Advanced Numerical Analysis Homework 2

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Throughout this homework, $\|\cdot\|$ denotes the ℓ_2-norm. Also if $\boldsymbol{x}$ and $\boldsymbol{y}$ are two column vectors in $\mathbb{R}^n$, then we write $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \boldsymbol{y}$.

1 Problem 1

Exercise 1. Let $a_0, a_1, \ldots, a_n$ be $n+1$ equispaced points on $[-1,1]$, where $a_0 = -1$ and $a_n = 1$. Assemble these $n+1$ values into a column vector $\boldsymbol{u}$, and use MATLAB's vander to generate Vandermonde matrices A from vector $\boldsymbol{u}$ for $n = 9, 19, 29, 39$. Let $\boldsymbol{x} = (1, 1, \ldots, 1)^\top$ and $\boldsymbol{b} = A\boldsymbol{x}$. Pretend that we do not know $\boldsymbol{x}$ and use numerical algorithms to solve this linear system for $\boldsymbol{x}$. Let $\widehat{\boldsymbol{x}}$ be the computed solution. Compute the relative forward errors $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\| / \|\boldsymbol{x}\|$ and the smallest relative backward errors

$$\frac{\|\boldsymbol{b} - A\widehat{\boldsymbol{x}}\|}{\|A\|\|\widehat{\boldsymbol{x}}\|} = \min\left\{ \frac{\|\delta A\|}{\|A\|} \mid (A + \delta A)\widehat{\boldsymbol{x}} = \boldsymbol{b} \right\},$$

where $\|\cdot\|$ denotes the ℓ_2-norm, for the following:

1. GEPP (MATLAB's backslash);
2. QR factorization of A;
3. Cramer's rule;
4. A^{-1} multiplied by $\boldsymbol{b}$;
5. GE without pivoting.

Comment on the forward/backward stability of these methods.

Solution 1. 1. We work in MATLAB below:

```
n = [9,19,29,39];
ForwardErrors = zeros(4,4);
BackwardErrors = zeros(4,4);

for k = 1:4
  u=(-1:2/n(k):1)';
  x = ones(n(k)+1,1);
  A = vander(u);
  b = A*x;
  [Q,R] = qr(A);

  xh = zeros(n(k)+1,4);
  xh(:,1) = A\b;
  xh(:,2) = R\(Q'*b);
  for j = 1:length(A)
      C = A;
      C(:,j) = b;
      xh(j,3) = det(C)/det(A);
  end
```

```
  xh(:,4) = inv(A)*b;

  for j = 1:4
     ForwardErrors(k,j) = norm(xh(:,j)-x)/norm(x);
     BackwardErrors(k,j) = norm(b-A*xh(:,j))/(norm(A)*norm(xh(:,j)));
  end;
end;
```

We see that GEPP and QR factorization are backward stable, however the other three algorithms are not.

2 Problem 2

Exercise 2. Consider the eigenvalue problem $Av = \lambda v$. Let $(\widehat{\lambda}, \widehat{v})$ be a computed eigenpair, which is assumed to be the exact eigenpair of a perturbed matrix $A + \delta A$. Show that the minimum ℓ_2-norm of all such δA is

$$\frac{\|A\widehat{v} - \widehat{\lambda}\widehat{v}\|}{\|\widehat{v}\|}, \tag{1}$$

and find a particular δA whose ℓ_2-norm is the minimum. (Note that this result can help us experimentally determine if an eigenvalue algorithm is backward stable).

Solution 2. Given such δA, we have $\delta A\widehat{v} = \widehat{\lambda}\widehat{v} - A\widehat{v}$. Therefore since $\|\delta A\|\|\widehat{v}\| \geq \|\delta A\widehat{v}\|$, we see that

$$\|\delta A\| \geq \frac{\|A\widehat{v} - \widehat{\lambda}\widehat{v}\|}{\|\widehat{v}\|}.$$

The norm is minimized when

$$\delta A = \frac{(\widehat{\lambda}\widehat{v} - A\widehat{v})\widehat{v}^\top}{\|v\|^2}.$$

3 Problem 3

Exercise 3. Give a proof that the worst-case growth factor $\rho_n = 2^{n-1}$ for GEPP. Compared to $\rho_n \leq Cn^{\frac{1}{2}+\frac{1}{4}\ln n}$ with complete pivoting and $\rho_n \leq 1.5n^{\frac{3}{4}\ln n}$ with rook pivoting, this is much larger. However, we construct matrices with random elements, each are independent samples from the normal distribution of mean 0 and standard deviation $\frac{1}{\sqrt{n}}$ ($\mathrm{A} = \mathrm{randn(n,n)/sqrt(n)}$). Let $n = 32, 64, \ldots, 512$, and for each n, repeat the experiment 1000 times. Find the percentage of experiments when $\rho_n > \sqrt{n}$. Make brief comments on the chance of having a large ρ_n.

4 Problem 4

Exercise 4. Though pivoting is needed for factorizing general matrices, it is not needed for symmetric positive definite and diagonally dominant matrices.

1. For a symmetric positive definite matrix $A = (a_{ij})$, with the one-step Cholesky factorization

$$A = \begin{pmatrix} a_{11} & \boldsymbol{w}^\top \\ \boldsymbol{w} & K \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{\boldsymbol{w}}{\sqrt{a_{11}}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & K - \frac{\boldsymbol{w}\boldsymbol{w}^\top}{a_{11}} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{\boldsymbol{w}^\top}{\sqrt{a_{11}}} \\ 0 & I \end{pmatrix} = R_1^\top A_1 R_1,$$

   show that the submatrix $K - (\boldsymbol{w}\boldsymbol{w}^\top)/a_{11}$ is symmetric positive definite. Consequently, the factorization can be completed without break-down. Then, show that $\|R\| = \|A\|^{1/2}$, which means the element in R are uniformly bounded by that of $\|A\|$. Explain why this observation leads to the backward stability of Cholesky factorization.

2. Suppose that $A = \begin{pmatrix} \alpha & w^\top \\ v & C \end{pmatrix}$ is column diagonally dominant, with one-step LU factorization

$$A = \begin{pmatrix} 1 & 0 \\ \frac{v}{\alpha} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - \frac{vw^\top}{\alpha} \end{pmatrix} \begin{pmatrix} \alpha & w^\top \\ 0 & I \end{pmatrix}.$$

Show that the sub-matrix $C - (vw^\top)/\alpha$ is also column diagonally dominant, and no pivoting is needed.

Solution 3. 1. Clearly both K and $-(ww^\top)/a_{11}$ are symmetric, so their sum $K - (ww^\top)/a_{11}$ is symmetric also. To see that it is positive-definite, observe that for nonzero $x \in \mathbb{R}^{n-1}$ where $x = (x_2, \dots, x_n)^\top$, positive-definiteness of A implies

$$\begin{aligned} 0 &\le (x_1, x^\top) \begin{pmatrix} a_{11} & w^\top \\ w & K \end{pmatrix} \begin{pmatrix} x_1 \\ x \end{pmatrix} \\ &= a_{11}x_1^2 + x_1 2\langle w, x\rangle + x^\top K x. \end{aligned}$$

In particular, setting $x_1 = -\langle w, x\rangle / a_{11}$ gives us

$$x^\top \left(K - \frac{w^\top w}{a_{11}} \right) x = x^\top K x - \frac{\langle w, x\rangle^2}{a_{11}} \ge 0,$$

which implies $K - (ww^\top)/a_{11}$ is positive-definite.

Now we show that $\|R\|^2 = \|A\| = \|R^\top R\|$. On the one hand we have $\|R^\top R\| \le \|R^\top\|\|R\| = \|R\|^2$. For the reverse inequality, let $x \in \mathbb{R}^n$ such that $\|x\| = 1$. Then

$$\begin{aligned} \|Rx\|^2 &= \langle Rx, Rx\rangle \\ &= \langle x, R^\top R x\rangle \\ &\le \|x\|\|R^\top R x\| \\ &= \|R^\top R x\|, \end{aligned}$$

where we used Cauchy-Schwarz to get from the second line to the third line. In particular, this implies

$$\begin{aligned} \|R\|^2 &= \sup\{\|Rx\|^2 \mid \|x\| = 1\} \\ &\le \sup\{\|R^\top R x\| \mid \|x\| = 1\} \\ &= \|R^\top R\|. \end{aligned}$$

Thus we have $\|R\|^2 = \|A\| = \|R^\top R\|$. Now recall from class that as long as the growth factor

$$e_n = \frac{\max\limits_{1\le i,j,k\le n} |a_{ij}^{(k)}|}{\max\limits_{1\le i,j,k\le n} |a_{ij}|}$$

does not approach ∞ as $\varepsilon \to 0$, we will have backward stability. Thus since the element in R are uniformly bounded by that of $\|A\|$, we know that the growth factor is bounded above as $\varepsilon \to 0$, thus we have backward stability.

2. Let $2 \le i \le n$. Since A is diagonally dominant, we obtain the inequalities (corresponding to first row and ith row of A):

$$1 - \sum_{j\ne i} \left| \frac{a_{1j}}{\alpha} \right| \ge \left| \frac{a_{1i}}{\alpha} \right| \quad \text{and} \quad |a_{ii}| - |a_{i1}| \ge \sum_{j\ne i} |a_{ij}|.$$

Therefore we have

$$\begin{aligned}
\left| a_{ii} - \frac{a_{i1}a_{1i}}{\alpha} \right| &\geq |a_{ii}| - |a_{i1}| \left| \frac{a_{1i}}{\alpha} \right| \\
&\geq |a_{ii}| - |a_{i1}| \left(1 - \sum_{j \neq i} \left| \frac{a_{1j}}{\alpha} \right| \right) \\
&= |a_{ii}| - |a_{i1}| + \sum_{j \neq i} \left| \frac{a_{i1}a_{1j}}{\alpha} \right| \\
&\geq \sum_{j \neq i} |a_{ij}| + \sum_{j \neq i} \left| \frac{a_{i1}a_{1j}}{\alpha} \right| \\
&\geq \sum_{j \neq i} \left| a_{ij} - \frac{a_{i1}a_{1j}}{\alpha} \right| .
\end{aligned}$$

It follows that $C - (\boldsymbol{v}\boldsymbol{w}^\top)/\alpha$ is also diagonally dominant.