Mod Two Homology and Cohomology

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1 Simplicial Complexes

Definition 1.1. A **simplicial complex** K consists of

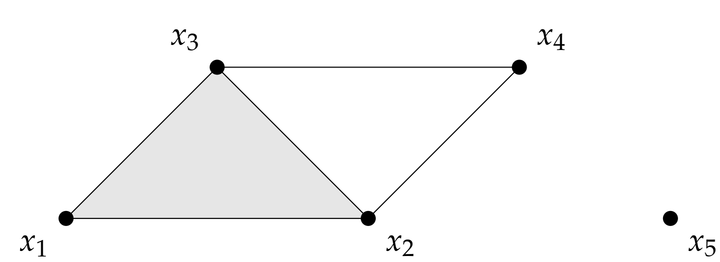
- A set $V(K)$, the set of **vertices** of K.
- A set $\mathcal{S}(K)$ of finite nonempty subsets of $V(K)$ which is closed under containment: if $\sigma \in S(K)$ and $\sigma \supset \tau$, then $\tau \in S(K)$. We require that $\{v\} \in S(K)$ for all $v \in V(K)$.

An element σ of $S(K)$ is called a **simplex** of K. If $|\sigma| = m+1$, we say that σ is of **dimension** m or that σ is an m-simplex. The set of m-simplexes of K is denoted $S_m(K)$. The set $S_0(K)$ of 0-simplexes is in bijection with $V(K)$, and we usually identify $v \in V(K)$ with $\{v\} \in S_0(K)$. We say that K is of **dimension** $\leq n$ if $S_m(K) = \emptyset$ for $m > n$, and that K is of **dimension** n or (n-dimensional) if it is of dimension $\leq n$ but not of dimension $\leq n-1$. A simplicial complex of dimension ≤ 1 is called a **simplicial graph**. A simplicial complex K is called **finite** if $V(K)$ is a finite set.

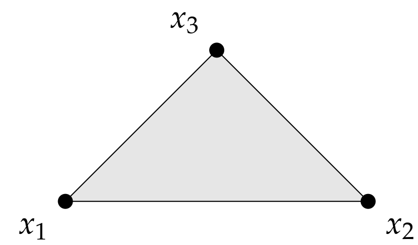
If $\sigma \in S(K)$ and $\tau \subset \sigma$, we say that τ is a **face** of σ. As $S(K)$ is closed under inclusion, it is determined by its subset $S(K)_{\max}$ of **maximal** simplexes (if K is finite dimensional). A **subcomplex** L of K is a simplicial complex such that $V(L) \subset V(K)$ and $S(L) \subset S(K)$. If $U \subset S(K)$, we denote by $\overline{U}$ the subcomplex generated by U, i.e. the smallest subcomplex of K such that $U \subset S(\overline{U})$. The m**-skeleton** K^m of K is the subcomplex of K generated by the union of $\mathcal{S}_k(K)$ for $k \leq m$.

Let $\sigma \in \mathcal{S}(K)$. We denote by $\overline{\sigma}$ (or $\mathcal{K}_\sigma$) the subcomplex of $\mathcal{K}$ formed by σ and all its faces. The subcomplex $\dot{\sigma}$ of $\overline{\sigma}$ generated by the proper faces of σ is called the **boundary** of σ.

Example 1.1. Let $\mathcal{K}$ be the simplical complex with $V(\mathcal{K}) = \{x_1, x_2, x_3, x_4, x_5\}$ and $S(\mathcal{K})_{\max} = \{x_1x_2x_3, x_2x_4, x_3x_4, x_5\}$, where we use the monomial notation $x_{i_1}x_{i_2}\cdots x_{i_k}$ to mean $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. We may visualize $\mathcal{K}$ as



The subcomplex $\mathcal{K}_{x_1x_2x_3}$ of $\mathcal{K}$ can be visualized as



1.1 Geometric Realization

Let $\mathcal{K}$ be a simplicial complex. The **geometric realization** $|\mathcal{K}|$ of $\mathcal{K}$ is, as a set, defined by

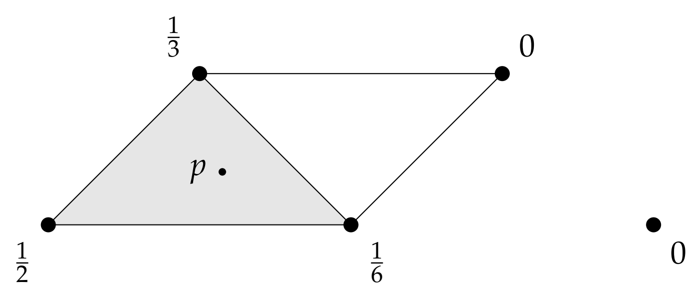
$$|\mathcal{K}| := \left\{ p : V(\mathcal{K}) \to [0,1] \mid \sum_{x \in V(\mathcal{K})} p(x) = 1 \text{ and } p^{-1}((0,1]) \in S(\mathcal{K}) \right\}$$

The condition $p^{-1}((0,1]) \in S(\mathcal{K})$ says that the set of all $x \in \mathcal{V}(K)$ such that $p(x) \neq 0$ must form a simplex of K. There is a distance on $|K|$ defined by

$$d(p,q) = \sqrt{\sum_{x \in V(\mathcal{K})} (p(x) - q(x))^2},$$

which defined the metric topology on $|\mathcal{K}|$. The set $|\mathcal{K}|$ with the metric topology is denoted by $|\mathcal{K}|_d$. For instance, if $\sigma \in S_m(\mathcal{K})$, then $|\mathcal{K}_\sigma|_d$ is isometric to the standard Euclidean simplex $\Delta^m = \{(a_0, \ldots, a_m) \in \mathbb{R}^{m+1} \mid a_i \geq 0 \text{ and } \sum a_i = 1\}$.

Example 1.2. Let $\mathcal{K}$ be the simplical complex as in Example (1.4). We can visualize a function $p \in |\mathcal{K}|$ by attaching a number in $(0,1]$ to each vertex likeso:



We can actually think of p here as the vector $v = \frac{1}{2}e_1 + \frac{1}{6}e_2 + \frac{1}{3}e_3 \in \mathbb{R}^3$, where e_i denote the standard basis. The distance function then is just the normal euclidean distance function ($d(v,w) = \|v - w\|$).

A more used topology for $|\mathcal{K}|$ is the **weak topology**, for which $A \subset |\mathcal{K}|$ is closed if and only if $A \cap |\mathcal{K}_\sigma|_d$ is closed in $|\mathcal{K}_\sigma|_d$ for all $\sigma \in S(\mathcal{K})$. The notation $|\mathcal{K}|$ stands for the set $|\mathcal{K}|$ endowed with the weak topology. A map f from $|\mathcal{K}|$ to a topological space X is then continuous if and only if its restriction to $|\mathcal{K}_\sigma|_d$ is continuous for each $\sigma \in S(\mathcal{K})$. In particular, the identity $|\mathcal{K}| \to |\mathcal{K}|_d$ is continuous, which implies that $|\mathcal{K}|$ is Hausdorff. The weak and the metric topology coincide if and only if $\mathcal{K}$ is locally finite, that is, each vertex is contained in a finite number of simplexes. When $\mathcal{K}$ is not locally finite, $|\mathcal{K}|$ is not metrizable.

1.2 Simplicial Join, Stars, and Links

1.2.1 Simplicial Join

Let $\mathcal{K}$ and $\mathcal{L}$ be simplicial complexes. Their **join** is the simplicial complex $\mathcal{K} \star \mathcal{L}$ defined by

$$\begin{aligned} V(\mathcal{K} \star \mathcal{L}) &= V(\mathcal{K}) \uplus V(\mathcal{L}) \\ S(\mathcal{K} \star \mathcal{L}) &= S(\mathcal{K}) \cup S(\mathcal{L}) \cup \{\sigma \cup \tau \mid \sigma \in S(\mathcal{K}) \text{ and } \tau \in S(\mathcal{L})\}. \end{aligned}$$

1.2.2 Stars and Links

Let $\mathcal{K}$ be a simplicial complex and $\sigma \in S(\mathcal{K})$. The **star St**$(\sigma)$ of σ is the subcomplex of $\mathcal{K}$ generated by all the simplexes containing σ. The **link Lk**(σ) of σ is the subcomplex of $\mathcal{K}$ formed by the simplexes $\tau \in S(\mathcal{K})$ such that $\tau \cap \sigma = \varnothing$ and $\tau \cup \sigma \in S(\mathcal{K})$. Thus, $\mathrm{Lk}(\sigma)$ is a subcomplex of $\mathrm{St}(\sigma)$ and

$$\mathrm{St}(\sigma) = \mathcal{K}_\sigma \star \mathrm{Lk}(\sigma).$$

Example 1.3. Let $\mathcal{K}$ be the simplical complex as in Example (1.4). Then

$$\begin{aligned} \mathrm{Lk}(x_1x_3)_{\max} &= \{x_2\} \\ \mathrm{Lk}(x_1)_{\max} &= \{x_2x_3\} \\ \mathrm{Lk}(x_2)_{\max} &= \{x_1x_3, x_4\} \\ \mathrm{Lk}(x_4)_{\max} &= \{x_2, x_3\} \\ \mathrm{Lk}(x_5)_{\max} &= \varnothing \end{aligned} \qquad \begin{aligned} \mathrm{St}(x_1x_3)_{\max} &= \{x_1x_2x_3\} \\ \mathrm{St}(x_1)_{\max} &= \{x_1x_2x_3\} \\ \mathrm{St}(x_2)_{\max} &= \{x_1x_2x_3, x_2x_4\} \\ \mathrm{St}(x_4)_{\max} &= \{x_3x_4, x_2x_4\} \\ \mathrm{St}(x_5)_{\max} &= \varnothing \end{aligned}$$

1.3 Simplicial Maps

Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes. A **simplicial map** $f : \mathcal{K} \to \mathcal{L}$ is a map $f : V(\mathcal{K}) \to V(\mathcal{L})$ such that the image of a simplex is a simplex:$\sigma \in S(\mathcal{K})$ implies $f(\sigma) \in S(\mathcal{L})$. Simplicial complexes and simplicial maps form a category, the **simplicial category**, denoted by **Simp**.

A simplicial map $f : \mathcal{K} \to \mathcal{L}$ induces a continuous map $|f| : |\mathcal{K}| \to |\mathcal{L}|$ defined, for $x \in V(\mathcal{L})$, by

$$|f|(p)(y) = \sum_{x \in f^{-1}(y)} p(x).$$

Example 1.4. Let $\mathcal{K}$ be the simplical complex as in Example (1.4), $\mathcal{L}$ be the simplical complex with $V(\mathcal{L}) = \{y_1, y_2, y_3\}$ and $S(\mathcal{L})_{\max} = \{y_1y_3, y_2\}$, and $\mathcal{M}$ be the simplicial complex with $V(\mathcal{M}) = \{z_1, z_2, z_3\}$ and $S(\mathcal{M})_{\max} = \{z_1z_2, z_1z_3, z_2z_3\}$. Then the maps $f : \mathcal{K} \to \mathcal{L}$ and $g : \mathcal{K} \to \mathcal{M}$ induced by

$$\begin{aligned} f(x_1) &= y_1 \\ f(x_2) &= y_3 \\ f(x_3) &= y_1 \\ f(x_4) &= y_3 \\ f(x_5) &= y_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} g(x_1) &= z_1 \\ g(x_2) &= z_2 \\ g(x_3) &= z_2 \\ g(x_4) &= z_3 \\ g(x_5) &= z_1 \end{aligned}$$

are simplicial maps.

- Triangulations

A **triangulation** of a topological space X is a homeomorphism $h : |\mathcal{K}| \to X$, where $\mathcal{K}$ is a simplicial complex. A topological space is **triangulable** if it admits a triangulation. A compact subspace A of $\mathbb{R}^n$ is a **convex cell** if it is the set of solutions of families of affine equations and inequalities

$$f_i = 0, \quad i = 1, \dots, r \quad \text{and} \quad g_j \geq 0, \quad j = 1, \dots s$$

A face B of A is a convex cell obtained by repacing some of the inequalities $g_j \geq 0$ by the set equations $g_j = 0$. For example, the standard Euclidean simplex $\Delta^2 \subset \mathbb{R}^3$ is a convex cell with

$$f_1 = x + y + z - 1, \quad g_1 = x, \quad g_2 = y, \quad \text{and} \quad g_3 = z$$

One face of Δ^2 is given by

$$f_1 = x + y + z - 1, \quad f_2 = x, \quad g_1 = y, \quad \text{and} \quad g_2 = z$$

Example 1.5. The real projective plane $\mathbb{RP}^2$ admits the following triangulation: Let

$$\begin{array}{llllll}
\ell_1 = x & \ell_4 = x - y & \ell_7 = x - y + z & a = [1:0:0] & d = [0:1:1] \\
\ell_2 = y & \ell_5 = x - z & \ell_8 = x + y - z & b = [0:1:0] & e = [1:1:0] \\
\ell_3 = z & \ell_6 = y - z & \ell_9 = -x + y + z & c = [0:0:1] & f = [1:0:1]
\end{array}$$

This gives us the following triangulation of $\mathbb{RP}^2$.