Homological Associativity of Differential Graded Algebras and Gröbner Bases

Michael Nelson

Abstract

We investigate associativity of multiplications on chain complexes over commutative noetherian rings from two perspectives. First, we introduce a natural associator subcomplex and show how its homology can detect associativity. Second, we use Gröbner bases to compute associators.

1 Introduction

In this paper, we study algebraic structures that we can attach to free resolutions. Our motivation is the following: let $(R, \mathfrak{m}, \Bbbk)$ be a local (or standard graded) commutative noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let $F = (F, \mathrm{d})$ be the minimal free resolution of R/I over R. The usual multiplication map $R/I \otimes_R R/I \to R/I$ can be lifted to a chain map $\mu \colon F \otimes_R F \to F$ defined by $a_1 \otimes a_2 \mapsto a_1 \star_\mu a_2$ where $a_1, a_2 \in F$ (where we simplify notation to $a_1 \star_\mu a_2 = a_1 a_2$ whenever μ is clear from context). Further, we can choose μ to be unital (with $1 \in F_0 = R$ being the identify element) and strictly graded-commutative; see Definition (2.1). In this case we call μ a **multiplication** on F, and when we equip F with this multiplication, we say F is a multiplicative differential graded algebra (MDG, for short). See Section 1 below for foundational material on MDG algebras and modules. It was first shown that F always possesses an MDG algebra structure by Buchsbaum and Eisenbud in [BE77], and in that paper they posed the following question:

Question 1.1: Does F possess the structure of a DG algebra? In other words, can μ be chosen such that it is associative?

One reason this question is interesting is that when we know the answer is "yes", then we gain a lot of information about the "shape" of F. For instance, Buchsbaum and Eisenbud proved that if we further assume R is a domain and we know that an associative multiplication on F exists, then one obtains important lower bounds of the Betti numbers $\beta_i = \beta_i^R(R/I)$. In particular, let $\boldsymbol{t} = t_1, \dots, t_g$ be a maximal R-sequence contained in I and let E be the Koszul algebra which resolves $R/\boldsymbol{t}$ over R. Any expression of the t_i in terms of the generators for I yields a canonical comparison map $E \to F$. Buchsbaum and Eisenbud showed that under these assumptions, this comparison map $E \to F$ is injective, hence we get the lower bound $\beta_i \geq \binom{g}{i}$ for each $i \leq g$. It turns out however, that the answer to Question 1.1 is that F need not have a DG algebra structure on it (see [Avr81, Kat19, Sri92] for counterexamples), so Buchsbaum and Eisenbud's proof of these lower bounds would fail in these cases. Nonetheless, these lower bounds are still conjectured to hold. It is known as the (local) Buchsbaum-Eisenbud-Horrocks (BEH) conjecture (see [Erm10, VW23, Wal17] for more on this topic):

Conjecture 1. *(BEH Conjecture). Let M be a nonzero R-module of finite projective dimension. Then we have*

$$\beta_i(M) \geq \binom{\operatorname{codim} M}{i}$$

for all i, where $\beta_i(M)$ is the ith Betti number of M and where $\operatorname{codim} M = \operatorname{height}(\operatorname{Ann} M)$.

One of the starting points for this paper is based on the observation that by slightly modifying Buchsbaum and Eisenbud's proof one can still obtain these lower bounds even in cases where it is known that we cannot choose μ to be associative. Indeed, we just need to find a multiplication μ on F together with a comparison map $\varphi \colon E \to F$ such that $\varphi \colon E \to F$ is multiplicative, meaning

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$$

for all $a_1, a_2 \in E$. The proof given by Buchsbaum and Eisenbud which shows $\varphi \colon E \to F$ is injective would still apply in this case. Furthermore, in their proof, Buchsbaum and Eisenbud used a property that the Koszul algebra E satisfies, namely that every nonzero DG ideal of E intersects the top degree E_g non-trivially. However there are many other MDG algebras which satisfy this property as well (the property being that every nonzero MDG ideal intersect the top degree non-trivially). Thus one can generalize this result even further by replacing $\boldsymbol{t}$ with an ideal J such that $\boldsymbol{t} \subseteq J \subseteq I$ and such that there exists a multiplication on the minimal free resolution G of R/J over R which satisfies this property. To see that we really do gain a new perspective here, we consider Example (3.4) where it is known that we cannot choose an associative multiplication μ on F yet we can find a multiplicative map $\varphi\colon T \to F$ where T is a Taylor resolution. In general, we would like to choose a multiplication which is as associative as possible. To this end, we pose the following question:

Question 1.2: Equip F with a multiplication μ giving it the structure of an MDG algebra. How can we measure the failure of F to be associative?

We answer this question 1.2 in by studying the maximal associative quotient of F. In short, in Subsection 3.1, we define the **associator** submodule of an MDG module X over an MDG algebra A to be the smallest MDG A-submodule containing all "associators" of X:

$$\langle X \rangle = \langle \{ (a_1 a_2)x - a_1(a_2 x) \mid a_1, a_2 \in A \text{ and } x \in X \} \rangle \subseteq X.$$

It is clear that if X is associative, then $\mathrm{H}(\langle X \rangle) = 0$. The first main result of this paper Theorem (3.1) shows that the converse holds under certain conditions. In Subsection 4.1, we exploit a criterion for exactness. We apply this criterion in our second main result, Theorem (4.1) to demonstrate associativity of exterior extensions. In the final section of this paper, we construct the symmetric DG algebra of an R-complex A which is centered at R (meaning $A_0 = R$ and $A_i = 0$ for all $i < 0$), denoted by $\mathrm{S}_R(A) = S$. This section contains our third result of the paper, namely Theorem (5.3), which says that if we fix a multiplication μ on A, then the quotient $A^{\mathrm{as}} := A/\langle A \rangle$ can be presented as a quotient of S by a DG S-ideal $\mathfrak{s} = \mathfrak{s}(\mu)$ which is constructed from μ in a functorial way. In particular, we can study MDG algebra structures on A by studying certain DG ideals of S. This presentation allows us to use Gröbner bases to help calculate A^{as} when working over an integral domain where we can see how associators naturally arise when performing Buchberger's algorithm to certain set of polynomials with respect to this monomial ordering.

This paper is organized into five sections, the first section being this introduction. In the second section, we work over an arbitrary commutative ring R and we define the category of MDG R-algebras. An MDG R-algebra A is essentially just a DG R-algebra except we don't require the associative rule to hold. We also define the category of MDG A-modules, where an MDG A-module X is essentially just a DG A-module except we do not require the associative rule to hold.

In the third section, we introduce tools which help us measure how far away MDG objects are from being DG objects. In particular, we define the associator of X to be the chain map $[\cdot]\colon A \otimes A \otimes X \to X$ defined on elementary tensors by

$$[a_1 \otimes a_2 \otimes x] = (a_1 a_2)x - a_1(a_2 x) = [a_1, a_2, x]$$

for all $a_1, a_2 \in A$ and $x \in X$, where we denote by $[\cdot,\cdot,\cdot]\colon A \times A \times X \to X$ to be the unique map corresponding to $[\cdot]$ via the universal mapping property of tensor products. We set $\langle X \rangle$ to be the smallest MDG A-submodule of X which contains the image of the associator of X. The quotient $X^{\mathrm{as}} := X/\langle X \rangle$ is called the maximal associative quotient of X: it plays a role analogous to the role of the maximal abelian quotient of a group. We study the homology of $\langle X \rangle$ as well as the homology of X^{as}. In this section we also define and study the multiplicator of a chain map $\varphi\colon X \to Y$, where X and Y are MDG A-modules. This is the chain map $[\cdot]_\varphi\colon A \otimes X \to Y$ defined on elementary tensors by

$$[a \otimes x]_\varphi = \varphi(ax) - a\varphi(x) = [a, x]$$

for all $a \in A$ and $x \in X$, where we denote by $[\cdot,\cdot]\colon A \times X \to Y$ to be the unique map corresponding to $[\cdot]_\varphi$ via the universal mapping property of tensor products.

In the fourth section, we turn our attention towards the associator functor which takes an MDG A-module X to the MDG A-module $\langle X \rangle$ and takes an MDG A-module homomorphism $\varphi\colon X \to Y$ to the restriction map $\varphi\colon \langle X \rangle \to \langle Y \rangle$. Given a short exact sequence

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{1}$$

of MDG A-modules, we obtain an induced sequence of MDG A-modules

$$0 \longrightarrow \langle X \rangle \xrightarrow{\varphi} \langle Y \rangle \xrightarrow{\psi} \langle Z \rangle \longrightarrow 0 \tag{2}$$

which is always exact at $\langle X \rangle$ and $\langle Z \rangle$ but not necessarily exact at $\langle Y \rangle$. In order to ensure exactness of (2) at $\langle Y \rangle$, we need to place a condition on (1); the condition being that Y is an **associative extension** of $\varphi(X)$, meaning $\varphi(X) \cap \langle Y \rangle = \langle \varphi(X) \rangle$. When this happens, we obtain a long exact sequence in homology:

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}\langle Z \rangle \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \mathrm{H}_i\langle X \rangle \arrow[r] & \mathrm{H}_i\langle Y \rangle \arrow[r] & \mathrm{H}_i\langle Z \rangle \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \mathrm{H}_{i-1}\langle X \rangle \arrow[r] & \cdots & \end{tikzcd} \tag{3}$$

We end this section with an application of this long exact sequence to certain exterior extensions. In a future paper, we would like to assign a finite number to a multiplication μ on a minimal free resolution F of a cyclic R-module over R where R is a local noetherian ring. This quantity should measure the failure for μ to being associative. We believe studying such exterior extensions will help us to move closer towards that goal.

In the final section of this paper, we construct the symmetric DG algebra of an R-complex A which is centered at R (meaning $A_0 = R$ and $A_i = 0$ for all $i < 0$), denoted by $\mathrm{S}_R(A) = S$. This section contains our third main result, namely Theorem (5.3), which says that if we fix a multiplication μ on A, then the maximal associative quotient of A can be presented as a quotient of S by a DG S-ideal $\mathfrak{s} = \mathfrak{s}(\mu)$ which is constructed from μ in a functorial way. In particular, we can study MDG algebra structures on A by studying certain DG ideals of S. This presentation also has interesting Gröbner basis applications in the case where R is a domain with fraction field K and F is an MDG R-algebra centered at R such that the underlying graded R-module of F is finite and free. Indeed, suppose that

$$F_+ = Re_1 + \cdots + Re_n$$

where $e_1, \dots, e_n$ is an ordered homogeneous basis of F_+ which is ordered in such a way that if $|e_{i'}| > |e_i|$, then $i' > i$, and let $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ be the free non-strict graded-commutative R-algebra generated by $e_1, \dots, e_n$. We will equip $K[\boldsymbol{e}] := K \otimes_R R[\boldsymbol{e}]$ with a specific monomial ordering and show how associators naturally arise when performing Buchberger's algorithm to certain set of polynomials with respect to this monomial ordering. We further demonstrate in Example (5.6) how, with the help of a computer algebra system like Singular, this monomial ordering can help us find associative multiplications on minimal free resolutions. For instance, we used Singular to find an associative multiplication on the minimal free resolution in Example (2.4).

2 MDG Algebras and MDG Modules

We begin by defining MDG algebras. After defining MDG algebras, we then motivate their study by explaining how they arise naturally in the study of minimal free resolutions of cyclic modules.

2.1 MDG Algebras

Let R be a commutative ring and let $A = (A, \mathrm{d})$ be an R-complex. We further equip A with a chain map $\mu \colon A \otimes_R A \to A$. We denote by $\star_\mu \colon A \times A \to A$ (or more simply by $\cdot$ if context is clear) to be the unique graded R-bilinear map which corresponds to μ via the universal mapping property of tensors products. Thus we have

$$\mu(a_1 \otimes a_2) = a_1 \star_\mu a_2 = a_1 a_2$$

for all $a_1, a_2 \in A$, where we further simplify the notation by writing $a_1 \star_\mu a_2 = a_1 a_2$ when context is clear. In order to simplify our notation in what follows, we often refer to the triple (A, d, μ) via its underlying graded R-module A, where we think of A as a graded R-module which is equipped with a differential $\mathrm{d} \colon A \to A$, giving it the structure of an R-complex, and which is further equipped with a chain map $\mu \colon A \otimes_R A \to A$. For instance, if μ satisfies a property (such as being associative), then we also say A satisfies that property.

Definition 2.1. With the notation as above, we make the following definitions:

1. We say A is **unital** if there exists $1 \in A$ such that $1a = a = a1$ for all $a \in A$.
2. We say A is **graded-commutative** if $a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_1$ for all homogeneous $a_1, a_2 \in A$.
3. We say A is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that $a^2 = 0$ for all elements $a \in A$ with $|a|$ odd.
4. We say A is **associative** if $(a_1 a_2)a_3 = a_1(a_2 a_3)$ for all for all $a_1, a_2, a_3 \in A$.

We say A is an **MDG R-algebra** if A is strictly graded-commutative and unital. We call μ the **multiplication** of A just as we call d the **differential** of A. We say A is **centered** at R if $A_0 = R$ and $A_i = 0$ for all $i < 0$. Suppose B is another MDG R-algebra and let $\varphi \colon A \to B$ be a function.

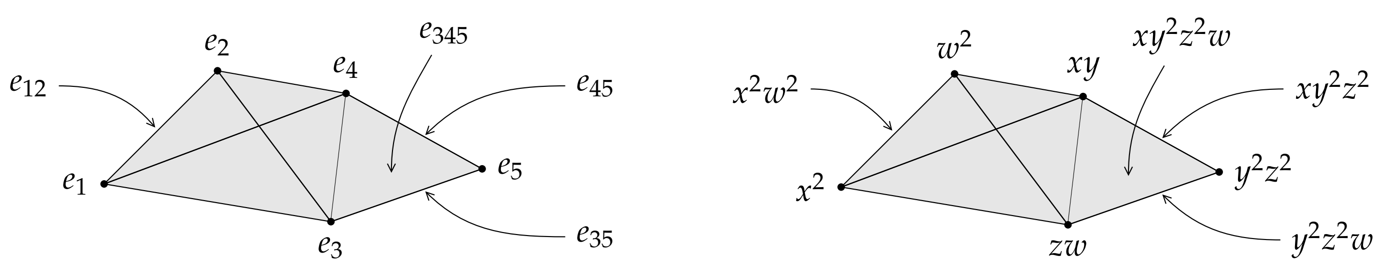
1. We say φ is **unital** if $\varphi(1) = 1$.
2. We say φ is **multiplicative** if $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in A$.

We say $\varphi \colon A \to B$ is an **MDG R-algebra homomorphism** if it is a chain map which is both unital and multiplicative. We denote by $\mathbf{MDG}_R$ to be the category of all MDG R-algebras and MDG R-algebra homomorphisms.

2.1.1 Examples of Multigraded MDG Algebras

In this subsubsection, we consider six examples of multigraded MDG algebras. The first two examples were considered in [Kat19] and [Avr81] respectively and were both shown to be examples of minimal free resolutions which do not admit DG algebra structures on them.

Example 2.1. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{K}} = \boldsymbol{m} = x^2, w^2, xy, zw, y^2z^2$ and let $F_{\mathrm{K}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:



In particular, the homogeneous components of F as a graded R-module are given by

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234},
\end{aligned}$$

and the differential d of F behaves just like the usual boundary map of the simplicial complex above except some monomials can show up as coefficients (so that the differential respects the multidegree). For instance, we have

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

For more details on this construction, see [BPS98]. We now wish to equip F with a multigraded multiplication $\mu_{\mathrm{K}} = \mu$ giving it the structure of a multigraded MDG algebra. Since μ respects the multigrading and satisfies Leibniz rule, we are forced to have:

$$\begin{aligned}
e_1 \star e_5 &= yz^2 e_{14} + xe_{45} & \qquad e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_2 &= e_{12} & e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_2 \star e_5 &= y^2 z e_{23} + we_{35} & e_1 \star e_{23} &= e_{123} \\
& & e_2 \star e_{14} &= -e_{124}
\end{aligned}$$

At this point however, one can conclude that F is not associative since

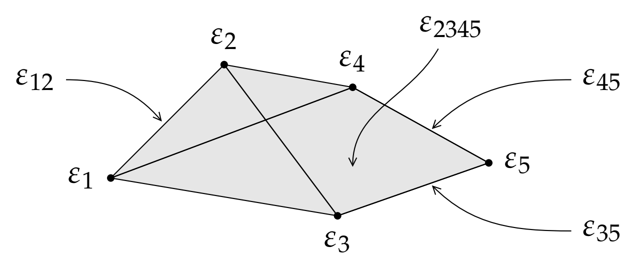
$$[e_1, e_5, e_2] := (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{4}$$

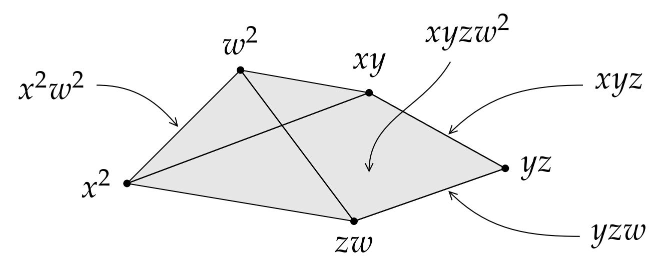
The multiplication is not uniquely determined on all pairs (e_σ, e_τ); for instance there are two possible ways in which μ is defined at the pair (e_5, e_{12}). We assume that μ is defined at (e_5, e_{12}) by

$$e_5 \star e_{12} = yz^2 e_{124} + xyze_{234} + xwe_{345}.$$

Finally, we would still like for μ to be as associative as possible even though we already know it is not associative at the triple (e_1, e_5, e_2). In particular, we want μ to be associative on all triples of the form $(e_\sigma, e_\sigma, e_\tau)$. It turns out this can be done and we will assume that μ is associative on all such triples.

Example 2.2. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{A}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz$ and let $F_{\mathrm{A}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled cellular complex below:





We write down the homogeneous components of F as a graded R-module below:

$$\begin{aligned}
F_0 &= R \\
F_1 &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_2 &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_3 &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_4 &= R\varepsilon_{12345}
\end{aligned}$$

The differential $\mathrm{d}_{\mathrm{A}} = \mathrm{d}$ is defined on the non-simplicial faces as below

$$\begin{aligned}
\mathrm{d}(\varepsilon_{12345}) &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\
\mathrm{d}(\varepsilon_{1345}) &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\
\mathrm{d}(\varepsilon_{2345}) &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}.
\end{aligned}$$

We obtain a multiplication μ_{A} on F_{A} from the one we constructed on F_{K} as follows: first note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{A}}$ induces a multigraded comparison map $\pi \colon F_{\mathrm{K}} \to F_{\mathrm{A}}$ defined by

$$\begin{aligned}
\pi(e_5) &= yz\varepsilon_5 & \pi(e_{345}) &= 0 \\
\pi(e_{35}) &= yz\varepsilon_{35} & \pi(e_{234}) &= \varepsilon_{2345} \\
\pi(e_{45}) &= yz\varepsilon_{45} & \pi(e_{134}) &= \varepsilon_{1345} \\
\pi(e_{34}) &= x\varepsilon_{35} - w\varepsilon_{45} & \pi(e_{1234}) &= \varepsilon_{12345}
\end{aligned}$$

and $\pi(e_\sigma) = \varepsilon_\sigma$ for the remaining homogeneous basis elements. Base changing to R_{yz}, we obtain quasi-isomorphisms $F_{\mathrm{A},yz} \to 0 \leftarrow F_{\mathrm{K},yz}$. In particular, there exists a comparison map $\iota \colon F_{\mathrm{A},yz} \to F_{\mathrm{K},yz}$ which splits comparison map $\pi \colon F_{\mathrm{K},yz} \to F_{\mathrm{A},yz}$. By considering the multigrading as well as the Leibniz rule, we see that

$$\begin{aligned}
\iota(\varepsilon_5) &= e_5/yz & \iota(\varepsilon_{2345}) &= -e_{234} + e_{345}/yz \\
\iota(\varepsilon_{35}) &= e_{35}/yz & \iota(\varepsilon_{1345}) &= e_{134} - e_{345}/yz \\
\iota(\varepsilon_{45}) &= e_{45}/yz & \iota(\varepsilon_{12345}) &= e_{1234}
\end{aligned}$$

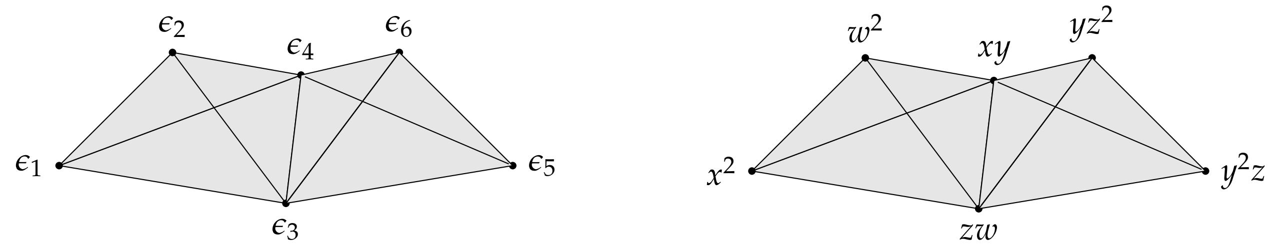
and $\iota(\varepsilon_\sigma) = e_\sigma$ for the remaining homogeneous basis elements. With this in mind, we define a multiplication μ_{A} on F_{A} by transporting the multiplication μ_{K} on $F_{\mathrm{K},yz}$ by setting $\mu_{\mathrm{A}} := \pi\mu_{\mathrm{K}}\iota^{\otimes 2}$. In other words, we have

$$\varepsilon_\sigma \star_{\mu_{\mathrm{A}}} \varepsilon_\tau = \pi(\iota(\varepsilon_\sigma) \star_{\mu_{\mathrm{K}}} \iota(\varepsilon_\tau)) \tag{5}$$

for all homogeneous basis elements $\varepsilon_\sigma, \varepsilon_\tau$ of $F_{\mathrm{A},yz}$. It is straightforward to check that μ_{A} restricts to a multiplication on F_{A} (the coefficients in (5) are in R). Note that μ_{A} is not associative since

$$[\varepsilon_1, \varepsilon_5, \varepsilon_2] = -\mathrm{d}(\varepsilon_{12345}) \neq 0.$$

Example 2.3. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{M}} = \boldsymbol{m} = x^2, w^2, zw, xy, y^2z, yz^2$ and let $F_{\mathrm{M}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ of R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:



We write down the homogeneous components of F as a graded R-module below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= R\epsilon_1 + R\epsilon_2 + R\epsilon_3 + R\epsilon_4 + R\epsilon_5 + R\epsilon_6 \\
F_2 &= R\epsilon_{12} + R\epsilon_{13} + R\epsilon_{14} + R\epsilon_{23} + R\epsilon_{24} + R\epsilon_{34} + R\epsilon_{35} + R\epsilon_{36} + R\epsilon_{45} + R\epsilon_{46} + R\epsilon_{56} \\
F_3 &= R\epsilon_{123} + R\epsilon_{124} + R\epsilon_{134} + R\epsilon_{234} + R\epsilon_{345} + R\epsilon_{346} + R\epsilon_{356} + R\epsilon_{456} \\
F_4 &= R\epsilon_{1234} + R\epsilon_{3456}.
\end{aligned}
$$

The canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{M}}$ induces multigraded comparison maps $\pi_\lambda \colon F_{\mathrm{K}} \to F_{\mathrm{M}}$ where $\lambda \in \Bbbk$ and where π_λ is defined by

$$
\begin{aligned}
\pi_\lambda(e_5) &= \lambda z\epsilon_5 + (1-\lambda)y\epsilon_6 \\
\pi_\lambda(e_{35}) &= \lambda z\epsilon_{35} + (1-\lambda)y\epsilon_{36} \\
\pi_\lambda(e_{45}) &= \lambda z\epsilon_{45} + (1-\lambda)y\epsilon_{46} \\
\pi_\lambda(e_{345}) &= \lambda z\epsilon_{345} + (1-\lambda)y\epsilon_{346}
\end{aligned}
$$

and $\pi_\lambda(e_\sigma) = \epsilon_\sigma$ for the remaining homogeneous basis elements. We will choose $\lambda = 1$ and view F_{K} as a subcomplex of F_{M} via $\pi = \pi_1$. We define a multigraded multiplication μ_{M} on F_{M} so that it extends the multiplication μ_{K} on F_{K}. Considerations of the Leibniz rule and the multigrading tells us that we are already forced to have:

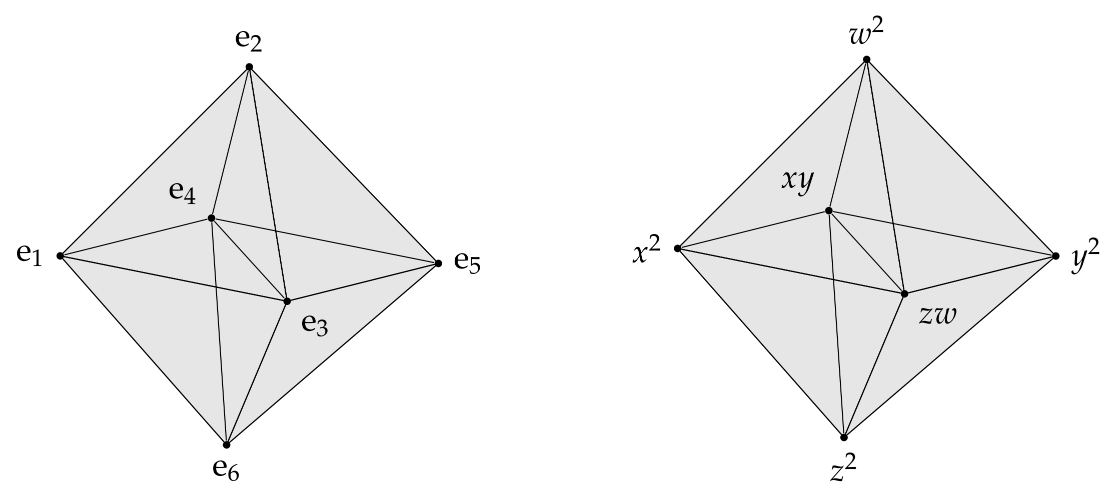
$$
\begin{aligned}
\epsilon_1 \star \epsilon_5 &= yz\epsilon_{14} + x\epsilon_{45} & \qquad \epsilon_1 \star \epsilon_6 &= z^2 e_{14} + xe_{46} \\
\epsilon_2 \star \epsilon_5 &= y^2\epsilon_{23} + w\epsilon_{35} & \qquad \epsilon_2 \star \epsilon_6 &= yz\epsilon_{23} + w\epsilon_{36} \\
\epsilon_2 \star \epsilon_{45} &= -y\epsilon_{234} + w\epsilon_{345} & \qquad \epsilon_2 \star \epsilon_{46} &= -z\epsilon_{234} + w\epsilon_{346} \\
\epsilon_1 \star \epsilon_{35} &= y\epsilon_{134} - x\epsilon_{345} & \qquad \epsilon_1 \star \epsilon_{36} &= z\epsilon_{134} - x\epsilon_{346}.
\end{aligned}
$$

In particular, μ_{K} is not associative (and in fact any multigraded multiplication on F_{M} is not associative) since:

$$
[\epsilon_1, \epsilon_5, \epsilon_2] = -y\mathrm{d}(\epsilon_{1234}) \neq 0 \quad \text{and} \quad [\epsilon_1, \epsilon_6, \epsilon_2] = -z\mathrm{d}(\epsilon_{1234}) \neq 0.
$$

On the other hand, since the multiplication of F_{M} extends the multiplication of F_{K}, we see that the comparison map $F_{\mathrm{K}} \to F_{\mathrm{M}}$ is multiplicative, and hence F_{K} is an MDG subalgebra of F_{M}.

Example 2.4. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{O}} = \boldsymbol{m} = x^2, w^2, zw, xy, y^2, z^2$ and let $F_{\mathrm{O}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:



We write down the homogeneous components of F as a graded R-module below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= R\mathrm{e}_1 + R\mathrm{e}_2 + R\mathrm{e}_3 + R\mathrm{e}_4 + R\mathrm{e}_5 + R\mathrm{e}_6 \\
F_2 &= R\mathrm{e}_{12} + R\mathrm{e}_{13} + R\mathrm{e}_{14} + R\mathrm{e}_{16} + R\mathrm{e}_{23} + R\mathrm{e}_{24} + R\mathrm{e}_{25} + R\mathrm{e}_{34} + R\mathrm{e}_{35} + R\mathrm{e}_{36} + R\mathrm{e}_{45} + R\mathrm{e}_{46} + R\mathrm{e}_{56} \\
F_3 &= R\mathrm{e}_{123} + R\mathrm{e}_{124} + R\mathrm{e}_{134} + R\mathrm{e}_{136} + R\mathrm{e}_{146} + R\mathrm{e}_{234} + R\mathrm{e}_{235} + R\mathrm{e}_{245} + R\mathrm{e}_{345} + R\mathrm{e}_{346} + R\mathrm{e}_{356} + R\mathrm{e}_{456} \\
F_4 &= R\mathrm{e}_{1234} + R\mathrm{e}_{1346} + R\mathrm{e}_{2345} + R\epsilon_{3456}.
\end{aligned}
$$

The canonical map $R/\boldsymbol{m}_{\mathrm{M}} \to R/\boldsymbol{m}_{\mathrm{O}}$ induces an injective multigraded comparison map $F_{\mathrm{M}} \to F_{\mathrm{O}}$ and we identify F_{M} with this subcomplex of F_{O}. This time it is not possible extend the multiplication of F_{M} to a multiplication on F_{O}. Indeed, assuming we could extend the multiplication, then we'd have

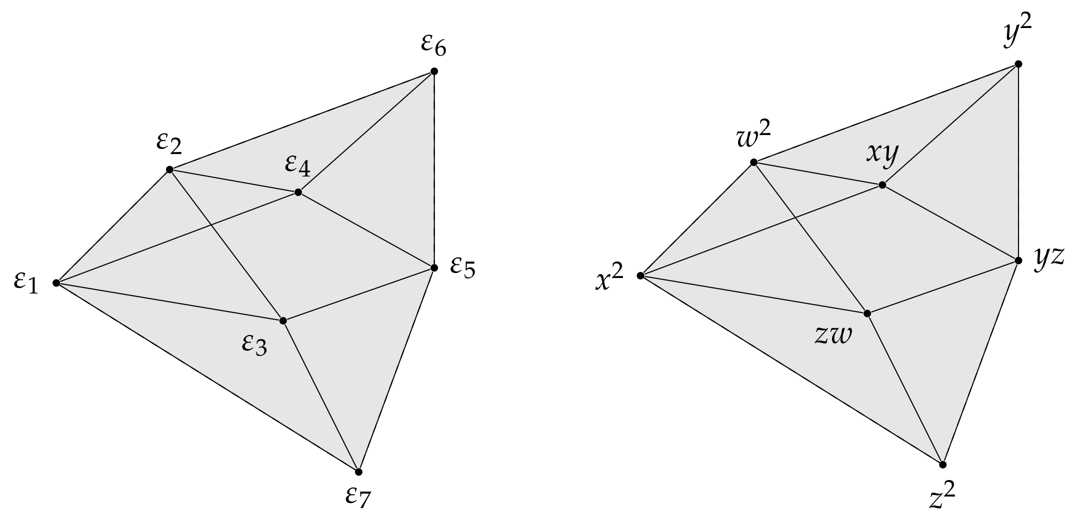
$$\begin{aligned} z(\mathrm{e}_2 \star \mathrm{e}_5) &= \mathrm{e}_2 \star (z\mathrm{e}_5) \\ &= \epsilon_2 \star \epsilon_5 \\ &= y^2\epsilon_{23} + w\epsilon_{35} \\ &= y^2\mathrm{e}_{23} + w\mathrm{e}_{35}, \end{aligned}$$

which would imply $\mathrm{e}_2 \star \mathrm{e}_5 = (y^2/z)\mathrm{e}_{23} + (w/z)\mathrm{e}_{35}$. However this is obviously not in F_{O} since the coefficients are not in R. On the other hand, it turns out that there is a better choice of a multigraded multiplication on F_{O} that we can use anyways: namely $\mathrm{e}_2 \star \mathrm{e}_5 = \mathrm{e}_{25}$. In fact, this is the only possible choice we can make if we want the multiplication to be multigraded. Similarly, we are forced to have $\mathrm{e}_1 \star \mathrm{e}_6 = \mathrm{e}_{16}$. Using the computer algebra system Singular, we found that this extends to an *associative* multigraded multiplication on F_{O} which has the following minimal presentation:

$$\begin{array}{lll}
\mathrm{e}_1^2 = 0 & \mathrm{e}_2 \star \mathrm{e}_5 = \mathrm{e}_{25} & \mathrm{e}_2 \star \mathrm{e}_{16} = -z\mathrm{e}_{123} - w\mathrm{e}_{136} \\
\mathrm{e}_2^2 = 0 & \mathrm{e}_2 \star \mathrm{e}_6 = z\mathrm{e}_{23} + w\mathrm{e}_{36} & \mathrm{e}_2 \star \mathrm{e}_{46} = \mathrm{e}_{234} + \mathrm{e}_{346} \\
\mathrm{e}_3^2 = 0 & \mathrm{e}_3 \star \mathrm{e}_4 = \mathrm{e}_{34} & \mathrm{e}_2 \star \mathrm{e}_{56} = -z\mathrm{e}_{235} + w\mathrm{e}_{356} \\
\mathrm{e}_4^2 = 0 & \mathrm{e}_3 \star \mathrm{e}_5 = \mathrm{e}_{35} & \mathrm{e}_3 \star \mathrm{e}_{45} = \mathrm{e}_{345} \\
\mathrm{e}_5^2 = 0 & \mathrm{e}_3 \star \mathrm{e}_6 = z\mathrm{e}_{36} & \mathrm{e}_5 \star \mathrm{e}_{24} = y\mathrm{e}_{245} \\
\mathrm{e}_6^2 = 0 & \mathrm{e}_4 \star \mathrm{e}_5 = y\mathrm{e}_{45} & \mathrm{e}_6 \star \mathrm{e}_{13} = z\mathrm{e}_{136} \\
\mathrm{e}_1 \star \mathrm{e}_2 = \mathrm{e}_{12} & \mathrm{e}_4 \star \mathrm{e}_6 = \mathrm{e}_{46} & \mathrm{e}_6 \star \mathrm{e}_{34} = z\mathrm{e}_{346} \\
\mathrm{e}_1 \star \mathrm{e}_3 = \mathrm{e}_{13} & \mathrm{e}_5 \star \mathrm{e}_6 = \mathrm{e}_{56} & \mathrm{e}_6 \star \mathrm{e}_{35} = z\mathrm{e}_{356} \\
\mathrm{e}_1 \star \mathrm{e}_4 = x\mathrm{e}_{14} & \mathrm{e}_1 \star \mathrm{e}_{25} = y\mathrm{e}_{124} - x\mathrm{e}_{245} & \mathrm{e}_6 \star \mathrm{e}_{45} = \mathrm{e}_{456} \\
\mathrm{e}_1 \star \mathrm{e}_5 = y\mathrm{e}_{14} + x\mathrm{e}_{45} & \mathrm{e}_1 \star \mathrm{e}_{35} = y\mathrm{e}_{134} - x\mathrm{e}_{345} & \mathrm{e}_1 \star \mathrm{e}_{235} = y\mathrm{e}_{1234} + x\mathrm{e}_{2345} \\
\mathrm{e}_1 \star \mathrm{e}_6 = \mathrm{e}_{16} & \mathrm{e}_1 \star \mathrm{e}_{56} = y\mathrm{e}_{146} + x\mathrm{e}_{456} & \mathrm{e}_1 \star \mathrm{e}_{346} = x\mathrm{e}_{1346} \\
\mathrm{e}_2 \star \mathrm{e}_3 = w\mathrm{e}_{23} & & \mathrm{e}_1 \star \mathrm{e}_{356} = y\mathrm{e}_{1346} - x\mathrm{e}_{3456} \\
\mathrm{e}_2 \star \mathrm{e}_4 = \mathrm{e}_{24} & & \mathrm{e}_2 \star \mathrm{e}_{456} = z\mathrm{e}_{2345} + w\mathrm{e}_{3456}
\end{array}$$

In Example (5.6), we demonstrate how one can find associative multiplications like this using a computer algebra system like Singular.

Example 2.5. Let $R = \mathbb{k}[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{N}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz, y^2, z^2$, and let $F_{\mathrm{N}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled cellular complex below:



It is visibly clear that the map $R/\boldsymbol{m}_{\mathrm{A}} \to R/\boldsymbol{m}_{\mathrm{N}}$ induces a comparison map $\iota\colon F_{\mathrm{A}} \to F_{\mathrm{N}}$ defined by $\iota(\varepsilon_\sigma) = \varepsilon_\sigma$ for all homogeneous basis element ε_σ of F_{A} (in particular, there are no monomials showing up in the coefficients in this comparison map). Thus we run into the same problem as in Example (2.2), and so there is no way to choose a multigraded multiplication on F_{N} which is associative.

Example 2.6. Let $R = \mathbb{k}[x, y, z, w]$, let $m = xyzw$, let $\boldsymbol{m} = mx, my, mz, mw$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F is just the Taylor resolution with respect to $\boldsymbol{m}$ and is supported on the 3-simplex. Usually F comes equipped with an associative multiplication giving it the structure of a DG algebra, however we wish to consider a different multiplication μ which gives it the structure of a non-associative MDG algebra. In particular, this multiplication will start out as:

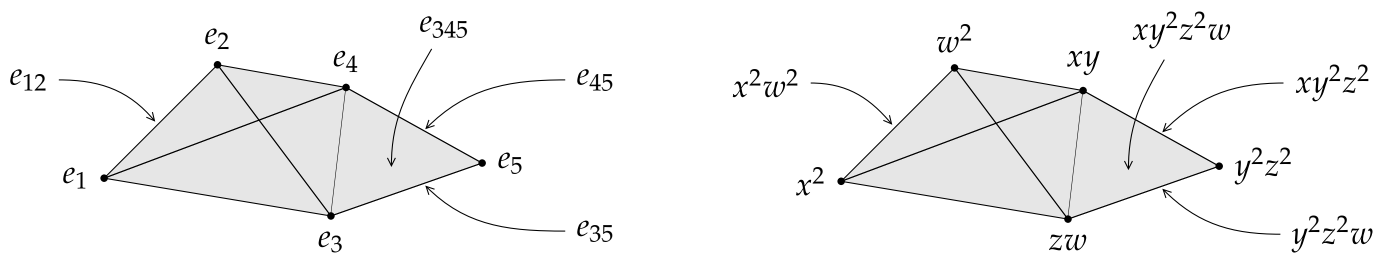
$$\begin{aligned}
e_1 \star e_2 &= xyzwe_{12} \\
e_1 \star e_3 &= xyz^2 e_{14} - x^2 yze_{34} \\
e_2 \star e_3 &= xyzwe_{23} \\
e_3 \star e_{12} &= xyzwe_{123} - xy^2 ze_{134} \\
e_2 \star e_{14} &= -xyzwe_{124} \\
e_2 \star e_{34} &= xyzwe_{234}
\end{aligned}$$

At this point, no matter how we extend this multiplication, it will not be associative since

$$[e_2, e_1, e_3] = x^2 y^2 z^2 w \mathrm{d}(e_{1234}) \neq 0.$$

The point we wish to emphasize here is that there is a "better" mutliplication that we can use on F anyways, namely the Taylor multiplication. In general we would like to find the best possible multiplication in the sense that it is as associative as possible.

Example 2.7. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = xu, xv, xw, yu, yv, yw, zu, zv, zw$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:



In particular, the homogeneous components of F as a graded R-module are given by

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234},
\end{aligned}$$

and the differential d of F behaves just like the usual boundary map of the simplicial complex above except some monomials can show up as coefficients (so that the differential respects the multidegree). For instance, we have

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

For more details on this construction, see [BPS98]. We now wish to equip F with a multigraded multiplication $\mu_{\mathrm{K}} = \mu$ giving it the structure of a multigraded MDG algebra. Since μ respects the multigrading and satisfies Leibniz rule, we are forced to have:

$$\begin{aligned}
e_1 \star e_5 &= yz^2 e_{14} + xe_{45} & \qquad e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_2 &= e_{12} & e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_2 \star e_5 &= y^2 ze_{23} + we_{35} & e_1 \star e_{23} &= e_{123} \\
& & e_2 \star e_{14} &= -e_{124}
\end{aligned}$$

At this point however, one can conclude that F is not associative since

$$[e_1, e_5, e_2] := (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{6}$$

The multiplication is not uniquely determined on all pairs (e_σ, e_τ); for instance there are two possible ways in which μ is defined at the pair (e_5, e_{12}). We assume that μ is defined at (e_5, e_{12}) by

$$e_5 \star e_{12} = yz^2 e_{124} + xyze_{234} + xwe_{345}.$$

Finally, we would still like for μ to be as associative as possible even though we already know it is not associative at the triple (e_1, e_5, e_2). In particular, we want μ to be associative on all triples of the form $(e_\sigma, e_\sigma, e_\tau)$. It turns out this can be done and we will assume that μ is associative on all such triples.

2.1.2 Multigraded Multiplications coming from the Taylor Algebra

In this subsubsection, we want to explain how all of the multigraded multiplications that we have considered thus far can be viewed as coming from a Taylor multiplication. Let $R = \Bbbk[x_1, \dots, x_d]$, let I be a monomial ideal in R, let F be the minimal free resolution of R/I over R, and let T be the Taylor algebra resolution of R/I over R. We denote the Taylor multiplication on T by ν_T. Let ν be a possibly different multiplication on T. We write T_ν to be the MDG R-algebra whose underlying R-complex is the same as the underlying complex of T but whose multiplication is ν. Since F is the minimal free resolution of R/I over R and since T is a free resolution of R/I over R, there exists multigraded chain maps $\iota\colon F \to T$ and $\pi\colon T \to F$ which lift the identity map $R/I \to R/I$ such that $\iota\colon F \to T$ is injective and is split by $\pi\colon T \to F$, meaning $\pi\iota = 1$. By identifying F with $\iota(F)$ if necessary, we may assume that $\iota\colon F \subseteq T$ is inclusion and that $\pi\colon T \to F$ is a projection, meaning $\pi\colon T \to F$ is a surjective chain map which satisfies $\pi^2 = \pi$, or equivalently, $\pi\colon T \to T$ is a chain map with $\operatorname{im}\pi = F$. Using the comparison maps $\iota\colon F \to T$ and $\pi\colon T \to F$, we can transport multiplications on F to multiplications on T and vice versa. Namely, given a multiplication μ on F, we set $\widetilde{\mu} := \iota\mu\pi^{\otimes 2}$. Similarly, given a multiplication ν on T, we set $\widetilde{\nu} := \pi\nu\iota^{\otimes 2}$. All of the multigraded multiplications that we've considered thus far are of the form $\widetilde{\nu}_T$. For instance:

Example 2.8. The multiplication μ in Example (2.7) is given by $\mu = \pi\nu_T\iota^{\otimes 2}$ where T is the Taylor algebra resolution of $R/\boldsymbol{m}_{\mathrm{K}}$ and where $\pi\colon T \to F$ is defined by

$$
\begin{aligned}
\pi(e_{15}) &= yz^2e_{14} + xe_{45} \\
\pi(e_{25}) &= y^2ze_{23} + we_{35} \\
\pi(e_{245}) &= -yze_{234} + we_{35} \\
\pi(e_{235}) &= 0 \\
\pi(e_{2345}) &= 0 \\
&\vdots
\end{aligned}
$$

and so on.

2.2 MDG Modules

We now want to define MDG A-modules where A is an MDG R-algebra.

Definition 2.2. Let X be an R-complex equipped with chain maps $\mu_{A,X}\colon A \otimes_R X \to X$ and $\mu_{X,A}\colon X \otimes_R A \to X$, denoted $a \otimes x \mapsto ax$ and $x \otimes a \mapsto xa$ respectively.

1. We say X is **unital** if $1x = x = x1$ for all $x \in X$.

2. We say X is **graded-commutative** if $ax = (-1)^{|a||x|}xa$ for all $a \in A$ homogeneous and $x \in X$ homogeneous. In this case, $\mu_{X,A}$ is completely determined by $\mu_{A,X}$, and thus we completely forget about it and write $\mu_X = \mu_{A,X}$.

3. We say X is **associative** if $a_1(a_2x) = (a_1a_2)x$ for all $a_1, a_2 \in A$ and $x \in X$.

We say X is an **MDG A-module** if it is graded-commutative, and the graded R-linear map

$$\overline{\mu}_X\colon \mathrm{H}(A) \otimes_R \mathrm{H}(X) \to \mathrm{H}(X)$$

induced by μ_X gives $\mathrm{H}(X)$ the structure of an associative graded-commutative $\mathrm{H}(A)$-module. We call μ_X the **A-scalar multiplication** of X. If X is also associative, then we say X is a **DG A-module**. Suppose Y is another MDG A-module and let $\varphi\colon X \to Y$ be a function. We say $\varphi\colon X \to Y$ is an **MDG A-module homomorphism** if it is a chain map which is also **multiplicative**, meaning

$$\varphi(ax) = a\varphi(x)$$

for all $a \in A$ and $x \in X$.

Example 2.9. Let A and B be MDG R-algebras and let $\varphi\colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we give B the structure of an MDG A-module by defining an A-scalar multiplication on B via

$$a \cdot b = \varphi(a)b$$

for all $a \in A$ and $b \in B$. Note that we need $\varphi(1) = 1$ in order for B to be unital as an MDG A-module. Also note that φ is an MDG A-module homomorphism if and only if it is an algebra homomorphism. Indeed, it is an

A-module homomorphism if and only if for all $a_1, a_2 \in A$ we have

$$\varphi(a_1 a_2) = a_1 \cdot \varphi(a_2) = \varphi(a_1)\varphi(a_2),$$

which is equivalent to saying φ is an algebra homomorphism (since we already have $\varphi(1) = 1$).

3 Associators and Multiplicators

In order to get a better understanding as to how far away MDG objects are from being DG objects, we need to discuss associators and multiplicators. Associators will help us measure the failure for an MDG A-module X to be associative, whereas multiplicators will help up measure the failure for a chain map $\varphi \colon X \to Y$ between MDG A-modules X and Y to be multiplicative.

3.1 Associators

We begin by defining associators. Throughout this subsection, let A be an MDG R-algebra and let X be an MDG A-module.

Definition 3.1. The **associator** of X is the chain map, denoted $[\cdot]_X$ (or more simply by $[\cdot]$ if X is understood from context), from $A \otimes_R A \otimes_R X$ to X defined by

$$[\cdot] := \mu(\mu \otimes 1 - 1 \otimes \mu).$$

Note that we use μ to denote both the multiplication μ_A on A and the A-scalar multiplication μ_X on X where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot] \colon A \times A \times X \to X$ to be the unique R-trilinear map which corresponds to $[\cdot]$ via the universal mapping property of tensor products. Thus we have

$$[a_1 \otimes a_2 \otimes x] = (a_1 a_2)x - a_1(a_2 x) = [a_1, a_2, x]$$

for all $a_1, a_2 \in A$ and $x \in X$.

3.1.1 Associator Identities

In order to familiarize ourselves with the associator we collect together some useful identities that the associator satisfies in this subsubsection:

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ we have the Leibniz rule

$$\mathrm{d}[a_1, a_2, x] = [\mathrm{d}a_1, a_2, x] + (-1)^{|a_1|}[a_1, \mathrm{d}a_2, x] + (-1)^{|a_1|+|a_2|}[a_1, a_2, \mathrm{d}x]. \tag{7}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1]. \tag{8}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = -(-1)^{|a_1||x|+|a_2||x|}[x, a_1, a_2] - (-1)^{|a_1||a_2|+|a_1||x|}[a_2, x, a_1] \tag{9}$$

- For all $a_1, a_2 \in A$ homogeneous and $x \in X$ homogeneous we have

$$[a_1, a_2, x] = (-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] \tag{10}$$

- For all $a_1, a_2, a_3 \in A$ and $x \in X$ we have

$$a_1[a_2, a_3, x] = [a_1 a_2, a_3, x] - [a_1, a_2 a_3, x] + [a_1, a_2, a_3 x] - [a_1, a_2, a_3]x \tag{11}$$

The way the signs in (8) show up can be interpreted as follows: in order to go from $[a_1, a_2, x]$ to $[x, a_2, a_1]$, we have to first swap a_1 with a_2 (this is where the $(-1)^{|a_1||a_2|}$ comes from), then swap a_1 with x (this is where the $(-1)^{|a_1||x|}$ comes from), and then finally swap a_2 with x (this is where the $(-1)^{|a_2||x|}$ comes from). We then obtain one extra minus sign by swapping terms in the associator at the final step:

$$\begin{aligned}
[a_1, a_2, x] &= (a_1 a_2)x - a_1(a_2 x) \\
&= (-1)^{|a_1|a_2|}(a_2 a_1)x - (-1)^{|a_2|||x|} a_1(x a_2) \\
&= (-1)^{|a_1||a_2|+|a_2||x|+|a_1||x|} x(a_2 a_1) - (-1)^{|a_2||x|+|a_1||x|+|a_1||a_2|}(x a_2)a_1 \\
&= (-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}(x(a_2 a_1) - (x a_2)a_1) \\
&= -(-1)^{|a_1||a_2|+|a_1||x|+|a_2||x|}[x, a_2, a_1].
\end{aligned}$$

A similar interpretation is also given to (9) and (10). For instance, in order to get from $[a_1, a_2, x]$ to $[x, a_1, a_2]$, we have to swap x with a_2 and then swap x with a_1 (this is where the $(-1)^{|a_1||x|+|a_2||x|}$ comes from). We do add an extra minus sign in (10) however since we never swap terms in the associator:

$$\begin{aligned}
(-1)^{|a_1||a_2|}[a_2, a_1, x] + (-1)^{|a_2||x|}[a_1, x, a_2] &= (a_1 a_2)x - (-1)^{|a_1||a_2|} a_2(a_1 x) + (-1)^{|a_2||x|}(a_1 x)a_2 - a_1(a_2 x) \\
&= (a_1 a_2)x - (-1)^{|a_1||a_2|} a_2(a_1 x) + (-1)^{|a_1|a_2|} a_2(a_1 x) - a_1(a_2 x) \\
&= (a_1 a_2)x - a_1(a_2 x) \\
&= [a_1, a_2, x].
\end{aligned}$$

3.1.2 Alternative MDG Modules

If X is not associative, then we are often interested in knowing whether or not X satisfies the following weaker property:

Definition 3.2. We say X is **alternative** if $[a, a, x] = 0$ for all $a \in A$ and $x \in X$.

In other words, X is alternative if for each $a \in A$ and $x \in X$, we have $a^2 x = a(ax)$. The reason behind the name "alternative" comes from the fact that in the case where $X = A$, then A is alternative if and only if the associator $[\cdot, \cdot, \cdot]$ is alternating.

Proposition 3.1. *Let $a \in A$ and $x \in X$ be homogeneous.*

1. *We have $[a, a, x] = 0$ if and only if $[x, a, a] = 0$.*

2. *If $[a, a, x] = 0$, then $[a, x, a] = 0$. The converse holds if $|a|$ is odd and* char $R \neq 2$.

3. *If $|a|$ is even, we have $[a, x, a] = 0$, and if $|a|$ is odd, we have $[a, x, a] = (-1)^{|x|} 2[a, a, x]$. In particular, if* char $R = 2$, *we always have $[a, x, a] = 0$.*

Proof. From identities (8) and (10) we obtain

$$\begin{aligned}
[a, a, x] &= -(-1)^{|a|}[x, a, a] \\
[a, x, a] &= (-1)^{|x||a|}(1 - (-1)^{|a|})[a, a, x].
\end{aligned}$$

In particular, we see that

$$[a, x, a] = \begin{cases} = (-1)^{|x|} 2[a, a, x] = -(-1)^{|x|} 2a(ax) & \text{if } a \text{ is odd} \\ 0 & \text{if } a \text{ is even} \end{cases} \tag{12}$$

Similarly we have

$$[a, a, x] = \begin{cases} (-1)^{|x|} \frac{1}{2}[a, x, a] & \text{if } a \text{ is odd and char } R \neq 2 \\ (-1)^{|a|}[x, a, a] & \text{if } a \text{ is even} \end{cases} \tag{13}$$

□

Remark 1. Suppose F is an MDG R-algebra whose underlying graded R-module is finite and free with $e_1, \dots, e_n$ being a homogeneous basis. In order to show F is alternative, it is *not* enough to check $[e_i, e_i, e_j] = 0$ for all e_i, e_j in the homogeneous basis. Indeed, even in this case, observe that if e_i and e_j are odd, then

$$\begin{aligned}[e_i + e_j, e_i + e_j, e_k] &= [e_i, e_i, e_k] + [e_i, e_j, e_k] + [e_j, e_i, e_k] + [e_j, e_j, e_k] \\ &= [e_i, e_j, e_k] + [e_j, e_i, e_k] \\ &= [e_i, e_j, e_k] - [e_i, e_j, e_k] + (-1)^{|e_k|}[e_j, e_k, e_i] \\ &= (-1)^{|e_k|}[e_j, e_k, e_i].\end{aligned}$$

Thus in order for F to be alternative, we certainly need $[a_1, a_2, a_3] = 0$ for all $a_1, a_2, a_3 \in F$ whenever both $|a_1|$ and $|a_3|$ are odd. For instance, consider the MDG R-algebra F_K given in Example (2.7). Then we have $[e_\sigma, e_\sigma, e_\tau] = 0$ for all $\sigma, \tau \in \Delta$, however F is not alternative since $[e_1, e_5, e_2] \neq 0$.

3.1.3 The Maximal Associative Quotient

Definition 3.3. The **associator R-subcomplex** of X, denoted $[X]$, is the R-subcomplex of X given by the image of the associator of X. Thus the underlying graded R-module of $[X]$ is

$$[X] = \operatorname{span}_R\{[a_1, a_2, x] \mid a_1, a_2 \in A \text{ and } x \in X\},$$

and the differential of $[X]$ is simply the restriction of the differential of X to $[X]$. The **associator A-submodule** of X, denoted $\langle X \rangle$, is defined to be the smallest A-submodule of X which contains $[X]$. The underlying graded R-module of $\langle X \rangle$ also has a simple description. Indeed, observe that

$$a_1(a_2[a_3, a_4, x]) = (a_1a_2)[a_3, a_4, x] - [a_1, a_2, [a_3, a_4, x]] \tag{14}$$

for all $a_1, a_2, a_3, a_4 \in A$ and $x \in X$. Using identities like (14) together with graded-commutativity, one can show that the underlying graded R-module of $\langle X \rangle$ is given by

$$\langle X \rangle = \operatorname{span}_R\{a_1[a_2, a_3, x] \mid a_1, a_2, a_3 \in A \text{ and } x \in X\}$$

The quotient $X^{\mathrm{as}} := X/\langle X \rangle$ is a DG A-module (i.e. an associative MDG A-module). We call X^{as} (together with its canonical quotient map $X \twoheadrightarrow X^{\mathrm{as}}$) the **maximal associative quotient** of X.

The maximal associative quotient of X satisfies the following universal mapping property:

Proposition 3.2. *Every* MDG *A-module homomorphism $\varphi\colon X \to Y$ in which Y is associative factors through a unique* MDG *A-module homomorphism $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$, meaning $\overline{\varphi}\rho = \varphi$ where $\rho\colon X \twoheadrightarrow X^{\mathrm{as}}$ is the canonical quotient map. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & X^{\mathrm{as}} \arrow[d, dashed, "\overline{\varphi}"] \\ & Y \end{tikzcd} \tag{15}$$

Remark 2. In other words, taking the maximal associative quotient extends to a functor from the category of all MDG A-modules to the category of all DG A-modules and this functor is left adjoint to the forgetful functor. In particular, the functor $(-)^{\mathrm{as}}$ preserves all colimits and the forgetful functor preserves all limits.

Proof. Indeed, suppose $\varphi\colon X \to Y$ is any MDG A-module homomorphism where Y is associative. In particular, we must have $[X] \subseteq \ker \varphi$, and since $\langle X \rangle$ is the smallest MDG A-submodule of X which contains $[X]$, it follows that $\langle X \rangle \subseteq \ker \varphi$. Thus the map $\overline{\varphi}\colon X^{\mathrm{as}} \to Y$ given by $\overline{\varphi}(\overline{x}) := \varphi(x)$ where $\overline{x} \in X^{\mathrm{as}}$ is well-defined. Furthermore, it is easy to see that $\overline{\varphi}$ is an MDG A-module homomorphism and the unique such one which makes the diagram (15) commute. □

3.1.4 Homological Associativity

Definition 3.4. The **associator homology** of X is the homology of the associator A-submodule of X. We often simplify notation and denote the associator homology of X by $\mathrm{H}\langle X \rangle$ instead of $\mathrm{H}(\langle X \rangle)$. We say X is **homologically associative** if $\mathrm{H}\langle X \rangle = 0$ and we say X is **homologically associative in degree** i if $\mathrm{H}_i\langle X \rangle = 0$. Similarly we say X is associative in degree if $\langle X \rangle_i = 0$.

Clearly, if X is associative, then X is homologically associative. The converse holds under certain conditions. This is the first main theorem given in the introduction.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a local ring, let A be an* MDG *R-algebra, and let X be an* MDG *A-module such that $\langle X \rangle$ is minimal (meaning $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$), and such that each $\langle X \rangle_i$ is a finitely generated R-module. If X is associative in degree i, then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $\langle X \rangle$ is also bounded below (meaning $\langle X \rangle_i = 0$ for $i \ll 0$), then X is associative if and only if X is homologically associative.*

Proof. Assume that X is associative in degree i. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume

$$\mathrm{H}_{i+1}\langle X \rangle = 0 \quad \text{and} \quad \langle X \rangle_{i+1} \neq 0.$$

Then by Nakayama's Lemma, we can find homogeneous $a_1, a_2, a_3 \in A$ and homogeneous $x \in X$ such that such that $a_1[a_2, a_3, x] \notin \mathfrak{m}\langle X \rangle_{i+1}$. Since $\langle X \rangle_i = 0$ by assumption, we have $\mathrm{d}(a_1[a_1, a_2, x]) = 0$. Also, since $\langle X \rangle$ is minimal, we have $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$. Thus $a_1[a_2, a_3, x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. □

Remark 3. We are often also interested in the homology of the maximal associative quotient of X as well. To this end, observe that the short exact sequence of MDG A-modules

$$0 \longrightarrow \langle X \rangle \longrightarrow X \longrightarrow X^{\mathrm{as}} \longrightarrow 0$$

induces a sequence of graded $\mathrm{H}(A)$-modules

$$\mathrm{H}\langle X \rangle \longrightarrow \mathrm{H}(X) \longrightarrow \mathrm{H}(X^{\mathrm{as}}) \xrightarrow{\overline{\mathrm{d}}} \Sigma\mathrm{H}\langle X \rangle \longrightarrow \Sigma\mathrm{H}(X)$$

which is exact at $\mathrm{H}\langle X \rangle$, $\mathrm{H}(X)$, and $\mathrm{H}(X^{\mathrm{as}})$ and where the connecting map $\overline{\mathrm{d}}\colon \mathrm{H}(X^{\mathrm{as}}) \to \Sigma\mathrm{H}\langle X \rangle$ is essentially defined in terms of the differential d of X, namely given $\overline{x} \in \mathrm{H}(X^{\mathrm{as}})$, we set $\overline{\mathrm{d}}\overline{x} = \overline{\mathrm{d}x}$. In particular, if $\mathrm{H}_i(X) = 0 = \mathrm{H}_{i-1}(X)$, then $\mathrm{H}_i(X^{\mathrm{as}}) \cong \mathrm{H}_{i-1}\langle X \rangle$.

Example 3.1. Let X be an MDG A-module. Assume that $(R, \mathfrak{m})$ is a local noetherian ring, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let F be the minimal R-free resolution of R/I. Equip F with a multiplication μ giving it the structure of an MDG R-algebra. Then

$$\mathrm{H}_i(F^{\mathrm{as}}) \cong \begin{cases} R/I & \text{if } i = 0 \\ \mathrm{H}_{i-1}\langle F \rangle & \text{else} \end{cases}$$

3.1.5 Computing Annihilators of the Associator Homology

In this subsubsection, we assume that R is an integral domain with quotient field K. We further assume that the underlying graded R-module of A is free. Recall that the A-scalar multiplication map $\mu_{\langle X \rangle}\colon A \otimes_R \langle X \rangle \to \langle X \rangle$ induces an $\mathrm{H}(A)$-scalar multiplication map $\overline{\mu}_{\langle X \rangle}\colon \mathrm{H}(A) \otimes_R \mathrm{H}\langle X \rangle \to \mathrm{H}\langle X \rangle$ which gives $\mathrm{H}\langle X \rangle$ an $\mathrm{H}(A)$-module structure. In particular, $\mathrm{d}A$ annihilates $\mathrm{H}\langle X \rangle$. However we can often find more annihilators of $\mathrm{H}\langle X \rangle$ than just the ones contained in $\mathrm{d}A$. Indeed, set

$$A_K = \{a/r \mid a \in A \text{ and } r \in R\backslash\{0\}\} \quad \text{and} \quad B = \{b \in A_K \mid b\langle X \rangle \subseteq \langle X \rangle\}.$$

Then A_K is an MDG K-algebra and B is an MDG subalgebra of A_K which contains A. Furthermore $\langle X \rangle$ is an MDG B-module (in fact B is the largest MDG subalgebra of A_K for which $\langle X \rangle$ is an MDG module over). In particular, $A \cap \mathrm{d}B$ annihilates $\mathrm{H}\langle X \rangle$. In general we have

$$\mathrm{d}A \subseteq A \cap \mathrm{d}B \subseteq A,$$

where the inclusions may be strict.

Example 3.2. Consider Example (2.7) where $R = \Bbbk[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and F is the minimal free resolution of $R/\boldsymbol{m}$ of R. Observe that

$$\begin{aligned}
\frac{e_1}{x}[e_1, e_5, e_2] &= \frac{1}{x}\left([e_1^2, e_5, e_2] - [e_1, e_1e_5, e_2] + [e_1, e_1, e_5e_2] - [e_1, e_1, e_5]e_2\right) \\
&= -\frac{1}{x}[e_1, e_1e_5, e_2] \\
&= -\frac{1}{x}[e_1, yz^2e_{14} + xe_{45}, e_2] \\
&= -\frac{yz^2}{x}[e_1, e_{14}, e_2] - [e_1, e_{45}, e_2] \\
&= -[e_1, e_{45}, e_2].
\end{aligned}$$

It follows that $\mathrm{d}(e_1/x) = x$ annihilates $\mathrm{H}\langle F\rangle$. Similar calculations likes this shows that $\langle x, y, z, w\rangle$ annihilates $\mathrm{H}\langle F\rangle$. It follows that

$$\mathrm{H}_i\langle F\rangle \cong \begin{cases} \Bbbk & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

One can interpret this as saying that the multiplication μ is very close to being associative (the failure for μ to being associative is reflected in the fact that $\dim_{\Bbbk}(\mathrm{H}\langle F\rangle) = 1$). Note that μ is not associative in homological degree 4 since

$$[e_1, e_{45}, e_2] = xyze_{1234} \neq 0.$$

In some sense however, the fact that the associator $[e_1, e_{45}, e_2]$ is nonzero isn't really a *new* obstruction to μ being associative. Indeed, one could argue that $[e_1, e_{45}, e_2]$ being nonzero is simply a consequence of $[e_1, e_5, e_2]$ being nonzero. More generally, in order for a nonzero element $\gamma \in \langle F\rangle$ to be considered an obstruction for μ to be associative, we should have $\mathrm{d}\gamma = 0$ (otherwise one could argue that γ being nonzero is simply a consequence of the associators in $\mathrm{d}\gamma$ being nonzero). Similarly, we shouldn't have $\gamma = \mathrm{d}\gamma'$ (otherwise one could argue that γ being nonzero is simply a consequence of γ' being nonzero). Thus the associators which really do contribute new obstructions for μ to be associative should be the ones which represent nonzero elements in homology. This is how we interpret the associator homology of F. In this case, we have precisely one nontrivial associator $[e_1, e_5, e_2]$ which represents a nonzero element in homology (all of the other nonzero associators are derived from the fact that $[e_1, e_5, e_2] \neq 0$).

Example 3.3. Consider Example (2.3) where $R = \Bbbk[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z, yz^2$, and F is the minimal free resolution of $R/\boldsymbol{m}$ of R. By performing similar calculations as in Example (3.3), one can show that

$$\mathrm{H}_i\langle F\rangle \cong \begin{cases} \Bbbk \oplus \Bbbk & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

3.1.6 The Nucleus

Definition 3.5. The **nuclear subcomplex** of X, denoted $\mathrm{N}(X)$, is the R-subcomplex of X given by

$$\mathrm{N}(X) := \{x \in \mathrm{X} \mid [a_1, a_2, x] = 0 \text{ for all } a_1, a_2 \in A\}.$$

Indeed, the Leibniz law implies $\mathrm{d}(\mathrm{N}(X)) \subseteq \mathrm{N}(X)$, so the differential of $\mathrm{N}(X)$ is simply the differential of X restricted to $\mathrm{N}(X)$. The **nucleus** of X, denoted $\mathrm{N}\langle X\rangle$, is defined to be the smallest MDG A-submodule of X which contains $\mathrm{N}(X)$. The nucleus of X plays a role that's similar to the center of a group G. In particular, every associative A-submodule of X is contained in $\mathrm{N}\langle X\rangle$. Note that $\mathrm{N}\langle X\rangle$ consists of all finite sums of the form $\sum a_i x_i$ where $a_i \in A$ and $x_i \in \mathrm{N}(X)$. We will also be interested in studying the **nuclear complex of** X **in** A, denoted $\mathrm{N}_A(X)$. This is the R-subcomplex of A given by

$$\mathrm{N}_A(X) := \{a \in A \mid [a, a', x] = 0 \text{ for all } a \in A \text{ and } x \in X\}.$$

Note that if $a_1, a_2 \in \mathrm{N}_A(X)$, then $a_1a_2 \in \mathrm{N}_A(X)$. However in general, if $a \in \mathrm{N}_A(X)$ and $b \in A$, then $[ab, c, x] = a[b, c, x]$ The **nucleus of** X **in** A, denoted $\mathrm{N}_A\langle X\rangle$, is defined to be the smallest MDG A-ideal which contains $\mathrm{N}_A(X)$. There's also the following weaker notion we may consider: we define the **middle nuclear complex of** X, denoted $\mathrm{M}(X)$, to be the R-subcomplex of X given by

$$\mathrm{M}(X) := \{x \in \mathrm{X} \mid [a_1, x, a_2] = 0 \text{ for all } a_1, a_2 \in A\}.$$

By combining (8) with (9), one can check that $\mathrm{N}(X) \subseteq \mathrm{M}(X)$, however this inclusion may be strict. Indeed, by combining the identities (8) with (9) we obtain the identity

$$[a_1, x, a_2] = (-1)^{|a_1||a_2|+|a_2||x|}((-1)^{|a_1||a_2|}[a_2, a_1, x] - [a_1, a_2, x]) \tag{16}$$

In particular, we have $x \in \mathrm{M}(X)$ if and only if $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a_1, a_2 \in A$. However just because we have $[a_1, a_2, x] = (-1)^{|a_1||a_2||}[a_2, a_1, x]$ for all $a, b \in A$ doesn't necessarily mean $[a_1, a_2, x] = 0$ for all $a_1, a_2 \in A$.

Proposition 3.3. *$\mathrm{N}(A)$ is a DG subalgebra of A.*

Proof. Clearly we have $1 \in \mathrm{N}(A)$. Let $a, a' \in \mathrm{N}(A)$. Then for each $a_1, a_2 \in A$, we have

$$[aa', a_1, a_2] = a[a', a_1, a_2] + [a, a'a_1, a_2] - [a, a', a_1a_2] + [a, a', a_1]a_2 = 0.$$

It follows that $aa' \in \mathrm{N}(A)$. Similarly, we have

$$[\mathrm{d}a, a_1, a_2] = \mathrm{d}[a, a_1, a_2] - (-1)^{|a|}[a, \mathrm{d}a_1, a_2] - (-1)^{|a|+|a_1|}[a, a_1, \mathrm{d}a_2] = 0.$$

It follows that $\mathrm{d}a \in \mathrm{N}(A)$. □

By using the identities (9), (10), and (11), one can show that every element in $\langle A \rangle$ can be expressed as the R-span of all elements of the form $a_1[a_2, a_3, a_4]$ where $|a_1| \leq |a_2|, |a_3|, |a_4|$. In fact, we can often do better than even this. Indeed, suppose $a_1 = az \neq 0$ for some homogeneous $a \in A$ with $|a| < |a_1|$ and homogeneous $z \in \mathrm{N}(A)$. Then we have $a_1[a_2, a_3, a_4] = a[za_2, a_3, a_4]$. It follows that we can express every element in $\langle A \rangle$ as an R-linear combination of elements of the form $a_1[a_2, a_3, a_4]$ where

$$|a_1| \leq \min\{|a_1|, |a_2|, |a_3|\} \quad \text{and} \quad a_1 \notin \mathrm{N}\langle A \rangle.$$

3.1.7 Associators up to Homotopy

Let I be an ideal of R and let F be a free resolution of R/I over R. A chain map $\mu \in F^{\otimes 2} \to F$ which lifts the multiplication map on R/I is unique up to homotopy. What this means is that if $\mu' \in F^{\otimes 2} \to F$ is another chain map which lifts the multiplication map on R/I, then there exists a graded R-linear map $h \colon F^{\otimes 2} \to F$ of degree one such that $\mu' = \mu_h$ where

$$\mu_h := \mu + \mathrm{d}h + h\mathrm{d}.$$

If both μ and μ_h are graded-commutative, then $h\sigma \colon F^{\otimes 2} \to F$ must be a chain map of degree 1, where $\sigma \colon F^{\otimes 2} \to F^{\otimes 2}$ is defined by

$$\sigma(a_1 \otimes a_2) = a_1 \otimes a_2 - (-1)^{|a_1||a_2|} a_2 \otimes a_1$$

for all homogeneous $a_1, a_2 \in F$. Indeed, since μ_h and μ are graded-commutative, we have

$$\begin{aligned} \mathrm{d}h\sigma + h\sigma\mathrm{d} &= \mathrm{d}h\sigma + h\mathrm{d}\sigma \\ &= (\mathrm{d}h + h\mathrm{d})\sigma \\ &= (\mu_h - \mu)\sigma \\ &= \mu_h\sigma - \mu\sigma \\ &= 0 - 0 \\ &= 0. \end{aligned}$$

Similarly, if both μ and μ_h are unital, then $h|_{F \otimes 1}$ and $h|_{1 \otimes F}$ must be chain maps of degree 1. Finally, note that the associator for μ_h is given by

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d} \tag{17}$$

where $H = \overline{[\cdot]}_{\mu,h} + [\cdot]_{h,\mu_h}$. Here, we set

$$\overline{[\cdot]}_{\mu,h} = \mu(h \otimes 1 - \overline{1} \otimes h) \quad \text{and} \quad [\cdot]_{h,\mu_h} = h(\mu_h \otimes 1 - 1 \otimes \mu_h)$$

where $\overline{1} \colon F \to F$ is the map defined by $\overline{1}(a) = (-1)^{|a|}a$ for all homogeneous $a \in A$. Note that we can break $[\cdot]_{h,\mu_h}$ further as

$$[\cdot]_{h,\mu_h} = [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}}$$

where

$$[\cdot]_{h,\mu} = h(\mu \otimes 1 - 1 \otimes \mu), \quad [\cdot]_{h,\mathrm{d}h} = h(\mathrm{d}h \otimes 1 - 1 \otimes \mathrm{d}h), \quad \text{and} \quad [\cdot]_{h,h\mathrm{d}} = h(h\mathrm{d} \otimes 1 - 1 \otimes h\mathrm{d}).$$

Theorem 3.2. *Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, yz$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure. In particular, any multiplication on F will be non-associative at the triple $(\varepsilon_1, \varepsilon_{45}, \varepsilon_2)$.*

Proof. Let μ be the usual multiplication and let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F. We claim that $[\varepsilon_1, \varepsilon_{45}, \varepsilon_5]_{\mu_h} \neq 0$. Indeed, the idea is that on the one hand we have $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_\mu = -x\varepsilon_{12345}$ but on the other hand we have

$$(\mathrm{d}H + H\mathrm{d})(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) \in IF$$

where H is the map described in (17) and where $I = \langle x^2, y, z, w \rangle$. In particular, $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_{\mu_h} \not\equiv 0$ modulo IF which implies $[\varepsilon_1, \varepsilon_{45}, \varepsilon_2]_{\mu_h} \neq 0$. To see this, first note that $\mathrm{d}H(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) = 0$, so we only need to show that

$$H\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) = (\overline{[\cdot]}_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) \in IF.$$

Now clearly we have

$$\operatorname{im}([\cdot]_{h,\mathrm{d}h})\mathrm{d}) \in \mathfrak{m}^2 F \subseteq IF \quad \text{and} \quad \operatorname{im}([\cdot]_{h,h\mathrm{d}})\mathrm{d}) \in \mathfrak{m}^2 F \subseteq IF,$$

where $\mathfrak{m} = \langle x, y, z, w\rangle$, since F is minimal and since the differential shows up twice in each case. Next note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) &\equiv x^2[1 \otimes \varepsilon_{45} \otimes \varepsilon_2]_{h,\mu} - x[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]_{h,\mu} + z[\varepsilon_1 \otimes \varepsilon_4 \otimes \varepsilon_2]_{h,\mu} + w^2[\varepsilon_1 \otimes \varepsilon_{45} \otimes 1]_{h,\mu} \\
&\equiv -x[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]_{h,\mu} \\
&\equiv -xh((z\varepsilon_{14} + x\varepsilon_{45}) \otimes \varepsilon_2 - \varepsilon_1 \otimes (z\varepsilon_{23} + y\varepsilon_{35})) \\
&\equiv 0.
\end{aligned}$$

Similarly in F/IF we have

$$\begin{aligned}
\overline{[\cdot]}_{\mu,h}\mathrm{d}(\varepsilon_1 \otimes \varepsilon_{45} \otimes \varepsilon_2) &\equiv x^2\overline{[1 \otimes \varepsilon_{45} \otimes \varepsilon_2]}_{\mu,h} - x\overline{[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]}_{\mu,h} + z\overline{[\varepsilon_1 \otimes \varepsilon_4 \otimes \varepsilon_2]}_{\mu,h} + w^2\overline{[\varepsilon_1 \otimes \varepsilon_{45} \otimes 1]}_{\mu,h} \\
&\equiv -x\overline{[\varepsilon_1 \otimes \varepsilon_5 \otimes \varepsilon_2]}_{\mu,h} \\
&\equiv 0
\end{aligned}$$

where we used the fact that $\varepsilon_1 F_3 \in \mathfrak{m}F_4$ and $\varepsilon_2 F_3 \in \mathfrak{m}F_4$. □

Theorem 3.3. *Let $R = \Bbbk[x, y, z, w]$ where $\operatorname{char}\Bbbk = 2$, let $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F does not admit a DG algebra structure. In particular, every MDG R-algebra will be non-associative at the triple (e_{12}, e_5, e_2).*

Proof. Let μ be the usual multiplication and let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F. We claim that $[e_{12}, e_5, e_2]_{\mu_h} \neq 0$. Indeed, first note that $[e_{12}, e_5, e_2]_\mu = x^2yze_{1234}$. We will show that

$$(\mathrm{d}H + H\mathrm{d})(e_{12} \otimes e_5 \otimes e_2) \in IF$$

where H is the map described in (17) and where $I = \langle x^3, y^2, z^2, w\rangle$. Again we have $\mathrm{d}H(e_{12} \otimes e_5 \otimes e_2) = 0$, so we only need to show that

$$H\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) = ([\cdot]_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) \in IF$$

First note in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mu}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,\mu} + w^2[e_1, e_5, e_2]_{h,\mu} + y^2z^2[e_{12}, 1, e_2]_{h,\mu} + w^2[e_{12}, e_5, 1]_{h,\mu} \\
&\equiv x^2[e_2, e_5, e_2]_{h,\mu} \\
&\equiv x^2h((y^2ze_{23} + we_{35}) \otimes e_2 + e_2 \otimes (y^2ze_{23} + we_{35})) \\
&\equiv 0
\end{aligned}$$

Next in F/IF we have

$$\begin{aligned}
[\cdot]_{\mu,h}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{\mu,h} + w^2[e_1, e_5, e_2]_{\mu,h} + y^2z^2[e_{12}, 1, e_2]_{\mu,h} + w^2[e_{12}, e_5, 1]_{\mu,h} \\
&\equiv x^2[e_2, e_5, e_2]_{\mu,h} \\
&\equiv x^2(e_2h(e_5 \otimes e_2) + h(e_2 \otimes e_5)e_2) \\
&\equiv 0,
\end{aligned}$$

where we used the fact that $e_2F_3 \in wF_3$. Next in F/IF we have

$$\begin{aligned}
[\cdot]_{h,h\mathrm{d}}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} + w^2[e_1, e_5, e_2]_{h,h\mathrm{d}} + y^2z^2[e_{12}, 1, e_2]_{h,h\mathrm{d}} + w^2[e_{12}, e_5, 1]_{h,h\mathrm{d}} \\
&\equiv x^2[e_2, e_5, e_2]_{h,h\mathrm{d}} \\
&\equiv x^2h(h\mathrm{d}(e_2 \otimes e_5) \otimes e_2 + e_2 \otimes h\mathrm{d}(e_5 \otimes e_2)) \\
&\equiv 0,
\end{aligned}$$

where we used the fact that $\mathrm{d}e_2 = w^2$ and $\mathrm{d}e_5 = y^2z^2$. Next in F/IF we have

$$\begin{aligned}
[\cdot]_{h,\mathrm{d}h}\mathrm{d}(e_{12} \otimes e_5 \otimes e_2) &\equiv x^2[e_2, e_5, e_2]_{h,\mathrm{d}h} + w^2[e_1, e_5, e_2]_{h,\mathrm{d}h} + y^2z^2[e_{12}, 1, e_2]_{h,\mathrm{d}h} + w^2[e_{12}, e_5, 1]_{h,\mathrm{d}h} \\
&\equiv x^2[e_2, e_5, e_2]_{h,\mathrm{d}h}
\end{aligned}$$

We claim that $[e_2, e_5, e_2]_{h,h\mathrm{d}} \in JF_4$ where $J = \langle w^2, y^2z^2\rangle$. Once we establish this, the proof will be complete as this implies $[e_2, e_5, e_2]_{h,\mathrm{d}h} \in IF$. Recall that for any $a_1, a_2 \in F$ we have

$$\mathrm{d}h(a_1 \otimes a_2) = \mathrm{d}h(a_2 \otimes a_1) + h\sigma\mathrm{d}(a_1 \otimes a_2).$$

In particular, in F/JF we have

$$\begin{aligned} \mathrm{d}[e_2, e_5, e_2]_{h,\mathrm{d}h} &\equiv \mathrm{d}h(\mathrm{d}h(e_2 \otimes e_5) \otimes e_2 + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv \mathrm{d}h(\mathrm{d}h(e_5 \otimes e_2) \otimes e_2 + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv \mathrm{d}h(e_2 \otimes \mathrm{d}h(e_5 \otimes e_2) + e_2 \otimes \mathrm{d}h(e_5 \otimes e_2)) \\ &\equiv 0. \end{aligned}$$

where we used the fact that $\mathrm{d}e_5 = y^2z^2$ and $\mathrm{d}e_2 = w^2$. Now note that

$$\mathrm{H}(F_4/JF_4) = \mathrm{Tor}_4^R(R/I, R/J) = 0$$

Thus we must have $[e_2, e_5, e_2]_{h,\mathrm{d}h} \in JF_4$. □

3.2 Multiplicators

Having discussed associators, we now wish to discuss multiplicators. Throughout this subsection, let A be an MDG R-algebra, let X be and Y be MDG A-modules, and let $\varphi\colon X \to Y$ be a chain map.

Definition 3.6. The are two types of multiplicators were are interested in:

1. The **multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi$, from $A \otimes_R X$ to Y defined by

   $$[\cdot]_\varphi := \varphi\mu - \mu(1 \otimes \varphi).$$

   Note that we use μ to denote both A-scalar multiplications μ_X and μ_Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot]_\varphi\colon A \times X \to Y$ (or more simply by $[\cdot,\cdot]$ if context is clear) to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot]_\varphi$). Thus we have

   $$[a \otimes x]_\varphi = \varphi(ax) - a\varphi(x) = [a, x]$$

   for all $a \in A$ and $x \in X$. We say φ is **multiplicative** if $[\cdot]_\varphi = 0$.

2. The **2-multiplicator** of φ is the chain map, denoted $[\cdot]_\varphi^{(2)}$, from $A \otimes_R A \otimes_R X$ to Y defined by

   $$[\cdot]_\varphi^{(2)} := \varphi[\cdot]_\mu - [\cdot]_\mu(1 \otimes 1 \otimes \varphi)$$

   where we write $[\cdot]_\mu$ to denote both the associator of X and the associator Y where context makes clear which multiplication μ refers to. We denote by $[\cdot,\cdot,\cdot]_\varphi\colon A \times X \to Y$ to be the unique graded R-bilinear map which corresponds to $[\cdot]_\varphi^{(2)}$ (in order to avoid confusion with the associator, we will *always* keep φ in the subscript of $[\cdot,\cdot,\cdot]_\varphi$). Thus we have

   $$[a_1 \otimes a_2 \otimes x]_\varphi^{(2)} = \varphi([a_1, a_2, x]) - [a_1, a_2, \varphi(x)] = [a_1, a_2, x]_\varphi$$

   for all $a_1, a_2 \in A$ and $x \in X$. We say φ is **2-multiplicative** if $[\cdot]_\varphi^{(2)} = 0$.

Remark 4. If A and B are MDG R-algebras and $\varphi\colon A \to B$ is a chain map such that $\varphi(1) = 1$, then we view B as an MDG A-module with the A-scalar multiplication defined by $a \cdot b = \varphi(a)b$. In this case, the multiplicator of φ has the form $[\cdot]_\varphi = \varphi\mu - \mu\varphi^{\otimes 2}$, or in other words

$$[a_1, a_2]_\varphi = \varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)$$

for all $a_1, a_2 \in A$. Similarly, the 2-multiplicator of φ has the form $[\cdot]_\varphi = \varphi[\cdot]_\mu - [\cdot]_\mu\varphi^{\otimes 3}$, or in other words

$$[a_1, a_2, a_3]_\varphi = \varphi[a_1, a_2, a_3] - [\varphi(a_1), \varphi(a_2), \varphi(a_3)]$$

for all $a_1, a_2, a_3 \in A$.

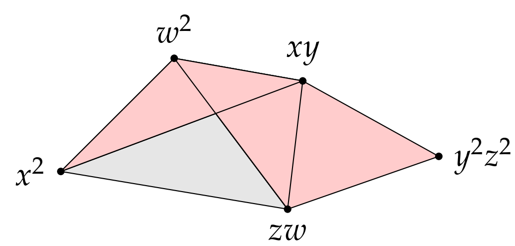
Example 3.4. Let us continue with Example (2.7) where $R = \Bbbk[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and F is the minimal free resolution of $R/\boldsymbol{m}$ over R. Let $\boldsymbol{m}' = x^2, w^2, y^2z^2$ and let E be the Koszul algebra which resolves $R/\boldsymbol{m}'$ over R. We denote the standard homogeneous basis of E by e'_σ and we denote the standard homogeneous basis of F by e_σ. Choose a chain map $\iota'\colon E \to F$ which lifts the projection $R/\boldsymbol{m}' \to R/\boldsymbol{m}$ such that ι' is unital and respects the multigrading. Then ι' being a chain map together with the fact that it is unital and respects the multigrading forces us to have

$$\begin{aligned} \iota'(e'_1) &= e_1 & \iota'(e'_{12}) &= e_{12} \\ \iota'(e'_2) &= e_2 & \iota'(e'_{13}) &= yz^2e_{14} + xe_{45} \\ \iota'(e'_3) &= e_5 & \iota'(e'_{23}) &= y^2ze_{23} + we_{35}. \end{aligned}$$

On the other hand, ι' can be defined at e'_{123} in two possible ways. Assume that it is defined by

$$\iota'(e'_{123}) = yz^2e_{124} + xyze_{234} - xwe_{345}.$$

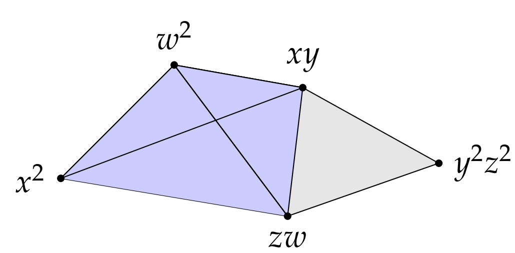
We can picture $\iota'(E')$ inside of F as being supported on the red-shaded subcomplex below:



We now ask: is ι' an MDG algebra homomorphism? The answer is no. Indeed, clearly this map is a chain map which fixes the identity element, however it is not multiplicative. In fact, it is not even 2-multiplicative. To see this, assume for a contradiction that it was 2-multiplicative. Then we would have

$$\begin{aligned} 0 &= \iota'(0) \\ &= \iota'([e'_1, e'_2, e'_3]) \\ &= [\iota'(e'_1), \iota'(e'_2), \iota'(e'_3)] \\ &= [e_1, e_2, e_5] \\ &\neq 0, \end{aligned}$$

which is a contradiction. Next let $\boldsymbol{m}'' = x^2, w^2, zw, xy$ and let T be the Taylor algebra which resolves $R/\boldsymbol{m}''$ over R. We denote the standard homogeneous basis of T by e''_σ. Choose a comparison map $\iota''\colon T \to F$ which lifts the projection $R/\boldsymbol{m}'' \to R/\boldsymbol{m}$ such that ι'' is unital and respects the multigrading. Then ι'' being a chain map together with the fact that it is unital and multigraded forces us to have $\iota''(e''_\sigma) = e_\sigma$ for all σ. We can picture $\iota''(T'')$ inside of F as being supported on the blue-shaded subcomplex below:



This time it is easy to check that ι'' is an MDG algebra homomorphism and we can give F the structure of an MDG T-module using ι'' in the usual way. Notice that F is *not* associative as a T-module, that is F is not a DG T-module since $[e_1, e_2, e_5] \neq 0$.

We now provide some context to Example (3.4) by discussing Buchsbaum and Eisenbud's strategy in proving Conjecture (1) in the case where R is a regular local ring and $M = R/I$ where I is an ideal of R of grade g. The idea goes as follows: let $\boldsymbol{t} = t_1, \dots, t_g$ be a maximal R-sequence contained in I, let E be the Koszul algebra resolution of $R/\boldsymbol{t}$ over R, and let F be the minimal free resolution of R/I over R. Choose a comparison map $\varphi\colon E \to F$ which lifts the canonical map $R/\boldsymbol{t} \to R/I$. The idea is that if F admits a DG algebra structure, then we can choose φ to be multiplicative, and we can use this to show that φ is injective which would imply Conjecture (1). Indeed, assume that F admits a DG algebra structure. To show φ is injective, we consider two steps:

Step 1: We first show $\varphi_g\colon E_g \to F_g$ is injective. Since $E_g \simeq R$ and every nonzero element of R is F_g-regular, it suffices to show that $\varphi_g \neq 0$. After applying $\mathrm{Hom}_R(-, R)$ to the following short exact sequence of R-modules

$$0 \longrightarrow I/\boldsymbol{t} \longrightarrow R/\boldsymbol{t} \longrightarrow R/I \longrightarrow 0 \qquad (18)$$

we obtain an induced map in Ext:

$$\cdots \longrightarrow \operatorname{Ext}_R^{g-1}(I/\boldsymbol{t}, R) \longrightarrow \operatorname{Ext}_R^{g}(R/I, R) \longrightarrow \operatorname{Ext}_R^{g}(R/\boldsymbol{t}, R) \longrightarrow \cdots \tag{19}$$

Note that $\boldsymbol{t}$ is a maximal R-sequence contained in $\langle \boldsymbol{t} \rangle \subseteq I$ of length g. It follows that from Ext characterization of depth that $\operatorname{Ext}_R^{g-1}(I/\boldsymbol{t}, R) = 0$ and $\operatorname{Ext}_R^{g}(R/I, R) \neq 0$. Thus the map

$$\varphi_g^\star \colon \operatorname{Ext}_R^{g}(R/I, R) \to \operatorname{Ext}_R^{g}(R/\boldsymbol{t}, R)$$

is nonzero. In particular, this implies $\varphi_g \neq 0$ which implies φ_g is injective.

Step 2: Let $\mathfrak{a} = \ker \varphi$ and assume for a contradiction that $\mathfrak{a} \neq 0$. Note that $\mathfrak{a}$ is a DG ideal of E since φ is multiplicative. Since every nonzero DG ideal of E intersects E_g nontrivially, we must have $\mathfrak{a}_g \neq 0$. However this contradicts the fact that $\mathfrak{a}_g = \ker \varphi_g = 0$ by the first step. Thus $\mathfrak{a} = 0$ which implies φ is injective.

Unfortunately, this strategy won't work in general since F need not have a DG algebra structure. Not all is lost however since, as we've discussed before, we only need to find a map $\varphi \colon E \to F$ which is multiplicative. However even then, Example (3.4) shows that this won't work in the multigraded case. At the same time however, Example (3.4) suggests that, at least in the monomial ideal case, one should replace E with a (possibly larger) Taylor algebra T which resolves R/J over R where $\boldsymbol{t} \subseteq J \subseteq I$ such that one can find a multiplicative chain map $\varphi \colon T \to F$ which lifts the canonical map $R/J \to R/I$.

Example 3.5. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, let F be the minimal free resolution of $R/\boldsymbol{m}$ over R, and let T be the Taylor algebra resolution of $R/\boldsymbol{m}$ over R. We denote the Taylor multiplication on T by ν. Recall that the multiplication μ on F described in Example (2.7) arises from the Taylor multiplication in the sense that there is a projection $\pi \colon T \to F$ such that $\mu = \pi\nu\iota^{\otimes 2}$ where $\iota \colon F \to T$ is the inclusion map. Observe that

$$\begin{aligned}
[e_1, e_{25}]_\pi &= \pi(e_1 \star_\nu e_{25}) - \pi(e_1) \star_\mu \pi(e_{25}) \\
&= \pi(e_{125}) - e_1 \star_\mu (y^2 z e_{23} + w e_{35}) \\
&= yz^2 e_{124} + xyz e_{234} + xw e_{345} - y^2 z e_{123} - yzw e_{134} - xw e_{345} \\
&= -yz \mathrm{d}(e_{1234}) \\
&= [e_1, e_5, e_2]_\mu \\
&\neq 0.
\end{aligned}$$

Thus $\pi \colon T \to F$ is not multiplicative.

3.2.1 Multiplicator Identities

We want to familiarize ourselves with the multiplicator of $\varphi \colon X \to Y$, so in this subsubsection we collect together some identities which the multiplicator satisfies:

- For all $a \in A$ homogeneous and $x \in X$, we have the Leibniz rule:

$$\mathrm{d}[a, x] = [\mathrm{d}a, x] + (-1)^{|a|}[a, \mathrm{d}x].$$

- For all $a \in A$ homogeneous and $x \in X$ homogeneous, we have

$$[a, x] = (-1)^{|a||x|}[x, a]. \tag{20}$$

- For all $a_1, a_2 \in A$ and $x \in X$, we have

$$a_1[a_2, x] - [a_1 a_2, x] + [a_1, a_2 x] = [a_1, a_2, x]_\varphi \tag{21}$$

Furthermore, if Z is another MDG A-module and $\psi \colon Y \to Z$ is another chain map, then for all $a \in A$ and $x \in X$, we have

$$[a, x]_{\psi\varphi} = \psi([a, x]_\varphi) + [a, \varphi(x)]_\psi \tag{22}$$

In particular, if ψ is multiplicative, then $\psi([Y]_\varphi) \subseteq [Z]_{\psi\varphi}$.

Remark 5. Let A and B be MDG R-algebras and let $\varphi \colon A \to B$ be a chain map such that $\varphi(1) = 1$. Then we can rewrite (21) as follows: for all $a_1, a_2, a_3 \in A$, we have

$$\varphi(a_1)[a_2, a_3] - [a_1a_2, a_3] + [a_1, a_2a_3] - [a_1, a_2]\varphi(a_3) = [\varphi(a_1), \varphi(a_2), \varphi(a_3)] - \varphi([a_1, a_2, a_3]) \tag{23}$$

Indeed, this follows from the fact that

$$[\varphi(a_1), \varphi(a_2), \varphi(a_3)] = [a_1, a_2, \varphi(a_3)] - [a_1, a_2]\varphi(a_3).$$

In this case, we also have $[a, a]_\varphi = 0$ for all $a \in A$ where $|a|$ is odd.

3.2.2 The Maximal Multiplicative Quotient

The **multiplicator complex** of φ, denoted $[Y]_\varphi$, is the R-subcomplex of Y given by $[Y]_\varphi := \operatorname{im} [\cdot]_\varphi$, so the underlying graded module of $[Y]_\varphi$

$$[Y]_\varphi := \operatorname{span}_R\{[a, x]_\varphi \mid a \in A \text{ and } x \in X\},$$

and the differential of $[Y]_\varphi$ is simply the restriction of the differential of Y to $[Y]_\varphi$. In order to avoid confusion with the associator complex, we will always write φ in the subscript of $[Y]_\varphi$. Even though the multiplicator complex of φ is closed under the differential, it need not be closed under A-scalar multiplication. In other words, if $a_1, a_2 \in A$ and $x \in X$, then it need not be the case that $a_1[a_2, x]_\varphi \in [Y]_\varphi$. We denote by $\langle Y \rangle_\varphi$ to be the MDG A-submodule of Y generated by $[Y]_\varphi$. In other words, $\langle Y \rangle_\varphi$ is the smallest MDG A-submodule of Y which contains $[Y]_\varphi$. Unlike the associator submodule, the multiplicator submodule is difficult to describe in terms of an R-span of elements. Indeed, as a first guess, one might think that $\langle Y \rangle_\varphi$ is given by

$$\operatorname{span}_R\{[a, x]_\varphi \mid a \in A \text{ and } x \in X\}. \tag{24}$$

However this is clearly incorrect in general as we may need to adjoin elements of the form $a_1[a_2, x]$ to (24). As a second guess, one might think that $\langle Y \rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi \mid a_1, a_2 \in A \text{ and } x \in X\}. \tag{25}$$

However this is not correct in general either since the identity

$$a_1(a_2[a_3, x]_\varphi) = (a_1a_2)[a_3, x]_\varphi - [a_1, a_2, [a_3, x]_\varphi]$$

tells us that should really adjoin elements of the form $a_1[a_2, a_3, [a_4, x]]$ to (25) as well. As a third guess, one might think that $\langle Y \rangle_\varphi$ is given by

$$\operatorname{span}_R\{a_1[a_2, x]_\varphi,\ a_1[a_2, a_3, [a_4, x]_\varphi] \mid a_1, a_2, a_3, a_4 \in A \text{ and } x \in X\}. \tag{26}$$

Again this is not correct in general since the identity

$$a_1(a_2[a_3, a_4, [a_5, x]_\varphi]) = (a_1a_2)[a_3, a_4, [a_5, x]] - [a_1, a_2, [a_3, a_4, [a_5, x]_\varphi]].$$

tells us that we should really adjoin elements of the form $a_1[a_2, a_3, [a_4, a_5, [a_6, x]_\varphi]]$ to (26) as well. The problem continues getting worse with no end in sight. It turns out however, that if φ is 2-multiplicative, then $\langle Y \rangle_\varphi$ given by (24).

Proposition 3.4. *If φ is 2-multiplicative, then for all $a_1, a_2, a_3 \in A$ and $x \in X$ we have*

$$a_1[a_2, x]_\varphi = [a_1a_2, x]_\varphi - [a_1, a_2x]_\varphi \quad \text{and} \quad [a_1, a_2, [a_3, x]_\varphi] = [[a_1, a_2, a_3], x]_\varphi - [a_1, [a_2, a_3, x]]_\varphi. \tag{27}$$

In particular, $\langle Y \rangle_\varphi$ is given by (24).

Proof. A straightforward calculation yields

$$a_1[a_2, a_3, x]_\varphi = [a_1a_2, a_3, x]_\varphi - [a_1, a_2a_3, x]_\varphi + [a_1, a_2, a_3x]_\varphi - [[a_1, a_2, a_3], x]_\varphi + [a_1, [a_2, a_3, x]]_\varphi - [a_1, a_2, [a_3, x]_\varphi].$$

Using this identity together with the identity (21), we see that if φ is 2-multiplicative, then we obtain (27). This implies all elements of the form $a_1[a_2, x]$ and $a_1[a_2, a_3, [a_4, x]]$ belong to (24). An easy induction argument shows that $\langle Y \rangle_\varphi$ is given by (24). $\square$

The quotient $Y/\langle Y \rangle_\varphi$ is an MDG A-module. We denote by $\pi \colon Y \to Y/\langle Y \rangle_\varphi$ to be the canonical quotient map. Note that both π and $\pi\varphi$ are multiplicative. Therefore (22) implies $[Y]_\varphi \subseteq \ker \pi$ which implies $\langle Y \rangle_\varphi \subseteq \ker \pi$. We call $Y/\langle Y \rangle_\varphi$ (together with its canonical quotient map π) the **maximal multiplicative quotient** of $\varphi \colon X \to Y$; it satisfies the following universal mapping property:

Proposition 3.5. *For all MDG A-modules Z and for all chain maps $\psi\colon Y\to Z$ where both ψ and $\psi\varphi$ are MDG A-module homomorphisms, there exists a unique MDG A-module homomorphism $\overline{\psi}\colon Y/\langle Y\rangle_\varphi\to Z$ such that $\overline{\psi}\pi=\psi$. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \arrow[dl, "\psi"'] \arrow[d, "\pi"] \\ Z & Y/\langle Y\rangle_\varphi \arrow[l, dashed, "\overline{\psi}"] \end{tikzcd} \tag{28}$$

Proof. Suppose $\psi\colon Y\to Z$ is such a map. Then (22) implies $[Y]_\varphi\subseteq\ker\psi$ which implies $\langle Y\rangle_\varphi\subseteq\ker\psi$. Thus the map $\overline{\psi}\colon Y/\langle Y\rangle_\varphi\to Z$ given by

$$\overline{\psi}(\overline{y}):=\psi(y),$$

where $\overline{y}\in Y/\langle Y\rangle_\varphi$ and where $y\in Y$ is a choice of an element in Y such that $\pi(y)=\overline{y}$, is well-defined. Furthermore, it is easy to check that $\overline{\psi}$ is an MDG A-module homomorphism and the unique such map which makes the diagram (43) commute. □

4 The Associator Functor

Let X and Y be MDG A-modules and let $\varphi\colon X\to Y$ be a chain map. If φ is multiplicative, then observe that for all $a_1,a_2,a_3\in A$ and $x\in X$, we have

$$\varphi(a_1[a_2,a_3,x])=a_1[a_2,a_3,\varphi(x)]. \tag{29}$$

Thus φ restricts to an MDG A-module homomorphism $\varphi\colon\langle X\rangle\to\langle Y\rangle$. In particular, we obtain a functor from the category of MDG A-module to itself which sends an MDG A-module X to the MDG associator submodule $\langle X\rangle$ and which sends an MDG A-module homomorphism $\varphi\colon X\to Y$ to its restriction $\varphi|_{\langle X\rangle}\colon\langle X\rangle\to\langle Y\rangle$. We call this the **associator functor**.

4.1 Failure of Exactness

The associator functor need not be exact. Indeed, let

$$0\longrightarrow X\xrightarrow{\varphi}Y\xrightarrow{\psi}Z\longrightarrow 0 \tag{30}$$

be a short exact sequence of MDG A-modules. Then we obtain an induced sequence of MDG A-modules

$$0\longrightarrow\langle X\rangle\xrightarrow{\varphi}\langle Y\rangle\xrightarrow{\psi}\langle Z\rangle\longrightarrow 0 \tag{31}$$

which is exact at $\langle X\rangle$ and $\langle Z\rangle$ but not necessarily exact at $\langle Y\rangle$. In order to ensure exactness of (31), we need to place a condition on (30). This leads us to consider the following definition:

Definition 4.1. Let X be an MDG A-submodule of Y. We say Y is an **associative extension** of X if

$$\langle X\rangle=X\cap\langle Y\rangle.$$

It is easy to see that (31) is a short exact sequence of MDG A-modules if and only if Y is an associative extension of $\varphi(X)$. In this case, we obtain a long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{i+1}\langle Z\rangle \arrow[dll, rounded corners] \\ \mathrm{H}_i\langle X\rangle \arrow[r] & \mathrm{H}_i\langle Y\rangle \arrow[r] & \mathrm{H}_i\langle Z\rangle \arrow[dll, rounded corners] \\ \mathrm{H}_{i-1}\langle X\rangle \arrow[r] & \cdots & \end{tikzcd} \tag{32}$$

An immediate consequence of this long exact sequence is the following theorem:

Theorem 4.1. *Let X be an MDG A-module and suppose Y is an associative extension of X. Then Y is homologically associative if and only if X and Y/X are homologically associative.*

4.2 An Application of the Long Exact Sequence

In this subsection, we give an application of the long exact sequence (32). Assume that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be the minimal free resolution of R/I over R, and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $F + eF$ is the minimal free resolution of $R/\langle I, r\rangle$ over R. Here, e is thought of as an exterior variable of degree 1, and the differential of the mapping cone is given by

$$\mathrm{d}(a + eb) = \mathrm{d}(a) + rb - e\mathrm{d}(b)$$

for all $a, b \in F$. Now equip F with a multiplication μ giving it the structure of an MDG algebra. We give $F + eF$ the structure of an MDG R-algebra by extending the multiplication on F to a multiplication on $F + eF$ by setting

$$(a + eb)(c + ed) = ac + e(bc + (-1)^{|a|}ad)$$

for all $a, b, c, d \in F$. In particular, note that $(eb)c = e(bc)$ for all $b, c \in F$, so e belongs to the nucleus of $F + eF$. We denote by $\iota \colon F \to F + eF$ to be the inclusion map. We can view $F + eF$ either as an MDG F-module or as an MDG R-algebra, thus we potentially have two different associator complexes to consider. It turns out however that these give rise to the same R-complex since e is in the nucleus of $F + eF$. This is the second main theorem from the introduction:

Theorem 4.2. *Let $\langle F + eF\rangle_F$ be the associator F-submodule of $F + eF$ and let $\langle F + eF\rangle$ be the associator $(F + eF)$-ideal of $F + eF$. Then*

$$\langle F + eF\rangle_F = \langle F\rangle + e\langle F\rangle = \langle F + eF\rangle. \tag{33}$$

In particular, $F + eF$ is an associative extension of F. More generally, suppose $\boldsymbol{r} = r_1, \dots, r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$. We set

$$F + \boldsymbol{e}F = F + \sum_{i=1}^{m} e_i F$$

to be minimal R-free resolution of $R/\langle I, \boldsymbol{r}\rangle$ obtained by iterating the mapping cone construction as above, where e_i is an exterior variable of degree 1 which satisfies $\mathrm{d}e_i = r_i$, and where we extend the multiplication of F to a multiplication on $F + \boldsymbol{e}F$ by extending it from $F + \sum_{i=1}^{k} e_i F$ to $F + \sum_{i=1}^{k+1} e_i F$ for each $1 \leq k < m$ as above. Then

$$\langle F + \boldsymbol{e}F\rangle_F = \langle F\rangle + \boldsymbol{e}\langle F\rangle = \langle F + \boldsymbol{e}F\rangle \tag{34}$$

where we set $\boldsymbol{e}\langle F\rangle := \sum_{i=1}^{m} e_i\langle F\rangle$. In particular, $F + \boldsymbol{e}F$ is an associative extension of F.

Proof. Since e is in the nucleus, we have $e[a, b, c] = [ea, b, c]$ for all $a, b, c \in F$. Similarly we have

$$\begin{aligned}
[a, b, ec] &= -(-1)^{|a||b|+|a||ec|+|ec||b|}[ec, b, a] \\
&= -(-1)^{|a||b|+|a||c|+|b||c|}[ec, b, a] \\
&= -(-1)^{|a||b|+|a||c|+|b||c|}e[c, b, a] \\
&= e[a, b, c]
\end{aligned}$$

for all $a, b, c \in F$. Similarly we have

$$\begin{aligned}
[a, eb, c] &= -(-1)^{|a||eb|+|a||c|}[eb, c, a] - (-1)^{|eb||c|+|a||c|}[c, a, eb] \\
&= e(-(-1)^{|a||eb|+|a||c|}[b, c, a] - (-1)^{|eb||c|+|a||c|}[c, a, b]) \\
&= e[a, b, c]
\end{aligned}$$

for all $a, b, c \in F$. Thus we have

$$\begin{aligned}
(a + ea')[b + eb', c + ec', d + ed'] &= (a + ea')[b, c, d] + (a + ea')(e[b', c', d']) \\
&= a[b, c, d] + ea'[b, c, d] + (-1)^{|a|}ea[b', c', d'] \\
&= a[b, c, d] + e(a'[b, c, d] + (-1)^{|a|}a[b', c', d'])
\end{aligned}$$

for all $a, b, c, d, a', b', c', d' \in F$. Thus we obtain (33). To see why (33) implies $F + eF$ is an associative extension of F, note that

$$F \cap \langle F + eF\rangle = F \cap (\langle F\rangle + e\langle F\rangle) = \langle F\rangle.$$

The last part of the theorem follows from induction. □

Theorem 4.3. *Let* $\varepsilon = \operatorname{lha}(F)$ *and let* $\delta = \operatorname{uha}(F)$*. Then* $\operatorname{lha}(F + eF) = \varepsilon$ *and*

$$\operatorname{uha}(F + eF) = \begin{cases} \delta & \textit{if } r \textit{ is } \mathrm{H}_\delta\langle F\rangle\textit{-regular} \\ \delta + 1 & \textit{otherwise} \end{cases} \tag{35}$$

Moreover, we have a short exact sequence of $R/\langle I, r\rangle$*-modules*

$$0 \longrightarrow \mathrm{H}_i\langle F\rangle/r\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + eF\rangle \longrightarrow 0 :_{\mathrm{H}_{i-1}\langle F\rangle} r \longrightarrow 0 \tag{36}$$

for each $i \in \mathbb{Z}$*. In particular, we have an isomorphism of* $R/\langle I, r\rangle$*-modules*

$$\mathrm{H}_\varepsilon\langle F\rangle/r\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F + eF\rangle.$$

Proof. Since $F + eF$ is an associative extension of F, we obtain a long exact sequence in homology:

$$\cdots \longrightarrow \mathrm{H}_i\langle F\rangle \xrightarrow{r} \mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + eF\rangle \longrightarrow \mathrm{H}_{i-1}\langle F\rangle \xrightarrow{r} \mathrm{H}_{i-1}\langle F\rangle \longrightarrow \cdots \tag{37}$$

We obtain (38) as well as (37) from this long exact sequence. We obtain $\operatorname{lha}(F + eF) = \varepsilon$ from the long exact sequence together with an application of Nakayama's lemma. $\square$

Corollary 1. *Suppose* $\boldsymbol{r} = r_1, \ldots, r_m$ *is a maximal* (R/I)*-regular sequence contained in* $\mathfrak{m}$ *and let* $F + \boldsymbol{e}F$ *be the corresponding* R*-free resolution of* $R/\langle I, \boldsymbol{r}\rangle$ *obtained by iterating the mapping cone construction. Then we obtain a short exact sequence of* $R/\langle I, \boldsymbol{r}\rangle$*-modules*

$$0 \longrightarrow \mathrm{H}_i\langle F\rangle/\boldsymbol{r}\mathrm{H}_i\langle F\rangle \longrightarrow \mathrm{H}_i\langle F + \boldsymbol{e}F\rangle \longrightarrow 0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r} \longrightarrow 0 \tag{38}$$

In particular, have an isomorphism of $R/\langle I, \boldsymbol{r}\rangle$*-modules:*

$$\mathrm{H}_\varepsilon\langle F\rangle/\boldsymbol{r}\mathrm{H}_\varepsilon\langle F\rangle \cong \mathrm{H}_\varepsilon\langle F + \boldsymbol{e}F\rangle.$$

We also have the length formula:

$$\ell(\mathrm{H}_i\langle F + \boldsymbol{e}F\rangle) = \ell(\mathrm{H}_i\langle F\rangle/\boldsymbol{r}\mathrm{H}_i\langle F\rangle) + \ell(0 :_{\mathrm{H}_{i-1}\langle F\rangle} \boldsymbol{r}),$$

here $\ell(-)$ *is the length function.*

5 The Symmetric DG Algebra

Let R be a commutative ring, let A be a $\mathbb{Z}$-graded R-module such that $A_0 = R$ which is also equipped with a $\mathbb{Z}$-linear differential $\mathrm{d}\colon A \to A$ giving it the structure of a chain complex. Note that the differential need not be R-linear and note that A may be nonzero in negative homological degree. In this section, we will construct the symmetric DG algebra of A, which we denote by $\mathrm{S}(A)$. After constructing the symmetric DG algebra in this general setting, we then specialize to the case we are mostly interesting in, namely that A is an R-complex centered at R meaning the differential of A is R-linear with $A_0 = R$ and $A_{<0} = 0$. In this case, we sometimes denote the symmetric DG algebra of A by $\mathrm{S}_R(A)$ with R in the subscript in order to emphasize that A is centered at R.

Before we give a rigorous construction of the symmetric DG algebra, we wish to help motivate the reader by giving an informal description of it in this special case where A is an R-complex centered at R. In this case, the underlying graded algebra of $S = \mathrm{S}_R(A)$ is the usual symmetric R-algebra $\mathrm{Sym}(A_+)$ where we view A_+ as just an R-module. However S obtains a bi-graded structure using homological degree and total degree: we have a decomposition of S into R-modules:

$$S = \bigoplus_{i \geq 0} S_i = \bigoplus_{m \geq 0} S^m = \bigoplus_{i,m \geq 0} S_i^m.$$

We refer to the i in the subscript as homological degree and we refer to the m in the superscript as total degree. We have $S_0 = S^0 = S_0^0 = R$ and $S^1 = A_+$. More generally, for $i, m \geq 1$, the R-module S_i^m is the R-span of all homogeneous elementary products of the form $\boldsymbol{a} = a_1 \cdots a_m$ where $a_1, \dots, a_m \in A_+$ are homogeneous (with respect to homological degree of course) such that

$$|\boldsymbol{a}| = |a_1| + \cdots + |a_m| = i.$$

In particular, note that $A = S^{\leq 1} = R + A_+$, thus we view A as being the total degree ≤ 1 part of S. The differential of A extends the differential of S in a natural way and is defined on homogeneous elementary products $\boldsymbol{a} = a_1 \cdots a_m$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \cdots \mathrm{d}(a_j) \cdots a_m. \tag{39}$$

If each of the a_j in (39) live in homological degree ≥ 2, then $\mathrm{d}\boldsymbol{a}$ and $\boldsymbol{a}$ has the same total degree, namely $\deg(\mathrm{d}\boldsymbol{a}) = m = \deg \boldsymbol{a}$. However if one of the a_j in (39) lives in homological degree 1, then $\deg(\mathrm{d}\boldsymbol{a}) = m - 1$. The diagram below illustrates how the differential acts on the bi-graded components:

$$\begin{tikzcd}[column sep=small]
A_i \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & S_i^{m-1} \arrow[dl] \arrow[dr] & & S_i^m \arrow[dl] \arrow[dr] & & S_i^{m+1} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \mathbb{K}_i \arrow[dl] \\
& A_{i-1} & & S_{i-1}^{m-2} & & S_{i-1}^{m-1} & & S_{i-1}^{m} & & S_{i-1}^{m+1} & & \mathbb{K}_{i-1} &
\end{tikzcd}$$

where we set $\mathbb{K}$ to be the koszul DG algebra induced by $\mathrm{d}\colon A_1 \to A_0$. Thus the differential of S connects the usual differential of A on the far left to a koszul differential on the far right. In order to keep track of how the differential operates on the bi-graded components, we express d as

$$\mathrm{d} = \eth + \partial,$$

where $\eth$ is the component of d which respects total degree and where ∂ is the component of d which drops total degree by 1. In the next example, we consider a free resolution of a cyclic module and work out what the symmetric DG algebra looks like in this case.

Example 5.1. Let $R = \Bbbk[x, y]$, let $\boldsymbol{m} = x^2, xy$, and let F be Taylor resolution of $R/\boldsymbol{m}$ over R. We write down the homogeneous components of F as a graded R-module as well as how the differential acts on the homogeneous basis below:

$$\begin{aligned}
F_0 &= R & \qquad \mathrm{d}e_1 &= x^2 \\
F_1 &= Re_1 + Re_2 & \mathrm{d}e_2 &= xy \\
F_2 &= Re_{12}, & \mathrm{d}e_{12} &= xe_2 - ye_1,
\end{aligned}$$

Note that the Taylor resolution usually comes equipped with a multiplication called the Taylor multiplication. Let us denote this by $\star$ so as not to confuse it with the multiplication $\cdot$ of the symmetric DG algebra $S = \mathrm{S}_R(F)$ of F. Now we write down the homogeneous components of S as a graded R-module (with respect to homological degree) below:

$$\begin{aligned}
S_0 &= R \\
S_1 &= Re_1 + Re_2 \\
S_2 &= Re_{12} + Re_1e_2 \\
S_3 &= Re_1e_{12} + Re_2e_{12} \\
S_4 &= Re_{12}^2 + Re_1e_2e_{12} \\
&\ \vdots \\
S_{2k-1} &= Re_1e_{12}^{k-1} + Re_2^{k-1} \\
S_{2k} &= Re_{12}^k + Re_1e_2e_{12}^{k-1} \\
S_{2k+1} &= Re_1e_{12}^k + Re_2e_{12}^k \\
&\ \vdots
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{d}(e_1e_2 - xe_{12}) &= \mathrm{d}(e_1e_2) - x\mathrm{d}(e_{12}) \\
&= \mathrm{d}(e_1)e_2 - e_1\mathrm{d}(e_2) - x(xe_2 - ye_1) \\
&= x^2e_2 - xye_1 - x^2e_2 + xye_1 \\
&= 0.
\end{aligned}$$

5.1 Construction of the Symmetric DG Algebra of A

We now provide a rigorous construction of $\mathrm{S}(A)$ in the general case where the differential of A need not be R-linear and where $A_{<0}$ is not necessarily zero. Our construction will occur in three steps:

Step 1: We define the **non-unital tensor DG algebra** of A to be

$$\mathrm{U}_{\mathbb{Z}}(A) := \bigoplus_{n=1}^{\infty} A^{\otimes n},$$

where the tensor product is taken as $\mathbb{Z}$-complexes. An elementary tensor in $U = \mathrm{U}_{\mathbb{Z}}(A)$ is denoted $\boldsymbol{a} = a_1 \otimes \cdots \otimes a_n$ where $a_1, \dots, a_n \in A$ and $n \geq 1$. The differential of U is denoted by d again to simplify notation and is defined on $\boldsymbol{a}$ by

$$\mathrm{d}\boldsymbol{a} = \sum_{j=1}^{n} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots \otimes a_n.$$

We say $\boldsymbol{a}$ is a homogeneous elementary tensors if each a_i is a homogeneous element in A. In this case, we set

$$|\boldsymbol{a}| = \sum_{i=1}^{n} |a_i| \quad \text{and} \quad \deg \boldsymbol{a} = \sum_{i=1}^{n} \deg a_i,$$

where deg is defined on elements $a \in A$ by

$$\deg a = \begin{cases} 1 & \text{if } a \in A_{>0} \\ 0 & \text{if } a \in R \\ -1 & \text{if } a \in A_{<0} \end{cases}$$

We call $|\boldsymbol{a}|$ the **homological degree** of $\boldsymbol{a}$ and we call $\deg \boldsymbol{a}$ the **total degree** of $\boldsymbol{a}$. With $|\cdot|$ and deg defined, we observe that U admits a bi-graded decomposition:

$$U = \bigoplus_{i \in \mathbb{Z}} U_i = \bigoplus_{m \in \mathbb{Z}} U^m = \bigoplus_{i,m \in \mathbb{Z}} U_i^m,$$

where the component U_i^m consists of all finite $\mathbb{Z}$-linear combinations of homogeneous elementary tensors $\boldsymbol{a} \in U$ such that $|\boldsymbol{a}| = i$ and $\deg \boldsymbol{a} = m$. We equip U with an associative (but not commutative nor unital) bi-graded $\mathbb{Z}$-bilinear multiplication which is defined on homogeneous elementary tensors by $(\boldsymbol{a}, \boldsymbol{a}') \mapsto \boldsymbol{a} \otimes \boldsymbol{a}'$ and is extended $\mathbb{Z}$-bilinearly everywhere else. This multiplication is easily seen to satisfy Leibniz rule, however note that U is not unital under this multiplication since $(1,1) \mapsto 1 \otimes 1 \neq 1$ (hence why we call this the *non-unital* tensor DG algebra). Also note that U already comes equipped with an R–scalar multiplication (from the R-module structure on A), denoted $(r, \boldsymbol{a}) \mapsto r\boldsymbol{a}$, however the multiplication of U only agrees with the R-scalar multiplication wherever they are both defined and vanish. To rectify this, let $\mathfrak{u} = \mathfrak{u}(A)$ be the U-ideal by all elements of the form

$$[r,a]_\mu = r \otimes a - ra \qquad [a,r]_\mu = a \otimes r - ar$$
$$[r,a]_{\mathrm{d}} = \mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a) \qquad [a,r]_{\mathrm{d}} = (-1)^{|a|} a \otimes \mathrm{d}r - \mathrm{d}(ar) + (\mathrm{d}a)r$$

where $r \in R$ and $a \in A$.

Lemma 5.1. *The differential maps* $\mathfrak{u}$ *to itself.*

Proof. Indeed, given $r \in R$ and $a \in A$, we have

$$\begin{aligned} \mathrm{d}[r,a]_\mu &= \mathrm{d}(r \otimes a) - \mathrm{d}(ra) \\ &= \mathrm{d}r \otimes a + r \otimes \mathrm{d}a - \mathrm{d}r \otimes a + r(\mathrm{d}a) + [r,a]_{\mathrm{d}} \\ &= r \otimes \mathrm{d}a + r(\mathrm{d}a) + [r,a]_{\mathrm{d}} \\ &= [r, \mathrm{d}a]_\mu + [r,a]_{\mathrm{d}} \\ &\in \mathfrak{u}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathrm{d}[r,a]_{\mathrm{d}} &= \mathrm{d}(\mathrm{d}r \otimes a - \mathrm{d}(ra) + r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r(\mathrm{d}a)) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}(r \otimes \mathrm{d}a - [r, \mathrm{d}a]_\mu) \\ &= -\mathrm{d}r \otimes \mathrm{d}a + \mathrm{d}r \otimes \mathrm{d}a - \mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -\mathrm{d}[r, \mathrm{d}a]_\mu \\ &= -[r, \mathrm{d}a]_{\mathrm{d}} \\ &\in \mathfrak{u}. \end{aligned}$$

Similar calculations show $\mathrm{d}[a,r]_\mu \in \mathfrak{u}$ and $\mathrm{d}[a,r]_\mathrm{d} \in \mathfrak{u}$. $\square$

Step 2: We define the **tensor DG algebra** of A to be the quotient

$$\mathrm{T}(A) := \mathrm{U}(A)/\mathfrak{u}(A).$$

The multiplication of $U = \mathrm{U}(A)$ induces a multiplication on $T = \mathrm{T}(A)$ which not only becomes unital but also agrees with the R-scalar multiplication on T where they are both defined. Since $\mathfrak{u} = \mathfrak{u}(A)$ is generated by elements which are homogeneous with respect to homological degree and since the differential of U maps $\mathfrak{u}$ to itself, it follows that the differential of U induces a differential on T, which we again denote by d again. This gives T the structure of a non-commutative (but unital) DG $\Bbbk$-algebra, where

$$\Bbbk = \{r \in R \mid \mathrm{d}r \otimes a = 0 \text{ for all } a \in A\}.$$

In other words, the differential of T satisfies Leibniz rule and is $\Bbbk$-linear. Note that the generator $[r,a]_\mu$ of $\mathfrak{u}$ is also homogeneous with respect to total degree, however the generators $[r,a]_\mathrm{d}$ is homogeneous with respect to total degree if and only if either $\mathrm{d}r \otimes a = 0$, or $\mathrm{d}(ra) = r\mathrm{d}a$, or $|a| \in \{0,1\}$. In particular, $\mathfrak{u}$ will be homogeneous with respect to total degree if A is an R-complex centered at R (which is a case we are interested in). In this case, T inherits from U a bi-graded R-algebra structure:

$$T = \bigoplus_{i\in\mathbb{Z}} T_i = \bigoplus_{m\in\mathbb{Z}} T^m = \bigoplus_{i,m\in\mathbb{Z}} T_i^m.$$

Example 5.2. Let us describe what the total degree m component of $T = \mathrm{T}_R(A)$ in the case where A is an R-complex centered at R. We have

$$\begin{aligned} T^0 &= R \\ T^1 &= \bigoplus_{1\le i} A_i \\ T^2 &= \bigoplus_{1\le i<j} ((A_i \otimes A_j) \oplus (A_j \otimes A_i)) \oplus \bigoplus_{1\le i} A_i^{\otimes 2} \end{aligned}$$

The component T^3 is slightly more complicated:

$$\bigoplus_{\substack{1\le i<j<k \\ \pi\in S_3}} (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(k)}) \oplus \bigoplus_{\substack{1\le i<j \\ \pi\in S_2}} ((A_{\pi(i)}^{\otimes 2} \otimes A_{\pi(j)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)} \otimes A_{\pi(i)}) \oplus (A_{\pi(i)} \otimes A_{\pi(j)}^{\otimes 2})) \oplus \bigoplus_{1\le i} A_i^{\otimes 3}.$$

More generally, there is an interpretation of T^m in terms of certain rooted trees.

Now let $\mathfrak{t} = \mathfrak{t}(A)$ be the T-ideal generated by all elements of the form

$$[a_1,a_2]_\sigma \colon = (-1)^{|a_1||a_2|} a_2 \otimes a_1 - a_1 \otimes a_2 \quad \text{and} \quad [a]_\tau := a \otimes a,$$

where $a, a_1, a_2 \in A$ are homogeneous and $|a|$ is odd.

Lemma 5.2. *The differential of T maps $\mathfrak{t}$ to itself.*

Proof. Indeed, if $a, a_1, a_2 \in A$ are homogeneous with $|a|$ odd, then we have

$$\mathrm{d}[a_1,a_2]_\sigma = [\mathrm{d}a_1,a_2]_\sigma + (-1)^{|a_1|}[a_1,\mathrm{d}a_2]_\sigma \in \mathfrak{t} \quad \text{and} \quad \mathrm{d}[a]_\tau = [\mathrm{d}a,a]_\sigma \in \mathfrak{t}.$$

$\square$

Step 3: We define the **symmetric DG algebra** of A to be the quotient

$$\mathrm{S}(A) := \mathrm{T}(A)/\mathfrak{t}(A)$$

The image of a homogeneous elementary tensor $a_1 \otimes \cdots \otimes a_m$ in $S = \mathrm{S}(A)$ is often denoted $a_1 \cdots a_n$ and is called a homogeneous elementary product. Since $\mathfrak{t} = \mathfrak{t}(A)$ is generated by elements which are homogeneous with respect to both homological degree and since the differential of $T = \mathrm{T}(A)$ maps $\mathfrak{t}$ to itself, we see that the differential of T induces a differential on S, which we again denote by d, giving it the structure of a strictly graded-commutative DG $\Bbbk$-algebra. Furthemore, if T inherits the bi-graded structure from U, then S inherits the bi-graded structure from T since $\mathfrak{t}$ is generated by elements which are homogeneous with respect to total degree.

5.2 Properties of the Symmetric DG Algebra

We now focus our attention to the case where A is an R-complex centered at R and we wish to study $S = \mathrm{S}_R(A)$ the symmetric DG R-algebra of A (note that we sometimes write R in the subscript of $\mathrm{S}_R(A)$ to emphasize that A and $S = \mathrm{S}_R(A)$ are centered at R). In this case, the underlying graded R-algebra of S is the usual symmetric algebra of A_+:

$$\mathrm{Sym}_R(A_+) = \frac{\bigoplus_{m\geq 0} A_+^{\otimes m}}{\langle\{[a_1, a_2]_\sigma, [a]_\tau\}\rangle},$$

where the tensor product is taken over R. Thus the symmetric DG algebra of A inherits all of the properties that are satisfied by the symmetric algebra of A_+ when we forget about the differential. For instance, recall that a bounded below R-complex is semiprojective if and only if its underlying graded R-module is projective as a graded R-module. In particular, if A is semiprojective, then S is semiprojective too. Thus if we assume that A is semiprojective *and* that there exists a chain map $\pi \colon S \to A$ which splits the inclusion map $\iota \colon A \hookrightarrow S$, then we can lift chains maps out of A along surjective quasiisomorphisms, meaning if $\varphi \colon A \to X$ is any chain map and $\tau \colon Y \to X$ is any surjective quasiisomorphism, then there exists a chain map $\widetilde{\varphi} \colon S \to Y$ such that $\tau\widetilde{\varphi} = \varphi$, moreover such a lift is unique up to homotopy. The assumption that A is semiprojective is mild whereas the assumption that there exists a chain map $S \to A$ which splits the inclusion map $A \hookrightarrow S$ is rather subtle. We will see that if A has a DG R-algebra structure on it, then there will be such a map $S \to A$.

Proposition 5.1. *Let R be a commutative ring and let A be an R-complex centered at R.*

1. *(Base Change) Let R' be an R-algebra. Then*

$$\mathrm{S}_R(A) \otimes_R R' = \mathrm{S}_{R'}(A \otimes_R R'). \tag{40}$$

2. *(Exact Sequences) Let*

$$B \longrightarrow A \longrightarrow A' \longrightarrow 0 \tag{41}$$

be an exact sequence of R-complexes where A' is centered at a cyclic R-algebra, say $R' = R/I$ for some ideal I of R. Then we obtain an exact sequence

$$\mathrm{S}_R(A) \otimes_R B \longrightarrow \mathrm{S}_R(A) \longrightarrow \mathrm{S}_{R'}(A') \longrightarrow 0 \tag{42}$$

.

3. *(Universal Mapping Property) For every chain map of the form $\varphi \colon A \to A'$, where A' is a DG algebra centered at a ring R' and where φ restricts to a ring homomorphism $\varphi_0 \colon R \to R'$, there exists a unique DG algebra homomorphism $\widetilde{\varphi} \colon \mathrm{S}_R(A) \to A'$ which extends $\varphi \colon A \to A'$, that is, such that $\widetilde{\varphi} \circ \iota = \varphi$ where $\iota \colon A \hookrightarrow \mathrm{S}_R(A)$ is the inclusion map. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{S}_R(A) \arrow[d, "\widetilde{\varphi}"] \\ & A' \end{tikzcd} \tag{43}$$

Remark 6. Strictly speaking, one should write $R \otimes_R R'$ in the subscript on the righthand side of Equation (40). However we may view R' as being the homological degree 0 part by identifying R' with $R \otimes_R R'$ via the canonical isomorphism $R' \simeq R \otimes_R R'$.

Proof. We only prove the third property since the first two properties are straightforward to show. Let $\varphi \colon A \to A'$ be such a chain map and denote $S = \mathrm{S}_R(A)$. We define $\widetilde{\varphi} \colon S \to A'$ by setting $\widetilde{\varphi}|_A = \varphi$ and

$$\widetilde{\varphi}(a_1 \cdots a_m) = \varphi(a_1) \cdots \varphi(a_m) \tag{44}$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$ and then extending it R-linearly everywhere else. By construction, $\widetilde{\varphi}$ is multiplicative and extends $\varphi \colon A \to A'$. Furthermore, $\widetilde{\varphi}$ is a chain map since it is a graded R-linear map which commutes with the differential. Indeed, we clearly have $\widetilde{\varphi}\mathrm{d}(1) = 0 = \mathrm{d}\widetilde{\varphi}(1)$, and for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, we have

$$\begin{aligned}
\widetilde{\varphi}\mathrm{d}(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \widetilde{\varphi}(a_1 \cdots \mathrm{d}(a_j) \cdots a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \varphi\mathrm{d}(a_j) \cdots \varphi(a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \mathrm{d}\varphi(a_j) \cdots \varphi(a_m) \\
&= \mathrm{d}(\varphi(a_1) \cdots \varphi(a_m)) \\
&= \mathrm{d}\widetilde{\varphi}(a_1 \cdots a_m).
\end{aligned}$$

Finally, if $\widehat{\varphi}\colon S \to A'$ were another DG algebra homomorphism which extended $\varphi\colon A \to B$, then we would have

$$\widetilde{\varphi}(a_1 \cdots a_m) = \widehat{\varphi}(a_1) \cdots \widehat{\varphi}(a_m) = \varphi(a_1) \cdots \varphi(a_m) = \widetilde{\varphi}(a_1 \cdots a_m)$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $S^{\geq 2}$, which implies $\widehat{\varphi} = \widetilde{\varphi}$. □

Definition 5.1. Let A and B be two R-complexes centered at R. We define their **wedge sum** $A \vee B$ to be the R-complex centered at R whose underlying graded R-module is given by

$$(A \vee B)_i = \begin{cases} A_i \oplus B_i & \text{if } i \geq 1 \\ R & \text{if } i = 0 \end{cases}$$

and whose differential is defined by

$$\mathrm{d}(a, b) = \begin{cases} (\mathrm{d}a, \mathrm{d}b) & \text{if } |a| = |b| \geq 2 \\ \mathrm{d}a - \mathrm{d}b & \text{if } |a| = |b| = 1 \end{cases}$$

Observe that

$$\mathrm{H}_i(A \vee B) = \begin{cases} R/(\mathrm{d}A_1 + \mathrm{d}B_1) & \text{if } i = 0 \\ (A_1 \times_R B_1)/(\mathrm{d}A_2 \oplus \mathrm{d}B_2) & \text{if } i = 1 \\ \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) & \text{if } i \geq 2 \end{cases}$$

Proposition 5.2. *Let A and B be two R-complexes centered at R. Then we have*

$$\mathrm{S}_R(A \vee B) = \mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B).$$

Proof. In terms of the underlying graded R-algebras, we have

$$\begin{aligned}
\mathrm{S}_R(A \vee B) &= \mathrm{Sym}_R(A_+ \oplus B_+) \\
&= \mathrm{Sym}_R(A_+) \otimes_R \mathrm{Sym}_R(B) \\
&= \mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B).
\end{aligned}$$

It is easy to check that the differential of $\mathrm{S}_R(A \vee B)$ is carried over to the differential of $\mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B)$ under this isomorphism (we write equality here because $\mathrm{S}_R(A) \otimes_R \mathrm{S}_R(B)$ satisfies the universal mapping property of the symmetric DG R-algebra of $A \vee B$. □

5.3 Presentation of the Maximal Associative Quotient

Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Equip A with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra. In particular, note that if $a_1, a_2 \in A_1$, then

$$a_1 a_2 \in S_2^2, \quad a_1 \star a_2 \in S_2^1, \quad \text{and} \quad [a_1, a_2] \in S_2,$$

where $[a_1, a_2] = a_1 \star a_2 - a_1 a_2$ is the multiplicator of the inclusion map $\iota\colon A \hookrightarrow S$ evaluated at $(a_1, a_2) \in A^2$. Let $\mathfrak{s} = \mathfrak{s}(\mu)$ be the S-ideal generated by all such multiplicators, so

$$\mathfrak{s} = \mathrm{span}_S\{[a_1, a_2] \mid a_1, a_2 \in A\}.$$

Also let $\pi\colon S \to S/\mathfrak{s}$ and $\pi^{\mathrm{as}}\colon A \twoheadrightarrow A^{\mathrm{as}}$ denote the canonical quotient maps. The universal mapping property of the symmetric DG algebra of A implies $\pi^{\mathrm{as}}\colon A \twoheadrightarrow A^{\mathrm{as}}$ extends uniquely to a DG algebra homomorphism $S \twoheadrightarrow A^{\mathrm{as}}$

which we again denote by π^{as}. We let $S^{\geq 2} = S/A$ be the R-complex whose underlying graded R-module is $S^{\geq 2}$ and whose differential $\mathrm{d}^{\geq 2}$ is defined by

$$\mathrm{d}^{\geq 2}|_{S^m} = \begin{cases} \eth|_{S^2} & \text{if } m = 2 \\ \mathrm{d}|_{S^m} & \text{if } m > 2. \end{cases}$$

We also let $\rho\colon S \twoheadrightarrow S/A = S^{\geq 2}$ be the canonical quotient map. We now present the third main theorem from the introduction.

Theorem 5.3. *With the notation as above, we have*

$$A^{\mathrm{as}} = \operatorname{coker}(\mathfrak{s} \hookrightarrow S) = S/\mathfrak{s}$$

More specifically, there is a unique isomorphism $A^{\mathrm{as}} \to S/\mathfrak{s}$ of DG *S-algebras (thus we are justified in writing $\pi\colon S \to A^{\mathrm{as}}$ to denote both $\pi^{\mathrm{as}}\colon S \to A^{\mathrm{as}}$ and $\pi\colon S \to S/\mathfrak{s}$ in order to simplify notation) In particular, this implies*

$$\langle A \rangle = A \cap \mathfrak{s} = \mathfrak{s}^{\leq 1} = \ker(\mathfrak{s} \to S^{\geq 2})$$

Thus we have the following canonically defined hexagonal-shaped diagram of R-complexes which is exact everywhere (in every direction) and which is natural in $A = (A, \mathrm{d}, \mu)$:

$$\begin{tikzcd}
 & S^{\geq 2} \arrow[r] & 0 \\
\mathfrak{s} \arrow[ur] \arrow[r, hook, "i"] & S \arrow[u, two heads, "\rho"] \arrow[r, two heads, "\pi"] & A^{\mathrm{as}} \arrow[u] \\
\mathfrak{s}^{\leq 1} \arrow[u, hook] \arrow[r, hook] & A \arrow[u, hook, "\iota"] \arrow[ur] &
\end{tikzcd} \tag{45}$$

where the blue arrows are DG *S-module homorphisms, where the green arrows are chain maps as R-complexes, and where the red arrows are* MDG *A-module homomorphisms. In particular, if $\mathrm{H}_+(A) = 0$, then $\mathrm{H}_+(S) = \mathrm{H}(S^{\geq 2})$ and we obtain a canonically defined sequence of graded $\mathrm{H}(S)$-modules:*

$$\mathrm{H}_+(\mathfrak{s}) \longrightarrow \mathrm{H}_+(S) \longrightarrow \mathrm{H}_+(A^{\mathrm{as}}) \longrightarrow \Sigma\mathrm{H}(\mathfrak{s}) \longrightarrow \Sigma\mathrm{H}(S) \tag{46}$$

which is natural in $A = (A, \mathrm{d}, \mu)$.

Remark 7. By "natural in $A = (A, \mathrm{d}, \mu)$" we mean that if R' is an R-algebra and $\varphi\colon A \to A'$ is an MDG R-algebra homomorphism where $A' = (A', \mathrm{d}', \mu')$ is an MDG R'-algebra centered at R', then we obtain canonically defined maps $S \to S'$ and $\mathfrak{s} \to \mathfrak{s}'$, where we set $S' = \mathrm{S}_{R'}(A')$ and $\mathfrak{s}' = \mathfrak{s}(\mu')$, which induces a map of hexagonal-shaped diagrams in which everything commutes. For instance, if $\mathrm{H}_+(A) = 0 = \mathrm{H}_+(A')$, then then we have a commutative diagram of graded $\mathrm{H}(\mathrm{S}')$-modules of the form:

$$\begin{tikzcd}
\mathrm{H}_+(\mathfrak{s}) \arrow[r] \arrow[d] & \mathrm{H}_+(S) \arrow[r] \arrow[d] & \mathrm{H}_+(A^{\mathrm{as}}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(\mathfrak{s}) \arrow[r] \arrow[d] & \Sigma\mathrm{H}(S) \arrow[d] \\
\mathrm{H}_+(\mathfrak{s}') \arrow[r] & \mathrm{H}_+(S') \arrow[r] & \mathrm{H}_+((A')^{\mathrm{as}}) \arrow[r] & \Sigma\mathrm{H}(\mathfrak{s}') \arrow[r] & \Sigma\mathrm{H}(S')
\end{tikzcd} \tag{47}$$

Proof. Observe that $\pi^{\mathrm{as}}\colon S \twoheadrightarrow A^{\mathrm{as}}$ satifies

$$\begin{aligned} \pi^{\mathrm{as}}[a_1, a_2] &= \pi^{\mathrm{as}}(a_1 \star a_2 - a_1 a_2) \\ &= \pi^{\mathrm{as}}(a_1 \star a_2) - \pi^{\mathrm{as}}(a_1 a_2) \\ &= \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2) - \pi^{\mathrm{as}}(a_1) \star \pi^{\mathrm{as}}(a_2) \\ &= 0. \end{aligned}$$

Thus the universal mapping property of the quotient $S/\mathfrak{s} = \operatorname{coker}(\mathfrak{s} \hookrightarrow S)$ implies there is a unique DG algebra homomorphism $\overline{\pi}^{\mathrm{as}}\colon S/\mathfrak{s} \to A^{\mathrm{as}}$ such that

$$\overline{\pi}^{\mathrm{as}} \circ \pi = \pi^{\mathrm{as}}.$$

Similarly, note that the composite $\pi \circ \iota\colon A \to S/\mathfrak{s}$ is an MDG algebra homomorphism which is surjective. Indeed, if $a_1 \cdots a_m$ is a homogeneous elementary tensor in S^m, then we have

$$a_1 a_2 a_3 \cdots a_m = ((\cdots(a_1 \star a_2) \star a_3) \star \cdots) \star a_m$$

in $S/\mathfrak{s}$. Thus every element in $S/\mathfrak{s}$ can be represented by an element in $A = S^1$ which implies $\pi\iota\colon A \twoheadrightarrow S/\mathfrak{s}$ is surjective as claimed. In particular, since $S/\mathfrak{s}$ is associative, it follows from the universal mapping property of the maximal associative quotient of A that there is a unique DG algebra homomorphism $\overline{\pi}\colon A^{\mathrm{as}} \to S/\mathfrak{s}$ such that

$$\pi \circ \iota = \overline{\pi} \circ \pi^{\mathrm{as}}.$$

Combining all of this together, we have a commutative diagram of MDG S-modules:

$$\begin{tikzcd}
S \arrow[r, two heads, "\pi"] \arrow[dr, dashed, "\pi^{\mathrm{as}}"] & S/\mathfrak{s} \arrow[d, dashed, shift left, "\overline{\pi}^{\mathrm{as}}"] \\
A \arrow[u, hook, "\iota"] \arrow[r, two heads, "\pi^{\mathrm{as}}"'] & A^{\mathrm{as}} \arrow[u, dashed, shift left, "\overline{\pi}"]
\end{tikzcd}$$

where the dashed arrows indicates uniqueness. □

Corollary 2. *Continuing with the notation as above, assume further that A is associative, so $A = A^{\mathrm{as}}$. Then the canonical map $\mathfrak{s} \to S^{\geq 2}$ defined on multiplicators by*

$$[a_1, a_2] \mapsto a_1 a_2$$

is an isomorphism of R-complexes. Let $\theta\colon S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s} \hookrightarrow S$ be the composite map where $S^{\geq 2} \xrightarrow{\simeq} \mathfrak{s}$ is the inverse isomorphism of the canonical map $\mathfrak{s} \to S^{\geq 2}$. We obtain a short exact sequence of R-complexes

$$0 \longrightarrow S^{\geq 2} \xrightarrow{\ \theta\ } S \xrightarrow{\ \pi\ } A \longrightarrow 0 \tag{48}$$

which is split by the inclusion map $\iota\colon A \to S$. Similarly, the short exact sequence of R-complexes

$$0 \longrightarrow A \xrightarrow{\ \iota\ } S \xrightarrow{\ \rho\ } S^{\geq 2} \longrightarrow 0 \tag{49}$$

is split by $\theta\colon S^{\geq 2} \to S$.

Corollary 3. *Let A be an R-complex centered at R and let $S = \mathrm{S}_R(A)$ be the symmetric DG algebra of A. Then a necessary condition for A to have a DG algebra structure is that the canonical short exact sequence of R-complexes*

$$0 \longrightarrow A \xrightarrow{\ \iota\ } S \xrightarrow{\ \rho\ } S^{\geq 2} \longrightarrow 0 \tag{50}$$

is split.

5.4 Homology of the Symmetric DG Algebra

Example 5.3. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2 \rangle$, and let F be the minimal free resolution of R/I over R as in Example (2.7). The homology of the symmetric DG algebra $S = \mathrm{S}_R(F)$ is complicated to describe, but it "knows" about multiplications on F. For instance, the polynomials below each represent a distinct elements which are linearly independent in $\mathrm{H}_2(S)$:

$$\begin{aligned}
f_{12} &= e_1e_2 - e_{12} \\
f_{13} &= e_1e_3 - e_{13} \\
f_{14} &= e_1e_4 - xe_{14} \\
f_{15} &= e_1e_5 - yz^2e_{14} - xe_{45} \\
f_{23} &= e_2e_3 - we_{23}.
\end{aligned}$$

More generally, for each $1 \leq i < j \leq 5$, the polynomial $f_{ij} = e_ie_j - e_i \star e_j$ represents another distinct element in homology and the collection $\{f_{ij}\}$ are all linearly independent in $\mathrm{H}_2(S)$. Note that $\mathrm{d}(e_1e_{14}) = yf_{14}$, so $y \in \mathrm{Ann}(\overline{f}_{14})$. Similar arguments show that $\mathrm{Ann}(\overline{f}_{14}) = \langle x, y, zw, w^2 \rangle = I : x$. On the other hand, one can show that $\mathrm{Ann}(\overline{f}_{12}) = I$. Furthermore, if we set $f_{1,23} = e_1e_{23} - e_{123}$, then we have $\mathrm{d}(f_{1,23}) = zf_{12} - wf_{13}$, so $z\overline{f}_{12} = w\overline{f}_{13}$. Finally, note that

$$f_{12}^2 = x^2e_{12}^2 - 2xe_1e_2e_{12} = \mathrm{d}(e_1e_{12}^2) \quad \text{and} \quad f_{13}^2 = e_{13}^2 - 2e_1e_3e_{13}.$$

In particular, $\overline{f}_{12}^2 = 0$ but $\overline{f}_{13}^2 \neq 0$ since the coefficient for e_{13}^2 is not in $\mathfrak{m} = \langle x, y, z, w \rangle$. More generally one can show that $\overline{f}_{13}^n \neq 0$ for all $n \geq 1$.

Example 5.4. Let us revisit Example (5.1) where $R = \Bbbk[x,y]$, $\boldsymbol{m} = x^2, xy$, F is the Taylor resolution of $R/\boldsymbol{m}$ over R, and S is the symmetric DG R-algebra of F. One can show that the homology of S is given by

$$\mathrm{H}_i(S) = \begin{cases} R/\langle k, x\rangle & \text{if } i = 2k+1 \text{ where } k \geq 1 \\ R/\langle x, y\rangle & \text{if } i = 2k \text{ where } k \geq 1 \\ R/\langle x^2, xy\rangle & \text{if } i = 0 \end{cases}$$

Furthermore, one can show that the underlying graded $\overline{R}$-algebra structure of $\mathrm{H}(S)$ looks like

$$\mathrm{H}(S) = (R/\boldsymbol{m})[\{f_{2k}, g_{2k+1} \mid k \geq 1\}]/\langle \{xf_{2k}, yf_{2k}, xg_{2k+1}, kg_{2k+1}, f_{2k}f_{2m}, f_{2k}g_{2m+1}, g_{2k+1}g_{2m+1} \mid k, m \geq 1\}\rangle,$$

where $f_{2k} = (e_1e_2 - xe_{12})^k/x^{k-1}$ and where $g_{2k+1} = \mathrm{d}(e_{12}^k)$ for each $k \geq 1$. On the other hand, let us treat e_{12} as a divided variable. Then with respect to the ordered bases $e_{12}^{(k)}, e_1e_2e_{12}^{(k-1)}$ for D_{2k} and $e_1e_{12}^{(k-1)}, e_2e_{12}^{(k-1)}$ for D_{2k+1}, the matrix representation of the differential looks like:

$$[\mathrm{d}_{2k}] = \begin{pmatrix} -y & -xy \\ x & x^2 \end{pmatrix} \quad \text{and} \quad [\mathrm{d}_{2k+1}] = \begin{pmatrix} x^2 & xy \\ -x & -y \end{pmatrix}.$$

In this case, one has $p_{2k} = xe_{12}^{(k)} - e_1e_2e_{12}^{(k-1)}$ and $q_{2k-1} = ye_1e_{12}^{(k-1)} - xe_2e_{12}^{(k-1)} = \mathrm{d}(e_{12}^{(k)})$ generating their respective kernels.

$$k!p_{2k} = f_{2k} \quad \text{and} \quad (k-1)!q_{2k-1} = g_{2k-1}.$$

In particular, for the divided algebra we have

$$\mathrm{H}_i(D) = \begin{cases} 0 & \text{if } i = 2k+1 \text{ where } k \geq 1 \\ R/\langle x, y\rangle & \text{if } i = 2k \text{ where } k \geq 1 \\ R/\langle x^2, xy\rangle & \text{if } i = 0 \end{cases}$$

and the underlying graded $\overline{R}$-algebra of $\mathrm{H}(D)$ looks like:

$$\mathrm{H}(D) = (R/\boldsymbol{m})[\{p_{2k} \mid k \geq 1\}]/\langle \{xp_{2k}, yp_{2k}, p_{2k}p_{2m} \mid k, m \geq 1\}$$

Example 5.5. Let $R = \Bbbk[x,y]$, let $I = \langle x, y\rangle$, and let F be Koszul resolution of $\Bbbk = R/I$. We write down the homogeneous components of F as a graded R-module as well as how the differential acts on the homogeneous basis below:

$$\begin{aligned} F_0 &= R & \mathrm{d}e_1 &= x \\ F_1 &= Re_1 + Re_2 & \mathrm{d}e_2 &= y \\ F_2 &= Re_{12}, & \mathrm{d}e_{12} &= xe_2 - ye_1, \end{aligned}$$

Let $S = \mathrm{S}_R(F)$ denote the symmetric DG R-algebra of F. The homogeneous components of S as a graded R-module (with respect to homological degree) looks the same as the previous example:

$$\begin{aligned} S_0 &= R \\ S_1 &= Re_1 + Re_2 \\ S_2 &= Re_{12} + Re_1e_2 \\ S_3 &= Re_1e_{12} + Re_2e_{12} \\ S_4 &= Re_{12}^2 + Re_1e_2e_{12} \\ &\vdots \\ S_{2k-1} &= Re_1e_{12}^{k-1} + Re_2^{k-1} \\ S_{2k} &= Re_{12}^k + Re_1e_2e_{12}^{k-1} \\ S_{2k+1} &= Re_1e_{12}^k + Re_2e_{12}^k \\ &\vdots \end{aligned}$$

where $2k \geq 1$. One can show that the homology of S is given by:

$$\mathrm{H}_i(S) = \begin{cases} 0 & \text{if } i = 2k+1 \text{ where } k \geq 0 \\ R/\langle x, y\rangle & \text{if } i = 2k \text{ where } k \geq 0 \end{cases}$$

Furthermore, one can show that the underlying graded $\Bbbk$-algebra structure of $\mathrm{H}(S)$ is just $\Bbbk[f_2]$ where $f_2 = e_{12} - e_1e_2$.

Proposition 5.3. *Let $R = (R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $F = (F, \mathrm{d})$ be the minimal free resolution of R/I over R where $I \subseteq \mathfrak{m}$. Equip F with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra and let $S = \mathrm{S}_R(F)$ be the symmetric DG R-algebra of F. Finally let*

$$f := [a_1, a_2] = a_1 a_2 - a_1 \star a_2,$$

where $a_1, a_2 \in F_1 \backslash \mathfrak{m} F_1$. Then f represents a nonzero element in $\mathrm{H}_2(S)$.

Proof. Clearly we have $\mathrm{d} f = 0$. Suppose that $\mathrm{d} g = f$ where $g \in S_3$. Let g^2 and g^3 be the components of g that lie in S_3^2 and S_3^3 respectively. Then in particular, we must have

$$a_1 a_2 = \partial g^3 + \eth g^2. \tag{51}$$

However this is a contradiction as minimality of F implies that the RHS of (51) lies in $\mathfrak{m} S$ however the LHS of (51) does not lie in $\mathfrak{m} S$ as $a_1, a_2 \notin \mathfrak{m} F$. $\square$

5.5 Symmetric Powers of Chain Complexes

In this subsection, we describe a construction given by Tchernev (in [Tch95]) and explain how it is related to our construction. In particular, let X be an R-complex. We construct the *non-unital* symmetric DG algebra of X over R, denoted $\mathrm{C}_R(X)$ as follows: we begin with the non-unital tensor DG algebra of X over R, given by

$$\mathrm{U}_R(X) = \bigoplus_{n=1}^{\infty} X^{\otimes n}$$

where the tensor product is taken as R-complexes. Just as before, an elementary tensor in $U = \mathrm{U}_R(A)$ is denoted $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_n$ where $x_1, \dots, x_n \in X$ and $n \geq 1$, and the differential of U is denoted by d again to simplify notation and is defined on $\boldsymbol{x}$ by

$$\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{n} (-1)^{|x_1| + \cdots + |x_{j-1}|} x_1 \otimes \cdots \otimes \mathrm{d}x_j \otimes \cdots \otimes x_n.$$

We say $\boldsymbol{x}$ is a homogeneous elementary tensor if each x_i is a homogeneous element in X. What is different this time is that we equip $U = \mathrm{U}_R(X)$ with a different bi-graded structure; namely we set

$$|\boldsymbol{x}| = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \deg \boldsymbol{x} = n.$$

Thus we make no distinction on whether or not $x_i \in X_0$ or $x_i \in X_{<0}$. With $|\cdot|$ and deg defined as above, we observe that U admits a bi-graded decomposition:

$$U = \bigoplus_{i \in \mathbb{Z}} U_i = \bigoplus_{n \geq 1} U^n = \bigoplus_{i,n} U_i^n,$$

where the component U_i^n consists of all finite R-linear combinations of homogeneous elementary tensors $\boldsymbol{x} \in U$ such that $|\boldsymbol{x}| = i$ and $\deg \boldsymbol{x} = n$. We equip U with an associative (but not commutative nor unital) bi-graded R-bilinear multiplication which is defined on homogeneous elementary tensors by $(\boldsymbol{x}, \boldsymbol{x}') \mapsto \boldsymbol{x} \otimes \boldsymbol{x}'$ and is extended R-bilinearly everywhere else. This multiplication is easily seen to satisfy Leibniz rule, however note that U is not unital under this multiplication since $(1,1) \mapsto 1 \otimes 1 \neq 1$ (hence why we call this the *non-unital* tensor DG algebra).

Next let $\mathfrak{c} = \mathfrak{c}(X)$ be the U-ideal generated by all elements of the form

$$[x_1, x_2]_\sigma := (-1)^{|x_1||x_2|} x_2 \otimes x_1 - x_1 \otimes x_2 \quad \text{and} \quad [x]_\tau := x \otimes x,$$

where $x, x_1, x_2 \in X$ are homogeneous and $|x|$ is odd. We then define the **non-unital symmetric DG algebra** of X over R to be the quotient

$$\mathrm{C}_R(X) := U/\mathfrak{c}.$$

Since the generators of $\mathfrak{c}$ are homogeneous with respect to both homological and total degree, we see that $C = \mathrm{C}_R(X)$ inherits a bi-graded structure from U. In particular, if X is a positive R-complex (meaning $X_i = 0$ for all $i < 0$), then one has $C_0^n = \mathrm{Sym}_R^n(X_0)$. In general, we call C^n the n**th symmetric power** of X. The second symmetric power and its properties were studied in [FST08]. The next proposition helps clarify how our construction is related to Tchernev's construction:

Proposition 5.4. *Let A be an R-complex centered at R. Denote $S = \mathrm{S}_R(A)$ and $C = \mathrm{C}_R(A)$. We have $S^{\leq n} \cong C^n$ as R-complexes.*

Proof. Define $\varphi_h \colon S^{\leq n} \to C^n$, called **homogenization**, as follows: let $f \in S^{\leq n}$ and express it as $f = \sum_{k=0}^{n} f^k$ where f^k is the total degree k component of f. We set

$$\varphi_h(f) = 1^{n-1} \otimes f^0 + \sum_{k=1}^{n} 1^{\otimes(n-k)} \otimes f^k.$$

Conversely, define $\varphi_d \colon C^n \to S^{\leq n}$, called **dehomogenization**, as follows: we set

$$\varphi_d(1^{\otimes k} \otimes \boldsymbol{a}) = \boldsymbol{a}$$

where $\boldsymbol{a} \in A_+^{\otimes(n-k)}$ is a homogeneous elementary tensor. We extend φ_d everywhere else R-linearly. It is straightforward to check that both φ_h and φ_d are chain maps and are inverse to each other. $\square$

Let X be an R-complex. Denote $C = \mathrm{C}_R(X)$, $\mathfrak{c} = \mathfrak{c}(X)$, and $U = \mathrm{U}_R(X)$. There's an alternative description of C^n which is often useful. Let $\sigma = (ij)$ be a transposition in the symmetric group Σ_n and let $\boldsymbol{x} = x_1 \otimes \cdots \otimes x_n$ be a homogeneous elementary tensor in U. We set

$$\sigma \boldsymbol{x} = \begin{cases} 0 & \text{if } x_i = x_j \text{ and } |x_i| \text{ is odd} \\ (-1)^{|x_i||x_j|} x_1 \otimes \cdots x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n & \text{else.} \end{cases} \tag{52}$$

Then (52) extends to an action of the symmetric group Σ_n on U^n. In other words, U^n has the structure of an $R[\Sigma_n]$-module. With this understood, we have $C^n = (U^n)_{\Sigma_n}$. If R contains $\mathbb{Q}$, then the short exact sequence of R-complexes

$$0 \longrightarrow \mathfrak{c} \longrightarrow U \longrightarrow C \longrightarrow 0 \tag{53}$$

is split exact with splitting map $C \to U$ defined on homogeneous elementary products by

$$x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(x_1 \otimes \cdots \otimes x_n).$$

In particular, we may identify C^n with the R-subcomplex of U^n which is fixed by Σ_n in this case.

Theorem 5.4. *Assume that $\mathbb{Q} \subseteq R$. Let $\varphi, \psi \colon X \to X'$ be chain maps of R-complexes. Denote $C = \mathrm{C}_R(X)$, $C' = \mathrm{C}_R(X')$, $U = \mathrm{U}_R(X)$, and $U' = \mathrm{U}_R(X')$, and identify C and C' with the R-subcomplexes of U and U' fixed by the symmetric groups. If φ is homotopic to ψ, then $\varphi^{\otimes n}$ is homotopic to $\psi^{\otimes n}$ for each n. Moreover, we can choose a homotopy $h^n \colon U^n \to U'^n$ from $\varphi^{\otimes n}$ to $\psi^{\otimes n}$ which restricts to a homotopy $h^n|_C \colon C^n \to C'^n$ from $\varphi^{\otimes n}|_C$ to $\psi^{\otimes}|_C$.*

Proof. Let h be a homotopy from φ to ψ. For $n = 1$, we set $h^1 = h$. The case where $n = 2$ was shown in [FST08]. More generally, we set

$$h^n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma \left(\sum_{k=1}^{n-1} (\varphi^{\otimes(n-k)} \otimes h \otimes \psi^{\otimes k}) \right).$$

One checks that h^n is a homotopy from $\varphi^{\otimes n}$ to $\psi^{\otimes n}$ and by construction is restricts to a map from C^n to C'^n. $\square$

Corollary 4. *Assume that $\mathbb{Q} \subseteq R$. Let $\varphi, \psi \colon A \to A'$ be chain maps of R-complexes centered at R. Denote $S = \mathrm{S}_R(A)$ and $S' = \mathrm{S}_R(A')$, and let $\widetilde{\varphi}, \widetilde{\psi} \colon S \to S'$ be the lifts of φ and ψ from the universal mapping property. If φ is homotopic to ψ, then $\widetilde{\varphi}$ is homotopic to $\widetilde{\psi}$.*

5.6 The Symmetric DG Algebra of a Finite Free Complex over an Integral Domain

Throughout this subsection, we assume that R is an integral domain with quotient field K. Let F be an R-complex centered at R such that the underlying graded R-module of F is finite and free. Let $e_1, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $i < j$, then $|e_i| \leq |e_j|$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $R[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $R[\boldsymbol{e}]$. The reason we do not want elements of odd degree to square to zero is because we will want to calculate Gröbner bases in $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is much simpler when we do not have any zerodivisors. In any case, one recovers the symmetric DG R-algebra of F as below:

$$R[\boldsymbol{e}] / \langle \{ e_i^2 \mid |e_i| \text{ is odd} \} \rangle \simeq \mathrm{S}_R(F).$$

Finally, equip F with a multiplication μ giving it the structure of an MDG algebra. Our goal is to compute the maximal associative quotient of F using the presentation given in Theorem (5.3) as well as the theory of Gröbner bases in $K[\boldsymbol{e}]$.

5.6.1 Monomials and Monomial Orderings

Before we can do this, we first need to introduce some notation for Gröbner basis applications in $K[\boldsymbol{e}]$. Our notation mostly follows [BE77] and [Mot10] however we introduce some of our own notation as well. A **monomial** in $K[\boldsymbol{e}]$ is an element of the form

$$e^{\boldsymbol{\alpha}} = e_1^{\alpha_1} \cdots e_n^{\alpha_n} \tag{54}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called the **multidegree** of $e^{\boldsymbol{\alpha}}$ and is denoted $\mathrm{multideg}(e^{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$. Similarly we define its **total degree**, denoted $\deg(e^{\boldsymbol{\alpha}})$, and its **homological degree**, denoted $|e^{\boldsymbol{\alpha}}|$, by

$$\deg(e^{\boldsymbol{\alpha}}) = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad |e^{\boldsymbol{\alpha}}| = \sum_{i=1}^{n} \alpha_i |e_i|.$$

By convention we set $e^{\mathbf{0}} = 1$ where $\mathbf{0} = (0, \dots, 0)$ is the zero vector in $\mathbb{N}^n$. Note how the ordering in (54) matters. In particular, if $i < j$ and both $|e_i|$ and $|e_j|$ are odd, then $e_j e_i$ is not a monomial in $K[\boldsymbol{e}]$ since it can be expressed as a non-trivial coefficient times a monomial:

$$e_j e_i = -e_i e_j.$$

On the other hand, if one of the e_i or e_j is even, then $e_j e_i$ is a monomial in $K[\boldsymbol{e}]$ since $e_j e_i = e_i e_j$. We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ (the notation for this monomial ordering in Singular is Wp(w)). More specifically, given two monomials $e^{\boldsymbol{\alpha}}$ and $e^{\boldsymbol{\beta}}$ in $K[\boldsymbol{e}]$, we say $e^{\boldsymbol{\beta}} > e^{\boldsymbol{\alpha}}$ if either

1. $|e^{\boldsymbol{\beta}}| > |e^{\boldsymbol{\alpha}}|$ or;
2. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and $\beta_1 > \alpha_1$ or;
3. $|e^{\boldsymbol{\beta}}| = |e^{\boldsymbol{\alpha}}|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Given a nonzero polynoimal $f \in K[\boldsymbol{e}]$, there exists unique $c_1, \dots, c_m \in K \backslash \{0\}$ and unique $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ where $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ for all $1 \leq i < j \leq m$ such that

$$f = c_1 e^{\boldsymbol{\alpha}_1} + \cdots + c_m e^{\boldsymbol{\alpha}_m} = \sum c_i e^{\boldsymbol{\alpha}_i} \tag{55}$$

The $c_i e^{\boldsymbol{\alpha}_i}$ in (55) are called the **terms** of f and the $e^{\boldsymbol{\alpha}_i}$ in (55) are called the **monomials** of f. By reindexing the $\boldsymbol{\alpha}_i$ if necessary, we may assume that $e^{\boldsymbol{\alpha}_1} > \cdots > e^{\boldsymbol{\alpha}_m}$. In this case, we call $c_1 e^{\boldsymbol{\alpha}_1}$ the **lead term** of f, we call $e^{\boldsymbol{\alpha}_1}$ the **lead monomial** of f, and we call c_1 the **lead coefficient** of f. We denote these, respectively, by

$$\mathrm{LT}(f) = c_1 e^{\boldsymbol{\alpha}_1}, \quad \mathrm{LM}(f) = e^{\boldsymbol{\alpha}_1}, \quad \text{and} \quad \mathrm{LC}(f) = c_1.$$

The **multidegree** of f is defined to be the multidegree of its lead monomial $e^{\boldsymbol{\alpha}_1}$ and is denoted $\mathrm{multideg}(f) = \boldsymbol{\alpha}_1$. The **total degree** of f is defined to be the maximum of the total degrees of its monomials and is denoted

$$\deg(f) = \max_{1 \leq i \leq m} \{\deg(e^{\boldsymbol{\alpha}_i})\}.$$

We say f is **homogeneous** of homological degree i if each of its monomials is homogeneous of homological degree i. In this case, we say f has **homological degree** i and we denote this by $|f| = i$.

Lemma 5.5. *For each $1 \leq i \leq j \leq n$, let $f_{ij} = e_i e_j - e_i \star e_j$. We have*

$$\mathrm{LT}(f_{ij}) = e_i e_j.$$

Proof. If $e_i \star e_j = 0$, then this is clear, otherwise let e_k be a monomial of $e_i \star e_j$. Since $\star$ respects homological degree, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > \max\{|e_i|, |e_j|\}$ since $|e_i|, |e_j| \geq 1$. This implies $k > \max\{i, j\}$ by our assumption on the ordering of $e_1, \dots, e_n$. Therefore since $|e_i e_j| = |e_k|$ and $k > \max\{i, j\}$, we see that $e_i e_j > e_k$. □

5.6.2 Gröbner Basis Calculations

Our goal is to use the theory of Gröbner bases to help us calculate

$$F^{\mathrm{as}} = \mathrm{S}_R(F)/\mathfrak{s}(\mu) \simeq R[\boldsymbol{e}]/\langle \{f_{ij}\} \rangle,$$

where $f_{ij} \in R[\boldsymbol{e}]$ are defined by

$$f_{ij} = e_i e_j - e_i \star e_j = e_i e_j - \sum_k c_{ij}^k e_k,$$

where the $c_{ij}^k \in R$ are the entries of the matrix representation of μ with respect to the ordered homogeneous basis $e_1, \dots, e_n$. In order to do this, we work over K instead of R sinc that is where the theory of Gröbner bases works best. Thus we wish to calculate:

$$F_K^{\text{as}} := F^{\text{as}} \otimes_R K \simeq K[\boldsymbol{e}]/\langle \{f_{ij}\} \rangle.$$

To this end, let $\mathcal{F} = \{f_{ij} \mid 1 \leq i, j \leq n\}$ and let $\mathfrak{a}$ be the $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. We wish to construct a left Gröbner basis for $\mathfrak{a}$ (which will turn out to be a two-sided Gröbner basis) via Buchberger's algorithm using the monomial ordering described above. Suppose f, g are two nonzero polynomials in $K[\boldsymbol{e}]$ with $\mathrm{LT}(f) = ce^{\boldsymbol{\alpha}}$ and $\mathrm{LT}(g) = de^{\boldsymbol{\beta}}$. Set $\boldsymbol{\gamma} = \mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and define the left **S-polynomial** of f and g to be

$$\mathrm{S}(f, g) = e^{\gamma - \boldsymbol{\alpha}} f \pm (c/d) e^{\gamma - \boldsymbol{\beta}} g \tag{56}$$

where the $\pm$ in (56) is chosen to be $+$ or $-$ depending on which sign will cancel out the lead terms. We begin Buchberger's algorithm by calculating the S-polynomials of all pairs of polynomials in $\mathcal{F}$. In other words, we calculate all S-polynomials of the form $\mathrm{S}(f_{kl}, f_{ij})$ where $1 \leq i, j, k, l \leq n$. Note that if $k > l$, then $f_{lk} = (-1)^{|e_k||e_l|} f_{kl}$ implies

$$\mathrm{S}(f_{lk}, f_{ij}) = (-1)^{|e_k||e_l|} \mathrm{S}(f_{kl}, f_{ij}) = \pm \mathrm{S}(f_{ij}, f_{lk}),$$

where the last equality follows from the fact that the lead coefficient of f_{ij} and f_{lk} is ± 1. Thus we may assume that $j \geq i$ and $l \geq k \geq i$. Obviously we have $\mathrm{S}(f_{ij}, f_{ij}) = 0$ for each i, j, however something interesting happens when we calculate the S-polynomial of f_{jk} and f_{ij} where $j > i$ and then divide this by $\mathcal{F}$ (where division by $\mathcal{F}$ means taking the left normal form of $\mathrm{S}(f_{jk}, f_{ij})$ with respect to $\mathcal{F}$ using the left normal form described in [GP02]). In particular, we obtain the associator $[e_i, e_j, e_k]$! Indeed, we have

$$\begin{aligned}
\mathrm{S}(f_{jk}, f_{ij}) &= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
&= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
&= \sum_l c_{ij}^l e_l e_k - \sum_l c_{jk}^l e_i e_l \\
&\to \sum_l c_{ij}^l e_l \star e_k - \sum_l c_{jk}^l e_i \star e_l \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k],
\end{aligned}$$

where in the fourth line we did division by $\mathcal{F}$ (note that if $[e_i, e_j, e_k] \neq 0$, then $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$). Next suppose that $j > i$, $l > k$, and $j \neq k$. Then we have

$$\begin{aligned}
\mathrm{S}(f_{kl}, f_{ij}) &= e_i e_j f_{kl} - f_{ij} e_k e_l \\
&= (e_i \star e_j) e_k e_l - e_i e_j (e_k \star e_l) \\
&\to (e_i \star e_j) \star (e_k \star e_l) - (e_i \star e_l) \star (e_k \star e_l) \\
&= 0
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Next, suppose that

$$f = c e_k + c' e_{k'} + \cdots + c'' e_{k''} \in \langle F \rangle$$

where $c, c', c'' \in R$ with $c \neq 0$ and where $\mathrm{LM}(f) = e_k$. Then we have

$$\begin{aligned}
\mathrm{S}(f, f_{jk}) &= e_j f - c f_{jk} \\
&= c' e_j e_{k'} + \cdots + c'' e_j e_{k''} + c e_j \star e_k \\
&\to c' e_j \star e_{k'} + \cdots + c'' e_j \star e_{k''} + c e_j \star e_k \\
&= e_j \star (c e_k + c' e_{k'} + \cdots + c'' e_{k''}) \\
&= e_j \star f \\
&\in \langle F \rangle
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Similarly, if $i \neq k \neq j$, then we have

$$\begin{aligned}
\mathrm{S}(f, f_{ij}) &= e_i e_j f - c f_{ij} e_k \\
&= c'(e_i e_j) e_{k'} + \cdots + c''(e_i e_j) e_{k''} + c(e_i \star e_j) e_k \\
&\to c'(e_i \star e_j) \star e_{k'} + \cdots + c''(e_i \star e_j) \star e_{k''} + c(e_i \star e_j) \star e_k \\
&= (e_i \star e_j) \star (c e_k + c' e_{k'} + \cdots + c'' e_{k''}) \\
&= (e_i \star e_j) \star f \\
&\in \langle F \rangle.
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Finally suppose that

$$g = de_m + d'e_{m'} + \cdots + d''e_{m''} \in \langle F \rangle$$

where $d, d', d'' \in R$ with $d \neq 0$ and where $\mathrm{LM}(g) = e_m$. If $k = m$, then we have

$$d\mathrm{S}(f, g) = cf - dg \in \langle F \rangle.$$

On the other hand, if $k \neq m$, then we have

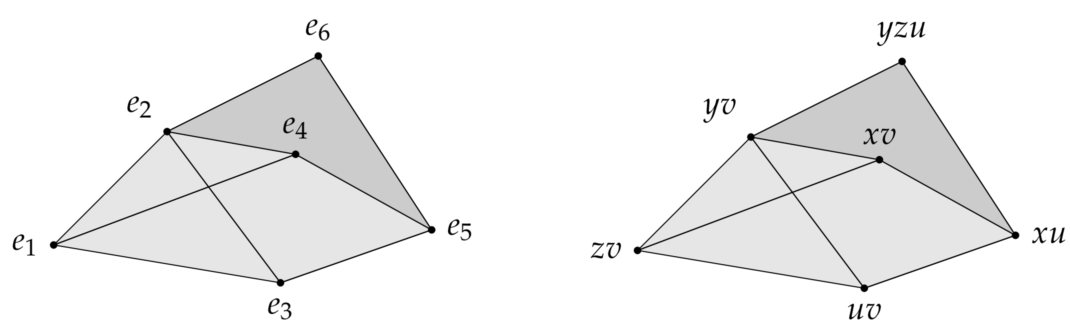
$$\begin{aligned} d\mathrm{S}(f, g) &= de_m f - cge_k \\ &= dc'e_m e_{k'} + \cdots + dc''e_m e_{k''} - cd'e_{m'}e_k - \cdots - cd''e_{m''}e_k \\ &\to dc'e_m \star e_{k'} + \cdots + dc''e_m \star e_{k''} - cd'e_{m'} \star e_k - \cdots - cd''e_{m''} \star e_k \\ &= de_m \star (c'e_{k'} + \cdots + c''e_{k''}) - c(d'e_{m'} + \cdots + d''e_{m''}) \star e_k \\ &= de_m \star (f - ce_k) - c(g - de_m) \star e_k \\ &= de_m \star f + cg \star e_k - dce_m \star e_k + cde_m \star e_k \\ &= de_m \star f + cg \star e_k \\ &\in \langle F \rangle. \end{aligned}$$

It follows that we can construct a Gröbner basis

$$\mathcal{G} := \mathcal{F} \cup \{g_1, \ldots, g_m\}$$

of $\mathfrak{a}$ such that the g_i all belong to $\langle F \rangle$.

Example 5.6. Let $R = \Bbbk[x, y, z, u, v]$, let $\boldsymbol{m} = zv, yv, uv, xv, xu, yzu$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F can be realized as the R-complex supported on the $\boldsymbol{m}$-labeled cellular complex pictured below:



We write down the homogeneous components of F as a graded module below:

$$\begin{aligned} F_0 &= R \\ F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 + Re_6 \\ F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{26} + Re_{35} + Re_{45} + Re_{56} \\ F_3 &= Re_{123} + Re_{124} + Re_{1345} + Re_{2345} + Re_{2456} \\ F_4 &= Re_{12345} \end{aligned}$$

We will use Singular to help us find an associative multigraded multiplication μ on F such that $e_\sigma^2 = 0$ for all σ. From multidegree and Leibniz rule considerations, we begin constructing μ as follows:

$$\begin{aligned} e_1 \star e_2 &= ve_{12} & e_3 \star e_5 &= ue_{35} \\ e_1 \star e_3 &= ve_{13} & e_3 \star e_6 &= -zue_{23} + ue_{26} \\ e_1 \star e_4 &= ve_{14} & e_4 \star e_5 &= xe_{45} \\ e_1 \star e_5 &= ue_{14} + ze_{45} & e_4 \star e_6 &= -zue_{24} + xe_{26} \\ e_1 \star e_6 &= zue_{12} + ze_{26} & e_5 \star e_6 &= ue_{56} \\ e_2 \star e_3 &= ve_{23} & e_1 \star e_{23} &= ve_{123} \\ e_2 \star e_4 &= ve_{24} & e_1 \star e_{24} &= ve_{124} \\ e_2 \star e_5 &= ue_{24} + ye_{45} & e_1 \star e_{35} &= -ve_{1345} \\ e_2 \star e_6 &= ye_{26} & e_1 \star e_{56} &= -uze_{124} + ze_{2456} \\ e_3 \star e_4 &= ve_{35} - ve_{45} & e_1 \star e_{2345} &= ve_{12345}. \end{aligned}$$

At this point, Singular can help us determine how we should define μ everywhere else. First we input the following code into Singular:

```
LIB "ncalg.lib";

intvec V = 1:6, 2:9, 3:5, 4:1;

ring A=(0,x,y,z,u,v),(e1,e2,e3,e4,e5,e6,
e12,e13,e14,e23,e24,e26,e35,e45,e56,
e123,e124,e1345,e2345,e2456,e12345),Wp(V);

matrix C[21][21]; matrix D[21][21]; int i; int j;
for (i=1; i<=21; i++) {for (j=1; j<=21; j++) {C[i,j]=(-1)^(V[i]*V[j]);}}
ncalgebra(C,D);

poly f(1)(2) = e1*e2 - v*e12;
poly f(1)(3) = e1*e3 - v*e13;
poly f(1)(4) = e1*e4 - v*e14;
poly f(1)(5) = e1*e5 - u*e14 - z*e45;
poly f(1)(6) = e1*e6 - zu*e12 - z*e26;
poly f(2)(3) = e2*e3 - v*e23;
poly f(2)(4) = e2*e4 - v*e24;
poly f(2)(5) = e2*e5 - u*e24 - y*e45;
poly f(2)(6) = e2*e6 - y*e26;
poly f(3)(4) = e3*e4 - v*e35 + v*e45;
poly f(3)(5) = e3*e5 - u*e35;
poly f(3)(6) = e3*e6 + zu*e23 - u*e26;
poly f(4)(5) = e4*e5 - x*e45;
poly f(4)(6) = e4*e6 + zu*e24 - x*e26;
poly f(5)(6) = e5*e6 - u*e56;
poly f(1)(23) = e1*e23 - v*e123;
poly f(1)(24) = e1*e24 - v*e124;
poly f(1)(35) = e1*e35 + v*e1345;
poly f(1)(56) = e1*e56 + uz*e124 - z*e2456;
poly f(1)(2345) = e1*e2345 - v*e12345;

list L = (e1,e2,e3,e4,e5,e6,
e12,e13,e14,e23,e24,e26,e35,e45,e56,
e123,e124,e1345,e2345,e2456,e12345);

ideal I; int i; for (i=1; i<=21; i++) {I = I + L[i]*L[i];}

I = I + f(1)(2),f(1)(3),f(1)(4),f(1)(5),f(1)(6),f(2)(3),f(2)(4),
f(2)(5),f(2)(6),f(3)(4),f(3)(5),f(3)(6),f(4)(5),f(4)(6),
f(5)(6),f(1)(23),f(1)(24),f(1)(35),f(1)(56),f(1)(2345);
```

To see that the multiplication is associative thus far, we calculate the Gröbner basis of I with respect to our fixed monomial ordering using the command std(I) in Singular. Singular gives us the following output:

```
_[1]=e6^2
_[2]=e5*e6+(-u)*e56
_[3]=e5^2
...
_[57]=e2*e56+(-y)*e2456
_[58]=e2*e45
_[59]=(z*u)*e2*e35+(-v)*e6*e35+(u*v)*e2456
_[60]=e2*e26
...
_[209]=e124*e12345
_[210]=e123*e12345
_[211]=e12345^2
```

where we omitted most of the Gröbner basis elements due to size constraints. Since the lead term of each polynomial showing up in the list has total degree > 1, we conclude that the multiplication we have defined so far is associative. Now observe that if we want the multiplication to continue being associative, then we need to define $e_2 \star e_{26} = 0$ since

$$\begin{aligned} ye_2 \star e_{26} &= e_2 \star (e_2 \star e_6) \\ &= (e_2 \star e_2) \star e_6 - [e_2, e_2, e_6] \\ &= -[e_2, e_2, e_6]. \end{aligned}$$

In fact, Singular already tells us this since it is computing the maximal associative quotient! In particular, setting I = std(I) and running the command reduce(e2∗e26 , I) outputs 0 in Singular which tells us that in the maximal associative quotient we have $e_1 \star e_{12} = 0$. Alternatively, we could simply read this off the list of polynomials that Singular outputted as the polynomial $e_2 \star e_{26}$ shows up in the Gröbner basis. Similarly, Singular tells us that we should define $e_2 \star e_{56} = -ye_{2456}$ since the polynomial $e_2 \star e_{56} - ye_{2456}$ shows up in the Gröbner basis. On the other hand, if we run the command reduce(e6∗e35 , I), then Singular outputs e6∗e35 which tells us that we still need to define $e_6 \star e_{35}$. Upon reflection of the multigrading and Leibniz rule, we define

$$e_6 \star e_{35} = -zue_{2345} + ue_{2456}.$$

Thus we add the polynomial poly f(6)(35) = e6∗e35 + zu∗e2345 − y∗e2456 to our ideal in the code. We observe that our multiplication is still associative by running the command std(I) and checking that none of the polynomials listed has lead term of total degree 1 again. Furthermore, running the command

```
for(i=1;i<=21;i++){for(j=i+1;j<=21;j++){reduce(L[i]*L[j],I);};};
```

shows that the multiplication is now defined everywhere. For instance, the command reduce(e12∗e35 , I) outputs (−v)∗e12345. This tells us that $e_{12} \star e_{35} = -ve_{12345}$.

Example 5.7. In Example (2.7) we calculate the associator $[e_1, e_5, e_2]$ using the following Singular code:

```
LIB "ncalg.lib";

intvec V = 1:3, 2:5, 3:5;

ring A=(0,x,y,z,w),(e1,e2,e5,
e12,e14,e23,e35,e45,
e123,e124,e134,e234,e345),Wp(V);

matrix C[13][13]; matrix D[13][13]; int i; int j;
for (i=1; i<=13; i++) {for (j=1; j<=13; j++) {C[i,j]=(-1)^(V[i]*V[j]);}}
ncalgebra(C,D);

poly f(1)(2) = e1*e2-e12;
poly f(1)(5) = e1*e5-yz2*e14-x*e45;
poly f(2)(5) = e2*e5-y2z*e23-w*e35;
poly f(2)(45) = e2*e45+yz*e234-w*e345;
poly f(1)(35) = e1*e35-yz*e134+x*e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14+e124;
poly S(1)(5)(2) = f(1)(5)*e2+e1*f(2)(5);

ideal I = f(1)(2),f(1)(5),f(2)(5),f(1)(23),f(1)(35),f(2)(14),f(2)(45);
reduce(S(1)(5)(2),I);

// [e1,e5,e2] = (y^2*z)*e123-(y*z^2)*e124+(y*z*w)*e134-(x*y*z)*e234
```

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