

# List of Schemes

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## Part I

# List of Algebraic Varieties

## 1 A Quartic Curve

Let  $A = \mathbb{Z}[x, y]/f$  where

$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \quad (1)$$

where we set  $u = y - x^2 + 5x - 5$  and  $v = y + x^2 - 5x + 5$ . Note that from the expression of  $f$  in (1) we see that  $u$  and  $v$  are units in  $A$ . Here we are describing  $A$  as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as  $A = \mathbb{Z}[y][\sqrt{g}]$  where

$$f = y^2 - (x - 1)(x - 2)(x - 3)(x - 4) = y^2 - g, \quad (2)$$

where  $g = (x - 1)(x - 2)(x - 3)(x - 4)$ . The expression of  $f$  in (2) is nice because we can read off information like the discriminant of  $A$  over  $\mathbb{Z}[y]$ . Basically from (2) we can read off useful information of  $A$  viewed as a finite module extension, whereas from (1) we can read off useful information of  $A$  viewed as a quotient. Both expressions give rise to the same ring  $A$  at the end of the day. Next we set  $X = \text{Spec } A$ . To get an idea of what  $X$  looks like, we consider the canonical morphism  $X \rightarrow \text{Spec } \mathbb{Z}$ . For each positive prime  $p$ , we obtain the fiber  $X_p = X_{\mathbb{F}_p}$  of this canonical morphism at the prime ideal  $\langle p \rangle$ :

$$X_p = \text{Spec}(\mathbb{F}_p \otimes_{\mathbb{Z}} A) = \text{Spec}(\mathbb{F}_p[x, y]/f).$$

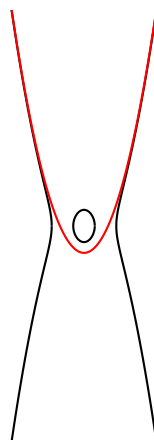
We also obtain the fiber  $X_0 = X_{\mathbb{Q}}$  of this canonical morphism at the generic point  $\langle 0 \rangle$ :

$$X_{\mathbb{Q}} = \text{Spec}(\mathbb{Q} \otimes_{\mathbb{Z}} A) = \text{Spec}(\mathbb{Q}[x, y]/f).$$

Note  $X_{\mathbb{Q}}$  is just the pullback of the morphism  $\text{Spec } \mathbb{Q} \rightarrow \text{Spec } \mathbb{Z}$  with respect to the canonical map  $X \rightarrow \text{Spec } \mathbb{Z}$ . We can specialize even further by setting  $X_K$  to be the pullback of the composite  $\text{Spec } K \rightarrow \text{Spec } \mathbb{Q} \rightarrow \text{Spec } \mathbb{Z}$  with respect to the canonical map  $X \rightarrow \text{Spec } \mathbb{Z}$ , where  $K/\mathbb{Q}$  is some field extension:

$$X_K = \text{Spec}(K \otimes_{\mathbb{Z}} A) = \text{Spec}(K[x, y]/f).$$

The closed points of  $X_K$  correspond to the maximal ideals of  $K[x, y]/f$ , and when  $K$  is algebraically closed, these correspond to the points of the variety  $V_K(f)$ . Note in general we have  $X_K(K) = X(K) \times \text{Aut } K$ , thus in particular  $X_{\mathbb{R}}(\mathbb{R}) = X(\mathbb{R}) = C$  where we can view  $C$  as the black curve below:



The thick black curve is  $C$  whereas the thick red curve is  $V_{\mathbb{R}}(u) = D$ . The closed points of  $X_{\mathbb{R}}$  correspond to the points of  $C$ : they have the form  $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$  where  $(a, b) \in \mathbb{R}^2$  such that  $f(a, b) = 0$  (i.e. such that  $(a, b) \in C$ ). There's also the generic point  $\eta \in X_{\mathbb{R}}$  corresponding to the  $0$  ideal, however this doesn't correspond to any point of  $C$ . Notice that  $C$  and  $D$  do not intersect: this is because  $u$  is a unit in  $A$  (and hence a unit in  $\mathbb{R} \otimes_{\mathbb{Z}} A$ ). The point is that  $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$  for all  $\mathfrak{p} \in X_{\mathbb{R}}$ .

If we equip  $X(\mathbb{R})$  with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology,  $X(\mathbb{R})$  is irreducible since  $f$  is irreducible over  $\mathbb{R}$ , so certainly  $X(\mathbb{R})$  is connected in the Zariski topology. The Jacobian matrix of  $f$  is given by

$$J_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that  $J_f(a, b) := J_f \bmod \mathfrak{m}_{a,b} \neq 0$  for all closed points  $\mathfrak{m}_{a,b} \in X(\mathbb{R})$ . It follows that  $X(\mathbb{R})$  is smooth. Alternatively, we have

$$df = \partial_x f dx + \partial_y f dy = (4x^3 + 30x^2 - 70x + 50)dx + 2ydy.$$

Thus when we set  $df = 0$ , then for  $y \neq 0$ , we have

$$\frac{dy}{dx} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \quad (3)$$

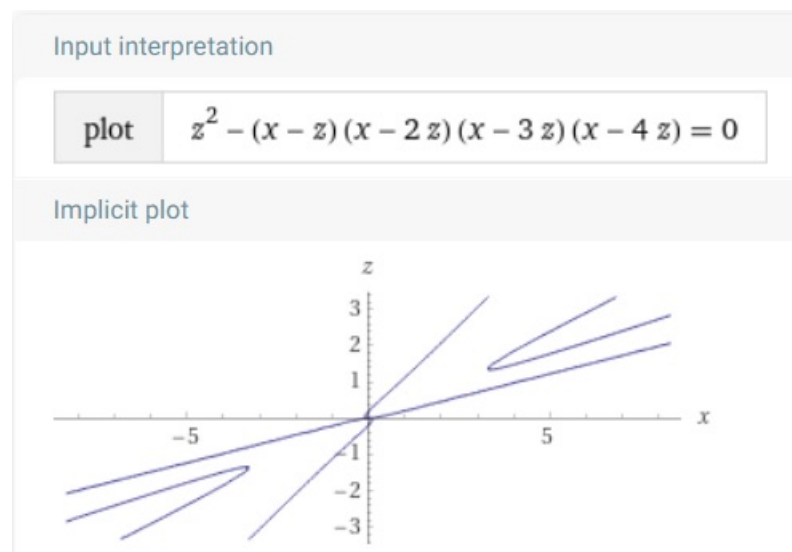
The DeRham complex of  $A$  is given by

$$\Omega_A := 0 \rightarrow A \rightarrow$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity  $[0 : 1 : 0]$ . To do this let  $\tilde{A} = \mathbb{Z}[x, z]/\tilde{f}$  where

$$\tilde{f} = \tilde{y}^2 - (\tilde{x} - \tilde{y})(\tilde{x} - 2\tilde{y})(x - 3\tilde{y})(x - 4\tilde{y}), \quad (4)$$

and set  $\tilde{X} = \text{Spec } \tilde{A}$ . To get an idea of what  $\tilde{X}_{\mathbb{R}}$  looks like, we shall look at its  $\mathbb{R}$ -valued points  $\tilde{X}_{\mathbb{R}}(\mathbb{R}) = V_{\mathbb{R}}(\tilde{f}) = \tilde{C}$  pictured below



The closed points of  $\tilde{X}_{\mathbb{R}}$  have the form  $\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}} = \langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle$  where  $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$  such that  $\tilde{f}(\tilde{a}, \tilde{b}) = 0$ . We have a ring isomorphism  $\tilde{\varphi}: \tilde{A}_{\tilde{y}} \rightarrow A_y$  given by  $\tilde{\varphi}(\tilde{x}) = x/y$  and  $\tilde{\varphi}(\tilde{y}) = 1/y$ , with inverse  $\varphi: A_y \rightarrow \tilde{A}_{\tilde{y}}$  given by  $\varphi(x) = \tilde{x}/\tilde{y}$  and  $\varphi(y) = 1/\tilde{y}$ . Notice that

$$\begin{aligned} \tilde{\varphi}(\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}) &= \varphi(\langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle) \\ &= \langle x/y - \tilde{a}, 1/y - \tilde{b} \rangle \\ &= \langle x - \tilde{a}y, 1 - \tilde{b}y \rangle \\ &= \langle x - \tilde{a}y, y - 1/\tilde{b} \rangle \\ &= \langle x - \tilde{a}/\tilde{b}, y - 1/\tilde{b} \rangle \\ &= \langle x - a, y - b \rangle \\ &= \mathfrak{m}_{a,b}, \end{aligned}$$

where we set  $a = \tilde{a}/\tilde{b}$  and  $b = 1/\tilde{b}$ . It follows that  ${}^a\tilde{\varphi}(\mathfrak{m}_{a,b}) = \tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}$ . Now observe that

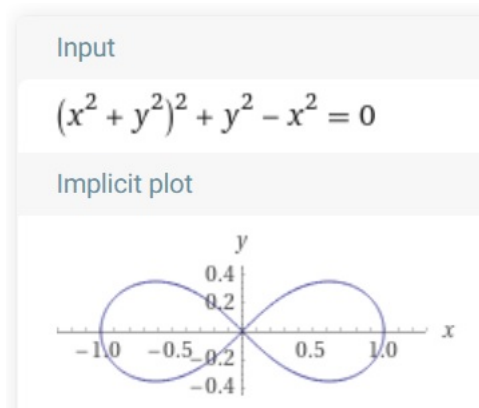
$$d\tilde{x} = \frac{ydx - xdy}{y^2} \quad \text{and} \quad d\tilde{y} = -\frac{dy}{y^2}.$$

## 2 The Lemniscate of Bernoulli

Let  $A = \mathbb{Z}[x, y]/f$  where

$$f = (x^2 + y^2)^2 + y^2 - x^2,$$

and we set  $X = \text{Spec } A$ . One can show that the set of integer solutions to the equation  $f = 0$  is given by  $\{(\pm 1, 0), (0, 0)\}$ . On the other hand, the  $\mathbb{R}$ -valued points  $X(\mathbb{R})$  can be visualized below



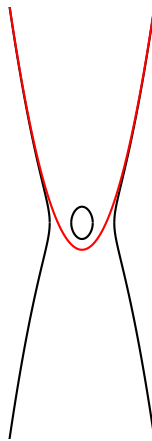
$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \quad (5)$$

where we set  $u = y - x^2 + 5x - 5$  and  $v = y + x^2 - 5x + 5$ . Note that from the expression of  $f$  in (1) we see that  $u$  and  $v$  are units in  $A$ . Here we are describing  $A$  as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as  $A = \mathbb{Z}[y][\sqrt{g}]$  where

$$f = y^2 - (x - 1)(x - 2)(x - 3)(x - 4) = y^2 - g, \quad (6)$$

where  $g = (x - 1)(x - 2)(x - 3)(x - 4)$ . The expression of  $f$  in (2) is nice because we can read off information like the discriminant of  $A$  over  $\mathbb{Z}[y]$ . Basically from (2) we can read off useful information of  $A$  viewed as a finite module extension, whereas from (1) we can read off useful information of  $A$  viewed as a quotient. Both expressions give rise to the same ring  $A$  at the end of the day.

Next we set  $X = \text{Spec } A$ . To get an idea of what  $X$  looks like, we first look at its  $\mathbb{R}$ -valued points:  $X(\mathbb{R}) = \text{Spec } \mathbb{R} \otimes_{\mathbb{Z}} A = \text{Spec } \mathbb{R}[x, y]/f$ . We can visualize the  $\mathbb{R}$ -valued points of  $X$  in the picture below:



The thick black curve is  $X(\mathbb{R}) = V_{\mathbb{R}}(f)$  whereas the thick red curve is  $V_{\mathbb{R}}(u)$ . Notice that  $V_{\mathbb{R}}(u)$  and  $X(\mathbb{R})$  do not intersect: this is because  $u$  is a unit in  $A$  (and hence a unit in  $\mathbb{R} \otimes_{\mathbb{Z}} A$ ). The point is that  $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$  for all  $\mathfrak{p} \in X$ . The closed points of  $X(\mathbb{R})$  have the form  $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$  where  $(a, b) \in \mathbb{R}^2$  such that  $f(a, b) = 0$ . There's also the generic point  $\eta \in X(\mathbb{R})$  corresponding to the 0 ideal.

If we equip  $X(\mathbb{R})$  with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology,  $X(\mathbb{R})$  is irreducible since  $f$  is irreducible over  $\mathbb{R}$ , so certainly  $X(\mathbb{R})$  is connected in the Zariski topology. The Jacobian matrix of  $f$  is given by

$$J_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that  $J_f(a, b) := J_f \bmod \mathfrak{m}_{a,b} \neq 0$  for all closed points  $\mathfrak{m}_{a,b} \in X(\mathbb{R})$ . It follows that  $X(\mathbb{R})$  is smooth. Alternatively, we have

$$df = \partial_x f dx + \partial_y f dy = (4x^3 + 30x^2 - 70x + 50)dx + 2ydy.$$

Thus when we set  $df = 0$ , then for  $y \neq 0$ , we have

$$\frac{dy}{dx} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \quad (7)$$

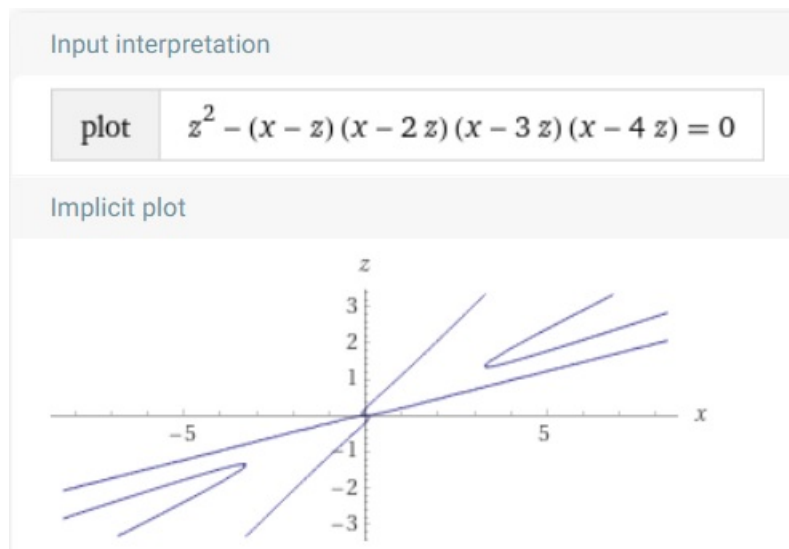
The DeRham complex of  $A$  is given by

$$\Omega_A := 0 \rightarrow A \rightarrow$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity  $[0 : 1 : 0]$ . To do this let  $\tilde{A} = \mathbb{Z}[x, z]/\tilde{f}$  where

$$\tilde{f} = \tilde{y}^2 - (\tilde{x} - \tilde{y})(\tilde{x} - 2\tilde{y})(x - 3\tilde{y})(x - 4\tilde{y}), \quad (8)$$

and set  $\tilde{X} = \text{Spec } \tilde{A}$ . We can visualize the  $\mathbb{R}$ -valued points of  $\tilde{X}$  in the picture below



The closed points of  $\tilde{X}(\mathbb{R})$  have the form  $\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}} = \langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle$  where  $(\tilde{a}, \tilde{b}) \in \mathbb{R}^2$  such that  $\tilde{f}(\tilde{a}, \tilde{b}) = 0$ . We have a ring isomorphism  $\tilde{\varphi}: \tilde{A}_{\tilde{y}} \rightarrow A_y$  given by  $\tilde{\varphi}(\tilde{x}) = x/y$  and  $\tilde{\varphi}(\tilde{y}) = 1/y$ , with inverse  $\varphi: A_y \rightarrow \tilde{A}_{\tilde{y}}$  given by  $\varphi(x) = \tilde{x}/\tilde{y}$  and  $\varphi(y) = 1/\tilde{y}$ . Notice that

$$\begin{aligned} \tilde{\varphi}(\tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}) &= \varphi(\langle \tilde{x} - \tilde{a}, \tilde{y} - \tilde{b} \rangle) \\ &= \langle x/y - \tilde{a}, 1/y - \tilde{b} \rangle \\ &= \langle x - \tilde{a}y, 1 - \tilde{b}y \rangle \\ &= \langle x - \tilde{a}y, y - 1/\tilde{b} \rangle \\ &= \langle x - \tilde{a}/\tilde{b}, y - 1/\tilde{b} \rangle \\ &= \langle x - a, y - b \rangle \\ &= \mathfrak{m}_{a,b}, \end{aligned}$$

where we set  $a = \tilde{a}/\tilde{b}$  and  $b = 1/\tilde{b}$ . It follows that  ${}^a\tilde{\varphi}(\mathfrak{m}_{a,b}) = \tilde{\mathfrak{m}}_{\tilde{a}, \tilde{b}}$ . Now observe that

$$d\tilde{x} = \frac{ydx - xdy}{y^2} \quad \text{and} \quad d\tilde{y} = -\frac{dy}{y^2}.$$

### 3 A Blowup Algebra

Let  $R = \mathbb{K}[x, y]/\langle y^2 - x^3 - x^2 \rangle$ , let  $Q = \langle \bar{x}, \bar{y} \rangle$  (we drop the overlines from  $\bar{x}$  and  $\bar{y}$  in just write  $x$  and  $y$  in order to simplify notation in what follows), and equip  $R$  with the  $Q$ -filtration making  $R = (Q^n)$  into a filtered ring.

Let  $\varphi: R[u, v] \rightarrow \text{bl}(R)$  be the unique surjective  $R$ -algebra homomorphism such that  $\varphi(u) = xt$  and  $\varphi(v) = yt$ . The kernel of  $\varphi$  is an ideal of  $R[u, v]$  which is homogeneous in the variables  $u, v$ :

$$\ker \varphi = \langle v^2 - (x+1)u^2, xv - yu \rangle.$$

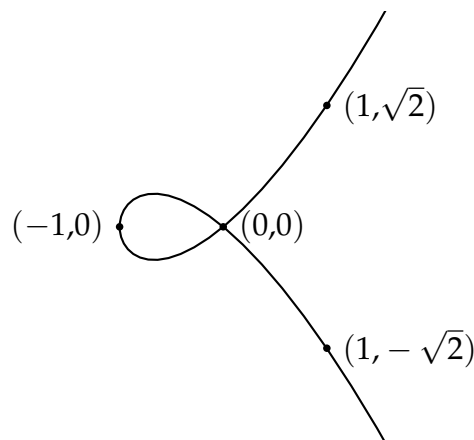
Thus we see that  $\text{bl}(R) \cong \mathbb{k}[x, y, u, v]/\mathfrak{a}$  where

$$\mathfrak{a} = \langle v^2 - (x+1)u^2, xv - yu, y^2 - x^3 - x^2 \rangle.$$

In particular,  $\text{bl}(R)$  corresponds to an algebraic subset  $Z \subseteq \mathbb{A}_{x,y}^2 \times \mathbb{P}_{u,v}^1$ . Let  $A = R[v]/\langle v^2 - (x+1), xv - y \rangle$ , so  $A$  corresponds to the affine open  $U = Z \cap (\mathbb{A}^2 \times D(u))$ . We can localize further by setting  $B = A_x = R[v]/\langle v - y/x \rangle$ , so  $B$  corresponds to the affine open  $V = Z \cap (D(x) \times D(u))$ . We have a canonical ring homomorphism  $\iota: R \rightarrow A$  where  $\iota$  is the inclusion map. Let us try to understand this homomorphism from a geometric point of view. Let  $X = V_{\mathbb{k}}(y^2 - x^3 - x^2)$  be affine algebraic subset of  $\mathbb{A}_{\mathbb{k}}^2$  defined by the equation  $y^2 = x^3 + x^2$ . The closed points of  $\text{Spec } R$  are in one-to-one correspondence with the points of  $V$ : they are all of the form

$$\mathfrak{p}_{(a,b)} = \langle x - a, y - b \rangle$$

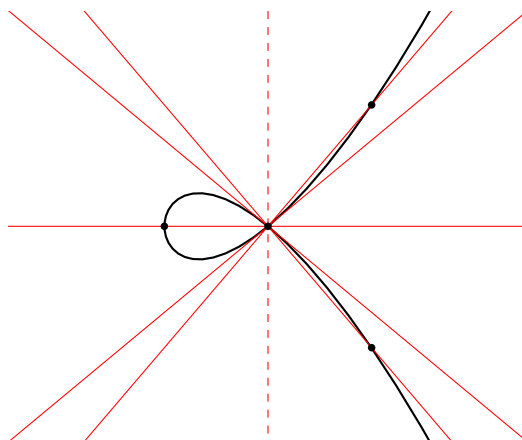
where  $(a, b) \in X$ , that is, where  $a, b \in \mathbb{k}$  such that  $b^2 = a^3 + a^2$ . If  $\mathbb{k} = \mathbb{R}$ , we can visualize the closed points of  $\text{Spec } R$  as below:



Note that  $\text{Spec } R$  also has a generic point  $\eta$  corresponding to the zero ideal of  $R$ . The closed points of  $\text{Spec } A$  correspond to the points of the affine open set  $U$ : they have the form

$$\mathfrak{p}_{(a,b),[1:t]} = \langle x - a, y - b, v - t \rangle$$

where  $a, b, t \in \mathbb{k}$  such that  $b^2 = a^3 + a^2$ ,  $at = b$ , and  $t^2 = a + 1$ . Note that if  $a \neq 0$ , then we automatically get  $t^2 = a + 1$ . If  $\mathbb{k} = \mathbb{R}$ , we can visualize the points of  $\text{Spec } A$  as below:



In particular, for  $a \neq 0$ , the prime  $\mathfrak{p}_{(a,b),[1:t]}$  corresponds to the point  $(a, b) \in X$  together with the unique line  $y = tx$  that passes through that point and the origin, where  $t$  represents the slope of that line. There are two points lying over the origin: namely  $\mathfrak{p}_{(0,0),[1:1]}$  and  $\mathfrak{p}_{(0,0),[1:-1]}$ , corresponding to the origin  $(0,0) \in V$  together with the lines  $y = x$  and  $y = -x$  respectively. The map  $\iota: R \rightarrow A$  induces a continuous map  ${}^a\iota: \text{Spec } A \rightarrow \text{Spec } R$  given by

$${}^a\iota(\mathfrak{p}_{(a,b),[1:t]}) = \mathfrak{p}_{(a,b)}.$$

This corresponds to the projection map  $\pi: U \rightarrow X$  given by

$$\pi(a, b, t) = (a, b).$$

Notice that in the image above there are “missing” points. For instance, we drew a vertical dashed line in the image above; it should correspond to the line  $x = 0$ , but it has nowhere to go under this projection. In fact, this missing line corresponds to the extra point in  $\text{Proj}(\text{bl}(R))$  which doesn’t belong to  $A$ .

**Definition 3.1.** A **hyperelliptic curve** is an algebraic curve of genus  $g > 1$ , given by an equation of the form

$$y^2 + h(x)y = f(x),$$

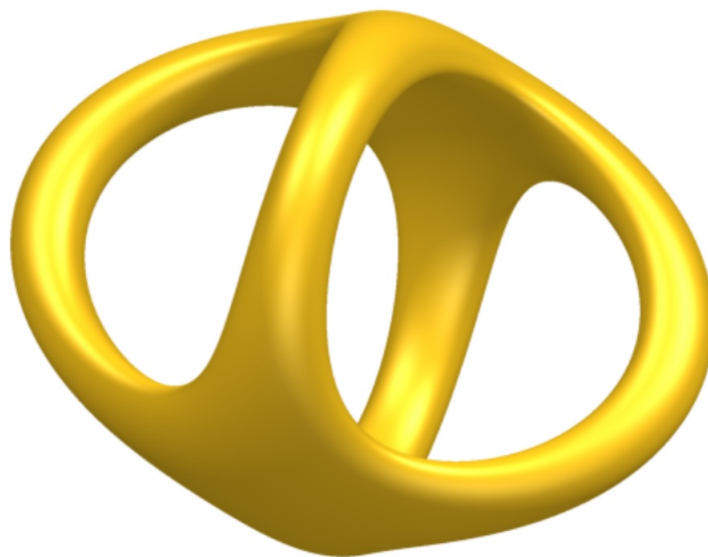
where  $f$  is a polynomial of degree  $n = 2g + 1 > 4$  or  $n = 2g + 2 > 4$  with  $n$  distinct roots and  $h(x)$  is a polynomial of degree  $< g + 2$  (if the characteristic of the ground field is not 2, one can take  $h(x) = 0$ ).

## 4 A Surface

Let  $a \in \mathbb{k}$  and let  $S_t = V_{\mathbb{k}}(f_t) \subseteq \mathbb{A}_{\mathbb{k}}^3$  where

$$f_t = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2 - t = g_1^2 + g_2^2 - t = \|g\|^2 - t$$

where  $g = (g_1, g_2)$ , where  $g_1 = x_1^2 + x_2^2 - 1$  and  $g_2 = x_2^2 + x_3^2 - 1$ . When  $\mathbb{k} = \mathbb{R}$  and  $t = 0.1$ , we can picture  $S_{0.1}$  as below:



The Jacobian matrix of  $f_t$  is given by

$$J_{f_t} = \begin{pmatrix} \partial_x f_t \\ \partial_y f_t \\ \partial_z f_t \end{pmatrix} = 4 \begin{pmatrix} x_1 g_1 \\ x_2 (g_1 + g_2) \\ x_3 g_2 \end{pmatrix}.$$

We write  $\Delta_t = V(J_{f_t}) = \{a \in \mathbb{A}_{\mathbb{k}}^3 \mid J_{f_t}(a) = 0\}$ . Given  $a \in \mathbb{A}_{\mathbb{k}}^3$ , we have  $a \in \Delta_t$  if and only if  $a = \mathbf{0}$  or  $a \in V_{\mathbb{k}}(g_1, g_2)$  (meaning  $g_1(a) = g_2(a) = 0$ ). In particular, if  $t \neq 0, 2$ , then  $S_t$  has no singular points since  $S_t \cap \Delta_t = \emptyset$  in this case. If  $t = 2$ , then  $\mathbf{0}$  is a singular point since  $\mathbf{0} \in S_2 \cap \Delta_2$ . If  $t = 0$ , then  $S_0$  has lots of singular points. For instance,  $\{(\pm 1, 0, \pm 1), (\pm 1, 0, \mp 1)\}$  are all singular points.

We can describe  $S_t$  as being the fibre at  $t \in \mathbb{k}$  with respect to the morphism of affine  $\mathbb{k}$ -schemes  $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$  (here we are indicating that the coordinate ring of  $\mathbb{A}_{\mathbb{k}, \tau}^1$  is given by  $\mathbb{k}[\tau]$ ) where  $S = \text{Spec}(\mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau)$  and where  $\pi$  corresponds to the morphism of  $\mathbb{k}$ -algebras  $\iota: \mathbb{k}[\tau] \rightarrow \mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau$  (which is just inclusion map). In particular, let  $\varepsilon_t: \text{Spec } \mathbb{k} \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$  be the morphism of affine  $\mathbb{k}$ -schemes which corresponds to the  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}[\tau] \rightarrow \mathbb{k}[\tau]/\langle \tau - t \rangle \simeq \mathbb{k}$  which sends  $\tau$  to  $t \in \mathbb{k}$ . Then  $S_t$  is the pullback of  $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$  with respect to  $\varepsilon_t: \text{Spec } \mathbb{k} \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$ . In particular, the corresponding  $\mathbb{k}$ -algebra of  $S_t$  is given by

$$\mathbb{k}[x_1, x_2, x_3]/f_t \simeq (\mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau) \otimes_{\mathbb{k}} \mathbb{k}[\tau]/\langle \tau - t \rangle.$$

Note that the morphism of affine  $\mathbb{k}$ -schemes  $\pi: S \rightarrow \mathbb{A}_{\mathbb{k}, \tau}^1$  is flat since the inclusion map of  $\mathbb{k}$ -algebras  $\iota: \mathbb{k}[\tau] \rightarrow \mathbb{k}[x_1, x_2, x_3, \tau]/f_\tau$  is flat.



## 5 An Elliptic Curve

We study the elliptic curve  $E$  defined by the equation  $y^2 = x^3 - 51$ . One calculates its discriminant to be  $\Delta = 2^4 \cdot 3^3 \cdot 51^2$ .

## 6 Degeneration to a Monomial Ideal

Let  $\mathbb{k}$  be a field, let  $R = \mathbb{k}[x, y]$ , let  $R' = \mathbb{k}[x', y']$ , and let  $S = \mathbb{k}[x, y, x', y']/J$  where

$$J = \langle x \rangle \langle x - x', y - y' \rangle = \langle x^2 - xx', xy - xy' \rangle.$$

We also set  $X = \operatorname{Spec} R$ ,  $X' = \operatorname{Spec} R'$ , and  $Y = \operatorname{Spec} S$ . Thus we have two morphisms of  $\mathbb{k}$ -schemes  $Y \rightarrow X$  and  $Y \rightarrow X'$  which correspond to the  $\mathbb{k}$ -algebra homomorphisms  $R \rightarrow S$  and  $R' \rightarrow S$  respectively. For each  $p = (a, b) \in \mathbb{k}^2$ , we set  $\mathfrak{m}_p = \langle x - a, y - b \rangle$ , and similarly for each  $p' = (a', b') \in \mathbb{k}^2$ , we set  $\mathfrak{m}'_{p'} = \langle x' - a', y' - b' \rangle$ . Let  $Y_p$  denote the fiber of  $Y$  over  $p$  and let  $Y'_{p'}$  denote the fiber of  $Y$  over  $p'$ . Then  $Y_p \simeq \mathbb{A}_{\mathbb{k}}^0$  whereas

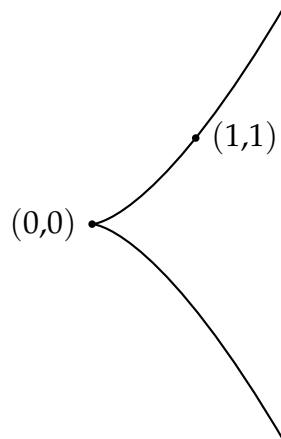
$$Y'_{p'} \simeq \begin{cases} \mathbb{A}_{\mathbb{k}}^1 \sqcup \mathbb{A}_{\mathbb{k}}^0 & \text{if } p' \neq 0 \\ \operatorname{Spec}(\mathbb{k}[x, y]/\langle x^2, xy \rangle) & \text{if } p' = 0. \end{cases}$$

## 7 Cuspidal Cubic

**Example 7.1.** Let  $\mathbb{k}$  be a field and let  $S = \mathbb{k}[x, y]/f$  where  $f = y^2 - x^3$ . Then we have

$$\Omega_{S/\mathbb{k}} = \frac{Sdx \oplus Sdy}{-3x^2dx + 2ydy}.$$

In order to better understand what kind of object  $\Omega_{S/\mathbb{k}}$  is, we digress a bit and explain how one should think  $S$  in terms of geometry. Let  $X = \operatorname{Spec} S$ . For each  $p = (a, b)$  in  $\mathbb{k}^2$  such that  $b^2 = a^3$ , we have a maximal ideal  $\mathfrak{m}_p = \langle x - a, y - b \rangle$  of  $S$  (or alternatively we can consider  $\mathfrak{m}_p$  as a closed point of  $X$ ) and we set  $\mathbb{k}_p := S/\mathfrak{m}_p \simeq \mathbb{k}$  to be the corresponding residue field (which is just  $\mathbb{k}$  but equipped with an  $S$ -module action coming from  $p$ ). If  $\mathbb{k}$  is algebraically closed, then these are all of the maximal ideals of  $S$ , however if  $\mathbb{k}$  is not algebraically closed, then there will be more maximal ideals than just this. For instance, suppose  $\mathbb{k} = \mathbb{R}$ . Then the set of all such closed points forms the curve below:



However  $X$  contains more closed points than just this (alternatively  $S$  contains more maximal ideals than just this). Indeed, for each  $p = (a, b)$  in  $\mathbb{C}^2$  such that  $b^2 = a^3$ , one gets an  $\mathbb{R}$ -algebra homomorphism  $e_p: S \rightarrow \mathbb{C}$  given by  $x \mapsto a$  and  $y \mapsto b$ . We call  $e_p$  a  **$\mathbb{C}$ -valued point** of  $S$  (or a  $\mathbb{C}$ -valued point of  $X$ ). For any such  $\mathbb{C}$ -valued point, we set  $\mathfrak{m}_p := \ker e_p$ . Then all maximal ideals of  $S$  are obtained this way (i.e. as the kernel of a  $\mathbb{C}$ -valued point). Furthermore, for two such points  $p, p'$ , we have  $\mathfrak{m}_p = \mathfrak{m}_{p'}$  if and only if  $e_{\sigma p} = e_{p'}$  for some  $\sigma \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ , where  $\sigma p = \sigma(a, b) = (\sigma a, \sigma b)$ . This holds more generally in the case where  $\mathbb{k} \neq \mathbb{R}$ . Indeed, choose an algebraic closure  $\bar{\mathbb{k}}$  of  $\mathbb{k}$ . Then we have natural bijections:

$$\{\text{maximal ideals of } S\} \simeq \{\text{closed points of } X\} \simeq \{\bar{\mathbb{k}}\text{-valued points of } X\}/\sim,$$

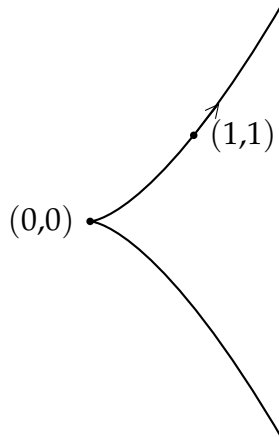
where  $p \sim p'$  if  $p = \sigma p'$  for some  $\sigma \in \text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$ . With this in mind, recall that for each closed point  $p$  of  $X$ , we have

$$\text{Hom}_S(\Omega_{S/\mathbb{k}}, \mathbb{k}_p) = \{\text{point derivations } \partial: S \rightarrow \mathbb{k}_p\}.$$

Thus we can think of  $\text{Hom}_S(\Omega_{S/\mathbb{k}}, \mathbb{k}_p)$  as the set of all tangent vectors at  $p$ . For instance, the point derivations at the origin  $\mathbf{0} = (0,0)$  correspond to all vectors  $v = (v_x, v_y) \in \mathbb{k}^2$  since  $v_x \tilde{\partial}_x|_0 + v_y \tilde{\partial}_y|_0$  vanishes on  $2ydy - 3x^2dx$ . On the other hand, the point derivations at the point  $p = (1,1)$  correspond to all vector  $v \in \mathbb{k}^2$  such that  $-3v_x + 2v_y = 0$  since

$$(v_x \tilde{\partial}_x|_p + v_y \tilde{\partial}_y|_p)(2ydy - 3x^2dx) = -3v_x + 2v_y = 0.$$

For instance, the point derivation  $(1/3)\tilde{\partial}_x|_p + (1/2)\tilde{\partial}_y|_p$  can be visualized on the curve as the tangent vector centered at  $(1,1)$  as below:



## 8 Parametrizing Field Extensions

Let  $\mathbb{k}$  be a field and fix an algebraic closure  $\bar{\mathbb{k}}$  of  $\mathbb{k}$ . Let

$$A = \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_n] / \langle y_1 - e_1, \dots, y_n - e_n \rangle = \mathbb{k}[x, y] / \langle y - e \rangle,$$

where  $e_i$  is the  $i$ th elementary symmetric polynomial:

$$e_i = \begin{cases} 1 & \text{if } k = 0 \\ \sum_{1 \leq j_1 < \dots < j_i \leq n} x_{j_1} \cdots x_{j_i} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

We view  $A$  as a  $\mathbb{k}[y]$ -algebra via the  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}[y] \rightarrow A$  which sends  $y_i$  to  $\bar{y}_i$ . Similarly, we view  $\mathbb{k}[x]$  as an  $A$ -algebra via the  $\mathbb{k}$ -algebra homomorphism  $A \rightarrow \mathbb{k}[x]$  which sends  $\bar{y}_i$  to  $e_i$ . Thus we have  $\mathbb{k}$ -algebra homomorphism  $\varphi: \mathbb{k}[y] \rightarrow \mathbb{k}[x]$  which sends  $y_i$  to  $e_i$ . Geometrically speaking, the  $\mathbb{k}$ -algebra homomorphism  $\varphi$  corresponds to the morphism of affine  $\mathbb{k}$ -schemes  $e: \mathbb{A}^n \rightarrow \mathbb{A}^n$ , where  $\mathbb{A}^n := \mathbb{A}_{\mathbb{k}}^n$ , which sends a  $\bar{\mathbb{k}}$ -valued point  $r = (r_1, \dots, r_n) \in \mathbb{A}^n(\bar{\mathbb{k}})$  to the  $\bar{\mathbb{k}}$ -valued point  $e(r) = (e_1(r), \dots, e_n(r)) \in \mathbb{A}^n(\bar{\mathbb{k}})$ . Then the  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}[x, y] \twoheadrightarrow A \rightarrow \mathbb{k}[x]$  corresponds to the graph of  $e$ :

$$\Gamma_e: \mathbb{A}^n \xrightarrow{\sim} X := \text{Spec } A \subset \text{Spec } (\mathbb{k}[x, y]) \simeq \mathbb{A}^n \times_{\mathbb{k}} \mathbb{A}^n,$$

which is given on  $\bar{\mathbb{k}}$ -valued points  $r \in \mathbb{A}^n(\bar{\mathbb{k}})$  by  $r \mapsto (r, e(r))$ . Finally the  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}[y] \rightarrow A$  corresponds to a projection map  $X \rightarrow \mathbb{A}^n$  which is given on  $\bar{\mathbb{k}}$ -valued points  $(r, c) \in X(\bar{\mathbb{k}})$  by  $(r, c) \mapsto c$ . Note that since the  $e_i$  are algebraically independent,  $\varphi$  induces an isomorphism of  $\mathbb{k}$ -algebras of  $\mathbb{k}[y]$  onto its image  $\mathbb{k}[e] = \mathbb{k}[e_1, \dots, e_n]$ . Thus we may identify  $\varphi: \mathbb{k}[y] \rightarrow \mathbb{k}[x]$  with  $\mathbb{k}[e] \subseteq \mathbb{k}[x]$ .

Now for each  $c = (c_1, \dots, c_n) \in \mathbb{A}^n(\bar{\mathbb{k}})$ , let  $e_c: \mathbb{k}[y] \twoheadrightarrow \mathbb{k}(c) \subseteq \bar{\mathbb{k}}$  be the  $\mathbb{k}$ -algebra homomorphism given by  $e_c(y_i) = c_i$ , let  $\mathfrak{m}_c = \ker e_c$ , and let  $\pi_c$  be the monic polynomial in  $\mathbb{k}(c)[t]$  given by

$$\pi_c := t^n + \sum_{i=1}^n (-1)^i c_i t^{n-i} = t^n + \sum_{i=1}^n (-1)^i e_i(r) t^{n-i} = \prod_{i=1}^n (t - r_i),$$

where  $r_i = r_{c,i}$  is the  $i$ th root of  $\pi_c$  in  $\bar{\mathbb{k}}$  (for each  $c$  we arbitrarily fix an ordering  $\mathbf{r}_c = \mathbf{r} = r_1, \dots, r_n$  of the roots of  $\pi_c$ , for instance, if  $\bar{\mathbb{k}} = \mathbb{C}$ , then we can order them likeso: given  $z = re^{i\theta}$  and  $z' = r'e^{i\theta'}$  are two nonzero complex numbers expressed in polarized form with  $r, r' > 0$  and  $\theta, \theta' \in [0, 2\pi)$ , then we say  $z \geq z'$  if either  $r > r'$  or  $|r| = |r'|$  and  $\theta > \theta'$ , and we extend this by setting  $z \geq 0$ ). Let  $G_c = \text{Gal}(\mathbb{k}(\mathbf{r}_c, c)/\mathbb{k}(c))$  and finally let

$$A_c = A \otimes_{\mathbb{k}[\mathbf{y}]} \mathbb{k}(c) \simeq \mathbb{k}[\mathbf{x}]/\langle c - e \rangle$$

be the fiber of  $A$  over  $\mathfrak{m}_c$ .

**Proposition 8.1.** *With the notation as above, we have a bijection*

$$G_c \backslash S_n \cong |\text{Spec } A_c|.$$

*Proof.* Then  $B_c$  is finite as a  $\mathbb{k}$ -vector space. Indeed, let  $\varphi: B_c \rightarrow \bar{\mathbb{k}}$  be a  $\mathbb{k}$ -algebra homomorphism. Then  $\varphi$  is completely determined by what it does to  $\bar{\mathbf{y}}$ , say  $\bar{\mathbf{y}} \mapsto \gamma$  where  $\gamma = (\gamma_1, \dots, \gamma_n) \in \bar{\mathbb{k}}$ . Note that in  $\mathbb{k}[\mathbf{y}, t]$  we have the polynomial identity:

$$\prod_{i=1}^n (t - y_i) = t^n + \sum_{i=1}^n (-1)^i e_i t^{n-i}.$$

In particular, since  $\bar{e} = c$  in  $B_c$ , this implies

$$\prod_{i=1}^n (t - \gamma_i) = t^n + \sum_{i=1}^n (-1)^i c_i t^{n-i} = \pi_c,$$

which implies  $\gamma = \rho \mathbf{r} = (r_{\rho(1)}, \dots, r_{\rho(n)})$  for some permutation  $\rho \in S_n$ . Without loss of generality, assume  $\varphi(\bar{\mathbf{y}}) = \mathbf{r}$ . Then every  $\mathbb{k}$ -algebra homomorphism  $B_c \rightarrow \bar{\mathbb{k}}$  must have the form  $\varphi\rho$  where  $\rho$  is a permutation of  $\bar{y}_1, \dots, \bar{y}_n$ . In particular, there are only finitely many  $\mathbb{k}$ -algebras  $B_c \rightarrow \bar{\mathbb{k}}$ , and each of them surjects onto  $L_c$ . The maximal ideals of  $B_c$  are precisely of the form  $\ker(\varphi\rho)$ . Furthermore, we have  $\ker(\varphi\rho) = \ker(\varphi\rho')$  if and only if  $\rho' = \sigma\rho$  where  $\sigma \in \text{Gal}(L_c/\mathbb{k})$  is viewed as the permutation of  $\bar{y}_1, \dots, \bar{y}_n$  which corresponds to how  $\sigma$  permutes the roots  $r_1, \dots, r_n$ . Thus the fiber over  $\mathfrak{m}_c$  is bijection with the quotient

$$\text{Gal}(L(c)/\mathbb{k}) \backslash S_n.$$

Now we projectivize everything. Let  $\tilde{A} = A[z]$  and let

$$\tilde{B} = A[\mathbf{y}, z]/\langle x_1 - e_1, zx_2 - e_2, \dots, z^{n-1}x_n - e_n \rangle.$$

Let  $f = (f_1, \dots, f_n): \mathbb{A}_{\bar{\mathbb{k}}}^n \rightarrow \mathbb{A}_{\bar{\mathbb{k}}}^n$  be the morphism given by  $f_i(\mathbf{r}) = e_i(\mathbf{r})$  for all  $\mathbf{r} = (r_1, \dots, r_n) \in \bar{\mathbb{k}}^n$ . For each  $c = (c_1, \dots, c_n) \in \bar{\mathbb{k}}$ , let  $\pi_c$  be the monic polynomial in  $\bar{\mathbb{k}}[t]$  given by

$$\pi_c = t^n + \sum_{i=1}^n (-1)^i c_i t^{n-i} = \prod_{i=1}^n (t - r_i)$$

where  $r_i = r_{i,c}$  is the  $i$ th root of  $\pi_c$  in  $\bar{\mathbb{k}}$  (for each  $c$  we arbitrarily fix an ordering  $\mathbf{r}_c = \mathbf{r} = (r_1, \dots, r_n)$  of the roots of  $\pi_c$ ). In particular  $f$  is an isomorphism.

Also let  $L(c) = \mathbb{k}(\mathbf{r})$  be the splitting field of  $\pi_c$  over  $\mathbb{k}$  contained in  $\bar{\mathbb{k}}$ . Note that if  $c' = (c'_1, \dots, c'_n) \in \bar{\mathbb{k}}^n$  with  $c \neq c'$ , then we may have  $L(c) = L(c')$  even though  $\pi_c \neq \pi_{c'}$  and  $\mathbf{r}_c \neq \mathbf{r}_{c'}$ . there exists a unique  $\mathbf{r} \in \text{map}$  is onto. Indeed, note that in  $\mathbb{k}[x, t]$  we have the polynomial identity:

$$\prod_{i=1}^n (t - x_i) = t^n + \sum_{i=1}^n (-1)^i e_i t^{n-i}.$$

Now given any closed point  $c = (c_1, \dots, c_n) \in \bar{\mathbb{k}}$ , form the monic polynomial in  $\bar{\mathbb{k}}[t]$ :

$$\pi_c = t^n + \sum_{i=1}^n (-1)^i c_i t^{n-i}.$$

Then Then in  $\bar{\mathbb{k}}[\pi_c = t^n + \sum_{i=1}^n (-1)^i c_i t^{n-i} = \prod_{i=1}^n (t - r_i)$

$$\prod_{i=1}^n (t - y_i) = t^n + \sum_{i=1}^n (-1)^i e_i t^{n-i}.$$

Algebraically speaking, the morphism  $f$  corresponds to the  $\mathbb{k}$ -algebra homomorphism  $\varphi: \overline{\mathbb{k}}[\mathbf{x}] \rightarrow \overline{\mathbb{k}}[\mathbf{y}]$  given by  $\varphi(x_i) = e_i$ . Note that  $\ker \varphi = 0$  since the  $e_i$  are algebraically independent, thus  $f(\mathbb{A}_{\mathbb{k}}^n)$

$$\overline{f(\mathbb{A}_{\mathbb{k}}^n)} = \mathbb{A}_{\mathbb{k}}^n$$

. is a we may also identify  $\varphi$  with the inclusion map

Alternatively, We factor  $f$  as

$$\mathbb{A}_{\mathbb{k}}^n \xrightarrow{\Gamma_f} \mathbb{A}_{\mathbb{k}}^n \times \mathbb{A}_{\mathbb{k}}^n \xrightarrow{\pi_2} \mathbb{A}_{\mathbb{k}}^n,$$

where the first morphism  $\Gamma_f$ , called the graph of  $f$ , takes  $\mathbf{r}$  to  $(\mathbf{r}, f(\mathbf{r}))$  and where the second morphism  $\pi_2$  is the projection map onto the second coordinate, that is, it takes  $(\mathbf{r}, \mathbf{c})$  to  $\mathbf{c}$ . Algebraically speaking, the morphism  $\Gamma_f$  corresponds to the  $\mathbb{k}$ -algebra homomorphism

$$\mathbb{k}[\mathbf{x}] \otimes_{\mathbb{k}} \mathbb{k}[\mathbf{y}] = \mathbb{k}[\mathbf{x}, \mathbf{y}] \rightarrow \mathbb{k}[\mathbf{x}]$$

□

## 9 Gluing

Consider the affine scheme

$$Z := \operatorname{Spec} \mathbb{k}[s, t, u] / \langle u^3 - st \rangle \cong \operatorname{Spec} \mathbb{k}[x^3, y^3, xy].$$

We can think of  $Z$  as the quotient of  $\mathbb{A}_{\mathbb{k}}^2$  by the group of third roots of unity with an isolated singularity at the origin. We resolve this singularity as follows: for  $i \in \{1, 2, 3\}$  let  $U_i = \operatorname{Spec} \mathbb{k}[u_i, v_i] \simeq \mathbb{A}_{\mathbb{k}}^2$ . We glue the  $U_i$  together via

$$\begin{array}{lll} u_2 = u_1^{-1} & u_3 = v_1^2 u_1 & u_3 = u_2^3 v_2^2 \\ v_2 = u_1^2 v_1 & v_3 = v_1^{-1} & v_3 = u_2^{-2} v_2^{-1}. \end{array}$$

More precisely, we have the following gluing datum:

$$\begin{aligned} U_2 \supset D(u_2) &:= U_{2,1} \xrightarrow[\simeq]{\varphi_{1,2}} U_{1,2} := D(u_1) \subset U_1 \\ U_3 \supset D(v_3) &:= U_{3,1} \xrightarrow[\simeq]{\varphi_{1,3}} U_{1,3} := D(v_1) \subset U_1 \\ U_2 \supset D(u_2 v_2) &:= U_{3,2} \xrightarrow[\simeq]{\varphi_{2,3}} U_{2,3} := D(u_3 v_3) \subset U_3 \end{aligned}$$

where

$$\begin{array}{lll} \varphi_{1,2}(u_2) = u_1^{-1} & \varphi_{1,3}(u_3) = v_1^2 u_1 & \varphi_{2,3}(u_3) = u_2^3 v_2^2 \\ \varphi_{1,2}(v_2) = u_1^2 v_1 & \varphi_{1,3}(v_3) = v_1^{-1} & \varphi_{2,3}(v_3) = u_2^{-2} v_2^{-1}. \end{array}$$

One checks that the  $\varphi_{i,j}$  satisfy the cocycle equation. For instance,

$$\begin{aligned} \varphi_{1,2} \varphi_{2,3}(u_3) &= \varphi_{1,2}(u_2^3 v_2^2) \\ &= u_1^{-3} (u_1^2 v_1)^2 \\ &= u_1 v_1^2 \\ &= \varphi_{1,3}(u_3). \end{aligned}$$

Let  $\tilde{Z}$  denote the scheme obtained by this gluing datum. Next, let

$$Z := \operatorname{Spec} \mathbb{k}[s, t, u] / \langle u^3 - st \rangle \cong \operatorname{Spec} \mathbb{k}[x^3, y^3, xy].$$

We can think of  $Z$  as the quotient of  $\mathbb{A}_{\mathbb{k}}^2$  by the group of third roots of unity. We have maps

$$\begin{aligned} U_1 &\rightarrow Z, & (u_1, v_1) &\mapsto (u_1 v_1^2, u_1^2 v_1, u_1 v_1) \\ U_2 &\rightarrow Z, & (u_2, v_2) &\mapsto (u_2^3 v_2^2, v_2, u_2 v_2) \\ U_3 &\rightarrow Z, & (u_3, v_3) &\mapsto (u_3, u_3^2 v_3^2, u_3 v_3), \end{aligned}$$

which glue to a morphism  $\pi: \tilde{Z} \rightarrow Z$ . One checks that the restriction  $\pi^{-1}(Z \setminus \{\mathbf{0}\}) \rightarrow Z \setminus \{\mathbf{0}\}$  is an isomorphism. The closed subscheme  $\pi^{-1}(\{\mathbf{0}\})$  (with the reduced scheme structure) can be identified with the union (inside a  $\mathbb{P}_{\mathbb{k}}^2$ ) of two projective lines intersecting in a single point.

## 10 Example

Let  $A = \mathbb{k}[x_0, x_1, x_2, x_3] / \langle f_1, f_2, f_3 \rangle = \mathbb{k}[\mathbf{x}] / \mathbf{f}$  where

$$\begin{aligned} f_1 &= x_0^2 \\ f_2 &= x_0 x_1 \\ f_3 &= x_0 x_2^d - x_1 x_3^d \end{aligned}$$

and let  $X = \text{Proj } A \subseteq \mathbb{P}^3$ .

## 11 Regularity not Preserved under Base Change

Let  $\mathbb{k} = \mathbb{F}_p(t)$ , let  $A = \mathbb{k}[x, y] / \langle y^2 - x^p + t \rangle$ , and let  $\mathfrak{m} = \langle y \rangle$ . Then  $\mathfrak{m}$  is a maximal ideal of  $A$  with residue field being  $A/\mathfrak{m} = \mathbb{F}_p(t') = \mathbb{k}'$  where we set  $t' = t^{1/p}$ . Note that  $A$  is regular at  $\mathfrak{m}$  since  $\dim A_{\mathfrak{m}} = 1$  and  $\mathfrak{m}_{\mathfrak{m}} = \langle y \rangle$ . Now let  $A' = A \otimes_{\mathbb{k}} \mathbb{k}'$  and let  $\mathfrak{m}'$  be the unique maximal of  $A'$  which lies over  $\mathfrak{m}$ . In other words,  $A' = \mathbb{k}'[x, y] / \langle y^2 - (x + t')^p \rangle$  and  $\mathfrak{m}' = \langle y, x + t' \rangle$ . This time  $A'$  is *not* regular at  $\mathfrak{m}'$  since  $\dim A'_{\mathfrak{m}'} = 1$  but  $\mathfrak{m}'_{\mathfrak{m}'}$  is minimally generated by two elements, namely  $y$  and  $x + t'$ . This shows that regularity is not preserved under base change. Another thing to observe is that  $A'$  is not integrally closed in its field of fractions. Indeed, let  $\alpha = y/(x + t')$  and note that  $\alpha^2 = (x + t')^{p-2} \in A'$  shows that  $\alpha$  is integral over  $A'$ . However  $\alpha \notin A'$ , so  $A'$  is not integrally closed. This is in contrast with  $A$  which is integrally closed in its field of fractions. This shows that normality isn't preserved under base change either.