Associators and Multiplicators

Let $\Bbbk$ be a field, let A be a (possibly non-associative) unital and graded-commutative $\Bbbk$-algebra with $A_0 = \Bbbk$. Let X and Y be (possibly non-associative) unital and graded-commutative A-modules such that restricting the A-scalar actions on X and Y to $\Bbbk$-scalar action gives them the structure of $\Bbbk$-vector spaces. For $a_1, a_2, a_3 \in A$ and $x \in X$, we set

$$[a_1, a_2, a_3] = (a_1 a_2) a_3 - a_1 (a_2 a_3) \quad \text{and} \quad [a_1, a_2, x] = (a_1 a_2) x - a_1 (a_2 x).$$

These are called **associators**; they measure failure of associativy. They give rise to graded $\Bbbk$-trilinear maps $A^3 \to A$ and $A^3 \to X$ respectively. Note that since X is a $\Bbbk$-vector space, we have $[c_1, c_2, x] = 0$ for all $c_1, c_2 \in \Bbbk$ and $x \in X$ by assumption. Next let $\varphi \colon X \to Y$ be a $\Bbbk$-linear map. For $a, a_1, a_2 \in A$ and $x \in X$, we set

$$[a, x]_\varphi = \varphi(ax) - a\varphi(x) \quad \text{and} \quad [a_1, a_2, x]_\varphi = \varphi([a_1, a_2, x]) - [a_1, a_2, \varphi(x)].$$

These are called **multiplicators** and 2-**multiplicators** respectively. They give rise to a graded $\Bbbk$-bilinear $A \times X \to Y$ and a graded $\Bbbk$-trilinear map $A^2 \times X \to Y$ respectively. The multiplicator of φ measures the failure for φ to being an A-module homomorphism. We always write φ in the subscript of the 2-multiplicator $[a_1, a_2, x]_\varphi$ in order to avoid confusion with the associator $[a_1, a_2, x]$. We have the following identities:

1. For all $a_1, a_2, a_3 \in A$ and $x \in X$ we have
$$a_1[a_2, a_3, x] = [a_1 a_2, a_3, x] - [a_1, a_2 a_3, x] + [a_1, a_2, a_3 x] - [a_1, a_2, a_3]x.$$
2. For all $a_1, a_2 \in A$ and $x \in X$, we have
$$a_1[a_2, x]_\varphi = [a_1 a_2, x]_\varphi - [a_1, a_2 x]_\varphi + [a_1, a_2, x]_\varphi.$$
3. For all $a_1, a_2, a_3 \in A$ and $x \in X$, we have
$$a_1[a_2, a_3, x]_\varphi = [a_1 a_2, a_3, x]_\varphi - [a_1, a_2 a_3, x]_\varphi + [a_1, a_2, a_3 x]_\varphi - [[a_1, a_2, a_3], x]_\varphi + [a_1, [a_2, a_3, x]]_\varphi - [a_1, a_2, [a_3, x]_\varphi].$$
4. In particular, if φ is 2-multiplicative, then we have
$$a_1[a_2, x]_\varphi = [a_1 a_2, x]_\varphi - [a_1, a_2 x]_\varphi \quad \text{and} \quad [a_1, a_2, [a_3, x]_\varphi] = [[a_1, a_2, a_3], x]_\varphi - [a_1, [a_2, a_3, x]]_\varphi.$$
5. If Z is another $\Bbbk$-vector space which is equipped with a (possibly non-associative) unital and graded-commutative A-scalar action, and $\psi \colon Y \to Z$ is a graded $\Bbbk$-linear map, then for all $a \in A$ and $x \in X$, we have
$$[a, x]_{\psi\varphi} = \psi([a, x]_\varphi) + [a, \varphi(x)]_\psi$$