Extensions

Let A be a noetherian domain which is integrally closed in its field of fractions K. Let L/K be a finite field extension with $n = [L : K]$ and let B be the integral closure of A in L. We want to know under what conditions is B a finitely generated A-module. The following proposition gives one such condition:

Proposition 0.1. *If L/K is separable, then B is a finitely generated A-module.*

Proof. We first define a symmetric non-denerate K-bilinear form $\langle \cdot, \cdot \rangle \colon L \times L \to K$ as follows: given $y, y' \in L$, we set

$$\langle y, y' \rangle := \operatorname{Tr}_{L/K}(yy').$$

Indeed, it is clearly symmetric and bilinear since the usual multiplication map on L is symmetric and K-bilinear and since the trace map is K-linear. Recall that $\operatorname{Tr}_{L/K} = 0$ if and only if L/K is nonseparable. Equivalently, $\operatorname{Tr}_{L/K}$ is onto if and only if L/K is separable. Since L/K is separable, there exists a $\widetilde{y} \in L$ such that $\operatorname{Tr}_{L/K}(\widetilde{y}) \neq 0$. In particular, if $y \neq 0$ is in L, then $\langle y, y^{-1}\widetilde{y} \rangle \neq 0$, hence $\langle \cdot, \cdot \rangle$ is non-degenerate as well. We claim that the trace map restricted to B lands in A. To see this, we first choose a finite extension L'/L such that L'/K is Galois. Then for each $b \in B$ we have

$$\operatorname{Tr}_{L/K}(b) = \sum_{\sigma \colon L \hookrightarrow L'} \sigma(b) \tag{1}$$

where the sum in L' is taken over all K-embeddings $\sigma \colon L \hookrightarrow L'$. Each $\sigma(b)$ is integral over A since b is integral over A, and thus the sum (1) is also integral over A. Since $\operatorname{Tr}_{L/K}(b) \in K$ and is integral over A, it follows that $\operatorname{Tr}_{L/K}(b) \in A$. Now for each $y \in L$, we obtain a K-linear map $\ell_y \colon L \to K$ where $\ell_y(y') = \langle y, y' \rangle$ for all $y' \in L$. Given an A-submodule M of L, we set

$$M^\vee = \{y \in L \mid \ell_y(M) \subseteq A\} = \{y \in L \mid \langle y, u \rangle \in A \text{ for all } u \in M\}.$$

Suppose that $e_1, \dots, e_n$ is a K-basis of L, and by rescaling the e_i if necessary, we may also assume that each e_i is in B. For each i, we let $e_i^\vee$ be the unique element in L such that

$$\langle e_i^\vee, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Indeed, $e_i^\vee$ is unique precisely because $\langle \cdot, \cdot \rangle$ is non-degenerate. If we set $F = \sum_i Ae_i$ to be the free A-module spanned by the e_i, then clearly we have $F^\vee = \sum_i Ae_i^\vee$. Furthermore we have inclusions:

$$F \subseteq B \subseteq B^\vee \subseteq F^\vee.$$

In particular, B is contained in a finitely generated A-module, and since A is noetherian, it follows that B is a finitely generated A-module. □

Remark 1. The condition stated in the proposition above is not the only condition that implies B is a finitely generated A-module. One can show that if A is a finitely generated $\Bbbk$-algebra where $\Bbbk$ is a field, then B is a finitely generated A-module. Similarly one can show that if A is a complete discrete valuation ring, then B is a finitely generated A-module.

For now on, we now assume that B is finitely generated as an A-module. We also assume that $\dim A = 1$, hence A is a Dedekind domain. This implies $\dim B = 1$ since B is integral over A, and thus B is a Dedekind domain too. In this case, if we are given a nonzero prime $\mathfrak{p}$ of A, then we have a decomposition

$$\mathfrak{p}B = \prod_{\mathfrak{q} \mid \mathfrak{p}} \mathfrak{q}^{e_\mathfrak{p}}$$

where the $e_\mathfrak{q} \in \mathbb{Z}_{\geq 0}$ are uniquely determined. Since there are only

Proposition 0.2.