Cohomology Homework

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In this homework, we will make use of the universal coefficient theorem for cohomology involving the Ext functor says that if G is an abelian group, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X), G) \longrightarrow \mathrm{H}^n(X; G) \xrightarrow{[[\cdot]]} \mathrm{Hom}(\mathrm{H}_n(X), G) \longrightarrow 0 \tag{1}$$

where $[[\cdot]]$ is defined as follows: if $[\varphi] \in \mathrm{H}^n(X; G)$ where $\varphi\colon C_n(X) \to G$ satisfies $\varphi\partial = 0$, and if $[a] \in \mathrm{H}_n(X)$ where $a \in C_n(X)$ satisfes $\partial(a) = 0$, then we set $[[\varphi]]$ to be the map from $\mathrm{H}_n(X) \to G$ given by

$$[[\varphi]][a] = \varphi(a).$$

This is well-defined since if $[\varphi] = [\varphi + \psi\partial]$ where $\psi\colon C_{n-1}(X) \to G$ and $[a] = [a + \partial(b)]$ where $b \in C_{n+1}(X)$, then we have

$$\begin{aligned} [[\varphi + \psi\partial]][a + \partial b] &= (\varphi + \psi\partial)(a + \partial b) \\ &= \varphi(a) + \varphi\partial(b) + \psi\partial(a) + \psi\partial\partial(b) \\ &= \varphi(a). \end{aligned}$$

Moreover, the map $[[\cdot]] = [[\cdot]]_X$ is *natural* in X. This means that if $f\colon X \to Y$ is a continuous map, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^n(Y; G) \arrow[r, "{[[\cdot]]_Y}"] \arrow[d, "\mathrm{H}(f^\star)"'] & \mathrm{Hom}(\mathrm{H}_n(Y), G) \arrow[d, "(\mathrm{H}(f_\star))^*"] \\ \mathrm{H}^n(X; G) \arrow[r, "{[[\cdot]]_X}"'] & \mathrm{Hom}(\mathrm{H}_n(X), G) \end{tikzcd}$$

Indeed, if $[a] \in \mathrm{H}_n(X)$ and $[\psi] \in \mathrm{H}^n(Y; G)$, then we have

$$\begin{aligned} ((f_\star)^*([[\psi]]_Y)[a] &= [[\psi]]_Y[f_\star(a)] \\ &= \psi(f_\star(a)) \\ &= (f^\star\psi)(a) \\ &= [[f^\star\psi]]_X[a]. \end{aligned}$$

It follows that $[[\cdot]]_X \circ \mathrm{H}(f^\star) = (\mathrm{H}(f_\star))^* \circ [[\cdot]]_Y$.

Remark 1. Note that in our notation, we use the $\star$ symbol to denote chain maps. For instance, a continuous map $f\colon X \to Y$ induces a chain map $f_\star\colon C_\star(X) \to C_\star(Y)$ which is defined on singular chains $a = \sum r_i\sigma_i \in C_\star(X)$ by

$$f_\star(a) = \sum r_i(f \circ \sigma).$$

This in turn induces a cochain map $f^\star\colon C^\star(Y) \to C^\star(X)$ which is defined by mapping the singular cochain $\varphi \in C^\star(Y)$ to the singular cochain $f^\star(\varphi) \in C^\star(X)$ which is defined on chains $a \in C_\star(X)$ by

$$f^\star(\varphi)(a) = \varphi(f_\star(a)).$$

Problem 1

Exercise 1. Let T be the torus, let K be the Klein bottle, and let P be the real projective plane.

1. Use the universal coefficient theorem to compute the cohomology of T, K, and P over $\mathbb{Z}$.
2. Use the definition to compute the simplicial cohomology of T, K, and K over $\mathbb{Z}$ using the Δ-complex structure on a square formed from two triangles.

Solution 1. When we set $G = \mathbb{Z}$ in (1), then the universal coefficient theorem takes the form:

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X), \mathbb{Z}) \longrightarrow \mathrm{H}^n(X) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(X), \mathbb{Z}) \longrightarrow 0 \tag{2}$$

We will use this short exact sequence to compute the cohomologies of T, K, and P over $\mathbb{Z}$. We first consider T. Recall that

$$\mathrm{H}_i(T) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

In each case, we have $\mathrm{Ext}^1(\mathrm{H}_{i-1}(T), \mathbb{Z}) = 0$ since $\mathrm{H}_{i-1}(T)$ is a free $\mathbb{Z}$-module for all i (note that 0 is the free module with empty set as basis). Therefore (2) gives us

$$\mathrm{H}^i(T) \simeq \mathrm{Hom}(\mathrm{H}_i(T), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

Now we first consider K. Recall that

$$\mathrm{H}_i(K) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

This time $\mathrm{H}_{i-1}(K)$ is free for all i *expect* $i = 2$. Therefore (2) gives us

$$\mathrm{H}^i(K) \simeq \mathrm{Hom}(\mathrm{H}_i(K), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = 1 \\ 0 & i \neq 0, 1, 2 \end{cases}$$

where we used the fact that

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}) &= \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &= \mathbb{Z} \oplus 0 \\ &= \mathbb{Z}. \end{aligned}$$

It remains to calculate $\mathrm{H}^2(K)$. In this case, the short exact sequence (2) gives us

$$0 \to \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{H}^2(K) \to 0 \to 0$$

where we used the fact that $\mathrm{H}_2(X) = 0$. Since Ext takes finite direct sums in the first variable to direct sum (more generally it takes direct sums in the first variable to products), we have

$$\begin{aligned} \mathrm{Ext}^1(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) &= \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &= 0 \oplus \mathbb{Z}/2\mathbb{Z} \\ &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Thus $\mathrm{H}^2(K) = \mathbb{Z}/2\mathbb{Z}$. Finally, we consider P. Recall that

$$\mathrm{H}_i(P) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

Again, $\mathrm{H}_{i-1}(P)$ is free for all i *expect* $i = 2$. Therefore (2) gives us

$$\mathrm{H}^i(P) \simeq \mathrm{Hom}(\mathrm{H}_i(P), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & i \neq 0, 2 \end{cases}$$

where we used the fact that $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$. It remains to calculate $\mathrm{H}^2(P)$. In this case, the short exact sequence (2) gives us

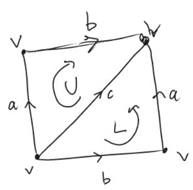
$$0 \to \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{H}^2(P) \to 0 \to 0$$

where we used the fact that $\mathrm{H}_2(P) = 0$. In particular, this imlpies $\mathrm{H}^2(P) = \mathbb{Z}/2\mathbb{Z}$.

2. First we calculate the cohomology of the Torus below:

smith normal form of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

Torus



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{c|cc} & U & L \\ \hline a & 1 & 1 \\ b & 1 & 1 \\ c & -1 & -1 \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{c|ccc} & a & b & c \\ \hline v & 0 & 0 & 0 \end{array}} \underset{0}{\mathbb{Z}} \to 0$$

$U \mapsto a+b-c$

$L \mapsto a+b-c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$$H_0(F^*) = \mathbb{Z}$$

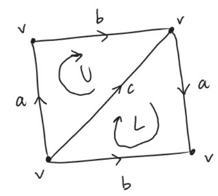
$$H_{-1}(F^*) = \ker \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{Z}^2$$

$$H_{-2}(F^*) = \mathbb{Z}^2 / \mathrm{im} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \cong \mathbb{Z}^2 / \mathrm{im} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{Z}$$

Next we calculate the cohomology of the Klein bottle below:

smith normal form of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$

Klein Bottle



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{cc} & \begin{array}{cc} U & L \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ v & \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{array}} \underset{0}{\mathbb{Z}} \to 0$$

$U \mapsto a + b - c$

$L \mapsto a - b + c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$H_0(F^*) = \mathbb{Z}$

$$H_{-1}(F^*) = \ker\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \ker\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \mathbb{Z}$$

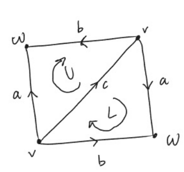
$$H_{-2}(F^*) = \mathbb{Z}^2 / \operatorname{im}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \mathbb{Z}^2 / \operatorname{im}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \mathbb{Z}/2\mathbb{Z}$$

Finally we calculate the cohomology of the real projective plane below:

smith normal form of $\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$

smith normal form of $\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Real Projective Plane



$$F = 0 \to \underset{2}{\mathbb{Z}^2} \xrightarrow{\begin{array}{cc} & \begin{array}{cc} U & L \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}} \underset{1}{\mathbb{Z}^3} \xrightarrow{\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} v \\ w \end{array} & \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array}} \underset{0}{\mathbb{Z}^2} \to 0$$

$U \mapsto a - b - c$

$L \mapsto a - b + c$

$$F^* = 0 \to \underset{0}{\mathbb{Z}^2} \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \underset{-1}{\mathbb{Z}^3} \xrightarrow{\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \underset{-2}{\mathbb{Z}^2} \to 0$$

$$H_0(F^*) = \ker\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} = \ker\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{Z}$$

$$H_{-1}(F^*) = \ker\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \Big/ \operatorname{im}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \cong \ker\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \Big/ \operatorname{im}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$H_{-2}(F^*) = \mathbb{Z}^2 / \operatorname{im}\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cong \mathbb{Z}^2 / \operatorname{im}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \mathbb{Z}/2\mathbb{Z}$$

Problem 2

Exercise 2. Show that if $f\colon S^n \to S^n$ has degree d, then $f^*\colon \mathrm{H}^n(S^n;G) \to \mathrm{H}^n(S^n;G)$ is multiplication by d map.

We prove this in a more general situation:

Exercise 3. Let $f\colon X \to X$ be a continuous map such that $\mathrm{H}(f_\star) = d$ where $d \in \mathbb{Z}$ (that is, $\mathrm{H}(f_\star)\colon \mathrm{H}_\star(X) \to \mathrm{H}_\star(X)$ is the multiplication by d map). Furthermore, assume that $\mathrm{H}_i(X)$ is free for all $i \in \mathbb{Z}$. Then $\mathrm{H}(f^\star) = d$ (that is, $\mathrm{H}(f^\star)\colon \mathrm{H}^\star(X;G) \to \mathrm{H}^\star(X;G)$ is the multiplication by d map).

Solution 2. Since each $\mathrm{H}_i(X)$ is free, the universal coefficient theorem gives us the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n(X;G) \arrow[r, "{[[\cdot]]}"] \arrow[d, "\mathrm{H}(f^\star)"'] & \mathrm{Hom}(\mathrm{H}_n(X),G) \arrow[d, "d^*"] \\
\mathrm{H}^n(X;G) \arrow[r, "{[[\cdot]]}"'] & \mathrm{Hom}(\mathrm{H}_n(X),G)
\end{tikzcd}$$

where $[[\,\cdot\,]]$ is an isomorphism. In particular, we have

$$\mathrm{H}(f^\star) = [[\cdot]]^{-1} \circ d^* \circ [[\cdot]].$$

Note that $d^* = d$ since all maps are $\mathbb{Z}$-linear and d is an integer. Next note that $d \circ [[\cdot]] = [[\cdot]] \circ d$ since d is an integer and $[[\cdot]]$ is a $\mathbb{Z}$-linear isomorphism. Thus we have

$$\begin{aligned}
\mathrm{H}(f^\star) &= [[\cdot]]^{-1} \circ d^* \circ [[\cdot]] \\
&= [[\cdot]]^{-1} \circ d \circ [[\cdot]] \\
&= [[\cdot]]^{-1} \circ [[\cdot]] \circ d \\
&= d.
\end{aligned}$$

Problem 3

Exercise 4. Use cup products over $\mathbb{Z}/2\mathbb{Z}$ to show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Solution 3. On the one hand, we have

$$\begin{aligned}
\mathrm{H}^\star(\mathbb{RP}^2 \vee S^3;\mathbb{Z}/2\mathbb{Z}) &= \mathrm{H}^\star(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \times \mathrm{H}^\star(S^3;\mathbb{Z}/2\mathbb{Z}) \\
&= \mathbb{F}_2[x]/\langle x^3\rangle \times \mathbb{F}_2[y]/\langle y^2\rangle,
\end{aligned}$$

where $|x| = 1$ and $|y| = 3$. On the other hand, we have

$$\mathrm{H}^\star(\mathbb{RP}^3;\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[z]/\langle z^4\rangle$$

where $|z| = 1$. These rings are not isomorphic. For instance,

$$\mathrm{Spec}(\mathbb{F}_2[x]/\langle x^3\rangle \times \mathbb{F}_2[y]/\langle y^2\rangle) = \{\langle \overline{x}\rangle, \langle \overline{y}\rangle, \langle \overline{x}, \overline{y}\rangle\}$$

consists of three points, however

$$\mathrm{Spec}(\mathbb{F}_2[z]/\langle z^4\rangle) = \{\langle \overline{z}\rangle\}$$

only has one point.

Appendix

We calculate $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ as follows: let F be the free $\mathbb{Z}$-complex below

$$F = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0,$$

where $F_0 = \mathbb{Z} = F_1$ and $F_i = 0$ for all $i \neq 0, 1$. Then F is a free resolution of $\mathbb{Z}/2\mathbb{Z}$. Next we set $F^\star := \mathrm{Hom}^\star(F, \mathbb{Z})$ (this is the hom-complex where

$$F_i^\star = \{\text{graded homomorphisms of degree } i \text{ from } F \text{ to } \mathbb{Z}\}.$$

In particular,

$$F_0^\star = \{\text{homomorphisms from } F_0 \text{ to } \mathbb{Z}\} = \mathbb{Z}$$
$$F_{-1}^\star = \{\text{homomorphisms from } F_1 \text{ to } \mathbb{Z}\} = \mathbb{Z}$$

and $F_{-1}^\star = 0$ for all $i \neq 0, -1$. The differential $\mathrm{d}_0^\star \colon F_0 \to F_{-1}$ is easily seen to be the multiplication by 2 map, so

$$F^\star = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0.$$

Finally we have

$$\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \mathrm{H}_{-1}(F^\star) = \mathbb{Z}/2\mathbb{Z}.$$