The Homological Conjectures

Introduction

In this note we write about the homological conjectures in commutative algebra.

The Zerodivisor Theorem

Let R be a local noetherian ring, let $r \in R$, and let M be a nonzero finitely generated R-module of finite projective dimension. The zerodivisor theorem states that if r is M-regular, then r is R-regular. To see why one might expect this result to be true, recall that the Auslander-Buchsbaum theorem states that if $\delta_R = \operatorname{depth} R$, $\delta_M = \operatorname{depth} M$, and $p_M = \operatorname{pd} M$, then $p_M < \infty$ implies

$$p_M + \delta_M = \delta_R.$$

In particular, if $\delta_M > 0$ then $\delta_R > 0$. The zerodivisor theorem refines this result by stating that if r realizes M has positive depth, then r also realizes R has positive depth too. In other words, if r is M-regular, then the Auslander-Buchsbaum theorem implies there exists an $r' \in \mathfrak{m}$ such that r' is R-regular (and we can even choose r' such that it is both R-regular and M-regular by the noetherian hypothesis together with prime avoidance). The zerodivisor theorem states that we can already choose $r' = r$.

To see how one could potentially prove this, we use induction on the projective dimension p of M. The base case $p = 0$ is trivial since in this case M is free. Assume that we have proven the theorem for some $p > 0$ and we wish to prove it in the case where $\operatorname{pd}_R M = p + 1$. To prove it in this case, we assume for a contradiction that that r is M-regular but that r is not R-regular. Thus there is an associated prime $\mathfrak{p} = 0 : a$ of R where $a \in R \backslash \{0\}$ such that $r \in \mathfrak{p}$ (and we choose $\mathfrak{p}$ to be minimal among the set of all associate primes which contain r). By localizing at $\mathfrak{p}$ if necessary, we may assume that $\mathfrak{p} = \mathfrak{m}$. In particular, this means that $\mathfrak{m} \notin \operatorname{Ass} M$, thus M doesn't contain a copy of $\Bbbk = R/\mathfrak{m}$ but R does contain a copy of $\Bbbk$. Note that necessarily a is also not R-regular since $ar = 0$ where $r \neq 0$. Furthermore note that necessarily we have $\operatorname{depth} R = 0$ (as every element in $\mathfrak{m}$ is a zerodivisor). From the Auslander-Buchsbaum formula we see that $p + 1 = \operatorname{depth} M$. Let's assume for a moment that a is M-regular and see how we might arrive at a contradiction

At some point we need to use the hypothesis that M has finite projective dimension as well as use the induction hypothesis. The idea is that if we can find an R-module N of projective dimension $\leq p$ such that r is N-regular, then r will be R-regular as well by induction. In particular, let $M_1, \dots, M_p$ be the syzygies of M. These have projective dimension $\leq p$, thus r is not M_i-regular for each $1 \leq i \leq p$. From the short exact sequence,

$$0 \to M_1 \to F_0 \to M \to 0$$

where F_0 is a finite free R-module, we see that

$$\mathfrak{m} \in \operatorname{Ass} R \subseteq \operatorname{Ass} M_1 \cup \operatorname{Ass} M,$$

and since $\mathfrak{m} \notin \operatorname{Ass} M$, we see that $\mathfrak{m} \in \operatorname{Ass} M_1$.

Let $I = \operatorname{Ann} M$ and note that

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Remark 1. If we can find an R-module N of projective dimension $\leq p$ such that r is N-regular, then r will be R-regular as well by induction. In particular, let $M_1, \dots, M_p$ be the syzygies of M. These have projective dimension $\leq p$, thus r is

Remark 2. Here's a potential generalization: let X be a finite R-complex and suppose r is $\mathrm{H}(X)$-regular. Then r is R-regular.

The Canonical Element Conjecture

Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $\boldsymbol{t} = t_1, \dots, t_d$ be a system of parameters for R, let F be a free resolution of $\Bbbk$ over R such that $F_0 = R$, and let $E = \mathbb{K}^R(\boldsymbol{t})$ be the Koszul complex with respect to $\boldsymbol{t}$ and R. Then the canonical map $R/\boldsymbol{t} \to \Bbbk$ can be lifted to a map $\varphi\colon E \to F$ which is unique up to homotopy. The canonical element conjecture states that no matter which choice of system of parameters we use or which lift we choose, the last map $\varphi_d\colon E_d \to F_d$ is not zero. One idea we can use to prove this is to find some R-module N and show that the induced map

$$\mathrm{Ext}^d_R(\Bbbk, N) = \mathrm{H}^d(\mathrm{Hom}^\star_R(F, N)) \xrightarrow{\mathrm{H}^d(\varphi^\star)} \mathrm{H}^d(\mathrm{Hom}^\star_R(E, N)) = \mathrm{Ext}^d_R(R/\boldsymbol{t}, N)$$

is not zero. Indeed, this would imply $\varphi_d \neq 0$, and it would also imply that if φ' were another homotopic lift of $R/\boldsymbol{t} \to \Bbbk$, then $\varphi'_d \neq 0$. We could do this if we could show $\mathrm{Ext}^d_R(\Bbbk, N) \neq 0$ and $\mathrm{Ext}^{d-1}_R(\mathfrak{m}/\boldsymbol{t}, N) = 0$ (or more generally $\mathrm{Ext}^{d-1}_R(\mathfrak{m}/\boldsymbol{t}, N) \to \mathrm{Ext}^d_R(\Bbbk, N)$ is the zero map). We will this is a consequence of the

Example 0.1. Let $R = K[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2\rangle$, and let $\boldsymbol{t} = t_1, t_2, t_3, t_4$ where

$$\begin{aligned} t_1 &= x^2 + w^2 \\ t_2 &= w^2 + zw \\ t_3 &= zw + xy \\ t_4 &= x^3 + w^3. \end{aligned}$$

Now when we apply $\mathrm{Hom}_R(-, R)$ to the following short exact sequence of R-modules

$$0 \longrightarrow I/\boldsymbol{t} \longrightarrow R/\boldsymbol{t} \longrightarrow R/I \longrightarrow 0 \tag{1}$$

and we obtain an induced map in Ext:

$$\cdots \longrightarrow \mathrm{Ext}^3_R(I/\boldsymbol{t}, R) \longrightarrow \mathrm{Ext}^4_R(R/I, R) \longrightarrow \mathrm{Ext}^4_R(R/\boldsymbol{t}, R) \longrightarrow \cdots \tag{2}$$

Note that $\boldsymbol{t}$ is an R-sequence contained in $\langle \boldsymbol{t}\rangle \subseteq I$ of length 4. It follows that from Ext characterization of depth that $\mathrm{Ext}^3_R(I/\boldsymbol{t}, R) = 0$ and $\mathrm{Ext}^4_R(R/I, R) \neq 0$. Thus the map

$$\mathrm{Ext}^4_R(R/I, R) \to \mathrm{Ext}^4_R(R/\boldsymbol{t}, R)$$

is not zero.

Existence of Balanced Big Cohen-Macaulay Modules Conjecture

Definition 0.1. Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring and let M be an R-module. We say M is **big Cohen-Macaulay module** if some system of parameters of R is a regular sequence on M. We say M is a **balanced big Cohen Macaulay module** if every system of parameters of R is a regular sequence on M.