Mathematical Programming

1 Convexity

1.1 Convex Sets

Definition 1.1. Let V be an $\mathbb{R}$-vector space and let C be a subset of V. We say C is **convex** if for all $t \in (0,1)$ and $x, y \in C$, we have $tx + (1-t)y \in C$.

Proposition 1.1. *Let V be an $\mathbb{R}$-vector space and let C be a convex subset of V. Then for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in C$, and $t_1, \dots, t_n \in (0,1)$ such that $\sum_{i=1}^n t_i = 1$, we have $\sum_{i=1}^n t_i x_i \in C$.*

Proof. Let $x = \sum_{i=1}^n t_i x_i$ and assume that n is minimal in the sense that if $x = \sum_{i'=1}^{n'} t'_{i'} x'_{i'}$ is another representation of x, where each $x'_{i'} \in C$ and $t'_{i'} \in (0,1)$ such that $\sum_{i'=1}^{n'} t'_{i'} = 1$, then we must have $n \leq n'$. Assume for a contradiction that $x \notin C$, so necessarily $n > 2$. Then observe that

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{t_i}{1-t_n}((1-t_n)x_i + t_n x_n) &= \sum_{i=1}^{n-1} t_i x_i + \left(\sum_{i=1}^{n-1} \frac{t_i}{1-t_n} \right) t_n x_n \\
&= \sum_{i=1}^{n-1} t_i x_i + \left(\frac{1-t_n}{1-t_n} \right) t_n x_n \\
&= \sum_{i=1}^{n-1} t_i x_i + t_n x_n \\
&= \sum_{i=1}^{n} t_i x_i \\
&= x,
\end{aligned}$$

gives another representation of x with $n-1$ terms, a contradiction. □

1.1.1 Convex Closure and Closed Convex Closure

Definition 1.2. Let V be an $\mathbb{R}$-vector space and let S be a subset of V. The **convex closure** of S is defined by

$$\operatorname{conv}(S) = \{tx + (1-t)y \mid t \in (0,1) \text{ and } x, y \in S\}.$$

Moreover, suppose $\|\cdot\|$ is a norm on V, so that $(V, \|\cdot\|)$ is a normed linear space. The **closed convex closure** of S is defined to be the smallest closed convex set which contains S and is denoted by $\overline{\operatorname{conv}}(S)$.

Proposition 1.2. *With the notation as above,* $\operatorname{conv}(S)$ *is the smallest convex set which contains S. Furthermore, we have* $\overline{\operatorname{conv}(S)} = \overline{\operatorname{conv}}(S)$.

Proof. Let us first show that $\operatorname{conv}(S)$ is in fact a convex set. Let $s, t, t' \in (0,1)$ and let $x, x', y, y' \in S$. Then observe that

$$s(tx + (1-t)y) + (1-s)(t'x' + (1-t')y') = stx + s(1-t)y + (1-s)t'x' + (1-s)(1-t')y' \in \operatorname{conv}(S),$$

where we used Proposition (1.1) together with the fact that

$$st + s(1-t) + (1-s)t' + (1-s)(1-t') = 1.$$

It follows that $\operatorname{conv}(S)$ is convex. It is also the smallest convex set which contains S since if C is a convex set which contains S, then we must have $tx + (1-t)y \in C$ for all $t \in (0,1)$ and $x, y \in S$, which implies $\operatorname{conv}(S) \subseteq C$.

Now we will show $\overline{\text{conv}(S)} = \overline{\text{conv}}(S)$. To see this, first note that since $\overline{\text{conv}}(S)$ is convex, we have $\text{conv}(S) \subseteq \overline{\text{conv}}(S)$, and hence

$$\begin{aligned}\overline{\text{conv}(S)} &\subseteq \overline{\overline{\text{conv}}(S)} \\ &= \overline{\text{conv}}(S).\end{aligned}$$

For the reverse inclusion, it suffices to show that $\overline{\text{conv}(S)}$ is convex, since then $\overline{\text{conv}(S)}$ would be a closed convex set, and so $\overline{\text{conv}}(S) \subseteq \overline{\text{conv}(S)}$ by definition of $\overline{\text{conv}}(S)$. In fact, we will show that the closure of a convex set is convex. To this end, suppose C is a convex set and let $t \in (0,1)$ and $x, y \in \overline{C}$. Choose sequences (x_n) and (y_n) in C such that $x_n \to x$ and $y_n \to y$. Then $(tx_n + (1-t)y_n)$ is a sequence in C (as C is convex) which converges to $tx + (1-t)y$. It follows that $tx + (1-t)y \in \overline{C}$, and hence $\overline{C}$ is convex. □

1.1.2 Convex Closure Preserves Minkowski Sum

Definition 1.3. Let V be an $\mathbb{R}$-vector space and let S_1, S_2 be subsets of V. We define the **Minkowski sum** of S_1 and S_2 to be the set

$$S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1 \text{ and } x_2 \in S_2\}.$$

Proposition 1.3. *Let V be an $\mathbb{R}$-vector space and let C_1, C_2 be convex subsets of V. Then $C_1 + C_2$ is convex.*

Proof. Let $t \in (0,1)$, let $c_1, c_1' \in C_1$, and let $c_2, c_2' \in C_2$. Then we have

$$t(c_1 + c_2) + (1-t)(c_1' + c_2') = (tc_1 + (1-t)c_1') + (tc_2 + (1-t)c_2') \in C_1 + C_2,$$

since both C_1 and C_2 are convex. It follows that $C_1 + C_2$ is convex. □

Proposition 1.4. *Let V be an $\mathbb{R}$-vector space and let S_1, S_2 be subsets of V. Then we have*

$$\text{conv}(S_1 + S_2) = \text{conv}(S_1) + \text{conv}(S_2).$$

Proof. Note that $\text{conv}(S_1) + \text{conv}(S_2)$ is a convex set which contains $S_1 + S_2$. Thus

$$\text{conv}(S_1 + S_2) \subseteq \text{conv}(S_1) + \text{conv}(S_2).$$

For the reverse inclusion, let $z_1 \in \text{conv}(S_1)$ and $z_2 \in \text{conv}(S_2)$ and express them as $z_1 = t_1x_1 + (1-t_1)y_1$ and $z_2 = t_2x_2 + (1-t_1)y_2$ where $x_1, y_1 \in S_1$, $x_2, y_2 \in S_2$, and $t_1, t_2 \in (0,1)$. Then note that

$$\begin{aligned}z_1 + z_2 &= t_1x_1 + (1-t_1)y_1 + t_2x_2 + (1-t_2)y_2 \\ &= t_1x_1 + t_2x_2 + y_1 - t_1y_1 + y_2 - t_2y_2 \\ &= (t_1 - t_2)(x_1 + y_2) + t_2(x_1 + x_2) + (1-t_1)(y_1 + y_2),\end{aligned}$$

where $(t_1 - t_2) + t_2 + (1 - t_1) = 1$ and where $x_1 + y_2, x_1 + x_2, y_1 + y_2 \in S_1 + S_2$. It follows that $z_1 + z_2 \in \text{conv}(S_1 + S_2)$. Thus we have the reverse inclusion

$$\text{conv}(S_1 + S_2) \supseteq \text{conv}(S_1) + \text{conv}(S_2).$$

□

1.2 Convex Functions

Definition 1.4. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is **convex** if for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $t \in (0,1)$ we have

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}).$$

We say f is **strictly convex** if for every $t \in (0,1)$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\boldsymbol{x} \neq \boldsymbol{y}$ we have

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) < tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}).$$

Here are some basic facts:

1. f is convex if and only if $\text{epi}(f) = \{(\boldsymbol{x}, c) \mid f(\boldsymbol{x}) \leq c\}$ is a convex set in $\mathbb{R}^{n+1}$.
2. If f_1 and f_2 are convex, then $f_1 + f_2$ is convex.
3. If $\{f_i\}_{i \in I}$ are all convex, then $\sup_{i \in I} f_i$ is convex.
4. If f is convex, then $\{f \leq c\}$ is a convex set for all $c \in \mathbb{R}$. The converse is not true in general.

1.2.1 Differentiable Convex Functions

In the definition of convex functions above, we have not assumed any regularity of f (apart from f only taking finite values). Indeed, one of the main advantages of the (rather extensive) theory of convex functions is that it allows to deal with non-differentiable functions using almost the same methods as we would use for differentiable functions. In particular, it is possible to introduce generalised notions of derivatives that in turn can be used for the characterisation and computation of solutions of optimisation problems. However, we will consider in the following differentiable convex functions, and we will study what the convexity of a function implies for its derivative.

Proposition 1.5. *Assume that the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then f is convex if and only if for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we have*

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{y} - \boldsymbol{x}). \tag{1}$$

Proof. Assume first that f is convex and suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\boldsymbol{x} \neq \boldsymbol{y}$. The convexity of f implies

$$f((\boldsymbol{x}+\boldsymbol{y})/2) \leq \frac{1}{2} f(\boldsymbol{x}) + \frac{1}{2} f(\boldsymbol{y}).$$

Setting $\boldsymbol{h} = \boldsymbol{y} - \boldsymbol{x}$, this inequality reads as

$$f(\boldsymbol{x} + \boldsymbol{h}/2) \leq \frac{1}{2} f(\boldsymbol{x}) + \frac{1}{2} f(\boldsymbol{x} + \boldsymbol{h}).$$

Using elementary transformations, this can be rearranged as

$$f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) \geq \frac{f(\boldsymbol{x}+\boldsymbol{h}/2) - f(\boldsymbol{x})}{1/2}.$$

Repeating this line argumentation with $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{h}/2$ instead of $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{h}$, we obtain

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) &\geq \frac{f(\boldsymbol{x}+\boldsymbol{h}/2) - f(\boldsymbol{x})}{1/2} \\ &\geq \frac{f(\boldsymbol{x}+\boldsymbol{h}/4) - f(\boldsymbol{x})}{1/4} \\ &\vdots \\ &\geq \frac{f(\boldsymbol{x}+2^{-k}\boldsymbol{h}) - f(\boldsymbol{x})}{2^{-k}} \end{aligned}$$

for all $k \in \mathbb{N}$. Now recall that the directional derivative of f at the point $\boldsymbol{x}$ in the direction $\boldsymbol{h}$ is defined as

$$\mathrm{D}f(\boldsymbol{x};\boldsymbol{h}) = \lim_{t \to 0} \frac{f(\boldsymbol{x}+t\boldsymbol{h}) - f(\boldsymbol{x})}{t}$$

and satisfies $\mathrm{D}f(\boldsymbol{x};\boldsymbol{h}) = \nabla f(\boldsymbol{x})^\top \boldsymbol{h}$. Thus taking the limit $k \to \infty$, we see that

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) &\geq \limsup_{k \to \infty} \frac{f(\boldsymbol{x}+2^{-k}\boldsymbol{h}) - f(\boldsymbol{x})}{2^{-k}} \\ &= \mathrm{D}(\boldsymbol{x};\boldsymbol{h}) \\ &= \nabla f(\boldsymbol{x})^\top \boldsymbol{h}. \end{aligned}$$

Replacing $\boldsymbol{h}$ by $\boldsymbol{y} - \boldsymbol{x}$ yields the required inequality.

Conversely, assume that (1) holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Let moreover $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{R}^n$ and $0 \leq t \leq 1$. Denote moreover

$$\boldsymbol{x} = t\boldsymbol{w} + (1-t)\boldsymbol{z}.$$

Then the inequality (1) implies that

$$\begin{aligned} f(\boldsymbol{w}) &\geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{w} - \boldsymbol{x}) \\ f(\boldsymbol{z}) &\geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top (\boldsymbol{z} - \boldsymbol{x}). \end{aligned}$$

Note moreover that

$$z - xw - x = (1-t)(w-z) \quad \text{and} \quad z - x = t(z-w).$$

Thus if we multiply the first line with t, the second line with $(1-t)$, and then add the two inequalities, we obtain

$$\begin{aligned} tf(w) + (1-t)f(z) &\geq f(x) + t\nabla f(x)^\top (1-t)(w-z) + (1-t)\nabla f(x)^\top t(z-w) \\ &= f(tw + (1-t)z). \end{aligned}$$

Since w and z were arbitrary, this proves convexity of f. □

Remark 1. Following basically the same proof as above and strategically replacing inequalities by strict inequalities, one can show that a differentiable function f is strictly convex, if and only

$$f(y) > f(x) + \nabla f(x)^\top (y-x)$$

whenever $x \neq y$.

As an immediate consequence of Proposition (1.5) one obtains the result that the first order necessary condition for a minimiser is, in the case of convex functions, also a sufficient condition. More precisely, the following holds:

Corollary 1. *Assume that $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable. Then x^* is a global minimiser of f if and only if $\nabla f(x^*) = 0$.*

Proof. First recall that the condition $\nabla f(x^*) = 0$ is, independent of the convexity of f, a necessary condition for x^* to be a global (and indeed already local) minimiser. Thus we only need to show that this condition actually implies that x^* *is* a global minimiser. Assume therefore that $\nabla f(x^*) = 0$ and let $y \in \mathbb{R}^n$. Then Proposition (1.5) implies that

$$\begin{aligned} f(y) &\geq f(x^*) + \nabla f(x^*)^\top (y - x^*) \\ &= f(x^*). \end{aligned}$$

Thus x^* is a global minimiser. □

1.2.2 Hessians of Convex Functions

Proposition 1.6. *A twice differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex if and only if the Hessian $\nabla^2(x)$ is positive semi-definite for all $x \in \mathbb{R}^n$.*

Proof. Assume first that f is convex and let $x \in \mathbb{R}^n$. Define the function $g \in \mathbb{R}^n \to \mathbb{R}$ by

$$g(y) = f(y) - \nabla f(x)^\top (y-x)$$

for all $y \in \mathbb{R}^n$. Note that g is a convex function since it is the sum of two convex functions. Moreover we have

$$\nabla g(y) = \nabla f(y) - \nabla f(x) \quad \text{and} \quad \nabla^2 g(y) = \nabla^2 f(y)$$

for all $y \in \mathbb{R}^n$. In particular, $\nabla g(x) = 0$. Thus x is a global minimizer of g. Now the second order necessary condition for a minimizer implies that $\nabla^2 g(x)$ is positive semi-definite. Since $\nabla^2 g(x) = \nabla^2 f(x)$ and x was arbitrary, this prove that the Hessian of f is positive semi-definite for all $x \in \mathbb{R}^n$.

Conversely, assume that the Hessian $\nabla^2 f(x)$ of f is positive semi-definite for all $x \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$. Then Taylor's theorem implies that

$$f(y) = f(x) + \nabla f(x)^\top (y-x) + \frac{1}{2}(y-x)^\top \nabla^2 f(x + t(y-x))(y-x)$$

for some $t \in [0,1]$. Since $\nabla^2 f$ is everywhere positive semi-definite, the quadratic term in this equation is always nonnegative. Thus we can estimate

$$f(y) \geq f(x) + \nabla f(x)^\top (y-x).$$

This implies f is convex. □

Example 1.1. Let $f(\boldsymbol{x}) = (x_1 - 2)^2 + (x_2 - 3)^2$. Then f is convex since

$$\mathrm{H}f(\boldsymbol{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which is positive semi-definite for all $\boldsymbol{x}$.

Theorem 1.1. *Let $U \subseteq V$ be an open set and let $f\colon U \to \mathbb{R}$ be a C^2 function. Suppose that a is a critical point for f in the sense that $\mathrm{D}f(a) = 0$ for some $a \in U$. Let $\mathrm{H}_f(a)\colon V \times V \to \mathbb{R}$ be the symmetric bilinear Hessian $\mathrm{D}^2 f(a)$, and let $q_{f,a}\colon V \to \mathbb{R}$ be the associated quadratic form. If $\mathrm{H}_f(a)$ is non-degenerate, then f has an isolated local minimum at a when $q_{f,a}$ is positive-definite, an isolated local maximum at a when $q_{f,a}$ is negative-definite, and neither a local minimum nor a maximum in the indefinite case.*

Proof. Replacing f with $f - f(a)$, we may assume that $f(a) = 0$. By Taylor's formula, for small h we have

$$\frac{f(a+h)}{\|h\|^2} = \frac{1}{2}\mathrm{H}_f(a)(\widehat{h}, \widehat{h}) + R_a(h) = q_{f,a}(\widehat{h}) + R_a(h)$$

where $R_a(h) \to 0$ as $h \to 0$ and $\widehat{h} = h/\|h\|$ is a unit vector pointing in the same direction as h. Thus, $f(a+h)/\|h\|^2$ is approximated by $q_{f,a}(\widehat{h})$ up to an error that ends to 0 locally uniformly in a as $h \to 0$. Provided that $q_{f,a}$ is non-degenerate, in the positive-definite case it is bounded below by some $c > 0$ on the unit sphere, and hence (depending on c) by taking h sufficiently small we get $f(a+h)/\|h\|^2 \geq c/2 > 0$. This shows that f has an isolated local minimum at a, and a similar argument gives an isolated local maximum at a if $q_{f,a}$ is negative-definite.

Now suppose that $q_{f,a}$ is indefinite. By the spectral theorem, if we choose the norm on V to come from an inner product, then the pairing $\mathrm{H}_f(a)$ is given by the inner product against an orthogonal linear map. Hence, in such cases we can find an orthonormal basis with respect to which $q_{f,a}$ is diagonalized, and so in the indefinite case there are lines on which the restriction of $q_{f,a}$ is positive-definite and there are lines on which the restriction of $q_{f,a}$ is negative-definite. Approaching a along such directions gives different types of behavior for f at a (isolated local minimum when approaching through the positive light cone for $q_{f,a}$, and an isolated local maximum when approaching through the negative light cone for $q_{f,a}$, provided the approach is not tangential to the null cone of vectors $v \in V$ for which $q_{f,a}(v) = 0$). This gives the familiar "saddle point" picture for the behavior of f, with the shape of the saddle governed by the eigenspace decomposition for the orthogonal map arising from the Hessian $\mathrm{H}_f(a)$ and the choice of inner product on V. $\square$

Example 1.2. Let $f(\boldsymbol{x}) = x_1^2 - 2x_1x_2 - x_2^2$ and let $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$. The gradient of f at a is given by

$$\nabla f(\boldsymbol{a}) = \begin{pmatrix} 2a_1 - 2a_2 \\ -2a_1 - 2a_2 \end{pmatrix}$$

The Hessian of f at $\boldsymbol{a}$ is given by

$$\mathrm{H}f(\boldsymbol{a}) = \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix}, \quad q_{f,\boldsymbol{a}} = 2x$$

In particular, the 0 is a critical point, and morever $\mathrm{H}f(0)$ diagonalizes as

$$\mathrm{H}f(0) = \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1-1-\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{4}(2+\sqrt{2}) \\ -\frac{1}{2\sqrt{2}} & \frac{1}{4}(2-\sqrt{2}) \end{pmatrix} = CDC^{-1}$$

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^\top(\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{x})^\top \nabla^2 f(\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}))(\boldsymbol{y} - \boldsymbol{x})$$

$$f(x+h) = f(x) + \mathrm{D}f(x)(h) + \frac{1}{2}\mathrm{D}^2 f(x+th)(h,h)$$

for some $t \in [0,1]$.

$$R_{p,a}(h) = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!}((\mathrm{D}^p f)(a+th) - \mathrm{D}^p f(a))(h^{(p)})\mathrm{d}t$$

By the second Fundamental Theorem of Calculus (applied componentwise using a basis of W, say) we have

$$\begin{aligned} f(a+h) &= f(a) + \int_0^1 \mathrm{D}f(a+th)(h)\mathrm{d}t \\ &= f(a) + \mathrm{D}f(a)(h) + \int_0^1 (\mathrm{D}f(a+th) - \mathrm{D}f(a)(h))\mathrm{d}t \end{aligned}$$