

List of Schemes

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Part I

List of Algebraic Varieties

1 A Quartic Curve

Let $A = \mathbb{Z}[x, y]/f$ where

$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \quad (1)$$

where we set $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$. Note that from the expression of f in (1) we see that u and v are units in A . Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g}]$ where

$$f = y^2 - (x - 1)(x - 2)(x - 3)(x - 4) = y^2 - g, \quad (2)$$

where $g = (x - 1)(x - 2)(x - 3)(x - 4)$. The expression of f in (2) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (2) we can read off useful information of A viewed as a finite module extension, whereas from (1) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day. Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we consider the canonical morphism $X \rightarrow \operatorname{Spec} \mathbb{Z}$. For each positive prime p , we obtain the fiber $X_p = X_{\mathbb{F}_p}$ of this canonical morphism at the prime ideal $\langle p \rangle$:

$$X_p = \operatorname{Spec}(\mathbb{F}_p \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(\mathbb{F}_p[x, y]/f).$$

We also obtain the fiber $X_0 = X_{\mathbb{Q}}$ of this canonical morphism at the generic point 0:

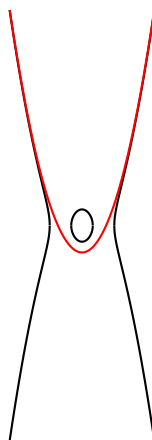
$$X_0 = \operatorname{Spec}(\mathbb{Q} \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(\mathbb{Q}[x, y]/f).$$

Note $X_{\mathbb{Q}}$ is just the pullback of the morphism $\operatorname{Spec} \mathbb{Q} \rightarrow \operatorname{Spec} \mathbb{Z}$ with respect to the canonical map $X \rightarrow \operatorname{Spec} \mathbb{Z}$. We can specialize even further by setting X_K to be the pullback of the composite $\operatorname{Spec} K \rightarrow \operatorname{Spec} \mathbb{Q} \rightarrow \operatorname{Spec} \mathbb{Z}$ with respect to the canonical map $X \rightarrow \operatorname{Spec} \mathbb{Z}$:

$$X_K = \operatorname{Spec}(K \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(K[x, y]/f).$$

The closed points of X_K correspond to the maximal ideals of $K[x, y]/f$, and when K is algebraically closed, these correspond to the points of the variety $V_K(f)$.

We now consider $X_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[x, y]/f)$, viewed as an \mathbb{R} -scheme (thus the canonical morphism is $X_{\mathbb{R}} \rightarrow \operatorname{Spec} \mathbb{R}$). To get an idea of what $X_{\mathbb{R}}$ looks like, we shall look at its \mathbb{R} -valued points $X_{\mathbb{R}}(\mathbb{R}) = V_{\mathbb{R}}(f) = C$ pictured below:



The thick black curve is C whereas the thick red curve is $V_{\mathbb{R}}(u) = D$. The closed points of $X_{\mathbb{R}}$ correspond to the points of C : they have the form $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$ where $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = 0$ (i.e. such that $(a, b) \in C$). There's also the generic point $\eta \in X_{\mathbb{R}}$ corresponding to the 0 ideal, however this doesn't correspond to any point of C . Notice that C and D do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X_{\mathbb{R}}$.

If we equip $X(\mathbb{R})$ with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology, $X(\mathbb{R})$ is irreducible since f is irreducible over \mathbb{R} , so certainly $X(\mathbb{R})$ is connected in the Zariski topology. The Jacobian matrix of f is given by

$$J_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that $J_f(a, b) := J_f \bmod \mathfrak{m}_{a,b} \neq 0$ for all closed points $\mathfrak{m}_{a,b} \in X(\mathbb{R})$. It follows that $X(\mathbb{R})$ is smooth. Alternatively, we have

$$df = \partial_x f dx + \partial_y f dy = (4x^3 + 30x^2 - 70x + 50)dx + 2ydy.$$

Thus when we set $df = 0$, then for $y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \quad (3)$$

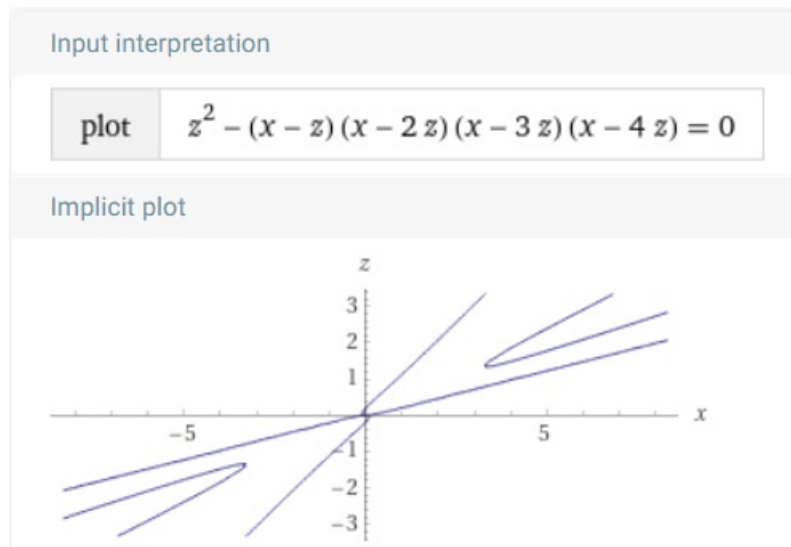
The DeRham complex of A is given by

$$\Omega_A := 0 \rightarrow A \rightarrow$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity $[0 : 1 : 0]$. To do this let $\tilde{A} = \mathbb{Z}[x, z]/\tilde{f}$ where

$$\tilde{f} = \tilde{y}^2 - (\tilde{x} - \tilde{y})(\tilde{x} - 2\tilde{y})(x - 3\tilde{y})(x - 4\tilde{y}), \quad (4)$$

and set $\tilde{X} = \text{Spec } \tilde{A}$. To get an idea of what $\tilde{X}_{\mathbb{R}}$ looks like, we shall look at its \mathbb{R} -valued points $\tilde{X}_{\mathbb{R}}(\mathbb{R}) = V_{\mathbb{R}}(\tilde{f}) = \tilde{C}$ pictured below



The closed points of $\tilde{X}_{\mathbb{R}}$ have the form $\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}} = \langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle$ where $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ such that $\tilde{f}(\tilde{a}, \tilde{b}) = 0$. We have a ring isomorphism $\tilde{\varphi}: \tilde{A}_{\tilde{y}} \rightarrow A_y$ given by $\tilde{\varphi}(\tilde{x}) = x/y$ and $\tilde{\varphi}(\tilde{y}) = 1/y$, with inverse $\varphi: A_y \rightarrow \tilde{A}_{\tilde{y}}$ given by $\varphi(x) = \tilde{x}/\tilde{y}$ and $\varphi(y) = 1/\tilde{y}$. Notice that

$$\begin{aligned} \tilde{\varphi}(\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}) &= \varphi(\langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle) \\ &= \langle x/y - \tilde{a}, 1/y - \tilde{b} \rangle \\ &= \langle x - \tilde{a}y, 1 - \tilde{b}y \rangle \\ &= \langle x - \tilde{a}y, y - 1/\tilde{b} \rangle \\ &= \langle x - \tilde{a}/\tilde{b}, y - 1/\tilde{b} \rangle \\ &= \langle x - a, y - b \rangle \\ &= \mathfrak{m}_{a,b}, \end{aligned}$$

where we set $a = \tilde{a}/\tilde{b}$ and $b = 1/\tilde{b}$. It follows that ${}^a\tilde{\varphi}(\mathfrak{m}_{a,b}) = \tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}$. Now observe that

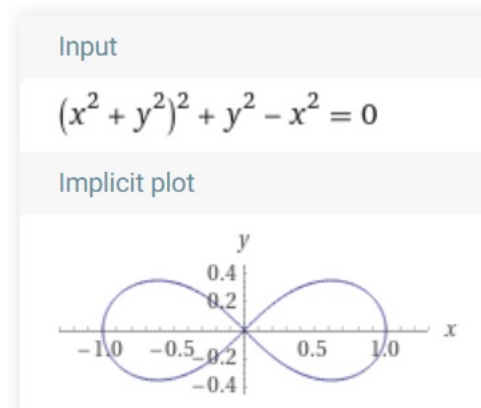
$$d\tilde{x} = \frac{ydx - xdy}{y^2} \quad \text{and} \quad d\tilde{y} = -\frac{dy}{y^2}.$$

2 The Lemniscate of Bernoulli

Let $A = \mathbb{Z}[x, y]/f$ where

$$f = (x^2 + y^2)^2 + y^2 - x^2,$$

and we set $X = \operatorname{Spec} A$. One can show that the set of integer solutions to the equation $f = 0$ is given by $\{(\pm 1, 0), (0, 0)\}$. On the other hand, the \mathbb{R} -valued points $X(\mathbb{R})$ can be visualized below



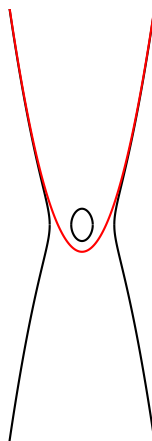
$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \quad (5)$$

where we set $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$. Note that from the expression of f in (1) we see that u and v are units in A . Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g}]$ where

$$f = y^2 - (x - 1)(x - 2)(x - 3)(x - 4) = y^2 - g, \quad (6)$$

where $g = (x - 1)(x - 2)(x - 3)(x - 4)$. The expression of f in (2) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (2) we can read off useful information of A viewed as a finite module extension, whereas from (1) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day.

Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we first look at its \mathbb{R} -valued points: $X(\mathbb{R}) = \operatorname{Spec} \mathbb{R} \otimes_{\mathbb{Z}} A = \operatorname{Spec} \mathbb{R}[x, y]/f$. We can visualize the \mathbb{R} -valued points of X in the picture below:



The thick black curve is $X(\mathbb{R}) = V_{\mathbb{R}}(f)$ whereas the thick red curve is $V_{\mathbb{R}}(u)$. Notice that $V_{\mathbb{R}}(u)$ and $X(\mathbb{R})$ do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X$. The closed points of $X(\mathbb{R})$ have the form $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$ where $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = 0$. There's also the generic point $\eta \in X(\mathbb{R})$ corresponding to the 0 ideal.

If we equip $X(\mathbb{R})$ with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology, $X(\mathbb{R})$ is irreducible since f is irreducible over \mathbb{R} , so certainly $X(\mathbb{R})$ is connected in the Zariski topology. The Jacobian matrix of f is given by

$$J_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that $J_f(a, b) := J_f \bmod \mathfrak{m}_{a,b} \neq 0$ for all closed points $\mathfrak{m}_{a,b} \in X(\mathbb{R})$. It follows that $X(\mathbb{R})$ is smooth. Alternatively, we have

$$df = \partial_x f dx + \partial_y f dy = (4x^3 + 30x^2 - 70x + 50)dx + 2ydy.$$

Thus when we set $df = 0$, then for $y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \quad (7)$$

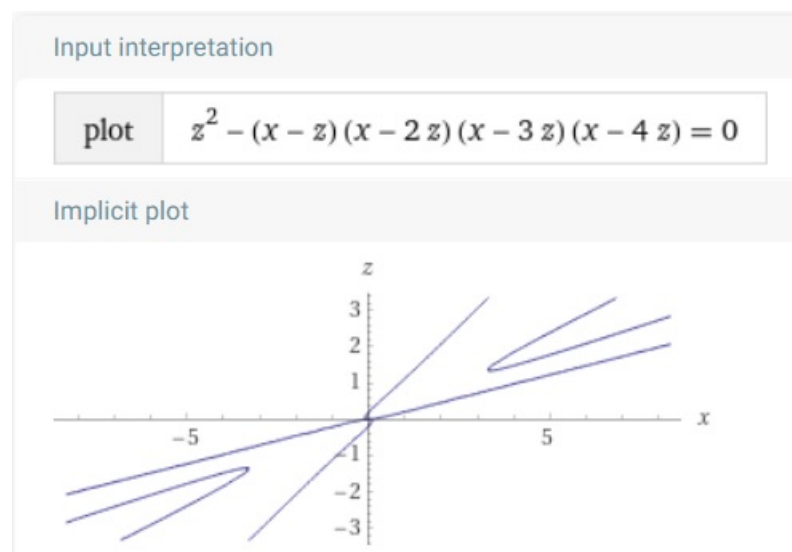
The DeRham complex of A is given by

$$\Omega_A := 0 \rightarrow A \rightarrow$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity $[0 : 1 : 0]$. To do this let $\tilde{A} = \mathbb{Z}[x, z]/\tilde{f}$ where

$$\tilde{f} = \tilde{y}^2 - (\tilde{x} - \tilde{y})(\tilde{x} - 2\tilde{y})(x - 3\tilde{y})(x - 4\tilde{y}), \quad (8)$$

and set $\tilde{X} = \text{Spec } \tilde{A}$. We can visualize the \mathbb{R} -valued points of \tilde{X} in the picture below



The closed points of $\tilde{X}(\mathbb{R})$ have the form $\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}} = \langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle$ where $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$ such that $\tilde{f}(\tilde{a}, \tilde{b}) = 0$. We have a ring isomorphism $\tilde{\varphi}: \tilde{A}_{\tilde{y}} \rightarrow A_y$ given by $\tilde{\varphi}(\tilde{x}) = x/y$ and $\tilde{\varphi}(\tilde{y}) = 1/y$, with inverse $\varphi: A_y \rightarrow \tilde{A}_{\tilde{y}}$ given by $\varphi(x) = \tilde{x}/\tilde{y}$ and $\varphi(y) = 1/\tilde{y}$. Notice that

$$\begin{aligned} \tilde{\varphi}(\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}) &= \varphi(\langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle) \\ &= \langle x/y - \tilde{a}, 1/y - \tilde{b} \rangle \\ &= \langle x - \tilde{a}y, 1 - \tilde{b}y \rangle \\ &= \langle x - \tilde{a}y, y - 1/\tilde{b} \rangle \\ &= \langle x - \tilde{a}/\tilde{b}, y - 1/\tilde{b} \rangle \\ &= \langle x - a, y - b \rangle \\ &= \mathfrak{m}_{a,b}, \end{aligned}$$

where we set $a = \tilde{a}/\tilde{b}$ and $b = 1/\tilde{b}$. It follows that ${}^a\tilde{\varphi}(\mathfrak{m}_{a,b}) = \tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}$. Now observe that

$$d\tilde{x} = \frac{ydx - xdy}{y^2} \quad \text{and} \quad d\tilde{y} = -\frac{dy}{y^2}.$$

3 A Blowup Algebra

Let $R = \mathbb{K}[x, y]/\langle y^2 - x^3 - x^2 \rangle$, let $Q = \langle \bar{x}, \bar{y} \rangle$ (we drop the overlines from \bar{x} and \bar{y} in just write x and y in order to simplify notation in what follows), and equip R with the Q -filtration making $R = (Q^n)$ into a filtered ring. Let

$$\varphi: R[u, v] \rightarrow \text{bl}(R)$$

be the unique surjective R -algebra homomorphism such that $\varphi(u) = xt$ and $\varphi(v) = yt$. The kernel of φ is an ideal of $R[u, v]$ which is homogeneous in the variables u, v :

$$\ker \varphi = \langle v^2 - (x+1)u^2, xv - yu \rangle.$$

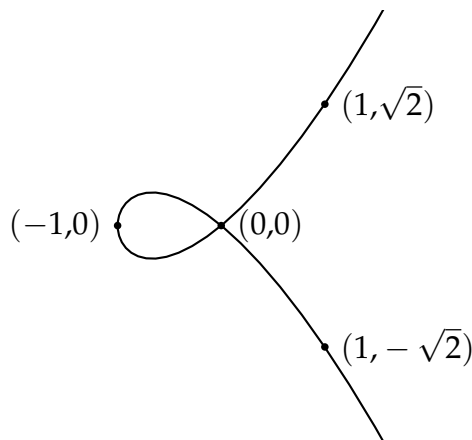
Thus we see that $\text{bl}(R) \cong \mathbb{k}[x, y, u, v]/\mathfrak{a}$ where

$$\mathfrak{a} = \langle v^2 - (x+1)u^2, xv - yu, y^2 - x^3 - x^2 \rangle.$$

In particular, $\text{bl}(R)$ corresponds to an algebraic subset $Z \subseteq \mathbb{A}_{x,y}^2 \times \mathbb{P}_{u,v}^1$. Let $A = R[v]/\langle v^2 - (x+1)u^2, xv - yu \rangle$, so A corresponds to the affine open $U = \mathbb{A}^2 \times D(u)$. We can localize further by setting $B = A_x = R[v]/\langle v - x/y \rangle$, so B corresponds to the affine open $D(x) \times D(u)$. We have a canonical ring homomorphism $\iota: R \rightarrow A$ where ι is the inclusion map. Let us try to understand this homomorphism from a geometric point of view. Let $V = V_K(y^2 - x^3 - x^2)$ be affine algebraic subset of $\mathbb{A}^2(K)$ defined by the equation $y^2 = x^3 + x^2$. The closed points of $\text{Spec } R$ are in one-to-one correspondence with the points of V : they are all of the form

$$\mathfrak{p}_{(a,b)} = \langle x - a, y - b \rangle$$

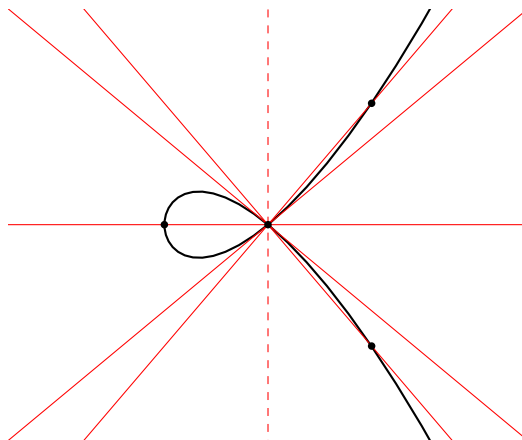
where $(a, b) \in V$, that is, where $a, b \in \mathbb{k}$ such that $b^2 = a^3 + a^2$. If $\mathbb{k} = \mathbb{R}$, we can visualize the closed points of $\text{Spec } R$ as below:



Note that $\text{Spec } R$ also has a generic point η corresponding to the zero ideal of R . The closed points of $\text{Spec } A$ correspond to the points of the affine open set U : they have the form

$$\mathfrak{p}_{(a,b),[1:t]} = \langle x - a, y - b, v - t \rangle$$

where $a, b, t \in \mathbb{k}$ such that $b^2 = a^3 + a^2$, $at = b$, and $t^2 = a + 1$. Note that if $a \neq 0$, then we automatically get $t^2 = a + 1$. If $\mathbb{k} = \mathbb{R}$, we can visualize the points of $\text{Spec } A$ as below:



In particular, for $a \neq 0$, the prime $\mathfrak{p}_{(a,b),[1:t]}$ corresponds to the point $(a, b) \in V$ together with the unique line $y = tx$ that passes through that point and the origin, where t represents the slope of that line. There are two points lying over the origin: namely $\mathfrak{p}_{(0,0),[1:1]}$ and $\mathfrak{p}_{(0,0),[1:-1]}$, corresponding to the origin $(0,0) \in V$ together with the lines $y = x$ and $y = -x$ respectively. The map $\iota: R \rightarrow A$ induces a continuous map ${}^a\iota: \text{Spec } A \rightarrow \text{Spec } R$ given by

$${}^a\iota(\mathfrak{p}_{(a,b),[1:t]}) = \mathfrak{p}_{(a,b)}.$$

This corresponds to the projection map $\pi: U \rightarrow V$ given by

$$\pi(a, b, t) = (a, b).$$

Notice that in the image above there are “missing” points. For instance, we drew a vertical dashed line in the image above; it should correspond to the line $x = 0$, but it has nowhere to go under this projection. In fact, this missing line corresponds to the extra point in $\text{Proj}(\text{bl}(R))$ which doesn’t belong to A .

Definition 3.1. A **hyperelliptic curve** is an algebraic curve of genus $g > 1$, given by an equation of the form

$$y^2 + h(x)y = f(x),$$

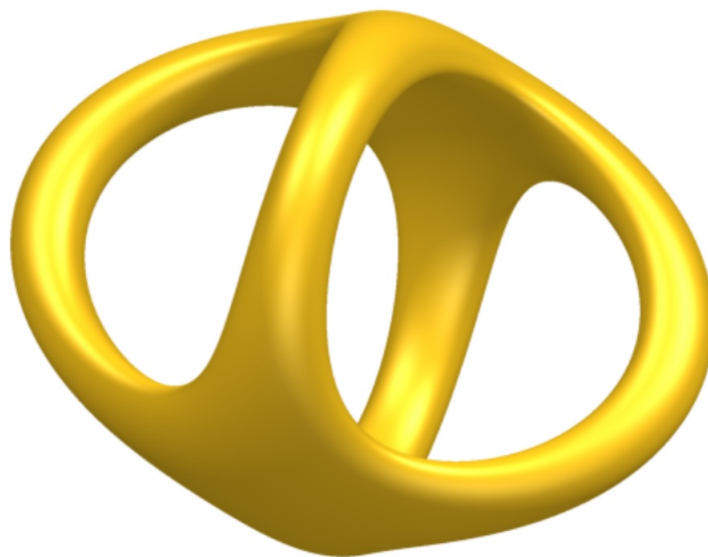
where f is a polynomial of degree $n = 2g + 1 > 4$ or $n = 2g + 2 > 4$ with n distinct roots and $h(x)$ is a polynomial of degree $< g + 2$ (if the characteristic of the ground field is not 2, one can take $h(x) = 0$).

4 A Surface

Let $a \in \mathbb{k}$ and let $S_t = V_{\mathbb{k}}(f_t) \subseteq \mathbb{A}_{\mathbb{k}}^3$ where

$$f_t = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2 - t = g_1^2 + g_2^2 - t = \|g\|^2 - t$$

where $g = (g_1, g_2)$, where $g_1 = x_1^2 + x_2^2 - 1$ and $g_2 = x_2^2 + x_3^2 - 1$. When $\mathbb{k} = \mathbb{R}$ and $t = 0.1$, we can picture $S_{0.1}$ as below:



The Jacobian matrix of f_t is given by

$$J_{f_t} = \begin{pmatrix} \partial_x f_t \\ \partial_y f_t \\ \partial_z f_t \end{pmatrix} = 4 \begin{pmatrix} x_1 g_1 \\ x_2 (g_1 + g_2) \\ x_3 g_2 \end{pmatrix}.$$

We write $\Delta_t = V(J_{f_t}) = \{a \in \mathbb{A}_{\mathbb{k}}^3 \mid J_{f_t}(a) = 0\}$. Given $a \in \mathbb{A}_{\mathbb{k}}^3$, we have $a \in \Delta_t$ if and only if $a = \mathbf{0}$ or $a \in V_{\mathbb{k}}(g_1, g_2)$ (meaning $g_1(a) = g_2(a) = 0$). In particular, if $t \neq 0, 2$, then S_t has no singular points since $S_t \cap \Delta_t = \emptyset$ in this case. If $t = 2$, then $\mathbf{0}$ is a singular point since $\mathbf{0} \in S_2 \cap \Delta_2$. If $t = 0$, then S_0 has lots of singular points. For instance, $\{(\pm 1, 0, \pm 1), (\pm 1, 0, \mp 1)\}$ are all singular points.

We can describe S_t as being the fibre at $t \in \mathbb{k}$ with respect to the morphism of affine \mathbb{k} -schemes $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$ (here we are indicating that the coordinate ring of $\mathbb{A}_{\mathbb{k}, \tau}^1$ is given by $\mathbb{k}[\tau]$) where $S = \text{Spec}(\mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau)$ and where π corresponds to the morphism of \mathbb{k} -algebras $\iota: \mathbb{k}[\tau] \rightarrow \mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau$ (which is just inclusion map). In particular, let $\varepsilon_t: \text{Spec } \mathbb{k} \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$ be the morphism of affine \mathbb{k} -schemes which corresponds to the \mathbb{k} -algebra homomorphism $\mathbb{k}[\tau] \rightarrow \mathbb{k}[\tau]/\langle \tau - t \rangle \simeq \mathbb{k}$ which sends τ to $t \in \mathbb{k}$. Then S_t is the pullback of $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$ with respect to $\varepsilon_t: \text{Spec } \mathbb{k} \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$. In particular, the corresponding \mathbb{k} -algebra of S_t is given by

$$\mathbb{k}[x_1, x_2, x_3]/f_t \simeq (\mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau) \otimes_{\mathbb{k}} \mathbb{k}[\tau]/\langle \tau - t \rangle.$$

Note that the morphism of affine \mathbb{k} -schemes $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$ is flat since the inclusion map of \mathbb{k} -algebras $\iota: \mathbb{k}[\tau] \rightarrow \mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau$ is flat.

5 An Elliptic Curve

We study the Elliptic curve E defined by the equation $y^2 = x^3 - 51$. In particular, set $f = y^2 - x^3 + 51$. Then

$$f = y^2 - x^3 + 51$$