Avramov Obstruction Notes

Let $f\colon R \to S$ be a finite local ring homomorphism such that the induced map on their common residue field κ is identity and let M be a finitely generated S-module. Let F be an MDG R-algebra resolution of S such that F is minimal. Next let X be any R-free resolution of M. Then the usual S-module structure on M induces an MDG F-module structure on X as follows: we choose a left scalar-multiplication $\mu_X\colon F \otimes_R X \to X$, denoted $a \otimes x \mapsto a \star_{\mu_X} x = ax$, which extends the usual left scalar-multiplication $S \otimes_R M \to M$ such that μ_X is unital, meaning $1x = x$ for all $x \in X$. Note that μ_X is unique up to homotopy. From the left scalar-multiplication, we obtain a right scalar multiplication $X \otimes_R F \to X$ by setting

$$xa := (-1)^{|a||x|}ax$$

for all $a \in F$ and $x \in X$. Then equipping X with these scalar-multiplication maps gives it the structure of an MDG F-module. We say μ_X gives X the structure of an MDG F-module in this case. If μ_X happens to be associative, then we say μ_X gives X the structure of a DG F-module.

Note that the map μ_X induces a map

$$\mathrm{H}(F) \otimes_R \mathrm{H}(X) \to \mathrm{H}(X), \tag{1}$$

which is given by $\overline{a} \otimes \overline{x} \mapsto \overline{a \star_{\mu_X} x} = \overline{ax}$, where $\overline{a} \in \mathrm{H}(F)$ and $\overline{x} \in \mathrm{H}(X)$. Since homotopic chain maps induces the same map in homology, the map (1) does not depend on the choice of μ_X (which is unique up to homotopy).

$$\mathrm{Tor}^R(S,\kappa) \otimes \mathrm{Tor}^R(M,\kappa) \to \mathrm{Tor}^R(M,\kappa). \tag{2}$$

Indeed, we have $\mathrm{H}(F) = \mathrm{Tor}^R(S,\kappa)$ and $\mathrm{H}(X) = \mathrm{Tor}^R(M,\kappa)$. Thus to give (2), it suffices to describe

$$\mathrm{H}(F) \otimes \mathrm{H}(X) \to \mathrm{H}(X),$$

which is given by $\overline{a} \otimes \overline{x} \mapsto \overline{ax}$, where $\overline{a} \in \mathrm{H}(F)$ and $\overline{x} \in \mathrm{H}(X)$.

Avramov considers the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Tor}^R_+(S,\kappa) \otimes \mathrm{Tor}^R(M,\kappa) \arrow[r] \arrow[d] & \mathrm{Tor}^R(M,\kappa) \arrow[d] \\
\mathrm{Tor}^S_+(S,\kappa) \otimes \mathrm{Tor}^S(M,\kappa) \arrow[r, "0"] & \mathrm{Tor}^S(M,\kappa)
\end{tikzcd}$$

The map ψ is induced by the map $F \otimes_R X \to X$. In particular, note that $\mathrm{H}_+(F) = \mathrm{Tor}^R_+(S,\kappa)$, and $\mathrm{H}(X) = \mathrm{Tor}^R(M,\kappa)$. This gives a canonical map of graded κ-vector spaces:

$$\frac{\mathrm{Tor}^R(M,\kappa)}{\mathrm{Tor}^R_+(S,\kappa)\mathrm{Tor}^R(M,\kappa)} \to \mathrm{Tor}^S(M,\kappa).$$

The kernel of this map is denoted $\mathrm{o}^f(M)$ and is called the **obstruction to the existence of multiplicative sructure** (on the minimal R-free resolution of M).

0.1 Buchsbaum and Eisenbud Conjecture

Suppose I is an ideal of R and $\boldsymbol{x} = x_1, \dots, x_g$ is an R-regular sequence contained in I. Then we consider $S = R/\langle \boldsymbol{x} \rangle$ and $M = R/I$. In this case, we can choose F to be the koszul algebra $\mathcal{K}(\boldsymbol{x})$ (in particular F is associative). Any expression of the x_i in terms of the generators for I yields a canonocal comparison map $F \to X$. With this notation in mind, Buchsbaum and Eisenbud made the following conjecture:

Corollary. *X can be given the structure of a DG F-module such that the comparison map $F \to X$ is a DG F-module homomorphism.*

The reason why this conjecture is interesting is because it's validity would imply important lower bounds for the ranks of the syzygies of R/I (where R is assumed to be a domain).

0.2 Avramov's Obstruction

Theorem 0.1. *Suppose the minimal R-free resolution F of S has the structure of a DG algebra. If* $o^f(M) \neq 0$*, then no DG F-module structure exists on the minimal R-free resolution X of M. In particular, in for X to possess the structure of a DG F-module, it is necessary that we have* $o^f(M) = 0$*.*