Probability Theory Homework 2

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Problem 5

Solution 1. labelsol Define

$$\mathcal{G} = \{B \in \mathcal{B}(\mathbb{R}) \mid \text{for all } \varepsilon > 0 \text{ there exists a finite union of intervals } A_\varepsilon \text{ such that } P(A\Delta B) < \varepsilon\}.$$

We show that $\mathcal{G}$ is a σ-algebra which contains all intervals. It will then follows that $\mathcal{G} = \mathcal{B}(\mathbb{R})$ since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra which contains all intervals. First note that if I is an interval and $\varepsilon > 0$, then $P(I\Delta I) = 0 < \varepsilon$. Thus $\mathcal{G}$ contains all intervals. Now we show $\mathcal{G}$ is closed under complements. Suppose $B \in \mathcal{G}$ and let $\varepsilon > 0$. Choose a finite union of intervals A such that $P(A\Delta B) < \varepsilon$. Then observe that A^c is a finite union of intervals and

$$P(A^c \Delta B^c) = P(A\Delta B) < \varepsilon$$

It follows that $B^c \in \mathcal{G}$, hence $\mathcal{G}$ is closed under complements. Finally we show that $\mathcal{G}$ is closed under countable unions. Let (B_n) be a sequence of sets in $\mathcal{G}$ and let $\varepsilon > 0$. By disjointifying (B_n) if necessary, we may assume that (B_n) is pairwise disjoint. Since

$$\sum_{n=1}^{\infty} P(B_n) = P\left(\sum_{n=1}^{\infty} B_n\right) \leq 1,$$

we know that there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} P(B_n) < \frac{\varepsilon}{2}.$$

Choose a sequence of finite unions of intervals (A_n) such that $P(A_n \Delta B_n) < \varepsilon/2N$ for each $1 \leq n \leq N$ and such that $A_n = \emptyset$ for all $n \geq N+1$. Then observe that $\bigcup_{n=1}^{N} A_n$ is a finite union of intervals, and

$$\begin{aligned}
P\left(\left(\bigcup_{n=1}^{N} A_n\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right)\right) &\leq P\left(\bigcup_{n=1}^{\infty} (A_n \Delta B_n)\right) \\
&\leq P\left(\bigcup_{n=1}^{N} (A_n \Delta B_n)\right) + P\left(\sum_{n=N+1}^{\infty} B_n\right) \\
&\leq \sum_{n=1}^{N} P(A_n \Delta B_n) + \sum_{n=N+1}^{\infty} P(B_n) \\
&< \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} \\
&< \varepsilon,
\end{aligned}$$

where we used Lemma(**??**) together with monotonicity of P to obtain the first inequality. It follows that $\mathcal{G}$ is a σ-algebra, and thus we are done.

Lemma 0.1. *Let (A_n) and (B_n) be two sequences of sets. Then*

$$\left(\bigcup_{m=1}^{\infty} A_m\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \Delta B_n)$$

Proof. We have

$$\begin{aligned}
\left(\bigcup_{m=1}^{\infty} A_m\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right) &= \left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right) \Big\backslash \left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) \\
&= \left(\bigcup_{n=1}^{\infty} (A_n \cup B_n)\right) \Big\backslash \left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (A_m \cap B_n)\right) \\
&\subseteq \left(\bigcup_{n=1}^{\infty} (A_n \cup B_n)\right) \Big\backslash \left(\bigcup_{n=1}^{\infty} (A_n \cap B_n)\right) \\
&\subseteq \bigcup_{n=1}^{\infty} (A_n \cup B_n) \backslash (A_n \cap B_n) \\
&= \bigcup_{n=1}^{\infty} (A_n \Delta B_n).
\end{aligned}$$

□

Problem 8

Solution 2. labelsol Clearly $P_1 \neq P_2$ since $P_1(\{a\}) = 1/6 \neq 1/3 = P_2(\{a\})$. However a calculation shows P_1 and P_2 agree on $\mathcal{C}$. Indeed,

$$\begin{aligned}
P_1(\{a,b\}) &= P_1(\{a\}) + P_1(\{b\}) \\
&= \frac{1}{6} + \frac{1}{3} \\
&= \frac{1}{3} + \frac{1}{6} \\
&= P_2(\{a\}) + P_2(\{b\}) \\
&= P_2(\{a,b\})
\end{aligned}$$

Similarly

$$\begin{aligned}
P_1(\{d,c\}) &= P_1(\{d\}) + P_1(\{c\}) \\
&= \frac{1}{6} + \frac{1}{3} \\
&= \frac{1}{3} + \frac{1}{6} \\
&= P_2(\{d\}) + P_2(\{c\}) \\
&= P_2(\{d,c\})
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_1(\{a,c\}) &= P_1(\{a\}) + P_1(\{c\}) \\
&= \frac{1}{6} + \frac{1}{3} \\
&= \frac{1}{3} + \frac{1}{6} \\
&= P_2(\{a\}) + P_2(\{c\}) \\
&= P_2(\{a,c\})
\end{aligned}$$

Similarly,

$$\begin{aligned} P_1(\{b,d\}) &= P_1(\{b\}) + P_1(\{d\}) \\ &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{6} + \frac{1}{3} \\ &= P_2(\{b\}) + P_2(\{d\}) \\ &= P_2(\{b,d\}) \end{aligned}$$

Also note that $\mathcal{C}$ generates $\mathcal{P}(\Omega)$ since $\mathcal{C}$ contains all singletons:

$$\begin{aligned} \{a\} &= \{a,b\} \cap \{a,c\} \\ \{b\} &= \{a,b\} \cap \{b,d\} \\ \{c\} &= \{a,c\} \cap \{d,c\} \\ \{d\} &= \{d,c\} \cap \{b,d\}. \end{aligned}$$

We have $F_r^{\leftarrow}(F(s)) \geq s$ and F is constant on the interval $[s, F_r^{\leftarrow}(F(s))]$. Indeed, if $t \in [s, F_r^{\leftarrow} F(s))]$, then $F(t) \geq F(s)$ since F is nondecreasing. Also note that

$$F_r^{\leftarrow}(F(s)) = \inf\{t \mid F(t) > F(s)\},$$

and since F is right continuous at s, we must have $F(F_r^{\leftarrow}(F(s))) = s$.

Problem 17

Solution 3. labelsol We first show $F_r^{\leftarrow}$ is right continuous. First we observe that for each $s \in \mathbb{R}$, we have $F_r^{\leftarrow}(F(s)) \geq s$ and F is constant on the interval $[s, F_r^{\leftarrow}(F(s))]$. Indeed, if $t \in [s, F_r^{\leftarrow} F(s))]$, then $F(t) \geq F(s)$ since F is nondecreasing. Also note that

$$F_r^{\leftarrow}(F(s)) = \inf\{t \mid F(t) > F(s)\},$$

and since F is right continuous at s, we must have $F(F_r^{\leftarrow}(F(s))) = F(s)$.

Now let $y \in (0,1)$.

Case 1: Suppose that $y \neq F(s)$ for any $s \in \mathbb{R}$. Since F is a distribution function, the output value y occurs at a jump discontinuity of F, say at s. In particular, $y < F(s)$, and for any $z \in (y, F(s))$, we have

$$F_r^{\leftarrow}(z) = s = F_r^{\leftarrow}(y).$$

Thus as $z \to y$ from the right, we see that $F_r^{\leftarrow}(z) \to F_r^{\leftarrow}(y)$ from the right. It follows that $F_r^{\leftarrow}$ is right continuous at y.

Case 2: Suppose that $y = F(s)$ for some $s \in \mathbb{R}$. Since F is right continuous and non-decreasing, we can choose s to be $F_r^{\leftarrow}(y)$, so set $s = F_r^{\leftarrow}(y)$. Since F is non-decreasing and right continuous at s, there exists an interval $[s, s+\varepsilon)$ such that F is continuous on $[s, s+\varepsilon)$ and, by construction, we have $F(t) > F(s)$ for all $t \in (s, s+\varepsilon)$ (if there was a $t \in [s, s+\varepsilon)$ such that $F(t) = F(s) = y$, then $t = \inf\{u \mid F(u) > y\} = s$. In particular, for $z \in [y, F(s+\varepsilon))$, we have

$$\begin{aligned} z \to y \text{ from the right} &\implies F(F_r^{\leftarrow}(z)) \to F(F_r^{\leftarrow}(y)) \text{ from the right} \\ &\implies F_r^{\leftarrow}(z) \to F_r^{\leftarrow}(y) \text{ from the right} \\ &\implies F_r^{\leftarrow} \text{ is right continuous at } y. \end{aligned}$$

For the second part of the problem, note that $F_r^{\leftarrow}(y) \neq F_l^{\leftarrow}(y)$ if and only if F takes the value y on an interval I. There are only countably many such y since F is non-decreasing and right continuous. Thus $\lambda\{y \mid F_r^{\leftarrow}(y) \neq F_l^{\leftarrow}(y)\} = 0$.

Problem 19

Problem 19.a

Solution 4. labelsol Note that $\emptyset \in \mathcal{B}^*$ since the emptyset is clearly neglible. In particular, $\mathcal{B}^* \supseteq \mathcal{B}$ since we can write $A = A \cup \emptyset$ for every $A \in \mathcal{B}$. Next we show $\mathcal{B}^*$ is closed under complements. Let $A \cup M \in \mathcal{B}^*$ where $A \in \mathcal{B}$ and $M \in \mathcal{N}$. Choose $N \in \mathcal{B}$ such that $M \subseteq N$ and $P(N) = 0$. Then

$$\begin{aligned}(A \cup M)^c &= A^c \cap M^c \\ &= (A^c \cap N^c) \cup (A^c \cap M^c \backslash N^c) \\ &= (A^c \cap N^c) \cup (A^c \cap N/M)\end{aligned}$$

where $A^c \cap N^c \in \mathcal{A}$ and where $A^c \cap (N/M) \in \mathcal{M}$ since $A^c \cap (N/M) \subseteq N$ and $P(N) = 0$. Thus $\mathcal{B}^*$ is closed under complements. Finally we show $\mathcal{B}^*$ is closed under countable unions. Let $(A_n \cup M_n)$ be a sequence of sets in $\mathcal{B}^*$. For each $n \in \mathbb{N}$ choose $N_n \in \mathcal{B}$ such that $M_n \subseteq N_n$ and $P(N_n) = 0$. Then observe that

$$\bigcup_{n=1}^{\infty}(A_n \cup M_n) = A \cup M$$

where $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$ and where $M = \bigcup_{n=1}^{\infty} M_n \in \mathcal{N}$ since $\bigcup_{n=1}^{\infty} M_n \subseteq \bigcup_{n=1}^{\infty} N_n \in \mathcal{B}$ and

$$\begin{aligned}P\left(\bigcup_{n=1}^{\infty} N_n\right) &\leq \sum_{n=1}^{\infty} P(N_n) \\ &= \sum_{n=1}^{\infty} 0 \\ &= 0.\end{aligned}$$

Thus $\mathcal{B}^*$ is a σ-algebra.

Problem 19.b

Solution 5. labelsol Choose $N_1, N_2 \in \mathcal{B}$ such that $M_1 \subseteq N_1$, $M_2 \subseteq N_2$, and $P(N_1) = 0 = P(N_2)$. Then observe that

$$\begin{aligned}P(A_1) &= P(A_1) + P(N_1) \\ &\geq P(A_1 \cup N_1) \\ &\geq P(A_2),\end{aligned}$$

where we used the face that $A_1 \cup N_1 \supseteq A_2$. A similar calculation shows $P(A_2) \geq P(A_1)$. Thus $P(A_1) = P(A_2)$.

Problem 19.c

Solution 6. labelsol The previous exercise shows us that P^* is well-defined. Furthermore, $P^*|_{\mathcal{B}} = P$ since if $A \in \mathcal{B}$, then $P^*(A) = P^*(A \cup \emptyset) = P(A)$. Finally, suppose $(A_n \cup M_n)$ is a sequence of pairwise disjoint sets in $\mathcal{B}^*$. Then

$$\begin{aligned}P^*\left(\bigcup_{n=1}^{\infty}(A_n \cup M_n)\right) &= P^*(A \cup M) \\ &= P(A) \\ &= P\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} P(A_n) \\ &= \sum_{n=1}^{\infty} P^*(A_n \cup M_n).\end{aligned}$$

It follows that P^* is a measure which extends P.

Problem 19.d

Solution 7. labelsol Observe that $B = A_1 \cup (B \backslash A_1)$ where $A_1 \in \mathcal{B}$ and $B \backslash A_1 \in \mathcal{N}$ since $B \backslash A_1 \subseteq A_2 \backslash A_1$ and $P(A_1 \backslash A_1) = 0$.

Problem 19.e

Solution 8. labelsol Let B be a neglible set. Choose $N \in \mathcal{B}$ such that $P(N) = 0$. Then $\emptyset \subseteq B \subseteq N$ implies $B \in \mathcal{B}^*$ by the previous exercise. Furthermore we have $P^*(B) \leq P(N) = 0$ by monotonicity. Therefore every neglible set is a null set, that is, $\mathcal{B}^*$ is complete.

Problem 19.f

Solution 9. labelsol By removing the p_k's such that $p_k = 0$ if necessary, we may assume that $p_k > 0$ for all $k \in \mathbb{N}$. Let $D = \{a_k \mid k \in \mathbb{N}\}$. Then the null sets are the sets $A \in \mathcal{B}$ which are disjoint from D. In fact, there is a *largest* null set, namely D^c, and subset of D^c is neglible. Conversely, every neglible set is a subset of D^c. Now every subset of D is already in $\mathcal{B}$, and thus since every subset of Ω can be expressed as $A \cup M$ where $A \subseteq D$ and $M \subseteq D^c$, we see that the completion of $\mathcal{B}$ is $\mathcal{P}(\Omega)$.

Problem 19.g

Solution 10. labelsol

Problem 19.h

Solution 11. labelsol Yes, $(\Omega, \mathcal{B}^*, P^*)$ must be the minimal extension. To see this, suppose $(\Omega, \mathcal{B}', P')$ is another complete extention of $(\Omega, \mathcal{B}, P)$. Let $A \cup M \in \mathcal{B}^*$ where $A \in \mathcal{B}$ and $M \in \mathcal{N}$. Choose $N \in \mathcal{B}$ such that $P(N) = 0$. Then note that $P^*(N) = 0 = P'(N)$. Thus M is neglible when considered as a set in $\mathcal{B}'$ or in $\mathcal{B}^*$. Since both $\mathcal{B}'$ and $\mathcal{B}^*$ are complete, this implies $M \in \mathcal{B}' \cap \mathcal{B}^*$ and $P^*(M) = 0 = P'(M)$. In particular, $A \cup M \in B'$ and $P'(A \cup M) = P(A) = P^*(A \cup M)$. It follows that $(\Omega, \mathcal{B}', P')$ is a complete extension of $(\Omega, \mathcal{B}^*, P^*)$.