MATH 8100 Fall 2021 Homework 5 and 6 due November 16, 2021.

Guidance for writing your assignment:
a) make sure that your writing is clear and legible
b) start your answer to each part of a problem on a new page
c) wherever appropriate, underline or rewrite the final answer
d) show all work for full credit

Please make every effort to follow this guidance and facilitate the reading of your assignment. The assignments that do not follow this guidance will be returned.

5. This question belongs to Homework 5.

a) (10 points) Let $\mathbf{x}$ be feasible but not a basic feasible solution for $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Prove that the columns of A corresponding to the nonzero entries of $\mathbf{x}$ are linearly dependent.

b) (10 points) Let $\mathbf{x}$ be a feasible point of $X = \{\mathbf{x} \in R^n\text{: } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ that is not an extreme point. Prove that there exists a vector $\mathbf{p} \in R^n$, $\mathbf{p} \neq \mathbf{0}$ such that

$$A\mathbf{p} = \mathbf{0}$$
$$p_i = 0 \text{ if } x_i = 0.$$

(Hint: Use the result from part a).)

Homework 6 starts here

100 points (LPs part 2: Simplex Algorithm)

1. (10 points) Consider the following linear program (LP):

$$\begin{array}{lll} \text{minimize} & x_1 - x_2 & \\ \text{s.t.} & x_1 - 2x_2 & \leq 4 \\ & -x_1 + x_2 & \leq 3 \\ & x_1 \text{ and } x_2 \geq 0 & \end{array}$$

The set of all optimal solutions to this problem is $X^* = \{ \mathbf{x} \in R^4\text{: } \mathbf{x} = \mathbf{x}(t) = \mathbf{x}^1 + t\,\mathbf{d}, t \geq 0 \}$, where $\mathbf{x}^1 = (0\ \ 3\ \ 10\ \ 0)^T$ is an extreme point and $\mathbf{d} = (1\ \ 1\ \ 1\ \ 0)^T$ is a recession direction of the feasible set for this LP.

Solve this problem in the Simplex tableau format. In the final tableau clearly identify the entries that allow you to conclude that X^* is the complete optimal solution set for this LP.

2. (20 points) Consider the following LP given in the Simplex tableau:

X1	X2	X3	X4	X5	X6	RHS
0	**0**	**1**	**0**	**0**	**0**	**-5**
1	1	1	0	0	-1	15
0	-1	0	0	1	0	5
0	0	-1	1	0	-1	10

Continue the tableau format and, based on the experience you gained in Question 1 above, find the set of all optimal solutions to this LP. Make sure that you write this set in an appropriate form.

3. (30 points) The following tableau represents a linear program in standard form.

x_1	x_2	x_3	x_4	x_5	RHS
2	0	***c***	0	***e***	6
3	0	-1	1	0	***a***
-1	0	***d***	0	1	2
4	***b***	-2	0	0	5

Give the most general conditions* on the parameters ***a, b, c, d***, and ***e*** so that the tableau represents

i) a nondegenerate basic feasible solution
ii) a degenerate basic feasible solution
iii) a basic feasible solution and unbounded feasible region
iv) a unique optimal basic feasible solution
v) an optimal basic feasible solution and optimal ray
vi) unbounded objective function value

Report the answers in the table:

	a	***b***	***c***	***d***	***e***
i)					
ii)					
iii)					
iv)					
v)					
vi)					

* "The most general conditions" means, for example, ***a*** >0 or ***a*** $\geq$ 0 or ***a*** free or ***a*** = 0, etc.

4. (10 points) Prove the following: Let $X = \{ \mathbf{x} \in R^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \}$. If and extreme point of X has more than one basis representation, it is degenerate.

5. (20 points) Use the tableau format and solve the following linear program with the Two-Phase Method. Clearly show all steps of this method.

$$\begin{array}{llrcl}
\text{minimize} & & x_1 - 2x_2 & & \\
\text{s.t.} & & x_1 + x_2 - x_3 & = & 2 \\
& & -x_1 + x_2 + x_4 & = & 4 \\
& & x_1, \; x_2, \; x_3, \; x_4 & \geq & 0
\end{array}$$

6. (10 points) For some linear program (LP), the associated artificial LP has been solved and produced the following tableau:

	x_1	x_2	x_3	a_1	a_2	RHS
z'	0	1	3	0	3	0
a_1	0	–1	–3	1	–2	0
x_1	1	1	2	0	1	2

Continue the Two-Phase Method and find an initial partial tableau for Phase 2 for that LP.