Fibers

Definition 0.1. Let S be an R-algebra and let $\mathfrak{p}$ be a prime ideal of R. We define the **fiber of** S **over** $\mathfrak{p}$ to be the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_R S$ where $\kappa(\mathfrak{p}) = K(R/\mathfrak{p})$ denotes the quotient field of $R/\mathfrak{p}$. In particular, if $\mathfrak{m}$ is a maximal ideal of R, then the fiber of S over $\mathfrak{m}$ is the $R/\mathfrak{m}$-algebra $R/\mathfrak{m} \otimes_R S \simeq S/\mathfrak{m}S$. If R is an integral domain with fraction field K, then **generic fiber of** S is the K-algebra $K \otimes_R S$.

Remark 1. Let $\iota \colon A \to B$ be an inclusion of $\Bbbk$-algebras where $\Bbbk$ is a field. Geometrically speaking, the inclusion map $\iota \colon A \to B$ of $\Bbbk$-algebras corresponds to the morphism $\pi \colon Y \to X$ of affine $\Bbbk$-schemes, where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and where π is defined by

$$\pi(\mathfrak{q}) = A \cap \mathfrak{q}$$

for all primes $\mathfrak{q}$ of B. If $\iota \colon A \to B$ is an integral extension, then π is surjective (this is referred to as the **lying over** property for integral extensions). Note that π is continuous with respect to the Zariski topology, for if $U := \mathrm{D}(a)$ is an open subset of X where $a \in A$, then

$$\pi^{-1}(U) = \pi^{-1}(\mathrm{D}(a)) = \mathrm{D}(\iota(a)) := V.$$

In other words, we have $a \notin A \cap \mathfrak{q}$ if and only if $a \notin \mathfrak{q}$ for all primes $\mathfrak{q}$ of B. Now, given a prime $\mathfrak{p}$ of A, the fiber of $\pi \colon Y \to X$ over $\mathfrak{p}$, denoted $Y_\mathfrak{p}$, is the pullback of $\pi \colon Y \to X$ with respect to the morphism $\varepsilon_\mathfrak{p} \colon \operatorname{Spec}(\kappa(\mathfrak{p})) \to X$ where $\varepsilon_\mathfrak{p}$ is the morphism which corresponds to the $\Bbbk$-algebra homomorphism $A \to \kappa(\mathfrak{p})$. In particular, $Y_\mathfrak{p}$ is an affine $\Bbbk$-scheme and the $\Bbbk$-algebra which corresponds to $Y_\mathfrak{p}$ is $\kappa(\mathfrak{p}) \otimes_A B$, which is precisely how we defined the fiber of B over $\mathfrak{p}$ in the first place.

Example 0.1. Let $R = \Bbbk[a] = \Bbbk[a_1, a_2, a_3]$ and let $S = R[x]/f = R[x_1, x_2]/f$ where $f = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2$. Also for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \Bbbk^3$, we set

$$\mathfrak{m}_\alpha = \langle a_1 - \alpha_1, a_2 - \alpha_2, a_3 - \alpha_3 \rangle \quad \text{and} \quad f_\alpha = \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2.$$

Then the fiber of S over $\mathfrak{m}_\alpha$ is the $\Bbbk$-algebra $S_\alpha := \Bbbk[a, x]/f_\alpha$. Geometrically speaking, the inclusion map $\iota \colon R \to S$ of $\Bbbk$-algebras corresponds to the projection $\pi \colon Y \to X$ of affine $\Bbbk$-schemes, where $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. The fiber of π over $\mathfrak{m}_\alpha$ is given by

$$\pi^{-1}(\{\mathfrak{m}_\alpha\}) = \mathrm{V}(f_\alpha) = \operatorname{Spec}(S_\alpha).$$

Example 0.2. Let $R = \Bbbk[t]$, let $S = R[x]/\langle x^2 - t \rangle$, and let $\mathfrak{p}_\tau = \langle t - \tau \rangle$ where $\tau \in \Bbbk$. Then for $\tau \neq 0$, the fiber of S over $\mathfrak{p}_\tau$ is $\Bbbk[x]/\langle x^2 - \tau \rangle \cong \Bbbk \times \Bbbk$. The fiber over $\mathfrak{p}_0$ is $S_0 := \Bbbk[x]/\langle x^2 \rangle$. Finally, the fiber over the zero ideal $\langle 0 \rangle$ is $\Bbbk(t)[x]/\langle x^2 - t \rangle$, a field of degree 2 over the residue field $\kappa(\langle 0 \rangle) = \Bbbk(t)$. We see that for each prime $\mathfrak{p}$, the fiber over $\mathfrak{p}$ is a vector space of dimension 2 over its residue field $\kappa(\mathfrak{p})$. In fact, S is a free R-module on the generators $(1, x)$. Thus $S \otimes_R N = N \oplus N$ for any R-module N, and it follows that S is flat.