Functions, Morphisms, and Varieties

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1 Functions on Affine Varieties

Definition 1.1. Let $X \subset \mathbb{A}^n$ be an affine variety. We call

$$A(X) := K[x_1, \dots, x_n]/\mathbf{I}(X)$$

the **coordinate ring** of X. The quotient field $K(X)$ of $A(X)$ is called the **field of rational functions** on X.

Remark. hfill

1. By an **affine variety** we mean an algebraic set which is irreducible.
2. Recall that $\mathbf{I}(X)$ is a prime ideal (since X is irreducible), hence $A(X)$ is an integral domain. So $K(X)$ is well-defined.

Definition 1.2. Let $X \subset \mathbb{A}^n$ be an affine variety and let $p \in X$ be a point. We call

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \mid f, g \in A(X) \text{ and } g(p) \neq 0 \right\} \subset K(X)$$

the **local ring** of X at the point p. If $U \subset X$ is a non-empty open subset, we set

$$\mathcal{O}_X(U) := \bigcap_{p \in U} \mathcal{O}_{X,p} \subset K(X).$$

This is a subring of $K(X)$. We call this the **ring of regular functions** on U.

Remark.

1. The set $\mathfrak{m}_{X,p} := \{f \in A(X) \mid f(p) = 0\}$ of all functions that vanish at p is a maximal ideal in $A(X)$. With this definition, $\mathcal{O}_{X,p}$ is just the localization of the ring $A(X)$ at the maximal ideal $\mathfrak{m}_{X,p}$.
2. Let X be an affine variety, $f \in A(X)$, and let $D(f)$ be the open subset in X given by $D(f) := \{p \in X \mid f(p) \neq 0\}$. Open subsets of this form are called **distinguished open subsets**. They form a basis in the Zariski topology. In fact, every open set can be written as a finite union of distinguished open subsets. Also, if $D(f_1)$ and $D(f_2)$ are two distinguished open subsets for two distinct polynomials f_1 and f_2, then $D(f_1) \cap D(f_2) = D(f_1 f_2)$.

It may be tempting to think that

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \mid f, g \in A(X) \text{ and } g(p) \neq 0 \text{ for all } p \in U \right\},$$

but this is not necessarily the case. For instance, let $X \subset \mathbb{A}^4$ be the variety defined by the equation $x_1 x_4 = x_2 x_3$. The function $\frac{x_1}{x_2}$ is defined on $D(x_2)$ and the function $\frac{x_3}{x_4}$ is defined on $D(x_4)$. By the equation of X, these two functions coincide where they are both defined; in other words

$$\frac{x_1}{x_2} = \frac{x_3}{x_4} \in K(X).$$

So this gives rise to a regular function on $D(x_2) \cup D(x_4)$, but there is no representation of this function as a quotient of two polynomials in $K[x_1, x_2, x_3, x_4]$ that works on all of $D(x_2) \cup D(x_4)$; we have to use different representations at different points.

On the other hand, it can be the case that

$$O_X(U) = \left\{ \frac{f}{g} \mid f, g \in A(X) \text{ and } g(p) \neq 0 \text{ for all } p \in U \right\}.$$

For instance, let $X = \mathbb{C}^2$ and $U = \mathbb{C}^2 \backslash \{0\}$. Suppose $\varphi \in O_X(U)$ and $p \in U$. Since $\varphi \in O_{X,p}$, we can write

$$\varphi = \frac{f_1}{f_2},$$

where $f_2(p) \neq 0$. We may assume f_1 and f_2 share no common factors. If f_2 is not a constant, then there exists another point $q \in U$ such that $f_2(q) = 0$. Since $\varphi \in O_{X,q}$, we must be able to write

$$\varphi = \frac{g_1}{g_2},$$

where $g_2(q) \neq 0$. However, the only way we can have

$$\frac{f_1}{f_2} = \frac{g_1}{g_2}$$

is if $g_1 = hf_1$ and $g_2 = hf_2$, where h is some polynomial. But this implies $g_2(q) = h(q)f_2(q) = 0$, which is a contradiction.

Lemma 1.1. *The following definition of regular functions is equivalent to the one of Definition (1.2):*

Let U be an open subset of an affine variety $X \subset \mathbb{A}^n$. A set-theoretic map $\varphi : U \to K$ is called regular at the point $p \in U$ if there is a neighborhood V of p in U such that there are polynomials $f, g \in K[x_1, \dots, x_n]$ with $g(q) \neq 0$ and $\varphi(q) = \frac{f(q)}{g(q)}$ for all $q \in V$. It is called regular on U if it is regular at every point in U.

Proof. It is obvious that an element of the ring of regular functions on U determines a regular function in the sense of the lemma. Indeed, let $r \in O_X(U)$ and let p be a point in U. Then $r \in O_X \subset O_{X,p}$ implies that it can be written as $r = \frac{f}{g}$ on $V := D(g) \cap U$, where $f, g \in A(X)$ and $g(p) \neq 0$. Since p is arbitrary, we are done.

Conversely, let $\varphi : U \to \mathbb{A}^1$ be a regular funciton in the sense of the lemma. Let $p \in U$ be any point. Then there are polynomials f, g such that $g(q) \neq 0$ and $\varphi(q) = \frac{f(q)}{g(q)}$ for all points q in some neighborhood V of p. We claim that $\frac{f}{g} \in K(X)$ is the element in the ring of regular functions that we seek.

In fact, all we have to show is that this element does not depend on the choices that we made. So let $p' \in U$ be another point (not necessarily distinct from p), and suppose that there are polynomials f', g' such that $\frac{f}{g} = \frac{f'}{g'}$ on some neighborhood V' of p'. Then $fg' = gf'$ on $V \cap V'$ and hence on X as $V \cap V'$ is dense in X (every open set is dense in an irreducible topological space). In other words, $fg' - gf' \in I(X)$, so it is zero in $A(X)$, i.e. $\frac{f}{g} = \frac{f'}{g'} \in K(X)$. □

Proposition 1.1. *Let $X \subset \mathbb{A}^n$ be an affine variety and let $f \in A(X)$. Then*

$$O_X(D(f)) = A(X)_f := \left\{ \frac{g}{f^r} \mid g \in A(X) \text{ and } r \geq 0 \right\}.$$

In particular, $O_X(X) = A(X)$, i.e. any regular function on X is polynomial (take $f = 1$).

Proof. It is obvious that $A(X)_f \subset O_X(D(f))$, so let us prove the converse. Let $\varphi \in O_X(D(f)) \subset K(X)$. Let $J = \{g \in A(X) \mid g\varphi \in A(X)\}$. This is an ideal in $A(X)$; we want to show that $f^r \in J$ for some r, for then $f^r\varphi = h$ for some $h \in A(X)$, and we'd have $\varphi = \frac{h}{f^r}$. The reason we work with ideals is because our argument will use Nullstellensatz which is a theorem about ideals.

For any $p \in D(f)$, we know that $\varphi \in O_{X,p}$, so $\varphi = \frac{h}{g}$ with $g \neq 0$ in a neighborhood of p. In particular, $g \in J$, so J contains an element not vanishing at p. Since p is arbitrary, this means that $\mathbf{V}(\widetilde{J}) \subset \mathbf{V}(\mathbf{I}(X) + \widetilde{f})$, where $\widetilde{f} \in K[x_1, \dots, x_n]$ is a representative of f and $\widetilde{J}$ is the inverse image of J under the natural quotient map $K[x_1, \dots, x_n] \to A(X)$. It follows that $\mathbf{I}(\mathbf{V}(\mathbf{I}(X) + \widetilde{f})) \subset \mathbf{I}(\mathbf{V}(\widetilde{J}))$, which implies $\widetilde{f} \in \mathbf{I}(\mathbf{V}(\widetilde{J}))$. Therefore $\widetilde{f}^r \in \widetilde{J}$ for some r by Nullstellensatz, and so $f^r \in J$. □

Remark. Note that we needed to use Nullstellensatz here. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1(\mathbb{R})$, but not polynomial.

Example 1.1. Probably the easiest case of an open subset of an affine variety X that is not of the form $D(f)$ is the complement $U = \mathbb{C}^2 \backslash \{0\}$ of the origin in the affine plane. Let us compute $O_{\mathbb{C}^2}(U)$. By definition, any element $\varphi \in O_{\mathbb{C}^2}(U) \subset \mathbb{C}(x,y)$ is *globally* the quotient $\varphi = \frac{f}{g}$ of two *polynomials* $f, g \in \mathbb{C}[x,y]$. The condition we have to satisfy is that $g(a,b) \neq 0$ for all $(a,b) \neq (0,0)$. To understand why this is the case, pick $(a,b) \neq (0,0)$ such that $g(a,b) = 0$. Let's be sure that there is not another representation for φ: Suppose $\varphi = \frac{p}{q}$ for two other polynomials $p, q \in \mathbb{C}[x,y]$ such that $q(a,b) \neq 0$. Since $\frac{f}{g} = \frac{p}{q}$, we have $f(a,b)q(a,b) = p(a,b)g(a,b) = 0$. Since $q(a,b) \neq 0$ and K is field, we must therefore have $f(a,b) = 0$.

So what we've shown here is that in order for $\frac{f}{g} \in O_{\mathbb{C}^2,(a,b)}$, we need either $g(a,b) \neq 0$ or if $g(a,b) = 0$ then $f(a,b) = 0$. to make sense, we'd need $\mathbf{V}(g) \subset \mathbf{V}(f)$. $g(a,b) = 0$

$$
\begin{aligned}
f(a,b)q(a,b) &= p(a,b)g(a,b) \\
\frac{f}{g} &= \frac{p}{q},
\end{aligned}
$$

which implies .

We claim that this implies that g is constant.

We know already by the Nullstellensatz that there is no non-constant polynomial that has empty zero locus in $\mathbb{C}^2$, so we can assume that g vanishes on $(0,0)$. If we write g as

$$g(x,y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n,$$

this means that $f_0(0) = 0$. We claim that $f_0(x)$ must be of the form x^m for some m. In fact:

- If f_0 is the zero polynomial, the $g(x,y)$ contains y as a factor and hence the whole x-axis in its zero locus,
- If f_0 contains more than one monomial, f_0 has a zero $x_0 \neq 0$, and hence $g(x_0, 0) = 0$.

So $g(x,y)$ is of the form

$$g(x,y) = x^m + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n.$$

Now set $y = \varepsilon$ for some small ε. As $g(x,0) = x^m$ and all f_i are continuous, the restriction $g(x,\varepsilon)$ cannot be the zero or a constant polynomial. Hence $g(x,\varepsilon)$ vanishes for some x, which is a contradiction.

Let S be an integral domain. Suppose that g_1, g_2 form a regular sequence in S. This means that the map from S to S given by multiplication by g_1 and the map from S/g_1 to S/g_1 given by multiplication by $\overline{g}_2$ are both injective. Now let f_1, f_2 be two elements in S such that $\frac{g_1}{g_2} = \frac{f_1}{f_2}$. Then this implies $f_1 g_2 = f_2 g_1$. Since g_1, g_2 form a regular sequence in S, we must have $f_1 = hg_1$ for some $h \in S$. Hence,

$$
\begin{aligned}
0 &= f_1 g_2 - f_2 g_1 \\
&= hg_1 g_2 - f_2 g_1 \\
&= g_1(hg_2 - f_2),
\end{aligned}
$$

which implies $f_2 = hg_2$. Therefore

$$\frac{g_1}{g_2} = \frac{f_1}{f_2} = \frac{hg_1}{hg_2}.$$

So what we've shown is that if g_1, g_2 form a regular sequence in S, then there are no nontrivial ways of writing $\frac{g_1}{g_2}$: they are all of the form $\frac{hg_1}{hg_2}$ where $h \in S$.

When does g_1, g_2 fail to form a regular sequence in S? Well, S is an integral domain, so the map from S to S given by multiplication of g_1 is injective. The only place where it can fail is if the map from S/g_1 to S/g_1 given by multiplication of $\overline{g}_2$ is not injective. This means that there exists f_1, f_2 in S such that $f_1 g_2 = f_2 g_1$ and f_1 is not of the form hg_1 for some $h \in S$. At the same time, this also implies that f_1, f_2 fails to form a regular sequence in S as well. This pair (f_1, f_2) represents a nontrivial element in $H_1(g_1, g_2; S)$, where $H_1(g_1, g_2; S)$ is the first homology in the Koszul complex $\mathcal{K}(g_1, g_2; S)$.•

$$\mathcal{K}(g_1, g_2; S)_\bullet : \qquad 0 \longrightarrow S \xrightarrow{\begin{pmatrix} g_2 \\ -g_1 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} g_1 & g_2 \end{pmatrix}} S \longrightarrow 0$$

Now let $I = \langle g_1, g_2 \rangle$. The maximal length r of an S-sequence $f_1, \dots, f_r \in I$ is called the I-**depth** of S. We denote this by $\operatorname{depth}(I, S)$. One can show that

$$\operatorname{depth}(I, M) = 2 - \sup\{i \mid H_i(g_1, g_2; S) \neq 0\}.$$

Since S is an integral domain, we must have $H_2(g_1, g_2; S) = 0$. Thus, if g_1, g_2 fail to form a regular sequence in S, then $\operatorname{depth}(I, M) = 1$.

For instance, let $K = \mathbb{Q}(\sqrt{-5})$ with ring of integers being $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. The sequence $2, 1 + \sqrt{-5}$ does not form a regular sequence in $\mathcal{O}_K$. Indeed, we have

$$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

and neither 2 divides $1 - \sqrt{-5}$ nor $1 + \sqrt{-5}$ divides 3.
Let $\mathfrak{p} = \langle 2, 1 + \sqrt{-5} \rangle$ and $\mathfrak{q} = \langle 3, 1 - \sqrt{-5} \rangle$.

$$\frac{2}{1 + \sqrt{-5}} = \frac{1 - \sqrt{-5}}{3}$$

2 Sheaves

We have seen that regular functions on affine varieties are defined in terms of local properties: they are set-theoretic functions that can locally be written as quotients of polynomials. Local constructions of function-like objects occur in many places in algebraic geometry (and also in many other "topological" fields of mathematics), so we should formalize the ideal of such objects.

Definition 2.1. A **presheaf** $\mathcal{F}$ of rings on a topological space X assigns to each open set U in X a ring $\mathcal{F}(U)$, and to every pair of nested open subsets $U \subset V$ of X, a ring homomorphism $\mathcal{F}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$, called the **restriction map**, such that

1. $\mathcal{F}(\emptyset) = 0$,
2. $\mathcal{F}_U^U$ is the identity map for all open sets U in X,
3. $\mathcal{F}_U^V \circ \mathcal{F}_V^W = \mathcal{F}_U^W$ for all open sets $U \subset V \subset W$ in X.

The elements $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U; elements of $\mathcal{F}(X)$ are called **global sections**. The restriction maps $\mathcal{F}_U^V$ are written as $f \mapsto f\mid_U$. A presheaf $\mathcal{F}$ of rings is called a **sheaf** of rings if it satisfies the following glueing property: if $U \subset X$ is an open set, $\{U_i\}$ an open cover of U ,and $f_i \in \mathcal{F}(U_i)$ are sections for all i such that $f_i \mid_{U_i \cap U_j} = f_j \mid_{U_i \cap U_j}$ for all i, j, then there is a unique $f \in \mathcal{F}(U)$ such that $f_{|U_i} = f_i$ for all i.

2.0.1 Presheaves that are not Sheaves

Example 2.1. Let $\mathcal{F}$ be the presheaf on $\mathbb{R}$ defined as follows:

$$\mathcal{F}(U) = \begin{cases} \mathbb{Q} & \text{if} U = \mathbb{R} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

locally, this presheaf is just $\{0\}$, so where did this extra element 1 come from? It can't be a sheaf. It would, however, be a sheaf if we got rid of the element 1.

Example 2.2. Let F be the presheaf of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Why is this not a sheaf? Hint: Try gluing a bunch of continuous bounded functions to form a continuous unbounded function on $\mathbb{R}$. We can turn this thing into a sheaf if we adjoin continuous unbounded functions.

Note: Notice in the previous two examples we either needed to remove elements to turn a presheaf into a sheaf, or we needed to adjoin elements to turn a presheaf into a sheaf. This process of removing or adjoining elements of a presheaf to obtain a sheaf is known as **sheafification**.

Example 2.3. If $X \subset \mathbb{A}^n$ is an affine variety, then the rings $O_X(U)$ of regular functions on open subsets of X (with the obvious restriction maps $O_X(V) \to O_X(U)$ for $U \subset V$) form a sheaf of rings O_X, called the **sheaf of regular functions** or **structure sheaf** of X. In fact, all defining properties of presheaves are obvious, and the glueing property of sheaves is easily seen from the description of regular functions in Lemma (1.1).

Example 2.4. Here are some examples from other fields of Mathematics: Let $X = \mathbb{R}^n$, and for any open subset $U \subset X$, let $\mathcal{F}(U)$ be the ring of continuous functions on U. Together with the obvious restriction maps, these rings $\mathcal{F}(U)$ form a sheaf, the **sheaf of continuous functions**. In the same way we can define the sheaf of k times differentiable functions, analytic functions, holomorphic functions on $\mathbb{C}^n$, and so on. The same definitions can be applied if X is a real or complex manifold instead of just $\mathbb{R}^n$ or $\mathbb{C}^n$.

In all these examples, the sheaves just defined "are" precisely the functions that are considered to be morphisms in the corresponding category (for example, in complex analysis the morphisms are just the holomorphic maps). This is usually expressed in the following way: a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X is called a **ringed space**. The sheaf $\mathcal{F}$ is then called the structure sheaf of this ringed space. (Although being general, this terminology will usually only be applied if $\mathcal{F}$ actually has an interpretation as the space of functions that are considered to be morphisms in the corresponding category.)

Definition 2.2. Let X be a topological space, $p \in X$, and $\mathcal{F}$ a (pre-)sheaf on X. Consider pairs (U, φ) where U is an open neighborhood of p and $\varphi \in \mathcal{F}(U)$ a section of $\mathcal{F}$ over U. We call two such pairs (U, φ) and (U', φ') equivalent if there is an open neighborhood V of p with $V \subset U \cap U'$ such that $\varphi \mid_V = \varphi \mid_V$. The set of all such pairs modulo this equivalence relation is called the **stalk** $\mathcal{F}_p$ of $\mathcal{F}$ at p, its elements are called **germs** of $\mathcal{F}$.

Definition 2.3. Let (X, O_X) and (Y, O_Y) be ringed spaces whose structure sheaves O_X and O_Y are sheaves of K-valued functions. Let $f : X \to Y$ be a set-theoretic map.

1. If $\varphi : U \to k$ is a K-valued function on an open subset U of Y, the composition $\varphi \circ f$ from $f^{-1}(U)$ to K is again a set-theoretic function. It is denoted by $f^*\varphi$ and is called the **pull-back** of φ.
2. The map f is called a **morphism** if it is continuous, and if it pulls back regular functions to regular functions, i.e. if $f^*O_Y(U) \subset O_X(f^{-1}(U))$ for all open $U \subset Y$.

We now want to show that for affine varieties, the situation is a lot easier: we actually do not have to deal with open subsets, but it suffices to check the pull-back property on *global* functions only:

Lemma 2.1. *Let $f : X \to Y$ be a continuous map between affine varieties. Then the following are equivalent.*

1. *f is a morphism, i.e. f pulls back regular functions on open subsets to regular functions on open subsets.*
2. *For every $\varphi \in O_Y(Y)$ we have $f^*\varphi \in O_X(X)$, i.e. f pulls back* global *regular functions to* global *regular functions.*
3. *For every $p \in X$ and every $\varphi \in O_{Y,f(p)}$, we have $f^*\varphi \in O_{X,p}$, i.e. f pulls back germs of regular functions to germs of regular functions.*

Proof.

(1) $\Longrightarrow$ (2): Trivial

(3) $\Longrightarrow$ (1): Follows immediately from the definition of $O_Y(U)$ and $O_X(f^{-1}(U))$ as intersections of local rings.

(2) $\Longrightarrow$ (3): Let $\varphi \in O_{Y,f(p)}$ be the germ of a regular function on Y. We write $\varphi = \frac{g}{h}$ with $g, h \in A(Y) = O_Y(Y)$ and $h(f(p)) \neq 0$. By (2), f^*g and f^*h are global regular functions in $A(X) = O_X(X)$, hence $f^*\varphi = \frac{f^*g}{f^*h} \in O_{X,p}$, since we have $h(f(p)) \neq 0$.

□

Example 2.5. Let $X = \mathbb{A}^1$ be the affine line with coordinate x, and let $Y = \mathbb{A}^1$ be the affine line with coordinate y. Consider the set-theoretic map

$$f : X \to Y, \qquad x \mapsto y = x^2.$$

We claim that this is a morphism. In fact, by Lemma (2.1) we just need to show that f pulls back polynomials in $k[y]$ to polynomials in $k[x]$. But this is obvious, as the pull-back of a polynomial $\varphi(y) \in k[y]$ is just $\varphi(x^2)$ (i.e. we substitute x^2 for y in φ). This is still a polynomial, so it is in $k[x]$.

2.1 Polynomial Maps

Definition 2.4. Let $X \subseteq \mathbb{A}_K^m$ and $Y \subseteq \mathbb{A}_K^n$ be affine varieties. A function $f : X \to Y$ is said to be a **polynomial mapping** (or **regular mapping**) if there exist polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_m]$ such that

$$f(a_1, \ldots, a_m) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in X$. We say that the n-tuple of polynomials

$$(f_1, \ldots, f_n) \in (K[x_1, \ldots, x_m])^n$$

represents f. The f_i are the **components** of this representation.

Remark.

1. To say that f is a polynomial mapping from $X \subseteq \mathbb{A}_K^m$ to $Y \subseteq \mathbb{A}_K^n$ represented by $(f_1, \ldots, f_n)$ means that $(f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m))$ must satisfy the defining equations of Y for all points $(a_1, \ldots, a_m) \in X$.
2. Of particular interest is the case $Y = \mathbb{A}_K^1$, where f simply becomes a scalar **polynomial function** defined on X.
3. A polynomial map from X to Y is continuous with respect to the Zariski topology. Indeed, if $D(g)$ is a basic open set in Y, then $f^{-1}(D(g)) = D(f^*g)$. Since f is continuous and pulls back polynomials to polynomials, Lemma (2.1) implies that f is a morphism.

Let's go over some examples of polynomial mappings. We typically use x, y, z and u, v, w to denote coordinate functions, a, b, c and α, β, γ to denote numbers, and p, q to denote points.

Example 2.6. Let $X = \mathbf{V}(y - x^2, z - x^3)$ be the twisted cubic in $\mathbb{R}^3$. Then

$$X = \left\{ (a, b, c) \in K^3 \mid b = a^2 \text{ and } c = a^3 \right\}.$$

Now let $Y = \mathbf{V}(v - u - u^2)$ in $\mathbb{R}^2$. Then

$$Y = \left\{ (\alpha, \beta) \in K^2 \mid \beta = \alpha + \alpha^2 \right\}$$

We want to describe a polynomial map from X to Y. We claim that $f : X \to Y$ be given by mapping $(a, b, c) \in X$ to $(ab, a^2b^2 + c) := (\alpha, \beta) \in Y$ is such a map. We need to make sure that this is well-defined. In particular, we need to check that the point (α, β) really does belong to Y. We verify this by showing that the point satisfies the defining equation of Y: Indeed, we have $c - ab = 0$, since $(a, b, c) \in X$ and $z - xy \in \langle y - x^2, z - x^3 \rangle$. Therefore

$$\begin{aligned}
(v - u - u^2)(\alpha, \beta) &= \alpha - \beta - \beta^2 \\
&= (c + a^2b^2) - (ab) - (ab)^2 \\
&= c - ab \\
&= 0.
\end{aligned}$$

Thus f is indeed a polynomial map from X to Y. The map f is represented by (f_1, f_2), where

$$\begin{aligned}
f_1 &= xy \\
f_2 &= x^2y^2 + z.
\end{aligned}$$

Observe that the map f induces a map

$$f^* : K[u, v] / \langle v - u - u^2 \rangle \to K[x, y, z] / \langle y - x^2, z - x^3 \rangle,$$

where f^* is the K-algebra map induced by mapping

$$\begin{aligned}
u &\mapsto xy \\
v &\mapsto x^2y^2 + z
\end{aligned}$$

We give some more examples of morphisms of affine algebraic sets

Example 2.7.

1. The map $\mathbb{A}^1(k) \to V(T_2 - T_1^2) \subset \mathbb{A}^2(k)$, given by $x \mapsto (x^2, x)$, is a morphism of affine algebraic sets. It is even an isomorphism with inverse morphism $(x, y) \mapsto y$.

2. The map $\mathbb{A}^1(k) \to V(T_2^2 - T_1^2(T_1 + 1))$, given by $x \mapsto (x^2 - 1, x(x^2 - 1))$, is a morphism of affine algebraic sets. For $\operatorname{char}(k) \neq 2$, it is not bijective: 1 and -1 are both mapped to the origin $(0, 0)$. In $\operatorname{char}(k) = 2$, it is bijective but not an isomorphism.

3. We identify the space $M_n(k)$ of $(n \times n)$-matrices with $\mathbb{A}^{n^2}(k)$, thus giving $M_n(k)$ the structure of an affine algebraic set. Then sending a matrix $A \in M_n(k)$ to its determinant $\det(A)$ is a morphism $M_n(k) \to \mathbb{A}^1(k)$ of affine algebraic sets.

4. For $k = \mathbb{C}$, consider the exponential function $\exp : \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$. This is *not* a morphism of algebraic sets. Indeed, this map is not even continuous with respect to the Zariski topology: The preimage of 1 is $\{2\pi i n \mid n \in \mathbb{Z}\}$, which is not closed in the Zariski topology.

Proposition 2.1. *Let $X \subseteq \mathbb{A}^m_K$ be an affine variety. Then*

1. *f and g in $K[x_1, \ldots, x_m]$ represent the same polynomial function on X if and only if $f - g \in \mathbf{I}(X)$.*

2. *$(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ represent the same polynomial mapping from X to $\mathbb{A}^n_K$ if and only if $f_i - g_i \in \mathbf{I}(V)$ for each i.*

Proof.

1. If $f - g = h \in \mathbf{I}(V)$, then for any point $p = (a_1, \ldots, a_m) \in V$, we have $f(p) - g(p) = h(p) = 0$. Hence, f and g represent the same function on V. Conversely, if f and g represent the same function, then, at every $p \in V$, we have $f(p) - g(p) = 0$. Thus, $f - g \in \mathbf{I}(V)$ by definition.

2. Follows directly from (1).

$\square$

Lemma 2.2. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. There is a one-to-one correspondence between morphisms $f : X \to Y$ and K-algebra homomorphisms $f^* : A(Y) \to A(X)$.*

Proof. Any morphism $f : X \to Y$ determines a K-algebra homomorphism $f^* : O_Y(Y) = A(Y) \to O_X(X) = A(X)$ by definition. Conversely, if

$$\varphi : K[y_1, \ldots, y_m]/\mathbf{I}(Y) \to K[x_1, \ldots, x_n]/\mathbf{I}(X)$$

is any K-algebra homomorphism, then it determines a polynomial map $f = (f_1, \ldots, f_m) : X \to Y$ by $f_i = \varphi(y_i)$, and hence a morphism. $\square$

Definition 2.5. An **isomorphism** is defined to be a morphism $f : X \to Y$ that has an inverse, i.e. a morphism $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Warning: An isomorphism of affine varieties is *not* the same as a bijective morphism. For example, let $X \subset \mathbb{A}^2$ be the curve given by the equation $x^2 = y^3$ and consider the map

$$f : \mathbb{A}^1 \to X \qquad t \mapsto (t^3, t^2).$$

This is a morphism as it is given by polynomials, and it is bijective as the inverse is given by

$$f^{-1} : X \to \mathbb{A}^1 \qquad (x, y) \mapsto \begin{cases} \frac{x}{y} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

But if f was an isomorphism, the corresponding K-algebra homomorphism

$$K[x, y]/\langle x^2 - y^3 \rangle \to K[t] \qquad x \mapsto t^3 \text{ and } y \mapsto t^2$$

would have to be an isomorphism. This is obviously not the case, as the image of this algebra homomorphism contains no linear polynomials.

2.1.1 Products of Affine Varieties

Definition 2.6. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties with ideals $\mathbf{I}(X) \subset K[x_1, \ldots, x_n]$ and $\mathbf{I}(Y) \subset K[y_1, \ldots, y_m]$. We define the **product** $X \times Y$ of X and Y to be the set

$$X \times Y = \{(p, q) \in \mathbb{A}^n \times \mathbb{A}^m \mid p \in X \text{ and } q \in Y\} \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}.$$

Obviously, this is an algebraic set in $\mathbb{A}^{n+m}$, with ideal

$$\mathbf{I}(X \times Y) = \mathbf{I}(X) + \mathbf{I}(Y) \subset K[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

Lemma 2.3. *If X and Y are affine varieties, then so is $X \times Y$.*

Proof. For simplicity, let us just write x for the collection of the x_i, and y for the collection of the y_i. We need to show that $\mathbf{I}(X \times Y)$ is prime. So let $f, g \in K[x, y]$ be polynomial functions such that $fg \in \mathbf{I}(X \times Y)$; we have to show that either f or g vanishes on all of $X \times Y$, i.e. that $X \times Y \subset \mathbf{V}(f)$ or $X \times Y \subset \mathbf{V}(g)$.

So let us assume that $X \times Y \not\subset \mathbf{V}(f)$, i.e. there is a point $(p, q) \in (X \times Y) \backslash \mathbf{V}(f)$. Denote by $f(\cdot, q) \in K[x]$ the polynomial obtained from $f \in K[x, y]$ by plugging in the coordinates of q for y. First, we will show that we can find an open set U in X such that $U \times Y \subset \mathbf{V}(g)$. In fact, this open set U will just be $X \backslash \mathbf{V}(f(\cdot, q))$: For all $p' \in X \backslash \mathbf{V}(f(\cdot, q))$ we must have

$$Y \subset \mathbf{V}(f(p', \cdot)g(p', \cdot)) = \mathbf{V}(f(p', \cdot)) \cup \mathbf{V}(g(p', \cdot)).$$

As Y is irreducible and $Y \not\subset \mathbf{V}(f(p', \cdot))$ by choice of p', it follows that $Y \subset \mathbf{V}(g(p', \cdot))$.

This is true for all $p' \in X \backslash \mathbf{V}(f(\cdot, q))$ by the choice of p', so we conclude that $(X \backslash \mathbf{V}(f(\cdot, q)) \times Y \subset \mathbf{V}(g)$. But as $\mathbf{V}(g)$ is closed, it must in fact contain the closure of $(X \backslash \mathbf{V}(f(\cdot, q)) \times Y$ as well, which is just $X \times Y$ as X is irreducible and $X \backslash \mathbf{V}(f(\cdot, q))$ a non-empty open subset of X. □

The obvious projection maps $\pi_X : X \times Y \to X$, where $(p, q) \mapsto p$, and $\pi_Y : X \times Y \to Y$, where $(p, q) \mapsto q$, are given by (trivial) polynomial maps and are therefore morphisms. The important main property of the product $X \times Y$ is the following:

Lemma 2.4. *Let X and Y be affine varieties. Then the product $X \times Y$ satisfies the following universal property: for every affine variety Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there is a unique morphism $h : Z \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.*

Remark. In other words, giving a morphism $Z \to X \times Y$ is "the same" as giving two morphisms $Z \to X$ and $Z \to Y$.

Proof. Let A be the coordinate ring of Z. Then by Lemma (2.2), the morphism $f : Z \to X$ is given by a K-algebra homomorphism $f^* : K[x_1, \ldots, x_n]/\mathbf{I}(X) \to A$. This in turn is determined by giving the images $f_i := f^*(x_i)$ of the generators x_i, satisfying the relations of I. The same is true for g, which is determined by the images $g_i := g^*(y_i) \in A$.

Now it is obvious that the elements f_i and g_i determine a K-algebra homomorphism

$$K[x_1, \ldots, x_n, y_1, \ldots, y_m]/(\mathbf{I}(X) + \mathbf{I}(Y)) \to A,$$

which determines a morphism $h : Z \to X \times Y$.

To show uniqueness, just note that the relations $f = \pi_X \circ h$ and $g = \pi_Y \circ h$ imply immediately that h must be given set-theoretically by $h(p) = (f(p), g(p))$ for all $p \in Z$. □

Lemma (2.2) allows us to associate an affine variety up to isomorphism to any finitely generated K-algebra that is a domain: if A is such an algebra, let $x_1, \ldots, x_n$ be generators of A, so that $A = K[x_1, \ldots, x_n]/I$ for some ideal I. Let X be the affine variety in $\mathbb{A}^n$ defined by the ideal I; by the lemma it is defined up to isomorphism. Therefore we should make a (very minor) redefinition of the term "affine variety" to allow for objects that are isomorphic to an affine variety in the old sense, but that do not come with an intrinsic description as the zero locus of some polynomials in affine space:

Definition 2.7. A ringed space (X, O_X) is called an **affine variety** over K if

1. X is irreducible,

2. O_X is a sheaf of K-valued functions,

3. X is isomorphic to an affine variety.

Here is an example of an affine variety in this new sense, although it is not a priori given as the zero locus of some polynomials in affine space:

Lemma 2.5. *Let X be an affine variety and $f \in A(X)$. Then the ringed space $(D(f), O_X\,|_{D(f)})$ is isomorphic to an affine variety with coordinate ring $A(X)_f$.*

Proof. Let $X \subset \mathbb{A}^n$ be an affine variety and let $\widetilde{f} \in K[x_1, \ldots, x_n]$ be a representative of f. Let $J \subset K[x_1, \ldots, x_n, t]$ be the ideal generated by $\mathbf{I}(X)$ and the function $1 - t\widetilde{f}$. We claim that the ringed space $(D(f), O_X\,|_{D(f)})$ is isomorphic to the affine variety

$$\mathbf{V}(J) = \left\{ \left(p, \frac{1}{\widetilde{f}(p)} \right) \mid p \in X \right\} \subset \mathbb{A}^{n+1}.$$

Consider the projection map $\pi : \mathbf{V}(J) \to X$ given by $\pi(p, \lambda) = p$. This is a morphism with image $D(f)$ and inverse morphism $\pi^{-1}(p) = \left(p, \frac{1}{\widetilde{f}(p)} \right)$, hence π is an isomorphism. It is obvious that $A(\mathbf{V}(J)) = A(X)_f$. □

2.2 Prevarieties

We have just seen that even open subsets of affine varieties are themselves affine varities, provided that the open subset is the complement of the zero locus of a single polynomial equation.

On the other hand, Example (1.1) shows that not all open subsets of affine varieties are themselves isomorphic to affine varieties. Indeed, if $U = \mathbb{C}^2 \backslash \{0\}$, then we've seen that $O_U(U) = K[x, y]$. So if U was an affine variety, its coordinate ring must be $K[x, y]$, which is the same as the coordinate ring of $\mathbb{C}^2$. By Lemma (2.2) this means that U and $\mathbb{C}^2$ would have to be isomorphic, with the isomorphism given by the identity map. Obviously, this is not true. Hence U is not an affine variety. It can however be covered by two open subsets $D(x)$ and $D(y)$, which are both affine by Lemma (2.5). This leads us to the idea of **patching** affine varieties together.

Every open subset of an affine variety can be written as a finite union of distinguished open subsets. Indeed, every closed subset has the form $\mathbf{V}(f_1, \ldots, f_n) = \mathbf{V}(f_1) \cap \cdots \cap \mathbf{V}(f_n)$, and so every open subset has the form $D(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n)$. This leads us to the idea that we should study objects that are not affine varieties themselves, but rather can be covered by (finitely many) affine varieties.

Definition 2.8. A **prevariety** is a ringed space (X, O_X) such that

1. X is irreducible,

2. O_X is a sheaf of K-valued functions,

3. there is a finite open cover $\{U_i\}$ of X such that $(U_i, O_X\,|_{U_i})$ is an affine variety for all i.

An open subset $U \subset X$ of a prevariety such that $(U, O_X\,|_U)$ is isomorphic to an affine variety is called an **affine open set**.

Remark. As before, a morphism o prevarieties is just a morphism as ringed spaces.

The most general way to construct prevarieties is to take some affine varieties and patch them together:

Example 2.8. Let X_1 and X_2 be prevarieties, $U_1 \subset X_1$ and $U_2 \subset X_2$ be non-empty open subsets, and let $f : (U_1, O_{X_1}\,|_{U_1}) \to (U_2, O_{X_2}\,|_{U_2})$ be an isomorphism. Then we can define a prevariety X, obtained by **gluing** X_1 and X_2 along U_1 and U_2 via the isomorphism φ:

- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation $p \sim \varphi(p)$ for all $p \in U_1$.

- As a topological space, we endow X with the so-called **quotient topology** induced by the above equivalence relation, i.e. we say that a subset $U \subset X$ is open if and only if $i_1^{-1}(U) \subset X_1$ and $i_2^{-1}(U) \subset X_2$ are both open, with $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ being the obvious inclusion maps.

- As a ringed space, we define the structure sheaf O_X by

$$O_X(U) = \{(\varphi_1, \varphi_2) \mid \varphi_1 \in O_{X_1}(U \cap X_1),\ \varphi_2 \in O_{X_2}(U \cap X_2), \text{ and } \varphi_1 = \varphi_2 \text{ on the overlap (i.e. } f^*(\varphi_2 \mid_{i_2^{-1}(U)\cap U_2}) = \varphi_1 \mid_{i_1^{-1}(U)\cap U_1})\}$$

Example 2.9. Let $X_1 = X_2 = \mathbb{A}^1$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $x \mapsto \frac{1}{x}$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $K = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$.
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X.

Example 2.10. Let X be the complex affine curve

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2 \widetilde{x}^4 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x, y) \mapsto (\widetilde{x}, \widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right)$$

$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x}, \widetilde{y}) \mapsto (x, y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^n}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

Example 2.11. Let $f : \mathbb{P}^1 \to \mathbb{A}^1$ be a morphism. We claim that f must be constant. In fact, consider the restriction $f\mid_{\mathbb{A}^1}$ of f to the open affine subset $\mathbb{A}^1 \subset \mathbb{P}^1$. By definition, the restriction of a morphism is again a morphism, so $f\mid_{\mathbb{A}^1}$ must be of the form $x \mapsto y = p(x)$ for some polynomial $p \in K[x]$. Now consider the second patch of $\mathbb{P}^1$ with coordinate $\widetilde{x} = \frac{1}{x}$. Applying this coordinate change, we see that $f\mid_{\mathbb{P}^1\backslash\{0\}}$ is given by $\widetilde{x} \mapsto p\left(\frac{1}{\widetilde{x}}\right)$. But this must be a morphism too, i.e. $p\left(\frac{1}{\widetilde{x}}\right)$ must be a polynomial in $\widetilde{x}$. This is only tru if p is a constant.