Algebraic Topology Homework 1

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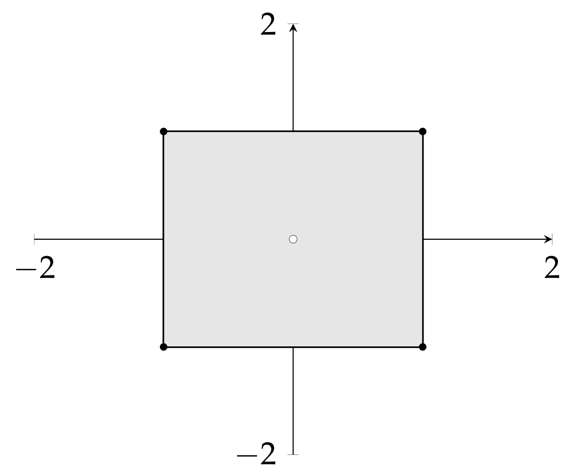
Problem 1

Exercise 1. Construct an explicit deformation retraction of the torus T with one point deleted onto a graph G consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution 1. Let $\|\cdot\|_\infty$ denote the sup norm on $\mathbb{R}^2$ defined by $\|x\|_\infty = \max\{x_1, x_2\}$ for all $x \in \mathbb{R}^2$. Note that the sup norm induces the same topology as the usual Euclidean norm does (in particular, $\|\cdot\|_\infty \colon \mathbb{R}^2 \to \mathbb{R}$ is continuous). Now set

$$X = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\} \backslash \{0\} \quad \text{and} \quad A = \{x \in \mathbb{R}^2 \mid \|x\|_\infty = 1\}.$$

We illustrate X and A below: X is the grey shaded region (including the borders) whereas A is the black shaded borders of the square.



We define $F \colon X \times I \to X$ by

$$F(x,t) = (1-t)x + t(x/\|x\|_\infty).$$

Note that $f_0(x) := F(x,0) = x$ and $f_1(x) = F(x,1) = x/\|x\|_\infty$. In particular, $f_0 = 1_X$ and f_1 is a retraction. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in A$, then $\|z\|_\infty = 1$, and thus $F(z,t) = z$ for all $t \in I$.

Next we identity T with the quotient space $[X] := X/\sim$ where $\sim$ is defined by

$$(-1,b) \sim (1,b) \text{ and } (a,-1) \sim (a,1)$$

for all $a, b \in [-1,1]$. Similarly we identify G with the quotient space $[A] := A/\sim$. Note that if $x \sim y$, then $F(x,t) \sim F(y,t)$ for all $t \in I$. Thus F induces a continuous map $[F] \colon [X] \times I \to [X]$. It is easy to see that $[F]$ is a deformation retract of $[X]$ onto $[A]$ since it inherits these properties from F.

Problem 2

Exercise 2. Construct an explicit deformation retraction of $X = \mathbb{R}^n \backslash \{0\}$ onto S^{n-1}.

Solution 2. Define $F \colon X \times I \to X$ by

$$F(x,t) = (1-t)x + t(x/\|x\|)$$

where $\|\cdot\|$ is the usual Euclidean norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$. Note that $f_0 = 1_X$ and f_1 is a retraction map. Moreover, since F is continuous at all points in its domain, we see that F is a deformation retraction of X onto A. In fact, F is a *strong* deformation retraction since if $z \in S^n$, then $\|x\| = 1$, and thus $F(x,t) = x$ for all $t \in I$.

Problem 3

To solve this problem (as well as the next problem), we will make use of the following lemma which says homotopies pass through the composition operation:

Lemma 0.1. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous functions which are homotopic to $f'\colon X \to Y$ and $g'\colon Y \to Z$ respectively (denoted $f \sim f'$ and $g \sim g'$). Then $gf \sim g'f'$ (where $gf = g \circ f$ and $g'f' = g' \circ f'$ denotes composition).*

Proof. Let $F\colon X \times I \to Y$ be a homotopy from f to f' and let $G\colon Y \times I \to Z$ be a homotopy from g to g'. Thus

$$\begin{aligned} F(x,0) &= f(x) \\ F(x,1) &= f'(x) \\ G(y,0) &= g(y) \\ G(y,1) &= g'(y) \end{aligned}$$

Define $H\colon X \times I \to Z$ by $H(x,t) = G(F(x,t),t)$. We can think of H as the composite map $X \times I \to Y \times I \to Z$ where the map $X \times I \to Y \times I$ sending (x,t) to $(F(x,t),t)$ is continuous since each component function is continuous and where the map $Y \times I \to Z$ sending (y,t) to $G(y,t)$ is continuous since G is a homotopy. Therefore, H is a continuous map. Furthermore it is straightforward to check that $H(-,0) = gf$ and $H(-,1) = g'f'$. Thus H is a homotopy from gf to $g'f'$, that is, $gf \sim g'f'$. □

Remark 1. Let $f_1, f_1'\colon X_1 \to X_2$, and $f_2, f_2'\colon X_2 \to X_3$, and $f_3, f_3'\colon X_3 \to X_4$ be continuous functions such that $f_1 \sim f_1'$, and $f_2 \sim f_2'$, and $f_3 \sim f_3'$. Write $f = f_3 f_2$ and $f' = f_3' f_2'$. By the lemma above, we have $f \sim f'$, which implies

$$\begin{aligned} f_3 f_2 f_1 &= (f_3 f_2) f_1 \\ &= f f_1 \\ &\sim f' f_1' \\ &= (f_3' f_2') f_1' \\ &= f_3' f_2' f_1'. \end{aligned}$$

This shows that we may replace a function in a composite with a homotopic map without having to worry about associativy.

Now we state and solve problem 3:

Exercise 3. Prove the following:

1. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

2. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

3. Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution 3. 1. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be homotopy equivalences with homotopy inverses $\widetilde{f}\colon Y \to X$ and $\widetilde{g}\colon Z \to Y$ respectively. Thus we have $\widetilde{f} f \sim 1_X$, $f\widetilde{f} \sim 1_Y$, $\widetilde{g} g \sim 1_Y$, and $g\widetilde{g} \sim 1_Z$. In particular, this implies

$$\begin{aligned} (gf)(\widetilde{f}\widetilde{g}) &= g(f\widetilde{f})\widetilde{g} \\ &\sim g 1_Y \widetilde{g} \\ &= g\widetilde{g} \\ &\sim 1_Z \end{aligned}$$

A similar computation gives us $(\widetilde{f}\widetilde{g})(gf) \sim 1_X$. It follows that $gf\colon X \to Z$ is a homotopy equivalence. In particular, this says that if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$ (this shows that $\sim$ is transitive; that $\sim$ is reflexive and symmetric is obvious).

2. Let $f, g, h \colon X \to Y$ be continuous functions such that $f \sim g$ and $g \sim h$, say $F \colon X \times I \to Y$ is a homotopy from f to g and $G \colon X \times I \to Y$ is a homotopy from g to h. Define $H \colon X \times I \to Y$ by

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \le t \le 1/2 \\ G(x, 2t-1) & 1/2 \le t \le 1 \end{cases}$$

Clearly H is continuous. Furthermore, we have $H(-,0) = f$, $H(-,1/2) = g$, and $H(-,1) = h$. In particular, H is a homotopy from f to h. It follows that $\sim$ is transitive (that $\sim$ is reflexive and symmetric is obvious).

3. Let $f \colon X \to Y$ be a homotopy equivalence with $\widetilde{f} \colon Y \to X$ being its homotopy inverse and suppose $f' \colon X \to Y$ is a map which is homotopic to f. Then by the lemma above, we have $1_Y \sim f\widetilde{f} \sim f'\widetilde{f}$ and $1_X \sim \widetilde{f}f \sim \widetilde{f}f'$. This shows that f' is a homotopy equivalence as well.

Problem 4

Exercise 4. A deformation retraction in the weak sense of a space X to a subspace A is a homotopy $f_t \colon X \to X$ such that $f_0 = 1_X$, $f_1(X) \subseteq A$, and $f_t(A) \subseteq A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $\iota \colon A \to X$ is a homotopy equivalence.

Solution 4. Define $r \colon X \to A$ by $r(x) = f_1(x)$ (thus $\iota r = f_1$). We claim that r is the homotopy inverse to ι. Indeed, we have $r\iota \sim 1_A$ since the map $R \colon A \times I \to A$ given by $R(a,t) = f_t(a)$ is a homotopy from 1_A to $r\iota$ (notice we needed the fact that $f_t(A) \subseteq A$ in order for this map to make sense). On the other hand, we have $\iota r \sim 1_X$ since $F \colon X \times I \to X$ is a homotopy from 1_X to ιr.