Thesis

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Part I
Preliminary Material

1 Gröbner Bases

Throughout this section, let K be a field, and let S denote the polynomial ring $K[x_1, \dots, x_n]$. In this section, we state all of our lemmas, propositions, and theorems without proof. All of the proofs can be found in **[?]** and **[?]**.

1.1 Monomials and Polynomials in S

A **monomial** m in S is a product in S of the form

$$m = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where all of the exponents $\alpha_1, \dots, \alpha_n$ are nonnegative integers. Sometimes we will use the notation x^α to denote a monomial, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers. Note that $x^\alpha = 1$ when $\alpha = (0, \dots, 0)$. If $m = x^\alpha$ is a monomial in S then the **degree** of m, denoted $\deg(m)$ or $|x^\alpha|$, is the sum $\alpha_1 + \cdots + \alpha_n$.

A **polynomial** f in S is a finite linear combination of monomials. We will write a polynomial f in the form

$$f = \sum_\alpha a_\alpha x^\alpha, \quad a_\alpha \in K,$$

where the sum is over a finite number of n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$. We call a_α the **coefficient** of the monomial x^α. If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a **term** of f. The **total degree** of $f \neq 0$, denoted $\deg(f)$, is the maximum $|\alpha|$ such that the coefficient a_α is nonzero.

Remark. If we replace the field K with a ring R, then the same terminology applies to $R[x_1, \dots, x_n]$. For instance, a **monomial** m in $R[x_1, \dots, x_n]$ is a product of the form $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and etc...

1.1.1 Monomial Orderings on S

A **monomial ordering** on S is a total ordering $>$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, a total ordering on the set of monomials x^α, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta,$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$. We say $>$ is a **global monomial ordering** if $x^\alpha > 1$ for all $\alpha \neq 0$.

Remark. By a total ordering, we mean for all distinct pairs of monomials x^α and x^β, we either have $x^\alpha > x^\beta$ or $x^\beta > x^\alpha$. This property is used in induction arguments.

Lemma 1.1. *Let $>$ be a monomial ordering, then the following conditions are equivalent.*

1. *$>$ is a well-ordering, i.e. every nonempty set of monomials has a smallest element, or equivalently, every decreasing sequence*

$$x^{\alpha(1)} > x^{\alpha(2)} > x^{\alpha(3)} > \cdots$$

eventually terminates.

2. *$x_i > 1$ for $i = 1, \dots, n$.*
3. *$>$ is global.*
4. *$\alpha \geq_{nat} \beta$ and $\alpha \neq \beta$ implies $x^\alpha > x^\beta$, where $\geq_{nat}$ is a partial order on $\mathbb{Z}^n_{\geq 0}$ defined by*

$$(\alpha_1, \dots, \alpha_n) \geq_{nat} (\beta_1, \dots, \beta_n) \text{ if and only if } \alpha_i \geq \beta_i \text{ for all } i.$$

1.1.2 Examples of Monomial Orderings

We now describe some important examples of global monomial orderings: Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$.

1. (Lexicographical ordering): We say $x^\alpha >_{lp} x^\beta$ if
$$\text{there exists } 1 \leq i \leq n \text{ such that } \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$
2. (Degree reverse lexicographical ordering) We say $x^\alpha >_{dp} x^\beta$ if
$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and there exists } 1 \leq i \leq n \text{ such that } \alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i.$$
3. (Degree lexicographical ordering) We say $x^\alpha >_{Dp} x^\beta$ if
$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and there exists } 1 \leq i \leq n \text{ such that } \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

Example 1.1. With respect to the lexicographical ordering on $K[x, y, z]$, we have $x^3y^2z >_{lp} x^3yz^3$ and $xy^2z >_{lp} xyz^2$. With respect to the degree reverse lexicographical ordering on $K[x, y, z]$, we have $x^2y^2z^2 >_{dp} x^3yz^3$ and $z^2 >_{dp} x$. With respect to the degree lexicographical ordering on $K[x, y, z]$, we have $x^3yz^3 >_{Dp} x^2y^2z^2$ and $z^2 >_{Dp} x$.

1.1.3 Multidegree, Leading Coefficients, Leading Monomials, and Leading Terms

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a nonzero polynomial in $K[x_1, \ldots, x_n]$ and let $>$ be a monomial order.

1. The **multidegree** of f is
$$\text{multdeg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \mid c_\alpha \neq 0).$$
2. The **leading coefficient** of f is
$$\text{LC}(f) = c_{\text{multdeg}(f)} \in K.$$
3. The **leading monomial** of f is
$$\text{LM}(f) = x^{\text{multdeg}(f)}.$$
4. The **leading term** of f is
$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

Example 1.2. Let $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2$. With respect to lexicographical ordering we have
$$\begin{aligned} \text{multdeg}(f) &= (3, 0, 0) \\ \text{LC}(f) &= -5 \\ \text{LM}(f) &= x^3 \\ \text{LT}(f) &= -5x^3. \end{aligned}$$
With respect to degree reverse lexicographical ordering we have
$$\begin{aligned} \text{multdeg}(f) &= (2, 0, 2) \\ \text{LC}(f) &= 7 \\ \text{LM}(f) &= x^2z^2 \\ \text{LT}(f) &= 7x^2z^2. \end{aligned}$$

1.2 Monomial Ideals

An ideal $I \subseteq S$ is a called a **monomial ideal** if there is a subset $A \subset \mathbb{Z}^n_{\geq 0}$ (possibly infinite) such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, where $h_\alpha \in K[x_1, \ldots, x_n]$. In this case, we write $I = \langle x^\alpha \mid \alpha \in A \rangle$.

Example 1.3. An example of a monomial ideal is given by $I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle \subseteq K[x, y]$. A nontrivial example of a monomial ideal is given by $J = \langle f_1, f_2, f_3, f_4 \rangle = \langle x^2 + x^2y^3, -x^2y^3 + y^3, x^4, y^6 \rangle$. It's easy to see that $J \subset \langle x^2, y^3 \rangle$. For the reverse inclusion, note that
$$\begin{aligned} x^2 &= f_1 - x^2 f_2 - y^3 f_3 \\ y^3 &= f_1 + y^3 f_2 - x^2 f_4. \end{aligned}$$
So $\langle x^2, y^3 \rangle \subset J$. Therefore $J = \langle x^2, y^3 \rangle$.

1.2.1 Monomials Ideals are Finitely-Generated

The next theorem tells us that monomials ideals are finitely generated.

Theorem 1.2. *(Dickson's Lemma.) Let $I = \langle x^\alpha \mid \alpha \in A \rangle$ be a monomial ideal. Then I can be written as $I = \langle x^{\alpha(1)}, \dots, x^{\alpha(s)} \rangle$ where $\alpha(1), \dots, \alpha(s) \in A$.*

1.3 Hilbert Basis Theorem

Throughout the rest of this section, fix a monomial ordering on S.

1.3.1 Lead Term Ideal

Let I be a nonzero ideal in S.

1. We denote by $\mathrm{LT}(I)$ the set of leading terms of nonzero elements of I. Thus,

$$\mathrm{LT}(I) = \{cx^\alpha \mid \text{there exists } f \in I \setminus \{0\} \text{ with } \mathrm{LT}(f) = cx^\alpha\}.$$

2. We denote by $\langle \mathrm{LT}(I) \rangle$ be the ideal generated by the elements of $\mathrm{LT}(I)$.

It is easy to see that $\langle \mathrm{LT}(I) \rangle$ is a monomial ideal. Therefore Theorem (1.2) implies that it is finitely-generated. Thus, there are $g_1, \dots, g_t \in I$ such that $\mathrm{LT}(I) = \langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle$. If we are given an arbitrary finite generating set for I, say $I = \langle f_1, \dots, f_s \rangle$, then $\langle \mathrm{LT}(f_1), \dots, \mathrm{LT}(f_s) \rangle$ and $\langle \mathrm{LT}(I) \rangle$ may be *different* ideals. To see this, consider the following example.

Example 1.4. Let $I = \langle f_1, f_2 \rangle$, where $f_1 = x^3 - 2xy$ and $f_2 = x^2y - 2y^2 + x$, and use grlex ordering on monomials in $K[x, y]$. Then

$$x \cdot (x^2y - 2y^2 + x) - y \cdot (x^3 - 2xy) = x^2,$$

so that $x^2 \in I$. Thus $x^2 = \mathrm{LT}(x^2) \in \langle \mathrm{LT}(I) \rangle$. However x^2 is not divisible by $\mathrm{LT}(f_1) = x^3$ or $\mathrm{LT}(f_2) = x^2y$, so that $x^2 \notin \langle \mathrm{LT}(f_1), \mathrm{LT}(f_2) \rangle$.

1.3.2 Hilbert Basis Theorem

Theorem 1.3. *(Hilbert Basis Theorem). Let I be an ideal in S. Then I is finitely-generated.*

1.4 Gröbner Bases

Let I be a nonzero ideal in S. A finite subset $G = \{g_1, \dots, g_t\}$ is said to be a **reduced Gröbner basis** if

1. $\langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle$
2. $\mathrm{LC}(g) = 1$ for all $g \in G$.
3. For all $g \in G$, no monomial of g lies in $\langle \mathrm{LT}(G \setminus \{g\} \rangle$.

Let I be an ideal in S and let $G = \{g_1, \dots, g_t\}$ be the reduced Gröbner basis for I. Then given a polynomial f in S, it can be shown that there are unique polynomials $\pi(f)$ and f^G in S such that $f = \pi(f) + f^G$ and no term of f^G is divisible by any of $\mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t)$. We call f^G the **normal form of f with respect to** G. It follows from uniqueness of f^G and $\pi(f)$ that taking the normal form of a polynomial is a K-linear map:

$$c_1 f_1^G + c_2 f_2^G = (c_1 f_1 + c_2 f_2)^G \tag{1}$$

for all $c_1, c_2 \in K$ and $f_1, f_2 \in S$. We will denote this map as $-^G$. An important property of $-^G$ is that it preserves homogeneity. The details can be found in \cite{GP08} and \cite{CLO15}.

2 Graded Rings and Modules

2.1 Graded Rings

A **graded ring** R is a ring together with a direct sum decomposition

$$R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R_i,$$

where the R_i are abelian groups which satisfies the condition that if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$. The R_i are called **homogeneous components** of R and the elements of R_i are called **homogeneous elements** of **degree** i. If r is a homogeneous element in R, then we denote the degree of r as $\deg(r)$. When we say "Let R be a graded ring", we denote the homogeneous components of R as R_i.

Remark. The condition that $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$ is equivalent to the condition that $R_i R_j \subset R_{i+j}$.

Example 2.1. An important example of a graded ring is a ring R endowed with the **trivial grading**: The homogoneeous components of R being $R_0 := R$ and $R_i := 0$ for all $i > 0$. If R is a field, then will *always* assume that R is a graded ring endowed with the trivial grading.

Example 2.2. Let R be a ring and let Q be an ideal in R. The **associated graded ring of** R **with respect to** Q is

$$\mathrm{Gr}_Q(R) := \bigoplus_{i=0}^{\infty} Q^i / Q^{i+1}.$$

Multiplication in $\mathrm{Gr}_Q(R)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$.

2.1.1 Weighted Polynomial Rings

Let $w := (w_1, \dots, w_n)$ be an n-tuple of positive integers. We define the **weighted polynomial ring** S_w with respect to the **weighted vector** w to be the polynomial ring $R[x_1, \dots, x_n]$ endowed with the unique grading such that $\deg(x_\lambda) = \alpha_\lambda$ for all $\lambda = 1, \dots, n$. We define the **weighted degree** of a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S_w, denoted $\deg_w(m)$, to be

$$\deg_w(m) := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

This grading gives S_w the structure of a graded ring, where the homogeneous components are given by

$$(S_w)_i := \mathrm{Span}_R \langle m \in S_w \mid m \text{ is monomial of weighted degree } i \rangle.$$

Remark. If $w = (1, \dots, 1)$, then we recover the polynomial ring $R[x_1, \dots, x_n]$ with the usual grading. If the context is clear, we simply use the letter S to denote this graded ring.

Example 2.3. Let K be a field and let S_w denote the weighted polynomial ring $K[x, y, z]$ with respect to the weighted vector $w := (1, 2, 3)$. The first few homogeneous components of S_w start out as

$$\begin{aligned}
(S_w)_0 &= K \\
(S_w)_1 &= Kx \\
(S_w)_2 &= Kx^2 + Ky \\
(S_w)_3 &= Kx^3 + Kxy + Kz \\
&\vdots
\end{aligned}$$

2.2 Graded R-Modules

Let R be a graded ring. An R-module M, together with a direct sum decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

into abelian groups M_i is called a **graded** R**-module** if $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. The M_i are called **homogeneous components** of M and the elements of M_i are called **homogeneous** of **degree** i. If m is a homogeneous element in M, then we denote the degree of m as $\deg(m)$. When we say "Let M be a graded R-module", then the homogeneous components of M are denoted M_i.

Remark. Unlike in the case of graded rings, we do *not* usually assume that $M_i = 0$ for $i < 0$.

Example 2.4. Here's an important example of a graded R-module where we do not necessarily have $M_i = 0$ for $i < 0$: If M is a graded R-module, then for $j \in \mathbb{Z}$, we define the j**'th twist** or the j**'th shift** of M to be the graded R-module

$$M(j) := \bigoplus_{i \in \mathbb{Z}} M(j)_i$$

where $M(j)_i := M_{i+j}$.

2.2.1 Graded R-Submodules

Lemma 2.1. *Let M be a graded R-module and $N \subset M$ a submodule. The following conditions are equivalent:*

1. *N is graded R-module whose homogeneous components are $M_i \cap N$.*
2. *N is generated by homogeneous elements.*
3. *Let $m = \sum m_i$ with $m_i \in M_i$. Then $m \in N$ if and only if $m_i \in N$ for all $i \in \mathbb{Z}$.*

Proof. The proof is straightforward and can be found in \cite{GP08}. □

A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (2.1) is called a **graded** (or **homogeneous**) R-submodule.

Example 2.5. Let K be a field, S_w be the polynoimal ring $K[x, y, z]$ with respect to the weight $w = (5, 6, 15)$, and let $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ be an ideal S_w. Then I is a homogeneous ideal in S_w.

Remark. Let R be a graded ring, and let I be a homogeneous ideal in R. Then the quotient R/I has an induced structure as a graded ring, where the homogeneous component of R/I is

$$(R/I)_i := (R_i + I)/I \cong R_i/I \cap R_i$$

2.2.2 Homomorphisms of Graded R-Modules

Let M and N be graded R-modules. A homomorphism $\varphi : M \to N$ is called **homogeneous** (or **graded**) of degree j if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If φ is homogeneous of degree zero then we will simply say φ is **homogeneous**.

Example 2.6. Let R denote the polynomial ring $K[x, y, z, t]$ with the natural grading. Then the matrix

$$U := \begin{pmatrix} x + y + z & w^2 - x^2 & x^3 \\ 1 & x & xy + z^2 \end{pmatrix}$$

defines a homomorphism $U : R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$ which is graded of degree zero.

2.3 Graded R-Algebras

Let R be a graded ring and let A be an R-algebra. We say A is a **graded R-algebra** if A is graded as a ring and $A_0 = R$.

Remark. We do not require A to be a *commutative* ring.

Example 2.7. Let Q be an ideal in R. The **blowup algebra of Q in R** is the R-algebra

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong R \oplus Q \oplus Q^2 \oplus Q^3 \oplus \cdots .$$

The multiplication in $B_Q(A)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$.

2.3.1 Homomorphisms of Graded R-Algebras

Let A and A' be graded R-algebras. We say $\varphi : A \to A'$ is an R-algebra homomorphism if

1. φ is a homomorphism when viewed as a map of R-modules. In other words,

   $$\varphi(r_1a_1 + r_2a_2) = r_1\varphi(a_1) + r_2\varphi(a_2)$$

   for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$.
2. φ preserves the algebra structure. In other words

   $$\varphi(ab) = \varphi(a)\varphi(b)$$

   for all $a, b \in A$.

Moreover, we say φ is **graded** if φ is a graded homomorphism when viewed as a map of graded R-modules.

2.3.2 Finitely-Generated Graded R-Algebras

An graded R-algebra A is said to be **finitely-generated** if it is finitely-generated as an R-algebra. The next proposition gives a classification of all finitely-generated commutative R-algebras.

Proposition 2.1. *Every finitely-generated commutative graded R-algebra is isomorphic to S_w/I, where S_w denotes the polynomial ring $R[x_1,\dots,x_n]$ with respect to the weighted vector $w \in \mathbb{Z}^n_{\geq 0}$ and I is a homogeneous ideal in S_w.*

Proof. Let A be a finitely-generated commutative R-algebra with generators $a_1,\dots,a_n$. Then for each $\lambda = 1,\dots,n$ we have $a_\lambda \in A_{w_\lambda}$, where $w_\lambda \in \mathbb{Z}_{\geq 0}$. Let $\varphi : S_w \to A$ be the unique morphism of graded R-algebras such that $\varphi(x_\lambda) = a_\lambda$ for all $\lambda = 1,\dots,n$. Then A is isomorphic to $S_w/\mathrm{Ker}(\varphi)$ as graded R-algebras. □

2.3.3 Algorithmic Computations in the R-algebra S/I using Gröbner Bases

Let K be a field, S denote the polynomials ring $K[x_1,\dots,x_n]$, and I be a homogeneous ideal in S. Then S/I is a graded K-algebra, where the homogeneous component S_i is the K-vector space of all homogeneous polynomials $f \in S$ of degree i. Now fix a monomial ordering and let G be the reduced Gröbner basis of I with respect to this ordering. Define

$$S_I := \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I)\rangle)$$

There is an obvious decompostion of S_I into K-vector spaces $(S_I)_i$, where

$$(S_I)_i = \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I)\rangle \text{ and } \deg(x^\alpha) = i).$$

In fact, S/I and S_I are isomorphic as graded K-modules. The isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. Indeed, K-linearity follows from (1), and the grading is preserved since $-^G$ preserves homogeneity. This makes S/I isomorphic to S_I as graded K-modules. Using this isomorphism, we can carry multiplication from S/I over to S_I to turn S_I into a graded K-algebra: For $f_1, f_2 \in S_I$, we define multiplication as

$$f_1 \cdot f_2 = (f_1 f_2)^G. \tag{2}$$

Defining multilpication this way makes S_I isomorphic to S/I as graded K-algebras. For computational purposes, it is easier to work with S_I rather than S/I.

Example 2.8. Consider $S = K[x,y]$ and $I = \langle xy^2 + y^3, x^3 + x^2y\rangle$. Then $G = \{xy^2 + y^3, x^3 + x^2y\}$ is the reduced Gröbner basis with respect to graded reverse lexicographical order. Thus $\mathrm{LT}(I) = \langle xy^2, x^3\rangle$. Let's do some computations in S_I. First, let's write the first few homogeneous terms of S_I:

$$\begin{aligned}
(S_I)_0 &= K \\
(S_I)_1 &= Kx + Ky \\
(S_I)_2 &= Kx^2 + Kxy + Ky^2 \\
(S_I)_3 &= Kx^2y + Ky^3 \\
(S_I)_4 &= Ky^4 \\
(S_I)_5 &= Ky^5 \\
&\vdots
\end{aligned}$$

Next, we multiply some elements together in S_I in the multiplication table below

$\cdot$	x	y	y^3
x^2y	y^4	y^4	y^6
x^2	x^2y	x^2y	y^5
x	x^2	xy	y^4

Example 2.9. Consider $S = K[x,y]$ and $I = \langle xy + y^2, x^3\rangle$. We first use Singular to compute a Gröbner basis G of I with respect to graded reverse lexicographical ordering. We obtain $G = \{g_1, g_2, g_3\}$. where $g_1 = xy + y^2$, $g_2 = x^3$, and $g_3 = y^4$. Then the first few homogeneous components of I, S/I and S_I are given below

$$\begin{array}{lll}
I_0 = 0 & (S/I)_0 = K \cdot \overline{1} & (S_I)_0 = K \\
I_1 = 0 & (S/I)_1 = K\overline{x} + K\overline{y} & (S_I)_1 = Kx + Ky \\
I_2 = Kg_1 & (S/I)_2 = K\overline{x}^2 + K\overline{y}^2 & (S_I)_2 = Kx^2 + Ky^2 \\
I_3 = Kxg_1 + Kyg_1 + Kg_2 & (S/I)_3 = K\overline{y}^3 & (S_I)_3 = Ky^3 \\
I_4 = S_4 & (S/I)_4 = 0 & (S_I)_4 = 0 \\
\vdots & \vdots & \vdots
\end{array}$$

3 Homological Algebra

Throughout this section, let R be a ring.

3.1 Chain Complexes over R

A **chain complex** (A, d) **over** R, or simply a **chain complex** if the base ring R is understood from context, is a sequence of R-modules A_i and morphisms $d_i : A_i \to A_{i-1}$

$$(A, d) := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\operatorname{Ker}(d_i) \supset \operatorname{Im}(d_{i+1})$. With this in mind, we define the i**th homology of the chain complex** (A, d) to be

$$H_i(A, d) := \operatorname{Ker}(d_i)/\operatorname{Im}(d_{i+1}).$$

Let (A, d) and (A', d') be two chain complexes. A **chain map** $\varphi : (A, d) \to (A', d')$ is a sequence of R-module homomoprhisms $\varphi_i : A_i \to A'_i$ such that $d'_i\varphi_i = \varphi_{i-1}d'_i$ for all $i \in \mathbb{Z}$. We can view a chain map visually as illustrated in the diagram below:

$$\begin{tikzcd}
(A,d) := \cdots \arrow[r] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\varphi_{i+1}"] & A_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"] & A_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \\
(A',d') := \cdots \arrow[r] & A'_{i+1} \arrow[r, "d'_{i+1}"] & A'_i \arrow[r, "d'_i"] & A'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

3.1.1 Simplifying Notation

To simplify notation in what follows, we think of R as a trivially graded ring. If (A, d) is a chain complex over R, then we think of (A, d) as a graded R-module A together with a graded endomorphism $d : A \to A$ of degree -1 such that $d^2 = 0$. We think of d_i as being the restriction of d to A_i and we often refer to d as the **differential**. An element in $\operatorname{Ker}(d)$ is called a **cycle** of (A, d) and an element in $\operatorname{Im}(d)$ is called a **boundary** of (A, d). We define the **homology** of (A, d) to be

$$H(A, d) := \operatorname{Ker}(d)/\operatorname{Im}(d)$$

Note that $H(A, d) = \bigoplus_{i\in\mathbb{Z}} H_i(A, d)$. We sometimes write $H(A)$ instead of $H(A, d)$ if the differential is understood from context.

Let (A, d) and (A', d') be chain complexes. A chain map $\varphi : (A, d) \to (A', d')$ can be thought of as a homogeneous homomorphism of graded R-modules such that $\varphi d = d'\varphi$.

3.1.2 Homotopy Equivalence

Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). We say φ is **homotopic** to ψ if there is a graded homomorphism $h : A \to A'$ of degree 1 such that $\varphi - \psi = d'h + hd$.

Proposition 3.1. *Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). Then φ and ψ induce the same map on homology.*

Proof. The proof is straightforward and can be found in [**?**]. □

3.2 Exact Sequences of Chain Complexes over R

Let (A, d), (A', d'), and (A'', d'') be chain complexes and let $\varphi : (A', d') \to (A, d)$ and $\psi : (A, d) \to (A'', d'')$ be chain maps. Then we say that

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

is a **short exact sequence** of chain complexes if the following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d'_{i+2}"] & \vdots \arrow[d, "d_{i+2}"] & \vdots \arrow[d, "d''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "d'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "d_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "d''_{i+1}"] & 0 \\
0 \arrow[r] & A'_{i} \arrow[r, "\varphi_{i}"] \arrow[d, "d'_{i}"] & A_{i} \arrow[r, "\psi_{i}"] \arrow[d, "d_{i}"] & A''_{i} \arrow[r] \arrow[d, "d''_{i}"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "d'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "d_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "d''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given such a short exact sequence, we get induced maps $\varphi_i : H_i(A') \to H_i(A)$ and $\psi_i : H_i(A) \to H_i(A'')$, and **connecting homomorphisms** $\gamma_i : H_i(A'') \to H_{i-1}(A')$ which gives rise a long exact sequence in homology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{i+1}(A'') \arrow[dll, rounded corners, "\gamma_{i+1}"] \\
H_i(A') \arrow[r, "\varphi_i"] & H_i(A) \arrow[r, "\psi_i"] & H_i(A'') \arrow[dll, rounded corners, "\gamma_i"] \\
H_{i-1}(A') \arrow[r, "\varphi_{i-1}"] & H_{i-1}(A) \arrow[r, "\psi_{i-1}"] & \cdots
\end{tikzcd}$$

Remark. It is a nice exercise in homological algebra to work out the details of the connecting map.

3.3 Differential Graded R-Algebras

A **differential graded R-algebra** is a chain complex (A, d) such that A is a graded R-algebra and the differential d satisfies the **Leibniz law** with respect to this algebra structure:

$$d(ab) = d(a)b + (-1)^{\deg(a)} a d(b). \tag{3}$$

for all $a, b \in A$. We say that the differential graded R-algebra is **commutative** if $ab = (-1)^{\deg(a)\deg(b)} ba$. We say that the differential graded R-algebra is **strictly commutative** if in addition $a^2 = 0$ for $\deg(a)$ odd.

3.3.1 Homomorphisms of Differential Graded R-Algebras

Let (A, d) and (A', d') be differential graded R-algebras. We say $\varphi : (A, d) \to (A', d')$ is **homomorphism of differential graded** R-algebras if φ is both a chain map and an R-algebra homomorphism.

3.3.2 Differential Graded A-Modules

Let (A, d) be a differential graded R-algebra. A **differential graded A-module** (M, d) is a chain complex (M, d) over R such that M is an A-module and such that the differential d satisfies the **Leibniz law** with respect to the algebra structure in A:

$$d(am) = d(a)m + (-1)^{\deg(a)} a d(m). \tag{4}$$

for all $a \in A$ and $m \in M$.

3.3.3 Obtaining a Differential Graded A-Module from a Chain Complex over R

Let (A, d_A) be a differential graded R-algebra. If we start with a chain complex over R, then we can construct a differential graded A-module. Indeed, suppose that (B, d_B) is a chain complex over R. Then $A \otimes_R B$ is an A-module and a graded R-module whose homogeneous component in degree k is

$$(A \otimes_R B)_k := \bigoplus_{i+j=k} A_i \otimes_R B_j.$$

We define a differential d on $A \otimes_R B$ by first defining it on the elementary tensors as

$$d(a \otimes b) := d_A(a) \otimes b + (-1)^{\deg(a)} a \otimes d_B(b),$$

for all $a \in A$ and $b \in B$, and then extending it R-linearly everywhere else. A straightforward calculation shows that $d^2 = 0$ and that the differential satisfies Leibniz law (4). Moreover, if B is a differential graded R-algebra, then $A \otimes_R B$ can realized as a differential graded A-algebra and a differential graded B-algebra. Multiplication in $A \otimes_R B$ is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'.$$

for all $a, a' \in A$ and $b, b \in B$.

Remark. In particular, if M is an R-module endowed with the trivial grading, then $(A \otimes_R M, d)$ is a differential graded A-module where the homogeneous componenet of degree k in $A \otimes_R M$ is $(A \otimes_R M)_k := A_k \otimes_R M$, and d acts on elementary tensors as $d(a \otimes m) = d(a) \otimes m$.

3.4 Exterior Algebras and Koszul Complexes

3.4.1 Exterior Algebras

Let R be a ring and M an R-module. For $k \geq 2$, the kth **exteror power** of M, denoted $\Lambda^k(M)$, is the R-module $M^{\otimes k}/J_k$ where J_k is the submodule of $M^{\otimes k}$ spanned by all $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for $i \neq j$. For any $m_1, \ldots, m_k \in M$, the coset of $m_1 \otimes \cdots \otimes m_k$ in $\Lambda^k(M)$ is denoted $m_1 \wedge \cdots \wedge m_k$. For completeness, we set $\Lambda^0(M) = R$ and $\Lambda^1(M) = M$. A general element in $\Lambda^k(M)$ will be denoted as ω or η. Since $M^{\otimes k}$ is spanned by tensors $m_1 \otimes \cdots \otimes m_k$, the quotient module $M^{\otimes k}/J_k = \Lambda^k(M)$ is spanned by their images $m_1 \wedge \cdots \wedge m_k$. That is, any $\omega \in \Lambda^k(M)$ is a finite R-linear combination

$$\omega = \sum r_{i_1,\ldots,i_k} m_{i_1} \wedge \cdots \wedge m_{i_k},$$

where there coefficients $r_{i_1,\ldots,i_k}$ are in R and the m_i's are in M. We call $m_1 \wedge \cdots \wedge m_k$ an **elementary wedge product**. Since $r(m_1 \wedge \cdots \wedge m_k) = (rm_1) \wedge \cdots \wedge m_k$, every element of $\Lambda^k(M)$ is a sum (not just a linear combination) of elementary wedge products.

We define the **exterior algebra** of M to be

$$\Lambda(M) := \bigoplus_{k \geq 0} \Lambda^k(M),$$

where the multiplication rule given by the wedge product. The exterior algebra of M is a graded R-algebra, where the degree k homogeneous component is $\Lambda^k(M)$. If R does not have characteristic 2, then the exterior algebra of M is **skew commutative**. This means that if ω_1 and ω_2 are homogeneous elements, then

$$\omega_1 \wedge \omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \omega_2 \wedge \omega_1.$$

The construction of $\Lambda(M)$ is functioral in M. This means that if N is another R-module and $\varphi : M \to N$ is an R-module homomorphism. Then φ induces a graded R-algebra homomorphism $\wedge\varphi : \Lambda(M) \to \Lambda(N)$, where $\wedge\varphi$ takes the elementary wedge product $m_1 \wedge \cdots \wedge m_k$ in $\Lambda(M)$ and maps it to the wedge product $\varphi(m_1) \wedge \cdots \wedge \varphi(m_k)$ in $\Lambda(N)$. We will write $\wedge^k\varphi$ to be the induced R-module homomorphism from $\Lambda^k(M)$ to $\Lambda^k(N)$. In particular, if N is free of rank n, then $\Lambda^n(N) \cong R$, and if $\varphi : N \to N$ is an R-module homomorphism, then $\wedge^n\varphi$ is multiplication by the determinant of any matrix representing φ.

Example 3.1. Let R be a ring, $M = Rx_1 \oplus Rx_2 \oplus Rx_3 \cong R^3$, and let $\varphi : M \to M$ be the R-module homomorphism induced by setting $\varphi(x_\mu) = \sum_{\lambda=1}^n a_{\lambda\mu} x_\lambda$ for $1 \leq \lambda, \mu \leq 3$. The matrix representation of φ with respect to the ordered basis $\beta_1 = \{x_1, x_2, x_3\}$ is given by

$$[\varphi]_{\beta_1}^{\beta_1} := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

To calculate $\wedge^2\varphi$, we need to see how it acts on the basis vectors $x_\lambda \wedge x_\mu$ where $1 \leq \lambda < \mu \leq 3$:

$$\begin{aligned}
\varphi(x_1) \wedge \varphi(x_2) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \\
&= (a_{11}a_{22} - a_{21}a_{12})x_1 \wedge x_2 + (a_{11}a_{32} - a_{31}a_{12})x_1 \wedge x_3 + (a_{21}a_{32} - a_{31}a_{22})x_2 \wedge x_3 \\
\\
\varphi(x_1) \wedge \varphi(x_3) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{11}a_{23} - a_{21}a_{13})x_1 \wedge x_2 + (a_{11}a_{33} - a_{31}a_{13})x_1 \wedge x_3 + (a_{21}a_{33} - a_{31}a_{23})x_2 \wedge x_3 \\
\\
\varphi(x_2) \wedge \varphi(x_3) &= (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{12}a_{23} - a_{22}a_{13})x_1 \wedge x_2 + (a_{12}a_{33} - a_{32}a_{13})x_1 \wedge x_3 + (a_{22}a_{33} - a_{32}a_{23})x_2 \wedge x_3.
\end{aligned}$$

So the matrix representation of $\wedge^2\varphi$ with respect to the ordered basis $\beta_2 = \{x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3\}$ is

$$[\varphi]_{\beta_2}^{\beta_2} = \begin{pmatrix} a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} & a_{12}a_{33} - a_{32}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & a_{21}a_{33} - a_{31}a_{23} & a_{22}a_{33} - a_{32}a_{23} \end{pmatrix}$$

To calculate $\wedge^3\varphi$, we need to see how it acts on the basis vector $x_1 \wedge x_2 \wedge x_3$:

$$\begin{aligned}
\varphi(x_1) \wedge \varphi(x_2) \wedge \varphi(x_3) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13})x_1 \wedge x_2 \wedge x_3 \\
&= \det\left([\varphi]_{\beta_1}^{\beta_1}\right) e_1 \wedge e_2 \wedge e_3.
\end{aligned}$$

3.4.2 Koszul Complexes

Let R be a ring, M an R-module, and $\varphi : M \to R$ an R-module homomorphism. The assignment

$$(m_1, \dots, m_k) \mapsto \sum_{i=1}^{k} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

defines an alternating n-linear map $M^k \to \Lambda^{k-1}(M)$. By the universal property of the kth exterior power, there exists an R-linear map $d_\varphi^{(k)} : \Lambda^k(M) \to \Lambda^{k-1}(M)$ with

$$d_\varphi^{(k)}(m_1 \wedge \cdots \wedge m_k) = \sum_{i=1}^{n} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

for all $m_1, \dots, m_k \in L$. The collection of the maps $d_\varphi^{(k)}$ defines a graded R-homomorphism

$$d_\varphi : \Lambda(M) \to \Lambda(M)$$

of degree -1. A straightforward calculation shows that d_φ gives $\Lambda(M)$ the structure of a differential graded R-algebra. This differential graded R-algebra is called the **Koszul complex** of φ and is denoted $\mathcal{K}_\bullet(\varphi)$. The **dual Koszul complex** of φ, denoted $\mathcal{K}^\bullet(\varphi)$, is the chain complex over R whose underlying graded R-module is $\mathrm{Hom}_R(\mathcal{K}_\bullet(\varphi), R)$ and whose differential is $d^\star$, where $d^\star$ is obtained by applying the functor $\mathrm{Hom}_R(-, R)$ to d.

Example 3.2. Let R be a ring of characteristic 2, S denote the polynomial ring $R[x_1, \dots, x_n]$, and let $\varphi : S_1 := \bigoplus_{\lambda=1}^n Rx_\lambda \to R$ be the unique R-linear map such that $\varphi(x_\lambda) = r_\lambda \in R$ for all $\lambda = 1, \dots, n$. Then $\Lambda(S_1)$ is isomorphic to $S/\langle x_1^2, \dots, x_n^2 \rangle$ as graded R-algebras. Using this isomorphism, we give $S/\langle x_1^2, \dots, x_n^2 \rangle$ the structure of a differential graded R-algebra by carrying over the differential d_φ for $\Lambda(S_1)$ to a differential d for $S/\langle x_1^2, \dots, x_n^2 \rangle$. A straightforward calculation shows that $d = \sum_{\lambda=1}^n r_\lambda \partial_{x_\lambda}$. We denote this Koszul complex as $\mathcal{K}(r_1, \dots, r_n)$.

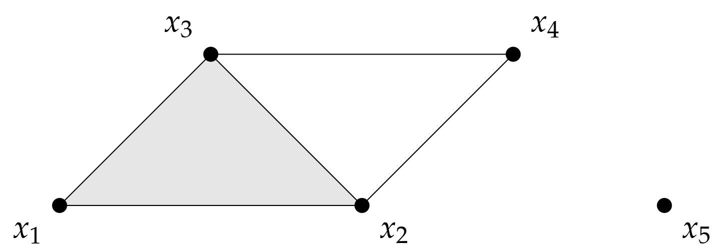
4 Simplicial Complexes

A **simplicial complex** Δ on the set $\{x_1, \dots, x_n\}$ is a collection of subsets of $\{x_1, \dots, x_n\}$ such that

1. The simplicial complex Δ contains all singletons: $\{x_\lambda\} \in \Delta$ for all $\lambda = 1, \dots, n$.
2. The simplicial complex Δ is closed under containment: if $\sigma \subseteq \{x_1, \dots, x_n\}$ and $\tau \supset \sigma$, then $\tau \in \Delta$.

An element of a simplicial complex is called a **face** or **simplex**, and a simplex of Δ not properly contained in another simplex of Δ is called a **facet**. A simplex $\sigma \in \Delta$ of cardinality $i+1$ is called an i-dimensional face or an i-face of Δ. The empty set $\emptyset$, is the unique face of dimension -1, as long as Δ is not the **void complex** $\{\}$ consisting of no subsets of $\{1,\ldots,n\}$. The **dimension** of Δ, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces (or $-\infty$ if $\Delta = \{\}$).

Example 4.1. The simplicial complex Δ on $\{x_1,x_2,x_3,x_4,x_5\}$ consisting of all subsets of $\{x_1,x_2,x_3\}$, $\{x_2,x_4\}$, $\{x_3,x_4\}$, and $\{x_4\}$ is pictured below



4.1 Simplicial Homology

Let Δ be a simplicial complex on $\{x_1,\ldots,x_n\}$. For $i \in \mathbb{Z}$, let

$$S_i(\Delta) := \mathrm{Span}_K(\sigma \in \Delta \mid \dim(\sigma) = i) \quad \text{and} \quad S(\Delta) := \bigoplus_{i \in \mathbb{Z}} S_i(\Delta).$$

Then $S(\Delta)$ is a graded K-module. Let $\partial : S(\Delta) \to S(\Delta)$ be the unique graded endomorphism of degree -1 such that

$$\partial(\sigma) = \sum_{\lambda \in \sigma} \sigma \backslash \{\lambda\}.$$

By a direct calculation, we have $\partial^2 = 0$, and so $(S(\Delta), \partial)$ forms a chain complex over K; it is called the (**augmented** or **reduced**) **chain complex of** Δ **over** K. The ith homology of $(S(\Delta),\partial)$ is called the i**th reduced homology** of Δ over K, and is commonly denoted as $\widetilde{H}_i(\Delta, K)$.

Example 4.2. For Δ as in Example (4.1), we have

$$\begin{aligned}
S_2(\Delta) &= \{\{1,2,3\}\}\\
S_1(\Delta) &= \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}\\
S_0(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\}\\
S_{-1}(\Delta) &= \{\emptyset\}
\end{aligned}$$

Choosing bases for the $S_i(\Delta)$ as suggested by the ordering of the faces listed above, the chain complex for Δ becomes

$$0 \longrightarrow K \xrightarrow[\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}]{} K^5 \xrightarrow[\begin{pmatrix}1&1&0&0&0\\1&0&1&1&0\\0&1&1&0&1\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}]{} K^5 \xrightarrow[\begin{pmatrix}1&1&1&1&1\end{pmatrix}]{} K \longrightarrow 0$$

For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} + e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1,1,1,0,0)$. The mapping ∂_1 has rank 3, so $\widetilde{H}_0(\Delta;K) \cong \widetilde{H}_1(\Delta;K) \cong K$ and the other homology groups are 0. Geometrically, $\widetilde{H}_0(\Delta;K)$ is nontrivial since Δ is disconnected and $\widetilde{H}_1(\Delta;K)$ is nontrivial since Δ contains a triangle which is not the boundary of an element of Δ.

Part II

Homological Constructions over a Field of Characteristic 2

Throughout this section, let K be a field of characteristic 2, S denote the polynomial ring $K[x_1,\ldots,x_n]$, I be a homogeneous ideal in S, and let G be the reduced Gröbner basis for I with respect to some fixed monomial ordering.

5 Constructing the Chain Complexes (S, d), (S_I, d), and $(I, \underline{d})$.

5.1 Construction of (S, d)

Let $d : S \to S$ be the graded K-linear map of degree -1 given by $d := \sum_{j=1}^{n} \partial_{x_j}$. Since K has characteristic 2, we have $d^2 = 0$. Indeed, it suffices to show that $d^2(m) = 0$ for all monomials m in S. So let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. Then

$$\begin{aligned} d^2(m) &= \left(\sum_{k=1}^{n} \partial_{x_k} \right)^2 \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) \\ &= \left(\sum_{k=1}^{n} \partial_{x_k}^2 \right) \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) \\ &= \sum_{k=1}^{\infty} \alpha_k (\alpha_k - 1) x_1^{\alpha_k - 2} \\ &= 0. \end{aligned}$$

Thus the differential d gives the graded K-module S the structure of a chain complex over K.

5.2 Construction of (S_I, d)

Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree i in S. We denote

$$[m]_o = \{1 \leq \lambda \leq n \mid \alpha_\lambda \text{ is odd}\} \qquad \text{and} \qquad [m]_e = \{1 \leq \mu \leq n \mid \alpha_\mu \text{ is even}\}$$

Using this notation, we can express the differential in another way:

$$d(m) = \sum_{\lambda \in [m]_o} x_\lambda^{-1} m.$$

This makes it clear that the differential d maps S_I into S_I. Indeed, if m is not in $\mathrm{LT}(I)$, then every term $x_\lambda^{-1} m$ of $d(m)$ is not in $\mathrm{LT}(I)$ either. Thus the differential d gives the graded K-module S_I the structure of a chain complex over K.

5.3 Construction of $(S/I, \overline{d})$

Definition 5.1. We say I is d-**stable** if d maps I into I.

Suppose I is d-stable. Then the differential $d : S \to S$ induces a graded linear map of degree -1, denoted $\overline{d} : S/I \to S/I$, where

$$\overline{d}(\overline{f}) = \overline{d(f)} \text{ for all } f \in S.$$

Indeed, the map $\overline{d}$ is well-defined since d is I-stable. To see why, let $f + g$ and f, where $g \in I$ and $f \in S$, be two different representatives of a class in S/I, i.e. $\overline{f + g} = \overline{f} \in S/I$. Then

$$\begin{aligned} \overline{d}\left(\overline{f + g}\right) &= \overline{d(f + g)} \\ &= \overline{d(f) + d(g)} \\ &= \overline{d(f)} \\ &= \overline{d}(\overline{f}). \end{aligned}$$

where $\overline{d(f) + d(g)} = \overline{d(f)}$ since $d(g) \in I$.

Moreover, the differential $\overline{d}$ gives S/I the structure of a differential graded K-algebra. Indeed, $\overline{d}$ is a graded linear map of degree -1 such that $\overline{d}^2 = 0$ and such that $\overline{d}$ satisfies Leibniz law. This is because $\overline{d}$ inherits all of the properties from d. For instance, to see that $\overline{d}$ satisfies Leibniz law, let $\overline{f}_1$ and $\overline{f}_2$ be in S/I. Then

$$\begin{aligned} \overline{d}(\overline{f_1 f_2}) &= \overline{d(f_1 f_2)} \\ &= \overline{d(f_1) f_2 + f_1 d(f_2)} \\ &= \overline{d(f_1) f_2} + \overline{f_1 d(f_2)} \\ &= \overline{d}(\overline{f_1}) \overline{f_2} + \overline{f_1} \overline{d}(\overline{f_2}). \end{aligned}$$

Thus if I is d-stable, then the differential $\overline{d}$ gives the graded K-algebra S/I the structure of a differential graded K-algebra.

5.4 Construction of $(I, \underline{d})$

Our final construction involves the graded K-module I. Let $\underline{d} : I \to I$ be the graded K-linear map of degree -1 given by

$$\underline{d}(f) := d(f) + d(f)^G = \pi(d(f))$$

for all $f \in I$. Then $\underline{d}^2 = 0$. Indeed, for all $f \in I$, we have

$$\begin{aligned}
\underline{d}(\underline{d}(f)) &= \underline{d}(d(f) + d(f)^G) \\
&= d(d(f) + d(f)^G) + d(d(f) + d(f)^G)^G \\
&= d(d(f)^G) + d(d(f)^G)^G \\
&= d(d(f)^G) + d(d(f)^G) \\
&= 0,
\end{aligned}$$

where $d(d(f)^G)^G = d(d(f)^G)$ since every term in $d(d(f)^G)$ is not in I. Thus the differential $\underline{d}$ gives the graded K-module I the structure of a chain complex over K.

Remark. Let J be a homogeneous ideal in S such that $I \supset J$. If J is $\underline{d}$-stable, then the differential $\underline{d}$ gives the graded K-module I/J the structure of a chain complex over K, which we denote by $(I/J, \underline{d})$.

6 Differential Graded K-Algebras

Since d is defined in terms of partial derivatives, it is clear that d satisfies Leibniz law. Thus (S, d) is more than just a chain complex over K; it is a differential graded K-algebra. Since S_I is a graded K-algebra, it is natural wonder if (S_I, d) is also a differential graded K-algebra. A quick counterexample shows that this is not necessarily the case:

Example 6.1. Consider $S = K[x]$ and $I = \langle x^5 \rangle$. Then

$$\begin{aligned}
d(x \cdot x^4) &= d((x^5)^G) \\
&= d(0) \\
&= 0,
\end{aligned}$$

but

$$\begin{aligned}
d(x) \cdot x^4 + x \cdot d(x^4) &= 1 \cdot x^4 + x \cdot 0 \\
&= (x^4)^G + 0^G \\
&= x^4,
\end{aligned}$$

so $d(x \cdot x^5) \neq d(x) \cdot x^4 + x \cdot d(x^4)$.

6.1 When (S_I, d) Has the Structure of a Differential Graded K-Algebra

The next theorem tells us precisely when (S_I, d) is a differential graded K-algebra.

Theorem 6.1. *(S_I, d) is a differential graded K-algebra if and only if $d(g) = 0$ for all $g \in G$.*

Proof. Assume that $d(g) = 0$ for all $g \in G$. We first prove that $d(f^G) = d(f)^G$ for all $f \in S$. Let $f \in S$. From the division algorithm, we have $f = g_1 q_1 + \cdots + g_r q_r + f^G$ for some $q_1, \dots, q_r \in S$. Thus

$$\begin{aligned}
d(f) &= d(g_1 q_1 + \cdots + g_r q_r + f^G) \\
&= d(g_1 q_1) + \cdots + d(g_r q_r) + d(f^G) \\
&= g_1 d(q_1) + \cdots + g_r d(q_r) + d(f^G).
\end{aligned}$$

Since $g_1 d(h_1) + \cdots + g_r d(h_r) \in I$ and no term of $d(f^G)$ is divisible by any element of $\mathrm{LT}(I)$, it follows from uniqueness of normal forms that $d(f^G) = d(f)^G$.

Now we show that this implies that (S_I, d) is a differential graded K-algebra. Let $f_1, f_2 \in S_I$. Then

$$\begin{aligned}
d(f_1 \cdot f_2) &= d((f_1 f_2)^G) \\
&= (d(f_1 f_2))^G \\
&= (d(f_1) f_2 + f_1 d(f_2))^G \\
&= (d(f_1) f_2)^G + (f_1 d(f_2))^G \\
&= d(f_1) \cdot f_2 + f_1 \cdot d(f_2).
\end{aligned}$$

Therefore (S_I, d) is a differential graded K-algebra.

Now we prove the converse. Assume (S_I, d) is a differential graded K-algebra. Let $g \in G$ and let m be the lead term of g. We may assume g is not a constant (otherwise we'd clearly have $d(g) = 0$). Thus, there exists some x_λ such that x_λ divides m. Then on the one hand, we have

$$\begin{aligned} d(x_\lambda \cdot x_\lambda^{-1} m) &= d(m^G) \\ &= d(g+m) \\ &= d(g) + d(m), \end{aligned}$$

since $m^G = g + m$. On the other hand, we have

$$\begin{aligned} d(x_\lambda) \cdot x_\lambda^{-1} m + x_\lambda \cdot d(x_\lambda^{-1} m) &= (x_\lambda^{-1} m)^G + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m + d(m))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m)^G + d(m)^G \\ &= x_\lambda^{-1} m + x_\lambda^{-1} m + d(m)^G \\ &= d(m), \end{aligned}$$

since $(x_\lambda^{-1} m)^G = x_\lambda^{-1} m$ and $d(m)^G = d(m)$ (every term of $d(m)$ does not lie in $\langle \mathrm{LT}(G) \rangle$). Since (S_I, d) is a differential graded K-algebra, we must have $d(g) = 0$. This establishes this theorem. □

Remark. We should note that the identity $x_\lambda d(x_\lambda^{-1} m) = x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)) = x_\lambda^{-1} m + d(m)$ follows since d satisfies Leibniz law not just in S, but also in $S[x_1^{-1}, \ldots, x_n^{-1}]$. Again, this is because d is defined in terms of partial derivatives.

Example 6.2. Going back to Example (2.8), where $S = K[x, y]$, $I = \langle xy^2 + y^3, x^3 + x^2 y \rangle$, and $G = \{xy^2 + y^3, x^3 + x^2 y\}$. We have $d(xy^2 + y^3) = d(x^3 + x^2 y) = 0$. Therefore Proposition (**??**) implies (S_I, d) is a differential graded K-algebra.

Now we want to show that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra, and moreover, they are isomorphic to each other.

Lemma 6.2. *Let I be a homogeneous ideal in the polynomial ring S, and let $G = \{g_1, g_2, \ldots, g_r\}$ be the reduced Gröbner basis for I. Then $d(g) = d(g)^G$ for all $g \in G$.*

Proof. Let $g \in G$. If $d(g) = 0$, then clearly we have $d(g) = d(g)^G$, so assume $d(g) \neq 0$. We need to prove that $d(g) = d(g)^G$. This is equivalent to saying that no term of $d(g)$ belongs to $\langle \mathrm{LT}(G) \rangle := \langle \mathrm{LT}(g_1), \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_r) \rangle$, since G is a Gröbner basis. Every term in of $d(g)$ has the form $x_\lambda^{-1} m$ where m is some term of g. It is easy to see that this term cannot belong to $\langle \mathrm{LT}(G) \rangle$. Indeed, if $x_\lambda^{-1} m \in \langle \mathrm{LT}(G) \rangle$, then $m \in \langle \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_r) \rangle$, and this contradicts the fact that G is a *reduced* Gröbner basis. □

Proposition 6.1. *I is d-stable if and only if $d(g) \in I$ for all $g \in G$.*

Proof. One direction is trivial, so let's prove the other direction. Suppose $d(g) \in I$ for all $g \in G$ and let $f \in I$. Since G generates I, we can write $f = \sum_{\lambda=1}^r q_\lambda g_\lambda$ for some $q_1, \ldots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(f) &= d\left(\sum_{\lambda=1}^r q_\lambda g_\lambda \right) \\ &= \sum_{\lambda=1}^r d(q_\lambda g_\lambda) \\ &= \sum_{\lambda=1}^r (d(q_\lambda) g_\lambda + q_\lambda d(g_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. □

Combining Lemma (6.2) and Proposition (10.1), we find that that $d(g) = 0$ for all $g \in G$ if and only if $d(g) \in I$ for all $g \in G$ if and only if I is d-stable. Combining this with Theorem (6.1), we find that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra. Now we will show that they are in fact isomorphic to each other.

Theorem 6.3. *Suppose I is d-stable. Then (S_I, d) is isomorphic to $(S/I, \overline{d})$ as differential graded K-algebras.*

Proof. Recall that S/I is isomorphic to S_I as graded K-algebras, where the isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. It remains to show that this isomorphism respects the differential graded algebra structure. In particular, we need to show that $d(f^G) = d(f)^G$ for all $f \in S$. This was already proven in Theorem (6.1). $\square$

6.2 More Differential Graded K-algebras

Proposition 6.2. *Suppose I is d-stable and let g be a homogeneous polynomial such that $d(g) = 0$. Then $(S_{\langle I,g\rangle}, d)$ and $(S_{I:g}, d)$ are differential graded K-algebras.*

Proof. We just need to show that $\langle I, g\rangle$ and $I : g$ are both d-stable. Since $d(g) = 0$, it follows that $\langle I, g\rangle$ is d-stable. To prove that $I : g$ is d-stable, let $f \in I : g$. Then since $fg \in I$, $d(g) = 0$, and I is d-stable, it follows that

$$d(f)g = d(f)g + fd(g) = d(fg) \in I$$

Therefore $d(f) \in I : g$, which implies that $I : g$ is d-stable. $\square$

Example 6.3. Consider $S = K[x, y, z]$, $g = x^2y + x^2z$, and $I = \langle f_1, f_2, f_3\rangle$ where

$$\begin{aligned} f_1 &= xy + xz + yz \\ f_2 &= x^4y + x^5 \\ f_3 &= y^3 + y^2z \end{aligned}$$

Then $d(f_1) = d(f_2) = d(f_3) = 0$ implies that (S_I, d) is a differential graded K-algebra. The reduced Gröbner basis for I with respect to graded lexicographical ordering is $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$\begin{aligned} g_1 &= xy + xz + yz \\ g_2 &= y^3 + y^2z \\ g_3 &= y^2z^2 \\ g_4 &= xz^4 + yz^4 \\ g_5 &= x^5 + x^4z + x^3z^2 + x^2z^3 \\ g_6 &= x^4z^2 \end{aligned}$$

Since $d(g) = 0$, we know that $(S_{\langle I,g\rangle}, d)$ and $(S_{I:g}, d)$ are also differential graded K-algebras. The reduced Gröbner basis for $I : g$ with respect to graded lexicographical ordering is $G'' = \{g''_1, g''_2, g''_3\}$, where

$$\begin{aligned} g''_1 &= y + z \\ g''_2 &= z^2 \\ g''_3 &= x^3 + x^2z \end{aligned}$$

and the reduced Gröbner basis for $\langle I, g\rangle$ with respect to graded lexicographical ordering is $G' = \{g'_1, g'_2, g'_3, g'_4, g'_5\}$, where

$$\begin{aligned} g'_1 &= xy + xz + yz \\ g'_2 &= y^3 + y^2z \\ g'_3 &= xz^2 + yz^2 \\ g'_4 &= y^2z^2 \\ g'_5 &= x^5 + x^4z + x^3z^2 + x^2z^3 \end{aligned}$$

7 Constructing the Cochain Complexes (S, δ), (S_I, δ), and (I, δ)

For a K-vector space V, let $V^\star := \mathrm{Hom}_K(V, K)$. We refer to $V^\star$ as the **dual** of V. If $\varphi : V \to W$ is a K-linear map from the vector space V to the vector space W, then we denote the K-linear map $\mathrm{Hom}_K(\varphi, K) : \mathrm{Hom}_K(W, K) \to \mathrm{Hom}_K(V, K)$ simply as $\varphi^\star$ and call it the **dual** of φ.

7.1 Construction of (S, δ) and (I, δ)

The duals of S_I, S, and I are all graded K-modules, where the homogeneous components are simply the duals of $(S_I)_i$, S_i, and I_i respectively. In fact, S_I, S/I, S, and I are all isomorphic as graded K-modules to their duals: To get an isomorphism from S_i to $S_i^\star$, we map the monomial $x^\alpha \in S_i$ to the element $\underline{x}^\alpha \in S_i^\star$, where $\underline{x}^\alpha$ is defined on the monomial $x^\beta \in S_i$ as

$$\underline{x}^\alpha(x^\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

and is extended linearly everywhere else. The isomorphisms $(S_I)_i^\star \cong (S_I)_i$ and $I_i^\star \cong I_i$ are induced from this isomorphism.

We can describe $d^\star$ as follows: Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree $i-1$ in S. Then

$$d^\star(\underline{m}) = \sum_{\mu \in [m]_e} \underline{x^\mu m}.$$

Using the isomorphism from S to $S^\star$ described above, we pull $d^\star$ back to a map on S and we denote this map as δ and call it the **codifferential**. Thus, for each monomial $m \in S$, we have

$$\delta(m) = \sum_{\mu \in [m]_e} x_\mu m. \tag{5}$$

It is clear that δ is a graded endomorphism of S of degree 1 such that $\delta^2 = 0$, and thus gives S the structure of a cochain complex over K.

Remark. Note that (S, δ) is not a differential graded K-algebra with respect to the usual multiplication maps. For instance, consider $S = K[x, y]$. Then $\delta(xy) = 0$ but $\delta(x)y + x\delta(y) = x^2y + xy^2$. Later on we will introduce a product, called **cup product**, which will give (S, δ) the structure of a differential graded K-algebra with respect to this product.

7.2 Construction of (I, δ)

Using the description of δ in (5) and the fact that I is an ideal, we see that δ restricts to a map $\delta : I \to I$, which is also a graded endomorphism of S of degree 1 such that $\delta^2 = 0$, and thus gives I the structure of a cochain complex over K.

7.2.1 Kronecker Pairing

When we identify monomials $\underline{x}^\alpha$ in $S^\star$ with monomials x^α in S_i, we are forgetting the way monomials in $S_i^\star$ act on monomials in S_i. To make up for this, we introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on S_i called the **Kronecker pairing**: For monomials x^α and x^β in S_i, we set

$$\langle x^\alpha, x^\beta \rangle = \underline{x}^\alpha(x^\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}.$$

Then we extend this linearly to $\langle \cdot, \cdot \rangle : S_i \times S_i \to K$. From the way we constructed δ, we have for all $f_1, f_2 \in S_i$, we have $\langle \delta(f_1), f_2 \rangle = \langle f_1, d(f_2) \rangle$.

7.3 Construction of (S_I, δ)

Let $\underline{\delta} : S_I \to S_I$ be the graded K-linear map of degree 1 given by

$$\underline{\delta}(f) := \delta(f)^G$$

for all $f \in S_I$. The map $\underline{\delta}$ gives the graded K-module S_I the structure of a cochain complex over K.

7.4 Cup and Cap Product

We introduce some notation. Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. If $\alpha_\lambda > 0$, then we say x_λ is in the **support** of m. We denote by $\mathrm{supp}(m)$ to be the set of all x_λ in the support of m. We will assume that $x_1 > x_2 > \cdots > x_n$. We say x_λ is the **last nonzero coordinate** of m if x_λ is the smallest element in $\mathrm{supp}(m)$. We say x_λ is the **first nonzero coordinate** of m if x_λ is the largest element in $\mathrm{supp}(m)$.

7.4.1 Cup Product

Definition 7.1. Let m_1 and m_2 be monomials S_i and S_j respectively. The **cup product** of m_1 and m_2 is

$$m_1 \smile m_2 = \begin{cases} \frac{m_1 m_2}{x_\lambda} & \text{if } x_\lambda \text{ is the last nonzero coordinate of } m_1 \text{ and the first nonzero coordinate of } m_2 \\ 0 & \text{otherwise} \end{cases}$$

This extends to a linear map $\smile: S_i \times S_j \to S_{i+j-1}$ which we call the cup product.

Example 7.1. Let $S = K[x, y, z]$. Then

$$\begin{aligned}(x^2y + xy^2) \smile (x^5 + y^4z) &= x^2y \smile x^5 + xy^2 \smile x^5 + x^2y \smile y^4z + xy^2 \smile y^4z \\ &= x^2y \smile y^4z + xy^2 \smile y^4z \\ &= x^2y^4z + xy^5z.\end{aligned}$$

Proposition 7.1. *Let m_1 and m_2 be two monomials in S. Then*

$$\delta(m_1 \smile m_2) = \delta(m_1) \smile m_2 + m_1 \smile \delta(m_2). \tag{6}$$

Proof. Let x_{λ_1} be the last nonzero coordinate of m_1 and let x_{λ_2} be the first nonzero coordinate of m_2. First assume that $x_{\lambda_1} > x_{\lambda_2}$. Then $m_1 \smile m_2 = 0$, and this implies $\delta(m_1 \smile m_2) = 0$. Also, the last nonzero coordinate of every monomial in $\delta(m_1)$ will be greater than or equal to x_{λ_1} which is strictly greater than x_{λ_2}. Therefore $\delta(m_1) \smile m_2 = 0$. Similarly, the first nonzero in $\delta(m_2)$ will be smaller than or equal to x_{λ_2} which is strictly smaller than x_{λ_1}. Therefore $m_1 \smile \delta(m_2) = 0$. So we trivially have (6) in this case.

Now we assume $x_{\lambda_2} > x_{\lambda_1}$. Then $m_1 \smile m_2 = 0$, and this implies $\delta(m_1 \smile m_2) = 0$. Also, since $x_{\lambda_1} \in [m_2]_e$ and $x_{\lambda_2} \in [m_1]_e$, we will have $\delta(m_1) \smile m_2 = m_1 m_2$ and $m_1 \smile \delta(m_2) = m_1 m_2$. Adding everything together, we get

$$\delta(m_1 \smile m_2) = 0 = \delta(m_1) \smile m_2 + m_1 \smile \delta(m_2).$$

Finally, assume $x_{\lambda_1} = x_{\lambda_2}$. Let's denote this common variable as x_μ. On the one hand, we have

$$\delta(m_1 \smile m_2) = \delta\left(\frac{m_1 m_2}{x_\mu}\right) = \sum_{\substack{x_\lambda \in [m_1]_e \\ x_\lambda \leq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu} + \sum_{\substack{x_\lambda \in [m_2]_e \\ x_\lambda \geq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu}$$

On the other hand, we have

$$\delta(m_1) \smile m_2 = \sum_{\substack{x_\lambda \in [m_1]_e \\ x_\lambda \leq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu} \qquad \text{and} \qquad m_1 \smile \delta(m_2) = \sum_{\substack{x_\lambda \in [m_2]_e \\ x_\lambda \geq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu}.$$

Combining these together gives the desired result. ☐

7.4.2 Cap Product

Definition 7.2. Let m_1 and m_2 be monomials S_i and S_j respectively. The **cap product** of m_1 and m_2 is

$$m_1 \frown m_2 = \begin{cases} x_\lambda \frac{m_2}{m_1} & \text{if } m_1 \mid m_2,\ x_\lambda \text{ is the last nonzero coordinate of } m_1, \text{ and } x_\lambda \text{ is the first nonzero coordinate of } \frac{x_\lambda m_2}{m_1}. \\ 0 & \text{otherwise} \end{cases}$$

The cap product extends linearly to a map $\frown: S_i \times S_j \to S_{i-j+1}$.

8 Topological Interpretation of $H(S_I)$

In this section, we give a topological interpretation of $H(S_I)$. Since $H(S_{\mathrm{LT}(I)}) \cong H(S_I)$, we only need to consider the case where I is a monomial ideal. Thus, throughout this section, we will assume that I is a monomial ideal.

8.1 Homology Calculations

Proposition 8.1. *Suppose I is d-stable. Then $H(S_I) = 0$.*

Proof. Let f be a homogeneous polynomial in S_I such that $d(f) = 0$. Then for any $x_\lambda \in (S_I)_1$, we have

$$d(x_\lambda f) = d(x_\lambda) f + x_\lambda d(f) = f.$$

Therefore $\mathrm{Ker}(d) = \mathrm{Im}(d)$, hence $H(S_I) = 0$. ☐

Remark. Taking $I = 0$ shows that $H(S) = 0$.

Proposition 8.2. *Let J be a monomial ideal such that $I \supset J$ and such that J is d-stable. Then the differential d induces isomorphisms $H_i(I/J) \cong H_{i-1}(S_I)$ for all $i > 0$.*

Proof. First we show that $\underline{d}\pi = \pi d$. For all $f \in S$, we have

$$\begin{aligned}
\underline{d}(\pi(f)) &= \underline{d}(f + f^G) \\
&= d(f + f^G) + d(f + f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G) \\
&= d(f) + d(f)^G \\
&= \pi(d(f)),
\end{aligned}$$

where $d(f^G)^G = d(f^G)$ because no term in $d(f^G)$ lies in $\mathrm{LT}(I)$.

Therefore we have a short exact sequence of chain complexes over K:

$$0 \longrightarrow (S_I, d) \longrightarrow (S_J, d) \xrightarrow{\pi} (I/J, \underline{d}) \longrightarrow 0.$$

From this, we obtain a long exact sequence in homology, which gives for each $i > 0$, the following short exact sequences:

$$0 = H_i(S_J) \longrightarrow H_i(I/J) \xrightarrow{d} H_{i-1}(S_I) \longrightarrow H_{i-1}(S_J) = 0.$$

where d is obtained from the connecting map. In more detail, d maps the element $[f] \in H_i(I/J)$ to the element $[d(f)] \in H_{i-1}(S_I)$. □

Proposition 8.3. *The differential d induces isomorphisms $H_i(I) \cong H_{i-1}(S_I)$ for all $i > 0$.*

Proof. First we show that $\underline{d}\pi = \pi d$. For all $f \in S$, we have

$$\begin{aligned}
\underline{d}(\pi(f)) &= \underline{d}(f + f^G) \\
&= d(f + f^G) + d(f + f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G)^G \\
&= d(f) + d(f^G) + d(f)^G + d(f^G) \\
&= d(f) + d(f)^G \\
&= \pi(d(f)),
\end{aligned}$$

where $d(f^G)^G = d(f^G)$ because no term in $d(f^G)$ lies in $\mathrm{LT}(I)$.

Therefore we have a short exact sequence of chain complexes over K:

$$0 \longrightarrow (S_I, d) \longrightarrow (S, d) \xrightarrow{\pi} (I, \underline{d}) \longrightarrow 0.$$

From this, we obtain a long exact sequence in homology, which gives for each $i > 0$, the following short exact sequences:

$$0 = H_i(S) \longrightarrow H_i(I) \xrightarrow{d} H_{i-1}(S_I) \longrightarrow H_{i-1}(S) = 0.$$

where d is obtained from the connecting map. In more detail, d maps the element $[f] \in H_i(I)$ to the element $[d(f)] \in H_{i-1}(S_I)$. □

8.2 Decomposing $H_i(S_I)$

Let g be a homogeneous polynomial of degree j and let G' be the reduced Gröbner basis for $\langle I, g\rangle$ with respect to our fixed monomial ordering. In Commutative Algebra, we learn about the following short exact sequence of graded S-modules

$$\begin{tikzcd} 0 \arrow[r] & (S/(I:g))(-j) \arrow[r, "g"] & S/I \arrow[r] & S/\langle I,g\rangle \arrow[r] & 0 \\ & \overline{f} \arrow[r, mapsto] & \overline{fg} & & \end{tikzcd}$$

We want to use this short exact sequence to our advantage. First, using the isomorphisms $S_{I:g} \cong S/(I:g)$, $S_I \cong S/I$, and $S_{\langle I,g\rangle} \cong S/\langle I,g\rangle$, we get, for each i, a short exact sequence of K-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & (S_{I:g})_{j-i} \arrow[r, "\cdot g"] & (S_I)_i \arrow[r, "\_^{G'}"] & (S_{\langle I,g\rangle})_i \arrow[r] & 0 \\ & f \arrow[r, mapsto] & (fg)^G & & \\ & & f \arrow[r, mapsto] & f^{G'} & \end{tikzcd}$$

or in other words, a short exact sequence of graded K-vector spaces

$$\begin{tikzcd} 0 \arrow[r] & (S_{I:g})(-j) \arrow[r, "\cdot g"] & S_I \arrow[r, "\_^{G'}"] & S_{\langle I,g\rangle} \arrow[r] & 0 \end{tikzcd}$$

We want to know under what conditions this becomes a short exact sequence of chain complexes over K, that is, when does the following diagram commute?

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & (S_{I:g})_{j-i} \arrow[r, "\cdot g"] \arrow[d, "d"'] & (S_I)_i \arrow[r, "\_^{G'}"] \arrow[d, "d"'] & (S_{\langle I,g\rangle})_i \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & (S_{I:g})_{j-i-1} \arrow[r, "\cdot g"] \arrow[d] & (S_I)_{i-1} \arrow[r, "\_^{G'}"] \arrow[d] & (S_{\langle I,g\rangle})_{i-1} \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

After some thought, we find that the conditions which need to be satisfied are the following:

$$(gd(m))^G = d((gm)^G) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I:g) \tag{7}$$

$$d(m)^{G'} = d(m^{G'}) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I) \tag{8}$$

For the moment, let's assume that these conditions are satisfied so that we have a short exact sequence of chain complexes. Then by the usual argument, the short exact sequence of chain complexes gives rise to a long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & H_{i+1}(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\ H_{i-j}(S_{I:g}) \arrow[r, "\cdot g"] & H_i(S_I) \arrow[r, "\_^{G'}"] & H_i(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\ H_{i-j-1}(S_{I:g}) \arrow[r, "\cdot g"] & H_{i-1}(S_I) \arrow[r, "\_^{G'}"] & \cdots \end{tikzcd}$$

It's easy to see that the connecting maps λ all induce the zero map. So in fact, we get for each i, the short exact sequence of K-vector spaces:

$$0 \longrightarrow H_{i-j}(S_{I:g}) \xrightarrow{\cdot g} H_i(S_I) \xrightarrow{-^{G'}} H_i(S_{\langle I,g \rangle}) \longrightarrow 0,$$

and since the inclusion map $S_{\langle I,g \rangle} \hookrightarrow S_I$ splits the map $-^{G'}$, we obtain the following isomorphism

$$H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g \rangle}) \cong H_i(S_I) \tag{9}$$

where we map the representative (f_1, f_2) in $H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g \rangle})$ to the representative $gf_1 + f_2$ in $H_i(S_I)$.

8.2.1 Decomposing $H_i(S_I)$ in a Special Case and an Example

We will now discuss a special case of when the conditions in Theorem (**??**) are satisfied. Consider the case where I is a monomial ideal and g is a monomial of degree j which is not in I. Then condition (7) is satisfied since if m is not in $I : g$, then gm is not in I, and so $(gm)^G = gm$ which implies $(gd(m))^G = gd(m)$.

For condition (8) first assume that m is not in $\langle I, g \rangle$. Then then $m^{G'} = m$, which implies $d(m)^{G'} = d(m) = d(m^{G'})$. Thus condition (8) is satisfied in this case. Now assume that $m = g$. Then $m^{G'} = 0$, which implies $d(m^{G'}) = 0$. Thus, we must have $d(g) = 0$ in order for condition (8) to be satisfied in this case. So assume $d(g) = 0$ and consider the final case where $m = m_1 g$. Since $d(g) = 0$, we obtain $d(m)^{G'} = (d(m_1)g)^{G'} = 0$, and thus (8) is satisfied in this case as well.

In the next example, we show how we can apply Theorem (**??**) recursively. In what follows, we frequently use the notation I, g to mean $\langle I, g \rangle$ and $I : g$ to mean $I : \langle g \rangle$. For example, $I, g_1 : g_2 = \langle I, g_1 \rangle : \langle g_2 \rangle$, and $I : g_1, g_2 = \langle (I : g_1), \langle g_2 \rangle \rangle$, and so on. We also note that $I : g_1 : g_2 = I : g_1 g_2$.

Example 8.1. Consider $S = K[x, y, z]$ and $I = \langle x^3 y, yz^3 \rangle$. Then $d(x^2) = d(z^2) = 0$, and so

$$\begin{aligned} H_i(S_I) &= x^2 H_{i-2}(S_{I:x^2}) \oplus H_i(S_{I,x^2}) \\ &= x^2(z^2 H_{i-4}(S_{I:x^2 z^2}) \oplus H_{i-2}(S_{I:x^2,z^2})) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \\ &= x^2 z^2 H_{i-4}(S_{I:x^2 z^2}) \oplus x^2 H_{i-2}(S_{I:x^2,z^2}) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \end{aligned}$$

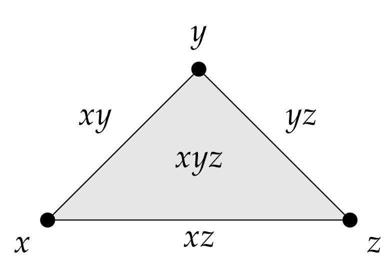
We calculate

$$\begin{aligned} I : x^2 z^2 &= \langle xy, yz \rangle \\ I, x^2 : z^2 &= \langle x^2, yz \rangle \\ I : x^2, z^2 &= \langle xy, z^2 \rangle \\ I, x^2, z^2 &= \langle x^2, z^2 \rangle \end{aligned}$$

The only part which has nontrivial homology is $S_{I:x^2 z^2}$. Thus, $H_5(S_I) = [d(x^3 y z^2)]K$ and $H_i(S_I) = 0$ for all $i \neq 5$.

8.3 Reinterpreting Simplicial Complexes

We want to reinterpret the theory simplicial complexes using the language of monomials. There is a bijection between the set of subsets of $\{x_1, \ldots, x_n\}$ and the set of squarefree monomials in the variables $x_1, \ldots, x_n$. Indeed, if m is a squarefree monomial, then the corresponding subset of $\{x_1, \ldots, x_n\}$ is $\mathrm{supp}(m)$. Moreover, if m and m' are squarefree monomials, then m divides m' if and only if $\mathrm{supp}(m) \subseteq \mathrm{supp}(m')$. Here's how we think of the squarefree monomials in x, y, z sit on the 2-simplex:



8.4 Stanley-Reisner Rings

8.4.1 Stanley-Reisner Ring

Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. We denote by I_Δ to be the ideal of nonfaces of Δ, that is, I_Δ is generated by the squarefree monomials m in S which are not in Δ. We define the **Stanley-Reisner ring** $K[\Delta]$ of the simplicial complex Δ to be the K-algebra $K[\Delta] := S/I_\Delta$. We will also denote by I_Δ^{sq} to mean $I_\Delta^{\mathrm{sq}} := \langle I_\Delta, x_1^2, \ldots, x_n^2 \rangle$.

Conversely, if I is a squarefree monomial ideal in S. Then we denote by Δ_I the simplicial complex on $\{x_1, \dots, x_n\}$ whose ideal of nonfaces is I, that is, Δ_I consists of all squarefree monomials which do not belong to I.

Lemma 8.1. *Let I be a monomial ideal. If $H(S_I) = 0$, then $H(S_{I,x_\lambda^2}) = H(S_{I:x_\lambda^2}) = 0$.*

Proof. From Theorem (**??**), we have a decomposition

$$H_i(S_I) \cong x_\lambda^2 H_{i-2}(S_{I:x_\lambda^2}) \oplus H_i(S_{I,x_\lambda^2})$$

for all $i \in \mathbb{Z}$. Thus, $H(S_I) = 0$ implies $H(S_{I,x_\lambda^2}) = H(S_{I:x_\lambda^2}) = 0$. □

Lemma 8.2. *Let Δ be a simplicial complex on $\{x_1, \dots, x_n\}$. Then*

$$H(S_{I_\Delta, x_1^2, \dots, x_{\lambda-1}^2 : x_\lambda^2}) = 0$$

for all $\lambda = 1, \dots, n$.

Proof. We prove this by induction. For the base case, let f represent an element in $H(S_{I_\Delta : x_1^2})$ and write f in terms of its monomial basis as

$$f = \sum_{\lambda=1}^{s} a_\lambda m_\lambda,$$

where $a_\lambda \in K$ for all $\lambda = 1, \dots, s$. Since I_Δ is a squarefree monomial ideal, we have $I_\Delta : x_1^2 = I_\Delta : x_1$. We claim that $x_1 f \in S_{I_\Delta : x_1^2}$. Indeed, for all $\lambda = 1, \dots, s$, we have $x_1 m_\lambda \in S_{I_\Delta : x_1} = S_{I_\Delta : x_1^2}$. This implies our claim. Now since $x_1 f \in S_{I_\Delta : x_1^2}$ and $d(x_1 f) = f$, it follows that f represents the zero element in $H(S_{I_\Delta : x_1^2})$.

Now assume that $H(S_{I_\Delta, x_1^2, \dots, x_{\lambda-1}^2 : x_\lambda^2}) = 0$ for some $1 \leq \lambda < n$. We prove that this implies $H(S_{I_\Delta, x_1^2, \dots, x_\lambda^2 : x_{\lambda+1}^2}) = 0$. First note that

$$H(S_{I_\Delta, x_1^2, \dots, x_\lambda^2 : x_{\lambda+1}^2}) = H(S_{I_\Delta : x_{\lambda+1}^2, x_1^2, \dots, x_\lambda^2}) = 0,$$

The same argument in the paragraph above implies $H(S_{I_\Delta : x_{\lambda+1}^2}) = 0$. Now we inductively apply Lemma (8.1) to get

$$H(S_{I_\Delta : x_{\lambda+1}^2, x_1^2, \dots, x_\lambda^2}) = 0.$$

□

Theorem 8.3. *Let Δ be a simplicial complex on $\{x_1, \dots, x_n\}$. Then*

$$H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$$

for all $i \in \mathbb{Z}$.

Proof. Let us first show that $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$. The map $\varphi : S_{I_\Delta^{\mathrm{sq}}} \to S(\Delta)$, given by $\varphi(m) = [m]_o$ for all monomials $m \in S_{I_\Delta^{\mathrm{sq}}}$, is a graded isomorphism of degree -1. Moreover, it is easy to check that $\varphi d = \partial \varphi$. Thus φ induces an isomorphism $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$.

Now we will prove that $H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{\mathrm{sq}}})$. We do this by combining Theorem (**??**) and Lemma (8.2). We have

$$\begin{aligned}
H_i(S_{I_\Delta}) &\cong x_1^2 H_{i-2}(S_{I_\Delta : x_1^2}) \oplus H_i(S_{I_\Delta, x_1^2}) \\
&\cong H_i(S_{I_\Delta, x_1^2}) \\
&\cong x_2^2 H_i(S_{I_\Delta, x_1^2 : x_2^2}) \oplus H_i(S_{I_\Delta, x_1^2, x_2^2}) \\
&\cong H_i(S_{I_\Delta, x_1^2, x_2^2}) \\
&\ \ \vdots \\
&\cong H_i(S_{I_\Delta^{\mathrm{sq}}})
\end{aligned}$$

for all $i \in \mathbb{Z}$. □

Example 8.2. Consider $S = K[x_1, x_2, x_3, x_4, x_5]$ and $I_\Delta = \langle x_1x_4, x_1x_5, x_2x_5, x_2x_3x_4, x_3x_5, x_4x_5 \rangle$. Then S/I_Δ is the Stanley-Reisner ring of the simplex Δ given in Example (4.1). Let's write down each homogeneous piece side by side:

$$
\begin{aligned}
S_2(\Delta) &= K\{1,2,3\} & (S_{I_\Delta^{\mathrm{sq}}})_3 &= Kx_1x_2x_3 \\
S_1(\Delta) &= K\{1,3\}+K\{1,2\}+K\{2,3\}+K\{2,4\}+K\{3,4\} & (S_{I_\Delta^{\mathrm{sq}}})_2 &= Kx_1x_3+Kx_1x_2+Kx_2x_3+Kx_2x_4+Kx_3x_4 \\
S_0(\Delta) &= K\{1\}+K\{2\}+K\{3\}+K\{4\}+K\{5\} & (S_{I_\Delta^{\mathrm{sq}}})_1 &= Kx_1+Kx_2+Kx_3+Kx_4+Kx_5 \\
S_{-1}(\Delta) &= K\cdot\varnothing & (S_{I_\Delta^{\mathrm{sq}}})_0 &= K\cdot 1
\end{aligned}
$$

Part III
Homological Constructions over a Ring of Characteristic 2

Throughout this chapter, let R be a ring of characteristic 2.

9 Constructing All Finitely-Generated Differential Graded R-Algebras

Theorem 9.1. *Let S_w denote the weighted polynomial ring $R[x_1,\dots,x_n]$ with respect to the weighted vector $w = (w_1,\dots,w_n)$. Define the map*

$$d := \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda},$$

where f_λ is a nonzero homogeneous polynomial in S_w of weighted degree $w_\lambda - 1$ for all $\lambda = 1,\dots,n$. Then

1. *d is a graded endomorphism $d : S_w \to S_w$ of degree -1 which satisfies Leibniz law.*
2. *Moreover, let $I \subset S_w$ be any d-stable homogeneous ideal such that $d(f_\lambda) \in I$ for all $\lambda = 1,\dots,n$. Then d induces a map $\overline{d} : S_w/I \to S_w/I$, given by $\overline{d}(\overline{f}) = \overline{d(f)}$ for all $\overline{f} \in S_w/I$, and $(S_w/I, \overline{d})$ is a differential graded R-algebra.*

Proof. We first show that d is a graded endomorphism $d : S_w \to S_w$ of degree -1 which satisfies Leibniz law:

- R-linearity: We have

$$
\begin{aligned}
d(r_1g_1 + r_2g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(r_1g_1 + r_2g_2) \\
&= \sum_{\lambda=1}^{n} f_\lambda (r_1\partial_{x_\lambda}(g_1) + r_2\partial_{x_\lambda}(g_2)) \\
&= r_1 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1) + r_2 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2)) \\
&= r_1 d(g_1) + r_2 d(g_2),
\end{aligned}
$$

  for all $r_1, r_2 \in R$ and $g_1, g_2 \in S_w$.

- Leibniz law: We have

$$
\begin{aligned}
d(g_1g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1g_2) \\
&= \sum_{\lambda=1}^{n} f_\lambda (\partial_{x_\lambda}(g_1)g_2 + g_1\partial_{x_\lambda}(g_2)) \\
&= \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1)\right) g_2 + g_1 \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2))\right) \\
&= d(g_1)g_2 + g_1 d(g_2),
\end{aligned}
$$

  for all $g_1, g_2 \in S_w$.

- Graded of degree -1: By R-linearity, we only need to check this on monomials. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of weighted degree i. A term in $d(x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ has the form $\alpha_\lambda f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}$ where $\alpha_\lambda \equiv 1 \bmod 3$, and

$$\begin{aligned}\deg_w\left(\alpha_\lambda f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda-1} \cdots x_n^{\alpha_n}\right) &= \deg_w\left(f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda-1} \cdots x_n^{\alpha_n}\right) \\ &= \deg_w(f_\lambda) + \deg_w\left(x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda-1} \cdots x_n^{\alpha_n}\right) \\ &= w_\lambda - 1 + w_1\alpha_1 + \cdots + w_\lambda(\alpha_\lambda - 1) + \cdots + w_n\alpha_n \\ &= -1 + w_1\alpha_1 + \cdots + w_\lambda\alpha_\lambda + \cdots + w_n\alpha_n \\ &= -1 + i.\end{aligned}$$

So every term in $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ has weighted degree $-1+i$. This implies that d is graded of degree -1.

Now we will show that $(S_w/I, \overline{d})$ is a differential graded R-algebra. Since I is d-stable, the map $\overline{d}$ is well-defined. The map $\overline{d}$ inherits the properties of being a graded endomorphism of degree -1 which satisfies Leibniz law from d, thus we just need to show that $\overline{d}^2 = 0$, or in other words, that $d^2(g) \in I$ for all $g \in S_w$. So let $g \in S_w$. Then

$$\begin{aligned}d^2(g) &= d\left(\sum_{\lambda=1}^n f_\lambda \partial_{x_\lambda}(g)\right) \\ &= \sum_{\lambda=1}^n d(f_\lambda \partial_{x_\lambda}(g)) \\ &= \sum_{\lambda=1}^n d(f_\lambda)\partial_{x_\lambda}(g) + f_\lambda d(\partial_{x_\lambda}(g))) \\ &= \sum_{\lambda=1}^n d(f_\lambda)\partial_{x_\lambda}(g) \in I,\end{aligned}$$

where we used the fact that $\partial_{x_\lambda}^2 = 0$ and $\partial_{x_\mu}\partial_{x_\lambda} = \partial_{x_\lambda}\partial_{x_\mu}$ to conclude that

$$\begin{aligned}\sum_{\lambda=1}^n f_\lambda d(\partial_{x_\lambda}(g)) &= \sum_{\lambda=1}^n f_\lambda \sum_{\mu=1}^n f_\mu \partial_{x_\mu}(\partial_{x_\lambda}(g)) \\ &= 0.\end{aligned}$$

□

Remark.

1. We often denote this differential graded R-algebra as $(S_w/I, f_1, \ldots f_n)$ instead of $(S_w/I, \overline{d})$.
2. When we write "let $(S_w/I, f_1, \ldots f_n)$ be a differential graded R-algebra", it is understood that the conditions in Theorem (9.1) are satisfied. Note that I is a *proper* ideal of S_w.

Proposition 9.1. *Let $(S_w/I, f_1, \ldots, f_n)$ be a differential graded R-algebra and let g be a homogeneous polynomial in S of degree j such that $d(g)$ is in I. Then $(S_w/\langle I, g\rangle, f_1, \ldots, f_n)$ and $(S/(I : g), f_1, \ldots, f_n)$ are differential graded R-algebras.*

Proof. First note that $d(f_\lambda) \in I$ implies $d(f_\lambda) \in \langle I, g\rangle$ and $d(f_\lambda) \in I : g$ for all $\lambda = 1, \ldots, n$. So we just need to show that $\langle I, g\rangle$ and $I : g$ are d-stable. Since$d(g)$ is in I, Proposition (10.1) implies that $\langle I, g\rangle$ is d-stable. Therefore $S/\langle I, g\rangle$ is a differential graded R-algebra. To prove that $I : g$ is d-stable, let $f \in I : g$. Then since $fg \in I$ and I is d-stable, it follows that $d(fg) = d(f)g + fd(g) \in I$. Which implies $d(f)g \in I$, since $d(g) \in I$. Therefore $d(f) \in I : g$, which implies that $I : g$ is d-stable. □

9.1 Classification of all Finitely-Generated Commutative Differential Graded R-Algebras

Theorem 9.2. *Every finitely-generated commutative differential graded R-algebra is isomorphic to one described in Theorem (9.1).*

Proof. Let (A, d_A) be a finitely generated differential graded R-algebra with generators $a_1, \ldots, a_n$. Then for each $\lambda = 1, \ldots n$, we have $a_\lambda \in A_{w_\lambda}$, where $w_\lambda \in \mathbb{Z}_{\geq 0}$. Let S_w denote the weighted polynomial ring $R[x_1, \ldots, x_n]$ with respect to the weighted vector $w = (w_1, \ldots, w_n)$, and let $\varphi : S_w \to A$ be the unique morphism of graded

R-algebras such that $\varphi(x_\lambda) = a_\lambda$ for all $\lambda = 1, \dots, n$. Then A is isomorphic to $S_w/\mathrm{Ker}(\varphi)$ as graded R-algebras. Choose $f_\lambda \in S$ such that $\varphi(f_\lambda) = d_A(a_\lambda)$ and define the map $d : S_w \to S_w$ as

$$d := \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}.$$

Then d is a graded endomorphism of degree -1 which satisfies Leibniz law, by Theorem (9.1). We claim that $\mathrm{Ker}(\varphi)$ is d-stable and that $d(f_\lambda) \in \mathrm{Ker}(\varphi)$ for all $\lambda = 1, \dots, n$. We do this in two steps:

Step 1: We will show that $\varphi d = d_A \varphi$. It suffices to show that for all monomials m, we have $\varphi(d(m)) = d_A(\varphi(m))$. We prove this by induction on $\deg(m)$. For the base case $\deg(m) = 1$, we have $m = x_\lambda$ for some $\lambda \in \{1, \dots, n\}$. Then

$$\begin{aligned}
\varphi(d(x_\lambda)) &= \varphi(f_\lambda) \\
&= d_A(a_\lambda) \\
&= d_A(\varphi(x_\lambda)).
\end{aligned}$$

Now suppose that $\varphi(d(m)) = d_A(\varphi(m))$ for all monomials m in S of degree less than i, where $i > 1$. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S whose degree is $i + 1$. We may assume that $\alpha_1, \alpha_\lambda \geq 1$ for some $\lambda \in \{1, \dots, n\}$. Then using Leibniz law together with induction, we obtain

$$\begin{aligned}
\varphi(d(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})) &= \varphi(d(x_1^{\alpha_1}) x_2^{\alpha_2} \cdots x_n^{\alpha_n} + x_1^{\alpha_1} d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\
&= \varphi(d(x_1^{\alpha_1}) \varphi(x_2^{\alpha_2} \cdots x_n^{\alpha_n}) + \varphi(x_1^{\alpha_1}) \varphi(d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\
&= \varphi(d(x_1^{\alpha_1})) a_2^{\alpha_2} \cdots a_n^{\alpha_n} + a_1^{\alpha_1} \varphi(d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\
&= d_A(a_1^{\alpha_1}) a_2^{\alpha_2} \cdots a_n^{\alpha_n} + a_1^{\alpha_1} d_A(a_2^{\alpha_2} \cdots a_n^{\alpha_n}) \\
&= d_A(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}) \\
&= d_A(\varphi(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})).
\end{aligned}$$

This establishes Step 1.

Step 2: We show that $\mathrm{Ker}(\varphi)$ is d-stable and that $d(f_\lambda) \in \mathrm{Ker}(\varphi)$ for all $\lambda = 1, \dots, n$. Let $g \in \mathrm{Ker}(\varphi)$. Then by Step 1, we have

$$\begin{aligned}
\varphi(d(f)) &= d_A(\varphi(f)) \\
&= d_A(0) \\
&= 0.
\end{aligned}$$

Thus $d(f) \in \mathrm{Ker}(\varphi)$, which implies $\mathrm{Ker}(\varphi)$ is d-stable. Step 1 also implies

$$\begin{aligned}
\varphi(d(f_\lambda)) &= d_A(\varphi(f_\lambda)) \\
&= d_A(d_A(f_\lambda)) \\
&= 0,
\end{aligned}$$

for all $\lambda = 1, \dots, n$.

Now Theorem (9.1) implies that $(S_w/\mathrm{Ker}(\varphi), \overline{d})$ is a differential graded R-algebra. Moreover, Step 1 implies $\varphi : (S_w/\mathrm{Ker}(\varphi), \overline{d}) \to (A, d_A)$ is an isomorphism of differential graded R-algebras. □

10 Constructing the Differential Graded R-algebra $(S/I, r_1, \dots, r_n)$

We now want to consider some special cases of Theorem (9.1). In particular, we want to consider the case where the weighted vector is $w = (1, \dots, 1)$. We will write S to denote the polynomial ring $R[x_1, \dots, x_n]$ equipped with this grading. Let $r_1, \dots, r_n$ be nonzero elements in R, and define $d : S \to S$ by

$$d := \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda}.$$

Since $d(r_\lambda) = 0$ for all $\lambda = 1, \dots, n$, it follows from Theorem (9.1) that $(S, r_1, \dots, r_n)$ is a differential graded R-algebra. Moreover, if I is a d-stable ideal, then $(S/I, r_1, \dots, r_n)$ is a differential graded R-algebra. The next proposition gives a necessary and sufficient condition for a finitely generated ideal I to be d-stable.

Proposition 10.1. *Let I be a homogeneous ideal in S. Then I is d-stable if and only if for some generating set $F = \{f_1, \dots, f_r\}$ of I, we have $d(f_\lambda) \in I$ for all $\lambda = 1, \dots, r$.*

Proof. One direction is trivial, so let's prove the other direction. Let $F = \{f_1, \dots, f_r\}$ be a generating set for I such that $d(f_\lambda) \in I$ for all $\lambda = 1, \dots, r$ and let $f \in I$. Since $\{f_1, \dots, f_r\}$ generates I, we can write $f = \sum_{\lambda=1}^r q_\lambda f_\lambda$ for some $q_1, \dots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(f) &= d\left(\sum_{\lambda=1}^r q_\lambda f_\lambda\right) \\ &= \sum_{\lambda=1}^r d(q_\lambda f_\lambda) \\ &= \sum_{\lambda=1}^r (d(q_\lambda) f_\lambda + q_\lambda d(f_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. □

10.1 Koszul Complex

Recall from Example (3.2) that the Koszul complex $\mathcal{K}(r_1, \dots, r_n)$ is a differential graded R-algebra. Indeed, $\mathcal{K}(r_1, \dots, r_n)$ is isomorphic to the differential graded R-algebra $(S/I, r_1, \dots, r_n)$, where I is generated by $\{x_1^2, \dots, x_n^2\}$.C learly I is d-stable since $d(x_\lambda^2) = 0$ for all $\lambda = 1, \dots, n$.

Example 10.1. Let $R = \mathbb{F}_2[x, y]/\langle xy \rangle$ and let $r_1 = x$ and $r_2 = y$. Then $S = R[u, v]$ has a differential graded R-algebra structure with the differential d given by

$$d := x\partial_u + y\partial_v.$$

Using graded lexicographical ordering on the monomials, we can explicitly write S as a chain complex over R using matrices as the linear maps:

$$\cdots \longrightarrow R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & y & 0 \\ 0 & x & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

Now let I be the homogeneous ideal in S generated by $\{x^2, y^2\}$. Then $(S/I, r_1, r_2)$ is isomorphic to the Koszul complex $\mathcal{K}(r_1, r_2)$. Using graded lexicographical ordering on the monomials, we can explicitly write S/I as a chain complex over R using matrices as the linear maps:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

10.2 Blowup algebras

Proposition 10.2. *Let Q be a finitely generated ideal in R with generating set $\{a_1, \dots, a_n\}$. Then the blowup algebra $B_Q(R)$ can be given the structure of differential graded R-algebra.*

Proof. Let $\varphi : S \to B_Q(R)$ be the unique graded R-algebra homomorphism such that $\varphi(x_\lambda) = ta_\lambda$ for all $\lambda = 1, \dots, n$ and let $d := \sum_{\lambda=1}^n a_\lambda \partial_\lambda$. We claim that $\mathrm{Ker}(\varphi)$ is d-stable. Indeed, let $f \in \mathrm{Ker}(\varphi)$. Since $\mathrm{Ker}(\varphi)$ is homogeneous, we may assume that f is homogeneous. Write f and $d(f)$ in terms of the monomial basis:

$$f = \sum_{\lambda=1}^r b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}} \quad \text{and} \quad d(f) = \sum_{\substack{1 \le \mu \le n \\ 1 \le \lambda \le r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}.$$

where $b_\lambda \in R$ and $\alpha_{\mu\lambda} \in \mathbb{Z}_{\ge 0}$ for all $\lambda = 1, \dots, r$ and $\mu = 1, \dots n$. Observe that

$$\begin{aligned} 0 &= \varphi(f) \\ &= \varphi\left(\sum_{\lambda=1}^r b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}}\right) \\ &= \sum_{\lambda=1}^r b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_n)^{\alpha_{n\lambda}} \\ &= t^i \left(\sum_{\lambda=1}^r b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n^{\alpha_{n\lambda}}\right) \end{aligned}$$

implies that $\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n{}^{\alpha_{n\lambda}} = 0$. Therefore

$$\begin{aligned}
\varphi(d(f)) &= \varphi\left(\sum_{\substack{1\leq\mu\leq n\\ 1\leq\lambda\leq r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}\right) \\
&= \sum_{\substack{1\leq\mu\leq n\\ 1\leq\lambda\leq r}} \alpha_{\mu\lambda} a_\mu b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_\mu)^{\alpha_{\mu\lambda}-1} \cdots \varphi(x_n)^{\alpha_{n\lambda}} \\
&= t^{i-1}\left(\sum_{\substack{1\leq\mu\leq n\\ 1\leq\lambda\leq r}} \alpha_{\mu\lambda} a_\mu b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_\mu^{\alpha_{\mu\lambda}-1} \cdots a_n^{\alpha_{n\lambda}}\right) \\
&= t^{i-1}\left(\left(\sum_{\mu=1}^{n} \alpha_{\mu\lambda}\right)\left(\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n{}^{\alpha_{n\lambda}}\right)\right) \\
&= 0.
\end{aligned}$$

Therefore $(S/\mathrm{Ker}(\varphi), a_1, \dots, a_n)$ is a differential graded R-algebra where $S/\mathrm{Ker}(\varphi) \cong B_Q(R)$. $\square$

Remark. It isn't too difficult to show that this differential graded R-algebra is $(B_Q(R), \partial_t)$, where ∂_t is defined in the obvious way.

Example 10.2. Let $R = \mathbb{F}_2[x,y]/\langle y^2 + x^3 + x^2\rangle$, $\mathfrak{m}$ be the maximal ideal in R generated by $\{\overline{x}, \overline{y}\}$, S denote the polynomial ring $R[u,v]$, and $d = \overline{x}\partial_u + \overline{y}\partial_v$. There is a surjective R-algebra homomorphism from S to the blowup algebra at $\mathfrak{m}$ given by

$$\varphi : S := \mathbb{F}_2[x,y,u,v]/\langle y^2 + x^3 + x^2\rangle \to B_{\mathfrak{m}}(R),$$

where φ is induced by $\varphi(u) = t\overline{x}$ and $v \mapsto t\overline{y}$. Using Singular, we find that the kernel of φ is an ideal which is homogeneous in the variables u, v, and is generated by the set $\{f_1, f_2, f_3\}$, where

$$\begin{aligned}
f_1 &= \overline{x}v + \overline{y}u \\
f_2 &= \overline{x}u^2 + u^2 + v^2 \\
f_3 &= \overline{x}^2 u + \overline{x}u + \overline{y}v
\end{aligned}$$

Note that $d(f_1) = d(f_2) = d(f_3) \in \mathrm{Ker}(\varphi)$. It follows from Proposition (10.1) that $\mathrm{Ker}(\varphi)$ is d-stable, which we already knew from Proposition (10.2).

10.3 Homology Calculations

Proposition 10.3. *Let $(S/I, r_1, \dots, r_n)$ be a differential graded R-algebra. Suppose that there are $t_1, \dots, t_n \in R$ such*

$$\sum_{\lambda=1}^{n} t_\lambda r_\lambda = 1. \tag{10}$$

Then $H(S/I, r_1, \dots, r_n) = 0$.

Proof. First note that $\sum_{\lambda=1}^{n} t_\lambda x_\lambda \notin I$, otherwise $d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) = 1 \notin I$ would imply that I is not d-stable. Let f be a homogeneous polynomial of degree i such $d(f) \in I$; so f represents a cycle of $(S/I, \overline{d})$. Then

$$\begin{aligned}
d\left(\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f\right) &= d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\
&= \left(\sum_{\lambda=1}^{n} t_\lambda r_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\
&= f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f) \\
&\equiv f \bmod I.
\end{aligned}$$

thus $\mathrm{Ker}(\overline{d}) = \mathrm{Im}(\overline{d})$, which proves the claim. $\square$

Remark.

1. By setting $I = 0$, we also find that $H(S) = 0$.
2. The condition (10) is equivalent to saying that $\{r_1, \dots, r_n\}$ generates the unit ideal.

10.3.1 Long Exact Sequence

It is straightforward to check that

$$\begin{tikzcd} 0 \arrow[r] & (S_w(-j)/(I:g),\overline{d}) \arrow[r, "\cdot g"] & (S/I,\overline{d}) \arrow[r] & (S/\langle I,g\rangle,\overline{d}) \arrow[r] & 0 \\ & \overline{f} \arrow[r, mapsto] & \overline{fg} & & \end{tikzcd} \tag{11}$$

is short exact sequence of chain complexes. The short exact sequence (11) gives rise to a long exact sequence in homology:

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{i+1}(S_w/\langle I,g\rangle) \arrow[dll, "\lambda", rounded corners] \\ H_{i-j}(S_w/(I:g)) \arrow[r, "\cdot g"] & H_i(S_w/I) \arrow[r] & H_i(S_w/\langle I,g\rangle) \arrow[dll, "\lambda", rounded corners] & \\ H_{i-j-1}(S_w/(I:g)) \arrow[r, "\cdot g"] & H_{i-1}(S_w/I) \arrow[r] & \cdots & \end{tikzcd}$$

Let us work out the details of the connecting map: Let $\overline{f}$ be a homogeneous element in $S_w/\langle I, g\rangle$ which represents a class in $H_i(S_w/\langle I, g\rangle)$. In particular, $f \in S$ and $d(f) \in \langle I, g\rangle$. We lift $\overline{f} \in S_w/\langle I, g\rangle$ to S_w/I and then apply d to get $\overline{d(f)} \in S_w/I$. Since $d(f) \in \langle I, g\rangle$, we can write $d(f) = p + gq$ where $p \in I$. Thus, $\overline{d(f)} = \overline{gq}$, and this pulls back to $\overline{q}$ in $S_w/(I : g)$.

11 Extra

11.1 Classifying d-Stable Ideals

Let $(R[x_1, \ldots, x_n]/I, r_1, \ldots, r_n)$ be a differential graded R-algebra. Suppose that there are $t_1, \ldots, t_m \in R$ such that $\langle r_1, \ldots, r_n\rangle = \langle t_1, \ldots, t_m\rangle$ and $(R[y_1, \ldots, y_m]/I, t_1, \ldots, t_m)$ is also a differential graded R-algebra. Then for all $1 \le \lambda \le n$ and $1 \le \mu \le n$, there are $a_{\lambda\mu}$ and $b_{\lambda\mu}$ in R such that

$$r_\lambda = \sum_{\mu=1}^{m} a_{\lambda\mu} t_\mu \text{ and } t_\mu = \sum_{\lambda=1}^{n} b_{\lambda\mu} r_\lambda.$$

Let $\varphi : R[x_1, \ldots, x_n] \to R[y_1, \ldots, y_m]$ be the unique graded R-algebra homomorphism such that $\varphi(x_\lambda) = \sum_{\mu=1}^{m} a_{\lambda\mu} y_\mu$ for all $\lambda = 1, \ldots, n$. Then φ induces a graded R-algebra homomorphism $\overline{\varphi} : R[x_1, \ldots, x_n]/I \to R[y_1, \ldots, y_m]/\langle\varphi(I)\rangle$ which in turn induces a homomorphism of differential graded R-algebras $\overline{\varphi} : (R[x_1, \ldots, x_n]/I, r_1, \ldots, r_n) \to (R[y_1, \ldots, y_m]/\langle\varphi(I)\rangle, t_1, \ldots, t_m)$. Indeed, let us denote the differentials as

$$d_r := \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda} \text{ and } d_t := \sum_{\mu=1}^{m} t_\mu \partial_{y_\mu}.$$

We first show that $\varphi d_r = d_t \varphi$. It is enough to show that $\varphi d_r(x_\lambda) = d_t \varphi(x_\lambda)$ for all $\lambda = 1, \ldots, n$. We have

$$\begin{aligned} d_t\varphi(x_\lambda) &= d_t\left(\sum_{\mu=1}^{m} a_{\lambda\mu} y_\mu\right) \\ &= \sum_{\mu=1}^{m} a_{\lambda\mu} t_\mu \\ &= r_\lambda \\ &= d_r(x_\lambda) \\ &= \varphi(d_r(x_\lambda)). \end{aligned}$$

Now we show that $(R[y_1,\dots,y_m]/\langle\varphi(I)\rangle, t_1,\dots,t_m)$ is a differential graded R-algebra. We do this by showing that $\langle\varphi(I)\rangle$ is d_t-stable. Let $\sum_{\kappa=1}^{r} g_\kappa\varphi(f_\kappa) \in \varphi(I)$. Then

$$\begin{aligned} d_t\left(\sum_{\kappa=1}^{r} g_\kappa\varphi(f_\kappa)\right) &= \sum_{\kappa=1}^{r} d_t(g_\kappa)\varphi(f_\kappa) + \sum_{\kappa=1}^{r} g_\kappa d_t(\varphi(f_\kappa)) \\ &= \sum_{\kappa=1}^{r} d_t(g_\kappa)\varphi(f_\kappa) + \sum_{\kappa=1}^{r} g_\kappa\varphi(d_r(f_\kappa)) \in \langle\varphi(I)\rangle. \end{aligned}$$

Similarly, let $\psi : R[y_1,\dots,y_m] \to R[x_1,\dots,x_n]$ be the unique graded R-algebra homomorphism such that $\psi(y_\mu) = \sum_{\lambda=1}^{n} b_{\lambda\mu}x_\lambda$ for all $\mu = 1,\dots,m$. Then φ induces a graded R-algebra homomorphism $\overline{\psi} : R[y_1,\dots,y_m]/\langle\varphi(I)\rangle \to R[x_1,\dots,x_n]/\langle\psi(\varphi(I))\rangle$ which in turn induces a homomorphism of differential graded R-algebras $\overline{\psi}(R[y_1,\dots,y_m]/\langle\varphi(I)\rangle, t_1,\dots,t_m) \to (R[x_1,\dots,x_n]/\langle\psi(\varphi(I))\rangle, r_1,\dots,r_n)$.

11.1.1 Evalutation Map

Let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra such that I is contained in $\langle x_1,\dots,x_n\rangle$. Let $Q = \langle r_1,\dots,r_n\rangle$ and $\mathrm{Ev}_r : S \to R$ be the unique R-algebra homomorphism such that $\mathrm{Ev}_r(x_\lambda) = r_\lambda$ for all $\lambda = 1,\dots,n$. We are interested in the ideal $\mathrm{Ev}_r(I)$ in R. Clearly we have $\mathrm{Ev}_r(I) \subset Q$. Suppose $a \in Q\backslash\mathrm{Ev}_r(I)$. Then $a = \sum_{\lambda=1}^{n} a_\lambda r_\lambda$ for some $a_\lambda \in R$. This implies $x := \sum_{\lambda=1}^{n} a_\lambda x_\lambda \notin I$. Now $J = I + \langle x, a\rangle$ is an ideal strictly larger than I such that J is d-stable $\mathrm{Ev}_r(J)$ is strictly larger than $\mathrm{Ev}_r(I)$. This implies that we can find an ideal I such that $\mathrm{Ev}_r(I) = Q$.

Proposition 11.1. *Let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra and let $Q = \langle r_1,\dots,r_n\rangle$ be an ideal in R. Suppose that $Ev_r(I) = 0$. Then there exists a unique homomorphism φ which makes the following diagram commute*

$$\begin{tikzcd} & B_Q(R) \arrow[dr, "Ev_1"] & \\ S/I \arrow[ur, dashed, "\varphi"] \arrow[rr, "Ev_r"'] & & R \end{tikzcd}$$

11.1.2 Tensor product of differential graded R-algebras

Let $(R[x_1,\dots,x_n]/I, d_r)$ and $(R[y_1,\dots,y_m]/J, d_t)$ be two differential graded R-algebras, where

$$d_r := \sum_{\lambda=1}^{n} r_\lambda\partial_{x_\lambda} \text{ and } d_t := \sum_{\mu=1}^{m} t_\mu\partial_{y_\mu}.$$

for $r_\lambda, t_\mu \in R$ for all $\lambda = 1,\dots,n$ and $\mu = 1,\dots,m$. Then their tensor product over R is

$$(R[x_1,\dots,x_n]/I, d_r) \otimes_R (R[y_1,\dots,y_m]/J, d_t) \cong (R[x_1,\dots,x_n,y_1,\dots,y_m]/(I+J), d_r + d_t).$$

Example 11.1. The Koszul complex $\mathcal{K}(r_1,\dots,r_n)$ can be realized as a tensor product:

$$\mathcal{K}(r_1,\dots,r_n) \cong \mathcal{K}(r_1) \otimes \cdots \otimes \mathcal{K}(r_n).$$

Let M be an R-module, and let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra. Recall that $(M \otimes_R S/I, d)$ is an (S/I)-module.