MATH 8100 Fall 2021 Homework 4 due October 18, 2021.

100 points (NLP methods; LP modeling)

Guidance for writing your assignment:
a) make sure that your writing is clear and legible
b) start your answer to each part of a problem on a new page
c) wherever appropriate, underline or rewrite the final answer
d) show all work for full credit

Please make every effort to follow this guidance and facilitate the reading of your assignment. The assignments that do not follow this guidance will be returned.

1. Consider the unconstrained optimization problem:

$$\text{minimize} \qquad f(\mathbf{x}) = 3x_1^2 + 3x_2^2 - 2x_1x_2 + 2x_1 - 6x_2$$

a) Let the initial point $\mathbf{x}^1 = (1, 2)^T$. Perform one iteration of

i) (15 points) the steepest descent method using the negative gradient of the objective function as the search direction. Report $\mathbf{x}^2$.

ii) (15 points) Newton's method and report $\mathbf{x}^2$.

b) (2 points) Compare the points $\mathbf{x}^2$ you found in parts i) and ii) above. What do you observe?

c) (5 points) What is the meaning of the points $\mathbf{x}^2$ for the minimization problem?

2. Consider the constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) = x_1^2 + x_2^2 \\ \text{subject to} & h(\mathbf{x}) = x_1 + x_2 - 1 = 0 \end{array}$$

a) (3 points) Use the penalty function $\Psi(\mathbf{x}) = (h(\mathbf{x}))^2$ and formulate the Approximation Problem (AP).

b) (10 points) Find an optimal solution and the optimal objective value to the AP.

c) (5 points) Do not solve the original problem but using the results of part b), give an optimal solution and the optimal objective value to the original problem.

3. (25 points) A factory has two different machines, each of which can be used to manufacture the same three different products. The following table gives the number of units of product *k* that can be produced by machine *i* if machine *i* is used the entire working day to make product *k*.

		product		
		1	2	3
machine	1	8	2	9
	2	3	5	6

If a machine is used for a fraction of the day to produce some product, that fraction of an entire day's output of that product is produced, and no production is lost in switching from one product to another. Thus, for example, if machine 1 is used for half a day to produce product 1 and then makes product 2 for the remainder of the day, it will produce 4 units of product 1 and 1 unit of product 2.

All three products can be produced in noninteger amounts. However, products 1, 2, and 3 must be produced in the preassigned proportions ½, ¼, ¼, respectively; that is, the quantity of product 1 must be ½ the total quantity of all products produced, and equal quantities of products 2 and 3 must be made.

Clearly define the decision variables and formulate an LP that will determine what fraction of the day each machine should be used to produce each product so as to maximize the total quantity of products produced.

4. (10 points) Consider the following homogenous system of equations $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix. This system has always a trivial solution $\mathbf{x} = (0, \ldots, 0) \in R^n$, but may also have nontrivial solutions ($\mathbf{x} \neq \mathbf{0}$). Clearly define the decision variables and formulate an LP whose optimal solution indicates whether or not this linear system has nonnegative solution(s) ($x_j \geq 0, j = 1, \ldots, n$) ?

5. (10 points) Formulate an LP for finding a vector satisfying

$$4x_1 + x_2 \leq 5 \quad \text{and} \quad x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

and having the maximum of

$$2x_1 - x_2 \quad \text{and} \quad -3x_1 + 2x_2$$

as small as possible.