# Tor-Persistence

## Introduction

Let $R$ be a commutative noetherian ring. Recall that a finitely generated $R$-module $M$ has finite projective dimension if $\mathrm{Tor}_i^R(M,N)=0$ for $i \gg 0$ for each finitely generated $R$-module $N$. Indeed, first note that $\mathrm{Tor}_i^R(M,N)=0$ if and only if

$$\mathrm{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) \simeq \mathrm{Tor}_i^R(M,N)_\mathfrak{p}=0$$

for all prime ideals $\mathfrak{p}$ of $R$. Thus by replacing $R$, $M$, and $N$ with $R_\mathfrak{p}$, $M_\mathfrak{p}$, and $N_\mathfrak{p}$ if necessary, we may assume that $R=(R,\mathfrak{m},\Bbbk)$ is local. Now let $F$ be the minimal $R$-free resolution of $M$. Thus

$$\mathrm{Tor}_i^R(M,N)=\mathrm{H}_i(F\otimes_R N).$$

We first prove the easy direction: suppose $M$ has finite projective dimension, say $\mathrm{pd}_R M=p$. This means that $F_p \neq 0$ and $F_i=0$ for all $i>p$. In particular that $(F\otimes_R N)_i=0$ for all $i>p$, which implies $\mathrm{Tor}_i^R(M,N)=0$ for all $i>p$. Now we prove the harder direction: suppose $\mathrm{Tor}_i^R(M,N)=0$ for $i\gg 0$ for each finitely generated $R$-module $N$. In particular, we have $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i\gg 0$. This implies $\mathrm{H}_i(F_\Bbbk)=0$ for $i \gg 0$ where we set $F_\Bbbk := F\otimes_R \Bbbk$. However $F$ is *minimal*, thus $\mathrm{d}_\Bbbk=0$, where $\mathrm{d}_\Bbbk$ is the differential of $F_\Bbbk$. Thus we have $\mathrm{H}_i(F_\Bbbk)=F_{i,\Bbbk}:=F_i\otimes_R\Bbbk$ and this implies $F_i\otimes_R \Bbbk=0$ for $i\gg 0$ which implies $F_i=0$ for $i\gg 0$ by Nakayama's lemma (here is where we used the fact that $R$ is noetherian and $M$ is finitely generated).

Now suppose that the only thing we knew was that $\mathrm{Tor}_i^R(M,M)=0$ for $i\gg 0$. Can we still conclude that the projective dimension of $M$ is finite? This is an open question in general, however it is known to be true for various rings $R$: we call such rings **Tor-persistent**. It is natural to wonder if in fact every commutative noetherian ring is Tor-persistent. Note that

$$\mathrm{Tor}_i^R(M,M)=\mathrm{H}_i(F\otimes_R M)=\mathrm{H}_i(F^{\otimes 2})$$

where we denoted $F^{\otimes 2}=F\otimes_R F$. One of the main reasons why we could conclude that $M$ had finite projective dimension if $\mathrm{Tor}_i^R(M,\Bbbk)=0$ for $i\gg 0$ was because the homology of $F_\Bbbk$ was extremely simple: $\mathrm{H}(F_\Bbbk)=F_\Bbbk$. The homology of $F^{\otimes 2}$ is more complicated, thus even if we knew that $\mathrm{H}_i(F^{\otimes 2})=0$ for $i\gg 0$, it is not at all clear why this should imply that $F_i=0$ for $i \gg 0$. In order to prove this, one would presumably need to use the fact that $R$ is noetherian, $M$ is finitely generated, and $F$ is minimal.

### Tor-Persistence

In what follows, we assume $(R,\mathfrak{m},\Bbbk)$ is a local noetherian ring. Let $F$ be the minimal $R$-free resolution of the cyclic $R$-module $R/I$ where $I\subseteq \mathfrak{m}$ is an ideal of $R$. Choose a multiplication $\mu$ on $F$ giving it the structure of an MDG $R$-algebra. We denote $\mu(a_1\otimes a_2)=a_1a_2$ for all $a_1,a_2\in F$ in order to simplify notation in what follows. Define a chain map $\{\cdot\}_\mu\colon F^{\otimes 3}\to F^{\otimes 2}$ by the formula

$$\{a_1\otimes a_2\otimes a_3\}=a_1a_2\otimes a_3-a_1\otimes a_2a_3=\{a_1,a_2,a_3\},$$

where we remove the subscript $\mu$ from $\{\cdot\}_\mu$ when context is clear and where we set $\{\cdot,\cdot,\cdot\}\colon F^3\to F^{\otimes 2}$ to be the unique $R$-trilinear map corresponding to $\{\cdot\}$ via the universal mapping property of tensor products. Our goal is to determine what $\ker\{\cdot\}$ and $\mathrm{im}\,\{\cdot\}$ look like. First we consider $\mathrm{im}\,\{\cdot\}$. For each $a_1,a_2,a_3\in F$, we have

$$\begin{aligned}
\{a_1,a_2,1\} &= a_1a_2\otimes 1-a_1\otimes a_2\\
\{1,a_2,a_3\} &= a_2\otimes a_3-1\otimes a_2a_3\\
\{a_1,1,a_3\} &= 0\\
\{a,a,b\} &= a^2\otimes b-a\otimes ab
\end{aligned}$$

Thus if $ab = 0$, then $a \otimes b \in \operatorname{im}\{\cdot\}$. Furthermore we have $a \otimes 1 - 1 \otimes a \in \operatorname{im}\{\cdot\}$. Now suppose that

$$\{e_{i_1}, e_{i_2}, e_{i_3}\} = e_{i_1}e_{i_2} \otimes e_{i_3} - e_{i_1} \otimes e_{i_2}e_{i_3} = 0.$$

Then we must have $e_{i_1} = e_{i_1}e_{i_2}$ and $e_{i_3} = e_{i_2}e_{i_3}$. Or in other words, we must have $e_{i_1}(1 - e_{i_2}) = 0$ and $e_{i_3}(1 - e_{i_2}) = 0$. By considering homological degrees as well as using the fact that $R$ is local, one sees that the only solution to these equations is

$$\{(0, e_{i_2}, 0), (0, 1, e_{i_3}), (e_{i_1}, 1, 0), (e_{i_1}, 1, e_{i_3})\}.$$

In particular, this spans $F^{\oplus 3} \oplus F^{\otimes 2}$.

**Proposition 0.1.** *Suppose* $\mathrm{H}_i(F) = 0 = \mathrm{H}_i(F^{\otimes 2})$ *for* $i \gg 0$. *Then* $\mathrm{H}_i(F^{\otimes n}) = 0$ *for* $i \gg 0$ *for all* $n \geq 1$.

*Proof.* Consider the short exact sequence $0 \to F \to F^{\otimes 3} \to F^{\otimes 2} \to 0$. Actually this even shows $\operatorname{Tor}_+^R(S, S) = \mathrm{H}_+(F^{\otimes n})$ for all $n \geq 2$. □