BEH Conjecture

We prove the BEH conjecture in the following special case:

Theorem 0.1. *Let R be a regular local ring and let I be an ideal of R of grade g. Then for $i = 1, \dots, g$ we have*

$$\binom{g}{i} \leq \beta_i \tag{1}$$

where $\beta_i = \beta_i^R(R/I)$ is the ith Betti number of R/I over R.

Before we give a proof, we want to discuss Buchsbaum and Eisenbud's strategy in proving Theorem (0.1). The idea is as follows: let $\boldsymbol{t} = t_1, \dots, t_g$ be a maximal R-sequence contained in I, let E be the Koszul algebra resolution of $R/\boldsymbol{t}$ over R, and let F be the minimal free resolution of R/I over R. Choose a comparison map $\varphi \colon E \to F$ which lifts the canonical map $R/\boldsymbol{t} \to R/I$. The idea is that if F admits a DG algebra structure, then we can choose φ to be multiplicative, and we can use this to show that φ is injective which implies (1). Indeed, assume that F admits a DG algebra structure. To show φ is injective, we consider two steps:

Step 1: We first show $\varphi_g \colon E_g \to F_g$ is injective. Since $E_g \simeq R$ and every nonzero element of R is F_g-regular, it suffices to show that $\varphi_g \neq 0$. After applying $\mathrm{Hom}_R(-, R)$ to the following short exact sequence of R-modules

$$0 \longrightarrow I/\boldsymbol{t} \longrightarrow R/\boldsymbol{t} \longrightarrow R/I \longrightarrow 0 \tag{2}$$

we obtain an induced map in Ext:

$$\cdots \longrightarrow \mathrm{Ext}_R^{g-1}(I/\boldsymbol{t}, R) \longrightarrow \mathrm{Ext}_R^{g}(R/I, R) \longrightarrow \mathrm{Ext}_R^{g}(R/\boldsymbol{t}, R) \longrightarrow \cdots \tag{3}$$

Note that $\boldsymbol{t}$ is a maximal R-sequence contained in $\langle \boldsymbol{t} \rangle \subseteq I$ of length g. It follows that from Ext characterization of depth that $\mathrm{Ext}_R^{g-1}(I/\boldsymbol{t}, R) = 0$ and $\mathrm{Ext}_R^{g}(R/I, R) \neq 0$. Thus the map

$$\varphi_g^\star \colon \mathrm{Ext}_R^{g}(R/I, R) \to \mathrm{Ext}_R^{g}(R/\boldsymbol{t}, R)$$

is nonzero. In particular, this implies $\varphi_g \neq 0$ which implies φ_g is injective.

Step 2: Let $\mathfrak{a} = \ker \varphi$ and assume for a contradiction that $\mathfrak{a} \neq 0$. Note that $\mathfrak{a}$ is a DG ideal of E since φ is multiplicative. Since every nonzero DG ideal of E intersects E_g nontrivially, we must have $\mathfrak{a}_g \neq 0$. However this contradicts the fact that $\mathfrak{a}_g = \ker \varphi_g = 0$ by the first step. Thus $\mathfrak{a} = 0$ which implies φ is injective.

Unfortunately, this strategy won't work in general since F need not have a DG algebra structure. However not all is lost since at the end of the day we are just trying to prove the inequality (1) which only involves natural numbers. In particular, if we could replace $\varphi \colon E \to F$ with an appropriate map $\widetilde{\varphi} \colon \widetilde{E} \to \widetilde{F}$ where $\widetilde{E}$ and $\widetilde{F}$ have the same "shape" as E and F (i.e. $\beta_i(\widetilde{E}) = \beta_i(E)$ and $\beta_i(\widetilde{F}) = \beta_i(F)$ for all i) and where $\widetilde{\varphi}$, $\widetilde{E}$, and $\widetilde{F}$ have the necessary algebraic properties to prove that $\widetilde{\varphi} \colon \widetilde{E} \to \widetilde{F}$ is injective using Buchsbaum and Eisenbud's strategy, then we can prove the inequality (1). This is the strategy we take.

Proof. By a result from Katthän, there exists a nonzero $s \in R$ such that the minimal free resolution of R/sI over R admits a DG algebra structure. Choose such an s and let $\widetilde{E}$ (respectively $\widetilde{F}$) be the minimal free resolution of $R/s\boldsymbol{t}$ (respectively R/sI) over R. In particular, note that

$$\widetilde{E}_i = \begin{cases} sE_1 & \text{if } i = 1 \\ E_i & \text{if } i \neq 1 \end{cases} \qquad \text{and} \qquad \widetilde{F}_i = \begin{cases} sF_1 & \text{if } i = 1 \\ F_i & \text{if } i \neq 1 \end{cases}$$

We claim that $\widetilde{E}$ has a DG structure with the property that every nonzero DG ideal of $\widetilde{E}$ intersects $\widetilde{E}_g$ nontrivially. Indeed, let $\{e_\sigma\}$ denote the homogeneous basis of E as a graded R-module where $\sigma \subseteq \{1, \dots, g\}$. Then the homogeneous basis of $\widetilde{E}$ is given by $\{\widetilde{e}_\sigma\}$ where

$$\widetilde{e}_\sigma = \begin{cases} se_i & \text{if } \sigma = \{i\} \text{ for all } 1 \leq i \leq g \\ e_\sigma & \text{if } |\sigma| \geq 2. \end{cases}$$

If $\sigma = \{i_1, \dots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then we use the notation $e_\sigma = e_{i_1 \cdots i_k}$. For nonempty $\sigma, \tau \subseteq \{1, \dots, g\}$ we define

$$\widetilde{e}_\sigma \widetilde{e}_\tau = \begin{cases} 0 & \text{else} \\ s\widetilde{e}_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \varnothing \end{cases}$$

It is straightforward to check that this give $\widetilde{E}$ a DG algebra structure. Now suppose that $\mathfrak{a}$ is a nonzero DG ideal of $\widetilde{E}$. Let $f \in \mathfrak{a}_k$ for some $1 \leq k < n$ and let $r\widetilde{e}_\sigma$ be a term of f where $|\sigma| = k$ and $r \in R \backslash \{0\}$. Let $\tau = \{1, \dots, n\} \backslash \sigma$ and note that $\widetilde{e}_\tau f = rs\widetilde{e}_{1 \cdots n} \neq 0$, showing that $\mathfrak{a}_g \neq 0$. Next observe that the canonical map $R/s\boldsymbol{t} \to R/sI$ induces a comparison map $\widetilde{\varphi} \colon \widetilde{E} \to \widetilde{F}$ which may be chosen to be multiplicative. Indeed, if $\widetilde{\varphi}(\widetilde{e}_i) = \widetilde{a}_i$ for all $1 \leq i \leq g$, then for $k \geq 2$ we set

$$\widetilde{\varphi}(\widetilde{e}_{i_1 \cdots i_k}) = \frac{1}{s^{k-1}} \widetilde{a}_{i_1} \cdots \widetilde{a}_{i_k} \in F_k = \widetilde{F}_k$$

where we used the fact that $a_{i_j} \in sF_1$ for all $1 \leq j \leq k$. We now prove that $\widetilde{\varphi} \colon \widetilde{E} \to \widetilde{F}$ is injective. Just like before, we proceed in two steps:

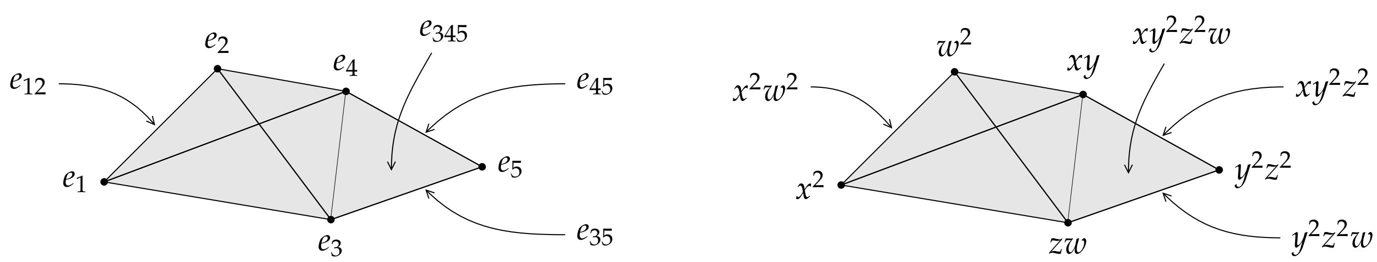
Step 1: We first show $\widetilde{\varphi}_g \colon \widetilde{E}_g \to \widetilde{F}_g$ is injective. Since $\widetilde{E}_g \simeq R$ and every nonzero element of R is $\widetilde{F}_g$-regular, it suffices to show that $\widetilde{\varphi}_g \neq 0$. We do this by showing that the induced map in homology

$$\widetilde{\varphi}_g^\star \colon \operatorname{Ext}_R^g(R/sI, R) \to \operatorname{Ext}_R^g(R/s\boldsymbol{t}, R)$$

is nonzero. Choose any comparison map $\varphi \colon E \to F$ which lifts the canonical map $R/\boldsymbol{t} \to R/I$. Note that for $g \geq 2$ we have $\operatorname{Ext}_R^g(R/s\boldsymbol{t}, R) = \operatorname{Ext}_R^g(R/\boldsymbol{t}, R)$ and $\operatorname{Ext}_R^g(R/sI, R) = \operatorname{Ext}_R^g(R/I, R)$. Furthermore, $\widetilde{\varphi}_g^\star$ and $\varphi_g^\star$ induce the same map in homology. Thus it suffices to show that $\varphi_g^\star$ induces a nonzero map in homology. However we've already shown this!

Step 2: Let $\widetilde{\mathfrak{a}} = \ker \widetilde{\varphi}$ and assume for a contradiction that $\widetilde{\mathfrak{a}} \neq 0$. Note that $\widetilde{\mathfrak{a}}$ is a DG ideal of $\widetilde{E}$ since $\widetilde{\varphi}$ is multiplicative. Since every nonzero DG ideal of E intersects E_g nontrivially, we must have $\widetilde{\mathfrak{a}}_g \neq 0$. However this contradicts the fact that $\widetilde{\mathfrak{a}}_g = \ker \widetilde{\varphi}_g = 0$ by the first step. Thus $\widetilde{\mathfrak{a}} = 0$ which implies $\widetilde{\varphi}$ is injective. $\square$

Example 0.1. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = x^2, w^2, xy, zw, y^2z^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:



We write down the homogeneous components of F as a graded R-module are given by

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234}.
\end{aligned}$$

Next let $\widetilde{F}$ be the minimal free resolution of $R/x\boldsymbol{m}$ over R. The homogeneous basis for $\widetilde{F}$ is denoted $\widetilde{e}_\sigma$. We define an associative multiplication on $\widetilde{F}$ as below:

$$
\begin{aligned}
\widetilde{e}_1 \star \widetilde{e}_2 &= x\widetilde{e}_{12} & \widetilde{e}_3 \star \widetilde{e}_5 &= xz\widetilde{e}_{35} \\
\widetilde{e}_1 \star \widetilde{e}_3 &= x\widetilde{e}_{13} & \widetilde{e}_4 \star \widetilde{e}_5 &= xy\widetilde{e}_{45} \\
\widetilde{e}_1 \star \widetilde{e}_4 &= x^2\widetilde{e}_{14} & \widetilde{e}_1 \star \widetilde{e}_{23} &= x\widetilde{e}_{123} \\
\widetilde{e}_1 \star \widetilde{e}_5 &= xyz^2\widetilde{e}_{14} + x^2\widetilde{e}_{45} & \widetilde{e}_1 \star \widetilde{e}_{24} &= x^2\widetilde{e}_{124} \\
\widetilde{e}_2 \star \widetilde{e}_3 &= xw\widetilde{e}_{23} & \widetilde{e}_1 \star \widetilde{e}_{34} &= x^2\widetilde{e}_{134} \\
\widetilde{e}_2 \star \widetilde{e}_4 &= x\widetilde{e}_{24} & \widetilde{e}_1 \star \widetilde{e}_{35} &= xyz\widetilde{e}_{134} - x^2\widetilde{e}_{345} \\
\widetilde{e}_2 \star \widetilde{e}_5 &= yz^2\widetilde{e}_{24} + w^2\widetilde{e}_{45} & \widetilde{e}_2 \star \widetilde{e}_{34} &= xw\widetilde{e}_{234} \\
\widetilde{e}_3 \star \widetilde{e}_4 &= x\widetilde{e}_{34} & \widetilde{e}_3 \star \widetilde{e}_{45} &= xz\widetilde{e}_{345} \\
\widetilde{e}_3 \star \widetilde{e}_5 &= xz\widetilde{e}_{35} & \widetilde{e}_1 \star \widetilde{e}_{234} &= x^2\widetilde{e}_{1234} \\
\widetilde{e}_4 \star \widetilde{e}_5 &= xy\widetilde{e}_{45} & \widetilde{e}_\sigma^2 &= 0 \text{ for all } \sigma
\end{aligned}
$$

Using a computer algebra system like Singular, one can check that this enough to define an associative multiplication on all of $\widetilde{F}$. Now let $\boldsymbol{t} = x^2, w^2, y^2z^2$, let E be the Koszul algebra resolution of $R/\boldsymbol{t}$ over R, and let $\widetilde{E}$ be the Taylor algebra resolution of $R/x\boldsymbol{t}$ over R. Let $\widetilde{\varepsilon}_\sigma$ denote the homogeneous basis of $\widetilde{E}$. If we try to define a map $\widetilde{\varphi}\colon \widetilde{E} \to \widetilde{F}$ as above, then we run into issues:

$$
\begin{aligned}
\widetilde{\varphi}(\widetilde{\varepsilon}_{23}) &= \frac{1}{x}\widetilde{\varphi}(\widetilde{\varepsilon}_2) \star \widetilde{\varphi}(\widetilde{\varepsilon}_3) \\
&= \frac{1}{x}(\widetilde{\varepsilon}_2 \star \widetilde{\varepsilon}_5) \\
&= (yz^2/x)\widetilde{e}_{24} + (w^2/x)\widetilde{e}_{45}
\end{aligned}
$$

which doesn't land in $\widetilde{F}$.