When a Graded Map is a Chain Map

Let (A, d) and (B, ∂) be R-complexes and let $\psi \colon \mathrm{H}(A) \to \mathrm{H}(B)$ be a graded R-linear map. Suppose that we could lift ψ to a graded R-linear map $\widetilde{\psi} \colon A \to B$ of the underlying graded R-modules. So $\widetilde{\psi}$ takes $\ker \mathrm{d}$ to $\ker \partial$ and it takes $\operatorname{im} \mathrm{d}$ to $\operatorname{im} \partial$ and $\mathrm{H}(\widetilde{\psi}) = \psi$. It's easy to see that $\widetilde{\psi}$ is a chain map if and only if $\operatorname{im}(\partial \widetilde{\psi} - \widetilde{\psi} \mathrm{d}) = 0$. If $\widetilde{\psi}$ is not a chain map, then can we adjust our R-complexes in a way so that it *induces* a chain map? It turns out that the answer is yes, and knowing that $\widetilde{\psi}$ induces $\psi \colon \mathrm{H}(A) \to \mathrm{H}(B)$ gives us a little more information about this induced chain map.

Proposition 0.1. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi \colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\overline{B} = B / \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ and let $\pi \colon B \to \overline{B}$ be the quotient map. Define $\overline{\partial} \colon \overline{B} \to \overline{B}$ by*

$$\overline{\partial}(\overline{b}) = \overline{\partial(b)}$$

for all $\overline{b} \in \overline{B}$. Then $(\overline{B}, \overline{\partial})$ is an R-complex and $\pi\varphi \colon A \to \overline{B}$ is a chain map. Moreover, if φ takes $\operatorname{im} \mathrm{d}$ *to* $\operatorname{im} \partial$*, then we have the following short exact sequence of graded R-modules and graded R-linear maps:*

$$0 \longrightarrow \mathrm{H}(B) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}(\overline{B}) \xrightarrow{\gamma} \operatorname{im}(\partial\varphi - \varphi\mathrm{d})(-1) \longrightarrow 0 \tag{1}$$

where γ is the connecting map coming from a long exact sequence in homology. In particular, φ is a chain map if and only if $\mathrm{H}(B) \cong \mathrm{H}(\overline{B})$.

Proof. Observe that $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ is a graded R-submodule of B since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on B induces a grading on $\overline{B}$ which makes π into a graded R-linear map. Therefore $\pi\varphi$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that $\overline{\partial}$ is well-defined, that is, we need to check that ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself. Let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Then

$$\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= (\partial\partial\varphi - \partial\varphi\mathrm{d})(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= (-\partial\varphi\mathrm{d} + \varphi\mathrm{d}\mathrm{d})(a) \\
&= (\partial\varphi - \varphi\mathrm{d})(-\mathrm{d}(a)) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d}).
\end{aligned}$$

Thus $\overline{\partial}$ is well-defined. Also $\overline{\partial}$ is an R-linear differential since it inherits these properties from ∂. Therefore $(\overline{B}, \overline{\partial})$ is an R-complex. Now let us check that $\pi\varphi$ is a chain map. To see this, we just need to show it commutes with the differentials since we've already shown that it is a graded R-linear map. Let $a \in A$. Then we have

$$\begin{aligned}
\overline{\partial}\pi\varphi(a) &= \overline{\partial}(\overline{\varphi(a)}) \\
&= \overline{\partial\varphi(a)} \\
&= \overline{\partial\varphi(a) - (\partial\varphi - \varphi\mathrm{d})(a)} \\
&= \overline{\partial\varphi(a) - \partial\varphi(a) + \varphi\mathrm{d}(a)} \\
&= \overline{\varphi\mathrm{d}(a)} \\
&= \pi\varphi\mathrm{d}(a).
\end{aligned}$$

Thus $\pi\varphi$ is a chain map.

Since ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself, it restricts to a differential on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. So we have a short exact sequence of R-complexes

$$0 \longrightarrow \operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \xrightarrow{\iota} B \xrightarrow{\pi} \overline{B} \longrightarrow 0 \tag{2}$$

where ι is the inclusion map. The short exact sequence (9) induces the following long exact sequence in homology

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{i+1}(\overline{B}) \arrow[dll, "\gamma_{i+1}"'] \\ \mathrm{H}_i(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(B) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_i(\overline{B}) \arrow[dll, "\gamma_i"'] \\ \mathrm{H}_{i-1}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(B) \arrow[r] & \cdots \end{tikzcd} \tag{3}$$

Let us work out the details of the connecting map γ. Let $[\overline{b}] \in \mathrm{H}_i(\overline{B})$, so $\overline{b} \in \overline{B}_i$ is a coset with $b \in B_i$ as a choice of representative, and this coset $\overline{b}$ represents an element in homology, meaning in particular that $\overline{\partial}(\overline{b}) = 0$, or in other words, that

$$\partial(b) = (\partial\varphi - \varphi\mathrm{d})(a) \tag{4}$$

for some $a \in A$. Then (4) implies that $(\partial\varphi - \varphi\mathrm{d})(a)$ is the unique element in $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ which maps to $\partial(b)$ (under the inclusion map). Therefore

$$\gamma_i[\overline{b}] = [(\partial\varphi - \varphi\mathrm{d})(a)].$$

Now suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. We claim that ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$, there exists a $b \in B$ such that $\varphi\mathrm{d}(a) = \partial(b)$. Then observe that

$$\begin{aligned} \partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a) \\ &= -\partial\varphi\mathrm{d}(a) \\ &= -\partial\partial(b) \\ &= 0. \end{aligned}$$

Thus ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. In particular,

$$\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \cong \mathrm{im}(\partial\varphi - \varphi\mathrm{d}).$$

Next we claim that $\mathrm{H}(\iota)$ is the zero map. Indeed, for any $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$, we choose $b \in B$ such that $\varphi\mathrm{d}(a) = \partial(b)$, then we have

$$\begin{aligned} (\partial\varphi - \varphi\mathrm{d})(a) &= \partial\varphi(a) - \varphi\mathrm{d}(a) \\ &= \partial\varphi(a) - \partial b \\ &= \partial(\varphi(a) - b) \\ &\in \mathrm{im}\,\partial. \end{aligned}$$

Therefore $\mathrm{H}(\iota)$ takes the coset in $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}))$ represented by $(\partial\varphi - \varphi\mathrm{d})(a)$ to the coset in $\mathrm{H}(B)$ represented by 0. Thus $\mathrm{H}(\iota)$ is the zero map as claimed. Combining everything together, we see that the long exact sequence (3) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(B) \xrightarrow{\mathrm{H}_i(\pi)} \mathrm{H}_i(\overline{B}) \xrightarrow{\gamma_i} \mathrm{im}(\partial_{i-1}\varphi_{i-1} - \varphi_{i-2}\mathrm{d}_{i-1}) \longrightarrow 0 \tag{5}$$

for all $i \in \mathbb{Z}$. In other words, (6) is a short exact sequence of graded R-modules. □

Corollary. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. Then φ is a chain map if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$.*

Corollary. *Indeed, φ is a chain map if and only if $\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) = 0$ if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$ by (6).*

Corollary. *Let (P, d) and (B, ∂) be R-complexes and let $\varphi\colon P \to B$ be a graded R-linear map of the underlying graded modules. Suppose P is a semiprojective R-complex and suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. Then φ is a chain map if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$.*

Proof. Indeed, φ is a chain map if and only if $\pi\colon B \to \overline{B}$ is a quasiisomorphism. Since π is surjective and P is semiprojective, there exists a chain map $\phi\colon P \to B$ such that $\pi\phi = \pi\varphi$. □

Dual Version

There is a dual version to Proposition (0.2). Let us state and prove it now.

Proposition 0.2. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\widetilde{A} = \ker(\partial\varphi - \varphi\mathrm{d})$ and let $\iota\colon \widetilde{A} \to A$ be the inclusion map. Define $\widetilde{\mathrm{d}}\colon \widetilde{A} \to \widetilde{A}$ by*

$$\widetilde{\mathrm{d}}(a) = \mathrm{d}(a)$$

for all $a \in \widetilde{A}$. Then $(\widetilde{A}, \widetilde{\mathrm{d}})$ is an R-complex and $\varphi\iota\colon \widetilde{A} \to B$ is a chain map. Moreover, if φ takes $\ker \mathrm{d}$ to $\ker \partial$ and takes $\operatorname{im} \mathrm{d}$ to $\operatorname{im} \partial$, then we have the following short exact sequence of graded R-modules and graded R-linear maps:

$$0 \longrightarrow \operatorname{im}(\partial\varphi - \varphi\mathrm{d})(-1) \xrightarrow{\gamma} \mathrm{H}(\widetilde{A}) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}(A) \longrightarrow 0 \tag{6}$$

where γ is the connecting map coming from a long exact sequence in homology. In particular, φ is a chain map if and only if $\mathrm{H}(\widetilde{A}) \cong \mathrm{H}(A)$.

Proof. Observe that $\widetilde{A}$ is a graded R-submodule of A since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on A induces a grading on $\widetilde{A}$ which makes ι into a graded R-linear map. Therefore $\varphi\iota$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that d restricted to $\widetilde{A}$ lands in $\widetilde{A}$. Suppose $a \in \widetilde{A}$. Thus $a \in A$ and $\partial\varphi(a) = \varphi\mathrm{d}(a)$. Then

$$\begin{aligned}(\partial\varphi - \varphi\mathrm{d})\mathrm{d}(a) &= \partial\varphi\mathrm{d}(a) - \varphi\mathrm{d}\mathrm{d}(a) \\ &= \partial\varphi\mathrm{d}(a) \\ &= \partial\partial\varphi(a) \\ &= 0.\end{aligned}$$

This implies $\mathrm{d}(a) \in \widetilde{A}$. Thus d restricted to $\widetilde{A}$ lands in $\widetilde{A}$. Clearly d is an R-linear differential. Therefore $(\widetilde{A}, \mathrm{d})$ is an R-complex. Now let us check that $\varphi\iota$ is a chain map. To see this, we just need to show it commutes with the differentials since we've already shown that it is a graded R-linear map. Let $a \in \widetilde{A}$ (so $a \in A$ and $\partial\varphi(a) = \varphi\mathrm{d}(a)$). Then we have

$$\begin{aligned}\partial\varphi\iota(a) &= \partial\varphi(a) \\ &= \varphi\mathrm{d}(a) \\ &= \varphi\iota\widetilde{\mathrm{d}}(a)\end{aligned}$$

Thus $\varphi\iota$ is a chain map.

We have a short exact sequence of R-complexes

$$0 \longrightarrow \widetilde{A} \xrightarrow{\iota} A \xrightarrow{\partial\varphi - \varphi\mathrm{d}} \Sigma\operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \longrightarrow 0 \tag{7}$$

where ι is the inclusion map. The short exact sequence (9) induces the following long exact sequence in homology

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_i(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \arrow[dll, "\lambda_i"'] \\ \mathrm{H}_i(\widetilde{A}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(A) \arrow[r, "\mathrm{H}_i(\partial\varphi - \varphi\mathrm{d})"] & \mathrm{H}_{i-1}(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \arrow[dll, "\lambda_{i-1}"'] \\ \mathrm{H}_{i-1}(\widetilde{A}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(A) \arrow[r] & \cdots \end{tikzcd} \tag{8}$$

Now suppose φ takes $\ker \mathrm{d}$ to $\ker \partial$. We claim that ∂ restricts to the zero map on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\operatorname{im} \mathrm{d}$ to $\operatorname{im} \partial$, there exists a $b \in B$ such that

$$\varphi\mathrm{d}(a) = \partial(b).$$

Choose such a $b \in B$. Then observe that

$$\begin{aligned}\partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a)\\ &= -\partial\varphi\mathrm{d}(a)\\ &= -\partial\partial(b)\\ &= 0.\end{aligned}$$

Thus ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. In particular, $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \cong \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$.

Next we claim that $\mathrm{H}(\partial\varphi - \varphi\mathrm{d})$ is the zero map. Indeed, let $[a] \in \mathrm{H}(A)$, so $a \in A$ and $\mathrm{d}(a) = 0$. Since φ takes $\ker \mathrm{d}$ to $\ker \partial$, we see that $\partial\varphi(a) = 0$. Therefore

$$\begin{aligned}\mathrm{H}(\partial\varphi - \varphi\mathrm{d})[a] &= [(\partial\varphi - \varphi\mathrm{d})(a)]\\ &= [\partial\varphi(a) - \varphi\mathrm{d}(a)]\\ &= [0].\end{aligned}$$

Thus $\mathrm{H}(\partial\varphi - \varphi\mathrm{d})$ is the zero map as claimed.

Combining everything together, we see that the long exact sequence (8) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \xrightarrow{\lambda_i} \mathrm{H}_i(\widetilde{A}) \xrightarrow{\iota_i} \mathrm{H}_i(A) \longrightarrow 0 \tag{9}$$

for all $i \in \mathbb{Z}$. In other words, (6) is a short exact sequence of graded R-modules. □

Homological Chain Maps

Let (A, d) and (B, ∂) be two R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map. We say φ is a **homological chain map** if $\mathrm{H}(\mathrm{im}(\varphi\mathrm{d} - \partial\varphi)) = 0$. From the long exact sequences constructed above, we see that if φ is a homological chain map, then both $\pi\colon B \to \overline{B}$ and $\iota\colon \widetilde{A} \to A$ are quasiisomorphisms. Conversely, if one of these maps is a quasiisomorphism, then φ is a homological chain map.

Proposition 0.3. *Keep the notation as above, and let P be a semiprojective R-complex and let E be a semiinjective R-complex. Assume that φ is a homological chain map.*

1. *For every chain map $\psi\colon P \to \overline{B}$, there exists a chain map $\widetilde{\psi}\colon P \to B$ such that $\pi\widetilde{\psi} = \psi$. In particular, if A is semiprojective, then there exists a chain map $\widetilde{\varphi}\colon A \to B$ such that $\pi\widetilde{\varphi} = \pi\varphi$. Moreover, if $\widetilde{\psi}'\colon P \to B$ is another chain map such that $\pi\widetilde{\psi}' \sim \psi$ (where $\sim$ means "homotopic to"), then $\widetilde{\psi} \sim \widetilde{\psi}'$.*

2. *For every chain map $\psi\colon \widetilde{A} \to E$, there exists a chain map $\widetilde{\psi}\colon A \to E$ such that $\widetilde{\psi}\iota = \psi$. In particular, if B is seminjective, then there exists a chain map $\widetilde{\varphi}\colon A \to B$ such that $\widetilde{\varphi}\iota = \varphi\iota$. Moreover, if $\widetilde{\psi}'\colon A \to E$ is another chain map such that $\widetilde{\psi}'\iota \sim \psi$, then $\widetilde{\psi} \sim \widetilde{\psi}'$.*

Applications

Let $\boldsymbol{x} = x_1, \ldots, x_n$, let $R = K[\boldsymbol{x}]$, let $\boldsymbol{m} = m_1, \ldots, m_r$ be monomials in R, and let Δ be a finite simplicial complex labeled by $\boldsymbol{m}$. In particular, Δ has r vertices which we label by $m_1, \ldots, m_r$. If σ is a face of Δ, then we label it by m_σ where $m_\sigma = \mathrm{lcm}(m_i \mid i \in \sigma)$. Let F be the R-complex induced by Δ. Thus the homogeneous component in homological degree $i \in \mathbb{Z}$ of the underlying graded R-module of F is given by

$$F_i := \begin{cases} \bigoplus\limits_{\dim\sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim\Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $\mathrm{d}(e_\varnothing) = 0$ and

$$\mathrm{d}(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma\setminus i}} e_{\sigma\setminus i}$$

for all $\sigma \in \Delta\backslash\{\varnothing\}$ where $\mathrm{pos}(i, \sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma\backslash i$ denotes the face obtained from σ by removing i. In the particular case where $\Delta = \Delta_{r-1}$ is the $(r-1)$-simplex, then F is just the Taylor resolution of $R/\boldsymbol{m}$ over R.

Let $\boldsymbol{m}' = m'_{1'}, \ldots, m'_{r'}$ be another list of monomial of R and let Δ' be a finite simplicial complex labeled by $\boldsymbol{m}'$. Let F' be the R-complex induced by Δ'. Any simplicial map $f\colon \Delta \to \Delta'$ which preserves dimensions will give rise to an induced map of R-complexes $\varphi_f\colon F \to F'$ given by $e_\sigma \mapsto e_{f(\sigma)}$ for all $\sigma \in \Delta$. In fact, φ_f is even a graded R-linear map, however it is usually not a chain map.

Example 0.1. Consider the case where $R = K[x, y, z]$, where $\boldsymbol{m} = m_1, m_2, m_3$, and where Δ is the 2-simplex labeled by $\boldsymbol{m}$ (so F is just the Taylor complex). Let $f \colon \Delta \to \Delta$ be the simplicial map corresponding to the permutation (12): so the induced graded R-linear map $\varphi = \varphi_f$ from $F \to F$ is given by

$$\begin{aligned}
\varphi(e_1) &= e_2 \\
\varphi(e_2) &= e_1 \\
\varphi(e_3) &= e_3 \\
\varphi(e_{12}) &= -e_{12} \\
\varphi(e_{13}) &= e_{23} \\
\varphi(e_{23}) &= e_{13} \\
\varphi(e_{123}) &= -e_{123.}
\end{aligned}$$

We want to determine what the map $\mathrm{d}\varphi - \varphi\mathrm{d}$ looks like. To do this, we just calculate how it acts on e_σ for each $\sigma \in \Delta$. Observe that

$$\begin{aligned}
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_1) &= \mathrm{d}\varphi(e_1) - \varphi\mathrm{d}(e_1) \\
&= \mathrm{d}(e_2) - \varphi(m_1) \\
&= m_2 - m_1
\end{aligned}$$

Similar computations show

$$\begin{aligned}
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_2) &= m_1 - m_2 \\
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_3) &= 0 \\
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_{12}) &= (m_2 - m_1)(e_1 + e_2) \\
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_{13}) &= (m_2 - m_1)e_3 \\
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_{23}) &= (m_1 - m_2)(e_1 + e_2) \\
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_{123}) &= (m_2 - m_1)(e_{23} + e_{13}).
\end{aligned}$$

$$\begin{aligned}
(\mathrm{d}\varphi - \varphi\mathrm{d})(e_{12}) &= -\mathrm{d}(e_{12}) - \varphi(\frac{m_{12}}{m_2}e_2 - \frac{m_{12}}{m_1}e_1) \\
&= -\mathrm{d}(e_{12}) - \frac{m_{12}}{m_2}e_1 + \frac{m_{12}}{m_1}e_2
\end{aligned}$$

More generally we have in mod 2 we have

$$\begin{aligned}
(\mathrm{d}\varphi + \varphi\mathrm{d})(e_\sigma) &= \mathrm{d}\varphi(e_\sigma) + \varphi\mathrm{d}(e_\sigma) \\
&= \mathrm{d}(e_{f(\sigma)}) + \sum_{i\in\sigma} \frac{m_\sigma}{m_{\sigma\setminus i}}\varphi(e_{\sigma\setminus i}) \\
&= \sum_{f(i)\in f(\sigma)} \frac{m_{f(\sigma)}}{m_{f(\sigma)\setminus f(i)}} e_{f(\sigma)\setminus f(i)} + \sum_{i\in\sigma} \frac{m_\sigma}{m_{\sigma\setminus i}} e_{f(\sigma\setminus i)} \\
&= \sum_{i\in\sigma} \left(\frac{m_{f(\sigma)}}{m_{f(\sigma)\setminus f(i)}} + \frac{m_\sigma}{m_{\sigma\setminus i}} \right) e_{f(\sigma)\setminus f(i)} \\
&= \sum_{i\in\sigma} \left(\frac{m_{\sigma\setminus i} m_{f(\sigma)} + m_\sigma m_{f(\sigma)\setminus f(i)}}{m_{\sigma\setminus i} m_{f(\sigma)\setminus f(i)}} \right) e_{f(\sigma)\setminus f(i)}
\end{aligned}$$

More generally we have in mod 2 we have (if we want φ to preserve the multigrading)

$$\begin{aligned}
(\mathrm{d}\varphi + \varphi\mathrm{d})(e_\sigma) &= \mathrm{d}\varphi(e_\sigma) + \varphi\mathrm{d}(e_\sigma) \\
&= \frac{m_\sigma}{m_{f(\sigma)}}\mathrm{d}(e_{f(\sigma)}) + \sum_{i\in\sigma} \frac{m_\sigma}{m_{\sigma\setminus i}}\varphi(e_{\sigma\setminus i}) \\
&= \sum_{f(i)\in f(\sigma)} \frac{m_\sigma}{m_{f(\sigma)}} \frac{m_{f(\sigma)}}{m_{f(\sigma)\setminus f(i)}} e_{f(\sigma)\setminus f(i)} + \sum_{i\in\sigma} \frac{m_\sigma}{m_{\sigma\setminus i}} \frac{m_{\sigma\setminus i}}{m_{f(\sigma\setminus i)}} e_{f(\sigma\setminus i)} \\
&= \sum_{i\in\sigma} \left(\frac{m_\sigma}{m_{f(\sigma)}} \frac{m_{f(\sigma)}}{m_{f(\sigma)\setminus f(i)}} + \frac{m_{\sigma\setminus i}}{m_{f(\sigma\setminus i)}} \frac{m_\sigma}{m_{\sigma\setminus i}} \right) e_{f(\sigma)\setminus f(i)} \\
&= \sum_{i\in\sigma} \left(\frac{m_\sigma}{m_{f(\sigma)\setminus f(i)}} + \frac{m_\sigma}{m_{f(\sigma\setminus i)}} \right) e_{f(\sigma)\setminus f(i)} \\
&= 0.
\end{aligned}$$

Example 0.2. Let $\underline{x} = x_1, \dots, x_n$ be a sequence of elements in R, let $\mathcal{K}(\underline{x})$ be the Koszul complex with respect to that sequence, and let π be a permutation of $[n]$. For any subset $\sigma \subseteq [n]$, we write $\sigma = \{\lambda_1, \dots, \lambda_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and we define $\pi \cdot \sigma$ to be the subset in $[n]$ defined by

$$\pi \cdot \sigma = \{\pi(\lambda_1), \dots, \pi(\lambda_k)\}.$$

We also define $\mathrm{sign}(\pi|_\sigma)$ to be the sign of the permutation which puts $(\pi(\lambda_1), \dots, \pi(\lambda_k))$ into the correct order. Then π induces a graded R-linear map $\pi \colon \mathcal{K}(\underline{x}) \to \mathcal{K}(\underline{x})$, uniquely determined by

$$\pi(e_\sigma) = (-1)^{\mathrm{sign}(\pi|_\sigma)} e_{\pi\cdot\sigma} \tag{10}$$

for all $\sigma \subseteq [n]$. If $\underline{x} = \underline{1}$, then (10) is a chain map. Indeed, we have

$$\begin{aligned}
\mathrm{d}_{\mathcal{K}(\underline{x})}\pi(e_\sigma) &= (-1)^{\mathrm{sign}(\pi|_\sigma)}\mathrm{d}_{\mathcal{K}(\underline{x})}(e_{\pi\cdot\sigma}) \\
&= \sum_{\pi(\lambda)\in\pi\cdot\sigma} (-1)^{\mathrm{sign}(\pi|_\sigma)}\langle \pi(\lambda), \sigma\backslash\pi(\lambda)\rangle e_{(\pi\cdot\sigma)\backslash\pi(\lambda)} \\
&= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle (-1)^{\mathrm{sign}(\pi|_{\sigma\backslash\lambda})} e_{\pi\cdot(\sigma\backslash\lambda)} \\
&= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \pi e_{\sigma\backslash\lambda} \\
&= \pi\mathrm{d}_{\mathcal{K}(\underline{x})}(e_\sigma)
\end{aligned}$$

for all $\sigma \subseteq [n]$.

Example 0.3. Let $R = K[x, y, z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi = (12)$. Throughout the following calculation, denote $\mathrm{d} = \mathrm{d}_{\mathcal{K}(\underline{f})}$. We first calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_1) &= \mathrm{d}\pi(e_1) - \pi\mathrm{d}(e_1) \\
&= \mathrm{d}(e_2) - \pi(f_1) \\
&= f_2 - f_1.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_2) &= \mathrm{d}\pi(e_2) - \pi\mathrm{d}(e_2) \\
&= \mathrm{d}(e_1) - \pi(f_2) \\
&= f_1 - f_2.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_3) &= \mathrm{d}\pi(e_3) - \pi\mathrm{d}(e_3) \\
&= \mathrm{d}(e_3) - \pi(f_3) \\
&= f_3 - f_3 \\
&= 0.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_{12}) &= \mathrm{d}\pi(e_{12}) - \pi\mathrm{d}(e_{12}) \\
&= \mathrm{d}(-e_{12}) - \pi(f_1e_2 - f_2e_1) \\
&= -f_1e_2 + f_2e_1 - f_1e_1 + f_2e_2 \\
&= (f_2 - f_1)(e_1 + e_2).
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_{13}) &= \mathrm{d}\pi(e_{13}) - \pi\mathrm{d}(e_{13}) \\
&= \mathrm{d}(e_{23}) - \pi(f_1e_3 - f_3e_1) \\
&= f_2e_3 - f_3e_2 - f_1e_3 + f_3e_2 \\
&= (f_2 - f_1)e_3.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}\pi - \pi\mathrm{d})(e_{23}) &= \mathrm{d}\pi(e_{23}) - \pi\mathrm{d}(e_{23}) \\
&= \mathrm{d}(e_{13}) - \pi(f_2e_3 - f_3e_2) \\
&= f_1e_3 - f_3e_1 - f_2e_3 + f_3e_1 \\
&= (f_1 - f_2)e_3.
\end{aligned}$$

Finally we calculate

$$\begin{aligned}(\mathrm{d}\pi - \pi\mathrm{d})(e_{123}) &= \mathrm{d}\pi(e_{123}) - \pi\mathrm{d}(e_{123}) \\ &= \mathrm{d}(-e_{123}) - \pi(f_1e_{23} - f_2e_{13} + f_3e_{12}) \\ &= -f_1e_{23} + f_2e_{13} - f_3e_{12} - f_1e_{13} + f_2e_{23} + f_3e_{12} \\ &= (f_2 - f_1)(e_{23} + e_{13}).\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathrm{im}(\mathrm{d}_0\pi_0 - \pi_{-1}\mathrm{d}_0) &= 0 \\ \mathrm{im}(\mathrm{d}_1\pi_1 - \pi_0\mathrm{d}_1) &= R(f_2 - f_1) \\ \mathrm{im}(\mathrm{d}_2\pi_2 - \pi_1\mathrm{d}_2) &= R(f_2 - f_1)(e_1 + e_2) + R(f_2 - f_1)e_3 \\ \mathrm{im}(\mathrm{d}_3\pi_3 - \pi_2\mathrm{d}_3) &= R(f_2 - f_1)(e_{23} + e_{13})\end{aligned}$$

Similarly we have

$$\begin{aligned}\ker(\mathrm{d}_0\pi_0 - \pi_{-1}\mathrm{d}_0) &= R \\ \ker(\mathrm{d}_1\pi_1 - \pi_0\mathrm{d}_1) &= R(e_1 + e_2) + Re_3 \\ \ker(\mathrm{d}_2\pi_2 - \pi_1\mathrm{d}_2) &= R(e_{23} + e_{13}) \\ \ker(\mathrm{d}_3\pi_3 - \pi_2\mathrm{d}_3) &= 0\end{aligned}$$

Example 0.4. Let $R = K[x,y,z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi = (12)$. We first calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) - \pi(f_1) \\ &= f_2 - f_1.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_2) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_2) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) - \pi(f_2) \\ &= f_1 - f_2.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_3) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_3) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) - \pi(f_3) \\ &= f_3 - f_3 \\ &= 0.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{12}) - \pi(f_1e_2 - f_2e_1) \\ &= -f_1e_2 + f_2e_1 - f_1e_1 + f_2e_2 \\ &= (f_2 - f_1)(e_1 + e_2).\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1e_3 - f_3e_1) \\ &= f_2e_3 - f_3e_2 - f_1e_3 + f_3e_2 \\ &= (f_2 - f_1)e_3.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) - \pi(f_2e_3 - f_3e_2) \\ &= f_1e_3 - f_3e_1 - f_2e_3 + f_3e_1 \\ &= (f_1 - f_2)e_3.\end{aligned}$$

Finally we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123}) - \pi(f_1e_{23} - f_2e_{13} + f_3e_{12}) \\ &= -f_1e_{23} + f_2e_{13} - f_3e_{12} - f_1e_{13} + f_2e_{23} + f_3e_{12} \\ &= (f_2 - f_1)(e_{23} + e_{13}).\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}) = \langle f_2 - f_1\rangle$$

Example 0.5. Let $R = K[x, y, z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi\colon \mathcal{K}(\underline{f}) \to \mathcal{K}(\underline{f})$ be a graded R-linear map. For each $1 \leq i < j \leq 3$, we have

$$\begin{aligned}\pi(1) &= f_0^0 \\ \pi(e_i) &= f_i^1e_1 + f_i^2e_2 + f_i^3e_3 \\ \pi(e_{ij}) &= f_{ij}^{12}e_{12} + f_{ij}^{13}e_{13} + f_{ij}^{23}e_{23} \\ \pi(e_{ijk}) &= f_{123}^{123}e_{123}\end{aligned}$$

where the f_i^k's and f_{ij}^{kl}'s are in R. Then we have

$$\begin{aligned}&(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(1) = f_0^0 \\ &(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_i) = f_i^1f_1 + f_i^2f_2 + f_i^3f_3 - f_0^0f_i \\ &(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{ij}) = (f_jf_i^1 - f_if_j^1 - f_{ij}^{13}f_3 - f_{ij}^{12}f_2)e_1 + (f_jf_i^2 - f_if_j^2 - f_{ij}^{23}f_3 + f_{ij}^{12}f_1)e_2 + (f_jf_i^3 - f_if_j^3 + f_{ij}^{23}f_j + f_{ij}^{13})e_3\end{aligned}$$

One can calculate more generally that

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) = f_1^1f_1 + f_1^2f_2 + f_1^3f_3 - f_1f_0$$

We first calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_1^1e_1 + f_1^2e_2 + f_1^3e_3) - \pi(f_1) \\ &= f_1^1f_1 + f_1^2f_2 + f_1^3f_3 - f_1f_0\end{aligned}$$

More generally we have

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_i) = f_i^1f_1 + f_i^2f_2 + f_i^3f_3 - f_if_0$$

for $i = 1, 2, 3$.

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_{12}^{12}e_{12} + f_{12}^{13}e_{13} + f_{12}^{23}e_{23}) - \pi(f_1e_2 - f_2e_1) \\ &= f_{12}^{12}(f_1e_2 - f_2e_1) + f_{12}^{13}(f_1e_3 - f_3e_1) + f_{12}^{23}(f_2e_3 - f_3e_2) - f_1(f_2^1e_1 + f_2^2e_2 + f_2^3e_3) + f_2(f_1^1e_1 + f_1^2e_2 + f_1^3e_3) \\ &= (f_2f_1^1 - f_1f_2^1 - f_{12}^{13}f_3 - f_{12}^{12}f_2)e_1 + (f_2f_1^2 - f_1f_2^2 - f_{12}^{23}f_3 + f_{12}^{12}f_1)e_2 + (f_2f_1^3 - f_1f_2^3 + f_{12}^{23}f_2 + f_{12}^{13})e_3\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1e_3 - f_3e_1) \\ &= f_2e_3 - f_3e_2 - f_1e_3 + f_3e_2 \\ &= (f_2 - f_1)e_3.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13})-\pi(f_2e_3-f_3e_2)\\ &= f_1e_3-f_3e_1-f_2e_3+f_3e_1\\ &= (f_1-f_2)e_3.\end{aligned}$$

Finally we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123})-\pi(f_1e_{23}-f_2e_{13}+f_3e_{12})\\ &= -f_1e_{23}+f_2e_{13}-f_3e_{12}-f_1e_{13}+f_2e_{23}+f_3e_{12}\\ &= (f_2-f_1)(e_{23}+e_{13}).\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})=\langle f_2-f_1\rangle$$

Example 0.6. Let $R=K[x,y,z]$, let $\underline{f}=f_1,f_2,f_3$ be a sequence of elements in R, and let $\pi=(12)$. We first calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1)-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1)\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_1^1e_1+f_1^2e_2+f_1^3e_3)-\pi(f_1)\\ &= f_1^1f_1+f_1^2f_2+f_1^3f_3-f_1f_0\end{aligned}$$

More generally we have

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_i)=f_i^1f_1+f_i^2f_2+f_i^3f_3-f_if_0$$

for $i=1,2,3$.

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_{12}^{12}e_{12}+f_{12}^{13}e_{13}+f_{12}^{23}e_{23})-\pi(f_1e_2-f_2e_1)\\ &= f_{12}^{12}(f_1e_2-f_2e_1)+f_{12}^{13}(f_1e_3-f_3e_1)+f_{12}^{23}(f_2e_3-f_3e_2)-f_1(f_2^1e_1+f_2^2e_2+f_2^3e_3)+f_2(f_1^1e_1+f_1^2e_2+f_1^3e_3)\\ &= (f_2f_1^1-f_1f_2^1-f_{12}^{13}f_3-f_{12}^{12}f_2)e_1+(f_2f_1^2-f_1f_2^2-f_{12}^{23}f_3+f_{12}^{12}f_1)e_2+(f_2f_1^3-f_1f_2^3+f_{12}^{23}f_2+f_{12}^{13})e_3\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23})-\pi(f_1e_3-f_3e_1)\\ &= f_2e_3-f_3e_2-f_1e_3+f_3e_2\\ &= (f_2-f_1)e_3.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13})-\pi(f_2e_3-f_3e_2)\\ &= f_1e_3-f_3e_1-f_2e_3+f_3e_1\\ &= (f_1-f_2)e_3.\end{aligned}$$

Finally we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123})-\pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123})\\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123})-\pi(f_1e_{23}-f_2e_{13}+f_3e_{12})\\ &= -f_1e_{23}+f_2e_{13}-f_3e_{12}-f_1e_{13}+f_2e_{23}+f_3e_{12}\\ &= (f_2-f_1)(e_{23}+e_{13}).\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi-\pi\mathrm{d}_{\mathcal{K}(\underline{f})})=\langle f_2-f_1\rangle$$

DG Algebras

Let (A, d) be an R-complex. A **graded-multiplication** on A is a graded R-linear map $\mathrm{m}\colon A \otimes_R A \to A$ of the underlying graded R-modules. The universal mapping property on graded tensor products tells us that there exists a unique graded R-bilinear map $B_{\mathrm{m}}\colon A \times A \to A$ such that

$$B_{\mathrm{m}}(a, b) = \mathrm{m}(a \otimes b)$$

for all $(a, b) \in A \times A$. However since B_{m} is *uniquely* determined by m, we often identify B_{m} with m and simply think of m as a graded R-bilinear map. In fact, we often drop m altogether and simply denote this multiplication map by

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i$$

for all $\sum a_i \otimes b_i \in A \otimes_R A$. At the end of the day, context will make everything clear.

Suppose m is a graded multiplication As the name of the definition suggests, a graded-multiplication on A must respect the grading. In particular, this means that if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. We can also impose other conditions on a graded-multiplication on A.

Definition 0.1. Let (A, d) be an R-complex and let m be a graded-multiplication on A.

1. We say a m is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.

2. We say m is **graded-commutative** if
$$ab = (-1)^i ba$$
for all $a \in A_i$ and $b \in B_j$ for all $i, j \in \mathbb{Z}$.

3. We say m is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all $a \in A_i$ for all i odd.

4. We say m is **unital** if there exists an $e \in A$ such that
$$ae = e = ea$$
for all $a \in A$.

5. We say a graded-multiplication satisfies **Leibniz law** if
$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$
for all $a \in A_i$ and $b \in B_j$ for all $i, j \in \mathbb{Z}$. This is equivalent to m being a chain map!

6. We say $(A, \mathrm{m}, \mathrm{d})$ is a **differential graded R-algebra** (or **DG R-algebra**) if m is a graded-multiplication on A which satisfies conditions 1-5.

We are often presented with the following scenario: we are given a graded-multiplication m on an R-complex (A, d) and would like to know if $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra. For instance, we may know that m satisfies the conditions 2-5 in Definition (0.1), which would reduce the question of whether $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra to the question of whether m is associative. On the other hand, we may know that m satisfies the conditions 1-4 in Definition (0.1), which would reduce the question of whether $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra to the question of whether m is chain map. Proposition (0.2) gives us some insight on how to proceed in this direction.

Proposition 0.4. *Let (A, d) be an R-complex and let* m *be a graded-multiplication on A which satisfies conditions 1-4 in Definition (0.1). Furthermore, suppose that*

$$\mathrm{d}(a)b + (-1)^i a\mathrm{d}(b) \in \operatorname{im} \mathrm{d} \tag{11}$$

for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$. Then $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra if and only if $\pi\colon A \to \overline{A}$ is a quasiisomorphism where

$$\overline{A} = A/\langle\{\mathrm{d}(ab) - \mathrm{d}(a)b - (-1)^i a\mathrm{d}(b)\}\rangle.$$

Proof. The condition (11) is equivalent to the condition that m takes $\operatorname{im} \mathrm{d}_A^{\otimes}$ to $\operatorname{im} \mathrm{d}$. □

When A and B are DG Algebras

Proposition 0.5. *With the notation above, suppose A and B are DG R-algebra and suppose $\varphi \colon A \to B$ is a graded R-algebra map. Then $\widetilde{A}$ and $\overline{B}$ are DG R-algebras.*

Proof. Let us first show $\widetilde{A}$ is a DG R-algebra. To do this, we just need to show that the multiplication map $\mathrm{B_m} \colon A \times A \to A$ restricted to $\widetilde{A} \times \widetilde{A}$ lands in $\widetilde{A}$. Let $a, a' \in \widetilde{A}$ (so $\partial\varphi(a) = \varphi\mathrm{d}(a)$ and $\partial\varphi(a') = \varphi\mathrm{d}(a')$). Then we have

$$\begin{aligned} \varphi\mathrm{d}(aa') &= \varphi(\mathrm{d}(a)a' + (-1)^{|a|}a\mathrm{d}(a')) \\ &= \varphi(\mathrm{d}(a))\varphi(a') + (-1)^{|a|}\varphi(a)\varphi(\mathrm{d}(a')) \\ &= \partial(\varphi(a))\varphi(a') + (-1)^{|a|}\varphi(a)\partial(\varphi(a')) \\ &= \partial(\varphi(a)\varphi(a')) \\ &= \partial\varphi(aa'). \end{aligned}$$

Thus $aa' \in \widetilde{A}$.

Now let us show $\overline{B}$ is a DG R-algebra. To do this, we just need to show that $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ is ∂-stable. This was shown above earlier.

□