Methodology

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convex BOP
nonconvex BOP with a connected Pareto set
nonconvex BOP with a disconnected Pareto set
your own BOP (show the details of your construction; avoid trivial problems)

Introduction

The weighted-sum method is a commonly used approach for solving biobjective optimization problems. It involves assigning weights to the objectives and then optimizing a weighted sum of the objectives. This method can be effective for problems with a convex Pareto set, but it may not produce accurate results for nonconvex problems.

The epsilon-constraint method is an alternative approach that involves adding constraints to the optimization problem. These constraints limit the maximum difference between the values of the two objectives. This method can produce more accurate results for nonconvex problems, but it may be computationally more expensive than the weighted-sum method.

In our project, we will implement both of these methods and apply them to four test BOPs. We will compare the effectiveness of the two methods using the two aforementioned criteria, and include the results in our report.

1 Non-convex BOP with non-connected Pareto set

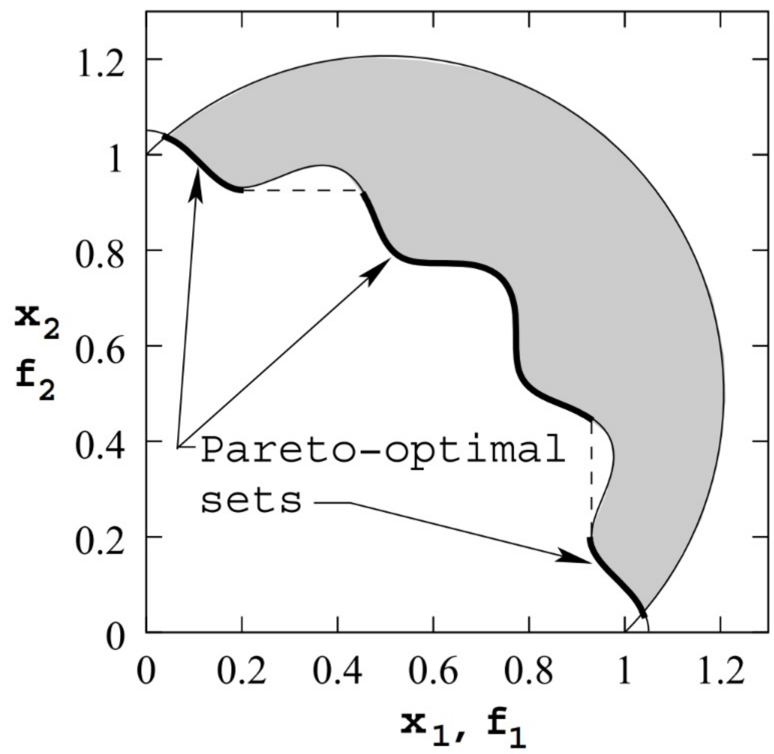
For this problem, let

$$\begin{aligned} f_1(\boldsymbol{x}) &= x_1 \\ f_2(\boldsymbol{x}) &= x_2 \\ c_1(\boldsymbol{x}) &= 1 + 0.1\cos\left(16\arctan\frac{x_1}{x_2}\right) - x_1^2 - x_2^2 \\ c_2(\boldsymbol{x}) &= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 0.5 \\ X &= \{\boldsymbol{x} \in [0,\pi]^2 \subseteq \mathbb{R}^2 \mid \boldsymbol{c}(\boldsymbol{x}) \leqq \boldsymbol{0}\}. \end{aligned}$$

For this problem, we consider the following BOP:

$$\begin{array}{ll} \text{minimize} & [f_1(x), f_2(x)] \\ \text{subject to} & \boldsymbol{x} \in X. \end{array} \tag{1}$$

This problem is considered as a test problem in [DPM02]. Since $x_1 = f_1$ and $x_2 = f_2$, the feasible and objective space coincide and is shown in the figure below:



In particular, the feasible set X is not convex set and the Pareto set Y_N is disconnected, so this is a nonconvex nonconnected Pareto BOP.

1.1 Weighted-Sum Method

We first solve (1) using the weighted-sum method. In other words, we solve the single objective problem

$$\begin{aligned} \text{minimize} \quad & F_w(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in X \\ & 0 \leq w \leq 1 \end{aligned} \tag{2}$$

where we set

$$F_w(\boldsymbol{x}) = wf_1(\boldsymbol{x}) + (1-w)f_2(\boldsymbol{x})$$

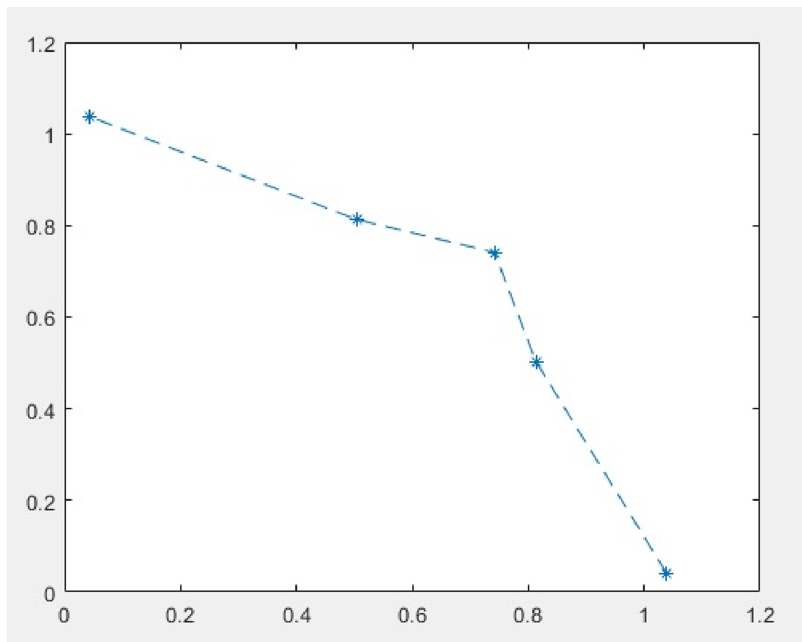
We will find optimals solutions to (2) using the MATLAB function x = fmincon(fun,xo,lb,ub,nonlcon) where the programming solver assumes that the programming has the form

$$\begin{aligned} \text{minimize} \quad & F_w(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{c}(\boldsymbol{x}) \leqq \boldsymbol{0} \\ & \boldsymbol{lb} \leqq \boldsymbol{x} \leqq \boldsymbol{ub}. \end{aligned}$$

The function starts at an initial guess $\boldsymbol{x}^0$ and attempts to find a *local* optimal solution $\widehat{\boldsymbol{x}}$. Further analysis will be needed in order to determine whether or not $\widehat{\boldsymbol{x}}$ is *global* optimal solution. Let us begin by using $\boldsymbol{x}^0 = (0.8, 0.8)^\top$ to be our initial guess. After setting up the correct MATLAB code (which is given in the Appendix), MATLAB gives us the following table:

w	exit flag	x1	x2	f1	f2
0	1	1.0384	0.041665	1.0384	0.041665
0.1	1	1.0384	0.041665	1.0384	0.041665
0.2	1	1.0384	0.041664	1.0384	0.041664
0.3	1	1.0384	0.041665	1.0384	0.041665
0.4	1	0.81296	0.50376	0.81296	0.50376
0.5	1	0.74162	0.74162	0.74162	0.74162
0.6	1	0.50376	0.81296	0.50376	0.81296
0.7	2	0.041665	1.0384	0.041665	1.0384
0.8	1	0.041664	1.0384	0.041664	1.0384
0.9	1	0.041665	1.0384	0.041665	1.0384
1	1	0.041665	1.0384	0.041665	1.0384

Let us explain what this table is telling us since we will be using the same format throughout the rest of the paper. We are using the weights $w = 0, 0.1, 0.2, \ldots, 1$ which is given in the first column in the table above. Consider the row corresponding to the weight $w = 0.5$. The (approximate) local optimal solution that we found corresponding to the weight $w = 0.5$ is given by $\widehat{\boldsymbol{x}} = (0.74162, 0.74162)^\top$. The f_1-value of $\widehat{\boldsymbol{x}}$ is $f_1(\widehat{\boldsymbol{x}}) = 0.74162$ and the f_2-value of $\widehat{\boldsymbol{x}}$ is $f_2(\widehat{\boldsymbol{x}}) = 0.74162$. Finally, the exit flag value is equal to 1, which tells us that $\widehat{\boldsymbol{x}}$ is a very good approximate to the local optimal solution. In general, positive exit flags correspond to successful outcomes, negative exit flags correspond to unsuccessful outcomes, and zero exit flag corresponds to the solver being halted by exceeding an iteration limit or limit on the number of function evaluations. This explains everything about the row corresponding to $w = 0.5$. The other rows in the table have similar interpretations as well. We now plot the objective values corresponding to the local optimal solutions that we found above:



In order to determine which of these points are guaranteed to be Pareto points, we will appeal to Proposition 3.9 on page 71 in the course text book which we will recall here (keeping the notation as given in the book):

Proposition 1.1. *Suppose $\widehat{\boldsymbol{x}}$ is a (global) optimal solution of the weighted sum optimization problem*

$$\min_{\boldsymbol{x} \in X} \sum_{k=1}^{p} w_k f_k(\boldsymbol{x}) \tag{3}$$

with $\boldsymbol{w} \in \mathbb{R}^p_{\geq}$. Then the following statement hold:

1. *If $\boldsymbol{w} \in \mathbb{R}^p_{\geq}$, then $\widehat{\boldsymbol{x}}$ is weakly efficient;*
2. *If $\boldsymbol{w} \in \mathbb{R}^p_{>}$ and $\mathcal{Y} := \boldsymbol{f}(X)$ is $\mathbb{R}^p_{\geqq}$-convex, then $\widehat{\boldsymbol{x}}$ is efficient (so $\widehat{\boldsymbol{y}} = \boldsymbol{f}(\widehat{\boldsymbol{x}})$ is a Pareto point or a non-dominated point);*
3. *If $\boldsymbol{w} \in \mathbb{R}^p_{\geq}$, $\mathcal{Y}$ is $\mathbb{R}^p_{\geqq}$-convex, and $\widehat{\boldsymbol{x}}$ is the* unique *optimal solution of (3) $\widehat{\boldsymbol{x}}$ is strictly efficient.*

Recall that this proposition only applies to *global* optimal solutions, and the MATLAB function that we used only calculates local optimal solutions. In fact, it's easy to check that if $0 \leq w < 1/2$ then there's a unique global optimal solution (approximately) at $\boldsymbol{a} := (1.0384, 0.041665)^\top$, if $w = 1/2$ then there are two global optimal solutions (approximately) at $\boldsymbol{a}$ and $\boldsymbol{b} := (0.041665, 1.0384)^\top$, and finally if $1/2 < w \leq 1$, then there's a unique global optimal solution (approximately) at $\boldsymbol{b}$. Thus the weighted-sum method for this problem will only guaranteee that these two intersection points are Pareto points.

1.2 Epsilon-Constraint Method

Next we solve (1) using the epsilon-constraint method. First let us recall Proposition 4.3, Proposition 4.4, and Theorem 4.5 on pages 99-100 in the course text book which we summarize in the proposition below (keeping the notation the same as given in the book):

Proposition 1.2. *For each $j = 1, \ldots, p$, we consider the ε-constraint problem:*

$$\begin{aligned} \textit{minimize} \quad & f_j(\boldsymbol{x}) \\ \textit{subject to} \quad & \boldsymbol{x} \in X \\ & f_k(\boldsymbol{x}) \leq \varepsilon_k \quad k = 1, \ldots, p \quad k \neq j \end{aligned} \tag{4}$$

where $\boldsymbol{\varepsilon} \in \mathbb{R}^p$. The following statements hold:

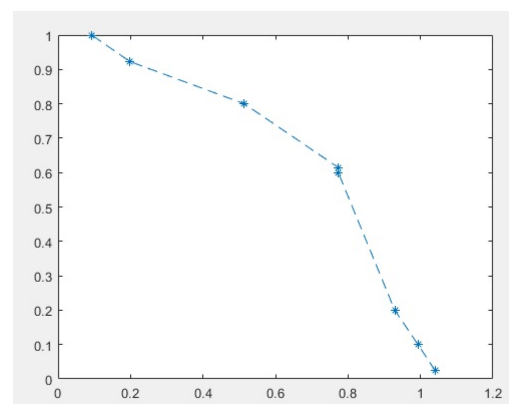
1. *If $\widehat{\boldsymbol{x}}$ is an optimal solution of (4) for some j, then $\widehat{\boldsymbol{x}}$ is weakly efficient;*
2. *If $\widehat{\boldsymbol{x}}$ is the unique optimal solution to (4) for some j, then $\widehat{\boldsymbol{x}}$ is strictly efficient (and hence $\widehat{\boldsymbol{x}}$ is efficient);*
3. *$\widehat{\boldsymbol{x}}$ is efficient if and only if there exists an $\widehat{\boldsymbol{\varepsilon}}$ such that $\widehat{\boldsymbol{x}}$ is an optimal solution of (4) for all j.*

In this case, we only need to consider the following epsilon-constraint problem

$$\begin{aligned} \text{minimize} \quad & f_1(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in X \\ & f_2(\boldsymbol{x}) - \varepsilon \leq 0 \end{aligned} \tag{5}$$

where $0.1 \leq \varepsilon \leq 1$. Indeed, it is straightforward to check that (5) will have a unique optimal solution for each such ε and so we can apply part 2 of Proposition (1.2). Furthermore, this unique optimal solution will be the only *local* optimal solution, so we can use the MATLAB function fmincon to obtain the optimal solutions to (5) for each such ε. Using the reference point $\boldsymbol{x}^0 = (0.8, 0.8)^\top$, MATLAB produces the following table and plot:

e	exit flag	x1	x2	f1	f2
0	-2	1.0406	0.023927	1.0406	0.023927
0.1	1	0.99324	0.1	0.99324	0.1
0.2	1	0.92905	0.19958	0.92905	0.19958
0.3	1	0.92905	0.19963	0.92905	0.19963
0.4	1	0.92905	0.19963	0.92905	0.19963
0.5	1	0.92905	0.19963	0.92905	0.19963
0.6	1	0.77315	0.6	0.77315	0.6
0.7	1	0.77308	0.61473	0.77308	0.61473
0.8	1	0.51394	0.8	0.51394	0.8
0.9	-2	0.1981	0.92292	0.1981	0.92292
1	1	0.093027	1	0.093027	1



Unlike the weighted-sum method, we can be

2 Non-convex BOP with connected Pareto set

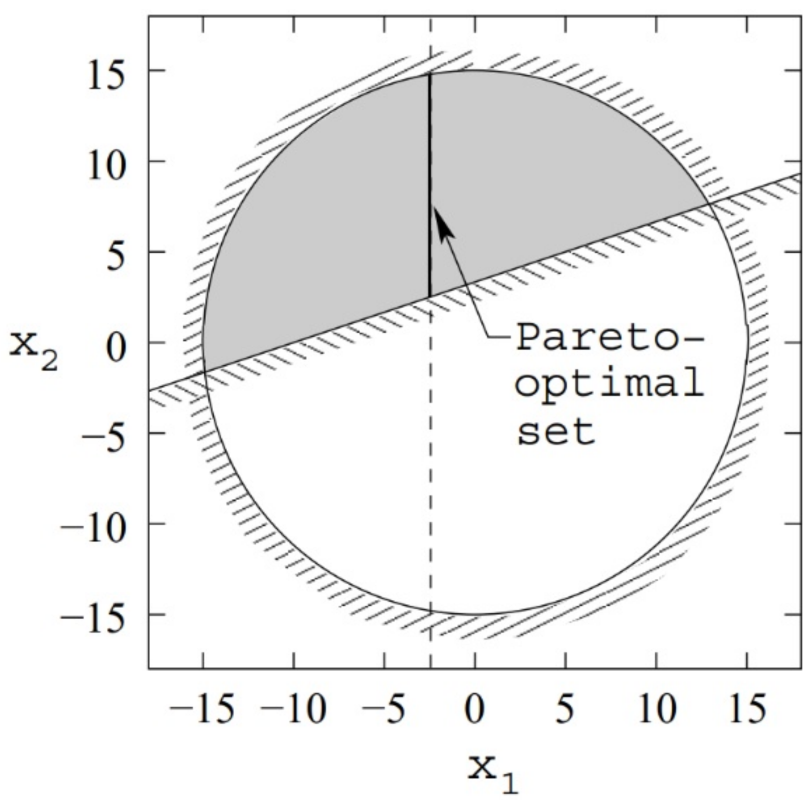
For this problem, let

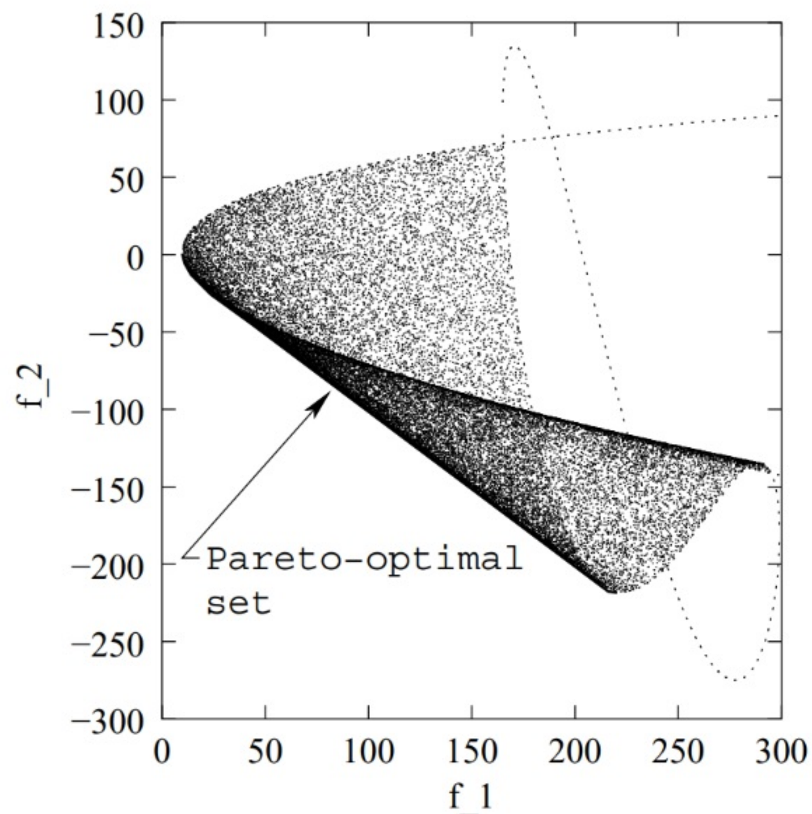
$$\begin{aligned} f_1(\boldsymbol{x}) &= 2 + (x_1 - 2)^2 + (x_2 - 1)^2 \\ f_2(\boldsymbol{x}) &= 9x_1 - (x_2 - 1)^2 \\ c_1(\boldsymbol{x}) &= x_1^2 + x_2^2 - 225 \\ c_2(\boldsymbol{x}) &= x_1 - 3x_2 + 10 \\ X &= \{\boldsymbol{x} \in [-20, 20]^2 \subseteq \mathbb{R}^2 \mid \boldsymbol{c}(\boldsymbol{x}) \leqq 0\}. \end{aligned}$$

For this problem, we consider the following BOP:

$$\begin{aligned} \text{minimize} \quad & [f_1(x), f_2(x)] \\ \text{subject to} \quad & \boldsymbol{x} \in X. \end{aligned} \tag{6}$$

This problem is considered as a test problem in [DPM02-1]. The feasible set X as well as the objective outcome set $Y = \boldsymbol{f}(X)$ is illustrated below:





Note that the Pareto set Y_N is connected. Also note that the matrix representation of the Hessian of f_1 is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and the matrix representation of the Hessian of f_2 is $\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$. In particular, f_1 is strictly convex, however f_2 is not convex (it is concave instead). Thus this is a nonconvex BOP whose Pareto set is connected.

2.1 Weighted-Sum Method

We first solve (6) using the weighted-sum method. In other words, we solve the single objective problem

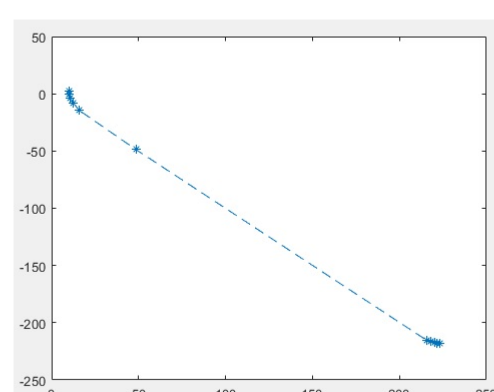
$$\begin{aligned} \text{minimize} \quad & F_w(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in X \\ & 0 \leq w \leq 1 \end{aligned} \tag{7}$$

where we set

$$F_w(\boldsymbol{x}) = wf_1(\boldsymbol{x}) + (1-w)f_2(\boldsymbol{x}).$$

Just like in the previous problem, we will find local optimal solutions to (7) (corresponding to each w) using the MATLAB function fmincon. After setting everything up in MATLAB, we obtain the following table and plot:

w	exit flag	x1	x2	f1	f2
0	1	-4.841	14.197	222.97	-217.74
0.1	1	-4.5615	14.29	221.67	-217.67
0.2	1	-4.2199	14.394	220.09	-217.38
0.3	1	-3.7922	14.513	218.14	-216.72
0.4	1	-3.2404	14.646	215.67	-215.37
0.5	1	-2.5	6.1226	48.491	-48.741
0.6	1	-1.2143	2.9286	16.051	-14.648
0.7	1	-0.35075	3.2164	12.439	-8.0692
0.8	1	0.26923	3.4231	10.867	-3.4482
0.9	1	0.73595	3.5787	10.247	-0.025852
1	1	1.1	3.7	10.1	2.61



Note however that all we can say is that these give weakly efficient solutions (with the exception of the point corresponding to weight $w = 0.5$). Indeed, $\mathcal{Y}$ is not $\mathbb{R}^2_{\geqq}$-convex and so we cannot apply the full strength of Proposition (1.1).

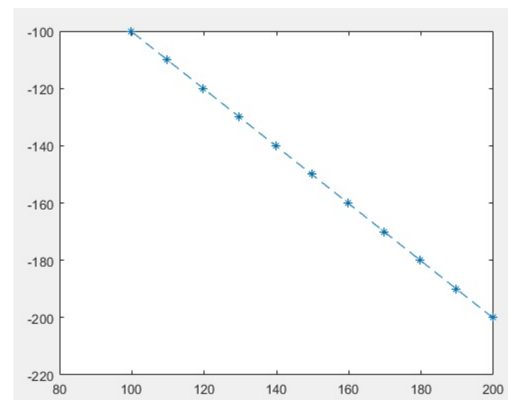
2.2 Epsilon-Constraint Method

Next we solve (6) using the epsilon-constraint method. In this case, we only need to consider the following epsilon-constraint problem

$$\begin{aligned} \text{minimize} \quad & f_2(\boldsymbol{x}) \\ \text{subject to} \quad & \boldsymbol{x} \in X \\ & f_1(\boldsymbol{x}) - \varepsilon \leq 0 \end{aligned} \tag{8}$$

where $-400 \leq \varepsilon \leq -100$. Then f_2 is convex in the new feasible region remains convex, so this just becomes a convex problem. Furthermore one can show that (**??**) has a unique (local and global) optimal solution in this case, and so we can apply part 2 of Proposition (1.2). Using the reference point $\boldsymbol{x}^0 = (-2.5, 8)^\top$, MATLAB produces the following table and plot:

e	exit flag	x1	x2	f1	f2
-200	1	-2.5	14.323	199.75	-200
-190	1	-2.5	13.942	189.75	-190
-180	1	-2.5	13.55	179.75	-180
-170	1	-2.5	13.145	169.75	-170
-160	1	-2.5	12.726	159.75	-160
-150	1	-2.5	12.292	149.75	-150
-140	1	-2.5	11.84	139.75	-140
-130	1	-2.5	11.368	129.75	-130
-120	1	-2.5	10.874	119.75	-120
-110	1	-2.5	10.354	109.75	-110
-100	1	-2.5	9.8034	99.75	-100



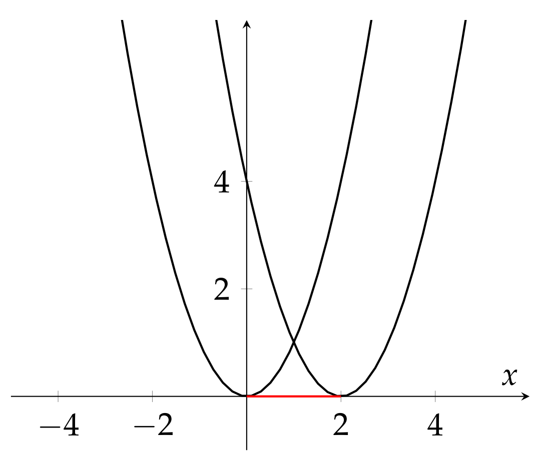
The epsilon-constraint method prevails over the the weighted-sum method yet again!

3 Convex BOP with connected Pareto set

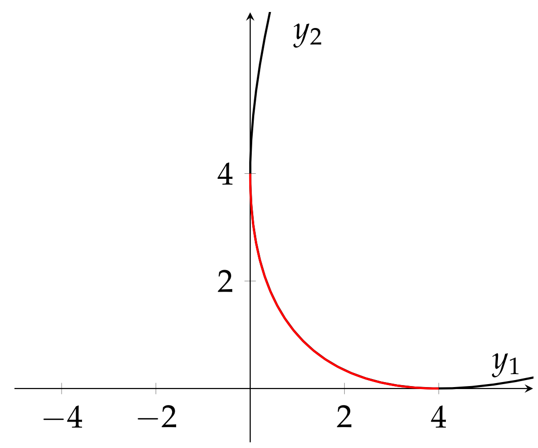
For this problem, let $f_1(x) = x^2$, let $f_2(x) = (x-2)^2$, and let $X = [-5, 5]$. We consider the following BOP:

$$\begin{aligned} \text{minimize} \quad & [f_1(x), f_2(x)] \\ \text{subject to} \quad & x \in X. \end{aligned} \tag{9}$$

This problem is considered as a test problem in [HHBW]. In the image below, we draw the graphs of f_1 and f_2, and we also draw the efficient set $X_E = [0, 2]$ (shaded in red):



Next we draw the outcome set $Y = \boldsymbol{f}(X)$ together with the Pareto front Y_N (shaded in red):



3.1 Weighted-Sum Method

We first solve (9) using the weighted-sum method. In other words, we solve the single objective problem

$$\begin{array}{ll} \text{minimize} & F_w(x) \\ \text{subject to} & x \in X \end{array} \tag{10}$$

where $0 \le w \le 1$ and where

$$F_w(x) = wf_1(x) + (1-w)f_2(x)$$

We can easily solve this BOP by hand. Indeed, note that

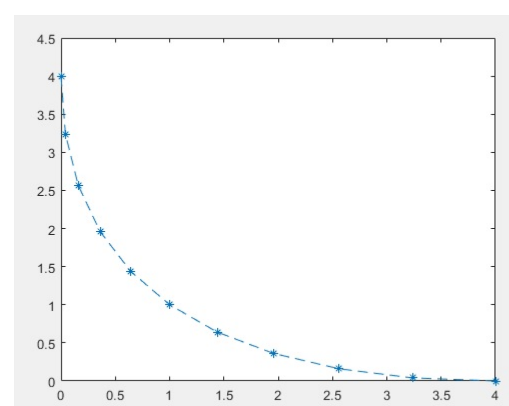
$$F_w'(x) = 2(2w + x - 2).$$

In particular, F_w has exactly one critical point: $c_w = 2(1-w)$. Furthermore observe that $F_w''(x) = 2$. In particular, the graph of F_w is a parabola whose global minimum occurs at the critical point $c_w = 2(1-w)$. The value of F_w at that critical point is given by

$$y_w := F_w(c_w) = 4w(1-w).$$

Thus we have found all optimal solutions to (10). We can verify our work using MATLAB which gives us the following table and plot:

w	exit flag	x	f1	f2
0	1	2	4	2.8847e-16
0.1	1	1.8	3.24	0.04
0.2	1	1.6	2.56	0.16
0.3	1	1.4	1.96	0.36
0.4	1	1.2	1.44	0.64
0.5	1	1	1	1
0.6	1	0.8	0.64	1.44
0.7	1	0.6	0.36	1.96
0.8	1	0.4	0.16	2.56
0.9	1	0.2	0.04	3.24
1	1	-7.1473e-09	5.1084e-17	4



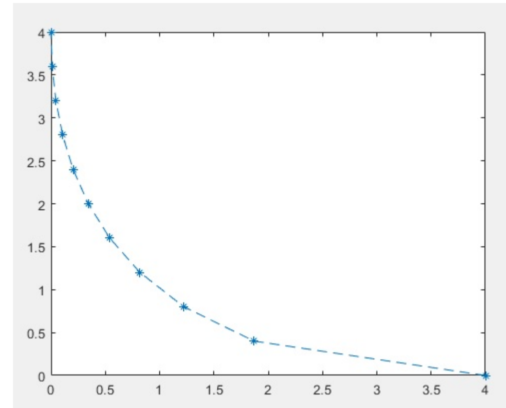
3.2 Epsilon-constraint Method

Next we solve (10) using the epsilon-constraint method. In this case, we only need to consider the following epsilon-constraint problem

$$\begin{array}{ll} \text{minimize} & f_1(x) \\ \text{subject to} & x \in X \\ & f_2(x) - \varepsilon \le 0 \end{array} \tag{11}$$

where $0 \le \varepsilon \le 4$ by the exact same argument as in the previous two cases. . Then f_2 is convex in the new feasible region remains convex, so this just becomes a convex problem. Furthermore one can show that (**??**) has a unique (local and global) optimal solution in this case, and so we can apply part 2 of Proposition (1.2). Using the reference point $x^0 = 1$, MATLAB produces the following table and plot:

e	exit flag	x	f1	f2
0	2	2	4	2.279e-16
0.4	1	1.3675	1.8702	0.4
0.8	1	1.1056	1.2223	0.8
1.2	1	0.90455	0.81822	1.2
1.6	1	0.73509	0.54036	1.6
2	1	0.58579	0.34315	2
2.4	1	0.45081	0.20323	2.4
2.8	1	0.32668	0.10672	2.8
3.2	1	0.21115	0.044582	3.2
3.6	1	0.10263	0.010534	3.6
4	1	0.00069373	4.8126e-07	3.9972



4 My BOP

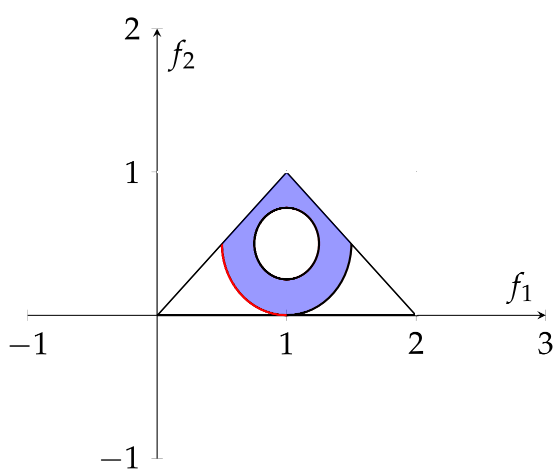
For this problem, we want to create our own BOP. Let $f_1 = f_1(\boldsymbol{x})$ and $f_2 = f_2(\boldsymbol{x})$ be functions on $\mathbb{R}^2$ which are to be determined, and let

$$\begin{aligned} e_1 &= f_2 \\ e_2 &= f_2 - f_1 \\ e_3 &= f_1 + f_2 - 2 \\ S &= (f_1 - 1)^2 + (f_2 - 1/2)^2 - 1/4 \\ s &= (f_1 - 1)^2 + (f_2 - 0.5)^2 - 1/16. \end{aligned}$$

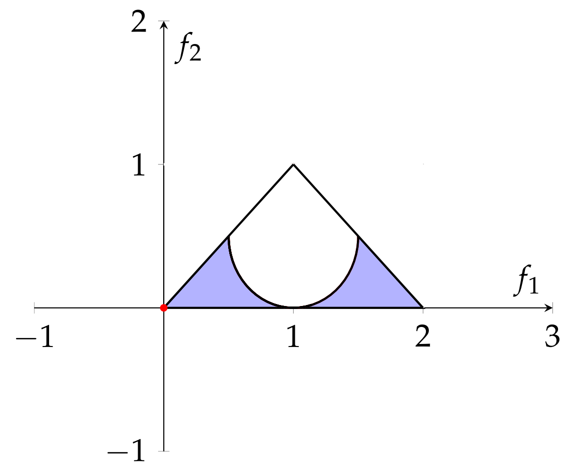
Consider the following BOP:

$$\begin{aligned} \text{minimize} \quad & [f_1(\boldsymbol{x}), f_2(\boldsymbol{x})] \\ \text{subject to} \quad & e_1 \geq 0 \\ & e_2 \leq 0 \\ & e_3 \leq 0 \\ & S \leq 0 \\ & s \geq 0 \\ & 0 \leqq \boldsymbol{x} \leqq 2 \end{aligned} \tag{12}$$

Thus the feasible region for this BOP is just $X = [0,2]^2$. We draw the outcome space $Y = \boldsymbol{f}(X)$ below shaded in blue together with the Pareto front Y_N shaded in red in the image below:



One reason why we think this BOP is interesting and could serve as a good test problem is because of the freedom we have in choosing the outcome set Y together with the Pareto front Y_N. For example, but changing $S \leq 0$ to $s \geq 0$ in (12), then instead we get this for Y and Y_N:



5 Appendix

We will find an optimal solution to this MILP problem using MATLAB, which has a built-in function whose purpose is to solve MILP problems like this. The syntax for this function is x = fmincon(fun,x0,Ain,bin,Aeq,beq,lb,ub,nonlcon), where the MILP solver assumes that the MILP has the form:

$$\begin{aligned} \text{minimize} \quad & F(\boldsymbol{x}) \\ \text{subject to} \quad & c(\boldsymbol{x}) \leq 0 \\ & c_{\text{eq}}(\boldsymbol{x}) = 0 \\ & A_{\text{in}}\boldsymbol{x} \leq \boldsymbol{b}_{\text{in}} \\ & A_{\text{eq}}\boldsymbol{x} = \boldsymbol{b}_{eq} \\ & \boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub} \\ & \boldsymbol{x}(\text{intcon}) \text{ are integers} \end{aligned}$$

where $c(\boldsymbol{x})$ and $c_{\text{eq}}(\boldsymbol{x})$ are functions that return vectors, where $\boldsymbol{b}_{\text{in}}$, $\boldsymbol{b}_{\text{eq}}$, $\boldsymbol{lb}$, $\boldsymbol{ub}$ are vectors, where A_{in} and A_{eq} are matrices, and where $F(\boldsymbol{x})$ is a function that returns a scalar.

```
w = 0.5;

function F = myfun(x)
f(1) = x(1);
f(2) = x(2);
F = w*f(1)+(1-w)*f(2);
end

function [c,ceq] = mycon(x)
c(1) = 1+0.1*cos(16*arctan(x(1)/x(2)))-x(1)^2-x(2)^2;
c(2) = (x(1)-0.5)^2 + (x(2)-0.5)^2 - 0.5;
ceq = [];
end

fun = @myfun;
nonlcon = @mycon;
x0 = [1,0.2];
lb = [0,0];
ub = [pi,pi];
```

References

[DPM02] Kalyanmoy Deb, Amrit Pratap, and T. Meyarivan. “Constrained Test Problems for Multi-Objective Evolutionary Optimization”. Page 4.

[DPM02-1] Kalyanmoy Deb, Amrit Pratap, and T. Meyarivan. “Constrained Test Problems for Multi-Objective Evolutionary Optimization”. Page 3.

[HHBW] Simon Huband, Philip Hingston, Luigi Barone, Lyndon While. A Review of Multi-Objective Test Problems and a Scalable Test Problem Toolkit”. Page 491.