Advanced Linear Programming Homework 5

Michael Nelson

Problem 1

For this problem, let $f(\boldsymbol{x}) = x_1^3 + 5x_1^2x_2 + 7x_1x_2^2 + 2x_2^3$ and let $\boldsymbol{x}^0 = (-2,3)^\top$.

Problem 1.a

Exercise 1. Calculate the gradient of f at $\boldsymbol{x}^0$.

Solution 1. We have

$$\begin{aligned}
\nabla f(\boldsymbol{x}^0) &= \begin{pmatrix} 3x_1^2 + 10x_1x_2 + 7x_2^2 \\ 5x_1^2 + 14x_1x_2 + 6x_2^2 \end{pmatrix}\Bigg|_{(-2,3)} \\
&= \begin{pmatrix} 12 - 60 + 63 \\ 20 - 84 + 54 \end{pmatrix} \\
&= \begin{pmatrix} 15 \\ -10 \end{pmatrix}
\end{aligned}$$

Problem 1.b

Exercise 2. Calculate the Hessian of f at $\boldsymbol{x}^0$.

Solution 2. We have

$$\begin{aligned}
\mathrm{H}_f(\boldsymbol{x}^0) &= \begin{pmatrix} 6x_1 + 10x_2 & 10x_1 + 14x_2 \\ 10x_1 + 14x_2 & 14x_1 + 12x_2 \end{pmatrix}\Bigg|_{(-2,3)} \\
&= \begin{pmatrix} -12 + 30 & -20 + 42 \\ -20 + 42 & -28 + 36 \end{pmatrix} \\
&= \begin{pmatrix} 18 & 22 \\ 22 & 8 \end{pmatrix}.
\end{aligned}$$

Problem 1.c

Exercise 3. Using the point $\boldsymbol{x}^0$ write the Taylor series expansion with three terms. Derive the resulting quadratic function.

Solution 3. Taylor series of f at $\boldsymbol{x}^0$ expressed with three terms is given by

$$\begin{aligned}
f(\boldsymbol{x}) &= Q(\boldsymbol{x}) + R(\boldsymbol{x}) \\
&= f(\boldsymbol{x}^0) + \nabla f(\boldsymbol{x}^0)^\top(\boldsymbol{x} - \boldsymbol{x}^0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^0)^\top \mathrm{H}f(\boldsymbol{x}^0)(\boldsymbol{x} - \boldsymbol{x}^0) + R(\boldsymbol{x}) \\
&= -20 + 15(x_1 + 2) - 10(x_2 - 3) + 9(x_1 + 2)^2 + 22(x_1 + 2)(x_2 - 3) + 4(x_2 - 3)^2 + R(\boldsymbol{x}) \\
&= -20 - 15x_1 + 10x_2 + 9x_1^2 + 22x_1x_2 + 4x_2^2 + R(\boldsymbol{x}).
\end{aligned}$$

Here, $Q(\boldsymbol{x})$ is the second degree Taylor polynomial of f at $\boldsymbol{x}^0$ (or the quadratic function) given by

$$Q(\boldsymbol{x}) = -20 - 15x_1 + 10x_2 + 9x_1^2 + 22x_1x_2 + 4x_2^2$$

and $R(\boldsymbol{x})$ is a remainder term.

Problem 1.d

Exercise 4. Find the approximate value of the function f at $\boldsymbol{x} = (-1.9, 3.2)$ using your work.

Solution 4. We have

$$\begin{aligned} f(-1.9, 3.2) &\approx Q(-1.9, 3.2) \\ &= -19.81 \end{aligned}$$

Problem 1.e

Exercise 5. Calculate the true value of the function f at $\boldsymbol{x} = (-1.9, 3.2)$ and compare it with the approximate value. What do you observe?

Solution 5. The true value is $f(-1.9, 3.2) = -19.755$. We notice that the approximate value is an underestimate of the true value. Also we notice that the remainder term is given by

$$\begin{aligned} R(-1.9, 3.2) &= f(-1.9, 3.2) - Q(-1.9, 3.2) \\ &= -19.755 + 19.81 \\ &= 0.055, \end{aligned}$$

In general, $R(\boldsymbol{x})$ tends to zero more rapidly than $\|\boldsymbol{x} - \boldsymbol{x}^0\|^2$ tends to zero as $\boldsymbol{x} \to \boldsymbol{x}^0$.

Problem 2

For this problem, let $f(\boldsymbol{x}) = (1/4)(x_1 - 2)^2 + (1/9)(x_2 - 3)^2$.

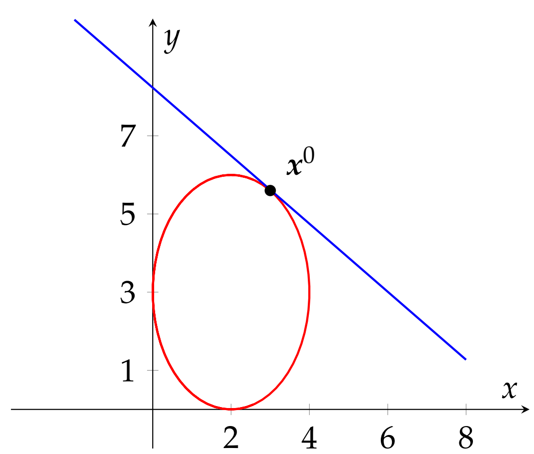
Problem 2.a

Exercise 6. Write the defining complete statement of the level curve of value 1. Clearly draw this level curve.

Solution 6. The level of curve of f with value 1 is given by

$$C = \{\boldsymbol{x} \in \mathbb{R}^2 \mid f(\boldsymbol{x}) = 1\}.$$

The curve C is just the ellipse centered at $(2, 3)$ with width 4 and with height 6. We draw this curve (in red) below and we also draw the point $\boldsymbol{x}^0 = (3, 3 + (3/2)\sqrt{3})^\top$ and the tangent line (in blue) of C at $\boldsymbol{x}^0$ as well:



Problem 2.b

Exercise 7. Calculate the gradient vector $\nabla f(\boldsymbol{x})$ at $\boldsymbol{x}^0 = (3, 3 + (3/2)\sqrt{3})^\top$

Solution 7. We have

$$\begin{aligned} \nabla f(\boldsymbol{x}^0) &= \begin{pmatrix} \frac{x_1 - 2}{2} \\ \frac{2(x_2 - 3)}{9} \end{pmatrix} \Bigg|_{(3, 3 + (3/2)\sqrt{3})} \\ &= \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \end{aligned}$$

Problem 2.c

Exercise 8. Derive the equation of the tangent line to the level curve of value 1 at $\boldsymbol{x}^0$. Show your work.

Solution 8. Let L be the tangent line of C at $\boldsymbol{x}^0$. Then L is given by

$$\begin{aligned} L &= \{\boldsymbol{x} \in \mathbb{R}^2 \mid \nabla f(\boldsymbol{x}^0)^\top (\boldsymbol{x} - \boldsymbol{x}^0) = 0\} \\ &= \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \frac{1}{2}(x_1 - 3) + \frac{\sqrt{3}}{3}\left(x_2 - \frac{6 + 3\sqrt{3}}{2}\right) = 0 \right\} \end{aligned}$$

In particular the equation of the tangent line is

$$\frac{1}{2}(x_1 - 3) + \frac{\sqrt{3}}{3}\left(x_2 - \frac{6 + 3\sqrt{3}}{2}\right) = 0.$$

Problem 3

For this problem, let $f(x) = xe^{-2x}$.

Problem 3.a

Exercise 9. Find all local/global minimizers and maximizers.

Solution 9. First note that f is a C^∞ function since it is a product of C^∞ functions (we only need that f is C^2 for this problem). First we find all possible places where f has a potential local extremum point. Since $f'(x) = e^{-2x}(1 - 2x)$, we have

$$\begin{aligned} f \text{ has a potential local extremum at } c \in \mathbb{R} &\iff f'(c) = 0 \\ &\iff e^{-2c}(1 - 2c) = 0 \\ &\iff 1 - 2c = 0 \\ &\iff c = 1/2. \end{aligned}$$

Thus f has a potential local extremum only at $c = 1/2$. To see if this is a genuine local extremum (and whether it is a local max or local min), we check concavity of f at $c = 1/2$. Since $f''(x) = 4e^{-2x}(x - 1)$, we have $f''(1/2) = -2/e < 0$. It follows that f is concave down at $c = 1/2$. Thus $c = 1/2$ is a local maximum. Since $c = 1/2$ is the only critical point of f, we also conclude that $c = 1/2$ is a global maximum of f too.

Problem 3.b

Exercise 10. Find all inflections points.

Solution 10. First we find all possible places where f has a potential inflection point:

$$\begin{aligned} f \text{ has a potential inflection point at } c \in \mathbb{R} &\iff f''(c) = 0 \\ &\iff 4e^{-2c}(c - 1) = 0 \\ &\iff c - 1 = 0 \\ &\iff c = 1. \end{aligned}$$

Thus f has a potential inflection point only at $c = 1$. To see if this is a genuine inflection point at $c = 1$, we need to check that f changes concavity at $c = 1$. Observe that if $x < 1$, then $f''(x) = 4e^{-2x}(x - 1) < 0$ (where we used the fact to e^{-2x} is always positive). This implies f is concave down for all $x < 1$. Similarly, observe that if $x > 1$, the $f''(x) = 4e^{-2x}(x - 1) > 0$. This implies f is concave up for all $x > 1$. Thus f does in fact change concavity at $c = 1$. It follows that f has an inflection point $c = 1$, and this is the only one.

Problem 4

Exercise 11. For a certain industrial process it is necessary to build a tank that has the shape of a circular cylinder of radius r and height h. Within the cylinder is a conical funnel equal in radius at its top to the radius of the cylinder, and having straight sides ending in a point of negligable radius in the center of the bottom face of the tank. The cylindrical tank, its circular bottom, and the cone are all to be fabricated from the same material, which weighs 3 lb/ft^2. The assembly is open on top and must not weigh more than 3000 lb when completed. Formulate a nonlinear program to find r and h so as to maximize the volume contained between the cylinder walls and the cone.

Solution 11. Let A_c represent the surface area (in square feet units) of the cone and let A_t represent the surface area of the tank. Then A_c and A_t can of as functions of r and h, expressed as

$$\begin{aligned} A_c &= \pi r\sqrt{r^2+h^2} \\ A_t &= \pi r^2 + 2\pi r h. \end{aligned}$$

The total weight of the cone (in pounds units) is

$$\text{total weight of cone } = \left(A_c \cdot \mathrm{ft}^2\right) \cdot \left(3 \cdot \frac{\mathrm{lb}}{\mathrm{ft}^2}\right) = 3A_c \cdot \mathrm{lb}.$$

Similarly, the total weight of the tank is $3A_t \cdot \mathrm{lb}$. To simplify notation, we may safely ignore the units in what follows.

Now let us determine what the constraints to this problem are. First, obviously r and h need to be positive, so $r, h \geq 0$. The other constraint that the problem tells us is we must have $3000 \geq 3A_c + 3A_t$. In other words, we must have

$$\begin{aligned} 1000 &\geq A_c + A_t \\ &= \pi r\sqrt{r^2+h^2} + \pi r^2 + 2\pi r h \\ &= \pi \left(r\sqrt{r^2+h^2} + r^2 + 2rh\right). \end{aligned}$$

If we set $g(r,h) = 1000/\pi - r\sqrt{r^2+h^2} - r^2 - 2rh$, then we must have $g(r,h) \geq 0$.

Next let us determine the objective function. Let V_c represent the volume inside the cone and let V_t represent the volume inside the tank. Again, these can be thought of as functions of r and h, expressed as

$$\begin{aligned} V_c &= \frac{1}{3}\pi r^2 h \\ V_t &= \pi r^2 h \end{aligned}$$

Thus if V represents the volume contained between the cylinder walls and the cone, then

$$\begin{aligned} V &= V_t - V_c \\ &= \pi r^2 h - \frac{1}{3}\pi r^2 h \\ &= \frac{2}{3}\pi r^2 h. \end{aligned}$$

The $V(r,h) = (2\pi/3)r^2 h$ is our objective function.

With the constraints and objective function determined, we can now formulate the nonlinear program to find r and h so as to maximize the volume contained between the cylinder walls and the cone:

$$\begin{array}{rl} \text{maximize} & V = (2\pi/3)r^2 h \\ \text{subject to} & g(r,h) \geq 0 \\ & r, h \geq 0. \end{array}$$

Problem 5

For this problem, let $f(\boldsymbol{x}) = 2x_1^2 + 2x_1x_2 + x_2^2 - x_2x_3 + 2x_3^2$.

Problem 5.a

Exercise 12. Is f convex?

Solution 12. Note that f is twice differentiable (in fact C^∞) and thus it is convex if and only if its Hessian $\mathrm{H}_f(\boldsymbol{x})$ is positive semi-definite for all $\boldsymbol{x} \in \mathbb{R}^3$. The Hessian of f is given by

$$\mathrm{H}_f(\boldsymbol{x}) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

The leading principal minors of $\mathrm{H}_f(\boldsymbol{x})$ are

$$D_1(\boldsymbol{x}) = 4, \qquad D_2(\boldsymbol{x}) = \left| \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \right| = 4, \qquad D_3(\boldsymbol{x}) = \left| \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix} \right| = 12.$$

Since $D_1(\boldsymbol{x}), D_2(\boldsymbol{x}), D_3(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$, we see that $\mathrm{H}_f(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \mathbb{R}^3$ which implies f is *strictly* convex everywhere in $\mathbb{R}^3$ (thus certainly f is convex).

Problem 5.b

Exercise 13. Is f strictly convex?

Solution 13. Yes, by our solution for 5.a.

Problem 6

For this problem, let $f(\boldsymbol{x}) = 2x_1^2 x_2^{-1}$ and let $A = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1, x_2 > 0\}$.

Problem 6.a

Exercise 14. Is f convex over A?

Solution 14. The Hessian of f is given by

$$\mathrm{H}_f(\boldsymbol{x}) = \begin{pmatrix} \frac{4}{x_2} & -\frac{4x_1}{x_2^2} \\ -\frac{4x_1}{x_2^2} & \frac{4x_1^2}{x_2^3} \end{pmatrix}$$

The principal minors of $\mathrm{H}_f(\boldsymbol{x})$ are

$$\Delta_1(\boldsymbol{x}) = \frac{4}{x_2}, \qquad \Delta_2(\boldsymbol{x}) = \frac{4x_1^2}{x_2^3}, \qquad \Delta_3(\boldsymbol{x}) = \left| \begin{pmatrix} \frac{4}{x_2} & -\frac{4x_1}{x_2^2} \\ -\frac{4x_1}{x_2^2} & \frac{4x_1^2}{x_2^3} \end{pmatrix} \right| = 0.$$

Since $\Delta_1(\boldsymbol{x}), \Delta_2(\boldsymbol{x}), \Delta_3(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in A$, we see that $\mathrm{H}_f(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in A$ which implies f is convex over A.

Problem 6.b

Exercise 15. Is f strictly convex over A?

Solution 15. No because $\Delta_3(\boldsymbol{x}) \not> 0$ for all $\boldsymbol{x} \in A$.