List of Schemes

Contents

Part I
List of Algebraic Varieties

1 A Quartic Curve

Let $A = \mathbb{Z}[x, y]/f$ where

$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \tag{1}$$

where we set $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$. Note that from the expression of f in (1) we see that u and v are units in A. Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g}]$ where

$$f = y^2 - (x-1)(x-2)(x-3)(x-4) = y^2 - g, \tag{2}$$

where $g = (x-1)(x-2)(x-3)(x-4)$. The expression of f in (2) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (2) we can read off useful information of A viewed as a finite module extension, whereas from (1) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day. Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we consider the canonical morphism $X \to \operatorname{Spec} \mathbb{Z}$. For each positive prime p, we obtain the fiber $X_p = X_{\mathbb{F}_p}$ of this canonical morphism at the prime ideal $\langle p \rangle$:

$$X_p = \operatorname{Spec}(\mathbb{F}_p \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(\mathbb{F}_p[x, y]/f).$$

We also obtain the fiber $X_0 = X_{\mathbb{Q}}$ of this canonical morphism at the generic point $\langle 0 \rangle$:

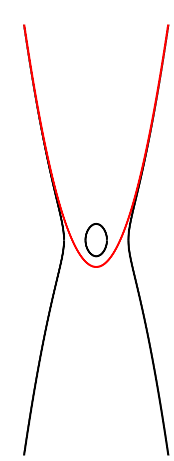
$$X_{\mathbb{Q}} = \operatorname{Spec}(\mathbb{Q} \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(\mathbb{Q}[x, y]/f).$$

Note $X_{\mathbb{Q}}$ is just the pullback of the morphism $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ with respect to the canonical map $X \to \operatorname{Spec} \mathbb{Z}$. We can specialize even further by setting X_K to be the pullback of the composite $\operatorname{Spec} K \to \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ with respect to the canonical map $X \to \operatorname{Spec} \mathbb{Z}$, where $K/\mathbb{Q}$ is some field extension:

$$X_K = \operatorname{Spec}(K \otimes_{\mathbb{Z}} A) = \operatorname{Spec}(K[x, y]/f).$$

The closed points of X_K correspond to the maximal ideals of $K[x, y]/f$, and when K is algebraically closed, these correspond to the points of the variety $\mathrm{V}_K(f)$.

We now consider $X_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[x, y]/f)$, viewed as an $\mathbb{R}$-scheme (thus the canonical morphism is $X_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R}$). To get an idea of what $X_{\mathbb{R}}$ looks like, we shall look at its $\mathbb{R}$-valued points $X_{\mathbb{R}}(\mathbb{R}) = \mathrm{V}_{\mathbb{R}}(f) = C$ pictured below:



The thick black curve is C whereas the thick red curve is $\mathrm{V}_{\mathbb{R}}(u) = D$. The closed points of $X_{\mathbb{R}}$ correspond to the points of C: they have the form $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$ where $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = 0$ (i.e. such that $(a, b) \in C$). There's also the generic point $\eta \in X_{\mathbb{R}}$ corresponding to the 0 ideal, however this doesn't correspond to any point of C. Notice that C and D do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X_{\mathbb{R}}$.

If we equip $X(\mathbb{R})$ with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology, $X(\mathbb{R})$ is irreducible since f is irreducible over $\mathbb{R}$, so certainly $X(\mathbb{R})$ is connected in the Zariski topology. The Jacobian matrix of f is given by

$$\mathrm{J}_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that $\mathrm{J}_f(a,b) := \mathrm{J}_f \bmod \mathfrak{m}_{a,b} \neq 0$ for all closed points $\mathfrak{m}_{a,b} \in X(\mathbb{R})$. It follows that $X(\mathbb{R})$ is smooth. Alternatively, we have

$$\mathrm{d}f = \partial_x f \mathrm{d}x + \partial_y f \mathrm{d}y = (4x^3 + 30x^2 - 70x + 50)\mathrm{d}x + 2y\mathrm{d}y.$$

Thus when we set $\mathrm{d}f = 0$, then for $y \neq 0$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \tag{3}$$

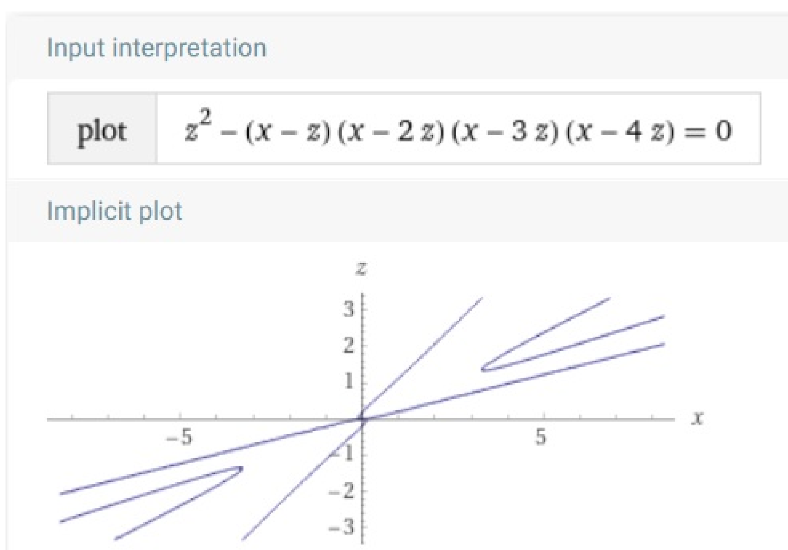
The DeRham complex of A is given by

$$\Omega_A := 0 \to A \to$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity $[0:1:0]$. To do this let $\widetilde{A} = \mathbb{Z}[x,z]/\widetilde{f}$ where

$$\widetilde{f} = \widetilde{y}^2 - (\widetilde{x} - \widetilde{y})(\widetilde{x} - 2\widetilde{y})(x - 3\widetilde{y})(x - 4\widetilde{y}), \tag{4}$$

and set $\widetilde{X} = \operatorname{Spec} \widetilde{A}$. To get an idea of what $\widetilde{X}_{\mathbb{R}}$ looks like, we shall look at its $\mathbb{R}$-valued points $\widetilde{X}_{\mathbb{R}}(\mathbb{R}) = \mathrm{V}_{\mathbb{R}}(\widetilde{f}) = \widetilde{C}$ pictured below



The closed points of $\widetilde{X}_{\mathbb{R}}$ have the form $\widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}} = \langle \widetilde{x} - \widetilde{a}, \widetilde{y} - \widetilde{b} \rangle$ where $(\widetilde{a}, \widetilde{b}) \in \mathbb{R}^2$ such that $\widetilde{f}(\widetilde{a}, \widetilde{b}) = 0$. We have a ring isomorphism $\widetilde{\varphi}\colon \widetilde{A}_{\widetilde{y}} \to A_y$ given by $\widetilde{\varphi}(\widetilde{x}) = x/y$ and $\widetilde{\varphi}(\widetilde{y}) = 1/y$, with inverse $\varphi\colon A_y \to \widetilde{A}_{\widetilde{y}}$ given by $\varphi(x) = \widetilde{x}/\widetilde{y}$ and $\varphi(y) = 1/\widetilde{y}$. Notice that

$$\begin{aligned}
\widetilde{\varphi}(\widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}}) &= \varphi(\langle \widetilde{x} - \widetilde{a}, \widetilde{y} - \widetilde{b} \rangle) \\
&= \langle x/y - \widetilde{a}, 1/y - \widetilde{b} \rangle \\
&= \langle x - \widetilde{a}y, 1 - \widetilde{b}y \rangle \\
&= \langle x - \widetilde{a}y, y - 1/\widetilde{b} \rangle \\
&= \langle x - \widetilde{a}/\widetilde{b}, y - 1/\widetilde{b} \rangle \\
&= \langle x - a, y - b \rangle \\
&= \mathfrak{m}_{a,b},
\end{aligned}$$

where we set $a = \widetilde{a}/\widetilde{b}$ and $b = 1/\widetilde{b}$. It follows that ${}^{a}\widetilde{\varphi}(\mathfrak{m}_{a,b}) = \widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}}$. Now observe that

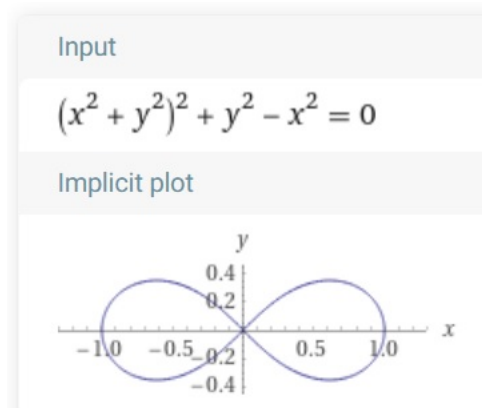
$$\mathrm{d}\widetilde{x} = \frac{y\mathrm{d}x - x\mathrm{d}y}{y^2} \quad \text{and} \quad \mathrm{d}\widetilde{y} = -\frac{\mathrm{d}y}{y^2}.$$

2 The Lemniscate of Bernoulli

Let $A = \mathbb{Z}[x,y]/f$ where

$$f = (x^2+y^2)^2 + y^2 - x^2,$$

and we set $X = \operatorname{Spec} A$. One can show that the set of integer solutions to the equation $f = 0$ is given by $\{(\pm 1, 0), (0,0)\}$. On the other hand, the $\mathbb{R}$-valued points $X(\mathbb{R})$ can be visualized below



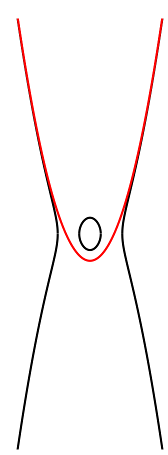
$$f = (y - x^2 + 5x - 5)(y + x^2 - 5x + 5) - 1 = uv - 1 \tag{5}$$

where we set $u = y - x^2 + 5x - 5$ and $v = y + x^2 - 5x + 5$. Note that from the expression of f in (1) we see that u and v are units in A. Here we are describing A as a quotient of a polynomial ring, but we can also describe it as a finite module extension over a polynomial ring. Namely, we can write it as $A = \mathbb{Z}[y][\sqrt{g}]$ where

$$f = y^2 - (x-1)(x-2)(x-3)(x-4) = y^2 - g, \tag{6}$$

where $g = (x-1)(x-2)(x-3)(x-4)$. The expression of f in (2) is nice because we can read off information like the discriminant of A over $\mathbb{Z}[y]$. Basically from (2) we can read off useful information of A viewed as a finite module extension, whereas from (1) we can read off useful information of A viewed as a quotient. Both expressions give rise to the same ring A at the end of the day.

Next we set $X = \operatorname{Spec} A$. To get an idea of what X looks like, we first look at its $\mathbb{R}$-valued points: $X(\mathbb{R}) = \operatorname{Spec} \mathbb{R} \otimes_{\mathbb{Z}} A = \operatorname{Spec} \mathbb{R}[x,y]/f$. We can visualize the $\mathbb{R}$-valued points of X in the picture below:



The thick black curve is $X(\mathbb{R}) = \mathrm{V}_{\mathbb{R}}(f)$ whereas the thick red curve is $\mathrm{V}_{\mathbb{R}}(u)$. Notice that $\mathrm{V}_{\mathbb{R}}(u)$ and $X(\mathbb{R})$ do not intersect: this is because u is a unit in A (and hence a unit in $\mathbb{R} \otimes_{\mathbb{Z}} A$). The point is that $u(\mathfrak{p}) := u \bmod \mathfrak{p} \neq 0$ for all $\mathfrak{p} \in X$. The closed points of $X(\mathbb{R})$ have the form $\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$ where $(a,b) \in \mathbb{R}^2$ such that $f(a,b) = 0$. There's also the generic point $\eta \in X(\mathbb{R})$ corresponding to the 0 ideal.

If we equip $X(\mathbb{R})$ with the Euclidean topology. Then we see that it is disconnected since

$$X(\mathbb{R}) = (\{u > 0\} \cap X(\mathbb{R})) \cup (\{u < 0\} \cap X(\mathbb{R})),$$

however in the Zariski topology, $X(\mathbb{R})$ is irreducible since f is irreducible over $\mathbb{R}$, so certainly $X(\mathbb{R})$ is connected in the Zariski topology. The Jacobian matrix of f is given by

$$\mathrm{J}_f = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = \begin{pmatrix} 4x^3 + 30x^2 - 70x + 50 \\ 2y \end{pmatrix}.$$

In particular, it's easy to see that $\mathrm{J}_f(a,b) := \mathrm{J}_f \bmod \mathfrak{m}_{a,b} \neq 0$ for all closed points $\mathfrak{m}_{a,b} \in X(\mathbb{R})$. It follows that $X(\mathbb{R})$ is smooth. Alternatively, we have

$$\mathrm{d}f = \partial_x f \mathrm{d}x + \partial_y f \mathrm{d}y = (4x^3 + 30x^2 - 70x + 50)\mathrm{d}x + 2y\mathrm{d}y.$$

Thus when we set $\mathrm{d}f = 0$, then for $y \neq 0$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4x^3 + 30x^2 - 70x + 50}{2y}. \tag{7}$$

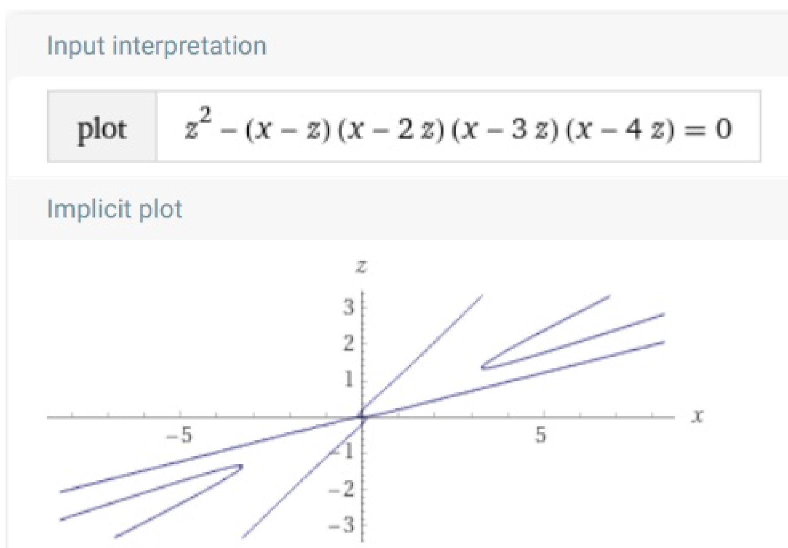
The DeRham complex of A is given by

$$\Omega_A := 0 \to A \to$$

Now, we want to see what the curve looks like at in affine neighborhood of the point at infinity $[0:1:0]$. To do this let $\widetilde{A} = \mathbb{Z}[x,z]/\widetilde{f}$ where

$$\widetilde{f} = \widetilde{y}^2 - (\widetilde{x} - \widetilde{y})(\widetilde{x} - 2\widetilde{y})(x - 3\widetilde{y})(x - 4\widetilde{y}), \tag{8}$$

and set $\widetilde{X} = \operatorname{Spec} \widetilde{A}$. We can visualize the $\mathbb{R}$-valued points of $\widetilde{X}$ in the picture below



The closed points of $\widetilde{X}(\mathbb{R})$ have the form $\widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}} = \langle \widetilde{x} - \widetilde{a}, \widetilde{y} - \widetilde{b} \rangle$ where $(\widetilde{a}, \widetilde{b}) \in \mathbb{R}^2$ such that $\widetilde{f}(\widetilde{a}, \widetilde{b}) = 0$. We have a ring isomorphism $\widetilde{\varphi} \colon \widetilde{A}_{\widetilde{y}} \to A_y$ given by $\widetilde{\varphi}(\widetilde{x}) = x/y$ and $\widetilde{\varphi}(\widetilde{y}) = 1/y$, with inverse $\varphi \colon A_y \to \widetilde{A}_{\widetilde{y}}$ given by $\varphi(x) = \widetilde{x}/\widetilde{y}$ and $\varphi(y) = 1/\widetilde{y}$. Notice that

$$\begin{aligned}
\widetilde{\varphi}(\widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}}) &= \varphi(\langle \widetilde{x} - \widetilde{a}, \widetilde{y} - \widetilde{b} \rangle) \\
&= \langle x/y - \widetilde{a}, 1/y - \widetilde{b} \rangle \\
&= \langle x - \widetilde{a}y, 1 - \widetilde{b}y \rangle \\
&= \langle x - \widetilde{a}y, y - 1/\widetilde{b} \rangle \\
&= \langle x - \widetilde{a}/\widetilde{b}, y - 1/\widetilde{b} \rangle \\
&= \langle x - a, y - b \rangle \\
&= \mathfrak{m}_{a,b},
\end{aligned}$$

where we set $a = \widetilde{a}/\widetilde{b}$ and $b = 1/\widetilde{b}$. It follows that ${}^{\mathrm{a}}\widetilde{\varphi}(\mathfrak{m}_{a,b}) = \widetilde{\mathfrak{m}}_{\widetilde{a},\widetilde{b}}$. Now observe that

$$\mathrm{d}\widetilde{x} = \frac{y\mathrm{d}x - x\mathrm{d}y}{y^2} \quad \text{and} \quad \mathrm{d}\widetilde{y} = -\frac{\mathrm{d}y}{y^2}.$$

3 A Blowup Algebra

Let $R = \Bbbk[x,y]/\langle y^2 - x^3 - x^2 \rangle$, let $Q = \langle \overline{x}, \overline{y} \rangle$ (we drop the overlines from $\overline{x}$ and $\overline{y}$ in just write x and y in order to simplify notation in what follows), and equip R with the Q-filtration making $R = (Q^n)$ into a filtered ring.

Let $\varphi\colon R[u,v] \to \mathrm{bl}(R)$ be the unique surjective R-algebra homomorphism such that $\varphi(u) = xt$ and $\varphi(v) = yt$. The kernel of φ is an ideal of $R[u,v]$ which is homogeneous in the variables u, v:

$$\ker \varphi = \langle v^2 - (x+1)u^2, xv - yu \rangle.$$

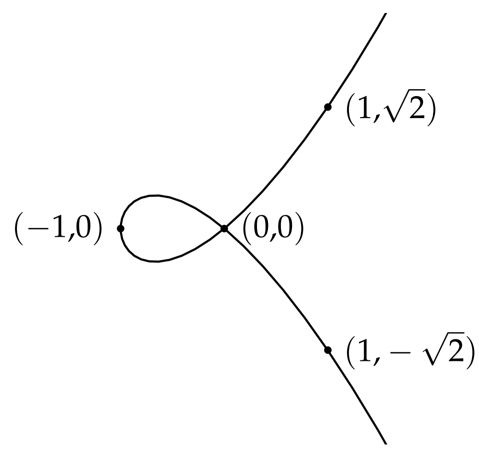
Thus we see that $\mathrm{bl}(R) \cong \Bbbk[x,y,u,v]/\mathfrak{a}$ where

$$\mathfrak{a} = \langle v^2 - (x+1)u^2, xv - yu, y^2 - x^3 - x^2 \rangle.$$

In particular, $\mathrm{bl}(R)$ corresponds to an algebraic subset $Z \subseteq \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$. Let $A = R[v]/\langle v^2 - (x+1), xv - y\rangle$, so A corresponds to the affine open $U = Z \cap (\mathbb{A}^2 \times \mathrm{D}(u))$. We can localize further by setting $B = A_x = R[v]/\langle v - y/x\rangle$, so B corresponds to the affine open $V = Z \cap (\mathrm{D}(x) \times \mathrm{D}(u))$. We have a canonical ring homomorphism $\iota\colon R \to A$ where ι is the inclusion map. Let us try to understand this homomorphism from a geometric point of view. Let $X = \mathrm{V}_{\Bbbk}(y^2 - x^3 - x^2)$ be affine algebraic subset of $\mathbb{A}^2_{\Bbbk}$ defined by the equation $y^2 = x^3 + x^2$. The closed points of $\operatorname{Spec} R$ are in one-to-one correspondence with the points of V: they are all of the form

$$\mathfrak{p}_{(a,b)} = \langle x - a, y - b \rangle$$

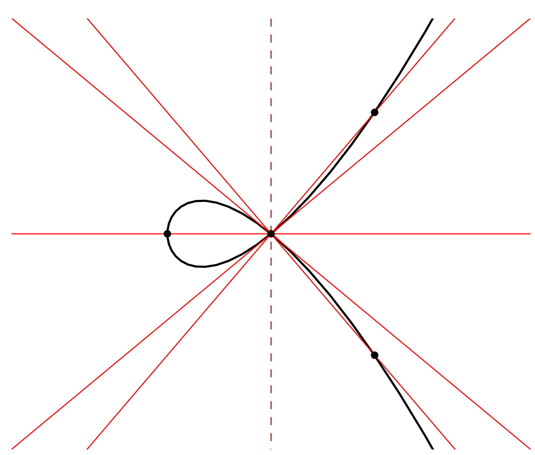
where $(a,b) \in X$, that is, where $a, b \in \Bbbk$ such that $b^2 = a^3 + a^2$. If $\Bbbk = \mathbb{R}$, we can visualize the closed points of $\operatorname{Spec} R$ as below:



Note that $\operatorname{Spec} R$ also has a generic point η corresponding to the zero ideal of R. The closed points of $\operatorname{Spec} A$ correspond to the points of the affine open set U: they have the form

$$\mathfrak{p}_{(a,b),[1:t]} = \langle x - a, y - b, v - t \rangle$$

where $a, b, t \in \Bbbk$ such that $b^2 = a^3 + a^2$, $at = b$, and $t^2 = a + 1$. Note that if $a \neq 0$, then we automatically get $t^2 = a + 1$. If $\Bbbk = \mathbb{R}$. we can visualize the points of $\operatorname{Spec} A$ as below:



In particular, for $a \neq 0$, the prime $\mathfrak{p}_{(a,b),[1:t]}$ corresponds to the point $(a,b) \in X$ together with the unique line $y = tx$ that passes through that point and the origin, where t represents the slope of that line. There are two points lying over the origin: namely $\mathfrak{p}_{(0,0),[1:1]}$ and $\mathfrak{p}_{(0,0),[1:-1]}$, corresponding to the origin $(0,0) \in V$ together with the lines $y = x$ and $y = -x$ respectively. The map $\iota\colon R \to A$ induces a continuous map ${}^{\mathrm{a}}\iota\colon \operatorname{Spec} A \to \operatorname{Spec} R$ given by

$${}^{\mathrm{a}}\iota(\mathfrak{p}_{(a,b),[1,t]}) = \mathfrak{p}_{(a,b)}.$$

This corresponds to the projection map $\pi\colon U \to X$ given by

$$\pi(a, b, t) = (a, b).$$

Notice that in the image above there are "missing" points. For instance, we drew a vertical dashed line in the image above; it should correspond to the line $x = 0$, but it has nowhere to go under this projection. In fact, this missing line corresponds to the extra point in $\mathrm{Proj}(\mathrm{bl}(R))$ which doesn't belong to A.

Definition 3.1. A **hyperellitpic curve** is an algebraic curve of genus $g > 1$, given by an equation of the form

$$y^2 + h(x)y = f(x),$$

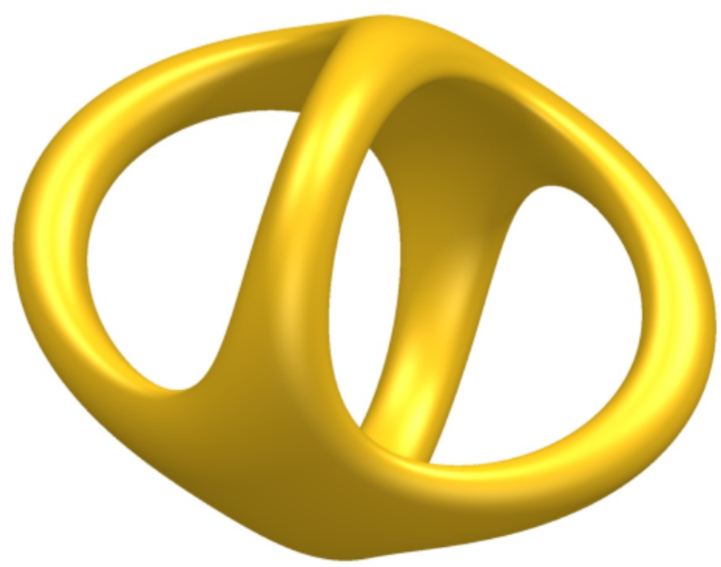
where f is a polynomial of degree $n = 2g + 1 > 4$ or $n = 2g + 2 > 4$ with n distinct roots and $h(x)$ is a polynomial of degree $< g + 2$ (if the characteristic of the ground field is not 2, one can take $h(x) = 0$).

4 A Surface

Let $a \in \Bbbk$ and let $S_t = \mathrm{V}_{\Bbbk}(f_t) \subseteq \mathbb{A}^3_{\Bbbk}$ where

$$f_t = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2 - t = g_1^2 + g_2^2 - t = \|\boldsymbol{g}\|^2 - t$$

where $\boldsymbol{g} = (g_1, g_2)$, where $g_1 = x_1^2 + x_2^2 - 1$ and $g_2 = x_2^2 + x_3^2 - 1$. When $\Bbbk = \mathbb{R}$ and $t = 0.1$, we can picture $S_{0.1}$ as below:



The Jacobian matrix of f_t is given by

$$\mathrm{J}_{f_t} = \begin{pmatrix} \partial_x f_t \\ \partial_y f_t \\ \partial_z f_t \end{pmatrix} = 4 \begin{pmatrix} x_1 g_1 \\ x_2(g_1 + g_2) \\ x_3 g_2 \end{pmatrix}.$$

We write $\Delta_t = \mathrm{V}(\mathrm{J}_{f_t}) = \{\boldsymbol{a} \in \mathbb{A}^3_{\Bbbk} \mid \mathrm{J}_{f_t}(\boldsymbol{a}) = 0\}$. Given $\boldsymbol{a} \in \mathbb{A}^3_{\Bbbk}$, we have $\boldsymbol{a} \in \Delta_t$ if and only if $\boldsymbol{a} = \mathbf{0}$ or $\boldsymbol{a} \in \mathrm{V}_{\Bbbk}(g_1, g_2)$ (meaning $g_1(\boldsymbol{a}) = g_2(\boldsymbol{a}) = 0$). In particular, if $t \neq 0, 2$, then S_t has no singular points since $S_t \cap \Delta_t = \emptyset$ in this case. If $t = 2$, then $\mathbf{0}$ is a singular point since $\mathbf{0} \in S_2 \cap \Delta_2$. If $t = 0$, then S_0 has lots of singular points. For instance, $\{(\pm 1, 0, \pm 1), (\pm 1, 0, \mp 1)\}$ are all singular points.

We can desribe S_t as being the fibre at $t \in \Bbbk$ with respect to the morphism of affine $\Bbbk$-schemes $\pi\colon S \to \mathbb{A}^1_{\Bbbk,\tau}$ (here we are indicating that the coordinate ring of $\mathbb{A}^1_{\Bbbk,\tau}$ is given by $\Bbbk[\tau]$) where $S = \mathrm{Spec}(\Bbbk[x_1, x_2, x_3, \tau]/f_\tau)$ and where π corresponds to the morphism of $\Bbbk$-algebras $\iota\colon \Bbbk[\tau] \to \Bbbk[x_1, x_2, x_3, \tau]/f_\tau$ (which is just inclusion map). In particular, let $\varepsilon_t\colon \mathrm{Spec}\, \Bbbk \to \mathbb{A}^1_{\Bbbk,\tau}$ be the morphism of affine $\Bbbk$-schemes which corresponds to the $\Bbbk$-algebra homomorphism $\Bbbk[\tau] \to \Bbbk[\tau]/\langle \tau - t \rangle \simeq \Bbbk$ which sends τ to $t \in \Bbbk$. Then S_t is the pullback of $\pi\colon S \to \mathbb{A}^1_{\Bbbk,\tau}$ with respect to $\varepsilon_t\colon \mathrm{Spec}\, \Bbbk \to \mathbb{A}^1_{\Bbbk,\tau}$. In particular, the corresponding $\Bbbk$-algebra of S_t is given by

$$\Bbbk[x_1, x_2, x_3]/f_t \simeq (\Bbbk[x_1, x_2, x_3, \tau]/f_\tau) \otimes_{\Bbbk} \Bbbk[\tau]/\langle \tau - t \rangle.$$

Note that the morphism of affine $\Bbbk$-schemes $\pi\colon S \to \mathbb{A}^1_{\Bbbk,\tau}$ is flat since the inclusion map of $\Bbbk$-algebras $\iota\colon \Bbbk[\tau] \to \Bbbk[x_1, x_2, x_3, \tau]/f_\tau$ is flat.

5 An Elliptic Curve

We study the elliptic curve E defind by the equation $y^2 = x^3 - 51$. One calculates its discriminant to be $\Delta = 2^4 \cdot 3^3 \cdot 51^2$.

6 Degeneration to a Monomial Ideal

Let $\Bbbk$ be a field, let $R = \Bbbk[x, y]$, let $R' = \Bbbk[x', y']$, and let $S = \Bbbk[x, y, x', y']/J$ where

$$J = \langle x \rangle \langle x - x', y - y' \rangle = \langle x^2 - xx', xy - xy' \rangle.$$

We also set $X = \operatorname{Spec} R$, $X' = \operatorname{Spec} R'$, and $Y = \operatorname{Spec} S$. Thus we have two morphisms of $\Bbbk$-schemes $Y \to X$ and $Y \to X'$ which correspond to the $\Bbbk$-algebra homomorphisms $R \to S$ and $R' \to S$ respectively. For each $p = (a, b) \in \Bbbk^2$, we set $\mathfrak{m}_p = \langle x - a, y - b \rangle$, and similarly for each $p' = (a', b') \in \Bbbk^2$, we set $\mathfrak{m}'_{p'} = \langle x' - a', y' - b' \rangle$. Let Y_p denote the fiber of Y over p and let $Y'_{p'}$ denote the fiber of Y over p'. Then $Y_p \simeq \mathbb{A}^0_{\Bbbk}$ whereas

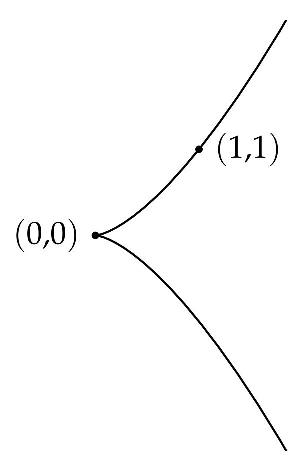
$$Y'_{p'} \simeq \begin{cases} \mathbb{A}^1_{\Bbbk} \sqcup \mathbb{A}^0_{\Bbbk} & \text{if } p' \neq 0 \\ \operatorname{Spec}(\Bbbk[x, y]/\langle x^2, xy \rangle) & \text{if } p = 0. \end{cases}$$

7 Cuspidal Cubic

Example 7.1. Let $\Bbbk$ be a field and let $S = \Bbbk[x, y]/f$ where $f = y^2 - x^3$. Then we have

$$\Omega_{S/\Bbbk} = \frac{S\mathrm{d}x \oplus S\mathrm{d}y}{-3x^2\mathrm{d}x + 2y\mathrm{d}y}.$$

In order to better understand what kind of object $\Omega_{S/\Bbbk}$ is, we digress a bit and explain how one should think S in terms of geometry. Let $X = \operatorname{Spec} S$. For each $\boldsymbol{p} = (a, b)$ in $\Bbbk^2$ such that $b^2 = a^3$, we have a maximal ideal $\mathfrak{m}_{\boldsymbol{p}} = \langle x - a, y - b \rangle$ of S (or alternatively we can consider $\mathfrak{m}_{\boldsymbol{p}}$ as a closed point of X) and we set $\Bbbk_{\boldsymbol{p}} := S/\mathfrak{m}_{\boldsymbol{p}} \simeq \Bbbk$ to be the corresponding residue field (which is just $\Bbbk$ but equipped with an S-module action coming from $\boldsymbol{p}$). If $\Bbbk$ is algebraically closed, then these are all of the maximal ideals of S, however if $\Bbbk$ is not algebraically closed, then there will be more maximal ideals than just this. For instance, suppose $\Bbbk = \mathbb{R}$. Then the set of all such closed points forms the curve below:



However X contains more closed points than just this (alternatively S contains more maximal ideals than just this). Indeed, for each $\boldsymbol{p} = (a, b)$ in $\mathbb{C}^2$ such that $b^2 = a^3$, one gets an $\mathbb{R}$-algebra homomorphism $e_{\boldsymbol{p}} \colon S \to \mathbb{C}$ given by $x \mapsto a$ and $y \mapsto b$. We call $e_{\boldsymbol{p}}$ a **$\mathbb{C}$-valued point** of S (or a $\mathbb{C}$-valued point of X). For any such $\mathbb{C}$-valued point, we set $\mathfrak{m}_{\boldsymbol{p}} := \ker e_{\boldsymbol{p}}$. Then all maximal ideals of S are obtained this way (i.e. as the kernel of a $\mathbb{C}$-valued point). Furthermore, for two such points $\boldsymbol{p}, \boldsymbol{p}'$, we have $\mathfrak{m}_{\boldsymbol{p}} = \mathfrak{m}_{\boldsymbol{p}'}$ if and only if $e_{\sigma \boldsymbol{p}} = e_{\boldsymbol{p}'}$ for some $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, where $\sigma \boldsymbol{p} = \sigma(a, b) = (\sigma a, \sigma b)$. This holds more generally in the case where $\Bbbk \neq \mathbb{R}$. Indeed, choose an algebraic closure $\overline{\Bbbk}$ of $\Bbbk$. Then we have natural bijections:

$$\{\text{maximal ideals of } S\} \simeq \{\text{closed points of } X\} \simeq \{\overline{\Bbbk}\text{-valued points of } X\}/\sim,$$

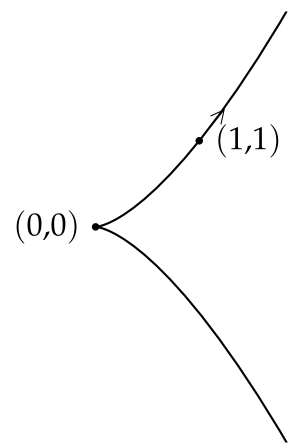
where $\boldsymbol{p} \sim \boldsymbol{p}'$ if $\boldsymbol{p} = \sigma\boldsymbol{p}'$ for some $\sigma \in \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$. With this in mind, recall that for each closed point $\boldsymbol{p}$ of X, we have

$$\mathrm{Hom}_S(\Omega_{S/\Bbbk}, \Bbbk_{\boldsymbol{p}}) = \{\text{point derivations } \partial\colon S \to \Bbbk_{\boldsymbol{p}}\}.$$

Thus we can think of $\mathrm{Hom}_S(\Omega_{S/\Bbbk}, \Bbbk_{\boldsymbol{p}})$ as the set of all tangent vectors at $\boldsymbol{p}$. For instance, the point derivations at the origin $\mathbf{0} = (0,0)$ correspond to all vectors $\boldsymbol{v} = (v_x, v_y) \in \Bbbk^2$ since $v_x\widetilde{\partial}_x|_{\mathbf{0}} + v_y\widetilde{\partial}_y|_{\mathbf{0}}$ vanishes on $2y\mathrm{d}y - 3x^2\mathrm{d}x$. On the other hand, the point derivations at the point $\boldsymbol{p} = (1,1)$ correspond to all vector $\boldsymbol{v} \in \Bbbk^2$ such that $-3v_x + 2v_y = 0$ since

$$(v_x\widetilde{\partial}_x|_{\boldsymbol{p}} + v_y\widetilde{\partial}_y|_{\boldsymbol{p}})(2y\mathrm{d}y - 3x^2\mathrm{d}x) = -3v_x + 2v_y = 0.$$

For instance, the point derivation $(1/3)\widetilde{\partial}_x|_{\boldsymbol{p}} + (1/2)\widetilde{\partial}_y|_{\boldsymbol{p}}$ can be visualized on the curve as the tangent vector centered at $(1,1)$ as below:



8 Parametrizing Field Extensions

Let $\Bbbk$ be a field and fix an algebraic closure $\overline{\Bbbk}$ of $\Bbbk$. Let

$$A = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]/\langle y_1 - e_1, \dots, y_n - e_n\rangle = \Bbbk[\boldsymbol{x}, \boldsymbol{y}]/\langle \boldsymbol{y} - \boldsymbol{e}\rangle,$$

where e_i is the ith elementary symmetric polynomial:

$$e_i = \begin{cases} 1 & \text{if } k = 0 \\ \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{j_1} \cdots x_{j_i} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

We view A as a $\Bbbk[\boldsymbol{y}]$-algebra via the $\Bbbk$-algebra homomorphism $\Bbbk[\boldsymbol{y}] \to A$ which sends y_i ot $\overline{y}_i$. Similarly, we view $\Bbbk[\boldsymbol{x}]$ as an A-algebra via the $\Bbbk$-algebra homomorphism $A \to \Bbbk[\boldsymbol{x}]$ which sends $\overline{y}_i$ to e_i. Thus we have $\Bbbk$-algebra homomorphism $\varphi\colon \Bbbk[\boldsymbol{y}] \to \Bbbk[\boldsymbol{x}]$ which sends y_i to e_i. Geometrically speaking, the $\Bbbk$-algebra homorphism φ corresponds to the morphism of affine schemes $e\colon \mathbb{A}^n_{\Bbbk} \to \mathbb{A}^n_{\Bbbk}$ which sends a $\overline{\Bbbk}$-valued point $\boldsymbol{r} = (r_1, \dots, r_n) \in \overline{\Bbbk}^n$ to the $\overline{\Bbbk}$-valued point $e(\boldsymbol{r}) = (e_1(\boldsymbol{r}), \dots, e_n(\boldsymbol{r})) \in \overline{\Bbbk}^n$. Then the $\Bbbk$-algebra homomorphism $\Bbbk[\boldsymbol{x}, \boldsymbol{y}] \twoheadrightarrow A \to \Bbbk[\boldsymbol{x}]$ corresponds to the morphism graph of e:

$$\Gamma_e\colon \mathbb{A}^n_{\Bbbk} \xrightarrow{\simeq} \mathrm{Spec}\, A \subset \mathrm{Spec}\,(\Bbbk[\boldsymbol{x}, \boldsymbol{y}]) \simeq \mathbb{A}^n_{\Bbbk} \times_{\mathrm{Spec}\,\Bbbk} \mathbb{A}^n_{\Bbbk},$$

which is given on $\overline{\Bbbk}$-valued points $\boldsymbol{r} \in \overline{\Bbbk}^n$ by $\boldsymbol{r} \mapsto (\boldsymbol{r}, e(\boldsymbol{r}))$. Finally the $\Bbbk$-algebra homomorphism $\Bbbk[\boldsymbol{y}] \to A$ corresponds to a projection map $\mathrm{Spec}\, A \to \mathbb{A}^n_{\Bbbk}$ which is given on $\overline{\Bbbk}$-valued points $(\boldsymbol{r}, \boldsymbol{c}) \in \overline{\Bbbk}^n \times \overline{\Bbbk}^n$ by $(\boldsymbol{r}, \boldsymbol{c}) \mapsto \boldsymbol{c}$. Note that since the e_i are algebraically independent, φ induces an isomorphism of $\Bbbk$-algebras of $\Bbbk[\boldsymbol{y}]$ onto its image $\Bbbk[\boldsymbol{e}] = \Bbbk[e_1, \dots, e_n]$. Thus we may identify $\varphi\colon \Bbbk[\boldsymbol{y}] \to \Bbbk[\boldsymbol{x}]$ with $\Bbbk[\boldsymbol{e}] \subseteq \Bbbk[\boldsymbol{x}]$.

For each $\boldsymbol{c} = (c_1, \dots, c_n) \in \overline{\Bbbk}$, let $\mathrm{e}_{\boldsymbol{c}}\colon \Bbbk[\boldsymbol{y}] \twoheadrightarrow \Bbbk(\boldsymbol{c}) \subseteq \overline{\Bbbk}$ be the $\Bbbk$-algebra homomorphism given by $\mathrm{e}_{\boldsymbol{c}}(y_i) = c_i$, let $\mathfrak{m}_{\boldsymbol{c}} = \ker \mathrm{e}_{\boldsymbol{c}}$, and let $\pi_{\boldsymbol{c}}$ be the monic polynomial in $\Bbbk(\boldsymbol{c})[t]$ given by

$$\pi_{\boldsymbol{c}} := t^n + \sum_{i=1}^{n} (-1)^i c_i t^{n-i} = t^n + \sum_{i=1}^{n} (-1)^i e_i(\boldsymbol{r}) t^{n-i} = \prod_{i=1}^{n} (t - r_i),$$

where $r_i = r_{c,i}$ is the ith root of $\pi_{\boldsymbol{c}}$ in $\overline{\Bbbk}$ (for each $\boldsymbol{c}$ we arbitrarily fix an ordering $\boldsymbol{r_c} = \boldsymbol{r} = r_1, \ldots, r_n$ of the roots of $\pi_{\boldsymbol{c}}$, for instance, if $\overline{\Bbbk} = \mathbb{C}$, then we can order them likeso: given $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$ are two nonzero complex numbers expressed in polarized form with $r, r' > 0$ and $\theta, \theta' \in [0, 2\pi)$, then we say $z \geq z'$ if either $r > r'$ or $|r| = |r'|$ and $\theta > \theta'$, and we extend this by setting $z \geq 0$). Let $G_{\boldsymbol{c}} = \mathrm{Gal}(\Bbbk(\boldsymbol{r_c}, \boldsymbol{c})/\Bbbk(\boldsymbol{c}))$ and finally let

$$A_{\boldsymbol{c}} = A \otimes_{\Bbbk[\boldsymbol{y}]} \Bbbk(\boldsymbol{c}) \simeq \Bbbk[\boldsymbol{x}]/\langle \boldsymbol{c} - \boldsymbol{e} \rangle$$

be the fiber of A over $\mathfrak{m}_{\boldsymbol{c}}$.

Proposition 8.1. *With the notation as above, we have a bijection*

$$G_{\boldsymbol{c}} \backslash S_n \cong |\mathrm{Spec}\, A_{\boldsymbol{c}}|.$$

Proof. Then $B_{\boldsymbol{c}}$ is finite as a $\Bbbk$-vector space. Indeed, let $\varphi \colon B_{\boldsymbol{c}} \to \overline{\Bbbk}$ be a $\Bbbk$-algebra homomorphism. Then φ is completely determined by what it does to $\overline{\boldsymbol{y}}$, say $\overline{\boldsymbol{y}} \mapsto \boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \overline{\Bbbk}$. Note that in $\Bbbk[\boldsymbol{y}, t]$ we have the polynomial identity:

$$\prod_{i=1}^{n}(t - y_i) = t^n + \sum_{i=1}^{n}(-1)^i e_i t^{n-i}.$$

In particular, since $\overline{\boldsymbol{e}} = \boldsymbol{c}$ in $B_{\boldsymbol{c}}$, this implies

$$\prod_{i=1}^{n}(t - \gamma_i) = t^n + \sum_{i=1}^{n}(-1)^i c_i t^{n-i} = \pi_{\boldsymbol{c}},$$

which implies $\boldsymbol{\gamma} = \rho\boldsymbol{r} = (r_{\rho(1)}, \ldots, r_{\rho(n)})$ for some permutation $\rho \in S_n$. Without loss of generality, assume $\varphi(\overline{\boldsymbol{y}}) = \boldsymbol{r}$. Then every $\Bbbk$-algebra homomorphism $B_{\boldsymbol{c}} \to \overline{\Bbbk}$ must have the form $\varphi\rho$ where ρ is a permutation of $\overline{y}_1, \ldots, \overline{y}_n$. In particular, there are only finitely many $\Bbbk$-algebras $B_{\boldsymbol{c}} \to \overline{\Bbbk}$, and each of them surjects onto $L_{\boldsymbol{c}}$. The maximal ideals of $B_{\boldsymbol{c}}$ are precisely of the form $\ker(\varphi\rho)$. Furthermore, we have $\ker(\varphi\rho) = \ker(\varphi\rho')$ if and only if $\rho' = \sigma\rho$ where $\sigma \in \mathrm{Gal}(L_{\boldsymbol{c}}/\Bbbk)$ is viewed as the permutation of $\overline{y}_1, \ldots, \overline{y}_n$ which corresponds to how σ permutes the roots $r_1, \ldots, r_n$. Thus the fiber over $\mathfrak{m}_{\boldsymbol{c}}$ is bijection with the quotient

$$\mathrm{Gal}(L(\boldsymbol{c})/\Bbbk) \backslash S_n.$$

Now we projectivize everything. Let $\widetilde{A} = A[z]$ and let

$$\widetilde{B} = A[\boldsymbol{y}, z]/\langle x_1 - e_1, zx_2 - e_2, \ldots, z^{n-1}x_n - e_n \rangle.$$

Let $f = (f_1, \ldots, f_n) \colon \mathbb{A}^n_{\overline{\Bbbk}} \to \mathbb{A}^n_{\overline{\Bbbk}}$ be the morphism given by $f_i(\boldsymbol{r}) = e_i(\boldsymbol{r})$ for all $\boldsymbol{r} = (r_1, \ldots, r_n) \in \overline{\Bbbk}^n$. For each $\boldsymbol{c} = (c_1, \ldots, c_n) \in \overline{\Bbbk}$, let $\pi_{\boldsymbol{c}}$ be the monic polynomial in $\overline{\Bbbk}[t]$ given by

$$\pi_{\boldsymbol{c}} = t^n + \sum_{i=1}^{n}(-1)^i c_i t^{n-i} = \prod_{i=1}^{n}(t - r_i)$$

where $r_i = r_{i,\boldsymbol{c}}$ is the ith root of $\pi_{\boldsymbol{c}}$ in $\overline{\Bbbk}$ (for each $\boldsymbol{c}$ we arbitrarily fix an ordering $\boldsymbol{r_c} = \boldsymbol{r} = (r_1, \ldots, r_n)$ of the roots of $\pi_{\boldsymbol{c}}$). In particular f is an isomorphism.
Also let $L(\boldsymbol{c}) = \Bbbk(\boldsymbol{r})$ be the splitting field of $\pi_{\boldsymbol{c}}$ over $\Bbbk$ contained in $\overline{\Bbbk}$. Note that if $\boldsymbol{c}' = (c'_1, \ldots, c'_n) \in \Bbbk^n$ with $\boldsymbol{c} \neq \boldsymbol{c}'$, then we may have $L(\boldsymbol{c}) = L(\boldsymbol{c}')$ even though $\pi_{\boldsymbol{c}} \neq \pi_{\boldsymbol{c}'}$ and $\boldsymbol{r_c} \neq \boldsymbol{r_{c'}}$. there exists a unique $\boldsymbol{r} \in$ map is onto. Indeed, note that in $\Bbbk[\boldsymbol{x}, t]$ we have the polynomial identity:

$$\prod_{i=1}^{n}(t - x_i) = t^n + \sum_{i=1}^{n}(-1)^i e_i t^{n-i}.$$

Now given any closed point $\boldsymbol{c} = (c_1, \ldots, c_n) \in \overline{\Bbbk}$, form the monic polynomial in $\overline{\Bbbk}[t]$:

$$\pi_{\boldsymbol{c}} = t^n + \sum_{i=1}^{n}(-1)^i c_i t^{n-i}.$$

Then Then in $\overline{\Bbbk}[$ $\pi_{\boldsymbol{c}} = t^n + \sum_{i=1}^{n}(-1)^i c_i t^{n-i} = \prod_{i=1}^{n}(t - r_i)$

$$\prod_{i=1}^{n}(t - y_i) = t^n + \sum_{i=1}^{n}(-1)^i e_i t^{n-i}.$$

Algebraically speaking, the morphism f corresponds to the $\Bbbk$-algebra homomorphism $\varphi\colon \overline{\Bbbk}[\boldsymbol{x}] \to \overline{\Bbbk}[\boldsymbol{y}]$ given by $\varphi(x_i) = e_i$. Note that $\ker \varphi = 0$ since the e_i are algebraically independent, thus $f(\mathbb{A}^n_{\Bbbk})$

$$\overline{f(\mathbb{A}^n_{\Bbbk})} = \mathbb{A}^n_{\Bbbk}$$

. is a we may also identify φ with the inclusion map
Alternatively, We factor f as

$$\mathbb{A}^n_{\Bbbk} \xrightarrow{\Gamma_f} \mathbb{A}^n_{\Bbbk} \times \mathbb{A}^n_{\Bbbk} \xrightarrow{\pi_2} \mathbb{A}^n_{\Bbbk},$$

where the first morphism Γ_f, called the graph of f, takes $\boldsymbol{r}$ to $(\boldsymbol{r}, f(\boldsymbol{r}))$ and where the second morphism π_2 is the projection map onto the second coordiante, that is, it takes $(\boldsymbol{r}, \boldsymbol{c})$ to $\boldsymbol{c}$. Algebraically speaking, the morphism Γ_f corresponds to the $\Bbbk$-algebra homomorphism

$$\Bbbk[\boldsymbol{x}] \otimes_{\Bbbk} \Bbbk[\boldsymbol{y}] = \Bbbk[\boldsymbol{x}, \boldsymbol{y}] \to \Bbbk[\boldsymbol{x}]$$

□

9 Gluing

Consider the affine scheme

$$Z := \operatorname{Spec} \Bbbk[s,t,u]/\langle u^3 - st\rangle \cong \operatorname{Spec} \Bbbk[x^3, y^3, xy].$$

We can think of Z as the quotient of $\mathbb{A}^2_{\Bbbk}$ by the group of third roots of unity with an isolated singularity at the origin. We resolve this singularity as follows: for $i \in \{1,2,3\}$ let $U_i = \operatorname{Spec} \Bbbk[u_i, v_i] \simeq \mathbb{A}^2_{\Bbbk}$. We glue the U_i together via

$$\begin{aligned} u_2 &= u_1^{-1} & u_3 &= v_1^2 u_1 & u_3 &= u_2^3 v_2^2 \\ v_2 &= u_1^2 v_1 & v_3 &= v_1^{-1} & v_3 &= u_2^{-2} v_2^{-1}. \end{aligned}$$

More precisely, we have the following gluing datum:

$$U_2 \supset \mathrm{D}(u_2) := U_{2,1} \xrightarrow[\simeq]{\varphi_{1,2}} U_{1,2} := \mathrm{D}(u_1) \subset U_1$$

$$U_3 \supset \mathrm{D}(v_3) := U_{3,1} \xrightarrow[\simeq]{\varphi_{1,3}} U_{1,3} := \mathrm{D}(v_1) \subset U_1$$

$$U_2 \supset \mathrm{D}(u_2 v_2) := U_{3,2} \xrightarrow[\simeq]{\varphi_{2,3}} U_{2,3} := \mathrm{D}(u_3 v_3) \subset U_3$$

where

$$\begin{aligned} \varphi_{1,2}(u_2) &= u_1^{-1} & \varphi_{1,3}(u_3) &= v_1^2 u_1 & \varphi_{2,3}(u_3) &= u_2^3 v_2^2 \\ \varphi_{1,2}(v_2) &= u_1^2 v_1 & \varphi_{1,3}(v_3) &= v_1^{-1} & \varphi_{2,3}(v_3) &= u_2^{-2} v_2^{-1}. \end{aligned}$$

One checks that the $\varphi_{i,j}$ satisfy the cocycle equation. For instance,

$$\begin{aligned} \varphi_{1,2}\varphi_{2,3}(u_3) &= \varphi_{1,2}(u_2^3 v_2^2) \\ &= u_1^{-3}(u_1^2 v_1)^2 \\ &= u_1 v_1^2 \\ &= \varphi_{1,3}(u_3). \end{aligned}$$

Let $\widetilde{Z}$ denote the scheme obtained by this gluing datum. Next, let

$$Z := \operatorname{Spec} \Bbbk[s,t,u]/\langle u^3 - st\rangle \cong \operatorname{Spec} \Bbbk[x^3, y^3, xy].$$

We can think of Z as the quotient of $\mathbb{A}^2_{\Bbbk}$ by the group of third roots of unity. We have maps

$$\begin{aligned} U_1 &\to Z, & (u_1, v_1) &\mapsto (u_1 v_1^2, u_1^2 v_1, u_1 v_1) \\ U_2 &\to Z, & (u_2, v_2) &\mapsto (u_2^3 v_2^2, v_2, u_2 v_2) \\ U_3 &\to Z, & (u_3, v_3) &\mapsto (u_3, u_3^2 v_2^3, u_3 v_3), \end{aligned}$$

which glue to a morphism $\pi\colon \widetilde{Z} \to Z$. One checks that the restriction $\pi^{-1}(Z\backslash\{\mathbf{0}\}) \to Z\backslash\{\mathbf{0}\}$ is an isomorphism. The closed subscheme $\pi^{-1}(\{\mathbf{0}\})$ (with the reduced scheme structure) can be identified with the union (inside a $\mathbb{P}^2_{\Bbbk}$) of two projective lines intersecting in a single point.

10 The Line With Two Origins