Advanced Linear Programming Homework 4

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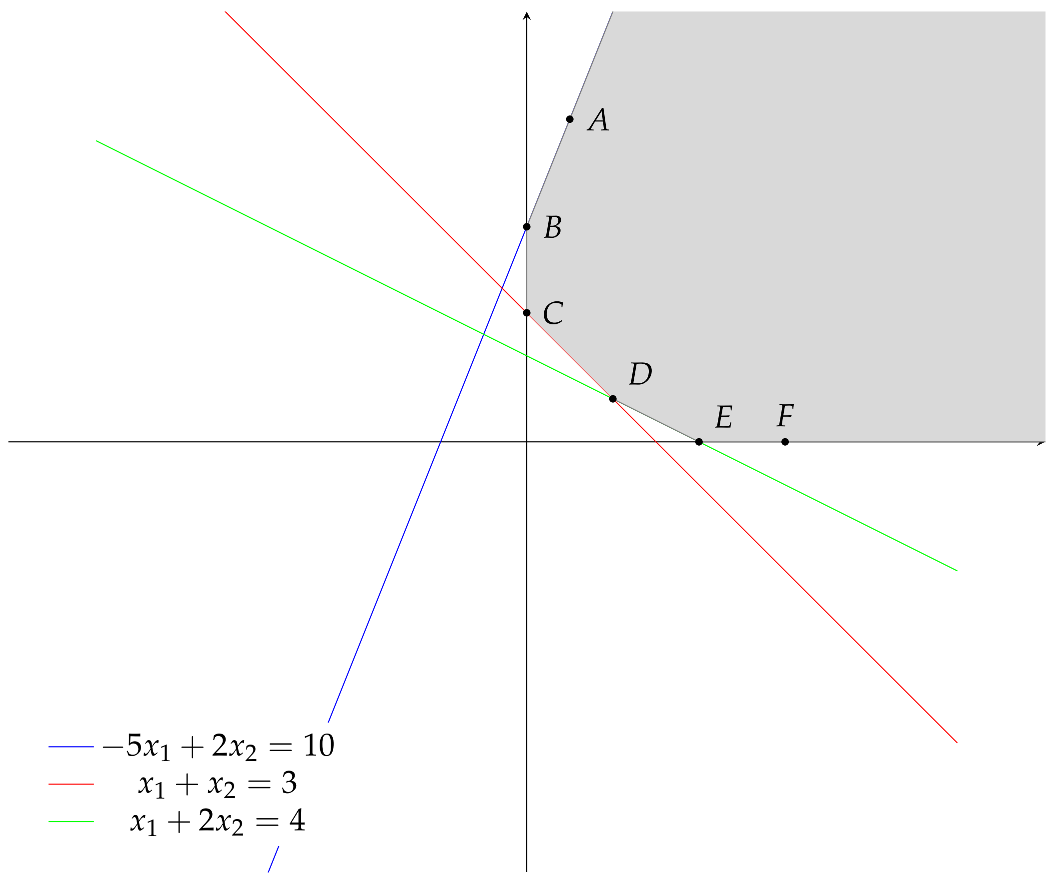
Problem 1

Problem 1.a

The feasible region is shaded in grey below (note that this region extends infinitely in the northeast direction).



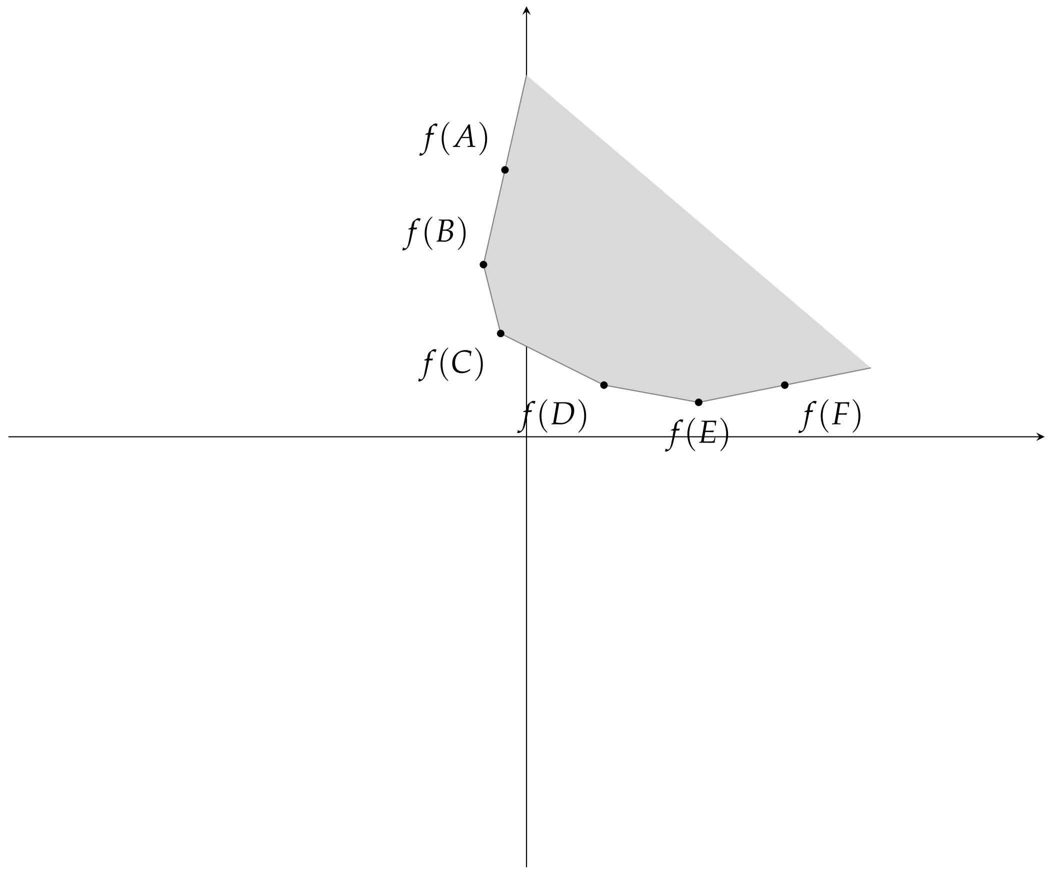
where $A = (1, 15/2)$ $B = (0, 5)$, $C = (0, 3)$, $D = (2, 1)$, $E = (4, 0)$, and $F = (6, 0)$.

Problem 1.b

We first calculate

$$
\begin{aligned}
f_1(A) &= -5/2 & \qquad f_2(A) &= 31 \\
f_1(B) &= -5 & \qquad f_2(B) &= 20 \\
f_1(C) &= -3 & \qquad f_2(C) &= 12 \\
f_1(D) &= 9 & \qquad f_2(D) &= 6 \\
f_1(E) &= 20 & \qquad f_2(E) &= 4 \\
f_1(F) &= 30 & \qquad f_2(F) &= 6
\end{aligned}
$$

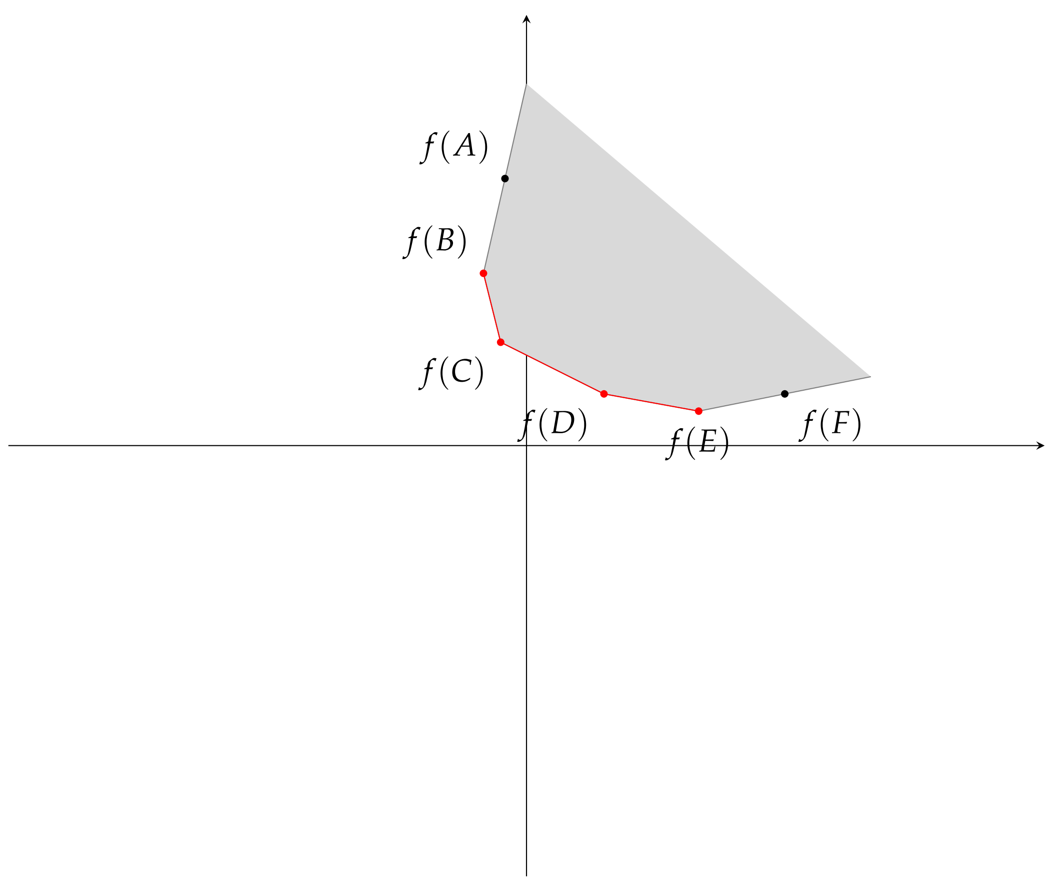
Next we plot these points shade the outcome set in gray below:



Note that the outcome set extends infinitely in the northeast direction again.

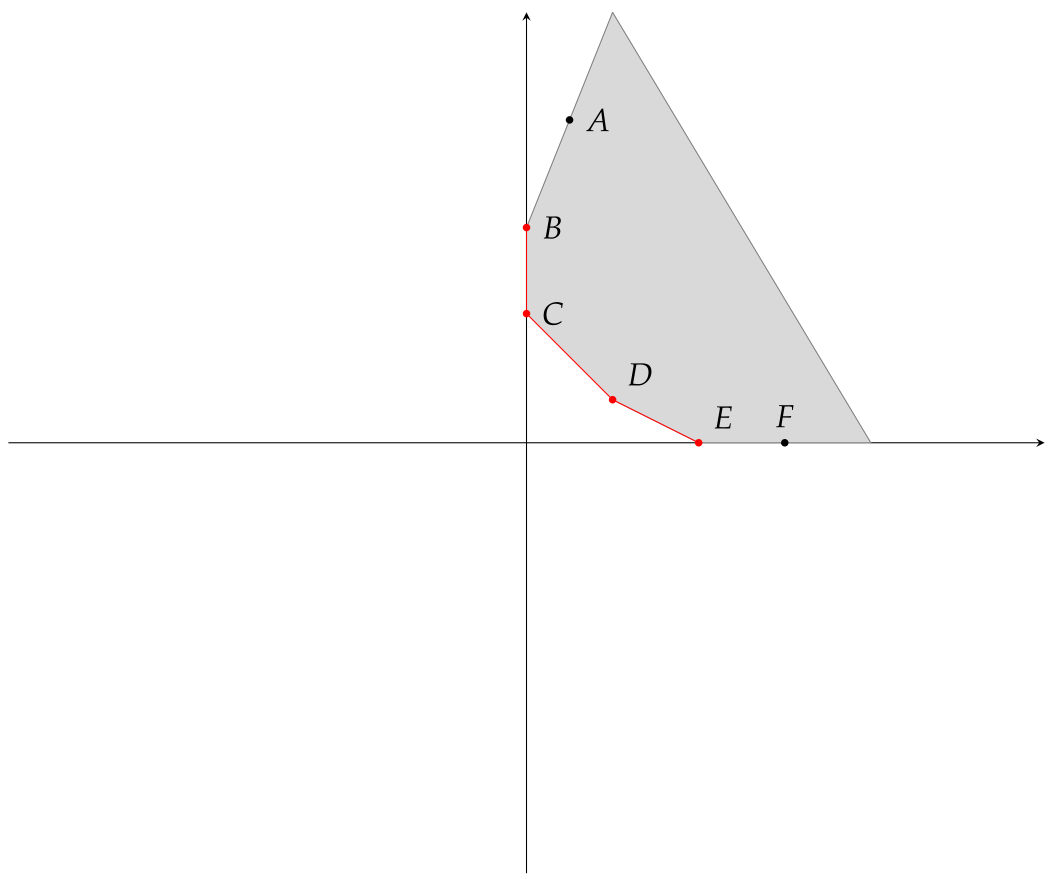
Problem 1.c

The Pareto points are $f(B)$, $f(C)$, $f(D)$, and $f(E)$. We shade the Pareto set in red below:



Problem 1.d

The efficient extreme points are B, C, D, and E. We shade the efficient set in red below:



Problem 2

Problem 2.a

The Grand objective function for the weight w is

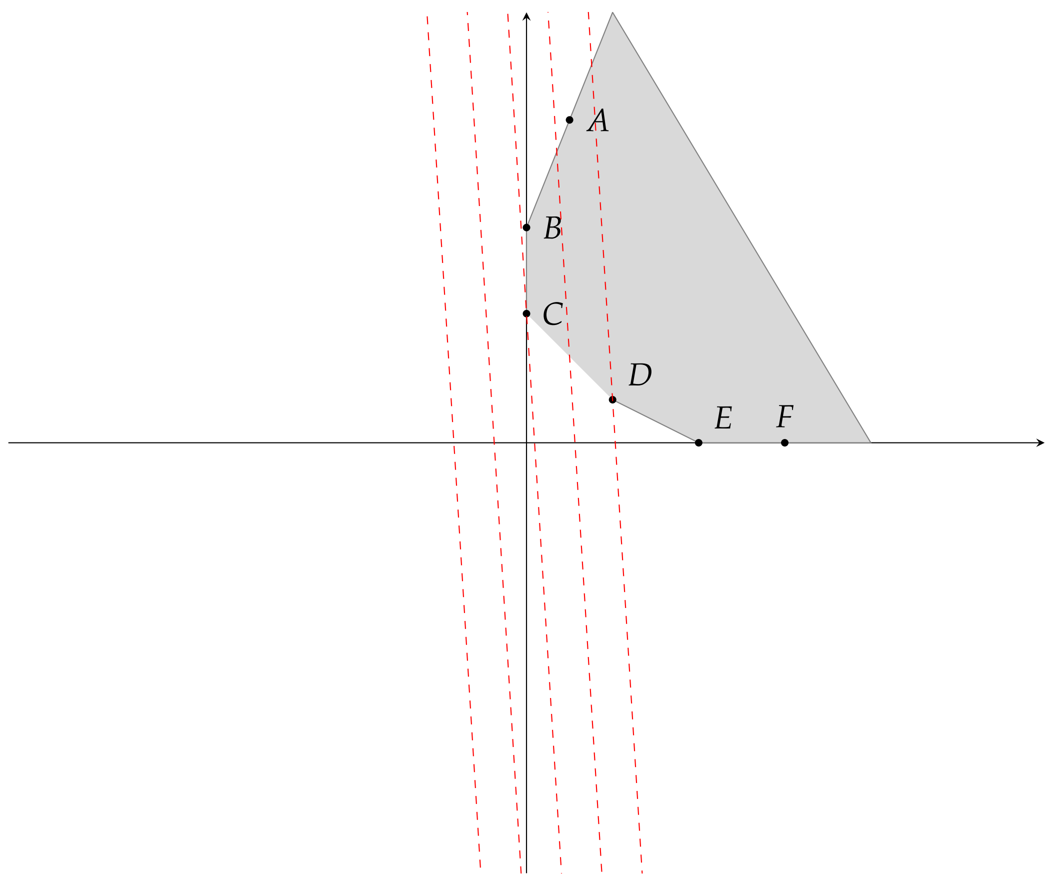
$$\begin{aligned} z_w &= f_1 + w f_2 \\ &= 5x_1 - x_2 + w(x_1 + 4x_2) \\ &= (5+w)x_1 + (4w-1)x_2. \end{aligned}$$

Then the weighted-sum problem with respect to weight w is

$$\begin{aligned} \text{minimize} z_w &= (5+w)x_1 + (4w-1)x_2 \\ \text{subject to } \boldsymbol{x} &\in X \end{aligned}$$

Problem 2.b

When $w = 1/3$, we have $z_{1/3} = (16/3)x_1 + (1/3)x_2$. Below we draw the feasible region X together with contours of the objective function $z_{1/3}$ (in the image below, we drew $\{z_{1/3} = 1\}$, $\{z_{1/3} = 6\}$, $\{z_{1/3} = 11\}$, $\{z_{1/3} = -9\}$, and $\{z_{1/3} = -4\}$).



Visually we see that $C = (0, 3)$ is the optimal solution for the weighted-sum problem.

Problem 2.c

Yes because the gradient of $z_{1/3}$ is $[16/3, 1/3]$ which points in feasible direction.

Problem 2.d

The gradient of $z_w = f_1 + w f_2$ is $\nabla z_w = [1, w]$. An optimal solution to $\mathrm{P}(w)$ is efficient for the BOLP if and only if the gradient of z_w points in a feasible direction, i.e. if and only if $0 \leq w \leq 25/2$.

Problem 3

Problem 3.a

The epsilon-constraint problem $\mathrm{P}(\varepsilon_1)$ is given by

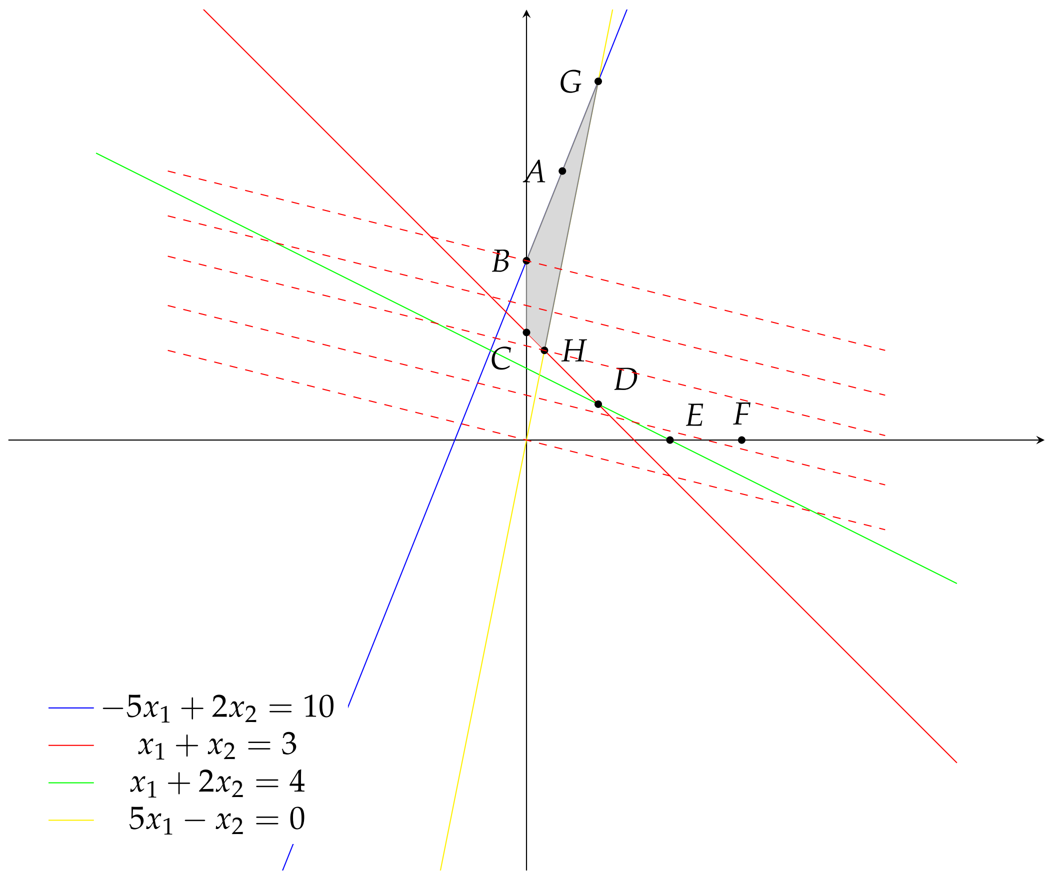
$$\begin{aligned} &\text{minimize } f_2(\boldsymbol{x}) = x_1 + 4x_2 \\ &\text{subject to } \boldsymbol{x} \in X \\ &\qquad\qquad f_1(\boldsymbol{x}) \leq \varepsilon_1 \\ &\qquad\qquad \varepsilon_1 \geq 0 \end{aligned}$$

Problem 3.b

Now suppose $\varepsilon_1 = 0$. Thus the epsilon-constraint problem has the form

$$\begin{aligned} &\text{minimize } f_2(\boldsymbol{x}) = x_1 + 4x_2 \\ &\text{subject to } -5x_1 + 2x_2 \leq 10 \\ &\qquad\qquad x_1 + x_2 \geq 3 \\ &\qquad\qquad x_1 + 2x_2 \geq 4 \\ &\qquad\qquad 5x_1 - x_2 \leq 0 \\ &\qquad\qquad x_1, x_2 \geq 0 \end{aligned}$$

The feasible region is shaded in grey below and we also plot the contours $\{f_2 = c\}$ for various c as red-dashed lines below:



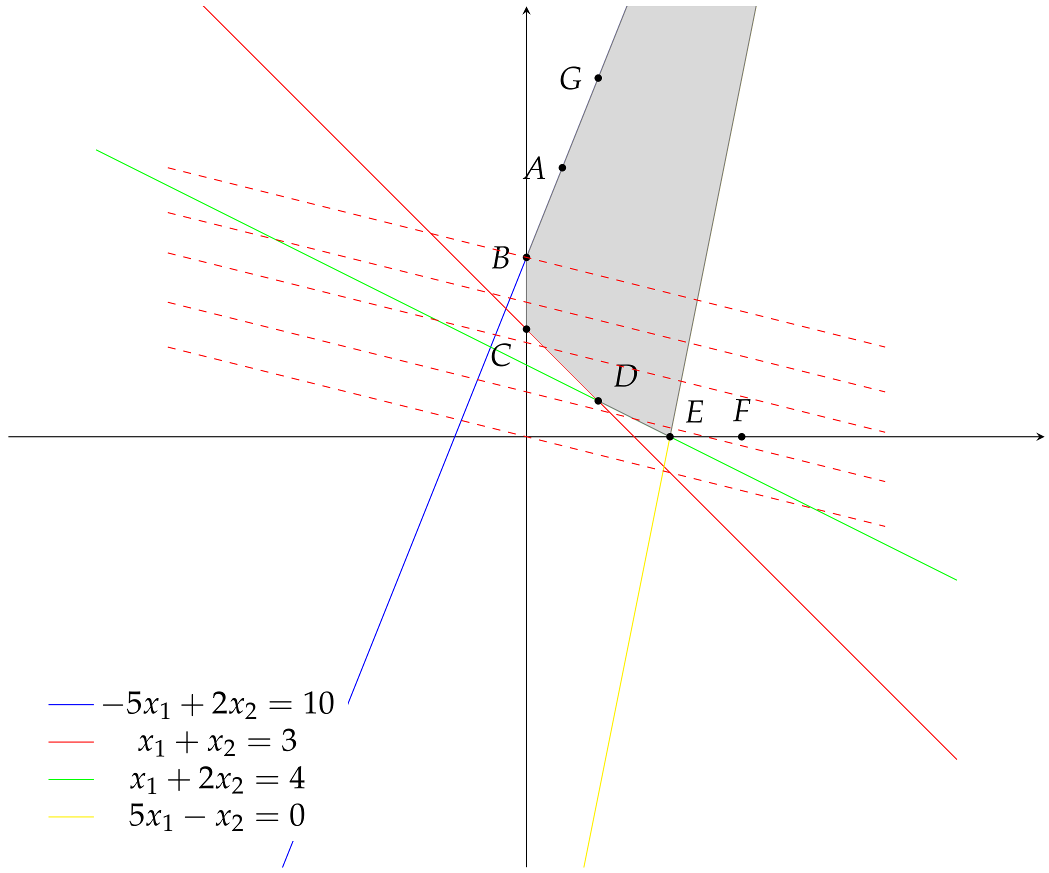
where $G = (2, 10)$ and $H = (1/2, 5/2)$. From this, we see that the optimal solution is H with objective value $f_2(H) = 21/2$.

Problem 3.c

Yes because the point H lies on the segment between adjacent extreme efficient points C and D.

Problem 3.d

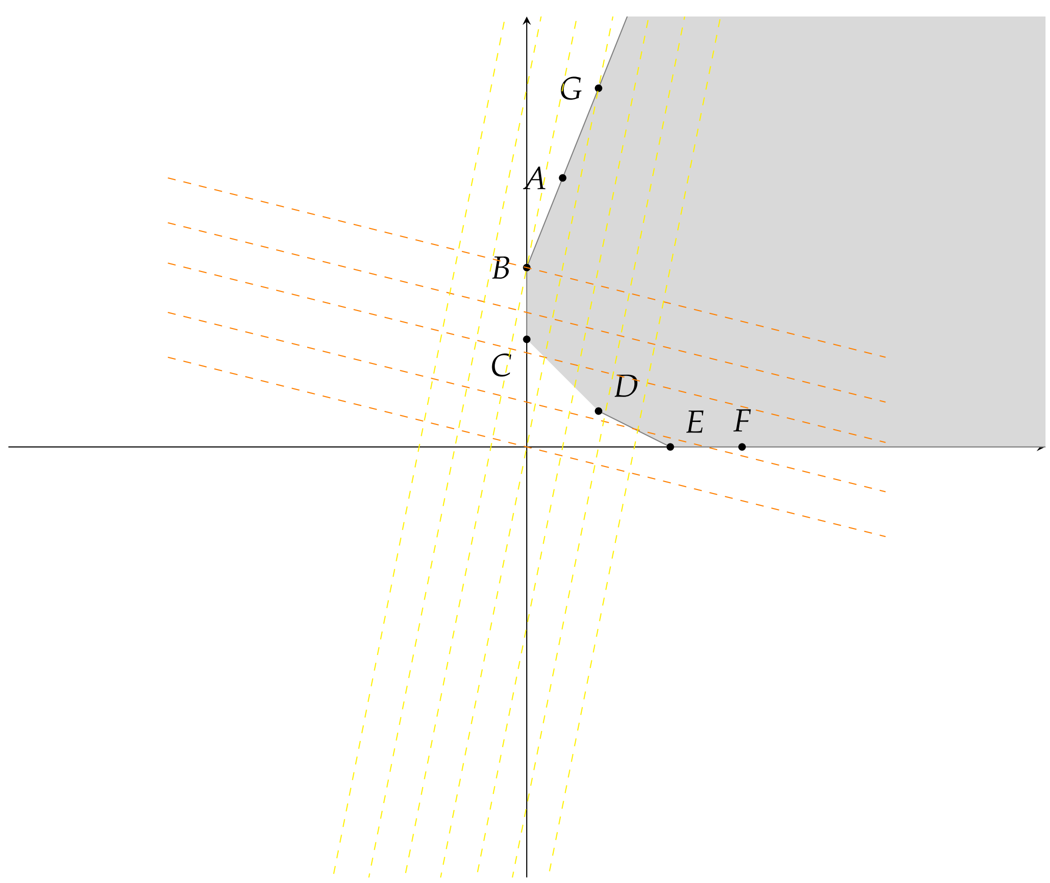
First we determine what ε needs to be in order for the line $5x_1 - x_2 = \varepsilon$ intersect E: since $E = (4, 0)$, we see that $\varepsilon = -20$. The feasible region is shaded in gray below:



Clearly E is the optimal solution which ius also an efficient solution for the original BOLP. Furthermore, it is easy to see that for any $\varepsilon < -20$, the point E is still the optimal solution. Next let's determine what ε needs to be in order for the line $5x_1 - x_2 = \varepsilon$ intersect B: since $B = (0, 6)$, we see that $\varepsilon = -6$. It is easy to see that if $\varepsilon > -6$, then the feasible region is empty and if $\varepsilon \leq 6$, then the optimal solution is always an efficient one too.

Problem 3.e

We plot the feasible region X together with the contours $\{f_1 = \varepsilon_1\}$ and $\{f_2 = \varepsilon_2\}$ for various ε_1 and ε_2:



Here, the yellow-dashed lines are the contours $\{f_1 = \varepsilon_1\}$ and the orange-dashed lines are the contours $\{f_2 = \varepsilon_2\}$.

Problem 4

Problem 4.a

Problem 4.b

Problem 4.c

Problem 4.d

Problem 4.e

Problem 5

Problem 5.a

Problem 5.b

Problem 5.b.i

Problem 5.b.ii