Mathematical Programming Homework 5

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Problem 1

Exercise 1. Reformulate the following linear program into Standard Form (SF).

$$\begin{aligned} \text{maximize} \quad & z = x_1 - x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 4 \\ & x_1 - x_3 \geq 2 \\ & x_1 + 2x_2 = 1 \\ & x_1 \text{ and } x_2 \text{ free}, \; x_3 \geq 0 \end{aligned}$$

For full credit, introduce as few new variables as possible and present the data of the final LP in the form:

$$\begin{aligned} \boldsymbol{x} &= \\ \boldsymbol{c} &= \\ A &= \\ \boldsymbol{b} &= \end{aligned}$$

Solution 1. To convert to a minimization problem, we multiply the objective function by -1:

$$\text{minimize } \widehat{z} = -x_1 + x_2 - 2x_3.$$

Suppose $\boldsymbol{x} = (x_1, x_2, x_3)$ is a feasible solution to this linear program. Observe that since $x_3 \geq 0$ and $x_1 - x_3 \geq 2$, we see that $x_1 \geq 2$. Furthermore, since $x_2 = (1 - x_1)/2$, we see that $x_2 \geq -1/2$. With this in mind, we make the following linear change of coordinates:

$$\begin{aligned} x_1' &= x_1 - 2 \\ x_2' &= x_2 + 1/2 \\ x_3' &= x_3 \end{aligned}$$

With these substitutions, the linear program becomes

$$\begin{aligned} \text{minimize} \quad & \widehat{z} = x_1' - x_2' + 2x_3' + 3/2 \\ \text{s.t.} \quad & 2x_1' + 3x_2' \leq 3/2 \\ & x_3' - x_1' \leq 0 \\ & x_1' + 2x_2' = 0 \\ & \boldsymbol{x}' \geq 0 \end{aligned}$$

We also want to remove the constant function in the objective function, so we set $z' = \widehat{z} + 3/2$. Next we introduce lack variables $s_1, s_2 \geq 0$ to convert the inequalities to an equalities:

$$\begin{aligned} 2x_1' + 3x_2' + s_1 &= 3/2 \\ x_3' - x_1' + s_2 &= 0 \end{aligned}$$

The linear program becomes

$$\begin{aligned} \text{minimize} \quad & z' = x_1' - x_2' + 2x_3' \\ \text{s.t.} \quad & 2x_1' + 3x_2' + s_1 = 3/2 \\ & x_3' - x_1' + s_2 = 0 \\ & x_1' + 2x_2' = 0 \\ & \boldsymbol{x}', s_1, s_2 \geq 0 \end{aligned}$$

Finally we set

$$A = \begin{pmatrix} 2 & 3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

and we clean our notation a bit and write $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) = (x_1', x_2', x_3', s_1, s_2)$ and $z = z'$. With these notational changes, the linear program becomes

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

which is in standard form.

Problem 2

Consider the following polyhedral set

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid -2x_1 + x_2 \leq 5,\ x_1/2 - x_2 \leq 2,\ x_1, x_2 \geq 0\}$$

Problem 2.a

Exercise 2. Use a method of your choice to find the set of all extreme points (EPs) in X.

Solution 2. Let $s_1, s_2 \geq 0$ and let

$$A = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1/2 & -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Then every extreme point of X corresponds to a basic feasible solution to

$$\begin{aligned} A\widehat{\boldsymbol{x}} &= \boldsymbol{b} \\ \widehat{\boldsymbol{x}} &\geq 0 \end{aligned}$$

where $\widehat{\boldsymbol{x}} = (\boldsymbol{x}, \boldsymbol{s})^\top = (x_1, x_2, s_1, s_2)$. We find all basic solutions and determine if they are feasible or not below:

1. The basis $\{x_1, x_2\}$ produces the basic solution $(-14/3, -13/3, 0, 0)^\top$, which is infeasible.
2. The basis $\{x_1, s_1\}$ produces the basic feasible solution $(4, 0, 13, 0)^\top$. It corresponds to the point $\boldsymbol{x}^1 = (4, 0)^\top$.
3. The basis $\{x_1, s_2\}$ produces the basic solution $(-5/2, 0, 0, 13/4)^\top$, which is infeasible.
4. The basis $\{x_2, s_1\}$ produces the basic solution $(-2, 0, 7, 0)^\top$, which is infeasible.
5. The basis $\{x_2, s_2\}$ produces the basic feasible solution $(0, 5, 0, 7)^\top$. It corresponds to the point $\boldsymbol{x}^2 = (0, 5)^\top$.
6. The basis $\{s_1, s_2\}$ produces the basic feasible solution $(0, 0, 5, 2)^\top$. It corresponds to the point $\boldsymbol{x}^3 = (0, 0)^\top$.

After considering all possibilities, we find that the extreme points are $\boldsymbol{x}^1 = (4, 0)^\top$, $\boldsymbol{x}^2 = (0, 5)^\top$, and $\boldsymbol{x}^3 = (0, 0)^\top$.

Problem 2.b

Exercise 3. Use an appropropiate algebraic derivation and find the set of all recession directions in X.

Solution 3. A direction of unboundness $\widehat{\boldsymbol{d}} = (\boldsymbol{d}, \boldsymbol{s})^\top = (d_1, d_2, s_1, s_2)$ must satisfy

$$\begin{aligned} A\widehat{\boldsymbol{d}} &= 0 \\ \widehat{\boldsymbol{d}} &\geq 0 \\ \widehat{\boldsymbol{d}} &\neq 0 \end{aligned} \tag{1}$$

Two such solutions are given by $\widehat{\boldsymbol{d}}^1 = (2,4,0,3)^\top$ and $\widehat{\boldsymbol{d}}^2 = (2,1,3,0)^\top$ corresponding to the directions $\boldsymbol{d}^1 = (2,4)$ and $\boldsymbol{d}^2 = (2,1)$ respectively. Note that since A has full rank, the dimension of its null space is 2, and since $\{\widehat{\boldsymbol{d}}^1, \widehat{\boldsymbol{d}}^2\}$ is linearly independent, we see that every $\widehat{\boldsymbol{d}} \in \mathbb{R}^4$ which satisfies $A\widehat{\boldsymbol{d}} = 0$ has the form

$$\widehat{\boldsymbol{d}} = t_1\widehat{\boldsymbol{d}}^1 + t_2\widehat{\boldsymbol{d}}^2$$

for some $t_1, t_2 \in \mathbb{R}$. If in addition we want $\widehat{\boldsymbol{d}} \geq 0$, then it's easy to see that we need $t_1, t_2 \geq 0$. Also if we want $\widehat{\boldsymbol{d}} \neq 0$, then we need at least one of t_1 or t_2 to be nonzero.

Problem 2.c

Exercise 4. What recession directions that you found in part b are extreme? Explain.

Solution 4. Let D be the set of all $\widehat{\boldsymbol{d}} \in \mathbb{R}^4$ which satisfy (1). Notice that D cannot have any extreme points. Indeed, if $\widehat{\boldsymbol{d}}$ is a direction of unboundness, then so too is $\widehat{\boldsymbol{d}}/2$, and since $\widehat{\boldsymbol{d}} = \widehat{\boldsymbol{d}}/2 + \widehat{\boldsymbol{d}}/2$, we see that $\widehat{\boldsymbol{d}}$ cannot be an extreme point of D. To rectify this issue, we set $D_1 = \{\widehat{\boldsymbol{d}} \in D \mid \|\widehat{\boldsymbol{d}}\| = 1\}$. Now it is easy to see that $\frac{1}{\sqrt{29}}\widehat{\boldsymbol{d}}^1$ and $\frac{1}{\sqrt{13}}\widehat{\boldsymbol{d}}^2$ are the extreme points of this set.

Problem 2.d

Exercise 5. Apply the Representation Theorem and find a representation for the point $\boldsymbol{x} = (4,6)^\top \in X$.

Solution 5. We have

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\ &= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= 2\boldsymbol{d}^2 + \frac{4}{5}\boldsymbol{x}^2 + \frac{1}{5}\boldsymbol{x}^3 \end{aligned}$$

which shows that $\boldsymbol{x}$ is a convex combination of the extreme points plus a direction of unboundness.

Problem 2.c

Exercise 6. Is your representation unique? Explain.

Solution 6. No because $\boldsymbol{x}$ doesn't belong to the convex hull of $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, and $\boldsymbol{x}^3$. For instance, another representation of $\boldsymbol{x}$ is given by

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} 4 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \frac{5}{10}\begin{pmatrix} 4 \\ 0 \end{pmatrix} + \frac{4}{10}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \boldsymbol{d}^1 + \frac{5}{10}\boldsymbol{x}^1 + \frac{4}{10}\boldsymbol{x}^2 + \frac{1}{10}\boldsymbol{x}^3 \end{aligned}$$

Problem 3

Exercise 7. Consider the linear program (LP) in $\mathbb{R}^2$:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in X \end{aligned}$$

where X is defined as in question 2 above. Apply a method for solving LP in $\mathbb{R}^2$ and find the set of all optimal solutions to this LP for each cost vectors below. Write each set in an appropriate form.

1. Let $\boldsymbol{a} = (1,1)^\top$
2. Let $\boldsymbol{b} = (2,-1)^\top$
3. Let $\boldsymbol{c} = (0,1)^\top$

Solution 7. Recall that the extreme points of X are $\boldsymbol{x}^1 = (4,0)^\top, \boldsymbol{x}^2 = (0,5)^\top$, and $\boldsymbol{x}^3 = (0,0)^\top$, and recall that the extreme directions of unboundedness of X are $\boldsymbol{d}^1 = (2,4)^\top$ and $\boldsymbol{d}^2 = (2,1)^\top$.

1. We first find the optimal solution for the cost vector $\boldsymbol{a} = (1,1)^\top$. Observe that $\boldsymbol{a}^\top \boldsymbol{d}^1 = 6 \geq 0$ and $\boldsymbol{a}^\top \boldsymbol{d}^2 = 3 \geq 0$. Therefore for any direction of unboundedness $\boldsymbol{d}$, we will have $\boldsymbol{a}^\top \boldsymbol{d} \geq 0$. It follows that the optimal solution will occur at one of the extreme points (or perhaps on an adjacent edge connecting two extreme points). A simple calculation shows

$$\begin{aligned} \boldsymbol{a}^\top \boldsymbol{x}^1 &= 4 \\ \boldsymbol{a}^\top \boldsymbol{x}^2 &= 5 \\ \boldsymbol{a}^\top \boldsymbol{x}^3 &= 0. \end{aligned}$$

It follows that the optimal solution for the cost vector $\boldsymbol{a}$ is $\boldsymbol{x}^3$ with optimal value given by $\boldsymbol{a}^\top \boldsymbol{x}^3 = 0$.

2. Now we find the optimal solution for the cost vector $\boldsymbol{b} = (2,-1)^\top$. Observe that $\boldsymbol{b}^\top \boldsymbol{d}^1 = 0 \geq 0$ and $\boldsymbol{b}^\top \boldsymbol{d}^2 = 3 \geq 0$. Therefore for any direction of unboundedness $\boldsymbol{d}$, we will have $\boldsymbol{b}^\top \boldsymbol{d} \geq 0$. It follows that the optimal solution will occur at one of the extreme points (or perhaps on an adjacent edge connecting two extreme points). A simple calculation shows

$$\begin{aligned} \boldsymbol{b}^\top \boldsymbol{x}^1 &= 8 \\ \boldsymbol{b}^\top \boldsymbol{x}^2 &= -5 \\ \boldsymbol{b}^\top \boldsymbol{x}^3 &= 0. \end{aligned}$$

It follows that the optimal solution for the cost vector $\boldsymbol{b}$ is $\boldsymbol{x}^2$ with optimal value given by $\boldsymbol{b}^\top \boldsymbol{x}^2 = -5$.

3. Finally we find the optimal solution for the cost vector $\boldsymbol{c} = (0,1)^\top$. Observe that $\boldsymbol{c}^\top \boldsymbol{d}^1 = 4 \geq 0$ and $\boldsymbol{c}^\top \boldsymbol{d}^2 = 1 \geq 0$. Therefore for any direction of unboundedness $\boldsymbol{d}$, we will have $\boldsymbol{c}^\top \boldsymbol{d} \geq 0$. It follows that the optimal solution will occur at one of the extreme points (or perhaps on an adjacent edge connecting two extreme points). A simple calculation shows

$$\begin{aligned} \boldsymbol{c}^\top \boldsymbol{x}^1 &= 0 \\ \boldsymbol{c}^\top \boldsymbol{x}^2 &= 5 \\ \boldsymbol{c}^\top \boldsymbol{x}^3 &= 0. \end{aligned}$$

It follows that the optimal solutions for the cost vector $\boldsymbol{c}$ occurs on the adjacent edge connecting $\boldsymbol{x}^1$ with $\boldsymbol{x}^3$, so they have the form $t_1 \boldsymbol{x}^1 + t_2 \boldsymbol{x}^3$ where $t_1, t_2 \geq 0$ satisfy $t_1 + t_2 = 1$. The optimal value at these optimal points is given by $\boldsymbol{c}^\top (t_1 \boldsymbol{x}^1 + t_2 \boldsymbol{x}^2) = 0$.

Problem 4

Exercise 8. Apply the Simplex Algorithm and find an optimal solution to the linear program. In every iteration, select the most negative reduced cost.

$$\begin{aligned} \min \quad & -6x_1 - 14x_2 - 13x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 + 2x_3 \leq 48 \\ & x_1 + 2x_2 + 4x_3 \leq 60 \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Solution 8. We introduce slack variables $x_4, x_5 \geq 0$ to convert the inequality constraints to equality constraints:

$$\begin{aligned} \min \quad & -6x_1 - 14x_2 - 13x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 + 2x_3 + x_4 = 48 \\ & x_1 + 2x_2 + 4x_3 + x_5 = 60 \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

where now $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)^\top$. Let

$$A = \begin{pmatrix} 2 & 4 & 2 & 1 & 0 \\ 1 & 2 & 4 & 0 & 1 \end{pmatrix}, \quad , \boldsymbol{b} = \begin{pmatrix} 48 \\ 60 \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} -6 \\ -14 \\ -13 \\ 0 \\ 0 \end{pmatrix}$$

Then in standard form, our linear program looks like:

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

We now perform the simplex algorithm as follows

First iteration of the simplex algorithm: We begin by using the slack variables as the initial basis. Let

$$\begin{aligned} \boldsymbol{x}_B &= (x_4, x_5)^\top & \boldsymbol{x}_N &= (x_1, x_2, x_3)^\top \\ \boldsymbol{c}_B &= (0, 0)^\top & \boldsymbol{c}_N &= (-6, -14, -13)^\top \\ B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & N &= \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix} \\ B^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & B^{-1}N &= \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix} \end{aligned}$$

Our current basic solution and our current objective value are given by

$$\begin{aligned} \widehat{\boldsymbol{b}} &= B^{-1}\boldsymbol{b} & \widehat{z} &= \boldsymbol{c}_B^\top \widehat{\boldsymbol{b}} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 48 \\ 60 \end{pmatrix} & &= (0, 0)^\top \begin{pmatrix} 48 \\ 60 \end{pmatrix} \\ &= (48, 60)^\top & &= 0 \end{aligned}$$

To determine whether or not this is optimal, we calculate the reduced cost vector:

$$\begin{aligned} \widehat{\boldsymbol{c}}_N^\top &= (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top B^{-1} N) \\ &= (-6, -14, -13) - (0, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix} \\ &= (-6, -14, -13). \end{aligned}$$

The components of $\widehat{\boldsymbol{c}}_N$ are negative, so the basis is not optimal. Since $(\widehat{\boldsymbol{c}}_N)_2$ is the more negative component, we select x_2 (the second nonbasic variable) as the entering variable. To find the leaving variable, we first calculate

$$\begin{aligned}\widehat{A}_2 &= B^{-1}A_2 \\ &= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}4 \\ 2\end{pmatrix} \\ &= \begin{pmatrix}4 \\ 2\end{pmatrix}\end{aligned}$$

where A_2 is the second column of the matrix A. Letting $\widehat{a}_{i,2}$ denote the ith entry in $\widehat{A}_2$, we then calculate

$$\begin{aligned}\overline{x}_2 &= \min_{1\leq i\leq 2}\left\{\frac{\widehat{b}_i}{\widehat{a}_{i,2}} \mid \widehat{a}_{i,2} > 0\right\} \\ &= \min\left\{\frac{48}{4}, \frac{60}{2}\right\} \\ &= 12.\end{aligned}$$

Thus the leaving variable is x_4 (the first basic variable) corresponding to the minimum of $\{\widehat{b}_i/\widehat{a}_{i,2} \mid \widehat{a}_{i,2} > 0\}$. This completes the first iteration of the simplex algorithm.

Second iteration of the simplex algorithm: We replace x_4 with x_2 in our basis, so let

$$\begin{aligned}\boldsymbol{x}_B &= (x_2, x_5)^\top & \boldsymbol{x}_N &= (x_1, x_4, x_3)^\top \\ \boldsymbol{c}_B &= (-14, 0)^\top & \boldsymbol{c}_N &= (-6, 0, -13)^\top \\ B &= \begin{pmatrix}4 & 0 \\ 2 & 1\end{pmatrix} & N &= \begin{pmatrix}2 & 1 & 2 \\ 1 & 0 & 4\end{pmatrix} \\ B^{-1} &= \begin{pmatrix}1/4 & 0 \\ -1/2 & 1\end{pmatrix} & B^{-1}N &= \begin{pmatrix}1/2 & 1/4 & 1/2 \\ 0 & -1/2 & 3\end{pmatrix}\end{aligned}$$

Our current basic solution and our current objective value are given by

$$\begin{aligned}\widehat{\boldsymbol{b}} &= B^{-1}\boldsymbol{b} & \widehat{z} &= \boldsymbol{c}_B^\top\widehat{\boldsymbol{b}} \\ &= \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}48 \\ 60\end{pmatrix} & &= (-14, 0)^\top\begin{pmatrix}12 \\ 36\end{pmatrix} \\ &= (12, 36)^\top & &= -168\end{aligned}$$

To determine whether or not this is optimal, we calculate the reduced cost vector:

$$\begin{aligned}\widehat{\boldsymbol{c}}_N^\top &= (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top B^{-1}N) \\ &= (-6, 0, -13) - (-14, 0)\begin{pmatrix}1/2 & 1/4 & 1/2 \\ 0 & -1/2 & 3\end{pmatrix} \\ &= (1, 7/2, -6).\end{aligned}$$

The third component of $\widehat{\boldsymbol{c}}_N$ is negative, so the basis is not optimal. Since this is the only negative component, we select x_3 (the third nonbasic variable) as the entering variable. To find the leaving variable, we first calculate

$$\begin{aligned}\widehat{A}_3 &= B^{-1}A_3 \\ &= \begin{pmatrix}1/4 & 0 \\ -1/2 & 1\end{pmatrix}\begin{pmatrix}2 \\ 4\end{pmatrix} \\ &= \begin{pmatrix}1/2 \\ 3\end{pmatrix}\end{aligned}$$

where A_3 is the second column of the matrix A. Letting $\widehat{a}_{i,3}$ denote the ith entry in $\widehat{A}_3$, we then calculate

$$\begin{aligned}\overline{x}_3 &= \min_{1\leq i\leq 2}\left\{\frac{\widehat{b}_i}{\widehat{a}_{i,3}} \mid \widehat{a}_{i,3} > 0\right\} \\ &= \min\left\{\frac{12}{1/2}, \frac{36}{3}\right\} \\ &= 12.\end{aligned}$$

Thus the leaving variable is x_5 (the second basic variable). This completes the second iteration of the simplex algorithm.

Third iteration of the simplex algorithm: We replace x_3 with x_5 in our basis, so let

$$\begin{aligned} \boldsymbol{x}_B &= (x_2, x_3)^\top & \boldsymbol{x}_N &= (x_1, x_4, x_5)^\top \\ \boldsymbol{c}_B &= (-14, -13)^\top & \boldsymbol{c}_N &= (-6, 0, 0)^\top \\ B &= \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} & N &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ B^{-1} &= \begin{pmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{pmatrix} & B^{-1}N &= \begin{pmatrix} 1/2 & 1/3 & -1/6 \\ 0 & -1/6 & 1/3 \end{pmatrix} \end{aligned}$$

Our current basic solution and our current objective value are given by

$$\begin{aligned} \widehat{\boldsymbol{b}} &= B^{-1}\boldsymbol{b} & \widehat{\boldsymbol{z}} &= \boldsymbol{c}_B^\top \widehat{\boldsymbol{b}} \\ &= \begin{pmatrix} 1/3 & -1/6 \\ -1/6 & 1/3 \end{pmatrix} \begin{pmatrix} 48 \\ 60 \end{pmatrix} & &= (-14, -13)^\top \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ &= (6, 12)^\top & &= -240 \end{aligned}$$

To determine whether or not this is optimal, we calculate the reduced cost vector:

$$\begin{aligned} \widehat{\boldsymbol{c}}_N^\top &= (\boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top B^{-1} N) \\ &= (-6, 0, 0) - (-14, -13) \begin{pmatrix} 1/2 & 1/3 & -1/6 \\ 0 & -1/6 & 1/3 \end{pmatrix} \\ &= (1, 5/2, 2). \end{aligned}$$

The cost vector consists of strictly positive entries. This means that the objective function will *increase* no matter which direction we choose: we have found a local minimum at $\boldsymbol{x}^* = (0, 6, 12)$ with objective value given by $z^* = -240$. In fact, this is a global minimum since linear programming problems are convex optimization problems.

Problem 5

Problem 5.a

Exercise 9. Let $\boldsymbol{x}$ be a feasible solution but not a basic feasible solution for $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$. Prove that the columns of A corresponding to the nonzero entries of $\boldsymbol{x}$ are linearly dependent.

Solution 9. According to the book, a point $\boldsymbol{x}$ is a **basic solution** if

1. $\boldsymbol{x}$ satisfies the equality constraints of the linear program
2. the columns of the constraint matrix corresponding to the nonzero components of $\boldsymbol{x}$ are linearly independent.

So by definition (according to the book), the columns of A corresponding to the nonzero entries of $\boldsymbol{x}$ are linearly dependent. To complete this problem though, I'll prove the following instead:

Proposition 0.1. *Let A be an $m \times n$ matrix with $m < n$ and let $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = 0$, $\boldsymbol{x} \geq 0$. Furthermore, suppose that $\boldsymbol{x}$ has k nonzero entries where $m < k \leq n$. Then the columns corresponding to the nonzero entries of $\boldsymbol{x}$ are linearly dependent.*

Proof. This follows from the fact that $\operatorname{rank} A \leq m$. Thus the maximum size of a set of linearly independent columns of A must be less than or equal to m as well. □

Problem 5.b

Exercise 10. Let $\boldsymbol{x}$ be a feasible point of $X = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$ that is not an extreme point. Prove that there exists a nonzero vector $\boldsymbol{p} \in \mathbb{R}^n$ such that

1. $A\boldsymbol{p} = 0$ and,
2. $p_i = 0$ if $x_i = 0$.

(Hint: use the result from part a).

Solution 10. Note that $\boldsymbol{x}$ is not a basic feasible solution since it is not an extreme point. Therefore the columns of A corresponding to the nonzero entries of $\boldsymbol{x}$ are linearly dependent and we can find a feasible direction $\boldsymbol{p}$ (where $\boldsymbol{p} \neq 0$) satisfying 1 and 2 above. For instance, suppose the first k entries of $\boldsymbol{x}$ are nonzero and the remaining $n-k$ entries are zero where $m < k \leq n$. Then the set of columns $\{A_1, \ldots, A_k\}$ is linearly dependent, say

$$a_1 A_1 + \cdots + a_k A_k = 0$$

for some $a_1, \ldots, a_k \in \mathbb{R}$ not all zero. By scaling the a_i if necessary, we can assume that $|a_i| < x_i$ for all $1 \leq i \leq k$. Then we can set $\boldsymbol{p} = (a_1, \ldots, a_k, 0, \ldots, 0)$, and $\boldsymbol{p}$ will satisfy 1 and 2 above (it will also be a feasible direction, meaning $\boldsymbol{x} + \boldsymbol{p}$ is feasible).