Avramov Obstruction Notes

Let $f\colon R \to S$ be a finite local ring homomorphism such that the induced map on their common residue field $\Bbbk$ is identity and let M be a finitely generated S-module. Let E be the minimal free resolution of S over R and let F be the minimal free resolution of M over R. Choose a multiplication μ on E giving it the structure of an MDG R-algebra and choose an E-scalar multiplication ν on F giving it the structure of an MDG E-module. Note that μ and ν induces graded R-linear maps

$$\mathrm{H}E \otimes_R \mathrm{H}E \to \mathrm{H}E \quad \text{and} \quad \mathrm{H}E \otimes_R \mathrm{H}F \to \mathrm{H}F, \tag{1}$$

which give $\mathrm{H}E$ the structure of a graded-commutative R-algebra and gives $\mathrm{H}F$ the structure of a graded-commutative $\mathrm{H}E$-module. The the graded R-linear maps (1) do not depend on the choice of μ and ν since they are unique up to homotopy. Now let us denote $E_\Bbbk = E \otimes_R \Bbbk$ and $F_\Bbbk = F \otimes_R \Bbbk$. Since the differential for $E_\Bbbk$ and $F_\Bbbk$ are zero, we have $\mathrm{H}E_\Bbbk = E_\Bbbk$ and $\mathrm{H}F_\Bbbk = F_\Bbbk$, thus the graded R-linear maps (1) become

$$E_\Bbbk \otimes_R E_\Bbbk \to E_\Bbbk \quad \text{and} \quad E_\Bbbk \otimes_R F_\Bbbk \to F_\Bbbk.$$

Alternatively, we can express this in terms of Tor:

$$\mathrm{Tor}^R(S,\Bbbk) \otimes \mathrm{Tor}^R(S,\Bbbk) \to \mathrm{Tor}^R(S,\Bbbk) \quad \text{and} \quad \mathrm{Tor}^R(S,\Bbbk) \otimes \mathrm{Tor}^R(M,\Bbbk) \to \mathrm{Tor}^R(M,\Bbbk). \tag{2}$$

Now consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Tor}^R_+(S,\Bbbk) \otimes \mathrm{Tor}^R(M,\Bbbk) \arrow[r] \arrow[d] & \mathrm{Tor}^R(M,\Bbbk) \arrow[d] \\
\mathrm{Tor}^S_+(S,\Bbbk) \otimes \mathrm{Tor}^S(M,\Bbbk) \arrow[r, "0"] & \mathrm{Tor}^S(M,\Bbbk)
\end{tikzcd}$$

This gives a canonical map of graded $\Bbbk$-vector spaces:

$$\frac{\mathrm{Tor}^R(M,\Bbbk)}{\mathrm{Tor}^R_+(S,\Bbbk)\mathrm{Tor}^R(M,\Bbbk)} \to \mathrm{Tor}^S(M,\Bbbk).$$

The kernel of this map is denoted $\mathrm{o}^f(M)$ and is called the **obstruction to the existence of multiplicative structure** (on the minimal free resolution of M over R).

Example 0.1. Let $R = \Bbbk[x,y,z,w]$, let $I = \langle x^2, w^2, zw, xy, yz\rangle$, and let $\boldsymbol{t} = x^2, w^2$. Here we consider $S = R/\boldsymbol{t}$ and $M = R/I$. Let $K = \mathcal{K}^R(x,y,z,w)$ be the koszul algebra R-resolution of $\Bbbk$, with koszul variables denoted $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and let

$$L = K\langle \delta_1, \delta_4 \mid \mathrm{d}(\delta_1) = x\varepsilon_1,\ \mathrm{d}(\delta_4) = w\varepsilon_4\rangle.$$

Then $L/\boldsymbol{t}L$ is the Tate-Zariski minimal algebra S-resolution of $\Bbbk$. Note that $L/\boldsymbol{t}L \otimes_S R/I \simeq L/IL$. In particular, we have

$$\mathrm{H}(K/IK) = \mathrm{Tor}^R(R/I,\Bbbk) \quad \text{and} \quad \mathrm{H}(L/IL) = \mathrm{Tor}^S(R/I,\Bbbk),$$

and the inclusion map $\iota\colon K/IK \hookrightarrow L/IK$ induces the map $\mathrm{Tor}^R(R/I,\Bbbk) \to \mathrm{Tor}^S(R/I,\Bbbk)$. Now observe that in L/IL, we have

$$xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = \mathrm{d}(w\delta_1\varepsilon_2\varepsilon_3\varepsilon_4 + y\delta_1\delta_4\varepsilon_3).$$

On the other hand, there exists no $\gamma \in K/IK$ such that $\mathrm{d}(\gamma) = xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ in K/IK. Thus, $xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ represents a nonzero element in $\mathrm{H}(K/IK)$, but $xw\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$ represents the zero element in $\mathrm{H}(L/IL)$.

0.1 Buchsbaum and Eisenbud Conjecture

Suppose I is an ideal of R and $\boldsymbol{x} = x_1, \ldots, x_g$ is an R-regular sequence contained in I. Then we consider $S = R/\langle \boldsymbol{x} \rangle$ and $M = R/I$. In this case, we can choose F to be the koszul algebra $\mathcal{K}(\boldsymbol{x})$ (in particular F is associative). Any expression of the x_i in terms of the generators for I yields a canonocal comparison map $F \to X$. With this notation in mind, Buchsbaum and Eisenbud made the following conjecture:

Corollary. *X can be given the structure of a DG F-module such that the comparison map $F \to X$ is a DG F-module homomorphism.*

The reason why this conjecture is interesting is because it's validity would imply important lower bounds for the ranks of the syzygies of R/I (where R is assumed to be a domain).

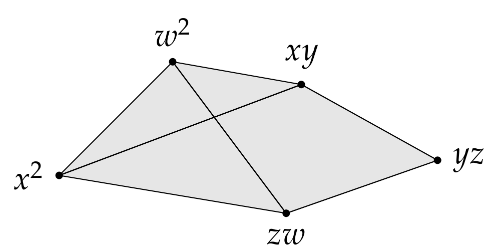
0.2 Avramov's Obstruction

Theorem 0.1. *Suppose the minimal R-free resolution F of S has the structure of a DG algebra. If $\mathrm{o}^f(M) \neq 0$, then no DG F-module structure exists on the minimal R-free resolution X of M. In particular, in for X to possess the structure of a DG F-module, it is necessary that we have $\mathrm{o}^f(M) = 0$.*

Example 0.2. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz \rangle$, and let $\boldsymbol{m} = x^2, w^2$. We set $S = R/\boldsymbol{m}$ and $M = R/I$. The are several complexes we consider: let

E be the Koszul algebra resolution of S over R
F be the minimal free resolution of M over R
T be the Taylor algebra resolution of M over R
K be the Koszul algebra resolution of $\Bbbk$ over R

We can visualize F as being supported on the $\boldsymbol{m}$-labeled cell complex as below:



Let's write down the homogeneous components of F as a graded R-module: we have

$$\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{1345} + Re_{2345} \\
F_4 &= Re_{12345}
\end{aligned}$$

We claim that F does not admit a DG E-module structure. We do this by showing $\mathrm{o}^{R \to S}(M) \neq 0$. In other words, we show that the kernel of the map

$$\frac{\mathrm{Tor}^R(M, \Bbbk)}{\mathrm{Tor}_+^R(S, \Bbbk)\mathrm{Tor}_+^R(M, \Bbbk)} \to \mathrm{Tor}^S(M, \Bbbk)$$

is nonzero. Observe the isomorphism of $\Bbbk$-algebras

$$\mathrm{H}(T_\Bbbk) \simeq \mathrm{Tor}^R(M, \Bbbk) \simeq \mathrm{H}(K_M),$$

where we denote $T_\Bbbk = T \otimes_R \Bbbk$ and $K_M = K \otimes_R M$ to simplify notation. Also observe that $\mathrm{Tor}_4^R(M, \Bbbk) \simeq \mathrm{H}_4(T_\Bbbk)$ is generated by the class of $e_{1234} \otimes 1$ and that

$$\begin{aligned}
\mathrm{Tor}_1^R(S, \Bbbk)\mathrm{Tor}_3^R(M, \Bbbk) &= E_{\Bbbk,1} F_{\Bbbk,3} \\
&\subseteq F_{\Bbbk,1} F_{\Bbbk,3} \\
&= 0.
\end{aligned}$$

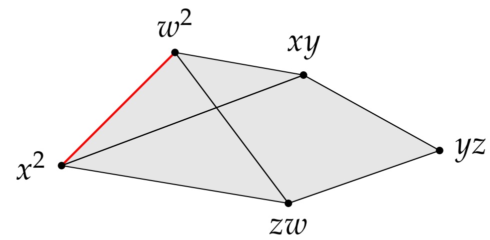
Thus it suffices to show that the kernel of $\operatorname{Tor}_4^R(M,\Bbbk) \to \operatorname{Tor}_4^S(M,\Bbbk)$ is nonzero. Note that this map is induced by the inclusion of complexes $K \subseteq L$ where

$$L = K\langle \delta_1, \delta_4 \mid \mathrm{d}(\delta_1) = x\varepsilon_1 \text{ and } \mathrm{d}(\delta_4) = w\varepsilon_4 \rangle.$$

Indeed, L is the Tate-Zariski minimal algebra resolution of $\Bbbk$ over S. Since $xw\varepsilon_{1234}$ is a non-zero element of $\mathrm{H}_4(K_M) \simeq \operatorname{Tor}_4^R(M,\Bbbk)$, the formula

$$xw\varepsilon_{1234} = \mathrm{d}(w\delta_1\varepsilon_{234} + y\delta_{14}\varepsilon_3) \in L$$

proves our claim.



We set $\{a_1, a_2, a_3\} := a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3$. Observe that

$$\mathrm{d}\{a_1, a_2, a_3\} = \{\mathrm{d}a_1, a_2, a_3\} + (-1)^{|a_1|}\{a_1, \mathrm{d}a_2, a_3\} + (-1)^{|a_1||a_2|}\{a_1, a_2, \mathrm{d}a_3\}.$$

The short exact sequence $0 \to V \to F^{\otimes 2} \xrightarrow{\mu} F \to 0$ induces isomorphisms $\mathrm{H}_+(V) = \operatorname{Tor}_+^R(R/I, R/I)$.

Proposition 0.1. *Assume that* $[a_1, a_2, a_3] \neq 0$ *represents a nontrivial element in* $\mathrm{H}[F]$ *such that* $[\mathrm{d}a_1, a_2, a_3] = [a_1, \mathrm{d}a_2, a_3] = [a_1, a_2, \mathrm{d}a_3] = 0$. *Then*

$$\mathrm{d}\{a_1, a_2, a_3\} = \{\mathrm{d}a_1, a_2, a_3\} + (-1)^{|a_1|}\{a_1, \mathrm{d}a_2, a_3\} + (-1)^{|a_1||a_2|}\{a_1, a_2, \mathrm{d}a_3\} \tag{3}$$

represents a nontrivial element in $\mathrm{H}_+(V)$.

Proof. Each term in the sum (3) belongs to V since $[\mathrm{d}a_1, a_2, a_3] = [a_1, \mathrm{d}a_2, a_3] = [a_1, a_2, \mathrm{d}a_3] = 0$, so $\mathrm{d}\{a_1, a_2, a_3\}$ certainly belongs to V. Furthermore, it is easy to see that $\mathrm{d}\{a_1, a_2, a_3\} \in \ker \mathrm{d}_V$. On the other hand, note that $\{a_1, a_2, a_3\} \notin V$ since $[a_1, a_2, a_3] \neq 0$. If $\mathrm{d}\{a_1, a_2, a_3\} = \mathrm{d}\tau$ for some $\tau \in V$, then $[a_1, a_2, a_3] = \mu\tau - \mu\{a_1, a_2, a_3\} \in \ker \mathrm{d}_F$, implies $[a_1, a_2, a_3] \in \operatorname{im} \mathrm{d}_F$ which is a contradiction. $\square$