Homework 5

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Problem 1

Exercise 1. Let $f_1(x) = x - 1$ and let $f_2(x) = (x-3)^2 + 1$. For this problem, we consider the following BOP:

$$\begin{array}{ll} \text{minimize} & [f_1(x), f_2(x)] \\ \text{subject to} & x \geq 1. \end{array} \tag{1}$$

1. Formulate the Benson problem $\mathrm{P}(\boldsymbol{\ell}, x^0)$.
2. Let $x^0 = 1$. Use any method of your choice and find an optimal solution to $\mathrm{P}(\boldsymbol{\ell}, x^0)$. What do you conclude?
3. Let $x^0 = 5$. Use any method of your choice and find an optimal solution to $\mathrm{P}(\boldsymbol{\ell}, x^0)$. What do you conclude?

Solution 1. 1. Benson's problem $\mathrm{P}(\boldsymbol{\ell}, x^0)$ is

$$\begin{array}{ll} \text{maximize} & \ell_1 + \ell_2 \\ \text{subject to} & f_1(x^0) - f_1(x) - \ell_1 = 0 \\ & f_2(x^0) - f_2(x) - \ell_2 = 0 \\ & \boldsymbol{\ell} \geqq 0 \\ & x \geq 1. \end{array}$$

Using the fact that $f_1(x) = x - 1$ and $f_2(x) = (x-3)^2 + 1$, we can rewrite this as

$$\begin{array}{ll} \text{maximize} & \ell_1 + \ell_2 \\ \text{subject to} & x^0 - x - \ell_1 = 0 \\ & (x^0 - 3)^2 - (x-3)^2 - \ell_2 = 0 \\ & \boldsymbol{\ell} \geqq 0 \\ & x \geq 1. \end{array}$$

2. Suppose $x^0 = 1$. Then Benson's problem simplifies to

$$\begin{array}{ll} \text{maximize} & \ell_1 + \ell_2 \\ \text{subject to} & 1 - x - \ell_1 = 0 \\ & 4 - (x-3)^2 - \ell_2 = 0 \\ & \boldsymbol{\ell} \geqq 0 \\ & x \geq 1. \end{array} \tag{2}$$

Since $\ell_1 = 1 - x$, $x \geq 1$, and $\ell_1 \geq 0$, we see that $x = 1$ and $\ell_1 = 0$. Then from $4 - (x-3)^2 - \ell_2 = 0$ and $x = 1$, we see that $\ell_2 = 0$. Therefore $x^* = 1$ is an optimal solution to (2), and since the optimal objective value is 0, it follows that $x^0 = 1$ is an efficient solution to (1).

3. Suppose $x^0 = 5$. Then Benson's problem simplifies to

$$\begin{array}{ll} \text{maximize} & \ell_1 + \ell_2 \\ \text{subject to} & 5 - x - \ell_1 = 0 \\ & 4 - (x-3)^2 - \ell_2 = 0 \\ & \boldsymbol{\ell} \geqq 0 \\ & x \geq 1. \end{array} \tag{3}$$

Observe that $\boldsymbol{\ell} \geq 0$ and $x \geq 1$ if and only if $1 \leq x \leq 4$. Therefore since $\ell_1 = 5 - x$ and $\ell_2 = 4 - (x-3)^2$, we can simplify (3) to

$$\begin{array}{ll} \text{maximize} & 9 - x - (x-3)^2 \\ \text{subject to} & 1 \leq x \leq 4 \end{array} \tag{4}$$

This is easily seen to have a global maximum at $x^* = 5/2$ with optimal objective value being $25/4$ (the graph of $9 - x - (x-3)^2$ is an upside down parabola with vertex $(5/2, 25/4)$). It follows that $x^* = 5/2$ is an efficient solution to (1).

Problem 2

Exercise 2. In this problem, we consider the MOP:

$$\min_{x \in X} (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})) \tag{5}$$

where $X \subseteq \mathbb{R}^n$. Let $\boldsymbol{w} \in \mathbb{R}^m_>$ and let $\boldsymbol{y}^{\mathrm{U}} = \boldsymbol{y}^{\mathrm{I}} - \boldsymbol{\varepsilon}$ be a utopian point of (5) where $\boldsymbol{y}^{\mathrm{I}}$ is the ideal point of (5) and where $\boldsymbol{\varepsilon} \in \mathbb{R}^m_>$. Consider the weighted-norm problem:

$$\min_{x \in X} \max_{1 \leq i \leq m} \left\{ \omega_i (f_i(\boldsymbol{x}) - y_i^{\mathrm{U}}) \right\} \tag{6}$$

Prove that if $\boldsymbol{x}^* \in X$ is a unique optimal solution to (6), then $\boldsymbol{x}^*$ is an efficient solution to MOP.

Solution 2. Assume for a contradiction that $\boldsymbol{x}^*$ is the unique optimal solution to (6) but that $\boldsymbol{x}^*$ is not an efficient solution to (5). Then there exists some $\boldsymbol{x}' \in X$ which is not equal to $\boldsymbol{x}^*$ such that $f(\boldsymbol{x}') \leq f(\boldsymbol{x}^*)$. Then we have that

$$\omega_i(f_i(\boldsymbol{x}') - y_i^{\mathrm{U}}) \leq \omega_i(f_i(\boldsymbol{x}^*) - y_i^{\mathrm{U}})$$

for each $1 \leq i \leq m$. Now suppose that

$$\max_{1 \leq i \leq m} \left\{ \omega_i (f_i(\boldsymbol{x}) - y_i^{\mathrm{U}}) \right\} = \omega_{i_0}(f_{i_0}(\boldsymbol{x}) - y_{i_0}^{\mathrm{U}})$$

for some $1 \leq i_0 \leq m$. Then we have

$$\begin{aligned} \max_{1 \leq i \leq m} \left\{ \omega_i (f_i(\boldsymbol{x}') - y_i^{\mathrm{U}}) \right\} &= \omega_{i_0}(f_{i_0}(\boldsymbol{x}') - y_{i_0}^{\mathrm{U}}) \\ &\leq \omega_{i_0}(f_{i_0}(\boldsymbol{x}^*) - y_{i_0}^{\mathrm{U}}) \\ &\leq \max_{1 \leq i \leq m} \left\{ \omega_i (f_i(\boldsymbol{x}^*) - y_i^{\mathrm{U}}) \right\}. \end{aligned}$$

However this contradicts our assumption that $\boldsymbol{x}^*$ is the unique optimal solution to (6).

Problem 3

Exercise 3. Let $\boldsymbol{w} = (1/4, 3/4)$ and $\rho = 1/4$.

1. Derive the equations of the set of points in $\mathbb{R}^2_{\geqq}$ whose augmented weighted Chebyshev distance 3 away from the origin $\mathbf{0}$ in $\mathbb{R}^2$.
2. Consider the modified weighted Chebyshev metric. Derive the equations of the set of points in $\mathbb{R}^2_{\geqq}$ of distance 3 away from the origin $\mathbf{0}$ in $\mathbb{R}^2$.
3. Draw both sets in the same figure. Find and plot all extreme points of each set. Make sure that your figure is clear and precise.

Solution 3. 1. Recall the augmented weighted Chebyshev norm (with respect to w and ρ) defined by

$$\|y\|_\infty^{w,\rho} = \max\{w_1|y_1|, w_2|y_2|\} + \rho(|y_1| + |y_2|)$$

for all $y \in \mathbb{R}^2$. Finding the set of all points in $\mathbb{R}^2_{\geqq}$ whose augmented weighted Chebyshev distance from the origin is 3 is equivalent to finding the set of all points in $\mathbb{R}^2_{\geqq}$ whose augmented weighted Chebyshev norm is 3. Now suppose $y \in \mathbb{R}^2_{\geqq}$ and let A be set of all points in $\mathbb{R}^2_{\geqq}$ whose augmented weighted Chebyshev norm is 3. Then we have

$$\begin{aligned} y \in A &\iff \max\{w_1y_1, w_2y_2\} + \rho(y_1 + y_2) = 3 \\ &\iff \max\left\{\frac{y_1}{4}, \frac{3y_2}{4}\right\} + \frac{y_1 + y_2}{4} = 3. \end{aligned}$$

Now note that

$$\begin{aligned} \max\left\{\frac{y_1}{4}, \frac{3y_2}{4}\right\} = \frac{y_1}{4} &\iff \frac{y_1}{4} \geq \frac{3y_2}{4} \\ &\iff y_1 \geq 3y_2 \end{aligned}$$

Thus if $y_1 \geq 3y_2$, then we have

$$\begin{aligned} y \in A &\iff \frac{y_1}{4} + \frac{y_1 + y_2}{4} = 3. \\ &\iff 2y_1 + y_2 - 12 = 0, \end{aligned}$$

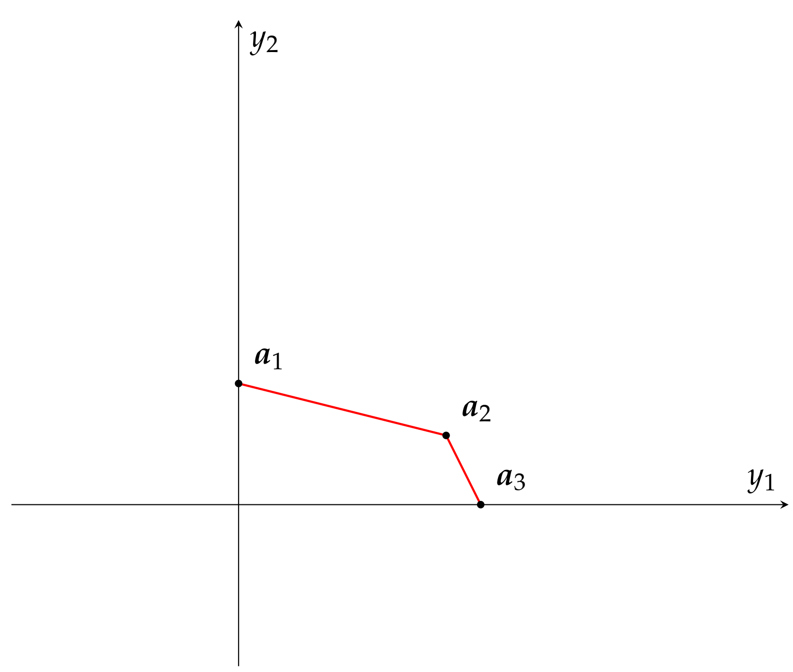
and if $y_1 < 3y_2$, then we have

$$\begin{aligned} y \in A &\iff \frac{3y_2}{4} + \frac{y_1 + y_2}{4} = 3. \\ &\iff y_1 + 4y_2 - 12 = 0. \end{aligned}$$

Therefore we conclude that

$$A = \{y \in \mathbb{R}^2_{\geqq} \mid y_1 \geq 3y_2 \text{ and } 2y_1 + y_2 - 12 = 0\} \cup \{y \in \mathbb{R}^2_{\geqq} \mid y_1 < 3y_2 \text{ and } y_1 + 4y_2 - 12 = 0\}.$$

Thet set A consists of two segments colored in red below:



where $a_1 = (0, 3)$, $a_2 = (36/7, 12/7)$, and $a_3 = (6, 0)$ are the extreme points.

2. Recall the modified weighted Chebyshev norm (with respect to w and ρ) is defined by

$$\|y\|_\infty^{w,\rho,\mathrm{m}} = \max\left\{w_1\left(|y_1| + \rho(|y_1| + |y_2|)\right), w_2\left(|y_2| + \rho(|y_1| + |y_2|)\right)\right\}$$

for all $\boldsymbol{y} \in \mathbb{R}^2$. Finding the set of all points in $\mathbb{R}^2_{\geqq}$ whose modified weighted Chebyshev distance from the origin is 3 is equivalent to finding the set of all points in $\mathbb{R}^2_{\geqq}$ whose modified weighted Chebyshev norm is 3. Suppose $\boldsymbol{y} \in \mathbb{R}^2_{\geqq}$ and let B be set of all points in $\mathbb{R}^2_{\geqq}$ whose modified weighted Chebyshev norm is 3. Then we have

$$\begin{aligned}
\boldsymbol{y} \in B &\iff \max\left\{w_1\left(y_1+\rho(y_1+y_2)\right), w_2\left(y_2+\rho(y_1+y_2)\right)\right\} = 3. \\
&\iff \max\left\{\frac{1}{4}\left(y_1+\frac{1}{4}(y_1+y_2)\right), \frac{3}{4}\left(y_2+\frac{1}{4}(y_1+y_2)\right)\right\} = 3. \\
&\iff \max\left\{\frac{5y_1+y_2}{16}, \frac{3y_1+15y_2}{16}\right\} = 3
\end{aligned}$$

Now note that

$$\begin{aligned}
\max\left\{\frac{5y_1+y_2}{16}, \frac{3y_1+15y_2}{16}\right\} = \frac{5y_1+y_2}{16} &\iff \frac{5y_1+y_2}{16} \geq \frac{3y_1+15y_2}{16} \\
&\iff 5y_1+y_2 \geq 3y_1+15y_2 \\
&\iff y_1 \geq 7y_2.
\end{aligned}$$

Thus if $y_1 \geq 7y_2$, then we have

$$\begin{aligned}
\boldsymbol{y} \in B &\iff \frac{5y_1+y_2}{16} = 3. \\
&\iff 5y_1+y_2-48 = 0,
\end{aligned}$$

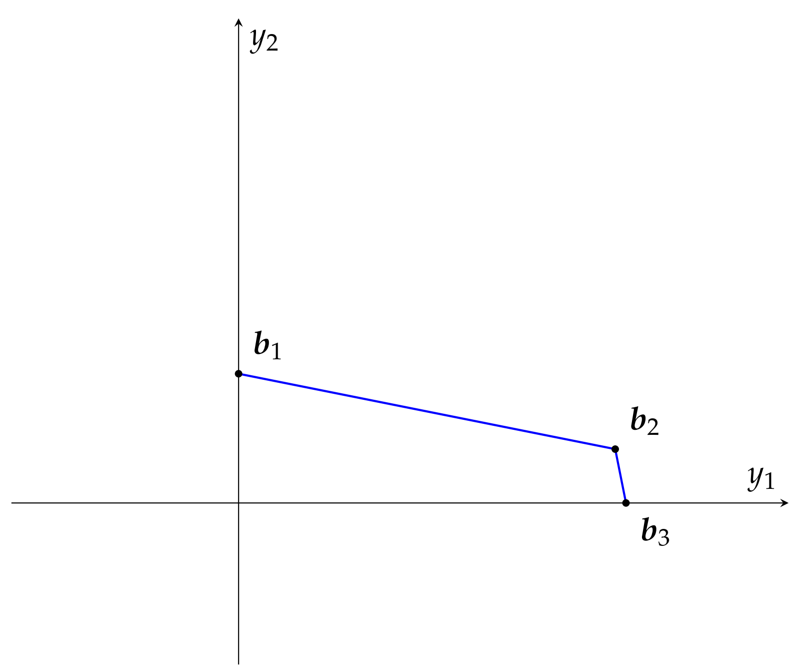
and if $y_1 < 7y_2$, then we have

$$\begin{aligned}
\boldsymbol{y} \in B &\iff \frac{3y_1+15y_2}{16} = 3. \\
&\iff 3y_1+15y_2-48 = 0,
\end{aligned}$$

Therefore we conclude that

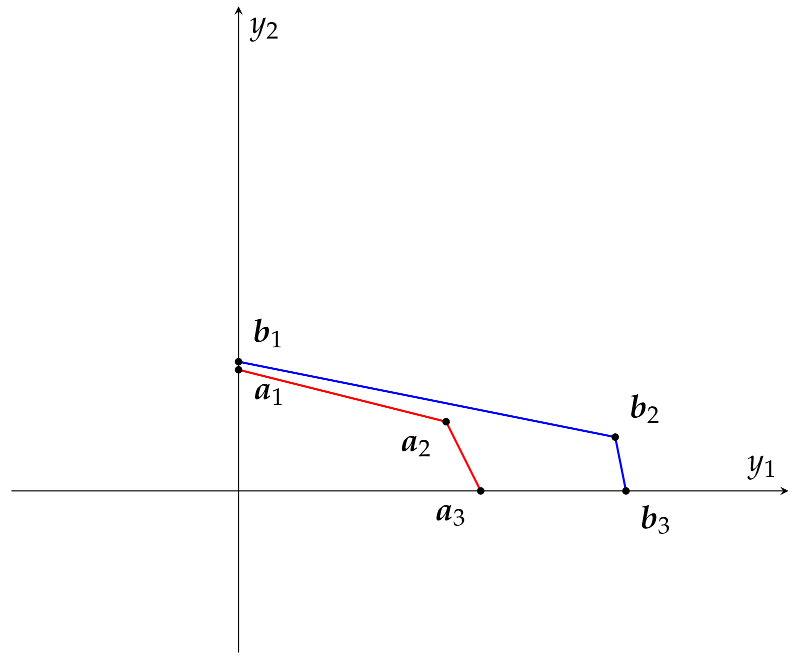
$$B = \{\boldsymbol{y} \in \mathbb{R}^2_{\geqq} \mid y_1 \geq 7y_2 \text{ and } 5y_1+y_2-48=0\} \cup \{\boldsymbol{y} \in \mathbb{R}^2_{\geqq} \mid y_1 < 7y_2 \text{ and } 3y_1+15y_2-48=0\}.$$

Thet set B consists of two segments colored in blue below:



where $\boldsymbol{b}_1 = (0, 48/15)$, $\boldsymbol{b}_2 = (28/3, 4/3)$, and $\boldsymbol{b}_3 = (48/5, 0)$ are the extreme points.

3. We draw both A and B together:



where $a_1 = (0, 3)$, $a_2 = (36/7, 12/7)$, and $a_3 = (6, 0)$ are the extreme points for A, and where $b_1 = (0, 48/15)$, $b_2 = (28/3, 4/3)$, and $b_3 = (48/5, 0)$ are the extreme points for B.

Problem 4

Exercise 4. Let $f_1(x) = 3x_1 + x_2$, let $f_2(x) = x_1 + 2x_2$, and let

$$X = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 7, 0 \leq x_1 \leq 5, 0 \leq x_2 \leq 5\}.$$

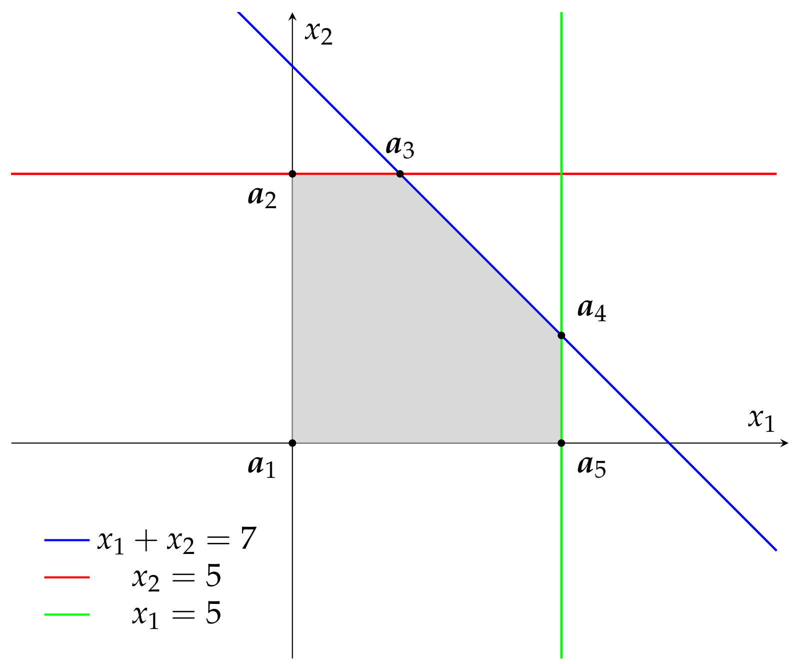
For this problem we consider the following BOP:

$$\begin{array}{ll} \text{maximize} & [f_1(x), f_2(x)] \\ \text{subject to} & x \in X. \end{array}$$

Use any method or software of your choice to answer the following questions.

1. Find the ideal point for this BOP and use it as the reference point in parts b), c), d) below.
2. Let $w = (1/4, 3/4)$ and $p = 1$. Formulate the weighted norm problem $\mathrm{P}(w, p)$ and find all optimal solutions to this problem.
3. Let $w = (1/4, 3/4)$ and $p = 2$. Formulate the weighted norm problem $\mathrm{P}(w, p)$ and find all optimal solutions to this problem.
4. Let $w = (1/4, 3/4)$ and $p = \infty$. Formulate the weighted norm problem $\mathrm{P}(w, p)$ and find all optimal solutions to this problem.
5. Compare the optimal solutions you obtained in parts 2-4. What do you observe?
6. Are the optimal solutions to the weighted norm problems $\mathrm{P}(w, p)$ efficient to the BOP? Explain.

Solution 4. 1. The feasible set X is the region shaded in grey below:



where the extreme points of X are

$$\begin{aligned}
\boldsymbol{a}_1 &= (0,0)\\
\boldsymbol{a}_2 &= (0,5)\\
\boldsymbol{a}_3 &= (2,5)\\
\boldsymbol{a}_4 &= (5,2)\\
\boldsymbol{a}_5 &= (5,0)
\end{aligned}$$

Since f_1 and f_2 are linear, we can determine where f_1 and f_2 are maximized by evaluating them at the extreme points and comparing them with each other. We find that f_1 is maximized at $\boldsymbol{a}_4$ with maximum objective value being $y_1^I := f_1(\boldsymbol{a}_4) = 17$ and f_2 is maximized at $\boldsymbol{a}_3$ with maximum objective value being $y_2^I := f_2(\boldsymbol{a}_3) = 12$. Therefore the ideal point is $\boldsymbol{y}^I = (y_1^I, y_2^I) = (17, 12)$.

2. The weighted norm problem is

$$\begin{aligned}
&\text{minimize} && \frac{1}{4}(17 - 3x_1 - x_2) + \frac{3}{4}(12 - x_1 - 2x_2)\\
&\text{subject to} && \boldsymbol{x} \in X.
\end{aligned}$$

We can simplify this as

$$\begin{aligned}
&\text{minimize} && \frac{1}{4}(53 - 6x_1 - 7x_2)\\
&\text{subject to} && \boldsymbol{x} \in X.
\end{aligned}$$

Since the objective is linear and the feasible set is bounded and convex, we can determine where the objective is minimized by evaluating it at the extreme points and comparing each value with each other. We find that the objective has a unique minimizer at $\boldsymbol{a}_3$.

3. The weighted norm problem is

$$\begin{aligned}
&\text{minimize} && \left(\frac{1}{4}(17 - 3x_1 - x_2)^2 + \frac{3}{4}(12 - x_1 - 2x_2)^2\right)^{1/2}\\
&\text{subject to} && \boldsymbol{x} \in X.
\end{aligned}$$

The optimal solutions to this problem will be the same as the optimal solutions to the following problem:

$$\begin{aligned}
&\text{minimize} && \frac{1}{4}(17 - 3x_1 - x_2)^2 + \frac{3}{4}(12 - x_1 - 2x_2)^2\\
&\text{subject to} && \boldsymbol{x} \in X.
\end{aligned}$$

Indeed, this follow from the fact that

$$\psi(\boldsymbol{x}) := \frac{1}{4}(17 - 3x_1 - x_2)^2 + \frac{3}{4}(12 - x_1 - 2x_2)^2$$

is always nonnegatve, being a sum of two squares. A quick calculation shows that, among the extreme points, ψ is minimized at $\boldsymbol{a}_4$ with objective value given by $\psi(\boldsymbol{a}_4) = 27/4$. We claim that $\boldsymbol{a}_4$ is the unique minimizer to the problem. Indeed, first note that the gradient of ψ at $\boldsymbol{a}_4$ is

$$\nabla\psi(\boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{a}_4} = \begin{pmatrix} \frac{3}{2}(4x_1 + 3x_2 - 29) \\ \frac{1}{2}(9x_1 + 13x_2 - 89) \end{pmatrix}\Big|_{\boldsymbol{x}=\boldsymbol{a}_4} = \begin{pmatrix} -9/2 \\ -9 \end{pmatrix}.$$

Furthermore, the tangent line of the level set $\{\psi = 27/4\}$ at the point $\boldsymbol{a}_4$ is precisely the blue line given by the equation $x_1 + x_2 = 7$. In particular, since ψ is globally strictly convex with positive Hessian and since the direction $(-9/2, -9)^\top$ points towards the feasible set from X, we see that moving along any direction from $\boldsymbol{a}_4$ to another point in the feasible set X will only increase the objective. It follows that $\boldsymbol{a}_4$ is the unique minimizer to our problem.

4. The weighted norm problem is

$$\begin{aligned} &\text{minimize} \quad \max\left\{\frac{1}{4}(17 - 3x_1 - x_2), \frac{3}{4}(12 - x_1 - 2x_2)\right\} \\ &\text{subject to} \quad \boldsymbol{x} \in X. \end{aligned} \tag{7}$$

Set $g_1(\boldsymbol{x}) = \frac{1}{4}(17 - 3x_1 - x_2)$ and set $g_2(\boldsymbol{x}) = \frac{3}{4}(12 - x_1 - 2x_2)$. Observe that

$$\begin{aligned} g_1(\boldsymbol{x}) \geq g_2(\boldsymbol{x}) &\iff \frac{1}{4}(17 - 3x_1 - x_2) \geq \frac{3}{4}(12 - x_1 - 2x_2) \\ &\iff 17 - 3x_1 - x_2 \geq 36 - 3x_1 - 6x_2 \\ &\iff x_2 \geq 19/5. \end{aligned}$$

In particular, if $x_2 \leq 19/5$, then the problem becomes

$$\begin{aligned} &\text{minimize} \quad \frac{1}{4}(17 - 3x_1 - x_2) \\ &\text{subject to} \quad \boldsymbol{x} \in X_{\leq 19/5} \end{aligned} \tag{8}$$

where $X_{\leq 19/5} = X \cap \{x_2 \leq 19/5\}$. Note that $X_{\leq 19/5}$ has the same extreme points as X except we replace $\boldsymbol{a}_2$ with $\boldsymbol{b}_2 = (0, 19/5)$ and we replace $\boldsymbol{a}_3$ with $\boldsymbol{b}_3 = (16/5, 19/5)$. After evaluating and comparing the objective at the extreme points, we find that (8) has a unique minimizer at $\boldsymbol{a}_4$ with objective value being 0. On the other hand, if $x_2 \geq 19/5$, then the problem becomes

$$\begin{aligned} &\text{minimize} \frac{3}{4}(12 - x_1 - 2x_2) \\ &\text{subject to} \quad \boldsymbol{x} \in X_{\geq 19/5} \end{aligned} \tag{9}$$

where $X_{\geq 19/5} = X \cap \{x_2 \geq 19/5\}$. Note that the extreme points of $X_{\geq 19/5}$ are $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$. After evaluating and comparing the objective at the extreme points, we find that (9) has a unique minimizer at $\boldsymbol{a}_4$ with objective value being 0. Altogether, we see that $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$ is the set of all optimal solutions to (7).

4. Each method gave us different answer, either $\boldsymbol{a}_3$, $\boldsymbol{a}_4$, or both $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$.

5. Yes because the first condition of Theorem 4.21 in the book by Ehrgott (the one that the class is referencing) holds for both part 2 and 3, showing that both $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$ are efficient solutions.

Problem 5

Exercise 5. Consider the lexicographic method and prove the following: If $\boldsymbol{x}^* \in X$ is an optimal solution to problem $\mathrm{P}(\Pi(p))$, then $\boldsymbol{x}^*$ is an efficient solution to the MOP.

Solution 5. Assume for a contradiction that $\boldsymbol{x}^* \in X$ is an optimal solution to $\mathrm{P}(\Pi(p))$ but is not efficient to the MOP. Then there exists $\boldsymbol{x} \in X$ such that $f(\boldsymbol{x}) \leq f(\boldsymbol{x}^*)$. In particular, there exists some $i \in \{1, \dots, p\}$ such that $f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*)$. Set

$$i_0 := \min\{i \mid f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*)\}.$$

Then $f_i(\boldsymbol{x}) = f_i(\boldsymbol{x}^*)$ for $i = 1, \dots, i_0 - 1$ and $f_{i_0}(\boldsymbol{x}) < f_{i_0}(\boldsymbol{x}^*)$ implies $f(\boldsymbol{x}) <_{\mathrm{lex}} f(\boldsymbol{x}^*)$ which contradicts our assumption that $\boldsymbol{x}^*$ an optimal solution to $\mathrm{P}(\Pi(p))$. Note that the same argument holds if we permute the indices by any permutation $\sigma \in S_p$.