Multiplicity and Koszul Homology

Lemma 0.1. *Let M be a finitely generated R-module and let I be an ideal of R. Then*

$$\sqrt{\mathrm{Ann}(M/IM)} = \sqrt{\langle I, \mathrm{Ann}\, M\rangle}.$$

Proof. To prove the equality on radicals, it suffices to show that a prime $\mathfrak{p}$ of R contains $\mathrm{Ann}(M/IM)$ if and only if it contains $\langle I, \mathrm{Ann}\, M\rangle$. Recall that for any finitely generated R-module N, we have $\mathrm{V}(\mathrm{Ann}\, N) = \mathrm{Supp}\, N$, or equivalently, $\mathfrak{p} \supseteq \mathrm{Ann}\, N$ if and only if $N_{\mathfrak{p}} \neq 0$. Thus since M is finitely generated (and hence M/IM is finitely generated too), we have

$$\begin{aligned}
\mathfrak{p} \supseteq \mathrm{Ann}(M/IM) &\iff M_{\mathfrak{p}}/I_{\mathfrak{p}}M_{\mathfrak{p}} \neq 0 \\
&\iff M_{\mathfrak{p}} \neq 0 \text{ and } I_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}} \\
&\iff \mathfrak{p} \supseteq \mathrm{Ann}\, M \text{ and } I \subseteq \mathfrak{p} \\
&\iff \mathfrak{p} \supseteq \langle \mathrm{Ann}\, M, I\rangle
\end{aligned}$$

□

Let $A = (A, \mathfrak{m}, \Bbbk)$ be a noetherian local ring, let $\boldsymbol{x} = x_1, \dots, x_r$ be a sequence contained in $\mathfrak{m}$, and let M be a finitely generated A-module such that $\ell(M/\boldsymbol{x}M) < \infty$ (equivalently, we have $\mathfrak{m} = \sqrt{\mathrm{Ann}(M/\boldsymbol{x}M)}$). We set $K = K(\boldsymbol{x}, M)$ to be koszul complex with respect to $\boldsymbol{x}$ and M and we denote its homology by $\mathrm{H}(\boldsymbol{x}, M)$. Recall that the A-module $\mathrm{H}_i(\boldsymbol{x}, M)$ is finitely generated and annihilated by $\langle \boldsymbol{x}, \mathrm{Ann}\, M\rangle$, hence they have finite length (indeed, we have $\mathfrak{m} = \sqrt{\mathrm{Ann}(M/\boldsymbol{x}M)} = \sqrt{\langle \boldsymbol{x}, \mathrm{Ann}\, M\rangle}$). We may therefore define the **Euler-Poincare characteristic**

$$\chi(\boldsymbol{x}, M) = \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(\boldsymbol{x}, M)).$$

On the other hand, we the Hilbert-Samuel polynomial $P_{\boldsymbol{x}}(M)$ has degree $\leq r$, and we have

$$P_{\boldsymbol{x}}(M, n) = \mathrm{e}_{\boldsymbol{x}}(M, r)\frac{n^r}{r!} + Q(n)$$

with $\deg Q < r$ and where $\mathrm{e}_{\boldsymbol{x}}(M, r) = \Delta^r P_{\boldsymbol{x}}(M)$ is the Hilbert-Samuel multiplicity.

Theorem 0.2. *We have $\chi(\boldsymbol{x}, M) = \mathrm{e}_{\boldsymbol{x}}(M, r)$.*

Proof. We prove this in several steps:

Step 1: To ease notation in what follows, we set $Q = \langle \boldsymbol{x} \rangle$. We first equip A with the standard Q-filtration $A = (Q^n)$ and view it as a filtered ring. Similarly, we equip M with the Q-filtration $M = (Q^n M)$ and view it as a filtered A-module. We now equip K with a Q-filtration as follows: for each $n \in \mathbb{N}$, let K^n be the R-subcomplex of K whose component in homological degree i

$$K_i^n = \begin{cases} Q^{n-i}K_i. & \text{if } 0 \leq i < n \\ K_i & \text{else} \end{cases}$$

Thus for example, we have

$$\begin{aligned}
K^0 &= M + \sum Me_i + \sum Me_{i,j} + \cdots \\
K^1 &= QM + \sum Me_i + \sum Me_{i,j} + \cdots \\
K^2 &= Q^2M + \sum QMe_i + \sum Me_{i,j} + \cdots \\
&\vdots
\end{aligned}$$

Notice that

$$
\begin{aligned}
K^0/K^1 &= M/QM \\
K^1/K^2 &= QM/Q^2M + \sum (M/QM)e_i \\
K^2/K^3 &= Q^2M/Q^3M + \sum (QM/Q^2M)e_i + \sum (M/QM)e_{i,j} \\
&\vdots
\end{aligned}
$$

In particular, we clearly have

$$
\begin{aligned}
\mathrm{gr}(K) &= \bigoplus_{n=0}^{\infty} K^n/K^{n+1} \\
&= \mathrm{gr}(M) + \sum \mathrm{gr}(M)e_i + \sum \mathrm{gr}(M)e_{i,j} \\
&= K(\boldsymbol{x}, \mathrm{gr}(M)).
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\chi(\boldsymbol{x}, M) &= \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(\boldsymbol{x}, M)) \\
&= \sum_{i=0}^{r} (-1)^i \ell(\mathrm{H}_i(K/K^n)) \\
&= \sum (-1)^i \ell(K_i/K_i^n) \\
&= \sum (-1)^i \ell \left(\bigoplus_{\binom{r}{i}} M/\boldsymbol{x}^{n-i}M \right) \\
&= \sum (-1)^i \binom{r}{i} \ell(M/\boldsymbol{x}^{n-i}M) \\
&= \mathrm{e}_{\boldsymbol{x}}(M, r).
\end{aligned}
$$

□