Mathematics Diary

Contents

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1.1 12/20/2022 - When $\Sigma(F/E)$ is the minimal free resolution of I/J over R

Lemma 1.1. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R. Let E be the minimal free resolution of R/J over R, let F be the minimal free resolution of R/I over R, and let $\varphi \colon E \to F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_{\Bbbk} := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_{\Bbbk}$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R.*

Proof. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_{\Bbbk} := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_{\Bbbk}$ are injective. Since $\varphi \colon E \to F$ is injective, we have a short exact sequence of R-complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \tag{1}$$

taking homology gives us a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}_{i+1}(F/E) \arrow[dll, rounded corners] \\
\mathrm{H}_i(E) \arrow[r] & \mathrm{H}_i(F) \arrow[r] & \mathrm{H}_i(F/E) \arrow[dll, rounded corners] \\
\mathrm{H}_{i-1}(E) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are resolutions we conclude that $\mathrm{H}_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $\mathrm{H}_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R-modules (1) with $\Bbbk$ over R gives us the long exact sequence in homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{i+1}(E,\Bbbk) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\mathrm{Tor}^R_i(E,\Bbbk) \arrow[r] & \mathrm{Tor}^R_i(F,\Bbbk) \arrow[r] & \mathrm{Tor}^R_i(F/E,\Bbbk) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\
\mathrm{Tor}^R_{i-1}(E,\Bbbk) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are free R-modules we conclude that $\mathrm{Tor}_i(F/E,\Bbbk) = 0$ for all $i \geq 1$. Since $\overline{\varphi}\colon E \otimes_R \Bbbk \to F \otimes_R \Bbbk$ is injective we conclude that $\mathrm{Tor}_1(F/E,\Bbbk) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $\mathrm{d}(F/E) \subseteq \mathfrak{m}(F/E)$). $\square$

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R-module M connecting maps

$$\mathrm{Tor}^R_{i+1}(R/I, M) \to \mathrm{Tor}^R_i(I/J, M) \quad \text{and} \quad \mathrm{Ext}^i_R(I/J, M) \to \mathrm{Ext}^{i+1}_R(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \to F/E \otimes_R M \quad \text{and} \quad \mathrm{Hom}^\star_R(F/E, M) \to \mathrm{Hom}^\star_R(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\overline{\varphi}\colon E_\Bbbk \to F_\Bbbk$ is injective, then already $\varphi\colon E \to F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of ideals

Let R be a commutative ring and let $\mathfrak{p}$ be an ideal of R. Recall the **height** of $\mathfrak{p}$ is defined to be the supremum of lengths of chains of primes which descend from $\mathfrak{p}$:

$$\operatorname{ht}\mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

When R is Noetherian, then Krull's prinicipal ideal theorem states that there exists an ideal $\langle \boldsymbol{x} \rangle = \langle x_1, \dots, x_c \rangle \subseteq \mathfrak{p}$ where $c = \operatorname{ht}\mathfrak{p}$ such that $\sqrt{\langle \boldsymbol{x} \rangle} = \mathfrak{p}$, and that if $\langle \boldsymbol{y} \rangle = \langle y_1, \dots, y_m \rangle$ is another ideal such that $\sqrt{\langle \boldsymbol{y} \rangle} = \mathfrak{p}$, then we must have $c \leq m$. If I is an ideal of R, then the **height** of I is defined to be the infimum of the heights of all primes which contain I:

$$\operatorname{ht} I = \inf\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. *Let I_1 and I_2 be ideals of R. Set $c = \operatorname{ht}(I_1 \cap I_2)$, set $c_1 = \operatorname{ht} I_1$, and set $c_2 = \operatorname{ht} I_2$.*

1. *If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.*
2. *We have $c = \min\{c_1, c_2\}$.*

Proof. 1. Let $\mathfrak{p}$ be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2. Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let $\mathfrak{p}$ be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since $\mathfrak{p}$ is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. $\square$

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1/20/2024 - $\mathrm{V}(\operatorname{Ann} M) = \mathrm{V}(\operatorname{Ann}(0 :_M x))$

Lemma 2.1. *Let R be a commutative ring, let M be an R-module, and let $x \in R$. Then*

$$\mathrm{V}(\operatorname{Ann}(0 :_M x)) = \mathrm{V}(\operatorname{Ann}(0 :_M x^2)).$$

Proof. Note that $0 :_M x \subseteq 0 :_M x^2$ implies $\operatorname{Ann}(0 :_M x^2) \supseteq \operatorname{Ann}(0 :_M x)$ which implies $\mathrm{V}(\operatorname{Ann}(0 :_M x^2)) \subseteq \mathrm{V}(\operatorname{Ann}(0 :_M x))$. For the reverse inclusion, suppose $\mathfrak{p}$ is a prime ideal of R which contains $\operatorname{Ann}(0 :_M x^2)$ and let $r \in \operatorname{Ann}(0 :_M x)$. We claim that $r^2 \in \operatorname{Ann}(0 :_M x^2)$. Indeed, if $u \in 0 :_M x^2$, then

$$\begin{aligned} x^2 u = 0 &\implies xu \in 0 :_M x \\ &\implies rxu = 0 \\ &\implies ru \in 0 :_M x \\ &\implies r^2 u = 0. \end{aligned}$$

Since u was arbitrary, we see that $r^2 \in \operatorname{Ann}(0 :_M x^2) \subseteq \mathfrak{p}$. However this implies $r \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime. Since r was arbitary, we see that $\operatorname{Ann}(0 :_M x) \subseteq \mathfrak{p}$. □

Corollary 1. *Let R be a commutative ring and let M be a finitely generated R-module. Assume that $x \in R$ acts nilpotently on M. Then*

$$\mathrm{V}(\operatorname{Ann}(M)) = \mathrm{V}(\operatorname{Ann}(0 :_M x)).$$

Proof. Since M is finitely generated, there exists an $n \in \mathbb{N}$ such that $M = 0 :_M x^n$. A straightforward induction on (**??**) gives us

$$\mathrm{V}(\operatorname{Ann}(M)) = \mathrm{V}(\operatorname{Ann}(0 :_M x^n)) = \mathrm{V}(\operatorname{Ann}(0 :_M x)).$$

□

1/21/2024 - Some subschemes of $\mathbb{P}^3$

Let $R = \Bbbk[x, y, z, w]$. We consider three cyclic R-algebras, namely $A = R/\boldsymbol{f} = R/\langle f_1, f_2, f_3\rangle$, $B = R/\boldsymbol{g} = R/\langle g_1, g_2, g_3\rangle$, and $C = R/\boldsymbol{h} = R/\langle h_1, h_2, h_3\rangle$ where

$$\begin{aligned} f_1 &= xy - zw & g_1 &= xz - y^2 & h_1 &= xz - y^2 \\ f_2 &= xz - yw & g_2 &= yw - z^2 & h_2 &= x^3 - yzw \\ f_3 &= xw - yz & g_3 &= xw - yz & h_3 &= x^2 y - z^2 w \end{aligned}$$

We want a geometric picture in mind when thinking of these rings, so let $X = \operatorname{Proj} A$, $Y = \operatorname{Proj} B$, and $Z = \operatorname{Proj} C$. First let us consider X. We can see that $X(\Bbbk)$ consists of 8 distinct points in $\mathbb{P}^3(\Bbbk)$ by calculating an irreducible primary decomposition for $\langle \boldsymbol{f} \rangle$. Indeed, an irredundant primary decomposition for $\langle \boldsymbol{f} \rangle$ is given by $\langle \boldsymbol{f} \rangle = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_8$ where

$$\begin{aligned} \mathfrak{p}_1 &= \langle y, z, w\rangle & \mathfrak{p}_5 &= \langle x + y, y + z, z + w\rangle \\ \mathfrak{p}_2 &= \langle x, z, w\rangle & \mathfrak{p}_6 &= \langle x + y, y - z, z + w\rangle \\ \mathfrak{p}_3 &= \langle x, y, w\rangle & \mathfrak{p}_7 &= \langle x + y, y - z, z - w\rangle \\ \mathfrak{p}_4 &= \langle x, y, z\rangle & \mathfrak{p}_8 &= \langle x - y, y - z, z - w\rangle. \end{aligned}$$

These primes correspond to the points

$$\begin{aligned} \boldsymbol{p}_1 &= [1 : 0 : 0 : 0] & \boldsymbol{p}_5 &= [-1 : 1 : -1 : 1] \\ \boldsymbol{p}_2 &= [0 : 1 : 0 : 0] & \boldsymbol{p}_6 &= [1 : -1 : -1 : 1] \\ \boldsymbol{p}_3 &= [0 : 0 : 1 : 0] & \boldsymbol{p}_7 &= [-1 : 1 : 1 : 1] \\ \boldsymbol{p}_4 &= [0 : 0 : 0 : 1] & \boldsymbol{p}_8 &= [1 : 1 : 1 : 1] \end{aligned}$$

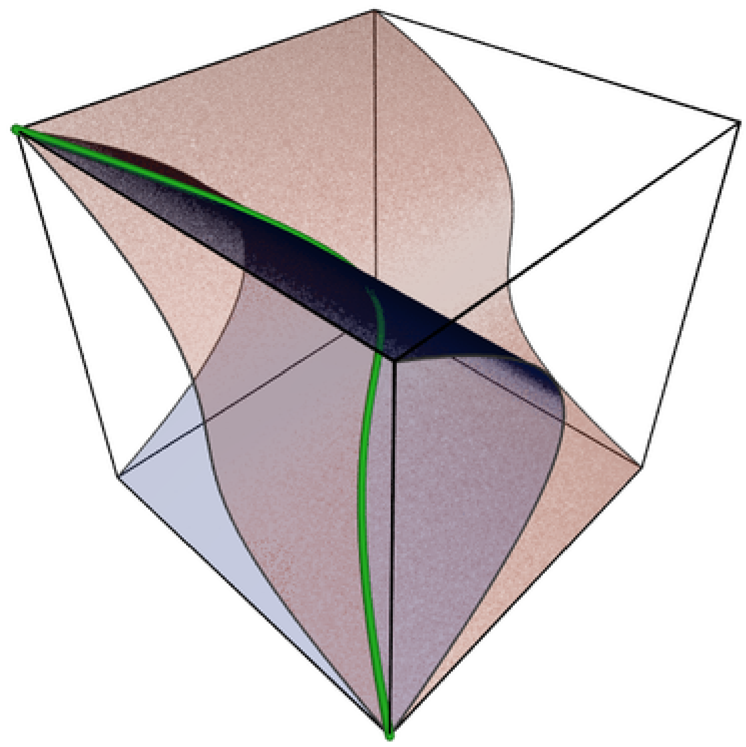
in $\mathbb{P}^3(\Bbbk)$. Note that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ are in linearly general position since the size 4 minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

are all nonzero. In other words, viewing $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ as vectors in $\Bbbk^4$, every subset of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8\}$ of size 4 is linearly independent. The Betti diagram of A over R is given by

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	-	3	-
2	-	-	-	1

Next we consider Y. In fact, Y is the twisted cubic. When $\Bbbk = \mathbb{R}$, we can visualize $Y(\Bbbk)$ as below:



In particular, $Y(\Bbbk)$ is the image of the map $\mathbb{P}^1(\Bbbk) \to \mathbb{P}^3(\Bbbk)$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Note that $\langle \boldsymbol{g} \rangle$ is a prime of height 2 and so $\langle \boldsymbol{g} \rangle$ can be generated up to radical by two homogeneous polynomials. In particular, we have $\langle \boldsymbol{g} \rangle = \sqrt{\langle g_1, g_4 \rangle}$ where $g_4 = zg_2 - wg_3$. However $\langle \boldsymbol{g} \rangle$ itself cannot be generated by only two polynomials; a minimum of three polynomials are needed. We can see this in Betti diagram of B over R:

	0	1	2
0	1	-	-
1	-	3	2

In particular, the Hilbert-Poincare series of B over R is given by

$$P(t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^4} = \frac{1 + 2t}{(1-t)^2} = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + \cdots .$$

Thus Y is the set-theoretic complete intersection of $\mathrm{V}(g_1)$ and $\mathrm{V}(g_4)$ however it is not a scheme-theoretic or ideal-theoretic complete intersection. Note also that $\langle \boldsymbol{g} \rangle$ corresponds to the ideal of size 2 minors of the matrix $\left(\begin{smallmatrix} x & y & z \\ y & z & w \end{smallmatrix}\right)$. Up to linear automorphism, the twisted cubic is the only irreducible curve of degree 3 not contained in a plane. Furthermore, any 6 points in linearly general position in $\mathbb{P}^3(\Bbbk)$ lie on a unique twisted cubic. However for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Consequently one can show that if W is a set of 7 points in linearly general position in $\mathbb{P}^3(\Bbbk)$, then there are only two distinct Betti diagrams possible for the homogeneous coordinate ring of W, namely

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	1	6	3

and

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

In the first case, the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the quadrics containing W is the ideal of the unique curve of degree 3 containing W, which is irreducible. Finally, let us write down the minimal free resolution of B over R:

$$R(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} R \longrightarrow 0$$

Now we consider Z. The Betti diagram of C over R is given by

	0	1	2
0	1	-	-
1	-	1	-
2	-	2	2

In particular, the Hilbert-Poincare series of C over R is given by

$$P(t) = \frac{1 - t^2 - 2t^3 + 2t^4}{(1-t)^4} = \frac{1 + 2t + 2t^2}{(1-t)^2} = 1 + 4t + 9t^2 + 14t^3 + 19t^4 + \cdots .$$

In particular, Z is an irreducible curve of degree 5 in $\mathbb{P}^3(\mathbb{k})$.

2.1 4/22/2024 - Lifting multiplication to a free module

Let A be a commutative ring and let B be a finite A-algebra. Then there exists a surjection $F \twoheadrightarrow B$ of A-modules where $F = A^{n+1}$ and where we assume $n \geq 0$ is minimal. We are interested in the question as to whether one can lift the associative and unital multiplication on B to an associative and unital multiplication on F. Let K be the kernel of the map $F \twoheadrightarrow B$. In what follows, all tensors products are taken over A.

Lemma 2.2. *The kernel of the map $F^{\otimes 2} \to B^{\otimes 2}$ is given by $K \otimes F + F \otimes K$.*

Proof. This is easily checked via a diagram chase in the diagram below which is exact everywhere and in all directions:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & K^{\otimes 2} \arrow[r] \arrow[d] & K \otimes F \arrow[r] \arrow[d] & K \otimes B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes K \arrow[r] \arrow[d] & F^{\otimes 2} \arrow[r] \arrow[d] & F \otimes B \arrow[r] \arrow[d] & 0 \\
 & B \otimes K \arrow[r] \arrow[d] & B \otimes F \arrow[r] \arrow[d] & B^{\otimes 2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Since $F^{\otimes 2}$ is free (hence projective), we can lift the composite map $F^{\otimes 2} \to B^{\otimes 2} \twoheadrightarrow B$ with respect to the map $F \twoheadrightarrow B$ to obtain an A-linear map $\mu\colon F^{\otimes 2} \to F$. Assume that A is a local noetherian ring. In this case, there exists a minimal generating set of B as an A-module of the form $\{b_0, b_1, \ldots, b_n\}$ where $b_0 = 1$. Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ be a basis for F as a free A-module and let $F \twoheadrightarrow B$ be the A-linear map defined by $\varepsilon_i \mapsto b_i$ for all i. For each i, j, we have

$$b_i b_j = \sum_k a_{ij}^k b_k$$

where the $a_{ij}^k \in A$ need not be unique. Since the multiplication on B is unital, we can choose the a_{ij}^k such that

$$a_{j0}^k = a_{0j}^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Furthermore, since the multiplication on B is commutative, we can also choose the a_{ij}^k such $a_{ij}^k = a_{ji}^k$. With these choices of a_{ij}^k in mind, we can define a commutative and unital multiplication μ on F which lifts the multiplication on B by

$$\varepsilon_i \varepsilon_j := \sum_k a_{ij}^k \varepsilon_k.$$

Note that this multiplication need not be associative. Indeed, since the multiplication on B is associative, we have

$$\begin{aligned} 0 &= [b_i, b_j, b_k] \\ &= (b_i b_j) b_k - b_i (b_j b_k) \\ &= \sum_l (a_{ij}^l b_l b_k - a_{jk}^l b_i b_l) \\ &= \sum_{l,m} (a_{ij}^l a_{lk}^m - a_{jk}^l a_{il}^m) b_m. \end{aligned}$$

However this need not imply that $\sum_l a_{ij}^l a_{lk}^m - a_{jk}^l a_{ik}^m = 0$ for all i, j, k, m (which is what we'd need in order for $[\varepsilon_i, \varepsilon_j, \varepsilon_k] = 0$).

Example 2.1. Let $A = \Bbbk[x_1, x_2]$ and let $B = A[e_1, e_2]/J$ where

$$J = \langle e_1^2 - x_1 e_1, e_2^2 - x_2 e_2, e_1 e_2 - x_2 e_1 - x_1 e_2, x_1 e_1 + x_2 e_2 - 1 \rangle.$$

Then B is a finite A-algebra with a minimal generating set of B as an A-module given by $\{\bar{e}_1, \bar{e}_2\}$. Furthermore, any minimal generating set of B as an A-module cannot contain 1. Now let $F_0 = A\varepsilon_1 \oplus A\varepsilon_2$ and consider the surjective A-module homomorphism $F_0 \twoheadrightarrow B$ given by $\varepsilon_i \mapsto e_i$. We can lift the multiplication on B to a multiplication on F_0 by setting $\varepsilon_1 \varepsilon_2 = x_1 \varepsilon_2 + x_2 \varepsilon_1$ and $\varepsilon_i^2 = x_i \varepsilon_i$ for $i = 1, 2$. However there is no identity element in F_0 with respect to this multiplication.

2.2 5/2/2024 - Colon ideal result

Let R be a noetherian ring, let I be an ideal of R, and let $r, r' \in R$. We have an R-linear map

$$\varphi \colon \langle I, r \rangle : r' \twoheadrightarrow (\langle I, r' \rangle : r)/(I : r)$$

defined as follows: if $a \in \langle I, r \rangle : r'$, then we have $ar' = br + x$ for some $b \in R$ and $x \in I$. The map is defined by sending a to the class of b in the quotient. It is straightforward to check that this is well-defined and surjective. Note if $b \in I : r$, then $ar' \in I : r'$. In particular, the kernel of φ is $I : r'$. Thus we've established an isomorphism

$$(\langle I, r \rangle : r'))/(I : r') \cong (\langle I, r' \rangle : r))/(I : r). \tag{2}$$

In particular, if $I : r' = I : r$, then we must have $\langle I, r \rangle : r' = \langle I, r' \rangle : r$. Now assume that $I : r = \mathfrak{p} = \langle I, r \rangle : r'$. Then (2) implies

$$\mathfrak{p}/(I : r') \cong (\langle I, r' \rangle : r)/\mathfrak{p}.$$

Example 2.2. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz \rangle$, let $r = yw$, and let $r' = y$. Then we have

$$\begin{aligned} I : r &= \langle x, z, w \rangle & \langle I, r' \rangle : r &= R \\ I : r' &= \langle x, z, w^2 \rangle & \langle I, r \rangle : r' &= \langle x, z, w \rangle. \end{aligned}$$

Now observe that $\langle I : r, r' \rangle \subseteq \langle I, r' \rangle : r$. Indeed, if $a \in \langle I : r, r' \rangle$, then we can express it as $a = b + cr'$ where $b \in I : r$ and $c \in R$. In particular, this means that $ar = br + cr'r \in \langle I, r' \rangle$, and hence $a \in \langle I, r' \rangle : r$.

2.3 5/20/2024 - Geometric description of finitely generated $\Bbbk$-algebra homomorphisms

Let $\Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \ldots, x_m]$, let $\Bbbk[\boldsymbol{y}] = \Bbbk[y_1, \ldots, y_n]$, and let $\varphi \colon \Bbbk[\boldsymbol{x}] \to \Bbbk[\boldsymbol{y}]$ be a $\Bbbk$-algebra homomorphism. Then the φ corresponds to the morphism of affine schemes $f \colon \mathbb{A}_\Bbbk^n \to \mathbb{A}_\Bbbk^m$ given by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in \mathbb{A}_\Bbbk^n$. We want to give a more geometric description of how f acts on the points of $\mathbb{A}_\Bbbk^n$, or in other words, how φ^{-1} acts on the prime ideals of $\Bbbk[\boldsymbol{y}]$. First, note that since $\Bbbk[\boldsymbol{y}]$ is Jacobson, we have

$$\varphi^{-1}(\mathfrak{q}) = \varphi^{-1}\left(\bigcap_{\substack{\mathfrak{n} \supseteq \mathfrak{q} \\ \mathfrak{n} \text{ maximal}}} \mathfrak{n} \right) = \bigcap_{\substack{\mathfrak{n} \supseteq \mathfrak{q} \\ \mathfrak{n} \text{ maximal}}} \varphi^{-1}(\mathfrak{n}).$$

Thus we will focus on the case where $\mathfrak{q} = \mathfrak{n}$ is a maximal ideal. First let's consider the maximal ideals of the form $\mathfrak{n}_{\boldsymbol{q}} = \langle y_1 - q_1, \dots, y_n - q_n \rangle$ where $\boldsymbol{q} \in \mathbb{A}^n_{\Bbbk}(\Bbbk) = \Bbbk^n$. To this end, for each $1 \leq i \leq m$ let $f_i = \varphi(x_i)$, and let $\boldsymbol{f} \colon \Bbbk^n \to \Bbbk^m$ be the map given by $\boldsymbol{f}(\boldsymbol{q}) = (f_1(\boldsymbol{q}), \dots, f_m(\boldsymbol{q}))$. Then we claim that

$$\varphi^{-1}(\mathfrak{n}_{\boldsymbol{q}}) = \mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} = \langle x_1 - f_1(\boldsymbol{q}), \dots, x_m - f_m(\boldsymbol{q}) \rangle.$$

Indeed, observe that

$$\begin{aligned}
\varphi(\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})}) &= \langle \varphi(x_1) - f_1(\boldsymbol{q}), \dots, \varphi(x_m) - f_m(\boldsymbol{q}) \rangle \\
&= \langle f_1 - f_1(\boldsymbol{q}), \dots, f_m - f_m(\boldsymbol{q}) \rangle \\
&\subseteq \mathfrak{n}_{\boldsymbol{q}}.
\end{aligned}$$

This shows that $\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} \subseteq \varphi^{-1}(\mathfrak{n}_{\boldsymbol{q}})$. We get the reverse inclusion from the fact that $\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})}$ is a maximal ideal of A. More generally, let $\mathfrak{n}$ be an arbitary maximal ideal of $\Bbbk[\boldsymbol{y}]$. Then there exists a maximal ideal of the form $\mathfrak{n}_{\boldsymbol{q}}$ of $\overline{\Bbbk}[\boldsymbol{y}]$, where $\boldsymbol{q} \in \overline{\Bbbk}^n$, which lies over $\mathfrak{n}$. Furthermore, there are only finitely many maximal ideals of $\overline{\Bbbk}[\boldsymbol{y}]$ which lie over $\mathfrak{n}$ and they all have the form $\mathfrak{n}_{\sigma\boldsymbol{q}}$ for some $\sigma \in \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ where $\sigma\boldsymbol{q} = (\sigma q_1, \dots, \sigma q_n)$ (this follows from a general proposition in commutative algebra which we state and proof at the end of this entry below). Then we have

$$\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} \cap \Bbbk[\boldsymbol{x}] := \mathfrak{m}.$$

Note this does not depend on the choice of maximal ideal which lies over $\mathfrak{n}$, for if $\mathfrak{n}_{\sigma\boldsymbol{q}}$ where another maximal ideal of $\overline{\Bbbk}[\boldsymbol{y}]$ which lies over $\mathfrak{n}$, then $\mathfrak{m}_{\boldsymbol{f}(\sigma\boldsymbol{q})} = \mathfrak{m}_{\sigma\boldsymbol{f}(\boldsymbol{q})}$ also lies over $\mathfrak{m}$.

Example 2.3. The maximal ideals $\mathfrak{n}_{i,\zeta_8}$, $\mathfrak{n}_{i,\zeta_8^5}$, $\mathfrak{n}_{-i,\zeta_8^3}$, and $\mathfrak{n}_{-i,\zeta_8^7}$ lie over $\mathfrak{n} = \langle y_1^2 + 1, y_2^2 + y_1 \rangle$.

Proposition 2.1. *Let A be an integral domain which is integrally closed in its field of fractions K, let L be a normal extension of K, let B be the integral closure of A in L, let G be the group of automorphisms of L over K, and let $\mathfrak{p}$ be a prime ideal of A. Then G acts transitively on the set of all primes of B which lie over $\mathfrak{p}$.*

Proof. We first consider the case where G is finite. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two prime ideals of B which lie over $\mathfrak{p}$. Then the $\sigma\mathfrak{q}$ (where $\sigma \in G$) is an ideal of B which lies over $\mathfrak{p}$ since B is integrally closed in L, and it suffices to show that $\mathfrak{q}'$ is contained in one of them, or equivalently, in their union by prime avoidance. Let $y \in \mathfrak{q}'$ and let $x = \prod \sigma y$ where the product runs over $\sigma \in G$. Note that x is fixed by G, thus since L/K is normal, it follows that there exists a power q of the characteristic of K such that $x^q \in K$. In particular, $x^q \in K \cap B = A$ since A is integrally closed. Thus $x^q \in \mathfrak{q}' \cap A = \mathfrak{p}$, which shows that x^q is contained in $\mathfrak{q}$. It follows that there exists a $\sigma \in G$ such that $\sigma y \in \mathfrak{q}$, whence $y \in \sigma^{-1}\mathfrak{q}$.

For the general case, assume $\mathfrak{q}$ and $\mathfrak{q}'$ lie over $\mathfrak{p}$. For every subfield E of L which is a finite normal extension over K, let G_E be the subset of G which consists of all $\sigma \in G$ which transform $\mathfrak{q} \cap E$ to $\mathfrak{q}' \cap E$. This is a closed subspace of G, hence compact since G is compact. Furthermore, each G_E is non-empty by what was shown above. As the G_E form a decreasing filtered family, their intersection is non-empty. □

2.4 5/21/2024 - Turning $\mathrm{Tor}^R(M_1, M_2)$ into an R-complex

Let R be a commutative ring, let M_1 and M_2 be R-modules, and set $T = \mathrm{Tor}^R(M_1, M_2)$. We can turn T into an R-complex as follows: choose projective resolutions F^1 of M_1 and F^2 of M_2 over R. Then $\mathrm{d} \otimes 1 \colon F^1 \otimes_R F^2 \to F^1 \otimes_R F^2$ is a chain map of degree -1, thus it induces a map in homology $\mathrm{d} \otimes 1 \colon T \to T$. Furthermore $(\mathrm{d} \otimes 1)^2 = 0$ and so $\mathrm{d} \otimes 1$ gives T an R-complex structure. There are map $\gamma_i^{31} \colon T_i^{31} \to T_{i-1}^{31}$ defined to be the composite

$$T_i^{31} \to T_i^{32} \to T_{i-1}^{12} \to T_{i-1}^{13} = T_{i-1}^{31}.$$

Similarly, we define $\gamma_i^{32} \colon T_i^{32} \to T_{i-1}^{32}$ to be the composite

$$T_i^{32} \to T_{i-1}^{12} \to T_{i-1}^{13} \to T_{i-1}^{23} = T_{i-1}^{32},$$

and we define $\gamma_i^{21} \colon T_i^{21} \to T_{i-1}^{21}$ to be the composite

$$T_i^{21} \to T_i^{31} \to T_i^{32} \to T_{i-1}^{12} = T_{i-1}^{21}$$

Actually I just realized these are all just the zero map.

2.5 5/29/2024 - Ext result of my paper

Proposition 2.2. *Let R be a regular local ring, let I be an ideal of R, let F be the minimal free resolution of R/I over R, and let $S = \mathrm{S}_R(F)$ be the symmetric DG algebra of F over R. There exists a surjective chain map $\pi\colon S \twoheadrightarrow F$ which splits the inclusion map $F \hookrightarrow S$.*

Proof. It suffices to show that $\mathrm{Ext}^1_R(S/F, F) = 0$. Note that the underlying graded R-module of S/F is just $S^{\geq 2}$. In particular, S/F is semi-projective, thus $\mathrm{Hom}^\star_R(S/F, -)$ preserves quasi-isomorphisms. It follows that

$$\mathrm{Ext}^1_R(S/F, F) = \mathrm{Ext}^1_R(S/F, R/I) = 0,$$

where the last part follows from the fact that R/I sits in homological degree 0 but $(S/F)_i = 0$ for all $i \leq 1$. □

Remark 3. Note that giving a surjective chain map $\pi\colon S \twoheadrightarrow F$ which splits the inclusion map is equivalent to giving chain maps $\pi^n\colon F^{\otimes n} \to F$ for each $n \geq 2$ such that each π^n is strictly commutative and such that for all $1 \leq i \leq n$ and for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in F_+$ we have

$$\pi^n(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n) = \pi^{n-1}(a_1, \ldots, a_{i-1}, a_i, \ldots, a_n).$$

For instance, if a_1, a_2, a_3 are homogeneous elements in F with $|a_1| = 1$ and $|a_2|, |a_3| \geq 2$, then we have

$$\mathrm{d}\pi^3(a_1, a_2, a_3) = r_1\pi^2(a_2, a_3) - \pi^3(a_1, \mathrm{d}a_2, a_3) + \pi^3(a_1, a_2, \mathrm{d}a_3),$$

where $r_1 = \mathrm{d}a_1$.

2.6 6/15/2024 - Associated primes of $\mathrm{Hom}_R(M, N)$

Today we prove the following result:

Proposition 2.3. *Let R be a noetherian ring and let M and N be R-modules such that M is finitely generated. Then*

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}\, M \cap \mathrm{Ass}\, N = \mathrm{V}(\mathrm{Ann}\, M) \cap \mathrm{Ass}\, N.$$

Proof. Let $\mathfrak{p}$ be an associated prime of $\mathrm{Hom}_R(M, N)$. Thus there exists an R-linear map $\varphi\colon M \to N$ such that $\mathfrak{p} = 0 : \varphi = \{a \in R \mid a\varphi = 0\}$. Let $u_1, \ldots, u_m$ be generators of M as an R-module and let $v_1, \ldots, v_m \in N$ be their respective images under φ. Then note that $a\varphi = 0$ if and only if $av_i = 0$ for all $1 \leq i \leq m$.

$$\begin{aligned} a \in \mathfrak{p} &\iff a\varphi = 0 \\ &\iff av_i = 0 \text{ for all } i \\ &\iff a \in \bigcap_{i=1}^m 0 : v_i. \end{aligned}$$

In particular we see that $\mathfrak{p} = \bigcap_{i=1}^m 0 : v_i$. Since $\mathfrak{p}$ is prime, we see that $\mathfrak{p} = 0 : v_i$ for some i, or in other words, $\mathfrak{p}$ is an associated prime of N. Next, assume for a contradiction that $M_\mathfrak{p} = 0$. Then for each i there exists an $s_i \in R \backslash \mathfrak{p}$ such that $s_i u_i = 0$. However this implies $s = s_1 \cdots s_n$ is in $\mathfrak{p}$ since $sv_i = \varphi(su_i) = 0$ for all i, which is a contradiction. Therefore $\mathfrak{p}$ is in the support of M. Thus far we have shown

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) \subseteq \mathrm{Supp}\, M \cap \mathrm{Ass}\, N.$$

For the converse direction, suppose $\mathfrak{p}$ is in the support of M and is an associated prime of N, so $M_\mathfrak{p} \neq 0$ and $\mathfrak{p} = 0 : v$ for some $v \in N$. Since $M_\mathfrak{p} \neq 0$, there exists an i such that $0 : u_i \subseteq \mathfrak{p} = 0 : v$. By reordering if necessary, we may assume that $0 : u_1 \subseteq \mathfrak{p} = 0 : v$. One would like to define an R-linear map $\varphi\colon M \to N$ such that $\varphi(u_1) = v$, but it's not clear how we should define it on the u_i for all $2 \leq i \leq m$. Let us cut to the chase and show how one usually proves this result: we have

$$\begin{aligned} \mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N)) &\iff \mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N)_\mathfrak{p}) \\ &\iff \mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \\ &\iff \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \neq 0 \\ &\iff \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}, N_\mathfrak{p}) \neq 0 \\ &\iff M_\mathfrak{p} \neq 0 \text{ and } \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), N_\mathfrak{p}) \neq 0 \\ &\iff \mathfrak{p} \in \mathrm{Supp}\, M \cap \mathrm{Ass}\, N, \end{aligned}$$

where in the second last if and only if we used the fact that $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is a finite dimensional $\kappa(\mathfrak{p})$ (so it is a direct sum of $\kappa(\mathfrak{p})$'s). Note that we needed Nakayama's lemma for the statement $M_\mathfrak{p} \neq 0$ if and only if $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0$, hence why we needed a noetherian hypothesis on R. The last equality comes from the fact that since M is finitely generated, we have $\operatorname{Supp} M = \mathrm{V}(\operatorname{Ann} M)$. □

Corollary 2. *Let R be a noetherian domain, let M be a finitely generated R-module, and let $M^\vee := \operatorname{Hom}_R(M, R)$ be the dual of M. If $M^\vee \neq 0$, then $\operatorname{Ass} M^\vee = \{0\}$.*

Remark 4. Note that if L and M are finitely generated R-modules, then tensor-hom adjointness implies

$$\begin{aligned}
\mathrm{V}(\operatorname{Ann}(L \otimes_R M)) \cap \operatorname{Ass} N &= \operatorname{Supp}(L \otimes_R M) \cap \operatorname{Ass} N \\
&= \operatorname{Ass}(\operatorname{Hom}_R(L \otimes_R M, N)) \\
&= \operatorname{Ass}(\operatorname{Hom}_R(L, \operatorname{Hom}_R(M, N))) \\
&= (\operatorname{Supp} L) \cap (\operatorname{Supp} M) \cap \operatorname{Ass} N \\
&= \mathrm{V}(\langle \operatorname{Ann} L, \operatorname{Ann} M \rangle) \cap \operatorname{Ass} N
\end{aligned}$$

for all R-modules N. In particular, we have

$$\mathrm{V}(\operatorname{Ann}(L \otimes_R M)) = \mathrm{V}(\operatorname{Ann} L) \cap \mathrm{V}(\operatorname{Ann} M) = \mathrm{V}(\langle \operatorname{Ann} L, \operatorname{Ann} M \rangle).$$

2.7 6/25/2024 - Inverse limit of $\cdots \to R \xrightarrow{r} R \xrightarrow{r} R$

Today I want to dicuss a result I was thinking about while driving to my parents house the other day. Let R be a ring and let $r \in R$. Consider the inverse system:

$$\mathcal{R} = \cdots \to R \xrightarrow{r} R \xrightarrow{r} R.$$

We set $A = \lim \mathcal{R}$. Then A consists of the set of all sequences (a_n) where $a_n \in R$ such that $r^m a_n = a_{n-m}$ for all $0 \leq m \leq n$. If R is an integral domain, then we can equivalently describe this as the set of all sequences (a_n) such that $r^n a_n = a_0$ for all $0 \leq n$. In particular, if $(a_n) \in A$, then we must have

$$a_m \in \bigcap_{n=1}^{\infty} \langle r \rangle^n := I.$$

for all $m \in \mathbb{N}$. Thus if $I = 0$, then necessarily $A = 0$. Krull's intersection theorem gives us $I = 0$ for many important rings that we care about. For example, if R is a noetherian local ring with maximal ideal $\mathfrak{m}$ and $r \in \mathfrak{m}$, then $I = 0$. Thus the inverse limit of the inverse system $\mathcal{R}$ would be 0 in this case. On the other hand, consider the direct system:

$$\mathcal{S} = R \xrightarrow{r} R \xrightarrow{r} R \to \cdots.$$

Then we have $R_r = \operatorname{colim} \mathcal{S}$. We have $R_r = 0$ if and only if r is nilpotent.

2.8 7/28/2024 - If $\mathrm{Z}G = 1$, then $\mathrm{Z}(\operatorname{Aut} G) = 1$

Here's a neat proposition in Group Theory that I proved involving the automorphism group of a centerless group.

Proposition 2.4. *Let G be a group such that $\mathrm{Z}G = 1$ and let $A = \operatorname{Aut} G$ be the automorphism group of G. The only automorphism of G which commutes with every inner automorphism of G is the identity automorphism. In particular, we have $\mathrm{Z}A = 1$.*

Proof. Suppose φ is an automorphism of G which commutes with every inner automorphism of G. Thus we have

$$\mathrm{c}_g \varphi = \varphi \mathrm{c}_g = \mathrm{c}_{\varphi g} \varphi$$

for all $g \in G$, or in other words, we have

$$g\varphi(x)g^{-1} = \varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$$

for all $x, g \in G$. Replacing x with $\varphi^{-1}x$ above and rearranging terms, we see that

$$(\varphi g)^{-1} g x = x (\varphi g)^{-1} g$$

for all $x, g \in G$. Since $\mathrm{Z}G = 1$, we must have $(\varphi g)^{-1} g = 1$, on in other words, $\varphi g = g$ for all $g \in G$. It follows that $\varphi = 1$. □

2.9 8/18/2024 - flatness and projectiveness are stable under composition

Today I updated the 5/20/2024 entry. In today's entry, I want to prove the following:

Proposition 2.5. *Let $A \to B$ be a ring homomorphism and let C be a B-module.*

1. *If B is A-flat and C is B-flat, then C is A-flat.*
2. *If B is A-projective and C is B-projective, then C is A-projective.*

Proof. Suppose $M \rightarrowtail M'$ is an injective A-module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{tikzcd} C \otimes_A M \arrow[r] \arrow[d, "\simeq"] & C \otimes_A M' \arrow[d, "\simeq"] \\ (C \otimes_B B) \otimes_A M \arrow[r] \arrow[d, "\simeq"] & (C \otimes_B B) \otimes_A M' \arrow[d, "\simeq"] \\ C \otimes_B (B \otimes_A M) \arrow[r, tail] & C \otimes_B (B \otimes_A M') \end{tikzcd}$$

The bottom arrow is injective since B is A-flat and C is B-flat. Therefore $C \otimes_A M \rightarrowtail C \otimes_A M'$ is injective; whence C is A-flat.

Now suppose that $M \twoheadrightarrow M'$ is a surjective A-module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{tikzcd} \mathrm{Hom}_A(C, M) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(C, M') \arrow[d, "\simeq"] \\ \mathrm{Hom}_A(C \otimes_B B, M) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(C \otimes_B B, M') \arrow[d, "\simeq"] \\ \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M)) \arrow[r, two heads] & \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M')) \end{tikzcd}$$

The bottom arrow is surjective since B is A-projective and C is B-projective. Therefore $\mathrm{Hom}_A(C, M) \rightarrowtail \mathrm{Hom}_A(C, M')$ is surjective; whence C is A-projective. $\square$

2.10 8/24/2024 - Connected integral domain has stalkwise local property

Proposition 2.6. *Let R be a connected commutative ring. Then R is an integral domain if and only if $R_{\mathfrak{p}}$ is an integral domain for each prime $\mathfrak{p}$ of R.*

The reason we need R to be connected is because the ring $R = K \times K$ where K is a field is clearly not an integral domain but the localization at each prime of R is isomorphic to K which is an integral domain.

2.11 8/30/2024 - Example

Today we study the following: let $\Bbbk$ be a field with characteristic $\neq 2$, let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, x_2]$, let $A = R[\boldsymbol{a}] = R[a_1, a_2, a_{11}^1, a_{11}^2, a_{12}^1, a_{12}^2, a_{22}^1, a_{22}^2]$, and let $B = A[\boldsymbol{e}]/\boldsymbol{f} = A[e_1, e_2]/\langle f_1, f_{11}, f_{12}, f_{22}\rangle$ where

$$\begin{aligned} f_1 &= -1 + a_1e_1 + a_2e_2, \\ f_{11} &= -e_1^2 + a_{11}^1e_1 + a_{11}^2e_2 \\ f_{12} &= -e_1e_2 + a_{12}^1e_1 + a_{12}^2e_2 \\ f_{22} &= -e_2^2 + a_{22}^1e_1 + a_{22}^2e_2 \end{aligned}$$

The Jacobian of B/A is given by

$$\mathrm{J}_{B/A} = \begin{pmatrix} a_1 & a_2 \\ a_{11}^1 - 2e_1 & a_{11}^2 \\ a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix},$$

and the Jacobian of B/R is given by

$$\mathrm{J}_{B/R} = \begin{pmatrix} e_1 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & e_1 & e_2 & 0 & 0 & 0 & 0 & a_{11}^1 - 2e_1 & a_{11}^2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & 0 & 0 & a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix}.$$

Let $\mathfrak{p}_{\boldsymbol{r}}$ be the prime ideal of A given by $\mathfrak{p}_{\boldsymbol{r}} = \langle \{a_{ij}^k - r_{ij}^k, a_i - r_i\} \rangle$ where $\boldsymbol{r} = (r_{11}^1, r_{11}^2, r_{12}^1, r_{12}^2, r_{22}^1, r_{22}^2, r_1, r_2) \in R^8$. Observe that

$$\begin{aligned} [e_i, e_j, e_k] &= (e_i e_j)e_k - e_i(e_j e_k) \\ &= \sum_l (a_{i,j}^l e_k e_l - a_{j,k}^l e_i e_l) \\ &= \sum_m \sum_l (a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m) e_m \\ &= \sum_m \sum_l b_{ijk}^{lm} e_m \\ &= \sum_m b_{ijk}^m e_m, \end{aligned}$$

where we set $b_{ijk}^{lm} = a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m$ and $b_{ijk}^m = \sum_l b_{ijk}^{lm}$. Let $J = \mathrm{J}_{B/A}(0)$. Then we have

$$b_{ijk}^{lm} - b_{ijk}^{ml} = J_{ij,kl}^{l,m} - J_{ij,kl}^{m,l}.$$

In particular, note that

$$b_{ijk}^m - b_{ijk}^l = \sum_l b_{ijk}^{lm} - \sum_m b_{ijk}^{ml}$$

Thus for instance we have

$$b_{112}^1 = a_{11}^1 a_{12}^1 - a_{12}^1 a_{11}^1 + a_{11}^2 a_{22}^1 - a_{12}^2 a_{12}^1$$

We have

$$1 = a_1 e_1 + a_2 e_2$$

implies

$$e_1 = a_1(a_{11}^1 e_1 + a_{11}^2 e_2) + a_2(a_{12}^1 e_1 + a_{12}^2 e_2)$$

imlpies

$$e_1 = (a_1 a_{11}^1 + a_2 a_{12}^1)e_1 + (a_1 a_{11}^2 + a_2 a_{12}^2)e_2 = b_1 e_1 + b_2 e_2$$

imlpies $(1 - b_1)e_1 = b_2 e_2$. We'd like to show that

$$e_1 = f + g(c_1 e_1 + c_2 e_2)$$

Suppose we have

$$\begin{aligned} e_1 &= a_{11} + a_{12}(c_1 e_1 + c_2 e_2) \\ e_2 &= a_{21} + a_{22}(c_1 e_1 + c_2 e_2) \end{aligned}$$

Rearranging terms, this implies

$$\begin{aligned} (1 - a_{12}c_1)e_1 - a_{12}c_2 e_2 &= a_{11} \\ (1 - a_{22}c_2)e_2 - a_{22}c_1 e_1 &= a_{21} \end{aligned}$$

This implies

$$a_{21}(1 - a_{12}c_1)e_1 - a_{21}a_{12}c_2 e_2 - a_{11}(1 - a_{22}c_2)e_2 + a_{11}a_{22}c_1 e_1 = 0$$

$$(a_{21}(1 - a_{12}c_1) + a_{11}a_{22}c_1)e_1 + (-a_{11}(1 - a_{22}c_2) - a_{21}a_{12}c_2)e_2 = 0$$

$$e_1 = a_{11}$$

$$ra_1 + xa_2$$

2.12 9/7/2024 - Example

Today we study the following: let $\Bbbk$ be a field with characteristic $\neq 2$, let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, x_2]$, let $A = R[\boldsymbol{a}] = R[a_1, a_2, a_{11}^1, a_{11}^2, a_{12}^1, a_{12}^2, a_{22}^1, a_{22}^2]$, and let $B = A[\boldsymbol{e}]/\boldsymbol{f} = A[e_1, e_2]/\langle f_1, f_{11}, f_{12}, f_{22}\rangle$ where

$$
\begin{aligned}
f_1 &= -1 + a_1 e_1 + a_2 e_2, \\
f_{11} &= -e_1^2 + a_{11}^1 e_1 + a_{11}^2 e_2 \\
f_{12} &= -e_1 e_2 + a_{12}^1 e_1 + a_{12}^2 e_2 \\
f_{22} &= -e_2^2 + a_{22}^1 e_1 + a_{22}^2 e_2
\end{aligned}
$$

The Jacobian of B/A is given by

$$
\mathrm{J}_{B/A} = \begin{pmatrix} a_1 & a_2 \\ a_{11}^1 - 2e_1 & a_{11}^2 \\ a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix},
$$

and the Jacobian of B/R is given by

$$
\mathrm{J}_{B/R} = \begin{pmatrix} e_1 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & e_1 & e_2 & 0 & 0 & 0 & 0 & a_{11}^1 - 2e_1 & a_{11}^2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & 0 & 0 & a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix}.
$$

Let $\mathfrak{p}_{\boldsymbol{r}}$ be the prime ideal of A given by $\mathfrak{p}_{\boldsymbol{r}} = \langle \{a_{ij}^k - r_{ij}^k, a_i - r_i\}\rangle$ where $\boldsymbol{r} = (r_{11}^1, r_{11}^2, r_{12}^1, r_{12}^2, r_{22}^1, r_{22}^2, r_1, r_2) \in R^8$. Observe that

$$
\begin{aligned}
[e_i, e_j, e_k] &= (e_i e_j) e_k - e_i (e_j e_k) \\
&= \sum_l (a_{i,j}^l e_k e_l - a_{j,k}^l e_i e_l) \\
&= \sum_m \sum_l (a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m) e_m \\
&= \sum_m \sum_l b_{ijk}^{lm} e_m \\
&= \sum_m b_{ijk}^m e_m,
\end{aligned}
$$

where we set $b_{ijk}^{lm} = a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m$ and $b_{ijk}^m = \sum_l b_{ijk}^{lm}$. Let $J = \mathrm{J}_{B/A}(0)$. Then we have

$$
b_{ijk}^{lm} - b_{ijk}^{ml} = J_{ij,kl}^{l,m} - J_{ij,kl}^{m,l}.
$$

In particular, note that

$$
b_{ijk}^m - b_{ijk}^l = \sum_l b_{ijk}^{lm} - \sum_m b_{ijk}^{ml}
$$

Thus for instance we have

$$
b_{112}^1 = a_{11}^1 a_{12}^1 - a_{12}^1 a_{11}^1 + a_{11}^2 a_{22}^1 - a_{12}^2 a_{12}^1
$$

We have

$$
1 = a_1 e_1 + a_2 e_2
$$

implies

$$
e_1 = a_1(a_{11}^1 e_1 + a_{11}^2 e_2) + a_2(a_{12}^1 e_1 + a_{12}^2 e_2)
$$

imlpies

$$
e_1 = (a_1 a_{11}^1 + a_2 a_{12}^1) e_1 + (a_1 a_{11}^2 + a_2 a_{12}^2) e_2 = b_1 e_1 + b_2 e_2
$$

imlpies $(1 - b_1) e_1 = b_2 e_2$. We'd like to show that

$$
e_1 = f + g(c_1 e_1 + c_2 e_2)
$$

Suppose we have

$$\begin{aligned} e_1 &= a_{11} + a_{12}(c_1 e_1 + c_2 e_2) \\ e_2 &= a_{21} + a_{22}(c_1 e_1 + c_2 e_2) \end{aligned}$$

Rearranging terms, this implies

$$\begin{aligned} (1 - a_{12}c_1)e_1 - a_{12}c_2e_2 &= a_{11} \\ (1 - a_{22}c_2)e_2 - a_{22}c_1e_1 &= a_{21} \end{aligned}$$

This implies

$$\begin{aligned} a_{21}(1 - a_{12}c_1)e_1 - a_{21}a_{12}c_2e_2 - a_{11}(1 - a_{22}c_2)e_2 + a_{11}a_{22}c_1e_1 &= 0 \\ (a_{21}(1 - a_{12}c_1) + a_{11}a_{22}c_1)e_1 + (-a_{11}(1 - a_{22}c_2) - a_{21}a_{12}c_2)e_2 &= 0 \\ e_1 &= a_{11} \end{aligned}$$

2.13 9/13/2024 - Determinants, Traces, and Free Resolutions

Let R be a commutative ring, let M be a projective stably free R-module, and let $\varphi\colon M \to M$ be an R-linear map. In particular, M admits a consisting of finite rank free modules. Let F be such a resolution. The map $\varphi\colon M \to M$ lifts to a chain map $\widetilde{\varphi}\colon F \to F$. For each i we set $\delta_i = \det \widetilde{\varphi}_i$ and we set $\tau_i = \operatorname{tr} \widetilde{\varphi}_i$ and we define

$$\delta := \prod_i \delta_i^{(-1)^i} \quad \text{and} \quad \tau := \sum_i (-1)^i \tau_i.$$

On the other hand, M is locally free, so there exists elements $s_1, \dots, s_n$ in R such that $\langle s_1, \dots, s_n \rangle = 1$ and $M_k := M_{s_k}$ is a free module over $R_k := R_{s_k}$ for all $1 \leq k \leq n$. The map $\varphi\colon M \to M$ induces an R-linear map $\varphi_k\colon M_k \to M_k$ for each k. For each k we set $d_k = \det \varphi_k$ and $t_k = \operatorname{tr} \varphi_k$. It is easy to see that for each $1 \leq k, k' \leq n$ we have $d_k = d_{k'}$ and $t_k = t_{k'}$ in $R_{k,k'} := R_{s_k s_{k'}}$. Therefore they glue to unique elements d and t in R.

Proposition 2.7. *With the notation as above, we have $d = \delta$ and $t = \tau$.*

Proof. It suffices to show that $\delta_k = d_k$ and $\tau_k = t_k$ for each k where δ_k and τ_k are the images of δ and τ in R_k. In this case, M_k is free and the augmented complex obtained by adjoining M_k in homological degree -1 to F is an exact complex consisting of finite free modules. By replacing R with R_k if necessary, we are reduced to the following problem: assume F is an exact complex of finite length consisting of finite free R-modules and let $\varphi\colon F \to F$ be a chain map. Then we have

$$1 = \prod_i \delta_i^{(-1)^i} \quad \text{and} \quad 0 = \sum_i (-1)^i \tau_i.$$

First we assume that $F_i = 0$ for all $i \in \mathbb{Z} \backslash \{0, 1, 2\}$. In this case, the chain map $\varphi\colon F \to F$ looks like

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r] \arrow[d, "\varphi_2"] & F_1 \arrow[r] \arrow[d, "\varphi_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ 0 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0 \end{tikzcd} \tag{3}$$

and we need to show that $\delta_1 = \delta_0 \delta_2$ and $\tau_1 = \tau_0 + \tau_2$. This short exact sequence splits and can be made to look like as below:

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r, "\iota"] \arrow[d, "\varphi_2"] & F_2 \oplus F_1 \arrow[r, "\pi"] \arrow[d, "\widehat{\varphi}_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ 0 \arrow[r] & F_2 \arrow[r, "\iota"] & F_2 \oplus F_1 \arrow[r, "\pi"] & F_0 \arrow[r] & 0 \end{tikzcd} \tag{4}$$

where $\iota\colon F_2 \to F_2 \oplus F_0$ and $\pi\colon F_2 \to F_2 \oplus F_0$ are the obvious inclusion and projection maps and where $\widehat{\varphi}_1$ satisfies $\delta_1 = \det \widehat{\varphi}_1$. Furthermore, the matrix represenation of $\widehat{\varphi}_1$ has the form

$$[\varphi_1] = \begin{pmatrix} [\varphi_2] & 0 \\ 0 & [\varphi_0] \end{pmatrix},$$

and so clearly we have $\delta_1 = \delta_0\delta_2$ in this case. Now suppose that $\varphi\colon F \to F$ starts out like

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \widetilde{\varphi_2}} & & \big\downarrow{\scriptstyle \varphi_1} & & \big\downarrow{\scriptstyle \varphi_0} & & \\ 0 & \longrightarrow & L & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \end{array} \tag{5}$$

where L is not necessarily free. Then an argument by induction of the length of the free resolution gives us the result. $\square$