Mathematical Programming Homework 4

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Problem 1

Let $f(\boldsymbol{x}) = 3x_1^2 + 3x_2^2 - 2x_1x_2 + 2x_1 - 6x_2$. Consider the unconstrained optimization problem:

$$\min \quad f(\boldsymbol{x}) \tag{1}$$

Problem 1.a

Exercise 1. Let the initial point $\boldsymbol{x}^1 = (1,2)^\top$. Perform one iteration of

1. the steepest descent method using the negative gradient of the objective function as the search direction and report $\boldsymbol{x}^2$.

2. Newton's method and report $\boldsymbol{x}^2$.

Solution 1. 1. We set $\boldsymbol{x}^2 = \boldsymbol{x}^1 - \gamma^1 \nabla f(\boldsymbol{x}^1)$ where the step size $\gamma^1 > 0$ is chosen to minimize $f(\boldsymbol{x}^1 - \gamma \nabla f(\boldsymbol{x}^1))$ over $\lambda \in \mathbb{R}_{>0}$. To find such λ^1, we first we calculate

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 6x_1 - 2x_2 + 2 \\ 6x_2 - 2x_1 - 6 \end{pmatrix}.$$

In particular, we have $\nabla f(\boldsymbol{x}^1) = (4,4)^\top$. If we set $\gamma^1 = 1/4$, then we have

$$\begin{aligned} \boldsymbol{x}^2 &= \boldsymbol{x}^1 - \gamma^1 \nabla f(\boldsymbol{x}^1) \\ &= (1,2)^\top - (1/4)(4,4)^\top \\ &= (1,2)^\top - (1,1)^\top \\ &= (1,0)^\top. \end{aligned}$$

It turns out that $\boldsymbol{x}^2$ is the global minimizer of f (we will show this in part c of this problem), thus our choice of λ^1 here certainly minimizes $f(\boldsymbol{x}^1 - \gamma \nabla f(\boldsymbol{x}^1))$ over $\lambda \in \mathbb{R}_{>0}$.

2. First we calculate

$$\mathrm{H}_f(\boldsymbol{x}) = \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix}.$$

Next we set

$$\begin{aligned} \boldsymbol{x}^2 &= \boldsymbol{x}^1 - \mathrm{H}_f^{-1}(\boldsymbol{x}^1) \nabla f(\boldsymbol{x}^1) \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{16} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{16} \begin{pmatrix} 16 \\ 16 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Problem 1.b

Exercise 2. Compare the points $\boldsymbol{x}^2$ you found in parts 1 and 2 above. What do you observe?

Solution 2. In each case, we obtained $\boldsymbol{x}^2 = (0,1)^\top$. The method of steepest descent took a little more work since it involved finding a step size λ^1.

Problem 1.c

Exercise 3. What is the meaning of the points $\boldsymbol{x}^2$ for the minimization problem.

Solution 3. We now show $\boldsymbol{x}^2$ is the global minimizer of f. First we find the critical points of f:

$$\begin{aligned} \nabla f(\boldsymbol{x}) = 0 &\iff \begin{pmatrix} 6x_1 - 2x_2 + 2 \\ 6x_2 - 2x_1 - 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff \boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

In particular, $\boldsymbol{x}^2 = (0,1)^\top$ is the only critical point of f. Next we observe that $\mathrm{H}_f(\boldsymbol{x}^2)$ is positive definite. It follows that $\boldsymbol{x}^2$ is the global minimum of f.

Problem 2

Let $f(\boldsymbol{x}) = x_1^2 + x_2^2$ and $h(\boldsymbol{x}) = x_1 + x_2 - 1$. Consider the constrained optimization problem

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & h(\boldsymbol{x}) = 0 \end{aligned}$$

Problem 2.a

Exercise 4. Use the penalty function $\Psi(\boldsymbol{x}) = (h(\boldsymbol{x}))^2$ and formulate the Approximation Problem (AP)

Solution 4. Let c be a positive constant (the penalty weight) and set $Q(\boldsymbol{x}; c) = f(\boldsymbol{x}) + c\Psi(\boldsymbol{x})$. Then the Approximation Problem is

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} Q(\boldsymbol{x}; c).$$

Problem 2.b

Exercise 5. Find an optimal solution and the optimal objective value to the AP.

Solution 5. First we calculate

$$\begin{aligned} Q(\boldsymbol{x}; c) &= x_1^2 + x_2^2 + c\,(x_1 + x_2 - 1)^2 \\ &= x_1^2 + x_2^2 + c(x_1^2 + x_2^2 + 1 + 2x_1x_2 - 2x_1 - 2x_2) \\ &= (1+c)x_1^2 + (1+c)x_2^2 + 2cx_1x_2 - 2cx_1 - 2cx_2 + c \end{aligned}$$

$$\begin{aligned} \nabla Q(\boldsymbol{x}; c) &= \begin{pmatrix} 2(1+c)x_1 + 2cx_2 - 2c \\ 2(1+c)x_2 + 2cx_1 - 2c \end{pmatrix} \\ &= 2\begin{pmatrix} (1+c)x_1 + cx_2 - c \\ (1+c)x_2 + cx_1 - c \end{pmatrix} \end{aligned}$$

$$\mathrm{H}_Q(\boldsymbol{x}; c) = 2\begin{pmatrix} 1+c & c \\ c & 1+c \end{pmatrix}$$

The eigenvalues of $\mathrm{H}_Q(\boldsymbol{x};c)$ are $\lambda_1 = 1$ and $\lambda_2 = 2c+1$. In particular, since c is a positive constant, we see that $\mathrm{H}_Q(\boldsymbol{x};c)$ is positive definite for all $\boldsymbol{x} \in \mathbb{R}^2$. It follows that $Q(\boldsymbol{x};c)$ is convex as a function in $\boldsymbol{x}$. In particular

$$\begin{aligned}
\boldsymbol{x}^* \text{ is a global minimizer of } Q(-;c) &\iff \nabla Q(\boldsymbol{x}^*;c) = 0 \\
&\iff \begin{pmatrix} (1+c)x_1^* + cx_2^* - c \\ (1+c)x_2^* + cx_1^* - c \end{pmatrix} = 0 \\
&\iff x_1^* = \frac{c}{1+2c} \quad \text{and} \quad x_2^* = \frac{c}{1+2c}
\end{aligned}$$

Thus the optimal objective value is

$$\begin{aligned}
Q(\boldsymbol{x}^*;c) &= \left(\frac{c}{1+2c}\right)^2 + \left(\frac{c}{1+2c}\right)^2 + c\left(\frac{c}{1+2c} + \frac{c}{1+2c} - 1\right)^2 \\
&= 2\left(\frac{c}{1+2c}\right)^2 + c\left(\frac{-1}{1+2c}\right)^2 \\
&= \frac{2c^2+c}{(1+2c)^2} \\
&= \frac{c(2c+1)}{(1+2c)^2} \\
&= \frac{c}{1+2c}.
\end{aligned}$$

Problem 2.c

Exercise 6. Do not solve the original problem, but using the results of part b, give an optimal solution and the optimal objective value to the original problem.

Solution 6. By taking $c \to \infty$, we see that an optimal solution to the original problem is $\boldsymbol{x}^* = (1/2, 1/2)^\top$ with optimal objective value given by $f(\boldsymbol{x}^*) = 1/2$.

Problem 3

For $\boldsymbol{x} \in \mathbb{R}^6$, we write its coordiantes as $\boldsymbol{x} = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})^\top$. Here, we think of x_{ij} as representing the fraction of time we use machine i to produce product j. Next set

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 8 & 2 & 9 & 3 & 5 & 6 \\ 4 & -1 & -9/2 & 3/2 & -5/2 & 3 \\ -2 & 3/4 & -9/4 & -3/4 & 15/4 & -3/2 \\ -2 & -1/2 & 27/4 & -3/4 & -5/4 & 9/2 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} 8 \\ 2 \\ 9 \\ 3 \\ 5 \\ 6 \end{pmatrix}$$

Then the linear program that will determine what fraction of the day each machine should be used to produce each product so as to maximize the total quantity of products produced is given by

$$\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & A\boldsymbol{x} = \boldsymbol{b} \\
& 0 \le \boldsymbol{x} \le 1.
\end{aligned}$$

Note that this linear program can be expressed in canonical form as

$$\begin{aligned}
\max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \begin{pmatrix} A \\ -A \\ -\mathrm{I} \end{pmatrix} \boldsymbol{x} \ge \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{b} \\ 1 \end{pmatrix} \\
& \boldsymbol{x} \ge 0.
\end{aligned}$$

Problem 4

Exercise 7. Consider the following homogenous system of equations $A\boldsymbol{x} = 0$, where A is an $m \times n$ matrix. This system has always a trivial solution $\boldsymbol{x} = 0$, but may also have nontrivial solutions ($\boldsymbol{x} \neq 0$). Clearly define the decision variables and formulate an LP whose optimal solution indicates whether or not this linear system has strictly positive solution(s) $\boldsymbol{x} > 0$.

Solution 7. Let $\boldsymbol{c}$ be the vector in $\mathbb{R}^n$ whose entries are all -1, so $\boldsymbol{c} = (-1, \ldots, -1)^\top$. Consider the following LP

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} = 0 \\ & 0 \leq \boldsymbol{x} \leq 1 \end{aligned}$$

Let $\boldsymbol{x}^*$ be an optimal solution to this linear program (such an optimal solution exists since the continuous function $\boldsymbol{c}^\top \boldsymbol{x}$ attains a minimum on the closed and bounded set $\ker A \cap \{0 \leq \boldsymbol{x} \leq 1\}$). We claim that the linear system $A\boldsymbol{x} = 0$ has strictly positive solutions if and only if $\boldsymbol{c}^\top \boldsymbol{x}^* < 0$. Indeed, if there are no strictly positive solutions to $A\boldsymbol{x} = 0$, then 0 is the only feasible solution of this LP, thus necessarily we have $\boldsymbol{x}^* = 0$ and $\boldsymbol{c}^\top \boldsymbol{x}^* = 0$. Converesly, suppose there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{y} > 0$ and $A\boldsymbol{y} = 0$. Choose $\lambda \in \mathbb{R}_{>0}$ such that $\boldsymbol{y}/\lambda \leq 1$. Then $\boldsymbol{c}^\top(\boldsymbol{y}/\lambda) < 0$, which implies $\boldsymbol{c}^\top \boldsymbol{x}^* < 0$.

Problem 5

Exercise 8. Formulate an LP for finding a vector satisfying

$$4x_1 + x_2 \leq 5, \quad x_1 \geq 0, \quad \text{and} \quad x_2 \geq 0$$

and having the maximum of

$$2x_1 - x_2 \text{ and } -3x_1 + 2x_2$$

as small as possible.

Solution 8. labelsol First, we note that

$$\max(2x_1 - x_2, -3x_1 + 2x_2) = \begin{cases} 2x_1 - x_2 & \text{if } -5x_1 + 3x_2 \leq 0 \\ -3x_1 + 2x_2 & \text{if } 5x_1 - 3x_2 \leq 0 \end{cases}$$

With this in mind, we consider two linear programs. First we define a linear program which we call LP: let $\boldsymbol{c} = (2, -1)^\top$, let $A = \left(\begin{smallmatrix} -5 & 3 \\ 4 & 1 \end{smallmatrix}\right)$, and let $\boldsymbol{b} = (0, 5)^\top$. Then LP is given by

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Observe that if $\boldsymbol{x}$ is a feasible solution of LP, then

$$\max(2x_1 - x_2, -3x_1 + 2x_2) = 2x_1 - x_2 = \boldsymbol{c}^\top \boldsymbol{x}.$$

The point $\boldsymbol{x}^* = \frac{5}{17}(5, 3)^\top$ is the only corner point for the feasible region of LP, thus it is an optimal solution for LP with optimal objective value being $\mathrm{c}^\top \boldsymbol{x}^* = 35/17$.

Next we define a linear program which we call LP′: let $\boldsymbol{c}' = (-3, 2)^\top$, let $A' = \left(\begin{smallmatrix} 5 & -3 \\ 4 & 1 \end{smallmatrix}\right)$, and let $\boldsymbol{b}' = (0, 5)^\top$. Then LP′ is given by

$$\begin{aligned} \min \quad & \boldsymbol{c}'^\top \boldsymbol{x} \\ \text{s.t.} \quad & A'\boldsymbol{x} \leq \boldsymbol{b}' \\ & \boldsymbol{x}' \geq 0 \end{aligned}$$

Observe that if $\boldsymbol{x}$ is a feasible solution of LP′, then

$$\max(2x_1 - x_2, -3x_1 + 2x_2) = -3x_1 + 2x_2 = \boldsymbol{c}'^\top \boldsymbol{x}.$$

The point $\boldsymbol{x}^* = \frac{5}{17}(5,3)^\top$ is the only corner point for the feasible region of LP', thus it is an optimal solution for LP' with optimal objective value being $c'^\top \boldsymbol{x}^* = -45/17$.

Thus $\boldsymbol{x}^* = \frac{5}{17}(5,3)^\top$ is a vector which satisfies

$$4x_1^* + x_2^* \leq 5, \quad \text{and} \quad \boldsymbol{x}^* \geq 0$$

and has

$$\max(2x_1^* - x_2^*, -3x_1^* + 2x_2^*) = 35/17$$

as small as possible.