Algebraic Topology Homework 5

Michael Nelson

Problem 1

Remark 1. In this problem, we are identifying S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Thus an element in S^1 has the form $\overline{\theta}$ where $\theta \in \mathbb{R}$.

Exercise 1. Does the Borsuk–Ulam theorem hold for the torus? In other words, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(\overline{\theta}, \overline{\vartheta}) \in S^1 \times S^1$ such that $f(\overline{\theta}, \overline{\vartheta}) = f(\overline{\theta + \pi}, \overline{\vartheta + \pi})$?

Solution 1. No: let $\iota_{r,R}\colon S^1 \times S^1 \to \mathbb{R}^3$ be the embedding of the torus in $\mathbb{R}^3$ given parametrically by

$$
\begin{aligned}
x(\overline{\theta}, \overline{\vartheta}) &= (R + r\cos\overline{\theta})\cos\overline{\vartheta} \\
y(\overline{\theta}, \overline{\vartheta}) &= (R + r\cos\overline{\theta})\sin\overline{\vartheta} \\
z(\overline{\theta}, \overline{\vartheta}) &= r\sin\overline{\theta}
\end{aligned}
$$

Here R is the distance from the center of the tube to the center of the torus and r is the radius of the tube. For this problem it doesn't matter what r and R are; we can set them both equal to 1 and denote $\iota = \iota_{1,1}$. Note that this map is well-defined since the cosine and sin functions are 2π-periodic. Next let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map given by $\pi(x, y, z) = (x, y)$. Clearly both ι and π are continuous, so the composite $f := \pi \circ \iota$ is also continuous. Furthermore, it is straightforward to check that $f(\overline{\theta}, \overline{\vartheta}) = f(\overline{\theta + \pi}, \overline{\vartheta + \pi})$ for any $(\overline{\theta}, \overline{\vartheta}) \in S^1 \times S^1$.

Problem 2

Exercise 2. Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk–Ulam theorem to show that there is one plane $\mathcal{P} \subseteq \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Solution 2. Step 1: Fix $\boldsymbol{s} \in S^2$ and let A be an arbitrary compact set in $\mathbb{R}^3$. We will find a plane with normal vector $\boldsymbol{s}$ which divides A into two pieces of equal measure. Let $t \in \mathbb{R}$, and let $P(\boldsymbol{s}, t)$ be the plane in $\mathbb{R}^3$ which passes through the point $t\boldsymbol{s}$ and with normal vector $\boldsymbol{s}$. Thus $P(\boldsymbol{s}, t)$ is given by

$$P(\boldsymbol{s}, t) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \ell(\boldsymbol{s}, t) = 0\}$$

where $\ell(\boldsymbol{s}, t) = s_1x_1 + s_2x_2 + s_3x_3 - t$. The plane $P(\boldsymbol{s}, t)$ partitions the compact set A into two pieces, namely $A = A^+(\boldsymbol{s}, t) \cup A^-(\boldsymbol{s}, t)$ where

$$A^+(\boldsymbol{s}, t) = \{\boldsymbol{a} \in A \mid \boldsymbol{a} \geq \ell(\boldsymbol{s}, t)\} \quad \text{and} \quad A^-(\boldsymbol{s}, t) = \{\boldsymbol{a} \in A \mid \boldsymbol{a} \leq \ell(\boldsymbol{s}, t)\}.$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(t) = \mathrm{m}(A^+(\boldsymbol{s}, t))$. It is easy to show that since A is bounded, the function f is continuous in t, and that there exists $T \in \mathbb{R}$ such that $f(-T) = 0$ and $f(T) = 1$. By the intermediate value theorem, there exists $t_0 \in [-T, T]$ such that $f(t_0) = 1/2$. Let $a = \inf\{t \in \mathbb{R} \mid f(t) = 1/2\}$ and let $b = \sup\{t \in \mathbb{R} \mid f(t) = 1/2\}$. We set $t_A(\boldsymbol{s}) = (a+b)/2$. Thus any plane of the form $P(\boldsymbol{s}, t)$, where $a \leq t \leq b$, divides A into two pieces, and the plane $P(\boldsymbol{s}, t_A(\boldsymbol{s}))$ is the one in the "middle" which divides A into two pieces of equal measure.

Step 2: For each $\boldsymbol{s} \in S^2$, let $P_i(\boldsymbol{s}, t_i(\boldsymbol{s}))$ be the "middle" plane which divides A_i into two pieces of equal measure where $t_i(\boldsymbol{s}) = t_{A_i}(\boldsymbol{s})$ for each $i = 1, 2, 3$. Define $\varphi\colon S^2 \to \mathbb{R}^2$ by

$$\varphi(\boldsymbol{s}) = (t_3(\boldsymbol{s}) - t_1(\boldsymbol{s}), t_3(\boldsymbol{s}) - t_2(\boldsymbol{s})).$$

This is a continuous map such that $\varphi(-\boldsymbol{s}) = -\varphi(\boldsymbol{s})$, so by Borsuk-Ulam, there exists $\boldsymbol{s}_0 \in S^2$ such that $\varphi(\boldsymbol{s}_0) = \varphi(-\boldsymbol{s}_0)$, which is equivalent to saying

$$t_1(\boldsymbol{s}_0) = t_2(\boldsymbol{s}_0) = t_3(\boldsymbol{s}_0).$$

In other words, $P_i(\boldsymbol{s}_0, t_i(\boldsymbol{s}_0))$ is the same plane for each $i = 1, 2, 3$.

Remark 2. I used https://math.stackexchange.com/questions/1166179/hatcher-exercise-9-chapter-1-using-borsuk-ulams-theorem as a reference for this solution.

Problem 3

Exercise 3. Show that there are no retractions $r\colon X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
3. $X = S^1 \times D^2$ and A the circle shown in the figure.
4. $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.
5. X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.
6. X the Möbius band and A its boundary circle.

Solution 3. First consider the most general case where X is an arbitrary topological space and where A is an arbitary subspace of X with $\iota\colon A \to X$ denoting the inclusion map. Suppose a retraction $r\colon X \to A$ exists. Since $\pi_1\colon \mathbf{Top} \to \mathbf{Gp}$ is a functor, we have

$$\begin{aligned} 1_{\pi_1(A)} &= \pi_1(1_A) \\ &= \pi_1(r \circ \iota) \\ &= \pi_1(r) \circ \pi_1(\iota). \end{aligned}$$

Thus we have the identity

$$1_{\pi_1(A)} = \pi_1(r) \circ \pi_1(\iota). \tag{1}$$

There are at least two ways we can obtain a contradiction from (1):

- If $\pi_1(A) \neq 0$ and $\pi_1(r)$ is not surjective, then $\pi_1(r) \circ \pi_1(\iota)$ is not surjective which contradicts (1).
- If $\pi_1(A) \neq 0$ and $\pi_1(\iota) = 0$, then $1_{\pi_1(A)} \neq 0 = \pi_1(r) \circ \pi_1(\iota)$ which contradicts (1).

We now consider the special cases:

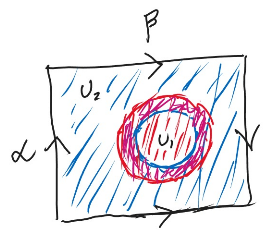
1. In this case, we have $\pi_1(A) = \mathbb{Z}$ and $\pi_1(\iota) = 0$ (since $\pi_1(X) = 0$), which contradicts (1).
2. In this case, we have $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(r)$ is not surjective (since $\pi_1(X) = \mathbb{Z}$), which contradicts (1).
3. In this case, we have $\pi_1(A) = \mathbb{Z} = \pi_1(X)$ and $\pi_1(\iota) = 2$. In particular, $\operatorname{im}(\pi_1(r) \circ \pi_1(\iota)) \subseteq 2\mathbb{Z}$ which contradicts (1).
4. In this case, we have $\pi_1(A) = \mathbb{Z} \star \mathbb{Z}$ and $\pi_1(\iota) = 0$ (since $\pi_1(X) = 0$), which contradicts (1).
5. In this case, we have $\pi_1(A) = \mathbb{Z} \star \mathbb{Z}$ and $\pi_1(r)$ is not surjective (since $\pi_1(X) = \mathbb{Z}$), which contradicts (1).
6. In this case, we have $\pi_1(A) = \mathbb{Z} = \pi_1(X)$ and $\pi_1(\iota) = 2$. In particular, $\operatorname{im}(\pi_1(r) \circ \pi_1(\iota)) \subseteq 2\mathbb{Z}$ which contradicts (1).

Problem 4

Exercise 4. Use van Kampen's theorem to compute the fundamental group of the Klein bottle and projective plane.

Solution 4. I wrote this solution down by hand:

Klein Bottle K



$U_1 \sim \bullet$ $\qquad \pi_1(U_1) = 1$

$U_2 \sim \infty$ $\qquad \pi_1(U_2) = \mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta \rangle$

$U_1 \cap U_2 \sim S^1$ $\qquad \pi_1(U_1 \cap U_2) = \mathbb{Z}$

$$\pi_1(K) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

$$= 1 *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z})$$

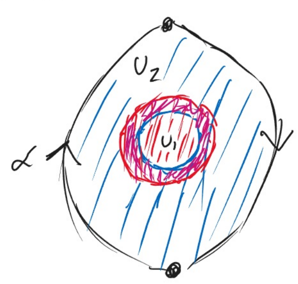
$$= \langle \alpha, \beta \mid 1 = \alpha\beta\alpha^{-1}\beta \rangle$$

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
\mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1(K) \\
 & \mathbb{Z} * \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

$1 \mapsto 1$

$1 \mapsto \alpha\beta\alpha^{-1}\beta$

Projective Plane $\mathbb{RP}^2$



$U_1 \sim \bullet$ $\qquad \pi_1(U_1) = 1$

$U_2 \sim S^1$ $\qquad \pi_1(U_2) = \mathbb{Z} = \langle \alpha \rangle$

$U_1 \cap U_2 \sim S^1$ $\qquad \pi_1(U_1 \cap U_2) = \mathbb{Z}$

$$\pi_1(K) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

$$= 1 *_{\mathbb{Z}} \mathbb{Z}$$

$$= \langle \alpha \mid \alpha^2 = 1 \rangle$$

$$\cong \mathbb{Z}/2\mathbb{Z}$$

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
\mathbb{Z} \arrow[ur] \arrow[dr] & & \pi_1(\mathbb{RP}^2) \\
 & \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

$1 \mapsto 1$

$1 \mapsto \alpha^2$