Derivatives

1 Introduction

In this document, I will show you how to compute derivatives in an algebraic and algorithmic manner. Let us recall the definition of the derivative of a function. Let $f(x)$ be a function. We say

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1.1 Product Rule

Recall that if $f(x)$ and $g(x)$ are two functions, then the derivative of their product is given by the **product rule:**

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)).$$

Let's calculate some derivatives of product functions.

Example 1.1. Suppose $f(x) = x^{2/3}(x^3 - 5x^2)$. We write $f(x)$ as a product of two functions,

$$g(x) = x^{2/3} \text{ and } h(x) = x^3 - 5x^2.$$

So $f(x) = g(x)h(x)$. We calculate

$$g'(x) = \frac{2}{3}x^{-1/3} \text{ and } h'(x) = 3x^2 - 10x.$$

Therefore

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$
$$= \left(\frac{2}{3}x^{-1/3}\right)\left(x^3 - 5x^2\right) + x^{2/3}(3x^2 - 10x).$$

This is the way the book may want you to do it. The way I calculate the derative of $f(x)$ is as follow

$$f'(x) = \frac{d}{dx}(f(x))$$
$$= \frac{d}{dx}(x^{2/3}(x^3 - 5x^2))$$
$$= \frac{d}{dx}(x^{2/3})(x^3 - 5x^2) + x^{2/3}\frac{d}{dx}(x^3 - 5x^2)$$
$$= \frac{2}{3}x^{-1/3}(x^3 - 5x^2) + x^{2/3}(3x^2 - 10x).$$

Example 1.2. Let's find the derivative of $e^{2x}\sqrt{x^3 - 5x^2}$:

$$\frac{d}{dx}\left(e^{2x}\sqrt{x^3 - 5x^2}\right) = \frac{d}{dx}(e^{2x})\sqrt{x^3 - 5x^2} + e^{2x}\frac{d}{dx}\left(\left(x^3 - 5x^2\right)^{1/2}\right)$$
$$= e^{2x}\frac{d}{dx}(2x)\sqrt{x^3 - 5x^2} + e^{2x} \cdot \frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}\frac{d}{dx}(x^3 - 5x^2)$$
$$= e^{2x} \cdot 2\sqrt{x^3 - 5x^2} + e^{2x} \cdot \frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}(3x^2 - 10x).$$

Example 1.3. Let's find the derivative of $(4x^2 - x + 1.5)(2(5^x))$:

$$\frac{d}{dx}\left((4x^2 - x + 1.5)(2(5^x))\right) = \frac{d}{dx}(4x^2 - x + 1.5) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot \frac{d}{dx}(2(5^x))$$

$$= (8x - 1) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot 2\frac{d}{dx}(5^x)$$

$$= (8x - 1) \cdot 2(5^x) + (4x^2 - x + 1.5) \cdot 2\ln(5)5^x.$$

Example 1.4. Let's find the derivative of $\frac{-2(3^x)}{\sqrt{x}}$:

$$\frac{d}{dx}\left(\frac{-2(3^x)}{\sqrt{x}}\right) = \frac{d}{dx}\left(-2(3^x) \cdot x^{-1/2}\right)$$

$$= \frac{d}{dx}(-2(3^x))) \cdot x^{-1/2} + -2(3^x) \cdot \frac{d}{dx}\left(x^{-1/2}\right)$$

$$= -2\ln(3)3^x x^{-1/2} + -2(3^x) \cdot \frac{-1}{2}x^{-3/2}.$$

Example 1.5. Let's find the derivative of $2.5x\sqrt{x^3 - x}$:

$$\frac{d}{dx}\left(2.5x\sqrt{x^3 - x}\right) = \frac{d}{dx}(2.5x) \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{d}{dx}\left(\sqrt{x^3 - x}\right)$$

$$= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{d}{dx}\left((x^3 - x)^{1/2}\right)$$

$$= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{1}{2} \cdot (x^3 - x)^{-1/2} \cdot \frac{d}{dx}(x^3 - x)$$

$$= 2.5 \cdot \sqrt{x^3 - x} + 2.5x \cdot \frac{1}{2} \cdot (x^3 - x)^{-1/2} \cdot (3x^2 - 1).$$

Example 1.6. Let's find the derivative of $(6x - 4)^5(2x + 1)$:

$$\frac{d}{dx}\left((6x - 4)^5(2x + 1)\right) = \frac{d}{dx}((6x - 4)^5) \cdot (2x + 1) + (6x - 4)^5 \cdot \frac{d}{dx}(2x + 1)$$

$$= 5 \cdot (6x - 4)^4 \cdot \frac{d}{dx}(6x - 4) \cdot (2x + 1) + (6x - 4)^5 \cdot 2$$

$$= 5 \cdot (6x - 4)^4 \cdot 6 \cdot (2x + 1) + (6x - 4)^5 \cdot 2.$$

Example 1.7. Let's find the derivative of $\frac{2x^3 + 7x}{3x - 5}$:

$$\frac{d}{dx}\left(\frac{2x^3 + 7x}{3x - 5}\right) = \frac{d}{dx}\left((2x^3 + 7x)(3x - 5)^{-1}\right)$$

$$= \frac{d}{dx}(2x^3 + 7x) \cdot (3x - 5)^{-1} + (2x^3 + 7x) \cdot \frac{d}{dx}\left((3x - 5)^{-1}\right)$$

$$= (6x^2 + 7)(3x - 5)^{-1} + (2x^3 + 7x) \cdot (-1) \cdot (3x - 5)^{-2} \cdot \frac{d}{dx}(3x - 5)$$

$$= (6x^2 + 7)(3x - 5)^{-1} - (2x^3 + 7x)(3x - 5)^{-2} \cdot 3.$$

Example 1.8. Let's find the derivative of $2(5^x)\ln(x)$:

$$\frac{d}{dx}\left(2(5^x)\ln(x)\right) = \frac{d}{dx}(2(5^x)) \cdot \ln(x) + 2(5^x) \cdot \frac{d}{dx}(\ln(x))$$

$$= 2\frac{d}{dx}(5^x) \cdot \ln(x) + 2(5^x) \cdot \frac{1}{x}$$

$$= 2 \cdot \ln(5) \cdot 5^x \cdot \ln(x) + 2(5^x) \cdot \frac{1}{x}.$$

So far we know how to compute the derivative of simple functions like x^5 or e^x. We also know how to compute the derivative of composite functions, like e^{x^5}: indeed we learned the chain rule last section:

$$\frac{d}{dx}\left(e^{x^5}\right) = e^{x^5} \cdot \frac{d}{dx}(x^5)$$

$$= e^{x^5} \cdot 5x^4.$$

Now we want to know how to calculate the derivative of a product of two functions like $x^5 e^x$.

The way that we will do it is via the product rule: The product rule says that if you have two functions $f(x)$ and $g(x)$, then the derivative of their product is given by

$$\frac{d}{dx}(f(x) \cdot g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x)).$$

So for example

$$\frac{d}{dx}(x^5 e^x) = \frac{d}{dx}(x^5)e^x + x^5 \frac{d}{dx}(e^x)$$
$$= 5x^4 e^x + x^5 e^x.$$

If we use the prime notation for the derative, then the product rule looks like this:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Let's looks at some examples.

Example 1.9. Suppose $g(x) = 5x^6$ and $h(x) = \ln(x)$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 5x^6 \ln(x).$$

The derivative of their product is given by

$$(g \cdot h)'(x) = \frac{d}{dx}((g \cdot h)(x))$$
$$= \frac{d}{dx}(5x^6 \ln(x))$$
$$= \frac{d}{dx}(5x^6) \ln(x) + 5x^6 \frac{d}{dx}(\ln(x))$$
$$= 30x^5 \ln(x) + 5x^6 \cdot \frac{1}{x}$$
$$= 30x^5 \ln(x) + 5x^5$$
$$= 5x^5 (6\ln(x) + 1).$$

Example 1.10. Suppose $g(x) = 2(3^x)$ and $h(x) = 3x^2 - 2x + 1$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 2(3^x)(3x^2 - 2x + 1).$$

The derivative of their product is given by

$$(g \cdot h)'(x) = \frac{d}{dx}((g \cdot h)(x))$$
$$= \frac{d}{dx}(2(3^x)(3x^2 - 2x + 1))$$
$$= \frac{d}{dx}(2(3^x))(3x^2 - 2x + 1) + 2(3^x)\frac{d}{dx}(3x^2 - 2x + 1)$$
$$= 2\frac{d}{dx}(3^x)(3x^2 - 2x + 1) + 2(3^x)\frac{d}{dx}(3x^2 - 2x + 1)$$
$$= 2\ln(3)3^x(3x^2 - 2x + 1) + 2(3^x)(6x - 2).$$