A criterion for uniform non-associativity

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Let $(R, \mathfrak{m})$ be a local noetherian ring, let I be an ideal of R, and let F be the minimal free resolution of R/I over R. Recall that by a multiplication on F we mean a chain map $\mu\colon F \otimes_R F \to F$ which is graded-commutative and unital, though not necessarily associative. One can show that every multiplication on F lifts the usual multiplication on R/I. In particular, all multiplications on F have the form $\mu_h := \mu + \mathrm{d}h + h\mathrm{d}$ where $h\colon F \otimes_R F \to F$ is a homotopy (i.e. a graded R-linear map of degree 1). It was shown by Buchsbaum and Eisenbud in [BE77] that there always exists a multiplication on F. In their work, they posed the question of whether there exists an *associative* multiplication on F. It was subsequently shown that this need not be the case (see [Avr81, Kat19, Sri92] for counterexamples). In this note, we show that if there exists a multiplication μ on F such that μ is not associative at some homogeneous triple (a_1, a_2, a_3) and such that some additional criteria are satisfied with respect to μ and the triple (a_1, a_2, a_3), then in fact *every* multiplication on F will also fail to be associative at the triple (a_1, a_2, a_3).

Before we state and prove the theorem below, we set up some notation. Equip F with a fixed multiplication μ. We simplify notation by writing $a_1 a_2 = \mu(a_1 \otimes a_2)$ where $a_1, a_2 \in F$. The associator for μ is a chain map $[\cdot]_\mu\colon F \otimes_R F \otimes_R F \to F$ given by $[\cdot]_\mu = \mu(\mu \otimes 1 - 1 \otimes \mu)$, and the corresponding R-trilinear map is denoted $[\cdot,\cdot,\cdot]_\mu\colon F^3 \to F$. In particular, given $a_1, a_2, a_3 \in F$ we have

$$[a_1 \otimes a_2 \otimes a_3]_\mu = [a_1, a_2, a_3]_\mu = (a_1 a_2) a_3 - a_1 (a_2 a_3).$$

Since μ is our fixed multiplication, we will simplify notation by dropping μ in the subscript of $[\cdot]_\mu$ and $[\cdot,\cdot,\cdot]_\mu$ in what follows. If μ_h is another multiplication on F where $h\colon F \otimes_R F \to F$ is a homotopy, then the associator for μ_h is given by

$$[\cdot]_{\mu_h} = [\cdot] + \mathrm{d}H + H\mathrm{d}, \tag{1}$$

where $H = [\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}$ and

$$[\cdot]_{\mu,h} = \mu(h \otimes 1 - 1 \otimes h), \quad \text{and} \quad [\cdot]_{h,\mu_h} = h(\mu_h \otimes 1 - 1 \otimes \mu_h). \tag{2}$$

We refer to $[\cdot]_{\mu,h}$ and $[\cdot]_{h,\mu_h}$ as generalized associators because they generalize the usual associator in the sense that $[\cdot]_{\mu,\mu} = [\cdot]_\mu$. Additional signs may appear in (2) when we evaluate these generalized associators to elements due to the Koszul sign rule. Note in (1) we simplified notation by using the same symbol d to denote either the differential $\mathrm{d}_{F^{\otimes 3}}$ of $F^{\otimes 3}$ or the differential d_F of F where context makes clear which differential the symbol d refers to (for instance, the d in $\mathrm{d}H$ clearly refers to d_F since H lands in F). This simplification in our notation will prove useful as we will be working with all three differentials d_F, $\mathrm{d}_{F^{\otimes 2}}$, and $\mathrm{d}_{F^{\otimes 3}}$ many times over. Finally, we can decompose the generalized associator $[\cdot]_{h,\mu_h}$ further into a sum of three generalized associators:

$$[\cdot]_{h,\mu_h} = [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}}.$$

The notation indicates how these generalized associators are defined (for example $[\cdot]_{h,h\mathrm{d}} = h(h\mathrm{d} \otimes 1 - 1 \otimes h\mathrm{d})$). With this notation set up, we are now ready to state and prove the theorem:

Theorem 0.1. *With the notation as above, suppose there exists homogeneous elements $a_1, a_2, a_3 \in F$ such that $\mathrm{d}a_1, \mathrm{d}a_2, \mathrm{d}a_3 \in \mathfrak{m}^2 F$, $a_1 a_2, a_2 a_3 \in \mathfrak{m}F$, and $[a_1, a_2, a_3] \notin \mathfrak{m}^2 F + a_1 \mathrm{d}F + a_3 \mathrm{d}F$. Then every multiplication on F is not associative at the triple (a_1, a_2, a_3).*

Proof. Let $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ be another multiplication on F where $h\colon F^{\otimes 2} \to F$ is a homotopy. The associator for μ_h is given by the formula (1). Let us analyze the $\mathrm{d}H + H\mathrm{d}$ term in the formula by evaluating it at $a_1 \otimes a_2 \otimes a_3$ in $\overline{F} := F/\mathfrak{m}^2$. In $\overline{F}$ we have

$$\begin{aligned}
(\mathrm{d}H + H\mathrm{d})(a_1 \otimes a_2 \otimes a_3) &\equiv \mathrm{d}H(a_1 \otimes a_2 \otimes a_3) \\
&= \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})(a_1 \otimes a_2 \otimes a_3) \\
&= \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu})(a_1 \otimes a_2 \otimes a_3) + \mathrm{d}([\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}})(a_1 \otimes a_2 \otimes a_3) \\
&\equiv \mathrm{d}([\cdot]_{\mu,h} + [\cdot]_{h,\mu})(a_1 \otimes a_2 \otimes a_3), \\
&\equiv \mathrm{d}[\cdot]_{\mu,h}(a_1 \otimes a_2 \otimes a_3) \\
&= \mathrm{d}(a_{1,2} a_3 - a_1 a_{2,3}) \\
&\equiv (\mathrm{d}a_{1,2}) a_3 - a_1 \mathrm{d}(a_{2,3}),
\end{aligned}$$

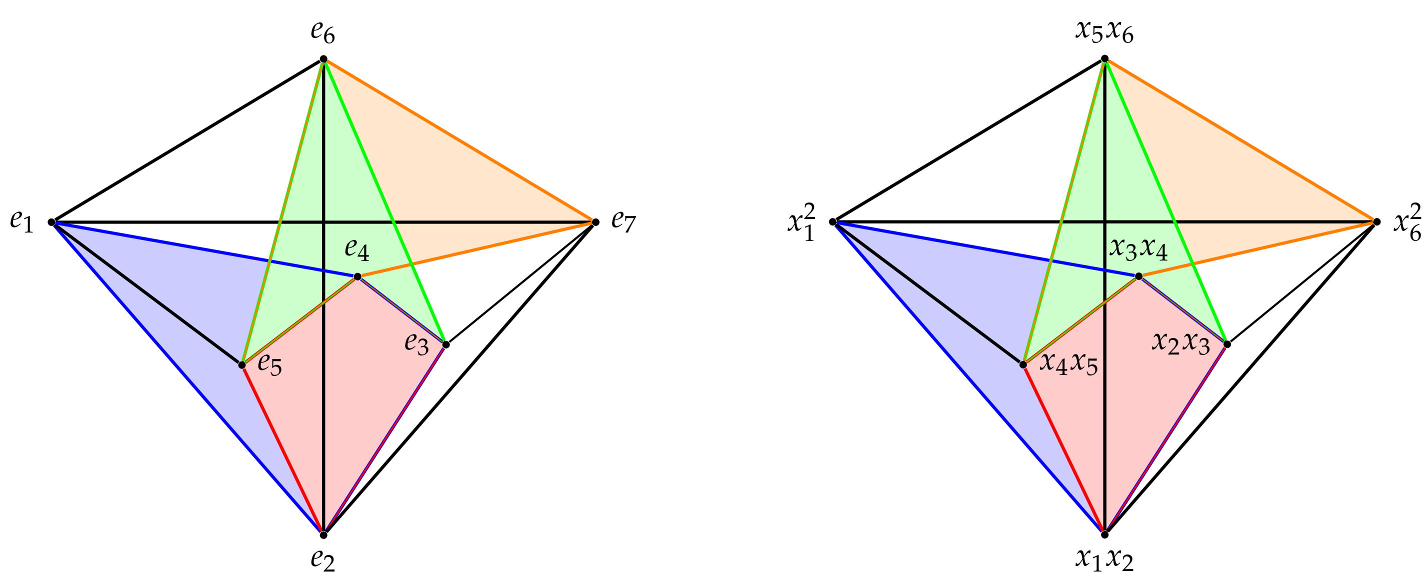
where we obtained the first line from the assumption that $\mathrm{d}a_1, \mathrm{d}a_2, \mathrm{d}a_3 \in \mathfrak{m}^2 F$, we obtained the fourth line from the third line from the assumption that $\mathrm{d}F \subseteq \mathfrak{m}F$ and the fact that the differential appears twice in the terms $\mathrm{d}[\cdot]_{h,\mathrm{d}h}$ and $\mathrm{d}[\cdot]_{h,h\mathrm{d}}$, we obtained the fifth line from the the fourth line using the assumption that $a_1 a_2, a_2 a_3 \in \mathfrak{m}F$, and we obtained the last two lines by denoting $a_{1,2} = h(a_1 \otimes a_2)$ and $a_{2,3} = h(a_2 \otimes a_3)$ and using the assumption that $\mathrm{d}a_1, \mathrm{d}a_3 \in \mathfrak{m}^2 F$. Therefore in $\overline{F}$, we have

$$[a_1, a_2, a_3]_{\mu_h} \equiv [a_1, a_2, a_3] - (\mathrm{d}a_{1,2})a_3 + a_1 \mathrm{d}(a_{2,3}) \neq 0,$$

where the last equality follows from the assumption that $[a_1, a_2, a_3] \notin \mathfrak{m}^2 F + a_1 \mathrm{d}F + a_3 \mathrm{d}F$. It follows that $[a_1, a_2, a_3]_{\mu_h} \neq 0$ in F, thus μ_h is not associative as the triple (a_1, a_2, a_3). □

We now give an example where we can apply this criterion. This example is based on and motivated by Avramov's example in [Avr81]. The minimal free resolution in his example was supported on a tetrahedron except with two of the triangular faces replaced with two square faces. We refer to this shape as an Avramov tetrahedron.

Example 0.1. Let $R = \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$, let $\boldsymbol{m} = x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F is the R-complex supported on the $\boldsymbol{m}$-labeled cellular complex below:



We label the homogeneous generators of F corresponding to the simplicial faces in the usual way. In particular, the complex in homological degree 1 consists of seven 0-simplices corresponding to the seven generators $e_1, \dots, e_7$ of F_1, and the complex in homological degree 2 consists of sixteen 1-simplices corresponding to the sixteen generators $e_{12}, e_{23}, \dots, e_{67}$ of F_2. The differential is defined on the generators corresponding to the simplicial faces via the Taylor rule (for example $\mathrm{d}e_1 = x_1^2$ and $\mathrm{d}e_{12} = x_2 e_1 - x_1 e_2$). The complex in homological degree 3 consists of thirteen 2-simplices and four squares (which we shaded in blue, red, green, and orange above). The differential on the squares is given by

$$\begin{aligned}
\mathrm{d}e_{1234} &= x_3x_4e_{12} + x_1x_4e_{23} - x_2e_{14} + x_1^2e_{34} \\
\mathrm{d}e_{2345} &= x_4x_5e_{23} + x_1x_5e_{34} - x_3e_{25} + x_1x_2e_{45} \\
\mathrm{d}e_{3456} &= x_5x_6e_{34} + x_2x_6e_{45} - x_4e_{36} + x_2x_3e_{56} \\
\mathrm{d}e_{4567} &= x_6^2e_{45} + x_3x_6e_{56} - x_5e_{47} + x_3x_4e_{67}
\end{aligned}$$

The complex in homological degree 4 consists of three 3-simplices, three Avramov tetrahedra, and two pyramids. The differential on the Avramov tetrahedra and pyramids is given by

$$\begin{aligned}
\mathrm{d}e_{12345} &= x_5e_{1234} - x_3e_{125} + x_2e_{145} - x_1e_{2345} \\
\mathrm{d}e_{23456} &= x_6e_{2345} - x_4e_{236} + x_3e_{256} - x_1e_{3456} \\
\mathrm{d}e_{34567} &= x_6e_{3456} - x_5e_{347} + x_4e_{367} - x_2e_{4567} \\
\mathrm{d}e_{123457} &= x_6^2e_{1234} - x_3x_4e_{127} - x_1x_4e_{237} + x_2e_{147} - x_1^2e_{347} \\
\mathrm{d}e_{134567} &= x_6^2e_{145} + x_3x_6e_{156} - x_5e_{147} + x_3x_4e_{167} - x_1^2e_{4567}
\end{aligned}$$

Finally, the complex in homological degree 5 consists of one 4-cell, and the differential on it is given by

$$\mathrm{d}e_{1234567} = x_6^2e_{12345} + x_3x_6e_{1256} + x_1x_6e_{23456} - x_5e_{123457} + x_3x_4e_{1267} + x_1x_4e_{2367} - x_2e_{134567} + x_1^2e_{34567}$$

Now equip F with a multigraded multiplication μ. Upon considerations of the Leibniz rule and multigrading, one can show that we already have three non-trivial associators corresponding to the three Avramov tetrahedra:

$$[e_1, e_3, e_5] = \mathrm{d}e_{12345}, \quad [e_2, e_4, e_6] = \mathrm{d}e_{23456}, \quad \text{and} \quad [e_3, e_5, e_7] = \mathrm{d}e_{34567}.$$

Furthermore, one can show that the criteria in Theorem (0.1) is satisfied with respect to μ and these three triples. It follows that *every* multiplication on F will also be non-associative at these three triples.

Recall that another multiplication on F has the form $\mu_h = \mu + \mathrm{d}h + h\mathrm{d}$ where $h : F^{\otimes 2} \to F$ is a homotopy. However h can't be just any graded R-linear map since both μ and μ_h are unital and graded-commutative. For instance, the fact that both μ and μ_h are unital means that $\mathrm{d}h(a \otimes 1) = h(\mathrm{d}a \otimes 1)$ for all $a \in F$. In order to get a better understanding of what h needs to satisfy, let us assume that $e_1, \ldots, e_n$ is a homogeneous basis of F as a graded module, ordered so that $i < j$ implies $|e_i| \leq |e_j|$, and let us simplify notation by writing $h_{i,j} = h(e_i \otimes e_j)$. Next let (r^i_j) be the matrix representation of d with respect to this basis. Therefore we have

$$\mathrm{d}e_j = \sum_i r^i_j e_i,$$

where necessarily we have $r^i_j = 0$ whenever $|e_i| \neq |e_j| - 1$. Since F is a resolution, one can show that the unitality conditions on μ and μ_h implies

$$\begin{array}{lll}
h_{1,1} = \mathrm{d}\varepsilon_{1,1} & h_{1,1} = \mathrm{d}\varepsilon_{1,1} & \\
h_{i,1} = r_i \varepsilon_{1,1} + \mathrm{d}\varepsilon_{i.1} & h_{1,i} = r_i \varepsilon_{1,1} + \mathrm{d}\varepsilon_{1,i} & h_{i,i} = r_i(\varepsilon_{i,1} - \varepsilon_{1,i}) + \mathrm{d}\varepsilon_{i,i} \\
h_{j,1} = \sum_i r^i_j \varepsilon_{i,1} + \mathrm{d}\varepsilon_{j,1} & h_{1,j} = \sum_i r^i_j \varepsilon_{1,i} + \mathrm{d}\varepsilon_{1,j} & h_{i_1,i_2} + h_{i_2,i_1} = r_{i_1}(\varepsilon_{i_1,1} - \varepsilon_{1,i_1}) + r_{i_2}(\varepsilon_{i_2,1} - \varepsilon_{1,i_2}) + \mathrm{d}\varepsilon_{i_1,i_2} \\
h_{k,1} = \sum_j r^j_k \varepsilon_{j,1} + \mathrm{d}\varepsilon_{k,1} & h_{1,k} = \sum_j r^j_k \varepsilon_{1,j} + \mathrm{d}\varepsilon_{1,k} & \\
\vdots & \vdots &
\end{array}$$

References

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