Algebraic Topology Homework 5

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Problem 1

Lemma 0.1. *Let X*

Exercise 1. Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$.

1. Compute the homology of X and Y and confirm that the homology is the same in every dimension.
2. Describe the universal covering spaces of X and Y.
3. Show that the universal covering spaces of X and Y do not have the same homology.

Solution 1. 1. We use Kunneth theorem which tells us that $\mathrm{H}(X) \simeq \mathrm{H}(S^1) \otimes \mathrm{H}(S^1)$ as graded modules. In particular, this implies

$$\mathrm{H}_i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

Next, note that the identified basepoint in the wedge sum $S^1 \vee S^1 \vee S^2$ is a deformation retract of open neighborhoods in S^1 and S^2. Thus one can use the Mayer-Vietoris sequence to deduce that $\widetilde{\mathrm{H}}(Y) \simeq \widetilde{\mathrm{H}}(S^1) \oplus \widetilde{\mathrm{H}}(S^1) \oplus \widetilde{\mathrm{H}}(S^2)$ as graded modules, where the tilde denoted "reduced homology". In particular, this implies

$$\mathrm{H}_i(Y) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{else} \end{cases}$$

where we use the fact that Y is connected so get $\mathrm{H}_0(Y) = \mathbb{Z}$.

2. Recall we have a homeomorphism $\mathbb{R}/\mathbb{Z} \simeq S^1$ defined by $\overline{x} \mapsto e^{2\pi i x}$. Thus it suffices to describe the universal covering space of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. The universal covering space of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ is given by $\pi \colon \mathbb{R}^2 \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ where π is canonical projection map defined by

$$\pi(\boldsymbol{x}) = \pi(x_1, x_2) = (\overline{x}_1, \overline{x}_2),$$

for all $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$.

Example 0.1. We have a right action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ given by

$$\boldsymbol{x} \cdot \boldsymbol{a} = (x_1 + a_1, x_2 + a_2) \tag{1}$$

for all $\boldsymbol{a} = (a_1, a_2) \in \mathbb{Z}^2$ and $\boldsymbol{x} \in (x_1, x_2) \in \mathbb{R}^2$.

1. The action is continuous as a map $\mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{R}^2$. Indeed, let $\boldsymbol{a} \in \mathbb{Z}^2$. The map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$(x_1, x_2) = \boldsymbol{x} \mapsto \boldsymbol{x} \cdot \boldsymbol{a} = (x_1 + a_1, x_2 + a_2)$$

is continuous since the component functions are continuous.

2. The action (**??**) is free since if $\boldsymbol{x} \cdot \boldsymbol{a} = \boldsymbol{x}$ implies $\boldsymbol{a} = 0$. T

3. The action (**??**) is also properly discontinuous. Indeed, given $\boldsymbol{x} \in \mathbb{R}^2$, choose

$$U_{\boldsymbol{x}} = \{\boldsymbol{y} \in \mathbb{R}^2 \mid \|\boldsymbol{y} - \boldsymbol{x}\|_\infty < 1/2\} = (x_1 - 1/2, x_1 + 1/2) \times (x_2 - 1/2, x_2 + 1/2),$$

that is, $U_{\boldsymbol{x}}$ is the open square centered at $\boldsymbol{x}$ whose sides have length 1. Then clearly $U_{\boldsymbol{x}} \cdot \boldsymbol{a}$ is disjoint from $U_{\boldsymbol{x}}$ for all $\boldsymbol{a} \in \mathbb{Z}^2 \backslash \{0\}$.

Problem 2

Exercise 2. Compute the homology groups $\mathrm{H}_n(X, A)$ in the following cases:

1. X is S^2 and A is a finite set of points in X.
2. X is $S^1 \times S^1$ and A is a finite set of points in X.
3. X is a surface of genus 2 and A is a loop that separates the two wholes (see Loop A in the figure on page 132 of Hatcher - Page 141 of the pdf document).
4. X is a surface of genus 2 and A is a loop that goes through one of the two holes (see Loop B in the figure on page 132 of Hatcher - Page 141 of the pdf document).

Solution 2.

Problem 3

Exercise 3. Compute the homologies of the following spaces:

1. The quotient of S^2 by identifying the north and south poles to a point.
2. The space $S^1 \times (S^1 \vee S^1)$. This space looks somewhat like a torus, but each of the radial slices is a figure-eight.
3. The quotient space formed from deleting two disjoint open disks in the interior of D^2 and identifying all three boundaries, preserving the clockwise orientations of the circles.

Solution 3.

Problem 4

Exercise 4. A map $f \colon S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is an **even map**. For this problem, you may assume that f has the property that there is some point $y \in S^n$ with finitely many preimages.

1. Prove that an even map $S^n \to S^n$ must have even degree.
2. Prove that when n is even, the degree of an even map must be 0.
3. Prove that when n is odd, there exist even maps of any given even degree.

Solution 4.