Homework 2

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Problem 1

Exercise 1. Prove the following statement: let $\prec$ be an irreflexive, transitive, and connected binary relation on $\mathbb{R}^n$ is negatively transitive.

Solution 1. Suppose $x \nprec y$ and $y \nprec z$. In order to show $\prec$ is negatively transitive, we need to show that $x \nprec z$ Since $x \nprec y$ and $y \nprec z$ and since $\prec$ is connected, we see that $z \prec y$ and $y \prec x$, and since $\prec$ is transitive, we see that $z \prec x$. It follows that $x \nprec z$ since $\prec$ is irreflexive.

Problem 2

Exercise 2. Let $n \geq 2$ and define a binary relation on $\mathbb{R}^n$ by

$$\boldsymbol{x} \preceq \boldsymbol{y} \text{ if and only if } \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Prove or disprive that this binary relation is:

1. A weak order on $\mathbb{R}^n$.
2. A total order on $\mathbb{R}^n$.

Solution 2. 1. Yes. It is transitive because $\leq$ is transitive on $\mathbb{R}$ and it is strongly connected because $\leq$ is strongly connected on $\mathbb{R}^n$.

2. No because it is not antisymmetric. Indeed consider the standard unit vectors $\boldsymbol{e}_1 = (1, 0, 0, \dots, 0)$ and $\boldsymbol{e}_2 = (0, 1, 0, \dots, 0)$. We have $\boldsymbol{e}_1 \preceq \boldsymbol{e}_2$ and $\boldsymbol{e}_2 \preceq \boldsymbol{e}_1$, however clearly $\boldsymbol{e}_1 \neq \boldsymbol{e}_2$.

Problem 3

Exercise 3. Let $n \geq 2$. The binary relation defined on $\mathbb{R}^n$ by

$$\boldsymbol{x} \prec \boldsymbol{y} \text{ if and only if } 0 < \boldsymbol{y} - \boldsymbol{x}$$

is referred to as the strict Pareto-min preference. Note that $0 < \boldsymbol{y} - \boldsymbol{x}$ means $x_i < y_i$ for all $1 \leq i \leq n$.

1. Prove or disprove that this preference is a strict partial order on $\mathbb{R}^n$.
2. Let the related indifference preference $\sim$ be defined as

   $$\boldsymbol{x} \sim \boldsymbol{y} \text{ if and only if } \boldsymbol{x} = \boldsymbol{y}.$$

   Derive the set $\{\prec\}$, $\{\sim\}$, $\{\succ\}$, and $\{?\}$.
3. Derive the corresponding sets P, D, I, and U. Graph these sets or $n = 2$.

Solution 3. 1. First note that it is strict since $>$ is strict on $\mathbb{R}^n$. To see that it is asymmetric, note that if $\boldsymbol{y} \succ \boldsymbol{x}$, then $\boldsymbol{y} - \boldsymbol{x} > 0$. In particular, $\boldsymbol{x} \not\succ \boldsymbol{y}$ since $\boldsymbol{x} - \boldsymbol{y} < 0$. To see that it is transitive, suppose $\boldsymbol{z} \succ \boldsymbol{y}$ and $\boldsymbol{y} \succ \boldsymbol{x}$. Thus $\boldsymbol{z} - \boldsymbol{y} > 0$ and $\boldsymbol{y} - \boldsymbol{x} > 0$. It follows that $\boldsymbol{z} - \boldsymbol{x} = (\boldsymbol{z} - \boldsymbol{y}) + (\boldsymbol{y} - \boldsymbol{z}) > 0$.

2. We have

$$\begin{aligned}
\{\prec\} &= \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_i < x_{i+n} \text{ for all } 1 \le i \le n\}. \\
\{\sim\} &= \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_i = x_{i+n} \text{ for all } 1 \le i \le n\} \\
\{\succ\} &= \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_i > x_{i+n} \text{ for all } 1 \le i \le n\}. \\
\{?\} &= \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_i < x_{i+n} \text{ for all } 1 \le i \le n\}.
\end{aligned}$$

Problem 4

Exercise 4. Let $n \geq 2$. The binary relation defined on $\mathbb{R}^n$ by $\boldsymbol{x} \prec \boldsymbol{y}$ if and only if there exists $1 \leq k \leq n$ such that $x_i = y_i$ for all $i < k$ and $x_k < y_k$ is referred to as the strict lexicographical-min preference.

1. Prove or disprove that this preference is a strict partial order on $\mathbb{R}^n$.

2. Let the related indifference preference $\sim$ be defined as

$$\boldsymbol{x} \sim \boldsymbol{y} \text{ if and only if } \boldsymbol{x} = \boldsymbol{y}.$$

   Derive the set $\{\prec\}$, $\{\sim\}$, $\{\succ\}$, and $\{?\}$.

3. Derive the corresponding sets P, D, I, and U. Graph these sets or $n = 2$.

Solution 4. 1. First note that it is strict and asymmetric since $>$ is strict and asymmetric on $\mathbb{R}^n$. To see that it is transitive, suppose $\boldsymbol{x} \prec \boldsymbol{y}$ and $\boldsymbol{y} \prec \boldsymbol{z}$. Choose k and m such that $x_i = y_i$ for all $i < k$ and $x_k = y_k$, and $y_j = z_j$ for all $j < m$ and $y_m = z_m$. Necessarily then we have $k \leq m$. It follows that $\boldsymbol{x} < \boldsymbol{z}$ and hence $\prec$ is transitive.

2. We have

$$\begin{aligned}
\{\prec\} &= \bigcup_{1 \le i \le n} \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_1 = x_{1+n}, \ldots, x_{i-1} = x_{i-1+n}, x_i < x_{i+n}\} \\
\{\sim\} &= \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_i = x_{i+n} \text{ for all } 1 \le i \le n\} \\
\{\succ\} &= \bigcup_{1 \le i \le n} \{\boldsymbol{x} \in \mathbb{R}^{2n} \mid x_1 = x_{1+n}, \ldots, x_{i-1} = x_{i-1+n}, x_i > x_{i+n}\} \\
\{?\} &= \mathbb{R}^{2n} \backslash (\{\prec\} \cup \{\sim\} \cup \{\succ\})
\end{aligned}$$

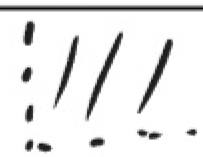
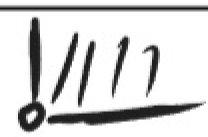
Problem 5

5. Fill the following tables (among others, use your results from questions 3 and 4 above):

a) (10 points) For each preference, that is defined by an appropriate binary relation, indicate whether or not this relation is reflexive, irreflexive, symmetric, antisymmetric, or transitive.

Preference (minimization)	Binary relation				
	reflexive	irreflexive	symmetric	antisymmetric	transitive
strict Pareto	no	yes	no	yes	yes
Pareto	no	yes	no	yes	yes
weak Pareto	yes	no	no	yes	yes
strict lexicographic	no	yes	no	yes	yes
lexicographic	yes	no	no	yes	yes

b) (10 points) For each preference, in the column labeled "figure" draw the associated <u>domination cone</u> in R^2, and in the other columns indicate whether or not this cone is closed, open, pointed, or convex.

Preference (minimization)	Cone				
	figure	closed	open	pointed	convex
strict Pareto		no	yes	yes	yes
Pareto		no	no	yes	yes
weak Pareto		yes	no	yes	yes
strict lexicographic		no	no	no	yes
lexicographic		no	no	no	yes