ALGEBRAIC TOPOLOGY: CLASS NOTES: SPRING 2022

MICHAEL BURR

1. Day 1: Introduction, motivation, and homotopies

1.1. **Motivation and overview.** Algebraic topology began as a technique to distinguish topological spaces. It has grown into a considerably more sophisticated area that can be used to explain many behaviors in mathematics.

Note 1.1. The basic idea of algebraic topology is to construct an algebraic object (such as a group, ring, or module) from a topological space. Two topological spaces which are equivalent should have the same constructed object. This gives a way to distinguish topological spaces. In particular, if the algebraic object associated to topological space X is not the same as the algebraic object associated to topological space Y, then X and Y are not equivalent topological spaces. In addition, these constructions are designed to be functorial, in other words, for any continuous map $f : X \to Y$, there is a corresponding map between the associated algebraic objects.

Fact 1.2. There are many (potentially surprising) topological facts whose proofs become easy when using algebraic topology:

- (Brouwer's fixed point theorem) Every continuous function from a closed disk in $\mathbb{R}^n$ to itself has a fixed point.
- (Borsuk-Ulam theorem) Suppose that f is a continuous function from the n-dimensional unit sphere to $\mathbb{R}^n$. There exists a point x in the unit sphere such that $f(x) = f(-x)$.
- (Hedgehog theorem) There does not exist a nowhere vanishing continuous tangent vector field on even-dimensional spheres.

Note 1.3. Algebraic topology can also be used to prove facts in algebra, and the proofs are sometimes quite easy when using topology (in contrast, algebraic proofs are either very hard or unknown):

- (Fundamental theorem of algebra) Every positive degree polynomial with complex coefficients has a complex root.
- (Nielsen-Schreier theorem) Every subgroup of a free group is free.

Note 1.4. Often, the existence (or nonexistence) of topological constructions can be expressed in terms of algebraic topology. For example, the Stiefel-Whitney numbers on a manifold of dimension n are values in $\mathbb{Z}/2$, which are computed by evaluating particular (simplicial) cohomology elements.

- Pontryagin proved that if M is the boundary of a manifold, then all of the Stiefel-Whitney numbers vanish.

Hence, these numbers describe an *obstruction* to a manifold being a boundary: For a fixed manifold M, we can calculate these numbers, and if any are nonzero, then M is not a boundary.

- On the other hand, Thom proved that if all of the Stiefel-Whitney numbers of a manifold M are zero, then M is a boundary of a manifold.

Therefore, the nonvanishing of the Stiefel-Whitney numbers is the only obstruction to a manifold being a boundary.

1.2. **Point-set topology review (topological spaces and continuous maps).** We briefly review some basic concepts from point-set topology. Additional concepts from point-set topology will be reviewed, as needed, throughout these notes.

Note 1.5. Loosely speaking, a topological space is a set with a list of its open sets.

The main source for these notes is Allen Hatcher's Algebraic Topology book [**?**].

Definition 1.6. Let X be a set and denote the power set of X by 2^X. A *topology* on X is a subset $\mathcal{T} \subseteq 2^X$ with the following properties:

- $\emptyset, X \in \mathcal{T}$.
- Arbitrary unions of elements of $\mathcal{T}$ is in $\mathcal{T}$.
- Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.

A *topological space* is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X.

Example 1.7. Let X be a set. The *discrete topology* on X is the collection $\mathcal{T} = 2^X$. In this topology, every subset of X is open.

Example 1.8. Let X be a set. The *indiscrete topology* on X is the collection $\mathcal{T} = \{\emptyset, X\}$. This topology has the fewest open sets.

Example 1.9. Let (X, d) be a metric space. In other words, X is a set and $d : X \times X \to \mathbb{R}$. Such that

- For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
- For all $x, y \in X$, $d(x, y) = d(y, x)$.
- (Triangle inequality) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

Let $x \in X$ and $\varepsilon > 0$. The *open ball* of radius ε and centered at x is $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. The *metric topology* on X is the collection $\mathcal{T}$ consisting of all arbitrary unions of open balls in X.

Definition 1.10. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$. The map f is *continuous* if for all open sets $U \in \mathcal{U}$, $f^{-1}(U) \in \mathcal{T}$. The map f is a *homeomorphism* if f is continuous, bijective, and f^{-1} is also continuous. Two topological spaces are *homeomorphic* if there is a homeomorphism between them.

Note 1.11. The definition of a continuous map can be stated, loosely, as the preimage of an open set is open. The definition of continuity using ε and δ from calculus is equivalent to the continuity definition when X and Y are real spaces using the (Euclidean) metric topology.

Fact 1.12. The homeomorphic property is an equivalence relation on topological spaces.

1.3. **Homotopies and homotopy equivalence.** In general, topologies can be very complicated and it is difficult to test if two topological spaces are homeomorphic. We will use a weaker notion of equivalence between spaces, called homotopy equivalence. The idea is to use continuous deformations of spaces.

Definition 1.13. Throughout these notes, we let $I = [0, 1]$ be the *unit interval*.

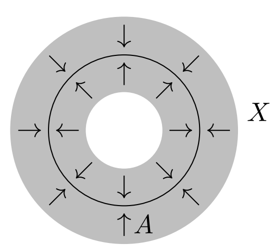
Definition 1.14. Let X be a topological space and $A \subseteq X$. A *deformation retraction* of X onto A is a continuous map $F : X \times I \to X$ such that:

- Define $f_0 := F(\cdot, 0)$. $f_0 = \mathbb{1}_X$, the identify function on X.
- Define $f_1 := F(\cdot, 1)$. $f_1(X) = A$.
- Define $f_t := F(\cdot, t)$. For all t, the restriction is $f_t|_A = \mathbb{1}_A$, the identify function on A.

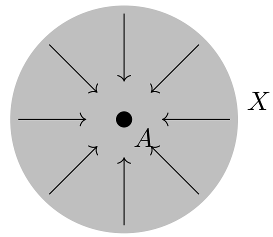
In this case, we say that A is a *deformation retract* of X.

Note 1.15. The conditions on a deformation retract, loosely speaking, mean that a deformation retraction is a deformation of the identity function to a map into A. The deforming maps are the maps $f_t := F(\cdot, t)$. Moreover, throughout this deformation, the points of A are fixed.

Example 1.16. An annulus can be retracted to a circle by moving the points radially inward or outward towards the circle.



Example 1.17. A disk can be retracted to a point by moving the points radially inward towards the center of the disk.



Example 1.18. Any $\mathbb{R}^n$ can be retracted to a point by moving the points of $\mathbb{R}^n$ inward. The points do not move at constant speed, but move faster the further they are from the origin.

Note 1.19. Deformation retractions are a good way to (initially) think about deformations of space, but they do leave some details to be desired. In particular, a deformation retract of a deformation retract might not be a deformation retract. Therefore, we use a generalization of this construction.

Definition 1.20. Let X and Y be topological spaces. A *homotopy* is a continuous map $F : X \times I \to Y$. Define $f_0 := F(\cdot, 0)$ and $f_1 := F(\cdot, 1)$. Then the two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are *homotopic*. We write this relationship as $f_0 \simeq f_1$.

Note 1.21. The homotopic property is an equivalence relation on continuous maps from X to Y.

Definition 1.22. Let X be a topological space and $A \subseteq X$. A *retraction* of X onto A is a continuous map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}_A$.

Note 1.23. A retraction $r : X \to X$ could alternately be defined as a continuous map where $r^2 = r$.

Example 1.24. A deformation retraction is a homotopy between $\mathbb{1}_X$ and r that is the identity on A.

Example 1.25. Not all retractions are deformation retractions. The simplest such example is the constant map where $A = \{p\}$ consists of a single point and r takes all points in X to p. We will see that for many spaces, such a map could not be a deformation retraction.

Definition 1.26. Let X and Y be topological spaces, $F : X \times I \to Y$ a homotopy, and $A \subseteq X$. We say that F is a homotopy *relative* to A if for all t, the map $f_t|_A$ is independent of A.

Example 1.27. A deformation retraction of X onto A is a homotopy from $\mathbb{1}_X$ to r relative to A.

Definition 1.28. Let X and Y be two topological spaces. A continuous map $f : X \to Y$ is a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq \mathbb{1}_X$. When such an f exists, we say that X and Y are *homotopy equivalent*.

Note 1.29. We do not need f and g to be inverses of each other, all that is needed is that their compositions can be deformed to the appropriate identity.

Example 1.30. Suppose that X is a deformation retraction onto A given by $r : X \to A$. Let $i : A \to X$ be the inclusion map. Then, $r \circ i = \mathbb{1}_A$ and $i \circ r \simeq \mathbb{1}_X$. Therefore, r is a homotopy equivalence between X and A.

Note 1.31. The homotopy equivalence property is an equivalence relation on topological spaces.

2. Day 2: Bases, the mapping cylinder, and contractibility

2.1. **Point-set topology review (bases and standard topologies).** In many cases, it is quite complicated to describe all of the open sets in a topology directly. Instead, we use smaller sets, which may be easier to describe, to generate the topology.

Definition 2.1. Let X be a topological space. A *basis* on X is a subset $\mathcal{B} \subseteq 2^X$ with the following properties:

- $\mathcal{B}$ covers X, that is, $\bigcup_{U \in \mathcal{B}} U = X$.
- For every $U_1, U_2 \in \mathcal{B}$, for all $x \in U_1 \cap U_2$, there exists a U_3 in $\mathcal{B}$ such that $x \in U_3 \subseteq U_1 \cap U_2$.

The topology *generated* by $\mathcal{B}$ consists of all arbitrary unions of elements of $\mathcal{B}$.

Note 2.2. The second condition for a basis states that whenever two elements of the basis intersect, the intersection can be written as a union of elements of $\mathcal{B}$. The condition is vacuously satisfied when U_1 and U_2 do not intersect. The topology generated by a basis is the smallest topology where the elements of $\mathcal{B}$ are open.

Example 2.3. A basis for a metric topology consists of all open balls.

Definition 2.4. Let $(X, \mathcal{T})$ be a topological space. This topological space is *second countable* if there is a countable basis $\mathcal{B}$ which generates the topology $\mathcal{T}$.

Note 2.5. We often restrict our attention to second countable spaces since they are relatively well-behaved.

Example 2.6. The space $\mathbb{R}^n$ with the metric topology is second countable when we use open balls centered a points in $\mathbb{Q}^n$ with rational radii.

Definition 2.7. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The *induced* topology on A is the topology whose open sets are of the form $U \cap A$ for some $U \in \mathcal{T}$.

Fact 2.8. Let $(X, \mathcal{T})$ be a topological space with basis $\mathcal{B}$ and $A \subseteq X$. A basis for the induced topology on A is given by the intersection of elements of $\mathcal{B}$ with A.

Definition 2.9. Let $(X, \mathcal{T})$ be a topological space and $q : X \to Y$ a surjective map. The *quotient* topology on Y is the topology whose open sets U are those sets for which $q^{-1}(U)$ is open in X. When Y has the quotient topology, q is called a quotient map.

Note 2.10. We can construct an equivalence relation on X where x_1 is equivalent to x_2 if and only if $q(x_1) = q(x_2)$. In this case, Y consists of the equivalence classes of this relation. We often visualize quotients by gluing or identifying points in the same class together.

Fact 2.11. If $q : X \to Y$ is an open quotient map, that is, the image of an open set is open, then a basis for the quotient topology is given by the image of a basis for the topology on X.

Note 2.12. Quotient spaces have the following universal property. Suppose that X, Y, and Z are topological spaces such that $q : X \to Y$ is a quotient and $f : X \to Z$ is continuous map such that if $x, y \in X$ are in the same equivalence class of the quotient, then $f(x) = f(y)$. Then, there is a unique continuous map $\widetilde{f} : Y \to Z$ such that $f = \widetilde{f} \circ q$. This is captured in the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Z \\ Y \arrow[ur, dashed, "\exists!\widetilde{f}"'] & \end{tikzcd}$$

Definition 2.13. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be two topological spaces. The *product* topology on $X \times Y$ is the topology generated by the basis consisting of sets of the form $U \times V$ where $U \in \mathcal{T}$ and $V \in \mathcal{S}$.

Note 2.14. In general, the product topology contains many more sets than products of open sets, so the basis identified in the product topology definition is not a topology.

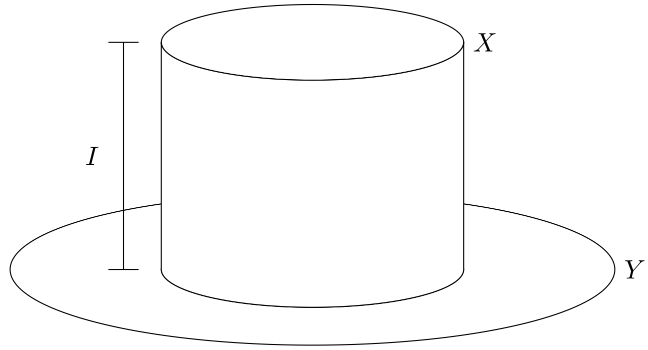
Fact 2.15. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topologies with bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Then a basis for the product topology consists of sets of the form $U \times V$ where $U \in \mathcal{B}$ and $V \in \mathcal{C}$.

2.2. **Mapping cylinder.** A common construction which generates deformation retracts is the mapping cylinder. We occasionally return to the ideas behind mapping cylinders later in these notes.

Definition 2.16. Let X and Y be topological spaces and $f : X \to Y$ a continuous map. The *mapping cylinder* M_f is the quotient space of the disjoint union $(X \times I) \coprod Y$ where $(x, 1)$ is identified with $f(x)$.

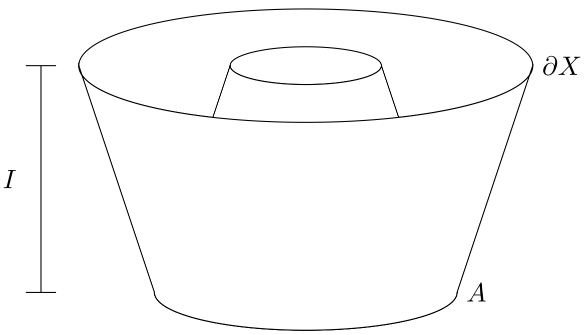
Note 2.17. The disjoint union of two sets A and B is the set containing a copy of A and a copy of B, duplicating any intersection between A and B. This can be constructed as $(A \times \{1\}) \cup (B \times \{2\})$ since any points in the intersection of A and B have distinct second coordinates.

Example 2.18. Schematically, the mapping cylinder looks like the following diagram.



Note 2.19. The mapping cylinder M_f deformation retracts onto Y by sliding the points in $X \times I$ along I. In other words, for a point (x, t), in the deformation retraction t increases to 1.

Example 2.20. In many cases, the mapping cylinder illustrates a deformation retraction. In Example 1.16, the boundary of the annulus is X and A is Y. In this case, the mapping cylinder looks like the following diagram.



If we imagine flattening this shape out, the result is the original annulus of Example 1.16.

Example 2.21. The mapping cylinder version of Example 1.17 is a cone.

Example 2.22. The reason that the construction in the previous two examples works is that the points on the boundary of X travel along disjoint paths to A. In general, not all deformation retracts can be expressed in terms of a mapping cylinder. The deformation retraction of a thickened "X" to a point by first retracting to X and then retracting the legs of X to the crossing point cannot be expressed as a mapping cylinder since some of the boundary points come together before the end of the deformation.

Fact 2.23. Suppose that X and Y are topological spaces. X and Y are homotopy equivalent if and only if there is a third space Z that contains both X and Y as deformation retracts.

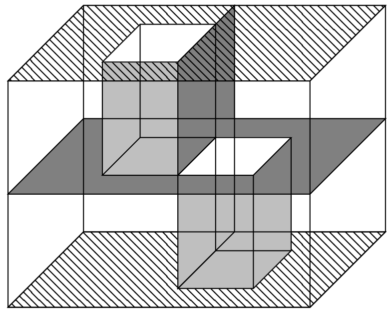
Note 2.24. The space Z in the previous fact can be taken to be the mapping cylinder of the map $f : X \to Y$. This space certainly retracts onto Y, but it takes a bit of work to show that it also retracts onto X.

2.3. **Contractible spaces.** Contractible spaces are ones that can be deformed to a point. One example of a contractible space for which the deformation is hard to visualize is the house with two rooms. The example consists of a house with two floors and a single room on each floor. The top floor is only accessible from beneath the house and the bottom floor is only accessible from the roof.

Definition 2.25. A map is *nullhomotopic* if it is homotopic to a constant map. A space which is homotopy equivalent to a point is *contractible*.

Note 2.26. A topological space X being contractible is equivalent to the identity $\mathbb{1}_X$ being nullhomotopic. Contractibility is slightly weaker than deformation retractibility to a point.

Example 2.27. The house with two rooms can be visualized in the following picture. The two floors are separated by the horizontal layer in gray. This divider has a hole in the middle to allow access to the bottom floor from the roof and access to the top floor from below the house. The access to the top and bottom floors is via a tube, depicted in light gray. These tubes are connected to the side of the house via a common wall, but it is important to note that the wall does not extend all the way across the house. If the wall were to connect all the way across the house, the top and bottom floors would consist of two rooms each. Since the wall doesn't extend, the top floor is connected, and, similarly, the bottom floor is connected.



It is quite challenging to see how this space is contractible directly (I don't know how to visualize it). One, however, can show that this is contractible by showing it is homotopy equivalent to another space, which, in turn, is homotopy equivalent to a point. First, thicken all the walls of the house by a small amount. The thickened house is homotopy equivalent to the original house because the thickened walls can be squished down to the thin walls. The thickened house, in turn, can be deformed from a 3-dimensional ball, by imagining the 3-dimensional ball as a made of clay, and we push down from above and up from below to hollow out the bottom and top of the house, respectively. A 3-dimensional ball has the same homotopy type as a point, as seen in Example 1.17.

3. Day 3: Weak topology and cell complexes

3.1. **Point-set topology review (weak topology).** The weak topology is one of the topologies that can be used for infinite unions of objects. This topology makes continuous maps on infinite unions straight-forward

Definition 3.1. Let (X^i) be a sequence of topological spaces where there is an inclusion map $\Phi_i : X^i \hookrightarrow X^{i+1}$ for each i. Let $X = \bigcup_i X^i$. The *weak* topology on X has open sets U provided $U \cap X^i$ is open in X^i for every i.

Fact 3.2. Let (X^i) be a sequence of topological spaces with inclusion maps $\Phi_i : X^i \hookrightarrow X^{i+1}$ and $X = \bigcup_i X^i$. Suppose that X has the weak topology. A map $f : X \to Y$ is continuous if and only if $f|_{X^i} : X^i \to Y$ is continuous for all i.

3.2. **Cell complexes.** One of the most common topological constructions are cell complexes (also called CW complexes). This construction provides a dimension-by-dimension construction of topological spaces. There is plenty of freedom in these constructions to arrive at interesting examples which are not too complicated.

Definition 3.3. A *cell complex* is constructed through the following iterative procedure:

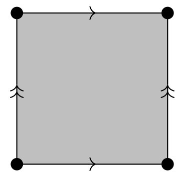
- X^0 is a discrete set of points, which are called 0*-cells*.
- The n*-skeleton* X^n is constructed iteratively from the $(n-1)$-skeleton X^{n-1} by attaching n-dimensional disks to X^{n-1}. More precisely, for an n-dimensional disk D^n_α, let $\varphi^n_\alpha : S^{n-1} \to X^{n-1}$ be a map from the boundary $S^{n-1} = \partial D^n$ of the disk to the $(n-1)$-skeleton. Then, X^n is the quotient of $X^{n-1} \coprod_\alpha D^n_\alpha$ under the identification of $x \in \partial D^n_\alpha$ with $\varphi^n_\alpha(x) \in X^{n-1}$.
- If we terminate the construction at some finite time, that is, $X = X^n$ for some n, then X has the appropriate quotient topology. Alternatively, the procedure may not terminate so $X = \bigcup_n X^n$ and is given the weak topology.

The image of the interior of D^n_α is called an n*-cell* and is denoted by e^n_α. For each D^n_α, the *characteristic* or *attaching* map is the extension of φ^n_α from S^{n-1}_α to D^n_α where $\Phi^n_\alpha : D^n_\alpha \to X$.

Example 3.4. A 1-dimensional cell complex, that is $X = X^1$ consists of points (zero cells) and segments (one cells) between them. In other words, X is a *graph*.

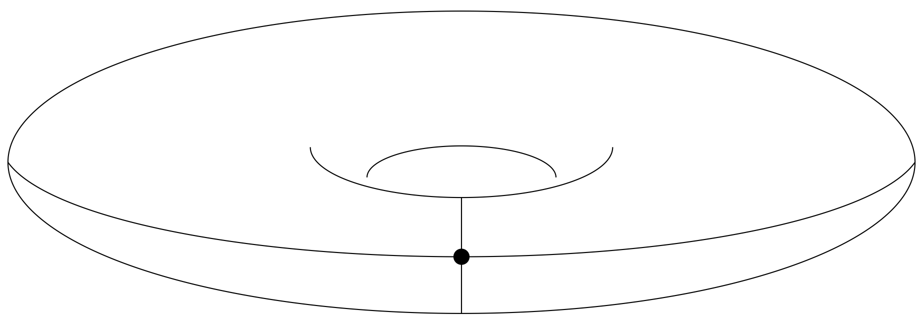
Example 3.5. An n-dimensional sphere can be formed as a cell complex by attaching an n-cell to a 0-cell. In particular, the entire boundary of an n-dimensional disk is mapped to a single point.

Example 3.6. One way to construct a torus is to identify the edges of a square as in the following diagram. The left and right sides of the square are glued to make a tube and the top and bottom edges of the tube are glued together to make a torus.



In the identification, the vertices on the left are identified with the corresponding vertices on the right. Similarly, the vertices on the top are identified with the corresponding vertices on the bottom. Therefore, the vertices on the ends of each side of the square are identified, so there is only one vertex in this construction. Therefore, X^0 consists of a single point. Since the top and bottom edges are identified and the left and right edges are identified, in the final torus, there are two edges, one for the horizontal edge and one for the vertical edge. Therefore, the 1-skeleton consists of a figure eight, as in the following picture.

In the torus, one of the figure eight loops corresponds to a circle around the outside of the torus and the other is a circle through the hole of the torus. When these two loops have been removed from the torus, the remaining 2-dimensional surface is the image of the interior of the square above and is the unique 2-cell of the torus.



Example 3.7. The real projective space $\mathbb{RP}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$. Since each line through the origin intersects the n-dimensional unit sphere in two antipodal points, real projective space can be constructed as a quotient of the sphere. This construction is equivalent to the quotient of the closed upper hemisphere D^n where antipodal points on the boundary are identified. In other words, as a cell complex, the boundary of D^n is glued via a quotient map to $\mathbb{RP}^{n-1}$. Thus, $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$ where each gluing map is the appropriate quotient map.

Note 3.8. Topologists often write real projective space as $\mathbb{RP}^n$, but algebraic geometers often write it as $\mathbb{P}^n_{\mathbb{R}}$.

Example 3.9. The complex projective space $\mathbb{CP}^n$ is the set of all complex lines through the origin in $\mathbb{C}^{n+1}$. There is a similar construction as for real projective space, but a bit more complicated. In particular, the complex projective space can be constructed as a quotient of S^{2n+1} in $\mathbb{C}^{n+1}$ by multiplication by complex numbers of magnitude one. Restricting a bit more, so that the last coordinate is real and nonnegative, we can write complex projective space as a quotient of D^{2n} where the boundary S^{2n-1} is mapped to $\mathbb{C}^{n-1}$ by the quotient map identified above. Thus, $\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$.

Example 3.10. The infinite dimensional real projective space is the infinite union $\mathbb{RP}^\infty = \bigcup \mathbb{RP}^n$, which is a cell complex. Then $\mathbb{RP}^\infty = e^0 \cup e^1 \cup \cdots$.

Example 3.11. The infinite dimensional complex projective space is the infinite union $\mathbb{CP}^\infty = \bigcup \mathbb{CP}^n$, which is a cell complex. Then $\mathbb{CP}^\infty = e^0 \cup e^2 \cup e^4 \cup \cdots$.

Definition 3.12. Let X be a cell complex, and $A \subseteq X$. A is a *subcomplex* if A is a closed union of cells of X. A pair (X, A) of a cell complex and a subcomplex is called a *CW pair*.

Note 3.13. By construction, each characteristic map of a cell of A has image in A, so A is, itself, a cell complex.

Example 3.14. Using the construction of S^n in Example 3.5, S^{n-1} is not a subcomplex of S^n. However, we may, alternately, construct S^n by gluing an upper and lower hemisphere to S^{n-1}. In this case, the equator of S^n is a subcomplex of S^n and is homeomorphic to S^{n-1}.

3.3. **Basic operations on cell complexes.** Many standard operations on sets and topological spaces can be applied to cell complexes to produce new cell complexes. We'll first explore the straight-forward constructions and then consider more sophisticated (but useful) constructions.

Definition 3.15. Let X and Y be two cell complexes. The product $X \times Y$ is composed of cells of the form $e^n_\alpha \times e^m_\beta$ where e^n_α is a cell of X and e^m_β is a cell of Y. The boundary of this cell is

$$\partial(e^n_\alpha \times e^m_\beta) = (\partial e^n_\alpha) \times e^m_\beta \cup e^n_\alpha \times (\partial e^m_\beta).$$

The attaching map for this cell is the product of the attaching maps $\Phi^n_\alpha \times \Phi^m_\beta$ restricted to the boundary of the cell. This cell complex is given the topology as a cell complex.

Note 3.16. The dimension of the cell $e^n_\alpha \times e^m_\beta$ in the product $X \times Y$ is $n + m$, so it appears in the $(n + m)$-skeleton.

Example 3.17. When $X = S^1$ and $Y = S^1$ are circles, then $X \times Y$ is a torus. Moreover, since X and Y each have a 0-cell and a 1-cell, the product has one 0-cell, two 1-cells, and one 2-cell, which are exactly the cells in the torus in Example 3.6.

Fact 3.18. The product topology on $X \times Y$ may have fewer open sets than the cell complex topology on $X \times Y$. The two topologies agree, however, if either X or Y has finitely many cells or both X and Y have countably many cells.

Definition 3.19. Let (X, A) be a CW pair (with $A \neq \emptyset$). Then X/A is a cell complex where the cells are the cells of X not in A along with a new 0-cell (corresponding to the cells in A). The attaching maps of these cells is the composition $q \circ \varphi^n_\alpha$ of the quotient map with the attaching map of e^n_α.

Note 3.20. The skeletons of the cell complex (X/A) have the form $(X/A)^n = X^n/A^n$.

Example 3.21. Suppose that the disk D^n is constructed as a cell complex by attaching an n-cell to S^{n-1}. Then (D^n, S^{n-1}) is a CW pair and the quotient D^n/S^{n-1} is the sphere S^n with the same cell complex structure as Example 3.5.

Example 3.22. Suppose that X is a cell complex. The *cone* $C(X)$ is the cell complex $(X \times I)/(X \times \{0\})$.

Note 3.23. The cone $C(X)$ takes the cylinder $X \times I$ and contracts one end of the cylinder to a point. The cone $C(X)$ of any cell complex X is always contractible by deforming to the point of the cone.

Example 3.24. Suppose that X is a cell complex. The *suspension* $S(X)$ is the cell complex $((X \times I)/(X \times \{0\}))/(X \times \{1\})$.

Note 3.25. The suspension $S(X)$ takes the cylinder $X \times I$ and contracts each end of the cylinder to a distinct point. The suspension becomes more important later in algebraic topology.

Fact 3.26. Suppose that X and Y are two cell complexes and $f : X \to Y$ is a continuous map. Then there is a suspended map $Sf : SX \to SY$ which takes $[(x, t)]$ to $[(f(x), t)]$. This map is well-defined when $t = 0$ or $t = 1$ since all points in SX and SY which have $t = 0$ are identified in both spaces, and similarly for $t = 1$.

- Next: Constructing new spaces
- After that: Criteria for homotopy equivalence
- Following that: Homotopy extension property