Goldbach Rings

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Abstract

Let $\Bbbk$ be a field. We introduce and study an infinite-dimensional $\Bbbk$-algebra G which we call the Goldbach ring. As the name suggests, the Goldbach ring is closely related to Goldbach's conjecture. Properties that G satisfies as a ring (such as whether or not it is an integral domain) may give us clues about Goldbach's conjecture itself.

1 Introduction

Let $\Bbbk$ be a field. We introduce and study an infinite dimensional $\Bbbk$-algebra which we call the Goldbach ring, which, as the name suggests, is related to Goldbach's conjecture:

Conjecture 1. *Every even integer ≥ 6 can be expressed as the sum of two odd primes.*

The Goldbach ring G is defined to be the quotient $G = R/I$ where

$$R = \Bbbk[x_p, x_{p+q} \mid p, q \text{ odd primes}] \quad \text{and} \quad I = \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\} \rangle$$

The Goldbach ring has the structure of a bi-graded $\Bbbk$-algebra meaning it can be decomposed as

$$G = \bigoplus_{n,d \geq 0} G_{n,d},$$

where the component $G_{n,d}$ in bi-degree $(n, d) \in \mathbb{Z}^2_{\geq 0}$ is a finite dimensional $\Bbbk$-vector space whose dimension we are interested in counting (see subsection 2.1 for the definition of $G_{n,d}$). Goldbach's conjecture itself is equivalent to the statement that $\dim_\Bbbk G_{2k,2} = 1$ for all $k \geq 3$, however this is just a restatment of Goldbach's conjecture; what's more interesting and new in our view is the following conjecture that we propose:

Conjecture 2. *We have*

$$\dim_\Bbbk G_{n,d} \leq 1$$

for all $n, d \in \mathbb{N}$.

A counter-example to Conjecture 2 would be the existence of odd primes $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$ such that

$$p_1 + \cdots + p_d = n = q_1 + \cdots + q_d$$

but $x_{p_1} \cdots x_{p_d} \neq x_{q_1} \cdots x_{q_d}$ in G. However we do not believe such a counter-example exists since in practice there are usually many ways to go from $x_{p_1} \cdots x_{p_d}$ to $x_{q_1} \cdots x_{q_d}$ by applying elementary Goldbach relations of the form $x_p x_q = x_{p+q}$. For instance, in $G_{36,4}$ we have $x_3^2 x_{11} x_{19} = x_5^2 x_{13}^2$ since

$$\begin{aligned} x_3^2 x_{11} x_{19} &= x_3 x_{11} x_{22} \\ &= x_3 x_5 x_{11} x_{17} \\ &= x_5 x_{11} x_{20} \\ &= x_5 x_7 x_{11} x_{13} \\ &= x_5 x_{13} x_{18} \\ &= x_5^2 x_{13}^2. \end{aligned}$$

Note there are other paths we could have taken to get from $x_3^2 x_{11} x_{19}$ to $x_5^2 x_{13}^2$ however it turns out that this is the shortest path. Ultimately any attempt towards a solution to Conjecture 2 will involve tools and techniques from analytic number theory. What we find interesting is that Conjecture 2 also seems to involve a lot of commutative algebra. For example, if Conjecture 2 is true, then it would imply that G is an integral domain. Conversely, one can show that if G is an integral domain and Conjecture 2 holds for n, d sufficiently large, then Conjecture 2 is true.

2 Goldbach Rings

Let $\mathcal{A}$ be a subset of the positive odd integers and set $\mathcal{B} := \mathcal{A} + \mathcal{A} = \{a_1 + a_2 \mid a_1, a_2 \in \mathcal{A}\}$. We set

$$\begin{aligned} R &= \Bbbk[x_a, x_b \mid a \in \mathcal{A}, b \in \mathcal{B}] \\ I &= \langle x_{a_1}x_{a_2} - x_{a_1+a_2} \mid a_1, a_2 \in \mathcal{A}\rangle \\ G &= R/I. \end{aligned}$$

We will refer to G as the **Goldbach ring supported on** $\mathcal{A}$ or just as a Goldbach ring if $\mathcal{A}$ is understood from context. Let $\mathcal{M}$ be the set of all monomials in R. There are two ways we can represent monomials both of which are convenient for our purposes. The first way is as a finite product of the indeterminates $\{x_a, x_b \mid a \in \mathcal{A}$ and $b \in \mathcal{B}\}$, that is, a monomial can be expressed in the form

$$x_{\boldsymbol{a}}x_{\boldsymbol{b}} := x_{a_1} \cdots x_{a_r} x_{b_1} \cdots x_{b_s}$$

where $\boldsymbol{a} = a_1, \dots, a_r$ is a (not necessarily distinct) sequence of elements in $\mathcal{A}$ and $\boldsymbol{b} = b_1, \dots, b_s$ is a sequence of (not necessarily distinct) elements in $\mathcal{B}$. We will use this way of representing monomials when describing the bi-graded structure on R. The second way of representing monomials is described as follows: given a function $\alpha\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, we define its **support**, denoted $\operatorname{supp}\alpha$, to be the set

$$\operatorname{supp}\alpha = \{m \in \mathbb{N} \mid \alpha(m) \neq 0\}.$$

Let $\mathcal{F}$ be the set of all functions $\alpha\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that $\operatorname{supp}\alpha$ is finite and is contained in $\mathcal{A} \cup \mathcal{B}$. There is a bijection from $\mathcal{F}$ to $\mathcal{M}$ given by assigning to $\alpha \in \mathcal{F}$ the monomial

$$x^\alpha := \prod_{m \in \mathbb{N}} x_m^{\alpha(m)} = \prod_{m \in \operatorname{supp}\alpha} x_m^{\alpha(m)}.$$

For instance, if $\alpha\colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is defined by

$$\alpha(m) = \begin{cases} 2 & \text{if } m = 3 \\ 2 & \text{if } m = 6 \\ 4 & \text{if } m = 11 \\ 0 & \text{if } m \in \mathbb{N}\backslash\{3, 6, 11\}. \end{cases}$$

Then $x^\alpha = x_3^2 x_6^2 x_{11}^4$ and $\operatorname{supp} x^\alpha = \{3, 6, 11\}$. This second way of expressing monimals gives us a cleaner way of expressing nonzero polynomials in R; namely, every nonzero polynomial $f \in R$ can be expressed in the form

$$f = c_1 x^{\alpha_1} + \cdots + c_n x^{\alpha_n}$$

for unique $c_1, \dots c_n \in \Bbbk$ and for unique $\alpha_1, \dots, \alpha_n \in \mathcal{F}$.

2.1 Bi-Graded $\Bbbk$-Structures on R and G

We give R and G bi-graded $\Bbbk$-structures as follows: we define $\deg_1\colon \mathcal{M} \to \mathbb{N}$ and $\deg_2\colon \mathcal{M} \to \mathbb{N}$ by

$$\deg_1(x_{\boldsymbol{a}}x_{\boldsymbol{b}}) = \sum_{i=1}^r a_i + \sum_{j=1}^s b_j \quad \text{and} \quad \deg_2(x_{\boldsymbol{a}}x_{\boldsymbol{b}}) = r + 2s.$$

For each $n, d \in \mathbb{N}$, we set

$$R_n = \operatorname{span}_\Bbbk\{x^\alpha \in \mathcal{M} \mid \deg_1(x^\alpha) = n\} \quad \text{and} \quad R_{n,d} = \operatorname{span}_\Bbbk\{x^\alpha \in \mathcal{M} \mid \deg_1(x^\alpha) = n \text{ and } \deg_2(x^\alpha) = d\}.$$

Then we have a decomposition of R into $\Bbbk$-vector spaces:

$$R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_n = \bigoplus_{n,d \in \mathbb{Z}_{\geq 0}} R_{n,d},$$

which gives R a bi-graded $\Bbbk$-structure. Since I is homogeneous with respect to this bi-grading, G inherits the bi-graded $\Bbbk$-structure induced by the one on R:

$$G = \bigoplus_n G_n = \bigoplus_{n,d} G_{n,d}.$$

We set $\Delta_{n,d} = \dim_\Bbbk R_{n,d}$ and $\delta_{n,d} = \dim_\Bbbk G_{n,d}$. Thus $\Delta_{n,d}$ counts the number of ways we can express n as a sum

$$n = a_1 + \cdots + a_r + b_1 + \cdots + b_s$$

where $a_1, \dots, a_r \in \mathcal{A}$, $b_1, \dots, b_s \in \mathcal{B}$, and $d = r + 2s$. Whenever we have $\Delta_{n,d} \neq 0$, we say (n, d) is a **good pair**. When (n, d) is a good pair, we are interested in determining whether or not $\delta_{n,d} = 1$ or $\delta_{n,d} > 1$. See the beginning of Section 3 for an example of what $R_{n,d}$ and $G_{n,d}$ look like in the case where $\mathcal{A} = \{\text{odd positive primes}\}$.

2.2 Constructing the Minimal Free Resolution of G over R

For each $m \geq 1$, we set

$$\begin{aligned} R^m &= R \cap \Bbbk[x_1, \dots, x_m] \\ I^m &= \langle x_{a_1} x_{a_2} - x_{a_1+a_2} \mid a_1 + a_2 \leq m \rangle \\ G^m &= R^m / I^m. \end{aligned}$$

Note that R^m and G^m have bi-graded $\Bbbk$-structures:

$$R^m = \bigoplus_n R^m_n = \bigoplus_{n,d} R^m_{n,d} \quad \text{and} \quad G^m = \bigoplus_n G^m_n = \bigoplus_{n,d} G^m_{n,d}.$$

Note that if $x_{\boldsymbol{a}} x_{\boldsymbol{b}} \in R_n$, then $a_1 + \cdots + a_r + b_1 + \cdots + b_s = n$ implies that the a_i's and b_j's must all be less than or equal to n. Thus for all $m \geq n$ we have we have

$$R^m_n = R^n_n = R_n \quad \text{and} \quad G^m_n = G^n_n = G_n.$$

Thus we have directed systems (R^m) and (G^m) of bi-graded $\Bbbk$-algebras where the bi-graded components $R^m_{n,d}$ and $G^m_{n,d}$ in bi-degree (n, d) stabilizes to $R_{n,d}$ and $G_{n,d}$ respectively for m sufficiently large (for example $m \geq n$). It follows that

$$R = \varinjlim R^m \quad \text{and} \quad G = \varinjlim G^m$$

as bi-graded direct limits.

Next let F^m be the minimal free resolution of G^m over R^m. We set

$$\delta^m = \operatorname{depth}_{R^m} G^m \quad \text{and} \quad \rho^m = \operatorname{pd}_{R^m} G^m = \operatorname{length} F^m.$$

Note that these quantities are intrinsic to R^m and G^m (and possibly the characteristic of $\Bbbk$), and are not intrinsic R and G. Nevertheless, one can hope that they might give useful information for m sufficiently. For instance, by the Auslander-Buchsbaum formula we have

$$\rho^m + \delta^m = \pi_{\mathcal{A} \cup \mathcal{B}}(m) := \#\{x \in \mathcal{A} \cup \mathcal{B} \mid x \leq m\}. \tag{1}$$

The left-hand side of (1) is of interest in commutative algebra whereas the right-hand side of (1) is of interest in analytic number theory. Observe that F^m has the structure of a bi-graded $\Bbbk$-complex meaning we have a decomposition of $\Bbbk$-complexes:

$$F^m = \bigoplus_n F^m_n = \bigoplus_{n,d} F^m_{n,d},$$

where $F^m_{n,d}$ is a $\Bbbk$-subcomplex of F^m. In particular, the differential of F^m is homogeneous with respect to this bi-grading, thus

$$\bigoplus_{n,d} G^m_{n,d} = G^m = \mathrm{H}(F^m) = \bigoplus_{n,d} \mathrm{H}(F^m_{n,d}),$$

where we view G^m as a graded module concentrated in homological degree 0. In other words, we have

$$\mathrm{H}_i(F^m_{n,d}) = \begin{cases} G^m_{n,d} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases} \tag{2}$$

The ith bi-graded Betti number of G^m in bi-degree (n, d) is given by

$$\beta^m_{i,n,d} := \dim_\Bbbk \operatorname{Tor}^{R^m}_i (G^m, \Bbbk)_{n,d} = \dim_\Bbbk (F^m_{i,n,d}).$$

We also set $\rho^m_{n,d} = \operatorname{length} F^m_{n,d}$. The maps $G^m \to G^{m+1}$ induce bi-graded comparison maps $F^m \to F^{m+1}$ for all m. In general, these comparison maps are difficult to describe, however it turns out that as m tends towards infinity the sequence of $\Bbbk$-complexes $(F^m_{n,d})$, with n and d fixed, stabilizes. For instance, if n is odd, then we have $F^m_{n,d} = F^{n-3d+6}_{n,d}$ for all $m \geq n - 3d + 6$ (see (3.1) for how some of these $\Bbbk$-complexes look in the case where $\mathcal{A} = \{\text{positive odd primes}\}$). The idea is that the coefficients for the differential all belong to $\Bbbk[x_1, \dots, x_m]$ for some sufficiently large m. Thus if we define F to be the direct limit of bi-graded $\Bbbk$-complexes

$$F := \varinjlim F^m,$$

then F is a free resolution of G over R which has the following bi-graded $\Bbbk$-complex structure:

$$F = \bigoplus_{n,d} F_{n,d} = \bigoplus_{n,d} F^m_{n,d}.$$

where m is a sufficiently large integer depending on n and d. In particular, we see that $\beta^m_{i,n,d} = \beta_{i,n,d}$ where

$$\beta_{i,n,d} := \dim_{\Bbbk} \mathrm{Tor}^R_i(G,\Bbbk)_{n,d} = \dim_{\Bbbk}(F_{i,n,d})$$

is the ith bi-graded Betti number of G in bi-degree (n,d). Similarly, $\rho^m_{n,d} = \rho_{n,d}$ where

$$\rho_{n,d} = \sup\{i \mid \mathrm{Tor}^R_i(G,\Bbbk)_{n,d} \neq 0\} = \mathrm{length}(F_{n,d}).$$

Unlike the quantities δ^m and ρ^m, the quantities $\beta^m_{i,n,d}$ and $\rho^m_{n,d}$ actually intrinsic to R and G (and possibly depend on the characteristic of $\Bbbk$ as well) when m is sufficiently large.

Proposition 2.1. *We have*

$$\delta_{n,d} = \Delta_{n,d} - \sum_{i=1}^{\infty}(-1)^i \beta_{i,n,d}, \tag{3}$$

Proof. The $\Bbbk$-complex $F_{n,d}$ an exact complex of finite length consisting finite dimensional $\Bbbk$-vector spaces. Thus we have $\chi(F_{n,d}) = 0$ where χ is the Euler characteristic of $F_{n,d}$. However this is exactly what (3) says. □

Proposition 2.2. *We have $G_{n,d} = \mathrm{H}_0(F_{n,d})$.*

Proof. This follows from (2) together with the fact that $F^m_{n,d} = F_{n,d}$ and $G^m_{n,d} = G_{n,d}$ for m sufficiently large. □

3 The Goldbach Ring

We now focus on the Goldbach ring that we are most interested in, namely where $\mathcal{A} = \{\text{positive odd primes}\}$. To get a feel for how this Goldbach ring looks, let us first write down the components of $R_n = \bigoplus R_{n,d}$ as $\Bbbk$-vector spaces for various n. For R_{18}, the components $R_{18,d}$ are given by

$$\begin{aligned}
R_{18,6} &= \Bbbk x_3^6 + \Bbbk x_3^4 x_6 + \Bbbk x_3^2 x_6^2 + \Bbbk x_6^3 \\
R_{18,4} &= \Bbbk x_3^2 x_5 x_7 + \Bbbk x_3 x_5^3 + \Bbbk x_3^2 x_{12} + \cdots + \Bbbk x_5 x_6 x_7 + \Bbbk x_6 x_{12} + \Bbbk x_8 x_{10} \\
R_{18,2} &= \Bbbk x_5 x_{13} + \Bbbk x_7 x_{11} + \Bbbk x_{18}
\end{aligned}$$

and $R_{18,d} = 0$ for all $d \neq 2,4,6$. For R_{17}, the components $R_{17,d}$ are given by

$$\begin{aligned}
R_{17,5} &= \Bbbk x_3^4 x_5 + \Bbbk x_3^3 x_8 + \Bbbk x_3^2 x_5 x_6 + \Bbbk x_3 x_6 x_8 + \Bbbk x_5 x_6^2 \\
R_{17,3} &= \Bbbk x_3^2 x_{11} + \Bbbk x_3 x_7^2 + \Bbbk x_5^2 x_7 + \Bbbk x_6 x_{11} + \Bbbk x_3 x_{14} + \Bbbk x_7 x_{10} + \Bbbk x_5 x_{12} \\
R_{17,1} &= \Bbbk x_{17}
\end{aligned}$$

and $R_{17,d} = 0$ for all $d \neq 1,3,5$. For instance, we see that $\Delta_{17,3} := \dim_{\Bbbk} R_{17,3} = 7$. The nonzero components for G_{17} and G_{18} are even simpler to describe, they are given by:

$$\begin{aligned}
G_{17,5} &= \Bbbk \overline{x}_3^4 \overline{x}_5 & \qquad G_{18,6} &= \Bbbk \overline{x}_3^6 \\
G_{17,3} &= \Bbbk \overline{x}_3^2 \overline{x}_{11} & \qquad G_{18,4} &= \Bbbk \overline{x}_3^2 \overline{x}_5 \overline{x}_7 \\
G_{17,1} &= \Bbbk \overline{x}_{17} & \qquad G_{17,1} &= \Bbbk \overline{x}_{17}
\end{aligned}$$

Thus Conjecture 2 holds at least in the case for all pairs of the form $(17,d)$ and $(18,d)$. In order to prove Conjecture 2, we would need to prove that for all good pairs (n,d), we can represent each basis element in $G_{n,d}$ by a monomial of the form $x_{\boldsymbol{p}} = x_{p_1} \cdots x_{p_d}$ where $\boldsymbol{p} = p_1, \ldots, p_d$ are d odd primes such that $n = p_1 + \cdots + p_d$. However if $x_{\boldsymbol{q}} = x_{q_1} \cdots x_{q_d}$ where $\boldsymbol{q} = q_1, \ldots, q_d$ are d odd primes such that $n = q_1 + \cdots + q_d$, then it is not obvious why $x_{\boldsymbol{p}}$ and $x_{\boldsymbol{q}}$ should represent the same basis element in $G_{n,d}$. Indeed, in $G_{27,3}$, we have $\overline{x}_3 \overline{x}_{11} \overline{x}_{13} = \overline{x}_5^2 \overline{x}_{17}$, however it takes some work to show this:

$$\begin{aligned}
\overline{x}_3 \overline{x}_{11} \overline{x}_{13} &= \overline{x}_{11} \overline{x}_{16} \\
&= \overline{x}_5 \overline{x}_{11} \overline{x}_{11} \\
&= \overline{x}_5 \overline{x}_{22} \\
&= \overline{x}_5^2 \overline{x}_{17}.
\end{aligned}$$

Note that at each step in the computation above, we are only allowed to use a relation of the form $\overline{x}_p\overline{x}_q = \overline{x}_{p+q}$. For another example, in $G_{36,4}$ we have $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19} = \overline{x}_5^2\overline{x}_{13}^2$ since

$$\begin{aligned}\overline{x}_3^2\overline{x}_{11}\overline{x}_{19} &= \overline{x}_3\overline{x}_{11}\overline{x}_{22}\\ &= \overline{x}_3\overline{x}_5\overline{x}_{11}\overline{x}_{17}\\ &= \overline{x}_5\overline{x}_{11}\overline{x}_{20}\\ &= \overline{x}_5\overline{x}_7\overline{x}_{11}\overline{x}_{13}\\ &= \overline{x}_5\overline{x}_{13}\overline{x}_{18}\\ &= \overline{x}_5^2\overline{x}_{13}^2.\end{aligned}$$

The path we took to get from $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19}$ to $\overline{x}_5^2\overline{x}_{13}^2$ was longer than the path we took to get from $\overline{x}_3\overline{x}_{11}\overline{x}_{13}$ to $\overline{x}_5^2\overline{x}_{17}$, so one can imagine that for n and d large, the path from $x_{\boldsymbol{p}}$ to $x_{\boldsymbol{q}}$ may be even longer. Nevertheless, the reason we believe Conjecture 2 to be true is that there are *more* ways to get from $\overline{x}_3^2\overline{x}_{11}\overline{x}_{19}$ to $\overline{x}_5^2\overline{x}_{13}^2$ than there are to get from $\overline{x}_3\overline{x}_{11}\overline{x}_{13}$ to $\overline{x}_5^2\overline{x}_{17}$, and hence for n and d large, our intuition tells us that there should be many ways to get from $\overline{x}_{\boldsymbol{p}}$ to $\overline{x}_{\boldsymbol{q}}$ (as there are many such relations of the form $\overline{x}_p\overline{x}_q = \overline{x}_{p+q}$). In order to prove Conjecture 2, we only need to find *one* path from $\overline{x}_{\boldsymbol{p}}$ to $\overline{x}_{\boldsymbol{q}}$.

3.1 Is the Goldbach Ring an Integral Domain?

If Conjecture 2 is true, then G has a nice property as a ring:

Proposition 3.1. *Assume Conjecture 2 is true. Then G is an integral domain.*

Proof. Let $f \in G_{n,d} = \Bbbk\overline{x}^{\alpha}$ and $f' \in G_{n',d'} = \Bbbk\overline{x}^{\alpha'}$ such that $ff' = 0$. Express f and f' as

$$f = c\overline{x}^{\alpha} \quad \text{and} \quad f' = c'\overline{x}^{\alpha'}.$$

Then clearly since $\overline{x}^{\alpha+\alpha'} \neq 0$, so we must have $cc' = 0$, which implies either $c = 0$ or $c' = 0$ which implies either $f = 0$ or $f' = 0$. □

Remark 1. Note that for m sufficiently large, G^m tends to have lots of zerodivisors. For instance, in G^{16} we have $\overline{x}_3\overline{x}_5\overline{x}_{13} = \overline{x}_5^2\overline{x}_{11} = \overline{x}_3\overline{x}_7\overline{x}_{11}$ which implies

$$\overline{x}_3(\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}) = 0.$$

Since $\overline{x}_3 \neq 0$ and $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11} \neq 0$, we see that $\overline{x}_3$ and $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}$ form a zerodivisor pair. The ring homomorphism $G^{16} \to G^{18}$ kills this zerodivisor pair by sending $\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}$ to 0, however we pick up another zero-divisor pair in G^{20}: namely $\overline{x}_3$ and $\overline{x}_{11}\overline{x}_{11} - \overline{x}_5\overline{x}_{17}$. Indeed, in G^{20} we have

$$\begin{aligned}\overline{x}_3\overline{x}_{11}\overline{x}_{11} &= \overline{x}_7\overline{x}_7\overline{x}_{11}\\ &= \overline{x}_5\overline{x}_7\overline{x}_{13}\\ &= \overline{x}_3\overline{x}_5\overline{x}_{17},\end{aligned}$$

but $\overline{x}_{11}\overline{x}_{11} - \overline{x}_5\overline{x}_{17} \neq 0$ in G^{20}.

Thus we see a necessary condition for Conjecture 2 to be true is that G is an integral domain.

Proposition 3.2. *Assume G is an integral domain and that Conjecture 2 is true for all sufficiently large n and d. Then Conjecture 2 is true for all (n, d).*

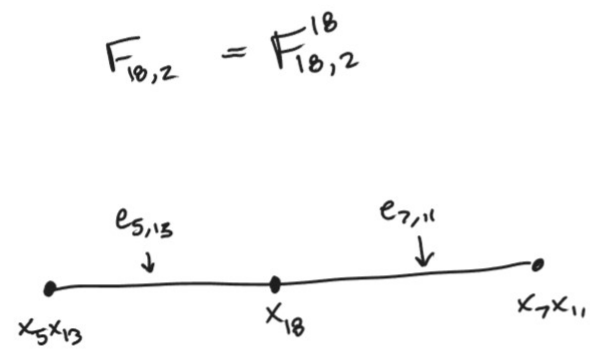
Proof. Assume that the conjecture is true for all pairs (n, d) with d sufficiently large. Let $x_{\boldsymbol{p}} = x_{p_1} \cdots x_{p_{d-1}}$ and $x_{\boldsymbol{q}} = x_{q_1} \cdots x_{q_{d-1}}$ such that $p_1, \dots, p_{d-1}$ and $q_1, \dots, q_{d-1}$ are odd primes which satisfy $p_1 + \cdots + p_{d-1} = n = q_1 + \cdots + q_{d-1}$. Choose an odd prime p such that the conjecture holds for $(p + n, d)$. Then we have

$$\overline{x}_p(\overline{x}_{\boldsymbol{p}} - \overline{x}_{\boldsymbol{q}}) = 0.$$

However this implies $\overline{x}_{\boldsymbol{p}} = \overline{x}_{\boldsymbol{q}}$ since G is an integral domain. It follows that the conjecture holds for all $d - 1$. Now proceed by induction. □

3.1.1 Explicit Calculations of the $\Bbbk$-Complex $F_{n,d}$

Example 3.1. The $\Bbbk$-complex $F_{18,2}$ can be realized as being supported on the labeled simplicial complex below:



In particular, we have

$$F_{0,18,2} = \Bbbk x_5x_{13} + \Bbbk x_7x_{11} + \Bbbk x_{18}$$

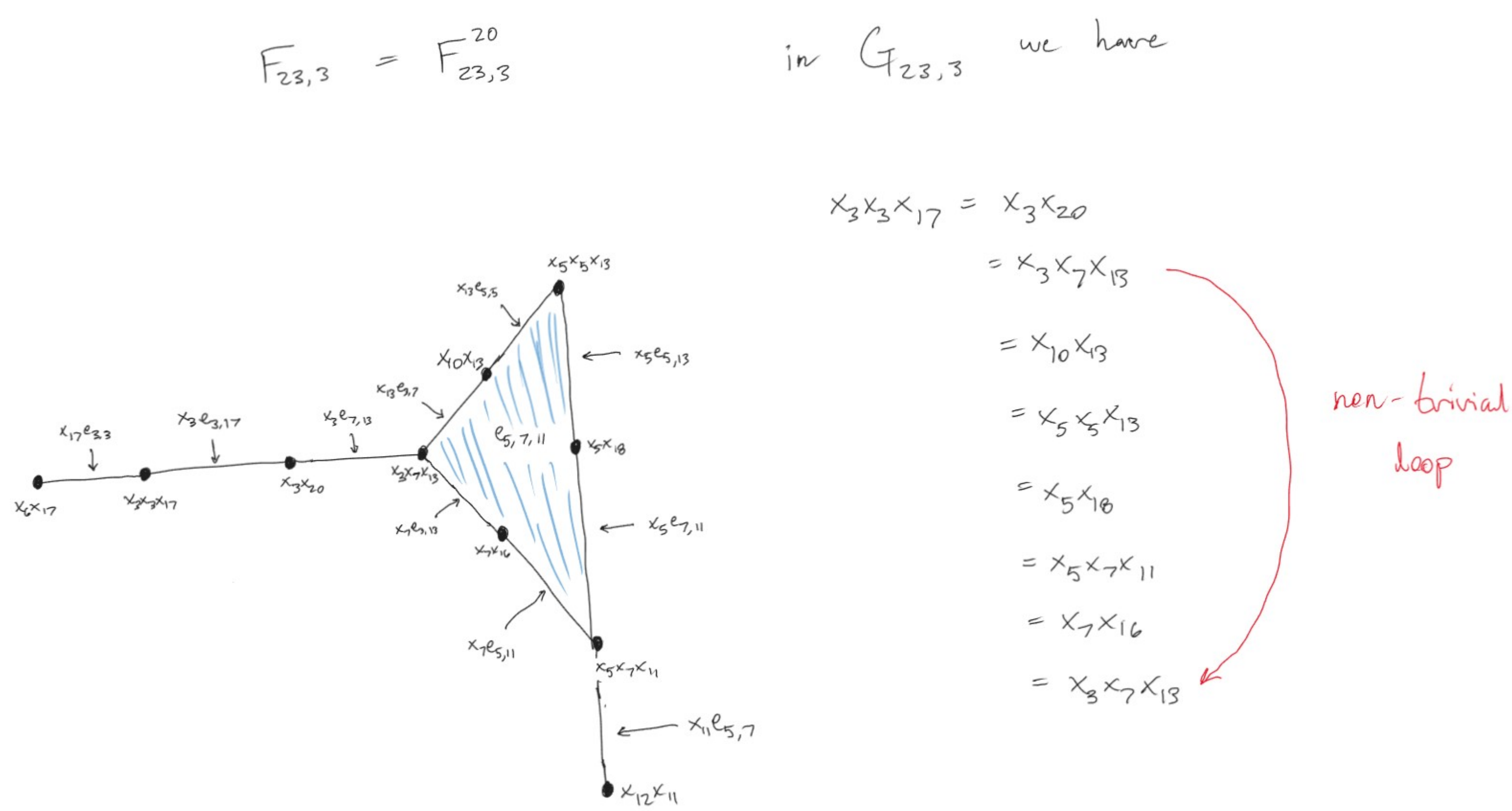
where the monomials x_5x_{13}, x_7x_{11}, and x_{18} correspond to the vertices of the simplicial complex, and we have

$$F_{1,18,2} = \Bbbk e_{5,13} + \Bbbk e_{7,11}$$

correspond to the edges of the simplicial complex. The differential is defined as if it were the usual boundary map, namely

$$\mathrm{d}(e_{5,13}) = x_5x_{13} - x_{18} \quad \text{and} \quad \mathrm{d}(e_{7,11}) = x_7x_{11} - x_{18}.$$

The $\Bbbk$-complex $F_{23,3}$ is supported on the labeled simplicial complex below.



For instance, we have

$$F_{0,23,3} = \Bbbk x_3^2x_{17} + \Bbbk x_3x_7x_{13} + \Bbbk x_5x_7x_{11} + \Bbbk x_5^2x_{13} + \Bbbk x_6x_{17} + \Bbbk x_{10}x_{13} + \Bbbk x_3x_{20} + \Bbbk x_7x_{16} + \Bbbk x_{12}x_{11} + \Bbbk x_5x_{18},$$

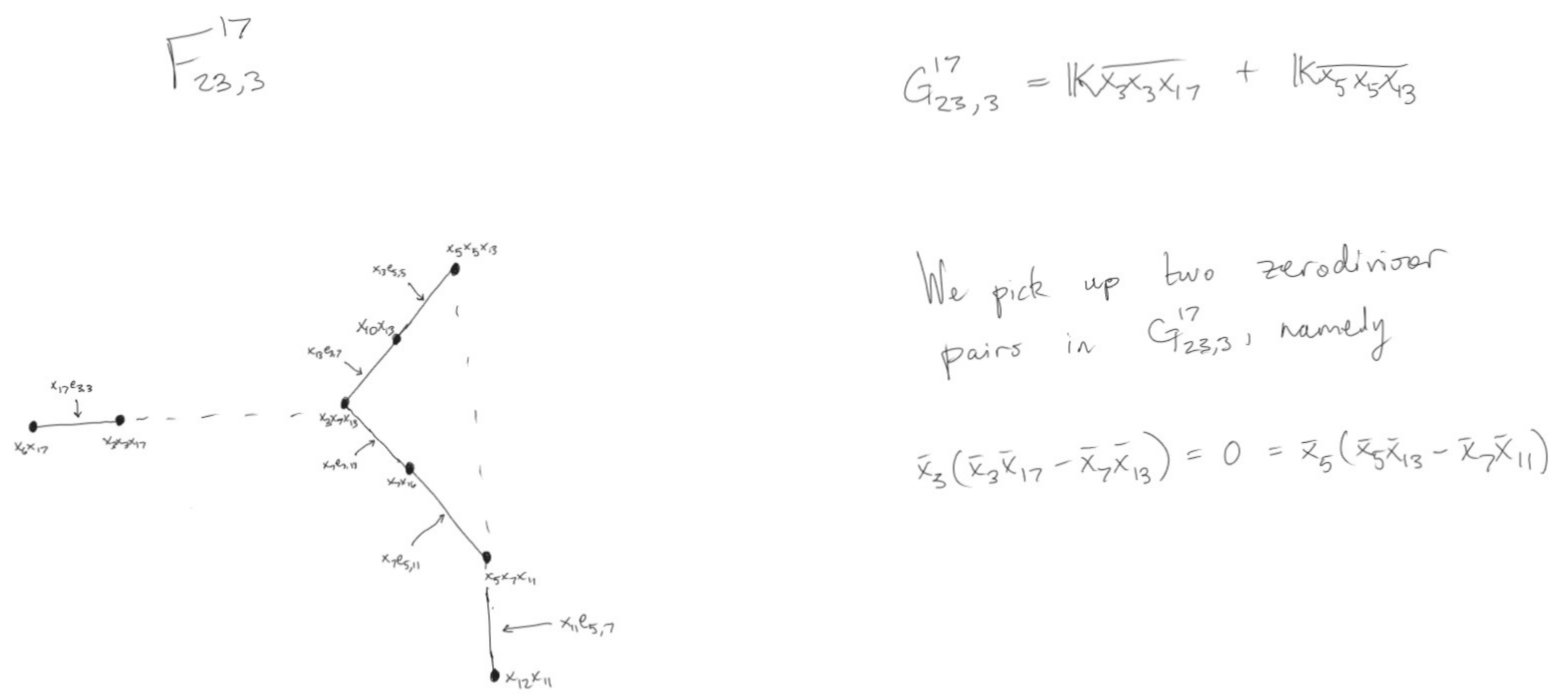
where each monomial corresponds to a vertex on the simplicial complex above. Similarly, the edges of the simplicial complex above correspond to the basis elements of $F_{1,23,3}$ and the triangular face corresponds to the basis element in $F_{2,23,3}$. Again, the differential is defined as if it were a boundary map with the extra condition that $\mathrm{d}(x_n) = 0$ for all n. Thus for example we have

$$\mathrm{d}(x_{13}e_{3,7}) = x_{13}\mathrm{d}(e_{3,7}) = x_{13}(x_3x_7 - x_{10}).$$

In homological degree 2, the differential is defined by

$$\mathrm{d}(e_{5,7,11}) = x_5(e_{7,11} - e_{5,13}) + x_7(e_{3,13} - e_{5,11}) + x_{13}(e_{5,5} - e_{3,7}).$$

Note that $F_{23,3} = F^m_{23,3}$ for all $m \geq 20$ since no there are no indeterminates x_n with $n > 20$ that show up in the differential. On the other hand, the $\Bbbk$-complex $F^{17}_{23,3}$ is supported on the labeled simplicial complex below:



Notice how we needed to delete the vertices labeled x_3x_{20} and x_5x_{18} and this resulted in a simplicial complex with two connected components corresponding to the fact that $\dim_{\Bbbk} G^{17}_{23,3} = 2$. Furthermore, we also pick up two zerodivisor pairs in $G^{17}_{23,3}$, namely

$$\overline{x}_3(\overline{x}_3\overline{x}_{17} - \overline{x}_7\overline{x}_{13}) = 0 = \overline{x}_5(\overline{x}_5\overline{x}_{13} - \overline{x}_7\overline{x}_{11}).$$

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