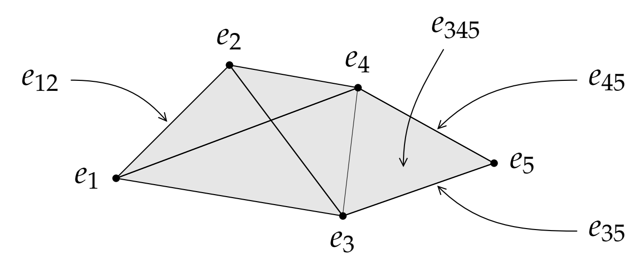
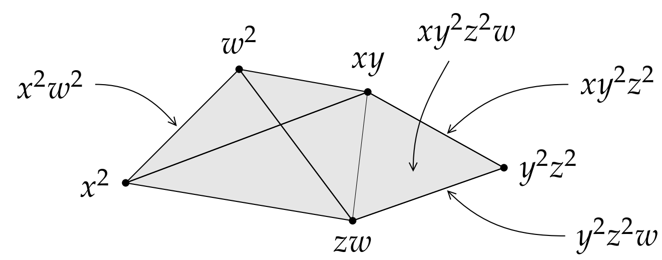
# Multigraded MDG Algebras

**Example 0.1.** ([?]) Let $\Delta_K = \Delta$ be the simplicial complex whose vertex set is $\{e_1, e_2, e_3, e_4, e_5\}$ and whose faces consists of all subsets of $e_{1234} = \{e_1, e_2, e_3, e_4\}$ and $e_{345} = \{e_3, e_4, e_5\}$, pictured below:




Let $\boldsymbol{m}_K = \boldsymbol{m} = x^2, w^2, xy, zw, y^2z^2$. Then we obtain an $\boldsymbol{m}$-labeled simplicial complex $\Delta = (\Delta, \boldsymbol{m})$ which is pictured below:




Let $F_K = F$ be the multigraded $R$-complex induced by $\Delta$. Thus the homogeneous components of $F$ as a graded $R$-module look like:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{34} + Re_{35} + Re_{45} \\
F_3 &= Re_{123} + Re_{124} + Re_{134} + Re_{234} + Re_{345} \\
F_4 &= Re_{1234}
\end{aligned}
$$

The differential $\mathrm{d}_K = \mathrm{d}$ of $F$ behaves just like the usual boundary map of the simplicial complex above except some monomials can show up as coefficients (which makes it so that the differential respects the multidegree). For instance, we have

$$\mathrm{d}(e_{1234}) = -ye_{123} + ze_{124} - we_{134} + xe_{234}.$$

Now equip $F$ with a multiplication $\mu_K = \mu$ which respects the multigrading, giving it the structure of a multigraded MDG algebra. Since $\mu$ respects the multigrading and satisfies Leibniz law, we are forced to have:

$$
\begin{aligned}
e_1 \star e_5 &= yz^2e_{14} + xe_{45} \\
e_1 \star e_2 &= e_{12} \\
e_2 \star e_5 &= y^2ze_{23} + we_{35} \\
e_2 \star e_{45} &= -yze_{234} + we_{345} \\
e_1 \star e_{35} &= yze_{134} - xe_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= -e_{124}
\end{aligned}
$$

At this point however, one can conclude that $F$ is not associative since

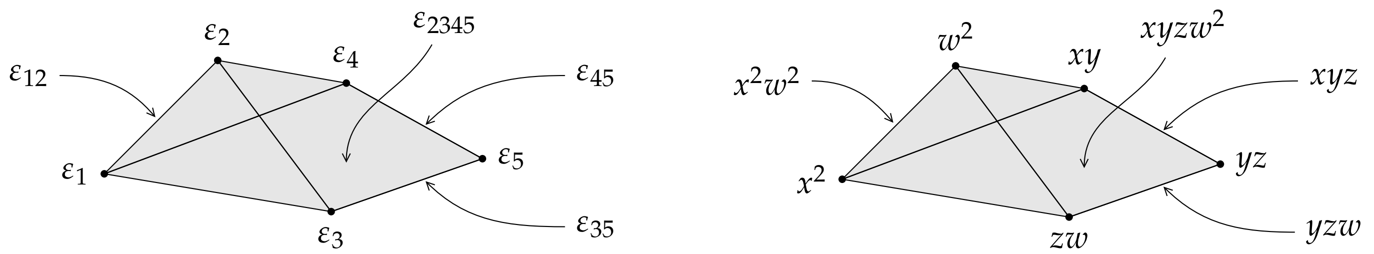
$$[e_1, e_5, e_2] := (e_1 \star e_5) \star e_2 - e_1 \star (e_5 \star e_2) = -yz\mathrm{d}(e_{1234}) \neq 0. \tag{1}$$

The multiplication isn't uniquely determined on all pairs $(e_\sigma, e_\tau)$; for instance there are two possible ways in which $\mu$ is defined at the pair $(e_5, e_{12})$. We assume that $\mu$ is defined at $(e_5, e_{12})$ by

$$e_5 \star e_{12} = yz^2e_{124} + xyze_{234} + xwe_{345}.$$

Finally, we would still like for $\mu$ to be as associative as possible (even though we already know it's not associative at the triple $(e_1, e_5, e_2)$). In particular, we want $\mu$ to be associative on all triples of the form $(e_\sigma, e_\sigma, e_\tau)$. It turns out this can be done and we will assume that $\mu$ is associative on all such triples.

**Example 0.2.** ([?]) Let $\boldsymbol{m}_{\mathrm{A}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz$, and let $F_{\mathrm{A}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. Then $F$ can be realized as the $R$-complex induced by the $\boldsymbol{m}$-labeled cellular complex pictured below:




Let's write the homogeneous components of $F$ as a graded module: we have

$$\begin{aligned}
F_0 &= R \\
F_1 &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_2 &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_3 &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_4 &= R\varepsilon_{12345}
\end{aligned}$$

The differential $\mathrm{d}_{\mathrm{A}} = \mathrm{d}$ on the non-simplicial faces as below

$$\begin{aligned}
\mathrm{d}(\varepsilon_{12345}) &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\
\mathrm{d}(\varepsilon_{1345}) &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\
\mathrm{d}(\varepsilon_{2345}) &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}.
\end{aligned}$$

We obtain a multiplication $\mu_{\mathrm{A}}$ on $F_{\mathrm{A}}$ from the one we constructed on $F_{\mathrm{K}}$ as follows: first note that the canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{A}}$ induces a multigraded comparison map $\pi\colon F_{\mathrm{K}} \to F_{\mathrm{A}}$ defined by

$$\begin{aligned}
\pi(e_5) &= yz\varepsilon_5 & \pi(e_{345}) &= 0 \\
\pi(e_{35}) &= yz\varepsilon_{35} & \pi(e_{234}) &= \varepsilon_{2345} \\
\pi(e_{45}) &= yz\varepsilon_{45} & \pi(e_{134}) &= \varepsilon_{1345} \\
\pi(e_{34}) &= x\varepsilon_{35} - w\varepsilon_{45} & \pi(e_{1234}) &= \varepsilon_{12345}
\end{aligned}$$

and $\pi(e_\sigma) = \varepsilon_\sigma$ for the remaining homogeneous basis elements. This map is locally invertible. Indeed, by base changing to $R_{yz}$, we obtain quasi-isomorphisms $F_{\mathrm{A},yz} \to 0 \leftarrow F_{\mathrm{K},yz}$. In particular, there exists a comparison map $\iota\colon F_{\mathrm{A},yz} \to F_{\mathrm{K},yz}$ which splits comparison map $\pi\colon F_{\mathrm{K},yz} \to F_{\mathrm{A},yz}$. By considering the multigrading as well as the Leibniz law, we see that

$$\begin{aligned}
\iota(\varepsilon_5) &= e_5/yz & \iota(\varepsilon_{2345}) &= -e_{234} + e_{345}/yz \\
\iota(\varepsilon_{35}) &= e_{35}/yz & \iota(\varepsilon_{1345}) &= e_{134} - e_{345}/yz \\
\iota(\varepsilon_{45}) &= e_{45}/yz & \iota(\varepsilon_{12345}) &= e_{1234}
\end{aligned}$$

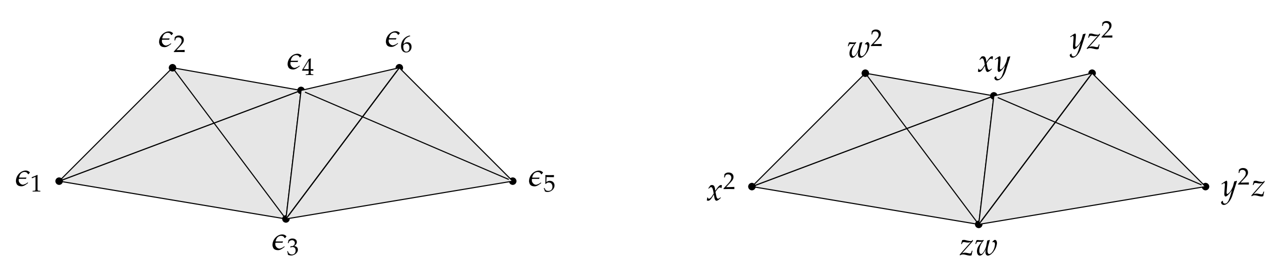
and $\iota(\varepsilon_\sigma) = e_\sigma$ for the remaining homogeneous basis elements. With this in mind, we define a multiplication $\mu_{\mathrm{A}}$ on $F_{\mathrm{A}}$ using the multiplication $\mu_{\mathrm{K}}$ on $F_{\mathrm{K},yz}$ by setting $\mu_{\mathrm{A}} = \pi\mu_{\mathrm{K}}\iota^{\otimes 2}$. In other words, we have

$$\varepsilon_\sigma \star_{\mu_{\mathrm{A}}} \varepsilon_\tau = \pi(\iota(\varepsilon_\sigma) \star_{\mu_{\mathrm{K}}} \iota(\varepsilon_\tau)) \tag{2}$$

for all homogeneous basis elements $\varepsilon_\sigma, \varepsilon_\tau$ of $F_{\mathrm{A},yz}$. It is straightforward to check that $\mu_{\mathrm{A}}$ restricts to a multiplication on $F_{\mathrm{A}}$ (the coefficients in (2) are in $R$). Note that $\mu_{\mathrm{A}}$ is not associative since

$$[\varepsilon_1, \varepsilon_5, \varepsilon_2] = -\mathrm{d}(\varepsilon_{1234}) \neq 0.$$

**Example 0.3.** Let $\boldsymbol{m}_{\mathrm{M}} = \boldsymbol{m} = x^2, w^2, zw, xy, y^2z, yz^2$, and let $F_{\mathrm{M}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ of $R$. Then $F$ can be realized as the $R$-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:



The homogeneous components of $F$ as a graded $R$-module are given below:

$$
\begin{aligned}
&F_0 = R\\
&F_1 = R\epsilon_1 + R\epsilon_2 + R\epsilon_3 + R\epsilon_4 + R\epsilon_5 + R\epsilon_6\\
&F_2 = R\epsilon_{12} + R\epsilon_{13} + R\epsilon_{14} + R\epsilon_{23} + R\epsilon_{24} + R\epsilon_{34} + R\epsilon_{35} + R\epsilon_{36} + R\epsilon_{45} + R\epsilon_{46} + R\epsilon_{56}\\
&F_3 = R\epsilon_{123} + R\epsilon_{124} + R\epsilon_{134} + R\epsilon_{234} + R\epsilon_{345} + R\epsilon_{346} + R\epsilon_{356} + R\epsilon_{456}\\
&F_4 = R\epsilon_{1234} + R\epsilon_{3456}.
\end{aligned}
$$

The canonical map $R/\boldsymbol{m}_{\mathrm{K}} \to R/\boldsymbol{m}_{\mathrm{M}}$ induces multigraded comparison maps $\pi_\lambda \colon F_{\mathrm{K}} \to F_{\mathrm{M}}$ where $\lambda \in \Bbbk$ and where $\pi_\lambda$ is defined by

$$
\begin{aligned}
\pi_\lambda(e_5) &= \lambda z\epsilon_5 + (1-\lambda)y\epsilon_6\\
\pi_\lambda(e_{35}) &= \lambda z\epsilon_{35} + (1-\lambda)y\epsilon_{36}\\
\pi_\lambda(e_{45}) &= \lambda z\epsilon_{45} + (1-\lambda)y\epsilon_{46}\\
\pi_\lambda(e_{345}) &= \lambda z\epsilon_{345} + (1-\lambda)y\epsilon_{346}
\end{aligned}
$$

and $\pi_\lambda(e_\sigma) = \epsilon_\sigma$ for the remaining homogeneous basis elements. We will choose $\lambda = 1$ and view $F_{\mathrm{K}}$ as a subcomplex of $F_{\mathrm{M}}$ via $\pi = \pi_1$. We define a multigraded multiplication $\mu_{\mathrm{M}}$ on $F_{\mathrm{M}}$ so that it extends the multiplication $\mu_{\mathrm{K}}$ on $F_{\mathrm{K}}$. Considerations of the Leibniz and multigrading tells us that we are already forced to have:

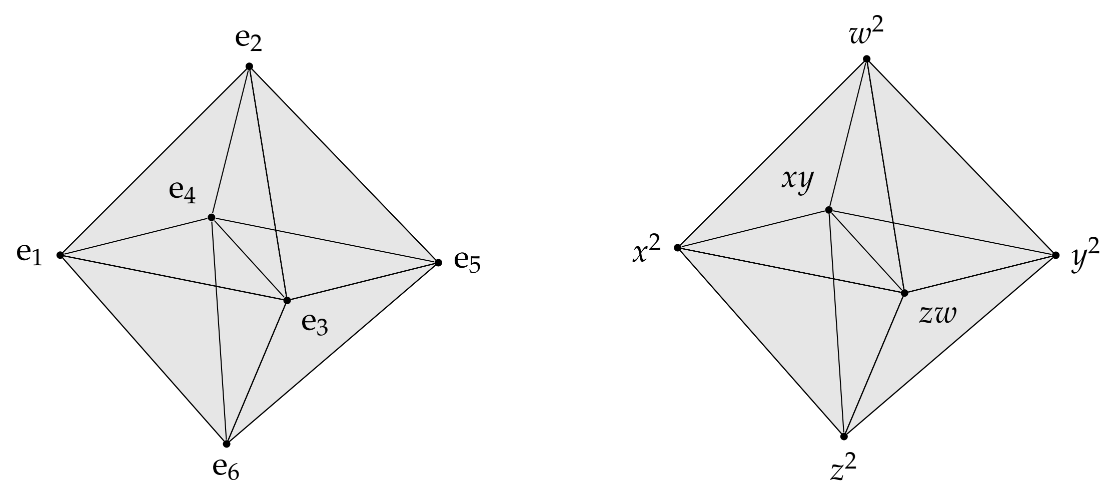
$$
\begin{aligned}
\epsilon_1 \star \epsilon_5 &= yz\epsilon_{14} + x\epsilon_{45} & \qquad \epsilon_1 \star \epsilon_6 &= z^2 e_{14} + xe_{46}\\
\epsilon_2 \star \epsilon_5 &= y^2\epsilon_{23} + w\epsilon_{35} & \epsilon_2 \star \epsilon_6 &= yz\epsilon_{23} + w\epsilon_{36}\\
\epsilon_2 \star \epsilon_{45} &= -y\epsilon_{234} + w\epsilon_{345} & \epsilon_2 \star \epsilon_{46} &= -ze_{234} + w\epsilon_{346}\\
\epsilon_1 \star \epsilon_{35} &= y\epsilon_{134} - x\epsilon_{345} & \epsilon_1 \star \epsilon_{36} &= z\epsilon_{134} - x\epsilon_{346}.
\end{aligned}
$$

In particular, $\mu_{\mathrm{K}}$ is not associative (and in fact any multigraded multiplication on $F_{\mathrm{M}}$ is not associative) since we will always have:

$$[\epsilon_1, \epsilon_5, \epsilon_2] = -y\mathrm{d}(\epsilon_{1234}) \neq 0 \quad \text{and} \quad [\epsilon_1, \epsilon_6, \epsilon_2] = -z\mathrm{d}(\epsilon_{1234}) \neq 0.$$

On the other hand, since the multiplication of $F_{\mathrm{M}}$ extends the multiplication of $F_{\mathrm{K}}$, we see that the comparison map $F_{\mathrm{K}} \to F_{\mathrm{M}}$ is multiplicative, and hence $F_{\mathrm{K}}$ is an MDG subalgebra of $F_{\mathrm{M}}$.

**Example 0.4.** Let $R = \Bbbk[x,y,z,w]$, let $\boldsymbol{m} = \boldsymbol{m}_{\mathrm{O}} = x^2, w^2, zw, xy, y^2, z^2$, and let $F_{\mathrm{O}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. Then $F$ can be realized as the $R$-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:




The homogeneous components of $F$ as a graded $R$-module are given below:

$$
\begin{aligned}
&F_0 = R\\
&F_1 = R\mathrm{e}_1 + R\mathrm{e}_2 + R\mathrm{e}_3 + R\mathrm{e}_4 + R\mathrm{e}_5 + R\mathrm{e}_6\\
&F_2 = R\mathrm{e}_{12} + R\mathrm{e}_{13} + R\mathrm{e}_{14} + R\mathrm{e}_{16} + R\mathrm{e}_{23} + R\mathrm{e}_{24} + R\mathrm{e}_{25} + R\mathrm{e}_{34} + R\mathrm{e}_{35} + R\mathrm{e}_{36} + R\mathrm{e}_{45} + R\mathrm{e}_{46} + R\mathrm{e}_{56}\\
&F_3 = R\mathrm{e}_{123} + R\mathrm{e}_{124} + R\mathrm{e}_{134} + R\mathrm{e}_{136} + R\mathrm{e}_{146} + R\mathrm{e}_{234} + R\mathrm{e}_{235} + R\mathrm{e}_{245} + R\mathrm{e}_{345} + R\mathrm{e}_{346} + R\mathrm{e}_{356} + R\mathrm{e}_{456}\\
&F_4 = R\mathrm{e}_{1234} + R\mathrm{e}_{1346} + R\mathrm{e}_{2345} + R\epsilon_{3456}.
\end{aligned}
$$

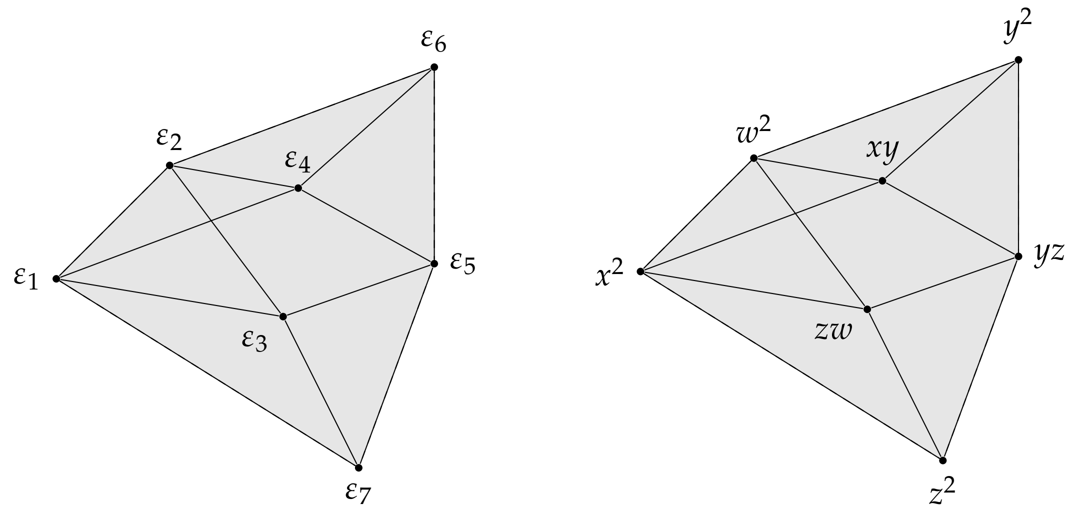
The canonical map $R/\boldsymbol{m}_{\mathrm{M}} \to R/\boldsymbol{m}_{\mathrm{O}}$ induces an injective multigraded comparison map $F_{\mathrm{M}} \to F_{\mathrm{O}}$ and we identify $F_{\mathrm{M}}$ with this subcomplex of $F_{\mathrm{O}}$. This time it is impossible extend the multiplication of $F_{\mathrm{M}}$ to a multigraded multiplication on $F_{\mathrm{O}}$. Indeed, assuming we could extend the multiplication, then

$$\begin{aligned} z(\mathrm{e}_2 \star \mathrm{e}_5) &= \mathrm{e}_2 \star (z\mathrm{e}_5) \\ &= \epsilon_2 \star \epsilon_5 \\ &= y^2\epsilon_{23} + w\epsilon_{35} \\ &= y^2\mathrm{e}_{23} + w\mathrm{e}_{35}, \end{aligned}$$

which would imply $\mathrm{e}_2 \star \mathrm{e}_5 = (y^2/z)\mathrm{e}_{23} + (w/z)\mathrm{e}_{35}$. However this is obviously not in $F_{\mathrm{O}}$ since the coefficients are not in $R$. On the other hand, it turns out that there is a better choice of multigraded multiplication that we can use on $F_{\mathrm{O}}$ anyways; namely namely $\mathrm{e}_2 \star \mathrm{e}_5 = \mathrm{e}_{25}$. In fact, this is the only possible choice we can make if we want the multiplication to be multigraded. Similarly, we are forced to have $\mathrm{e}_1 \star \mathrm{e}_6 = \mathrm{e}_{16}$. One can show that this extends to an *associative* multigraded multiplication on $F_{\mathrm{O}}$. We define it below on some of the homogeneous basis elements:

$$\begin{aligned} \mathrm{e}_1 \star \mathrm{e}_5 &= y\mathrm{e}_{14} + x\mathrm{e}_{45} & \qquad \mathrm{e}_2 \star \mathrm{e}_{46} &= -z\mathrm{e}_{234} + w\mathrm{e}_{346} \\ \mathrm{e}_2 \star \mathrm{e}_6 &= z\mathrm{e}_{23} + w\mathrm{e}_{35} & \mathrm{e}_2 \star \mathrm{e}_{56} &= -z\mathrm{e}_{235} + w\mathrm{e}_{356} \\ \mathrm{e}_1 \star \mathrm{e}_{25} &= y\mathrm{e}_{124} - x\mathrm{e}_{245} & \mathrm{e}_2 \star \mathrm{e}_{146} &= \mathrm{e}_{1234} + \mathrm{e}_{1346} \\ \mathrm{e}_1 \star \mathrm{e}_{35} &= y\mathrm{e}_{134} - x\mathrm{e}_{345} & \mathrm{e}_2 \star \mathrm{e}_{456} &= \mathrm{e}_{2345} + \mathrm{e}_{3456} \\ \mathrm{e}_1 \star \mathrm{e}_{56} &= y\mathrm{e}_{146} + x\mathrm{e}_{456} & \mathrm{e}_1 \star \mathrm{e}_{235} &= \mathrm{e}_{1234} + \mathrm{e}_{2345} \\ \mathrm{e}_2 \star \mathrm{e}_{16} &= -z\mathrm{e}_{123} - w\mathrm{e}_{136} & \mathrm{e}_1 \star \mathrm{e}_{356} &= \mathrm{e}_{1346} + \mathrm{e}_{3456} \end{aligned}$$

**Example 0.5.** Let $\boldsymbol{m}_{\mathrm{N}} = \boldsymbol{m} = x^2, w^2, zw, xy, yz, y^2, z^2$, and let $F_{\mathrm{N}} = F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. Then $F$ can be realized as the $R$-complex induced by the $\boldsymbol{m}$-labeled simplicial complex pictured below:




It is visibly clear that the map $R/\boldsymbol{m}_{\mathrm{A}} \to R/\boldsymbol{m}_{\mathrm{N}}$ induces a comparison map $\iota\colon F_{\mathrm{A}} \to F_{\mathrm{N}}$ defined by $\iota(\varepsilon_\sigma) = \varepsilon_\sigma$ for all homogeneous basis element $\varepsilon_\sigma$ of $F_{\mathrm{A}}$ (in particular, there are no monomials showing up in this comparison map). Thus we run into the same problem as in Example (0.2), and so there is no way to choose a multigraded multiplication on $F_{\mathrm{N}}$ which is associative.

**Example 0.6.** Let $m = xyzw$, let $\boldsymbol{m} = mx, my, mz, mw$, and let $F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. Then $F$ is just the Taylor resolution with respect to $\boldsymbol{m}$ and is supported on the 3-simplex. Usually $F$ comes equipped with an associative multiplication giving it the structure of a DG algebra, however we wish to consider a different multiplication $\mu$ which gives it the structure of a non-associative MDG algebra. In particular, this multiplication will start out as:

$$\begin{aligned} e_1 \star e_2 &= xyzwe_{12} \\ e_1 \star e_3 &= xyz^2e_{14} - x^2yze_{34} \\ e_2 \star e_3 &= xyzwe_{23} \\ e_1 \star e_{23} &= xyzwe_{123} + xy^2ze_{134} \\ e_2 \star e_{14} &= -xyzwe_{124} \\ e_2 \star e_{34} &= xyzwe_{234} \end{aligned}$$

At this point, no matter how we extend this multiplication, it won't be associative since

$$[e_1, e_2, e_3] = x^2y^2z^2w\mathrm{d}(e_{1234}).$$

#### 0.0.1 Multigraded Multiplications coming from the Taylor Algebra

In this subsubsection, we want to explain how all of the multigraded multiplications that we've considered in the examples above come from a Taylor multiplication in the following sense: let $R = \Bbbk[x_1, \dots, x_d]$, let $I$ be a monomial ideal in $R$, let $F$ be the minimal $R$-free resolution of $R/I$, and let $T$ be the Taylor algebra resolution of $R/I$. The Taylor multiplication is denoted $\nu_T$. Let $\nu$ be a possibly different multiplication on $T$. We write $T_\nu$ to be the MDG $R$-algebra whose underlying $R$-complex is the same as the underlying complex of $T$ but whose multiplication is $\nu$. Since $F$ is the minimal $R$-free resolution of $R/I$ and since $T$ is an $R$-free resolution of $R/I$, there exists multigraded chain maps $\iota\colon F \to T$ and $\pi\colon T \to F$ which lift the identity map $R/I \to R/I$ such that $\iota\colon F \to T$ is injective and is split by $\pi\colon T \to F$, meaning $\pi\iota = 1$. By identifying $F$ with $\iota(F)$ if necessary, we may assume that $\iota\colon F \subseteq T$ is inclusion and that $\pi\colon T \to F$ is a **projection**, meaning $\pi\colon T \to F$ is a surjective chain map which satisfies $\pi^2 = \pi$, or alternatively, $\pi\colon T \to T$ is a chain map with $\operatorname{im}\pi = F$. In what follows, we fix $\iota\colon F \subseteq T$ once and for all and we denote by $\mathcal{P}(T, F)$ to be the set of all projections $\pi\colon T \to F$. For each $\mu \in \operatorname{Mult}(F)$, we denote by $\operatorname{Mult}(T/\mu)$ to be the set of all multiplications on $T$ which extends $\mu$:

$$\operatorname{Mult}(T/\mu) = \{\nu \in \operatorname{Mult}(T) \mid \nu|_{F^{\otimes 2}} = \nu\iota^{\otimes 2} = \mu\}.$$

Observe that if $\pi \in \mathcal{P}(T, F)$ and $\nu \in \operatorname{Mult}(T/\mu)$, then $\pi\nu \in \operatorname{Mult}(T/\mu)$. Indeed, $\pi\nu$ is clearly a multiplication on $T$. Furthermore, since $\pi$ is a projective and since $\mu$ lands in $F$, we have $\pi\mu = \mu$. Therefore

$$\pi\nu\iota^{\otimes 2} = \pi\mu = \mu,$$

so $\pi\nu$ restricts to $\mu$ as well. Next, observe that if $\pi \in \mathcal{P}(T, F)$ and $\mu \in \operatorname{Mult}(F)$, then $\widehat{\mu}_\pi := \mu\pi^{\otimes 2} \in \operatorname{Mult}(T/\mu)$. We call $\widehat{\mu} = \widehat{\mu}_\pi$ the **trivial extension** of $\mu$ with respect to $\pi$ for the following reasons: first note that for each $\nu \in \operatorname{Mult}(T/\mu)$, the inclusion map $\iota\colon F_\mu \subseteq T_\nu$ is multiplicative since $\nu\iota^{\otimes 2} = \mu = \iota\mu$, however $\pi\colon T_\nu \to F_\mu$ need not be multiplicative in general. In the case of the trivial extension $\widehat{\mu}$ however, $\pi\colon T_{\widehat{\mu}} \to F_\mu$ is multiplicative since

$$\pi\widehat{\mu} = \pi\mu\pi^{\otimes 2} = \mu\pi^{\otimes 2}.$$

Next, note that since $\pi\colon T \to F$ splits the inclusion $\iota\colon F \subseteq T$, we obtain isomorphism $\theta_\pi\colon T \simeq F \oplus H$ of $R$-complexes, where $H = \ker\pi$ is a trivial $R$-complex with $H_0 = 0 = H_1$, and where $\theta_\pi = (\pi, 1 - \pi)$. There's an obvious multiplication that we can give $F \oplus H$, namely $\mu \oplus 0$, where $0\colon H \otimes H \to H$ is the zero map. Equip $F \oplus H$ with this multiplication. We claim that $\theta_\pi\colon T_{\widehat{\mu}} \to F \oplus H$ is multiplicative, and hence an isomorphism of MDG $R$-algebras. Indeed, we have

$$\begin{aligned}
\theta_\pi\widehat{\mu} &= (\pi\widehat{\mu}, (1-\pi)\widehat{\mu}) \\
&= (\pi\widehat{\mu}, \widehat{\mu} - \pi\widehat{\mu}) \\
&= (\widehat{\mu}, \widehat{\mu} - \widehat{\mu}) \\
&= (\widehat{\mu}, 0) \\
&= (\mu\pi^{\otimes 2}, 0) \\
&= (\mu \oplus 0)(\pi^{\otimes 2}, 1 - \pi^{\otimes 2}) \\
&= (\mu \oplus 0)\theta_\pi^{\otimes 2}.
\end{aligned}$$

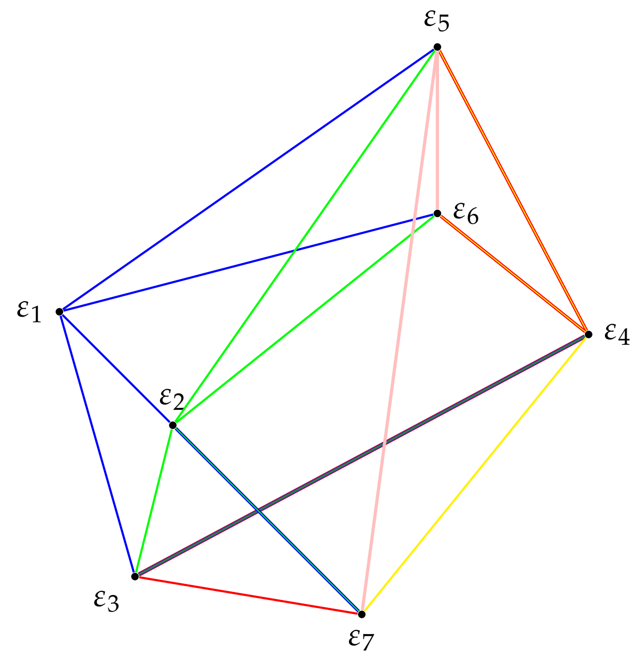
In particular, every $b \in T$ can expressed in the form $b = a + c$ for unique $a \in F$ and unique $c \in H$. If $b_1, b_2 \in T$ have the unique expressions $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$, then we have $b_1 \star_\nu b_2 = a_1 \star_\mu a_2$.

**Example 0.7.** The multiplication $\mu$ in Example (0.1) is given by $\mu = \pi\nu_T\iota^{\otimes 2}$ where $T$ is the Taylor algebra resolution of $R/\boldsymbol{m}_{\mathrm{M}}$ and where $\pi\colon T \to F$ is defined by

$$\begin{aligned}
\pi(e_{15}) &= yz^2e_{14} + xe_{45} \\
\pi(e_{25}) &= y^2ze_{23} + we_{35} \\
\pi(e_{245}) &= -yze_{234} + we_{35} \\
\pi(e_{235}) &= 0 \\
\pi(e_{2345}) &= 0 \\
&\vdots
\end{aligned}$$

and so on.

**Example 0.8.** Let $R = \Bbbk[x, y, z, u, v, w]$, let $I = \langle w^3, zw^2, uvw, xuv, x^2y, x^3, y^2z^2\rangle$, and let $F$ be the minimal free resolution of $R/I$ over $R$. One can visualize $F$ as below:




Now choose a multigraded multiplication $\mu$ on $F$ giving it the structure of a multigraded MDG algebra. In low homological degrees, we are forced to have certain multiplications. For instance, we must have

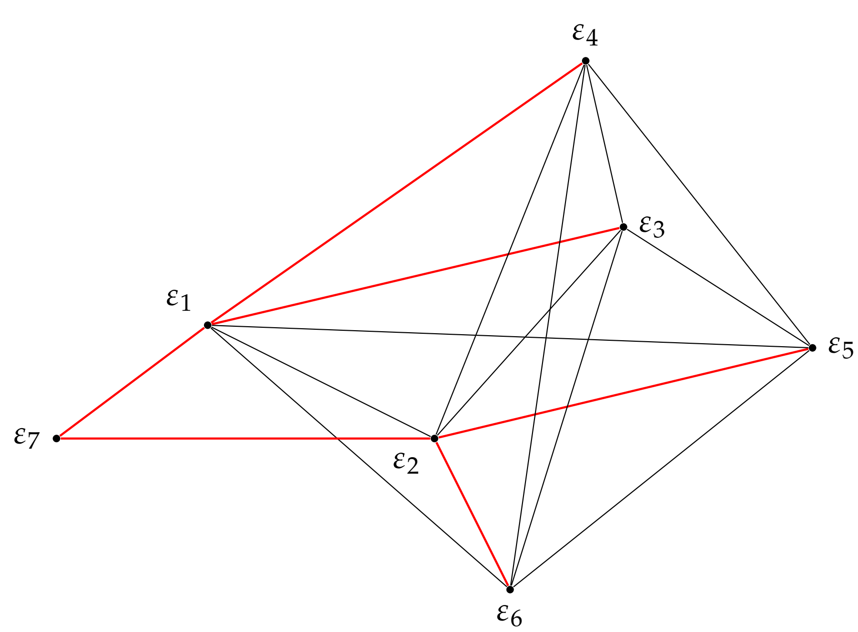
$$\varepsilon_1\varepsilon_5 = \varepsilon_{15}$$
$$\varepsilon_1\varepsilon_2 = w^2\varepsilon_{12}$$
$$\vdots$$

and so on. In the image above, the edges colored in blue correspond to multiplications with $\varepsilon_1$. The edges colored in green correspond to multiplications with $\varepsilon_2$. The edges colored in red correspond to multiplications with $\varepsilon_3$. The multiplications colored in yellow correspond to multiplications with $\varepsilon_4$. Finally the edges colored in pink correspond to multiplications with $\varepsilon_5$, $\varepsilon_6$, and $\varepsilon_7$. The associated primes of $R/I$ are

$$\begin{aligned}
\mathfrak{p}_1 &= \langle x, z, w\rangle \\
\mathfrak{p}_2 &= \langle x, y, w\rangle \\
\mathfrak{p}_3 &= \langle x, z, v, w\rangle \\
\mathfrak{p}_4 &= \langle x, y, v, w\rangle \\
\mathfrak{p}_5 &= \langle x, z, u, w\rangle \\
\mathfrak{p}_6 &= \langle x, y, u, w\rangle \\
\mathfrak{p}_7 &= \langle x, y, z, v, w\rangle \\
\mathfrak{p}_8 &= \langle x, y, z, u, w\rangle
\end{aligned}$$

We have $\dim(R/I) = 3$ and $\operatorname{depth}(R/I) = 1$. Clearly $u - v$ is an $(R/I)$-regular element. We have $R/\langle I, u-v\rangle \cong S/J$ where $S = \Bbbk[x, y, z, u, w]$ and where $J = \langle x^3, w^3, x^2y, zw^2, y^2z^2, xu^2, u^2w\rangle$.

**Example 0.9.** Let $R = \Bbbk[x, y, z, u, v, w]$, let $I = \langle vw, xy, w^3, u^2w, x^2z, x^3, y^2z^2uv\rangle$, and let $F$ be the minimal free resolution of $R/I$ over $R$. One can visualize $F$ as below:



The associated primes of $R/I$ are

$$\begin{aligned}
\mathfrak{p}_1 &= \langle x, v, w \rangle \\
\mathfrak{p}_2 &= \langle x, u, w \rangle \\
\mathfrak{p}_3 &= \langle x, z, w \rangle \\
\mathfrak{p}_4 &= \langle x, y, w \rangle \\
\mathfrak{p}_5 &= \langle x, u, v, w \rangle \\
\mathfrak{p}_6 &= \langle x, y, z, w \rangle \\
\mathfrak{p}_7 &= \langle x, y, u, v, w \rangle \\
\mathfrak{p}_8 &= \langle x, y, z, u, v, w \rangle
\end{aligned}$$

We have $\dim(R/I) = 3$ and $\operatorname{depth}(R/I) = 0$. We will have

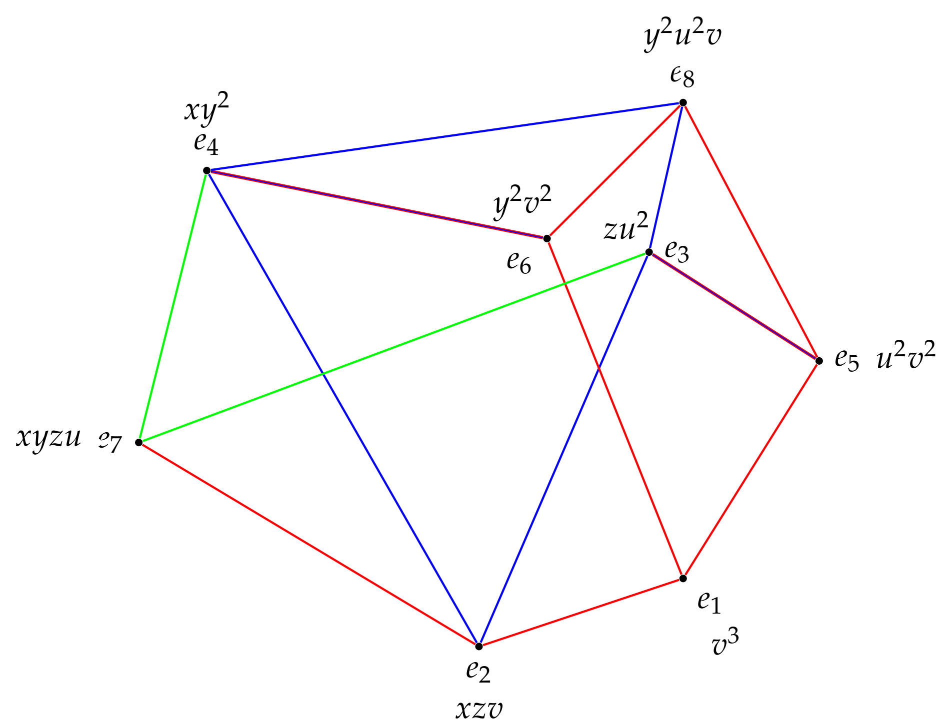
$$\begin{aligned}
[e_3, e_7, e_6] &= yz^2u\mathrm{d}(e_{1236}) \\
[e_3, e_7, e_5] &= yz^2u\mathrm{d}(e_{1235}) \\
[e_4, e_7, e_5] &= yz^2u\mathrm{d}(e_{1245}) \\
[e_4, e_7, e_6] &= yz^2u\mathrm{d}(e_{1246})
\end{aligned}$$

In the minimal free resolution, we have

$$\mathrm{d}(e_{127}) = yz^2ue_{12} - xe_{17} + we_{27}.$$

If we set $J = \langle I, yz^2u \rangle$, then the minimal free resolution of $R/J$ over $R$ becomes symmetric. In particular, note that $J = \langle vw, xy, w^3, u^2w, x^2z, x^3, yz^2u \rangle$.

**Example 0.10.** Let $R = \mathbb{k}[x, y, z, u, v]$, let $\boldsymbol{m} = v^3, xzv, zu^2, xy^2, u^2v^2, y^2v^2, xyzu, y^2u^2v$, and let $F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. One can visualize $F$ as below:




The associated primes of $R/I$ are

$$\begin{aligned}
\mathfrak{p}_1 &= \langle y, u, v \rangle \\
\mathfrak{p}_2 &= \langle x, u, v \rangle \\
\mathfrak{p}_3 &= \langle y, z, w \rangle \\
\mathfrak{p}_4 &= \langle x, z, v \rangle \\
\mathfrak{p}_5 &= \langle y, z, u \rangle \\
\mathfrak{p}_6 &= \langle x, y, u, v \rangle
\end{aligned}$$

We have $\dim(R/I) = 2$ and $\operatorname{depth}(R/I) = 0$. We will have

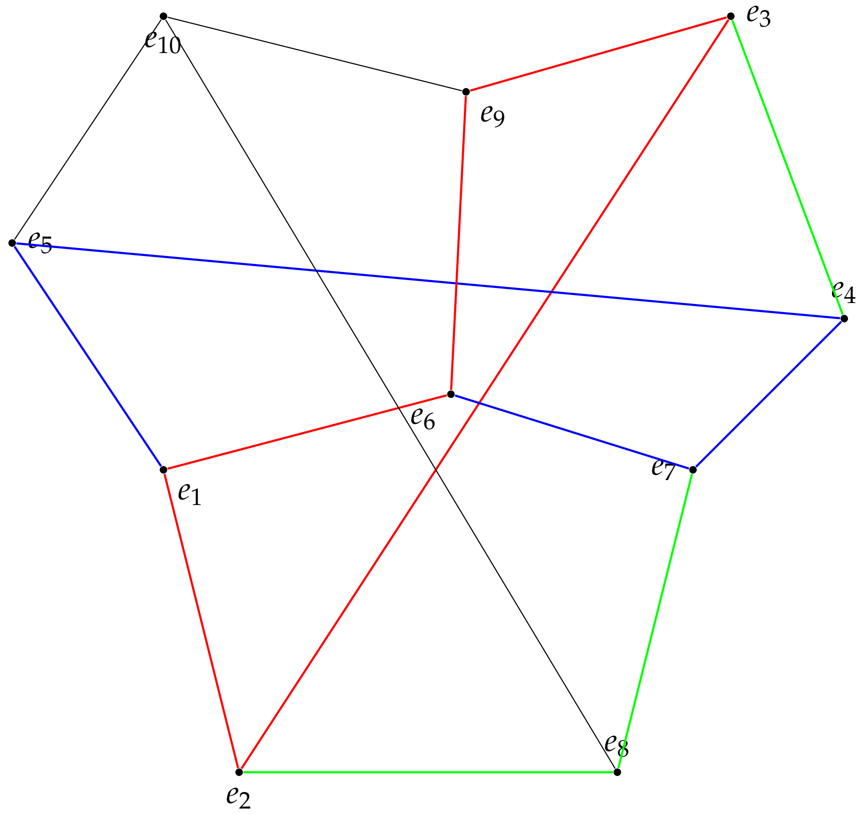
$$\begin{aligned}
[e_3, e_7, e_6] &= yz^2u\mathrm{d}(e_{1236}) \\
[e_3, e_7, e_5] &= yz^2u\mathrm{d}(e_{1235}) \\
[e_4, e_7, e_5] &= yz^2u\mathrm{d}(e_{1245}) \\
[e_4, e_7, e_6] &= yz^2u\mathrm{d}(e_{1246})
\end{aligned}$$

In the minimal free resolution, we have

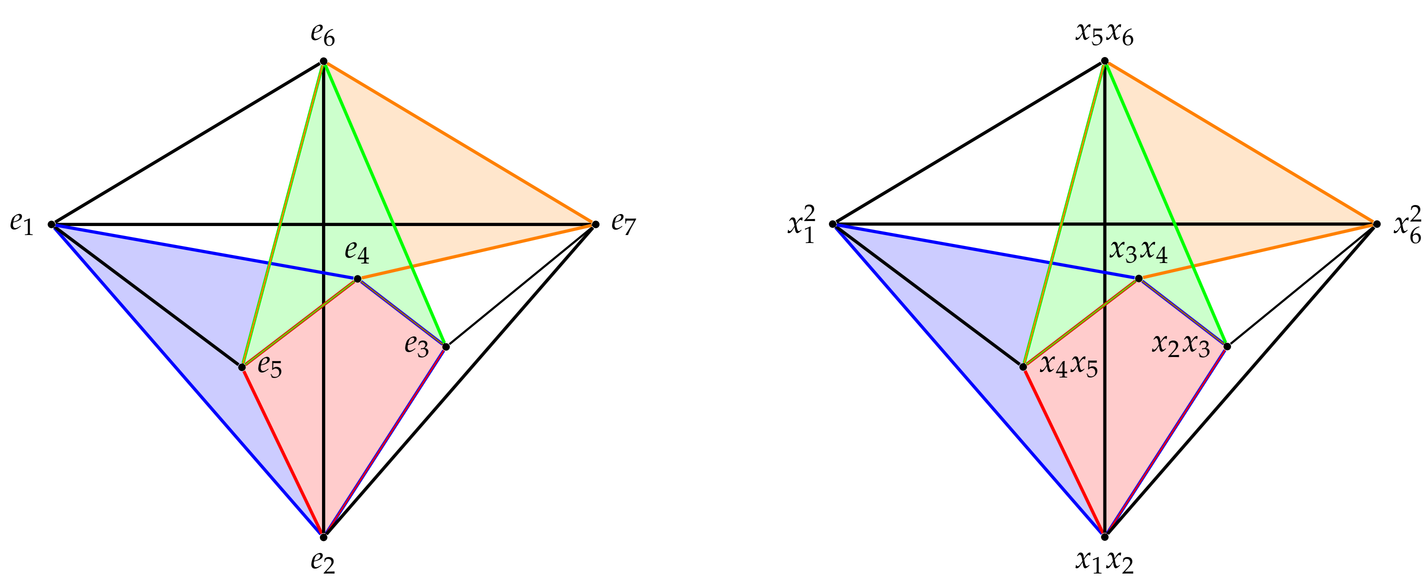
$$\mathrm{d}(e_{127}) = yz^2ue_{12} - xe_{17} + we_{27}.$$

If we set $J = \langle I, yz^2u \rangle$, then the minimal free resolution of $R/J$ over $R$ becomes symmetric. In particular, note that $J = \langle vw, xy, w^3, u^2w, x^2z, x^3, yz^2u \rangle$.

**Example 0.11.** Let $R = \Bbbk[a,b,c,d,e,f]$, let $\boldsymbol{m} = abc, abe, acf, ade, adf, bcd, bdf, bef, cde, cef$, and let $F$ be the minimal free resolution of $R/\boldsymbol{m}$ over $R$. One can visualize $F$ as below:




**Example 0.12.** Let $R = \Bbbk[x_1,x_2,x_3,x_4,x_5,x_6]$, let $\boldsymbol{m} = x_1^2, x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6^2$, and let $F$ be the minimal free resolution of $R/\boldsymbol{m}$ of $R$. One can visualize $F$ as being supported on the $\boldsymbol{m}$-labeled cellular complex below:




The complex in homological degree 1 consists of seven 0-simplices and the complex in homological degree 2 consists of sixteen 1-simplices. The differential is defined on all simplices via the Taylor rule (for example $\mathrm{d}e_1 = x_1^2$ and $\mathrm{d}e_{12} = x_2e_1 - x_1e_2$). The complex in homological degree 3 consists of twelve 2-simplices and four squares (which we shaded in blue, red, green, and orange above). The differential on the squares is given by

$$\begin{aligned}
\mathrm{d}e_{1234} &= x_3x_4e_{12} + x_1x_4e_{23} - x_2e_{14} + x_1^2e_{34} \\
\mathrm{d}e_{2345} &= x_4x_5e_{23} + x_1x_5e_{34} - x_3e_{25} + x_1x_2e_{45} \\
\mathrm{d}e_{3456} &= x_5x_6e_{34} + x_2x_6e_{45} - x_4e_{36} + x_2x_3e_{56} \\
\mathrm{d}e_{4567} &= x_6^2e_{45} + x_3x_6e_{56} - x_5e_{47} + x_3x_4e_{67}
\end{aligned}$$

The complex in homological degree 4 consists of three 3-simplices, three Avramov tetrahedra, and two pyramids. The differential on the Avramov tetrahedra and pyramids is given by

$$
\begin{aligned}
\mathrm{d}e_{12345} &= x_5e_{1234} - x_3e_{125} + x_2e_{145} - x_1e_{2345} \\
\mathrm{d}e_{23456} &= x_6e_{2345} - x_4e_{236} + x_3e_{256} - x_1e_{3456} \\
\mathrm{d}e_{34567} &= x_6e_{3456} - x_5e_{347} + x_4e_{367} - x_2e_{4567} \\
\mathrm{d}e_{123457} &= x_6^2e_{1234} - x_3x_4e_{127} - x_1x_4e_{237} + x_2e_{147} - x_1^2e_{347} \\
\mathrm{d}e_{134567} &= x_6^2e_{145} + x_3x_6e_{156} - x_5e_{147} + x_3x_4e_{167} - x_1^2e_{4567}
\end{aligned}
$$

Finally, the complex in homological degree 5 consists of one 4-cell, and the differential on it is given by

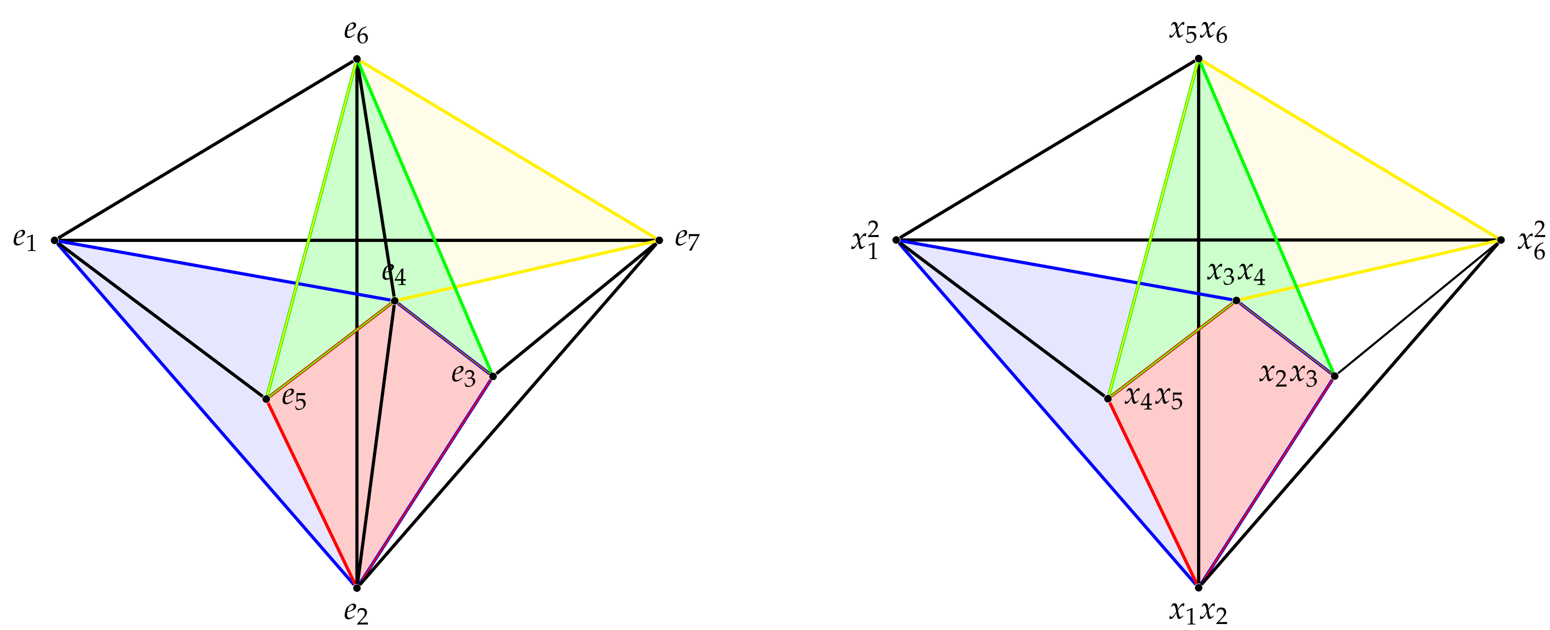
$$
\mathrm{d}e_{1234567} = x_6^2e_{12345} + x_3x_6e_{1256} + x_1x_6e_{23456} - x_5e_{123457} + x_3x_4e_{1267} + x_1x_4e_{2367} - x_2e_{134567} + x_1^2e_{34567}
$$

We have

$$
\begin{aligned}
e_1e_{36} &= e_{126} + e_{236} \\
e_1e_{26} &= e_{126} \\
e_1e_{37} &= e_{127} + e_{237} \\
e_1e_{27} &= e_{127} \\
e_1e_{237} &= 0 \\
e_1e_{2345} &= e_{12345} \\
e_1e_{3456} &= e_{23456} + e_{12345} + e_{1256} \\
e_1e_{4567} &= e_{14567} \\
e_1e_{367} &=
\end{aligned}
$$

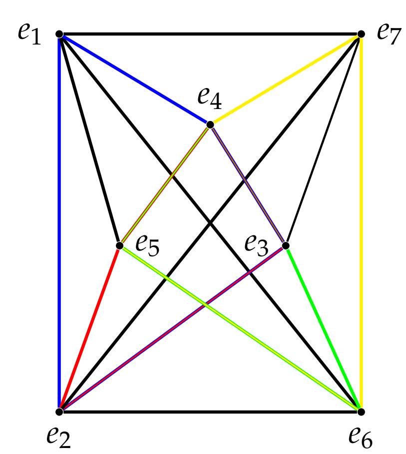
We have

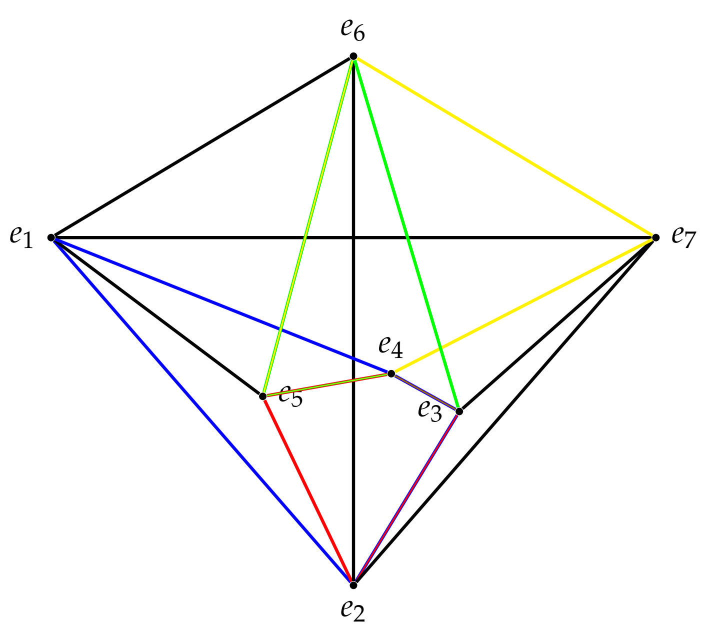
**Example 0.13.** Let $R = \Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$, let $\boldsymbol{m} = x_1^2, x_1x_2, x_2^2x_3^2, x_3x_4, x_4^2x_5^2, x_5x_6, x_6^2$, and let $F$ be the minimal free resolution of $R/\boldsymbol{m}$ of $R$. One can visualize $F$ as being supported on the $\boldsymbol{m}$-labeled cellular complex below:



The homogeneous components of $F$ as a graded $R$-module are given below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 + Re_6 + Re_7 \\
F_2 &= Re_{12} + Re_{14} + Re_{15} + Re_{16} + Re_{17} + Re_{23} + Re_{25} + Re_{26} + Re_{27} \\
&\quad + Re_{34} + Re_{36} + Re_{37} + Re_{45} + Re_{47} + Re_{56} + Re_{57} \\
F_3 &= Re_{1234} + Re_{2345} + Re_{3456} + Re_{4567} + Re_{125} + Re_{126} + Re_{127} + Re_{145} \\
&\quad + Re_{147} + Re_{156} + Re_{167} + Re_{236} + Re_{237} + Re_{256} + Re_{267} + Re_{347} + Re_{367} \\
F_4 &= Re_{12345} + Re_{1256} + Re_{23456} + Re_{1267} + Re_{2367} + Re_{34567} + Re_{12347} + Re_{14567} \\
F_5 &= Re_{1234567}
\end{aligned}
$$




Let $R = \Bbbk[\boldsymbol{x}] = \Bbbk[x_1, \ldots, x_n]$ and let $\langle \boldsymbol{u} \rangle = \langle u_1, \ldots, u_s \rangle$ be a monomial ideal of $R$.

**Proposition 0.1.** *For each $k \geq 3$, let $R_k = \Bbbk[x_1, x_2, \ldots]$, let $I_k = \langle x_1^2, x_1x_2, \ldots, x_{k-1}x_k, x_k^2 \rangle$ and let $a_k = \#\mathrm{Ass}(R/I_k)$. Then*

$$a_k = a_{k-2} + a_{k-3}$$

*with $a_0 = 1$ and $a_1 = 0 = a_2$.*