# Goldbach Rings

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**Abstract**

Let $\Bbbk$ be a field. We introduce and study an interesting infinite-dimensional $\Bbbk$-algebra $G$ which we call the Goldbach ring. As the name suggests, the Goldbach ring is closely related to Goldbach's conjecture. Properties that $G$ satisfies as a ring (such as whether or not it is an integral domain) may give us clues about Goldbach's conjecture itself.

## 1  Introduction

Let $\Bbbk$ be a field. We introduce and study an interesting infinite-dimensional $\Bbbk$-algebra which we call the Goldbach ring, which, as the name suggests, is closely related to Goldbach's conjecture. The Goldbach ring $G$ is defined to be the quotient $G = R/I$ where

$$R = \Bbbk[\{x_p, x_{p+q} \mid p, q \text{ odd primes}\}]$$
$$I = \langle\{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\}\rangle$$

The Goldbach ring has the structure of a bi-graded $\Bbbk$-algebra meaning it can be decomposed as

$$G = \bigoplus_{n,d \geq 0} G_{n,d},$$

where the component $G_{n,d}$ in bi-degree $(n,d) \in \mathbb{N}^2$ is a finite-dimensional $\Bbbk$-vector space whose dimension we are interested in counting. For instance, Goldbach's conjecture is equivalent to the statement that $\dim_{\Bbbk} G_{2k,2} = 1$ for all $k \geq 3$. However this is really just a restatement of Goldbach's conjecture; what's more interesting and new in our view is the following conjecture which seems to hold in small examples:

**Conjecture 1.** *We have*

$$\dim_{\Bbbk} G_{n,d} \leq 1$$

*for all $n, d \in \mathbb{N}$.*

A counter-example to Conjecture (1) would be the existence of odd primes $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$ such that

$$p_1 + \cdots + p_d = n = q_1 + \cdots + q_d$$

but $x_{p_1} \cdots x_{p_d} \neq x_{q_1} \cdots x_{q_d}$ in $G$. However we do not believe such a counter-example exists since. Indeed, based on our initial calculations, it seems that there are usually many ways to go from $x_{p_1} \cdots x_{p_d}$ to $x_{q_1} \cdots x_{q_d}$ by applying elementary Goldbach relations of the form $x_p x_q = x_{p+q}$. For another example, in $G_{36,4}$ we have $x_3^2 x_{11} x_{19} = x_5^2 x_{13}^2$ since

$$
\begin{aligned}
x_3^2 x_{11} x_{19} &= x_3 x_{11} x_{22} \\
&= x_3 x_5 x_{11} x_{17} \\
&= x_5 x_{11} x_{20} \\
&= x_5 x_7 x_{11} x_{13} \\
&= x_5 x_{13} x_{18} \\
&= x_5^2 x_{13}^2.
\end{aligned}
$$

There are many other paths we can take from $x_3^2 x_{11} x_{19}$ to $x_5^2 x_{13}^2$, however it turns out that this is the shortest path. Ultimately a solution to Conjecture (1) will involve tools and techniques from analytic number theory. What we find interesting is that Conjecture (1) also seems to involve a lot of commutative algebra as well. For example, if Conjecture (1) is true, then it woud imply that $G$ is an integral domain. Conversely, one can show that if $G$ is an integral domain and Conjecture (1) holds for $n, d$ sufficiently large, then Conjecture (1) is true.

A deeper relationship between Conjecture ([1]), anlaytic number theory, and commutative algebra is realized when one studies $G$ as a direct limit

$$G = \varinjlim G^m$$

of bi-graded noetherian $\Bbbk$-algebras $G^m = R^m/I^m$, where

$$R^m = \Bbbk[x_1, \ldots, x_m] \cap R$$
$$I^m = \Bbbk[x_1, \ldots, x_m] \cap I.$$

Indeed, for each $m$, we denote by $\delta(m)$ and $\rho(m)$ to be the $R^m$-depth and $R^m$-projective dimension of $G^m$ respectively. Then the Auslander-Buchsbaum formula implies

$$\delta(2m) + \rho(2m) = \pi(2m) + m - \kappa(2m) - 3, \tag{1}$$

where $\pi(2m)$ is the usual prime-counting function which counts the number of primes $\leq 2m$ and where $\kappa(2m)$ counts then number of positive even numbers $\leq 2m$ that are coutner-examples to Goldbach's conjecture.

## 2   $\mathcal{A}$-Supported Goldbach Rings

Let $\mathcal{A}$ be a subset of the positive odd integers and set $\mathcal{C} := \mathcal{A} + \mathcal{A} = \{a + b \mid a, b \in \mathcal{A}\}$. We set

$$R_{\mathcal{A}} = \Bbbk[\{x_a, x_c \mid a \in \mathcal{A}, c \in \mathcal{C}\}]$$
$$I_{\mathcal{A}} = \langle \{x_a x_b - x_{a+b} \mid a, b \in \mathcal{A}\} \rangle$$
$$G_{\mathcal{A}} = R_{\mathcal{A}}/I_{\mathcal{A}}.$$

We will refer to $G_{\mathcal{A}}$ as the $\mathcal{A}$-**supported Goldbach ring**. We simplify our notation by writing $\{x_a, x_c\}$ to denote the set $\{x_a, x_c \mid a \in \mathcal{A}, c \in \mathcal{C}\}$. Similarly we write $\{x_a x_b - x_{a+b}\}$ to denote the set $\{x_a x_b - x_{a+b} \mid a, b \in \mathcal{A}\}$. We often simplify our notation even further by dropping $\mathcal{A}$ from our notation whenever it is clear from context. For instance, we write "$G$" instead of "$G_{\mathcal{A}}$" when it's understood that $G$ is the $\mathcal{A}$-supported Goldbach ring. Similarly, if we write "let $G$ be the $\mathcal{A}$-supported Golbach ring", then it's understood that $\mathcal{A}$ is a subset of the positive odd integers and that $\mathcal{C} = \mathcal{A} + \mathcal{A}$.

### 2.1   Representing Monomials

We will denote by $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$ to be the set of all monomials in $R = R_{\mathcal{A}}$. There are two ways we can represent monomials in $R$. The first way is as a finite product of the indeterminates $\{x_a, x_c\}$, namely, a monomial can be expressed in the form

$$x_{\boldsymbol{a}} x_{\boldsymbol{c}} := x_{a_1} \cdots x_{a_r} x_{c_1} \cdots x_{c_s}$$

where $\boldsymbol{a} = a_1, \ldots, a_r$ is a sequence of elements in $\mathcal{A}$ (not necessarily distinct, but often we assume $a_1 \leq \cdots \leq a_r$) and $\boldsymbol{c} = c_1, \ldots, c_s$ is a sequence of elements in $\mathcal{C}$ (again not necessarily distinct, but often we assume $c_1 \leq \cdots \leq c_s$). We will use this way of representing monomials to give $R$ a nice bi-graded structure. The second way of representing monomials is described as follows: given a function $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, we define its **support**, denoted $\operatorname{supp} \alpha$, to be the set

$$\operatorname{supp} \alpha = \{m \in \mathbb{N} \mid \alpha(m) \neq 0\}.$$

We denote by $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$ to be the set

$$\mathcal{F} = \{\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0} \mid \operatorname{supp} \alpha \text{ is finite and contained in } \{x_a, x_c\}\}.$$

Thus if $\alpha \in \mathcal{F}$, then $\alpha$ takes value $0$ zero almost everywhere, and the only places where it is nonzero is at an element in $\{x_a, x_c\}$. Then there is a bijection from $\mathcal{F}$ to $\mathcal{M}$ given by assigning $\alpha \in \mathcal{F}$ to the monomial

$$x^{\alpha} := \prod_{m \in \mathbb{N}} x_m^{\alpha(m)} = \prod_{m \in \operatorname{supp} \alpha} x_m^{\alpha(m)}.$$

For instance, suppose $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is defined by

$$\alpha(m) = \begin{cases} 3 & \text{if } m = 2 \\ 2 & \text{if } m = 6 \\ 4 & \text{if } m = 11 \\ 0 & \text{if } m \in \mathbb{N} \setminus \{2, 6, 11\} \end{cases}$$

Then $x^\alpha = x_3^3 x_6^2 x_{11}^4$ and supp $x^\alpha = \{2, 6, 11\}$. This second way of expressing monimals gives us a cleaner way of expressing nonzero polynomials in $R$, namely, every nonzero polynomial $f \in R$ can be expressed in the form

$$f = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}$$

for unique $a_1, \ldots a_n \in \Bbbk$ and for unique $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$. We often pass back and forth between functions $\alpha \in \mathcal{F}$ and monomials $x^\alpha \in \mathcal{M}$. For instance, given a monomial $x^\alpha \in \mathcal{M}$, we define its **support**, denoted supp $x^\alpha$, to be supp $x^\alpha = $ supp $\alpha$, and etc...

## 2.2 The Bi-Graded $\Bbbk$-Structure on $R$ and $G$

We give $R$ and $G$ a bi-graded $\Bbbk$-structure as follows: we define $\deg_1 : \mathcal{M} \to \mathbb{N}$ and $\deg_2 : \mathcal{M} \to \mathbb{N}$ by

$$\deg_1(x_a x_c) = \sum_{i=1}^{r} a_i + \sum_{j=1}^{s} c_j \quad \text{and} \quad \deg_2(x_a x_c) = r + 2s.$$

In particular, we have $\deg_1(x_a) = a$, $\deg_1(x_c) = c$, $\deg_2(x_a) = 1$, and $\deg_2(x_c) = 2$. For each $n, d \in \mathbb{N}$, we set

$$R_{n,d} = \operatorname{span}_\Bbbk \{x^\alpha \in \mathcal{M} \mid \deg_1(x^\alpha) = n \text{ and } \deg_2(x^\alpha) = d\}.$$

Then we have a decomposition of $R$ into $\Bbbk$-vector spaces:

$$R = \bigoplus_{n,d \in \mathbb{N}} R_{n,d},$$

which gives $R$ a bi-graded $\Bbbk$-structure. Since $I$ is homogeneous with respect to this bi-grading, $G$ inherits a bi-graded $\Bbbk$-structure, induced by the one on $R$:

$$G = \bigoplus_{n,d \in \mathbb{N}} G_{n,d}.$$

Thus $\dim_\Bbbk R_{n,d}$ counts the number of ways we can express $n$ as a sum

$$n = a_1 + \cdots + a_r + c_1 + \cdots + c_s$$

where $a_1, \ldots, a_r \in \mathcal{A}$, $c_1, \ldots, c_s \in \mathcal{C}$, and $d = r + s$. Whenever we have $\dim_\Bbbk R_{n,d} \geq 1$, then we say $(n, d)$ is a **good pair**. In this case, we are very interested in determining whether or not $\dim_\Bbbk G_{n,d} = 1$ or $\dim_\Bbbk G_{n,d} > 1$. Intuitively, we have $\dim_\Bbbk G_{n,d} = 1$ when $\mathcal{A}$ is sufficiently "dense" in $\mathbb{N}$ and we have $\dim_\Bbbk G_{n,d} > 1$ whenever $\mathcal{A}$ is very "sparse" in $\mathbb{N}$.

## 2.3 Constructing the Minimal $R$-Free Resolution of $G$

We now build the minimal $R$-free resolution of $G$ as follows: first, for each $m \geq 1$, we define the **$m$-th approximation** of $R$, $I$, and $G$ to be:

$$R^m = \Bbbk[\{x_a, x_c \mid a, c \leq m\}]$$
$$I^m = \langle \{x_a x_b - x_{a+b} \mid a + b \leq m\} \rangle$$
$$G^m = R^m / I^m.$$

Again, $R^m$ and $G^m$ have bi-graded $\Bbbk$-structures:

$$R^m = \bigoplus_{n,d} R^m_{n,d} \quad \text{and} \quad G^m = \bigoplus_{n,d} G^m_{n,d}.$$

Recall that if $x_a x_c \in R^m_n$, then we must have $a + c = n$ and $a, c \leq m$. In particular, if $m \geq n$ then we have $R^m_n = R^n_n = R_n$. Similarly, if $m \geq n$ then we have $G^m_n = G^n_n = G_n$. Thus we have directed systems

$$(R^m)_{m \geq 1} \quad \text{and} \quad (G^m)_{m \geq 1}$$

of bi-graded $\Bbbk$-algebras (with the obvious $\Bbbk$-algebra homomorphisms) where the components $R^m_{n,d}$ and $G^m_{n,d}$ in bi-graded degree $(n, d)$ stabilizes to $R_{n,d}$ and $G_{n,d}$ respectively whenever $m$ is sufficiently large (for example $m \geq n$). It follows that

$$R = \varinjlim R^m \quad \text{and} \quad G = \varinjlim G^m$$

as bi-graded direct limits.

Next we let $F^m = F_{\mathcal{A}}^m$ be the minimal bi-graded $R^m$-free resolution of $G^m$ (where $F^m$ is necessarily finite since $R^m$ and $G^m$ are noetherian). We set

$$\delta(m) = \delta_{\mathcal{A}}(m) := \operatorname{depth}_{R^m} G^m \quad \text{and} \quad \rho(m) = \rho_{\mathcal{A}}(m) := \operatorname{pd}_{R^m} G^m = \operatorname{length} F^m.$$

Note that these quantities are intrinsic to $R^m$ and $G^m$ (and not $R$ and $G$). By the Auslander-Buchsbaum formula we have

$$\rho(m) + \delta(m) = \pi_{\mathcal{A} \cup \mathcal{C}}(m) := \#\{a, c \in \mathcal{A} \cup \mathcal{C} \mid a, c \leq m\}. \tag{2}$$

Note that $F^m$ has the structure of a bi-graded $\Bbbk$-complex, meaning we have a decomposition of $\Bbbk$-complexes:

$$F^m = \bigoplus_{n,d} F_{n,d}^m,$$

where $F_{n,d}^m$ is a finite $\Bbbk$-subcomplex of $F^m$ which minimially resolves $G_{n,d}^m$, meaning the augmented complex

$$\widetilde{F}_{n,d}^m := \cdots \to F_{i,n,d}^m \to F_{i-1,n,d}^m \to \cdots \to F_{1,n,d}^m \to R_{n,d}^m \to G_{n,d}^m \to 0$$

is exact and where the $i$-th Betti number of $G^m$ in bi-degree $(n, d)$ is given by

$$\beta_{i,n,d}^m := \dim_{\Bbbk} \operatorname{Tor}_i^{R^m}(G^m, \Bbbk)_{n,d} = \dim_{\Bbbk}(F_{i,n,d}^m).$$

The canonical map $G^m \to G^{m+1}$ induces an injective comparison map $F^m \to F^{m+1}$ which we may choose to respect the bi-graded structure. Furthermore, since $R_n^m = R_n^n$ and $G_n^m = G_n^n$ whenever $m \geq n$, we see that $F_n^m = F_n^n$ whenever $m \geq n$. Thus if we *define* $F = F_{\mathcal{A}}$ to be the direct limit of bi-graded $\Bbbk$-complexes

$$F := \varinjlim F^m,$$

then $F$ is a bi-graded $R$-free resolution of $G$ which has the following bi-graded $\Bbbk$-complex structure:

$$F = \bigoplus_{n,d} F_{n,d} = \bigoplus_{n,d} F_{n,d}^n.$$

In particular, if $m$ is sufficiently large, then we see that $\beta_{i,n,d}^m = \beta_{i,n,d}$ where

$$\beta_{i,n,d} := \dim_{\Bbbk} \operatorname{Tor}_i^R(G, \Bbbk)_{n,d} = \dim_{\Bbbk}(F_{i,n,d})$$

is the $i$th Betti number of $G$ in bi-degree $(n, d)$. Thus, unlike the quantities $\delta(m)$ and $\rho(m)$, the quantitiy $\beta_{i,n,d}^m$ is actually *intrinsic* to $R$ and $G$ (and not just $R^m$ and $G^m$) when $m$ is sufficiently large.

**Theorem 2.1.** *We have* $\dim_{\Bbbk} G_{n,d} = \chi(F_{n,d})$. *In other words, we have*

$$\dim_{\Bbbk} G_{n,d} = \dim_{\Bbbk} R_{n,d} - \sum_{i=1}^{\infty} (-1)^i \beta_{i,n,d} = \sum_{i=0}^{\infty} (-1)^i \dim_{\Bbbk} \operatorname{Tor}_i^R(G, \Bbbk)_{n,d}, \tag{3}$$

*where the sum on the right (3) is finite.*

# 3  The Goldbach Ring

We now consider the case where $\mathcal{A} = \{\text{positive odd primes}\}$. In this case, we have

$$R = \Bbbk[\{x_p, x_{2k} \mid p \text{ odd prime and } k \in \mathbb{Z}_{\geq 3}\}]$$
$$I = \langle\{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\}\rangle$$
$$G = R/I.$$

For obvious reasons, we call $G$ the **Goldbach ring.** The homogeneous components of the form $R_{18,d}$ looks like:

$$\vdots = \vdots$$
$$R_{18,7} = 0$$
$$R_{18,6} = \Bbbk x_3^6 + \Bbbk x_3^4 x_6 + \Bbbk x_3^2 x_6^2 + \Bbbk x_6^3$$
$$R_{18,5} = 0$$
$$R_{18,4} = \Bbbk x_3^2 x_5 x_7 + \Bbbk x_3 x_5^3 + \Bbbk x_3^2 x_{12} + \cdots + \Bbbk x_5 x_6 x_7 + \Bbbk x_6 x_{12} + \Bbbk x_8 x_{10}$$
$$R_{18,3} = 0$$
$$R_{18,2} = \Bbbk x_5 x_{13} + \Bbbk x_7 x_{11} + \Bbbk x_{18}$$
$$R_{18,1} = 0$$
$$\vdots = \vdots$$

Similarly, the homogeneous components of the form $R_{17,d}$ looks like:

$$\vdots = \vdots$$
$$R_{17,6} = 0$$
$$R_{17,5} = \Bbbk x_3^4 x_5 + \Bbbk x_3^3 x_8 + \Bbbk x_3^2 x_5 x_6 + \Bbbk x_3 x_6 x_8 + \Bbbk x_5 x_6^2$$
$$R_{17,4} = 0$$
$$R_{17,3} = \Bbbk x_3^2 x_{11} + \Bbbk x_3 x_7^2 + \Bbbk x_5^2 x_7 + \Bbbk x_6 x_{11} + \Bbbk x_3 x_{14} + \Bbbk x_7 x_{10} + \Bbbk x_5 x_{12}$$
$$R_{17,2} = 0$$
$$R_{17,1} = \Bbbk x_{17}$$
$$R_{17,0} = 0$$
$$\vdots = \vdots$$

Staring at the homogeneous components above, we see that $\dim_{\Bbbk} R_{18,4} = 9$ and $\dim_{\Bbbk} R_{17,3} = 7$. More generally, $\dim_{\Bbbk} R_{n,d}$ counts the number of ways we can express $n$ as a sum:

$$n = p_1 + \cdots + p_r + 2(k_1 + \cdots + k_s), \tag{4}$$

where $p_1, \ldots,, p_r$ are odd primes, $k_1, \ldots, k_s \geq 3$, and $d = r + 2s$. Here are some basic facts about $\Delta_{n,d}$:

1. Assume $n$ is even.
$$\text{we have } \begin{cases} \dim_{\Bbbk} R_{n,d} \geq 1 & \text{if } d \text{ is even and } 2 \leq d \leq \lfloor n/3 \rfloor \\ \dim_{\Bbbk} R_{n,d} = 0 & \text{else} \end{cases}$$

   Indeed, if $d$ is even and satisfies $2 \leq d \leq \lfloor n/3 \rfloor$, then we have $\dim_{\Bbbk} R_{n,d} \geq 1$ since we have the decomposition $n = (n - 6d) + 6d$.

2. Assume $n$ is odd.
$$\text{we have } \begin{cases} \dim_{\Bbbk} R_{n,d} \geq 1 & \text{if } d \text{ is odd and } 3 \leq d \leq \lfloor n/3 \rfloor \\ \dim_{\Bbbk} R_{p,1} = 1 & \text{if } p \text{ is odd prime} \\ \dim_{\Bbbk} R_{n,d} = 0 & \text{else} \end{cases}$$

   Indeed, if $d$ is odd and satisfies $3 \leq d \leq \lfloor n/3 \rfloor$, then we have $\dim_{\Bbbk} R_{n,d} \geq 1$ since we have the decomposition $n = (n - 3 - 6d) + 6d + 3$.

Next, the homogeneous components of the form $G_{17,d}$ and $G_{18,d}$ looks like:

$$\vdots = \vdots \qquad\qquad \vdots = \vdots$$
$$G_{17,6} = 0 \qquad\qquad G_{18,6} = \Bbbk \overline{x}_3^6$$
$$G_{17,5} = \Bbbk \overline{x}_3^4 \overline{x}_5 \qquad\qquad G_{18,5} = 0$$
$$G_{17,4} = 0 \qquad\qquad G_{18,4} = \Bbbk \overline{x}_3^2 \overline{x}_5 \overline{x}_7$$
$$G_{17,3} = \Bbbk \overline{x}_3^2 \overline{x}_{11} \qquad\qquad G_{18,3} = 0$$
$$G_{17,2} = 0 \qquad\qquad G_{18,2} = \Bbbk \overline{x}_5 \overline{x}_{13}$$
$$G_{17,1} = \Bbbk \overline{x}_{17} \qquad\qquad G_{18,1} = 0$$
$$\vdots = \vdots \qquad\qquad \vdots = \vdots$$

From what we've seen above, it is *very* tempting to consider the following conjecture:

**Conjecture 2.** *If $n > 0$ is even, then*

$$\text{we have } \begin{cases} \dim_{\Bbbk} G_{n,d} = 1 & \text{if } d \text{ is even and } 2 \leq d \leq \lfloor n/3 \rfloor \\ \dim_{\Bbbk} G_{n,d} = 0 & \text{else} \end{cases}$$

*If $n$ is odd, then*

$$\text{we have } \begin{cases} \dim_{\Bbbk} G_{n,d} = 1 & \text{if } d \text{ is odd and } 3 \leq d \leq \lfloor n/3 \rfloor \\ \dim_{\Bbbk} G_{p,1} = 1 & \text{if } p \text{ is odd prime} \\ \delta_{n,d} = 0 & \text{else} \end{cases}$$

If Conjecture ([2]) is true, then $G$ has a nice property as a ring:

**Proposition 3.1.** *Assume Conjecture ([2]) is true. Then G is an integral domain.*

*Proof.* Let $f, g \in G_{n,d} = \Bbbk \bar{x}^\alpha$ such that $fg = 0$ and express $f$ and $g$ as

$$f = a\bar{x}^\alpha \quad \text{and} \quad g = b\bar{x}^\alpha.$$

Then clearly since $\bar{x}^{2\alpha} \neq 0$, we must have $ab = 0$, which implies either $a = 0$ or $b = 0$ which implies either $f = 0$ or $g = 0$. □

**Example 3.1.** In $G^{46}$, we have

$$
\begin{aligned}
x_{23}(x_{19}x_{29} - x_{17}x_{31}) &= x_{23}x_{19}x_{29} - x_{23}x_{17}x_{31} \\
&= x_{11}x_{31}x_{29} - x_{11}x_{29}x_{31} \\
&= 0,
\end{aligned}
$$

however $x_{19}x_{29} \neq x_{17}x_{31}$ in $G^{46}$. It follows that $x_{23}$ is a zerodivisor in $G^{46}$. Similarly, we have

$$
\begin{aligned}
x_3(x_{19}x_{29} - x_{17}x_{31}) &= x_3 x_{19}x_{29} - x_3 x_{17}x_{31} \\
&= x_5 x_{17}x_{29} - x_5 x_{17}x_{29} \\
&= 0.
\end{aligned}
$$

It follows that $x_3$ is also a zero divisor in $G^{46}$. Note that $x_{19}x_{29} = x_{17}x_{31}$ in $G^{48}$, however $x_3$ remains a zerodivisor in $G^{48}$ since

$$
\begin{aligned}
x_3(x_{29}x_{31} - x_{23}x_{37}) &= x_3 x_{29}x_{31} - x_3 x_{23}x_{37} \\
&= x_{11}x_{29}x_{23} - x_{11}x_{23}x_{29} \\
&= 0,
\end{aligned}
$$

and $x_{29}x_{31} \neq x_{23}x_{37}$ in $G^{48}$. Similar calculations like this show that $x_3, x_5, x_7, x_{11}, x_{13}, x_{17},$ and $x_{19}$ are all zerodivisors in $G^{46}$. On the other hand, using Singular we find that a maximal $G^{46}$-regular sequence is given by $x_{29}, x_{31}, x_{37}, x_{41}, x_{43}$.

### 3.0.1 Explicit Calculations of the $\Bbbk$-Complex $F_{n,d}$

**Example 3.2.** Let's describe $\widetilde{F}_{18,2}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$
\begin{aligned}
\widetilde{F}_{1,18,2} &= \Bbbk e_{5,13} + \Bbbk e_{7,11} \\
\widetilde{F}_{0,18,2} &= R_{18,2} = \Bbbk x_5 x_{13} + \Bbbk x_7 x_{11} + \Bbbk x_{18} \\
\widetilde{F}_{-1,18,2} &= G_{18,2} = \Bbbk \bar{x}_5 \bar{x}_{13},
\end{aligned}
$$

and $\widetilde{F}_{i,18,2} = 0$ for all $i \neq -1, 0, 1$. The differential is the unique $R$-linear map defined by $d(e_{5,13}) = x_5 x_{13} - x_{18}$ and $d(e_{7,11}) = x_7 x_{13} - x_{18}$. After choosing ordered bases, we can express $\widetilde{F}_{18,2}$ in the form

$$
0 \longrightarrow \Bbbk^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \Bbbk^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}} G_{18,2} \longrightarrow 0
$$

Thus we have

$$
\begin{aligned}
\dim_{\Bbbk} G_{18,2} &= \chi(F_{18,2}) \\
&= 3 - 2 \\
&= 1.
\end{aligned}
$$

.

Next, let's describe $\widetilde{F}_{23,3}$ as a $\Bbbk$-complex. First, as a graded $\Bbbk$-vector space, we have

$$
\begin{aligned}
\widetilde{F}_{2,23,3} &= \Bbbk e_{5,7,11} \\
\widetilde{F}_{1,23,3} &= \Bbbk x_{13} e_{3,7} + \Bbbk x_{13} e_{5,5} + \Bbbk x_7 e_{3,13} + \Bbbk x_7 e_{5,11} + \Bbbk x_5 e_{7,11} + \Bbbk x_5 e_{5,13} \\
\widetilde{F}_{0,23,3} &= R_{23,3} = \Bbbk x_{13} x_{10} + \Bbbk x_7 x_{16} + \Bbbk x_5 x_{18} + \Bbbk x_5 x_7 x_{11} + \Bbbk x_3 x_7 x_{13} + \Bbbk x_5^2 x_{13} \\
\widetilde{F}_{-1,23,3} &= G_{23,2} = \Bbbk \bar{x}_5 \bar{x}_7 \bar{x}_{11}
\end{aligned}
$$

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and $\widetilde{F}_{i,23,3} = 0$ for all $i \neq -1, 0, 1, 2$. The differential is the unique $R$-linear map defined by

$$d(e_{5,7,11}) = x_5 e_{7,11} - x_5 e_{5,13} + x_{13} e_{5,5} - x_{13} e_{3,7} + x_7 e_{3,13} - x_7 e_{5,11}$$
$$d(e_{3,7}) = x_3 x_7 - x_{10}$$
$$d(e_{5,5}) = x_5 x_5 - x_{10}$$
$$d(e_{3,13}) = x_3 x_{13} - x_{16}$$
$$d(e_{5,11}) = x_5 x_{11} - x_{16}$$
$$d(e_{7,11}) = x_7 x_{11} - x_{18}$$
$$d(e_{5,13}) = x_5 x_{13} - x_{18}.$$

After choosing ordered basis, we can express $\widetilde{F}_{23,3}$ in the form

$$0 \longrightarrow \Bbbk \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}} \Bbbk^6 \xrightarrow{M} \Bbbk^6 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}} G_{23,3} \longrightarrow 0$$

where $M$ is a matrix whose entries are either $-1$, $0$, or $1$. Thus we have

$$\dim_{\Bbbk} G_{23,3} = \chi(F_{23,3})$$
$$= 6 - 6 + 1$$
$$= 1.$$

## 3.1 Re-interpreting the Conjecture

From Theorem (2.1), we can express Conjecture (2) in another form:

**Conjecture 3.** *Assume $(n, d)$ is a good pair. Then*

$$\sum_{i=0}^{\infty} (-1)^i \dim_{\Bbbk} \mathrm{Tor}_i^R(G, \Bbbk)_{n,d} = 1.$$