

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1 Problem 1

Exercise 1. Show that SOR fails to converge for any matrix with $\omega \leq 0$ or $\omega \geq 2$.

Solution 1. Recall the SOR iteration is

$$x^{(k+1)} = (D - \omega E)^{-1}((1-\omega)D + \omega F)x^{(k)} + \omega b).$$

Multiplying both sides by $D - \omega E$ gives us:

$$(D - \omega E)x^{(k+1)} = ((1-\omega)D + \omega F)x^{(k)} + \omega b.$$

Then multiplying both sides by D^{-1} gives us

$$(1 - \omega D^{-1}E)x^{(k+1)} = ((1-\omega)I + \omega D^{-1}F)x^{(k)} + \omega D^{-1}b.$$

Thus the SOR iteration matrix is given by

$$(1 - \omega D^{-1}E)^{-1}((1-\omega)I + \omega D^{-1}F).$$

Now let $\{\lambda_i\}$ denote the eigenvalues of the SOR iteration matrix. Then

$$|\lambda_1 \cdots \lambda_n| = \left|\det((1-\omega)I + \omega D^{-1}F)\right| = |1-\omega|^n.$$

Therefore at least one eigenvalue λ_i must exist such that $|\lambda_i| \geq |1-\omega|$. In particular, in order for convergence to hold, we must have $|1-\omega| < 1$. In other words, we must have $0 < \omega < 2$.

2 Problem 2

Exercise 2. Consider an $n \times n$ symmetric tridiagonal matrix of the form

$$T(\alpha) = \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & -1 \\ & & & -1 & \alpha \end{pmatrix},$$

where α is a real parameter.

1. Verify that the eigenvalues of $T(\alpha)$ are given by

   $$\lambda_k = \alpha - 2\cos\frac{k\pi}{n+1},$$

   where $k = 1, \ldots, n$. Also verify that the eigenvector associated with λ_k is

   $$v_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots, \sin\frac{nk\pi}{n+1}\right)^\top.$$

   Under what condition on α is $T(\alpha)$ positive-definite?

2. Let $\alpha = 2$. Show that $T(2)$ is obtained by setting up a uniform mesh on $[a, b]$, namely,

$$x_k = a + k\frac{b-a}{n+1} \quad 0 \le k \le n+1$$

and applying the 2nd order centered finite difference approximation for the 1D Poisson equation $-u''(x) = f(x)$ with Dirichlet boundary condition $u(a) = u_0$ and $u(b) = u_{n+1}$ (both values given).

3. Does the Jacobi iteration converge for $T(2)$? If so,what is the convergence factor?

4. Does Gauss-Seidel converge for $T(2)$? If so, what is the convergence factor?

5. For which values of ω does the SOR iteration converge for $T(2)$?

Solution 2. 1. Write $T = T(\alpha)$, $v = v_k$, $\lambda = \lambda_k$, and $\theta = k\pi/(n+1)$ in order to simplify notation. Note that for each $j \ge 1$, we have the following trigonometric identity:

$$2\cos\theta\sin(j\theta) = \sin((j-1)\theta) + \sin((j+1)\theta). \tag{2.1}$$

Indeed, we have

$$\begin{aligned}
\frac{\sin((j-1)\theta) + \sin((j+1)\theta)}{\sin(j\theta)} &= \frac{e^{i(j-1)\theta} - e^{-i(j-1)\theta} + e^{i(j+1)\theta} - e^{-i(j+1)\theta}}{e^{ij\theta} - e^{-ij\theta}} \\
&= \frac{e^{ij\theta}e^{-i\theta} - e^{-ij\theta}e^{i\theta} + e^{ij\theta}e^{i\theta} - e^{-ij\theta}e^{-i\theta}}{e^{ij\theta} - e^{-ij\theta}} \\
&= \frac{\left(e^{ij\theta} - e^{-ij\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)}{e^{ij\theta} - e^{-ij\theta}} \\
&= e^{i\theta} + e^{-i\theta} \\
&= 2\cos\theta.
\end{aligned}$$

In particular, note that in the case were $j = 1$, (2.1) simplifies to the usual double sine angle formula, and in the case where $j = n$, then (2.1) simplifies to:

$$2\cos\sin(n\theta) = \sin((n-1)\theta).$$

Therefore we have:

$$\begin{aligned}
Tv &= \begin{pmatrix} \alpha & -1 & & & \\ -1 & \alpha & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \alpha & -1 \\ & & & -1 & \alpha \end{pmatrix} \begin{pmatrix} \sin\theta \\ \vdots \\ \sin j\theta \\ \vdots \\ \sin n\theta \end{pmatrix} \\
&= \begin{pmatrix} \alpha\sin\theta - \sin 2\theta \\ \vdots \\ -\sin((j-1)\theta) + \alpha\sin j\theta - \sin((j+1)\theta) \\ \vdots \\ -\sin((n-1)\theta) + \alpha\sin n\theta \end{pmatrix} \\
&= \begin{pmatrix} \alpha\sin\theta - 2\cos\theta\sin\theta \\ \vdots \\ \alpha\sin j\theta + 2\cos\theta\sin(j\theta) \\ \vdots \\ -2\cos\sin(n\theta) + \alpha\sin n\theta \end{pmatrix} \\
&= \begin{pmatrix} (\alpha - 2\cos\theta)\sin\theta \\ \vdots \\ (\alpha - 2\cos\theta)\sin(j\theta) \\ \vdots \\ (\alpha - 2\cos\theta)\sin n\theta \end{pmatrix} \\
&= \lambda v.
\end{aligned}$$

It follows that the eigenvalues of $T(\alpha)$ are λ_k for each $1 \le k \le n$ (indeed the λ_k are all distinct from each other, and since T is $n \times n$, these must be all of the eigenvalues of T). Note that

$$\begin{aligned} T(\alpha) \text{ is positive-definite} &\iff \lambda_k > 0 \text{ for all } k \\ &\iff \alpha > 2\cos\theta_k \text{ for all } k \\ &\iff \alpha > 2\cos\theta_1. \end{aligned}$$

2. We now consider $\alpha = 2$. We also simplify life by considering $a = 0$ and $b = 1$ and we assume $f(0) = 0 = f(1)$. In this case, we have $\boldsymbol{x} = (x_1, \dots, x_{n+1})^\top$ where $x_k = k/(n+1)$ for each $0 \le k \le n+1$. Note that

$$\begin{aligned} -Tf(\boldsymbol{x}) &= \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_k) \\ \vdots \\ f(x_n) \end{pmatrix} \\ &= \begin{pmatrix} f(x_2) - 2f(x_1) \\ \vdots \\ f(x_{k-1}) - 2f(x_k) + f(x_{k+1}) \\ \vdots \\ f(x_{n-1}) - 2f(x_n) \end{pmatrix} \\ &\approx f''(\boldsymbol{x})/(n+1). \end{aligned}$$

In particular, T is obtained by setting up a 2nd order finite difference approximation for the Poisson equation

$$-u''(x) = f(x)$$

with Dirichlet boundary condition $u(0) = 0 = u(1)$.

3. Note that T is symmetric and positive-definite (it is positive-definite because $2 \ge 2\cos\theta$ for any θ). Let

$$C = C_\omega = 1 - (\omega/2)T = (3\omega/2) \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} = -(3\omega/2)S,$$

where we set $S = T(0)$. Then

$$\begin{aligned} \text{The Jacobi method converges} &\iff \rho(C) < 1 && \rho \text{ is spectral radius} \\ &\iff 0 < \omega < 4/\lambda_{\max}(T) \\ &\iff 0 < \omega < 2/\left(\cos\frac{k\pi}{n+1}\right). \end{aligned}$$

(Yes because $T(2)$ is diagonally dominant.

4. Yes because T is symmetric positive-definite.

5. SOR convergence is guaranteed if $0 < \omega < 2$ since $T(2)$ is positive-definite.