# Mathematical Programming Homework 3

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## Problem 1

Let

$$\begin{aligned} f(x_1, x_2) &= (x_1-2)^2 + (x_2-3)^2 \\ g_1(x_1, x_2) &= x_1/2 + x_2 - 3/2 \\ g_2(x_1, x_2) &= x_2 - 1 \end{aligned}$$

Consider the following inequality constrained nonlinear optimization problem (NLP1):

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g_1(\boldsymbol{x}) \leq 0 \\ & g_2(\boldsymbol{x}) \leq 0 \end{array}$$

### Problem 1.a

**Exercise 1.** Find all solutions to the KKT FONC for this NLP1.

**Solution 1.** A KKT point for this NLP1 is a pair $(\boldsymbol{x}, \boldsymbol{\mu})$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \boldsymbol{\mu}^\top \nabla \boldsymbol{g}(\boldsymbol{x}) &= 0 && \text{Stationary} \\ \boldsymbol{g}(\boldsymbol{x}) &\leq 0 && \text{Feasibility} \\ \boldsymbol{\mu} &\geq 0 && \text{Nonnegativity} \\ \boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{x}) &= 0 && \text{Complementary slackness} \end{aligned}$$

Suppose $(\boldsymbol{x}, \boldsymbol{\mu}) = (x_1, x_2, \mu_1, \mu_2)$ is a KKT point. Then from the stationary equations, we have

$$\begin{aligned} 2(x_1-2) + \mu_1/2 &= 0 \\ 2(x_2-3) + \mu_1 + \mu_2 &= 0 \end{aligned}$$

Solving for $\mu_1$ and $\mu_2$ in terms of $x_1$ and $x_2$ gives us:

$$\begin{aligned} \mu_1 &= 8 - 4x_1 \\ \mu_2 &= -2 + 4x_1 - 2x_2 \end{aligned}$$

Observe that $\mu_1 g_1(\boldsymbol{x}) = 0$ and $\mu_2 g_2(\boldsymbol{x}) = 0$ gives us four cases to consider:

**Case 1:** Suppose $\mu_1 = 0$ and $\mu_2 = 0$. Then

$$\begin{aligned} 0 &= 8 - 4x_1 \\ 0 &= -2 + 4x_1 - 2x_2, \end{aligned}$$

which implies $x_1 = 2$ and $x_2 = 3$. In this case $g_1(2,3) = 5/2$, which violates feasibility. Therefore $(2,3,0,0)$ is not a KKT point.

**Case 2:** Suppose $\mu_1 = 0$ and $g_2(\boldsymbol{x}) = 0$. Then

$$\begin{aligned} 0 &= 8 - 4x_1 \\ 0 &= x_2 - 1, \end{aligned}$$

which implies $x_1 = 2$ and $x_2 = 1$. In this case $\mu_2 = -4$, which violates nongegativity. Therefore $(2, 1, 0, -4)$ is not a KKT point.

**Case 3:** Suppose $g_1(\boldsymbol{x}) = 0$ and $\mu_2 = 0$. Then

$$\begin{aligned} 0 &= x_1/2 + x_2 - 3/2 \\ 0 &= -2 + 4x_1 - 2x_2, \end{aligned}$$

which implies $x_1 = 1$ and $x_2 = 1$. In this case, $\mu_1 = 4$ and $g_2(1,1) = 0$, and everything is satisfied. Therefore $(1, 1, 4, 0)$ is a KKT point.

**Case 4:** Suppose $g_1(\boldsymbol{x}) = 0$ and $g_2(\boldsymbol{x}) = 0$. Then

$$\begin{aligned} 0 &= x_1/2 + x_2 - 3/2 \\ 0 &= x_2 - 1, \end{aligned}$$

which implies $x_1 = 1$ and $x_2 = 1$. In this case, $\mu_1 = 4$ and $\mu_2 = 0$. So we obtain the same KKT point $(1, 1, 4, 0)$.

From our calculations above, we see that the only KKT point is $(\boldsymbol{x}, \boldsymbol{\mu}) = (1, 1, 4, 0)$.

### Problem 1.b

**Exercise 2.** Find an optimal solution $\boldsymbol{x}^*$ to this NLP1 and clearly explain why $\boldsymbol{x}^*$ is optimal

**Solution 2.** Note that $f$ is a convex function since its Hessian

$$\mathrm{H}(f)(\boldsymbol{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive semi-definite for all $\boldsymbol{x} \in \mathbb{R}^n$. Simiarly, $g_1$ and $g_2$ are convex functions since they are affine. Also $f$, $g_1$, and $g_2$ are all differentiable as well. Therefore setting $\boldsymbol{x}^* = (1, 1)$ and $\boldsymbol{\mu}^* = (4, 0)$, then since $(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$ is a KKT point, we see that $\boldsymbol{x}^*$ is a global minimum.

### Problem 1.c and 1.d

**Exercise 3.** Without any calculations or geometric arguments, but based only on your answer to part a and b, give an optimal solution $\boldsymbol{x}^{**}$ to the following NLP2:

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) \\ \text{subject to} & g_1(x_1, x_2) \leq 0 \end{array}$$

Clearly explain why $\boldsymbol{x}^{**}$ is optimal for NLP2.

**Solution 3.** Note that the equations which describe a KKT point of NLP2 correspond to setting $\mu_2 = 0$ in the equations which describe a KKT point of NLP1. In particular, the KKT point $(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = (1, 1, 4, 0)$ for NLP1 corresponds to a KKT point $(\boldsymbol{x}^{**}, \boldsymbol{\mu}^{**}) = (1, 1, 4)$ for NLP2, where $\boldsymbol{x}^{**} = (1, 1)$ and $\boldsymbol{\mu}^{**} = 4$. Again, since $f$ and $g_1$ are convex and differentiable, it follows that $\boldsymbol{x}^{**}$ is a global minimum for NLP2.

## Problem 2

Let

$$\begin{aligned} f(x_1, x_2) &= x_1 - x_2 \\ h(x_1, x_2) &= x_1^2 + x_2^2 - 1 \end{aligned}$$

Consider the following equality constrained nonlinear optimization problem (NLP):

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & h(\boldsymbol{x}) = 0 \end{array}$$

### Problem 2.a

**Exercise 4.** Find all solutions to the KKT FONC for this NLP.

**Solution 4.** A KKT point for this NLP is a pair $(\boldsymbol{x}, \lambda)$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \lambda \nabla h(\boldsymbol{x}) &= 0 && \text{Stationary} \\ h(\boldsymbol{x}) &= 0 && \text{Feasibility} \end{aligned}$$

Suppose $(\boldsymbol{x}, \lambda) = (x_1, x_2, \lambda)$ is a KKT point. Then we have

$$\begin{aligned} 1 + 2\lambda x_1 &= 0 \\ -1 + 2\lambda x_2 &= 0 \\ x_1^2 + x_2^2 - 1 &= 0 \end{aligned}$$

From the first two equations, we obtain $x_1 = -x_2$. Then from the third equation, we obtain $2x_1^2 = 1$. In other words, $\boldsymbol{x} = (\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ and $\lambda = \mp\sqrt{2}/2$. So there are two KKT points for this NLP.

### Problem 2.b

**Exercise 5.** Find all points that satisfy the KKT FOSC for this NLP.

**Solution 5.** We cannot apply FOSC here because $h$ is not affine.

### Problem 2.c

**Exercise 6.** Regardless of your answer in part b, find an optimal solution $\boldsymbol{x}^*$ to this NLP.

**Solution 6.** We first note that every point $\boldsymbol{x}$ which is feasible is also regular. Indeed, we have $\nabla h(\boldsymbol{x}) = (2x_1, 2x_2)^\top \neq 0$ (since $(0,0)$ is not feasible). Therefore if $\boldsymbol{x}$ is a local minimum of $f$, then there exists a unique $\lambda$ such that $(\boldsymbol{x}, \lambda)$ is a KKT point. There are only two possible cases, namely $\boldsymbol{x}^\pm = (\pm\sqrt{2}/2, \pm\sqrt{2}/2)$ and $\lambda^\pm = \mp\sqrt{2}/2$ . A quick calculation shows that

$$\begin{aligned} f(\boldsymbol{x}^+) &= 0 \\ &> -\sqrt{2} \\ &= f(\boldsymbol{x}^-). \end{aligned}$$

Finally note that $f$ being continuous function defined on the compact domain $\{h = 0\}$, we see that $f$ must attain a maximum value and a minimum value. Clearly $f$ attains a maximum value at $\boldsymbol{x}^+$ and attains a minimum value at $\boldsymbol{x}^-$.

## Problem 3

Let

$$\begin{aligned} f(x_1, x_2) &= x_1 + x_2 \\ g(x_1, x_2) &= x_1^2 + x_2^2 - 4 \end{aligned}$$

Consider the following inequality constrained nonlinear optimization problem (INLP):

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g(\boldsymbol{x}) \leq 0 \\ & -g(\boldsymbol{x}) \leq 0 \end{array}$$

Note that this NLP is equivalent to the following equality constrained nonlinear optimization problem (ENLP):

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & g(\boldsymbol{x}) = 0 \end{array}$$

## Problem 3.a

**Exercise 7.** Solve this INLP geometrically. Let $\boldsymbol{x}^*$ be an optimal solution you found.

**Solution 7.** The equation $g(\boldsymbol{x}) = 4$ describes a circle in the plane which is centered at 0 and has radius 2. Every point on this circle has the form $(2\cos\theta, 2\sin\theta)$ for a unique $\theta \in [0, 2\pi)$. Thus

$$\min_{g(\boldsymbol{x})=4} f(\boldsymbol{x}) = \min_{\theta\in[0,2\pi)} 2(\cos\theta + \sin\theta) = \min_{\theta\in[0,2\pi)} h(\theta)$$

where we set $h(\theta) = 2(\cos\theta + \sin\theta)$. For $\theta \in [0, 2\pi)$, observe that

$$\begin{aligned} h'(\theta) = 0 &\iff 2(\cos\theta - \sin\theta) = 0 \\ &\iff \cos\theta = \sin\theta \\ &\iff \theta = \pi/4 \text{ or } \theta = 5\pi/4. \end{aligned}$$

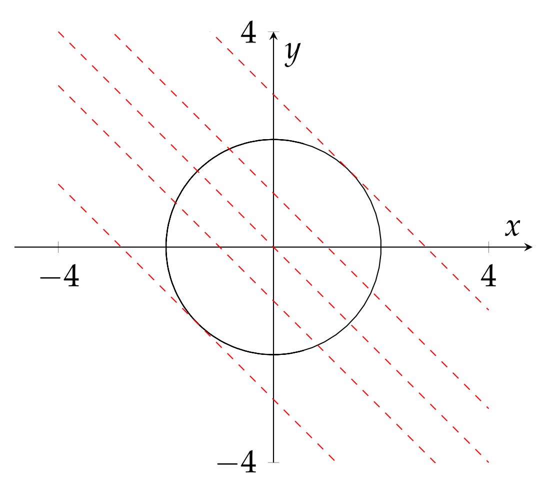
Thus the only critical points of $h$ (excluding the boundary points) are $\theta = \pi/4$ and $\theta = 5\pi/4$. Furthermore, notice that

$$\begin{aligned} h(0) &= 1 \\ h(\pi/4) &= 2\sqrt{2} \\ h(5\pi/4) &= -2\sqrt{2} \end{aligned}$$

It follows that $h$ has a global minimum at $\theta = 5\pi/4$. In other words, $f$ has a global minimum at $\boldsymbol{x}^* = (-\sqrt{2}, -\sqrt{2})^\top$.

Alternative geometric solution:

**Solution 8.** Below we've graphed the circle of radius centered at 0 and of radius 2 (that is, the feasible region), together with countours of the objective function $f$ for various values $c$ (in the image below, we drew $\{f = 2\sqrt{2}\}$, $\{f = 1\}$, $\{f = 0\}$, $\{f = -1\}$, and $\{f = -2\sqrt{2}\}$).




Clearly, the function $f$ will be minimized with value $c$ when the contour $\{f = c\}$ intersects the circle $\{g = 0\}$ tangentially. To determine which $c$-value does $\{f = c\}$ intersect $\{g = 0\}$ tangentially, we first solve the system of equations below:

$$\begin{aligned} x^2 + y^2 - 4 &= 0 \\ x + y - c &= 0 \end{aligned}$$

From these two equations, we obtain

$$y^2 - 2cy + c^2 - 4 = 0.$$

The discriminant $\Delta$ of $y^2 - 2cy + c^2 - 4$ is

$$\begin{aligned} \Delta &= 4c^2 - 8(c^2 - 4) \\ &= -4c^2 + 32. \end{aligned}$$

We have $\Delta = 0$ if and only if $c = \pm 2\sqrt{2}$. These two $c$-values correspond to the two contour lines which intersect the circle tangentially. The smaller $c$-value is clearly the min and the larger $c$-value is clearly the max.

### Problem 3.b

**Exercise 8.** Check whether $\boldsymbol{x}^*$ is a regular point of the constraints of this ENLP/INLP.

**Solution 9.** The INLP system has no regular points since $\{\nabla g(\boldsymbol{x}), -\nabla g(\boldsymbol{x})\}$ is always linearly dependent for all $\boldsymbol{x}$. On the other hand, $\boldsymbol{x}^*$ is a regular point for the ENLP system. Indeed, we have $\nabla g(\boldsymbol{x}) = (2x_1, 2x_2)^\top$ for all $\boldsymbol{x}$, in particular $\nabla g(\boldsymbol{x}^*) \neq 0$.

### Problem 3.c

**Exercise 9.** Check whether KKT FONC hold at $\boldsymbol{x}^*$. Explain.

**Solution 10.** A KKT point for the ENLP is a pair $(\boldsymbol{x}, \lambda)$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \lambda \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &= 0 && \text{Feasibility} \end{aligned}$$

Suppose $(\boldsymbol{x}, \lambda) = (x_1, x_2, \lambda)$ is a KKT point. Then we have

$$\begin{aligned} 1 + 2\lambda x_1 &= 0 \\ 1 + 2\lambda x_2 &= 0 \\ x_1^2 + x_2^2 - 4 &= 0 \end{aligned}$$

From the first two equations, we obtain $x_1 = x_2$. Then from the third equation, we obtain $2x_1^2 = 4$. In other words, $\boldsymbol{x} = (\pm\sqrt{2}, \pm\sqrt{2})$ and $\lambda = \mp\sqrt{2}/4$. So there are two KKT points for ENLP. One of the KKT point is $(\boldsymbol{x}^*, \lambda^*)$ where $\boldsymbol{x}^* = (-\sqrt{2}, -\sqrt{2})^\top$ and $\lambda^* = -\sqrt{2}/4$. Therefore KKT FONC hold at $\boldsymbol{x}^*$.

A KKT point for the INLP is a pair $(\boldsymbol{x}, \boldsymbol{\mu})$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \boldsymbol{\mu}^\top \nabla \boldsymbol{g}(\boldsymbol{x}) &= 0 && \text{Stationary} \\ \boldsymbol{g}(\boldsymbol{x}) &\leq 0 && \text{Feasibility} \\ \boldsymbol{\mu} &\geq 0 && \text{Nonnegativity} \\ \boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{x}) &= 0 && \text{Complementary slackness} \end{aligned}$$

Suppose $(\boldsymbol{x}, \boldsymbol{\mu}) = (x_1, x_2, \mu_1, \mu_2)$ is a KKT point. Then we have

$$\begin{aligned} 1 + 2\mu_1 x_1 - 2\mu_2 x_1 &= 0 \\ 1 + 2\mu_1 x_2 - 2\mu_2 x_2 &= 0 \\ x_1^2 + x_2^2 - 4 &= 0 \end{aligned}$$

From the first two equations, we obtain $x_1 = x_2$. Then from the third equation, we obtain $2x_1^2 = 4$. In other words, $\boldsymbol{x} = (\pm\sqrt{2}, \pm\sqrt{2})$ and $\mu_1 - \mu_2 = \mp\sqrt{2}/4$. There are many KKT points for NLP. One of the KKT points is $(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$ where $\boldsymbol{x}^* = (-\sqrt{2}, -\sqrt{2})^\top$ and $\boldsymbol{\mu}^* = (0, \sqrt{2}/4)$. Therefore KKT FONC hold at $\boldsymbol{x}^*$.

### Problem 3.d

**Exercise 10.** Check whether KKT FOSC hold at $\boldsymbol{x}^*$. Explain

**Solution 11.** The KKT FOSC do not hold for the INLT because $-g(\boldsymbol{x})$ is not convex. Similary, the KKT FOSC does not hold for the ENLT because $g(\boldsymbol{x})$ is not affine.

## Problem 4

Let

$$\begin{aligned} f(x_1, x_2) &= x_1^2/2 + x_2^2/2 + x_1 \\ g(x_1, x_2) &= -x_1 \end{aligned}$$

Consider the following nonlinear optimization problem (NLP) and the primal problem (PP):

$$\begin{aligned} &\text{minimize} && f(\boldsymbol{x}) \\ &\text{subject to} && g(\boldsymbol{x}) \leq 0 \end{aligned}$$

## Problem 4.a

**Exercise 11.** Find an optimal solution and the optimal objective value to the PP.

**Solution 12.** Both $f$ and $g$ are convex and differentiable, so it suffices to find a KKT point for PP. A point $(\boldsymbol{x}, \mu)$ is a KKT point for PP if

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \mu \nabla g(\boldsymbol{x}) &= 0 && \text{Stationary} \\ g(\boldsymbol{x}) &\leq 0 && \text{Feasibility} \\ \mu &\geq 0 && \text{Nonnegativity} \\ \mu g(\boldsymbol{x}) &= 0 && \text{Complementary slackness} \end{aligned}$$

Suppose $(\boldsymbol{x}, \mu) = (x_1, x_2, \mu)$ is a KKT point. Then we have

$$\begin{aligned} x_1 + 1 - \mu &= 0 \\ x_2 &= 0 \\ -x_1 &\leq 0 \end{aligned}$$

If $\mu = 0$, then $x_1 = -1$, which contradicts the fact that $-x_1 \leq 0$. Thus we must have $g_1(\boldsymbol{x}) = 0$, which implies $x_1 = 0$, $x_2 = 0$, and $\mu = 1$. Thus a KKT point for PP is given by $(\boldsymbol{x}^*, \mu^*) = (0, 0, 1)$ where $\boldsymbol{x}^* = (0, 0)$ and $\mu = 1$. Hence $\boldsymbol{x}^*$ is a global minimizer for this NLP with optimal value being $f(\boldsymbol{x}^*) = 0$.

## Problem 4.b and 4.c

**Exercise 12.** Derive the Lagrangian Dual Problem (DP) (in dual variables only). Check whether the Strong Duality Theorem holds. Explain why.

**Solution 13.** The Langrangian is

$$L(\boldsymbol{x}, \mu) = f(\boldsymbol{x}) + \mu g(\boldsymbol{x}) = x_1^2/2 + x_2^2/2 + (1 - \mu)x_1.$$

Observe that $L(\boldsymbol{x}, \mu)$ is a convex and differentiable function with $\mu$ fixed. Also observe

$$\begin{aligned} \boldsymbol{x}^* \text{ is a global minimizer of } L(-, \mu) &\iff \nabla_{\boldsymbol{x}} L(\boldsymbol{x}^*, \mu) = 0 \\ &\iff \begin{pmatrix} x_1^* + 1 - \mu \\ x_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff x_1^* = \mu - 1 \text{ and } x_2^* = 0. \end{aligned}$$

Therefore if we define

$$\mathrm{d}(\mu) = \min_{\boldsymbol{x} \in \mathbb{R}^n} L(\boldsymbol{x}, \mu),$$

then we see that $\mathrm{d}(\mu) = -(\mu - 1)^2/2$. The Lagrandian dual of PP then is

$$d^* = \max_{\mu \in \mathbb{R}_{\geq 0}} \min_{x \in \mathbb{R}^n} L(x, \mu) = \max_{\mu \in \mathbb{R}_{\geq 0}} \mathrm{d}(\mu) = 0,$$

where the max is attained at $\mu = 0$. Setting

$$p^* = \min_{g(\boldsymbol{x}) \leq 0} f(\boldsymbol{x}) = 0.$$

Then we see that the duality gap $p^* - d^* = 0$ is zero. Therefore the strong duality theorem holds.

## Problem 4.d

**Exercise 13.** Identify a saddle point of the Lagrange function.

**Solution 14.** Since we have strong duality, a saddle point is given by $(\boldsymbol{x}^*, \mu^*)$ where $\boldsymbol{x}^* = (0, 0)$ and $\mu^* = 0$.

## Problem 5

Let $\boldsymbol{b} \in \mathbb{R}^m$, let $\boldsymbol{c} \in \mathbb{R}^n$, let $f, g_j \colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{x} = \sum_{j=1}^n c_j x_j$ and $g_j(\boldsymbol{x}) = -x_j$ for all $1 \leq j \leq n$, and let $A$ be an $m \times n$ matrix. Consider the following optimization problem referred to as the linear program (LP):

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}) \leq 0 \\ & A\boldsymbol{x} = \boldsymbol{b} \end{array}$$

For each $1 \leq i \leq m$, we define $h_i \colon \mathbb{R}^n \to \mathbb{R}$ by $h_i(\boldsymbol{x}) = \sum_{j=1}^n a_{ij} x_j$ where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Thus this LP can be expressed as

$$\begin{array}{lll} \text{minimize} & f(\boldsymbol{x}) & \\ \text{subject to} & g_j(\boldsymbol{x}) \leq 0 & \text{for all } 1 \leq j \leq n \\ & h_i(\boldsymbol{x}) = b_i & \text{for all } 1 \leq i \leq m \end{array}$$

### Problem 5.a and 5.b

**Exercise 14.** Apply the theory of nonlinear optimization to this LP and write the complete KKT FONC for optimality for this LP. Are the conditions sufficient? Explain why.

**Solution 15.** A KKT point for LP is a triple $(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ which satisfies the following:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \boldsymbol{\mu}^\top \nabla \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{\lambda}^\top \nabla \boldsymbol{h}(\boldsymbol{x}) &= 0 && \text{Stationary} \\ \boldsymbol{g}(\boldsymbol{x}) &\leq 0 && \text{Feasibility} \\ \boldsymbol{h}(\boldsymbol{x}) &= 0 && \text{Feasibility} \\ \boldsymbol{\mu} &\geq 0 && \text{Nonnegativity} \\ \boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{x}) &= 0 && \text{Complementary slackness} \end{aligned}$$

Suppose that $(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is a KKT point for LP. Then we have

$$\begin{aligned} c_1 - \mu_1 + \lambda_1 a_{11} + \lambda_2 a_{21} + \cdots + \lambda_m a_{m1} &= 0 \\ c_2 - \mu_2 + \lambda_1 a_{12} + \lambda_2 a_{22} + \cdots + \lambda_m a_{m2} &= 0 \\ &\vdots \\ c_n - \mu_n + \lambda_1 a_{1n} + \lambda_2 a_{2n} + \cdots + \lambda_m a_{mn} &= 0 \end{aligned}$$

Alternatively, in matrix form this says

$$\lambda^\top A = \boldsymbol{\mu}^\top - \boldsymbol{c}^\top.$$

If we apply $\boldsymbol{g}(\boldsymbol{x}) = -\boldsymbol{x}$ to both sides, we obtain

$$\begin{aligned} -\boldsymbol{\lambda}^\top \boldsymbol{b} &= -\lambda^\top A\boldsymbol{x} \\ &= -\boldsymbol{\mu}^\top \boldsymbol{x} + \boldsymbol{c}^\top \boldsymbol{x} \\ &= \boldsymbol{c}^\top \boldsymbol{x} \\ &= f(\boldsymbol{x}). \end{aligned}$$

In particular, if $(\boldsymbol{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ is a KKT point of LP, then since $f$ and the $g_j$ are all convex, differentiable functions, and the $h_i$ are all affine functions, we see that $\boldsymbol{x}$ is optimal with optimal objective value being given by $f(\boldsymbol{x}) = -\lambda^\top \boldsymbol{b}$.

## Problem 5.b

**Exercise 15.** Are the conditions you wrote in part a sufficient? Explain why.

**Solution 16.** Yes, because $f$ and the $g_j$ are all convex, differentiable functions, and the $h_i$ are all affine functions. Therefore KKT FOSC holds.

## Problem 5.c

**Exercise 16.** Treating this LP is a primal problem, relax the equality constraint, and derive the dual problem (in dual variables only!).

**Solution 17.** The Langrangian is

$$L(\boldsymbol{x}, \boldsymbol{\mu}) = f(\boldsymbol{x}) + \boldsymbol{\mu}^\top \boldsymbol{g}(\boldsymbol{x}) = (\boldsymbol{c} - \boldsymbol{\mu})^\top \boldsymbol{x}.$$

Observe that $L(\boldsymbol{x}, \boldsymbol{\mu})$ is a convex and differentiable function with $\mu$ fixed. Also observe

$$\begin{aligned} \boldsymbol{x}^* \text{ is a global minimizer of } L(-, \boldsymbol{\mu}) &\iff \nabla_{\boldsymbol{x}} L(\boldsymbol{x}^*, \boldsymbol{\mu}) = 0 \\ &\iff \begin{pmatrix} c_1 - \mu_1 \\ \vdots \\ c_n - \mu_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ &\iff c_i = \mu_i \text{ for all } 1 \leq i \leq n \end{aligned}$$

Therefore if we define

$$\mathrm{d}(\boldsymbol{\mu}) = \min_{\boldsymbol{x} \in \mathbb{R}^n} L(\boldsymbol{x}, \boldsymbol{\mu}),$$

then we see that

$$\mathrm{d}(\boldsymbol{\mu}) = \begin{cases} 0 & \text{if } \boldsymbol{\mu} = \boldsymbol{c} \\ -\infty & \text{else} \end{cases}$$

The Lagrandian dual of PP then is

$$d^* = \max_{\boldsymbol{\mu} \in \mathbb{R}_{\geq 0}} \min_{\boldsymbol{x} \in \mathbb{R}^n} L(\boldsymbol{x}, \boldsymbol{\mu}) = \max_{\boldsymbol{\mu} \in \mathbb{R}_{\geq 0}} \mathrm{d}(\boldsymbol{\mu}) = 0.$$