Algebro-Geometric Classification

Let $\Bbbk$ be a commutative ring and let F be a finite free graded $\Bbbk$-module such that $F_0 = \Bbbk$, $F_i = 0$ for all $i < 0$, and $F_+ \neq 0$. In this note, we give an algebro-geometric classification of various structures we can attach to F. We begin by classifying all $\Bbbk$-complex structures on F which fixed the identity element $1 \in \Bbbk = F_0$.

Classifying $\Bbbk$-Complex Structures on F

Let us state up front what we wish to prove:

Theorem 0.1. *We have the following bijection of sets:*

$$\left\{ \mathrm{GL}_n(\Bbbk)\text{-orbits of } h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of } \Bbbk\text{-complex} \\ \text{structures on } F \text{ with fixed identity} \end{array} \right\}$$

where $\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)$ is a $\Bbbk$-algebra (to be constructed below) and where

$$h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) := \mathrm{Hom}_{\Bbbk\text{-}alg}(\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F), \Bbbk)$$

is the $\Bbbk$-valued points of $\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)$. Two $\Bbbk$-complex structures (F, d) and (F, d') on F are said to be isomorphic with fixed identity if there exists a chain map $\varphi \colon F \to F$ such that $\varphi(1) = 1$.

The proof of this theorem mostly involves setting up our notation which will be used later on when we wish to classify other algebraic structures on F.

Proof. Let d be a $\Bbbk$-linear differential on F, meaning $\mathrm{d} \colon F \to F$ is a graded $\Bbbk$-linear map of degree -1 which satisfies $\mathrm{d}^2 = 0$. Choose an ordered homogeneous basis $\boldsymbol{e} = (e_0, e_1, \dots, e_n)$ of F where we set $e_0 = 1$ and let $\boldsymbol{d} = (d^i_j)$ be the matrix representation of the differential d with respect to the ordered homogeneous basis $\boldsymbol{e}$. Thus we have $\mathrm{d}\boldsymbol{e} = \boldsymbol{e}\boldsymbol{d}$ where $\mathrm{d}\boldsymbol{e} = (0, \mathrm{d}e_1, \dots, \mathrm{d}e_n)$ and $\boldsymbol{e}\boldsymbol{d}$ is the product of the row vector $\boldsymbol{e}$ on the left with the matrix $\boldsymbol{d}$ on the right. Alternatively we could express this in terms of the matrix entries of $\boldsymbol{d}$: for each $0 \leq j \leq n$ we have

$$\mathrm{d}e_j = \sum_{0 \leq i \leq n} d^i_j e_i.$$

Note that since d is graded of degree -1, we necessarily have $d^i_j = 0$ whenever $|e_i| \neq |e_j| - 1$. Also note that since $\mathrm{d}^2 = 0$, we have $\boldsymbol{d}^2 = 0$. Again we can express this in terms of matrix entries of $\boldsymbol{d}$: for each $0 \leq i, j \leq n$ we have

$$\sum_{0 \leq \iota \leq n} d^{\iota}_j d^i_{\iota} = 0 \tag{1}$$

Now consider the following polynomial ring following polynomial ring

$$\Bbbk[\boldsymbol{D}] = \Bbbk[\{D^i_j \mid 0 \leq i, j \leq n\}]$$

where the D^i_j are coordinates which correspond to the matrix entries of $\boldsymbol{d}$. Let $\mathrm{e}_{\boldsymbol{d}} \colon \Bbbk[\boldsymbol{D}] \twoheadrightarrow \Bbbk$ be the $\Bbbk$-algebra homomorphism given by $\mathrm{e}_{\boldsymbol{d}}(\boldsymbol{D}) = \boldsymbol{d}$ and set $\mathfrak{q}_{\boldsymbol{d}} = \langle \boldsymbol{D} - \boldsymbol{d} \rangle$ to be the kernel of this evaluation map: it is the $\Bbbk[\boldsymbol{D}]$-ideal generated by $D^i_j - d^i_j$ for all $0 \leq i, j \leq n$. Note that if $\Bbbk$ is an integral domain, then $\mathfrak{q}_{\boldsymbol{d}}$ is a prime ideal since $\Bbbk[\boldsymbol{D}]/\mathfrak{q}_{\boldsymbol{d}} \cong \Bbbk$, and if $\Bbbk$ is a field, then $\mathfrak{q}_{\boldsymbol{d}}$ is a maximal ideal of $\Bbbk[\boldsymbol{D}]$ and $\Bbbk \to \Bbbk[\boldsymbol{D}]/\mathfrak{q}_{\boldsymbol{d}}$ is a finite extension of fields. For each $0 \leq i, j \leq n$ we define the quadratic polynomials $\Delta^i_j \in \Bbbk[\boldsymbol{D}]$ by:

$$\Delta^i_j := \sum_{0 \leq \iota \leq n} D^{\iota}_j D^i_{\iota}.$$

Then we see that the evaluation map $\mathrm{e}_{\boldsymbol{d}} \colon \Bbbk[\boldsymbol{D}] \twoheadrightarrow R$ factors through a unique $\Bbbk$-algebra homomorphism $\overline{\mathrm{e}}_{\boldsymbol{d}} \colon \mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) \twoheadrightarrow \Bbbk$ where we set

$$\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) := \Bbbk[\boldsymbol{D}]/\langle \boldsymbol{\Delta} \rangle \cup \langle \{ D^i_j \mid |e_i| \neq |e_j| - 1 \} \rangle$$

where we set $\boldsymbol{\Delta} = (\Delta^i_j)$. Conversely, suppose $\mathrm{e}_{\boldsymbol{r}} \colon \Bbbk[\boldsymbol{D}] \twoheadrightarrow \Bbbk$ is another $\Bbbk$-algebra homomorphism where $\mathrm{e}_{\boldsymbol{r}}(\boldsymbol{D}) = \boldsymbol{r}$ where $\boldsymbol{r} = (r^i_j)$. Then we define a differential $\mathrm{d}_{\boldsymbol{r}}$ on F by $\mathrm{d}_{\boldsymbol{r}}\boldsymbol{e} := \boldsymbol{e}\boldsymbol{r}$. Thus if we set $\mathrm{Diff}_{\Bbbk}(F)$ be the set of all $\Bbbk$-linear differentials on F, then we have a bijection of sets:

$$h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) := \mathrm{Hom}_{\Bbbk\text{-alg}}(\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F), \Bbbk) \simeq \mathrm{Diff}_{\Bbbk}(F).$$

Now suppose that $\boldsymbol{e}' = (1, e'_1, \dots, e'_n)$ is another ordered homogeneous basis of F. Thus there is a graded $\Bbbk$-linear isomorphism $\varphi \colon F \to F$ such that $\varphi\boldsymbol{e} = \boldsymbol{e}'$. Let $\widetilde{\gamma}_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_\varphi \end{pmatrix}$ be the matrix representation of φ with respect to $\boldsymbol{e}$ where $\gamma_\varphi \in \mathrm{GL}_n(\Bbbk)$. Thus we have $\varphi\boldsymbol{e} = \boldsymbol{e}' = \boldsymbol{e}\widetilde{\gamma}_\varphi$. Then the matrix representation of d in the $\boldsymbol{e}'$ coordinates is given by $\boldsymbol{d}' = \widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi$ since

$$\begin{aligned} \mathrm{d}\boldsymbol{e}' &= \mathrm{d}\boldsymbol{e}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}\boldsymbol{d}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}'\widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi \\ &= \boldsymbol{e}'\boldsymbol{d}'. \end{aligned}$$

Thus we see that $\mathrm{GL}_n(\Bbbk)$ acts on $h_{\mathrm{A}^{\mathrm{d}}_{R}(F)}(\Bbbk)$ by conjugation $\mathrm{e}_{\boldsymbol{d}} \mapsto \mathrm{e}_{\widetilde{\gamma}_\varphi^{-1}\boldsymbol{d}\widetilde{\gamma}_\varphi}$. On the other hand, if we define $\mathrm{d}' \colon F \to F$ by $\mathrm{d}' = \varphi^{-1}\mathrm{d}\varphi$, then we obtain $\mathrm{d}'\boldsymbol{e} = \boldsymbol{e}\boldsymbol{d}'$, hence d' is the differential on F whose matrix representation with respect to our original ordered basis $\boldsymbol{e}$ is $\boldsymbol{d}'$. In particular, $\mathrm{e}_{\boldsymbol{d}}$ and $\mathrm{e}_{\boldsymbol{d}'}$ belong to the same $\mathrm{GL}_n(\Bbbk)$-orbit in $h_{\mathrm{A}^{\mathrm{d}}_{R}(F)}(\Bbbk)$ if and only if the corresponding differentials d and d' give isomorphic $\Bbbk$-complex structures on F with fixed identity. □

Base Change

Suppose that R is a $\Bbbk$-algebra. Then $G := F \otimes_{\Bbbk} A$ is a finite free graded R-module with $G_0 \simeq R$, $G_i = 0$ for all $i < 0$, and $G_+ \neq 0$. We set

$$\mathrm{A}^{\mathrm{d}}_{R}(G) := \mathrm{A}^{\mathrm{d}}_{\Bbbk}(F) \otimes_{\Bbbk} R \simeq R[\boldsymbol{D}]/\langle \boldsymbol{\Delta} \rangle \cup \langle \{ D^i_j \mid |e_i| \neq |e_j| - 1 \} \rangle.$$

It is clear that we have an inclusion of sets $h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(R) \subseteq h_{\mathrm{A}^{\mathrm{d}}_{R}(G)}(R)$.

Proposition 0.1. *Let $G = \mathrm{Aut}(R/\Bbbk)$. Then G acts on $h_{\mathrm{A}^{\mathrm{d}}_{R}(G)}(R)$ and the set of all fixed points is precisely $h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(R)$.*

Classifying Other Algebraic Structures on F

Let $\lambda \colon F \to F$ and $\mu \colon F \otimes_R F \to F$ be graded R-linear maps. With F equipped with λ and μ as above, we make the following definitions:

1. We say F is **unital** if $\lambda(1) = 1$ and $\mu(1 \otimes a) = a = \mu(a \otimes 1)$ for all $a \in F$.
2. We say F is **graded-commutative** (or μ is **graded-commutative**) if

   $$ab = (-1)^{|a||b|}ba$$

   for all homogeneous $a, b \in F$. We say it is **strictly graded-commutative** if it is graded-commutative and satisfies the additional property that

   $$a^2 = 0$$

   for all homogeneous $a \in F$ whenever $|a|$ is odd.
3. We say F is **multiplicative** (or λ is **μ-multiplicative**) if it satisfies the **multiplicative law**:

   $$\lambda(ab) = \lambda(a)\lambda(b)$$

   for all $a, b \in F$

4. We say F is **hom-associative** (or μ is λ**-associative**) if it satisfies the **hom-associative law**:

$$(ab)\lambda(c) = \lambda(a)(bc)$$

for all $a, b, c \in F$.

5. We say F is **permutative** (or μ is λ**-permutative**) if it satisfies the **permutative law**:

$$(\lambda(a)\lambda(b))\lambda(cd) = \lambda(ab)(\lambda(c)\lambda(d)) \tag{2}$$

for all $a, b, c, d \in F$.

Why are we interested in these definitions? Basically we view permutativity as a mixture between hom-associativity and multiplicativity.

Proposition 0.2. *Let $F = (F, \mathrm{d}, \lambda, \mu)$ be an MLDG algebra.*

1. *If F is multiplicative, then F is permutative. The converse is true if F is unital.*

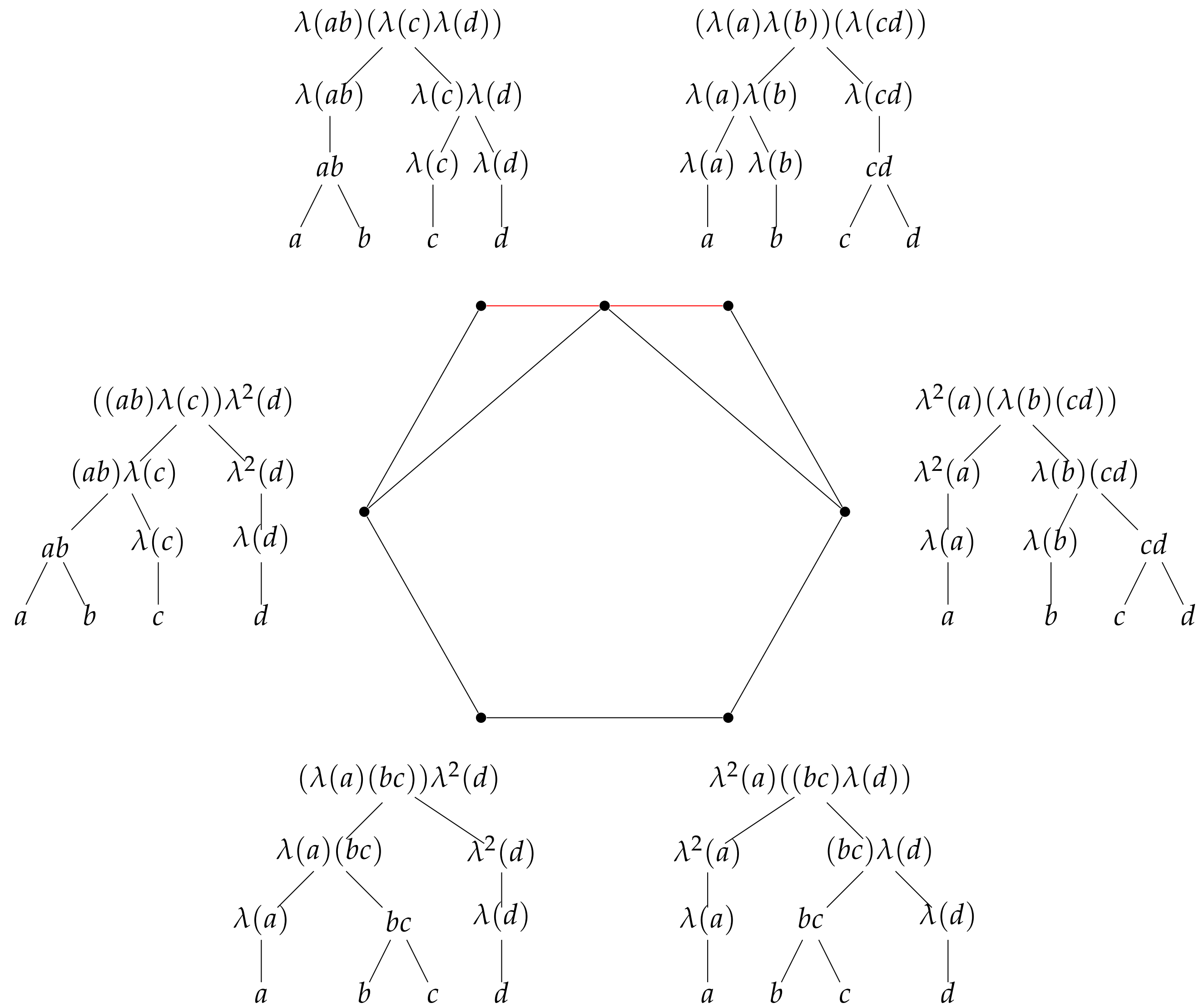
2. *If F is hom-associative, then F is permutative. In particular, if F is unital, then hom-associativity implies multiplicativity.*

Proof. 1. It is clear that if F is multiplicative, then F is permutative. Now suppose that F is unital and permutative. Then setting $c = 1 = d$ in (2) shows that F is multiplicative. In the general case where λ is not necessarily unital, we have $\lambda(1) = e$ where $e \in F_0$. In this case, the permutative law would imply that e associates with all of the other elements, and furthermore it would tell us that $e^2\lambda(ab) = e\lambda(a)\lambda(b)$ for all $a, b \in A$ (which is not quite the same as F being multiplicative).

2. Suppose F is hom-associative. Then for all $a, b, c, d \in F$, we have

$$\begin{aligned}
\lambda(ab)(\lambda(c)\lambda(d)) &= ((ab)\lambda(c))\lambda^2(d) \\
&= (\lambda(a)(bc))\lambda^2(d) \\
&= \lambda^2(a)((bc)\lambda(d)) \\
&= \lambda^2(a)(\lambda(b)(cd)) \\
&= (\lambda(a)\lambda(b))\lambda(cd).
\end{aligned}$$

There's a cute way to visualize this by tracing the edges of the permutohedron (the hexagon) below:



Note that the red edge "collapses" to the associahedra (the pentagon) if $\lambda = 1$. $\square$

Example 0.1. Let $\lambda \in R\backslash\{0\}$ and let A be an MLDG R-algebra with $\lambda_A = \mathrm{m}_\lambda$ being the multilpication by λ map given by $a \mapsto \lambda a$. Recall that A is an R-algebra, so in particular the element λ belongs to the nucleus of A. It follows that A is permutative since

$$\lambda(ab)(\lambda(c)\lambda(d)) = \lambda^3((ab)(cd)) = (\lambda(a)\lambda(b))\lambda(cd).$$

On the other hand, A is not necessarily hom-associative. Indeed, we have

$$\lambda(a)(bc) = (ab)\lambda(c) \iff \lambda[a,b,c] = 0$$

for all $a, b, c \in A$ and the righthand side need not be zero. In particular, A is hom-associative if and only if $\lambda \in \mathrm{Ann}([A])$. Similarly, A is not necessarily multiplicative. Indeed, we have

$$\lambda(ab) = \lambda(a)\lambda(b) \iff \lambda(1-\lambda)ab = 0$$

for all $a, b \in A$. If we assume that R is local and that $\lambda \in \mathfrak{m}$, then $1 - \lambda$ is a unit. In this case, we see that A is multiplicative if and only if $\lambda \in \mathrm{Ann}(\mathrm{im}\,\mu)$.

Classifying MLDG $\Bbbk$-Algebras on F

We now repeat the same procedure that we did when classifying $\Bbbk$-complex structures on F. Let $\boldsymbol{\lambda} = (\ell^i_j)$ and let $\boldsymbol{m} = (m^k_{i,j})$ be their matrix representations with respect to $\boldsymbol{e}$ respectively. Thus we have $\lambda\boldsymbol{e} = \boldsymbol{e}\boldsymbol{\lambda}$ we have $\mu(\boldsymbol{e} \otimes \boldsymbol{e}) = \boldsymbol{e} \otimes \boldsymbol{m}\boldsymbol{e}$. In terms of the matrix entries, these are given by

$$\lambda(e_j) = \sum_i \ell^i_j e_i \quad \text{and} \quad \mu(e_i \otimes e_j) = \sum_k m^k_{i,j} e_k.$$

We express the algebraic laws introduced above in terms of coordinates in the table below:

Algebraic Law	Equation
Graded Law	$G^k_{\boldsymbol{i}} = M^k_{i_1,i_2}$ if $\lvert e_i\rvert + \lvert e_j\rvert \neq \lvert e_k\rvert$ (else $G^k_{\boldsymbol{i}} = 0$)
Graded-Commutative Law	$\Gamma^k_{\boldsymbol{i}} = M^k_{i_1,i_2} - (-1)^{\lvert e_{i_1}\rvert\lvert e_{i_2}\rvert} M^k_{i_2,i_1}$
Leibniz Law	$\Lambda^k_{\boldsymbol{i}} = \sum_j (M^j_{i_1,i_2} D^k_j - D^j_{i_1} M^k_{j,i_2} - (-1)^{\lvert e_{i_1}\rvert\lvert e_{i_2}\rvert} D^j_{i_2} M^k_{i_1,j})$
Multiplicative Law	$\Theta^k_{\boldsymbol{i}} = \sum_j M^j_{i_1,i_2} L^k_j - \sum_{\boldsymbol{j}} L^{j_1}_{i_1} L^{j_2}_{i_2} M^k_{j_1,j_2}$
Hom-Associative Law	$H^k_{\boldsymbol{i}} = \sum_{\boldsymbol{j}} (M^{j_1}_{i_1,i_2} L^{j_2}_{i_3} M^k_{j_1,j_2} - M^{j_1}_{i_2,i_3} L^{j_2}_{i_1} M^k_{j_2,j_1})$
Permutative Law	$P^k_{\boldsymbol{i}} = \sum_{\boldsymbol{j}} (M^{j_1}_{i_1,i_2} L^{j_2}_{i_3} L^{j_3}_{i_4} M^k_{j_4,j_5} - M^{j_1}_{i_3,i_4} L^{j_2}_{i_1} L^{j_3}_{i_2} M^k_{j_5,j_4}) L^{j_4}_{j_1} M^{j_5}_{j_2,j_3}$

In particular, note that $P^k_{\boldsymbol{i}}$ is a sum over $2n^5$ terms Let $R = \mathbb{Z}[\boldsymbol{M}, \boldsymbol{L}]$, let $I_{\mathrm{P}} = \langle \boldsymbol{P} \rangle$, and let $I_{\mathrm{H}} = \langle \boldsymbol{H} \rangle$. Here we write $\boldsymbol{P} = \{P^k_{\boldsymbol{i}}\}$ and $\boldsymbol{H} = \{H^k_{\boldsymbol{i}}\}$, so $\boldsymbol{P}$ consists of n^5 polynomials and $\boldsymbol{H}$ consists of n^4 polynomials. Note also that each polynomial in $\boldsymbol{P}$ is a sum over $2n^5$ terms and each polynomial in $\boldsymbol{H}$ is a sum of $2n^2$ terms.
Each
The ideal I_{P} contains Note that Furthermore set $X_{\mathrm{P}} = \mathrm{Proj}(R/I)$ and set $X_{\mathrm{H}} = \mathrm{Proj}(R/J)$. Note that $X_{\mathrm{P}}, X_{\mathrm{H}} \subseteq \mathbb{P}^N$ where $N = n^2(n+3)/2$.
Thus A and B are graded rings and B is an A-algebra.

$$A_P = \mathbb{Z}[\boldsymbol{M}, \boldsymbol{L}]/\boldsymbol{\Pi}$$
$$A_H = \mathbb{Z}[\boldsymbol{M}, \boldsymbol{L}]/\langle \{\Pi^k_{\boldsymbol{i}}\} \rangle$$

$$\mathrm{A}^{(n)} = \mathrm{A} = \mathbb{Z}[\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{D}]/\langle \boldsymbol{G}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{\Theta}, \boldsymbol{H}, \boldsymbol{\Pi} \rangle.$$

where $F_+ = \mathbb{Z}^n$.

Theorem 0.2. *We have the following bijection of sets:*

$$\left\{ \mathrm{GL}_n(\Bbbk)\text{-orbits of } h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes of } \Bbbk\text{-complex} \\ \text{structures on } F \text{ with fixed identity} \end{array} \right\}$$

where $\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)$ is a $\Bbbk$-algebra (to be constructed below) and where

$$h_{\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)}(\Bbbk) := \mathrm{Hom}_{\Bbbk\text{-}alg}(\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F), \Bbbk)$$

is the $\Bbbk$-valued points of $\mathrm{A}^{\mathrm{d}}_{\Bbbk}(F)$. Two $\Bbbk$-complex structures (F, d) and (F, d') on F are said to be isomorphic with fixed identity if there exists a chain map $\varphi\colon F \to F$ such that $\varphi(1) = 1$.

Classifying MDG $\Bbbk$-Algebras on F

We now fix a differential d on F giving it the structure of a $\Bbbk$-complex and we are interested in giving an algebro-geometric classification all multiplications on F (up to isomorphism). Let $\mu \in \mathrm{Mult}(F)$ and let $\boldsymbol{m} = (m^k_{i_1,i_2})$ be its matrix representation with respect to $\boldsymbol{e}$. Thus we have $\mu(\boldsymbol{e}^\top \otimes \boldsymbol{e}) = \boldsymbol{e}^\top \boldsymbol{m}\boldsymbol{e}$. In terms of the matrix entries, these are given by

$$\mu(e_{i_1} \otimes e_{i_2}) = \sum_k m^k_{i_1,i_2} e_k.$$

for all $1 \leq i_1, i_2 \leq n$. Furthermore, let $\varepsilon \in \mathbb{N} \cup \{\infty\}$ and assume that μ is **ε-associative** meaning it is associative in homological degree i for all $i < \varepsilon$ (thus ∞-associative means associative). In the table below, we translate the algebraic laws which μ satisfies into equations which $\boldsymbol{m}$ satisfies:

Algebraic Law	Equation
Graded	$G^k_{\boldsymbol{i}} = M^k_{i_1,i_2}$ if $\lvert e_i\rvert + \lvert e_j\rvert \neq \lvert e_k\rvert$ (else $G^k_{\boldsymbol{i}} = 0$)
Graded-Commutative Law	$\Gamma^k_{\boldsymbol{i}} = M^k_{i_1,i_2} - (-1)^{\lvert e_{i_1}\rvert\lvert e_{i_2}\rvert} M^k_{i_2,i_1}$
Leibniz Law	$\Lambda^k_{\boldsymbol{i}} = \sum_j (M^j_{i_1,i_2} d^k_j - d^j_{i_1} M^k_{j,i_2} - (-1)^{\lvert e_{i_1}\rvert\lvert e_{i_2}\rvert} d^j_{i_2} M^k_{i_1,j})$
ε-Associative Law	$H^k_{\varepsilon,\boldsymbol{i}} = \sum_j (M^j_{i_1,i_2} M^k_{j,i_3} - M^j_{i_2,i_3} M^k_{i_1,j})$ if $\lvert e_{i_1}\rvert + \lvert e_{i_2}\rvert + \lvert e_{i_3}\rvert < \varepsilon$ (else $H^k_{\varepsilon,\boldsymbol{i}} = 0$)

We set $A_\varepsilon = \Bbbk[\boldsymbol{M}]/\langle \boldsymbol{G}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{H}_\varepsilon\rangle$ and we set $X_\varepsilon = \operatorname{Spec} A_\varepsilon$.

Theorem 0.3. *We have the following bijection of sets:*

$$\{\mathrm{GL}_n(\Bbbk)\text{-orbits of } \operatorname{Hom}_{\Bbbk\text{-alg}}(A_\varepsilon, R)\} \longleftrightarrow \{\text{ isomorphism classes of } \varepsilon\text{-associative multiplications on } F\ \}.$$

Thus the $\mathrm{GL}_n(\Bbbk)$-orbits of the $\Bbbk$-valued points of X_ε are in bijection $[\mathrm{Mult}_\varepsilon(F)] := \mathrm{Mult}_\varepsilon(F)/\!\sim$ where $\mathrm{Mult}_\varepsilon(F)$ denotes the set of all ε-associative multiplications on F and where $\sim$ is the isomorphism equivalence relation.

Now suppose that R is a $\Bbbk$-algebra. Then $G := R \otimes_\Bbbk F$ is a finite free graded R-module with $G_0 \simeq R$, $G_i = 0$ for all $i < 0$, and $G_+ \neq 0$. We set

$$\mathrm{A}^{\mathrm{d}}_R(G) := \mathrm{A}^{\mathrm{d}}_\Bbbk(F) \otimes_\Bbbk R \simeq R[\boldsymbol{D}]/\langle \boldsymbol{\Delta}\rangle \cup \langle \{D^i_j \mid \lvert e_i\rvert \neq \lvert e_j\rvert - 1\}\rangle.$$

It is clear that we have an inclusion of sets $h_{\mathrm{A}^{\mathrm{d}}_\Bbbk(F)}(R) \subseteq h_{\mathrm{A}^{\mathrm{d}}_R(G)}(R)$.

Proposition 0.3. *Let $G = \operatorname{Aut}(R/\Bbbk)$. Then G acts on $h_{\mathrm{A}^{\mathrm{d}}_R(G)}(R)$ and the set of all fixed points is precisely $h_{\mathrm{A}^{\mathrm{d}}_\Bbbk(F)}(R)$.*

We set

$$\mathrm{A}^{(n)} = \mathrm{A} = \mathbb{Z}[\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{D}]/\langle \boldsymbol{G}, \Gamma, \boldsymbol{\Lambda}, \Theta, \boldsymbol{H}, \boldsymbol{\Pi}\rangle$$

where $F_+ = \mathbb{Z}^n$. Then we have

$$\mathrm{A} \otimes_\mathbb{Z} R = R[\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{D}]/\langle \boldsymbol{G}, \Gamma, \boldsymbol{\Lambda}, \Theta, \boldsymbol{H}, \boldsymbol{\Pi}\rangle$$

which classifies MLDG structures on $F \otimes_\mathbb{Z} R = R^n$. Similarly,

$$\mathrm{A} \otimes_\mathbb{Z} \mathbb{Q} = \mathrm{A}_\mathbb{Q} = \mathbb{Q}[\boldsymbol{M}, \boldsymbol{L}, \boldsymbol{D}]/\langle \boldsymbol{G}, \Gamma, \boldsymbol{\Lambda}, \Theta, \boldsymbol{H}, \boldsymbol{\Pi}\rangle$$

We are interested in

$$\mathrm{h}_\mathbb{Q}(K) := \operatorname{Hom}_{\mathbb{Q}\text{-alg}}(\mathrm{A}_\mathbb{Q}, K)$$

where $K/\mathbb{Q}$ is a finite extension of degree n. Then $\mathrm{h}_\mathbb{Q}(K)$ classifies MLDG $\mathbb{Q}$-algebras.
Consider

$$\mathbb{Z}_p[x, y, z]/\langle x^2, p^2, pz, xy, y^2z^2\rangle = (\mathbb{Z}/p^2)[x, y, z]/\langle x^2, pz, xy, y^2z^2\rangle$$

Consider

$$\mathbb{Z}_p[x, y, z]/\langle x^2, w^2, zw, px, p^2z^2\rangle$$