

Tor-Persistence

Introduction

Let R be a commutative noetherian ring. Recall that a finitely generated R -module M has finite projective dimension if $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$ for each finitely generated R -module N . Indeed, first note that $\mathrm{Tor}_i^R(M, N) = 0$ if and only if

$$\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \simeq \mathrm{Tor}_i^R(M, N)_{\mathfrak{p}} = 0$$

for all prime ideals \mathfrak{p} of R . Thus by replacing R , M , and N with $R_{\mathfrak{p}}$, $M_{\mathfrak{p}}$, and $N_{\mathfrak{p}}$ if necessary, we may assume that $R = (R, \mathfrak{m}, \mathbb{k})$ is local. Now let F be the minimal R -free resolution of M . Thus

$$\mathrm{Tor}_i^R(M, N) = H_i(F \otimes_R N).$$

We first prove the easy direction: suppose M has finite projective dimension, say $\mathrm{pd}_R M = p$. This means that $F_p \neq 0$ and $F_i = 0$ for all $i > p$. In particular that $(F \otimes_R N)_i = 0$ for all $i > p$, which implies $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > p$. Now we prove the harder direction: suppose $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$ for each finitely generated R -module N . In particular, we have $\mathrm{Tor}_i^R(M, \mathbb{k}) = 0$ for $i \gg 0$. This implies $H_i(F_{\mathbb{k}}) = 0$ for $i \gg 0$ where we set $F_{\mathbb{k}} := F \otimes_R \mathbb{k}$. However F is *minimal*, thus $d_{\mathbb{k}} = 0$, where $d_{\mathbb{k}}$ is the differential of $F_{\mathbb{k}}$. Thus we have $H_i(F_{\mathbb{k}}) = F_{i, \mathbb{k}} := F_i \otimes_R \mathbb{k}$ and this implies $F_i \otimes_R \mathbb{k} = 0$ for $i \gg 0$ which implies $F_i = 0$ for $i \gg 0$ by Nakayama's lemma (here is where we used the fact that R is noetherian and M is finitely generated).

Now suppose that the only thing we knew was that $\mathrm{Tor}_i^R(M, M) = 0$ for $i \gg 0$. Can we still conclude that the projective dimension of M is finite? This is an open question in general, however it is known to be true for various rings R : we call such rings **Tor-persistent**. It is natural to wonder if in fact every commutative noetherian ring is Tor-persistent. Note that

$$\mathrm{Tor}_i^R(M, M) = H_i(F \otimes_R M) = H_i(F^{\otimes 2})$$

where we denoted $F^{\otimes 2} = F \otimes_R F$. One of the main reasons why we could conclude that M had finite projective dimension if $\mathrm{Tor}_i^R(M, \mathbb{k}) = 0$ for $i \gg 0$ was because the homology of $F_{\mathbb{k}}$ was extremely simple: $H(F_{\mathbb{k}}) = F_{\mathbb{k}}$. The homology of $F^{\otimes 2}$ is more complicated, thus even if we knew that $H_i(F^{\otimes 2}) = 0$ for $i \gg 0$, it is not at all clear why this should imply that $F_i = 0$ for $i \gg 0$. In order to prove this, one would presumably need to use the fact that R is noetherian, M is finitely generated, and F is minimal.

Reduction to Depth Zero

Lemma 0.1. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherian ring and let M be a finitely generated R -module such that*

$$\mathrm{Tor}_i^R(M, M) = 0$$

for all $i \gg 0$. Suppose that $x \in \mathfrak{m}$ is an R -regular and M -regular element. Then

$$\mathrm{Tor}_i^{R/x}(M/x, M/x) = 0$$

for all $i \gg 0$ and M has finite projective dimension if and only if M/x has finite projective dimension.

Proof. Consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/x \longrightarrow 0 \tag{1}$$

After tensoring (1) with M , we obtain a long exact sequence of Tor modules

$$\begin{array}{c}
\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(M, M/x) \\
\downarrow \\
\mathrm{Tor}_i^R(M, M) \xrightarrow{x} \mathrm{Tor}_i^R(M, M) \longrightarrow \mathrm{Tor}_i^R(M, M/x) \\
\downarrow \\
\mathrm{Tor}_{i-1}^R(M, M) \longrightarrow \cdots
\end{array}$$

In particular, since $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \gg 0$, we see that $\mathrm{Tor}_i^R(M, M/x) = 0$ for all $i \gg 0$. Similarly, after tensoring (1) with M/x , we obtain the long exact sequence of Tor modules

$$\begin{array}{c}
\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(M/x, M/x) \\
\downarrow \\
\mathrm{Tor}_i^R(M, M/x) \xrightarrow{x} \mathrm{Tor}_i^R(M, M/x) \longrightarrow \mathrm{Tor}_i^R(M/x, M/x) \\
\downarrow \\
\mathrm{Tor}_{i-1}^R(M, M/x) \longrightarrow \cdots
\end{array}$$

Since $\mathrm{Tor}_i^R(M, M/x) = 0$ for all $i \gg 0$, we see that $\mathrm{Tor}_i^R(M/x, M/x) = 0$ for all $i \gg 0$. Now suppose F is the minimal free resolution of M over R . Then F/x is the minimal free resolution of M/x over R/x and $C(x) = F \oplus eF$ is the minimal free resolution of M over R . In particular, note that

$$\begin{aligned}
\mathrm{Tor}_i^R(M/x, M/x) &= \mathrm{H}_i((F \oplus eF) \otimes_R M/x) \\
&= \mathrm{H}_i((F/x) \otimes_R M \oplus e((F/x) \otimes_R M)) \\
&= \mathrm{H}_i(F/x \otimes_{R/x} M/x) \oplus \mathrm{H}_{i+1}(F/x \otimes_{R/x} M/x) \\
&= \mathrm{Tor}_i^{R/x}(M/x, M/x) \oplus \mathrm{Tor}_{i-1}^{R/x}(M/x, M/x)
\end{aligned}$$

where we used the fact that $de = 0$ in F/x . Thus we see that since $\mathrm{Tor}_i^R(M/x, M/x) = 0$ for all $i \gg 0$, we have $\mathrm{Tor}_i^{R/x}(M/x, M/x) = 0$ for all $i \gg 0$. Finally, note that

$$\mathrm{pd}_R(M) = \mathrm{length}(F) = \mathrm{length}(F/x) = \mathrm{pd}_{R/x}(M/x).$$

□

The lemma above tells us that we can reduce to the case where either $\mathrm{depth} R = 0$ or $\mathrm{depth} M = 0$. Furthermore, note that anytime short exact sequence of the form

$$0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0 \quad (2)$$

, then virtually by the same argument as in the lemma above, if $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \gg 0$, then $\mathrm{Tor}_i^R(E, E) = 0$ for all $i \gg 0$. The R -module E is called an extension of M by M . The isomorphism classes of extensions of M by M is in bijection with $\mathrm{Ext}_R^1(M, M)$.

Lemma 0.2. *Let E be an extension of M by M . Then $\mathrm{pd}_R(E) = \mathrm{pd}_R(M)$.*

Proof. After tensoring (2) by \mathbb{k} , we obtain the long exact sequence of Tor modules

$$\begin{array}{c}
\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(M, \mathbb{k}) \\
\downarrow \\
\mathrm{Tor}_i^R(M, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(E, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(M, \mathbb{k}) \\
\downarrow \\
\mathrm{Tor}_{i-1}^R(M, \mathbb{k}) \longrightarrow \cdots
\end{array}$$

In particular, suppose $p = \mathrm{pd}_R(M)$. we see that

$$\mathrm{pd}_R(M) = \sup\{\mathrm{Tor}_i^R(M, \mathbb{k}) \neq 0 \mid i \in \mathbb{N}\}$$

□

Tor-Persistence

In what follows, we assume $(R, \mathfrak{m}, \mathbb{k})$ is a local noetherian ring. Let F be the minimal R -free resolution of the cyclic R -module R/I where $I \subseteq \mathfrak{m}$ is an ideal of R . Choose a multiplication μ on F giving it the structure of an MDG R -algebra. We denote $\mu(a_1 \otimes a_2) = a_1 a_2$ for all $a_1, a_2 \in F$ in order to simplify notation in what follows. Define a chain map $\{\cdot\}_\mu: F^{\otimes 3} \rightarrow F^{\otimes 2}$ by the formula

$$\{a_1 \otimes a_2 \otimes a_3\} = a_1 a_2 \otimes a_3 - a_1 \otimes a_2 a_3 = \{a_1, a_2, a_3\},$$

where we remove the subscript μ from $\{\cdot\}_\mu$ when context is clear and where we set $\{\cdot, \cdot, \cdot\}: F^3 \rightarrow F^{\otimes 2}$ to be the unique R -trilinear map corresponding to $\{\cdot\}$ via the universal mapping property of tensor products. Our goal is to determine what $\ker\{\cdot\}$ and $\mathrm{im}\{\cdot\}$ look like. First we consider $\mathrm{im}\{\cdot\}$. For each $a_1, a_2, a_3 \in F$, we have

$$\begin{aligned}
\{a_1, a_2, 1\} &= a_1 a_2 \otimes 1 - a_1 \otimes a_2 \\
\{1, a_2, a_3\} &= a_2 \otimes a_3 - 1 \otimes a_2 a_3 \\
\{a_1, 1, a_3\} &= 0 \\
\{a, a, b\} &= a^2 \otimes b - a \otimes ab
\end{aligned}$$

Thus if $ab = 0$, then $a \otimes b \in \mathrm{im}\{\cdot\}$. Furthermore we have $a \otimes 1 - 1 \otimes a \in \mathrm{im}\{\cdot\}$. Now suppose that

$$\{e_{i_1}, e_{i_2}, e_{i_3}\} = e_{i_1} e_{i_2} \otimes e_{i_3} - e_{i_1} \otimes e_{i_2} e_{i_3} = 0.$$

Then we must have $e_{i_1} = e_{i_1} e_{i_2}$ and $e_{i_3} = e_{i_2} e_{i_3}$. Or in other words, we must have $e_{i_1}(1 - e_{i_2}) = 0$ and $e_{i_3}(1 - e_{i_2}) = 0$. By considering homological degrees as well as using the fact that R is local, one sees that the only solution to these equations is

$$\{(0, e_{i_2}, 0), (0, 1, e_{i_3}), (e_{i_1}, 1, 0), (e_{i_1}, 1, e_{i_3})\}.$$

In particular, this spans $F^{\otimes 3} \oplus F^{\otimes 2}$.

Proposition 0.1. Suppose $H_i(F) = 0 = H_i(F^{\otimes 2})$ for $i \gg 0$. Then $H_i(F^{\otimes n}) = 0$ for $i \gg 0$ for all $n \geq 1$.

Proof. Consider the short exact sequence $0 \rightarrow F \rightarrow F^{\otimes 3} \rightarrow F^{\otimes 2} \rightarrow 0$. Actually this even shows $\mathrm{Tor}_+^R(S, S) = H_+(F^{\otimes n})$ for all $n \geq 2$. □