Gröbner MDG

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Throughout this subsection, we assume that R is an integral domain with quotient field K. Let F be an R-free resolution of a cyclic R-module with $F_0 = R$ such that the underlying graded R-module of F is a finite and free as an R-module. Let $e_1, \dots, e_n$ be an ordered homogeneous basis of F_+ as a graded R-module which is ordered in such a way that if $|e_{i'}| > |e_i|$, then $i' > i$. We denote by $R[\boldsymbol{e}] = R[e_1, \dots, e_n]$ to be the free *non-strict* graded-commutative R-algebra generated by $e_1, \dots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $R[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $R[\boldsymbol{e}]$. The reason we do not allow elements of odd degree to square to zero is because we will want to calculate the Gröbner basis of an ideal in $K[\boldsymbol{e}]$, and the theory of Gröbner bases for $K[\boldsymbol{e}]$ is simpler when we don't have any zerodivisors. In any case, it is straightforward to check that

$$R[\boldsymbol{e}]/\langle \{e_i^2 \mid |e_i| \text{ is odd}\}\rangle \simeq \mathrm{S}(F).$$

Finally, let $(\mu, \star)$ be a multiplication of F. Our goal is to compute the maximal associative quotient of F using the presentation given in Theorem (**??**) as well as the theory of Gröbner bases in $K[\boldsymbol{e}]$. We need to introduce some notation for Gröbner basis applications in $K[\boldsymbol{e}]$. Our notation mostly follows **[?]** however we introduce some of our own notation as well.

0.0.1 Monomials and Monomial Orderings in $K[\boldsymbol{e}]$

A **monomial** in $K[\boldsymbol{e}]$ is an element of the form

$$e^{\boldsymbol{\alpha}} = e_1^{\alpha_1} \cdots e_n^{\alpha_n} \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called the **multidegree** of $e^{\boldsymbol{\alpha}}$ and is denoted $\mathrm{multideg}(e^{\boldsymbol{\alpha}}) = \boldsymbol{\alpha}$. Similarly we define its **total degree**, denoted $\deg(e^{\boldsymbol{\alpha}})$, and its **homological degree** denoted $|e^{\boldsymbol{\alpha}}|$, by

$$\deg(e^{\boldsymbol{\alpha}}) = \sum_{i=1}^n \alpha_i \quad \text{and} \quad |e^{\boldsymbol{\alpha}}| = \sum_{i=1}^n \alpha_i |e_i|.$$

By convention we set $e^{\mathbf{0}} = 1$ where $\mathbf{0} = (0, \dots, 0)$ is the zero vector in $\mathbb{N}^n$. We define the **support** of $e^{\boldsymbol{\alpha}}$, denoted $\mathrm{supp}(e^{\boldsymbol{\alpha}})$, to be the set

$$\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_i \mid e_i \text{ divides } e^{\boldsymbol{\alpha}}\} = \{e_i \mid \alpha_i \neq 0\}.$$

Note that if the support of $e^{\boldsymbol{\alpha}}$ is empty if and only if $e^{\boldsymbol{\alpha}} = 1$. If $e^{\boldsymbol{\alpha}}$ has non-empty support, then we define its **initial variable** and **terminal variable** to be the elements e_i and e_k where

$$i = \inf\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\} \quad \text{and} \quad \max\{j \mid e_j \in \mathrm{supp}(e^{\boldsymbol{\alpha}})\}.$$

For instance, suppose that $\mathrm{supp}(e^{\boldsymbol{\alpha}}) = \{e_{i_1}, \dots, e_{i_k}\}$ where $1 \leq i_1 < \cdots < i_k \leq n$, then can express (1) as

$$e^{\boldsymbol{\alpha}} = e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_k}.$$

Then e_{i_1} is the initial variable of $e^{\boldsymbol{\alpha}}$ and e_{i_k} is the terminal variable of $e^{\boldsymbol{\alpha}}$. Note how the ordering matters. In particular, if $i < j$ and both $|e_i|$ and $|e_j|$ are odd, then $e_j e_i$ is not a monomial in $K[\boldsymbol{e}]$ since it can be expressed as a non-trivial coefficient times a monomial:

$$e_j e_i = -e_i e_j.$$

On the other hand, if one of the e_i or e_j is even, then $e_j e_i$ is a monomial in $K[\boldsymbol{e}]$ since $e_j e_i = e_i e_j$. We equip $K[\boldsymbol{e}]$ with a weighted lexicographical ordering $>$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ (the notation for this monomial ordering in Singular is Wp(w)). More specifically, given two monomials $e^{\boldsymbol{\alpha}}$ and $e^{\boldsymbol{\beta}}$ in $K[\boldsymbol{e}]$, we say $e^{\boldsymbol{\beta}} > e^{\boldsymbol{\alpha}}$ if either

1. $|e^{\boldsymbol{\beta}}| > |e^{\boldsymbol{\alpha}}|$ or;

2. $|e^{\beta}| = |e^{\alpha}|$ and $\beta_1 > \alpha_1$ or;

3. $|e^{\beta}| = |e^{\alpha}|$ and there exists $1 < j \leq n$ such that $\beta_j > \alpha_j$ and $\beta_i = \alpha_i$ for all $1 \leq i < j$.

Given a nonzero polynoimal $f \in K[\boldsymbol{e}]$, there exists unique $c_1, \dots, c_m \in K \backslash \{0\}$ and unique $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_m \in \mathbb{N}^n$ where $\boldsymbol{\alpha}_i \neq \boldsymbol{\alpha}_j$ for all $1 \leq i < j \leq m$ such that

$$f = c_1 e^{\boldsymbol{\alpha}_1} + \cdots + c_m e^{\boldsymbol{\alpha}_m} = \sum c_i e^{\boldsymbol{\alpha}_i} \tag{2}$$

The $c_i e^{\boldsymbol{\alpha}_i}$ in (2) are called the **terms** of f, and the $e^{\boldsymbol{\alpha}_i}$ in (2) are called the **monomials** of f. By reindexing the $\boldsymbol{\alpha}_i$ if necessary, we may assume that $e^{\boldsymbol{\alpha}_1} > \cdots > e^{\boldsymbol{\alpha}_m}$. In this case, we call $c_1 e^{\boldsymbol{\alpha}_1}$ the **lead term** of f, we call $e^{\boldsymbol{\alpha}_1}$ the **lead monomial** of f, and we call c_1 the **lead coefficient** of f. We denote these, respectively, by

$$\mathrm{LT}(f) = c_1 e^{\boldsymbol{\alpha}_1}, \quad \mathrm{LM}(f) = e^{\boldsymbol{\alpha}_1}, \quad \text{and} \quad \mathrm{LC}(f) = c_1.$$

The **multidegree** of f is defined to be the multidegree of its lead monomial $e^{\boldsymbol{\alpha}_1}$ and is denoted $\mathrm{multideg}(f) = \boldsymbol{\alpha}_1$. The **total degree** of f is defined to be the maximum of the total degrees of its monomials and is denoted

$$\deg(f) = \max_{1 \leq i \leq m} \{\deg(e^{\boldsymbol{\alpha}_i})\}.$$

We say f is **homogeneous** of homological degree i if each of its monomials is homogeneous of homological degree i. In this case, we say f has **homological degree** i and we denote this by $|f| = i$.

Proposition 0.1. *For each $1 \leq i, j \leq n$, let $f_{ij} = -[e_i, e_j] = e_i e_j - e_i \star e_j$. We have*

$$\mathrm{LT}(f_{ij}) = e_i e_j.$$

Proof. If $e_i \star e_j = 0$, then this is clear, otherwise term of $e_i \star e_j$ has the form $r_{i,j}^k e_k$ for some k where $r_{i,j}^k \neq 0$. Since $\star$ respects homological degree, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > |e_i|$ and $|e_k| > |e_j|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ and $k > j$ by our assumption on the ordering of $e_1, \dots, e_n$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j > e_k$. $\square$

0.0.2 Gröbner Basis Calculations

The inclusion map $R \subseteq K$ induces an inclusion map $F \to F_K$ where $F_K = \{a/r \mid a \in F \text{ and } r \in R \backslash \{0\}\}$. For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in $R[\boldsymbol{e}] \subseteq K[\boldsymbol{e}]$ defined by $f_{i,j} := -[e_i, e_j]$. Thus we have

$$f_{i,j} = e_i e_j - e_i \star e_j = e_i e_j - \sum_k r_{i,j}^k e_k,$$

where the $r_{i,j}^k$ are the entries of the matrix representation of μ with respect to the ordered homogeneous basis $e_1, \dots, e_n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$, let $\mathfrak{b}$ be the $R[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$, and let $\mathfrak{b}_K$ be the $K[\boldsymbol{e}]$-ideal generated by $\mathcal{F}$. Note that if e_i is odd, then $f_{i,i} = e_i^2$ since $\star$ is strictly graded-commutative, thus $R[\boldsymbol{e}]/\mathfrak{b} \simeq F/\langle F \rangle$ and $K[\boldsymbol{e}]/\mathfrak{b}_K \simeq F_K/\langle F_K \rangle$ by Theorem (**??**).

Recall that $K[\boldsymbol{e}]$ comes equipped with a monomial ordering which we defined earlier. We wish to construct a left Gröbner basis for $\mathfrak{b}_K$ (which will turn out to be a two-sided Gröbner basis) using this monomial ordering via Buchberger's algorithm (as described in [**?**]). Suppose f, g are two nonzero polynomials in $K[\boldsymbol{e}]$ with $\mathrm{LT}(f) = re^{\boldsymbol{\alpha}}$ and $\mathrm{LT}(g) = se^{\boldsymbol{\beta}}$. Set $\boldsymbol{\gamma} = \mathrm{lcm}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and the left **S-polynomial** of f and g to be

$$\mathrm{S}(f, g) = e^{\boldsymbol{\gamma} - \boldsymbol{\alpha}} f \pm (r/s) e^{\boldsymbol{\gamma} - \boldsymbol{\beta}} g \tag{3}$$

where the $\pm$ in (3) is chosen to be $+$ or $-$, depending on which sign will cancel out the lead terms. We begin Buchberger's algorithm by calculating the S-polynomials of all pairs of polynomials in $\mathcal{F}$. In other words, we calculate all S-polynomials of the form $\mathrm{S}(f_{k,l}, f_{i,j})$ where $1 \leq i, j, k, l \leq n$. Note that if $k > l$, then

$$f_{l,k} = (-1)^{|e_k||e_l|} f_{k,l},$$

which implies

$$\mathrm{S}(f_{l,k}, f_{i,j}) = (-1)^{|e_k||e_l|} \mathrm{S}(f_{k,l}, f_{i,j}) = \pm \mathrm{S}(f_{i,j}, f_{k,l}).$$

Similarly, if $i \geq k$, then

$$\mathrm{S}(f_{i,j}, f_{l,k}) = \pm \mathrm{S}(f_{k,l}, f_{i,j}).$$

Thus we may assume that $j \geq i$ and $l \geq k \geq i$. Obviously we have $\mathrm{S}(f_{i,j}, f_{i,j}) = 0$ for each i, j, however something interesting happens when we calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $j > i$ and then divide this by $\mathcal{F}$ (where division by $\mathcal{F}$ means taking the left normal form of $\mathrm{S}(f_{j,k}, f_{i,j})$ with respect to $\mathcal{F}$ using the left normal form described in [**?**]). We have

$$\begin{aligned}
\mathrm{S}(f_{j,k}, f_{i,j}) &= e_i(e_je_k - e_j \star e_k) - (e_ie_j - e_i \star e_j)e_k \\
&= (e_i \star e_j)e_k - e_i(e_j \star e_k) \\
&= \sum_l r_{i,j}^l e_l e_k - \sum_l r_{j,k}^l e_i e_l \\
&\to \sum_l r_{i,j}^l e_l \star e_k - \sum_l r_{j,k}^l e_i \star e_l \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k],
\end{aligned}$$

where in the fourth line we did division by $\mathcal{F}$ (note that if $[e_i, e_j, e_k] \neq 0$, then $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$). Finally if $j > i$, $l > k$, and $j \neq k$, then we have

$$\begin{aligned}
\mathrm{S}(f_{k,l}, f_{i,j}) &= e_ie_jf_{k,l} - f_{i,j}e_ke_l \\
&= (e_i \star e_j)e_ke_l - e_ie_j(e_k \star e_l) \\
&\to (e_i \star e_j) \star (e_k \star e_l) - (e_i \star e_l) \star (e_k \star e_l) \\
&= 0
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Next, suppose that

$$f = re_k + r'e_{k'} + \cdots + r''e_{k''} \in \langle F \rangle$$

where $r, r', r'' \in R$ with $r \neq 0$ and where $\mathrm{LM}(f) = e_k$. Then we have

$$\begin{aligned}
\mathrm{S}(f, f_{j,k}) &= e_jf - rf_{j,k} \\
&= r'e_je_{k'} + \cdots + r''e_je_{k''} + re_j \star e_k \\
&\to r'e_j \star e_{k'} + \cdots + r''e_j \star e_{k''} + re_j \star e_k \\
&= e_j \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\
&= e_j \star f \\
&\in \langle F \rangle
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Similarly, we have if $i \neq k \neq j$, then we have

$$\begin{aligned}
\mathrm{S}(f, f_{i,j}) &= e_ie_jf - rf_{i,j}e_k \\
&= r'(e_ie_j)e_{k'} + \cdots + r''(e_ie_j)e_{k''} + r(e_i \star e_j)e_k \\
&\to r'(e_i \star e_j) \star e_{k'} + \cdots + r''(e_i \star e_j) \star e_{k''} + r(e_i \star e_j) \star e_k \\
&= (e_i \star e_j) \star (re_k + r'e_{k'} + \cdots + r''e_{k''}) \\
&= (e_i \star e_j) \star f \\
&\in \langle F \rangle.
\end{aligned}$$

where in the third line we did division by $\mathcal{F}$. Finally suppose that

$$g = se_m + s'e_{m'} + \cdots + s''e_{m''} \in \langle F \rangle$$

where $s, s', s'' \in R$ with $s \neq 0$ and where $\mathrm{LM}(g) = e_m$. If $k = m$, then we have

$$s\mathrm{S}(f, g) = sf - rg \in \langle F \rangle.$$

On the other hand, if $k \neq m$, then we have

$$\begin{aligned}
s\mathrm{S}(f, g) &= se_mf - rge_k \\
&= sr'e_me_{k'} + \cdots + sr''e_me_{k''} - rs'e_{m'}e_k - \cdots - rs''e_{m''}e_k \\
&\to sr'e_m \star e_{k'} + \cdots + sr''e_m \star e_{k''} - rs'e_{m'} \star e_k - \cdots - rs''e_{m''} \star e_k \\
&= se_m \star (r'e_{k'} + \cdots + r''e_{k''}) - r(s'e_{m'} + \cdots + s''e_{m''}) \star e_k \\
&= se_m \star (f - re_k) - r(g - se_m) \star e_k \\
&= se_m \star f + rg \star e_k - sre_m \star e_k + rse_m \star e_k \\
&= se_m \star f + rg \star e_k \\
&\in \langle F \rangle.
\end{aligned}$$

It follows that we can construct a Gröbner basis

$$\mathcal{G} := \mathcal{F} \cup \{g_1, \ldots, g_m\}$$

of $\mathfrak{b}_K$ such that the g_i all belong to $\langle F \rangle$.