Symmetric DG Algebra

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1 The Symmetric DG Algebra

Let A be an R-complex centered at R (thus $A_0 = R$ and $A_i = 0$ for all $i < 0$). In this section, we will construct the symmetric DG R-algebra of A, which we denote by $\mathrm{S}_R(A) = \mathrm{S}(A)$. Before we give a rigorous construction of it, we wish to describe it informally first in order to help motivate the reader. The underlying R-algebra of $\mathrm{S}(A)$ is the usual symmetric R-algebra $\mathrm{Sym}(A_+)$ where we view A_+ as just an R-module. However $\mathrm{S}(A)$ obtains a bi-graded structure using homological degree as follows: we can decompose $\mathrm{S}(A)$ into R-modules as:

$$\mathrm{S}(A) = \bigoplus_{i \geq 0} \mathrm{S}_i(A) = \bigoplus_{m \geq 0} \mathrm{S}^m(A) = \bigoplus_{i,m \geq 0} \mathrm{S}_i^m(A)$$

We refer to the i in the subscript as **homological degree** and we refer to the m in the superscript as **total degree**. The R-module $\mathrm{S}_i^m(A)$ can be described as follows: we have

$$\mathrm{S}_0(A) = \mathrm{S}^0(A) = \mathrm{S}_0^0(A) = R \quad \text{and} \quad \mathrm{S}^1(A) = A_+.$$

More generally, for $i, m \geq 1$, the R-module $\mathrm{S}_i^m(A)$ is the R-span of all homogeneous elementary products of the form $a_1 \cdots a_m$ where $a_1, \dots, a_m \in A_+$ are homogeneous such that

$$|a_1| + \cdots + |a_m| = i.$$

In particular, note that $A = \mathrm{S}^{\leq 1}(A) = R + A_+$. We let $\iota \colon A \subseteq \mathrm{S}(A)$ denote the inclusion map. The differential of $\mathrm{S}(A)$ extends the differential of A and is defined on homogeneous elementary products of the form $a_1 \cdots a_m$ where $a_1, \dots, a_m \in A_+$ are homogeneous by

$$\mathrm{d}(a_1 \cdots a_m) = \sum_{j=1}^{m} (-1)^{|a_1| + \cdots + |a_{j-1}|} a_1 \cdots \mathrm{d}(a_j) \cdots a_m.$$

In the next example, we consider an R-free resolution F of a cyclic R-module and we work out what $\mathrm{S}(F)$ looks like.

Example 1.1. Let $R = \Bbbk[x, y]$, let $I = \langle x^2, xy \rangle$, and let F be Taylor resolution of R/I. Let's write down the homogeneous components of F as a graded R-module: we have

$$\begin{aligned} F_0 &= R \\ F_1 &= Re_1 + Re_2 \\ F_2 &= Re_{12}, \end{aligned}$$

and if $i \notin \{0, 1, 2\}$, then $F_i = 0$. The differential of F is defined on the homogeneous basis elements by

$$\begin{aligned} \mathrm{d}(e_1) &= x^2 \\ \mathrm{d}(e_2) &= xy \\ \mathrm{d}(e_{12}) &= xe_2 - ye_1. \end{aligned}$$

Note that the Taylor resolution usually comes equipped with a multiplication called the Taylor multiplication. Let us denote this by $\star$ so as not to confuse it with the multiplication $\cdot$ of $\mathrm{S}(F)$. Now let's write down the homogeneous components of $\mathrm{S}(F)$ as a graded R-module (with respect to homological degree): we have

$$\begin{aligned} \mathrm{S}_0(F) &= R \\ \mathrm{S}_1(F) &= Re_1 + Re_2 \\ \mathrm{S}_2(F) &= Re_{12} + Re_1e_2 \\ \mathrm{S}_3(F) &= Re_1e_{12} + Re_2e_{12} \\ \mathrm{S}_4(F) &= Re_{12}^2 + Re_1e_2e_{12} \\ &\vdots \end{aligned}$$

Note that $\mathrm{S}_4^3(F) = Re_1e_2e_{12}$ and $\mathrm{S}_4^2(F) = Re_{12}^2$. Also note that

$$\begin{aligned}
\mathrm{d}(e_1e_2 - e_1 \star e_2) &= \mathrm{d}(e_1e_2 - xe_{12}) \\
&= \mathrm{d}(e_1)e_2 - e_1\mathrm{d}(e_2) - x\mathrm{d}(e_{12}) \\
&= x^2e_2 - xye_1 - x(xe_2 - ye_1) \\
&= x^2e_2 - xye_1 - x^2e_2 + xye_1 \\
&= 0.
\end{aligned}$$

1.1 Construction of the Symmetric DG Algebra of A

We now provide a rigorous construction of $\mathrm{S}(A)$. This will occur in three steps:

Step 1: We define the **non-unital tensor algebra** of A to be the associative, graded, and non-unital R-algebra

$$\mathrm{U}(A) = \bigoplus_{i,k,m \geq 0} \mathrm{U}_i^{k,m}(A).$$

The component $\mathrm{U}_i^{k,m}(A)$ consists of all finite R-linear combinations of elementary tensors of the form

$$1^{\otimes k} \otimes a_1 \otimes \cdots \otimes a_m = 1 \otimes \cdots \otimes 1 \otimes a_1 \otimes \cdots \otimes a_m \tag{1}$$

where $a_1, \dots, a_m \in A_+$ are homogeneous such that

$$|a_1| + \cdots + |a_m| = i$$

We think of (1) as being graded of total degree m by setting $\deg(1) = 0$ and $\deg(a) = 1$ for all $a \in A_+$ and extending this multiplicatively. The multiplication of $\mathrm{U}(A)$ is defined on such elementary tensors by

$$(1^{\otimes k} \otimes a_1 \otimes \cdots \otimes a_m) \otimes (1^{\otimes k'} \otimes a_1' \otimes \cdots \otimes a_{m'}') \mapsto 1^{\otimes (k+k')} \otimes a_1 \otimes \cdots \otimes a_m \otimes a_1' \otimes \cdots \otimes a_{m'}'$$

and is extended R-linearly everywhere else. In particular, note that $\mathrm{U}(A)$ is not unital since $a \otimes 1 = 1 \otimes a \neq a$ for all nonzero $a \in A$. We set $\mathfrak{t}$ to be the $\mathrm{U}(A)$-ideal generated by all elements of the form $1 \otimes a - a$ where $a \in A$.

Step 2: We define the **tensor algebra** of A to be the associative, graded, and unital R-algebra given by the quotient

$$\mathrm{T}(A) := \mathrm{U}(A)/\mathfrak{t}.$$

The image of the elementary tensor (1) in $\mathrm{T}(A)$ is denoted by $a_1 \otimes \cdots \otimes a_m$ and will be referred to as a homogeneous elementary tensor. Since $\mathfrak{t}$ is generated by elements of the form $1 \otimes a - a$, which are homogeneous with respect to the homological degree and total degree, we see that $\mathrm{T}(A)$ is an associative and unital R-algebra which is bi-graded with respect to homological degree and total degree. In particular, we have $\mathrm{T}_0(A) = R = T^0(A)$, and for $m \geq 1$, the component of $\mathrm{T}(A)$ in total degree m is given by

$$\mathrm{T}^m(A) = A_+^{\otimes m}$$

where the tensor product is taken over R. On the other hand, for $i \geq 1$, the component of $\mathrm{T}(A)$ in homological degree i consists of the R-span of all homogeneous elementary tensors of the form $a_1 \otimes \cdots \otimes a_m$ where $m \geq 1$ and where $a_1, \dots, a_m$ are homogeneous elements in A_+ such that

$$|a_1| + \cdots + |a_m| = i.$$

We set $\mathfrak{s}$ to be the $\mathrm{T}(A)$-ideal generated by all elements of the form

$$[a_1, a_2]_\sigma :\, = (-1)^{|a_1||a_2|} a_2 \otimes a_1 - a_1 \otimes a_2 \quad \text{and} \quad [a]_\tau := a \otimes a,$$

where $a, a_1, a_2 \in A$ are homogeneous and $|a|$ is odd.

Step 3: We define the **symmetric algebra** of A to be the associative, strictly graded-commutative, and unital R-algebra given by the quotient

$$\mathrm{S}(A) := \mathrm{T}(A)/\mathfrak{s}.$$

The image of a homogeneous elementary tensor $a_1 \otimes \cdots \otimes a_m$ in $\mathrm{T}(A)$ will be denoted by $a_1 \cdots a_m$ in $\mathrm{S}(A)$ and we refer to $a_1 \cdots a_m$ as a homogeneous elementary product. Since $\mathfrak{s}$ is generated by elements which are homogeneous with respect to both homological degree and total degree, we see that $\mathrm{S}(A)$ inherits from $\mathrm{T}(A)$

the structure of a bi-graded associative R-algebra which is also strictly graded-commutative with respect to homological degree.

We now want to show that the differential of A can be extended to a differential on $\mathrm{S}(A)$ giving it the structure of a DG R-algebra centered at R.

Theorem 1.1. *The differential of A extends to a differential on $\mathrm{S}(A)$ giving it the structure of a DG R-algebra centered at R. Moreover, $\mathrm{S}(A)$ satisfies the following universal mapping property: for every chain map $\varphi\colon A \to B$ such that $\varphi(1) = 1$ where B is a DG R-algebra centered at R, there exists a unique DG R-algebra homomorphism $\widetilde{\varphi}\colon \mathrm{S}(A) \to B$ such that $\widetilde{\varphi}\iota = \varphi$. We express this in terms of a commutative diagram as below:*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \mathrm{S}(A) \arrow[d, "\widetilde{\varphi}"] \\ & B \end{tikzcd} \tag{2}$$

Proof. Let d be the differential of A. We first extend d to an an R-linear map $\mathrm{U}(A) \to \mathrm{U}(A)$, which we denote by d again, which is graded of degree -1 with respect to homological degree as follows: for all homogeneous elementary tensors of the form (1), we set

$$\mathrm{d}(1^{\otimes k} \otimes a_1 \otimes \cdots \otimes a_m) = 1^{\otimes k} \otimes \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots \otimes a_m,$$

and we extend d R-linearly everywhere else. It is clear that d is R-linear, graded of degree -1 with respect to the homological degree, and that $\mathrm{d}|_A$ is the differential of A. Furthermore, for any elementary tensor of the form (1), we have

$$\begin{aligned} \mathrm{d}^2(1^{\otimes k} \otimes a_1 \otimes \cdots \otimes a_m) &= 1^{\otimes k} \otimes \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \mathrm{d}(a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots a_m) \\ &= 1^{\otimes k} \otimes \sum_{1 \leq i < j \leq m} (-1)^{|a_i|+\cdots+|a_{j-1}|} (a_1 \otimes \cdots \otimes \mathrm{d}a_i \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots a_m) \\ &= 1^{\otimes k} \otimes \sum_{1 \leq j < k \leq m} (-1)^{|a_j|+\cdots+|a_{k-1}|} (a_1 \otimes \cdots \otimes \mathrm{d}a_j \otimes \cdots \otimes \mathrm{d}a_k \otimes \cdots a_m) \\ &= 0. \end{aligned}$$

It follows that $\mathrm{d}^2 = 0$, and thus d is indeed a differential. Observe that the differential maps $\mathfrak{t}$ to itself since if $a \in A$, then we have

$$\mathrm{d}(1 \otimes a - a) = 1 \otimes \mathrm{d}a - \mathrm{d}a \in \mathfrak{t}.$$

Thus d induces a differential on $\mathrm{T}(A)$, which we again denote by d. Similarly, observe that d maps $\mathfrak{s}$ to itself since if $a, a_1, a_2 \in A_+$ are homogeneous with $|a|$ odd, then we have

$$\mathrm{d}[a_1, a_2]_\sigma = [\mathrm{d}a_1, a_2]_\sigma + (-1)^{|a_1|}[a_1, \mathrm{d}a_2]_\sigma \in \mathfrak{s} \quad \text{and} \quad \mathrm{d}[a]_\tau = [\mathrm{d}a, a]_\sigma \in \mathfrak{s}$$

Thus the differential d induces a differential on $\mathrm{S}(A)$, which we again denote by d, giving $\mathrm{S}(A)$ the structure of a DG R-algebra centered at R.

Now suppose that $\varphi\colon A \to B$ is a chain map such that $\varphi(1) = 1$ where B is a DG R-algebra centered at R. We define $\widetilde{\varphi}\colon \mathrm{S}(A) \to B$ by setting $\widetilde{\varphi}(1) = 1$ and

$$\widetilde{\varphi}(a_1 \cdots a_m) = \varphi(a_1) \cdots \varphi(a_m) \tag{3}$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $\mathrm{S}(A)$ and then extending it R-linearly everywhere else. By construction, $\widetilde{\varphi}$ is multiplicative and satisfies $\widetilde{\varphi}(1) = 1$. It also clearly extends $\varphi\colon A \to B$. Furthermore, $\widetilde{\varphi}$ is a chain map since it is a graded R-linear map which commutes with the differential. Indeed, we clearly have

$\widetilde{\varphi}\mathrm{d}(1) = 0 = \mathrm{d}\widetilde{\varphi}(1)$, and for all homogeneous elementary products $a_1 \cdots a_m$ in $\mathrm{S}(A)$, we have

$$\begin{aligned}
\widetilde{\varphi}\mathrm{d}(a_1 \cdots a_m) &= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \widetilde{\varphi}(a_1 \cdots \mathrm{d}(a_j) \cdots a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \varphi\mathrm{d}(a_j) \cdots \varphi(a_m) \\
&= \sum_{j=1}^{m} (-1)^{|a_1|+\cdots+|a_{j-1}|} \varphi(a_1) \cdots \mathrm{d}\varphi(a_j) \cdots \varphi(a_m) \\
&= \mathrm{d}(\varphi(a_1) \cdots \varphi(a_m)) \\
&= \mathrm{d}\widetilde{\varphi}(a_1 \cdots a_m).
\end{aligned}$$

Finally, if $\widetilde{\varphi}' \colon \mathrm{S}(A) \to B$ were another DG R-algebra homomorphism which extended $\varphi \colon A \to B$, then we'd have

$$\widetilde{\varphi}'(a_1 \cdots a_m) = \widetilde{\varphi}'(a_1) \cdots \widetilde{\varphi}'(a_m) = \varphi(a_1) \cdots \varphi(a_m) = \widetilde{\varphi}(a_1 \cdots a_m)$$

for all homogeneous elementary products $a_1 \cdots a_m$ in $\mathrm{S}(A)$, which implies $\widetilde{\varphi}' = \widetilde{\varphi}$. □

1.2 A Presentation of the Maximal Associative Quotient

We now equip A with a multiplication $(\mu, \star)$ giving it the structure of an MDG R-algebra. In particular, note that if $a_1, a_2 \in A_1$, then

$$a_1 a_2 \in \mathrm{S}_2^2(A), \quad a_1 \star a_2 \in \mathrm{S}_2^1(A), \quad \text{and} \quad a_1 a_2 - a_1 \star a_2 \in \mathrm{S}_2(A)$$

Also note that the multiplicator of the inclusion $\iota \colon A \subseteq \mathrm{S}(A)$ has the form

$$[a_1, a_2]_\iota = \iota(a_1 \star a_2) - \iota(a_1)\iota(a_2) = a_1 \star a_2 - a_1 a_2$$

for all $a_1, a_2 \in A$. Let $\mathfrak{b}$ be the $\mathrm{S}(A)$-ideal generated by the multiplicator complex $[\mathrm{S}(A)]_\iota$. Since $\mathrm{S}(A)$ is associative, we have

$$\mathfrak{b} = \mathrm{span}_B\{[a_1, a_2]_\iota \mid a_1, a_2 \in A\}.$$

Finally let

$$\rho_1 \colon A \to A/\langle A \rangle \quad \text{and} \quad \rho_2 \colon \mathrm{S}(A) \to \mathrm{S}(A)/\mathfrak{b}$$

denote the corresponding quotient maps.

Theorem 1.2. *With the notation as above, we have $\langle A \rangle = A \cap \mathfrak{b}$. In particular, the composite $\rho_2\iota \colon A \to \mathrm{S}(A) \to \mathrm{S}(A)/\mathfrak{b}$ induces an isomorphism*

$$A/\langle A \rangle \simeq \mathrm{S}(A)/\mathfrak{b}$$

of DG R-algebras which is natural in A.

Proof. Note that the composite map $\rho_2\iota \colon A \to \mathrm{S}(A) \to \mathrm{S}(A)/\mathfrak{b}$ is a surjective MDG R-algebra homomorphism. Since $\mathrm{S}(A)/\mathfrak{b}$ is associative, it follows from the universal mapping property of the maximal associative quotient of A that $\ker(\rho_2\iota) = A \cap \mathfrak{b}$ contains $\langle A \rangle$. Conversely, since $A/\langle A \rangle$ is associative, it follows from the universal mapping property of the symmetric DG R-algebra of A that there exists a unique DG R-algebra homomorphism $\widetilde{\rho}_1 \colon \mathrm{S}(A) \to A/\langle A \rangle$ which extends $\rho_1 \colon A \to A/\langle A \rangle$. In particular, note that for $a_1, a_2 \in A$ we have

$$\begin{aligned}
\widetilde{\rho}_1[a_1, a_2]_\iota &= \widetilde{\rho}_1(a_1 \star a_2 - a_1 a_2) \\
&= \rho_1(a_1 \star a_2) - \widetilde{\rho}_1(a_1 a_2) \\
&= \rho_1(a_1) \star \rho_1(a_2) - \rho_1(a_1) \star \rho_1(a_2) \\
&= 0.
\end{aligned}$$

since $\rho_1 \colon A \to A/\langle A \rangle$ is multiplicative. It follows that $\mathfrak{b} \subseteq \ker \widetilde{\rho}_1$, and since $A \cap \ker \widetilde{\rho}_1 = \ker \rho_1 = \langle A \rangle$, it follows that $A \cap \mathfrak{b} \subseteq \langle A \rangle$.

Finally, the ismorphism is natural in A in the sense that if R' is and R-algebra and $\varphi \colon A \to A'$ is an MDG R-algebra homomorphism where A' is an MDG R'-algebra centered at R'. Then we obtain a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \mathrm{S}_R(A) \arrow[r, "\pi"] \arrow[d, "\widetilde{\varphi}"'] & \mathrm{S}_R(A)/\mathfrak{b} \arrow[d, "\overline{\varphi}"'] \\
A' \arrow[r, "\iota'"'] & \mathrm{S}_{R'}(A') \arrow[r, "\pi'"'] & \mathrm{S}_{R'}(A')/\mathfrak{b}'
\end{tikzcd}$$

where we set $\mathfrak{b}'$ to be the DG $\mathrm{S}_{R'}(A')$-ideal generated by the multiplicator complex $[\mathrm{S}_{R'}(A')]_{\iota'}$. Indeed, the map $\widetilde{\varphi}\colon \mathrm{S}_R(A) \to \mathrm{S}_{R'}(A')$ is the unique DG R-algebra which extends the composite $\iota'\varphi\colon A \to \mathrm{S}_{R'}(A')$. Since $\widetilde{\varphi}$ is multiplicative, it takes $[\mathrm{S}_R(A)]_\iota$ to $[\mathrm{S}_{R'}(A')]_{\iota'}$ and thus takes $\mathfrak{b}$ to $\mathfrak{b}'$. In particular, it induces a well-defined map

$$A/\langle A\rangle \simeq \mathrm{S}_R(A)/\mathfrak{b} \xrightarrow{\overline{\varphi}} \mathrm{S}_{R'}(A')/\mathfrak{b}' \simeq A'/\langle A'\rangle.$$

. □

1.3 The Symmetric DG Algebra of a Finite Free Resolution

Throughout this subsection, we assume that R is an integral domain with quotient field K. Let F be an R-free resolution of a cyclic R-module with $F_0 = R$ such that the underlying graded R-module of F is a finite and free as an R-module. Let $e_1, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $|e_{i'}| > |e_i|$, then $i' > i$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $R[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $R[\boldsymbol{e}]$. The reason we do not allow elements of odd degree to square to zero is because we will want to calculate the Gröbner basis of an ideal in $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is simpler when we don't have any zerodivisors. In any case, it is straightforward to check that

$$R[\boldsymbol{e}]/\langle \{e_i^2 \mid |e_i| \text{ is odd}\}\rangle \simeq \mathrm{S}(F).$$

Finally, let $(\mu, \star)$ be a multiplication of F. Our goal is to compute the maximal associative quotient of F using the presentation given in Theorem (1.2) as well as the theory of Gröbner bases in $K[\boldsymbol{e}]$. We need to introduce some notation for Gröbner basis applications in $K[\boldsymbol{e}]$. Our notation mostly follows [GP02] however we introduce some of our own notation as well.

1.3.1 Monomials and Monomial Orderings in $K[\boldsymbol{e}]$

A **monomial** in $K[\boldsymbol{e}]$ is an element of the form

$$e^{\boldsymbol{\alpha}} = e_1^{\alpha_1} \cdots e_n^{\alpha_n} \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called the **multidegree** of $e^{\boldsymbol{\alpha}}$ and is denoted $\mathrm{multideg}(e^{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$. Similarly we define its **total degree**, denoted $\deg(e^{\boldsymbol{\alpha}})$, and its **homological degree** denoted $|e^{\boldsymbol{\alpha}}|$, by

$$\deg(e^{\boldsymbol{\alpha}}) = \sum_{i=1}^n \alpha_i \quad \text{and} \quad |e^{\boldsymbol{\alpha}}| = \sum_{i=1}^n \alpha_i |e_i|.$$

By convention we set $e^{\boldsymbol{0}} = 1$ where $\boldsymbol{0} = (0, \dots, 0)$ is the zero vector in $\mathbb{N}^n$. We define the **support** of $e^{\boldsymbol{\alpha}}$, denoted $\mathrm{supp}(e^{\boldsymbol{\alpha}})$, to be the set

$$\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_i \mid e_i \text{ divides } e^{\boldsymbol{\alpha}}\} = \{e_i \mid \alpha_i \neq 0\}.$$

Note that if the support of $e^{\boldsymbol{\alpha}}$ is empty if and only if $e^{\boldsymbol{\alpha}} = 1$. If $e^{\boldsymbol{\alpha}}$ has non-empty support, then we define its **initial variable** and **terminal variable** to be the elements e_i and e_k where

$$i = \inf\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\} \quad \text{and} \quad \max\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\}.$$

For instance, suppose that $\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_{i_1}, \dots, e_{i_k}\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then can express (4) as

$$e^{\boldsymbol{\alpha}} = e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_k}.$$

Then e_{i_1} is the initial variable of $e^{\boldsymbol{\alpha}}$ and e_{i_k} is the terminal variable of $e^{\boldsymbol{\alpha}}$. Note how the ordering matters. In particular, if $i < j$ and both $|e_i|$ and $|e_j|$ are odd, then $e_j e_i$ is not a monomial in $K[\boldsymbol{e}]$ since it can be expressed as a non-trivial coefficient times a monomial:

$$e_j e_i = -e_i e_j.$$

On the other hand, if one of the e_i or e_j is even, then $e_j e_i$ is a monomial in $K[\boldsymbol{e}]$ since $e_j e_i = e_i e_j$. We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ (the notation for this monomial ordering in Singular is Wp(w)). More specifically, given two monomials $e^{\boldsymbol{\alpha}}$ and $e^{\boldsymbol{\beta}}$ in $K[\boldsymbol{e}]$, we say $e^{\boldsymbol{\beta}} > e^{\boldsymbol{\alpha}}$ if either

1. $|e^{\beta}| > |e^{\alpha}|$ or;

2. $|e^{\beta}| = |e^{\alpha}|$ and $\beta_1 > \alpha_1$ or;

3. $|e^{\beta}| = |e^{\alpha}|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Given a nonzero polynoimal $f \in K[\boldsymbol{e}]$, there exists unique $c_1, \dots, c_m \in K \backslash \{0\}$ and unique $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ where $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ for all $1 \leq i < j \leq m$ such that

$$f = c_1 e^{\boldsymbol{\alpha}_1} + \cdots + c_m e^{\boldsymbol{\alpha}_m} = \sum c_i e^{\boldsymbol{\alpha}_i} \tag{5}$$

The $c_i e^{\boldsymbol{\alpha}_i}$ in (5) are called the **terms** of f, and the $e^{\boldsymbol{\alpha}_i}$ in (5) are called the **monomials** of f. By reindexing the $\boldsymbol{\alpha}_i$ if necessary, we may assume that $e^{\boldsymbol{\alpha}_1} > \cdots > e^{\boldsymbol{\alpha}_m}$. In this case, we call $c_1 e^{\boldsymbol{\alpha}_1}$ the **lead term** of f, we call $e^{\boldsymbol{\alpha}_1}$ the **lead monomial** of f, and we call c_1 the **lead coefficient** of f. We denote these, respectively, by

$$\mathrm{LT}(f) = c_1 e^{\boldsymbol{\alpha}_1}, \quad \mathrm{LM}(f) = e^{\boldsymbol{\alpha}_1}, \quad \text{and} \quad \mathrm{LC}(f) = c_1.$$

The **multidegree** of f is defined to be the multidegree of its lead monomial $e^{\boldsymbol{\alpha}_1}$ and is denoted $\mathrm{multideg}(f) = \boldsymbol{\alpha}_1$. The **total degree** of f is defined to be the maximum of the total degrees of its monomials and is denoted

$$\deg(f) = \max_{1 \leq i \leq m} \{\deg(e^{\boldsymbol{\alpha}_i})\}.$$

We say f is **homogeneous** of homological degree i if each of its monomials is homogeneous of homological degree i. In this case, we say f has **homological degree** i and we denote this by $|f| = i$.

Proposition 1.1. *For each $1 \leq i, j \leq n$, let $f_{ij} = -[e_i, e_j] = e_i e_j - e_i \star e_j$. We have*

$$\mathrm{LT}(f_{ij}) = e_i e_j.$$

Proof. If $e_i \star e_j = 0$, then this is clear, otherwise term of $e_i \star e_j$ has the form $r_{i,j}^k e_k$ for some k where $r_{i,j}^k \neq 0$. Since $\star$ respects homological degree, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > |e_i|$ and $|e_k| > |e_j|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ and $k > j$ by our assumption on the ordering of $e_1, \dots, e_n$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j > e_k$. □

1.3.2 Gröbner Basis Calculations

The inclusion map $R \subseteq K$ induces an inclusion map $F \to F_K$ where $F_K = \{a/r \mid a \in F \text{ and } r \in R \backslash \{0\}\}$. For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in $R[\boldsymbol{e}] \subseteq K[\boldsymbol{e}]$ defined by $f_{i,j} := -[e_i, e_j]$. Thus we have

$$f_{i,j} = e_i e_j - e_i \star e_j = e_i e_j - \sum_k r_{i,j}^k e_k,$$

where the $r_{i,j}^k$ are the entries of the matrix representation of μ with respect to the ordered homogeneous basis $e_1, \dots, e_n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$, let $\mathfrak{b}$ be the $R[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$, and let $\mathfrak{b}_K$ be the $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. Note that if e_i is odd, then $f_{i,i} = e_i^2$ since $\star$ is strictly graded-commutative, thus $R[\boldsymbol{e}]/\mathfrak{b} \simeq F/\langle F \rangle$ and $K[\boldsymbol{e}]/\mathfrak{b}_K \simeq F_K/\langle F_K \rangle$ by Theorem (1.2).

Recall that $K[\boldsymbol{e}]$ comes equipped with a monomial ordering which we defined earlier. We wish to construct a left Gröbner basis for $\mathfrak{b}_K$ (which will turn out to be a two-sided Gröbner basis) using this monomial ordering via Buchberger's algorithm (as described in [GP02]). Suppose f, g are two nonzero polynomials in $K[\boldsymbol{e}]$ with $\mathrm{LT}(f) = r e^{\boldsymbol{\alpha}}$ and $\mathrm{LT}(g) = s e^{\boldsymbol{\beta}}$. Set $\boldsymbol{\gamma} = \mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and the left S**-polynomial** of f and g to be

$$\mathrm{S}(f, g) = e^{\boldsymbol{\gamma} - \boldsymbol{\alpha}} f \pm (r/s) e^{\boldsymbol{\gamma} - \boldsymbol{\beta}} g \tag{6}$$

where the $\pm$ in (6) is chosen to be $+$ or $-$, depending on which sign will cancel out the lead terms. We begin Buchberger's algorithm by calculating the S-polynomials of all pairs of polynomials in $\mathcal{F}$. In other words, we calculate all S-polynomials of the form $\mathrm{S}(f_{k,l}, f_{i,j})$ where $1 \leq i, j, k, l \leq n$. Note that if $k > l$, then

$$f_{l,k} = (-1)^{|e_k||e_l|} f_{k,l},$$

which implies

$$\mathrm{S}(f_{l,k}, f_{i,j}) = (-1)^{|e_k||e_l|} \mathrm{S}(f_{k,l}, f_{i,j}) = \pm \mathrm{S}(f_{i,j}, f_{k,l}).$$

Similarly, if $i \geq k$, then

$$\mathrm{S}(f_{i,j}, f_{l,k}) = \pm \mathrm{S}(f_{k,l}, f_{i,j}).$$

Thus we may assume that $j \geq i$ and $l \geq k \geq i$. Obviously we have $\mathrm{S}(f_{i,j}, f_{i,j}) = 0$ for each i, j, however something interesting happens when we calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $j > i$ and then divide this by $\mathcal{F}$ (where division by $\mathcal{F}$ means taking the left normal form of $\mathrm{S}(f_{j,k}, f_{i,j})$ with respect to $\mathcal{F}$ using the left normal form described in [GP02]). We have

$$\begin{aligned}
\mathrm{S}(f_{j,k}, f_{i,j}) &= e_i(e_je_k - e_j \star e_k) - (e_ie_j - e_i \star e_j)e_k \\
&= (e_i \star e_j)e_k - e_i(e_j \star e_k) \\
&= \sum_l r^l_{i,j} e_l e_k - \sum_l r^l_{j,k} e_i e_l \\
&\rightarrow \sum_l r^l_{i,j} e_l \star e_k - \sum_l r^l_{j,k} e_i \star e_l \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k],
\end{aligned}$$

where in the fourth line we did division by $\mathcal{F}$ (note that if $[e_i, e_j, e_k] \neq 0$, then $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$). Finally if $j > i$, $l > k$, and $j \neq k$, then we have

$$\begin{aligned}
\mathrm{S}(f_{k,l}, f_{i,j}) &= e_ie_jf_{k,l} - f_{i,j}e_ke_l \\
&= (e_i \star e_j)e_ke_l - e_ie_j(e_k \star e_l) \\
&\rightarrow (e_i \star e_j) \star (e_k \star e_l) - (e_i \star e_l) \star (e_k \star e_l) \\
&= 0
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Next, suppose that

$$f = re_k + r'e_{k'} + \cdots + r''e_{k''} \in \langle F \rangle$$

where $r, r', r'' \in R$ with $r \neq 0$ and where $\mathrm{LM}(f) = e_k$. Then we have

$$\begin{aligned}
\mathrm{S}(f, f_{j,k}) &= e_jf - rf_{j,k} \\
&= r'e_je_{k'} + \cdots + r''e_je_{k''} + re_j \star e_k \\
&\rightarrow r'e_j \star e_{k'} + \cdots + r''e_j \star e_{k''} + re_j \star e_k \\
&= e_j \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\
&= e_j \star f \\
&\in \langle F \rangle
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Similarly, we have if $i \neq k \neq j$, then we have

$$\begin{aligned}
\mathrm{S}(f, f_{i,j}) &= e_ie_jf - rf_{i,j}e_k \\
&= r'(e_ie_j)e_{k'} + \cdots + r''(e_ie_j)e_{k''} + r(e_i \star e_j)e_k \\
&\rightarrow r'(e_i \star e_j) \star e_{k'} + \cdots + r''(e_i \star e_j) \star e_{k''} + r(e_i \star e_j) \star e_k \\
&= (e_i \star e_j) \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\
&= (e_i \star e_j) \star f \\
&\in \langle F \rangle.
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Finally suppose that

$$g = se_m + s'e_{m'} + \cdots + s''e_{m''} \in \langle F \rangle$$

where $s, s', s'' \in R$ with $s \neq 0$ and where $\mathrm{LM}(g) = e_m$. If $k = m$, then we have

$$s\mathrm{S}(f, g) = sf - rg \in \langle F \rangle.$$

On the other hand, if $k \neq m$, then we have

$$\begin{aligned}
s\mathrm{S}(f, g) &= se_mf - rge_k \\
&= sr'e_me_{k'} + \cdots + sr''e_me_{k''} - rs'e_{m'}e_k - \cdots - rs''e_{m''}e_k \\
&\rightarrow sr'e_m \star e_{k'} + \cdots + sr''e_m \star e_{k''} - rs'e_{m'} \star e_k - \cdots - rs''e_{m''} \star e_k \\
&= se_m \star (r'e_{k'} + \cdots + r''e_{k''}) - r(s'e_{m'} + \cdots + s''e_{m''}) \star e_k \\
&= se_m \star (f - re_k) - r(g - se_m) \star e_k \\
&= se_m \star f + rg \star e_k - sre_m \star e_k + rse_m \star e_k \\
&= se_m \star f + rg \star e_k \\
&\in \langle F \rangle.
\end{aligned}$$

It follows that we can construct a Gröbner basis

$$\mathcal{G} := \mathcal{F} \cup \{g_1, \dots, g_m\}$$

of $\mathfrak{b}_K$ such that the g_i all belong to $\langle F \rangle$.

References

[GP02] Gert-Martin Greuel and Gerhard Pfister, A Singular Introduction to Commutative Algebra, second ed.