Algebra Exercises

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Definition 0.1. Let A be a ring. We say A is **antilocal** if it satisfies the following property: for all units u of A, either $1+u=0$ or $1+u$ is a unit.

Proposition 0.1. *Let A be an antilocal ring. Then $K = A^\times \cup \{0\}$ is a field. Moreover, A is a reduced K-algebra with K being the largest field contained in A.*

Proof. Clearly $1 \in K$. Also, given $u, v \in K$ we have

$$u + v = u(1 + v/u) = \begin{cases} 0 & \text{if } u = -v \\ \text{nonzero unit} & \text{else} \end{cases}$$

It follows that K is a field, and hence A is a K-algebra. In fact, K is the largest field contained in A (if K' was another field contained in A, then $K' \subseteq A^\times \subseteq K$). Furthermore, note that A doesn't contain any nilpotents since a niplotent plus a unit is a unit (if $\varepsilon^n = 0$ and $uv = 1$, then $(u+\varepsilon)\sum_{i=1}^{n-1} v^i \varepsilon^{i-1} = 1$). It follows that A is a reduced K-algebra. $\square$

Here are several examples and nonexamples of antilocal rings:

1. The ring $A = \mathbb{Q}[x]/\langle x^2 \rangle$ is not antilocal since it contains a nilpotent. In particular, we have $(1-x)(1+x) = 1$ in A, and we have
$$A \cong \mathbb{Q} \oplus \mathbb{Q}\varepsilon \quad \text{and} \quad A^\times \cong \mathbb{Q}^\times \oplus \mathbb{Q}\varepsilon$$
where $\varepsilon^2 = 0$.

2. The ring $A = \mathbb{Q}[x]/\langle x^2 - 1 \rangle$ is not antilocal. In particular, observe that
$$A \cong \mathbb{Q}[x]/\langle x-1 \rangle \times \mathbb{Q}[x]/\langle x+1 \rangle \quad \text{and} \quad A^\times \cong \mathbb{Q}^\times \times \mathbb{Q}^\times.$$

3. The ring $A = \mathbb{R}[x]/\langle x^2 + 1 \rangle$ is antilocal. In particular, observe that
$$A \cong \mathbb{C} \quad \text{and} \quad A^\times \cong \mathbb{C}^\times.$$

4. The ring $A = \mathbb{R}[x,y]/\langle x^2 - y^2 - 1 \rangle$ is not antilocal since $(x+y)(x-y) = 1$ and $x+y \neq 0 \neq x-y$ in A. In particular, observe that
$$A \cong \mathbb{R}[u,v]/\langle uv - 1 \rangle \cong \mathbb{R}[u, 1/u] \quad \text{and} \quad A^\times \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{R}u^n.$$
via the map given by $u \mapsto x+y$ and $v \mapsto x-y$. We can describe A as such:
$$A \cong \mathbb{R}[t][\sqrt{1+t^2}] \quad \text{and} \quad A^\times.$$

5. The ring $A = \mathbb{R}[x,y]/\langle x^2 + y^2 - 1 \rangle$ is antilocal, however
$$B := \mathbb{C} \otimes_\mathbb{R} A \simeq \mathbb{C}[x,y]/\langle x^2 + y^2 - 1 \rangle \cong \mathbb{C}[y]/\langle \sqrt{1-x^2} \rangle$$
is not antilocal since $(x+iy)(x-iy) = 1$ and $x+iy \neq 0 \neq x-iy$ in B. Note that $B \simeq \mathbb{C}[u, 1/u]$.

6. The ring $A = \mathbb{C}[x,y]/\langle y^2 - x^3 - 1 \rangle$ is antilocal.

7. The ring $A = \mathbb{R}[x,y,z]/\langle x^2 - y^2 - z^2 \rangle$ is antilocal.

Proposition 0.2. *Let $A = K[\mathbf{x}]/\mathfrak{p}$ be a K-algebra where $\mathfrak{p}$ is a homogeneous prime ideal. Then $A^\times = K$; in particular, A is antilocal.*

Proof. Suppose $\overline{uv} = 1$ where $u, v \in K[\mathbf{x}]$ both having degree ≥ 1. Then we have $uv = 1 + p$ where $p \in \mathfrak{p}$. In particular, if we express u and v in terms of their homogeneous components in decreasing order, say as $u = u_{i_m} + u_{i_{m-1}} + \cdots + u_{i_1}$ and $v = v_{j_n} + v_{j_{n-1}} + \cdots + v_1$, then we see that $u_{i_m} v_{j_n} \in \mathfrak{p}$. It follows that either u_{i_m} or v_{j_n} belongs to $\mathfrak{p}$, and so by an induction argument on the $m+n$ terms, we see that $u, v \in K$. $\square$

Proposition 0.3. *Let A be an antilocal ring with $\mathbb{Q} = A^\times \cup \{0\}$. Let K be a number field and set $B = L \otimes_K A$. Then B is antilocal with $B = L^\times \cup \{0\}$.*

Proof. Let $\alpha \in \mathcal{O}_K$ and

$$f(X) = X^n + c_{n-1}X^{n-1} + \cdots c_1 X + c_0$$

where $c_0, \ldots, c_{n-1}, c_n \in K$. Let α be a root of f in a splitting field L/K where we may assume that n is minimal and let $B = K \otimes_{\mathbb{Q}} A$ (in particular, $\alpha \in B$ is integral over A). Then we have

$$-c_0^{-1}\alpha(\alpha^{n-1} + c_{n-1}\alpha^{n-2} + c_1) = 1.$$

By minimality of n, we see that α is a unit in B. $\square$

Proposition 0.4. *Let A be an antilocal ring with $K = A^\times \cup \{0\}$. Let K be a number field and set $B = L \otimes_K A$. Then B is antilocal with $B = L^\times \cup \{0\}$.*

Proof. Let $\alpha \in \mathcal{O}_K$ and

$$f(X) = X^n + c_{n-1}X^{n-1} + \cdots c_1 X + c_0$$

where $c_0, \ldots, c_{n-1}, c_n \in K$. Let α be a root of f in a splitting field L/K where we may assume that n is minimal and let $B = K \otimes_{\mathbb{Q}} A$ (in particular, $\alpha \in B$ is integral over A). Then we have

$$-c_0^{-1}\alpha(\alpha^{n-1} + c_{n-1}\alpha^{n-2} + c_1) = 1.$$

By minimality of n, we see that α is a unit in B. $\square$

0.1 A Quartic

In this subsection, we go over an example of a quartic curve which will demonstrate many of the concepts introduced above. Let $A = \mathbb{Z}[x,y]/\langle f(x,y) \rangle$ where

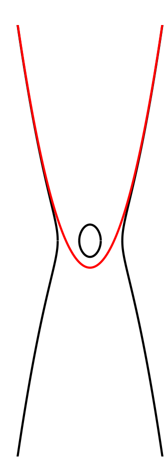
$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 \tag{1}$$

Note that from the expression of f in (1) we see that $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$ are units in A. Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g(x)}]$ where

$$f = y^2 - (x-1)(x-2)(x-3)(x-4) = y^2 - g(x). \tag{2}$$

The expression of f in (2) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (2) we can read off useful information of A viewed as a finite module extension, whereas from (1) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day.

Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we first look at its $\mathbb{R}$-valued points: $X(\mathbb{R}) = \operatorname{Spec} \mathbb{R} \otimes_{\mathbb{Z}} A = \operatorname{Spec} \mathbb{R}[x,y]/f$. We can visualize the $\mathbb{R}$-valued points of X in the picture below:



The thick black curve is $X(\mathbb{R}) = \mathrm{V}_{\mathbb{R}}(f)$ whereas the thick red curve is $\mathrm{V}_{\mathbb{R}}(u)$. Notice that $\mathrm{V}_{\mathbb{R}}(u)$ and $X(\mathbb{R})$ do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X$. Note that the closed points of $X(\mathbb{R})$ have the form $\mathfrak{p}_{a,b} = \langle x - a, y - b \rangle$ where $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = 0$. There's also the generic point $\eta \in X(\mathbb{R})$ corresponding to the 0 ideal.

Now let $p(x) = x^2 - 5x + 5$, so $u = y - p$ and $v = y + p$. The existence of u and v tells us that A is not antilocal (if you look at the curves $\mathrm{V}_{\mathbb{R}}(u)$ and $\mathrm{V}_{\mathbb{R}}(f)$ in $\mathbb{R}^2$, you'll see that they just barely miss each other), however we can still ask: how far away is A from being antilocal? If we add u and v together, we obtain $u + v = 2y$, which is not a unit in A since the line $\mathrm{V}_{\mathbb{R}}(y)$ intersects the curve $\mathrm{V}_{\mathbb{R}}(f)$ at four points (you could also see this by plugging in $y = 0$ in (1) above).

1 Almost antilocal rings

For p large, the p-adic integers $\mathbb{Z}_p$ is very close to being an antilocal ring. Indeed, if u and v are units of $\mathbb{Z}_p$, then the probability that $u + v$ is a unit is $(p - 2)/(p - 1)$. So it's almost as if you could treat $\mathbb{Z}_p$ as a K-algebra when p is large. In other words, if we set $K = \mathbb{Z}_p^\times \cup \{0\}$, then K is very close to being a field.