Methodology

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Introduction

In this project, we study [JTMW99].

The Weighted Quadratic Approach

In the weighted quadratic approach a quadratic function of the objective functions

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{f}(\boldsymbol{x})^\top Q \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{q}^\top \boldsymbol{f}(\boldsymbol{x}) \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{1}$$

where Q is a $p \times p$ matrix and where $\boldsymbol{q}$ is a vector in $\mathbb{R}^p$.

Theorem 0.1. *Under conditions of a quadratic Lagrangian duality, if $\widehat{\boldsymbol{x}} \in X$ is efficient, then there exist a symmetric $p \times p$ matrix Q and a vector $\boldsymbol{q} \in \mathbb{R}^p$ such that $\widehat{\boldsymbol{x}}$ is an optimal solution to (1)*

MOP Formulation

Let $\boldsymbol{f} \colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued objective function. Thus we have

$$\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x}))^\top$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, where $f_k \colon \mathbb{R}^n \to \mathbb{R}$ are the component objective functions of $\boldsymbol{f}$ for $1 \leq k \leq m$. Next, let $\boldsymbol{h} \colon \mathbb{R}^n \to \mathbb{R}^p$ be a vector-valued constraint function. Thus we have

$$\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}), \dots, h_p(\boldsymbol{x}))^\top$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, where $h_j \colon \mathbb{R}^n \to \mathbb{R}$ are the component constraint functions of $\boldsymbol{h}$ for $1 \leq j \leq m$. Finally, let

$$X = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{h}(\boldsymbol{x}) = 0\}.$$

We consider the following multi objective program (MOP) below:

$$\begin{aligned} &\text{maximize} \quad \boldsymbol{f}(\boldsymbol{x}) \\ &\text{subject to} \quad \boldsymbol{x} \in X \end{aligned} \tag{2}$$

A point $\boldsymbol{x}^0 \in X$ is called an efficient solution of this MOP if there is no other point $\boldsymbol{x} \in X$ such that $\boldsymbol{f}(\boldsymbol{x}) \geq \boldsymbol{f}(\boldsymbol{x}^0)$ with strict inequality holding for at least one component (i.e. $f_k(\boldsymbol{x}) > f_k(\boldsymbol{x}^0)$ for some $1 \leq k \leq m$).

Tchebycheff Approach

We first use the Tchebycheff approach to find the efficient solutions to (**??**). Let $\boldsymbol{\lambda} \in \mathbb{R}^m$ be a weight vector and let $\boldsymbol{z}^* \in \mathbb{R}^m$ be the ideal point whose i-th component is given by $z_i^* = \max_{x \in X} f_i(\boldsymbol{x})$. Now consider the problem

$$\min_{x \in X} \max_{1 \leq i \leq m} \{\lambda_i (z_i^* - f_i(\boldsymbol{x}))\} \tag{3}$$

where $\boldsymbol{\lambda} \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. All efficient solutions of (**??**) can be found as optimal solutions of (**??**) by adjusting the $\boldsymbol{\lambda}$-values.

Primal Problem

We can rewrite (5) as

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}, \alpha) = 0 \\ & \boldsymbol{x} \in X \end{array} \tag{4}$$

where we set

$$\boldsymbol{g}(\boldsymbol{x}, \alpha) = \boldsymbol{\lambda}^\top (\boldsymbol{z}^* - \boldsymbol{f}(\boldsymbol{x})) - \alpha.$$

Next, let A be the $m \times m$ diagonal matrix whose diagonal entries are all equal to a, and define the augmented Lagrange function:

$$L_Q(x, \alpha, a, y) = \alpha + \boldsymbol{g}(\boldsymbol{x}, \alpha)^\top A \boldsymbol{g}(\boldsymbol{x}, \alpha) + \boldsymbol{y}^\top \boldsymbol{g}(\boldsymbol{x}, \alpha),$$

and the following dual program:

$$\max_{a > 0, \boldsymbol{y}} \min_{\boldsymbol{x} \in X, \alpha} L_Q(\boldsymbol{x}, \alpha, a, \boldsymbol{y}). \tag{5}$$

Then subject to ceratin conditions, program (**??**) has an optimal solution $(\boldsymbol{x}, \alpha)$ if and only if its dual (**??**) has an optimal solution $(a, \boldsymbol{y})$, and in this case the objective values of both programs are equal. Furthermore, $L_Q(\boldsymbol{x}, \alpha, a, \boldsymbol{y})$ has a saddle point in the primal variables $(\boldsymbol{x}, \alpha)$ and the dual variables $(a, \boldsymbol{y})$.

Main Result

Theorem 0.2.

Conclusion

References

References

[JTMW99] JØRGEN TIND1 and MARGARET M. WIECEK. “Augmented Lagrangian and Tchebycheff Approaches in Multiple Objective Programming”. In: Journal of Global Optimization 14: 251–266, 1999