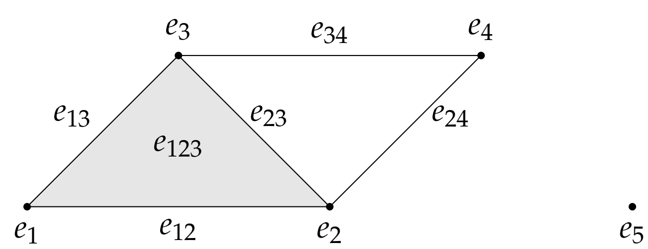
Associativity Test Using Gröbner Bases

Michael Nelson

Introduction

Introduction

Let Δ be a finite simplicial complex and let K be a field of characteristic 2 (we only assume characteristic 2 for simplicity in what follows). Attached to Δ is a graded K-complex F_Δ whose homogeneous component of degree $k \in \mathbb{N}$ is the K-span of all $(k-1)$-faces of Δ. For instance, if Δ is the simplicial complex below,



then the homogeneous components of F_Δ are given by:

$$
\begin{aligned}
F_{\Delta,0} &= Ke_\varnothing \\
F_{\Delta,1} &= Ke_1 + Ke_2 + Ke_3 + Ke_4 + Ke_5 \\
F_{\Delta,2} &= Ke_{12} + Ke_{13} + Ke_{23} + Ke_{24} + Ke_{34} \\
F_{\Delta,3} &= Ke_{123}.
\end{aligned}
$$

Note that we often write $e_\varnothing = 1 = e_0$ and we think of F_Δ as a graded K-vector space with $F_{\Delta,0} = K$. Now let us equip F_Δ with a **graded-multiplication** $\star$, where by a graded-multiplication, we mean that $\star$ is a binary operator on F_Δ which satisfies the following properties:

1. $\star$ is unital with 1 being the unit;
2. $\star$ is K-bilinear;
3. $\star$ is commutative;
4. $\star$ respects the grading meaning that if α, β are homogeneous elements of F_Δ, then $\alpha \star \beta$ is homogeneous and
$$|\alpha \star \beta| = |\alpha| + |\beta|,$$
where $|\cdot|$ denote the homogeneous degree of an element in F_Δ.

Given such a graded-multiplication F_Δ, it is natural to wonder whether or not $\star$ is associative, meaning

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$$

for all $\alpha, \beta, \gamma \in F_\Delta$. In this note, we will determine whether or not $\star$ is associative using tools from the theory of Gröbner bases.

Setting up our Notation

We begin in a slightly more general context. Let F be a positively graded K-vector space which is finite dimensional as a K-vector space. Being a positively graded K–vector space means we have a decomposition

$$F = \bigoplus_{m \in \mathbb{N}} F_m,$$

where each F_m is a K-vector space, called the **homogeneous component** of F in **homological degree** m. If $\alpha \in F_m$, then we set $|\alpha| = m$ and say α is **homogeneous** and that it has **homological degree** m. We assume that $F_0 = K$ and $F_+ := \bigoplus_{m>0} F_m \neq 0$. Let $(e_1 \dots, e_n)$ be an ordered homogeneous basis of F_+ which is ordered in such a way if $|e_j| > |e_i|$, then $j > i$.

Next let $\star\colon F \times F \to F$ be a graded K-bilinear map on F such that $\star$ is unital with $1 \in K = F_0$ being the identity (meaning $\alpha \star 1 = \alpha = 1 \star \alpha$ for all $\alpha \in F$) and such that $\star$ is graded-commutative (meaning $\alpha \star \beta = (-1)^{|\alpha||\beta|}\beta \star \alpha$ for all $\alpha, \beta \in F$. For each $0 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=0}^{n} c_{i,j}^{k} e_k,$$

where the $c_{i,j}^k \in K$ are called the **structured K-coefficients** of $\star$ (they are uniquely determined by $\star$).

Next let S be the polynomial ring $K[e_1, \dots, e_n]$. Monomials in S are expressed in the form

$$e^{\boldsymbol{a}} = e_1^{a_1} \cdots e_n^{a_n},$$

where $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ and where $e^{(0,\dots,0)} = 1$. Given a monomial $e^{\boldsymbol{a}}$ in S, we define its **degree**, denoted $\deg(e^{\boldsymbol{a}})$, and its **homological degree**, denoted $|e^{\boldsymbol{a}}|$, by

$$\deg(e^{\boldsymbol{a}}) = \sum_{i=1}^{n} a_i \quad \text{and} \quad |e^{\boldsymbol{a}}| = \sum_{i=1}^{n} a_i |e_i|.$$

For each $m \in \mathbb{N}$, we set

$$S_m = \operatorname{span}_K \{ e^{\boldsymbol{a}} \mid \deg(e^{\boldsymbol{a}}) = m \}.$$

Clearly we have $S = \bigoplus_{m \geq 0} S_m$. We identity F with $S_0 + S_1 = K + \sum_{i=1}^{n} K e_i$. In order to keep notation consistent, we write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to $\star$, and we shall write $\alpha\beta$ to denote their multiplication with respect to the usual multiplication $\cdot$ in S. In particular, note that $\deg(e_i \star e_j) = 1$, $\deg(e_i e_j) = 2$, and $|e_i \star e_j| = |e_i| + |e_j| = |e_i e_j|$.

Finally, for each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

We let $\mathcal{F} = \{ f_{i,j} \mid 1 \leq i, j \leq n \}$ and let I be the ideal of S generated by $\mathcal{F}$. We equip S with a weighted lexicographical ordering $>_w$ with respect to the weighted vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials $e^{\boldsymbol{a}}$ and $e^{\boldsymbol{b}}$ in S, we say $e^{\boldsymbol{a}} >_w e^{\boldsymbol{b}}$ if either

1. $|e^{\boldsymbol{a}}| > |e^{\boldsymbol{b}}|$ or;
2. $|e^{\boldsymbol{a}}| = |e^{\boldsymbol{b}}|$ and there exists $1 \leq i \leq n$ such that $a_i > b_i$ and $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_{i-1} = b_{i-1}$.

Observe that for each $1 \leq i, j \leq n$, we have $\operatorname{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since $\star$ is graded, we have $|e_k| = |e_i| + |e_j| = |e_i e_j|$. It follows that $|e_k| > |e_i|$ and $|e_k| > |e_j|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ and $k > j$ by our assumption on the ordering of $(e_1, \dots, e_n)$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

Associator

The **associator** of $\star$ is the graded K-trilinear map $[\cdot,\cdot,\cdot]\colon F^3 \to F$ defined by

$$[\alpha, \beta, \gamma] = (\alpha\beta)\gamma - \alpha(\beta\gamma)$$

for all $\alpha, \beta, \gamma \in F$. Clearly, $\star$ is associative if and only if $[\alpha, \beta, \gamma] = 0$ for all $\alpha, \beta, \gamma \in F$. Using the fact that $\star$ is graded-commutative, we obtain the identities

- For all $\alpha, \beta, \gamma \in F$ homogeneous we have

$$[\alpha, \beta, \gamma] = -(-1)^{|\alpha||\beta| + |\alpha||\gamma| + |\beta||\gamma|} [\gamma, \beta, \alpha]. \tag{1}$$

- For all $\alpha, \beta, \gamma \in F$ homogeneous we have

$$[\alpha, \beta, \gamma] = -(-1)^{|\alpha||\gamma| + |\beta||\gamma|} [\gamma, \alpha, \beta] - (-1)^{|\alpha||\beta| + |\alpha||\gamma|} [b, x, a] \tag{2}$$

Proposition 0.1. *For all $\alpha, \beta \in F$ homogeneous, we have*

$$[\alpha, \beta, \alpha] = (-1)^{|\alpha||\beta||}(1 - (-1)^{|\alpha|})[\beta, \alpha, \alpha].$$

Proof. We combine (1) with (2) to get

$$\begin{aligned}
[\alpha, \beta, \alpha] &= -(-1)^{|\alpha||\beta|+|\alpha|}[\alpha, \alpha, \beta] - (-1)^{|\alpha||\beta|+|\alpha|}[x, a, a] \\
&= (-1)^{|\alpha||\beta|}[\beta, \alpha, \alpha] - (-1)^{|\alpha||\beta|+|\alpha|}[\beta, \alpha, \alpha] \\
&= (-1)^{|\alpha||\beta||}(1 - (-1)^{|\alpha|})[\beta, \alpha, \alpha].
\end{aligned}$$

□

Corollary 1. *Suppose that $[\alpha, \beta, \gamma] = 0$ whenever $\alpha \neq \gamma$. Then $\star$ is associative.*

Proof. It suffices to check that $[\alpha, \beta, \alpha] = 0$ for all $\alpha, \beta \in F$. Graded-commutativity of $\star$ implies $[\alpha, \alpha, \alpha] = 0$, so we may assume that $\beta \neq \alpha$. Then the hypothesis together with Proposition (0.1) implies

$$[\alpha, \beta, \alpha] = (-1)^{|\alpha||\beta||}(1 - (-1)^{|\alpha|})[\beta, \alpha, \alpha] = 0.$$

□

The Main Theorem

We are now ready to state and prove the main theorem:

Theorem 0.1. *The following statements are equivalent:*

1. *$\star$ is associative.*
2. *$\mathcal{F}$ is a Gröbner basis.*
3. *$\{e^a \mid e^a \notin \mathrm{LT}(I)\} = \{e_1, \ldots, e_n\}$.*

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $i \neq k$. We have

$$\begin{aligned}
S_{i,j,k} &:= \mathrm{S}(f_{j,k}, f_{i,j}) \\
&= e_i f_{j,k} - f_{i,j} e_k \\
&= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
&= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
&= \left(\sum_l c_{i,j}^l e_l \right) e_k - e_i \left(\sum_l c_{j,k}^l e_l \right) \\
&= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l.
\end{aligned}$$

Now we divide $S_{i,j,k}$ by $\mathcal{F}$. We have

$$\begin{aligned}
S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l (e_l e_k - e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l) \\
&= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
&= \left(\sum_l c_{i,j}^l e_l \right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l \right) \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&:= [e_i, e_j, e_k].
\end{aligned}$$

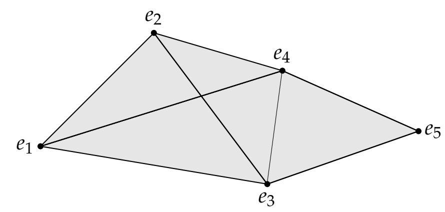
Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$. It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. Next, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $i \neq k$, $i \neq l$, $j \neq k$, and $j \neq l$. We have

$$\begin{aligned}
S_{i,j,k,l} &:= \mathrm{S}(f_{k,l}, f_{i,j}) \\
&= e_i e_j f_{j,k} - f_{i,j} e_k e_l \\
&= (f_{i,j} + e_i \star e_j) f_{j,k} - f_{i,j}(f_{k,l} + e_k \star e_l) \\
&= (e_i \star e_j) f_{j,k} - f_{i,j}(e_k \star e_l).
\end{aligned}$$

It follows that $S^{\mathcal{F}}_{i,j,k,l} = 0$. Obviously we have $\mathrm{S}(f_{i,i}, f_{i,i})$ for each $1 \leq i \leq n$. Now the equivalence of statements 1 and 2 follows immediately from Buchberger's Criterion. $\square$

Remark 1. Note that the proof gives an algorithm for calculating associators; namely to calculate $[e_i, e_j, e_k]$, we first calculate the S-polynomial $S_{i,j,k} = \mathrm{S}(f_{j,k}, f_{i,j})$, and then we reduce $S_{i,j,k}$ with respect to $\mathcal{F}$ in the obvious way. In Singular, this can be calculated using the reduce command.

Example 0.1. Let Δ be the simplicial complex below



and let F be the corresponding graded $\mathbb{F}_2$-vector space induced by Δ. Let's write the homogeneous components of F as a graded $\mathbb{F}_2$-vector space

$$
\begin{aligned}
F_0 &= \mathbb{F}_2 \\
F_1 &= \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5 \\
F_2 &= \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45} \\
F_3 &= \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345} \\
F_4 &= \mathbb{F}_2 e_{1234}
\end{aligned}
$$

Let $\star$ be a graded-multiplication on F such that

$$
\begin{aligned}
e_1 \star e_5 &= e_{14} - e_{45} \\
e_2 \star e_5 &= e_{23} - e_{35} \\
e_2 \star e_{45} &= e_{234} - e_{345} \\
e_1 \star e_{35} &= e_{134} - e_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= e_{124}.
\end{aligned}
$$

Then $\star$ is not associative since $[e_1, e_5, e_2] = -e_{123} + e_{124} - e_{234} + e_{134}$. We used Singular to calculate this associator using the code below:

```
intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=0,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5-e14+e45;
poly f(2)(5) = e2*e5-e23+e35;
poly f(2)(45) = e2*e45-e234+e345;
poly f(1)(35) = e1*e35-e134+e345;
poly f(1)(23) = e1*e23-e123;
poly f(2)(14) = e2*e14-e124;
poly S(1)(5)(2) = e1*f(2)(5)-e2*f(1)(5);

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);
reduce(S(1)(5)(2),I); // calculates associator [e1,e5,e2].

// -e123+e124-e234+e134
```

See [Avr81] for more details

References

[BE77] D. A. Buchsbaum and D. Eisenbud. "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3". In: Amer. J. Math. 99.3 (1977), pp. 447–485.

[Avr81] L. L. Avramov. "Obstructions to the Existence of Multiplicative Structures on Minimal Free Resolutions". In: Amer. J. Math. 103.1 (1981), pp. 1–31.

[Luk26] Lukas Katthän. "The structure of DGA resolutions of monomial ideals". In: Preprint (2016). arXiv:1610.06526