

Mathematics Diary

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1.1 12/20/2022

Lemma 1.1. *Let $(R, \mathfrak{m}, \mathbb{k})$ be a local noetherian ring, let $J \subseteq I \subseteq \mathfrak{m}$ be ideals of R . Let E be the minimal free resolution of R/J over R , let F be the minimal free resolution of R/I over R , and let $\varphi: E \rightarrow F$ be a comparison map which lifts the canonical surjective map $R/J \twoheadrightarrow R/I$. Assume both $\varphi: E \rightarrow F$ and $\bar{\varphi}: E_{\mathbb{k}} := E \otimes_R \mathbb{k} \rightarrow F \otimes_R \mathbb{k} := F_{\mathbb{k}}$ are injective. Then $\Sigma(F/E)$ is the minimal free resolution of I/J over R .*

Proof. Assume both $\varphi: E \rightarrow F$ and $\bar{\varphi}: E_{\mathbb{K}} := E \otimes_R \mathbb{K} \rightarrow F \otimes_R \mathbb{K} := F_{\mathbb{K}}$ are injective. Since $\varphi: E \rightarrow F$ is injective, we have a short exact sequence of R -complexes

$$0 \longrightarrow E \xrightarrow{\varphi} F \longrightarrow F/E \longrightarrow 0 \quad (1)$$

taking homology gives us a long exact sequence

$$\begin{array}{c} \dots \longrightarrow H_{i+1}(F/E) \longrightarrow \dots \\ \downarrow \qquad \qquad \qquad \downarrow \\ H_i(E) \longrightarrow H_i(F) \longrightarrow H_i(F/E) \longrightarrow \dots \\ \downarrow \qquad \qquad \qquad \downarrow \\ H_{i-1}(E) \longrightarrow \dots \end{array}$$

Since E and F are resolutions we conclude that $H_i(F/E) = 0$ for all $i \neq 1$. Since $R/J \twoheadrightarrow R/I$ is surjective we conclude that $H_1(F/E) = I/J$. To see that F/E is free, note that tensoring the short exact sequence of graded R -modules (1) with \mathbb{k} over R gives us the long exact sequence in homology

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & \mathrm{Tor}_{i+1}^R(E, \mathbb{k}) \longrightarrow \\
 & & & & & & \uparrow \\
 & & & & & & \mathrm{Tor}_i^R(E, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(F, \mathbb{k}) \longrightarrow \mathrm{Tor}_i^R(F/E, \mathbb{k}) \longrightarrow \\
 & & & & & & \uparrow \\
 & & & & & & \mathrm{Tor}_{i-1}^R(E, \mathbb{k}) \longrightarrow \cdots
 \end{array}$$

Since E and F are free R -modules we conclude that $\text{Tor}_i(F/E, \mathbb{k}) = 0$ for all $i \geq 1$. Since $\bar{\varphi}: E \otimes_R \mathbb{k} \rightarrow F \otimes_R \mathbb{k}$ is injective we conclude that $\text{Tor}_1(F/E, \mathbb{k}) = 0$. In particular, F/E must be free. Finally, F/E is minimal since the differential d on F induces a minimal differential on F/E (i.e. $d(F/E) \subseteq \mathfrak{m}(F/E)$). \square

Remark 1. Under the assumptions of Lemma (1.1), we see that for any R -module M connecting maps

$$\text{Tor}_{i+1}^R(R/I, M) \rightarrow \text{Tor}_i^R(I/J, M) \quad \text{and} \quad \text{Ext}_R^i(I/J, M) \rightarrow \text{Ext}_R^{i+1}(R/I, M)$$

are represented by the chain maps

$$F \otimes_R M \rightarrow F/E \otimes_R M \quad \text{and} \quad \text{Hom}_R^*(F/E, M) \rightarrow \text{Hom}_R^*(F, M)$$

respectively.

Remark 2. Note that under the assumptions we are working with, if $\bar{\varphi}: E_{\mathbb{k}} \rightarrow F_{\mathbb{k}}$ is injective, then already $\varphi: E \rightarrow F$ is injective. The converse need not hold.

1.2 12/21/2023 - Heights of Ideals

Let R be a commutative ring and let \mathfrak{p} be an ideal of R . Recall the **height** of \mathfrak{p} is defined to be the supremum of lengths of chains of primes which descend from \mathfrak{p} :

$$\text{ht } \mathfrak{p} = \sup\{c \in \mathbb{N} \mid \mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_c\}.$$

When R is Noetherian, then Krull's principal ideal theorem states that there exists an ideal $\langle x \rangle = \langle x_1, \dots, x_c \rangle \subseteq \mathfrak{p}$ where $c = \text{ht } \mathfrak{p}$ such that $\sqrt{\langle x \rangle} = \mathfrak{p}$, and that if $\langle y \rangle = \langle y_1, \dots, y_m \rangle$ is another ideal such that $\sqrt{\langle y \rangle} = \mathfrak{p}$, then we must have $c \leq m$. If I is an ideal of R , then the **height** of I is defined to be the infimum of the heights of all primes which contain I :

$$\text{ht } I = \inf\{\text{ht } \mathfrak{p} \mid \mathfrak{p} \supseteq I\}.$$

Lemma 1.2. Let I_1 and I_2 be ideals of R . Set $c = \text{ht}(I_1 \cap I_2)$, set $c_1 = \text{ht } I_1$, and set $c_2 = \text{ht } I_2$.

1. If $I_1 \subseteq I_2$, then $c_1 \leq c_2$.
2. We have $c = \min\{c_1, c_2\}$.

Proof. 1. Let \mathfrak{p} be a prime which contains I_2 whose height is minimal among all heights of primes which contain I_2 . Since $I_1 \subseteq I_2$, we see that $I_1 \subseteq \mathfrak{p}$ also. In particular, it follows that $c_1 \leq c_2$.

2. Note that $I_1 \cap I_2 \subseteq I_1$ implies $c \leq c_1$. Similarly, $I_1 \cap I_2 \subseteq I_2$ implies $c \leq c_2$. It follows that $c \leq \min\{c_1, c_2\}$. Conversely, let \mathfrak{p} be a prime which contains $I_1 \cap I_2$ whose height is minimal among all heights of primes which contain $I_1 \cap I_2$. Then $\mathfrak{p} \supseteq I_1 \cap I_2$ implies either $\mathfrak{p} \supseteq I_1$ or $\mathfrak{p} \supseteq I_2$ since \mathfrak{p} is a prime. In particular it follows that either $c \geq c_1$ or $c \geq c_2$ or equivalently $c \geq \min\{c_1, c_2\}$. \square

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1/20/2024 - $V(\text{Ann } M) = V(\text{Ann}(0 :_M x))$

Lemma 2.1. Let R be a commutative ring, let M be an R -module, and let $x \in R$. Then

$$V(\text{Ann}(0 :_M x)) = V(\text{Ann}(0 :_M x^2)).$$

Proof. Note that $0 :_M x \subseteq 0 :_M x^2$ implies $\text{Ann}(0 :_M x^2) \supseteq \text{Ann}(0 :_M x)$ which implies $V(\text{Ann}(0 :_M x^2)) \subseteq V(\text{Ann}(0 :_M x))$. For the reverse inclusion, suppose \mathfrak{p} is a prime ideal of R which contains $\text{Ann}(0 :_M x^2)$ and let $r \in \text{Ann}(0 :_M x)$. We claim that $r^2 \in \text{Ann}(0 :_M x^2)$. Indeed, if $u \in 0 :_M x^2$, then

$$\begin{aligned} x^2 u = 0 &\implies xu \in 0 :_M x \\ &\implies rxu = 0 \\ &\implies ru \in 0 :_M x \\ &\implies r^2 u = 0. \end{aligned}$$

Since u was arbitrary, we see that $r^2 \in \text{Ann}(0 :_M x^2) \subseteq \mathfrak{p}$. However this implies $r \in \mathfrak{p}$ since \mathfrak{p} is a prime. Since r was arbitrary, we see that $\text{Ann}(0 :_M x) \subseteq \mathfrak{p}$. \square

Corollary 1. *Let R be a commutative ring and let M be a finitely generated R -module. Assume that $x \in R$ acts nilpotently on M . Then*

$$V(\text{Ann}(M)) = V(\text{Ann}(0 :_M x)).$$

Proof. Since M is finitely generated, there exists an $n \in \mathbb{N}$ such that $M = 0 :_M x^n$. A straightforward induction on $(?)$ gives us

$$V(\text{Ann}(M)) = V(\text{Ann}(0 :_M x^n)) = V(\text{Ann}(0 :_M x)).$$

\square

1/21/2024 Some subschemes of \mathbb{P}^3

Let $R = \mathbb{k}[x_1, x_2, x_3, x_4] = \mathbb{k}[x]$. We consider three cyclic R -algebras, namely $A = R/\mathbf{f} = R/\langle f_1, f_2, f_3 \rangle$, $B = R/\mathbf{g} = R/\langle g_1, g_2, g_3 \rangle$, and $C = R/\mathbf{h} = R/\langle h_1, h_2, h_3 \rangle$ where

$$\begin{array}{lll} f_1 = xy - zw & g_1 = xz - y^2 & h_1 = xz - y^2 \\ f_2 = xz - yw & g_2 = yw - z^2 & h_2 = x^3 - yzw \\ f_3 = xw - yz & g_3 = xw - yz & h_3 = x^2y - z^2w \end{array}$$

We want a geometric picture in mind when thinking of these rings, so let $X = \text{Proj } A$, $Y = \text{Proj } B$, and $Z = \text{Proj } C$. First let us consider X . We can see that X is 8 distinct points in \mathbb{P}^3 by calculating an irreducible primary decomposition for $I = \langle \mathbf{f} \rangle$. Indeed, an irredundant primary decomposition for $\langle \mathbf{f} \rangle$ is given by

$$\langle \mathbf{f} \rangle = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \mathfrak{p}_4 \cap \mathfrak{p}_5 \cap \mathfrak{p}_6 \cap \mathfrak{p}_7 \cap \mathfrak{p}_8,$$

where

$$\begin{array}{ll} \mathfrak{p}_1 = \langle y, z, w \rangle & \mathfrak{p}_5 = \langle x + y, y + z, z + w \rangle \\ \mathfrak{p}_2 = \langle x, z, w \rangle & \mathfrak{p}_6 = \langle x + y, y - z, z + w \rangle \\ \mathfrak{p}_3 = \langle x, y, w \rangle & \mathfrak{p}_7 = \langle x + y, y - z, z - w \rangle \\ \mathfrak{p}_4 = \langle x, y, z \rangle & \mathfrak{p}_8 = \langle x - y, y - z, z - w \rangle. \end{array}$$

These primes correspond to the points

$$\begin{array}{ll} \mathbf{p}_1 = [1 : 0 : 0 : 0] & \mathbf{p}_5 = [-1 : 1 : -1 : 1] \\ \mathbf{p}_2 = [0 : 1 : 0 : 0] & \mathbf{p}_6 = [1 : -1 : -1 : 1] \\ \mathbf{p}_3 = [0 : 0 : 1 : 0] & \mathbf{p}_7 = [-1 : 1 : 1 : 1] \\ \mathbf{p}_4 = [0 : 0 : 0 : 1] & \mathbf{p}_8 = [1 : 1 : 1 : 1] \end{array}$$

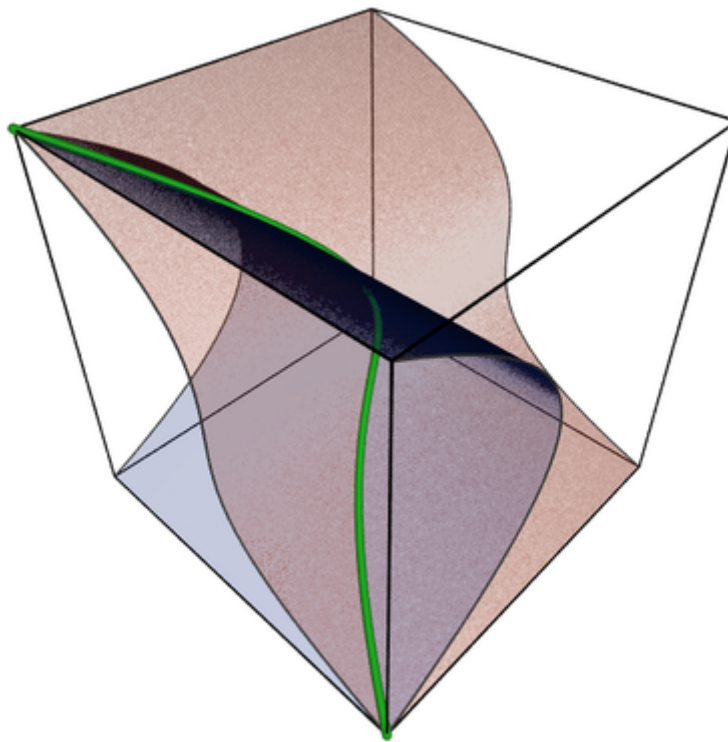
in \mathbb{P}^3 . Note that $\mathbf{p}_1, \dots, \mathbf{p}_8$ are in linearly general position since the size k minors of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

are all nonzero for all $1 \leq k \leq 4$. The Betti diagram of A over R is given by

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & - & - \\ 2 & - & - & 4 & 2 \end{array}$$

Next we consider Y . In fact, Y is the twisted cubic:



In particular, Y is the image of the map $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. Note that $\langle g \rangle$ is a prime of height 2 and so $\langle g \rangle$ can be generated up to radical by two homogeneous polynomials. In particular, we have $\langle g \rangle = \sqrt{\langle g_1, g_4 \rangle}$ where $g_4 = zg_2 - wg_3$. However $\langle g \rangle$ itself cannot be generated by only two polynomials; a minimum of three polynomials are needed. We can see this in Betti diagram of B over R :

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & - & - \\ 1 & - & 3 & 2 \end{array}$$

In particular, the Hilbert-Poincare series of B over R is given by

$$P(t) = \frac{1 - 3t^2 + 2t^3}{(1 - t)^4} = \frac{1 + 2t}{(1 - t)^2} = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + \dots$$

Thus Y is the set-theoretic complete intersection of $V(g_1)$ and $V(g_4)$ however it is not a scheme-theoretic or ideal-theoretic complete intersection. Note also that $\langle g \rangle$ corresponds to the ideal of size 2 minors of the matrix $\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$. Up to linear automorphism, the twisted cubic is the only irreducible curve of degree 3 not contained in a plane. Furthermore, any 6 points in linearly general position in \mathbb{P}^3 lie on a unique twisted cubic. However for a twisted cubic to pass through 7 points, the seventh must lie on the twisted cubic determined by the first 6. Consequently one can show that if W is a set of 7 points in linearly general position in \mathbb{P}^3 , then there are only two distinct Betti diagrams possible for the homogeneous coordinate ring of W , namely

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & - & - \\ 2 & - & 1 & 6 & 3 \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 3 & 2 & - \\ 2 & - & 3 & 6 & 3 \end{array}$$

In the first case, the points do not lie on any curve of degree 3. In the second case, the ideal J generated by the quadrics containing W is the ideal of the unique curve of degree 3 containing W , which is irreducible. Finally, let us write down the minimal free resolution of B over R :

$$R(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} R \longrightarrow 0$$

Now we consider Z . The Betti diagram of C over R is given by

	0	1	2
0	1	-	-
1	-	1	-
2	-	2	2

In particular, the Hilbert-Poincare series of C over R is given by

$$P(t) = \frac{1 - t^2 - 2t^3 + 2t^4}{(1 - t)^4} = \frac{1 + 2t + 2t^2}{(1 - t)^2} = 1 + 4t + 9t^2 + 14t^3 + 19t^4 + \dots.$$

In particular, Z is an irreducible curve of degree 5 in \mathbb{P}^3 .