Tensor-Hom Adjointness and its Applications

Let B be an A-algebra, let X and Y be B-modules, and let Z be an A-module. Define a map

$$(-)^\diamond\colon \mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z)) \to \mathrm{Hom}_A(X \otimes_B Y, Z)$$

as follows: for all $\varphi \in \mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z))$ we set $\varphi^\diamond$ to be the unique linear map defined on elementary tensors by $x \otimes y \in X \otimes_B Y$ by

$$\varphi^\diamond(x \otimes y) := (\varphi x)y, \tag{1}$$

where we are using the notational convention $\varphi x = \varphi(x)$ in order to simplify our notation in what follows. Note that (3) is well-defined since the map $(x, y) \mapsto (\varphi x)y$ is B-bilinear. Indeed, additivity in one argument while the other is fixed is obvious. Also φ is B-linear by assumption, so $(\varphi(bx))y = (b(\varphi x))y$, and φx is B-linear because $\mathrm{Hom}_A(Y, Z)$ is given the structure of a B-module using the fact that Y is a B-module; namely $(b(\varphi x))y := (\varphi x)(by)$. Finally $(-)^\diamond$ is B-linear because both φ and φx are B-linear and because $\mathrm{Hom}_A(X \otimes_B Y, Z)$ is given the structure of a B-module using the fact that Y is a B-module; namely

$$(b(\varphi^\diamond))(x \otimes y) := \varphi^\diamond(x \otimes by) = (\varphi x)(by) = (b(\varphi x))y = (\varphi(bx))y = ((b\varphi)x)y =: (b\varphi)^\diamond(x \otimes y). \tag{2}$$

Notice that b never appeared outside all of the parenthesis in (2): every term in (2) is an element of Z, which is an A-module! Next we define a map

$$(-)_\diamond\colon \mathrm{Hom}_A(X \otimes_B Y, Z) \to \mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z))$$

as follows: for all $\psi \in \mathrm{Hom}_A(X \otimes_B Y, Z)$ we set $\psi_\diamond$ to be the unique B-linear map such that for all $x \in X$ and $y \in Y$ we have

$$(\psi_\diamond x)y := \psi(x \otimes y) \tag{3}$$

Note that (3) is well-defined since the map $(x, y) \mapsto (\psi_\diamond x)y$ is B-bilinear. Thus for instance, the following is a perfectly legitimate computation:

$$\begin{aligned}
((b\psi + \widetilde{\psi})_\diamond x)y &= (b\psi + \widetilde{\psi})(x \otimes y) \\
&= (b\psi)(x \otimes y) + \widetilde{\psi}(x \otimes y) \\
&= \psi(x \otimes by) + \widetilde{\psi}(x \otimes y) \\
&= (\psi_\diamond x)(by) + (\widetilde{\psi}_\diamond x)y \\
&= (b(\psi_\diamond x)y + (\widetilde{\psi}_\diamond x)y \\
&= ((b\psi)_\diamond x)y + (\widetilde{\psi}_\diamond x)y.
\end{aligned}$$

Again, b never appears outside the parenthesis in the computation above because each of these elements belongs to Z. Thus $(-)_\diamond$ and $(-)^\diamond$ are both B-module homomorphisms. In fact, we get something much stronger!

Theorem 0.1. *The map $(-)^\diamond$ is an isomorphism which is natural in X, Y, and Z, with the map $(-)_\diamond$ being its inverse. In particular, the functor $- \otimes_B Y$ is left adjoint to the functor $\mathrm{Hom}_B(X, -)$, and thus $- \otimes_B X$ preserves all colimits and $\mathrm{Hom}_A(X, -)$ preserves all limits.*

Intuitively, one thinks of $\varphi^\diamond(x \otimes y) = (\varphi x)y$ as applying the "associative law" where the diamond in the superscript tells us that we can "pull back" the parenthesis. Similarly, one thinks of $(\psi_\diamond x)y = \psi(x \otimes y)$ as applying the "associative law" where the diamond in the subscript tells us that we can "push forward" the parenthesis. With this in in mind, it is very easy to see why $(-)^\diamond$ and $(-)_\diamond$ are inverse to each other: we are just applying the associative law! Indeed, we have

$$((\varphi^\diamond)_\diamond x)y = \varphi^\diamond(x \otimes y) = (\varphi x)y \quad \text{and} \quad (\psi_\diamond)^\diamond(x \otimes y) = (\psi_\diamond x)y = \psi(x \otimes y). \tag{4}$$

In particular, one should note that the reason why $(-)_\diamond$ and $(-)^\diamond$ are inverse to each other is precisely due to the way we defined them in the first place. Another added benefit that we get when using this notation is that when we write an interpretable string using the symbols $\{\diamond, (,), \varphi, \psi, \phi, x, y, z\}$, then it becomes visibly clear how we could interpret this string, where we consider a string interpretable if we can obtain a new string without any diamond symbols by applying the associative law a finite number of times to the original string. For instance, the string $\varphi_\diamond(x \otimes y)$ is uninterpretable in our language since we can't "pullback" the parenethesis and remove the diamond in the subscript. On the other hand, the string $\phi^\diamond(\psi x \otimes (\varphi_\diamond x)y)$ is interpretable: if we apply the associative law one time, we can remove the subscript diamond and obtain $\phi^\diamond(\psi x \otimes \varphi(x \otimes y))$. If we apply the associative law again, we can remove the superscript diamond and obtain $(\psi(\varphi x))\phi(x \otimes y)$. Since this string doesn't contain any diamonds, we can give a reasonable interpretation to it. For instance, ψ can be thought of as a map in $\mathrm{Hom}_B(L, \mathrm{Hom}_A(M, N))$, which maps the element $\varphi x \in L$ to the map $\psi(\varphi x) \in \mathrm{Hom}_A(M, N)$ whose value at $\phi(x \otimes y)$ is $(\psi(\varphi x))\phi(x \otimes y)$.

Proof. We've have already shown that $(-)^\diamond$ is a B-linear isomorphism with $(-)_\diamond$ being its inverse. It remains to show that $(-)^\diamond$ (or equivalently $(-)_\diamond$) is natural in X, Y, and Z. But our simple description of $(-)^\diamond$ makes this completely obvious! For instance, naturality in X means that if we have an R-module homomorphism $\lambda\colon X \to X'$, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z)) \arrow[r, "(-)^\diamond"] \arrow[d, "\lambda^*"'] & \mathrm{Hom}_A(X \otimes_B Y, Z) \arrow[d, "(\lambda\otimes 1)^*"] \\
\mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z)) \arrow[r, "(-)^\diamond"] & \mathrm{Hom}_A(X \otimes_B Y, Z)
\end{tikzcd}$$

Where $(-)^\diamond$ is defined on $\mathrm{Hom}_B(X', \mathrm{Hom}_A(Y, Z))$ essentially the same way that it was defind on $\mathrm{Hom}_B(X, \mathrm{Hom}_A(Y, Z))$. Furthermore, the diagram above commutes since if $\varphi \in \mathrm{Hom}_B(X', \mathrm{Hom}_A(Y, Z))$, then we have

$$\begin{aligned}
(\lambda^*\varphi)^\diamond(x \otimes y) &= ((\lambda^*\varphi)x)y \\
&= (\varphi(\lambda x))y \\
&= \varphi^\diamond(\lambda x \otimes y) \\
&= ((\lambda \otimes 1)^*(\varphi^\diamond))(x \otimes y).
\end{aligned}$$

The point to remember in the computation above is that all we are doing here is applying universal algrebraic rules like "commutativity" and "associativity", so it's perfectly reasonable that these become natural isomorphisms. □

0.0.1 General Version of Tensor-Hom Adjunction

Let B be an A-algebra, let X be an A-module and let Y and Z be B-modules. Note that Y and Z are given the structure of an A-module using the ring homomorphim $A \to B$, thus they are naturally A-modules. There is another version of tensor-hom which we would like to describe now. We claim that exists a canonical isomorphisms

$$(-)^\diamond\colon \mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z)) \to \mathrm{Hom}_B(X \otimes_A Y, Z) \quad \text{and} \quad (-)_\diamond\colon \mathrm{Hom}_B(X \otimes_A Y, Z) \to \mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z))$$

as B-modules, both of which are natural in X, Y, and Z. Notice that the rings have swapped positions this time. We give $\mathrm{Hom}_B(Y, Z)$ the structure of an A-module using the fact that Y and Z are A-modules; namely $(a\varphi)y := \varphi(ay) := a(\varphi y)$. Similarly we give $\mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z))$ the structure of a B-module using the fact $\mathrm{Hom}_B(Y, Z)$ and Z are B-modules; namely $((b\psi)x)y := (b(\psi x))y = (\psi x)(by) = b((\psi x)y)$. Finally we give $X \otimes_A Y$ the structure of a B-module using the fact that Y is a B-module. With all of this in mind, we define

$$\varphi^\diamond(x \otimes y) = (\varphi x)y \quad \text{and} \quad (\psi_\diamond x)y = \psi(x \otimes y).$$

These maps still work since all maps involved are B-linear maps. Here is a much more general version of the tensor-hom adjunction:

Theorem 0.2. *Let A, B, and C be three different rings (each of which is not necessarily-commutative). Let X be an (A,B)-bimodule (so A acts on the left of X and B acts on the right of X), let Y be a (B,C)-bimodule, and let Z be an (A,C)-bimodule.*

1. *We have canonical isomorphisms*

$$(-)^{\diamond}\colon \mathrm{Hom}_B(X,\mathrm{Hom}_C(Y,Z)) \to \mathrm{Hom}_C(X\otimes_B Y,Z) \quad \textit{and} \quad (-)_{\diamond}\colon \mathrm{Hom}_C(X\otimes_B Y,Z) \simeq \mathrm{Hom}_B(X,\mathrm{Hom}_C(Y,Z))$$

as (A,A)-bimodules, natural in X, Y, and Z, defined by

$$(\psi^{\diamond}x)y = \psi(x\otimes y) \quad \textit{and} \quad (\varphi_{\diamond}x)y = \varphi(x\otimes y).$$

2. *We have canonical isomorphisms*

$$(-)^{\diamond}\colon \mathrm{Hom}_B(Y,\mathrm{Hom}_A(X,Z)) \to \mathrm{Hom}_A(X\otimes_B Y,Z) \quad \textit{and} \quad (-)_{\diamond}\colon \mathrm{Hom}_A(X\otimes_B Y,Z) \simeq \mathrm{Hom}_B(Y,\mathrm{Hom}_A(X,Z))$$

as (C,C)-bimodules, natural in X, Y, and Z, defined by

$$(\psi^{\diamond}y)x = \psi(x\otimes y) \quad \textit{and} \quad (\varphi_{\diamond}x)y = \varphi(x\otimes y)$$

Note that first tensor-hom adjunction has the form $\mathrm{Hom}(X\otimes Y,Z)\simeq \mathrm{Hom}(X,\mathrm{Hom}(Y,Z))$ whereas the second tensor-hom adjunction has the form $\mathrm{Hom}(X\otimes Y,Z)\simeq \mathrm{Hom}(Y,\mathrm{Hom}(X,Z))$ where we note the letters X and Y getting swapped. In the case where we are working over commutative rings, then we have $X\otimes Y\simeq Y\otimes X$, so we swapping can be fixed by just relabeling things. The important to remember, is that tensor-hom should look something like $\mathrm{Hom}_{(-)}(X\otimes_{(-)} Y,Z)\simeq \mathrm{Hom}_{(-)}(X,\mathrm{Hom}_{(-)}(Y,Z))$ where we place a ring in the spots $(-)$ only where they make sense. For instance, $\mathrm{Hom}_C(X,\mathrm{Hom}_B(Y,Z))$ doesn't make sense because X is not a (left or right) C-module and there's no canonical way to give it the structure of a C-module, so it doens't make sense to talk about C-linear maps from X to $\mathrm{Hom}_B(Y,Z)$. Another thing to consider is that there are two ways of giving $\mathrm{Hom}_A(X,Z)$ an A-module structure: we can give it a left A-module structure via $(a\varphi)x := \varphi(ax)$ and we can also give it a right A-module structure via $(\varphi a)(x) := (\varphi x)a$, so $\mathrm{Hom}_A(X,Z)$ can be viewed as an (A,A)-bimodule. Also, $\mathrm{Hom}_B(Y,\mathrm{Hom}_A(X,Z))$ is a (C,C)-bimodule via $((c\psi)y)x := c((\psi y)x)$ and $((\psi c)y)x = (\psi(yc))x$.

0.0.2 Transporting Projective/Injective Modules over one Ring to Another

Let B be an A-algebra. We can use the tensor-hom adjunction to transport injective A-modules to injective B-modules as follows:

Proposition 0.1. *Let E be an injective A-module, and let P a projective B-module. Then $\mathrm{Hom}_A(P,E)$ is an injective B-module.*

Proof. The functor $\mathrm{Hom}_B(-,\mathrm{Hom}_A(P,E))$ is exact if and only if the functor $\mathrm{Hom}_A(-\otimes_B P,E)$ is exact by tensor-hom adjunction. Now notice that the functor $-\otimes_B P$ is exact since P is projective (and hence flat), and the functor $\mathrm{Hom}_A(-,E)$ is exact since E is injective. Thus $\mathrm{Hom}_A(-\otimes_B P,E)$ is a composition of exact functors, and so it must be exact too. □

We can also transport injective B-modules down to injective A-modules:

Proposition 0.2. *Let E be an injective B-module and let P be a B-module which is also a projective A-module. Then $\mathrm{Hom}_B(P,E)$ is an injective A-module.*

Proof. The functor $\mathrm{Hom}_A(-,\mathrm{Hom}_B(P,E))$ is exact if and only if the functor $\mathrm{Hom}_B(-\otimes_A P,E)$ is exact by tensor-hom adjunction. Now notice that the functor $-\otimes_A P$ is exact since P is a projective A-module (and hence flat), and the functor $\mathrm{Hom}_B(-,E)$ is exact since E is injective. Thus $\mathrm{Hom}_A(-\otimes_B P,E)$ is a composition of exact functors, and so it must be exact too. □

Now let's see how to transport projective modules; namely if we have a projective A-module and a projective B-module, then we can tensor them together to obtain another projective B-module.

Proposition 0.3. *Let P be a projective A-module and Q be a projective B-module. Then $P\otimes_A Q$ is a projective B-module.*

Proof. It suffices to show that $\mathrm{Hom}_B(P\otimes_A Q,-)$ is exact. Let

$$M_1\to M_2\to M_3\to 0$$

be a short exact sequence of B-modules. Then since Q is a projective B-module, the induced sequence

$$0 \to \mathrm{Hom}_B(Q, M_1) \to \mathrm{Hom}_B(Q, M_2) \to \mathrm{Hom}_B(Q, M_3) \to 0$$

is exact. Then since P is a projective A-module, the induced sequence

$$0 \to \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_1)) \to \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_2)) \to \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_3)) \to 0$$

is exact. By tensor-hom adjointness, we have a commutative diagram[1]

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_1)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_2)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A(P, \mathrm{Hom}_B(Q, M_3)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_B(P \otimes_A Q, M_1) \arrow[r] & \mathrm{Hom}_B(P \otimes_A Q, M_2) \arrow[r] & \mathrm{Hom}_B(P \otimes_A Q, M_3) \arrow[r] & 0
\end{tikzcd}$$

where the columns are isomorphisms and where the top row is exact. It follows from the 3×3 lemma that the bottom row is exact too. □

Essentially by the same argument works in the reverse direction too (though under more restrictions):

Proposition 0.4. *Let Q be a projective B-module and let P be a B-module which is projective as an A-module. Then $Q \otimes_B$ is a projective A-module.*

Proof. It suffices to show that $\mathrm{Hom}_A(Q \otimes_B P, -)$ is exact. Let

$$M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of A-modules. Then since P is a projective A-module, the induced sequence

$$0 \to \mathrm{Hom}_A(P, M_1) \to \mathrm{Hom}_A(P, M_2) \to \mathrm{Hom}_A(P, M_3) \to 0$$

is exact. Then since Q is a projective B-module, the induced sequence

$$0 \to \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_1)) \to \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_2)) \to \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_3)) \to 0$$

is exact. By tensor-hom adjointness, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_1)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_2)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, M_3)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_A(Q \otimes_B P, M_1) \arrow[r] & \mathrm{Hom}_A(Q \otimes_B P, M_2) \arrow[r] & \mathrm{Hom}_A(Q \otimes_B P, M_3) \arrow[r] & 0
\end{tikzcd}$$

where the columns are isomorphisms and where the top row is exact. It follows from the 3×3 lemma that the bottom row is exact too. □

0.0.3 Base Change in Ext

Let B be an A-algebra. We are often presented with the situation where we are working over the ring A and would like to change our base ring to B (and vice-versa). For instance, perhaps we know something about Ext_A and would like to use this information to obtain something about Ext_B. One way we can do this is to use tensor-hom adjointness:

Proposition 0.5. *Assume B is a flat A-algebra. Then there is a canonical isomorphism of graded B-modules*

$$\mathrm{Ext}_B(M \otimes_A B, N) \to \mathrm{Ext}_A(M, N) \tag{5}$$

which is natural in M and N.

[1]Note how we need naturality in the third argument to get a commutative diagram.

Proof. Let F be a projective resolution of M over A. Then $F \otimes_A B$ is a B-complex whose underlying graded module is projective. Furthermore, since B is flat, we have $\mathrm{H}_+(F \otimes B) = 0$ and $\mathrm{H}_0(F \otimes B) = M \otimes B$. Therefore $F \otimes B$ is a projective resolution of $M \otimes B$ over B (note that if B were not a flat A-algebra, then we'd have $\mathrm{H}_+(F \otimes B) = \mathrm{Ext}_A^+(M, B)$ which doesn't necessarily vanish). So to define the map (5), it suffices to define a quasiisomorphism

$$\mathrm{Hom}_B(F \otimes_A B, N) \to \mathrm{Hom}_A(F, N) \tag{6}$$

of B-complexes natural in F and N. Once this chain map is defined, we can then pass it through homology to get the map (5). In fact, we will construct an isomorphism (6) of B-complexes! This is much *stronger* than merely being a quasiisomoprhism. Consider the composition

$$\mathrm{Hom}_B(F \otimes_A B, N) \xrightarrow[\simeq]{(-)_\diamond} \mathrm{Hom}_A(F, \mathrm{Hom}_B(B, N)) \xrightarrow[\simeq]{[-]} \mathrm{Hom}_A(F, N).$$

The way the composite of this map looks on the elements can be seen as follows: let $\varphi\colon F \otimes_A B \to N$ be an i-chain B-map (that is, a chain map of degree i of B-complexes). From φ, we obtain the i-chain B-map $\varphi_\diamond\colon F \to \mathrm{Hom}_B(B, N)$ where $\varphi_\diamond$ is defined on elements by $(\varphi_\diamond \alpha)b = \varphi(\alpha \otimes b)$ where $\alpha \in F$ and $b \in B$. From $\varphi_\diamond$ we obtain the i-chain A-map $[\varphi_\diamond]\colon F \to N$ where $[\varphi_\diamond]$ is defined on elements by $[\varphi_\diamond](\alpha) = (\varphi_\diamond \alpha)1$ for all $\alpha \in F$. We then extend $[\varphi_\diamond]$ to a B-linear map using the fact that N is a B-module; namely

$$(b[\varphi_\diamond])(\alpha) := b([\varphi]_\diamond \alpha) = b((\varphi\alpha)1) = b\varphi(\alpha \otimes 1) = \varphi(\alpha \otimes b).$$

Composing these maps gives us a chain map of B-complexes (6). We already know that $(-)_\diamond$ is isomorphism of B-complexes, natural in F and M. It is easy to see that $[\cdot]$ is also an isomorphism of B-complexes, natural in F and N. Thus their composite, denoted $\varphi \mapsto [\varphi_\diamond]$, is an isomorphism of B-complexes, natural in F and N.

Finally, we need to discuss why naturality is important. Suppose we have an A-linear map $\lambda\colon M \to M'$. Let F' be a projective resolution of M' over A and lift λ to a comparison map $\lambda\colon F \to F'$. We obtain a diagram

$$\begin{tikzcd}
\mathrm{Hom}(F' \otimes B, N) \arrow[r, "{[(-)_\diamond]}"] & \mathrm{Hom}(F', N) \\
\mathrm{Hom}(F \otimes B, N) \arrow[r, "{[(-)_\diamond]}"] \arrow[u, "(\lambda\otimes 1)^*"] & \mathrm{Hom}(F, N) \arrow[u, "\lambda^*"']
\end{tikzcd} \tag{7}$$

which is commutative on the nose since $[(-)_\diamond]$ is a natural isomorphism. Thus when we take this diagram in homology, we obtain

$$\begin{tikzcd}
\mathrm{Ext}_B(M' \otimes B, N) \arrow[r] & \mathrm{Ext}_A(M', N) \\
\mathrm{Ext}_B(M \otimes B, N) \arrow[r] \arrow[u] & \mathrm{Ext}_A(M, N) \arrow[u]
\end{tikzcd}$$

which is again commutative on the nose. Thus the isomorphism $\mathrm{Ext}_B(M \otimes_A B, N) \to \mathrm{Ext}_A(M, N)$ is natural in M (and similarly in N), but keep in mind that we only required the diagram (7) to be commutative up to homotopy in order to bet naturality in M for Ext. □

Proposition 0.6. *Let B be an A-algebra, let Q be a projective B-module which is flat as an A-module, and let N be a B-module. Then we have*

$$\mathrm{Ext}_B(M \otimes_A Q, N) \to \mathrm{Ext}_A(M, \mathrm{Ext}_B(Q, N)) \tag{8}$$

which is natural in M and N.

Proof. Note that since Q is a projective B-module, we have $\mathrm{Ext}_B(Q, N) = \mathrm{Hom}_B(Q, N)$ as graded modules. Let X be a projective resolution of M over A. Then $X \otimes_A Q$ is a B-complex whose underlying graded module is projective (by the base change formula) and such that $\mathrm{H}_+(X \otimes Q) = \mathrm{H}_+(X) = 0$ and $\mathrm{H}_+(X \otimes Q) = M \otimes_A Q$. Thus $X \otimes_A Q$ is a projective resolution of $M \otimes_A Q$ over B. So to define the map (8), it suffices to define a quasiisomorphism

$$\mathrm{Hom}_B(F \otimes_A Q, N) \to \mathrm{Hom}_A(M, \mathrm{Hom}_B(Q, N)) \tag{9}$$

of B-complexes natural in F, Q, and N. Once this chain map is defined, we can then pass it through homology to get the map (5). In fact, we will construct an isomorphism (6) of B-complexes! This is much *stronger* than merely being a quasiisomoprhism. Consider the composition

$$\mathrm{Hom}_B(F \otimes_A Q, N) \to \mathrm{Hom}_A(F, \mathrm{Hom}_B(Q, N)).$$

The way the composite of this map looks on the elements can be seen as follows: let $\varphi\colon F \otimes_A Q \to N$ be an i-chain B-map. From φ, we obtain the i-chain B-map $\varphi_\diamond\colon F \to \mathrm{Hom}_B(Q, N)$ where $\varphi_\diamond$ is defined on elements by $(\varphi_\diamond \alpha)q = \varphi(\alpha \otimes q)$ where $\alpha \in F$ and $q \in Q$. We already know that $(-)_\diamond$ is isomorphism of B-complexes, natural in F, Q, and M, so we are done. □

0.0.4 Tensor Product of Projective is Projective

Let B be an A-algebra, let X be an A-module and let Y and Z be B-modules. Note that Y and Z are given the structure of an A-module using the ring homomorphim $A \to B$, thus they are naturally A-modules. There is another version of tensor-hom which we would like to describe. We claim that exists an isomorphism of B-modules

$$\mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z)) \to \mathrm{Hom}_B(X \otimes_A Y, Z)$$

which is natural in X, Y, and Z. Notice that the rings have swapped positions this time. We give $\mathrm{Hom}_B(Y, Z)$ the structure of an A-module using the fact that Y and Z are A-modules; namely $(a\varphi)y := \varphi(ay) := a(\varphi y)$. Similarly we give $\mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z))$ the structure of a B-module using the fact $\mathrm{Hom}_B(Y, Z)$ and Z are B-modules; namely $((b\psi)x)y := (\psi x)(by) = b((\psi x)y)$. Finally we give $X \otimes_A Y$ the structure of a B-module using the fact that Y is a B-module. With all of this in mind, we could try to define this map via $(-)^\diamond$ again and set

$$\varphi^\diamond(x \otimes y) = (\varphi x)y$$

for all $\varphi \in \mathrm{Hom}_A(X, \mathrm{Hom}_B(Y, Z))$ and for all $x \in X$ and $y \in Y$. This map still works since all maps involved are B-linear maps. Here's the most general version:

Theorem 0.3. *Let A, B, and C be three different rings (each of which is not necessarily-commutative). Let X be an (A, B)-bimodule (so A acts on the left of X and B acts on the right of X), let Y be a (B, C)-bimodule, and let Z be an (A, C)-bimodule.*

1. *We have canonical isomorphisms*

   $$(-)^\diamond\colon \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y, Z)) \to \mathrm{Hom}_C(X \otimes_B Y, Z) \quad \text{and} \quad (-)_\diamond\colon \mathrm{Hom}_C(X \otimes_B Y, Z) \simeq \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y, Z))$$

   as (A, A)-bimodules, natural in X, Y, and Z, defined by

   $$(\psi^\diamond x)y = \psi(x \otimes y) \quad \text{and} \quad (\varphi_\diamond x)y = \varphi(x \otimes y).$$

2. *We have canonical isomorphisms*

   $$(-)^\diamond\colon \mathrm{Hom}_B(Y, \mathrm{Hom}_A(X, Z)) \to \mathrm{Hom}_A(X \otimes_B Y, Z) \quad \text{and} \quad (-)_\diamond\colon \mathrm{Hom}_A(X \otimes_B Y, Z) \simeq \mathrm{Hom}_B(Y, \mathrm{Hom}_A(X, Z))$$

   as (C, C)-bimodules, natural in X, Y, and Z, defined by

   $$(\psi^\diamond y)x = \psi(x \otimes y) \quad \text{and} \quad (\varphi_\diamond x)y = \varphi(x \otimes y)$$

Note that first tensor-hom adjunction has the form $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ whereas the second tensor-hom adjunction has the form $\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(Y, \mathrm{Hom}(X, Z))$ where we note the letters X and Y getting swapped. In the case where we are working over commutative rings, then we have $X \otimes Y \simeq Y \otimes X$, so we swapping can be fixed by just relabeling things. The important to remember, is that tensor-hom should look something like $\mathrm{Hom}_{(-)}(X \otimes_{(-)} Y, Z) \simeq \mathrm{Hom}_{(-)}(X, \mathrm{Hom}_{(-)}(Y, Z))$ where we place a ring in the spots $(-)$ only where they make sense. For instance, $\mathrm{Hom}_C(X, \mathrm{Hom}_B(Y, Z))$ doesn't make sense because X is not a (left or right) C-module and there's no canonical way to give it the structure of a C-module, so it doens't make sense to talk about C-linear maps from X to $\mathrm{Hom}_B(Y, Z)$. Another thing to consider is that there are two ways of giving $\mathrm{Hom}_A(X, Z)$ an A-module structure: we can give it a left A-module structure via $(a\varphi)x := \varphi(ax)$ and we can also give it a right A-module structure via $(\varphi a)(x) := (\varphi x)a$, so $\mathrm{Hom}_A(X, Z)$ can be viewed as an (A, A)-bimodule. Also, $\mathrm{Hom}_B(Y, \mathrm{Hom}_A(X, Z))$ is a (C, C)-bimodule via $((c\psi)y)x := c((\psi y)x)$ and $((\psi c)y)x = (\psi(yc))x$.