Algebraic Topology

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1 Simplices

Let $n \in \mathbb{Z}_{\geq 0}$ and let $V = (V, \|\cdot\|)$ be an $(n+1)$-dimensional normed vector space over $\mathbb{R}$. Let $\boldsymbol{v} = v_0, \dots, v_n$ be an ordered list of $n+1$ vectors $v_0, \dots, v_n \in V$ such that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. The **n-simplex** $[\boldsymbol{v}] = [v_0, \dots, v_n]$ (or **simplex** if n is understood from context) is the defined to be the convex closure of $\{v_1, \dots, v_n\}$. In other words, $[\boldsymbol{v}]$ is the set of all convex combinations of the v_i:

$$[\boldsymbol{v}] = \left\{ \sum_{i=0}^{n} t_i v_i \mid t_i \in [0,1] \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

In the case where $V = \mathbb{R}^{n+1}$, we call $[\boldsymbol{v}]$ a **Euclidean** n-simplex. Let $\boldsymbol{e} = e_0, \dots, e_n$ be the standard ordered basis of $\mathbb{R}^{n+1}$, where where $e_i = (0, \dots, 1, \dots, 0)$ is the column vector with entry 1 in the ith spot and entry 0 everywhere else, and let $\boldsymbol{t} = t_0, \dots, t_n$ be the corresponding standard coordinates of $\mathbb{R}^{n+1}$. The n-simplex $[\boldsymbol{e}]$ is given a special name: it is called the **n-dimensional standard simplex**, and is denoted by $\Delta^n := [\boldsymbol{e}]$. Thus

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \in [0,1] \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

Note that for every $v \in [\boldsymbol{v}]$, there is a unique point $(t_0, \dots, t_n) \in \Delta^n$ such that $v = \sum t_i v_i$. The map $\phi = \phi_{\boldsymbol{v}}$ from $[\boldsymbol{v}]$ to Δ^n sending a point $v \in [\boldsymbol{v}]$ to the point $(t_0, \dots, t_n) \in \Delta^n$ (uniquely determined by v) for each $v \in [\boldsymbol{v}]$ is also seen to be the unique linear homeomorphism such that $\phi(\boldsymbol{v}) = \boldsymbol{e}$ (that is, such that $\phi(v_i) = e_i$ for all $0 \leq i \leq n$). The coefficients t_i are called the **barycentric coordinates** of v.

Observe that for each $0 \leq i_0 < \cdots < i_k \leq n$, the k-simplex $[v_{i_0}, \dots, v_{i_k}]$ is contained in $[\boldsymbol{v}]$. If $0 \leq i \leq n$, then $[v_i] = v_i$ is called a **vertex** of $[\boldsymbol{v}]$. If $1 \leq i < j \leq n$, then $[v_i, v_j]$ is called an **edge** of $[\boldsymbol{v}]$. If $1 \leq i < j < k \leq n$, then $[v_i, v_j, v_k]$ is called a **face** of $[\boldsymbol{v}]$. More generally, if $0 \leq i_0 < \cdots < i_k \leq n$, then $[v_{i_0}, \dots, v_{i_k}]$ is called a **k-face** of $[\boldsymbol{v}]$. The collection of all k-faces of $[\boldsymbol{v}]$ as k ranges from 1 to n has the structure of an abstract simplicial complex. Moreover, the ordering of $\boldsymbol{v}$ induces an orientation on each k-face of $[\boldsymbol{v}]$, and the canonical map $\phi_{\boldsymbol{v}} \colon [\boldsymbol{v}] \to \Delta^n$ preserves this orientation since it preserves the orderings on $\boldsymbol{v}$ and $\boldsymbol{e}$.

1.1 Singular Homology

Let X be a topological space. An **n-simplex** of X is a continuous map of the form $\sigma \colon [\boldsymbol{v}] \to X$ where $[\boldsymbol{v}]$ is an n-simplex of an $(n+1)$-dimensional normed vector space over $\mathbb{R}$. In the case where $[\boldsymbol{v}] = \Delta^n$, then we call σ a **singular n-simplex** of X. Note that an n-simplex $\sigma \colon [\boldsymbol{v}] \to X$ of X determines a unique singular n-simplex $\widetilde{\sigma} \colon \Delta^n \to X$ of X given by $\widetilde{\sigma} = \sigma \circ \phi_{\boldsymbol{v}}^{-1}$ where $\phi_{\boldsymbol{v}} \colon [\boldsymbol{v}] \to \Delta^n$ is uniquely determined by the data $\boldsymbol{v}$. We often pass from σ to $\widetilde{\sigma}$ without comment. The set of all singular n-simplices in X is denoted is denoted by $\Sigma_n(X)$ and collection of all simplices of X is denoted $\Sigma(X)$. In particular, note that

$$\Sigma(X) = \bigcup_{n=0}^{\infty} \Sigma_n(X).$$

Let R be a ring. We define an R-complex, called the **singular chain complex of X with coefficients in R**, denoted by $C(X;R) = (C(X;R), \partial)$, as follows: the underlying graded R-module of $C(X;R)$ is given by

$$C(X;R) = \bigoplus_{n=0}^{\infty} C_n(X;R) \quad \text{where} \quad C_n(X;R) = \bigoplus_{\sigma \in \Sigma_n(X)} R\sigma.$$

The elements of $C_n(X;R)$ are called the **singular n-chains of X with coefficients in R**. The differential of $C(X;R)$ is defined on singular n-chains $\sigma \in C_n(X;R)$ by

$$\partial(\sigma) = \sum_{i=0}^{n}(-1)^i \sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]} \tag{1}$$

and is extended R-linearly everywhere else. Note that $\sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}$ is technically *not* an element of Σ because, even though the map $\sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}$ makes perfect since as a map from the $(n-1)$-simplex $[e_0,\dots,\widehat{e}_i,\dots,e_n]$ to the space X, it is not a singular chain of X since $[e_0,\dots,\widehat{e}_i,\dots,e_n]$ is not the standard $(n-1)$-simplex Δ^{n-1}. So technically speaking, the expression (1) makes no sense (as the differential ∂ needs to be a map from $C(X;R)$ to itself). The key however is this: let $\widehat{\boldsymbol{e}}_i = e_0,\dots,e_{i-1},e_i,\dots,e_n$ (so $[e_0,\dots,\widehat{e}_i,\dots,e_n] = [\widehat{\boldsymbol{e}}_i]$) and let $\widetilde{\boldsymbol{e}} = \widetilde{e}_0,\dots,\widetilde{e}_{n-1}$ denote the standard ordered basis of $\mathbb{R}^n$ (so $\Delta^{n-1} = [\widetilde{\boldsymbol{e}}]$), then we know that there is a linear homeomorphism $\phi_{\widehat{\boldsymbol{e}}_i} \colon [\widehat{\boldsymbol{e}}_i] \to \Delta^{n-1}$ such that $\phi_{\widehat{\boldsymbol{e}}_i}(\widehat{\boldsymbol{e}}_i) = \widetilde{\boldsymbol{e}}$ and that this map is *uniquely determined* by the data $\widehat{\boldsymbol{e}}_i$. Then the expression

$$\partial(\sigma) = \sum_{i=0}^{n}(-1)^i (\sigma|_{[\widehat{\boldsymbol{e}}_i]} \circ \phi_{\widehat{\boldsymbol{e}}_i}^{-1}) \tag{2}$$

makes perect sense because $\sigma|_{[\widehat{\boldsymbol{e}}_i]} \circ \phi_{\widehat{\boldsymbol{e}}_i}^{-1}$ is a continuous map from Δ^{n-1} to X, and the maps $\phi_{\widehat{\boldsymbol{e}}_i}$ are uniquely determined by the data $\widehat{\boldsymbol{e}}_i$. So the expression (1) is implicitly understood to be the expression (2). The reason why we can safely ignore $\phi_{\widehat{\boldsymbol{e}}_i}^{-1}$ and use the expression (1), is because $\phi_{\widehat{\boldsymbol{e}}_i}^{-1}$ is completely determined by the data $\widehat{\boldsymbol{e}}_i$ which is already present in (1). For instance, let's show that $\partial^2 = 0$ using the expression (1): we have

$$\begin{aligned}
\partial^2(\sigma) &= \sum_{0 \le i \le n} (-1)^i \partial(\sigma|_{[\widehat{\boldsymbol{e}}_i]}) \\
&= \sum_{0 \le i \le n} (-1)^i \left(\sum_{0 \le j < i} (-1)^j \sigma|_{[\widehat{\boldsymbol{e}}_{j,i}]} + \sum_{i < j \le n} (-1)^{j+1} \sigma|_{[\widehat{\boldsymbol{e}}_{i,j}]} \right) \\
&= \sum_{0 \le j < i \le n} (-1)^{i+j} \sigma|_{[\widehat{\boldsymbol{e}}_{j,i}]} + \sum_{0 \le i < j \le n} (-1)^{i+j+1} \sigma|_{[\widehat{\boldsymbol{e}}_{i,j}]} \\
&= 0.
\end{aligned}$$

Now we show $\partial^2 = 0$ using the expression (2): we have

$$\begin{aligned}
\partial^2(\sigma) &= \sum_{0 \le i \le n} (-1)^i \partial(\sigma|_{[\widehat{\boldsymbol{e}}_i]} \circ \phi_{\widehat{\boldsymbol{e}}_i}^{-1}) \\
&= \sum_{0 \le i \le n} (-1)^i \left(\sum_{0 \le j < i} (-1)^j (\sigma|_{[\widehat{\boldsymbol{e}}_{j,i}]} \circ \phi_{[\widehat{\boldsymbol{e}}_{j,i}]}^{-1}) + \sum_{i < j \le n} (-1)^{j+1} (\sigma|_{[\widehat{\boldsymbol{e}}_{i,j}]} \circ \phi_{[\widehat{\boldsymbol{e}}_{i,j}]}^{-1}) \right) \\
&= \sum_{0 \le j < i \le n} (-1)^{i+j} (\sigma|_{[\widehat{\boldsymbol{e}}_{j,i}]} \circ \phi_{[\widehat{\boldsymbol{e}}_{j,i}]}^{-1}) + \sum_{0 \le i < j \le n} (-1)^{i+j+1} (\sigma|_{[\widehat{\boldsymbol{e}}_{i,j}]} \circ \phi_{[\widehat{\boldsymbol{e}}_{i,j}]}^{-1}) \\
&= 0.
\end{aligned}$$

As you can see, it makes essentially no difference to the computation of $\partial^2 = 0$ whether we included the $\widehat{\phi}_{[\widehat{\boldsymbol{e}}_i]}$ or not, though including the $\widehat{\phi}_{[\widehat{\boldsymbol{e}}_i]}$ made the computation look uglier.

The homology of $C(X;R)$ is called the **singular homology of X with coefficients in R**, and is denoted by $\mathrm{H}^{\mathrm{sing}}(X;R)$. In the case where $R = \mathbb{Z}$, then we simplify our notation as follows: we write $C(X;R) = C(X)$ and call this the **singular chain complex of X** and we write $\mathrm{H}^{\mathrm{sing}}(X;R) = \mathrm{H}^{\mathrm{sing}}(X)$ and call this the **singular homology of X**. Note that if $\varphi \colon R \to S$ is a ring homomorphism, then we have a canonical isomorphism of R-complexe

$$S \otimes_R C(X;R) \simeq C(X;S)$$

defined on elementary tensors by $s \otimes \sigma \mapsto s\sigma$ for all $s \in S$ and $\sigma \in \Sigma(X)$. In particular, we see that

$$\mathrm{H}(S \otimes_R C(X;R)) \simeq \mathrm{H}^{\mathrm{sing}}(X;S).$$

Keep in mind though that the lefthand side is not isomorphic to $S \otimes_R \mathrm{H}(X;R)$. However if $R = \mathbb{Z}$, then there is a theorem from homological algebra, called the **universal coefficient theorem**, which says that we have a short exact sequence

$$0 \to S \otimes_R \mathrm{H}_n(X) \to \mathrm{H}^{\mathrm{sing}}(X;S) \to \mathrm{Tor}_1(S, \mathrm{H}_{n-1}(X)) \to 0,$$

which splits (though not naturally). This is actually a theorem in homological algebra called the universal coefficient theorem.

Example 1.1. In this example, we compute the singular homology of S^1. Notice that for $n \ge 2$, we have $\Sigma_2(S^1)$

1.1.1 Reduced Homology

Let us work out the
It is often very convenient to have a slightly modified version of singular homology for which a point has trivial singular homology groups in all dimensions, including zero. This is done by defining the **reduced homology groups** $\widetilde{H}_n(X)$ to be the homology groups of the augmented chain complex

$$\cdots \longrightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} R \longrightarrow 0$$

where R sits in degree -1 and $\varepsilon\left(\sum_i r_i\sigma_i\right) = \sum_i r_i$. Here we had better require X to be nonempty in order to avoid a nontrivial homology group in dimension -1. Since $\varepsilon\partial_1 = 0$, ε vanishes on $\mathrm{Im}(\partial_1)$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \cong \widetilde{H}_0(X) \oplus R$. It is clear that $H_n(X) = \widetilde{H}_n(X)$ for all $n > 0$.

1.1.2 Homotopy Invariance

Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon C(X;R) \to C(Y;R)$ to be the unique graded homomorphism of R-modules such $f_*(\sigma) = f \circ \sigma$ for all $\sigma \in \Sigma(X)$. We claim that f_* is more than just a graded homomorphism: it is a chain map of R-complexes. Indeed, we have

$$\begin{aligned}
\partial f_*(\sigma) &= \partial(f \circ \sigma) \\
&= \sum_{0 \leq i \leq n} (-1)^i (f \circ \sigma)|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]} \\
&= \sum_{0 \leq i \leq n} (-1)^i \left(f \circ \sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}\right) \\
&= f_*\left(\sum_{0 \leq i \leq n} (-1)^i \sigma|_{[e_0,\dots,\widehat{e}_i,\dots,e_n]}\right) \\
&= f_*\partial(\sigma).
\end{aligned}$$

If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous maps, then clearly we have $(f \circ g)_* = f_* \circ g_*$. In particular, we obtain a covariant functor $\mathrm{C}\colon \mathbf{Top} \to \mathbf{Chain}_R$ from the category of topological spaces to the category of R-complexes, which sends a topological space X to the chain complex $C(X)$ and which sends a continuous map $f\colon X \to Y$ to the chain map $f_*\colon C(X) \to C(Y)$. We call C the **singular chain functor**. Note that we also have a homology functor $\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$ which sends an R-complex A to its homology $\mathrm{H}(A)$ and which sends a chain map $\varphi\colon A \to B$ to the induced map in homology $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(B)$. The **singular homology functor** is the composition of these two functors.

Recall that two continuous maps $f, g\colon X \to Y$ are said to be homotopic to each other (as continuous functions), denoted $f \sim g$, if there exists a continuous map $H\colon X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case, we say H is a homotopy form f to g. Similarly, recall that two chain maps $\varphi, \psi\colon A \to B$ are said to be homotopic to each other (as chain maps), denoted $\varphi \sim \psi$, if there exists a graded R-linear map $h\colon X \to Y$ of degree 1 such that $\varphi - \psi = \mathrm{d}_Y h + h\mathrm{d}_X$. In both **Top** and $\mathbf{Comp}_R$, the homotopy relation $\sim$ is easily seen to be an equivalence relation. We want to now show that the singular chain functor preserves these homotopy equivance relations:

Proposition 1.1. *Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions such that $f \sim g$ as continuous functions. Then $f_* \sim g_*$ as chain maps.*

Proof. Let $H\colon X \times I \to Y$ be a homotopy from f to g (so H is continuous and $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$). In order to show $f_* \sim g_*$, we need to find a graded homomorphism $h\colon C(X) \to C(Y)$ of degree 1 such that $f_* = g_* + \partial h + h\partial$ which is equivalent to showing

$$\partial h(\sigma) = g_*(\sigma) - f_*(\sigma) + h\partial(\sigma) \tag{3}$$

for all $\sigma \in \Sigma(X)$. With that in mind, let $\sigma \in \Sigma_n(X)$; so we want to find a graded homomorphism $h\colon C(X) \to C(Y)$ of degree 1 such that (3) holds. The idea is that h should somehow come from H. Consider the composite function $H \circ (\sigma \times 1) = H(\sigma \times 1)$ which is a map from $\Delta^n \times I$ to Y. Both $\Delta^n \times \{0\}$ and $\Delta^n \times \{1\}$ are n-simplices (even Euclidean n-simplices), write them as $\Delta^n \times \{0\} = [v_0, \dots, v_n] = [\boldsymbol{v}]$ and $\Delta^n \times \{1\} = [w_0, \dots, w_n] = [\boldsymbol{w}]$. Observe that

$$H(\sigma \times 1)(x,0) = f(\sigma(x)) \quad \text{and} \quad H(\sigma \times 1)(x,1) = g(\sigma(x))$$

for all $x \in X$. It follows that $H(\sigma \times 1)|_{[\boldsymbol{v}]} = f_*(\sigma)$ and $H(\sigma \times 1)_{[\boldsymbol{w}]} = g_*(\sigma)$. If $n = 0$, then $\Delta^0 \times I$ is a simplex (namely $\Delta^0 \times I = [v_0, w_0]$) and thus $H(\sigma \times 1)$ is a singular chain (up to the unique linear homeomorphism determined by (v_0, w_0)), and so if we defined $h(\sigma) = H(\sigma \times 1)$, then (3) is satisfed since $\partial(\sigma) = 0$. If $n > 0$, then

$\Delta^n \times I$ is not a simplex, so this idea doesn't work. The key however, is that we can subdivide $\Delta^n \times I$ into simplices in a nice way to make the $n = 0$ case work more generally. The idea is to pass from $[v]$ to $[w]$ by interpolating a sequence of n-simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. Thus the first step is to move $[v] = [v_0, \ldots, v_n]$ up to $[v_{n-1}, w_1] = [v_0, \ldots, v_{n-1}, w_n]$, then the second step is to move this up to $[v_{n-2}, w_2] = [v_0, \ldots, v_{n-2}, w_{n-1}, w_n]$, and so on. In the typical step $[v_i, w_{n-i}] = [v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ moves up to $[v_{i-1}, w_{n-i+1}] = [v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$. The region between these two n-simplices is exactly the $(n+1)$-simplex $[v_i, w_i, w_{n-i}]$ which has $[v_i, w_{n-i}]$ as its lower face and $[v_{i-1}, w_{n-i+1}]$ as its upper face. Altogether, $\Delta^n \times I$ is the union of $(n+1)$-simplices $[v_i, w_i, w_{n-i}]$, each intersecting the next in an n-simplex face. With this understood, we define

$$h(\sigma) = \sum_{0 \le i \le n} (-1)^i H(\sigma \times 1)_{[v_i, w_i, w_{n-i}]}.$$

Given a homotopy $H\colon X \times I \to Y$ from f to g and a singular simplex $\sigma\colon \Delta^n \to X$, we can form the composition $H \circ (\sigma \times 1)$, which is a map from $\Delta^n \times I$ to Y. Using this, we can define **prism operators** $P\colon C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$h(\sigma) = \sum_{0 \le i \le n} (-1)^i H \circ (\sigma \times 1)|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}.$$

It is straightforward to check that h satisfies (3), giving us our desired homotopy from f_* to g_*. □

Corollary. *If f and g are homotopically equivalent as continuous functions, then $f_\#$ and $g_\#$ induce the same map on homology.*

1.2 Exact Sequences and Excision

Let X be a topological space and let A a subspace of X. Then the inclusion map $\iota\colon A \to X$ induces a chain map $\iota_*\colon C(A) \to C(X)$. Let $C(X, A)$ be the cokernel of this map: $C(X, A) = C(X)/C(A)$. The homology of $C(X, A)$ is called **relative homology** and is denoted $\mathrm{H}(X, A)$. By considering the definition of the relative boundary map we see that:

- Elements of $\mathrm{H}(X, A)$ are represented by **relative cycles**: chains $\alpha \in C(X)$ such that $\partial\alpha \in C(A)$.
- A relative cycle α is trivial in $\mathrm{H}(X, A)$ if and only if it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in C(X)$ and $\gamma \in C(A)$.

The quotient $S_n(X)/S_n(A)$ could also be viewed as a subgroup of $S_n(X)$, the subgroup with basis the singular n-simplices $\sigma\colon \Delta^n \to X$ whose image is not contained in A. However, the boundary map does not take this subgroup of $S_n(X)$ to the corresponding subgroup of $S_{n-1}(X)$, so it is usually better to regard $S_n(X, A)$ as a quotient rather than a subgroup of $S_n(X)$.

Example 1.2. In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n)$, the maps $H_i(D^n, \partial D^n) \to \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the remaining terms $\widetilde{H}_i(D^n)$ are zero for all i. Thus we obtain the calculation

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.3. Applying the long exact sequence of reduced homology groups to a pair (X, x_0) with $x_0 \in X$ yields isomorphisms $H_n(X, x_0) \cong \widetilde{H}_n(X)$ for all n since $\widetilde{H}_n(x_0) \cong 0$ for all n.

1.2.1 Excision

Theorem 1.1. *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X \backslash Z, A \backslash Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \backslash Z, A \backslash Z) \to H_n(X, A)$ for all n. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.*

The translation between the two versions is obtained by setting $B = X \backslash Z$ and $Z = X \backslash B$. Then $A \cap B = A \backslash Z$ and the condition $\overline{Z} \subset \mathrm{int}(A)$ is equivalent to $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ since $X \backslash \mathrm{int}(B) = \overline{Z}$.

For a spae X, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $S_n^{\mathcal{U}}(X)$ be the subgroup of $S_n(X)$ consisting of chains $\sum_i r_i \sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$. The boundary map ∂ takes $S_n^{\mathcal{U}}(X)$ to $S_n^{\mathcal{U}}(X)$, so the groups $S_n^{\mathcal{U}}(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^{\mathcal{U}}(X)$.

Proposition 1.2. *The inclusion $\iota\colon S_n^{\mathcal{U}}(X) \hookrightarrow S_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho\colon S_n(X) \to S_n^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence ι induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.*

Proof. The barycentric subdivision process will be performed at four levels, beginning with the most geometric and becoming increasingly algebraic.

(1) Barycentric Subdivision of Simplices: The points of a simplex $[v_0, \ldots, v_n]$ are the linear combinations $\sum_i t_i v_i$ with $\sum t_i = 1$ and $t_i \in [0,1]$ for each i. The **barycenter** or 'center of gravity' of the simplex $[v_0, \ldots, v_n]$ is the point $b = \sum t_i v_i$ whose barycentric coordinates t_i are all equal, namely $t_i = 1/(n+1)$ for each i. The **barycentric subdivision** of $[v_0, \ldots, v_n]$ is the decomposition of $[v_0, \ldots, v_n]$ into the n-simplices $[b, w_0, \ldots, w_{n-1}]$ where, inductively, $[w_0, \ldots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$. The induction starts with the case $n = 0$ when the barycentric subdivision of $[v_0]$ is defined to be just $[v_0]$ itself. It follows from the inductive definition that the vertices of simplices in the barycentric subdivision of $[v_0, \ldots, v_n]$ are exactly the barycenters of all the k-dimensional faces $[v_{i_0}, \ldots, v_{i_k}]$ of $[v_0, \ldots, v_n]$ for $0 \leq k \leq n$. When $k = 0$ this gives the original vertices v_i since the barycenter of 0-simplex is itself. The barycenter of $[v_{i_0}, \ldots, v_{i_k}]$ has barycentric coordinates $t_i = 1/(k+1)$ for $i = i_0, \ldots, i_k$ and $t_i = 0$ otherwise.

The n-simplices of the barycentric subdivision of Δ^n, together with all their faces, do in fact form a Δ-complex structure on Δ^n, indeed a simplicial complex structure, though we shall not need to know this in what follows.

A fact we will need is that the diameter of each simplex of the barycentric subdivision of $[v_0, \ldots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \ldots, v_n]$. Here the diameter of a simplex is by definition the maximum distance between any two of its points, and we are using the metric from the ambient Euclidean space $\mathbb{R}^m$ containing $[v_0, \ldots, v_n]$. The diameter of a simplex equals the maximum distance between any of its vertices because the distance between the points v and $\sum t_i v_i$ of $[v_0, \ldots, v_n]$ satisfies the inequality

$$\begin{aligned} \left| v - \sum_{i=0}^{n} t_i v_i \right| | &= \left| \sum_{i=0}^{n} t_i (v - v_i) \right| \\ &\leq \sum_{i=0}^{n} t_i |v - v_i| \\ &\leq \sum_{i=0}^{n} t_i \max_{0 \leq j \leq n} |v - v_j| \\ &= \max_{0 \leq j \leq n} |v - v_j|. \end{aligned}$$

The significance of the factor $n/(n+1)$ is that by repeated barycentric subdivision we can produce simplices of arbitrarily small diameter since $(n/(n+1))^r$ approaches 0 as r goes to infinity. It is important that the bound $n/(n+1)$ does not depend on the shape of the simplex since repeated barycentric subdivision produces simplices of many different shapes.

To obtain the bound $n/(n+1)$ on the ratio of diameters, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex $[w_0, \ldots, w_n]$ of the barycentric subdivision of $[v_0, \ldots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \ldots, v_n]$.

(2) Barycentric Subdivision of Linear Chains. The main part of the proof will be to construct a subdivision operator $\mathcal{S}\colon S_n(X) \to S_n(X)$ and show that this is chain homotopic to the identity map. First we will construct $\mathcal{S}$ and the chain homotopy in a more restricted linear setting.

For a convex set Y in some Euclidean space, the linear maps $\Delta^n \to Y$ generate a subgroup of $S_n(Y)$ that we denote $L_n(Y)$, the **linear chains**. Note that $L(Y)$ is ∂-stable, so the linear chains form a subcomplex of $(S(Y), \partial)$. We can uniquely designate a linear map $\lambda\colon \Delta^n \to Y$ by $[w_0, \ldots, w_n]$ where w_i is the image under λ of the ith vertex of Δ^n. Indeed, by linearity we have $\lambda(\sum t_i e_i) = \sum t_i \lambda(e_i)$. To avoid having to make exceptions for 0-simplices, it will be convenient to augment the complex $(L(Y), \partial)$ by setting $L_{-1}(Y) = R$ generated by the empty simplex $[\varnothing]$, with $\partial[w_0] = [\varnothing]$ for all 0-simplices $[w_0]$.

Each point $b \in Y$ determines a graded homomorphism $b\colon L(Y) \to L(Y)$ of degree 1, defined on basis elements by $b\left([w_0, \ldots, w_n]\right) = [b, w_0, \ldots, w_n]$. Geometrically, the homomorphism b can be regarded as a cone operator, sending a linear chain to the cone having the linear chain as the base of the cone and the point b as the tip of the cone. Applying the usual formula for ∂, we obtain the relation

$$\begin{aligned} \partial b([w_0, \ldots, w_n]) &= \partial[b, w_0, \ldots, w_n]) \\ &= [w_0, \ldots, w_n] - b(\partial[w_0, \ldots, w_n]). \end{aligned}$$

By linearity it follows that $\partial b(\alpha) = \alpha - b(\partial\alpha)$ for all $\alpha \in L(Y)$. This expresses algebraically the geometric fact that the boundary of a cone consists of its base together with the cone on the boundary of its base. The relation $\partial b(\alpha) = \alpha - b(\partial\alpha)$ can be rewritten as

$$\partial b + b\partial = 1,$$

so b is a chain homotopy between the identity map and the zero map of the augmented chain complex $(L(Y), \partial)$.

Now we define a graded homomorphism $\mathcal{S}\colon L(Y) \to L(Y)$ by induction on n. Let $\lambda\colon \Delta^n \to Y$ be a generator of $L(Y)$ and let b_λ be the image of the barycenter of Δ^n under λ. Then the inductive formula for $\mathcal{S}$ is

$$\mathcal{S}(\lambda) = b_\lambda(\mathcal{S}(\partial\lambda)),$$

where $b_\lambda\colon L(Y) \to L(Y)$ is the cone operator defined in the preceding paragraph. The induction starts with $\mathcal{S}([\emptyset]) = [\emptyset]$, so $\mathcal{S}$ is the identity on $L_{-1}(Y)$. To get a feel for the map $\mathcal{S}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned} \mathcal{S}[w_0] &= w_0\,(\mathcal{S}(\partial[w_0])) \\ &= w_0(\mathcal{S}[\emptyset]) \\ &= w_0[\emptyset] \\ &= [w_0]. \end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter b_{01}. Then

$$\begin{aligned} \mathcal{S}[w_0, w_1] &= b_{01}\,(\mathcal{S}(\partial[w_0, w_1])) \\ &= b_{01}(\mathcal{S}[w_1] - \mathcal{S}[w_0]) \\ &= b_{01}([w_1] - [w_0]) \\ &= [b_{01}, w_1] - [b_{01}, w_0]. \end{aligned}$$

Now let $[w_0, w_1, w_2] \in L_2(Y)$ with barycenter b_{012}. Then

$$\begin{aligned} \mathcal{S}[w_0, w_1, w_2] &= b_{012}\,(\mathcal{S}(\partial[w_0, w_1, w_2])) \\ &= b_{012}(\mathcal{S}[w_1, w_2] - \mathcal{S}[w_0, w_2] + \mathcal{S}[w_0, w_1]) \\ &= b_{012}([b_{12}, w_2] - [b_{12}, w_1] - [b_{02}, w_2] + [b_{02}, w_0] + [b_{01}, w_1] - [b_{01}, w_0]) \\ &= [b_{012}, b_{12}, w_2] - [b_{012}, b_{12}, w_1] + [b_{012}, b_{02}, w_0] - [b_{012}, b_{02}, w_2] + [b_{012}, b_{01}, w_1] - [b_{012}, b_{01}, w_0], \end{aligned}$$

where b_{12}, b_{02}, and b_{01} are the barycenters for the simplices $[w_1, w_2]$, $[w_0, w_2]$, and $[w_0, w_1]$ respectively. In general, when λ is an embedding, with image a genuine n-simplex $[w_0, \ldots, w_n]$, then $\mathcal{S}(\lambda)$ is the sum of the n-simplices in the barycentric subdivision of $[w_0, \ldots, w_n]$, with certain signs that could be computed explicitly.

Let us check that $\mathcal{S}\colon L(Y) \to L(Y)$ is a chain map, i.e. $\partial\mathcal{S} = \mathcal{S}\partial$. Since $\mathcal{S} = 1$ on $L_0(Y)$ and $L_{-1}(Y)$, we certainly have $\partial\mathcal{S} = \mathcal{S}\partial$ on $L_0(Y)$. The result for larger n is given by the following calculation, in which we omit some parentheses to unclutter the formulas:

$$\begin{aligned} \partial\mathcal{S}\lambda &= \partial b_\lambda(\mathcal{S}\partial\lambda) \\ &= (1 - b_\lambda\partial)(\mathcal{S}\partial\lambda) \\ &= \mathcal{S}\partial\lambda - b_\lambda\partial(\mathcal{S}\partial\lambda) \\ &= \mathcal{S}\partial\lambda - b_\lambda\mathcal{S}(\partial\partial\lambda) \\ &= \mathcal{S}\partial\lambda, \end{aligned}$$

where $\partial S(\partial\lambda) = S\partial(\partial\lambda)$ follows by induction on n.

We next build a chain homotopy $\mathcal{T}\colon L(Y) \to L(Y)$ between $\mathcal{S}$ and the identity. We define $\mathcal{T}$ on $L_n(Y)$ inductively by setting $\mathcal{T} = 0$ for $n = -1$ and let $\mathcal{T}\lambda = b_\lambda(\lambda - \mathcal{T}\partial\lambda)$ for $n \geq 0$. The induction starts with $\mathcal{T}[\emptyset] = 0$. To get a feel for the map $\mathcal{T}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned} \mathcal{T}[w_0] &= w_0\,([w_0] - \mathcal{T}\partial[w_0]) \\ &= w_0\,([w_0] - \mathcal{T}[\emptyset]) \\ &= [w_0, w_0]. \end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter b_{01}. Then

$$\begin{aligned} \mathcal{T}[w_0, w_1] &= b_{01}\,([w_0, w_1] - \mathcal{T}\partial[w_0, w_1]) \\ &= b_{01}\,([w_0, w_1] - \mathcal{T}[w_1] + \mathcal{T}[w_0]) \\ &= [b_{01}, w_0, w_1] - [b_{01}, w_1, w_1] + [b_{01}, w_0, w_0]. \end{aligned}$$

The geometric motivation for this formula is an inductively defined subdivision of $\Delta^n \times I$ obtained by joining all simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$. What $\mathcal{T}$ actually does is take the image of this subdivision under the projection $\Delta^n \times I \to \Delta^n$.

The chain homotopy formula $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ is trivial on $L_{-1}(Y)$ where $\mathcal{T} = 0$ and $\mathcal{S} = 1$. Verifying the formula on $L_n(Y)$ with $n \geq 0$ is done by the calculation

$$\begin{aligned}
\partial\mathcal{T}\lambda &= \partial b_\lambda(\lambda - \mathcal{T}\partial\lambda) \\
&= (1 - b_\lambda\partial)(\lambda - \mathcal{T}\partial\lambda) \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\mathcal{T}\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda(1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\lambda - b_\lambda\mathcal{S}\partial\lambda - b_\lambda\mathcal{T}\partial\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\mathcal{S}\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - \mathcal{S}\lambda.
\end{aligned}$$

where $\partial\mathcal{T}\partial\lambda = (1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda$ follows by induction on n. Now we discard $L_{-1}(Y)$ and the relation $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ still holds since $\mathcal{T}$ was zero on $L_{-1}(Y)$.

(3) Barycentric Subdivision of General Chains. Define $\mathcal{S}\colon S_n(X) \to S_n(X)$ by setting $\mathcal{S}\sigma = \sigma_\#\mathcal{S}\Delta^n$ for a singular n-simplex $\sigma\colon \Delta^n \to X$. Since $\mathcal{S}\Delta^n$ is the sum of the n-simplices in the barycentric subdivision of Δ^n, with certain signs, $\mathcal{S}\sigma$ is the corresponding signed sum of the restrictions of σ to the n-simplices of the barycentric subdivision of Δ^n. For example, if $\sigma \in S_1(X)$, then

$$\begin{aligned}
\mathcal{S}\sigma &= \sigma_\#\mathcal{S}[e_0, e_1] \\
&= \sigma \circ ([b, e_1] - [e_0, b]) \\
&= \sigma|_{[b,e_1]} - \sigma|_{[e_0,b]},
\end{aligned}$$

where $b = (e_0 + e_1)/2$ is the barycenter of $[e_0, e_1]$.

The operator $\mathcal{S}$ is a chain map since

$$\begin{aligned}
\partial\mathcal{S}\sigma &= \partial\sigma_\#\mathcal{S}\Delta^n \\
&= \sigma_\#\partial\mathcal{S}\Delta^n \\
&= \sigma_\#\mathcal{S}\partial\Delta^n \\
&= \sigma_\# S(\sum_i (-1)^i \Delta_i^n) \\
&= \sum_i (-1)^i \sigma_\# S\Delta_i^n \\
&= \sum_i (-1)^i S(\sigma|_{\Delta_i^n}) \\
&= S\left(\sum_i (-1)^i \sigma|_{\Delta_i^n}\right) \\
&= S(\partial\sigma).
\end{aligned}$$

where Δ_i is the ith face of Δ^n.

In similar fashion we define $T\colon S_n(X) \to S_n(X)$ by $T\sigma = \sigma_\# T\Delta^n$, and this gives a chain homotopy between S and the identity, since the formula $\partial T + T\partial = 1 - S$ holds by the calculation

$$\begin{aligned}
\partial T\sigma &= \partial\sigma_\# T\Delta^n \\
&= \sigma_\#\partial T\Delta^n \\
&= \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n) \\
&= \sigma - S\sigma - \sigma_\# T\partial\Delta^n \\
&= \sigma - S\sigma - T(\partial\sigma)
\end{aligned}$$

where the last equality follows just as in the previous displayed calculation, with S replaced by T.

(4) Iterated Barycentric Subdivision. A chain homotopy between 1 and the iterate S^m is given by the operator

$D_m = \sum_{0\le i<m} TS^i$ since

$$\begin{aligned}\partial D_m + D_m\partial &= \sum_{0\le i<m}(\partial TS^i + TS^i\partial)\\ &= \sum_{0\le i<m}(\partial TS^i + T\partial S^i)\\ &= \sum_{0\le i<m}(\partial T + T\partial)S^i\\ &= \sum_{0\le i<m}(1-S)S^i\\ &= \sum_{0\le i<m}(S^i - S^{i+1})\\ &= 1 - S^m.\end{aligned}$$

For each singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an m such that $S^m(\sigma)$ lies in $S_n^{\mathcal{U}}(X)$ since the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of Δ^n by the open sets $\sigma^{-1}(\mathrm{int}(U_j))$ if m is large enough. (Recall that a Lebesgue number for an open cover of a compact metric space is a number $\varepsilon > 0$ such that every set of diameter less than ε lies in some set of the cover; such a number exists by an elementary compactness argument). We cannot expect the same number m to work for all σ's, so let us define $m(\sigma)$ to be the smallest m such that $S^m(\sigma)$ is in $S_n^{\mathcal{U}}(X)$.

We now define $D\colon S_n(X) \to S_{n+1}(X)$ by setting $D\sigma = D_{m(\sigma)}\sigma$ for each singular n-simplex $\sigma\colon \Delta^n \to X$. For this D we would like to find a chain map $\rho\colon S_n(X) \to S_n(X)$ with image in $S_n^{\mathcal{U}}(X)$ satisfying the chain homotopy equation

$$\partial D + D\partial = 1 - \rho. \tag{4}$$

A quick way to do this is to simply regard this equation as defining ρ, so we let $\rho = 1 - \partial D - D\partial$. It follows easily that ρ is a chain map since

$$\begin{aligned}\partial\rho(\sigma) &= \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma\\ &= \partial\sigma - \partial D\partial\sigma\\ &= \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma\\ &= \rho(\partial\sigma).\end{aligned}$$

To check that ρ takes $S_n(X)$ to $S_n^{\mathcal{U}}(X)$, we compute $\rho(\sigma)$ more explicitly:

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial D\sigma - D(\partial\sigma)\\ &= \sigma - \partial D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)\\ &= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma).\end{aligned}$$

The term $S^{m(\sigma)}\sigma$ lies in $S_n^{\mathcal{U}(X)}$ by the definition of $m(\sigma)$. The remaining terms $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ for σ_j the restriction of σ to a face of Δ^n, so $m(\sigma_j) \le m(\sigma)$ and hence the difference $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ consists of terms $TS^i(\sigma_j)$ with $i \ge m(\sigma_j)$, and these terms lie in $S_n^{\mathcal{U}}(X)$ since T takes $S_{n-1}^{\mathcal{U}}(X)$ to $S_n^{\mathcal{U}}(X)$.

View ρ as a chain map $S_n(X) \to S_n^{\mathcal{U}}(X)$, the equation (4) says that $\partial D + D\partial = 1 - \iota\rho$ for $\iota\colon S_n^{\mathcal{U}}(X) \hookrightarrow S_n(X)$ the inclusion. Furthermore, $\rho\iota = 1$ since D is identically zero on $S_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ if σ is in $S_n^{\mathcal{U}}(X)$, hence the summation defining $D\sigma$ is empty. Thus we have shown that ρ is a chain homotopy inverse for ι.

□

Let R be a ring and let M be an R-module. A derivation is a map $d\colon R \to M$ such that

1.3 Singular Cohomology

Let R be a ring and N and R-module. If M is a graded R-module, then we set $\mathrm{Hom}_R(M,N)_{\mathrm{gr}}$ to be the graded R-module whose homogeneous component in degree n is $M_n := \mathrm{Hom}_R(M_n, N)$. If (M, d) is a chain complex over R, where M is considered a graded R-module and d is considered a graded endomorphism $d\colon M \to M$ of degree -1, then we obtain a cochain complex over R given by $(\mathrm{Hom}_R(M,N)_{\mathrm{gr}}, d_*)$, where if $\psi \in \mathrm{Hom}_R(M_{n-1}, N)$ then $d_*(\psi) = \psi \circ d \in \mathrm{Hom}_R(M_n, N)$.

In particular, we obtain a cochain complex $(\mathrm{Hom}_R(S(X), N)_{\mathrm{gr}}, \partial_*)$ called the **singular cochain complex of** X **over** R **with values in** N. Elements in $S_n(X)^\vee$ are called **singular** n**-cochains** and the nth cohomology, called the **singular cohomology of** X **over** R, is denoted $H^n_{\text{sing}}(X, R)$. For notational purposes, we denote $\delta := \partial^\vee_{\text{gr}}$ and $S^n(X, R) := S_n(X, R)^\vee$. We can work out δ explicitly as follows: if $\psi \in S^n(X)$, then $\delta(\psi) \in S^{n+1}(X)$ is given by

$$\delta(\psi)(\sigma) = \psi(\partial(\sigma)) = \sum_i (-1)^i \psi(\sigma_i)$$

for all $\sigma \in S_{n+1}(X)$.

1.3.1 Delta Complex

A Δ**-complex** structure on a space X is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with n depending on the index α, such that:

1. The restriction $\sigma_\alpha|_{\Delta^n \backslash \partial\Delta^n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\Delta^n \backslash \partial\Delta^n}$.
2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism betweem them that preserves the ordering of the vertices.
3. A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Among other things, this last condition rules out trivialities like regarding all the points of X are individual vertices.

2 $C(X, Y)$

Let X and Y be topological spaces. We define $C(X, Y)$ to be the set of all continuous maps from X to Y. We endow $C(X, Y)$ with a topology, called the **compact-open topology**, where a subbase of $C(X, Y)$ is given by the collection of all sets of the form

$$\mathrm{B}(K, V) = \{f \in C(X, Y) \mid f(K) \subseteq V\}$$

where $K \subseteq X$ is compact and where $V \subseteq Y$ is open. In particular, if $\Omega \subseteq C(X, Y)$ is open and $f \in \Omega$, then we can find a $\mathrm{B}(K, V)$ such $f \in \mathrm{B}(K, V) \subseteq \Omega$. In other words, we can find $K \subseteq X$ compact and $V \subseteq Y$ open such that $f(K) \subseteq V$ and for any $g \in C(X, Y)$ with the property that $g(K) \subseteq V$, we have $g \in \Omega$. Whenever we write "let $\mathrm{B}(K, V) \subseteq C(X, Y)$ be a basic open", then as long as context is clear, it will be understood that $K \subseteq X$ is compact and $V \subseteq Y$ is open. We will try to be as consistent with our notation as possible. For instance, we typically use Δ, K, L, M to denote compact sets, Ω, U, V, W to denote open sets, and Γ, E, F, G to denote closed sets. Furthermore, we will try to be as lexicographically consistent with our notation as possible (for instance, $K \subseteq X$, $L \subseteq Y$, $M \subseteq Z$ is lexicographically consistent, whereas $V \subseteq X$, $W \subseteq Y$, and $U \subseteq Z$ is not). Even though there's no guarantee that we will adhere to this principal at all times, we will always try to be as clear as possible from context.

Now we assume that X and Y are locally compact Hausdorff spaces. Let $f \in \mathrm{B}(K, V)$. Note that since Y is locally compact, we can find a compact neighborhood of L, say We want to show that there exists $K' \subseteq K$ compact, $V' \subseteq V$ open, and $L' \subseteq Y$ compact such that

$$f(K') \subseteq V' \subseteq L'.$$

Indeed, for
such that there exists wish to replace K and V with K' and V' such that let $x \in K$, and set $y = f(x)$. First we wish to replace K with another compact set $K' \subseteq X$ and we wish to replace V with another open set $V' \subseteq Y$ such that

$$K' \subseteq K, \quad \text{and} \quad f(K') \subseteq V' \subseteq \widetilde{L}$$

$K' \subseteq K$ and V with V'
Choose a compact neighborhood of y, say $y \in V_y \subseteq L_y$. Note that $V \cap V_y \subseteq V$ is a smaller neighborhood of y which is contained in the compact L_y. The collection $\{V \cap V_y\}$ covers L, so we can extract a finite subcovering $L = (V \cap V_{y_1}) \cup \cdots \cup (V \cap V_{y_n})$.

Now suppose X and Y are Hausdorff and let $f \in \mathrm{B}(K, V)$. Since f is continuous, we see that $L := f(K)$ is compact, and since Y is Hausdorff, this implies $L \subseteq Y$ is closed. For each $y \in L$, choose a compact neighborhood of y, say $y \in V_y \subseteq L_y$. Note that $\{V_y\}_{y \in L}$ is a covering of L, so we can extract a finite subcovering, say

$$L \subseteq \bigcup_{i=1}^n V_{y_i} \subseteq \bigcup_{i=1}^n L_{y_i}.$$

Set $V_0 = \bigcup_{i=1}^n V_{y_i}$ and set $L_0 = \bigcup_{i=1}^n L_{y_i}$, so $f(K) \subseteq V_0 \subseteq L_0$ (The open V_0 is smushed inbetween two compacts!). Thus we have shown that we can replace $\mathrm{B}(K,V)$ with a smaller open neighborhood of f, namely $\mathrm{B}(K,V_0)$, which has the property that V_0 is contained in a compact set. Furthermore, the difference $V_0 \backslash K = V_{0,1} \cup \cdots \cup V_{0,k}$ can be covered by finitely many opens In particular, if $g(K) \subseteq V$Furthermore, notice that if $g \in \mathrm{B}(K,V_0)$, then
Even better, there exist finiely many Now we note that the difference $L_0 \backslash V_0$ is open since L_0 is closed (Y is Hausdorff) and V_0 is open. By the usual argument, we can again extract
In particular, note that $f \in \mathrm{B}(K,V_0) \subseteq \mathrm{B}(K,V)$. In fact, we can do better. Observe that the difference $\widetilde{V} := V_0 \backslash L$
Thus the difference $\widetilde{V} := V \backslash L$ is open in Y since the difference between an open with a closed is open. Then $\widetilde{U} := f^{-1}(\widetilde{V})$ is open in X and disjoint from K, and moreover we have $f(\widetilde{U} \cup K) = V$. Now for each $y \in V$, choose $V_y \subseteq Y$ open and $L_y \subseteq L$ compact such that $y \in V_y \subseteq L_y$. Then $\{V_y\}_{y \in V}$ is a covering of L, so we can extract a finite subcovering, say $L = \bigcup_{j=1}^n V_{y_j}$. Then

Note that if $K' \subseteq X$ is compact and $K' \subseteq K$, then $\mathrm{B}(K',V) \subseteq \mathrm{B}(K,V)$ is a smaller subbasic open neighborhood of f, and if $V' \subseteq Y$ is open such that $L \subseteq V' \subseteq V$, then $\mathrm{B}(K,V) \subseteq \mathrm{B}(K,V')$ is also a smaller subbasic open neighborhood of f. Now fix a point $x \in K$ and choose a compact neighborhood of x, say $x \in U_0 \subseteq K_0$.
. iwe can obtain a smaller subbasic open neighborhood of f by replacing K with any $K' \subseteq X$ compact such that if we replace V with any $V' \subseteq Y$ open such that $L \subseteq V' \subseteq V$, then we obtain a smaller subbasic neighborhood of f, namely $\mathrm{B}(K,V')$. to get a smaller subbasic set $\mathrm{B}(L,V')$
Now fix $x \in K$ and denote $y = f(x)$. Since Y is locally Hausdorff, there exists $V_y \subseteq Y$ open and $L_y \subseteq Y$ compact such that $y \in V_y \subseteq L_y$.
Since Y is *locally* compact,

Suppose $f \in \mathrm{B}(K,V) \cap \mathrm{B}(K',V')$, so $f(K) \subseteq V \subseteq \overline{V}$ and $f(K') \subseteq V' \subseteq \overline{V}'$ with $\overline{V}$ and $\overline{V}'$ compact. Then $f(K \cup K') \subseteq V \cup V' \subseteq \overline{V} \cup \overline{V}'$. It follows that If $g(K'') \not\subseteq V$, then
If $g(K \cup K') \subseteq V \cap V'$, then certainly We want to find K'' and V'' such that $f(K'') \subseteq V''$ and if $g(K'') \subseteq V''$ then $g(K) \subseteq V \subseteq L_0$ and $g(K') \subseteq V' \subseteq L_0'$. Let's try K''

Given $f \in C(X,Y)$, we have

$$\begin{aligned} f \in \mathrm{B}(K,V) \cap \mathrm{B}(K',V') &\iff f \in \mathrm{B}(K,V) \text{ and } f \in \mathrm{B}(K,V') \\ &\iff f(K) \subseteq V \text{ and } f(K) \subseteq V' \\ &\iff f(K) \subseteq V \cap V' \\ &\iff f \in \mathrm{B}(K, V \cap V'). \end{aligned}$$

It follows that $\mathrm{B}(K,V) \cap \mathrm{B}(K,V') = \mathrm{B}(K, V \cap V')$ where $V \cup V' \subseteq Y$ is open. However, note that $\{\mathrm{B}(K,V)\}$ is not necessarily a basis for $C(X,Y)$. To see why, consider two subbasic opens $\mathrm{B}(K,V)$ and $\mathrm{B}(K',V')$ and let $f \in \mathrm{B}(K,V) \cap \mathrm{B}(K',V')$, so $f(K) \subseteq V$ and $f(K') \subseteq V'$. Note $K \cap K'$ is compact since X is Hausdorff and so $L = f(K \cap K')$ is compact since f is continuous. Thus for any $V'' \subseteq Y$ open such that $L \subseteq V''$, we have $f \in \mathrm{B}(K,V'')$.

B(

that X is HausdorffSince Y is locally compact Hausdorff, Then it's easy to see that $f(K \cap K') \subseteq V \cap V'$ and $f(K \cup K') \subseteq V \cup V'$.
and if $g(K \cap K') \subseteq V \cap V'$, then $g(K) \subseteq V$ and $g(K') \subseteq V'$, so $g \in \mathrm{B}(K,V) \cap \mathrm{B}(K',V')$. and
closed under finite intersections; all we can say in general is that $\mathrm{B}(K,V) \cap \mathrm{B}(K',V') \subseteq \mathrm{B}(K \cup K', V \cup V')$. In particular, $\{\mathrm{B}(K,V)\}$ is not necessarily a basis for $C(X,Y)$. If we want $\{\mathrm{B}(K,V)\}$ to be a basis, then we need impose a condition on X. One such conditon is X being locally Hausdorff. Indeed, if X is locally Hausdorff, and $x \in K \cap K' x \in \mathrm{B}(K,V) \cap \mathrm{B}(K',V')$ (so only forms a subbasis for the compact-open topology; it does not necesssarily form a basis. Indeed, the problem is that we may not be able to express $C(X,Y)$ as a union of sets of the form $\mathrm{B}(K,V)$. In other words, there may be continuous functions $f\colon X \to Y$ such that there does not exist a compact set $K \subseteq X$ and an open set $Y \subseteq V$ such that $f(K) \subseteq Y$.
first problem is that we may not have $C(X,Y) = \mathrm{B}(K,V)$ for some $K \subseteq X$ compact and $V \subseteq Y$ open.
, so that Note that the collection of all sets of the form $\mathrm{B}(K,V)$

A natural space to consider is the iterated space $C(X, C(Y,Z))$. The basic opens in this space have the form $\mathrm{B}(K,\Omega)$ (where again it is implicitly understood that $K \subseteq X$ is compact and $\Omega \subseteq C(Y,Z)$ is open). Write Ω as a union of the basic opens of $C(Y,Z)$, say $\Omega = \bigcup_{i \in I} \mathrm{B}(L_i, W_i)$. We claim that

$$\mathrm{B}(K,\Omega) = \bigcup_{i \in I} \mathrm{B}(K, \mathrm{B}(L_i, W_i)).$$

Indeed, given $f \in C(X, C(Y,Z))$, we have

$$\begin{aligned} f \in \bigcup_{i \in I} \mathrm{B}(K, \mathrm{B}(L_i, W_i)) &\iff f \in \mathrm{B}(K, \mathrm{B}(L_i, W_i)) \text{ for some } i \\ &\iff f(K) \subseteq \mathrm{B}(L_i, W_i) \text{ for some } i \\ &\iff f(K) \subseteq \bigcup_{i \in I} \mathrm{B}(L_i, W_i) \\ &\iff f(K) \subseteq \Omega \\ &\iff f \in \mathrm{B}(K, \Omega) \end{aligned}$$

Thus the sets of the form $\mathrm{B}(K, \mathrm{B}(L, W))$ serve as a basis for $C(X, C(Y,Z))$. Another natural space to consider is $C(X \times Y, Z)$. The basic opens of this space have the form $\mathrm{B}(\Delta, W)$ where $\Delta \subseteq X \times Y$ is compact. Note that Δ has the form $\Delta = K \times L$ where $K \subseteq X$ and $L \subseteq Y$ are compact (namely $K = \pi_1(X \times Y)$ and $L = \pi_2(X \times Y)$). So the sets of the form $\mathrm{B}(K \times L, W)$ serve as a basis for $C(X \times Y, Z)$.

Proposition 2.1. *Define a map* $(-)^{\diamond} \colon C(X, C(Y,Z)) \to C(X \times Y, Z)$ *by*

$$f^{\diamond}(x,y) := (f(x))(y) \tag{5}$$

where $f \in C(X, C(Y,Z))$*, where* $x \in X$*, and* $y \in Y$*. If* X, Y*, and* Z *are Hausdorff and locally compact, then* Φ *is a homeomorphism. In this case, we can write* $(Z^Y)^X = Z^{X \times Y}$*; this formula provides a justification for the notation* Y^X *and is called the* ***exponential law****.*

Note that in (5), f is a continuous function from X to $C(Y,Z)$, so it takes an element x and spits out another continuous function $f(x) \colon Y \to Z$, which takes an element y and spits out and element $(f(x))(y)$. We can express this whole process by simply writing $(f(x))(y)$ where the parethenthesis gives us context of what is what; for instance, the fact that we have a parenthesis surrounding $f(x)$ in $(f(x))(y)$ should tell you that $f(x)$ is a function of y. Before we give a proof of this proposition, we want to point out that $(-)_{\diamond}$ is already a bijection as a set-theoretic function, with inverse $(-)_{\diamond} \colon C(X \times Y, Z) \to C(X, C(Y,Z))$ defined by

$$(g_{\diamond}(x))(y) := g(x,y) \tag{6}$$

where $g \in C(X \times Y, Z)$, where $x \in X$, and where $y \in Y$. Again, the parenthesis in (6) should tell us how to interpret this equation. One should think of the maps $(-)^{\diamond}$ and $(-)_{\diamond}$ as applying some sort of associative law. To see this, we first simplify our notation and write (5) and (6) as

$$f^{\diamond}(x,y) := (fx)y \quad \text{and} \quad (g_{\diamond}x)y := g(x,y) \tag{7}$$

instead (this is similar to the notation used in linear analysis where we typically write Tx instead of $T(x)$ for a linear map T and a vector x). Intuitively, one thinks of $f^{\diamond}(x,y) = (fx)y$ as applying the "associative law" where the diamond in the superscript tells us that we can "pull back" the parenthesis. Similarly, one thinks of $(g_{\diamond}x)y = g(x,y)$ as applying the "associative law" where the diamond in the subscript tells us that we can "push forward" the parenthesis. With this notational simplification in mind, it is very easy to see why $(-)^{\diamond}$ and $(-)_{\diamond}$ are inverse to each other as set-theoretic functions: we are just applying the "associative law": we have

$$((f^{\diamond})_{\diamond}x)y = f^{\diamond}(x,y) = f(x)y \quad \text{and} \quad (g_{\diamond})^{\diamond}(x,y) = (g_{\diamond}x)y = g(x,y) \tag{8}$$

where context makes it clear how to interpret all of the symbols in (8) In particular, one should note that the reason why $(-)_{\diamond}$ and $(-)^{\diamond}$ are inverse to each other is precisely due to the way we defined them in the first place. Another added benefit that we get when using this notation is that when we write an interpretable string using the symbols $\{\diamond, (,), f, g, h, x, y, z\}$, then it becomes visibly clear how we could interpret this string, where we consider a string interpretable if we can obtain a new string without any diamond symbols by applying the associative law a finite number of times to the original string. For instance, the string $f_{\diamond}(x,y)$ is uninterpretable in our language since we can't "pullback" the parenethesis and remove the diamond in the subscript. On the other had, the string $h^{\diamond}(gx, (f_{\diamond}x)y)$ is interpretable: if we apply the "associative law" one time, we can remove the subscript diamond and obtain $h^{\diamond}(gx, f(x,y))$. If we apply the associative law again, we can remove the superscript diamond and obtain $(h(gx))f(x,y)$. Since this string doesn't contain any diamonds, we can give a reasonable interpretation to it. For instance, h can be thought of as a function in $C(A_1, C(A_2, A_3))$, with maps the element $gx \in A_1$ to the function $h(gx) \in C(A_2, A_3)$ whose value at $f(x,y) \in A_2$ is $(h(gx))f(x,y)$. Let us now prove Proposition (2.1).

Proof. It suffices to show that both $(-)^\diamond$ and $(-)_\diamond$ are continuous. Observe that

$$\begin{aligned} g_\diamond \in \mathrm{B}(K, \mathrm{B}(L, W)) &\iff g_\diamond K \subseteq \mathrm{B}(L, W) \\ &\iff g_\diamond x \in \mathrm{B}(L, W) \text{ for all } x \in K \\ &\iff (g_\diamond x)y \in W \text{ for all } y \in L \text{ for all } x \in K \\ &\iff g(x, y) \in W \text{ for all } y \in L \text{ for all } x \in K \\ &\iff g(K \times L) \in W \\ &\iff g \in \mathrm{B}(K \times L, W). \end{aligned}$$

Similarly, we have

$$f^\diamond \in \mathrm{B}(K \times L, W) \iff f^\diamond$$

First let us show $(-)_\diamond$ is continuous , say at $g \in C(X \times Y, Z)$. Choose a basic open $\mathrm{B}(K, \mathrm{B}(L, W)) \subseteq C(X(Y, Z))$ such that $g_\diamond \in \mathrm{B}(K, \mathrm{B}(L, W))$, so $(g_\diamond x)y = g(x, y) \subseteq W$ for all $x \in K$ and $y \in L$. Observe that

$$\begin{aligned} \widetilde{g}_\diamond \in \mathrm{B}(K \times L, W) &\iff (\widetilde{g}_\diamond x)y \in W \text{ for all } x \in K \text{ and } y \in L \\ &\iff \widetilde{g}(x, y) \in W \text{ for all } x \in K \text{ and } y \in L \\ &\iff \widetilde{g} \in \mathrm{B}(K \times L, W) \end{aligned}$$

if $\widetilde{g} \in \mathrm{B}(K \times L, W)$, then $\widetilde{g}_\diamond \in \mathrm{B}(K, \mathrm{B}(L, W))$ if and only if $\widetilde{g}_\diamond(x, y) \subseteq W$ for all

$$f \in \mathrm{B}($$

$$\begin{aligned} f_\diamond \in \mathrm{B}(K, \mathrm{B}(L, W)) &\iff f_\diamond K \subseteq \mathrm{B}(L, W) \\ &\iff (f_\diamond K)L \subseteq W \\ &\iff (f_\diamond(x))y \subseteq W \text{ for all } x \in K \text{ and } y \in L \\ &\iff f(x, y) \subseteq W \text{ for all } x \in K \text{ and } y \in L \end{aligned}$$

First let us show that $(-)^\diamond$ is continuous. We do this by showing it is continuous at the point $f \in C(X, C(Y, Z))$. Let $\Omega_2 \subseteq C(X \times Y, Z)$ be an open neighborhood of $f^\diamond$ where, by replacing Ω_2 with a smaller open if necessary, we can assume it has the form $\Omega_2 = \mathrm{B}(K_1 \times K_2, V_3)$ where $K_1 \subseteq X$ is compact, $K_2 \subseteq Y$ is compact, and $V_3 \subseteq Z$ is open. We need to find a basic open $\mathrm{B}(K_1, \mathrm{B}(K_2, V_3))$ $K \subseteq X$ compact and $\mathrm{B}(K$

□