Research Statement

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Introduction

My research focuses on algebraic structures that we can attach to free resolutions. In particular, I'm motivated by the following problem: let R be a commutative ring, let I be an ideal of R, and let F be a free resolution of R/I over R such that $F_0 = R$, where by a free resolution we mean F is a sequence of free R-modules and R-linear maps

$$F = \cdots \longrightarrow F_{i+1} \xrightarrow{\mathrm{d}_{i+1}} F_i \xrightarrow{\mathrm{d}_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\mathrm{d}_1} F_0 \xrightarrow{\mathrm{d}_0} 0 \longrightarrow \cdots$$

such that $\mathrm{d}_{i+1}\mathrm{d}_i = 0$ for all $i \in \mathbb{Z}$ and such that the ith homology of F is given by

$$\mathrm{H}_i(F) := \ker \mathrm{d}_i / \operatorname{im} \mathrm{d}_i = \begin{cases} R/I & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

We may equivalently think of F as a free graded R-module equipped with a graded R-linear map $\mathrm{d}\colon F \to F$ of degree -1 such that $\mathrm{d}^2 = 0$ and such that the homology of F is given by

$$\mathrm{H}(F) := \ker \mathrm{d} / \operatorname{im} \mathrm{d} = R/I,$$

where we view R/I as a graded R-module which sits in degree 0. A multiplication on F is a chain map $\mu\colon F \otimes_R F \to F$, denoted $a_1 \otimes a_2 \mapsto a_1 a_2$ where $a_1, a_2 \in F$, such that μ is strictly graded-commutative and unital, though not necessarily associative. In particular, μ being a chain map means it satisfies the Leibniz Rule which says

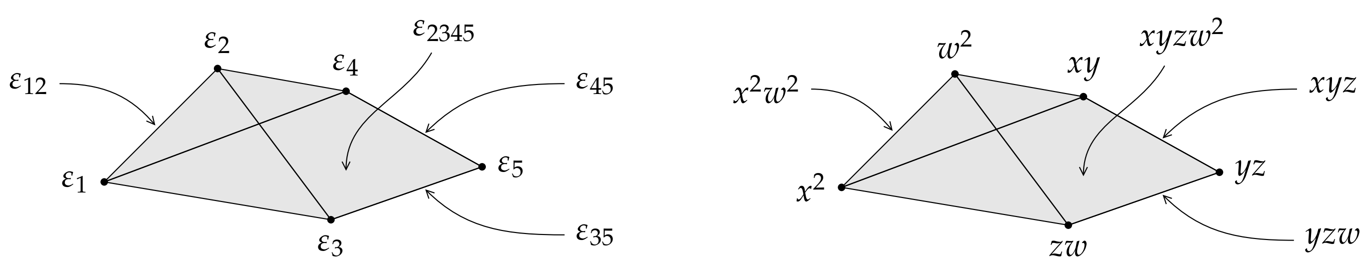
$$\mathrm{d}(a_1 a_2) = \mathrm{d}(a_1) a_2 + (-1)^{|a_1|} a_1 \mathrm{d}(a_2)$$

for all homogeneous $a_1, a_2 \in F$. Similarly, μ being graded-commutative means

$$a_1 a_2 = (-1)^{|a_1||a_2|} a_2 a_2 \quad \text{and} \quad 1a = a = a1$$

for all homogeneous $a, a_1, a_2 \in F$. When we equip F with a multiplication, we refer to it as an MDG algebra (here M stands for multiplication, D stands for differential, and G stands for grading). We say F is a DG algebra when the multiplication is associative since this is the terminology used throughout the literature. In order to conceptualize these ideas, we consider the following example:

Example. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, yz$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled cellular complex below:



We write down the homogeneous components of F as a graded R-module below:

$$\begin{aligned}
F_0 &= R \\
F_1 &= R\varepsilon_1 + R\varepsilon_2 + R\varepsilon_3 + R\varepsilon_4 + R\varepsilon_5 \\
F_2 &= R\varepsilon_{12} + R\varepsilon_{13} + R\varepsilon_{14} + R\varepsilon_{23} + R\varepsilon_{24} + R\varepsilon_{35} + R\varepsilon_{45} \\
F_3 &= R\varepsilon_{123} + R\varepsilon_{124} + R\varepsilon_{1345} + R\varepsilon_{2345} \\
F_4 &= R\varepsilon_{12345}.
\end{aligned}$$

The differential d of F behaves just like the usual boundary map on the cellular complex except some monomials can show up as coefficients (so that the differential respects the multidegree). For instance, on the non-simplicial faces, d is given by

$$\begin{aligned} \mathrm{d}(\varepsilon_{12345}) &= x\varepsilon_{2345} - z\varepsilon_{124} + w\varepsilon_{1345} - y\varepsilon_{123} \\ \mathrm{d}(\varepsilon_{1345}) &= x^2\varepsilon_{35} - xw\varepsilon_{45} - zw\varepsilon_{14} + y\varepsilon_{13} \\ \mathrm{d}(\varepsilon_{2345}) &= xw\varepsilon_{35} - w^2\varepsilon_{45} - z\varepsilon_{24} + xy\varepsilon_{23}. \end{aligned}$$

We now want to equip F with a multiplication μ which respects the multigrading. The Leibniz rule and the multigrading require us to have

$$\begin{aligned} \varepsilon_1 \star \varepsilon_5 &= z\varepsilon_{14} + x\varepsilon_{45} & \qquad \varepsilon_2 \star \varepsilon_{45} &= \varepsilon_{2345} \\ \varepsilon_1 \star \varepsilon_2 &= \varepsilon_{12} & \varepsilon_1 \star \varepsilon_{35} &= \varepsilon_{1345} \\ \varepsilon_2 \star \varepsilon_5 &= y\varepsilon_{23} + w\varepsilon_{35} & \varepsilon_1 \star \varepsilon_{23} &= \varepsilon_{123} \\ & & \varepsilon_2 \star \varepsilon_{14} &= -\varepsilon_{124} \end{aligned}$$

At this point however, one can conclude that F is not associative since

$$[\varepsilon_1, \varepsilon_2, \varepsilon_5] := (\varepsilon_1 \star \varepsilon_2) \star \varepsilon_5 - \varepsilon_1 \star (\varepsilon_2 \star \varepsilon_5) = \mathrm{d}(\varepsilon_{12345}) \neq 0.$$

It was shown by Buchsbaum and Eisenbud [BE77] that there always exists a multiplication on F. In their work, they posed the following question:

Question A: Can μ be chosen to be associative? In other words, does there exist a DG algebra structure on F?

One reason this question is interesting is that when we know the answer is "yes", then we gain some insight about the structure of F. For instance, Buchsbaum and Eisenbud [BE77] proved that in the case where R is a local noetherian domain and F is the minimal free resolution of R/I over R, and we know that an associative multiplication on F exists, then one obtains important lower bounds of the Betti numbers β_i of R/I. In particular, let $\boldsymbol{t} = t_1, \dots, t_g$ be a maximal R-sequence contained in I and let $E = \mathcal{K}(\boldsymbol{t})$ be the Koszul R-algebra resolution of $R/\boldsymbol{t}$. Any expression of the t_i in terms of the generators for I yields a canonical comparison map $E \to F$. Buchsbaum and Eisenbud showed that under these assumptions, this comparison map $E \to F$ is injective, hence we get the lower bound $\binom{m}{i} \leq \beta_i$ for each $i \leq g$. Other applications in the literature include [Avr81, Kat19, Van22]. Unfortunately, the answer to Question A turns out to be "not always" and several counterexamples have since been found [Avr81, Kat19, Sri92].

One of the starting points for my research is based on the observation that one can still obtain Buchsbaum and Eisenbud's lower bounds even in cases where it is known that we can't choose μ to be associative. Indeed, we just need to find a multiplication μ on F together with a comparison map $\varphi\colon E \to F$ such that $\varphi\colon E \to F$ is multiplicative, meaning

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$$

for all $a_1, a_2 \in E$. The proof given in [BE77] which shows $\varphi\colon E \to F$ is injective would still apply in this case. Furthermore, in their proof, Buchsbaum and Eisenbud used a property that the Koszul algebra E satisfies, namely that every nonzero DG ideal of E intersects the top degree E_g non-trivially. However there are many other (not necessarily associative) DG algebras which satisfy this property (the property being that their nonzero DG ideals intersect the top degree non trivially). Thus one may be able to generalize this result even further by replacing $\boldsymbol{t}$ with an ideal J such that $\boldsymbol{t} \subseteq J \subseteq I$ and such that there exists a multiplication on the minimal free resolution G of R/J over R which satisfies this property. Thus a study of DG algebras which are not necessarily associative can still be fruitful.

In general we would like to choose μ such that it is as associative as possible. To this end, we pose the following question:

Question B: Given a multiplication μ on F, how can we measure the failure of μ to be associative?

I have summarized how to attack this problem from two different angles in the following two subsections. The reader may find it helpful to keep example we discussed above in mind when thinking about the ideas discussed in the following two subsections.

The Maximal Associative Quotient

The first angle is to study the maximal associative quotient $F^{\mathrm{as}} := F/\langle F\rangle$ of F where we set

$$\langle F\rangle = \mathrm{span}_R\{a_1((a_2a_3)a_4 - a_2(a_3a_4)) \mid a_1, a_2, a_3, a_4 \in F\}.$$

The maximal associative quotient of F plays a similar role as the maximal abelian quotient G^{ab} plays for a group G. In particular, F^{as} satisfies the following universal mapping property: every multiplicative map $\varphi\colon F \to G$ where G is an associative DG algebra factors uniquely through the maximal associative quotient. Note that μ being associative is equivalent to the condition that $\langle F\rangle = 0$. It turns out that under further assumptions (local/noetherian/minimal) the condition that $\langle F\rangle = 0$ is equivalent to the condition that $\mathrm{H}(\langle F\rangle) = 0$. This is one of the theorems I proved in my paper [Nel24]:

Theorem. *Assume $(R, \mathfrak{m})$ is a local noetherian ring, that $I \subseteq \mathfrak{m}$, and that F is minimal. Then the following conditions are equivalent:*

1. *F is associative, that is $\langle F\rangle = 0$.*
2. *F is homologically associative, that is $\mathrm{H}(\langle F\rangle) = 0$.*

Under these local/noetherian/minimal assumptions, it makes more sense to study $\mathrm{H}(\langle F\rangle)$ than it does to study $\langle F\rangle$. For instance, $\mathrm{H}(\langle F\rangle)$ is naturally an R/I-module, and in some cases $\mathrm{H}(\langle F\rangle)$ will have finite length (e.g., it will have finite length if R/I is artinian). One of my main research goals is to assign a number to the multiplication μ which tells us how non-associative μ is. In cases where $\mathrm{H}(\langle F\rangle)$ has finite length, then its length gives us information about how non-associative μ is; the higher its length, the more non-associative it is. In my research, I have found a way to measure the failure of non-associativity of μ even in cases where $\mathrm{H}(\langle F\rangle)$ does not have finite length. Here is the rough sketch of the idea: suppose $r \in \mathfrak{m}$ is an R/I-regular element. Then the mapping cone $F + eF$ is the minimal free resolution of $R/\langle I, r\rangle$ over R. Here, e is thought of as an exterior variable of degree 1. The differential of the mapping cone is given by

$$\mathrm{d}(a + eb) = \mathrm{d}(a) + rb - e\mathrm{d}(b)$$

for all $a, b \in F$. We extend the multiplication μ on F to a multiplication on $F + eF$ by setting

$$(a + eb)(c + ed) = ac + e(bc + (-1)^{|a|}ad)$$

for all $a, b, c, d \in F$. In particular, note that $(eb)c = e(bc)$ for all $b, c \in F$, so e belongs to the nucleus of $F + eF$ (i.e., it associates with all other elements). More generally, suppose $\boldsymbol{r} = r_1, \dots, r_m$ is a maximal (R/I)-regular sequence contained in $\mathfrak{m}$ and let

$$F + \boldsymbol{e}F := F + e_1F + \cdots + e_mF$$

be the corresponding free resolution of $R/\langle I, \boldsymbol{r}\rangle$ over R. Then we have the following theorem [Nel24]:

Theorem. *Let $\varepsilon = \inf\{i \mid \mathrm{H}_i(\langle F\rangle) \neq 0\}$. Then we have an isomorphism of $R/\langle I, \boldsymbol{r}\rangle$-modules:*

$$\mathrm{H}_\varepsilon(\langle F\rangle)/\boldsymbol{r}\mathrm{H}_\varepsilon(\langle F\rangle) \cong \mathrm{H}_\varepsilon(\langle F + \boldsymbol{e}F\rangle)$$

Thus in the case where R/I is Cohen-Macaulay, the idea is to measure how non-associative μ is by passing to the artinian ring $R/\langle I, \boldsymbol{r}\rangle$ and using the length function on $\mathrm{H}_\varepsilon(\langle F + \boldsymbol{e}F\rangle)$ instead. I am currently working on fleshing out the details of this idea.

Uniform Non-Associativity

The second angle is to take advantage of the fact that *all* multiplications on F have the form $\mu_h := \mu + \mathrm{d}h + h\mathrm{d}$ where $h\colon F \otimes_R F \to F$ is a homotopy (i.e., a graded R-linear map of degree 1) and where μ is a particular choice of a multiplication. A somewhat surprising fact is that if there exists a exists a multiplication μ on F such that μ is not associative at some homogeneous triple (a_1, a_2, a_3) and such that some additional criteria are satisfied, then *every* multiplication on F will also fail to be associative at that triple as well. The theorem below is one of my main results in this direction [Nel24]:

Theorem. *Equip F with a fixed multiplication μ (which is thought of as "nice" in the sense that it is as associative as possible). Suppose there exists homogeneous elements $a_1, a_2, a_3 \in F$ such that $\mathrm{d}a_1, \mathrm{d}a_2, \mathrm{d}a_3 \in \mathfrak{m}^2F$, $a_1a_2, a_2a_3 \in \mathfrak{m}F$, and $(a_1a_2)a_3 - a_1(a_2a_3) \notin \mathfrak{m}^2F + a_1\mathrm{d}F + a_3\mathrm{d}F$. Then every multiplication on F is not associative at the triple (a_1, a_2, a_3).*

I want to briefly sketch the main idea behind the proof of this result. To do so, I need to set up some notation. The associator for μ is a chain map $[\cdot]_\mu \colon F \otimes_R F \otimes_R F \to F$ given by $[\cdot]_\mu = \mu(\mu \otimes 1 - 1 \otimes \mu)$, and the corresponding R-trilinear map is denoted $[\cdot,\cdot,\cdot]_\mu \colon F^3 \to F$. In particular, given $a_1, a_2, a_3 \in F$ we have

$$[a_1 \otimes a_2 \otimes a_3]_\mu = [a_1, a_2, a_3]_\mu = (a_1 a_2) a_3 - a_1 (a_2 a_3).$$

If μ_h is another multiplication on F where $h \colon F \otimes_R F \to F$ is a homotopy, then the associator for μ_h is given by

$$[\cdot]_{\mu_h} = [\cdot]_\mu + \mathrm{d}H + H\mathrm{d}, \tag{1}$$

where $H = [\cdot]_{\mu,h} + [\cdot]_{h,\mu_h}$ and

$$[\cdot]_{\mu,h} = \mu(h \otimes 1 - 1 \otimes h), \quad \text{and} \quad [\cdot]_{h,\mu_h} = h(\mu_h \otimes 1 - 1 \otimes \mu_h). \tag{2}$$

We refer to $[\cdot]_{\mu,h}$ and $[\cdot]_{h,\mu_h}$ as generalized associators because they generalize the usual associator in the sense that $[\cdot]_{\mu,\mu} = [\cdot]_\mu$. Additional signs may appear in (2) when we evaluate these generalized associators to elements due to the Koszul sign rule. Note in (1) we simplified notation by using the same symbol d to denote either the differential $\mathrm{d}_{F^{\otimes 3}}$ of $F^{\otimes 3}$ or the differential d_F of F where context makes clear which differential the symbol d refers to (for instance, the d in $\mathrm{d}H$ clearly refers to d_F since H lands in F). Finally, we can decompose the generalized associator $[\cdot]_{h,\mu_h}$ further into a sum of three generalized associators:

$$[\cdot]_{h,\mu_h} = [\cdot]_{h,\mu} + [\cdot]_{h,\mathrm{d}h} + [\cdot]_{h,h\mathrm{d}}.$$

The notation indicates how these generalized associators are defined (for example $[\cdot]_{h,h\mathrm{d}} = h(h\mathrm{d} \otimes 1 - 1 \otimes h\mathrm{d})$). With this notation set up, the idea behind the proof is that the conditions in the theorem guarantee that we will not be able to cancel out the associator $[a_1, a_2, a_3]_\mu$ using the extra term $(\mathrm{d}H + H\mathrm{d})(a_1 \otimes a_2 \otimes a_3)$, and so μ_h will also fail to be associative at that triple as well. Since all multiplications have the form μ_h for some $h \colon F \otimes_R F \to F$, it follows that all multiplications on F will fail to be associative at that triple.

Applications

The final question we pose relates to applications.

Question C: Is there a practical way to construct multiplications on F which are as associative as possible with the help of a computer algebra system like Macaulay2 or Singular?

The answer to this is "yes!", and in fact this works in the more general setup: assume R is a domain with fraction field K and let F be a finite free graded R-module such that $F_0 = R$ and

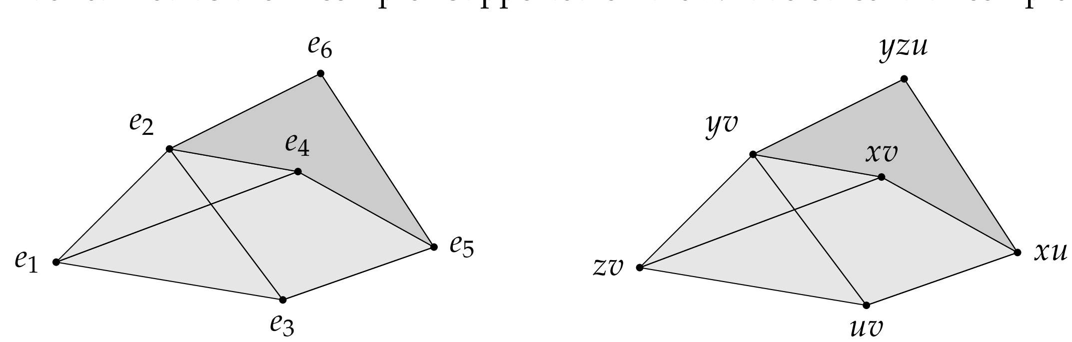
$$F_+ = Re_1 + \cdots + Re_n$$

where $e_1, \ldots, e_n$ is an ordered homogeneous basis of F_+ which is ordered in such a way that if $|e_{i'}| > |e_i|$, then $i' > i$. Let $K[\boldsymbol{e}] = K[e_1, \ldots, e_n]$ be the free non-strict graded-commutative K-algebra generated by $e_1, \ldots, e_n$. In particular, if e_i and e_j are distinct, then we have

$$e_i e_j = (-1)^{|e_i||e_j|} e_j e_i,$$

in $K[\boldsymbol{e}]$, however elements of odd degree do not square to zero in $K[\boldsymbol{e}]$. If we equip $K[\boldsymbol{e}]$ with a specific monomial ordering (namely weighted lexicographical), then one can show how associators naturally arise when performing Buchberger's algorithm to a certain set of polynomials with respect to this monomial ordering. Let us demonstrate this in the following example:

Example. Let $R = \Bbbk[x, y, z, u, v]$, let $\boldsymbol{m} = zv, yv, uv, xv, xu, yzu$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. Then F can be realized as the R-complex supported on the $\boldsymbol{m}$-labeled cellular complex pictured below:



We write down the homogeneous components of F as a graded module below:

$$
\begin{aligned}
F_0 &= R \\
F_1 &= Re_1 + Re_2 + Re_3 + Re_4 + Re_5 + Re_6 \\
F_2 &= Re_{12} + Re_{13} + Re_{14} + Re_{23} + Re_{24} + Re_{26} + Re_{35} + Re_{45} + Re_{56} \\
F_3 &= Re_{123} + Re_{124} + Re_{1345} + Re_{2345} + Re_{2456} \\
F_4 &= Re_{12345}
\end{aligned}
$$

We will use Singular to help us find an associative multigraded multiplication μ on F such that $e_\sigma^2 = 0$ for all σ. From multidegree and Leibniz rule considerations, we begin constructing μ as follows:

$$
\begin{aligned}
e_1 \star e_2 &= ve_{12} & e_3 \star e_5 &= ue_{35} \\
e_1 \star e_3 &= ve_{13} & e_3 \star e_6 &= -zue_{23} + ue_{26} \\
e_1 \star e_4 &= ve_{14} & e_4 \star e_5 &= xe_{45} \\
e_1 \star e_5 &= ue_{14} + ze_{45} & e_4 \star e_6 &= -zue_{24} + xe_{26} \\
e_1 \star e_6 &= zue_{12} + ze_{26} & e_5 \star e_6 &= ue_{56} \\
e_2 \star e_3 &= ve_{23} & e_1 \star e_{23} &= ve_{123} \\
e_2 \star e_4 &= ve_{24} & e_1 \star e_{24} &= ve_{124} \\
e_2 \star e_5 &= ue_{24} + ye_{45} & e_1 \star e_{35} &= -ve_{1345} \\
e_2 \star e_6 &= ye_{26} & e_1 \star e_{56} &= -uze_{124} + ze_{2456} \\
e_3 \star e_4 &= ve_{35} - ve_{45} & e_1 \star e_{2345} &= ve_{12345}.
\end{aligned}
$$

At this point, Singular can help us determine how we should define μ everywhere else. First we input the following code into Singular:

```
LIB "ncalg.lib";

intvec V = 1:6, 2:9, 3:5, 4:1;

ring A=(0,x,y,z,u,v),(e1,e2,e3,e4,e5,e6,
e12,e13,e14,e23,e24,e26,e35,e45,e56,
e123,e124,e1345,e2345,e2456,e12345),Wp(V);

matrix C[21][21]; matrix D[21][21]; int i; int j;
for (i=1; i<=21; i++) {for (j=1; j<=21; j++) {C[i,j]=(-1)^(V[i]*V[j]);}}
ncalgebra(C,D);

poly f(1)(2) = e1*e2 - v*e12;
poly f(1)(3) = e1*e3 - v*e13;
poly f(1)(4) = e1*e4 - v*e14;
poly f(1)(5) = e1*e5 - u*e14 - z*e45;
poly f(1)(6) = e1*e6 - zu*e12 - z*e26;
poly f(2)(3) = e2*e3 - v*e23;
poly f(2)(4) = e2*e4 - v*e24;
poly f(2)(5) = e2*e5 - u*e24 - y*e45;
poly f(2)(6) = e2*e6 - y*e26;
poly f(3)(4) = e3*e4 - v*e35 + v*e45;
poly f(3)(5) = e3*e5 - u*e35;
poly f(3)(6) = e3*e6 + zu*e23 - u*e26;
poly f(4)(5) = e4*e5 - x*e45;
poly f(4)(6) = e4*e6 + zu*e24 - x*e26;
poly f(5)(6) = e5*e6 - u*e56;
poly f(1)(23) = e1*e23 - v*e123;
poly f(1)(24) = e1*e24 - v*e124;
poly f(1)(35) = e1*e35 + v*e1345;
poly f(1)(56) = e1*e56 + uz*e124 - z*e2456;
poly f(1)(2345) = e1*e2345 - v*e12345;

list L = (e1,e2,e3,e4,e5,e6,
e12,e13,e14,e23,e24,e26,e35,e45,e56,
e123,e124,e1345,e2345,e2456,e12345);

ideal I; int i; for (i=1; i<=21; i++) {I = I + L[i]*L[i];}

I = I + f(1)(2),f(1)(3),f(1)(4),f(1)(5),f(1)(6),f(2)(3),f(2)(4),
f(2)(5),f(2)(6),f(3)(4),f(3)(5),f(3)(6),f(4)(5),f(4)(6),
f(5)(6),f(1)(23),f(1)(24),f(1)(35),f(1)(56),f(1)(2345);
```

To see that the multiplication is associative thus far, we calculate the Gröbner basis of I with respect to our fixed monomial ordering using the command std(I) in Singular. Singular gives us the following output:

```
_[1]=e6^2
_[2]=e5*e6+(-u)*e56
_[3]=e5^2
...
_[57]=e2*e56+(-y)*e2456
_[58]=e2*e45
_[59]=(z*u)*e2*e35+(-v)*e6*e35+(u*v)*e2456
_[60]=e2*e26
...
_[209]=e124*e12345
_[210]=e123*e12345
_[211]=e12345^2
```

where we omitted most of the Gröbner basis elements due to size constraints. Since the lead term of each polynomial showing up in the list has total degree > 1, we conclude that the multiplication we have defined so far is associative. Now observe that if we want the multiplication to continue being associative, then we need to define $e_2 \star e_{26} = 0$ since

$$\begin{aligned} ye_2 \star e_{26} &= e_2 \star (e_2 \star e_6) \\ &= (e_2 \star e_2) \star e_6 - [e_2, e_2, e_6] \\ &= -[e_2, e_2, e_6]. \end{aligned}$$

In fact, Singular already tells us this since it is computing the maximal associative quotient! In particular, setting I = std(I) and running the command reduce(e2*e26 , I) outputs 0 in Singular which tells us that in the maximal associative quotient we have $e_1 \star e_{12} = 0$. Alternatively, we could simply read this off the list of polynomials that Singular outputted as the polynomial $e_2 \star e_{26}$ shows up in the Gröbner basis. Similarly, Singular tells us that we should define $e_2 \star e_{56} = -ye_{2456}$ since the polynomial $e_2 \star e_{56} - ye_{2456}$ shows up in the Gröbner basis. On the other hand, if we run the command reduce(e6*e35 , I), then Singular outputs e6*e35 which tells us that we still need to define $e_6 \star e_{35}$. Upon reflection of the multigrading and Leibniz rule, we define

$$e_6 \star e_{35} = -zue_{2345} + ue_{2456}.$$

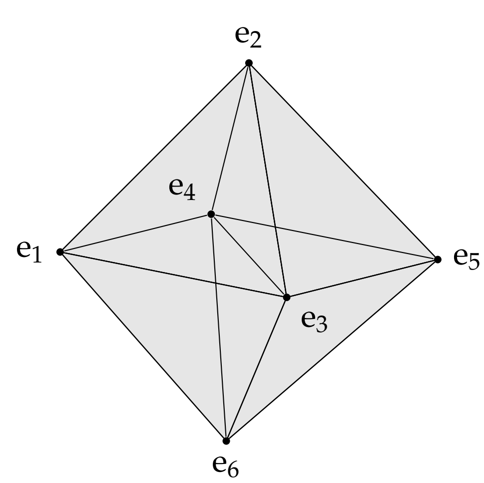
Thus we add the polynomial poly f(6)(35) = e6*e35 + zu*e2345 − y*e2456 to our ideal in the code. We observe that our multiplication is still associative by running the command std(I) and checking that none of the polynomials listed has lead term of total degree 1 again. Furthermore, running the command

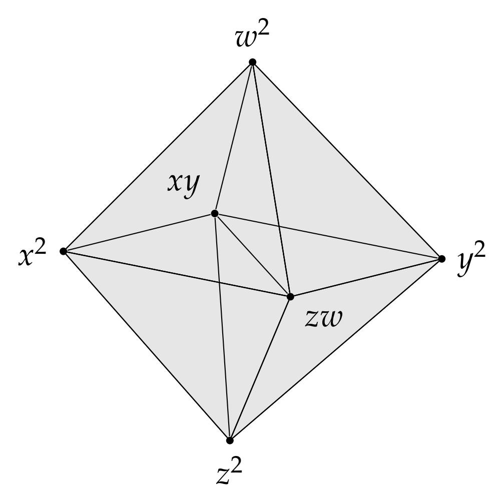
```
for(i=1;i<=21;i++){for(j=i+1;j<=21;j++){reduce(L[i]*L[j],I);};};
```

shows that the multiplication is now defined everywhere. For instance, the command reduce(e12*e35 , I) outputs (−v)*e12345. This tells us that $e_{12} \star e_{35} = -ve_{12345}$.

I'd like to end this research statement with one particularly nice example where I found a DG algebra structure on a minimal free resolution with the help of the computer algebra system Singular.

Example. Let $R = \Bbbk[x, y, z, w]$, let $\boldsymbol{m} = x^2, w^2, zw, xy, y^2, z^2$, and let F be the minimal free resolution of $R/\boldsymbol{m}$ over R. One can visualize F as being supported on the $\boldsymbol{m}$-labeled simplicial complex below:





We write down the homogeneous components of F as a graded R-module below:

$F_0 = R$
$F_1 = Re_1 + Re_2 + Re_3 + Re_4 + Re_5 + Re_6$
$F_2 = Re_{12} + Re_{13} + Re_{14} + Re_{16} + Re_{23} + Re_{24} + Re_{25} + Re_{34} + Re_{35} + Re_{36} + Re_{45} + Re_{46} + Re_{56}$
$F_3 = Re_{123} + Re_{124} + Re_{134} + Re_{136} + Re_{146} + Re_{234} + Re_{235} + Re_{245} + Re_{345} + Re_{346} + Re_{356} + Re_{456}$
$F_4 = Re_{1234} + Re_{1346} + Re_{2345} + Re_{3456}.$

Using the computer algebra system Singular, we found an *associative* multigraded multiplication on F which has the following minimal presentation:

$$
\begin{array}{lll}
e_1^2 = 0 & e_2 \star e_5 = e_{25} & e_2 \star e_{16} = -ze_{123} - we_{136} \\
e_2^2 = 0 & e_2 \star e_6 = ze_{23} + we_{36} & e_2 \star e_{46} = e_{234} + e_{346} \\
e_3^2 = 0 & e_3 \star e_4 = e_{34} & e_2 \star e_{56} = -ze_{235} + we_{356} \\
e_4^2 = 0 & e_3 \star e_5 = e_{35} & e_3 \star e_{45} = e_{345} \\
e_5^2 = 0 & e_3 \star e_6 = ze_{36} & e_5 \star e_{24} = ye_{245} \\
e_6^2 = 0 & e_4 \star e_5 = ye_{45} & e_6 \star e_{13} = ze_{136} \\
e_1 \star e_2 = e_{12} & e_4 \star e_6 = e_{46} & e_6 \star e_{34} = ze_{346} \\
e_1 \star e_3 = e_{13} & e_5 \star e_6 = e_{56} & e_6 \star e_{35} = ze_{356} \\
e_1 \star e_4 = xe_{14} & e_1 \star e_{25} = ye_{124} - xe_{245} & e_6 \star e_{45} = e_{456} \\
e_1 \star e_5 = ye_{14} + xe_{45} & e_1 \star e_{35} = ye_{134} - xe_{345} & e_1 \star e_{235} = ye_{1234} + xe_{2345} \\
e_1 \star e_6 = e_{16} & e_1 \star e_{56} = ye_{146} + xe_{456} & e_1 \star e_{346} = xe_{1346} \\
e_2 \star e_3 = we_{23} & & e_1 \star e_{356} = ye_{1346} - xe_{3456} \\
e_2 \star e_4 = e_{24} & & e_2 \star e_{456} = ze_{2345} + we_{3456}
\end{array}
$$

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