Hom-Complex

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Definition 0.1. Let X and Y be two R-complexes. We define their **hom-complex** $\mathrm{Hom}^{\star}_{R}(X,Y)$ to be the R-complex whose underlying graded R-module has homogeneous component in degree $i \in \mathbb{Z}$ given by

$$\mathrm{Hom}^{\star}_{R,i}(X,Y) = \{\varphi\colon X \to Y \mid \varphi \text{ is a graded } R\text{-linear of degree } i\}.$$

whose differential, denoted $\mathrm{d}^{\star}_{X,Y}$ is defined by

$$\mathrm{d}^{\star}_{X,Y}(\varphi) = \mathrm{d}_Y\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}_X. \tag{1}$$

for all homogeneous $\varphi \in \mathrm{Hom}^{\star}_{R}(X,Y)$.

If the ring R is understood from context, then we simplify our notation by saying "$\varphi\colon X \to Y$ is an i-map" to mean "$\varphi\colon X \to Y$ is a graded R-linear map of degree i. If in addition, φ commutes with the differentials (or equivalently $\mathrm{d}^{\star}(\varphi) = 0$), then we say φ is an i-chain map. If X and Y are understood from context, then we simplify our notation even more by dropping X and Y in the subscripts of $\mathrm{d}^{\star}_{X,Y}$, d_Y, and d_X. With this notational convention in mind, we may rewrite (1) in a much cleaner format:

$$\mathrm{d}^{\star}(\varphi) = \mathrm{d}\varphi - (-1)^{|\varphi|}\varphi\mathrm{d} \tag{2}$$

The sign $-(-1)^{|\varphi|}$ in (2) may seem a little unusual at first glance. Indeed, the differential for the tensor compex $X \otimes_R Y$ is defined by

$$\mathrm{d}^{\otimes}(x \otimes y) = \mathrm{d}(x) \otimes y + (-1)^{|x|} x \otimes \mathrm{d}(y)$$

for all homogeneous $x \in X$ and $y \in Y$. In fact , if we had replaced $-(-1)^{|\varphi|}$ in (1) with $(-1)^{|\varphi|}$, then we would still still obtain a differential. So why should we change things up here? One of the reasons is that it allows us to interpret $\mathrm{d}^{\star}(\varphi)$ as measuring the failure of the i-map φ to be an i-chain map. Indeed, φ is an i-chain map if and only if $\mathrm{d}\varphi = (-1)^{|\varphi|}\varphi\mathrm{d}$ if and $\varphi \in \ker \mathrm{d}^{\star}$. Furthermore, two i-chain maps φ and ψ are homotopy equivalent if and only if there exists an $(i+1)$-map ϕ such that $\varphi - \psi = \mathrm{d}\phi + (-1)^{|\phi|}\phi\mathrm{d}$ if and only if $\varphi - \psi \in \operatorname{im} \mathrm{d}^{\star}$. Thus the homology of the hom-complex has a really nice interpretation:

$$\mathrm{H}_i(\mathrm{Hom}^{\star}_{R}(X,Y)) = \{\text{homotopy classes of } i\text{-chain maps } X \to Y\}.$$

This is probably the most important reason we use the $-(-1)^{|\varphi|}$ in (2).

When an i-map is a chain map

Let A and B be R-complexes and suppose that $\varphi\colon A \to B$ is a graded R-linear map. Then $\mathrm{d}^{\star}(\varphi)\colon A \to \Sigma B$ is a chain map since

$$\begin{aligned}
\mathrm{d}_{\Sigma B}\mathrm{d}^{\star}(\varphi) &= \mathrm{d}_{\Sigma B}(\mathrm{d}_B\varphi - \varphi\mathrm{d}_A) \\
&= -\mathrm{d}_{\Sigma B}\varphi\mathrm{d} \\
&= \mathrm{d}_B\varphi\mathrm{d}_A \\
&= (\mathrm{d}_B\varphi - \varphi\mathrm{d}_A)\mathrm{d}_A.
\end{aligned}$$

So we have a short exact sequence of R-complexes

$$0 \longrightarrow A \xrightarrow{\mathrm{d}^{\star}(\varphi)} \Sigma B \xrightarrow{\pi} \Sigma B/\mathrm{im}(\mathrm{d}^{\star}(\varphi)) \longrightarrow 0 \tag{3}$$

which in turn induces the following long exact sequence in homology:

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_i(B/\mathrm{im}(\mathrm{d}^\star(\varphi))) \arrow[dlll, rounded corners, "\gamma"] \\
\mathrm{H}_i(A) \arrow[r, "\mathrm{d}\varphi"] & \mathrm{H}_{i-1}(B) \arrow[r] & \mathrm{H}_{i-1}(B/\mathrm{im}(\mathrm{d}^\star(\varphi))) \arrow[dll, rounded corners, "\gamma"] & \\
\mathrm{H}_{i-1}(A) \arrow[r] & \cdots & &
\end{tikzcd}
\tag{4}
$$

Here, the connecting map $\gamma\colon \mathrm{H}(B/\mathrm{im}(\mathrm{d}^\star(\varphi))) \to \mathrm{H}(A)$ is constructed as follows: let $\beta = [\bar{b}] \in \mathrm{H}(\Sigma B/\mathrm{im}(\mathrm{d}^\star(\varphi)))$ where $\bar{b} \in \Sigma B/\mathrm{im}(\mathrm{d}^\star(\varphi))$ and $b \in \Sigma B$ satisfies $\mathrm{d}_{\Sigma B}(b) = 0$, that is

$$
-\mathrm{d}(b) = (\mathrm{d}^\star(\varphi))(a) \tag{5}
$$

for some $a \in A$. We then lift $\bar{b}$ to $b \in B$ and then apply the differential $\mathrm{d}_{\Sigma B}$ to get $\mathrm{d}_{\Sigma B}(b) = -\mathrm{d}(b) \in B$. Then (5) tells us that $\gamma(\beta) = a$. In fact, it is clear from (5) that if φ is a chain map, then $\mathrm{d}^\star(\varphi)$ vanishses everywhere which implies γ vanishes everywhere. On the other hand, if for some $a \in A$ we have $\varphi\mathrm{d}(a) \neq \mathrm{d}\varphi(a)$, then $(\mathrm{d}^\star(\varphi))(a) \neq 0$. If we can find a $b \in B$ such that $\mathrm{d}(b) = (\mathrm{d}^\star(\varphi))(a)$, then γ will take a nonzero value at b. If we cannot find such a $b \in B$, then γ will vanish everywhere, however we will still have $(\mathrm{d}^\star(\varphi))(a) \neq 0$. Finally, observe that if $\varphi(\ker \mathrm{d}) \subseteq \ker \mathrm{d}$, then the map $\mathrm{d}\varphi\colon \mathrm{H}(A) \to \mathrm{H}(\Sigma B)$ vanishes, in which case we obtain a short exact sequence of graded R-modules

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{H}(B) \arrow[r] & \mathrm{H}(B/\mathrm{im}(\mathrm{d}^\star(\varphi))) \arrow[r, "\gamma"] & \mathrm{H}(A) \arrow[r] & 0
\end{tikzcd}
\tag{6}
$$

Definition 0.2. We say $\varphi\colon A \to B$ is **homologically** a chain map if $\mathrm{H}(\mathrm{im}(\mathrm{d}^\star(B))) = 0$. In particular, if $\varphi(\ker \mathrm{d}) \subseteq \ker \mathrm{d}$ and is homologically a chain map, then (6) implies $\varphi\colon A \to B$ is homologically a chain map if and only if $\gamma = 0$.

Functorial Properties of the Hom-Complex

Let A, A', B, and B' be R-complexes and let $\varphi\colon A \to B$ and $\phi\colon A' \to B'$ be i-chain maps. Then we get induced i-chain maps

$$
\phi_*\colon \mathrm{Hom}_R^\star(A, A') \to \mathrm{Hom}_R^\star(A, B') \quad \text{and} \quad \varphi^*\colon \mathrm{Hom}_R^\star(B, B') \to \mathrm{Hom}_R^\star(A, B')
$$

given by

$$
\phi_*(\alpha) = \phi\alpha \quad \text{and} \quad \varphi^*(\beta) = \beta\varphi
$$

for all $\alpha \in \mathrm{Hom}_R^\star(A, A')$ and $\beta \in \mathrm{Hom}_R^\star(B, B')$. Furthermore, the following diagram commutes

$$
\begin{tikzcd}
\mathrm{Hom}_R^\star(A, A') \arrow[r, "\varphi^*"] \arrow[d, "\phi_*"'] & \mathrm{Hom}_R^\star(B, A') \arrow[d, "\phi_*"] \\
\mathrm{Hom}_R^\star(A, B') \arrow[r, "\varphi^*"'] & \mathrm{Hom}_R^\star(B, B')
\end{tikzcd}
\tag{7}
$$

In particular, $\mathrm{Hom}_R^\star(-,-)$ is a bifunctor:

$$
\mathrm{Hom}_R^\star(-,-)\colon \mathbf{Comp}_R^{\mathrm{op}} \times \mathbf{Comp}_R \to \mathbf{Comp}_R.
$$

The bifunctor $\mathrm{Hom}_R^\star(-,-)$ shares almost all of the same properties as the usual bifunctor $\mathrm{Hom}_R^\star(-,-)$, where

$$
\mathrm{Hom}_R(-,-)\colon \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \to \mathbf{Mod}_R.
$$

For instance, $\mathrm{Hom}_R^\star(-,-)$ is a left-exact covariant functor when the first argument is fixed and is a left-exact contravariant functor when the second argument is fixed.