Probability Theory Homework 1

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Problem 3

Solution 1. labelsol Suppose $x \in \limsup(A_n \cup B_n)$. Then x belongs to infinitely many of the $A_n \cup B_n$'s. In particular, this implies x belongs to infinitely many of the A_n's or it belongs to infinitely many of the B_n's. Without loss of generality, say x belongs to infinitely many of the A_n's. In other words, $x \in \limsup A_n$. It follows that

$$\limsup(A_n \cup B_n) \subseteq \limsup A_n \cup \limsup B_n.$$

Conversely, suppose $x \in \limsup A_n \cup \limsup B_n$. Then either $x \in \limsup A_n$ or $x \in \limsup B_n$. Without loss of generality, say $x \in \limsup A_n$. Then x belongs to infinitely many of the A_n's. This implies x belongs to infinitely many of the $A_n \cup B_n$'s. In other words, $x \in \limsup(A_n \cup B_n)$. It follows that

$$\limsup(A_n \cup B_n) \supseteq \limsup A_n \cup \limsup B_n.$$

Now we answer the second part of the question. Suppose $A_n \to A$ and $B_n \to B$ where

$$\liminf A_n = A = \limsup A_n \quad \text{and} \quad \liminf B_n = B = \limsup B_n.$$

Then observe that

$$\begin{aligned} A \cup B &= \liminf A_n \cup \liminf B_n \\ &\subseteq \liminf(A_n \cup B_n) \\ &\subseteq \limsup(A_n \cup B_n) \\ &= \limsup A_n \cup \limsup B_n \\ &= A \cup B. \end{aligned}$$

It follows that $A_n \cup B_n \to A \cup B$. Similarly, we have

$$\begin{aligned} A_n \cap B_n &= (A_n^c \cup B_n^c)^c \\ &\to A^c \cup B^c \\ &= (A \cap B)^c. \end{aligned}$$

Problem 5

Solution 2. labelsol For each $k, n \in \mathbb{N}$, set $A_{k,n} = \{\omega \mid f_n(\omega) - f(\omega) \geq 1/k\}$. We want to show that

$$\{\omega \mid f_n(\omega) \not\to f(\omega)\} = \bigcup_{k=1}^{\infty} \limsup_{n\to\infty}(A_{k,n}).$$

To see this, observe that

$$\begin{aligned} \omega \in \{\omega \mid f_n(\omega) \not\to f(\omega)\} &\iff \exists k \in \mathbb{N} \text{ such that } |f_n(\omega) - f(\omega)| \geq 1/k \text{ for infinitely many } n \in \mathbb{N} \\ &\iff \exists k \in \mathbb{N} \text{ such that } \omega \in A_{k,n} \text{ for infinitely many } n \in \mathbb{N} \\ &\iff \exists k \in \mathbb{N} \text{ such that } \omega \in \limsup_{n\to\infty}(A_{k,n}) \\ &\iff \omega \in \bigcup_{k=1}^{\infty} \limsup_{n\to\infty}(A_{k,n}). \end{aligned}$$

Problem 10

Solution 3. labelsol We have

$$\begin{aligned} 1_{A_n} \to 1_A &\iff \liminf 1_{A_n} = \limsup 1_{A_n} \\ &\iff 1_{\liminf A_n} = 1_{\limsup A_n} \\ &\iff \liminf A_n = \limsup A_n \\ &\iff A_n \to A \end{aligned}$$

Problem 24

Solution 4. labelsol First we show $\mathcal{A}$ is a field. We have $\emptyset \in \mathcal{A}$ since $\emptyset$ is finite. Clearly $\mathcal{A}$ is closed under compliments since $A \in \mathcal{A}$ implies either A or A^c is finite which implies $A^c \in \mathcal{A}$. It remains to show that $\mathcal{A}$ is closed under finite intersections. Let $A, B \in \mathcal{A}$ and suppose that $A \cap B$ is infinite. We must show that $(A \cap B)^c = A^c \cup B^c$ is finite. In other words, we need to show that both A^c and B^c are finite. Assume for a contradiction that A^c is infinite. Then A must be finite since $A \in \mathcal{A}$. But this implies $A \cap B$ is finite, which is a contradiction. Thus A^c must be finite. Similarly, we can prove by contradiction that B^c is finite too. This shows that $\mathcal{A}$ is a field. To see that $\mathcal{A}$ is not a σ-field, consider the set of all positive even numbers A. If $\mathcal{A}$ were a σ-field, then we must have $A \in \mathcal{A}$ since it can be expressed as a countable union of singleton sets (with each singleton set clearly belonging to $\mathcal{A}$):

$$A = \bigcup_{n \in \mathbb{N}} \{2n\}.$$

However $A \notin \mathcal{A}$ since both A and A^c are infinite:A^c is the set of all positive odd integers).

Problem 26

Problem 26.a

Solution 5. labelsol Let $\mathcal{B} = \{\bigcup_{i \in I} A_i \mid I \subseteq [k]\}$ where $[k] = \{1, \dots, k\}$. First note that $\mathcal{B} \subseteq \mathcal{A}(\mathcal{C})$ since $\mathcal{A}(\mathcal{C})$ contains all finite unions of members of $\mathcal{C}$. To show $\mathcal{A}(\mathcal{C}) \subseteq \mathcal{B}$, it suffices to show that $\mathcal{B}$ is an algebra. We have $\emptyset \in \mathcal{B}$. Given $\bigcup_{i \in I} A_i$ and $\bigcup_{j \in J} A_j$ in $\mathcal{B}$, we have

$$\begin{aligned} \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} A_j\right) &= \bigcup_{i \in I, j \in J} A_i \cap A_j \\ &= \bigcup_{i \in I \cap J} A_i, \end{aligned}$$

since $A_i \cap A_j = \emptyset$ for all $i \neq j$. It follows that $\mathcal{B}$ is closed under finite intersections. Also, denoting $I^* = [k] \backslash I$, we have

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcup_{i^* \in I^*} A_{i^*}$$

since $\mathcal{C}$ forms a partition of Ω. It follows that $\mathcal{B}$ is closed under complements. Therefore $\mathcal{B}$ is algebra, and since $\mathcal{B}$ contains $\mathcal{C}$, we see that $\mathcal{A}(\mathcal{C}) \subseteq \mathcal{B}$.

Problem 26.b

Solution 6. labelsol This σ-algebra generated by $\mathcal{C}$ is precisely $\mathcal{A}(\mathcal{C})$. This is because $\mathcal{A}(\mathcal{C})$ contains only finitely many members. In particular, a countable union of members of $\mathcal{A}(\mathcal{C})$ can be reexpressed as a finite union of members of $\mathcal{A}(\mathcal{C})$. For instance,

$$\bigcup_{n \in \mathbb{N}} \bigcup_{i \in I_n} A_i = \bigcup_{i \in I} A_i$$

where $I = \bigcup_{n \in \mathbb{N}} I_n$.

Problem 26.c

Solution 7. labelsol The induced σ-algebra is given by $\mathcal{D} = \{\bigcup_{i\in I} A_i \mid I \subseteq \mathbb{N}\}$. Indeed, we have $\emptyset \in \mathcal{D}$. Let $\{\bigcup_{i\in I_n} A_i\}_{n\in\mathbb{N}}$ be a countable collection of members of $\mathcal{D}$. Then we have

$$\bigcup_{n\in\mathbb{N}} \bigcup_{i\in I_n} A_i = \bigcup_{i\in I} A_i$$

where $I = \bigcup_{n\in\mathbb{N}} I_n$. Thus $\mathcal{D}$ is closed under countable unions. Finally, given $\bigcup_{i\in I} A_i$ in $\mathcal{D}$, set $I^* = \mathbb{N}\backslash I$ and observe that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcup_{i^*\in I^*} A_{i^*}$$

since $\{\mathcal{A}_1, \mathcal{A}_2 \dots\}$ forms a countable partition of Ω.

Problem 26.d

Solution 8. labelsol The atoms of the collection of half intervals are of the form $(n, n+1]$ where $n \in \mathbb{Z}$. For the converse of part a of this problem, note that the atoms of $\mathcal{A}$ are disjoint from one another: if A and B are distinct atoms of $\mathcal{A}$, then $A \cap B \in \mathcal{A}$ and $A \cap B \subseteq A$ implies $A \cap B = \emptyset$. Moreover, since $\mathcal{A}$ is finite, it has finitely many atoms, say $A_1, \dots, A_n$, and we must have $\Omega = A_1 \cup \cdots \cup A_n$. In particular, $\{A_1, \dots, A_n\}$ forms a finite partition of Ω.

Problem 31

Solution 9. labelsol Assume for a contradiction that $\mathcal{G}$ is countably generated, say by $\mathcal{C} = \{A_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, either A_n is countable or A_n^c is countable. By replacing A_n with its complement if necessary, we may assume that each A_n is countable. Let $X = \bigcup_{n=1}^{\infty} A_n$ and define $\mathcal{H}$ to be the collection of sets B such that either $B \subseteq X$ or $B^c \subseteq X$. By construction, we have

$$\mathcal{C} \subseteq \mathcal{H} \subset \mathcal{G},$$

where the inclusion $\mathcal{H} \subset \mathcal{G}$ is strict since for any $\omega \in \Omega\backslash X$ we have $\{\omega\} \in \mathcal{G}\backslash\mathcal{H}$ (such an ω exists since Ω is uncountable). We claim that $\mathcal{H}$ is a σ-algebra (which will contradict our assumption that $\mathcal{G}$ is the smallest σ-algebra which contains $\mathcal{C}$) . Indeed, we clearly have $\emptyset \in \mathcal{H}$. Also $\mathcal{H}$ is clearly closed under complements. To show $\mathcal{H}$ is closed under countable unions, let $\{B_n\}$ be a countable collection of members of $\mathcal{H}$. If each B_n is contained in X, then their union is contained in X; hence in this case their union belongs to $\mathcal{H}$. On the other hand, if one of the B_n's is not contained in X, say B_1, then B_1^c is contained in X, and in this case we have

$$\left(\bigcup_{n=1}^{\infty} B_n\right)^c = \bigcap_{n=1}^{\infty} B_n^c \subseteq X.$$

This also implies that their union belongs to $\mathcal{H}$. Thus $\mathcal{H}$ is closed under countable unions.

Problem 32

Solution 10. labelsol The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ contains all countable subsets of $\mathbb{R}$. In particular, if we define $\mathcal{G}$ to be the σ-algebra consisting of subsets $A \subseteq \mathbb{R}$ such that either A or A^c is countable, then $\mathcal{G} \subseteq \mathcal{B}(\mathbb{R})$. However, $\mathcal{B}(\mathbb{R})$ is countably generated but $\mathcal{G}$ is not countable generated (by problem 31). To see why $\mathcal{B}(\mathbb{R})$ is countable generated, for each $q, r \in \mathbb{Q}$ and $n \in \mathbb{N}$, let

$$\mathrm{B}_{1/n}(q) = \{x \in \mathbb{R} \mid |x - q| < 1/n\}$$

Then the collection

$$\mathcal{C} = \{\mathrm{B}_{1/n}(q) \mid n \in \mathbb{N} \text{ and } q \in \mathbb{Q}\}$$

forms a countable basis for the usual topology on $\mathbb{R}$. In particular, if U be an open subset of $\mathbb{R}$, then we can express U as a union of the form

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

where $B_\lambda \in \mathcal{C}$ and where the index set Λ is *countable.* In particular, since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra which contains all open sets, we see that $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$.