Mathematical Programming Homework 6

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Problem 1

Exercise 1. Consider the following linear program (LP):

$$\begin{aligned} \text{minimize} \quad & z = x_1 - x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq 4 \\ & -x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The set of all optimal solutions to this problem is

$$X^* = \{\boldsymbol{x} \in \mathbb{R}^m \mid \boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}^1 + t\boldsymbol{d}, t \geq 0\}$$

where $\boldsymbol{x}^1 = (0, 3, 10, 0)^\top$ is an extreme point and $\boldsymbol{d} = (1, 1, 1, 0)^\top$ is a recession direction of the feasible set for this LP. Solve this problem in the Simplex tableua format. In the final tableau, clearly identify the entires that allow you to conclude that X^* is the complete optimal solution set for this LP.

Solution 1. First we introduce slack variables $x_3, x_4 \geq 0$ and convert the LP into standard form:

$$\begin{aligned} \text{minimize} \quad & z = x_1 - x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 + x_3 = 4 \\ & -x_1 + x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The initial tablue looks like:

basic	x_1	x_2	x_3	x_4	rhs
$-z$	1	-1	0	0	0
x_3	1	-2	1	0	4
x_4	-1	1	0	1	3

The reduced cost for x_2 is negative, and since it is the only variable for which the reduced cost is negative, we select it as our entering variable. By the ratio test, we see that we can choose x_4 to be our leaving variable. We use the entry corresponding to (x_4, x_2) as our pivot entry and we transform the tableau using elemenatry row operations so that the column corresponding to x_2 looks like $(0, 0, 1)^\top$; we obtain

basic	x_1	x_2	x_3	x_4	rhs
$-z$	0	0	0	1	3
x_3	-1	0	1	2	10
x_2	-1	1	0	1	3

The reduced costs of the nonbasic varibles are all positive, so the current basis is optimal. Thus we have one optimal solution which can be read off the tableau as: $z = -3$, $x_3 = 10$, $x_2 = 3$ and the nonbasic variables x_1 and x_4 being zero. In particular, this optimal solution corresponds to the extreme point $\boldsymbol{x}^1 = (0, 3, 10, 0)^\top$.

Now observe that the reduced cost for x_1 is zero. This implies that the objective value will not increase if we increase x_1. Furthermore, since the remaining entries in the x_1 column are all negative, we see that the basic variables will increase as x_1 increases. In particular, if $x_1 = t$ where $t \geq 0$, then $x_2 = 3 + t$, $x_3 = 10 + t$, and $z = -3$. It follows that all of the optimal solutions are given by $\boldsymbol{x}^1 + t\boldsymbol{d}$ where $\boldsymbol{x}^1 = (0, 3, 10, 0)^\top$, $\boldsymbol{d} = (1, 1, 1, 0)$, and $t \geq 0$.

Problem 2

Exercise 2. Consider the following LP given in the Simplex tableau:

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z$	0	0	1	0	0	0	-5
x_1	1	1	1	0	0	-1	15
x_5	0	-1	0	0	1	0	5
x_4	0	0	-1	1	0	-1	10

Continue the tableau format and, based on the experience you gained in question 1 above, find the set of all optimal solutions to this LP. Make sure that you write this set in an appropriate form.

Solution 2. The reduced costs of the nonbasic varibles are all nonnegative, so the current basis is optimal. Note that the reduced cost for the nonbasic variable x_2 is zero and the entry corresponding to (x_1, x_2) is positive. Thus if we use x_2 as entering variable and x_1 as a leaving variable, then the objective value will remain unchanged (and hence optimal). Pivoting gives us

basic	x_1	x_2	x_3	x_4	x_5	x_6	rhs
$-z$	0	0	1	0	0	0	-5
x_2	1	1	1	0	0	-1	15
x_5	1	0	1	0	1	-1	20
x_4	0	0	-1	1	0	-1	10

The reduced cost for the nonbasic variable x_1 is zero. Note that we can't use x_5 here as a leaving variable (if we try to pivot at (x_5, x_1), we'd obtain an infeasible solution). It follows that the set of all optimal solutions to this LP is given by all convex combinations of

$$\boldsymbol{x}^1 = (15, 0, 0, 10, 5) \quad \text{and} \quad \boldsymbol{x}^2 = (0, 15, 0, 10, 20).$$

Problem 3

Exercise 3. The following tableau represents a linear program in standard form:

basic	x_1	x_2	x_3	x_4	x_5	rhs
$-z$	2	0	c	0	e	6
x_4	3	0	-1	1	0	a
x_5	-1	0	d	0	1	2
	4	b	-2	0	0	5

Give the most general conditions on the parameters a, b, c, d, and e such that the tableau represents

1. a nondegenerate basic feasible solution
2. a degenerate basic feasible solution
3. a basic feasible solution and unbounded feasible region
4. a unique optimal basic feasible solution
5. an optimal basic feasible solution and optimal ray
6. unbounded objective function value

where by "most general conditions", we mean, for example, $a > 0$, $a \geq 0$, a free, or $a = 0$, etc... Report the answers in the table below:

	a	b	c	d	e
1					
2					
3					
4					
5					
6					

Solution 3. We report the answers in the table below:

	a	b	c	d	e
1	$a \geq 0$	$b = 1$	d free	c free	e free
2	$a \geq 0$	$b = 1$	d free	c free	e free
3	$a \geq 0$	$b = 1$	c free	$d \leq 0$	e free
4	$a \geq 0$	$b = 1$	$c > 0$	d free	$e = 0$
5	$a \geq 0$	$b = 1$	$c = 0$	$d \leq 0$	$e > 0$
6	a free	b free	$c < 0$	$d \leq 0$	$e \geq 0$

Problem 4

Exercise 4. Prove the following: let $X = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$. If an extreme point of X has more than one basis representation, then it is degenerate.

Solution 4. Suppose $\boldsymbol{x}_B$ and $\boldsymbol{x}_{B'}$ are two distinct bases which give the same basic solution $\boldsymbol{x}$ and assume for a contradiction that $\boldsymbol{x}$ is not degenerate. This means that every meaning every one of the m variables of $\boldsymbol{x}$ is nonzero. Thus every variable in $\boldsymbol{x}_B$ and $\boldsymbol{x}_{B'}$ is nonzero. Since $\boldsymbol{x}_B \neq \boldsymbol{x}_{B'}$, there exists a variable in $\boldsymbol{x}_B$ which is not in $\boldsymbol{x}_{B'}$. However this implies at least $m+1$ variables of $\boldsymbol{x}$ are nonzero, which is a contradiction. Thus $\boldsymbol{x}$ is degenerate.

Problem 5

Exercise 5. Use the tableau format and solve the following linear program with the Two-Phase Method. Clearly show all steps of this method.

$$\begin{aligned} \text{minimize} \quad & z = x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 = 2 \\ & -x_1 + x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution 5. We introduce an artifical variable $a_1 \geq 0$ to the constraint that doesn't have a slack variable and we wish to minimize $z' = a_1$. The tableau for the problem with artificial variables is

basic	x_1	x_2	x_3	x_4	a_1	rhs
$-z'$	0	0	0	0	1	0
a_1	1	1	-1	0	1	2
x_4	-1	1	0	1	0	4

The top row entry for a_1 is not zero, so z' is not expressed only in terms of the nonbasic variables. To fix this we perform an elementary row operation to obtain

basic	x_1	x_2	x_3	x_4	a_1	rhs
$-z'$	-1	-1	1	0	0	-2
a_1	1	1	-1	0	1	2
x_4	-1	1	0	1	0	4

The reduced cost for both x_1 and x_2 is negative, so this basis is not optimal. We will choose x_1 as our entering variable. The ratio test then indicates that a_1 should be our leaving variable. So we use the entry corresponding to (x_1, a_1) as our pivot entry and we transform the tableau using elemenatry row operations so that the column corresponding to x_1 looks like $(0, 1, 0)^\top$. We also remove a_1 from the problem. The resulting tableau looks like:

basic	x_1	x_2	x_3	x_4	rhs
$-z'$	0	0	0	0	0
x_1	1	1	-1	0	2
x_4	0	2	-1	1	6

The current basis does not involve any artificial variables and the objective value is zero, so this is a feasible point for the constraints of the original problem. This completes phase 1.

Now we proceed to phase 2 and use the intial basic feasible solution that we found in phase 1 and we want to minimize the objective $z = x_1 - 2x_2$:

basic	x_1	x_2	x_3	x_4	rhs
$-z$	1	-2	0	0	0
x_1	1	1	-1	0	2
x_4	0	2	-1	1	6

The reduced cost for x_1 is not zero, so the problem must be expressed in standard form before the simplex method is used. To do this, we just apply an elementary row operation to the tableau to obtain:

basic	x_1	x_2	x_3	x_4	rhs
$-z$	0	-3	1	0	-2
x_1	1	1	-1	0	2
x_4	0	2	-1	1	6

The reduced cost for x_2 is negative, so the basis is not optimal. The ratio test tells us that we should use x_1 as our leaving variable. Pivoting gives us

basic	x_1	x_2	x_3	x_4	rhs
$-z$	3	0	-2	0	4
x_2	1	1	-1	0	2
x_4	-2	0	1	1	2

The reduced cost for x_3 is negative, so the basis is not optimal. The ratio test tells us that we should use x_4 as our leaving variable. Pivoting gives us

basic	x_1	x_2	x_3	x_4	rhs
$-z$	-1	0	0	2	2
x_2	-1	1	0	1	4
x_3	-2	0	1	1	2

It follows that the problem is unbounded.

Problem 6

Exercise 6. For some linear program (LP), the associated artificial LP has been solved and produced the following tableau:

basic	x_1	x_2	x_3	a_1	a_2	rhs
$-z'$	0	1	3	0	3	0
a_1	0	-1	-3	1	-2	0
x_1	1	1	2	0	1	2

Continue the Two-Phase Method and find an intial partial tableau for Phase 2 for that LP.

Solution 6. At the current point the values of the variables are $(x_1, x_2, x_3, a_1, a_2)^\top = (2, 0, 0, 0, 0)^\top$. Thus we can use the basis $x_B = (x_1, x_2)$. Then in the x_B basis, we have

basic	x_1	x_2	x_3	a_1	a_2	rhs
$-z'$	0	0	0	1	1	0
x_2	0	1	3	-1	2	0
x_1	1	0	-1	1	-1	2