MDG Associator Homology

Let R be a commutative ring, let $A = (A, \mathrm{d}, \mu)$ be an MDG algebra centered at R, and let $T = \mathrm{T}(A)$ be the tensor DG algebra of A. Define $\delta \colon T \to T$ to be the chain map given by

$$\delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n-1} (-1)^{k-1} a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots a_n.$$

For instance, we have

$$\begin{aligned}
\delta(a_1 \otimes a_2) &= a_1 a_2 \\
\delta(a_1 \otimes a_2 \otimes a_3) &= a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 \\
\delta(a_1 \otimes a_2 \otimes a_3 \otimes a_4) &= a_1 a_2 \otimes a_3 \otimes a_4 - a_1 \otimes a_2 a_3 \otimes a_4 + a_1 \otimes a_2 \otimes a_3 a_4.
\end{aligned}$$

Observe that

$$\begin{aligned}
\delta^2(a_1 \otimes a_2 \otimes a_3) &= [a_1, a_2, a_3] \\
\delta^2(a_1 \otimes a_2 \otimes a_3 \otimes a_4) &= [a_1, a_2, a_3] \otimes a_4 + a_1 \otimes [a_2, a_3, a_4] \\
\delta^3(a_1 \otimes a_2 \otimes a_3 \otimes a_4) &= [a_1 a_2, a_3, a_4] - [a_1, a_2 a_3, a_4] + [a_1, a_2, a_3 a_4]
\end{aligned}$$

In particular, one can show that $\delta^2 = 0$ if and only if A is associative. Since δ is a chain map, we obtain a sequence of differentials $\mathrm{d}^{(n)} := \mathrm{d}\delta^n$ on T.