Associativity Test Using Gröbner Bases

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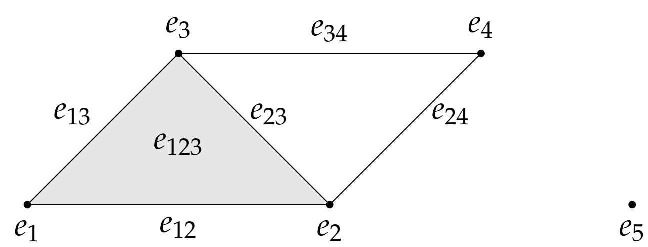
Introduction

Let Δ be a finite simplicial complex and let K be a field of characteristic 2 (we only assume characteristic 2 for simplicity in what follows). Attached to Δ is a graded K-complex F_Δ whose homogeneous component of degree $k \in \mathbb{N}$ is the K-span of all $(k-1)$-faces of Δ. For instance, if Δ is the simplicial complex below,



then the homogeneous components of F_Δ are given by:

$$
\begin{aligned}
F_{\Delta,0} &= Ke_\varnothing \\
F_{\Delta,1} &= Ke_1 + Ke_2 + Ke_3 + Ke_4 + Ke_5 \\
F_{\Delta,2} &= Ke_{12} + Ke_{13} + Ke_{23} + Ke_{24} + Ke_{34} \\
F_{\Delta,3} &= Ke_{123}.
\end{aligned}
$$

Note that we often write $e_\varnothing = 1 = e_0$ and we think of F_Δ as a graded K-vector space with $F_{\Delta,0} = K$. Now let us equip F_Δ with a **graded-multiplication** $\star$, where by a graded-multiplication, we mean that $\star$ is a binary operator on F_Δ which satisfies the following properties:

1. $\star$ is unital with 1 being the unit;
2. $\star$ is K-bilinear;
3. $\star$ is commutative;
4. $\star$ respects the grading meaning that if α, β are homogeneous elements of F_Δ, then $\alpha \star \beta$ is homogeneous and
$$|\alpha \star \beta| = |\alpha| + |\beta|,$$
where $|\cdot|$ denote the homogeneous degree of an element in F_Δ.

Given such a graded-multiplication F_Δ, it is natural to wonder whether or not $\star$ is associative, meaning

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$$

for all $\alpha, \beta, \gamma \in F_\Delta$. In this note, we will determine whether or not $\star$ is associative using tools from the theory of Gröbner bases.

Setting up our Notation

We begin in a slightly more general context. Let F be a graded K-vector space and let $\star$ be a graded-multiplication on F. Let $n \geq 1$ and assume that $(e_0, e_1, \ldots, e_n)$ is an ordered homogeneous basis of F such that

1. $e_0 = 1$;
2. $|e_i| \geq 1$ for all $1 \leq i \leq n$,
3. if $|e_j| > |e_i|$, then $j > i$.

For each $0 \le i, j \le n$, we have

$$e_i \star e_j = \sum_{k=0}^{n} c_{i,j}^k e_k,$$

where $c_{i,j}^k \in K$ for each k. Let S be the weighted polynomial ring $K[e_1, \dots, e_n]$ where e_i is weighted of degree $|e_i|$ for each $1 \le i \le n$. A monomial of S has the form

$$e^{\boldsymbol{a}} = e_1^{a_1} \cdots e_n^{a_n}$$

where $\boldsymbol{a} \in \mathbb{N}^n$ and where we identify the monomial $e^{(0,\dots,0)}$ with 1 in this notation. Given a monomial $e^{\boldsymbol{a}}$, we define its **degree**, denoted $\deg(e^{\boldsymbol{a}})$, and its **weighted degree**, denoted $|e^{\boldsymbol{a}}|$, by

$$\deg(e^{\boldsymbol{a}}) = \sum_{i=1}^{n} a_i \quad \text{and} \quad |e^{\boldsymbol{a}}| = \sum_{i=1}^{n} a_i |e_i|.$$

For each $k \in \mathbb{N}$, we shall write

$$S_k = \operatorname{span}_K\{e^{\boldsymbol{a}} \mid \deg(e^{\boldsymbol{a}}) = k\}.$$

We identity F with $S_0 + S_1 = K + \sum_{i=1}^{n} Ke_i$. In order to keep notation consistent, we shall write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to $\star$, and we shall write $\alpha\beta$ to denote their multiplication with respect to $\cdot$ in S. In particular, note that $\deg(e_i \star e_j) = 1$, $\deg(e_i e_j) = 2$, and $|e_i \star e_j| = |e_i| + |e_j| = |e_i e_j|$.

For each $1 \le i, j \le n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Note that since both $\star$ and $\cdot$ are commutative, we have $f_{i,j} = f_{j,i}$ for all $1 \le i, j \le n$. Let

$$\mathcal{F} = \{f_{i,j} \mid 1 \le i, j \le n\}$$

and let I be the ideal of S generated by $\mathcal{F}$. We equip S with a weighted lexicographic ordering $>_w$ with respect to the weight vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials $e^{\boldsymbol{a}}$ and $e^{\boldsymbol{b}}$ in S, we say $e^{\boldsymbol{a}} >_w e^{\boldsymbol{b}}$ if either

1. $|e^{\boldsymbol{a}}| > |e^{\boldsymbol{b}}|$ or;
2. $|e^{\boldsymbol{a}}| = |e^{\boldsymbol{b}}|$ and there exists $1 \le i \le n$ such that $\alpha_i > \beta_i$ and $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\dots$, $\alpha_{i-1} = \beta_{i-1}$.

Observe that for each $1 \le i \le j \le n$, we have $\mathrm{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since $\star$ is graded, we must have $|e_i e_j| = |e_i| + |e_j| = |e_k|$. It follows that $|e_k| > |e_i|$ since $|e_i|, |e_j| \ge 1$. This implies $k > i$ by our assumption on $(e_1, \dots, e_n)$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

The Main Theorem

Before we state and prove the main theorem, let us introduce one more piece of notation. We denote

$$\mathcal{M} = \{e^{\boldsymbol{a}} \mid e^{\boldsymbol{a}} \notin \mathrm{LT}(I)\}.$$

Since $\mathrm{LT}(f_{i,j}) = e_i e_j$ for all $1 \le i, j \le n$, we see that $\mathcal{M}$ is a subset of $\{e_1, \dots, e_n\}$. Now we are ready to state and prove the main theorem:

Theorem 0.1. *The following statements are equivalent:*

1. *$\star$ is associative.*
2. *$\mathcal{F}$ is a Gröbner basis.*
3. *$\mathcal{M} = \{e_1, \dots, e_n\}$.*

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq n$. We have

$$\begin{aligned} S_{i,j,k} &:= \mathrm{S}(f_{j,k}, f_{i,j}) \\ &= e_i f_{j,k} - f_{i,j} e_k \\ &= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\ &= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\ &= \left(\sum_l c_{i,j}^l e_l\right) e_k - e_i \left(\sum_l c_{j,k}^l e_l\right) \\ &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l. \end{aligned}$$

Now we divide $S_{i,j,k}$ by $\mathcal{F}$:

$$\begin{aligned} S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} \\ &= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) + \sum_l c_{j,k}^l (f_{i,l} - e_i e_l) \\ &= \sum_l c_{i,j}^l (e_l e_k - e_l e_k + e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l - e_i e_l) \\ &= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\ &= \left(\sum_l c_{i,j}^l e_l\right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l\right) \\ &= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\ &= [e_i, e_j, e_k]. \end{aligned}$$

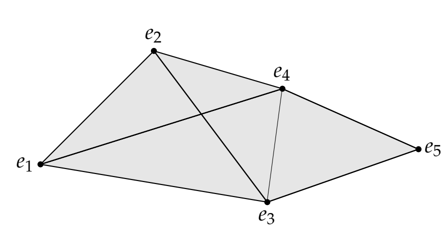
Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$. It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. A straightforward comptutation also shows that $\mathrm{S}(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \leq i \leq n$. Finally, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq l \leq n$. We have

$$\begin{aligned} S_{i,j,k,l} &:= \mathrm{S}(f_{k,l}, f_{i,j}) \\ &= e_i e_j f_{j,k} - f_{i,j} e_k e_l \\ &= (f_{i,j} + e_i \star e_j) f_{j,k} - f_{i,j}(f_{k,l} + e_k \star e_l) \\ &= (e_i \star e_j) f_{j,k} - f_{i,j}(e_k \star e_l). \end{aligned}$$

From this, it's easy to see that $S_{i,j,k,l}^{\mathcal{F}} = 0$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. □

Remark 1. Note that the proof gives an algorithm for calculating associators. In Singular, this can be calculated using the reduce command.

Example 0.1. Let Δ be the simplicial complex below



and let F be the corresponding graded $\mathbb{F}_2$-vector space induced by Δ. Let's write the homogeneous components of F as a graded $\mathbb{F}_2$-vector space

$$\begin{aligned} F_0 &= \mathbb{F}_2 \\ F_1 &= \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5 \\ F_2 &= \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45} \\ F_3 &= \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345} \\ F_4 &= \mathbb{F}_2 e_{1234} \end{aligned}$$

Let $\star$ be a graded-multiplication on F such that

$$\begin{aligned}
e_1 \star e_5 &= e_{14} + e_{45} \\
e_2 \star e_5 &= e_{23} + e_{35} \\
e_2 \star e_{45} &= e_{234} + e_{345} \\
e_1 \star e_{35} &= e_{134} + e_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= e_{124}.
\end{aligned}$$

Then $\star$ is not associative since

$$\begin{aligned}
[e_1, e_5, e_2] &= (e_1 e_5) e_2 + e_1 (e_5 e_2) \\
&= e_{123} + e_{124} + e_{234} + e_{134} \\
&\neq 0.
\end{aligned}$$

We used Singular to calculate this associator as follows:

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intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);

poly S(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(S(1)(5)(2),I); // calculates associator [e1,e5,e2].

// e123+e124+e234+e134
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