# Mathematical Programming Homework 1

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## Problem 1

**Exercise 1.** Which of the following collection of vectors form a basis in $\mathbb{R}^3$, span $\mathbb{R}^3$, or neither? Explain why.

1. $a^1 = (1,2,1)^\top$, $a^2 = (-1,0,-1)^\top$, $a^3 = (0,0,1)^\top$
2. $b^1 = (1,3,2)^\top$, $b^2 = (1,0,5)^\top$
3. $c^1 = (-1,2,3)^\top$, $c^2 = (0,1,0)^\top$, $c^3 = (1,2,3)^\top$, $c^4 = (-3,2,4)^\top$

**Solution 1.** 1. The collection of vectors $\{a^1, a^2, a^3\}$ forms a basis for $\mathbb{R}^3$ since it is a linearly independent set of size 3. To see that it is linearly independent, observe that the $3 \times 3$ matrix whose columns are $a^1$, $a^2$, and $a^3$ has nonzero determinant:

$$\begin{aligned}\left|\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}\right| &= 1 \cdot \left|\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\right| - (-1) \cdot \left|\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\right| + 0 \cdot \left|\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}\right| \\ &= 0 + 2 + 0 \\ &= 2 \\ &\neq 0. \end{aligned}$$

2. The collection of vectors $\{b^1, b^2, b^3\}$ does not span $\mathbb{R}^3$ (and hence cannot form a basis) since it consists of just two vectors: a spanning set of $\mathbb{R}^3$ must contain at least three vectors. For instance, the vector $(0,0,1)^\top$ does not belong to $\text{span}\{b^1, b^2\}$.

3. The collection $\{c^1, c^2, c^3, c^4\}$ cannot form a basis of $\mathbb{R}^3$ since it is not linearly independent. A linearly independent set in $\mathbb{R}^3$ must contain at most three vectors. On the other hand, the collection $\{c^1, c^2, c^3, c^4\}$ spans $\mathbb{R}^3$. To see this, it suffices to show that the collection $\{c^1, c^2, c^3\}$ forms a basis, and showing this comes to down to showing that a certain matrix has nonzero determinant:

$$\begin{aligned}\left|\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 3 \end{pmatrix}\right| &= -1 \cdot \left|\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\right| - 0 \cdot \left|\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\right| + 1 \cdot \left|\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}\right| \\ &= -3 - 3 \\ &= -6 \\ &\neq 0. \end{aligned}$$

## Problem 2

**Exercise 2.** Let $a^1 = (-1,2,0)^\top$, $a^2 = (3,2,5)^\top$, $a^3 = (5/2,3,5)^\top$ be vectors in $\mathbb{R}^3$. Are these vectors linearly independent? Do they span $\mathbb{R}^3$? Explain why.

**Solution 2.** We claim that $\{a^1, a^2, a^3\}$ is not linearly independent. In particular, it does not span $\mathbb{R}^3$ since a spanning set of $\mathbb{R}^3$ must contain at least three vectors which form a linearly independent set. Showing $\{a^1, a^2, a^3\}$

is linearly dependent comes down to showing that the $3 \times 3$ matrix whose columns are $a^1$, $a^2$, and $a^3$ has zero determinant:

$$\begin{aligned} \left|\begin{pmatrix} -1 & 3 & 5/2 \\ 2 & 2 & 3 \\ 0 & 5 & 5 \end{pmatrix}\right| &= -1 \cdot \left|\begin{pmatrix} 2 & 3 \\ 5 & 5 \end{pmatrix}\right| - 3 \cdot \left|\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}\right| + 5/2 \cdot \left|\begin{pmatrix} 2 & 2 \\ 0 & 5 \end{pmatrix}\right| \\ &= 5 - 30 + 25 \\ &= 0. \end{aligned}$$

Thus $\{a^1, a^2, a^3\}$ is linearly dependent.

# Problem 3

**Exercise 3.** Let $a^1 = (1,0,0)^\top$, $a^2 = (0,1,0)^\top$, $a^3 = (1,5,3)^\top$ be vectors in $\mathbb{R}^3$.

1. Show that these vectors form a basis for $\mathbb{R}^3$.
2. Let $a^2$ be replaced by $a^4 = (0,1,1)^\top$. Does the new set of vectors form a basis for $\mathbb{R}^3$? Explain why.

**Solution 3.** We claim that $\{a^1, a^2, a^3\}$ is a basis. To see this, it suffices to show that $\{a^1, a^2, a^3\}$ is linearly independent since any linearly independent set of size 3 forms a basis in $\mathbb{R}^3$. Thus showing $\{a^1, a^2, a^3\}$ forms a basis comes down to showing that the $3 \times 3$ matrix whose columns are $a^1$, $a^2$, and $a^3$ has nonzero determinant:

$$\begin{aligned} \left|\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}\right| &= 1 \cdot \left|\begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}\right| - 0 \cdot \left|\begin{pmatrix} 0 & 5 \\ 0 & 3 \end{pmatrix}\right| + 1 \cdot \left|\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right| \\ &= 3 - 0 + 0 \\ &= 3 \\ &\neq 0 \end{aligned}$$

Thus $\{a^1, a^2, a^3\}$ forms a basis.

2. Yes, by the same reason as in 1:

$$\begin{aligned} \left|\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}\right| &= 1 \cdot \left|\begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}\right| - 0 \cdot \left|\begin{pmatrix} 0 & 5 \\ 0 & 3 \end{pmatrix}\right| + 1 \cdot \left|\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right| \\ &= -2 - 0 + 0 \\ &= -2 \\ &\neq 0 \end{aligned}$$

# Problem 4

**Exercise 4.** Find the rank of the following matrix:

$$\begin{pmatrix} 1 & 3 & -1 & 2 & 1 \\ 1 & 2 & -3 & 2 & 2 \\ 1 & 4 & 1 & 2 & -1 \\ 1 & 5 & 3 & 2 & 1 \end{pmatrix}.$$

**Solution 4.** We perform Gaussian elimination and reduce the matrix to row echelon form:

$$
\begin{aligned}
\begin{pmatrix} 1 & 3 & -1 & 2 & 1 \\ 1 & 2 & -3 & 2 & 2 \\ 1 & 4 & 1 & 2 & -1 \\ 1 & 5 & 3 & 2 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 3 & -1 & 2 & 1 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 2 & 4 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 3 & -1 & 2 & 1 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & -7 & 2 & 4 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & -7 & 2 & 0 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & 0 & -7 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

From the reduced row echelon form, we see that the rank is 3.

## Problem 5

**Exercise 5.** Consider the following system of linear equations:

$$
\begin{aligned}
-x_1 + 2x_2 + x_3 + x_4 - 2x_5 &= 4 \\
x_1 - 2x_2 + 2x_4 - x_5 &= 3.
\end{aligned}
$$

1. Find all solutions to this system.
2. Find all basic solutions to this system.

**Solution 5.** 1. We first write the equations using a matrix and reduce this matrix to row echelon form:

$$
\begin{aligned}
\begin{pmatrix} -1 & 2 & 1 & 1 & -2 & 4 \\ 1 & -2 & 0 & 2 & -1 & 3 \end{pmatrix} &\to \begin{pmatrix} 1 & -2 & 0 & 2 & -1 & 3 \\ -1 & 2 & 1 & 1 & -2 & 4 \end{pmatrix} \\
&\to \begin{pmatrix} 1 & -2 & 0 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -3 & 7 \end{pmatrix}
\end{aligned}
$$

From this we obtain the following equivalent set of linear equations:

$$
\begin{aligned}
x_1 - 2x_2 + 2x_4 - x_5 &= 3 \\
x_3 + 3x_4 - 3x_5 &= 7.
\end{aligned}
$$

Here, $x_2, x_4$, and $x_5$ are free parameters, in particular every solution to the set of equations above has the form

$$
\begin{pmatrix} 3 + 2x_2 - 2x_4 + x_5 \\ x_2 \\ 7 - 3x_4 + 3x_5 \\ x_4 \\ x_5 \end{pmatrix}
$$

where $x_2, x_4, x_5 \in \mathbb{R}$.

2. We can use the row echelon form of the matrix to find all basic solutions:

$$(3,0,7,0,0)^\top$$

$$(-5/3,0,0,7/3,0)^\top$$
$$(2/3,0,0,0,-7/3)^\top$$
$$(0,-3/2,7,0,0)^\top$$
$$(0,-23/4,0,-7/3,0)^\top$$
$$(0,-1/3,0,0,-7/3)^\top$$
$$(0,0,5/2,3/2,0)^\top$$
$$(0,0,-2,0,-3)^\top$$
$$(0,0,0,2/3,-5/3)^\top$$

# Problem 6

**Exercise 6.** Prove that a hyperplane in $\mathbb{R}^n$ is a convex set.

**Solution 6.** Let $L$ be a hyperplane in $\mathbb{R}^n$, so $L = \ker \ell$ for some linear funcitonal $\ell \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$. Let $x, y \in L$ and let $t \in (0,1)$. Then observe that

$$\begin{aligned}\ell(tx + (1-t)y) &= t\ell(x) + (1-t)\ell(y) \\ &= t \cdot 0 + (1-t) \cdot 0 \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

It follows that $tx + (1-t)y \in L$, which implies $L$ is convex.

Note that translated hyperplanes are convex as well. Indeed, a translated hyperplane has the form $L + v$ where $L$ is a hyperplane and where $v \in \mathbb{R}^n$. Given $x + v, y + v \in L + v$ where $x, y \in L$, and given $t \in (0,1)$, we have

$$\begin{aligned}t(x+v) + (1-t)(y+v) &= tx + tv + (1-t)y + (1-t)v \\ &= tx + (1-t)v + tv + v - tv \\ &= tx + (1-t)v + v \\ &\in L + v.\end{aligned}$$

Thus translated hyperplanes are convex too.