MATH 8190 Fall 2022 Homework 1 due September 19, 2022

Biobjective case, convex analysis

(100 points = 100%)

Guidance for writing your assignment:
a) make sure that your writing is legible and clear
b) wherever appropriate, underline or rewrite the final answer
c) clearly separate your work for subsequent questions
d) submit your work on Canvas as one pdf file

Please make every effort to follow this guidance and facilitate the reading of your assignment. The assignments that do not follow this guidance will be returned.

1. Consider the following biobjective program (BOP):

$$\min \ [\, f_1(x) = -(x+1)^{1/2},\ f_2(x) = x^2 - 4x + 5 \,]$$
$$\text{s.t. } x \geq 0$$

a) (6 points) Derive the formula representing the outcome set Y in $\mathbb{R}^2$ for this BOP.
b) (2 points) Graph the outcome set Y.
c) (5 points) Identify and mark the Pareto-nondominated outcomes in Y.
d) (3 points) Find the Pareto-efficient solutions in X.
e) (4 points) Find the ideal point.

2. Consider the following biobjective program (BOP):

$$\max \ [\, f_1(\mathbf{x}) = x_1 - 3x_2,\ f_2(\mathbf{x}) = -4x_1 + x_2 \,]$$
$$\text{s.t. } \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{R}^2 : g_j(x) \leq 0,\ j = 1, \ldots, 4,\ x_1 \geq 0,\ x_2 \geq 0\}$$

where

$$g_1(\mathbf{x}) = -x_1 + x_2 - 7/2$$
$$g_2(\mathbf{x}) = x_1 + x_2 - 11/2$$
$$g_3(\mathbf{x}) = 2x_1 + x_2 - 9$$
$$g_4(\mathbf{x}) = x_1 - 4$$

a) (3 points) Graph the feasible set X in the decision space.
b) (7 points) Graph the outcome set Y in the objective space $\mathbb{R}^2$. Explain what mathematical property you used to draw Y.
c) (5 points) Identify and mark the Pareto-nondominated outcomes in Y.
d) (5 points) Identify and mark the Pareto-efficient solutions in X.
e) (5 points) Find and graph the ideal point.

3. Let C_1 and C_2 be finite cones in R^p and C_1^* and C_2^* be their dual cones, respectively. Prove the following:

a) (5 points) If $C_1 \subseteq C_2$ then $C_2^* \subseteq C_1^*$
b) (10 points) $(C_1 + C_2)^* = C_1^* \cap C_2^*$

4. Derive the formula representing the polar cone of the cone generated by

a) (5 points) the vector (2, 3) in $\mathbb{R}^2$
b) (10 points) the vectors (4, 1) and (4, –1) in $\mathbb{R}^2$

5.

a) (5 points) Let C be a polyhedral cone defined as $C = \{x \in \mathbb{R}^2: Ax \geqq 0\}$, where $A = \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$. Derive the generator form for this cone.

b) (10 points) Let C be a polyhedral cone defined as $C = \{x \in \mathbb{R}^3: x = B\lambda, \lambda \geqq 0\}$, where $B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$. Derive the inequality form for this cone.

6. Graphically find A – B, where

a) (5 points) $A = \{\mathbf{x} \in \mathbb{R}^2: x_1^2 + x_2^2 \leq 9\}$, $B = \mathbb{R}^2_{\geq} \cup (-\mathbb{R}^2_{\geq})$
b) (5 points) A is a set in $\mathbb{R}^2$ and has the shape of a thick letter U rotated 45 degrees to the right and $B = \mathbb{R}^2_{\geq}$
Make sure your pictures are neat and accurate.