Research Statement

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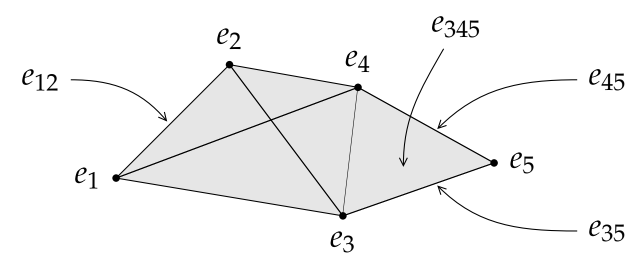
Introduction

My research focuses on free resolutions. More specifically, let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian (or standard graded) ring, let I be an ideal of R, and let $F = (F, \mathrm{d})$ be the minimal R-free resolution of R/I. The usual multiplication map $\mathrm{m}\colon R/I \otimes_R R/I \to R/I$ can be lifted to a chain map $\mu\colon F \otimes_R F \to F$, denoted $a_1 \otimes a_2 \mapsto a_1 \star_\mu a_2 = a_1 a_2$ where $a_1, a_2 \in F$ (where we make the further simplification $a_1 \star_\mu a_2 = a_1 a_2$ whenever μ is clear from context). Up to homotopy, μ is unital, strictly graded-commutative, and associative. It is clear that we can always choose μ to be unital on the nose (with $1 \in F$ being the identity element). It was shown in [BE77] that μ can even be chosen to be strictly graded-commutative on the nose as well. The first part of my research has been dedicated to the following question, which we call Question 1:

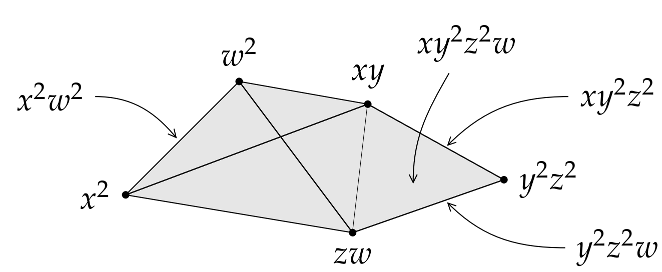
Question 1: Can μ be chosen such that it is associative on the nose?

The reason this question is interesting is because we gain a lot of information about the "shape" of F when we know the answer to that question is "yes". Indeed, it was shown in [BE77] that if we assume R is a domain and I is perfect, and we know that an associative multiplication on F exists, then one obtains important lower bounds of the betti numbers β_i of R/I. In particular, let $\boldsymbol{t} = t_1, \dots, t_m$ be a maximal R-sequence contained in I and let $E = \mathcal{K}(\boldsymbol{t})$ be the koszul R-algebra resolution of $R/\boldsymbol{t}$. Then the natural map $R/\boldsymbol{t} \to R/I$ induces an algebra homomorphism $E \to F$ which can be shown to be injective, whence we get the lower bound $\binom{m}{i} \leq \beta_i$ for each i. With this in mind, Buchsbaum and Eisenbud conjectured that the answer to Question 1 was always "yes". However this conjecture turned out to be false (see [Avr81]), and many counterexamples have been found ever since.

Example 0.1. Let Δ be the simplicial complex whose vertex set is $\{e_1, e_2, e_3, e_4, e_5\}$ and whose faces consists of all subsets of $e_{1234} = \{e_1, e_2, e_3, e_4\}$ and $e_{345} = \{e_3, e_4, e_5\}$, pictured below:



Next suppose $R = \Bbbk[x, y, z, w]$ and let $\boldsymbol{m}_{\mathrm{K}} = x^2, w^2, xy, zw, y^2z^2$. Then we obtain an $\boldsymbol{m}_{\mathrm{K}}$-labeled simplicial complex $\Delta = (\Delta, \boldsymbol{m}_{\mathrm{K}})$ which is pictured below:



Let F_{K} be the $\mathbb{N}^4$-graded R-complex induced by Δ (see the Appendix for details on how this is constructed). Let's write down the homogeneous components of F_{K} as a graded module: we have

$$
\begin{aligned}
F_{\mathrm{K},0} &= R\\
F_{\mathrm{K},1} &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5\\
F_{\mathrm{K},2} &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45}\\
F_{\mathrm{K},3} &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345}\\
F_{\mathrm{K},4} &= Re_{1234}
\end{aligned}
$$

The differential $\mathrm{d}\colon F_{\mathrm{K}} \to F_{\mathrm{K}}$ behaves just like the usual boundary map except some monomials can show up as coefficients. For instance,

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

Now, choose a multiplication μ on F_{K} which respects the multigrading. Since μ respects the multigrading and satisfies Leibniz law, we are forced to have

$$
\begin{aligned}
e_1 \star e_5 &= yz^2 e_{14} + xe_{45}\\
e_1 \star e_2 &= e_{12}\\
e_2 \star e_5 &= y^2 z e_{23} + we_{35}\\
e_2 \star e_{45} &= -yze_{234} + we_{345}\\
e_1 \star e_{35} &= yze_{134} - xe_{345}\\
e_1 \star e_{23} &= e_{123}\\
e_2 \star e_{14} &= -e_{124}
\end{aligned}
$$

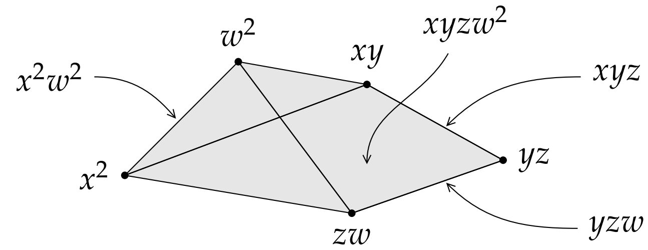
At this point however, one can conclude that F_{K} is not associative since

$$[e_1, e_5, e_2] = (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{1}$$

One can work (1) out by hand, however one of the main results of our research is a method for calculating associators like (1) using tools from the theory of Gröbner bases. For instance, we used the following Singular code below to calculate the associator $[e_1, e_5, e_2]$:

```
LIB "ncalg.lib";

intvec v= 1:3, 2:5, 3:5;
ring A=(0,x,y,z,w),(e1,e2,e5,e12,e14,e23,e35,e45,e123,e124,e134,e234,e345),Wp(v);

matrix C[13][13]; matrix D[13][13]; int i; int j;
for (i=1; i<=13; i++) {for (j=1; j<=13; j++) {C[i,j]=(-1)^(v[i]*v[j]);}}
ncalgebra(C,D);

poly f(1)(5) = e1*e5-yz2*e14-x*e45;
poly f(1)(2) = e1*e2-e12;
poly f(2)(5) = e2*e5-y2z*e23-w*e35;
poly f(2)(45) = e2*e45+yz*e234-w*e345;
poly f(1)(35) = e1*e35-yz*e134+x*e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14+e124;
poly S(1)(5)(2) = f(1)(5)*e2+e1*f(2)(5);

ideal I = f(2)(14),f(2)(45),f(1)(23),f(1)(35),f(2)(5),f(1)(5);
reduce(S(1)(5)(2),b);

// [e1,e5,e2] = (-y^2*z)*e123+(y*z^2)*e124+(-y*z*w)*e134+(x*y*z)*e234
```

Example 0.2. Let $R = \mathbb{k}[x, y, z, w]$, let $\boldsymbol{m}_{\mathrm{A}} = x^2, w^2, xy, zw, yz$, and let F_{A} be the minimal R-free resolution of $R/\boldsymbol{m}_{\mathrm{A}}$. Then F_{A} can be realized as the R-complex induced by the $\boldsymbol{m}_{\mathrm{A}}$-labeled cellular complex pictured below:



Let's write down the homogeneous components of F_{A} as a graded module: we have

$$\begin{aligned}
F_{\mathrm{A},0} &= R \\
F_{\mathrm{A},1} &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_{\mathrm{A},2} &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_{\mathrm{A},3} &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_{\mathrm{A},4} &= R\varepsilon_{12345}
\end{aligned}$$

The differential $\mathrm{d}\colon F_{\mathrm{A}} \to F_{\mathrm{A}}$ on the non-simplex faces are given below

$$\begin{aligned}
\mathrm{d}\varepsilon_{12345} &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\
\mathrm{d}\varepsilon_{1345} &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\
\mathrm{d}\varepsilon_{2345} &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}.
\end{aligned}$$

Note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{A}}$ induces a comparison map $\varphi\colon F_{\mathrm{K}} \to F_{\mathrm{A}}$ which we can choose to be multigraded. We have

$$\begin{aligned}
\varphi(e_5) &= yz\varepsilon_5 \\
\varphi(e_{35}) &= yz\varepsilon_{35} \\
\varphi(e_{45}) &= yz\varepsilon_{45} \\
\varphi(e_{34}) &= x\varepsilon_{35} - w\varepsilon_{45} \\
\varphi(e_{345}) &= 0 \\
\varphi(e_{234}) &= \varepsilon_{2345} \\
\varphi(e_{134}) &= \varepsilon_{1345} \\
\varphi(e_{1234}) &= \varepsilon_{12345}
\end{aligned}$$

and we have $\varphi(e_\sigma) = \varepsilon_\sigma$ for all other $e_\sigma \in F$. On the other hand, the multiplication by yz map $R/\boldsymbol{m}_{\mathrm{A}} \to R/\boldsymbol{m}_{\mathrm{K}}$ induces a comparison map $\psi\colon F_{\mathrm{A}} \to F_{\mathrm{K}}$ which we can choose to be multigraded. We have

$$\begin{aligned}
\psi(\varepsilon_5) &= e_5 \\
\psi(\varepsilon_{35}) &= e_{35} \\
\psi(\varepsilon_{45}) &= e_{45} \\
\psi(\varepsilon_{2345}) &= yze_{234} - e_{345} \\
\psi(\varepsilon_{1345}) &= yze_{134} - e_{345}
\end{aligned}$$

and we have $\psi(\varepsilon_\sigma) = yze_\sigma$ for all other $e_\sigma \in F_{\mathrm{A}}$. We define

$$\varepsilon_\sigma \star \varepsilon_\tau := \varphi(\psi(\varepsilon_\sigma) \star \psi(\varepsilon_\tau))$$

for all $\varepsilon_\sigma, \varepsilon_\tau$ in the homogeneous basis of F_{A}.

Measuring the Failure for μ to being Associative

We now equip F with a fixed multiplication μ (which is assumed to be unital and strictly graded-commutative on the nose). We simplify our notation by referring to the triple (F, d, μ) via its underlying graded R-module F, where we think of F as a graded R-module which is equipped with a differential $\mathrm{d}\colon F \to F$, giving it the structure of an R-complex, and which is further equipped with a chain map $\mu\colon F \otimes_R F \to F$. For instance, if μ

satisfies a property (such as being associative), then we also say F satisfies that property. With this notation in mind, we are interested in the following question:

Question 2: How can we measure how far away F is from being associative?

There are several approaches to answering this question and they all involve the maximal associative quotient of F. In order to explain this further, we make the following definitions:

1. The **associator** of F is the chain map, denoted $[\cdot]_\mu$ (or more simply by $[\cdot]$ if μ is understood from context), from $F \otimes_R F \otimes_R F$ to F defined by
$$[\cdot] := \mu(\mu \otimes 1 - 1 \otimes \mu).$$
We denote by $[\cdot,\cdot,\cdot]\colon F^3 \to F$ to be the unique R-trilinear map which corresponds to $[\cdot]$ via the universal mapping property of tensor products. Thus we have
$$[a_1 \otimes a_2 \otimes a_3] = (a_1 \star a_2) \star a_3 - a_1 \star (a_2 \star a_3) = [a_1, a_2, a_3]$$
for all $a_1, a_2, a_3 \in F$.

2. The **associator R-subcomplex** of F, denoted $[F]$, is the R-subcomplex of F given by the image of the associator of μ. Thus the underlying graded R-module of $[F]$ is
$$[F] = \mathrm{span}_R\{[a_1, a_2, a_3] \mid a_1, a_2, a_3 \in F\},$$
and the differential of $[F]$ is simply the restriction of the differential of F to $[F]$.

3. The **associator F-submodule** of F, denoted $\langle F \rangle$, is defined to be the smallest F-submodule of F which contains $[F]$. The underlying graded R-module of $\langle F \rangle$ also has a simple description. Indeed, observe that
$$a_1(a_2[a_3, a_4, a_5]) = (a_1a_2)[a_3, a_4, a_5] - [a_1, a_2, [a_3, a_4, a_5]] \tag{2}$$
for all $a_1, a_2, a_3, a_4, a_5 \in F$. Using identities like (5) together with graded-commutativity, one can show that the underlying graded R-module of $\langle F \rangle$ is given by
$$\langle F \rangle = \mathrm{span}_R\{a_1[a_2, a_3, a_4] \mid a_1, a_2, a_3, a_4 \in F\}.$$

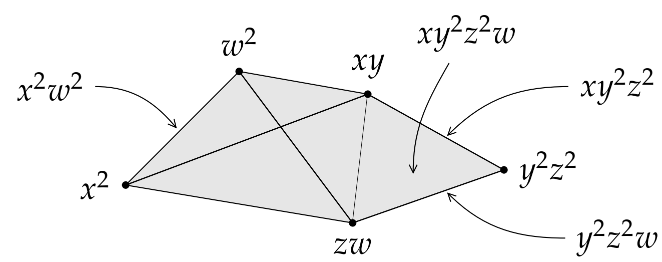
4. The **maximal associative quotient** of F is the quotient $F/\langle F \rangle$.

Theorem 0.1. *With notation as above, we have the following:*

1. *I kills $\mathrm{H}(F/\langle F \rangle)$. In other words, $\mathrm{H}(F/\langle F \rangle)$ is an (R/I)-module.*

2. *F is associative if and only if $\mathrm{H}(F/\langle F \rangle) = 0$.*

Example 0.3. Let us revisit Example (0.5) where $R = \Bbbk[x, y, z, w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and where F is the R-complex incuded by the labeled simplicial complex pictured below:



Choose a multiplication μ on F which respects the multigrading. Recall that since μ respects the multigrading and satisfies Leibniz law, we are forced to have

$$\begin{aligned}
e_1 \star e_5 &= yz^2e_{14} + xe_{45}\\
e_1 \star e_2 &= e_{12}\\
e_2 \star e_5 &= y^2ze_{23} + we_{35}\\
e_2 \star e_{45} &= -yze_{234} + we_{345}\\
e_1 \star e_{35} &= yze_{134} - xe_{345}\\
e_1 \star e_{23} &= e_{123}\\
e_2 \star e_{14} &= -e_{124}
\end{aligned}$$

We want to calculate the associator homology of F. Observe that

$$\begin{aligned}
\frac{e_1}{x}[e_1,e_5,e_2] &= \frac{1}{x}\left([e_1^2,e_5,e_2]-[e_1,e_1e_5,e_2]+[e_1,e_1,e_5e_2]-[e_1,e_1,e_5]e_2\right)\\
&= -\frac{1}{x}[e_1,e_1e_5,e_2]\\
&= -\frac{1}{x}[e_1,yz^2e_{14}+xe_{45},e_2]\\
&= -\frac{yz^2}{x}[e_1,e_{14},e_2]-[e_1,e_{45},e_2]\\
&= -[e_1,e_{45},e_2].
\end{aligned}$$

It follows that $\mathrm{d}(e_1/x)=x$ annihilates $\mathrm{H}\langle F\rangle$. Similar calculations likes this shows that $\mathfrak{m}=\langle x,y,z,w\rangle$ annihilates $\mathrm{H}\langle F\rangle$. It follows that

$$\mathrm{H}_i\langle F\rangle \simeq \begin{cases}\Bbbk & \text{if } i=3\\ 0 & \text{else}\end{cases}$$

One can interpret this as saying that the multiplication μ is very close to being associative. Homologically speaking, the failure for μ to being associative is reflected in the fact that $\ell(\mathrm{H}\langle F\rangle)=1$. Note however that μ is not associative in homological degree 4 since

$$[e_1,e_{45},e_2]=xyze_{1234}\neq 0.$$

In particular we have $\mathrm{uha}(F)=\mathrm{lha}(F)=3$, but $\mathrm{ua}(F)=4$. In some sense however, the nonzero associator $[e_1,e_{45},e_2]$ isn't really anything *new*. Indeed, we obtained the nonzero associator $[e_1,e_{45},e_2]$ from the nonzero associator $[e_1,e_5,e_2]$, so one could argue that $[e_1,e_{45},e_2]$ being nonzero is simply a direct consequence of $[e_1,e_5,e_2]$ being nonzero. More generally, a nonzero associator $\chi\in\langle F\rangle$ should only be thought of as contributing something new towards the failure for μ to being associative if $\mathrm{d}(\chi)=0$ (otherwise one could argue that χ being nonzero is simply a direct consequence of the associators in $\mathrm{d}(\chi)$ being nonzero). Similarly, if $\chi=\mathrm{d}(\chi')$ for some other nonzero aasociator $\chi'\in\langle F\rangle$, then again χ isn't contributing anything new towards the failure for μ to being associative, since one could argue that χ being nonzero is a direct consequence of χ' being nonzero. Thus the associators which really do contribute something new towards the failure for μ being associative should be the ones which represent nonzero elements in homology. In this case, we have precisely one nontrivial nonzero associator $[e_1,e_5,e_2]$ which represents a nontrivial element in homology (all other nonzero associators can be derived from the fact that $[e_1,e_5,e_2]\neq 0$).

Presentation of the Maximal Associative Quotient

In this section, we will construct the symmetric DG algebra of F, which we denote by $\mathrm{S}(F)$. The underlying R-module of $\mathrm{S}(F)$ has a bi-graded structure, more specifically, we can decompose $\mathrm{S}(F)$ into R-modules as:

$$\mathrm{S}(F)=\bigoplus_{i\geq 0}\mathrm{S}_i(F)=\bigoplus_{m\geq 0}\mathrm{S}^m(F)=\bigoplus_{i,m\geq 0}\mathrm{S}_i^m(F)$$

We refer to the i in the subscript as **homological degree** and we refer to the m in the superscript as **total degree**. The R-module $\mathrm{S}_i^m(F)$ can be described as follows: first we have

$$\mathrm{S}_0(F)=\mathrm{S}^0(F)=\mathrm{S}_0^0(F)=R.$$

Next, for $i,m\geq 1$, the R-module $\mathrm{S}_i^m(F)$ is the R-span of all elementary products of the form $a_1\cdots a_m$ where $a_1,\dots,a_m\in F_+$ are homogeneous such that

$$|a_1|+\cdots+|a_m|=i.$$

We identify A with its image in $\mathrm{S}(F)$ and let $\iota\colon F\to\mathrm{S}(F)$ denote the inclusion map. Thus we have

$$F=\mathrm{S}^0(F)+\mathrm{S}^1(F)=R+F_+.$$

The differential of $\mathrm{S}(F)$ extends the differential of F and is defined on elementary products of the form $a_1\cdots a_m$ where $a_1,\dots,a_m\in A_+$ are homogeneous by

$$\mathrm{d}(a_1\cdots a_m)=\sum_{j=1}^{m}(-1)^{|a_1|+\cdots+|a_{j-1}|}a_1\cdots\mathrm{d}(a_j)\cdots a_m.$$

Example 0.4. Let $R = \Bbbk[x,y]$, let $I = \langle x^2, xy \rangle$, and let F be Taylor resolution of R/I. Let's write down the homogeneous components of F as a graded R-module: we have

$$\begin{aligned} F_0 &= R \\ F_1 &= Re_1 + Re_2 \\ F_2 &= Re_{12}, \end{aligned}$$

and if $i \notin \{0,1,2\}$, then $F_i = 0$. The differential of F is defined on the homogeneous basis elements by

$$\begin{aligned} \mathrm{d}(e_1) &= x^2 \\ \mathrm{d}(e_2) &= xy \\ \mathrm{d}(e_{12}) &= xe_2 - ye_1. \end{aligned}$$

Note that the Taylor resolution usually comes equipped with a multiplication called the Taylor multiplication. Let us denote this by $\star$ so as not to confuse it with the multiplication $\cdot$ of $\mathrm{S}(F)$. Now let's write down the homogeneous components of $\mathrm{S}(F)$ as a graded R-module (with respect to homological degree): we have

$$\begin{aligned} \mathrm{S}_0(F) &= R \\ \mathrm{S}_1(F) &= Re_1 + Re_2 \\ \mathrm{S}_2(F) &= Re_{12} + Re_1e_2 \\ \mathrm{S}_3(F) &= Re_1e_{12} + Re_2e_{12} \\ \mathrm{S}_4(F) &= Re_{12}^2 + Re_1e_2e_{12} \\ &\vdots \end{aligned}$$

Note that $\mathrm{S}_4^3(F) = Re_1e_2e_{12}$ and $\mathrm{S}_4^2(F) = Re_{12}^2$. Also note that

$$\begin{aligned} \mathrm{d}(e_1e_2 - e_1 \star e_2) &= \mathrm{d}(e_1e_2 - xe_{12}) \\ &= \mathrm{d}(e_1)e_2 - e_1\mathrm{d}(e_2) - x\mathrm{d}(e_{12}) \\ &= x^2e_2 - xye_1 - x(xe_2 - ye_1) \\ &= x^2e_2 - xye_1 - x^2e_2 + xye_1 \\ &= 0. \end{aligned}$$

Note that the multiplicator of $\iota\colon F \to \mathrm{S}(F)$ has the form

$$[a_1, a_2] = \iota(a_1 \star a_2) - \iota(a_1)\iota(a_2) = a_1 \star a_2 - a_1a_2$$

for all $a_1, a_2 \in A$. Let $\mathfrak{b}$ be the DG $\mathrm{S}(A)$-ideal generated by the multiplicator complex $[B]_\iota$. Since B is associative, we have

$$\mathfrak{b} = \mathrm{span}_B\{[a_1, a_2] \mid a_1, a_2 \in A\}.$$

Let $\rho_1\colon A \to A/\langle A \rangle$ and $\rho_2\colon \mathrm{S}(A) \to \mathrm{S}(A)/\mathfrak{b}$ denote the corresponding quotient maps.

Theorem 0.2. *With the notation as above, we have $\langle F \rangle = F \cap \mathfrak{b}$. In particular, the composite $\rho_2\iota\colon F \to \mathrm{S}(F) \to \mathrm{S}(F)/\mathfrak{b}$ induces an isomorphism $F/\langle F \rangle \simeq \mathrm{S}(F)/\mathfrak{b}$ of DG R-algebras.*

An Application Using Gröbner Bases

Throughout this subsection, we assume that R is an integral domain with quotient field K and we further assume that the underlying graded R-module of F is a finite and free. Let $e_1, e_2, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $|e_{i'}| > |e_i|$, then $i' > i$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_ie_j = (-1)^{|e_i||e_j|}e_je_i,$$

in $R[\boldsymbol{e}]$, however odd elements do not square to zero in $R[\boldsymbol{e}]$. The reason we do not allow odd elements to square to zero is because later on we want to calculate the Gröbner basis of an ideal of $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is simpler when we don't have any zerodivisors. We identity F with $R + Re_1 + \cdots + Re_n$ and let $\iota\colon F \to R[\boldsymbol{e}]$ denote the inclusion map. We extend the differential of F to a differential on $R[\boldsymbol{e}]$. For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in $R[e]$ defined by $f_{i,j} := -[e_i, e_j]$. Thus we have

$$f_{i,j} = e_i e_j - \sum_k r_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let $\mathfrak{b}$ be the DG $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $(|e_1|, \ldots, |e_n|)$. More specifically, given two monomials e^α and e^β in $K[\boldsymbol{e}]$, we say $e^\beta > e^\alpha$ if either

1. $|e^\beta| > |e^\alpha|$ or;
2. $|e^\beta| = |e^\alpha|$ and $\beta_1 > \alpha_1$ or;
3. $|e^\beta| = |e^\alpha|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Finally let $\mathcal{G}$ be the Gröbner basis of $\mathfrak{b}$ obtained by applying Buchberger's algorithm to $\mathcal{F}$.

Theorem 0.3. *We have the following:*

1. $R[\boldsymbol{e}]/\mathfrak{b} \simeq F/\langle F \rangle$.
2. $K[\boldsymbol{e}]/\mathfrak{b}_K \simeq F_K/\langle F_K \rangle$
3. $\mathcal{G} \cap F$

Using the Gröbner basis we constructed above, we can measure the failure for F to being associative in degree i. In particular, observe that

$$\begin{aligned} \operatorname{rank}_R(F_i/\langle F \rangle_i) &= \dim_K((F_K/\langle F_K \rangle)_i) \\ &= \dim_K(K[\boldsymbol{e}]_i/\mathfrak{b}_{K,i}) \\ &= \dim_K(F_i) - \#\{e_j \mid |e_j = \mathrm{LM}(f) \text{ for some } f \in \langle F \rangle_i\} \\ &= \operatorname{rank}_R(F_i) - \#\{e_j \mid |e_j = \mathrm{LM}(f) \text{ for some } f \in \langle F \rangle_i\}. \end{aligned}$$

Thus we have

$$\operatorname{rank}_R(F_i) - \operatorname{rank}_R(F_i/\langle F \rangle_i) = \#\{e_j \mid |e_j = \mathrm{LM}(f) \text{ for some } f \in \langle F \rangle_i\}$$

just as we did before giving $R[\boldsymbol{e}]$ the structure of a DG R-algebra so that $\iota\colon F \to R[\boldsymbol{e}]$ can be viewed as a chain map which satisfies $\iota(1) = 1$.

Clearly, =

With the lower associative index of X and the lower homological associative index of X defined, we see after analyzing the proof of Theorem (0.4), that if R is local, $\langle X \rangle$ is minimal and bounded below, and each $\langle X \rangle_i$ is finitely generated as an R-module, then we have $\mathrm{la}\langle X \rangle = \mathrm{lha}\langle X \rangle$. On the other hand, even if these conditions are satisfied, we often have $\mathrm{ua}\langle X \rangle > \mathrm{uha}\langle X \rangle$. For instance, we will see in Example (0.3) that $\mathrm{ua}\langle F \rangle = 4$ and $\mathrm{uha}\langle F \rangle = 3$. In the case that we're mostly interested in, R is a local noetherian ring and F is the minimal free R-resolution of R/I. In this case, we are interested in the quantity:

$$\mathrm{a}(R/I) := \inf_{\mu \in \mathrm{Mult}(F)} \{\mathrm{uha}\langle F_\mu \rangle - \mathrm{lha}\langle F_\mu \rangle + 1\},$$

where F_μ denotes F equipped with the multiplication μ. We call $\mathrm{a}(R/I)$ the **associative index** of R/I. One can think of $\mathrm{a}(R/I)$ as measuring the failure to put a DG algebra structure on F. In particular, there exists a DG algebra structure on F if and only if $\mathrm{a}(R/I) = 0$. In Example (0.3), we have $\mathrm{a}(R/I) = 1$. Thus there is no DG algebra structure on F in this case, but the fact that $\mathrm{a}(R/I) = 1$ tells us that we can get extremely close.

Remark 1. Let X be an MDG A-module. Then the short exact sequence of graded $\mathrm{H}(A)$-modules

$$0 \longrightarrow \langle X \rangle \xrightarrow{\iota} X \xrightarrow{\rho} X/\langle X \rangle \longrightarrow 0$$

induces a long exact sequence of R-modules:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(X) \arrow[r] & \mathrm{H}_{i+1}(X/\langle X \rangle) \arrow[dll, "\mathrm{d}_i"'] \\
\mathrm{H}_i\langle X \rangle \arrow[r] & \mathrm{H}_i(X) \arrow[r] & \mathrm{H}_i(X/\langle X \rangle) \arrow[dll, "\mathrm{d}_{i-1}"'] \\
\mathrm{H}_{i-1}\langle X \rangle \arrow[r] & \mathrm{H}_{i-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{3}$$

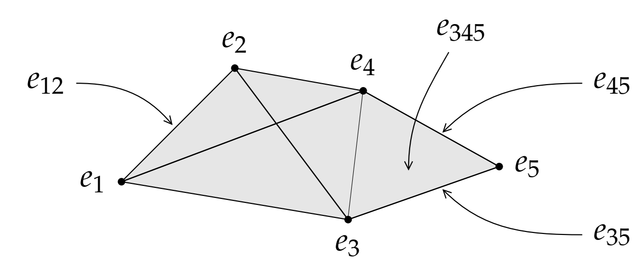
where the connecting map is induced by the differential $\mathrm{d}\colon X \to X$. In particular, we obtain a sequence of graded $\mathrm{H}(A)$-modules:

$$\mathrm{H}(X) \xrightarrow{\rho} \mathrm{H}(X/\langle X \rangle) \xrightarrow{\mathrm{d}} \mathrm{H}\langle X \rangle(-1) \xrightarrow{\iota} \mathrm{H}(X)(-1)$$

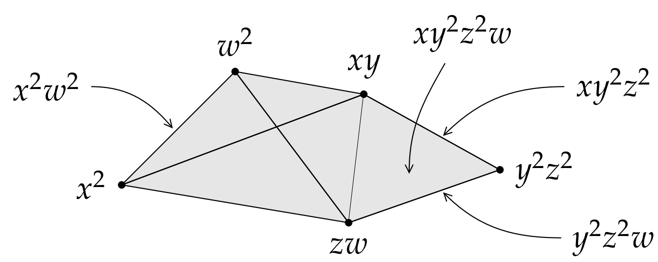
which is exact at $\mathrm{H}(X/\langle X \rangle)$ and $\mathrm{H}\langle X \rangle(-1)$.

We will go over a counterexample (first shown to be a counterexample in [Luk26]) in a moment.

Example 0.5. Let Δ be the simplicial complex whose vertex set is $\{e_1, e_2, e_3, e_4, e_5\}$ and whose faces consists of all subsets of $e_{1234} = \{e_1, e_2, e_3, e_4\}$ and $e_{345} = \{e_3, e_4, e_5\}$, pictured below:



Next suppose $R = \Bbbk[x, y, z, w]$ and let $\boldsymbol{m} = x^2, w^2, xy, zw, y^2z^2$. Then we obtain an $\boldsymbol{m}$-labeled simplicial complex $\Delta = (\Delta, \boldsymbol{m})$ which is pictured below:



Let F be the R-complex induced by Δ (see the Appendix for . Let's write down the homogeneous components of F as a graded module: we have

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234}
\end{aligned}$$

The differential $\mathrm{d}\colon F \to F$ behaves just like the usual boundary map except some monomials can show up as coefficients. For instance,

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

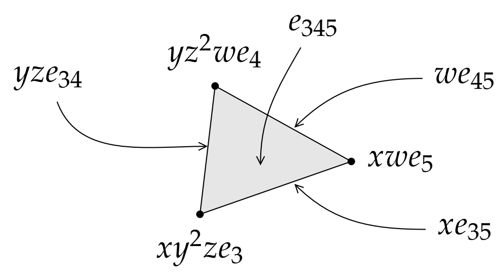
Next let $\boldsymbol{\alpha} = (1, 2, 2, 1)$. As a $\Bbbk$-vector space, $F_{\boldsymbol{\alpha}}$ looks like:

$$F_{\boldsymbol{\alpha}} = \Bbbk + \Bbbk xy^2ze_3 + \Bbbk yz^2we_4 + \Bbbk xwe_5 + \Bbbk yze_{34} + \Bbbk xe_{35} + \Bbbk we_{45} + \Bbbk e_{345}.$$

However $F_{\boldsymbol{\alpha}}$ is more than just a $\Bbbk$-vector space: it has the structure of a $\Bbbk$-complex. Let's write down the homogeneous components of $F_{\boldsymbol{\alpha}}$ as a graded $\Bbbk$-vector space

$$\begin{aligned}
F_{0,\boldsymbol{\alpha}} &= \Bbbk \\
F_{1,\boldsymbol{\alpha}} &= \Bbbk xy^2ze_3 + \Bbbk yz^2we_4 + \Bbbk xwe_5 \\
F_{2,\boldsymbol{\alpha}} &= \Bbbk yze_{34} + \Bbbk xe_{35} + \Bbbk we_{45} \\
F_{3,\boldsymbol{\alpha}} &= \Bbbk e_{345}
\end{aligned}$$

we think of this complex as corresponding to $\Delta_{\boldsymbol{\alpha}}$ pictured below



Now, choose a multiplication μ on F which respects the multigrading. Since μ respects the multigrading and satisfies Leibniz law, we are forced to have

$$\begin{aligned}
e_1 \star e_5 &= yz^2e_{14} + xe_{45} \\
e_1 \star e_2 &= e_{12} \\
e_2 \star e_5 &= y^2ze_{23} + we_{35} \\
e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= -e_{124}
\end{aligned}$$

At this point however, one can conclude that F is not associative since

$$[e_1, e_5, e_2] = (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{4}$$

One can work (**??**) out by hand, however one of the main results of this paper is a method for calculating associators like (**??**) using tools from the theory of Gröbner bases. For instance, we used the following Singular code below to calculate the associator $[e_1, e_5, e_2]$:

```
LIB "ncalg.lib";

intvec v= 1:3, 2:5, 3:5;
ring A=(0,x,y,z,w),(e1,e2,e5,e12,e14,e23,e35,e45,e123,e124,e134,e234,e345),Wp(v);

matrix C[13][13]; matrix D[13][13]; int i; int j;
for (i=1; i<=13; i++) {for (j=1; j<=13; j++) {C[i,j]=(-1)^(v[i]*v[j]);}}
ncalgebra(C,D);

poly f(1)(5) = e1*e5-yz2*e14-x*e45;
poly f(1)(2) = e1*e2-e12;
poly f(2)(5) = e2*e5-y2z*e23-w*e35;
poly f(2)(45) = e2*e45+yz*e234-w*e345;
poly f(1)(35) = e1*e35-yz*e134+x*e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14+e124;
poly S(1)(5)(2) = f(1)(5)*e2+e1*f(2)(5);

ideal I = f(2)(14),f(2)(45),f(1)(23),f(1)(35),f(2)(5),f(1)(5);
reduce(S(1)(5)(2),b);

// [e1,e5,e2] = (y^2*z)*e123+(-x*y*z^2)*e124+(y*z*w)*e134+(-x*y*z)*e234


(-y^2*z)*e123+(y*z^2)*e124+(-y*z*w)*e134+(x*y*z)*e234
```

In any case, we will call μ a **multiplication on** F when it is unital and strictly graded-commutative (though not necessarily associative), and we will call $F = (F, \mathrm{d}, \mu)$ an **MDG** R**-algebra**. The "M" stands for multiplication, the "D" stands for differential, and the "G" stands for grading; this explains our terminology. If μ also satisfies the associativity axiom, then we will also call F a **DG** R**-algebra**.

Question 2

We are next led to the following question:

Question 2: Given a multiplication μ on F, can we provide a "good" measure as to how far away μ is from being associative?

Question 2 has different answers, depending on what "good" means. We provide a possible answer by studying the homology of the image of the associator map as well as studying the maximal associative quotient of μ. The **associator** of μ is the chain map, denoted $[\cdot]_\mu$ (or more simply by $[\cdot]$ if μ is understood from context), from $F \otimes_R F \otimes_R F$ to F defined by

$$[\cdot] := \mu(\mu \otimes 1 - 1 \otimes \mu).$$

We denote by $[\cdot,\cdot,\cdot]\colon F^3 \to F$ to be the unique R-trilinear map which corresponds to $[\cdot]$ via the universal mapping property of tensor products. Thus we have

$$[a_1 \otimes a_2 \otimes a_3] = (a_1 \star a_2) \star a_3 - a_1 \star (a_2 \star a_3) = [a_1, a_2, a_3]$$

for all $a_1, a_2, a_3 \in F$. The **associator** R**-complex** of μ, denoted $[\mu]$, is the R-subcomplex of F given by the image of the associator of μ. Thus the underlying graded R-module of $[\mu]$ is

$$[\mu] = \mathrm{span}_R\{[a_1, a_2, a_3] \mid a_1, a_2, a_3 \in F\},$$

and the differential of $[\mu]$ is simply the restriction of the differential of F to $[\mu]$. The **associator** A**-submodule** of X, denoted $\langle X \rangle$, is defined to be the smallest A-submodule of X which contains $[X]$. The underlying graded R-module of $\langle X \rangle$ also has a simple description. Indeed, observe that

$$a_1(a_2[a_3, a_4, x]) = (a_1 a_2)[a_3, a_4, x] - [a_1, a_2, [a_3, a_4, x]] \tag{5}$$

for all $a_1, a_2, a_3, a_4 \in A$ and $x \in X$. Using identities like (5) together with graded-commutativity, one can show that the underlying graded R-module of $\langle X \rangle$ is given by

$$\langle X \rangle = \operatorname{span}_R \{ a_1[a_2, a_3, x] \mid a_1, a_2, a_3 \in A \text{ and } x \in X \}$$

The quotient $X/\langle X \rangle$ is an associative A-module. We denote by $\rho \colon X \to X/\langle X \rangle$ to be the canonical quotient map and we call $X/\langle X \rangle$ (together with its canonical quotient map ρ) the **maximal associative quotient** of X. It satisfies the following universal mapping property: every MDG A-module homomorphism $\varphi \colon X \to Y$ in which Y is associative factors through a unique MDG A-module homomorphism $\overline{\varphi} \colon X/\langle X \rangle \to Y$, meaning $\overline{\varphi}\rho = \varphi$. We express this in terms of a commutative diagram as below:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & X/\langle X \rangle \arrow[d, dashed, "\overline{\varphi}"] \\ & Y \end{tikzcd} \tag{6}$$

Indeed, suppose $\varphi \colon X \to Y$ is any MDG A-module homomorphism where Y is associative. In particular, we must have $[X] \subseteq \ker \varphi$, and since $\langle X \rangle$ is the smallest MDG A-submodule of X which contains $[X]$, it follows that $\langle X \rangle \subseteq \ker \varphi$. Thus the map $\overline{\varphi} \colon X/\langle X \rangle \to Y$ given by $\overline{\varphi}(\overline{x}) := \varphi(x)$ where $\overline{x} \in X/\langle X \rangle$ is well-defined. Furthermore, it is easy to see that $\overline{\varphi}$ is an MDG A-module homomorphism and the unique such one which makes the diagram (6) commute.

Homological Associativity

Definition 0.1. Let A be an MDG R-algebra and let X be an A-module. The **associator homology** of X is the homology of the associator A-submodule of X. We often simplify notation and denote the associator homology of X by $\mathrm{H}\langle X \rangle$ instead of $\mathrm{H}(\langle X \rangle)$. We say X is **homologically associative** if $\mathrm{H}\langle X \rangle = 0$ and we say X is homologically associative in degree i if $\mathrm{H}_i\langle X \rangle = 0$. Similarly we say X is associative in degree if $\langle X \rangle_i = 0$.

Clearly, if X is associative, then X is homologically associative. The converse holds under certain conditions.

Theorem 0.4. *Assume that $(R, \mathfrak{m})$ is a local ring, that $\langle X \rangle$ is minimal (meaning $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$), and that each $\langle X \rangle_i$ is a finitely generated R-module. If X is associative in degree i, then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $\langle X \rangle$ is also bounded below (meaning $\langle X \rangle_i = 0$ for $i \ll 0$), then X is associative if and only if X is homologically associative.*

Proof. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume

$$\mathrm{H}_{i+1}\langle X \rangle = 0 \quad \text{and} \quad \langle X \rangle_{i+1} \neq 0.$$

By Nakayama's Lemma, we can find homogeneous $a_1, a_2, a_3 \in A$ and homogeneous $x \in X$ such that $|a_1| + |a_2| + |a_3| + |x| = i+1$ and such that $a_1[a_2, a_3, x] \notin \mathfrak{m}\langle X \rangle_{i+1}$. Since $\langle X \rangle_i = 0$ by assumption, we have $\mathrm{d}(a_1[a_1, a_2, x]) = 0$. Also, since $\langle X \rangle$ is minimal, we have $\mathrm{d}\langle X \rangle \subseteq \mathfrak{m}\langle X \rangle$. Thus $a_1[a_2, a_3, x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. $\square$

The proof of Theorem (0.4) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition:

Definition 0.2. Let X be an MDG A-module.

1. Assume that $\langle X\rangle$ is bounded below. The **lower associative index** of X, denoted $\mathrm{la}\langle X\rangle$, is defined to be the smallest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\langle X\rangle_i \neq 0$ where we set $\mathrm{la}\langle X\rangle = \infty$ if X is associative. We extend this definition to case where $\langle X\rangle$ is not bounded below by setting $\mathrm{la}\langle X\rangle = -\infty$.

2. Assume that $\mathrm{H}\langle X\rangle$ is bounded below. The **lower homological associative index** of X, denoted $\mathrm{lha}\langle X\rangle$, is defined to be the smallest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\mathrm{H}_i\langle X\rangle \neq 0$ where we set $\mathrm{lha}\langle X\rangle = \infty$ if X is homologically associative. We extend this definition to case where $\mathrm{H}\langle X\rangle$ is not bounded below by setting $\mathrm{lha}\langle X\rangle = -\infty$.

3. Assume that $\langle X\rangle$ is bounded above. The **upper associative index** of X, denoted $\mathrm{ua}\langle X\rangle$, is defined to be the largest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\langle X\rangle_i \neq 0$ where we set $\mathrm{ua}\langle X\rangle = -\infty$ if X is associative. We extend this definition to case where $\langle X\rangle$ is not bounded above by setting $\mathrm{ua}\langle X\rangle = \infty$.

4. Assume that $\mathrm{H}\langle X\rangle$ is bounded above. The **upper homological associative index** of X, denoted $\mathrm{uha}\langle X\rangle$, is defined to be the largest $i \in \mathbb{Z}\cup\{\infty\}$ such that $\mathrm{H}_i\langle X\rangle \neq 0$ where we set $\mathrm{uha}\langle X\rangle = -\infty$ if X is homologically associative. We extend this definition to case where $\mathrm{H}\langle X\rangle$ is not bounded above by setting $\mathrm{uha}\langle X\rangle = \infty$.

With the lower associative index of X and the lower homological associative index of X defined, we see after analyzing the proof of Theorem (0.4), that if R is local, $\langle X\rangle$ is minimal and bounded below, and each $\langle X\rangle_i$ is finitely generated as an R-module, then we have $\mathrm{la}\langle X\rangle = \mathrm{lha}\langle X\rangle$. On the other hand, even if these conditions are satisfied, we often have $\mathrm{ua}\langle X\rangle > \mathrm{uha}\langle X\rangle$. For instance, we will see in Example (0.3) that $\mathrm{ua}\langle F\rangle = 4$ and $\mathrm{uha}\langle F\rangle = 3$. In the case that we're mostly interested in, R is a local noetherian ring and F is the minimal free R-resolution of R/I. In this case, we are interested in the quantity:

$$\mathrm{a}(R/I) := \inf_{\mu\in\mathrm{Mult}(F)}\{\mathrm{uha}\langle F_\mu\rangle - \mathrm{lha}\langle F_\mu\rangle + 1\},$$

where F_μ denotes F equipped with the multiplication μ. We call $\mathrm{a}(R/I)$ the **associative index** of R/I. One can think of $\mathrm{a}(R/I)$ as measuring the failure to put a DG algebra structure on F. In particular, there exists a DG algebra structure on F if and only if $\mathrm{a}(R/I) = 0$. In Example (0.3), we have $\mathrm{a}(R/I) = 1$. Thus there is no DG algebra structure on F in this case, but the fact that $\mathrm{a}(R/I) = 1$ tells us that we can get extremely close.

Remark 2. Let X be an MDG A-module. Then the short exact sequence of graded $\mathrm{H}(A)$-modules

$$0 \longrightarrow \langle X\rangle \xrightarrow{\iota} X \xrightarrow{\rho} X/\langle X\rangle \longrightarrow 0$$

induces a long exact sequence of R-modules:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(X) \arrow[r] & \mathrm{H}_{i+1}(X/\langle X\rangle) \arrow[dll, rounded corners, "\mathrm{d}_i"'] \\
\mathrm{H}_i\langle X\rangle \arrow[r] & \mathrm{H}_i(X) \arrow[r] & \mathrm{H}_i(X/\langle X\rangle) \arrow[dll, rounded corners, "\mathrm{d}_{i-1}"'] \\
\mathrm{H}_{i-1}\langle X\rangle \arrow[r] & \mathrm{H}_{i-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{7}$$

where the connecting map is induced by the differential $\mathrm{d}\colon X \to X$. In particular, we obtain a sequence of graded $\mathrm{H}(A)$-modules:

$$\mathrm{H}(X) \xrightarrow{\rho} \mathrm{H}(X/\langle X\rangle) \xrightarrow{\mathrm{d}} \mathrm{H}\langle X\rangle(-1) \xrightarrow{\iota} \mathrm{H}(X)(-1)$$

which is exact at $\mathrm{H}(X/\langle X\rangle)$ and $\mathrm{H}\langle X\rangle(-1)$.

Appendix

Before we dive into the theory of MDG R-algebras, we provide some motivation for their study by discussing a combinatorial setting where they show up. The following construction was first described in [BPS98]: let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_d]$ where $\Bbbk$ is a field and let $I = \langle \boldsymbol{m} \rangle = \langle m_1, \dots, m_r \rangle$ is a monomial ideal in R. For each subset $\sigma \subseteq \{1, \dots, r\}$, we denote $e_\sigma := \{e_i \mid i \in \sigma\}$ (thus $e_{123} = \{e_1, e_2, e_3\}$). We also set $m_\sigma := \mathrm{lcm}(m_i \mid i \in \sigma)$ and we set $\boldsymbol{\alpha}_\sigma \in \mathbb{Z}^n$ to be the exponent vector of m_σ. Let Δ be a finitely simplicial complex with r-vertices denoted $e_1, \dots, e_r$. The sequence of monomials $\boldsymbol{m}$ induces a labeling of the faces of Δ as follows: we label the vertices $e_1, \dots, e_r$ of Δ by the monomials $m_1, \dots, m_r$ (so e_i is labeled by m_i). More generally, if e_σ a face of Δ, then we label it by m_σ. With the faces labeled this way, we call Δ an **$\boldsymbol{m}$-labeled simplicial complex** (or a labeled simplicial complex if $\boldsymbol{m}$ is understood from context). Also, for each $\boldsymbol{\alpha} \in \mathbb{Z}^n$, let $\Delta_{\boldsymbol{\alpha}}$ be the subcomplex of Δ defined by

$$\Delta_{\boldsymbol{\alpha}} = \{\sigma \in \Delta \mid m_\sigma \text{ divides } x^{\boldsymbol{\alpha}}\}.$$

We often denote the faces of $\Delta_{\boldsymbol{\alpha}}$ by $(x^{\boldsymbol{\alpha}}/m_\sigma)e_\sigma$ instead of σ whenever context is clear.

Definition 0.3. We define an R-complex, denoted F_Δ (or more simply denoted F if Δ is understood from context) and called the **R-complex induced by** Δ as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R-module of F is given by

$$F_k := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $\mathrm{d}(e_\varnothing) = 0$ and

$$\mathrm{d}(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \setminus \{\varnothing\}$ where $\mathrm{pos}(i, \sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma \setminus i$ denotes the face obtained from σ by removing i. In the case where Δ is the r-simplex, we call F the **Taylor complex**.

Observe that F also has the structure of a **multigraded** $\Bbbk$-complex (or an $\mathbb{N}^n$-graded $\Bbbk$-complex) since the differential d respects the multigrading. In other words, we have a decomposition of $\Bbbk$-complexes

$$F = \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} F_{\boldsymbol{\alpha}},$$

where the $\Bbbk$-complex $F_{\boldsymbol{\alpha}}$ in multidegree $\boldsymbol{\alpha} \in \mathbb{N}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded $\Bbbk$-vector space is given by

$$F_{k,\boldsymbol{\alpha}} := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} \Bbbk \frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{\alpha}} \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $\mathrm{d}_{\boldsymbol{\alpha}}$ of $F_{\boldsymbol{\alpha}}$ is just the restriction of d to $F_{\boldsymbol{\alpha}}$. Notice that the differential behaves exactly like boundary map of $\Delta_{\boldsymbol{\alpha}}$ does:

$$\begin{aligned} \mathrm{d}_{\boldsymbol{\alpha}} \left(\frac{x^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma \right) &= \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma} \mathrm{d}(e_\sigma) \\ &= \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma} \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{\alpha}} m_\sigma}{m_\sigma m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_{\sigma \setminus i}} e_{\sigma \setminus i}. \end{aligned}$$

Thus if we define $\varphi_{\boldsymbol{\alpha}} \colon F_{\boldsymbol{\alpha}}(1) \to \mathcal{S}(\Delta_{\boldsymbol{\alpha}})$ to be the unique graded $\Bbbk$-linear isomorphism such that $\frac{\boldsymbol{x}^{\boldsymbol{\alpha}}}{m_\sigma} e_\sigma \mapsto \sigma$, then from the computation above, we see that $\mathrm{d}_{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} = \partial_{\boldsymbol{\alpha}} \mathrm{d}_{\boldsymbol{\alpha}}$, and hence $\varphi_{\boldsymbol{\alpha}}$ gives an isomorphism of $\Bbbk$-complexes

$\varphi\colon \Sigma^{-1}F_\alpha \simeq C(\Delta_\alpha;\Bbbk)$, where $C(\Delta_\alpha,\Bbbk)$ is the reduced chain complex of Δ_α over $\Bbbk$. In particular, this implies

$$\begin{aligned} \mathrm{H}(F) &= \ker \mathrm{d}/\operatorname{im}\mathrm{d} \\ &= \left(\bigoplus_{\alpha\in\mathbb{Z}^n} \ker \mathrm{d}_\alpha\right) / \left(\bigoplus_{\alpha\in\mathbb{Z}^n} \operatorname{im}\mathrm{d}_\alpha\right) \\ &\cong \bigoplus_{\alpha\in\mathbb{Z}^n} (\ker \mathrm{d}_\alpha/\operatorname{im}\mathrm{d}_\alpha) \\ &= \bigoplus_{\alpha\in\mathbb{Z}^n} \mathrm{H}(F_\alpha) \\ &\cong \bigoplus_{\alpha\in\mathbb{Z}^n} \widetilde{\mathrm{H}}(\Delta_\alpha,\Bbbk)(-1). \end{aligned}$$

In other words, we have

$$\mathrm{H}_i(F) \cong \bigoplus_{\alpha\in\mathbb{Z}^n} \mathrm{H}_i(F_\alpha) \cong \bigoplus_{\alpha\in\mathbb{Z}^n} \widetilde{\mathrm{H}}_{i-1}(\Delta;\Bbbk).$$

for all $i \in \mathbb{Z}$. From this we easily get the following theorem:

Theorem 0.5. *F is an R-free resolution of $R/\mathfrak{m}$ if and only if for all $\alpha \in \mathbb{Z}^n$ either Δ_α is the void complex or Δ_α is acyclic. In particular, the Taylor complex is an R-free resolution of $R/\mathfrak{m}$. Moreover, F is minimal if and only if $m_\sigma \neq m_{\sigma'}$ for every proper subface σ' of a face σ.*

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