Math 1070 Test 1 Review

September 19, 2023

This study guide is deisgned to help you be prepared for the test 1.

Inverse Trig Functions

Exercise 1. For this exercise, you need to complete the table below. I have filled in all of the information regarding the trig functions for you, however you should also verify that what I wrote down is correct. For instance, why is the domain of $\cot x$ equal to $\mathbb{R}\backslash\{n\pi \mid n \in \mathbb{Z}\}$? This is because $\cot x = \cos x / \sin x$, so $\cot x$ is defined whenever $\sin x \neq 0$, and $\sin x \neq 0$ whenever x is any real number except those of the form $n\pi$ where n is an integer. It is very important that *understand* why the domains and ranges for these functions are the way that they are. Finally, when filling this table out, please try to derive your answers on your own (rather than simply copying them from another source). To see what I mean, I'll show you how to derive the derivative of $\arcsin x$: All you have to do is use the fact that $\sin(\arcsin x) = x$ and take derivatives on both sides.

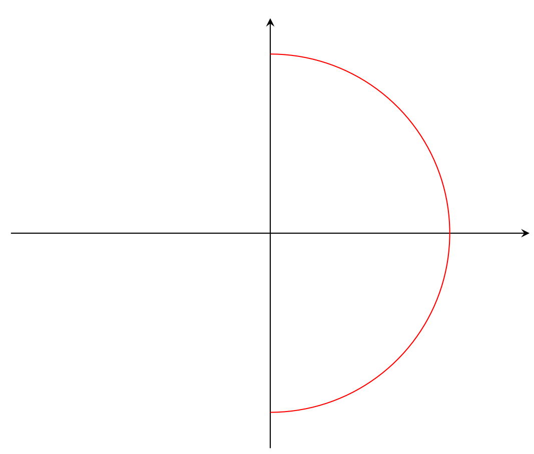
$$\begin{aligned}
\sin(\arcsin x) = x \implies & \frac{\mathrm{d}}{\mathrm{d}x}(\sin(\arcsin x)) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \\
\implies & \cos(\arcsin x)\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = 1 \\
\implies & \sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = 1 \\
\implies & \frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

Here we used the fact that $\cos(\arcsin x) = \sqrt{1-x^2}$. In fact, you may want to fill in the table in Exercise (4) before calculating these derivatives.

Function	Domain	Range	Derivative
$\sin x$	$\mathbb{R}$	$[-1,1]$	$\cos x$
$\cos x$	$\mathbb{R}$	$[-1,1]$	$-\sin x$
$\tan x$	$\mathbb{R}\backslash\{\frac{\pi}{2}+n\pi \mid n\in\mathbb{Z}\}$	$\mathbb{R}$	$\sec^2 x$
$\csc x$	$\mathbb{R}\backslash\{n\pi \mid n\in\mathbb{Z}\}$	$(-\infty,1]\cup[1,\infty)$	$-\csc x\cot x$
$\sec x$	$\mathbb{R}\backslash\{\frac{\pi}{2}+n\pi \mid n\in\mathbb{Z}\}$	$(-\infty,1]\cup[1,\infty)$	$\sec x\tan x$
$\cot x$	$\mathbb{R}\backslash\{n\pi \mid n\in\mathbb{Z}\}$	$\mathbb{R}$	$-\csc^2 x$
$\arcsin x$	$[-1,1]$	$[-\pi/2,\pi/2]$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$[-1,1]$	$[0,\pi]$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan x$	$\mathbb{R}$	$(-\pi/2,\pi/2)$	$\frac{1}{1+x^2}$
$\operatorname{arccsc} x$	$(-\infty,1]\cup[1,\infty)$	$[-\pi/2,0)\cup(0,\pi/2]$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arcsec} x$	$(-\infty,1]\cup[1,\infty)$	$[0,\pi/2)\cup(\pi/2,\pi]$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$\mathbb{R}$	$(0,\pi)$	$\frac{-1}{1+x^2}$

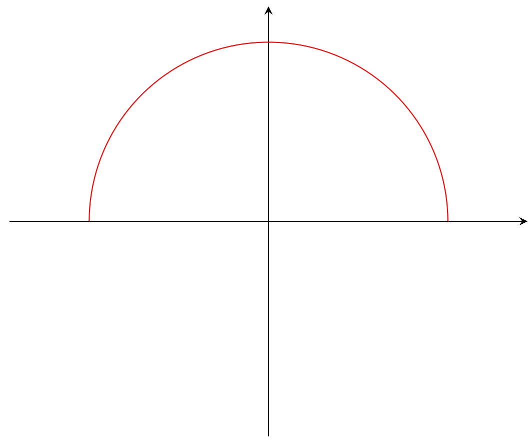
Solution 1. The domain of the inverse trig functions are easy to remember: they are just the ranges of their corresponding trig functions. Thus the domain of $\arcsin x$ is the range of $\sin x$, which is $[-1,1]$. Similarly, the domain of $\operatorname{arccot} x$ is the range of $\cot x$, which is $\mathbb{R}$. The domains of the other inverse trig functions are filled in the table above.

The ranges of the inverse trig functions are slightly more involved. One way to memorize them however is to group them together as follows: The ranges of inverse trig functions $\arcsin x$, $\operatorname{arccsc} x$, and $\arctan x$ all lie in the right semi-circle



The range of $\arcsin x$ consists all *all* angles in this right semi-circle, thus the range $\arcsin x$ is $[-\pi/2,\pi/2]$. The range of $\operatorname{arccsc} x$ consists all angles in this right semi-circle except the 0 angle, thus the range $\operatorname{arccsc} x$ is $[-\pi/2,0)\cup(0,\pi/2]$. Finally, the range of $\arctan x$ consists of all angles in this right semi-circle except the boundary angles $-\pi/2$ and $\pi/2$, thus the range of $\arctan x$ is $(-\pi/2,\pi/2)$.

The ranges of inverse trig functions $\arccos x$, $\operatorname{arcsec} x$, and $\operatorname{arccot} x$ all lie in the upper semi-circle



The range of $\arccos x$ consists all *all* angles in this upper semi-circle, thus the range $\arcsin x$ is $[0, \pi]$. The range of arcsec$x$ consists all angles in this upper semi-circle except the $\pi/2$ angle, thus the range $\operatorname{arcsec} x$ is $[0, \pi/2) \cup (\pi/2, \pi]$. Finally, the range of $\operatorname{arccot} x$ consists of all angles in this upper semi-circle except the boundary angles 0 and π, thus the range of $\arctan x$ is $(0, \pi)$.

For the last part of this problem, we calculate the derivatives of the inverse trig functions. In order to do this, you should first read the solutions to Exercise (4).

Remark 1. Note that $\arcsin x$ and $\sin^{-1} x$ denote the *same* function. Technically speaking, $\arcsin x$ is the more accurate notation here, however in this class you'll often see both notations used, so keep this in mind. Also make sure you understand that $\sin^{-1} x$ is the *inverse* function – it does not denote the function $(\sin x)^{-1} = 1/\sin x$. This notation is admittedly confusing since $\sin^2 x$ means $(\sin x)^2 = \sin x \cdot \sin x$.

Exercise 2. Evaluate the following:

$$\begin{aligned}
\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) &= \\
\arcsin\left(\frac{-\sqrt{3}}{2}\right) &= \\
\sec^{-1}(1) &= \\
\operatorname{arccot}(1) &= \\
\operatorname{arcsec}\left(\sqrt{2}\right) &= \\
\arcsin\left(\frac{1}{2}\right) &= \\
\cos^{-1}\left(\cos\left(\frac{-\pi}{6}\right)\right) &=
\end{aligned}$$

Exercise 3. Suppose $a \in (-\infty, -1] \cup [1, \infty)$. Explain why $\operatorname{arccsc}(a) = \arcsin(1/a)$. Similarly explain why $\operatorname{arcsec}(a) = \arccos(1/a)$.

Exercise 4. Complete the table below. I've filled in the first two rows for you (check that what I wrote down is accurate!). You should only have to use the two formulas

$$\sin^2 x + \cos^2 x = 1 \quad \text{and} \quad \sec^2 x - \tan^2 x = 1$$

to fill in what's left. Again, it's very important that you are able to derive this stuff on your own (rather than copying from another source).

Composite Function	Alternate Form	Domain	Range
$\cos(\arcsin x)$	$\cos(\arcsin x) = \sqrt{1-x^2}$	$[-1,1]$	$[0,1]$
$\tan(\arcsin x)$	$\tan(\arcsin x) = x/\sqrt{1-x^2}$	$(-1,1)$	$\mathbb{R}$
$\sec(\arcsin x)$	$\sec(\arcsin x) = 1/\sqrt{1-x^2}$		
$\csc(\arcsin x)$			
$\cot(\arcsin x)$			
$\sin(\arccos x)$			
$\tan(\arccos x)$			
$\tan(\operatorname{arcsec} x)$			
$\cot(\operatorname{arcsec} x)$			

Note that there is a subtle issue that you need to consider when filling in the last two rows. You should look up the answer on wolfram.com and try to understand why they express these composite functions in the way that they do.

Solution 2. Let's first calculate $\sec(\arcsin x)$. To do this, we first need to express $\sec x$ as a function in $\sin x$. In particular, we have

$$\begin{aligned} \sin^2 x + \cos^2 x = 1 &\implies \cos^2 x = 1 - \sin^2 x \\ &\implies \sec^2 x = \frac{1}{1-\sin^2 x} \\ &\implies \sec x = \pm\sqrt{\frac{1}{1-\sin^2 x}} \end{aligned}$$

Using this, we see that

$$\begin{aligned} \sec(\arcsin x) &= \pm\sqrt{\frac{1}{1-\sin^2(\arcsin x)}} \\ &= \pm\sqrt{\frac{1}{1-\sin(\arcsin x)\cdot\sin(\arcsin x)}} \\ &= \pm\sqrt{\frac{1}{1-x\cdot x}} \\ &= \pm\sqrt{\frac{1}{1-x^2}} \\ &= \pm\frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

Now we need to get rid of the $\pm$: recall that the range of $\arcsin x$ is $[-\pi/2, \pi/2]$, so if θ is an angle between $-\pi/2$ and $\pi/2$, then $\cos\theta$ is always nonnegative, and thus $\sec\theta$ is also nonnegative. This tells us that we can replace the $\pm$ with just $+$, so

$$\sec(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

Now let's calculate $\tan(\arccos x)$. This time, we need to express $\tan x$ as a function in $\cos x$. In particular, we have

$$\begin{aligned}
\sec^2 x - \tan^2 x = 1 &\implies \tan^2 x = \sec^2 x - 1 \\
&\implies \tan^2 x = \frac{1}{\cos^2 x} - 1 \\
&\implies \tan^2 x = \frac{1 - \cos^2 x}{\cos^2 x} \\
&\implies \tan x = \pm\sqrt{\frac{1 - \cos^2 x}{\cos^2 x}}.
\end{aligned}$$

Using this, we see that

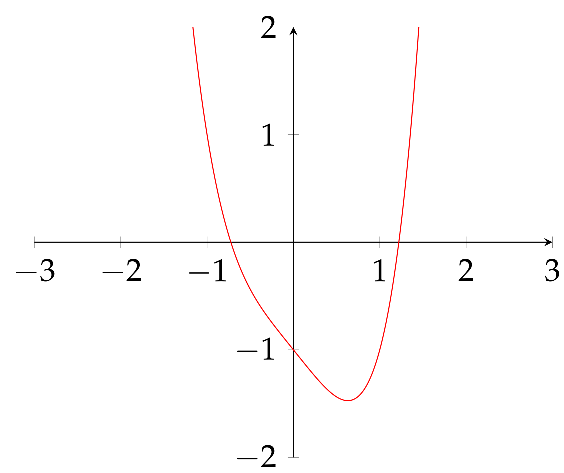
$$\begin{aligned}
\tan(\arccos x) &= \pm\sqrt{\frac{1 - \cos^2(\arccos x)}{\cos^2(\arccos x)}} \\
&= \pm\sqrt{\frac{1 - \cos(\arccos x) \cdot \cos(\arccos x)}{\cos(\arccos x) \cdot \cos(\arccos x)}} \\
&= \pm\sqrt{\frac{1 - x \cdot x}{x \cdot x}} \\
&= \pm\sqrt{\frac{1 - x^2}{x^2}} \\
&= \pm\frac{\sqrt{1 - x^2}}{x}.
\end{aligned}$$

Now we need to get rid of the $\pm$: recall that the range of $\arccos x$ is $[0, \pi]$, so if θ is an angle between 0 and π, then $\tan\theta$ is negative whenever $\theta \in (\pi/2, \pi)$ and $\tan\theta$ is nonnegative whenever $\theta \in [0, \pi/2]$. What this tell us is that we also need to consider the domain of $\arccos x$, which is $[-1, 1]$. In particular, note that if $x \in [-1, 0)$, then $\arccos x \in (\pi/2, \pi)$, which implies $\tan(\arccos x)$ is
always nonnegative, and thus $\sec\theta$ is also nonnegative. This tells us that we can replace the $\pm$ with just $+$, so

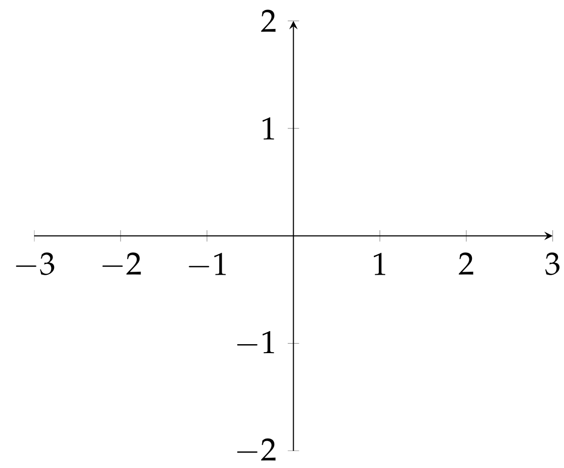
$$\sec(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}.$$

Graphs

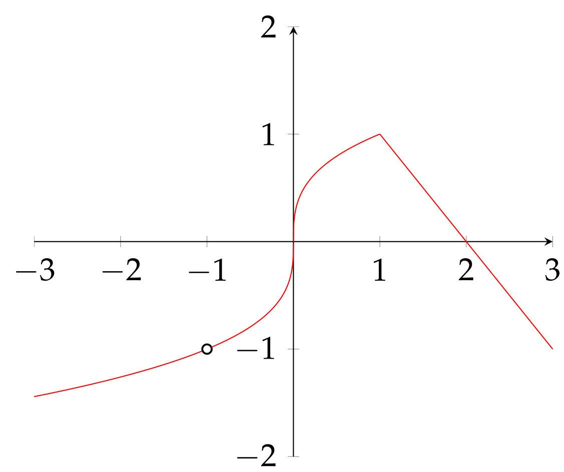
Exercise 5. Consider the function $f(x)$ whose graph is given below



Draw the graph of the derivative function $f'(x)$ below



Exercise 6. Consider the function $f(x)$ whose graph is given below

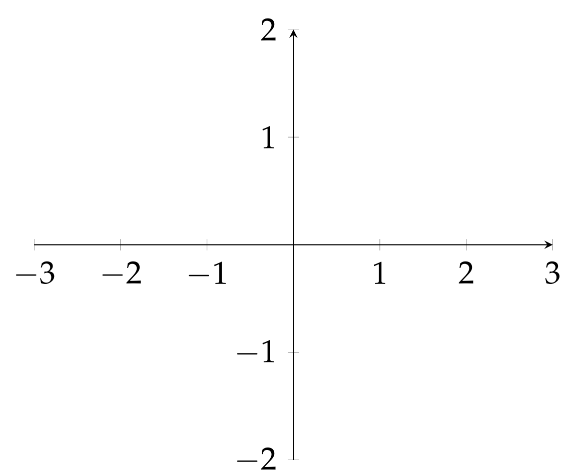


State all x-values where this function is not differentiable. State all x-values where this function is not continuous.

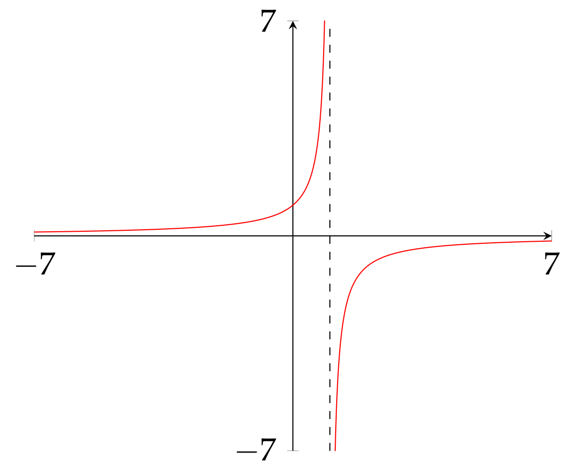
It is not differentiable at:

It is not continuous at:

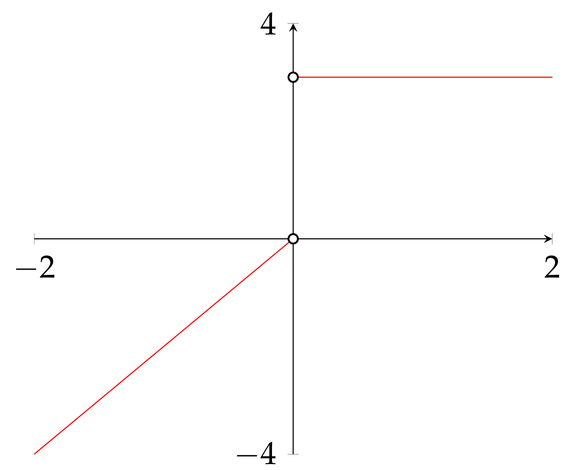
Now draw the graph of the derivative function $f'(x)$ below



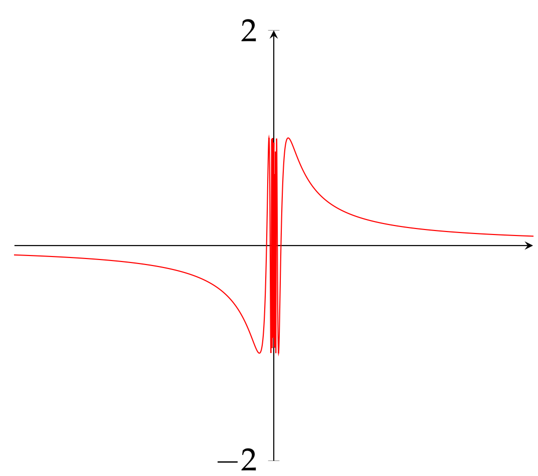
Exercise 7. Make sure you know the different types of discontinuities a function can have, namely jump, infinite, removable, and oscilatting discontinuities. What type of discontinuity is presented in the graph below?



What type of discontinuity is presented in the graph below?



What type of discontinuity is presented in the graph below?



What type of discontinuity is given by the function below?

$$f(x) = \frac{(x-3)(x+2)}{x+2}.$$

Write down the intervals on which $f(x)$ is continuous.

Derivatives and Limits

Exercise 8. List three reasons why a funciton $f(x)$ may not be differentiable at a point $x = a$.

Exercise 9. Find the derivative of $f(x) = -4x^2 + 3x - 2$ using the *limit* definition. Note that you'll see this type of question on the test. **Make sure you do this right and do not skip any steps!** I'll show you how to do it when it comes time to review this question in class, but you need to be able to write it on your own almost *exactly* the way that I will show you.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Exercise 10. Find the derivative of $f(x) = \frac{2}{x+1}$ using the *limit* definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Exercise 11. Find the derivative of the following functions (I'll do the first one for you).

$$p(x) = x^3 + 3x^2$$

$$\begin{aligned} p'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(p(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + \frac{\mathrm{d}}{\mathrm{d}x}(3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \\ &= 3x^{3-1} + 3 \cdot 2x^{2-1} \\ &= 3x^2 + 6x \end{aligned}$$

You don't have to do every step as I did above. If you feel comfortable, you can skip some steps like this:

$$p(x) = x^3 + 3x^2$$

$$\begin{aligned} p'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(p(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \\ &= 3x^2 + 6x. \end{aligned}$$

Now you do the rest

$$h(t) = e^{\sqrt{t}} - t^e + 5e$$

$$\begin{aligned} h'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(h(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\sqrt{t}} - t^e + 5e\right) \end{aligned}$$

$$f(x) = \tan(2^x e^{3x+1})$$

$$f'(x) =$$

$$g(x) = \log_5(2^{\cot(x)})$$

$$g'(x) =$$

$$h(x) = \sin^{-1}(\sec(x^2 - 2)) \qquad h'(x) =$$

$$f(x) = (\sin(4x))^{\sqrt{x}} \qquad f'(x) =$$

Finally, let $g(x)$ be a differentiable function and let $f(x) = \ln(g(2\sec x))$. Find $f'(x)$

Exercise 12. Evaluate the following limits. Do *not* use L'Hospital's rule! Note that some of these limits are secretly derivatives of simple functions in disguise.

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} =$$

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3} =$$

$$\lim_{x\to -2} \frac{-2 - x}{1 - \sqrt{x+3}} =$$

$$\lim_{x\to 0} \frac{\frac{1}{x+1} - 1}{x} =$$

$$\lim_{x\to 1} \frac{\ln(x)}{x - 1} =$$

$$\lim_{x\to 3^-} \frac{x+3}{|x-3|} =$$

Exercise 13. Let y be a function such that $\frac{dy}{dx} = \frac{y^2-x}{x}$. Find $\frac{d^2y}{dx^2}$.

Exercise 14. Let $f(x)$ be a function. What does it mean for the line $y = L$ to be a horizontal asymptote of $f(x)$? What does it mean for the line $x = a$ to be a vertical asymptote of $f(x)$?

Exercise 15. Let $f(x) = x^3 - 1$. What is the average rate of change of $f(x)$ over the interval $[2, 6]$.