First Fundamental Theorem of Calculus

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Theorem 0.1. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then there exists a function $F\colon [a,b] \to \mathbb{R}$ such that*

1. *F is uniformly continuous on the closed interval $[a,b]$.*
2. *F is differentiable on the open interval (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.*

Moreover, if $G\colon [a,b] \to \mathbb{R}$ is another function which is differentiable on the open interval (a,b) such that $G'(x) = f(x)$ for all $x \in (a,b)$, then $G - F = G(a)$.

Proof. We define $F\colon [a,b] \to \mathbb{R}$ by

$$F(x) := \int_a^x f(t)\mathrm{d}t$$

for all $x \in [a,b]$. Let us first prove (1): let $\varepsilon > 0$ and let $x, y \in [a,b]$. As f is continuous on a compact interval, there exists an $M \in \mathbb{R}$ such that $f(t) \leq M$ for all $t \in [a,b]$. We set $\delta = \varepsilon/M$. Then $|x-y| < \delta$ implies

$$\begin{aligned} |F(x) - F(y)| &= \left| \int_a^x f(t)\mathrm{d}t - \int_a^y f(t)\mathrm{d}t \right| \\ &= \left| \int_a^x f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t - \int_x^y f(t)\mathrm{d}t \right| \\ &= \left| \int_x^y f(t)\mathrm{d}t \right| \\ &\leq |x-y|\, M \\ &< \delta M \\ &= \varepsilon. \end{aligned}$$

This proves 1. Now we prove 2: Let $x \in (a,b)$. Then h sufficiently small, we have

$$\begin{aligned} F(x+h) - F(x) &= \int_x^{x+h} f(t)\mathrm{d}t \\ &= hf(x) + E(h) \end{aligned}$$

where $E(h) := \int_x^{x+h} f(t)\mathrm{d}t - hf(x)$ is the excess area. Observe that

$$|E(h)| \leq \left| h \left(\sup_{t \in [x,x+h]} f(t) - \inf_{t \in [x,x+h]} f(t) \right) \right|$$

In particular continuity of f at x, implies $\lim_{h \to 0}(E(h)/h) = 0$. Thus, if we let ψ be the function defined for small h given by $\psi(h) := E(h)/h$, then it follows that

$$F(x+h) - F(x) = hf(x) + h\psi(h),$$

which implies that F is differentiable at x with $F'(x) = f(x)$.

Finally, let $G\colon [a,b] \to \mathbb{R}$ be another function which is differentiable in the open interval (a,b) such that $G'(x) = f(x)$ for all $x \in (a,b)$. Then

$$(G-F)'(x) = f(x) - f(x) = 0$$

for all $x \in (a,b)$. It follows (from a consequence of the mean value theorem) that $G - F$ is constant on $[a,b]$. In particular,

$$(G-F)(a) = G(a)$$

implies $G - F = G(a)$. □

0.0.1 Consequences of the First Fundamental Theorem

Corollary. *Let f be a continuous real-valued function defined on the closed interval $[a,b]$ such that f is differentiable on the open interval (a,b). Suppose that*

$$f'(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \textit{for all } x \in [a,b],$$

where $a_0, \ldots, a_n \in \mathbb{R}$. Then

$$f(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + a_0 x + a_{-1},$$

for some $a_{-1} \in \mathbb{R}$.

Proof. Let $F\colon [a,b] \to \mathbb{R}$ be given by

$$F(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \cdots + a_0 x$$

for all $x \in [a,b]$. Then observe that both F and f are antiderivatives of f'. In particular, we must have $F - f = a_{-1}$, for some $a_{-1} \in \mathbb{R}$. □