Test 2 Review Solutions

Limits

Exercise 1. Let $f(x)$ be a function defined in a neighborhood of a real number a. Suppose $L = \lim_{x\to a^-} f(x)$ and $R = \lim_{x\to a^+} f(x)$. When does $\lim_{x\to a} f(x)$ exist?

Exercise 2. Evaluate the following limits. If the limit does not exist, then write DNE and explain why it does not exist.

$$\begin{aligned}\lim_{x\to -1} \sqrt{x^2}+1 &= \sqrt{(-1)^2}+1\\ &= \sqrt{1}+1\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{t\to 4} \frac{t^2-7t+12}{t-4} &= \lim_{t\to 4}\left(\frac{(t-4)(t-3)}{t-4}\right)\\ &= \lim_{t\to 4}(t-3)\\ &= 4-3\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to -3^+} \frac{3x}{\sqrt{x+3}} &= \frac{3\cdot(-3)}{\text{small positive}}\\ &= \frac{-9}{\text{small positive}}\\ &= -\infty\end{aligned}$$

Next we solve $\lim_{x\to 0}\frac{x}{\sin x}$. To solve this limit, we need to use the squeeze theorem:

$$\begin{aligned}1 \le \frac{x}{\sin x} \le 3-2\cos x &\implies \lim_{x\to 0}(1) \le \lim_{x\to 0}\left(\frac{x}{\sin x}\right) \le \lim_{x\to 0}(3-2\cos x)\\ &\implies 1 \le \lim_{x\to 0}\left(\frac{x}{\sin x}\right) \le 3-2\cos(0)\\ &\implies 1 \le \lim_{x\to 0}\left(\frac{x}{\sin x}\right) \le 1\\ &\implies \lim_{x\to 0}\left(\frac{x}{\sin x}\right) = 1.\end{aligned}$$

$$\begin{aligned}\lim_{t\to 4} \frac{t^2-7t+12}{t-4} &= \lim_{t\to 4}\left(\frac{(t-4)(t-3)}{t-4}\right)\\ &= \lim_{t\to 4}(t-3)\\ &= 4-3\\ &= 1\end{aligned}$$

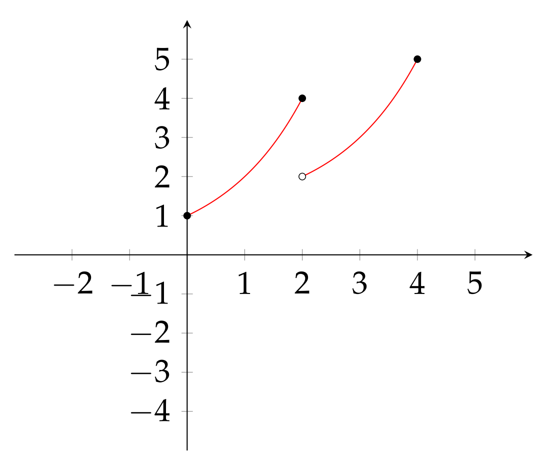
$$\begin{aligned}
\lim_{x\to 0^+} \sqrt{\sec x} &= \lim_{x\to 0^+} \frac{1}{\sqrt{\cos x}} \\
&= \frac{1}{\sqrt{\cos(0)}} \\
&= \frac{1}{\sqrt{1}} \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \left(\frac{\sqrt{x}-x^2}{2x+1}\right) &= \lim_{x\to\infty}\left(\frac{\sqrt{x}}{2x+1}\right) + \lim_{x\to\infty}\left(\frac{-x^2}{2x+1}\right) \\
&= \lim_{x\to\infty}\left(\frac{\sqrt{x}}{x}\frac{1}{2+1/x}\right) + \lim_{x\to\infty}\left(\frac{x^2}{x}\frac{-1}{2+1/x}\right) \\
&= \lim_{x\to\infty}\left(\frac{1}{\sqrt{x}}\cdot\frac{1}{2+1/x}\right) + \lim_{x\to\infty}\left(x\cdot\frac{-1}{2+1/x}\right) \\
&= 0\cdot\frac{1}{2+0} + (\text{big negative})\cdot\frac{-1}{2+0} \\
&= -\infty.
\end{aligned}$$

Exercise 3. Suppose $\lim_{x\to 3} f(x) = 6$ and $\lim_{x\to 3} g(x) = 2$. Evaluate

$$\begin{aligned}
\lim_{x\to 3}\left(\frac{(f(x)+g(x))^{1/3}}{f(x)g(x)}\right) &= \frac{(6+2)^{1/3}}{6\cdot 2} \\
&= \frac{8^{1/3}}{12} \\
&= \frac{2}{12} \\
&= \frac{1}{6}
\end{aligned}$$

Exercise 4. Consider the function $f(x)$ defined on the closed interval $[0,4]$ whose graph is given below

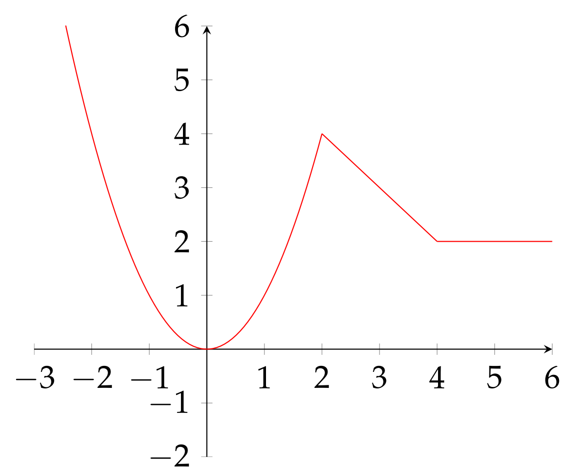


(3.a) Evaluate the following

$$\lim_{x \to 1^+} f(x) = 2$$
$$\lim_{x \to 2^-} f(x) = 4$$
$$\lim_{x \to 2^+} f(x) = 2$$
$$\lim_{x \to 2} f(x) \text{ DNE}$$
$$f(2) = 4$$
$$\lim_{x \to 4^-} f(x) = 5$$

(3.b) Is $f(x)$ continuous at $x = 2$? No because the limit doesn't exist there.

Exercise 5. Consider the function $f(x)$ defined on the whole real line whose graph is given below



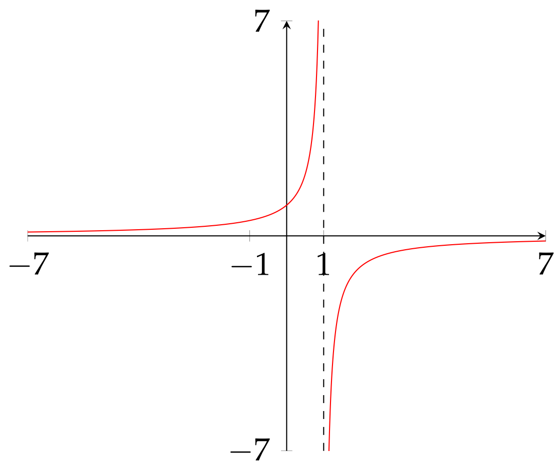
(4.a) Evaluate the following

$$\lim_{x \to 2^-} f(x) = 4$$
$$\lim_{x \to 2^+} f(x) = 4$$
$$\lim_{x \to 2} f(x) = 4$$
$$f(2) = 4$$
$$\lim_{x \to 4^-} f(x) = 2$$
$$\lim_{x \to 4^+} f(x) = 2$$
$$\lim_{x \to 4} f(x) = 2$$
$$f(4) = 2$$

(4.b) Is $f(x)$ continuous at $x = 2$? Yes because

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x).$$

Exercise 6. Let $f(x)$ be the function whose graph is given below



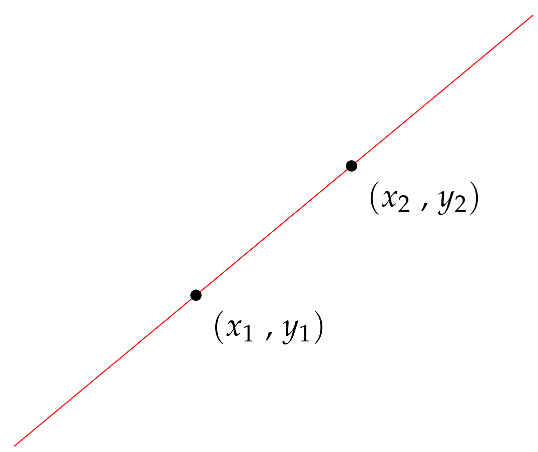
(5.a) Evaluate the following

$$\lim_{x\to 1^-} f(x) = \infty$$
$$\lim_{x\to 1^+} f(x) = -\infty$$
$$\lim_{x\to 1} f(x)\text{DNE}$$

(5.b) Is $f(x)$ continuous at $x = 2$? Yes.

Graphs

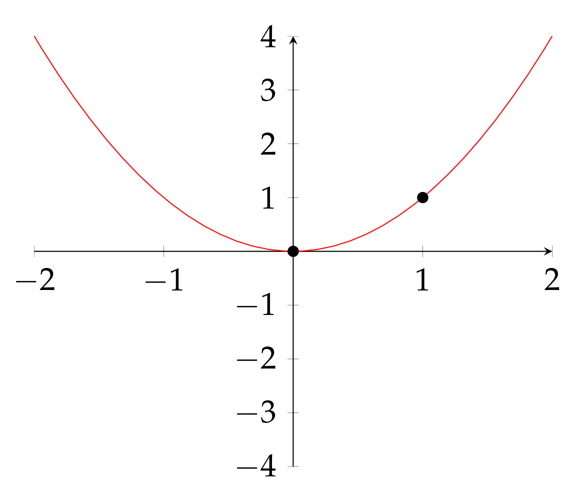
Exercise 7. Suppose a line passes through the points (x_1, y_1) and (x_2, y_2), as shown below:



What is the slope of this line? The slope of the line is given by the formula

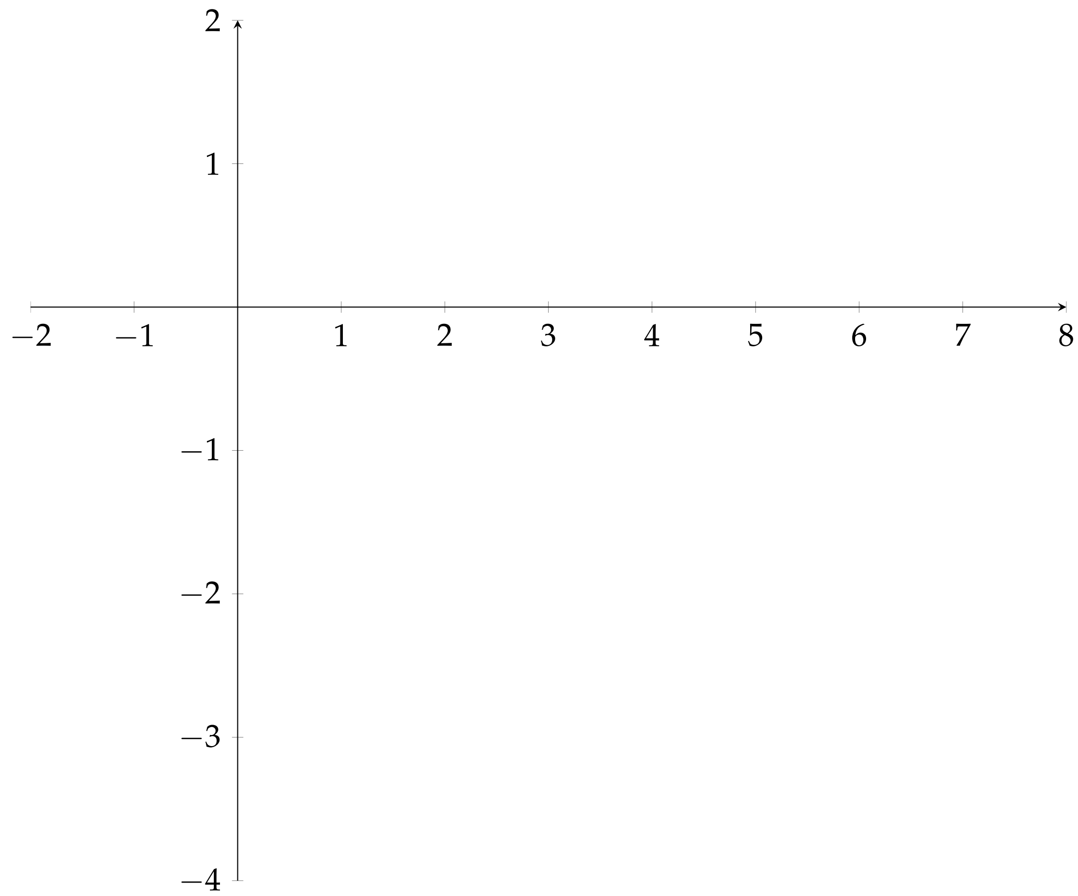
$$\frac{y_2 - y_1}{x_2 - x_1}$$

Exercise 8. Let $f(x)$ be the function given by the graph below:

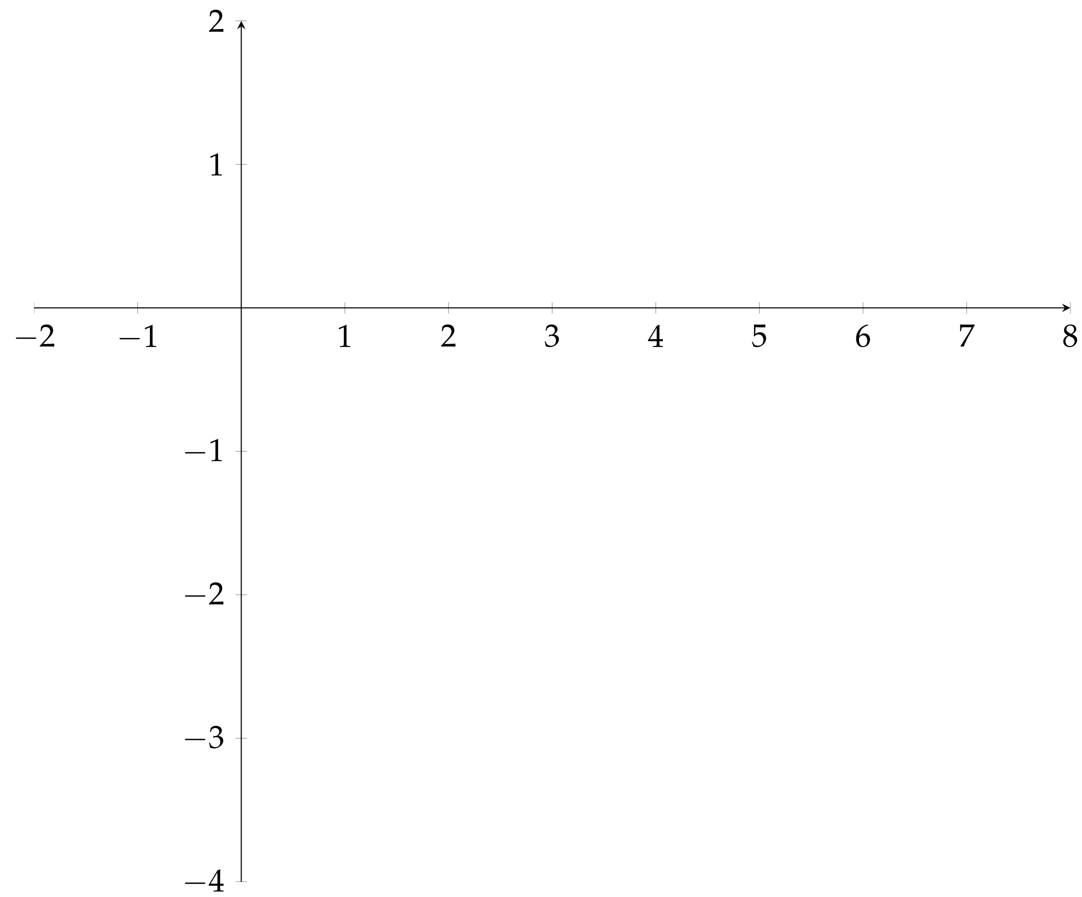


(2.a) Draw the tangent line to the graph of the function at the point $(1, 1)$ and find the slope of this line.
(2.b) Draw the tangent line to the graph of the function at the point $(0, 0)$ and find the slope of this line.
(2.c) Draw the secant line through the points $(0, 0)$ and $(1, 1)$ and find the slope of this line.
(2.d) What is the average velocity of this function from $x = 0$ to $x = 1$?
(2.e) What is the difference between a tangent line and a secant line?

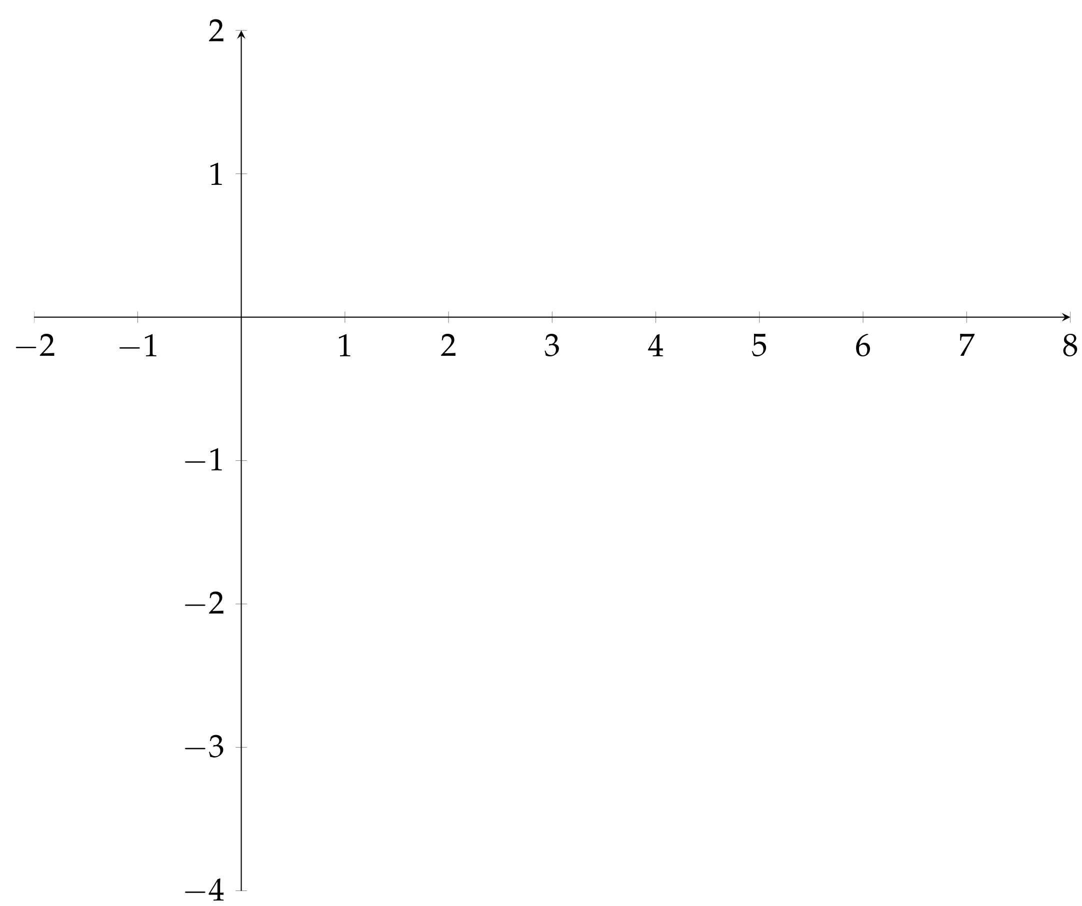
Exercise 9. You should definitinely know what the graphs of $\ln x$ and e^x look like. For this problem, let's focus on $\ln x$. First graph $\ln x$ below



Next graph $\ln(x - 1)$



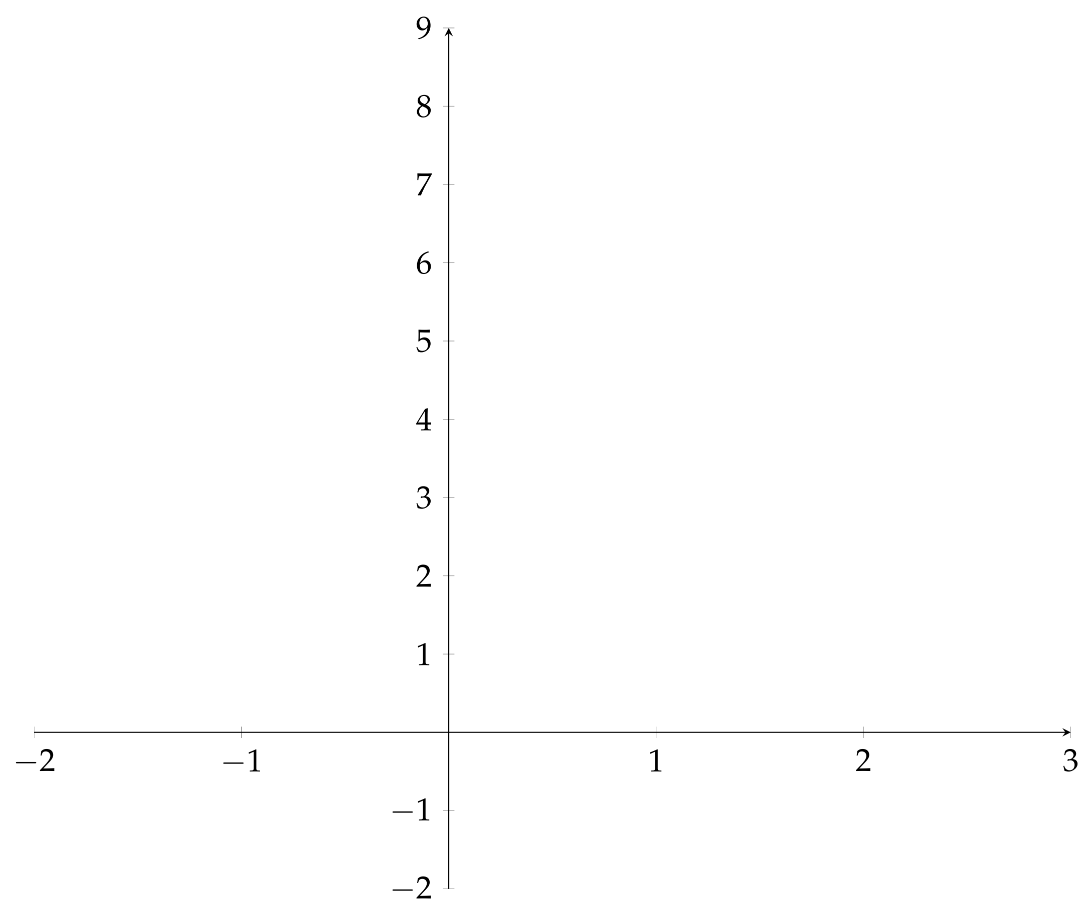
Next graph $\ln(x-1)-2$



What is the vertical asymptote of the function $\ln(x-1)-2$? Draw the vertical asymptote where it needs to be in the graph above.

Exercise 10. You should definitinely know what the graphs of $\ln x$ and e^x look like. For this problem, let's focus on e^x. First graph e^x below

Next graph e^{x+1}.



Next graph $e^{x+1} - 1$.

What is the horizontal asymptote of the function $e^{x+1} - 1$? Draw the horizontal asymptote where it needs to be in the graph above.

Algebra

Exercise 11. Let a, b, c be real numbers. Simplify the following expressions.

$$\frac{1}{x^a} = x^{-a}$$

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$\sqrt{x} = x^{1/2}$$

$$\sqrt[3]{x} = x^{1/3}$$

$$\frac{1}{\sqrt[3]{x^2}} = x^{-2/3}$$

$$\log_a(bc) = \log_a(b) + \log_a(c)$$

$$\log_a(b^c) = c\log_a(b)$$

$$\begin{aligned} e^{a\ln b} &= e^{\ln(b^a)} \\ &= b^a. \end{aligned}$$

Exercise 12. Suppose $f(x) = 2 - \ln x$ and $g(x) = x + \cos x$. Find

$$\begin{aligned} (f \circ g)(x) &= \\ (g \circ f)(x) &= \end{aligned}$$

Now suppose $f(x) = 3^x - x$ and $g(x) = \sec x + \tan x$. Find

$$\begin{aligned} (f \circ g)(x) &= \\ (g \circ f)(x) &= \end{aligned}$$

Now suppose $f(x) = e^{x+3}$. Find

$$f^{-1}(x) =$$

Exercise 13. Solve for x in the following equations

$$3^x = 9^{x^2}$$

$$\log_{125}(x+1) = \frac{1}{3}$$

$$\log_4(x^2+1) = \frac{1}{2}$$

$$e^{\log(2x+1)} = x^2 + 2$$

$$2^{2x} - 2^{x+1} + 1 = 0$$