Mathematics Diary

January 30, 2025

Contents

1 2023

X_{et} X_{et} $X_{\text{ét}}$ $X_{\text{Ét}}$ $\text{Ét}(X)$

1.1 12/20/2022 - When $\Sigma(F/E)$ is the minimal free resolution of I/J over R

Lemma 1.1. *Let $(R, \mathfrak{m}, \Bbbk)$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R. Let E be the minimal free resolution of R/J over R, let F be the minimal free resolution of R/I over R, and let $\varphi \colon E \to F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_\Bbbk := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_\Bbbk$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R.*

Proof. Assume both $\varphi \colon E \to F$ and $\overline{\varphi} \colon E_\Bbbk := E \otimes_R \Bbbk \to F \otimes_R \Bbbk := F_\Bbbk$ are injective. Since $\varphi \colon E \to F$ is injective, we have a short exact sequence of R-complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \tag{1}$$

taking homology gives us a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}_{i+1}(F/E) \arrow[dll, rounded corners] \\
\mathrm{H}_i(E) \arrow[r] & \mathrm{H}_i(F) \arrow[r] & \mathrm{H}_i(F/E) \arrow[dll, rounded corners] \\
\mathrm{H}_{i-1}(E) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are resolutions we conclude that $\mathrm{H}_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $\mathrm{H}_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R-modules (1) with $\Bbbk$ over R gives us the long exact sequence in homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{i+1}(E, \Bbbk) \arrow[dll, rounded corners] \\
\mathrm{Tor}^R_i(E, \Bbbk) \arrow[r] & \mathrm{Tor}^R_i(F, \Bbbk) \arrow[r] & \mathrm{Tor}^R_i(F/E, \Bbbk) \arrow[dll, rounded corners] \\
\mathrm{Tor}^R_{i-1}(E, \Bbbk) \arrow[r] & \cdots &
\end{tikzcd}$$

Since E and F are free R-modules we conclude that $\mathrm{Tor}_i(F/E, \Bbbk) = 0$ for all $i \geq 1$. Since $\overline{\varphi} \colon E \otimes_R \Bbbk \to F \otimes_R \Bbbk$ is injective we conclude that $\mathrm{Tor}_1(F/E, \Bbbk) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $\mathrm{d}(F/E) \subseteq \mathfrak{m}(F/E)$). □

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R-module M connecting maps

$$\mathrm{Tor}^R_{i+1}(R/I, M) \to \mathrm{Tor}^R_i(I/J, M) \quad \text{and} \quad \mathrm{Ext}^i_R(I/J, M) \to \mathrm{Ext}^{i+1}_R(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \to F/E \otimes_R M \quad \text{and} \quad \mathrm{Hom}^\star_R(F/E, M) \to \mathrm{Hom}^\star_R(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\overline{\varphi} \colon E_\Bbbk \to F_\Bbbk$ is injective, then already $\varphi \colon E \to F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of ideals

Let R be a commutative ring and let $\mathfrak{p}$ be an ideal of R. Recall the **height** of $\mathfrak{p}$ is defined to be the supremum of lengths of chains of primes which descend from $\mathfrak{p}$:

$$\mathrm{ht}\,\mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

When R is Noetherian, then Krull's prinicipal ideal theorem states that there exists an ideal $\langle \boldsymbol{x} \rangle = \langle x_1, \ldots, x_c \rangle \subseteq \mathfrak{p}$ where $c = \mathrm{ht}\,\mathfrak{p}$ such that $\sqrt{\langle \boldsymbol{x} \rangle} = \mathfrak{p}$, and that if $\langle \boldsymbol{y} \rangle = \langle y_1, \ldots, y_m \rangle$ is another ideal such that $\sqrt{\langle \boldsymbol{y} \rangle} = \mathfrak{p}$, then we must have $c \leq m$. If I is an ideal of R, then the **height** of I is defined to be the infimum of the heights of all primes which contain I:

$$\operatorname{ht} I = \inf\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. *Let I_1 and I_2 be ideals of R. Set $c = \operatorname{ht}(I_1 \cap I_2)$, set $c_1 = \operatorname{ht} I_1$, and set $c_2 = \operatorname{ht} I_2$.*

1. *If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.*
2. *We have $c = \min\{c_1, c_2\}$.*

Proof. 1. Let $\mathfrak{p}$ be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2. Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let $\mathfrak{p}$ be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since $\mathfrak{p}$ is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. $\square$

2 2024

1/20/2024 - $\mathrm{V}(\operatorname{Ann} M) = \mathrm{V}(\operatorname{Ann}(0 :_M x))$

Lemma 2.1. *Let R be a commutative ring, let M be an R-module, and let $x \in R$. Then*

$$\mathrm{V}(\operatorname{Ann}(0 :_M x)) = \mathrm{V}(\operatorname{Ann}(0 :_M x^2)).$$

Proof. Note that $0 :_M x \subseteq 0 :_M x^2$ implies $\operatorname{Ann}(0 :_M x^2) \supseteq \operatorname{Ann}(0 :_M x)$ which implies $\mathrm{V}(\operatorname{Ann}(0 :_M x^2)) \subseteq \mathrm{V}(\operatorname{Ann}(0 :_M x))$. For the reverse inclusion, suppose $\mathfrak{p}$ is a prime ideal of R which contains $\operatorname{Ann}(0 :_M x^2)$ and let $r \in \operatorname{Ann}(0 :_M x)$. We claim that $r^2 \in \operatorname{Ann}(0 :_M x^2)$. Indeed, if $u \in 0 :_M x^2$, then

$$\begin{aligned} x^2 u = 0 &\implies xu \in 0 :_M x \\ &\implies rxu = 0 \\ &\implies ru \in 0 :_M x \\ &\implies r^2 u = 0. \end{aligned}$$

Since u was arbitrary, we see that $r^2 \in \operatorname{Ann}(0 :_M x^2) \subseteq \mathfrak{p}$. However this implies $r \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime. Since r was arbitary, we see that $\operatorname{Ann}(0 :_M x) \subseteq \mathfrak{p}$. $\square$

Corollary 1. *Let R be a commutative ring and let M be a finitely generated R-module. Assume that $x \in R$ acts nilpotently on M. Then*

$$\mathrm{V}(\operatorname{Ann}(M)) = \mathrm{V}(\operatorname{Ann}(0 :_M x)).$$

Proof. Since M is finitely generated, there exists an $n \in \mathbb{N}$ such that $M = 0 :_M x^n$. A straightforward induction on (**??**) gives us

$$\mathrm{V}(\operatorname{Ann}(M)) = \mathrm{V}(\operatorname{Ann}(0 :_M x^n)) = \mathrm{V}(\operatorname{Ann}(0 :_M x)).$$

$\square$

1/21/2024 - Some subschemes of $\mathbb{P}^3$

Let $R = \Bbbk[x, y, z, w]$. We consider three cyclic R-algebras, namely $A = R/\boldsymbol{f} = R/\langle f_1, f_2, f_3\rangle$, $B = R/\boldsymbol{g} = R/\langle g_1, g_2, g_3\rangle$, and $C = R/\boldsymbol{h} = R/\langle h_1, h_2, h_3\rangle$ where

$$\begin{aligned} f_1 &= xy - zw & \qquad g_1 &= xz - y^2 & \qquad h_1 &= xz - y^2 \\ f_2 &= xz - yw & g_2 &= yw - z^2 & h_2 &= x^3 - yzw \\ f_3 &= xw - yz & g_3 &= xw - yz & h_3 &= x^2 y - z^2 w \end{aligned}$$

We want a geometric picture in mind when thinking of these rings, so let $X = \operatorname{Proj} A$, $Y = \operatorname{Proj} B$, and $Z = \operatorname{Proj} C$. First let us consider X. We can see that $X(\Bbbk)$ consists of 8 distinct points in $\mathbb{P}^3(\Bbbk)$ by calculating an irreducible primary decomposition for $\langle \boldsymbol{f}\rangle$. Indeed, an irredundant primary decomposition for $\langle \boldsymbol{f}\rangle$ is given by $\langle \boldsymbol{f}\rangle = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_8$ where

$$\begin{aligned} \mathfrak{p}_1 &= \langle y, z, w\rangle & \qquad \mathfrak{p}_5 &= \langle x + y, y + z, z + w\rangle \\ \mathfrak{p}_2 &= \langle x, z, w\rangle & \mathfrak{p}_6 &= \langle x + y, y - z, z + w\rangle \\ \mathfrak{p}_3 &= \langle x, y, w\rangle & \mathfrak{p}_7 &= \langle x + y, y - z, z - w\rangle \\ \mathfrak{p}_4 &= \langle x, y, z\rangle & \mathfrak{p}_8 &= \langle x - y, y - z, z - w\rangle. \end{aligned}$$

These primes correspond to the points

$$\begin{aligned} \boldsymbol{p}_1 &= [1:0:0:0] & \qquad \boldsymbol{p}_5 &= [-1:1:-1:1] \\ \boldsymbol{p}_2 &= [0:1:0:0] & \boldsymbol{p}_6 &= [1:-1:-1:1] \\ \boldsymbol{p}_3 &= [0:0:1:0] & \boldsymbol{p}_7 &= [-1:1:1:1] \\ \boldsymbol{p}_4 &= [0:0:0:1] & \boldsymbol{p}_8 &= [1:1:1:1] \end{aligned}$$

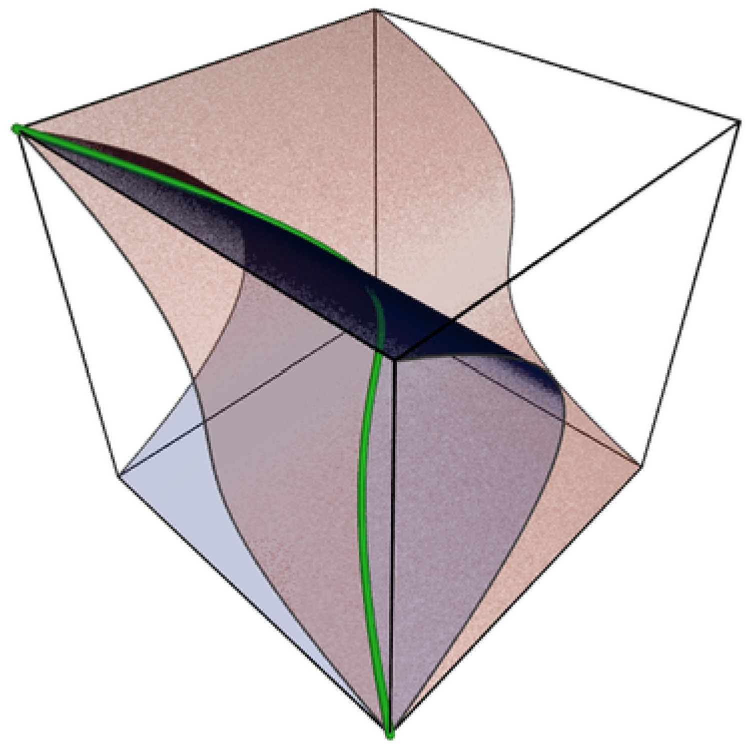
in $\mathbb{P}^3(\Bbbk)$. Note that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ are in linearly general position since the size 4 minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

are all nonzero. In other words, viewing $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8$ as vectors in $\Bbbk^4$, every subset of $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_8\}$ of size 4 is linearly independent. The Betti diagram of A over R is given by

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	-	3	-
2	-	-	-	1

Next we consider Y. In fact, Y is the twisted cubic. When $\Bbbk = \mathbb{R}$, we can visualize $Y(\Bbbk)$ as below:



In particular, $Y(\Bbbk)$ is the image of the map $\mathbb{P}^1(\Bbbk) \to \mathbb{P}^3(\Bbbk)$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Note that $\langle \boldsymbol{g} \rangle$ is a prime of height 2 and so $\langle \boldsymbol{g} \rangle$ can be generated up to radical by two homogeneous polynomials. In particular, we have $\langle \boldsymbol{g} \rangle = \sqrt{\langle g_1, g_4 \rangle}$ where $g_4 = zg_2 - wg_3$. However $\langle \boldsymbol{g} \rangle$ itself cannot be generated by only two polynomials; a minimum of three polynomials are needed. We can see this in Betti diagram of B over R:

	0	1	2
0	1	-	-
1	-	3	2

In particular, the Hilbert-Poincare series of B over R is given by

$$P(t) = \frac{1 - 3t^2 + 2t^3}{(1-t)^4} = \frac{1+2t}{(1-t)^2} = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + \cdots .$$

Thus Y is the set-theoretic complete intersection of $\mathrm{V}(g_1)$ and $\mathrm{V}(g_4)$ however it is not a scheme-theoretic or ideal-theoretic complete intersection. Note also that $\langle \boldsymbol{g} \rangle$ corresponds to the ideal of size 2 minors of the matrix $\left(\begin{smallmatrix} x & y & z \\ y & z & w \end{smallmatrix}\right)$. Up to linear automorphism, the twisted cubic is the only irreducible curve of degree 3 not contained in a plane. Furthermore, any 6 points in linearly general position in $\mathbb{P}^3(\Bbbk)$ lie on a unique twisted cubic. However for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Consequently one can show that if W is a set of 7 points in linearly general position in $\mathbb{P}^3(\Bbbk)$, then there are only two distinct Betti diagrams possible for the homogeneous coordinate ring of W, namely

	0	1	2	3
0	1	-	-	-
1	-	3	-	-
2	-	1	6	3

and

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

In the first case, the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the quadrics containing W is the ideal of the unique curve of degree 3 containing W, which is irreducible. Finally, let us write down the minimal free resolution of B over R:

$$R(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} R \longrightarrow 0$$

Now we consider Z. The Betti diagram of C over R is given by

	0	1	2
0	1	-	-
1	-	1	-
2	-	2	2

In particular, the Hilbert-Poincare series of C over R is given by

$$P(t) = \frac{1 - t^2 - 2t^3 + 2t^4}{(1-t)^4} = \frac{1 + 2t + 2t^2}{(1-t)^2} = 1 + 4t + 9t^2 + 14t^3 + 19t^4 + \cdots .$$

In particular, Z is an irreducible curve of degree 5 in $\mathbb{P}^3(\Bbbk)$.

2.1 4/22/2024 - Lifting multiplication to a free module

Let A be a commutative ring and let B be a finite A-algebra. Then there exists a surjection $F \twoheadrightarrow B$ of A-modules where $F = A^{n+1}$ and where we assume $n \geq 0$ is minimal. We are interested in the question as to whether one can lift the associative and unital multiplication on B to an associative and unital multiplication on F. Let K be the kernel of the map $F \twoheadrightarrow B$. In what follows, all tensors products are taken over A.

Lemma 2.2. *The kernel of the map $F^{\otimes 2} \to B^{\otimes 2}$ is given by $K \otimes F + F \otimes K$.*

Proof. This is easily checked via a diagram chase in the diagram below which is exact everywhere and in all directions:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & K^{\otimes 2} \arrow[r] \arrow[d] & K \otimes F \arrow[r] \arrow[d] & K \otimes B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \otimes K \arrow[r] \arrow[d] & F^{\otimes 2} \arrow[r] \arrow[d] & F \otimes B \arrow[r] \arrow[d] & 0 \\
 & B \otimes K \arrow[r] \arrow[d] & B \otimes F \arrow[r] \arrow[d] & B^{\otimes 2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Since $F^{\otimes 2}$ is free (hence projective), we can lift the composite map $F^{\otimes 2} \to B^{\otimes 2} \twoheadrightarrow B$ with respect to the map $F \twoheadrightarrow B$ to obtain an A-linear map $\mu \colon F^{\otimes 2} \to F$. Assume that A is a local noetherian ring. In this case, there exists a minimal generating set of B as an A-module of the form $\{b_0, b_1, \ldots, b_n\}$ where $b_0 = 1$. Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ be a basis for F as a free A-module and let $F \twoheadrightarrow B$ be the A-linear map defined by $\varepsilon_i \mapsto b_i$ for all i. For each i, j, we have

$$b_i b_j = \sum_k a_{ij}^k b_k$$

where the $a_{ij}^k \in A$ need not be unique. Since the multiplication on B is unital, we can choose the a_{ij}^k such that

$$a_{j0}^k = a_{0j}^k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Furthermore, since the multiplication on B is commutative, we can also choose the a_{ij}^k such $a_{ij}^k = a_{ji}^k$. With these choices of a_{ij}^k in mind, we can define a commutative and unital multiplication μ on F which lifts the multiplication on B by

$$\varepsilon_i \varepsilon_j := \sum_k a_{ij}^k \varepsilon_k.$$

Note that this multiplication need not be associative. Indeed, since the multiplication on B is associative, we have

$$\begin{aligned} 0 &= [b_i, b_j, b_k] \\ &= (b_i b_j) b_k - b_i (b_j b_k) \\ &= \sum_l (a_{ij}^l b_l b_k - a_{jk}^l b_i b_l) \\ &= \sum_{l,m} (a_{ij}^l a_{lk}^m - a_{jk}^l a_{il}^m) b_m. \end{aligned}$$

However this need not imply that $\sum_l a_{ij}^l a_{lk}^m - a_{jk}^l a_{ik}^m = 0$ for all i, j, k, m (which is what we'd need in order for $[\varepsilon_i, \varepsilon_j, \varepsilon_k] = 0$).

Proposition 2.1. *Let $A = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_m]$, let $B = A[\boldsymbol{e}] = A[e_1, \dots, e_n]$, and let $F = A + Ae_1 + \cdots + Ae_n$. Equip F with an A-linear commutative multiplication $\star$ such that 1 is the identity element and such that*

$$e_i \star e_j = \sum_{k=1}^{n} a_{ij}^k e_k$$

for all $1 \leq i, j \leq n$ where $a_{ij}^k \in A$. Set J to be the ideal of B generated by the polynomials $f_{ij} = e_i e_j - e_i \star e_j$ for all $1 \leq i \leq j \leq n$, and set $I = J \cap A$. Then the composite $F \hookrightarrow B \twoheadrightarrow B/J$ induces an isomorphism of A-algebras

$$F^{\mathrm{as}} \simeq B/J$$

where F^{as} is the maximal associative quotient of F. In particular, the multiplication $\star$ is associative if and only if B/J is a finite free A-module of rank $n+1$.

Proof. The map $\theta \colon F \to B/J$ factors as $\overline{\theta} \colon F^{\mathrm{as}}$ is clearly onto since B/J is generated by □

Example 2.1. Let $A = \Bbbk[x_1, x_2]$, let $B = A[e_1, e_2]$, let $F = A + Ae_1 + Ae_2$, let $\star$ be the A-linear commutative multiplication on F such that 1 is the identity element and such that

$$\begin{aligned} e_1 \star e_1 &= x_1 e_1 \\ e_1 \star e_2 &= x_2 e_1 + x_1 e_2 \\ e_2 \star e_2 &= x_2 e_2. \end{aligned}$$

and letting 1 be the identity element. We can determine whether or not this multiplication is associative in the following way: let $B = \Bbbk[x_1, x_2, e_1, e_2]$ and let J be the ideal of B given by $J = \langle f_{11}, f_{12}, f_{22} \rangle$ where

$$\begin{aligned} f_{11} &= e_1^2 - x_1 e_1 \\ f_{12} &= e_1 e_2 - x_2 e_1 - x_1 e_2 \\ f_{22} &= e_2^2 - x_2 e_2. \end{aligned}$$

Then the multiplication $\star$ is associative if and only if $J \cap A = 0$, in which case the map $F \hookrightarrow B \twoheadrightarrow B/J$ is an isomorphism of A-algebras.
be the ideal in
where 1 $B = A[e_1, e_2]/J$ where

$$J = \langle e_1^2 - x_1 e_1, e_2^2 - x_2 e_2, e_1 e_2 - x_2 e_1 - x_1 e_2, x_1 e_1 + x_2 e_2 - 1 \rangle.$$

Then B is a finite A-algebra with a minimal generating set of B as an A-module given by $\{\overline{e}_1, \overline{e}_2\}$. Furthermore, any minimal generating set of B as an A-module cannot contain 1. Now let $F_0 = A\varepsilon_1 \oplus A\varepsilon_2$ and consider the surjective A-module homomorphism $F_0 \twoheadrightarrow B$ given by $\varepsilon_i \mapsto e_i$. We can lift the multiplication on B to a multiplication on F_0 by setting $\varepsilon_1 \varepsilon_2 = x_1 \varepsilon_2 + x_2 \varepsilon_1$ and $\varepsilon_i^2 = x_i \varepsilon_i$ for $i = 1, 2$. However there is no identity element in F_0 with respect to this multiplication.

One has

$$\mathrm{J} = \mathrm{J}_{\boldsymbol{f}} = \begin{pmatrix} 2e_1 - a_{11}^1 & e_2 - a_{12}^1 & -a_{22}^1 \\ -a_{11}^2 & e_1 - a_{12}^2 & 2e_2 - a_{22}^2 \end{pmatrix}.$$

Let Δ_i be the size 2-minor of J corresponding to the 2×2 matrix obtained by removing the ith column. Then we have

$$\begin{aligned} \Delta_1 &= 2e_2^2 - (2a_{12}^1 + a_{22}^2)e_2 + a_{22}^1 e_1 + (a_{12}^1 a_{22}^2 - a_{12}^2 a_{22}^1) \\ \Delta_2 &= 4e_1 e_2 - 2a_{22}^2 e_1 - 2a_{11}^1 e_2 + (a_{11}^1 a_{22}^2 - a_{11}^2 a_{22}^1) \\ \Delta_3 &= 2e_1^2 - (2a_{12}^2 + a_{11}^1)e_1 + a_{11}^2 e_2 + (a_{11}^1 a_{12}^2 - a_{11}^2 a_{12}^1) \end{aligned}$$

Where we set $\delta_{112} =$

$$\begin{aligned} \sigma_{2,2}^{1,2} &= a_{12}^1 a_{12}^2 - a_{11}^2 a_{22}^1 \\ &= a_{12}^1 a_{12}^1 - a_{11}^1 a_{22}^1 \\ &= a_{12}^2 a_{12}^2 - a_{11}^2 a_{22}^2 \\ &= a_{11}^1 a_{12}^2 \end{aligned}$$

To ease notation, let us set $u_1 = a^1_{11}$, $u_2 = a^1_{12}$, $u_3 = a^1_{23}$, $v_1 = a^2_{11}$, $v_2 = a^2_{12}$, and $v_3 = a^2_{23}$, Furthermore, we set

$$\begin{aligned} \delta_{12} &= u_1v_2 - u_2v_1 & \sigma &= u_2^2 - u_1u_3 \\ \delta_{13} &= u_1v_3 - u_3v_1 & \tau &= v_2^2 - v_1v_3 \\ \delta_{23} &= u_2v_3 - u_3v_2 & \delta_{22} &= u_2v_2 - u_3v_1 \end{aligned}$$

We observe that if $1 \leq i < j \leq 3$, then δ_{ij} is the size-2 minor of the Jacobian matrix J corresponding to the 2×2 matrix whose columns are the ith and jth column of J. On the other hand, we have

$$[e_1, e_1, e_2] = -\delta_{22}e_1 + (\delta_{12} - \tau)e_2 \quad \text{and} \quad [e_1, e_2, e_2] = (\sigma - \delta_{23})e_1 + \delta_{22}e_2.$$

Note that since our multiplication is commutative, we already have

$$[e_1, e_2, e_1] = [e_1, e_1, e_1] = [e_2, e_1, e_2] = [e_2, e_2, e_2] = 0.$$

Similarly, we also have

$$[e_2, e_1, e_1] = -[e_1, e_1, e_2] \quad \text{and} \quad [e_2, e_2, e_1] = -[e_1, e_2, e_2].$$

Therefore our multiplication is associative if and only if

$$\begin{aligned} u_2v_2 &= u_3v_1 \\ u_1v_2 &= u_2v_1 - v_1v_3 + v_2^2 \\ u_2v_3 &= u_3v_2 - u_1u_3 + u_2^2 \end{aligned}$$

Now consider the ring

$$E = A[\boldsymbol{u}, \boldsymbol{v}]/P = A[u_1, u_2, u_3, v_1, v_2, v_3]/\langle \delta_{12} - \tau, \delta_{22}, \delta_{13} - \sigma \rangle.$$

when $A = \Bbbk$ is a field then P is prime and E has minimal free resolution of the form $\Bbbk[\boldsymbol{u}, \boldsymbol{v}]^2 \to \Bbbk[\boldsymbol{u}, \boldsymbol{v}]^3$. Next let $E' = A[\boldsymbol{u}, \boldsymbol{v}]/Q = E/\langle \delta_{12}, \delta_{13}, \delta_{23} \rangle$. When $A = \Bbbk$ is a field then Q is prime and E' has minimal free resolution of the form $\Bbbk[\boldsymbol{u}, \boldsymbol{v}]^3 \to \Bbbk[\boldsymbol{u}, \boldsymbol{v}]^8 \to \Bbbk[\boldsymbol{u}, \boldsymbol{v}]^6$

$$\begin{aligned} e_1 \star e_1 &= \alpha e_1 \\ e_1 \star e_2 &= \alpha e_2 \\ e_2 \star e_2 &= \beta e_1 + \gamma e_2. \end{aligned}$$

$$\begin{aligned} e_1 \star e_1 &= \alpha e_1 \\ e_1 \star e_2 &= 0 \\ e_2 \star e_2 &= \beta e_2. \end{aligned}$$

2.1.1 Case $n = 3$

We set $\delta^{l,m}_{i,j,k} = \delta^{lm}_{ijk} = a^l_{ij}a^m_{lk} - a^m_{il}a^l_{jk}$. Note that

$$\begin{aligned} \delta^{lm}_{ijk} &= a^l_{ij}a^m_{lk} - a^m_{il}a^l_{jk} \\ &= -(a^l_{kj}a^m_{li} - a^m_{kl}a^l_{ji}) \\ &= -\delta^{lm}_{kji}. \end{aligned}$$

Similarly we have $\delta^{ll}_{ilk} = 0$. Furthermore, we have

$$\delta^{lm}_{ijk} - \delta^{ln}_{ijk} = a^l_{ij}a^{mn}_{lk} + a^{nm}_{il}a^l_{jk}$$

where we set $a^{mn}_{lk} = a^m_{lk} - a^n_{lk}$ and $a^{nm}_{il} = a^n_{il} - a^m_{il}$. Thus

$$\begin{aligned} [e_1, e_2, e_3] = {} & ((a^1_{12}a^1_{13} - a^1_{11}a^1_{23}) + (a^2_{12}a^1_{23} - a^1_{12}a^2_{23}) + (a^3_{12}a^1_{33} - a^1_{13}a^3_{23}))e_1 \\ & + ((a^1_{12}a^2_{13} - a^2_{11}a^1_{23}) + (a^3_{12}a^2_{33} - a^2_{13}a^3_{23}))e_2 \\ & + ((a^1_{12}a^3_{13} - a^3_{11}a^1_{23}) + (a^2_{12}a^3_{23} - a^3_{12}a^2_{23}) + (a^3_{12}a^3_{33} - a^3_{13}a^3_{23}))e_3 \end{aligned}$$

or in other words

$$[e_1, e_2, e_3] = (\delta^{11}_{123} + \delta^{21}_{123} + \delta^{31}_{123})e_1 + (\delta^{12}_{123} + \delta^{32}_{123})e_2 + (\delta^{13}_{123} + \delta^{23}_{123} + \delta^{33}_{123})e_3$$

Thus

$$\begin{aligned} [e_1, e_2, e_3] = {} & (a^1_{12}(a^1_{13} - a^2_{23}) + a^1_{23}(a^2_{12} - a^1_{11}) + a^3_{12}a^1_{33} - a^1_{13}a^3_{23})e_1 \\ & + (a^2_{13}(a^1_{12} - a^3_{23}) + a^3_{12}a^2_{33} - a^2_{11}a^1_{23}))e_2 \\ & + (a^3_{12}(a^3_{33} - a^3_{12}) + a^3_{23}(a^2_{12} - a^3_{13}) + a^1_{12}a^3_{13} - a^3_{11}a^1_{23})e_3 \end{aligned}$$

or in other words

$$[e_1, e_2, e_3] = (\delta_{123}^{11} + \delta_{123}^{21} + \delta_{123}^{31})e_1 + (\delta_{123}^{12} + \delta_{123}^{32})e_2 + (\delta_{123}^{13} + \delta_{123}^{23} + \delta_{123}^{33})e_3$$

Thus

$$\begin{aligned}[e_1, e_2, e_2] = {} & ((a_{12}^1 a_{12}^1 - a_{11}^1 a_{22}^1) + (a_{12}^2 a_{22}^1 - a_{12}^1 a_{22}^2) + (a_{12}^3 a_{23}^1 - a_{13}^1 a_{22}^3))e_1 \\ & + ((a_{12}^1 a_{12}^2 - a_{11}^2 a_{22}^1) + (a_{12}^3 a_{23}^2 - a_{13}^2 a_{22}^3))e_2 \\ & + ((a_{12}^1 a_{12}^3 - a_{11}^3 a_{22}^1) + (a_{12}^2 a_{22}^3 - a_{12}^3 a_{22}^2) + (a_{12}^3 a_{23}^3 - a_{13}^3 a_{22}^3))e_3\end{aligned}$$

or in other words

$$[e_1, e_2, e_3] = (\delta_{123}^{11} + \delta_{123}^{21} + \delta_{123}^{31})e_1 + (\delta_{123}^{12} + \delta_{123}^{32})e_2 + (\delta_{123}^{13} + \delta_{123}^{23} + \delta_{123}^{33})e_3$$

2.1.2 New

Let $\gamma\colon F_+ \to F_+$ be an $\Bbbk$-linear isomorphism. Then we have $\gamma\boldsymbol{e} = \boldsymbol{e}C$ where $C \in \mathrm{GL}_n(\Bbbk)$ is the matrix which represents γ with respect to the ordered basis $\boldsymbol{e} = (e_1, \dots, e_n)$ which we view as a row vector with entries $e_1, \dots, e_n$. Similarly, if $\mu\colon F_+^{\otimes 2} \to F_+$ is an R-linear multiplication, then we have $\mu(\boldsymbol{e}^{\otimes 2}) = \boldsymbol{e}M$ where $M \in \mathrm{M}_{n\times n^2}(\Bbbk)$ is the matrix which represents μ with respect to the ordered basis $\boldsymbol{e}^{\otimes 2}$ and $\boldsymbol{e}$ where $\boldsymbol{e}^{\otimes 2} = (e_1e_1, e_1e_2, \dots, e_ie_j, \dots, e_ne_n)$ is the Kronecker product of $\boldsymbol{e}$ with itself. Then the R-linear map $\mu_\gamma\colon F_+^{\otimes 2} \to F_+$ defined by $\mu_\gamma := \gamma^{-1}\mu\gamma^{\otimes 2}$ satisfies

$$\begin{aligned}\gamma^{-1}\mu\gamma^{\otimes 2}\boldsymbol{e}^{\otimes 2} &= \gamma^{-1}\mu(\gamma\boldsymbol{e})^{\otimes 2} \\ &= \gamma^{-1}\mu(\boldsymbol{e}C)^{\otimes 2} \\ &= \gamma^{-1}\mu\boldsymbol{e}C^{\otimes 2} \\ &= \gamma^{-1}\boldsymbol{e}MC^{\otimes 2} \\ &= \boldsymbol{e}C^{-1}MC^{\otimes 2}.\end{aligned}$$

It follows that the matrix representation of μ_γ with respect to the ordered bases $\boldsymbol{e}\otimes\boldsymbol{e}$ and $\boldsymbol{e}$ is given by $C^{-1}MC^{\otimes 2}$. Note that if $\mu(\sigma - 1)$ is the commutator of μ where $\sigma\colon F_+^{\otimes 2} \to F_+$ is given by $\sigma(a \otimes b) = b \otimes a$, then the commutator for μ_γ is given by

$$\begin{aligned}\mu_\gamma(\sigma - 1) &= \gamma^{-1}\mu\gamma^{\otimes 2}\sigma - \gamma^{-1}\mu\gamma^{\otimes 2} \\ &= \gamma^{-1}\mu\sigma\gamma^{\otimes 2} - \gamma^{-1}\mu\gamma^{\otimes 2} \\ &= \gamma^{-1}\mu(\sigma - 1)\gamma^{\otimes 2}.\end{aligned}$$

In particular, μ is commutative if and only if μ_γ is commutative. Similarly, if $[\cdot]_\mu = \mu(\mu \otimes 1 - 1 \otimes \mu)$ is the associator for μ, then

$$\begin{aligned}[\cdot]_{\mu_\gamma} &= \mu_\gamma(\mu_\gamma \otimes 1 - 1 \otimes \mu_\gamma) \\ &= \gamma^{-1}\mu\gamma^{\otimes 2}(\gamma^{-1}\mu\gamma^{\otimes 2} \otimes 1 - 1 \otimes \gamma^{-1}\mu\gamma^{\otimes 2}) \\ &= \gamma^{-1}\mu(\mu\gamma^{\otimes 2} \otimes \gamma - \gamma \otimes \mu\gamma^{\otimes 2}) \\ &= \gamma^{-1}\mu(\mu \otimes 1 - 1 \otimes \mu)\gamma^{\otimes 3} \\ &= \gamma^{-1}[\cdot]_\mu\gamma^{\otimes 3}\end{aligned}$$

is the associator of μ_γ. Thus μ is associative if and only if μ_γ is associative. Now, we want to classify isomorphism classes of $\Bbbk$-algebra structures on F preserving a fixed smooth base point. By the discussion above, we see that this problem is equivalent to classifying all commutative associative $\Bbbk$-linear maps $\mu\colon F_+^{\otimes 2} \to F_+$ where μ and μ' are equivalent if $\mu' = \mu_\gamma = \gamma^{-1}\mu\gamma^{\otimes 2}$ for some $\Bbbk$-linear automorphism $\gamma\colon F_+ \to F_+$. On a side note, observe that if $e_{12}(\lambda) = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, then we have

$$e_{12}(\lambda)^{\otimes 2} = \begin{pmatrix} 1 & \lambda & \lambda & \lambda^2 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} = e_{34}(\lambda)e_{24}(\lambda)e_{14}(\lambda^2)e_{13}(\lambda)e_{12}(\lambda).$$

It follows that

$$\begin{aligned}e_{12}(-\lambda)Me_{12}(\lambda)^{\otimes 2} &= e_{12}(-\lambda)\begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{21}^1 & a_{22}^1 \\ a_{11}^2 & a_{12}^2 & a_{21}^2 & a_{22}^2 \end{pmatrix} e_{34}(\lambda)e_{24}(\lambda)e_{14}(\lambda^2)e_{13}(\lambda)e_{12}(\lambda) \\ &= e_{12}(-\lambda)\begin{pmatrix} a_{11}^1 & \lambda a_{11}^1 + a_{12}^1 & \lambda a_{11}^1 + a_{21}^1 & \lambda^2 a_{11}^1 + \lambda a_{12}^1 + \lambda a_{21}^1 + a_{22}^1 \\ a_{11}^2 & \lambda a_{11}^2 + a_{12}^2 & \lambda a_{12}^2 + a_{21}^2 & \lambda^2 a_{11}^2 + \lambda a_{12}^2 + \lambda a_{21}^2 + a_{22}^2 \end{pmatrix} \\ &= \left(\begin{matrix} a_{11}^1 - \lambda a_{11}^2 & a_{12}^1 + \lambda(a_{11}^1 - a_{12}^2) - \lambda^2 a_{11}^2 & \lambda a_{11}^1 + a_{21}^1 - \lambda^2 a_{12}^2 - \lambda a_{21}^2 & -\lambda^3 a_{11}^2 + \lambda^2 a_{11}^1 - \lambda^2 a_{12}^2 + \lambda a_{12}^1 - \lambda^2 a_{21}^2 + \lambda a_{21}^1 - \lambda a^2 \\ a_{11}^2 & a_{12}^2 + \lambda a_{11}^2 & (1+\lambda)a_{12}^2 & \lambda^2 a_{11}^2 + 2\lambda a_{12}^2 + a_{22}^2 \end{matrix}\right.\end{aligned}$$

Actually, let's think of it this way:

$$\begin{aligned}C^{-1}MC^{\otimes 2} &= C^{-1}(M_1,\dots,M_n)\begin{pmatrix} c_{11}C & \cdots & c_{1n}C \\ \vdots & \ddots & \vdots \\ c_{n1}C & \cdots & c_{nn}C\end{pmatrix} \\ &= C^{-1}((c_{11}M_1+\cdots c_{n1}M_n)C,\dots,(c_{1n}M_1+\cdots c_{nn}M_1)C),\end{aligned}$$

Now the fundamental fact is that

$$= -\mathrm{J}_{\boldsymbol{f}}(0) = M$$

and this is smooth at the origin if and only if M has full rank if and only if one of the maximal minors of M is nonzero. Now we have

$$\begin{aligned}C^{-1}MC^{\otimes 2} &= C^{-1}(M_1,\dots,M_n)\begin{pmatrix} c_{11}C & \cdots & c_{1n}C \\ \vdots & \ddots & \vdots \\ c_{n1}C & \cdots & c_{nn}C\end{pmatrix} \\ &= (C^{-1}(c_{11}M_1+\cdots c_{n1}M_n)C,\dots,C^{-1}(c_{1n}M_1+\cdots c_{nn}M_1)C),\end{aligned}$$

We must have $e_1 \star e_1 \neq 0 \neq e_2 \star e_2$ otherwise associativity forces non-smoothness. So we have

$$e_{21}(-\lambda)Me_{21}(\lambda)^{\otimes 2} = (e_{21}(-\lambda)(M_1+\lambda M_2)e_{21}(\lambda), e_{21}(-\lambda)M_2e_{21}(\lambda))$$

We have $d_\lambda^{\otimes 2} = d_{\lambda^2}$ Thus $d_{1/\lambda}Md_\lambda^{\otimes 2} = \lambda M$.

2.2 5/2/2024 - Colon ideal result

Let R be a noetherian ring, let I be an ideal of R, and let $r, r' \in R$. To simplify notation in what follows, we remove the bracket symbol $\langle\cdot\rangle$ when context is clear. For example, we write $I : r, r'$ to mean $\langle I : r, r'\rangle$. With this notation in mind, observe that we have an R-linear map

$$I, r : r' \twoheadrightarrow (I, r' : r)/(I : r)$$

defined as follows: if $a \in I, r : r'$, then we have $ar' = br + x$ for some $b \in R$ and $x \in I$. The map is defined by sending a to the class of b in the quotient. It is straightforward to check that this is well-defined and surjective. Note if $b \in I : r$, then $ar' \in I : r'$. In particular, the kernel of φ is $I : r'$. Thus we've established an isomorphism

$$(I, r : r')/(I : r') \cong (I, r' : r)/(I : r). \tag{2}$$

In particular, if $I : r' = I : r$, then we must have $I, r : r' = I, r' : r$.

Example 2.2. Let $R = \Bbbk[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, yz\rangle$, let $r = yw$, and let $r' = y$. Then we have

$$\begin{aligned} I : r &= \langle x, z, w\rangle & \qquad I, r' : r &= R \\ I : r' &= \langle x, z, w^2\rangle & \qquad I, r : r' &= \langle x, z, w\rangle.\end{aligned}$$

Thus we have

$$\langle x, z, w\rangle/\langle x, z, w^2\rangle = I, r : r'/I : r' \cong I, r' : r/I : r = R/\langle x, z, w\rangle.$$

Next we observe that

$$I : r, r' \subseteq I, r' : r.$$

Indeed, if $a \in I : r, r'$, then we can express it as $a = b + cr'$ where $b \in I : r$ and $c \in R$. In particular, this means that $ar = br + cr'r \in I, r'$, and hence $a \in I, r' : r$. Conversely, assume that that $\overline{r}, \overline{r}'$ forms a regular sequence in R/I. Then we have

$$I : r, r' \supseteq I, r' : r.$$

Indeed, if $a \in I, r' : r$, then we have $ra = x + r'b$ where $b \in R$ and $x \in I$. Since $\overline{r}, \overline{r}'$ forms a regular sequence in R/I, this implies $a = y + r'c$ for some $c \in R$ and $y \in I$ which implies $a \in I, r' \subseteq I : r, r'$.

Thus if $\mathfrak{p} = I : r$, $\mathfrak{p}' = I : r'$, and r, r' forms a regular sequence modulo I, then we have

$$\langle \mathfrak{p}', r\rangle/\mathfrak{p}' \cong \langle \mathfrak{p}, r'\rangle/\mathfrak{p}.$$

In the case where $\mathfrak{p} = \mathfrak{p}'$, this says $\mathfrak{p}, r = \mathfrak{p}, r'$.

2.3 5/20/2024 - Geometric description of finitely generated $\Bbbk$-algebra homomorphisms

Let $\Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \dots, x_m]$, let $\Bbbk[\boldsymbol{y}] = \Bbbk[y_1, \dots, y_n]$, and let $\varphi \colon \Bbbk[\boldsymbol{x}] \to \Bbbk[\boldsymbol{y}]$ be a $\Bbbk$-algebra homomorphism. Then the φ corresponds to the morphism of affine schemes $f \colon \mathbb{A}^n_{\Bbbk} \to \mathbb{A}^m_{\Bbbk}$ given by $f(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in \mathbb{A}^n_{\Bbbk}$. We want to give a more geometric description of how f acts on the points of $\mathbb{A}^n_{\Bbbk}$, or in other words, how φ^{-1} acts on the prime ideals of $\Bbbk[\boldsymbol{y}]$. First, note that since $\Bbbk[\boldsymbol{y}]$ is Jacobson, we have

$$\varphi^{-1}(\mathfrak{q}) = \varphi^{-1}\left(\bigcap_{\substack{\mathfrak{n} \supseteq \mathfrak{q} \\ \mathfrak{n} \text{ maximal}}} \mathfrak{n} \right) = \bigcap_{\substack{\mathfrak{n} \supseteq \mathfrak{q} \\ \mathfrak{n} \text{ maximal}}} \varphi^{-1}(\mathfrak{n}).$$

Thus we will focus on the case where $\mathfrak{q} = \mathfrak{n}$ is a maximal ideal. First let's consider the maximal ideals of the form $\mathfrak{n}_{\boldsymbol{q}} = \langle y_1 - q_1, \dots, y_n - q_n \rangle$ where $\boldsymbol{q} \in \mathbb{A}^n_{\Bbbk}(\Bbbk) = \Bbbk^n$. To this end, for each $1 \le i \le m$ let $f_i = \varphi(x_i)$, and let $\boldsymbol{f} \colon \Bbbk^n \to \Bbbk^m$ be the map given by $\boldsymbol{f}(\boldsymbol{q}) = (f_1(\boldsymbol{q}), \dots, f_m(\boldsymbol{q}))$. Then we claim that

$$\varphi^{-1}(\mathfrak{n}_{\boldsymbol{q}}) = \mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} = \langle x_1 - f_1(\boldsymbol{q}), \dots, x_m - f_m(\boldsymbol{q}) \rangle.$$

Indeed, observe that

$$\begin{aligned} \varphi(\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})}) &= \langle \varphi(x_1) - f_1(\boldsymbol{q}), \dots, \varphi(x_m) - f_m(\boldsymbol{q}) \rangle \\ &= \langle f_1 - f_1(\boldsymbol{q}), \dots, f_m - f_m(\boldsymbol{q}) \rangle \\ &\subseteq \mathfrak{n}_{\boldsymbol{q}}. \end{aligned}$$

This shows that $\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} \subseteq \varphi^{-1}(\mathfrak{n}_{\boldsymbol{q}})$. We get the reverse inclusion from the fact that $\mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})}$ is a maximal ideal of A. More generally, let $\mathfrak{n}$ be an arbitrary maximal ideal of $\Bbbk[\boldsymbol{y}]$. Then there exists a maximal ideal of the form $\mathfrak{n}_{\boldsymbol{q}}$ of $\overline{\Bbbk}[\boldsymbol{y}]$, where $\boldsymbol{q} \in \overline{\Bbbk}^n$, which lies over $\mathfrak{n}$. Furthermore, there are only finitely many maximal ideals of $\overline{\Bbbk}[\boldsymbol{y}]$ which lie over $\mathfrak{n}$ and they all have the form $\mathfrak{n}_{\sigma\boldsymbol{q}}$ for some $\sigma \in \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ where $\sigma\boldsymbol{q} = (\sigma q_1, \dots, \sigma q_n)$ (this follows from a general proposition in commutative algebra which we state and proof at the end of this entry below). Then we have

$$\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}_{\boldsymbol{f}(\boldsymbol{q})} \cap \Bbbk[\boldsymbol{x}] := \mathfrak{m}.$$

Note this does not depend on the choice of maximal ideal which lies over $\mathfrak{n}$, for if $\mathfrak{n}_{\sigma\boldsymbol{q}}$ where another maximal ideal of $\overline{\Bbbk}[\boldsymbol{y}]$ which lies over $\mathfrak{n}$, then $\mathfrak{m}_{\boldsymbol{f}(\sigma\boldsymbol{q})} = \mathfrak{m}_{\sigma\boldsymbol{f}(\boldsymbol{q})}$ also lies over $\mathfrak{m}$.

Example 2.3. The maximal ideals $\mathfrak{n}_{i,\zeta_8}$, $\mathfrak{n}_{i,\zeta_8^5}$, $\mathfrak{n}_{-i,\zeta_8^3}$, and $\mathfrak{n}_{-i,\zeta_8^7}$ lie over $\mathfrak{n} = \langle y_1^2 + 1, y_2^2 + y_1 \rangle$.

Proposition 2.2. *Let A be an integral domain which is integrally closed in its field of fractions K, let L be a normal extension of K, let B be the integral closure of A in L, let G be the group of automorphisms of L over K, and let $\mathfrak{p}$ be a prime ideal of A. Then G acts transitively on the set of all primes of B which lie over $\mathfrak{p}$.*

Proof. We first consider the case where G is finite. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two prime ideals of B which lie over $\mathfrak{p}$. Then the $\sigma\mathfrak{q}$ (where $\sigma \in G$) is an ideal of B which lies over $\mathfrak{p}$ since B is integrally closed in L, and it suffices to show that $\mathfrak{q}'$ is contained in one of them, or equivalently, in their union by prime avoidance. Let $y \in \mathfrak{q}'$ and let $x = \prod \sigma y$ where the product runs over $\sigma \in G$. Note that x is fixed by G, thus since L/K is normal, it follows that there exists a power q of the characteristic of K such that $x^q \in K$. In particular, $x^q \in K \cap B = A$ since A is integrally closed. Thus $x^q \in \mathfrak{q}' \cap A = \mathfrak{p}$, which shows that x^q is contained in $\mathfrak{q}$. It follows that there exists a $\sigma \in G$ such that $\sigma y \in \mathfrak{q}$, whence $y \in \sigma^{-1}\mathfrak{q}$.

For the general case, assume $\mathfrak{q}$ and $\mathfrak{q}'$ lie over $\mathfrak{p}$. For every subfield E of L which is a finite normal extension over K, let G_E be the subset of G which consists of all $\sigma \in G$ which transform $\mathfrak{q} \cap E$ to $\mathfrak{q}' \cap E$. This is a closed subspace of G, hence compact since G is compact. Furthermore, each G_E is non-empty by what was shown above. As the G_E form a decreasing filtered family, their intersection is non-empty. $\square$

2.4 5/21/2024 - Turning $\mathrm{Tor}^R(M_1, M_2)$ into an R-complex

Let R be a commutative ring, let M_1 and M_2 be R-modules, and set $T = \mathrm{Tor}^R(M_1, M_2)$. We can turn T into an R-complex as follows: choose projective resolutions F^1 of M_1 and F^2 of M_2 over R. Then $\mathrm{d} \otimes 1 \colon F^1 \otimes_R F^2 \to F^1 \otimes_R F^2$ is a chain map of degree -1, thus it induces a map in homology $\mathrm{d} \otimes 1 \colon T \to T$. Furthermore $(\mathrm{d} \otimes 1)^2 = 0$ and so $\mathrm{d} \otimes 1$ gives T an R-complex structure. There are map $\gamma_i^{31} \colon T_i^{31} \to T_{i-1}^{31}$ defined to be the composite

$$T_i^{31} \to T_i^{32} \to T_{i-1}^{12} \to T_{i-1}^{13} = T_{i-1}^{31}.$$

Similarly, we define $\gamma_i^{32} \colon T_i^{32} \to T_{i-1}^{32}$ to be the composite

$$T_i^{32} \to T_{i-1}^{12} \to T_{i-1}^{13} \to T_{i-1}^{23} = T_{i-1}^{32},$$

and we define $\gamma_i^{21} \colon T_i^{21} \to T_{i-1}^{21}$ to be the composite

$$T_i^{21} \to T_i^{31} \to T_i^{32} \to T_{i-1}^{12} = T_{i-1}^{21}$$

Actually I just realized these are all just the zero map.

2.5 5/29/2024 - Ext result of my paper

Proposition 2.3. *Let R be a regular local ring, let I be an ideal of R, let F be the minimal free resolution of R/I over R, and let $S = \mathrm{S}_R(F)$ be the symmetric DG algebra of F over R. There exists a surjective chain map $\pi\colon S \twoheadrightarrow F$ which splits the inclusion map $F \hookrightarrow S$.*

Proof. It suffices to show that $\mathrm{Ext}^1_R(S/F, F) = 0$. Note that the underlying graded R-module of S/F is just $S^{\geq 2}$. In particular, S/F is semi-projective, thus $\mathrm{Hom}^\star_R(S/F, -)$ preserves quasi-isomorphisms. It follows that

$$\mathrm{Ext}^1_R(S/F, F) = \mathrm{Ext}^1_R(S/F, R/I) = 0,$$

where the last part follows from the fact that R/I sits in homological degree 0 but $(S/F)_i = 0$ for all $i \leq 1$. $\square$

Remark 3. Note that giving a surjective chain map $\pi\colon S \twoheadrightarrow F$ which splits the inclusion map is equivalent to giving chain maps $\pi^n\colon F^{\otimes n} \to F$ for each $n \geq 2$ such that each π^n is strictly commutative and such that for all $1 \leq i \leq n$ and for all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \in F_+$ we have

$$\pi^n(a_1, \dots, a_{i-1}, 1, a_i, \dots, a_n) = \pi^{n-1}(a_1, \dots, a_{i-1}, a_i, \dots, a_n).$$

For instance, if a_1, a_2, a_3 are homogeneous elements in F with $|a_1| = 1$ and $|a_2|, |a_3| \geq 2$, then we have

$$\mathrm{d}\pi^3(a_1, a_2, a_3) = r_1\pi^2(a_2, a_3) - \pi^3(a_1, \mathrm{d}a_2, a_3) + \pi^3(a_1, a_2, \mathrm{d}a_3),$$

where $r_1 = \mathrm{d}a_1$.

2.6 6/15/2024 - Associated primes of $\mathrm{Hom}_R(M, N)$

Today we prove the following result:

Proposition 2.4. *Let R be a noetherian ring and let M and N be R-modules such that M is finitely generated. Then*

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}\, M \cap \mathrm{Ass}\, N = \mathrm{V}(\mathrm{Ann}\, M) \cap \mathrm{Ass}\, N.$$

Proof. Let $\mathfrak{p}$ be an associated prime of $\mathrm{Hom}_R(M, N)$. Thus there exists an R-linear map $\varphi\colon M \to N$ such that $\mathfrak{p} = 0 : \varphi = \{a \in R \mid a\varphi = 0\}$. Let $u_1, \dots, u_m$ be generators of M as an R-module and let $v_1, \dots, v_m \in N$ be their respective images under φ. Then note that $a\varphi = 0$ if and only if $av_i = 0$ for all $1 \leq i \leq m$.

$$\begin{aligned} a \in \mathfrak{p} &\iff a\varphi = 0 \\ &\iff av_i = 0 \text{ for all } i \\ &\iff a \in \bigcap_{i=1}^m 0 : v_i. \end{aligned}$$

In particular we see that $\mathfrak{p} = \bigcap_{i=1}^m 0 : v_i$. Since $\mathfrak{p}$ is prime, we see that $\mathfrak{p} = 0 : v_i$ for some i, or in other words, $\mathfrak{p}$ is an associated prime of N. Next, assume for a contradiction that $M_\mathfrak{p} = 0$. Then for each i there exists an $s_i \in R\backslash\mathfrak{p}$ such that $s_i u_i = 0$. However this implies $s = s_1 \cdots s_n$ is in $\mathfrak{p}$ since $sv_i = \varphi(su_i) = 0$ for all i, which is a contradiction. Therefore $\mathfrak{p}$ is in the support of M. Thus far we have shown

$$\mathrm{Ass}(\mathrm{Hom}_R(M, N)) \subseteq \mathrm{Supp}\, M \cap \mathrm{Ass}\, N.$$

For the converse direction, suppose $\mathfrak{p}$ is in the support of M and is an associated prime of N, so $M_\mathfrak{p} \neq 0$ and $\mathfrak{p} = 0 : v$ for some $v \in N$. Since $M_\mathfrak{p} \neq 0$, there exists an i such that $0 : u_i \subseteq \mathfrak{p} = 0 : v$. By reordering if necessary, we may assume that $0 : u_1 \subseteq \mathfrak{p} = 0 : v$. One would like to define an R-linear map $\varphi\colon M \to N$ such that $\varphi(u_1) = v$, but it's not clear how we should define it on the u_i for all $2 \leq i \leq m$. Let us cut to the chase and show how one usually proves this result: we have

$$\begin{aligned} \mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N)) &\iff \mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_R(M, N)_\mathfrak{p}) \\ &\iff \mathfrak{p}_\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \\ &\iff \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \neq 0 \\ &\iff \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}, N_\mathfrak{p}) \neq 0 \\ &\iff M_\mathfrak{p} \neq 0 \text{ and } \mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), N_\mathfrak{p}) \neq 0 \\ &\iff \mathfrak{p} \in \mathrm{Supp}\, M \cap \mathrm{Ass}\, N, \end{aligned}$$

where in the second last if and only if we used the fact that $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is a finite dimensional $\kappa(\mathfrak{p})$ (so it is a direct sum of $\kappa(\mathfrak{p})$'s). Note that we needed Nakayama's lemma for the statement $M_\mathfrak{p} \neq 0$ if and only if $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0$, hence why we needed a noetherian hypothesis on R. The last equality comes from the fact that since M is finitely generated, we have $\mathrm{Supp}\, M = \mathrm{V}(\mathrm{Ann}\, M)$. $\square$

Corollary 2. *Let R be a noetherian domain, let M be a finitely generated R-module, and let $M^\vee := \mathrm{Hom}_R(M,R)$ be the dual of M. If $M^\vee \neq 0$, then $\mathrm{Ass}\, M^\vee = \{0\}$.*

Remark 4. Note that if L and M are finitely generated R-modules, then tensor-hom adjointness implies

$$\begin{aligned}
\mathrm{V}(\mathrm{Ann}(L \otimes_R M)) \cap \mathrm{Ass}\, N &= \mathrm{Supp}(L \otimes_R M) \cap \mathrm{Ass}\, N \\
&= \mathrm{Ass}(\mathrm{Hom}_R(L \otimes_R M, N)) \\
&= \mathrm{Ass}(\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))) \\
&= (\mathrm{Supp}\, L) \cap (\mathrm{Supp}\, M) \cap \mathrm{Ass}\, N \\
&= \mathrm{V}(\langle \mathrm{Ann}\, L, \mathrm{Ann}\, M \rangle) \cap \mathrm{Ass}\, N
\end{aligned}$$

for all R-modules N. In particular, we have

$$\mathrm{V}(\mathrm{Ann}(L \otimes_R M)) = \mathrm{V}(\mathrm{Ann}\, L) \cap \mathrm{V}(\mathrm{Ann}\, M) = \mathrm{V}(\langle \mathrm{Ann}\, L, \mathrm{Ann}\, M \rangle).$$

2.7 6/25/2024 - Inverse limit

Today I want to dicuss a result I was thinking about while driving to my parents house the other day. Let R be a ring and let $r \in R$. Consider the inverse system:

$$\mathcal{R} = \cdots \to R \xrightarrow{r} R \xrightarrow{r} R.$$

We set $A = \lim \mathcal{R}$. Then A consists of the set of all sequences (a_n) where $a_n \in R$ such that $r^m a_n = a_{n-m}$ for all $0 \leq m \leq n$. If R is an integral domain, then we can equivalently describe this as the set of all sequences (a_n) such that $r^n a_n = a_0$ for all $0 \leq n$. In particular, if $(a_n) \in A$, then we must have

$$a_m \in \bigcap_{n=1}^{\infty} \langle r \rangle^n := I.$$

for all $m \in \mathbb{N}$. Thus if $I = 0$, then necessarily $A = 0$. Krull's intersection theorem gives us $I = 0$ for many important rings that we care about. For example, if R is a noetherian local ring with maximal ideal $\mathfrak{m}$ and $r \in \mathfrak{m}$, then $I = 0$. Thus the inverse limit of the inverse system $\mathcal{R}$ would be 0 in this case. On the other hand, consider the direct system:

$$\mathcal{S} = R \xrightarrow{r} R \xrightarrow{r} R \to \cdots.$$

Then we have $R_r = \mathrm{colim}\, \mathcal{S}$. We have $R_r = 0$ if and only if r is nilpotent.

2.8 7/28/2024 - If $\mathrm{Z}G = 1$, then $\mathrm{Z}(\mathrm{Aut}\, G) = 1$

Here's a neat proposition in Group Theory that I proved involving the automorphism group of a centerless group.

Proposition 2.5. *Let G be a group such that $\mathrm{Z}G = 1$ and let $A = \mathrm{Aut}\, G$ be the automorphism group of G. The only automorphism of G which commutes with every inner automorphism of G is the identity automorphism. In particular, we have $\mathrm{Z}A = 1$.*

Proof. Suppose φ is an automorphism of G which commutes with every inner automorphism of G. Thus we have

$$\mathrm{c}_g \varphi = \varphi \mathrm{c}_g = \mathrm{c}_{\varphi g} \varphi$$

for all $g \in G$, or in other words, we have

$$g\varphi(x)g^{-1} = \varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}$$

for all $x, g \in G$. Replacing x with $\varphi^{-1}x$ above and rearranging terms, we see that

$$(\varphi g)^{-1} g x = x (\varphi g)^{-1} g$$

for all $x, g \in G$. Since $\mathrm{Z}G = 1$, we must have $(\varphi g)^{-1} g = 1$, on in other words, $\varphi g = g$ for all $g \in G$. It follows that $\varphi = 1$. ∎

2.9 8/18/2024 - flatness and projectiveness are stable under composition

Today I updated the 5/20/2024 entry. In today's entry, I want to prove the following:

Proposition 2.6. *Let $A \to B$ be a ring homomorphism and let C be a B-module.*

1. *If B is A-flat and C is B-flat, then C is A-flat.*
2. *If B is A-projective and C is B-projective, then C is A-projective.*

Proof. Suppose $M \rightarrowtail M'$ is an injective A-module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{tikzcd}
C \otimes_A M \arrow[r] \arrow[d, "\simeq"] & C \otimes_A M' \arrow[d, "\simeq"] \\
(C \otimes_B B) \otimes_A M \arrow[r] \arrow[d, "\simeq"] & (C \otimes_B B) \otimes_A M' \arrow[d, "\simeq"] \\
C \otimes_B (B \otimes_A M) \arrow[r, tail] & C \otimes_B (B \otimes_A M')
\end{tikzcd}$$

The bottom arrow is injective since B is A-flat and C is B-flat. Therefore $C \otimes_A M \rightarrowtail C \otimes_A M'$ is injective; whence C is A-flat.

Now suppose that $M \twoheadrightarrow M'$ is a surjective A-module homomorphism. We have a commutative diagram whose vertical arrows are isomorphisms:

$$\begin{tikzcd}
\mathrm{Hom}_A(C, M) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(C, M') \arrow[d, "\simeq"] \\
\mathrm{Hom}_A(C \otimes_B B, M) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(C \otimes_B B, M') \arrow[d, "\simeq"] \\
\mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M)) \arrow[r, two heads] & \mathrm{Hom}_B(C, \mathrm{Hom}_A(B, M'))
\end{tikzcd}$$

The bottom arrow is surjective since B is A-projective and C is B-projective. Therefore $\mathrm{Hom}_A(C, M) \rightarrowtail \mathrm{Hom}_A(C, M')$ is surjective; whence C is A-projective. $\square$

2.10 8/24/2024 - Connected integral domain has stalkwise local property

Proposition 2.7. *Let R be a connected commutative ring. Then R is an integral domain if and only if $R_{\mathfrak{p}}$ is an integral domain for each prime $\mathfrak{p}$ of R.*

The reason we need R to be connected is because the ring $R = K \times K$ where K is a field is clearly not an integral domain but the localization at each prime of R is isomorphic to K which is an integral domain.

Proof. If R is an integral domain then it is clear that $R_{\mathfrak{p}}$ is an integral domain for all primes $\mathfrak{p}$ of R. Conversely assume for a contradiction that $R_{\mathfrak{p}}$ is an integral domain for all primes $\mathfrak{p}$ of R but that R is not an integral domain. Choose nonzero nonunits $x, y \in R$ such that $xy = 0$. Note that $\langle x, y \rangle = 1$ since otherwise there would exist a prime $\mathfrak{p}$ which contains $\langle x, y \rangle$ and then $R_{\mathfrak{p}}$ would not be a domain since $x, y \neq 0$ in $R_{\mathfrak{p}}$ yet $xy = 0$. Thus there exists $a, b \in R$ such that $ax + by = 1$. Replacing x with ax and y with by if necessary, we may assume that $x + y = 1$. Multiplying both sides of this equation by x then implies $x^2 = x$ which contradicts the assumption that R is connected (a connected ring contains no nonzero nonunit idempotents). $\square$

2.11 8/30/2024 - Example

Today we study the following: let $\Bbbk$ be a field with characteristic $\neq 2$, let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, x_2]$, let $A = R[\boldsymbol{a}] = R[a_1, a_2, a_{11}^1, a_{11}^2, a_{12}^1, a_{12}^2, a_{22}^1, a_{22}^2]$, and let $B = A[\boldsymbol{e}]/\boldsymbol{f} = A[e_1, e_2]/\langle f_1, f_{11}, f_{12}, f_{22} \rangle$ where

$$\begin{aligned}
f_1 &= -1 + a_1 e_1 + a_2 e_2, \\
f_{11} &= -e_1^2 + a_{11}^1 e_1 + a_{11}^2 e_2 \\
f_{12} &= -e_1 e_2 + a_{12}^1 e_1 + a_{12}^2 e_2 \\
f_{22} &= -e_2^2 + a_{22}^1 e_1 + a_{22}^2 e_2
\end{aligned}$$

The Jacobian of B/A is given by

$$\mathrm{J}_{B/A} = \begin{pmatrix} a_1 & a_2 \\ a_{11}^1 - 2e_1 & a_{11}^2 \\ a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix},$$

and the Jacobian of B/R is given by

$$\mathrm{J}_{B/R} = \begin{pmatrix} e_1 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & e_1 & e_2 & 0 & 0 & 0 & 0 & a_{11}^1 - 2e_1 & a_{11}^2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & 0 & 0 & a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix}.$$

Let $\mathfrak{p}_{\boldsymbol{r}}$ be the prime ideal of A given by $\mathfrak{p}_{\boldsymbol{r}} = \langle \{a_{ij}^k - r_{ij}^k, a_i - r_i\}\rangle$ where $\boldsymbol{r} = (r_{11}^1, r_{11}^2, r_{12}^1, r_{12}^2, r_{22}^1, r_{22}^2, r_1, r_2) \in R^8$. Observe that

$$\begin{aligned}
[e_i, e_j, e_k] &= (e_i e_j)e_k - e_i(e_j e_k) \\
&= \sum_l (a_{i,j}^l e_k e_l - a_{j,k}^l e_i e_l) \\
&= \sum_m \sum_l (a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m) e_m \\
&= \sum_m \sum_l b_{ijk}^{lm} e_m \\
&= \sum_m b_{ijk}^m e_m,
\end{aligned}$$

where we set $b_{ijk}^{lm} = a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m$ and $b_{ijk}^m = \sum_l b_{ijk}^{lm}$. Let $J = \mathrm{J}_{B/A}(0)$. Then we have

$$b_{ijk}^{lm} - b_{ijk}^{ml} = J_{ij,kl}^{l,m} - J_{ij,kl}^{m,l}.$$

In particular, note that

$$b_{ijk}^m - b_{ijk}^l = \sum_l b_{ijk}^{lm} - \sum_m b_{ijk}^{ml}$$

Thus for instance we have

$$b_{112}^1 = a_{11}^1 a_{12}^1 - a_{12}^1 a_{11}^1 + a_{11}^2 a_{22}^1 - a_{12}^2 a_{12}^1$$

We have

$$1 = a_1 e_1 + a_2 e_2$$

implies

$$e_1 = a_1(a_{11}^1 e_1 + a_{11}^2 e_2) + a_2(a_{12}^1 e_1 + a_{12}^2 e_2)$$

imlpies

$$e_1 = (a_1 a_{11}^1 + a_2 a_{12}^1)e_1 + (a_1 a_{11}^2 + a_2 a_{12}^2)e_2 = b_1 e_1 + b_2 e_2$$

imlpies $(1 - b_1)e_1 = b_2 e_2$. We'd like to show that

$$e_1 = f + g(c_1 e_1 + c_2 e_2)$$

Suppose we have

$$\begin{aligned}
e_1 &= a_{11} + a_{12}(c_1 e_1 + c_2 e_2) \\
e_2 &= a_{21} + a_{22}(c_1 e_1 + c_2 e_2)
\end{aligned}$$

Rearranging terms, this implies

$$\begin{aligned}
(1 - a_{12}c_1)e_1 - a_{12}c_2 e_2 &= a_{11} \\
(1 - a_{22}c_2)e_2 - a_{22}c_1 e_1 &= a_{21}
\end{aligned}$$

This implies

$$a_{21}(1 - a_{12}c_1)e_1 - a_{21}a_{12}c_2 e_2 - a_{11}(1 - a_{22}c_2)e_2 + a_{11}a_{22}c_1 e_1 = 0$$

$$(a_{21}(1 - a_{12}c_1) + a_{11}a_{22}c_1)e_1 + (-a_{11}(1 - a_{22}c_2) - a_{21}a_{12}c_2)e_2 = 0$$

$$e_1 = a_{11}$$

$$r a_1 + x a_2$$

2.12 9/7/2024 - Example

Today we study the following: let $\Bbbk$ be a field with characteristic $\neq 2$, let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, x_2]$, let $A = R[\boldsymbol{a}] = R[a_1, a_2, a_{11}^1, a_{11}^2, a_{12}^1, a_{12}^2, a_{22}^1, a_{22}^2]$, and let $B = A[\boldsymbol{e}]/\boldsymbol{f} = A[e_1, e_2]/\langle f_1, f_{11}, f_{12}, f_{22}\rangle$ where

$$\begin{aligned} f_1 &= -1 + a_1 e_1 + a_2 e_2, \\ f_{11} &= -e_1^2 + a_{11}^1 e_1 + a_{11}^2 e_2 \\ f_{12} &= -e_1 e_2 + a_{12}^1 e_1 + a_{12}^2 e_2 \\ f_{22} &= -e_2^2 + a_{22}^1 e_1 + a_{22}^2 e_2 \end{aligned}$$

The Jacobian of B/A is given by

$$\mathrm{J}_{B/A} = \begin{pmatrix} a_1 & a_2 \\ a_{11}^1 - 2e_1 & a_{11}^2 \\ a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix},$$

and the Jacobian of B/R is given by

$$\mathrm{J}_{B/R} = \begin{pmatrix} e_1 & e_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & e_1 & e_2 & 0 & 0 & 0 & 0 & a_{11}^1 - 2e_1 & a_{11}^2 \\ 0 & 0 & 0 & 0 & e_1 & e_2 & 0 & 0 & a_{12}^1 - e_2 & a_{12}^2 - e_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_1 & e_2 & a_{22}^1 & a_{22}^2 - 2e_2 \end{pmatrix}.$$

Let $\mathfrak{p}_{\boldsymbol{r}}$ be the prime ideal of A given by $\mathfrak{p}_{\boldsymbol{r}} = \langle\{a_{ij}^k - r_{ij}^k, a_i - r_i\}\rangle$ where $\boldsymbol{r} = (r_{11}^1, r_{11}^2, r_{12}^1, r_{12}^2, r_{22}^1, r_{22}^2, r_1, r_2) \in R^8$. Observe that

$$\begin{aligned} [e_i, e_j, e_k] &= (e_i e_j) e_k - e_i (e_j e_k) \\ &= \sum_l (a_{i,j}^l e_k e_l - a_{j,k}^l e_i e_l) \\ &= \sum_m \sum_l (a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m) e_m \\ &= \sum_m \sum_l b_{ijk}^{lm} e_m \\ &= \sum_m b_{ijk}^m e_m, \end{aligned}$$

where we set $b_{ijk}^{lm} = a_{i,j}^l a_{k.l}^m - a_{j,k}^l a_{i,l}^m$ and $b_{ijk}^m = \sum_l b_{ijk}^{lm}$. Let $J = \mathrm{J}_{B/A}(0)$. Then we have

$$b_{ijk}^{lm} - b_{ijk}^{ml} = J_{ij,kl}^{l,m} - J_{ij,kl}^{m,l}.$$

In particular, note that

$$b_{ijk}^m - b_{ijk}^l = \sum_l b_{ijk}^{lm} - \sum_m b_{ijk}^{ml}$$

Thus for instance we have

$$b_{112}^1 = a_{11}^1 a_{12}^1 - a_{12}^1 a_{11}^1 + a_{11}^2 a_{22}^1 - a_{12}^2 a_{12}^1$$

We have

$$1 = a_1 e_1 + a_2 e_2$$

implies

$$e_1 = a_1(a_{11}^1 e_1 + a_{11}^2 e_2) + a_2(a_{12}^1 e_1 + a_{12}^2 e_2)$$

imlpies

$$e_1 = (a_1 a_{11}^1 + a_2 a_{12}^1) e_1 + (a_1 a_{11}^2 + a_2 a_{12}^2) e_2 = b_1 e_1 + b_2 e_2$$

imlpies $(1 - b_1) e_1 = b_2 e_2$. We'd like to show that

$$e_1 = f + g(c_1 e_1 + c_2 e_2)$$

Suppose we have

$$\begin{aligned} e_1 &= a_{11} + a_{12}(c_1 e_1 + c_2 e_2) \\ e_2 &= a_{21} + a_{22}(c_1 e_1 + c_2 e_2) \end{aligned}$$

Rearranging terms, this implies

$$(1-a_{12}c_1)e_1 - a_{12}c_2e_2 = a_{11}$$
$$(1-a_{22}c_2)e_2 - a_{22}c_1e_1 = a_{21}$$

This implies

$$a_{21}(1-a_{12}c_1)e_1 - a_{21}a_{12}c_2e_2 - a_{11}(1-a_{22}c_2)e_2 + a_{11}a_{22}c_1e_1 = 0$$
$$(a_{21}(1-a_{12}c_1) + a_{11}a_{22}c_1)e_1 + (-a_{11}(1-a_{22}c_2) - a_{21}a_{12}c_2)e_2 = 0$$
$$e_1 = a_{11}$$

2.13 9/13/2024 - Determinants, Traces, and Free Resolutions

Let R be a commutative ring, let M be a projective stably free R-module, and let $\varphi\colon M \to M$ be an R-linear map. In particular, M admits a consisting of finite rank free modules. Let F be such a resolution. The map $\varphi\colon M \to M$ lifts to a chain map $\widetilde{\varphi}\colon F \to F$. For each i we set $\delta_i = \det \widetilde{\varphi}_i$ and we set $\tau_i = \operatorname{tr} \widetilde{\varphi}_i$ and we define

$$\delta := \prod_i \delta_i^{(-1)^i} \quad \text{and} \quad \tau := \sum_i (-1)^i \tau_i.$$

On the other hand, M is locally free, so there exists elements $s_1, \ldots, s_n$ in R such that $\langle s_1, \ldots, s_n \rangle = 1$ and $M_k := M_{s_k}$ is a free module over $R_k := R_{s_k}$ for all $1 \leq k \leq n$. The map $\varphi\colon M \to M$ induces an R-linear map $\varphi_k\colon M_k \to M_k$ for each k. For each k we set $d_k = \det \varphi_k$ and $t_k = \operatorname{tr} \varphi_k$. It is easy to see that for each $1 \leq k, k' \leq n$ we have $d_k = d_{k'}$ and $t_k = t_{k'}$ in $R_{k,k'} := R_{s_k s_{k'}}$. Therefore they glue to unique elements d and t in R.

Proposition 2.8. *With the notation as above, we have $d = \delta$ and $t = \tau$.*

Proof. It suffices to show that $\delta_k = d_k$ and $\tau_k = t_k$ for each k where δ_k and τ_k are the images of δ and τ in R_k. In this case, M_k is free and the augmented complex obtained by adjoining M_k in homological degree -1 to F is an exact complex consisting of finite free modules. By replacing R with R_k if necessary, we are reduced to the following problem: assume F is an exact complex of finite length consisting of finite free R-modules and let $\varphi\colon F \to F$ be a chain map. Then we have

$$1 = \prod_i \delta_i^{(-1)^i} \quad \text{and} \quad 0 = \sum_i (-1)^i \tau_i.$$

First we assume that $F_i = 0$ for all $i \in \mathbb{Z}\backslash\{0,1,2\}$. In this case, the chain map $\varphi\colon F \to F$ looks like

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r] \arrow[d, "\varphi_2"] & F_1 \arrow[r] \arrow[d, "\varphi_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ 0 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0 \end{tikzcd} \tag{3}$$

and we need to show that $\delta_1 = \delta_0\delta_2$ and $\tau_1 = \tau_0 + \tau_2$. This short exact sequence splits and can be made to look like as below:

$$\begin{tikzcd} 0 \arrow[r] & F_2 \arrow[r, "\iota"] \arrow[d, "\varphi_2"] & F_2 \oplus F_1 \arrow[r, "\pi"] \arrow[d, "\widehat{\varphi}_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ 0 \arrow[r] & F_2 \arrow[r, "\iota"] & F_2 \oplus F_1 \arrow[r, "\pi"] & F_0 \arrow[r] & 0 \end{tikzcd} \tag{4}$$

where $\iota\colon F_2 \to F_2 \oplus F_0$ and $\pi\colon F_2 \to F_2 \oplus F_0$ are the obvious inclusion and projection maps and where $\widehat{\varphi}_1$ satisfies $\delta_1 = \det\widehat{\varphi}_1$. Furthermore, the matrix represenation of $\widehat{\varphi}_1$ has the form

$$[\varphi_1] = \begin{pmatrix} [\varphi_2] & 0 \\ 0 & [\varphi_0] \end{pmatrix},$$

and so clearly we have $\delta_1 = \delta_0\delta_2$ in this case. Now suppose that $\varphi\colon F \to F$ starts out like

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\widetilde{\varphi_2}"] & F_1 \arrow[r] \arrow[d, "\varphi_1"] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ 0 \arrow[r] & L \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0 \end{tikzcd} \tag{5}$$

where L is not necessarily free. Then an argument by induction of the length of the free resolution gives us the result. $\square$

2.14 10/9/24

Let A be a commutative ring and let S be a multiplicatively closed subset of A. We want to factor the localization map $A \to A_S$ as $A \twoheadrightarrow \overline{A} \rightarrowtail \overline{A}_S = A_S$ where $A \twoheadrightarrow \overline{A}$ is surjective and where the localization map $\overline{A} \rightarrowtail \overline{A}_S = A_S$ is injective. To do this, let

$$I = \bigcup_{s \in S} 0 : s = \{x \in A \mid xs = 0 \text{ for some } x \in S\}$$

We note that I is an ideal since S is multiplicatively closed. Now set $\overline{A} = A/I$. Then the quotient map $A \to \overline{A}$ followed by the localization map $\overline{A} \to \overline{A}_S = A_S$ gives the desired factorization

2.15 12/8/2024 - Weak Dimension

Definition 2.1. Let M be a nonzero R-module. We say M has **weak dimension** d if there exists an R-module M' such that $\mathrm{Tor}_d^R(M, M') \neq 0$ and such that $\mathrm{Tor}_i^R(M, N) = 0$ for all R-modules N and for all $i > d$.

Proposition 2.9. *The weak dimension of M is the shortest length of a flat resolution of M over R.*

Proof. Assume M has weak dimension d and let $F = (F, \mathrm{d})$ be a flat resolution of M over R and set $M_i = \mathrm{im}\,\mathrm{d}_i$. Then observe that

$$\mathrm{Tor}_1^R(M_d, N) \cong \mathrm{Tor}_{d+1}^R(M, N) = 0$$

for all R-modules N. It follows that M_d is flat. It follows that

$$F' = \cdots \to 0 \to M_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0 \to \cdots$$

is a flat resolution of M over R of length d. □

2.16 12/10/2024 - Continuity of R-algebra homomorphisms

Below is a characterization of continuous R-algebra homomorphisms between R-algebras which are complete with respect to their ideals of definition.

Proposition 2.10. *Let A be an R-algebra complete with respect to an ideal $\mathfrak{a}$ and let B be an R-algebra complete with respect to an ideal $\mathfrak{b}$. Then an R-algebra homomorphism $\varphi\colon A \to B$ is continuous if and only if there exists a $k \in \mathbb{N}$ such that $\varphi(\mathfrak{a}^k) \subseteq \mathfrak{b}$. Furthermore, if $\mathfrak{b}$ is radical, then φ is continuous if and only if $\varphi(\mathfrak{a}) \subseteq \mathfrak{b}$.*

2.17 12/16/2024 - Another characterization of Hausdorff spaces

Proposition 2.11. *Let X be a Hausdorff space and let K and K' be two disjoint compact subspaces of X. Then there exists open sets $U \supseteq K$ and $U' \supseteq K'$ such that $U \cap U' = \emptyset$.*

Proof. For each $x \in K$, let U_x be an open neighborhood of x and let U'_x be an open neighborhood K' such that $U_x \cap U'_x = \emptyset$. The collection $\{U_x\}_{x \in K}$ forms a cover of K, hence admits a finite subcover of K, say $\{U_{x_1}, \dots, U_{x_n}\}$. Then we set

$$U = U_{x_1} \cup \cdots \cup U_{x_n} \quad \text{and} \quad U' = U'_{x_1} \cap \cdots \cap U'_{x_n}$$

to finish the job. □

2.18 12/17/2024 - Fiber product properties of schemes

Let $f\colon Y \to X$ be a morphism of schemes and let E be a subscheme of X. We want to give $f^{-1}(E)$ the structure of a scheme. The idea is to use fiber products which we already know exist in the category of schemes. In particular, we define $f^{-1}(E)$ to be the fiber product

$$f^{-1}(E) := Y \times_X E$$

with respect to the morphism $f\colon Y \to X$ and the inclusion $\iota\colon E \hookrightarrow X$. To see why this makes sense, note that the points of $Y \times_X E$ have the form (y, e, P) where $y \in Y$ and $e \in E$ such that $f(y) = e$ and where P is a prime ideal of $\kappa(y) \otimes_{\kappa(e)} \kappa(e) \simeq \kappa(y)$. In particular, we have $P = 0$, so the points of $Y \times_X E$ (viewed as a topological space) are naturally in bijection with the points of $f^{-1}(E)$ (viewed as a topological space). Note that if E' is another subscheme of X, then

$$E \cap E' = \iota^{-1}(E') := E \times_X E',$$

so this also gives us a way to define intersections of subschemes. For instance, if $X = \mathrm{Spec}\, A$, $E = \mathrm{Spec}(A/\mathfrak{a})$, and $E' = \mathrm{Spec}(A/\mathfrak{a}')$, then

$$\begin{aligned} E \cap E' &= \mathrm{Spec}(A/\mathfrak{a}) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{a}') \\ &= \mathrm{Spec}(A/\mathfrak{a} \otimes_A A/\mathfrak{a}') \\ &= \mathrm{Spec}(A/\langle \mathfrak{a}, \mathfrak{a}' \rangle). \end{aligned}$$

Let us now check that this definition of inverse image satisfies some of the usual properties that inverse images satisfy:

Lemma 2.3. *Let $f\colon Y \to X$ be a morphism of schemes and let E, E be subschemes of X. We have*

$$f^{-1}(E) \cap f^{-1}(E') = f^{-1}(E \cap E').$$

Proof. In other words, we want to show

$$(Y \times_X E) \times_{Y \times_X X} (Y \times_X E') = Y \times_X (E \times_X E'). \tag{6}$$

To prove this, we may work locally; assume that $X = \operatorname{Spec} A$, $E = \operatorname{Spec} B$, $E' = \operatorname{Spec} B'$, and $Y = \operatorname{Spec} C$, where A is a ring and B, B', C are A-algebras. Then we have

$$\begin{aligned}(C \otimes_A B) \otimes_{(C \otimes_A A)} (C \otimes_A B') &\simeq (C \otimes_A B) \otimes_C (C \otimes_A B') \\ &\simeq B \otimes_A C \otimes_C C \otimes_A B' \\ &\simeq B \otimes_A C \otimes_A B' \\ &\simeq C \otimes_A (B \otimes_A B').\end{aligned}$$

This gives us (6). □

Lemma 2.4. *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be morphisms of schemes and let E be a subscheme of X. We have*

$$g^{-1}(f^{-1}(E)) = (f \circ g)^{-1}(E).$$

Proof. In other words, we want to show

$$Z \times_Y (Y \times_X E) = Z \times_X E. \tag{7}$$

Again, we may work locally; assume that $X = \operatorname{Spec} A$, $E = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$, and $Z = \operatorname{Spec} D$. Then we have

$$\begin{aligned}D \otimes_C (C \otimes_A B) &\simeq (D \otimes_C C) \otimes_A B \\ &\simeq D \otimes_A B.\end{aligned}$$

This gives us (7). □

2.19 12/28/2024 - Exercise 10.6 in Eisenbud

Let $R \to R'$ be a map of local rings such that R is regular, and let $\mathfrak{p}$ be a prime ideal of R. Then one can show that

$$\operatorname{ht}(\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p}R'). \tag{8}$$

On the other hand, if R is not regular, then the inequality (8) need not hold. For instance: let $R = \Bbbk[x, y, s, t]/\langle xs - yt \rangle$, let $\mathfrak{p} = \langle s, t \rangle \subseteq R$, and let $R' = R/\langle x, y \rangle \simeq \Bbbk[s, t]$. Then in this case, one has $\operatorname{ht} \mathfrak{p} = 1$ but $\operatorname{ht}(\mathfrak{p}R') = 2$.

2.20 12/29/2024

Let $R = \mathbb{Z}[a]$ where $a^2 = -5$ and let $\mathfrak{p}$ be a prime ideal of R. One can show that the minimal free resolution F of $R/\mathfrak{p}$ over R has one of two forms:

1. $F = (R \to R)$
2. $F = (\cdots \to R^2 \to R^2 \to \cdots \to R^2 \to R^2 \to R)$

In the first case, $\mathfrak{p}$ is a principal ideal. In the second case, $\mathfrak{p}$ is not principal and the free resolution is infinite and 2-periodic. In the image below, I worked out how this resolution looks in the case where $\mathfrak{p} = \langle 3, 1 + a \rangle$:

$R = \mathbb{Z}[a]$ $\qquad a^2 = -5$

$\mathfrak{p} = \langle 3, b \rangle$ $\qquad b = 1 + a$

$\bar{b} = 1 - a = 2 - b$

$$\cdots \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -b & -2 \\ 3 & \bar{b} \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} \bar{b} & 2 \\ 3 & b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -b & -2 \\ 3 & \bar{b} \end{pmatrix}} R^2 \xrightarrow{(3,\ b)} R$$

In fact, F has the structure of a DG algebra; it is given by $F = R[u, v, w]$ where $|u| = 1 = |v|$ and $|w| = 2$ with differential d given by

$$\begin{aligned} \mathrm{d}u &= 3 \\ \mathrm{d}v &= b \\ \mathrm{d}w &= -2u + (2-b)v. \end{aligned}$$

Here, u and v are exterior variables and w is a divided power variable (so in particular we have $uv = -vu$, $u^2 = 0 = v^2$, and $\mathrm{d}w^2 = (\mathrm{d}w)w$). It turns out that for any ring of integers R and non-principal prime ideal $\mathfrak{p}$ of R, the minimal free resolution F of $R/\mathfrak{p}$ over R has the same form as the $\mathbb{Z}[a]$ case. The reason for this boils down to two facts:

1. Every ideal of R can be generated by at most two elements (this is true in any Dedekind domain).
2. Every element of the class group is represented by a prime ideal (by Chebotarev).

For instance, a non-principal prime ideal of R has the form $\mathfrak{p} = \langle p, \alpha\rangle$ with $\alpha \in R\backslash\mathbb{Z}$ and $p \mid a$ where we set $a := \mathrm{N}\alpha$. Then the DG algebra structure of F is given by $F = R[u, v, w]$ where $|u| = 1 = |v|$ and $|w| = 2$ with differential d given by

$$\begin{aligned} \mathrm{d}u &= p \\ \mathrm{d}v &= \alpha \\ \mathrm{d}w &= -(a/p)u + (a/\alpha)v. \end{aligned}$$

Next suppose R is the coordinate ring of the nodal cubic $R = \Bbbk[x, y]/\langle y^2 - x^3 - x^2\rangle$ and $\mathfrak{m} = \langle x, y\rangle$ is the maximal ideal of R corresponding to the singular point. Then the minimal free resolution F of $R/\mathfrak{m}$ over R has the same form as the ones above too: F is a DG algebra given by $F = R[u, v, w]$ where $|u| = 1 = |v|$ and $|w| = 2$ with differential d given by

$$\begin{aligned} \mathrm{d}u &= x \\ \mathrm{d}v &= y \\ \mathrm{d}w &= -(x^2 + x)u + yv. \end{aligned}$$

Finally, assume $R = \Bbbk[x, y]/(y^n + a_{n-1}y^{n-1} + \cdots + a_0)$ such that $a_0, \ldots, a_{n-1} \in \Bbbk[x]$ with $a_0 \in \langle x\rangle^2$, and let $\mathfrak{m} = \langle x, y\rangle \subseteq R$. Then the minimal free resolution F of $R/\mathfrak{m}$ over R has a DG algebra structure given by $F = R[u, v, w]$ where $|u| = 1 = |v|$ and $|w| = 2$ with differential d given by

$$\begin{aligned} \mathrm{d}u &= x \\ \mathrm{d}v &= y \\ \mathrm{d}w &= -(a_0/x)u + (a_0/y)v. \end{aligned}$$

Note that w being a divided powers variable means this: we adjoin indeterminants $w^{(n)}$ to $R[u, v, w]$ for $n \geq 1$ and define the differential on $w^{(n)}$ by $\mathrm{d}w^{(n)} = (\mathrm{d}w)w^{(n-1)}$. We then quotient out relations depending on the following cases:

1. If R has characteristic 0, then we quotient out by the relations $w^n - n!w^{(n)}$ for all $n \geq 1$.
2. If R has characteristic p, then we quotient out by the relations w^p and $w^n - n!w^{(n)}$ for $n = 1, \ldots, p-1$.

2.21 12/30/2024 - Compact Objects

Definition 2.2. Let X be an object in a category C which admits filtered colimits. We say X is **compact** if the functor $\mathrm{Hom}_C(X, -)$ commutes with filtered colimits, i.e., if (Y_i) is a filtered system of objects in C, then the natural map

$$\operatorname{colim} \mathrm{Hom}_C(X, Y_i) \to \mathrm{Hom}_C(X, \operatorname{colim} Y_i)$$

is a bijection.

Proposition 2.12. *Let R be a commutative ring. The compact objects of the category of all R-modules are precisely the finitely presented R-modules.*

Proof. Let M be a finitely presented R-module and let (N_i) be a filtered system of R-modules. We wish to show the canonical map

$$\operatorname{colim} \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \operatorname{colim} N_i) \tag{9}$$

is surjective since it is straightforward to check that it is injective. We first assume $M = F$ is finite free, say of rank n with ordered basis $e_1, \ldots, e_n$. Let $\varphi \in \mathrm{Hom}_R(F, \operatorname{colim} N_i)$. For each $1 \leq i \leq n$, we have $\varphi(e_i) = \overline{v_i}$ where $v_i \in N_k$

for some sufficiently large k (note that we can assume that each v_i is in N_k for some sufficiently large k since (N_i) is filtered). We then define $\overline{\psi} \in \operatorname{colim} \operatorname{Hom}_R(F, N_i)$ to be the element class represented by $\psi \in \operatorname{Hom}_R(F, N_k)$ which is defined by $\psi(e_i) = v_i$ for all $1 \leq i \leq n$. Then it is straightforward to check that the canonical map (9) sends $\overline{\psi}$ to φ, hence this map is surjective.

To handle the general case where M is finitely presented and not necessarily free, we choose a presentation $G \to F \to M \to 0$ of M which is exact everywhere and where both F and G are finite free R-modules. The we do the usual trick and apply

$$\operatorname{colim} \operatorname{Hom}_R(-, N_i) \to \operatorname{Hom}_R(-, \operatorname{colim} N_i)$$

to this exact sequence and argue that the map (9) is a bijection from the finite free case. □

Note that a special case of the proposition above is when M and N are R-modules and S is a multiplicatively closed subset of R, then we have a canonical map

$$\operatorname{colim}_{s \in S} \operatorname{Hom}_R(M, N_s) \to \operatorname{Hom}_R(M, N_S)$$

3 2025

3.1 1/1/2025

Let $(X, \mathcal{O}) = \operatorname{Proj} \Bbbk'[t_1, t_2, t_3] = \operatorname{Proj} A$, let $U_1 = \mathrm{D}_+(t_1)$, let $U_2 = \mathrm{D}_+(t_2)$, and let $U_3 = \mathrm{D}_+(t_3)$. Then we have

$$\begin{aligned} A_1 &:= \mathcal{O}(U_1) = \Bbbk[t_{21}, t_{31}] & \qquad B_1 &:= \mathcal{O}(2)(U_1) = \Bbbk[t_{21}, t_{31}]t_1^2 \\ A_2 &:= \mathcal{O}(U_2) = \Bbbk[t_{12}, t_{32}] & B_2 &:= \mathcal{O}(2)(U_2) = \Bbbk[t_{12}, t_{32}]t_2^2 \\ A_3 &:= \mathcal{O}(U_3) = \Bbbk[t_{13}, t_{23}] & B_3 &:= \mathcal{O}(2)(U_3) = \Bbbk[t_{13}, t_{23}]t_3^2 \end{aligned}$$

where we set $t_{ij} = t_i/t_j$. Note that $B_i \cong A_i$ as A_i-modules. Observe in $B_{12} := \mathcal{O}(2)(U_{12})$ we have

$$f(t_{21}, t_{31})t_1^2 = f(t_{12}^{-1}, t_{32}t_{12}^{-1})t_{12}^2 t_2^2$$

In particular, if we set $x = t_{21}$, $y = t_{31}$, $x' = t_{12}$, and $y' = t_{32}$, then transition map is given by

$$f(x, y) \mapsto f(1/x', y/x')x'^2.$$

i

Let $(X, \mathcal{O}) = \operatorname{Proj} \Bbbk'[t_1, t_2, t_3] = \operatorname{Proj} A$, let $U_1 = \mathrm{D}_+(t_1)$, let $U_2 = \mathrm{D}_+(t_2)$, and let $U_3 = \mathrm{D}_+(t_3)$. Then we have

$$\begin{aligned} A_1 &:= \mathcal{O}(U_1) = \Bbbk[t_{21}, t_{31}] & \qquad B_1 &:= \mathcal{O}(k)(U_1) = \Bbbk[t_{21}, t_{31}]t_1^k \\ A_2 &:= \mathcal{O}(U_2) = \Bbbk[t_{12}, t_{32}] & B_2 &:= \mathcal{O}(k)(U_2) = \Bbbk[t_{12}, t_{32}]t_2^k \\ A_3 &:= \mathcal{O}(U_3) = \Bbbk[t_{13}, t_{23}] & B_3 &:= \mathcal{O}(k)(U_3) = \Bbbk[t_{13}, t_{23}]t_3^k \end{aligned}$$

where we set $t_{ij} = t_i/t_j$. Note that $B_i \cong A_i$ as A_i-modules. Observe in $B_{12} := \mathcal{O}(k)(U_{12})$ we have

$$f(t_{21}, t_{31})t_1^k = f(t_{12}^{-1}, t_{32}t_{12}^{-1})t_{12}^k t_2^k$$

In particular, if we set $x = t_{21}$, $y = t_{31}$, $x' = t_{12}$, and $y' = t_{32}$, then transition map is given by

$$f(x, y) \mapsto f(1/x', y/x')x'^k.$$

3.2 1/2/2025

Here's a really cool fact I learned today. Let R be a local noetherian ring and let U be a nonzero finitely generated R-module. The following are equivalent:

1. U has depth 0.
2. $\operatorname{Hom}_R(\Bbbk, U) \neq 0$.
3. $\mathfrak{m}$ is an associated prime of U.

Now here's the cool part: let M be another nonzero finitely generated R-module. Recall that

$$\operatorname{Ass} \operatorname{Hom}_R(M, U) = \operatorname{Ass} U \cap \operatorname{Supp} M.$$

In particular, since M is not zero (so that $\mathfrak{m} \in \operatorname{Supp} M$) we see that $\operatorname{Hom}_R(M, U)$ has depth 0 if and only if U has depth 0. In particular, one has $\operatorname{Ass} V = \operatorname{Ass} U$ where $V = \operatorname{Hom}_R(U, U)$. One can also show that $\operatorname{Ann} U = \operatorname{Ann} V$. I'm beginning to appreciate the ext characterization of depth based on results like these. If R is a complete local noetherian ring, then there's the following tor characterization of flatness of U (hence freeness of U since we are assuming it is finitely generated):

1. U is free.

2. $\mathrm{Tor}_1^R(U, \Bbbk) = 0$.

Note that we also have a tor characterization of projective dimension which holds all local noetherian rings:

1. U has projective dimension $= \rho$

2. $\mathrm{Tor}_\rho^R(U, \Bbbk) \neq 0$ and $\mathrm{Tor}_i^R(U, \Bbbk) = 0$ for all $i > \rho$.

Note that for all local rings and finitely generated modules we have free = projective = flat. Thus in the complete case, we see that $\mathrm{Tor}_1^R(U, \Bbbk) = 0$ guarantees that $\mathrm{Tor}_i^R(U, \Bbbk) = 0$ for all $i \geq 1$.

Let me discuss another fun fact. Let $R = (R, \mathfrak{m}, \Bbbk)$ be a complete local noetherian ring and let

$$0 \longrightarrow U \xrightarrow{t} U \longrightarrow M \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules. Assume that $tU \subseteq \mathfrak{m}U$ and that M has finite projective dimension. Then for all i sufficiently large we have isomorphisms

$$\mathrm{Tor}_i^R(U, \Bbbk) \xrightarrow{t} \mathrm{Tor}_i^R(U, \Bbbk).$$

However the image of t is contained in $\mathfrak{m}\mathrm{Tor}_i^R(U, \Bbbk)$., thus by Nakayama's lemma we see that $\mathrm{Tor}_i^R(U, \Bbbk) = 0$ for all i sufficiently large, hence U has finite projective dimension.
Assume that both U and M are not zero and that tU is contained in $\mathfrak{m}U$. Then we obtain the exact sequence

$$\mathrm{Tor}_1^R(U, \Bbbk) \xrightarrow{t} \mathrm{Tor}_1^R(U, \Bbbk) \to \mathrm{Tor}_1^R(M, \Bbbk) \to U_\Bbbk \to 0.$$

where again the multiplication by t map is contained in $\mathfrak{m}\mathrm{Tor}_1^R(U, \Bbbk)$. It follows at once that M cannot be free since it would force $U_\Bbbk = 0$. On the other hand, suppose that $t^{-1}(\mathfrak{m}U) = \mathfrak{m}U$. Then we obtain the exact sequence

$$\mathrm{Tor}_1^R(U, \Bbbk) \xrightarrow{t} \mathrm{Tor}_1^R(U, \Bbbk) \to \mathrm{Tor}_1^R(M, \Bbbk) \to 0$$

If U is free, then M is free too.

3.3 1/3/2025

Proposition 3.1. *Let R be a local noetherian ring, let U be an R-module, and let F be the minimal free resolution of U over R. We set $V = \mathrm{Hom}_R(U, U)$, $G = \mathrm{Hom}_R^\star(F, U)$, and $\Gamma = \mathrm{Hom}_R^\star(F, F)$. Throughout we implicitly view U and V as graded modules concentrated in degree 0. Thus if A is an R-complex and we write $A = V$, then it is understood that $A_i = 0$ for all $i \neq 0$ and $A_i = V$.*

1. *We have $\mathrm{Ext}_R(U, U) = V$ if and only if G is the minimal free resolution of V over R if and only if $\mathrm{H}(\Gamma) = V$.*

2. *Recall that we have the biduality map $U \to \mathrm{Hom}_R(V, U)$, given by sending $u \in U$ to $\hat{u} \in \mathrm{Hom}_R(V, U)$ where $\hat{u}$ is defined by $\hat{u}(v) = v(u)$ for all $v \in V$. We also have biduality map of complexes $F \to \mathrm{Hom}_R^\star(G, U)$, given by sending $a \in F_i$ to the map $\hat{a} \in \mathrm{Hom}_R^\star(G, U)_i$ which itself is defined by*

$$\hat{a}(b) = \begin{cases} b(a) & \text{if } b \in G_{-i} = \mathrm{Hom}_R(F_i, U) \\ 0 & \text{else} \end{cases}$$

for all homogeneous $b \in G$. With this terminology in mind, we now assume that $\mathrm{Ext}_R(U, U) = V$. Then the biduality map $U \to \mathrm{Hom}_R(V, U)$ is an isomorphism if and only if the biduality map $F \to \mathrm{Hom}_R^\star(G, U)$ is a quasiisomorphism if and only if $\mathrm{Ext}_R(V, U) = U$.

3. *Recall that V obtains the structure of a (not necessarily commutative) ring where the product is given by composition of maps. If F has finite length so that Γ is semiprojective and $\mathrm{Ext}_R(U, U) = V$, then this product can be uniquely lifted up to homotopy to a product on Γ, though it is only associative up to homotopy. There is a canonical homothety map $V \to \mathrm{Hom}_R(V, V)$ obtained by sending $v \in V$ to the map $\omega_v \in \mathrm{Hom}_R(V, V)$ where $\omega_v(v') = v \circ v'$ for all $v' \in V$. With this in mind, we now assume that $\mathrm{Ext}_R(U, U) = V$ and $\mathrm{Ext}_R(V, U) = U$. Then the homothety map $V \to \mathrm{Hom}_R(V, V)$ is an isomorphism and we have $\mathrm{Ext}_R(V, V) = \mathrm{Hom}_R(V, V)$.*

Proof. 1. Clearly G is minimal. Furthermore, since $\mathrm{Hom}_R^\star(F, -)$ preserves the quasiisomorphism $F \xrightarrow{\simeq} U$, we have

$$\mathrm{Ext}_R(U, U) = \mathrm{H}(\mathrm{Hom}_R^\star(F, U)) \simeq \mathrm{H}(\mathrm{Hom}_R^\star(F, F)).$$

Therefore $\mathrm{Ext}_R(U, U) = \mathrm{H}(G) = \mathrm{H}(\Gamma)$.

2. The biduality map $U \to \operatorname{Hom}_R(V, U)$ can be constructed via the sequence of maps:

$$\begin{aligned} U &= \mathrm{H}(F) \\ &\to \mathrm{H}(\operatorname{Hom}_R^{\star}(G, U)) \\ &= \operatorname{Ext}_R(V, U) \end{aligned}$$

where the map on the second line is induced by the biduality map $F \to \operatorname{Hom}_R^{\star}(G, U)$.

3. The homothety map $V \to \operatorname{Hom}_R(V, V)$ can be constructed via the sequence of maps

$$\begin{aligned} V &= \mathrm{H}(\Gamma) \\ &= \mathrm{H}(\operatorname{Hom}_R^{\star}(F, F)) \\ &\xrightarrow{\simeq} \mathrm{H}(\operatorname{Hom}_R^{\star}(F, \operatorname{Hom}_R^{\star}(G, U))) \\ &\xrightarrow{\simeq} \mathrm{H}(\operatorname{Hom}_R^{\star}(F, \operatorname{Hom}_R^{\star}(G, F))) \\ &\simeq \mathrm{H}(\operatorname{Hom}_R^{\star}(F \otimes_R G, F)) \\ &\simeq \mathrm{H}(\operatorname{Hom}_R^{\star}(G, \operatorname{Hom}_R^{\star}(F, F))) \\ &= \mathrm{H}(\operatorname{Hom}_R^{\star}(G, \Gamma)) \\ &= \mathrm{H}(\operatorname{Hom}_R^{\star}(G, V)) \\ &= \operatorname{Ext}_R(V, V). \end{aligned}$$

□

3.4 1/4/2025 - Blowup algebra and associated graded rings

Let R be a commutative ring, let I be an ideal of R, and let M be a finitely generated R-module. We set A to be the blowup algebra and we set G to be the associated graded ring:

$$A = \mathrm{bl}_I R := R \oplus I \oplus I^2 \oplus \cdots \quad \text{and} \quad G = \mathrm{gr}_I R = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

We claim that the blowup algebra is R-flat if and only I is R-flat. Indeed, if A is R-flat, then I must be R-flat as well since it is a direct summand of A. Conversely, suppose that I is R-flat. Then the canonical map $I \otimes_R I \to I^2$ is an isomorphism. In particular, we see that I^2 is R-flat too since the tensor product of two flat R-modules is flat. More generally, we see that I^n is R-flat and hence A, being a direct sum of R-flat modules, is itself R-flat.

Remark 5. Suppose R is the valuation ring of a perfectoid field K. Then one has $\mathfrak{m} = \mathfrak{m}^2 = \mathfrak{m} \otimes_R \mathfrak{m}$. In particular, $\mathrm{bl}_{\mathfrak{m}} R = R \oplus \mathfrak{m}R[t]$ and $\mathrm{gr}_{\mathfrak{m}} R = \Bbbk$.

Note that we can view A as the subring $A = R[It]$ of the ring $R[t]$. There's another algebra that's interesting to consider, called the Rees algebra, and it is given by

$$B = R[t] \oplus It^{-1} \oplus I^2 t^{-2} \oplus \cdots \subseteq R[t.t^{-1}].$$

Again, this is flat as an R-module if and only if I is R-flat. On the other hand, B is also an $R[t]$-algebra via the inclusion map $R[t] \hookrightarrow B$ and it turns out that B is $R[t]$-flat (nevermind this isn't true). Note that the fiber of B over $\langle t \rangle$ is the associated graded ring G and all other fibers are isomorphic to R. In other words, G is the special fiber of a flat deformation with the general fiber being R.

Example 3.1. labelexample Let $R = \Bbbk[\varepsilon]$ where $\varepsilon^2 = 0$ and let $I = \langle \varepsilon \rangle$. Then the Rees algebra is given by

$$B = R[t] \oplus R\varepsilon/t$$

3.4.1 Q-filtrations

Let $R = (R, \mathfrak{m}, \Bbbk)$ be a local Cohen Macaulay ring of dimension d, let Q be an $\mathfrak{m}$-primary anbe an ideal of R, and equip R with the Q-filtration $R = (Q^n)$. Let $M = (M_n)$ be a finitely generated filtered R-module. Recall that this means $(M_n)_{n \in \mathbb{N}}$ is a descending sequence of R-submodules of M with $M_0 = M$ and $QM_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$. Then the famous Artin-Rees lemma tells us that M is stable: there exists $k \in \mathbb{N}$ such that $Q^m M_k = M_{k+m}$ for all $m \geq k$. Observe for all n we have the following exact sequence:

$$0 \longrightarrow (Q^n \cap \mathfrak{m}^{n+1})/Q^{n+1} \longrightarrow Q^n/Q^{n+1} \longrightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^n/Q^n \longrightarrow 0$$

The Artin-Rees lemma implies for n sufficiently large we have

$$\frac{Q^n\cap\mathfrak{m}^{n+1}}{Q^{n+1}}=\frac{(Q^n\cap\mathfrak{m}^{n-k})\mathfrak{m}^k}{Q^{n+1}}\subseteq\frac{(Q^n\cap\mathfrak{m}^{n-k})Q}{Q^{n+1}}=0,$$

where k is chosen such that $\mathfrak{m}^k\subseteq Q$. Therefore for n sufficiently large we obtain a short exact sequence

$$0\longrightarrow Q^n/Q^{n+1}\longrightarrow\mathfrak{m}^n/\mathfrak{m}^{n+1}\longrightarrow\mathfrak{m}^n/Q^n\longrightarrow 0$$

. Now

3.5 1/5/2025

Example 3.2. labelexample Let $R=\Bbbk[x,y]/\langle y^2-x^3\rangle$ and let $\mathfrak{m}=\langle x,y\rangle\subseteq R$. Then

$$\mathrm{bl}_{\mathfrak{m}}R\cong R[u,v]/\langle yu-xv,xu^2-v^2,x^2u-yv\rangle\quad\text{and}\quad\mathrm{gr}_{\mathfrak{m}}R\cong\Bbbk[u,v]/v^2$$

3.6 1/7/2025

Let $(R,\mathfrak{m})$ be a local ring and let $I\subseteq\mathfrak{m}$ be a finitely generated ideal of R such that $I^n=I$ for some n. Then I is principal. Here's the sketch of the proof: suppose $I=\langle a_1,a_2\rangle$. Then we have

$$a_1=r_0a_1^n+r_1a_1^{n-1}a_2+\cdots+r_{n-1}a_1a_2^{n-1}+r_na_2^n$$

where $r_0,r_1,\dots,r_n\in R$. This implies

$$ua_1=(1-r_0a_1^{n-1}-r_1a_1^{n-2}a_2-\cdots-r_{n-1}a_2^{n-1})a_1=r_na_2^n,$$

where $u=1-r_0a_1^{n-1}-r_1a_1^{n-2}a_2-\cdots-r_{n-1}a_2^{n-1}$ is a unit. Therefore $I=\langle a_2\rangle$. However then we have $ra_2^n=a_2$ which implies $a_1(1-ra_2^{n-1})=0$ which implies $a_2=0$. Okay, this is just nakayama's lemma...

Okay, here's something: suppose R is a noetherian ring and I is an ideal of R such that $I^2=I$. We claim that $I\cap J=IJ$ for all other ideals J of R. Indeed, we clearly have $IJ\subseteq I\cap J$. Conversely, by Artin-Rees, there exists an n such that

$$I\cap J=I^2\cap J=\cdots=I^n\cap J=I(I^{n-1}\cap J)\subseteq IJ.$$

In particular, this says that $R\to R/I$ is flat if and only if $I/I^2=\mathrm{Tor}_1^R(R/I,R/I)=0$ (at least for noetherian rings).

Actually I can say one more thing about this. If R is noetherian, then R/I is R-flat if and only if R/I is finite projective over R if and only if the quotient map $R\to R/I$ splits. If $R\to R/I$ splits, then we can write $R=I\oplus R/I$. Let $e=1-x$ be the image of $\overline{1}\in R/I$ under the splitting map $R/I\to R$. Then for each $y\in I$, we have $0=ye=y-yx$ implies $I=\langle x\rangle$ where $x^2=x$ is an idempotent. Thus for R noetherian, the only flat maps that look like $R\to R/I$ are of the form $R:=R_1\times R_2\to R_1$. This is really just a consequence of the fact that finite projective = finitely presented and flat = finite locally free.

One more thing before I get some shut eye. Let $A\to B$ be a ring homomorphism. Let $I=\ker(\mu\colon B^{\otimes 2}\to B)$ where μ is the multiplication map and where $B^{\otimes 2}=B\otimes_A B$. Recall that I is generated as a B-module by elements of the form $\mathrm{d}b:=b\otimes 1-1\otimes b$ for $b\in B$. If B is a finitely generated A-algebra, then one can show that I is finitely generated as a $B^{\otimes}$-ideal; if B is generated by $b_1,\dots,b_n$ as an A-algebra, then I is generated by $\mathrm{d}b_1,\dots,\mathrm{d}b_n$ as a $B^{\otimes}$-ideal.

3.7 1/9/2025 - If finitely presented module splits locally, then it splits globally.

Let R be a ring and let M be a finitely presented R-module. Thus there exists a surjective R-linear map $\varphi\colon R^n\twoheadrightarrow M$ for some $n\geq 0$. Recall that M is a projective R-module if and only if φ splits globally, meaning there exists $\phi\in\mathrm{Hom}_R(M,R^n)$ such that $\varphi\circ\phi=1$. On the other hand, suppose we knew that φ splits locally, meaning there exists $s_1,\dots,s_m\in R$ such that $\langle s_1,\dots,s_m\rangle=1$ and such that $\varphi_{s_i}\colon R^n_{s_i}\twoheadrightarrow M_{s_i}$ splits for each $1\leq i\leq m$. Can we then deduce that φ already splits globally? It turns out that the answer is yes:

Proposition 3.2. *With the notation as above, if φ splits locally, then it splits globally.*

Proof. Since $\varphi_{s_i}\colon R^n_{s_i}\twoheadrightarrow M_{s_i}$ splits, there exists an R_{s_i}-linear maps $\psi_i\colon M_{s_i}\to R^n_{s_i}$ such that $\varphi_{s_i}\circ\psi_i=1$ in $\mathrm{Hom}_{R_{s_i}}(M_{s_i},R^n_{s_i})$. Since M is finitely presented, there exists a sufficiently large $N\geq 0$ such that $\psi_i=\widetilde{\psi}_i/s_i^N$ for all $1\leq i\leq m$ where $\widetilde{\psi}_i\in\mathrm{Hom}_R(M,R^n)$. This implies $\varphi_{s_i}\circ\widetilde{\psi}_i=s_i^N$ in $\mathrm{Hom}_R(M,R^n)_{s_i}$ which further implies $s_i^{N'}\varphi\circ\widetilde{\psi}_i=s_i^{N+N'}$ in $\mathrm{Hom}_R(M,R^n)$ for some sufficiently large $N'\geq 0$. Replacing ϕ_i with $s_i^{N'}\widetilde{\psi}_i$ and replacing N with $N+N'$ if necessary, we may assume we have R-linear maps $\phi_i\colon M\to R^n$ such that

$$\varphi\circ\phi_i=s_i^N$$

for all $1 \le i \le m$. Note that $\langle s_1^N, \dots, s_m^N \rangle = 1$, so there exists $c_1, \dots, c_m \in R$ such that $c_1 s_1^N + \cdots + c_m s_m^N = 1$. Finally we set

$$\phi := c_1\phi_1 + \cdots + c_m\phi_m.$$

Then observe that

$$\begin{aligned} \varphi \circ \phi &= \varphi \circ (c_1\phi_1 + \cdots + c_m\phi_m) \\ &= c_1 s_1^N + \cdots + c_m s_m^N \\ &= 1. \end{aligned}$$

□

3.8 1/10/2025

Let $A = \mathbb{k}[\{x_{ij} \mid 1 \le 1, j \le n\}]/\langle \Delta - 1 \rangle$ where we set

$$\Delta = \det(x_{ij}) = \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \prod_{1 \le i \le n} x_{i\sigma(i)}.$$

Thus A is the ring associated to the affine $\mathbb{k}$-group SL_n.

3.9 1/11/2025 - Artinian ring with infinitely many ideals

Here's an interesting example of an artinian ring with infinitely many ideals. Let $\mathbb{k}$ be an infinite field and let $R = \mathbb{k}[x,y]/\langle x^2, xy, y^2 \rangle$. For each $c \in \mathbb{k}$, we set $I_c = \langle x - cy \rangle$. Then $\{I_c \mid c \in \mathbb{k}\}$ gives us infinitely many ideals of R. The way we think about this geometrically is that $X = \operatorname{Spec} R$ is a point in 2-space fattened up in all directions. The closed subschemes corresponding to I_c are then "lines" with various (infinitely many) slopes.

3.10 1/25/2025 - New

Let

$$0 \longrightarrow K \xrightarrow{\iota} P \xrightarrow{\varphi} M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow K' \xrightarrow{\iota'} P' \xrightarrow{\varphi'} M \longrightarrow 0$$

be two short exact sequences of R-modules where P and P' are projective R-modules.

3.11 1/26/2025

I'm leaving this here for now in case I need it again: A deeper relationship between analytic number theory, and commutative algebra is realized when one studies G as a direct limit

$$G = \varinjlim G^m$$

of bi-graded noetherian $\mathbb{k}$-algebras $G^m = R^m/I^m$, where

$$R^m = \mathbb{k}[x_1, \dots, x_m] \cap R \quad \text{and} \quad I^m = \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes and } p + q \le m\} \rangle$$

Indeed, for each m we denote by $\delta(m)$ and $\rho(m)$ to be the R^m-depth and R^m-projective dimension of G^m respectively. Then the Auslander-Buchsbaum formula implies

$$\delta(2m) + \rho(2m) = \pi(2m) + m - \kappa(2m) - 2, \tag{10}$$

where $\pi(2m)$ is the usual prime-counting function which counts the number of primes $\le 2m$ and where $\kappa(2m)$ counts then number of positive even numbers $\le 2m$ that are counter-examples to Goldbach's conjecture.

3.12 Expressing the Prime Counting Function in terms of Projective Dimension

In this subsection, we assume that Goldbach's conjecture is true. Let us recall some notation we developed in this case and explain what they look like here. For each $m \ge 1$, we have

$$\begin{aligned} R^m &= \mathbb{k}[\{x_p, x_{2k} \mid p, 2k \le m\}] \\ I^m &= \langle \{x_p x_q - x_{p+q} \mid p, q \le m\} \rangle \\ G^m &= R^m/I^m \\ F^m &\text{ is the minimal resolution of } G^m \text{ over } R \\ F &\text{ is the minimal resolution of } G \text{ over } R \\ \varepsilon(m) &= \operatorname{depth}_{R^m}(G^m) \\ \rho(m) &= \operatorname{pd}_{R^m}(G^m) = \operatorname{length}(F^m) \end{aligned}$$

Note that (**??**) has a very nice interpretation in this case. Indeed, we have

$$\rho(2m) + \varepsilon(2m) = \pi(2m) + m - 3, \tag{11}$$

where π is the usual prime-counting function. For m sufficiently large, we should have

$$\varepsilon(2m) = \#\{p \mid m \leq p \leq 2m\}.$$

The idea is that if $p_1, \ldots, p_d$ are all of the primes between m and $2m$, then it is easy to check that $\boldsymbol{x} = x_{p_1}, \ldots, x_{p_d}$ is a G^{2m}-regular sequence (the hard part is showing that this is in fact a maximal G^{2m}-regular sequence, however we will assume this is true for the moment). Thus we have $\pi(m) = \pi(2m) - \varepsilon(2m)$, and so we should be able to re-express (11) as

$$\pi(m) = \rho(2m) - m + 3. \tag{12}$$

For example, a calculation using a computer algebra program such as Singular shows $\rho(18) = 10$. Then since $\pi(9) = 4$, we have $4 = 10 - 9 + 3$, and thus (12) holds on the nose in this case.

3.13 1/30/2025

Let R be a commutative ring and let $x, y \in R$. Then the maps $x : y \xrightarrow{y} x \cap y$ and $y : x \xrightarrow{x} y \cap x$ induces an isomorphisms of R-modules

$$\frac{x : y}{x, (0 : y)} \simeq \frac{x \cap y}{xy} \simeq \frac{y : x}{y, (0 : x)}.$$

This can be shown by calculating $\mathrm{Tor}_1^R(R/x, R/y)$ in three ways.