Advanced Numerical Analysis Homework 5

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Throughout this homework, $\|\cdot\|$ denotes the ℓ_2-norm.

1 Problem 1

Exercise 1. Consider the matrix $A = \left(\begin{smallmatrix} -2 & 11 \\ -10 & 5 \end{smallmatrix}\right)$.

1. Determine, on paper, a real SVD of A in the form $A = U\Sigma V^\top$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.
2. List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.
3. What are the 1, 2, ∞, and Frobenius norms of A?
4. Find A^{-1} not directly, but via the SVD.
5. Find the eigenvalues λ_1, λ_2 of A.
6. Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.
7. What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

Solution 1. 1. First we find the singular values by computing the eigenvalues of $A^\top A$. Observe that

$$A^\top A = \begin{pmatrix} 104 & -72 \\ -72 & 146 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 - 250\lambda + 10000 = (\lambda - 50)(\lambda - 200).$$

Therefore the eigenvalues are $\lambda_1 = 200$ and $\lambda_2 = 50$, thus the singular values of A are $\sigma_1 = 10\sqrt{2}$ and $\sigma_2 = 5\sqrt{2}$. Next we find the right singular vectors (the columns of V) by finding an orthonormal set of eigenvectors of $A^\top A$. For $\lambda_1 = 200$, we have

$$A^\top A - 200 = \begin{pmatrix} -96 & -72 \\ -72 & -54 \end{pmatrix} \to \begin{pmatrix} 1 & 3/4 \\ 0 & 0 \end{pmatrix},$$

where the arrow denotes row reduction. It is easy to see that $v_1 = (-3/5, 4/5)$ is in the kernel of this matrix and that $\|v_1\| = 1$. For $\lambda_2 = 50$, we have

$$A^\top A - 50 = \begin{pmatrix} 54 & -72 \\ -72 & 96 \end{pmatrix} \to \begin{pmatrix} 1 & -4/3 \\ 0 & 0 \end{pmatrix},$$

where the arrow denotes row reduction. It is easy to see that $v_2 = (4/5, 3/5)$ is in the kernel of this matrix and that $\|v_2\| = 1$. Thus

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

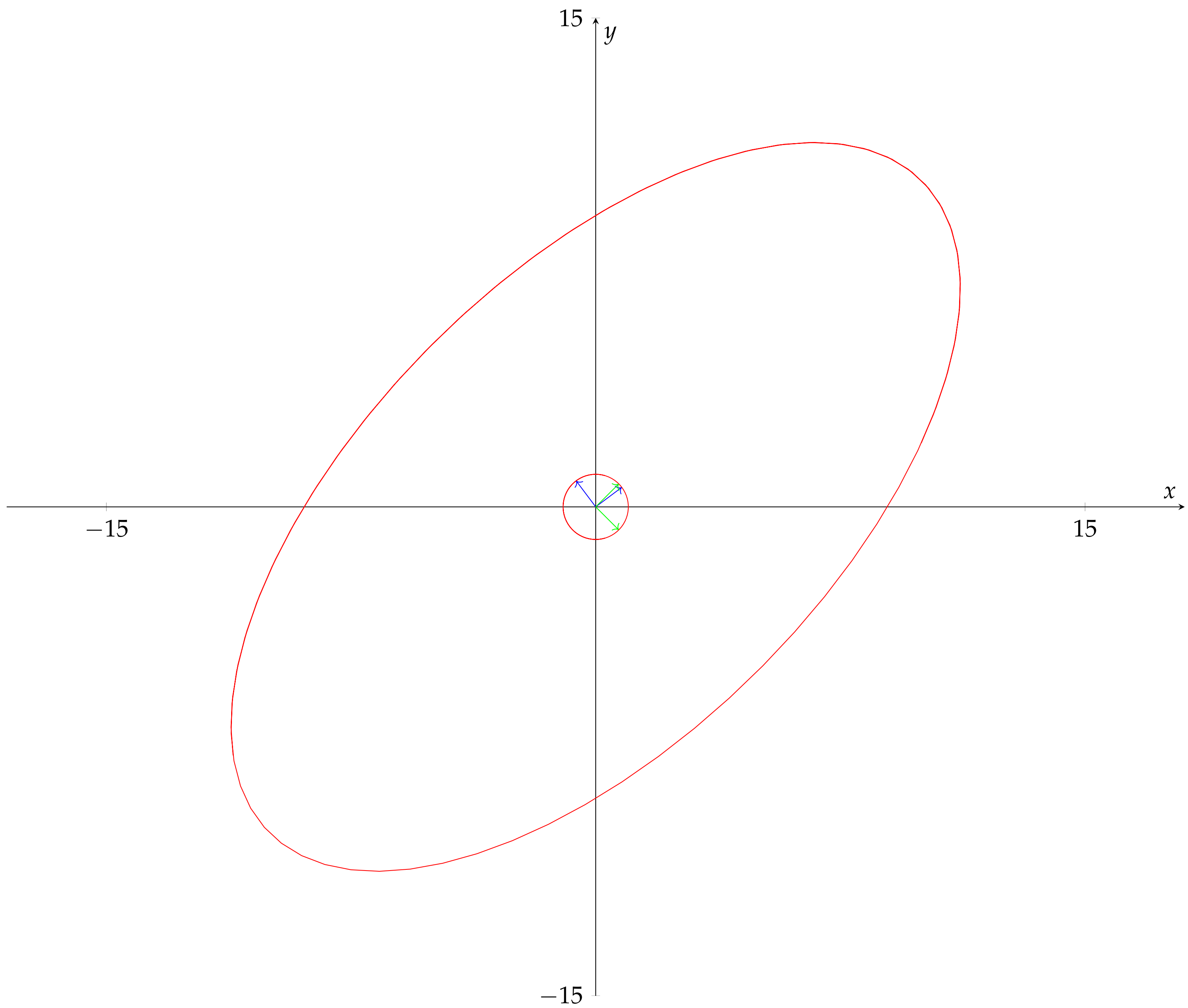
Finally, we compute U by the formula $u_i = \sigma_i^{-1} A v_i$. This gives us

$$U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

2. We have

$$
\begin{aligned}
\sigma_1 &= 10\sqrt{2} \\
\sigma_2 &= 5\sqrt{2} \\
u_1 &= (1/\sqrt{2}, 1/\sqrt{2})^\top \\
u_2 &= (1/\sqrt{2}, -1/\sqrt{2})^\top \\
v_1 &= (-3/5, 4/5) \\
v_2 &= (4/5, 3/5).
\end{aligned}
$$

Below we draw a picture of the circles of radius 5 centered at the origin in $\mathbb{R}^2$ and its image under A, together with the singular vectors:



where the green vectors are u_1 and u_2 and where the blue vectors are v_1 and v_2.

3. We have

$$
\begin{aligned}
\|A\|_1 &= \max\{|-2| + |-10|, |11| + |5|\} \\
&= \max\{12, 16\} \\
&= 16
\end{aligned}
\qquad
\begin{aligned}
\|A\|_\infty &= \max\{|-2| + |11|, |-10| + |5|\} \\
&= \max\{13, 15\} \\
&= 15.
\end{aligned}
$$

Similarly, we have

$$\begin{aligned} \|A\|_2 &= \|\sigma\|_\infty \\ &= \max\{\sigma_1, \sigma_2\} \\ &= 10\sqrt{2} \end{aligned} \qquad\qquad \begin{aligned} \|A\|_{\mathrm{F}} &= \|\sigma\|_2 \\ &= \sqrt{50+200} \\ &= 5\sqrt{10}. \end{aligned}$$

4. We have

$$\begin{aligned} A^{-1} &= (U\Sigma V^\top)^{-1} \\ &= V\Sigma^{-1}U^\top \\ &= \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix} \begin{pmatrix} 1/10\sqrt{2} & 0 \\ 0 & 1/5\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\ &= \frac{1}{100} \begin{pmatrix} 5 & -11/100 \\ 10 & -1/50 \end{pmatrix}. \end{aligned}$$

5. The characteristic polynomial of A is given by

$$\lambda^2 - 3\lambda + 100 = \left(\lambda - \left(\frac{3}{2} - i\frac{\sqrt{391}}{2}\right)\right)\left(\lambda - \left(\frac{3}{2} + i\frac{\sqrt{391}}{2}\right)\right).$$

Therefore the eigenvalues of A are $\lambda_1 = \frac{3}{2} - i\frac{\sqrt{391}}{2}$ and $\lambda_2 = \frac{3}{2} + i\frac{\sqrt{391}}{2}$.

6. We have

$$\begin{aligned} \det A &= -2 \cdot 5 - (-10) \cdot 11 \\ &= -10 + 110 \\ &= 100. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \lambda_1\lambda_2 &= \left(\frac{3}{2} - i\frac{\sqrt{391}}{2}\right)\left(\frac{3}{2} + i\frac{\sqrt{391}}{2}\right) \\ &= \frac{9}{4} + \frac{391}{4} \\ &= 100. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sigma_1\sigma_2 &= (10\sqrt{2}) \cdot (5\sqrt{2}) \\ &= 50 \cdot 2 \\ &= 100. \end{aligned}$$

7. The area of the ellipse is given by $\sigma_1\sigma_2\pi = 100\pi$.

2 Problem 2

Exercise 2. Solve the following:

1. If $A, E \in \mathbb{R}^{m\times n}$, show that

$$\sigma_{\max}(A) - \|E\| \le \sigma_{\max}(A+E) \le \sigma_{\max}(A) + \|E\|.$$

   Comment on the absolute condition number of $\|A\|$ as a function of A.

2. If $A \in \mathbb{R}^{m\times n}$ where $m > n$ and $z \in \mathbb{R}^m$, show that

$$\sigma_{\max}\begin{pmatrix} A & z \end{pmatrix} \geq \sigma_{\max}(A) \quad \text{and} \quad \sigma_{\min}\begin{pmatrix} A & z \end{pmatrix} \leq \sigma_{\min}(A).$$

Solution 2. 1. Recall that $\|A\| = \sigma_{\max}(A)$ and $\|A+E\| = \sigma_{\max}(A+E)$. Thus it suffices to show that

$$\|A\| - \|E\| \leq \|A+E\| \leq \|A\| + \|E\|.$$

However this follows from subadditivity of the norm $\|\cdot\|$. Indeed, we have

$$\|A+E\| \leq \|A\| + \|E\|.$$

Similarly, we have

$$\|A-E\| \leq \|A\| + \|E\|. \tag{2.1}$$

In particular, setting $A = A+E$ in (2.1) gives us

$$\|A\| - \|E\| \leq \|A+E\|.$$

2. We have

$$\begin{aligned}
\sigma_{\max}\begin{pmatrix} A & z \end{pmatrix} &= \max_{\|(\boldsymbol{x}, x_{m+1})^\top\|=1} \left\| \begin{pmatrix} A & z \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ x_{m+1} \end{pmatrix} \right\| \\
&\geq \max_{\|\boldsymbol{x}\|=1} \left\| \begin{pmatrix} A & z \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 0 \end{pmatrix} \right\| \\
&= \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| \\
&= \sigma_{\max}(A).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sigma_{\min}\begin{pmatrix} A & z \end{pmatrix} &= \min_{\|(x, x_{m+1})^\top\|=1} \left\| \begin{pmatrix} A & z \end{pmatrix} \begin{pmatrix} x \\ x_{m+1} \end{pmatrix} \right\| \\
&\leq \min_{\|x\|=1} \left\| \begin{pmatrix} A & z \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\| \\
&= \min_{\|x\|=1} \|Ax\| \\
&= \sigma_{\min}(A).
\end{aligned}$$

3 Problem 3

Exercise 3. Solve the following:

1. Show that if $A \in \mathbb{R}^{m\times n}$, then

$$\|A\|_F \leq \sqrt{\operatorname{rank}(A)}\|A\|.$$

2. Show that if $A \in \mathbb{R}^{m\times n}$ has rank n, then

$$\|A(A^\top A)^{-1}A^\top\| = 1.$$

Solution 3. 1. Let $k = \operatorname{rank} A$ and let $\sigma_1 \geq \cdots \geq \sigma_k$ be the nonzero singular values of A. Then we have

$$\begin{aligned}
\|A\|_F &= \sqrt{\sigma_1^2 + \cdots + \sigma_k^2} \\
&\leq \sqrt{\sigma_1^2 + \cdots + \sigma_1^2} \\
&= \sqrt{k}\sigma_1 \\
&= \sqrt{k}\|A\|.
\end{aligned}$$

2. Let $P = A(A^\top A)^{-1}A^\top$. Since A has Then note that

$$P^2 = A(A^\top A)^{-1}A^\top A(A^\top A)^{-1}A^\top = A(A^\top A)^{-1}A^\top = P.$$

Thus P is a projector. In particular, we have

$$\begin{aligned}\|Px\| &= \|P(Px)\| \\ &\leq \|P\|\|Px\|\end{aligned}$$

for all nonzero $x \in \mathbb{R}^n$, which implies $1 \leq \|P\|$. Furthermore, we have $P^\top = P$, thus P is an orthogonal projection. By the Pythagorean theorem, we have

$$\begin{aligned}\|x\|^2 &= \|Px\|^2 + \|x - Px\|^2 \\ &\geq \|Px\|^2,\end{aligned}$$

for all nonzero $x \in \mathbb{R}^n$. This implies $\|Px\| \leq x$ for all nonzero $x \in \mathbb{R}^n$ which implies $\|P\| \leq 1$.

4 Problem 4

Exercise 4. Solve the following.

1. Given $A \in \mathbb{R}^{n\times n}$, let $A = U\Sigma V^\top$ be an SVD of A, where $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Let $B = [U \operatorname{diag}(1, \dots, 1, -1)]\Sigma V^\top$ such that $\det B = -\det A$ and $\|A - B\|_F = 2\sigma_n$. Show that for any singular values $\sigma_1, \dots, \sigma_{n-1}$ ($\geq \sigma_n$), there exists $C \in \mathbb{R}^{n\times n}$ such that $\det C = \det B = -\det A$ and $\|A - C\|_F < \|A - B\|_F = 2\sigma_n$. (Hint: to construct C, modify σ_n and σ_{n-1} of A only (change the sign of one and keep the sign of the other, but make sure that their product does not change).
2. If P is an orthogonal projector, then $1 - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Solution 4. 1.

2. Let $U = 1 - 2P$. We have

$$\begin{aligned}\langle Ux, Uy\rangle &= \langle x - 2Px, y - 2Py\rangle \\ &= \langle x, y\rangle - 2\langle x, Py\rangle - 2\langle Px, y\rangle + 4\langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle x, PPy\rangle - 2\langle PPx, y\rangle + 4\langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, Py\rangle - 2\langle Px, Py\rangle + 4\langle Px, Py\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

It follows that U is unitary. Geometrically speaking, U is the reflection about the plane spanned by range(P).

5 Problem 5

Exercise 5. Solve the following.

1. Implement the Golub-Kahan (GK) bidiagonalization of a matrix. Test it on $F \in \mathbb{R}^{10\times 10}$ obtained as follows

```
rgn(default);
F = randn(10,10);
```

   Make sure that your bidiagonal matrix has the same singular values as F.

2. Generate a matrix $A \in \mathbb{R}^{(1024^2+1)\times 32}$ as follows

```
col = linspace(-1,1,1024*1024+1)';
A = col.^(0:31);
```

Apply Householder QR to A and get $R \in \mathbb{R}^{32\times 32}$, then apply GK to R and get bidiagonal $B \in \mathbb{R}^{32\times 32}$ (no need to retrieve Q for this problem). Compute the 5 largest and 5 smallest singular values of A from the eigenvalues of $\begin{pmatrix} 0 & B^\top \\ B & 0 \end{pmatrix}$. Compare these singular values with those computed by taking the square root of the 5 largest and 5 smallest eigenvalues of $A^\top A$. What conclusion do you draw? Is it a good idea to compute the eigenvalues of $\begin{pmatrix} 0 & A^\top \\ A & 0 \end{pmatrix}$ directly, and why?

Solution 5.