Probability Homework 1

Michael Nelson

Problem 1

We prove this by induction on $n \geq 2$. For the base case, $n = 2$, we have

$$\begin{aligned} P(A \cup B) &= P((A \backslash B) \cup B) \\ &= P(A \backslash B) + P(B) \\ &= P(A \backslash (A \cap B)) + P(B) \\ &= P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

Now suppose we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

for some $n \geq 2$ and for any events $A_1, \ldots, A_n$. Let $A_1, \ldots, A_n, A_{n+1}$ be any $n+1$ events. Then we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} A_i\right) &= P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) \\ &= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right) \\ &= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right) \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}) + P(A_{n+1}) - \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k} \cap A_{n+1}) \\ &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}) + P(A_{n+1}) + \sum_{k=1}^{n} (-1)^{k+2} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k} \cap A_{n+1}) \\ &= \sum_{k=1}^{n+1} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n+1} P(A_{i_1} \cap \cdots \cap A_{i_k}), \end{aligned}$$

where we used the base case step to get from the first line to the second line, and where we used the induction step to get from the third line to the fourth line.

Problem 2

We first calculate the probability of winning at the game of craps. When we first roll the two dice, the probability that they roll to 7 is given by

$$P(\text{roll is } 7) = \frac{6}{36}.$$

This is because there are $6^2 = 36$ possible ways to roll two dice and 6 of them add up to seven (namely $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$). A similar computation shows that the probabilities of rolling an eleven is given by

$$P(\text{sum is } 11) = \frac{2}{36}.$$

2/36. Thus the probability of winning on the first roll is $6/36 + 2/36 = 2/9$.

Now we consider the case when our initial roll is 4, 5, 6, 8, 9, or 10. Let us first focus on the case when the initial roll is 4. To determine the probability of winning when our initial roll is 4, we just need to determine the probability of rolling a 4 given that we've rolled either a 4 or a 7. We have

$$\begin{aligned} P(\text{roll is 4} \mid \text{roll is 4 or 7}) &= \frac{P((\text{roll is 4}) \text{ and } (\text{roll is 4 or 7}))}{P(\text{roll is 4 or 7})} \\ &= \frac{P(\text{roll is 4})}{P(\text{roll is 4 or 7})} \\ &= \frac{3/36}{3/36 + 6/36} \\ &= \frac{1}{3}. \end{aligned}$$

Thus the probability of winning when our first roll is a 4 is

$$\begin{aligned} P(\text{winning when initial roll is 4}) &= P(\text{initial roll is 4}) \cdot P(\text{roll is 4} \mid \text{roll is 4 or 7}) \\ &= \frac{3}{36} \cdot \frac{1}{3} \\ &= \frac{1}{36}. \end{aligned}$$

Similar calculations gives us

$$\begin{aligned} P(\text{winning when initial sum is 5}) &= \frac{2}{45} \\ P(\text{winning when initial sum is 6}) &= \frac{25}{396} \\ P(\text{winning when initial sum is 8}) &= \frac{25}{396} \\ P(\text{winning when initial sum is 9}) &= \frac{2}{45} \\ P(\text{winning when initial sum is 10}) &= \frac{1}{36} \end{aligned}$$

To find the probability of winning the game, we add everything together:

$$P(\text{winning game}) = \frac{2}{9} + \frac{1}{36} + \frac{2}{45} + \frac{25}{396} + \frac{25}{396} + \frac{2}{45} + \frac{1}{36} = \frac{244}{495}.$$

Throughout the rest of this problem, denote $\gamma = 244/495$.

Problem 2.a

Define the random variable

$$X = \{\text{number of games won}\}.$$

To find the PMF and CDF of X, we first calculate the probability of winning exactly k games. There are $\binom{10}{k}$ ways of winning exactly k games, and for a fixed choice of k games to win, the probability of winning those k games is given by

$$P(\text{fixed way of winning } k \text{ games}) = \gamma^k (1 - \gamma)^{10-k}$$

Thus the probability of winning exactly k games is given by

$$P(\text{winning exactly } k \text{ games}) = \binom{10}{k} \gamma^k (1 - \gamma)^{10-k}.$$

This implies the probability of winning less $\leq k$ games is given by

$$\begin{aligned} P(\text{winning } \leq k \text{ games}) &= \sum_{i=0}^{k} P(\text{winning exactly } i \text{ games}) \\ &= \sum_{i=0}^{k} \binom{10}{i} \gamma^i (1 - \gamma)^{10-i}. \end{aligned}$$

Now we can calculate the PMF and CDF of X. The PMF of X is given by

$$f_X(x) = P_X(X = x) = \begin{cases} \binom{10}{x}\gamma^x(1-\gamma)^{10-x} & \text{if } x = 0, 1, \ldots, 10 \\ 0 & \text{else} \end{cases}$$

The CDF of X is given by

$$F_X(x) = P_X(X \leq x) = \sum_{i=0}^{\lfloor x \rfloor} \binom{10}{i} \gamma^i (1-\gamma)^{10-i}.$$

Problem 2.b

Define the random variable

$$X = \{\text{number of games played until player wins}\}.$$

To find the PMF and CDF of X, we first calculate the probability of playing n games before winning (meaning they finally won during the nth game, so $n = 2$ means they lost the first game but won the second game). For $n = 1$, we have

$$P(\text{winning first game}) = \gamma.$$

More generally, we have

$$P(\text{playing } n \text{ games before winning}) = (1-\gamma)^{n-1}\gamma.$$

Now we can calculate the PMF and CDF of X. The PMF of X is given by

$$f_X(x) = P_X(X = x) = \begin{cases} (1-\gamma)^{x-1}\gamma & \text{if } x \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

The CDF of X is given by

$$F_X(x) = P_X(X \leq x) = \sum_{i=1}^{\lfloor x \rfloor} (1-\gamma)^{i-1}\gamma.$$

Problem 2.c

Define the random variable

$$X = \{\text{number of games played until player wins 6 times}\}.$$

To find the PMF and CDF of X, we first calculate the probability of playing n games before winning 6 times, meaning they finally won their 6th during the nth game. Clearly we need $n \geq 6$ because at least 6 games need to be played before 6 games are won. For $n = 6$, we have

$$P(\text{winning first six games}) = \gamma^6.$$

Now let us consider the case where $n > 6$. Then number of ways for a player to win their 6th game during the nth game is given by $\binom{n-1}{5}$. Indeed, the nth game needs to be won, and there are $\binom{n-1}{5}$ ways of winning exacty 5 of the first $n-1$ games. For a fixed way of winning the 6th during the nth (say win the first 5 games, lose the rest of the games up to the $(n-1)$st game, and then win the nth game), the probability is given by

$$P(\text{fixed way of winning 6th game during the } n\text{th game}) = \gamma^6(1-\gamma)^{n-6}$$

Therefore the probability of winning the 6th game during the nth game is given by

$$P(\text{winning 6th game during the } n\text{th game}) = \binom{n-1}{5}\gamma^6(1-\gamma)^{n-6}.$$

Now we can calculate the PMF and CDF of X. The PMF of X is given by

$$f_X(x) = P_X(X = x) = \begin{cases} \binom{x-1}{5}\gamma^6(1-\gamma)^{x-6} & \text{if } x \in \mathbb{Z}_{\geq 6} \\ 0 & \text{else} \end{cases}$$

The CDF of X is given by

$$F_X(x) = P_X(X \leq x) = \sum_{i=6}^{\lfloor x \rfloor} \binom{x-1}{5}\gamma^6(1-\gamma)^{x-6}.$$

Problem 3

Let $n \in \mathbb{N}$ and let S_n denote the symmetric. We say $\sigma \in S_n$ is a **derangement** if $\sigma(i) \neq i$ for all $1 \leq i \leq n$. We denote by D_n to be the set of all derangements of n elements. For $1 \leq i \leq n$ we define F_i to be the set of permutations of n objects that fix the ith object. Note that if $1 \leq i_1 < \cdots < i_k \leq n$, then $|F_{i_1} \cap \cdots \cap F_{i_k}| = |S_{n-k}| = (n-k)!$. Therefore by the inclusion-exclusion principle, we have

$$\begin{aligned}
|D_n| &= n! - |F_1 \cup \cdots \cup F_n| \\
&= n! - \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |F_{i_1} \cap \cdots \cap F_{i_k}| \\
&= n! - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! \\
&= n! - \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!} \\
&= n! \left(1 - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!} \right) \\
&= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.
\end{aligned}$$

Now suppose that n men at a party throw their hats in the center of the room. Define the random variable

$$X_n = \{\text{number of men who selected their own hat}\}.$$

Problem 3.a

We first calculate the probability that nobody selected their own hat. This is given by the number of derangements of the n hats divided by the number of permutations of the n hats:

$$P(\text{exactly 0 men selected their own hat}) = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

Now let us consider the probability that exactly k people selected their own hat where $1 \leq k \leq n$. To do this, we first count the number of permutations in S_n which fix exactly k objects. First let us consider the number of permutations in S_n which fixes exactly the first k objects. Any permutation in S_n which fixes exactly the first k objects must be a derangement on the last $n-k$ objects. In particular, the number of permutations in S_n which fixes exactly the first k objects is given $|D_{n-k}|$. By a similar argument, for any fixed choice of k objects, there are $|D_{n-k}|$ permutations in S_n which fix exactly these k objects. Since there $\binom{n}{k}$ choices of k objects to fix, we see that the number of permutations in S_n which fix exactly k objects is given by $\binom{n}{k}|D_{n-k}|$. Therefore the probability that exactly k men selected their own hat is given by

$$\begin{aligned}
P(\text{exactly } k \text{ men selected their own hat}) &= \frac{\binom{n}{k}|D_{n-k}|}{|S_n|} \\
&= \frac{n!}{k!(n-k)!}(n-k)! \sum_{i=0}^{(n-k)} (-1)^i \frac{1}{i!} \cdot \frac{1}{n!} \\
&= \frac{1}{k!} \sum_{i=0}^{(n-k)} (-1)^i \frac{1}{i!}
\end{aligned}$$

So the probability mass function of X_n is

$$f_{X_n}(x) = P(X_n = x) = \begin{cases} \frac{1}{x!} \sum_{i=0}^{(n-x)} (-1)^i \frac{1}{i!} & \text{if } x = 0, 1, 2, \ldots, n \\ 0 & \text{else} \end{cases}$$

Problem 3.b

By taking $n \to \infty$, we get

$$\begin{aligned} f_X(x) &= \lim_{n\to\infty} f_{X_n}(x) \\ &= \lim_{n\to\infty} \begin{cases} \frac{1}{x!}\sum_{i=0}^{(n-x)}(-1)^i\frac{1}{i!} & \text{if } x = 0,1,2,\dots,n \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{x!e} & \text{if } x \in \mathbb{Z}_{\geq 0} \\ 0 & \text{else} \end{cases} \end{aligned}$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} f_X(n) &= \sum_{n=0}^{\infty} \frac{1}{x!e} \\ &= \frac{1}{e}\sum_{n=0}^{\infty}\frac{1}{x!} \\ &= \frac{1}{e}\cdot e \\ &= 1. \end{aligned}$$

Problem 3.c

Problem 3.b suggests that a good approximation to f_{X_n} when n is large is given by the function $1/x!e$. Let us check how this approximation is when $n = 100$. Using wolfram alpha, we calculate

$$\begin{aligned} f_{X_{100}}(0) &\approx 0.368 \\ f_{X_{100}}(1) &\approx 0.368 \\ f_{X_{100}}(2) &\approx 0.184 \\ f_{X_{100}}(3) &\approx 0.061 \end{aligned}$$

Also using wolfram alpha, we calculate

$$\begin{aligned} \frac{1}{e} &\approx 0.368 \\ \frac{1}{e} &\approx 0.368 \\ \frac{1}{2e} &\approx 0.184 \\ \frac{1}{6e} &\approx 0.061. \end{aligned}$$

Thus for $n = 100$ and $x = 1,2,3$, the function $1/x!e$ is a good approximation to $f_{X_n}(x)$. In fact, for any $x = 1,2,3,\dots,100$, the function $1/x!e$ is a good approximation to $f_{X_n}(x)$. This is because

$$\begin{aligned} f_{X_{100}}(x) - \frac{1}{x!e} &= \frac{1}{x!}\left(\sum_{i=0}^{100-x}(-1)^i\frac{1}{i!} - \frac{1}{e}\right) \\ &= \frac{1}{x!}\sum_{i=100-x}^{\infty}(-1)^i\frac{1}{i!} \\ &\leq \frac{e}{x!}, \end{aligned}$$

and $x!$ grows very large very fast. More generally, for any n large, we have

$$f_{X_n}(x) - \frac{1}{x!e} \leq \frac{e}{x!}.$$

Problem 1.6

We first note that

$$\begin{aligned}
p_0 &= (1-u)(1-w) \\
p_1 &= (1-u)w + u(1-w) \\
p_2 &= uw.
\end{aligned}$$

Now suppose $p_0 = p_1 = p_2$. Since $p_2 = p_1$, we have

$$u + w - 2uw = uw.$$

In other words, $u + w = 3uw$. Since $p_0 = p_2$, we have

$$1 - u + w + uw = uw.$$

In other words, $u + w = 1$. Thus we have the equations

$$\begin{aligned}
3uw &= 1 \\
u + w &= 1
\end{aligned}$$

Rewrite the second equation as $u = 1 - w$ and plug this into the first equation to get

$$\begin{aligned}
1 &= 3(1-w)w \\
&= -w^2 + 3w.
\end{aligned}$$

In other words we hav

$$w^2 - 3w - 1 = 0 \tag{1}$$

The solutions to (1) are $w = (3-\sqrt{5})/2$ and $w = (3+\sqrt{5})/2$. We need $w \leq 1$, so the only solution that works is $w = (3-\sqrt{5})/2$. Since $u = 1 - w$, this imlpies $u = (-1+\sqrt{5})/2$.

Problem 1.7

Problem 1.7.a

First note that

$$P(\text{scoring 0 points}) = 1 - \frac{\text{Area of dart board}}{\text{Area of wall}} = 1 - \frac{\pi r^2}{A}.$$

For $i = 1, 2, 3, 4, 5$, we can calculate

$$\begin{aligned}
P(\text{scoring } i \text{ points}) &= \frac{\text{Area of region } i}{\text{Area of wall}} \\
&= \frac{\text{Area of region } i}{\text{Area of dart board}} \cdot \frac{\text{Area of dart board}}{\text{Area of wall}} \\
&= \frac{(6-i)^2 - (5-i)^2}{5^2} \cdot \frac{\pi r^2}{A}.
\end{aligned}$$

Problem 1.7.b

For $i = 1, 2, 3, 4, 5$, note that $\{\text{scoring } i \text{ points}\}$ is a subset of $\{\text{board is hit}\}$. Therefore we have

$$\begin{aligned}
P(\text{scoring } i \text{ points} \mid \text{board is hit}) &= \frac{P(\text{scoring } i \text{ points and board is hit})}{P(\text{board is hit})} \\
&= \frac{P(\text{scoring } i \text{ points})}{P(\text{board is hit})} \\
&= \frac{(6-i)^2 - (5-i)^2}{5^2}.
\end{aligned}$$

Problem 1.8

Problem 1.8.a

We have

$$\begin{aligned}
P(\text{scoring } i \text{ points}) &= \frac{\text{Area of region } i}{\text{Area of dartboard}} \\
&= \frac{\pi((6-i)r/5)^2 - \pi((6-i-1)r/5)^2}{\pi r^2} \\
&= \frac{((6-i)r/5)^2 - ((6-i-1)r/5)^2}{r^2} \\
&= \frac{r^2(6-i)/5)^2 - r^2((6-i-1)/5)^2}{r^2} \\
&= \frac{(6-i)^2 - (5-i)^2}{5^2}. \\
&= \frac{11-2i}{25}.
\end{aligned}$$

Problem 1.8.b

Let $j \geq i$ where $i, j \in \{1, 2, 3, 4, 5\}$. Then we have

$$\begin{aligned}
P(\text{scoring } j \text{ points}) \leq P(\text{scoring } i \text{ points}) &\iff \frac{(6-j)^2 - (5-j)^2}{5^2} \leq \frac{(6-i)^2 - (5-i)^2}{5^2} \\
&\iff (6-j)^2 - (5-j)^2 \leq (6-i)^2 - (5-i)^2 \\
&\iff 11 - 2j \leq 11 - 2i \\
&\iff -2j \leq -2i \\
&\iff j \geq i.
\end{aligned}$$

Thus P is a decreasing function of i.

Problem 1.8.c

We have

$$\begin{aligned}
\sum_{i=1}^{5} P(\text{scoring } i \text{ points}) &= \sum_{i=1}^{5} \frac{11-2i}{25} \\
&= \frac{1}{25} \sum_{i=1}^{5} (11-2i) \\
&= \frac{1}{25}(55 - 2 - 4 - 6 - 8 - 10) \\
&= \frac{1}{25} \cdot 25 \\
&= 1.
\end{aligned}$$

Problem 1.11

Problem 1.11.a

We have $\emptyset \in \mathcal{B}$. It is also closed under complements: $\emptyset^c = S \in \mathcal{B}$ and $S^c = \emptyset \in \mathcal{B}$. Finally it is closed under countable unions: $\emptyset \cup S = S \in \mathcal{B}$, $\emptyset \cup \emptyset = \emptyset \in \mathcal{B}$, and $S \cup S = S \in \mathcal{B}$.

Problem 1.11.b

We have $\emptyset \in \mathcal{B}$. It is also closed under complements: if $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ (since $\mathcal{B}$ contains *every* subset of S). Similarly, it is closed under countable unions for the same reasons ($\mathcal{B}$ contains *every* subset of S).

Problem 1.11.c

Let $\mathcal{M}$ and $\mathcal{N}$ be two σ-algebras of subsets of S. We will show $\mathcal{M} \cap \mathcal{N}$ is a σ-algebra of subsets of S. We have $\emptyset \in \mathcal{M} \cap \mathcal{N}$ since $\emptyset \in \mathcal{M}$ and $\emptyset \in \mathcal{N}$. It is also closed under complements: let $A \in \mathcal{M} \cap \mathcal{N}$. Then $A \in \mathcal{M}$ and $A \in \mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras, this implies $A^c \in \mathcal{M}$ and $A^c \in \mathcal{N}$. Thus $A^c \in \mathcal{M} \cap \mathcal{N}$. Finally, it is closed under countable unions: let (A_n) be a sequence of members of $\mathcal{M} \cap \mathcal{N}$. Then for each $n \in \mathbb{N}$, we have $A_n \in \mathcal{M}$ and $A_n \in \mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras, this implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}$. Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M} \cap \mathcal{N}$.

Problem 1.23

Let $0 \leq i \leq n$. First we calculate the probability of person 1 tossing i heads. This is given by

$$P_1(i \text{ heads}) = \frac{\binom{n}{i}}{2^n}$$

Similarly, the the probability of person 2 tossing i heads is given by

$$P_2(i \text{ heads}) = \frac{\binom{n}{i}}{2^n}$$

Since the number of heads which person 1 tosses is independent from the number of heads that person 2 tosses, we see that the probability of both person 1 and person 2 tossing i heads is given by

$$\begin{aligned} P(i \text{ heads}) &= P_1(i \text{ heads}) \cdot P_2(i \text{ heads}) \\ &= \frac{\binom{n}{i}}{2^n} \cdot \frac{\binom{n}{i}}{2^n} \\ &= 4^{-n} \binom{n}{i}^2. \end{aligned}$$

Finally, to find the probability of person 1 and person 2 tossing the same number of heads, we sum all of our probabilities over i:

$$\begin{aligned} P(\text{same number of heads}) &= \sum_{i=1}^{n} P(i \text{ heads}) \\ &= \sum_{i=1}^{n} 4^{-n} \binom{n}{i}^2 \\ &= 4^{-n} \sum_{i=1}^{n} \binom{n}{i}^2 \\ &= 4^{-n} \binom{2n}{n}. \end{aligned}$$

Problem 1.24

Problem 1.24.a

Let $n \in \mathbb{N}$. We first count the probability of player A winning at the nth game. For $n = 1$, we have clearly have

$$P(\text{winning first game}) = \frac{1}{2}.$$

For player A to win at the 2nd game, player A needs to flip tails, then player B needs to flip tails, and finally player A needs to flip heads. In other words, we need the sequence TTH when we flip a coin 3 times. The probability of getting this sequence is $1/2^3$. More generally, we have

$$P(\text{winning } n\text{th game}) = \frac{1}{2^{2n-1}}.$$

Now we we sum over all n to get

$$
\begin{aligned}
P(\text{winning the game}) &= \sum_{n=1}^{\infty} P(\text{winning } n\text{th game}) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{4^n} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} \\
&= \frac{1}{2} \left(\frac{1}{1-\frac{1}{4}} \right) \\
&= \frac{2}{3}.
\end{aligned}
$$

Problem 1.24.b

We do a similar computation as in 1.24.a. For $n = 1$, we have

$$P(\text{winning first game}) = p.$$

For player A to win at the 2nd game, player A needs to flip tails, then player B needs to flip tails, and finally player A needs to flip heads. In other words, we need the sequence TTH when we flip a coin 3 times. This time, the probability of getting this sequence is given by $(1-p)^2 p$. More generally, we have

$$P(\text{winning } n\text{th game}) = (1-p)^{2n-2} p.$$

Now we we sum over all n to get

$$
\begin{aligned}
P(\text{winning the game}) &= \sum_{n=1}^{\infty} P(\text{winning } n\text{th game}) \\
&= \sum_{n=1}^{\infty} (1-p)^{2(n-1)} p \\
&= \sum_{n=0}^{\infty} (1-p)^{2n} p \\
&= p \left(\frac{1}{1-(1-p)^2} \right) \\
&= \frac{p}{2p-p^2} \\
&= \frac{1}{2-p}.
\end{aligned}
$$

Problem 1.24.c

This is clear since if $0 < p < 1$, then we have

$$
\begin{aligned}
P(\text{winning the game}) &= \frac{1}{2-p} \\
&> \frac{1}{2}.
\end{aligned}
$$

Problem 1.37

Problem 1.37.a

First we calculate $P(\mathcal{W})$. We have

$$\begin{aligned}
P(\mathcal{W}) &= P(\text{warden says } B \text{ dies}) \\
&= P(\text{warden says } B \text{ dies and } A \text{ pardoned}) + P(\text{warden says } B \text{ dies and } C \text{ pardoned}) \\
&= \frac{\gamma}{3} + \frac{1}{3} \\
&= \frac{\gamma+1}{3}.
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
P(A \cap \mathcal{W}) &= P(\text{warden says } B \text{ dies and } A \text{ pardoned}) \\
&= \frac{\gamma}{3}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
P(A \mid \mathcal{W}) &= \frac{P(A \cap \mathcal{W})}{P(\mathcal{W})} \\
&= \frac{\gamma/3}{(\gamma+1)/3} \\
&= \frac{\gamma}{\gamma+1}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\gamma}{\gamma+1} \geq \frac{1}{3} &\iff 3\gamma \geq \gamma + 1 \\
&\iff 2\gamma \geq 1 \\
&\iff \gamma \geq \frac{1}{2}.
\end{aligned}$$

Problem 1.37.b

A's reasoning is correct. This is because

$$\begin{aligned}
P(C \mid \mathcal{W}) &= \frac{P(C \cap \mathcal{W})}{P(\mathcal{W})} \\
&= \frac{1/3}{1/2} \\
&= 2/3.
\end{aligned}$$

Problem 1.38

Problem 1.38.a

We have

$$\begin{aligned}
P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\
&= P(A \cap B) \\
&= P(A) + P(B) - P(A \cup B) \\
&= P(A) + 1 - 1 \\
&= P(A).
\end{aligned}$$

Problem 1.38.b

We have

$$\begin{aligned} P(B \mid A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{P(A)}{P(A)} \\ &= 1. \end{aligned}$$

Also, we have

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)}. \end{aligned}$$

Problem 1.38.c

We have

$$\begin{aligned} P(A \mid (A \cup B)) &= \frac{P(A \cap (A \cup B))}{P(A \cup B)} \\ &= \frac{P(A)}{P(A \cup B)} \\ &= \frac{P(A)}{P(A) + P(B)}. \end{aligned}$$

Problem 1.39

Problem 1.39.a

Suppose A and B are mutually exclusive and assume for a contradiction that they are also independent. Then we have

$$\begin{aligned} 0 &< P(A)P(B) \\ &= P(A \cap B) \\ &= P(A \cup B) - P(A) - P(B) \\ &= 0, \end{aligned}$$

which is a contradiciton.

Problem 1.39.b

Suppose A and B are independent and assume for a contradiction that they are mutually exclusive. Then we have

$$\begin{aligned} 0 &= P(A \cup B) - P(A) - P(B) \\ &= P(A \cap B) \\ &= P(A)P(B) \\ &> 0, \end{aligned}$$

which is a contradiction.

Problem 1.41

Problem 1.41.a

We first calculate

$$\begin{aligned} P(\text{dash received}) &= P(\text{dash received and dash sent}) + P(\text{dash received and dot sent}) \\ &= P(\text{dash received} \mid \text{dash sent})P(\text{dash sent}) + P(\text{dash received} \mid \text{dot sent})P(\text{dot sent}) \\ &= \frac{7}{8} \cdot \frac{4}{7} + \frac{1}{8} \cdot \frac{3}{7} \\ &= \frac{31}{56}. \end{aligned}$$

Thus using Bayes Rule, we have

$$\begin{aligned} P(\text{dash sent} \mid \text{dash received}) &= P(\text{dash received} \mid \text{dash sent})\frac{P(\text{dash sent})}{P(\text{dash received})} \\ &= \frac{7}{8} \cdot \frac{4/7}{31/56} \\ &= \frac{7}{8} \cdot \frac{4}{7} \cdot \frac{56}{31} \\ &= \frac{28}{31}. \end{aligned}$$

Problem 1.41.b

We have

$$\begin{aligned} P(\text{dash-dash sent} \mid \text{dot-dot received}) &= P(\text{dash sent} \mid \text{dot received}) \cdot P(\text{dash sent} \mid \text{dot received}) = \frac{4}{25} \cdot \frac{4}{25} \\ P(\text{dash-dot sent} \mid \text{dot-dot received}) &= P(\text{dash sent} \mid \text{dot received}) \cdot P(\text{dot sent} \mid \text{dot received}) = \frac{4}{25} \cdot \frac{21}{25} \\ P(\text{dot-dash sent} \mid \text{dot-dot received}) &= P(\text{dash sent} \mid \text{dot received}) \cdot P(\text{dash sent} \mid \text{dot received}) = \frac{21}{25} \cdot \frac{4}{25} \\ P(\text{dot-dot sent} \mid \text{dot-dot received}) &= P(\text{dot sent} \mid \text{dot received}) \cdot P(\text{dot sent} \mid \text{dot received}) = \frac{21}{25} \cdot \frac{21}{25} \end{aligned}$$

Problem 1.43

Problem 1.4.3.a

We will show

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \tag{2}$$

by induction on $n \geq 1$. The base case $n = 1$ is trivial. Now suppose we have shown (2) for some $n \geq 1$. Then using the inclusion-exclusion identity, we have
Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} A_i\right) &= P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) \\ &\leq P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) \\ &\leq \sum_{i=1}^{n+1} P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right) \\ &\leq \sum_{i=1}^{n+1} P(A_{n+1}). \end{aligned}$$

Taking $n \to \infty$ gives us

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

Problem 1.4.3.b and problem 1.4.3.c

For each $1 \leq k \leq n$, we set

$$P_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}).$$

The books asks us to show that $P_m \geq P_k$ given $m \geq k$. However I don't see how this is true. For example,

$$P_1 = \sum_{i=1}^{n} P(A_i) \geq P(A_1 \cap \cdots A_n) = P_n.$$

We also do not necessarily have $P_m \leq P_k$ given $m \geq k$. Indeed, consider the extreme case $A_1 = A_2 = A_3 = A_4 = A$. Then

$$P_1 = 4P(A) \leq 6P(A) = P_2.$$

Problem 1.45

Let $A \subseteq \mathcal{X}$. Then we have

$$\begin{aligned} P_X(X \in A) &= P(\{s \in S \mid X(s) \in A\}) \\ &\geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} P_X(X \in \mathcal{X}) &= P(\{s \in S \mid X(s) \in \mathcal{X}\}) \\ &= P(S) \\ &\geq 1. \end{aligned}$$

Finally, let (A_n) be a pairwise disjoint sequence in $\mathcal{X}$. Then we have

$$\begin{aligned} P_X\left(X \in \bigcup_{n=1}^{\infty} A_n\right) &= P\left(\left\{s \in S \mid X(s) \in \bigcup_{n=1}^{\infty} A_n\right\}\right) \\ &= P\left(\bigcup_{n=1}^{\infty} \{s \in S \mid X(s) \in A_n\}\right) \\ &= \sum_{n=1}^{\infty} P\left\{s \in S \mid X(s) \in A_n\right\} \\ &= \sum_{n=1}^{\infty} P_X(X \in A_n). \end{aligned}$$

Thus P_X satisfies the Kolmogorov axioms.

Problem 1.51

We first calculate the probability that no defectives found. There are $\binom{30}{4}$ ways of selecting 4 ovens. There are $\binom{26}{4}$ ways of selecting 4 nondefective ovens. Therefore

$$\begin{aligned} P(\text{no defective ovens found}) &= \frac{\binom{26}{4}}{\binom{30}{4}} \\ &= \frac{2990}{5481}. \end{aligned}$$

Now we calculate the probability of selecting exactly one defective oven. There are $\binom{4}{1}$ ways of selecting 1 defective oven and there are $\binom{26}{3}$ ways of selecting 3 nondefective ovens. Therefore

$$\begin{aligned} P(\text{exactly one defective oven found}) &= \frac{\binom{4}{1}\binom{26}{3}}{\binom{30}{4}} \\ &= \frac{2080}{5481}. \end{aligned}$$

Now we calculate the probability of selecting exactly two defective ovens. There are $\binom{4}{2}$ ways of selecting 2 defective ovens and there are $\binom{26}{2}$ ways of selecting 2 nondefective ovens. Therefore

$$\begin{aligned} P(\text{exactly two defective ovens found}) &= \frac{\binom{4}{2}\binom{26}{2}}{\binom{30}{4}} \\ &= \frac{130}{1827}. \end{aligned}$$

Now we calculate the probability of selecting exactly three defective ovens. There are $\binom{4}{3}$ ways of selecting 3 defective ovens and there are $\binom{26}{1}$ ways of selecting 3 nondefective ovens. Therefore

$$\begin{aligned} P(\text{exactly two defective ovens found}) &= \frac{\binom{4}{3}\binom{26}{1}}{\binom{30}{4}} \\ &= \frac{104}{27405}. \end{aligned}$$

Now we calculate the probability of selecting exactly four defective ovens. There are $\binom{4}{4}$ ways of selecting 4 defective ovens and there are $\binom{26}{0}$ ways of selecting 0 nondefective ovens. Therefore

$$\begin{aligned} P(\text{exactly two defective ovens found}) &= \frac{\binom{4}{4}\binom{26}{0}}{\binom{30}{4}} \\ &= \frac{1}{27405}. \end{aligned}$$

Therefore the PMF of X is given by

$$f_X(x) = P_X(X = x) = \begin{cases} \frac{\binom{4}{x}\binom{26}{4-x}}{\binom{30}{4}} & \text{if } x = 0, 1, 2, 3, 4 \\ 0 & \text{else} \end{cases}$$

The CDF of X is given by

$$F_X(x) = P_X(X \leq x) = \sum_{i=0}^{\lfloor x \rfloor} \frac{\binom{4}{\lfloor x \rfloor}\binom{26}{4-\lfloor x \rfloor}}{\binom{30}{4}}.$$

Problem 1.52

We check that the conditions in Theorem 1.6.5 from the book hold. First, let us check that g is nonnegative. This is clearly the case since if $x < x_0$, then $g(x_0) = 0$, and if $x \geq x_0$, then

$$\begin{aligned} g(x) &= \frac{f(x)}{1 - F(x_0)} \\ &\geq 0, \end{aligned}$$

where we used the fact that $f \geq 0$ and $F(x_0) < 1$. Finally, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\mathrm{d}x &= \int_{-\infty}^{x_0} g(x)\mathrm{d}x + \int_{x_0}^{\infty} g(x)\mathrm{d}x \\
&= \int_{-\infty}^{x_0} 0\mathrm{d}x + \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)}\mathrm{d}x \\
&= \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)\mathrm{d}x \\
&= \frac{1}{1-F(x_0)} \left(-\int_{-\infty}^{x_0} f(x)\mathrm{d}x + \int_{-\infty}^{\infty} f(x)\mathrm{d}x \right) \\
&= \frac{1}{1-F(x_0)} \left(-F(x_0)+1\right) \\
&= 1,
\end{aligned}$$

where we used the fact that f is a pdf.

Problem 1.53

Problem 1.53.a

We have

$$\begin{aligned}
\lim_{y\to\infty} F_Y(y) &= \lim_{y\to-\infty} \left(1 - \frac{1}{y^2}\right) \\
&= 1.
\end{aligned}$$

Similarly, $\lim_{y\to-\infty} F_Y(y) = 0$. Also F_Y is an increasing function. Indeed, if $z \geq y$, then $1/z^2 \leq 1/y^2$, and thus

$$\begin{aligned}
F_Y(z) &= 1 - \frac{1}{z^2} \\
&\geq 1 - \frac{1}{y^2} \\
&= F_Y(y).
\end{aligned}$$

Finally, the function $1 - 1/y^2$ is continuous on $(0, \infty)$. Thus F is continuous everywhere (it wasn't mentioned in the problem, but I believe F_Y is defined on all of $\mathbb{R}$ by extending it by $F_Y(y) = 0$ for all $y \leq 1$).

Problem 1.53.b

Suppose that $y > 1$. It follows from the fundamental theorem of calculus that

$$\begin{aligned}
f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) \\
&= \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - \frac{1}{y^2}\right) \\
&= \frac{2}{y^3}.
\end{aligned}$$

If we want to extend f_Y to all of $\mathbb{R}$, then we set $f_Y(y) = 0$ for all $y \leq 1$.

Problem 1.53.c

We have

$$\begin{aligned}
F_Z(y) &= P(Z \leq y) \\
&= P(10(Y-1) \leq y) \\
&= P\left(Y \leq \frac{y}{10} + 1\right) \\
&= F_Y\left(\frac{y}{10} + 1\right).
\end{aligned}$$

Problem 1.55

We calculate

$$\begin{aligned}
F_T(t) &= \int_0^t f_T(x)\mathrm{d}x \\
&= \int_0^t \frac{1}{1.5} e^{-x/(1.5)}\mathrm{d}x \\
&= \frac{1}{1.5}\int_0^t e^{-x/(1.5)}\mathrm{d}x \\
&= \frac{1}{1.5}\left(-1.5e^{-x/1.5}\Big|_0^t\right) \\
&= 1 - e^{-t/1.5}.
\end{aligned}$$