Advanced Numerical Analysis Homework 1

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1 Problem 1

Exercise 1. Solve the following:

1. Find the absolute and relative condition numbers of $f(x) = e^{-2x}$ and $g(x) = \ln^3 x$. For what values of x are these functions sensitive to perturbations?

2. Let $x_1, x_2 \in \mathbb{R}^+$ and let $f(\boldsymbol{x}) = f(x_1, x_2) = x_1^{x_2}$. Find the relative condition number of $f(\boldsymbol{x})$. For what range of values of x_1 and x_2 is the problem ill-conditioned?

Solution 1. 1. First we find the absolute condition number of f and g at x. Since f and g are differentiable everywhere they are defined, we see that

$$\begin{aligned} \widehat{\kappa}_f(x) &= |f'(x)| \\ &= |-2e^{-2x}| \\ &= 2e^{-2x} \end{aligned} \qquad\qquad \begin{aligned} \widehat{\kappa}_g(x) &= |g'(x)| \\ &= |(3/x)\ln^2 x| \\ &= \frac{3\ln^2 x}{|x|}. \end{aligned}$$

Next we find the relative condition numbers of f and g at x. Since f and g are differentiable everywhere they are defined, we see that

$$\begin{aligned} \kappa_f(x) &= \frac{|f'(x)||x|}{|f(x)|} \\ &= \frac{2e^{-2x}|x|}{e^{-2x}} \\ &= 2|x|. \end{aligned} \qquad\qquad \begin{aligned} \kappa_g(x) &= \frac{|g'(x)||x|}{|g(x)|} \\ &= \frac{\frac{3\ln^2 x}{|x|}|x|}{\left|\ln^3 x\right|} \\ &= \frac{3}{|\ln x|}. \end{aligned}$$

In particular, g is sensitive to perturbations when x is near 1.

2. Since f is differentiable we see that

$$\begin{aligned} \kappa_f(\boldsymbol{x}) &= \frac{\|\mathrm{J}_f(\boldsymbol{x})\|\|\boldsymbol{x}\|}{|f(\boldsymbol{x})|} \\ &= \frac{\left\|\begin{pmatrix} x_2 x_1^{x_2-1} & x_1^{x_2}\ln x_1 \end{pmatrix}\right\| \|\boldsymbol{x}\|}{|x_1^{x_2}|} \\ &= \frac{\left(\left|x_2 x_1^{x_2-1}\right| + |x_1^{x_2}\ln x_1|\right)\max\{|x_1|, |x_2|\}}{|x_1^{x_2}|}. \end{aligned}$$

2 Problem 2

Exercise 2. Consider the following recurrence:

$$x_{k+1} = 111 - \frac{1130 - \frac{3000}{x_{k-1}}}{x_k},$$

whose general solution is

$$x_k = \frac{100^{k+1}a + 6^{k+1}b + 5^{k+1}c}{100^k a + 6^k b + 5^k c},$$

where a, b, c depend on the initial values. Given $x_0 = 11/2$ and $x_1 = 61/11$, we have $a = 0$ and $b = 1 = c$. So in this case we have

$$x_k = \frac{6^{k+1} + 5^{k+1}}{6^k + 5^k},$$

1. Show that this gives a monotonically increasing sequence to the limit of value 6.
2. Implement this recurrence in MATLAB. Plot (x_k) and compare it with the exact solution. Explain any major discrepancies you see. What is the condition number of the limit of this particular sequence as a function of x_0 and x_1?

Solution 2. labelsol 1. To see that it is monotonically increasing, let $f_k(x, y) = (x^{k+1} + y^{k+1})/(x^k + y^k)$ and observe that

$$\partial_k f_k(x, y) = \partial_k \left(\frac{x^{k+1} + y^{k+1}}{x^k + y^k} \right) = \frac{(y - x)x^k y^k \ln(y/x)}{(x^k + y^k)^2}.$$

So in particular, we have

$$\partial_k f_k(5, 6) = \frac{5^k 6^k \ln(6/5)}{(5^k + 6^k)^2} > 0.$$

So in particular, $f_k(5, 6) = x_k$ is strictly increasing in k. Next observe that

$$\begin{aligned}
\lim_{k\to\infty} x_k &= \lim_{k\to\infty} \left(\frac{6^{k+1} + 5^{k+1}}{6^k + 5^k} \right) \\
&= \lim_{k\to\infty} \left(\frac{6^{k+1}}{6^k + 5^k} \right) + \lim_{k\to\infty} \left(\frac{5^{k+1}}{6^k + 5^k} \right) \\
&= \lim_{k\to\infty} \left(\frac{6^{k+1}}{6^k} \right) + \lim_{k\to\infty} \left(\frac{5^{k+1}}{6^k} \right) \\
&= \lim_{k\to\infty} \left(\frac{6^{k+1}}{6^k} \right) + \lim_{k\to\infty} \left(\frac{5^{k+1}}{6^k} \right) \\
&= 6,
\end{aligned}$$

where we used the fact that $\lim\limits_{k\to\infty} (6^k/5^k) = \infty$ (i.e. 6^k grows much faster than 5^k which is why we are allowed to remove 5^k from the denominator in the third line).

2.

3 Problem 3

Exercise 3. Let

$$p_{24}(x) = (x - 1)(x - 2) \cdots (x - 24) = x^{24} + a_{23}x^{23} + \cdots + a_1 x + a_0,$$

where

$$\begin{aligned} a_{14} &= 9.2447 \times 10^{16} \\ a_{15} &= -5.7006 \times 10^{15} \\ a_{16} &\approx 2.9089 \times 10^{14} \\ a_{17} &\approx -1.2191 \times 10^{13} \\ a_{18} &\approx 4.1491 \times 10^{11}. \end{aligned}$$

Evaluate the relative condition number of the kth root $x_k = k$ subject to the perturbation of a_k for $k = 14, 15, \ldots, 18$ and find the root that is most sensitive to perturbation of the corresponding coefficient. Use the attached MATLAB data file wilk24mc.mat containing the coefficients $a_{24}, a_{23}, \ldots, a_1, a_0$ and use MATLAB's roots function to find the roots. Compare with the true roots and comment on what you see.

Solution 3.