Goldbach Rings

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Abstract

Let $\Bbbk$ be a field. We introduce and study an interesting infinite-dimensional $\Bbbk$-algebra G which we call the Goldbach ring. As the name suggests, the Goldbach ring is closely related to Goldbach's conjecture. Properties that G satisfies as a ring (such as whether or not it is an integral domain) may give us clues about Goldbach's conjecture itself.

1 Introduction

Let $\Bbbk$ be a field. We introduce and study an interesting infinite-dimensional $\Bbbk$-algebra which we call the Goldbach ring, which, as the name suggests, is closely related to Goldbach's conjecture. The Goldbach ring G is defined to be the quotient $G = R/I$ where

$$\begin{aligned} R &= \Bbbk[\{x_p, x_{p+q} \mid p, q \text{ odd primes}\}] \\ I &= \langle\{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\}\rangle \end{aligned}$$

The Goldbach ring has the structure of a bi-graded $\Bbbk$-algebra meaning it can be decomposed as

$$G = \bigoplus_{n,d\geq 0} G_{n,d},$$

where the component $G_{n,d}$ in bi-degree $(n, d) \in \mathbb{N}^2$ is a finite-dimensional $\Bbbk$-vector space whose dimension we are interested in counting. For instance, Goldbach's conjecture is equivalent to the statement that $\dim_\Bbbk G_{2k,2} = 1$ for all $k \geq 3$. However this is really just a restatment of Goldbach's conjecture; what's more interesting and new in our view is the following conjecture which seems to hold in small examples:

Conjecture 1. *We have*

$$\dim_\Bbbk G_{n,d} \leq 1$$

for all $n, d \in \mathbb{N}$.

A counter-example to Conjecture (1) would be the existence of odd primes $p_1, \dots, p_d$ and $q_1, \dots, q_d$ such that

$$p_1 + \cdots + p_d = n = q_1 + \cdots + q_d$$

but $x_{p_1} \cdots x_{p_d} \neq x_{q_1} \cdots x_{q_d}$ in G. However we do not believe such a counter-example exists since. Indeed, based on our initial calculations, it seems that there are usually many ways to go from $x_{p_1} \cdots x_{p_d}$ to $x_{q_1} \cdots x_{q_d}$ by applying elementary Goldbach relations of the form $x_p x_q = x_{p+q}$. For another example, in $G_{36,4}$ we have $x_3^2 x_{11} x_{19} = x_5^2 x_{13}^2$ since

$$\begin{aligned} x_3^2 x_{11} x_{19} &= x_3 x_{11} x_{22} \\ &= x_3 x_5 x_{11} x_{17} \\ &= x_5 x_{11} x_{20} \\ &= x_5 x_7 x_{11} x_{13} \\ &= x_5 x_{13} x_{18} \\ &= x_5^2 x_{13}^2. \end{aligned}$$

There are many other paths we can take from $x_3^2 x_{11} x_{19}$ to $x_5^2 x_{13}^2$, however it turns out that this is the shortest path. Ultimately a solution to Conjecture (1) will involve tools and techniques from analytic number theory. What we find interesting is that Conjecture (1) also seems to involve a lot of commutative algebra as well. For example, if Conjecture (1) is true, then it woud imply that G is an integral domain. Conversely, one can show that if G is an integral domain and Conjecture (1) holds for n, d sufficiently large, then Conjecture (1) is true.

A deeper relationship between Conjecture (1), anlaytic number theory, and commutative algebra is realized when one studies G as a direct limit

$$G = \varinjlim G^m$$

of bi-graded noetherian $\Bbbk$-algebras $G^m = R^m/I^m$, where

$$\begin{aligned} R^m &= \Bbbk[x_1, \dots, x_m] \cap R \\ I^m &= \Bbbk[x_1, \dots, x_m] \cap I. \end{aligned}$$

Indeed, for each m, we denote by $\delta(m)$ and $\rho(m)$ to be the R^m-depth and R^m-projective dimension of G^m respectively. Then the Auslander-Buchsbaum formula implies

$$\delta(2m) + \rho(2m) = \pi(2m) + m - \kappa(2m) - 3, \tag{1}$$

where $\pi(2m)$ is the usual prime-counting function which counts the number of primes $\leq 2m$ and where $\kappa(2m)$ counts then number of positive even numbers $\leq 2m$ that are coutner-examples to Goldbach's conjecture.

2 $\mathcal{A}$-Supported Goldbach Rings

Let $\mathcal{A}$ be a subset of the positive odd integers and set $\mathcal{C} := \mathcal{A} + \mathcal{A} = \{a + b \mid a, b \in \mathcal{A}\}$. We set

$$\begin{aligned} R_{\mathcal{A}} &= \Bbbk[\{x_a, x_c \mid a \in \mathcal{A}, c \in \mathcal{C}\}] \\ I_{\mathcal{A}} &= \langle \{x_a x_b - x_{a+b} \mid a, b \in \mathcal{A}\} \rangle \\ G_{\mathcal{A}} &= R_{\mathcal{A}}/I_{\mathcal{A}}. \end{aligned}$$

We will refer to $G_{\mathcal{A}}$ as the **$\mathcal{A}$-supported Goldbach ring**. We simplify our notation by writing $\{x_a, x_c\}$ to denote the set $\{x_a, x_c \mid a \in \mathcal{A}, c \in \mathcal{C}\}$. Similarly we write $\{x_a x_b - x_{a+b}\}$ to denote the set $\{x_a x_b - x_{a+b} \mid a, b \in \mathcal{A}\}$. We often simplify our notation even further by dropping $\mathcal{A}$ from our notation whenever it is clear from context. For instance, we write "G" instead of "$G_{\mathcal{A}}$" when it's understood that G is the $\mathcal{A}$-supported Goldbach ring. Similarly, if we write "let G be the $\mathcal{A}$-supported Golbach ring", then it's understood that $\mathcal{A}$ is a subset of the positive odd integers and that $\mathcal{C} = \mathcal{A} + \mathcal{A}$.

2.1 Representing Monomials

We will denote by $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$ to be the set of all monomials in $R = R_{\mathcal{A}}$. There are two ways we can represent monomials in R. The first way is as a finite product of the indeterminates $\{x_a, x_c\}$, namely, a monomial can be expressed in the form

$$x_{\boldsymbol{a}} x_{\boldsymbol{c}} := x_{a_1} \cdots x_{a_r} x_{c_1} \cdots x_{c_s}$$

where $\boldsymbol{a} = a_1, \dots, a_r$ is a sequence of elements in $\mathcal{A}$ (not necessarily distinct, but often we assume $a_1 \leq \cdots \leq a_r$) and $\boldsymbol{c} = c_1, \dots, c_s$ is a sequence of elements in $\mathcal{C}$ (again not necessarily distinct, but often we assume $c_1 \leq \cdots \leq c_s$). We will use this way of representing monomials to give R a nice bi-graded structure. The second way of representing monomials is described as follows: given a function $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, we define its **support**, denoted $\operatorname{supp} \alpha$, to be the set

$$\operatorname{supp} \alpha = \{m \in \mathbb{N} \mid \alpha(m) \neq 0\}.$$

We denote by $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$ to be the set

$$\mathcal{F} = \{\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0} \mid \operatorname{supp} \alpha \text{ is finite and contained in } \{x_a, x_c\}\}.$$

Thus if $\alpha \in \mathcal{F}$, then α takes value 0 zero almost everywhere, and the only places where it is nonzero is at an element in $\{x_a, x_c\}$. Then there is a bijection from $\mathcal{F}$ to $\mathcal{M}$ given by assigning $\alpha \in \mathcal{F}$ to the monomial

$$x^\alpha := \prod_{m \in \mathbb{N}} x_m^{\alpha(m)} = \prod_{m \in \operatorname{supp} \alpha} x_m^{\alpha(m)}.$$

For instance, suppose $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is defined by

$$\alpha(m) = \begin{cases} 3 & \text{if } m = 2 \\ 2 & \text{if } m = 6 \\ 4 & \text{if } m = 11 \\ 0 & \text{if } m \in \mathbb{N} \backslash \{2, 6, 11\} \end{cases}$$

Then $x^\alpha = x_3^3 x_6^2 x_{11}^4$ and $\operatorname{supp} x^\alpha = \{2, 6, 11\}$. This second way of expressing monimals gives us a cleaner way of expressing nonzero polynomials in R, namely, every nonzero polynomial $f \in R$ can be expressed in the form

$$f = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}$$

for unique $a_1, \dots a_n \in \Bbbk$ and for unique $\alpha_1, \dots, \alpha_n \in \mathcal{F}$. We often pass back and forth between functions $\alpha \in \mathcal{F}$ and monomials $x^\alpha \in \mathcal{M}$. For instance, given a monomial $x^\alpha \in \mathcal{M}$, we define its **support**, denoted $\operatorname{supp} x^\alpha$, to be $\operatorname{supp} x^\alpha = \operatorname{supp} \alpha$, and etc...

2.2 The Bi-Graded $\Bbbk$-Structure on R and G

We give R and G a bi-graded $\Bbbk$-structure as follows: we define $\deg_1 \colon \mathcal{M} \to \mathbb{N}$ and $\deg_2 \colon \mathcal{M} \to \mathbb{N}$ by

$$\deg_1(x_{\boldsymbol{a}} x_{\boldsymbol{c}}) = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} c_j \quad \text{and} \quad \deg_2(x_{\boldsymbol{a}} x_{\boldsymbol{c}}) = r + 2s.$$

In particular, we have $\deg_1(x_a) = a$, $\deg_1(x_c) = c$, $\deg_2(x_a) = 1$, and $\deg_2(x_c) = 2$. For each $n, d \in \mathbb{N}$, we set

$$R_{n,d} = \operatorname{span}_\Bbbk \{x^\alpha \in \mathcal{M} \mid \deg_1(x^\alpha) = n \text{ and } \deg_2(x^\alpha) = d\}.$$

Then we have a decomposition of R into $\Bbbk$-vector spaces:

$$R = \bigoplus_{n,d \in \mathbb{N}} R_{n,d},$$

which gives R a bi-graded $\Bbbk$-structure. Since I is homogeneous with respect to this bi-grading, G inherits a bi-graded $\Bbbk$-structure, induced by the one on R:

$$G = \bigoplus_{n,d \in \mathbb{N}} G_{n,d}.$$

Thus $\dim_\Bbbk R_{n,d}$ counts the number of ways we can express n as a sum

$$n = a_1 + \cdots + a_r + c_1 + \cdots + c_s$$

where $a_1, \dots, a_r \in \mathcal{A}$, $c_1, \dots, c_s \in \mathcal{C}$, and $d = r + s$. Whenever we have $\dim_\Bbbk R_{n,d} \geq 1$, then we say (n, d) is a **good pair**. In this case, we are very interested in determining whether or not $\dim_\Bbbk G_{n,d} = 1$ or $\dim_\Bbbk G_{n,d} > 1$. Intuitively, we have $\dim_\Bbbk G_{n,d} = 1$ when $\mathcal{A}$ is sufficiently "dense" in $\mathbb{N}$ and we have $\dim_\Bbbk G_{n,d} > 1$ whenever $\mathcal{A}$ is very "sparse" in $\mathbb{N}$.

2.3 Constructing the Minimal R-Free Resolution of G

We now build the minimal R-free resolution of G as follows: first, for each $m \geq 1$, we define the **m-th approximation** of R, I, and G to be:

$$\begin{aligned} R^m &= \Bbbk[\{x_a, x_c \mid a, c \leq m\}] \\ I^m &= \langle \{x_a x_b - x_{a+b} \mid a + b \leq m\} \rangle \\ G^m &= R^m / I^m. \end{aligned}$$

Again, R^m and G^m have bi-graded $\Bbbk$-structures:

$$R^m = \bigoplus_{n,d} R^m_{n,d} \quad \text{and} \quad G^m = \bigoplus_{n,d} G^m_{n,d}.$$

Recall that if $x_a x_c \in R^m_n$, then we must have $a + c = n$ and $a, c \leq m$. In particular, if $m \geq n$ then we have $R^m_n = R^n_n = R_n$. Similarly, if $m \geq n$ then we have $G^m_n = G^n_n = G_n$. Thus we have directed systems

$$(R^m)_{m \geq 1} \quad \text{and} \quad (G^m)_{m \geq 1}$$

of bi-graded $\Bbbk$-algebras (with the obvious $\Bbbk$-algebra homomorphisms) where the components $R^m_{n,d}$ and $G^m_{n,d}$ in bi-graded degree (n, d) stabilizes to $R_{n,d}$ and $G_{n,d}$ respectively whenever m is sufficiently large (for example $m \geq n$). It follows that

$$R = \varinjlim R^m \quad \text{and} \quad G = \varinjlim G^m$$

as bi-graded direct limits.

Next we let $F^m = F^m_{\mathcal{A}}$ be the minimal bi-graded R^m-free resolution of G^m (where F^m is necessarily finite since R^m and G^m are noetherian). We set

$$\delta(m) = \delta_{\mathcal{A}}(m) := \operatorname{depth}_{R^m} G^m \quad \text{and} \quad \rho(m) = \rho_{\mathcal{A}}(m) := \operatorname{pd}_{R^m} G^m = \operatorname{length} F^m.$$

Note that these quantities are intrinsic to R^m and G^m (and not R and G). By the Auslander-Buchsbaum formula we have

$$\rho(m) + \delta(m) = \pi_{\mathcal{A}\cup\mathcal{C}}(m) := \#\{a, c \in \mathcal{A} \cup \mathcal{C} \mid a, c \leq m\}. \tag{2}$$

Note that F^m has the structure of a bi-graded $\mathbb{k}$-complex, meaning we have a decomposition of $\mathbb{k}$-complexes:

$$F^m = \bigoplus_{n,d} F^m_{n,d},$$

where $F^m_{n,d}$ is a finite $\mathbb{k}$-subcomplex of F^m which minimially resolves $G^m_{n,d}$, meaning the augmented complex

$$\widetilde{F}^m_{n,d} := \cdots \to F^m_{i,n,d} \to F^m_{i-1,n,d} \to \cdots \to F^m_{1,n,d} \to R^m_{n,d} \to G^m_{n,d} \to 0$$

is exact and where the i-th Betti number of G^m in bi-degree (n,d) is given by

$$\beta^m_{i,n,d} := \dim_{\mathbb{k}} \operatorname{Tor}^{R^m}_i (G^m, \mathbb{k})_{n,d} = \dim_{\mathbb{k}}(F^m_{i,n,d}).$$

The canonical map $G^m \to G^{m+1}$ induces an injective comparison map $F^m \to F^{m+1}$ which we may choose to respect the bi-graded structure. Furthermore, since $R^m_n = R^n_n$ and $G^m_n = G^n_n$ whenever $m \geq n$, we see that $F^m_n = F^n_n$ whenever $m \geq n$. Thus if we *define* $F = F_{\mathcal{A}}$ to be the direct limit of bi-graded $\mathbb{k}$-complexes

$$F := \varinjlim F^m,$$

then F is a bi-graded R-free resolution of G which has the following bi-graded $\mathbb{k}$-complex structure:

$$F = \bigoplus_{n,d} F_{n,d} = \bigoplus_{n,d} F^n_{n,d}.$$

In particular, if m is sufficiently large, then we see that $\beta^m_{i,n,d} = \beta_{i,n,d}$ where

$$\beta_{i,n,d} := \dim_{\mathbb{k}} \operatorname{Tor}^R_i (G, \mathbb{k})_{n,d} = \dim_{\mathbb{k}}(F_{i,n,d})$$

is the ith Betti number of G in bi-degree (n,d). Thus, unlike the quantities $\delta(m)$ and $\rho(m)$, the quantitiy $\beta^m_{i,n,d}$ is actually *intrinsic* to R and G (and not just R^m and G^m) when m is sufficiently large.

Theorem 2.1. *We have* $\dim_{\mathbb{k}} G_{n,d} = \chi(F_{n,d})$. *In other words, we have*

$$\dim_{\mathbb{k}} G_{n,d} = \dim_{\mathbb{k}} R_{n,d} - \sum_{i=1}^{\infty} (-1)^i \beta_{i,n,d} = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{k}} \operatorname{Tor}^R_i (G, \mathbb{k})_{n,d}, \tag{3}$$

where the sum on the right (3) is finite.

3 The Goldbach Ring

We now consider the case where $\mathcal{A} = \{\text{positive odd primes}\}$. In this case, we have

$$\begin{aligned} R &= \mathbb{k}[\{x_p, x_{2k} \mid p \text{ odd prime and } k \in \mathbb{Z}_{\geq 3}\}] \\ I &= \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\} \rangle \\ G &= R/I. \end{aligned}$$

For obvious reasons, we call G the **Goldbach ring.** The homogeneous components of the form $R_{18,d}$ looks like:

$$\begin{aligned} \vdots &= \vdots \\ R_{18,7} &= 0 \\ R_{18,6} &= \mathbb{k}x_3^6 + \mathbb{k}x_3^4 x_6 + \mathbb{k}x_3^2 x_6^2 + \mathbb{k}x_6^3 \\ R_{18,5} &= 0 \\ R_{18,4} &= \mathbb{k}x_3^2 x_5 x_7 + \mathbb{k}x_3 x_5^3 + \mathbb{k}x_3^2 x_{12} + \cdots + \mathbb{k}x_5 x_6 x_7 + \mathbb{k}x_6 x_{12} + \mathbb{k}x_8 x_{10} \\ R_{18,3} &= 0 \\ R_{18,2} &= \mathbb{k}x_5 x_{13} + \mathbb{k}x_7 x_{11} + \mathbb{k}x_{18} \\ R_{18,1} &= 0 \\ \vdots &= \vdots \end{aligned}$$

Similarly, the homogeneous components of the form $R_{17,d}$ looks like:

$$\begin{aligned}
\vdots &= \vdots \\
R_{17,6} &= 0 \\
R_{17,5} &= \Bbbk x_3^4 x_5 + \Bbbk x_3^3 x_8 + \Bbbk x_3^2 x_5 x_6 + \Bbbk x_3 x_6 x_8 + \Bbbk x_5 x_6^2 \\
R_{17,4} &= 0 \\
R_{17,3} &= \Bbbk x_3^2 x_{11} + \Bbbk x_3 x_7^2 + \Bbbk x_5^2 x_7 + \Bbbk x_6 x_{11} + \Bbbk x_3 x_{14} + \Bbbk x_7 x_{10} + \Bbbk x_5 x_{12} \\
R_{17,2} &= 0 \\
R_{17,1} &= \Bbbk x_{17} \\
R_{17,0} &= 0 \\
\vdots &= \vdots
\end{aligned}$$

Staring at the homogeneous components above, we see that $\dim_\Bbbk R_{18,4} = 9$ and $\dim_\Bbbk R_{17,3} = 7$. More generally, $\dim_\Bbbk R_{n,d}$ counts the number of ways we can express n as a sum:

$$n = p_1 + \cdots + p_r + 2(k_1 + \cdots + k_s), \tag{4}$$

where $p_1, \dots, , p_r$ are odd primes, $k_1, \dots, k_s \geq 3$, and $d = r + 2s$. Here are some basic facts about $\Delta_{n,d}$:

1. Assume n is even.
$$\text{we have } \begin{cases} \dim_\Bbbk R_{n,d} \geq 1 & \text{if } d \text{ is even and } 2 \leq d \leq \lfloor n/3 \rfloor \\ \dim_\Bbbk R_{n,d} = 0 & \text{else} \end{cases}$$
Indeed, if d is even and satisfies $2 \leq d \leq \lfloor n/3 \rfloor$, then we have $\dim_\Bbbk R_{n,d} \geq 1$ since we have the decomposition $n = (n - 6d) + 6d$.

2. Assume n is odd.
$$\text{we have } \begin{cases} \dim_\Bbbk R_{n,d} \geq 1 & \text{if } d \text{ is odd and } 3 \leq d \leq \lfloor n/3 \rfloor \\ \dim_\Bbbk R_{p,1} = 1 & \text{if } p \text{ is odd prime} \\ \dim_\Bbbk R_{n,d} = 0 & \text{else} \end{cases}$$
Indeed, if d is odd and satisfies $3 \leq d \leq \lfloor n/3 \rfloor$, then we have $\dim_\Bbbk R_{n,d} \geq 1$ since we have the decomposition $n = (n - 3 - 6d) + 6d + 3$.

Next, the homogeneous components of the form $G_{17,d}$ and $G_{18,d}$ looks like:

$$\begin{aligned}
\vdots &= \vdots & \qquad \vdots &= \vdots \\
G_{17,6} &= 0 & G_{18,6} &= \Bbbk \overline{x}_3^6 \\
G_{17,5} &= \Bbbk \overline{x}_3^4 \overline{x}_5 & G_{18,5} &= 0 \\
G_{17,4} &= 0 & G_{18,4} &= \Bbbk \overline{x}_3^2 \overline{x}_5 \overline{x}_7 \\
G_{17,3} &= \Bbbk \overline{x}_3^2 \overline{x}_{11} & G_{18,3} &= 0 \\
G_{17,2} &= 0 & G_{18,2} &= \Bbbk \overline{x}_5 \overline{x}_{13} \\
G_{17,1} &= \Bbbk \overline{x}_{17} & G_{18,1} &= 0 \\
\vdots &= \vdots & \vdots &= \vdots
\end{aligned}$$

From what we've seen above, it is *very* tempting to consider the following conjecture:

Conjecture 2. *If $n > 0$ is even, then*

$$\textit{we have } \begin{cases} \dim_\Bbbk G_{n,d} = 1 & \textit{if } d \textit{ is even and } 2 \leq d \leq \lfloor n/3 \rfloor \\ \dim_\Bbbk G_{n,d} = 0 & \textit{else} \end{cases}$$

If n is odd, then

$$\textit{we have } \begin{cases} \dim_\Bbbk G_{n,d} = 1 & \textit{if } d \textit{ is odd and } 3 \leq d \leq \lfloor n/3 \rfloor \\ \dim_\Bbbk G_{p,1} = 1 & \textit{if } p \textit{ is odd prime} \\ \dim_\Bbbk G_{n,d} = 0 & \textit{else} \end{cases}$$

If Conjecture (2) is true, then G has a nice property as a ring:

Proposition 3.1. *Assume Conjecture (2) is true. Then G is an integral domain.*

Proof. Let $f, g \in G_{n,d} = \Bbbk\overline{x}^{\alpha}$ such that $fg = 0$ and express f and g as

$$f = a\overline{x}^{\alpha} \quad \text{and} \quad g = b\overline{x}^{\alpha}.$$

Then clearly since $\overline{x}^{2\alpha} \neq 0$, we must have $ab = 0$, which implies either $a = 0$ or $b = 0$ which implies either $f = 0$ or $g = 0$. $\square$

Example 3.1. In G^{46}, we have

$$\begin{aligned} x_{23}(x_{19}x_{29} - x_{17}x_{31}) &= x_{23}x_{19}x_{29} - x_{23}x_{17}x_{31} \\ &= x_{11}x_{31}x_{29} - x_{11}x_{29}x_{31} \\ &= 0, \end{aligned}$$

however $x_{19}x_{29} \neq x_{17}x_{31}$ in G^{46}. It follows that x_{23} is a zerodivisor in G^{46}. Similarly, we have

$$\begin{aligned} x_3(x_{19}x_{29} - x_{17}x_{31}) &= x_3x_{19}x_{29} - x_3x_{17}x_{31} \\ &= x_5x_{17}x_{29} - x_5x_{17}x_{29} \\ &= 0. \end{aligned}$$

It follows that x_3 is also a zero divisor in G^{46}. Note that $x_{19}x_{29} = x_{17}x_{31}$ in G^{48}, however x_3 remains a zerodivisor in G^{48} since

$$\begin{aligned} x_3(x_{29}x_{31} - x_{23}x_{37}) &= x_3x_{29}x_{31} - x_3x_{23}x_{37} \\ &= x_{11}x_{29}x_{23} - x_{11}x_{23}x_{29} \\ &= 0, \end{aligned}$$

and $x_{29}x_{31} \neq x_{23}x_{37}$ in G^{48}. Similar calculations like this show that $x_3, x_5, x_7, x_{11}, x_{13}, x_{17}$, and x_{19} are all zero-divisors in G^{46}. On the other hand, using Singular we find that a maximal G^{46}-regular sequence is given by $x_{29}, x_{31}, x_{37}, x_{41}, x_{43}$.

3.0.1 Explicit Calculations of the $\Bbbk$-Complex $F_{n,d}$

Example 3.2. Let's describe $\widetilde{F}_{18,2}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$\begin{aligned} \widetilde{F}_{1,18,2} &= \Bbbk e_{5,13} + \Bbbk e_{7,11} \\ \widetilde{F}_{0,18,2} &= R_{18,2} = \Bbbk x_5x_{13} + \Bbbk x_7x_{11} + \Bbbk x_{18} \\ \widetilde{F}_{-1,18,2} &= G_{18,2} = \Bbbk\overline{x}_5\overline{x}_{13}, \end{aligned}$$

and $\widetilde{F}_{i,18,2} = 0$ for all $i \neq -1, 0, 1$. The differential is the unique R-linear map defined by $\mathrm{d}(e_{5,13}) = x_5x_{13} - x_{18}$ and $\mathrm{d}(e_{7,11}) = x_7x_{13} - x_{18}$. After choosing ordered bases, we can express $\widetilde{F}_{18,2}$ in the form

$$0 \longrightarrow \Bbbk^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} G_{18,2} \longrightarrow 0$$

Thus we have

$$\begin{aligned} \dim_{\Bbbk} G_{18,2} &= \chi(F_{18,2}) \\ &= 3 - 2 \\ &= 1. \end{aligned}$$

.

Next, let's describe $\widetilde{F}_{23,3}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$\begin{aligned} \widetilde{F}_{2,23,3} &= \Bbbk e_{5,7,11} \\ \widetilde{F}_{1,23,3} &= \Bbbk x_{13}e_{3,7} + \Bbbk x_{13}e_{5,5} + \Bbbk x_7e_{3,13} + \Bbbk x_7e_{5,11} + \Bbbk x_5e_{7,11} + \Bbbk x_5e_{5,13} \\ \widetilde{F}_{0,23,3} &= R_{23,3} = \Bbbk x_{13}x_{10} + \Bbbk x_7x_{16} + \Bbbk x_5x_{18} + \Bbbk x_5x_7x_{11} + \Bbbk x_3x_7x_{13} + \Bbbk x_5^2x_{13} \\ \widetilde{F}_{-1,23,3} &= G_{23,2} = \Bbbk\overline{x}_5\overline{x}_7\overline{x}_{11} \end{aligned}$$

and $\widetilde{F}_{i,23,3} = 0$ for all $i \neq -1, 0, 1, 2$. The differential is the unique R-linear map defined by

$$\begin{aligned}
\mathrm{d}(e_{5,7,11}) &= x_5 e_{7,11} - x_5 e_{5,13} + x_{13} e_{5,5} - x_{13} e_{3,7} + x_7 e_{3,13} - x_7 e_{5,11} \\
\mathrm{d}(e_{3,7}) &= x_3 x_7 - x_{10} \\
\mathrm{d}(e_{5,5}) &= x_5 x_5 - x_{10} \\
\mathrm{d}(e_{3,13}) &= x_3 x_{13} - x_{16} \\
\mathrm{d}(e_{5,11}) &= x_5 x_{11} - x_{16} \\
\mathrm{d}(e_{7,11}) &= x_7 x_{11} - x_{18} \\
\mathrm{d}(e_{5,13}) &= x_5 x_{13} - x_{18}.
\end{aligned}$$

After choosing ordered basis, we can express $\widetilde{F}_{23,3}$ in the form

$$0 \longrightarrow \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}} \Bbbk^6 \xrightarrow{M} \Bbbk^6 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}} G_{23,3} \longrightarrow 0$$

where M is a matrix whose entries are either -1, 0, or 1. Thus we have

$$\begin{aligned}
\dim_{\Bbbk} G_{23,3} &= \chi(F_{23,3}) \\
&= 6 - 6 + 1 \\
&= 1.
\end{aligned}$$

3.1 Re-interpreting the Conjecture

From Theorem (2.1), we can express Conjecture (2) in another form:

Conjecture 3. *Assume (n, d) is a good pair. Then*

$$\sum_{i=0}^{\infty} (-1)^i \dim_{\Bbbk} \operatorname{Tor}_i^R(G, \Bbbk)_{n,d} = 1.$$