

Mathematics Diary

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1 2023

1.1 12/20/2022 - When $\Sigma(F/E)$ is the minimal free resolution of I/J over R

Lemma 1.1. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R . Let E be the minimal free resolution of R/J over R , let F be the minimal free resolution of R/I over R , and let $\varphi: E \rightarrow F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi: E \rightarrow F$ and $\bar{\varphi}: E_{\mathbb{k}} := E \otimes_R \mathbb{k} \rightarrow F \otimes_R \mathbb{k} := F_{\mathbb{k}}$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R .*

Proof. Assume both $\varphi: E \rightarrow F$ and $\bar{\varphi}: E_{\mathbb{k}} := E \otimes_R \mathbb{k} \rightarrow F \otimes_R \mathbb{k} := F_{\mathbb{k}}$ are injective. Since $\varphi: E \rightarrow F$ is injective, we have a short exact sequence of R -complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \quad (1)$$

taking homology gives us a long exact sequence

$$\begin{array}{c} \dots \longrightarrow H_{i+1}(F/E) \longrightarrow \\ \quad \downarrow \qquad \qquad \qquad \downarrow \\ \hookrightarrow H_i(E) \longrightarrow H_i(F) \longrightarrow H_i(F/E) \longrightarrow \\ \quad \downarrow \qquad \qquad \qquad \downarrow \\ \hookrightarrow H_{i-1}(E) \longrightarrow \dots \end{array}$$

Since E and F are resolutions we conclude that $H_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $H_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R -modules (1) with \mathbb{k} over R gives us the long exact sequence in homology

$$\begin{array}{c}
\cdots \longrightarrow \mathrm{Tor}_{i+1}^R(E, \mathbb{k}) \\
\downarrow \\
\mathrm{Tor}_i^R(E, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(F, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(F/E, \mathbb{k}) \\
\downarrow \\
\mathrm{Tor}_{i-1}^R(E, \mathbb{k}) \longrightarrow \cdots
\end{array}$$

Since E and F are free R -modules we conclude that $\mathrm{Tor}_i(F/E, \mathbb{k}) = 0$ for all $i \geq 1$. Since $\bar{\varphi}: E \otimes_R \mathbb{k} \rightarrow F \otimes_R \mathbb{k}$ is injective we conclude that $\mathrm{Tor}_1(F/E, \mathbb{k}) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $d(F/E) \subseteq \mathfrak{m}(F/E)$). \square

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R -module M connecting maps

$$\mathrm{Tor}_{i+1}^R(R/I, M) \rightarrow \mathrm{Tor}_i^R(I/J, M) \quad \text{and} \quad \mathrm{Ext}_R^i(I/J, M) \rightarrow \mathrm{Ext}_R^{i+1}(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \rightarrow F/E \otimes_R M \quad \text{and} \quad \mathrm{Hom}_R^*(F/E, M) \rightarrow \mathrm{Hom}_R^*(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\bar{\varphi}: E_{\mathbb{k}} \rightarrow F_{\mathbb{k}}$ is injective, then already $\varphi: E \rightarrow F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of ideals

Let R be a commutative ring and let \mathfrak{p} be an ideal of R . Recall the **height** of \mathfrak{p} is defined to be the supremum of lengths of chains of primes which descend from \mathfrak{p} :

$$\mathrm{ht} \mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

When R is Noetherian, then Krull's principal ideal theorem states that there exists an ideal $\langle x \rangle = \langle x_1, \dots, x_c \rangle \subseteq \mathfrak{p}$ where $c = \mathrm{ht} \mathfrak{p}$ such that $\sqrt{\langle x \rangle} = \mathfrak{p}$, and that if $\langle y \rangle = \langle y_1, \dots, y_m \rangle$ is another ideal such that $\sqrt{\langle y \rangle} = \mathfrak{p}$, then we must have $c \leq m$. If I is an ideal of R , then the **height** of I is defined to be the infimum of the heights of all primes which contain I :

$$\mathrm{ht} I = \inf\{\mathrm{ht} \mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. Let I_1 and I_2 be ideals of R . Set $c = \mathrm{ht}(I_1 \cap I_2)$, set $c_1 = \mathrm{ht} I_1$, and set $c_2 = \mathrm{ht} I_2$.

1. If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.
2. We have $c = \min\{c_1, c_2\}$.

Proof. 1. Let \mathfrak{p} be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2 . Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let \mathfrak{p} be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since \mathfrak{p} is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. \square

2 2024

1/20/2024 - $V(\mathrm{Ann} M) = V(\mathrm{Ann}(0 :_M x))$

Lemma 2.1. Let R be a commutative ring, let M be an R -module, and let $x \in R$. Then

$$V(\mathrm{Ann}(0 :_M x)) = V(\mathrm{Ann}(0 :_M x^2)).$$

Proof. Note that $0 :_M x \subseteq 0 :_M x^2$ implies $\text{Ann}(0 :_M x^2) \supseteq \text{Ann}(0 :_M x)$ which implies $V(\text{Ann}(0 :_M x^2)) \subseteq V(\text{Ann}(0 :_M x))$. For the reverse inclusion, suppose \mathfrak{p} is a prime ideal of R which contains $\text{Ann}(0 :_M x^2)$ and let $r \in \text{Ann}(0 :_M x)$. We claim that $r^2 \in \text{Ann}(0 :_M x^2)$. Indeed, if $u \in 0 :_M x^2$, then

$$\begin{aligned} x^2 u = 0 &\implies xu \in 0 :_M x \\ &\implies rxu = 0 \\ &\implies ru \in 0 :_M x \\ &\implies r^2 u = 0. \end{aligned}$$

Since u was arbitrary, we see that $r^2 \in \text{Ann}(0 :_M x^2) \subseteq \mathfrak{p}$. However this implies $r \in \mathfrak{p}$ since \mathfrak{p} is a prime. Since r was arbitrary, we see that $\text{Ann}(0 :_M x) \subseteq \mathfrak{p}$. \square

Corollary 1. *Let R be a commutative ring and let M be a finitely generated R -module. Assume that $x \in R$ acts nilpotently on M . Then*

$$V(\text{Ann}(M)) = V(\text{Ann}(0 :_M x)).$$

Proof. Since M is finitely generated, there exists an $n \in \mathbb{N}$ such that $M = 0 :_M x^n$. A straightforward induction on $(?)$ gives us

$$V(\text{Ann}(M)) = V(\text{Ann}(0 :_M x^n)) = V(\text{Ann}(0 :_M x)).$$

\square

1/21/2024 - Some subschemes of \mathbb{P}^3

Let $R = \mathbb{k}[x, y, z, w]$. We consider three cyclic R -algebras, namely $A = R/\mathbf{f} = R/\langle f_1, f_2, f_3 \rangle$, $B = R/\mathbf{g} = R/\langle g_1, g_2, g_3 \rangle$, and $C = R/\mathbf{h} = R/\langle h_1, h_2, h_3 \rangle$ where

$$\begin{array}{lll} f_1 = xy - zw & g_1 = xz - y^2 & h_1 = xz - y^2 \\ f_2 = xz - yw & g_2 = yw - z^2 & h_2 = x^3 - yzw \\ f_3 = xw - yz & g_3 = xw - yz & h_3 = x^2 y - z^2 w \end{array}$$

We want a geometric picture in mind when thinking of these rings, so let $X = \text{Proj } A$, $Y = \text{Proj } B$, and $Z = \text{Proj } C$. First let us consider X . We can see that $X(\mathbb{k})$ consists of 8 distinct points in $\mathbb{P}^3(\mathbb{k})$ by calculating an irreducible primary decomposition for $\langle \mathbf{f} \rangle$. Indeed, an irredundant primary decomposition for $\langle \mathbf{f} \rangle$ is given by $\langle \mathbf{f} \rangle = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_8$ where

$$\begin{array}{ll} \mathfrak{p}_1 = \langle y, z, w \rangle & \mathfrak{p}_5 = \langle x + y, y + z, z + w \rangle \\ \mathfrak{p}_2 = \langle x, z, w \rangle & \mathfrak{p}_6 = \langle x + y, y - z, z + w \rangle \\ \mathfrak{p}_3 = \langle x, y, w \rangle & \mathfrak{p}_7 = \langle x + y, y - z, z - w \rangle \\ \mathfrak{p}_4 = \langle x, y, z \rangle & \mathfrak{p}_8 = \langle x - y, y - z, z - w \rangle. \end{array}$$

These primes correspond to the points

$$\begin{array}{ll} \mathbf{p}_1 = [1 : 0 : 0 : 0] & \mathbf{p}_5 = [-1 : 1 : -1 : 1] \\ \mathbf{p}_2 = [0 : 1 : 0 : 0] & \mathbf{p}_6 = [1 : -1 : -1 : 1] \\ \mathbf{p}_3 = [0 : 0 : 1 : 0] & \mathbf{p}_7 = [-1 : 1 : 1 : 1] \\ \mathbf{p}_4 = [0 : 0 : 0 : 1] & \mathbf{p}_8 = [1 : 1 : 1 : 1] \end{array}$$

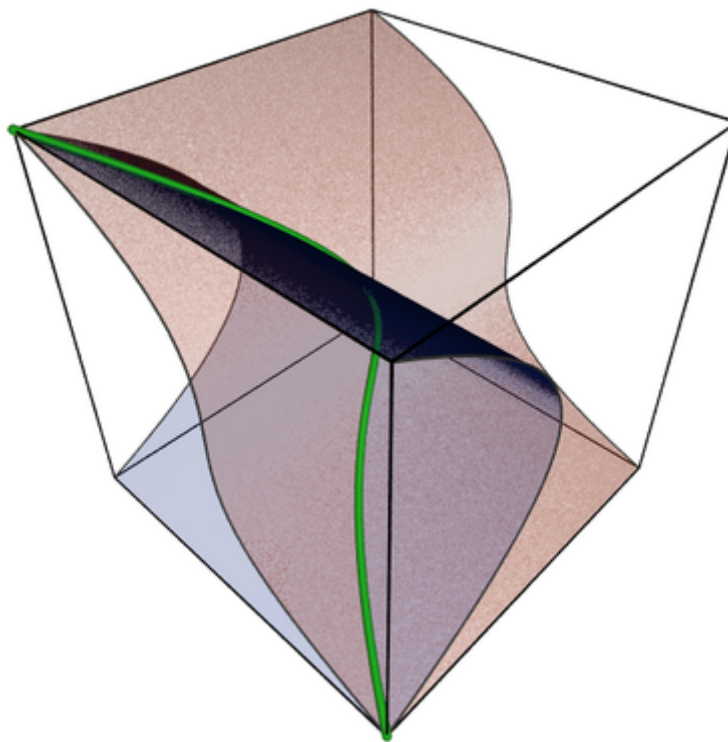
in $\mathbb{P}^3(\mathbb{k})$. Note that $\mathbf{p}_1, \dots, \mathbf{p}_8$ are in linearly general position since the size 4 minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

are all nonzero. In other words, viewing $\mathbf{p}_1, \dots, \mathbf{p}_8$ as vectors in \mathbb{k}^4 , every subset of $\{\mathbf{p}_1, \dots, \mathbf{p}_8\}$ of size 4 is linearly independent. The Betti diagram of A over R is given by

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & - & - \\ 2 & - & - & 3 & - \\ 2 & - & - & - & 1 \end{array}$$

Next we consider Y . In fact, Y is the twisted cubic. When $\mathbb{k} = \mathbb{R}$, we can visualize $Y(\mathbb{k})$ as below:



In particular, $Y(\mathbb{k})$ is the image of the map $\mathbb{P}^1(\mathbb{k}) \rightarrow \mathbb{P}^3(\mathbb{k})$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Note that $\langle g \rangle$ is a prime of height 2 and so $\langle g \rangle$ can be generated up to radical by two homogeneous polynomials. In particular, we have $\langle g \rangle = \sqrt{\langle g_1, g_4 \rangle}$ where $g_4 = zg_2 - wg_3$. However $\langle g \rangle$ itself cannot be generated by only two polynomials; a minimum of three polynomials are needed. We can see this in Betti diagram of B over R :

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & - & - \\ 1 & - & 3 & 2 \end{array}$$

In particular, the Hilbert-Poincare series of B over R is given by

$$P(t) = \frac{1 - 3t^2 + 2t^3}{(1 - t)^4} = \frac{1 + 2t}{(1 - t)^2} = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + \dots$$

Thus Y is the set-theoretic complete intersection of $V(g_1)$ and $V(g_4)$ however it is not a scheme-theoretic or ideal-theoretic complete intersection. Note also that $\langle g \rangle$ corresponds to the ideal of size 2 minors of the matrix $\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$. Up to linear automorphism, the twisted cubic is the only irreducible curve of degree 3 not contained in a plane. Furthermore, any 6 points in linearly general position in $\mathbb{P}^3(\mathbb{k})$ lie on a unique twisted cubic. However for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Consequently one can show that if W is a set of 7 points in linearly general position in $\mathbb{P}^3(\mathbb{k})$, then there are only two distinct Betti diagrams possible for the homogeneous coordinate ring of W , namely

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & - & - \\ 2 & - & 1 & 6 & 3 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & 2 & - \\ 2 & - & 3 & 6 & 3 \end{array}$$

In the first case, the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the quadrics containing W is the ideal of the unique curve of degree 3 containing W , which is irreducible. Finally, let us write down the minimal free resolution of B over R :

$$R(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} R \longrightarrow 0$$

Now we consider Z . The Betti diagram of C over R is given by

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & - & - \\ 1 & - & 1 & - \\ 2 & - & 2 & 2 \end{array}$$

In particular, the Hilbert-Poincare series of C over R is given by

$$P(t) = \frac{1 - t^2 - 2t^3 + 2t^4}{(1 - t)^4} = \frac{1 + 2t + 2t^2}{(1 - t)^2} = 1 + 4t + 9t^2 + 14t^3 + 19t^4 + \dots$$

In particular, Z is an irreducible curve of degree 5 in $\mathbb{P}^3(\mathbb{k})$.

2.1 4/22/2024 - Lifting multiplication to a free module

Let A be a commutative ring and let B be an A -algebra which is finite as an A -module. Then there exists a surjection $F \twoheadrightarrow B$ of A -modules where $F = A^{n+1}$ where we assume $n \geq 0$ is minimal. We are interested in the question as to whether one can lift the multiplication on B to a multiplication on F . Let K be the kernel of the map $F \twoheadrightarrow B$. In what follows, all tensor products are taken over A .

Lemma 2.2. *The kernel of the map $F^{\otimes 2} \rightarrow B^{\otimes 2}$ is given by $K \otimes F + F \otimes K$.*

Proof. This is easily checked via a diagram chase in the diagram below which is exact everywhere and in all directions:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & K^{\otimes 2} & \longrightarrow & K \otimes F & \longrightarrow & K \otimes B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F \otimes K & \longrightarrow & F^{\otimes 2} & \longrightarrow & F \otimes B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & B \otimes K & \longrightarrow & B \otimes F & \longrightarrow & B^{\otimes 2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

□

Since $F^{\otimes 2}$ is free (hence projective), we can lift the composite map $F^{\otimes 2} \rightarrow B^{\otimes 2} \twoheadrightarrow B$ with respect to the map $F \twoheadrightarrow B$ to obtain an A -linear map $\mu: F^{\otimes 2} \rightarrow F$. Assume that A is a local noetherian ring. In this case, there exists a minimal generating set of B as an A -module of the form $\{b_0, b_1, \dots, b_n\}$ where $b_0 = 1$. Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ be a basis for F as a free A -module and let $F \twoheadrightarrow B$ be the A -linear map defined by $\varepsilon_i \mapsto b_i$ for all i . For each i, j , we have

$$b_i b_j = \sum_k a_{ij}^k b_k$$

where the $a_{ij}^k \in A$ need not be unique. Since the multiplication on B is unital, we can choose the a_{ij}^k such that

$$a_{j0}^k = a_{0j}^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Furthermore, since the multiplication on B is commutative, we can also choose the a_{ij}^k such $a_{ij}^k = a_{ji}^k$. With these choices of a_{ij}^k in mind, we can define a commutative and unital multiplication μ on F which lifts the multiplication on B by

$$\varepsilon_i \varepsilon_j := \sum_k a_{ij}^k \varepsilon_k.$$

Note that this multiplication need not be associative. Indeed, since the multiplication on B is associative, we have

$$\begin{aligned} [b_i, b_j, b_k] &= (b_i b_j) b_k - b_i (b_j b_k) \\ &= \sum_l (a_{ij}^l b_l b_k - a_{jk}^l b_i b_l) \\ &= \sum_{l,m} (a_{ij}^l a_{lk}^m - a_{jk}^l a_{il}^m) b_m. \end{aligned}$$

However this need not imply that $a_{ij}^l a_{lk}^m - a_{jk}^l a_{il}^m = 0$ for all i, j, k, l, m (which is what we'd need in order for $[\varepsilon_i, \varepsilon_j, \varepsilon_k] = 0$).

2.2 5/2/2024 - Colon ideal result

Let R be a noetherian ring, let I be an ideal of R , and let $r, r' \in R$. We have an R -linear map

$$\varphi: \langle I, r \rangle : r' \twoheadrightarrow (\langle I, r' \rangle : r) / (I : r)$$

defined as follows: if $a \in \langle I, r \rangle : r'$, then we have $ar' = br + x$ for some $b \in R$ and $x \in I$. The map is defined by sending a to the class of b in the quotient. It is straightforward to check that this is well-defined and surjective. Note if $b \in I : r$, then $ar' \in I : r'$. In particular, the kernel of φ is $I : r'$. Thus we've established an isomorphism

$$(\langle I, r \rangle : r') / (I : r') \cong (\langle I, r' \rangle : r) / (I : r). \quad (2)$$

In particular, if $I : r' = I : r$, then we must have $\langle I, r \rangle : r' = \langle I, r' \rangle : r$. Now assume that $I : r = \mathfrak{p} = \langle I, r \rangle : r'$. Then (2) implies

$$\mathfrak{p} / (I : r') \cong (\langle I, r' \rangle : r) / \mathfrak{p}.$$

Example 2.1. Let $R = \mathbb{k}[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz \rangle$, let $r = yw$, and let $r' = y$. Then we have

$$\begin{aligned} I : r &= \langle x, z, w \rangle & \langle I, r' \rangle : r &= R \\ I : r' &= \langle x, z, w^2 \rangle & \langle I, r \rangle : r' &= \langle x, z, w \rangle. \end{aligned}$$

Now observe that $\langle I : r, r' \rangle \subseteq \langle I, r' \rangle : r$. Indeed, if $a \in \langle I : r, r' \rangle$, then we can express it as $a = b + cr'$ where $b \in I : r$ and $c \in R$. In particular, this means that $ar = br + cr'r \in \langle I, r' \rangle$, and hence $a \in \langle I, r' \rangle : r$.

2.3 5/20/2024 - Geometric description of finitely generated \mathbb{k} -algebra homomorphisms

Let $\mathbb{k}[x] = \mathbb{k}[x_1, \dots, x_m]$, let $\mathbb{k}[y] = \mathbb{k}[y_1, \dots, y_n]$, and let $\varphi: \mathbb{k}[x] \rightarrow \mathbb{k}[y]$ be a \mathbb{k} -algebra homomorphism. Then the φ corresponds to the morphism of affine schemes $f: \mathbb{A}_{\mathbb{k}}^n \rightarrow \mathbb{A}_{\mathbb{k}}^m$ given by $f(q) = \varphi^{-1}(q)$ for all $q \in \mathbb{A}_{\mathbb{k}}^n$. We want to give a more geometric description of how f acts on the points of $\mathbb{A}_{\mathbb{k}}^n$, or in other words, how φ^{-1} acts on the prime ideals of $\mathbb{k}[y]$. First, note that since $\mathbb{k}[y]$ is Jacobson, we have

$$\varphi^{-1}(q) = \varphi^{-1} \left(\bigcap_{\substack{\mathfrak{n} \supseteq q \\ \mathfrak{n} \text{ maximal}}} \mathfrak{n} \right) = \bigcap_{\substack{\mathfrak{n} \supseteq q \\ \mathfrak{n} \text{ maximal}}} \varphi^{-1}(\mathfrak{n}).$$

Thus we may reduce to the case where $q = \mathfrak{n}$ is a maximal ideal. First let's consider the maximal ideals of the form $\mathfrak{n}_q = \langle y_1 - q_1, \dots, y_n - q_n \rangle$ where $q \in \mathbb{A}_{\mathbb{k}}^n(\mathbb{k}) = \mathbb{k}^n$. To this end, for each $1 \leq i \leq m$ let $f_i = \varphi(x_i)$, and let $f: \mathbb{k}^n \rightarrow \mathbb{k}^m$ be the map given by $f(q) = (f_1(q), \dots, f_m(q))$. Then we claim that

$$\varphi^{-1}(\mathfrak{n}_q) = \mathfrak{m}_{f(q)} = \langle x_1 - f_1(q), \dots, x_m - f_m(q) \rangle.$$

Indeed, observe that

$$\begin{aligned} \varphi(\mathfrak{m}_{f(q)}) &= \langle \varphi(x_1) - f_1(q), \dots, \varphi(x_m) - f_m(q) \rangle \\ &= \langle f_1 - f_1(q), \dots, f_m - f_m(q) \rangle \\ &\subseteq \mathfrak{n}_q. \end{aligned}$$

This shows that $\mathfrak{m}_{f(q)} \subseteq \varphi^{-1}(\mathfrak{n}_q)$. We get the reverse inclusion from the fact that $\mathfrak{m}_{f(q)}$ is a maximal ideal of A . More generally, let \mathfrak{n} be an maximal ideal of $\mathbb{k}[y]$. Then there exists a maximal ideal of the form \mathfrak{n}_q of $\overline{\mathbb{k}}[y]$, where $q \in \overline{\mathbb{k}}^n$, which lies over \mathfrak{n} . Furthermore, there are only finitely many maximal ideals of $\overline{\mathbb{k}}[y]$ which lie over \mathfrak{n} and they all have the form $\mathfrak{n}_{\sigma q}$ for some $\sigma \in \text{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ where $\sigma q = (\sigma q_1, \dots, \sigma q_n)$ (this follows from a general proposition in commutative algebra which we state and proof at the end of this entry below). Then we have

$$\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}_{f(q)} \cap \mathbb{k}[x] := \mathfrak{m}.$$

Note this does not depend on the choice of maximal ideal which lies over \mathfrak{n} , for if $\mathfrak{n}_{\sigma q}$ where another maximal ideal of $\overline{\mathbb{k}}[y]$ which lies over \mathfrak{n} , then $\mathfrak{m}_{f(\sigma q)} = \mathfrak{m}_{\sigma f(q)}$ also lies over \mathfrak{m} .

Example 2.2. The maximal ideals $\mathfrak{n}_{i, \zeta_8}$, $\mathfrak{n}_{i, \zeta_8^5}$, $\mathfrak{n}_{-i, \zeta_8^3}$, and $\mathfrak{n}_{-i, \zeta_8^7}$ lie over $\mathfrak{n} = \langle y_1^2 + 1, y_2^2 + y_1 \rangle$.

Proposition 2.1. *Let A be an integral domain which is integrally closed in its field of fractions K , let L be a normal extension of K , let B be the integral closure of A in L , let G be the group of automorphisms of L over K , and let \mathfrak{p} be a prime ideal of A . Then G acts transitively on the set of all primes of B which lie over \mathfrak{p} .*

Proof. We first consider the case where G is finite. Let \mathfrak{q} and \mathfrak{q}' be two prime ideals of B which lie over \mathfrak{p} . Then the $\sigma \mathfrak{q}$ (where $\sigma \in G$) is an ideal of B which lies over \mathfrak{p} since B is integrally closed in L , and it suffices to show that \mathfrak{q}' is contained in one of them, or equivalently, in their union by prime avoidance. Let $y \in \mathfrak{q}'$ and let $x = \prod \sigma y$ where the product runs over $\sigma \in G$. Note that x is fixed by G , thus since L/K is normal, it follows that there exists a power q of the characteristic of K such that $x^q \in K$. In particular, $x^q \in K \cap B = A$ since A is integrally closed. Thus $x^q \in \mathfrak{q}' \cap A = \mathfrak{p}$, which shows that x^q is contained in \mathfrak{q} . It follows that there exists a $\sigma \in G$ such that $\sigma y \in \mathfrak{q}$, whence $y \in \sigma^{-1} \mathfrak{q}$.

For the general case, assume \mathfrak{q} and \mathfrak{q}' lie over \mathfrak{p} . For every subfield E of L which is a finite normal extension over K , let G_E be the subset of G which consists of all $\sigma \in G$ which transform $\mathfrak{q} \cap E$ to $\mathfrak{q}' \cap E$. This is a closed subspace of G , hence compact since G is compact. Furthermore, each G_E is non-empty by what was shown above. As the G_E form a decreasing filtered family, their intersection is non-empty. \square

2.4 5/21/2024 - Turning $\text{Tor}^R(M_1, M_2)$ into an R -complex

Let R be a commutative ring, let M_1 and M_2 be R -modules, and set $T = \text{Tor}^R(M_1, M_2)$. We can turn T into an R -complex as follows: choose projective resolutions F^1 of M_1 and F^2 of M_2 over R . Then $d \otimes 1: F^1 \otimes_R F^2 \rightarrow F^1 \otimes_R F^2$ is a chain map of degree -1 , thus it induces a map in homology $d \otimes 1: T \rightarrow T$. Furthermore $(d \otimes 1)^2 = 0$ and so $d \otimes 1$ gives T an R -complex structure. There are map $\gamma_i^{31}: T_i^{31} \rightarrow T_{i-1}^{31}$ defined to be the composite

$$T_i^{31} \rightarrow T_i^{32} \rightarrow T_{i-1}^{12} \rightarrow T_{i-1}^{13} = T_{i-1}^{31}.$$

Similarly, we define $\gamma_i^{32}: T_i^{32} \rightarrow T_{i-1}^{32}$ to be the composite

$$T_i^{32} \rightarrow T_{i-1}^{12} \rightarrow T_{i-1}^{13} \rightarrow T_{i-1}^{23} = T_{i-1}^{32},$$

and we define $\gamma_i^{21}: T_i^{21} \rightarrow T_{i-1}^{21}$ to be the composite

$$T_i^{21} \rightarrow T_i^{31} \rightarrow T_i^{32} \rightarrow T_{i-1}^{12} = T_{i-1}^{21}$$

Actually I just realized these are all just the zero map.

2.5 5/29/2024 - Ext result of my paper

Proposition 2.2. *Let R be a regular local ring, let I be an ideal of R , let F be the minimal free resolution of R/I over R , and let $S = S_R(F)$ be the symmetric DG algebra of F over R . There exists a surjective chain map $\pi: S \rightarrow F$ which splits the inclusion map $F \hookrightarrow S$.*

Proof. It suffices to show that $\text{Ext}_R^1(S/F, F) = 0$. Note that the underlying graded R -module of S/F is just $S^{\geq 2}$. In particular, S/F is semi-projective, thus $\text{Hom}_R^*(S/F, -)$ preserves quasi-isomorphisms. It follows that

$$\text{Ext}_R^1(S/F, F) = \text{Ext}_R^1(S/F, R/I) = 0,$$

where the last part follows from the fact that R/I sits in homological degree 0 but $(S/F)_i = 0$ for all $i \leq 1$. \square

Remark 3. Note that giving a surjective chain map $\pi: S \twoheadrightarrow F$ which splits the inclusion map is equivalent to giving chain maps $\pi^n: F^{\otimes n} \rightarrow F$ for each $n \geq 2$ such that each π^n is strictly commutative and such that for all $1 \leq i \leq n$ and for all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in F_+$ we have

$$\pi^n(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_n) = \pi^{n-1}(a_1, \dots, a_{i-1}, a_i, \dots, a_n).$$

For instance, if a_1, a_2, a_3 are homogeneous elements in F with $|a_1| = 1$ and $|a_2|, |a_3| \geq 2$, then we have

$$d\pi^3(a_1, a_2, a_3) = r_1 \pi^2(a_2, a_3) - \pi^3(a_1, da_2, a_3) + \pi^3(a_1, a_2, da_3),$$

where $r_1 = da_1$.

2.6 6/15/2024 - Associated primes of $\text{Hom}_R(M, N)$

Today we prove the following result:

Proposition 2.3. *Let R be a noetherian ring and let M and N be R -modules such that M is finitely generated. Then*

$$\text{Ass}(\text{Hom}_R(M, N)) = \text{Supp } M \cap \text{Ass } N = V(\text{Ann } M) \cap \text{Ass } N.$$

Proof. Let \mathfrak{p} be an associated prime of $\text{Hom}_R(M, N)$. Thus there exists an R -linear map $\varphi: M \rightarrow N$ such that $\mathfrak{p} = 0 : \varphi = \{a \in R \mid a\varphi = 0\}$. Let u_1, \dots, u_m be generators of M as an R -module and let $v_1, \dots, v_m \in N$ be their respective images under φ . Then note that $a\varphi = 0$ if and only if $av_i = 0$ for all $1 \leq i \leq m$.

$$\begin{aligned} a \in \mathfrak{p} &\iff a\varphi = 0 \\ &\iff av_i = 0 \text{ for all } i \\ &\iff a \in \bigcap_{i=1}^m 0 : v_i. \end{aligned}$$

In particular we see that $\mathfrak{p} = \bigcap_{i=1}^m 0 : v_i$. Since \mathfrak{p} is prime, we see that $\mathfrak{p} = 0 : v_i$ for some i , or in other words, \mathfrak{p} is an associated prime of N . Next, assume for a contradiction that $M_{\mathfrak{p}} = 0$. Then for each i there exists an $s_i \in R \setminus \mathfrak{p}$ such that $s_i u_i = 0$. However this implies $s = s_1 \cdots s_n$ is in \mathfrak{p} since $sv_i = \varphi(su_i) = 0$ for all i , which is a contradiction. Therefore \mathfrak{p} is in the support of M . Thus far we have shown

$$\text{Ass}(\text{Hom}_R(M, N)) \subseteq \text{Supp } M \cap \text{Ass } N.$$

For the converse direction, suppose \mathfrak{p} is in the support of M and is an associated prime of N , so $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} = 0 : v$ for some $v \in N$. Since $M_{\mathfrak{p}} \neq 0$, there exists an i such that $0 : u_i \subseteq \mathfrak{p} = 0 : v$. By reordering if necessary, we may assume that $0 : u_1 \subseteq \mathfrak{p} = 0 : v$. One would like to define an R -linear map $\varphi: M \rightarrow N$ such that $\varphi(u_1) = v$, but it's not clear how we should define it on the u_i for all $2 \leq i \leq m$. Let us cut to the chase and show how one usually proves this result: we have

$$\begin{aligned} \mathfrak{p} \in \text{Ass}(\text{Hom}_R(M, N)) &\iff \mathfrak{p}_{\mathfrak{p}} \in \text{Ass}(\text{Hom}_R(M, N)_{\mathfrak{p}}) \\ &\iff \mathfrak{p}_{\mathfrak{p}} \in \text{Ass}(\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})) \\ &\iff \text{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})) \neq 0 \\ &\iff \text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0 \\ &\iff M_{\mathfrak{p}} \neq 0 \text{ and } \text{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), N_{\mathfrak{p}}) \neq 0 \\ &\iff \mathfrak{p} \in \text{Supp } M \cap \text{Ass } N, \end{aligned}$$

where in the second last if and only if we used the fact that $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is a finite dimensional $\kappa(\mathfrak{p})$ (so it is a direct sum of $\kappa(\mathfrak{p})$'s). Note that we needed Nakayama's lemma for the statement $M_{\mathfrak{p}} \neq 0$ if and only if $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \neq 0$, hence why we needed a noetherian hypothesis on R . The last equality comes from the fact that since M is finitely generated, we have $\text{Supp } M = V(\text{Ann } M)$. \square

Corollary 2. *Let R be a noetherian domain, let M be a finitely generated R -module, and let $M^{\vee} := \text{Hom}_R(M, R)$ be the dual of M . If $M^{\vee} \neq 0$, then $\text{Ass } M^{\vee} = \{0\}$.*

Remark 4. Note that if L and M are finitely generated R -modules, then tensor-hom adjointness implies

$$\begin{aligned} V(\text{Ann}(L \otimes_R M)) \cap \text{Ass } N &= \text{Supp}(L \otimes_R M) \cap \text{Ass } N \\ &= \text{Ass}(\text{Hom}_R(L \otimes_R M, N)) \\ &= \text{Ass}(\text{Hom}_R(L, \text{Hom}_R(M, N))) \\ &= (\text{Supp } L) \cap (\text{Supp } M) \cap \text{Ass } N \\ &= V(\langle \text{Ann } L, \text{Ann } M \rangle) \cap \text{Ass } N \end{aligned}$$

for all R -modules N . In particular, we have

$$V(\text{Ann}(L \otimes_R M)) = V(\text{Ann } L) \cap V(\text{Ann } M) = V(\langle \text{Ann } L, \text{Ann } M \rangle).$$

2.7 6/25/2024 - Inverse limit of $\cdots \rightarrow R \xrightarrow{r} R \xrightarrow{r} R$

Today I want to discuss a result I was thinking about while driving to my parents house the other day. Let R be a ring and let $r \in R$. Consider the inverse system:

$$\mathcal{R} = \cdots \rightarrow R \xrightarrow{r} R \xrightarrow{r} R.$$

We set $A = \lim \mathcal{R}$. Then A consists of the set of all sequences (a_n) where $a_n \in R$ such that $r^m a_n = a_{n-m}$ for all $0 \leq m \leq n$. If R is an integral domain, then we can equivalently describe this as the set of all sequences (a_n) such that $r^n a_n = a_0$ for all $0 \leq n$. In particular, if $(a_n) \in A$, then we must have

$$a_m \in \bigcap_{n=1}^{\infty} \langle r \rangle^n := I.$$

for all $m \in \mathbb{N}$. Thus if $I = 0$, then necessarily $A = 0$. Krull's intersection theorem gives us $I = 0$ for many important rings that we care about. For example, if R is a noetherian local ring with maximal ideal \mathfrak{m} and $r \in \mathfrak{m}$, then $I = 0$. Thus the inverse limit of the inverse system \mathcal{R} would be 0 in this case. On the other hand, consider the direct system:

$$\mathcal{S} = R \xrightarrow{r} R \xrightarrow{r} R \rightarrow \cdots.$$

Then we have $R_r = \text{colim } \mathcal{S}$. We have $R_r = 0$ if and only if r is nilpotent.

2.8 7/28/2024 - If $ZG = 1$, then $Z(\text{Aut } G) = 1$

Here's a neat proposition in Group Theory that I proved involving the automorphism group of a centerless group.

Proposition 2.4. *Let G be a group such that $ZG = 1$ and let $A = \text{Aut } G$ be the automorphism group of G . The only automorphism of G which commutes with every inner automorphism of G is the identity automorphism. In particular, we have $ZA = 1$.*

Proof. Suppose φ is an automorphism of G which commutes with every inner automorphism of G . Thus we have

$$c_g \varphi = \varphi c_g = c_{\varphi g} \varphi$$

for all $g \in G$, or in other words, we have

$$g\varphi(x)g^{-1} = \varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$$

for all $x, g \in G$. Replacing x with $\varphi^{-1}x$ above and rearranging terms, we see that

$$(\varphi g)^{-1}gx = x(\varphi g)^{-1}g$$

for all $x, g \in G$. Since $ZG = 1$, we must have $(\varphi g)^{-1}g = 1$, or in other words, $\varphi g = g$ for all $g \in G$. It follows that $\varphi = 1$. \square

2.9 8/18/2024 - flatness and projectiveness are stable under composition

Today I updated the 5/20/2024 entry. In today's entry, I want to prove the following:

Proposition 2.5. *Let $A \rightarrow B$ be a ring homomorphism and let C be a B -module.*

1. *If B is A -flat and C is B -flat, then C is A -flat.*
2. *If B is A -projective and C is B -projective, then C is A -projective.*

Proof. Suppose $M \hookrightarrow M'$ is an injective A -module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{array}{ccc}
 C \otimes_A M & \longrightarrow & C \otimes_A M' \\
 \downarrow \simeq & & \downarrow \simeq \\
 (C \otimes_B B) \otimes_A M & \longrightarrow & (C \otimes_B B) \otimes_A M' \\
 \downarrow \simeq & & \downarrow \simeq \\
 C \otimes_B (B \otimes_A M) & \hookrightarrow & C \otimes_B (B \otimes_A M')
 \end{array}$$

The bottom arrow is injective since B is A -flat and C is B -flat. Therefore $C \otimes_A M \hookrightarrow C \otimes_A M'$ is injective; whence C is A -flat.

Now suppose that $M \twoheadrightarrow M'$ is a surjective A -module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{array}{ccc}
 \mathrm{Hom}_A(C, M) & \longrightarrow & \mathrm{Hom}_A(C, M') \\
 \downarrow \simeq & & \downarrow \simeq \\
 \mathrm{Hom}_A(C \otimes_B B, M) & \longrightarrow & \mathrm{Hom}_A(C \otimes_B B, M') \\
 \downarrow \simeq & & \downarrow \simeq \\
 \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M)) & \twoheadrightarrow & \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M'))
 \end{array}$$

The bottom arrow is surjective since B is A -projective and C is B -projective. Therefore $\mathrm{Hom}_A(C, M) \twoheadrightarrow \mathrm{Hom}_A(C, M')$ is surjective; whence C is A -projective. \square

2.10 8/24/2024 - Connected integral domain has stalkwise local property

Proposition 2.6. *Let R be a connected commutative ring. Then R is an integral domain if and only if $R_{\mathfrak{p}}$ is an integral domain for each prime \mathfrak{p} of R .*

The reason we need R to be connected is because the ring $R = K \times K$ where K is a field is clearly not an integral domain but the localization at each prime of R is isomorphic to K which is an integral domain.