Algebraic Topology Homework 2

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Problem 6

Problem 6.a

Lemma 0.1. *Let B be a deformation retact of C and let A be a deformation retract of B. Then A is a deformation retract of C.*

Proof. Choose deformation retractions $G\colon C \times I \to C$ and $F\colon B \times I \to B$; so F and G are continuous maps such that $G(-,0) = 1_C$, $F(-,0) = 1_B$, and $G(-,1) = s$ and $F(-,1) = r$ are both retracts: we view s as map $s\colon C \to B$ such that $s(b) = b$ for all $b \in B$ and we view r as a map $r\colon B \to A$ such that $r(a) = a$ for all $a \in A$. Note that rs is a map from C to A such that $rs(a) = a$ for all $a \in A$, i.e. rs is a retraction of C onto A. Let ι denote the inclusion map $\iota\colon B \to C$ and define $\widetilde{F}\colon C \times I \to C$ be defined by $\widetilde{F}(c,t) = \iota F(s(c),t)$. Finally, to get a deformation retraction with respect to $A \subseteq C$, we glue $\widetilde{F}$ and and G together as follows: define $H\colon C \times I \to C$ by

$$H(c,t) = \begin{cases} G(c,2t) & 0 \leq t \leq 1/2 \\ \widetilde{F}(c,2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Then H is continuous and satisfies $H(-,0) = 1_C$ and $H(-,1) = rs$. Thus A is a deformation retract of C with H being a deformation retraction. □

Exercise 1. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segmnet $[0,1] \times \{0\}$ together with all the verticle segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Solution 1. Let $A = [0,1] \times \{0\}$ and let $\boldsymbol{a} = (a,0)$ be a point in A. We show X deformation retracts to A then we show A deformation retracts to $\boldsymbol{a}$. First we show X deformation rectracts to A. Define $F\colon X \times I \to X$ as follows: let $\boldsymbol{x} = (x_1, x_2)$ be a point in X. If $x_2 = 0$, then we set $F(\boldsymbol{x}, t) = 0$ for all $t \in I$. Otherwise, x_1 is rational and $0 < x_2 \leq 1 - x_1$. In this case, we set

$$F(\boldsymbol{x}, t) = (1-t)(x_1, x_2) + t(x_1, 0)$$

for all $t \in I$. Then F is a homotopy from 1_X to a retraction map $r\colon X \to A$. In fact, it's easy to see that F is a *strong* deformation retraction. This shows that X deformation retracts to A. Now we show that A deformation retracts to $\boldsymbol{a}$. Define $G\colon A \times I \to A$ as follows: let $\boldsymbol{x} = (x,0)$ be a point in A. We set

$$G(\boldsymbol{x}, t) = (1-t)\boldsymbol{x} + t\boldsymbol{a}$$

for all $t \in I$. Then it's easy to see that G is a deformation rectraction which shows that A deformation retracts to $\boldsymbol{a}$.

Now we show X does not deformation retract to any other point in the segment $[0,1] \times \{0\}$.

Problem 6.b

Exercise 2. Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below (see Hatcher exercise 0.6.b). Show that Y is contractible but does not deformation retract onto any point.

Solution 2.

Problem 6.c

Exercise 3. Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segmnet $[0,1] \times \{0\}$ together with all the verticle segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point.

Solution 3.

Problem 9

Exercise 4. Show that a retract of a contractible space is contractible.

Solution 4. Let A be a retract of a contractible space X. Thus we have a continuous map $r \colon X \to A$ such that $r \circ \iota = 1_A$ where $\iota \colon A \to X$ is the inclusion map. Since X is contractible, there exists $z \in X$ such that $1_X \sim c_z$ where $c_z \colon X \to X$ is the constant map defined by $c_z(x) = z$ for all $x \in X$. We claim that A is contractible with $1_A \sim c_{r(z)}$. Indeed, let $F \colon X \times I \to I$ be a homotopy from 1_X to c_z; so F is continuous and $F(-,0) = 1_X$ and $F(-,1) = c_z$. Let G be the composite map

$$A \times I \xrightarrow{\iota \times 1} X \times I \xrightarrow{F} X \xrightarrow{r} A.$$

Concretely $G(a,t) = r(F(a,t))$ for all $a \in A$ and $t \in I$. Then G is continuous (being a composite of continuous functions) and we have $G(-,0) = 1_A$ and $G(-,1) = c_{r(z)}$.

Problem 10

Exercise 5. Show that a space X is contractible iff every map $f \colon X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f \colon Y \to X$ is nullhomotopic.

Solution 5. It suffices to show the first part of the exercise since the proof of the second part is almost identical to the proof of the first part. Suppose X is contractible and let $f \colon X \to Y$ be an arbitrary continuous map. Since X is contractible, there exists $z \in X$ such that $1_X \sim c_z$. Choose a homotopy from 1_X to c_z, say $F \colon X \times I \to X$. Let G be the composite map

$$X \times I \xrightarrow{F} X \xrightarrow{f} Y.$$

Concretely, $G(x,t) = f(F(x,t))$ for all $(x,t) \in X \times I$. Then G is continuous (being a composite of continuous functions) and we have $G(-,0) = f$ and $G(-,1) = c_{f(z)}$. Thus f is nullhomotopic. Conversely, suppose every continuous map $f \colon X \to Y$ is nullhomotopic. Then in particular, $1_X \colon X \to X$ is nullhomotopic. However this implies X is contractible, by definition.

Problem 12

Before we solve this exercise, we introduce some terminology as well as state and prove a lemma. Let X be a topological space. We denote by $\pi_0(X)$ to be the set of path-connected components of X. We write $[x] \in \pi_0(X)$ for the equivalence class with $x \in X$ as a particular choice of representative. Thus if $x' \in [x]$, then there exists a path $\gamma \colon I \to X$ form x to x', i.e. such that $\gamma(0) = x$ and $\gamma(1) = x'$. Next let $f \colon X \to Y$ be a continuous map. Define a map $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$ as follows: given $[x] \in \pi_0(X)$, we set

$$\pi_0(f)[x] := [f(x)].$$

To see that this is well-defined, let $x' \in [x]$ be another representative. Choose a path $\gamma \colon I \to X$ from x to x'. Then $f\gamma$ is path from $f(x)$ to $f(x')$. Indeed, it is continuous since it is a composite of two continuous functions and we have $f\gamma(0) = f(x)$ and $f\gamma(1) = f(x')$ (note we are using the notation $f\gamma$ to mean $f \circ \gamma$). It is straightforward to check that we obtain a functor $\pi_0 \colon \mathbf{Top} \to \mathbf{Set}$ which takes a topological space X to the set $\pi_0(X)$ and which takes a continuous map $f \colon X \to Y$ to the function $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$. In particular, this means that π_0 preserves compositions: if $g \colon Y \to Z$ is another continuous map, then we have $\pi_0(gf) = \pi_0(g)\pi_0(f)$. Similarly, this means π_0 preserves identities: we have $\pi_0(1_X) = 1_{\pi_0(X)}$. The functor π_0 turns out to be invariant under homotopy:

Lemma 0.2. *Let $f, g \colon X \to Y$ be two continuous maps such that $f \sim g$. Then $\pi_0(f) = \pi_0(g)$.*

Proof. Choose a homotopy from f to g, say $F\colon X \times I \to Y$; so F is continuous and $F(-,0) = f$ and $F(-,1) = g$. Let $[x_0]$ be a path-connected component in X. Then observe that $F(x_0,-)\colon I \to Y$ is a path from $f(x_0)$ to $g(x_0)$ by our assumption of F (the map $F(x_0,-)\colon I \to Y$ is deifned by sending $t \in I$ to $F(x_0,t)$; so x_0 is fixed and t varies). In particular, it follows that $[f(x_0)] = [g(x_0)]$. Since $[x_0]$ was arbitrary, it follows that $\pi_0(f) = \pi_0(g)$. $\square$

Now we solve the exercise:

Exercise 6. Show that a homotopy equivalence $f\colon X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X .

Solution 6. Let $f\colon X \to Y$ and $g\colon Y \to X$ be continuous functions such that $fg \sim 1_Y$ and $gf \sim 1_X$. It follows from the fact that π_0 is a homotopy invariant functor that $\pi_0(f)\pi_0(g) = 1_Y$ and $\pi_0(g)\pi_0(f) = 1_X$. In other words, $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection with $\pi_0(g)$ being its inverse. For the second part of the exercise, let $P \subseteq X$ be a connected component of X. Then $f(P)$ is contained in a connected component of Y since f is continuous, say $f(P) \subseteq Q$. Similarly, $g(Q)$ is contained in a connected component of X. Since $gf \sim 1_X$, we must have $g(Q) \subseteq P$.