Algebraic Topology Homework 3

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Problem 1

Exercise 1. Show that $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $fg \sim 1_Y$ and $hf \sim 1_X$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Solution 1. labelsol Observe that $h \sim hfg \sim g$. In particular, we have $fg \sim 1_Y$ and $gf \sim hf \sim 1_X$, thus g is a homotopic inverse of f. It follows that $f\colon X \to Y$ is a homotopy equivalence. For the generalization, suppose $u\colon X \to X$ is a homotopic inverse of $hf\colon X \to X$ and suppose $v\colon Y \to Y$ is a homotopic inverse of $fg\colon Y \to Y$. Then observe that $1_Y \sim (fg)v = f(gv)$ and $1_X \sim u(hf) = (uh)f$. It follows from the previous case that f is a homotopy equivalence.

Problem 2

Exercise 2. Show that S^∞ is contractible.

Solution 2. labelsol Recall that S^∞ is the unit circle
Let $T\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the shift operator defined as follows: given a sequence $\boldsymbol{x} = (x_1, x_2, x_3, \dots,)$ in $\mathbb{R}^\infty$, we set $T(\boldsymbol{x}) = (0, x_1, x_2, \dots)$. Define $F\colon S^\infty \times I \to S^\infty$ by

$$F(\boldsymbol{x}, \lambda) = \frac{(1-\lambda+\lambda T)\boldsymbol{x}}{\|(1-\lambda+\lambda T)\boldsymbol{x}\|} \tag{1}$$

where $\boldsymbol{x} \in S^\infty$ and where $\lambda \in I$. Observe that F is continuous since $1-\lambda+\lambda T$ is a bounded linear operator and since the denominator (1) is never zero. Furthermore observe that $F(-,0) = 1_{S^\infty}$ and $F(-,1) = T$. Thus F is a homotopy from 1_{S^∞} to T. Next let $\boldsymbol{e} = (1, 0, 0, \dots)$ and define $G\colon S^\infty \times I \to S^\infty$ by

$$G(\boldsymbol{x}, \lambda) = (1-\lambda)T\boldsymbol{x} + \lambda \boldsymbol{e}.$$

Clearly G is a homotopy from T to the constant map $c_{\boldsymbol{e}}$. Thus $1_{S^\infty} \sim T \sim c_{\boldsymbol{e}}$ which means S^∞ is contractible.

Problem 3

Exercise 3. Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.

Solution 3. labelsol Let X be a CW complex and suppose $X = A \cup B$ with A and B two contractible subcomplexes of X such that $Z = A \cap B$ is also contractible. Since Z is a contractible subspace of X, we have $X \sim X/Z$. If we can show X/Z is contractible, then it will follow that X is contractible since contractibility is preserved under homotopy equivalences. Thus by replacing X with X/Z if necessary, we may assume that $A \cap B = \{z\}$ is a singleton set. Since A is a contractible subspace of X, we have $X \sim X/A = B$. Finally since B is contractible, it follows that X is contractible.

Remark 1. Note that we are appealing to propositions 0.16 and 0.17 in Hatcher here which says if (X, A) is a CW pair, then (X, A) has the homotopy extension property and thus $X \to X/A$ is a homotopy equivalence. This need not hold for arbitrary topological spaces $A \subseteq X$.

Problem 4

Exercise 4. Read the proof of Proposition 0.19 in Hatcher's book and explain the square figure that's part of the proof on Page 17.

Solution 4. labelsol The square represents the homotopy of homotopies $K\colon X \times I \times I \to X$ denoted $(x, t, u) \mapsto K(x, t, u)$. The square is basically the parameter domain $I \times I$ for the pairs (t, u) with t-axis horizontal and u-axis vertical. Each point (t, u) in the square represents a continuous function $K(-, t, u)\colon X \to X$. For instance, the bottom left corner of the square represents the continuous function $K(-, 0, 0) = g_1 f$ and the bottom right corner of the square represents the continuous function $K(-, 1, 0) = 1_X$. The bottom edge of the square represents the homotopy $K(-, -, 0) = F$ where $F\colon X \times I \to X$ is the homotopy from $g_1 f$ to 1_X defined by

$$F(-, t) = \begin{cases} g_{1-2t} f & 0 \leq t \leq 1/2 \\ h_{2t-1} & 1/2 \leq t \leq 1 \end{cases}$$

More generally for each $u \in I$, the line segment $\{u\} \times I$ represents a homotopy $K(-, -, u)$ from $g_1 f$ to 1_X. This is why we call K a homotopy of homotopies.