Multiplicity and Koszul Homology

Lemma 0.1. *Let M be a finitely generated R-module and let I be an ideal of R. Then*

$$\sqrt{\operatorname{Ann}(M/IM)} = \sqrt{\langle I, \operatorname{Ann} M\rangle}.$$

Proof. To prove the equality on radicals, it suffices to show that a prime $\mathfrak{p}$ of R contains $\operatorname{Ann}(M/IM)$ if and only if it contains $\langle I, \operatorname{Ann} M\rangle$. Recall that for any finitely generated R-module N, we have $\mathrm{V}(\operatorname{Ann} N) = \operatorname{Supp} N$, or equivalently, $\mathfrak{p} \supseteq \operatorname{Ann} N$ if and only if $N_{\mathfrak{p}} \neq 0$. Thus since M is finitely generated (and hence M/IM is finitely generated too), we have

$$\begin{aligned}
\mathfrak{p} \supseteq \operatorname{Ann}(M/IM) &\iff M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}} \neq 0 \\
&\iff M_{\mathfrak{p}} \neq 0 \text{ and } I_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}} \\
&\iff \mathfrak{p} \supseteq \operatorname{Ann} M \text{ and } I \subseteq \mathfrak{p} \\
&\iff \mathfrak{p} \supseteq \langle \operatorname{Ann} M, I\rangle
\end{aligned}$$

□

Let $A = (A, \mathfrak{m}, \Bbbk)$ be a noetherian local ring, let $\boldsymbol{x} = x_1, \dots, x_r$ be a sequence contained in $\mathfrak{m}$, and let M be a finitely generated A-module such that $\ell(M/\boldsymbol{x}M) < \infty$ (equivalently, we have $\mathfrak{m} = \sqrt{\operatorname{Ann}(M/\boldsymbol{x}M)}$). We set $K = K(\boldsymbol{x}, M)$ to be koszul complex with respect to $\boldsymbol{x}$ and M and we denote its homology by $\mathrm{H}(\boldsymbol{x}, M)$. Recall that the A-module $\mathrm{H}_i(\boldsymbol{x}, M)$ is finitely generated and annihilated by $\langle \boldsymbol{x}, \operatorname{Ann} M\rangle$, hence they have finite length (indeed, we have $\mathfrak{m} = \sqrt{\operatorname{Ann}(M/\boldsymbol{x}M)} = \sqrt{\langle \boldsymbol{x}, \operatorname{Ann} M\rangle}$). We may therefore define the **Euler-Poincare characteristic**

$$\chi(\boldsymbol{x}, M) = \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(\boldsymbol{x}, M)).$$

On the other hand, we the Hilbert-Samuel polynomial $P_{\boldsymbol{x}}(M)$ has degree $\leq r$, and we have

$$P_{\boldsymbol{x}}(M, n) = \mathrm{e}_{\boldsymbol{x}}(M, r)\frac{n^r}{r!} + Q(n)$$

with $\deg Q < r$ and where $\mathrm{e}_{\boldsymbol{x}}(M, r) = \Delta^r P_{\boldsymbol{x}}(M)$ is the Hilbert-Samuel multiplicity.

Theorem 0.2. *We have $\chi(\boldsymbol{x}, M) = \mathrm{e}_{\boldsymbol{x}}(M, r)$.*

Proof. We prove this in several steps:

Step 1: To ease notation in what follows, we set $Q = \langle \boldsymbol{x}\rangle$. We first equip A with the standard Q-filtration $A = (Q^n)$ and view it as a filtered ring. Similarly, we equip M with the Q-filtration $M = (Q^n M)$ and view it as a filtered A-module. We now equip K with a Q-filtration as follows: for each $n \in \mathbb{N}$, let K^n be the R-subcomplex of K whose component in homological degree i

$$K_i^n = \begin{cases} Q^{n-i}K_i. & \text{if } 0 \leq i < n \\ K_i & \text{else} \end{cases}$$

Thus for example, we have

$$\begin{aligned}
K^0 &= M + \sum Me_i + \sum Me_{i,j} + \cdots \\
K^1 &= QM + \sum Me_i + \sum Me_{i,j} + \cdots \\
K^2 &= Q^2M + \sum QMe_i + \sum Me_{i,j} + \cdots \\
&\vdots
\end{aligned}$$

Notice that

$$\begin{aligned}
K^0/K^1 &= M/QM \\
K^1/K^2 &= QM/Q^2M + \sum (M/QM)e_i \\
K^2/K^3 &= Q^2M/Q^3M + \sum (QM/Q^2M)e_i + \sum (M/QM)e_{i,j} \\
&\vdots
\end{aligned}$$

In particular, we clearly have

$$\begin{aligned}
\mathrm{gr}(K) &= \bigoplus_{n=0}^{\infty} K^n/K^{n+1} \\
&= \mathrm{gr}(M) + \sum \mathrm{gr}(M)e_i + \sum \mathrm{gr}(M)e_{i,j} \\
&= K(\boldsymbol{x}, \mathrm{gr}(M)).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\chi(\boldsymbol{x}, M) &= \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(\boldsymbol{x}, M)) \\
&= \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(K/K^n)) \\
&= \sum (-1)^i \ell(K_i/K_i^n) \\
&= \sum (-1)^i \ell\left(\bigoplus_{\binom{r}{i}} M/\boldsymbol{x}^{n-i}M\right) \\
&= \sum (-1)^i \binom{r}{i} \ell(M/\boldsymbol{x}^{n-i}M) \\
&= \mathrm{e}_{\boldsymbol{x}}(M, r).
\end{aligned}$$

□

0.1 Extra

Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let M be a nonzero finitely generated R-module of dimension d, and let $\boldsymbol{r} = r_1, \dots, r_d$ be a system of parameters for M. By definition, this means $\boldsymbol{r}$ is a sequence contained in $\mathfrak{m}$ such that $M/\boldsymbol{r}M$ has finite length, or equivalently, such that

$$\mathfrak{m} = \sqrt{\langle \mathrm{Ann}(M/\boldsymbol{r}M)\rangle} = \sqrt{Q},$$

where $Q = \langle \boldsymbol{r}, \mathrm{Ann}\, M\rangle$. There's a beautiful formula due to Serre which expresses the Hilbert multiplicity of M with respect to Q as an alternating sum of lengths of Koszul homology modules. To explain this, first let's recall how the Hilbert multiplicity of M with respect to Q is defined: let (M_n) be any Q-stable filtration of M (for example, we can pick $M_n = \langle \boldsymbol{r}\rangle^n M = Q^n M$). Then the Hilbert-Samuel function with respect (M_n) is the function $f_{(M_n)} = f \colon \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \ell_R(M/M_n) = \sum_{i=0}^{n-1} \ell_{R/Q}(M_i/M_{i+1}).$$

For n sufficiently large, we have $f(n) = P(n)$ where $P = P_{Q,M}$ is a polynomial whose lead coefficient is $e/d!$. Here, $e = e_{Q,M}$ is called the **Hilbert multiplicity** of M with respect to Q. It depends on the choice of Q (which itself depends on the choice of $\boldsymbol{r}$ assuming M is fixed), however it doesn't depend on the choice of stable Q-filtration (M_n).

On the other hand, the Euler-Poincare characteristic with respect to $\boldsymbol{r}$ and M is the alternating sum:

$$\chi(\boldsymbol{r}, M) = \sum_{i=0}^{\infty} (-1)^i \ell_{R/Q}(\mathrm{H}_i(\boldsymbol{r}, M)) = \sum_{i=0}^{d} (-1)^i \ell_{R/Q}(\mathrm{H}_i(\boldsymbol{r}, M)), \tag{1}$$

where $\mathrm{H}(\boldsymbol{r}, M)$ is the homology of the Koszul complex $\mathcal{K}(\boldsymbol{r}, M) = \mathcal{K}(\boldsymbol{r}) \otimes_R M$. Note that if $\boldsymbol{r}$ is an R-sequence, then we have

$$\mathrm{H}(\boldsymbol{r}, M) = \mathrm{Tor}_R(R/\boldsymbol{r}, M)$$

since $\mathcal{K}(\boldsymbol{r})$ is an R-free resolution of $R/\boldsymbol{r}$ in this case. So if $\boldsymbol{r}$ is an R-sequence, then we can re-express (1) as

$$\chi(\boldsymbol{r}, M) = \sum_{i=0}^{\infty} (-1)^i \ell_{R/Q}(\mathrm{Tor}_i^R(R/\boldsymbol{r}, M)).$$

More generally, let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R and set $I = \mathfrak{p} + \mathfrak{q}$. We define the **intersection multiplicity** of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ to be the quantity:

$$\chi(R/\mathfrak{p}, R/\mathfrak{q}) := \sum_{i=0}^{\infty} (-1)^i \ell_{R/I}(\mathrm{Tor}_i^R(R/\mathfrak{p}, R/\mathfrak{q})).$$

Note that this only makes sense when I is $\mathfrak{m}$-primary. If $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim R$, then it is an open conjecture that $\chi(R/I, R/J) > 0$.