Advanced Numerical Analysis Homework 1

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1 Problem 1

Exercise 1. Solve the following:

1. Find the absolute and relative condition numbers of $f(x) = e^{-2x}$ and $g(x) = \ln^3 x$. For what values of x are these functions sensitive to perturbations?

2. Let $x_1, x_2 \in \mathbb{R}^+$ and let $f(\boldsymbol{x}) = f(x_1, x_2) = x_1^{x_2}$. Find the relative condition number of $f(\boldsymbol{x})$. For what range of values of x_1 and x_2 is the problem ill-conditioned?

Solution 1. 1. First we find the absolute condition number of f and g at x. Since f and g are differentiable everywhere they are defined, we see that

$$
\begin{aligned}
\widehat{\kappa}_f(x) &= |f'(x)| \\
&= |-2e^{-2x}| \\
&= 2e^{-2x}
\end{aligned}
\qquad\qquad
\begin{aligned}
\widehat{\kappa}_g(x) &= |g'(x)| \\
&= |(3/x)\ln^2 x| \\
&= \frac{3\ln^2 x}{|x|}.
\end{aligned}
$$

Next we find the relative condition numbers of f and g at x. Since f and g are differentiable everywhere they are defined, we see tha

$$
\begin{aligned}
\kappa_f(x) &= \frac{|f'(x)||x|}{|f(x)|} \\
&= \frac{2e^{-2x}|x|}{e^{-2x}} \\
&= 2|x|.
\end{aligned}
\qquad\qquad
\begin{aligned}
\kappa_g(x) &= \frac{|g'(x)||x|}{|g(x)|} \\
&= \frac{\frac{3\ln^2 x}{|x|}|x|}{\left|\ln^3 x\right|} \\
&= \frac{3}{|\ln x|}.
\end{aligned}
$$

Recall that f and g are sensitive to perturbations (in a relative sense) at x whenever the (relative) condition number κ at x is large. In particular, f is sensitive to perturbations (in a relative sense) when $|x|$ is large (say $|x| \geq 10^6$) and g is sensitive to perturbations (in a relative sense) when x is near 1.

2. In this problem, we use the ℓ_∞ norm $\|\cdot\|_\infty = \|\cdot\|$. Since f is differentiable we see that

$$
\begin{aligned}
\kappa_f(\boldsymbol{x}) &= \frac{\|\mathrm{J}_f(\boldsymbol{x})\|\|\boldsymbol{x}\|}{|f(\boldsymbol{x})|} \\
&= \frac{\left\|\begin{pmatrix} x_2 x_1^{x_2-1} & x_1^{x_2}\ln x_1 \end{pmatrix}\right\| \|\boldsymbol{x}\|}{|x_1^{x_2}|} \\
&= \frac{\left(\left|x_2 x_1^{x_2-1}\right| + |x_1^{x_2}\ln x_1|\right)\max\{|x_1|, |x_2|\}}{|x_1^{x_2}|.} \\
&= \frac{\left(x_2 x_1^{x_2-1} + x_1^{x_2}|\ln x_1|\right)\max\{x_1, x_2\}}{x_1^{x_2}.}, \\
&= \left(\frac{x_2}{x_1} + |\ln x_1|\right)\max\{x_1, x_2\},
\end{aligned}
$$

where we used the fact that $x_1, x_2 \geq 0$ to remove some of the absolute values. Now we find the range of values of x_1 and x_2 for which the problem is ill-conditioned (or in other words, when $\kappa_f(\boldsymbol{x})$ is large). We split this into two cases:

Case 1: Assume $x_2 \geq x_1$. Then

$$\kappa_f(\boldsymbol{x}) = \frac{x_2^2}{x_1} + x_2|\ln x_1|.$$

If x_1 is large (say $x_1 \geq 10^6$), then x_2 will be large as well (since $x_2 \geq x_1$), and in this case the term $x_2|\ln x_1|$ will be large which will cause $\kappa_f(\boldsymbol{x})$ to be large as well. If x_2 is large and x_1 is small (say $x_2 \geq 10^6$ and $x_1 \leq 1$), then the term x_2^2/x_1 will be large which again will contribute to a large value of $\kappa_f(\boldsymbol{x})$. Finally, if x_2 is small and x_1 is not much smaller than x_2, then $\kappa_f(\boldsymbol{x})$ will be small (if x_1 is much smaller than x_2, then both terms in $\kappa_f(\boldsymbol{x})$ will be large which will cause $\kappa_f(\boldsymbol{x})$ to be large).

Case 2: Assume $x_1 \geq x_2$. Then

$$\kappa_f(\boldsymbol{x}) = x_2 + x_1|\ln x_1|.$$

If x_2 is large (say $x_2 \geq 10^6$), then x_1 will be large as well (since $x_1 \geq x_2$), and each term in $\kappa_f(\boldsymbol{x})$ will be large which implies $\kappa_f(\boldsymbol{x})$ will be large. If x_1 is large (say $x_1 \geq 10^6$), then the term $x_1 \ln|x_1|$ will be large and this will contribute to a large value of $\kappa_f(\boldsymbol{x})$, regardless of whether or not x_2 is small or not. Finally, if x_1 is small (say $x_1 \leq 1$), then x_2 will be small as well since $x_2 \leq x_1$, and in this case, both terms x_2 and $x_1|\ln x_1|$ in $\kappa_f(\boldsymbol{x})$ will be small, so overall $\kappa_f(\boldsymbol{x})$ will be small.

2 Problem 2

Exercise 2. Consider the following recurrence:

$$x_{k+1} = 111 - \frac{1130 - \frac{3000}{x_{k-1}}}{x_k},$$

whose general solution is

$$x_k = \frac{100^{k+1}a + 6^{k+1}b + 5^{k+1}c}{100^k a + 6^k b + 5^k c}, \tag{1}$$

where a, b, c depend on the initial values. Given $x_0 = 11/2$ and $x_1 = 61/11$, we have $a = 0$ and $b = 1 = c$. So in this case we have

$$x_k = \frac{6^{k+1} + 5^{k+1}}{6^k + 5^k},$$

1. Show that this gives a monotonically increasing sequence to the limit of value 6.
2. Implement this recurrence in MATLAB. Plot (x_k) and compare it with the exact solution. Explain any major discrepancies you see. What is the condition number of the limit of this particular sequence as a function of x_0 and x_1?

Solution 2. 1. To see that it is monotonically increasing, let

$$f_k(x, y) = \frac{x^{k+1} + y^{k+1}}{x^k + y^k}$$

and observe that

$$\partial_k f_k(x, y) = \partial_k \left(\frac{x^{k+1} + y^{k+1}}{x^k + y^k} \right) = \frac{(y - x)x^k y^k \ln(y/x)}{(x^k + y^k)^2}.$$

In particular, we have

$$\partial_k f_k(5, 6) = \frac{5^k 6^k \ln(6/5)}{(5^k + 6^k)^2} > 0.$$

This implies $f_k(5,6) = x_k$ is strictly increasing in k. Next observe that

$$\begin{aligned}\lim_{k\to\infty} x_k &= \lim_{k\to\infty}\left(\frac{6^{k+1}+5^{k+1}}{6^k+5^k}\right)\\ &= \lim_{k\to\infty}\left(\frac{6^{k+1}}{6^k+5^k}\right) + \lim_{k\to\infty}\left(\frac{5^{k+1}}{6^k+5^k}\right)\\ &= \lim_{k\to\infty}\left(\frac{6^{k+1}}{6^k}\right) + \lim_{k\to\infty}\left(\frac{5^{k+1}}{6^k}\right)\\ &= 6+0\\ &= 6,\end{aligned}$$

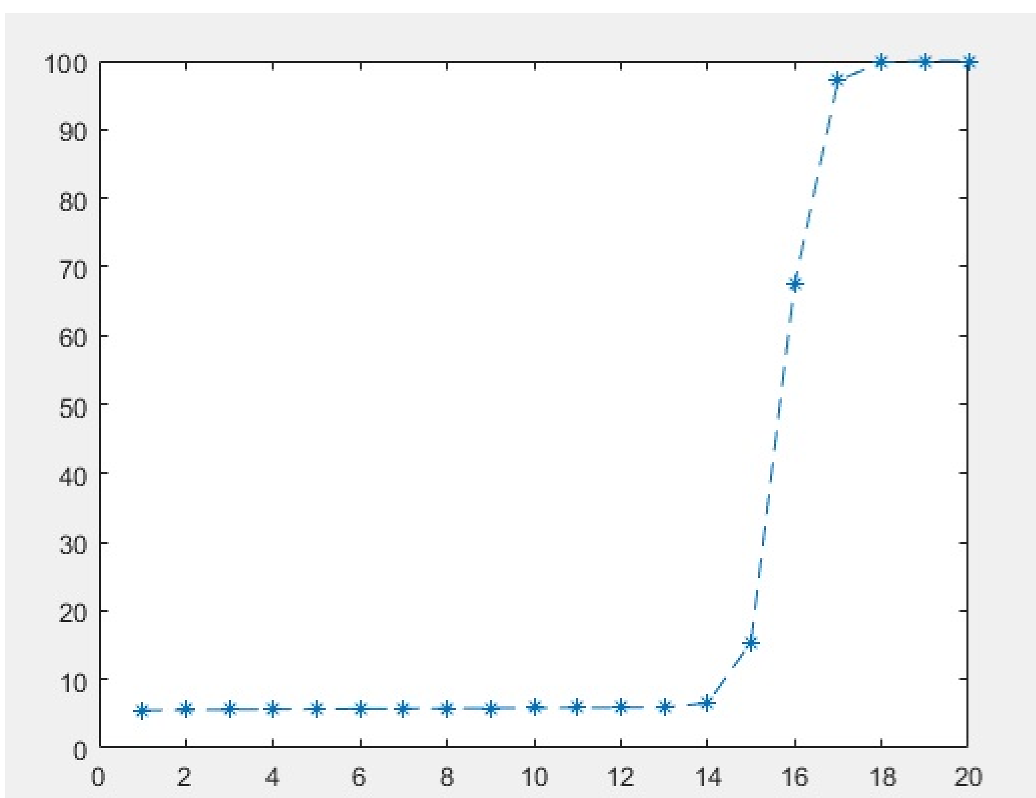
where we used the fact that $\lim_{k\to\infty}(6^k/5^k) = \infty$ (i.e. 6^k grows much faster than 5^k which is why we are allowed to remove 5^k from the denominator in the third line).

2. We implement the following code in MATLAB:

```
t(0) = 1; t(1) = 2; x(0) = 11/2; x(1) = 61/11;
for i = 2:20
    t(i) = i;   x(i) = 111 - (1130 - 3000/x(i-2))/x(i-1);
end;

plot(t,x,'*--')
```

MATLAB gives us the following plot:



So initially, MATLAB's plot seems as if it's converging to 6 (the true limit), however around $t = 14$ it starts to converge to 100. To see what's going on, let L be the limit of the sequence (x_k) where x_k is given in the fomula (1). Then observe that

$$L = \begin{cases} 100 & \text{if } a \neq 0\\ 6 & \text{if } a = 0 \text{ and } b \neq 0\\ 5 & \text{if } a = b = 0 \text{ and } c \neq 0.\end{cases}$$

The key is that MATLAB does not store $x_1 = 61/11$ as it's true value because $61/11 = 5.\overline{54}$ is a number with a repeated decimal. Instead MATLAB stores x_1 at $\widehat{x}_1 = 5.4545\ldots456$ (a finite number of digits after the decimal). If we replace $x_1 = 61/11$ with $\widehat{x}_1$ as MATLAB does, then we will find that $\widehat{a} \neq 0$. In this case, the limit $\widehat{L}$ will be 100.

Now let us find the relative condition number of L as a function of $\boldsymbol{x} = (x_0, x_1)$. In particular, we will find the relative condition number of L at the point $\boldsymbol{p} := (p_0, p_1) := (11/2, 61/11)$. Recall that this is given by

$$\widehat{\kappa}(\boldsymbol{p}) := \frac{\|\boldsymbol{p}\|}{|L(\boldsymbol{p})|} \lim_{\delta \to 0} \sup_{h \leq \delta} \frac{|L(\boldsymbol{p}+\boldsymbol{h}) - L(\boldsymbol{p})|}{\|\boldsymbol{h}\|}. \tag{2}$$

Note that for each $\delta > 0$, we can always find an $\boldsymbol{h} \in \mathbb{R}^2$ such that $\|\boldsymbol{h}\| \leq \delta$ and $L(\boldsymbol{p}+\boldsymbol{h}) = 100$. Indeed, the point is that we can always an find $\boldsymbol{h} = (h_0, h_1)^\top$ such that $\|\boldsymbol{h}\| \leq \delta$ and such that when we solve for the system of equations

$$p_0 + h_0 = \frac{100a + 6b + 5c}{a + b + c} \quad \text{and} \quad p_1 + h_1 = \frac{10000a + 36b + 25c}{100a + 6b + 5c}, \tag{3}$$

we will have $a \neq 0$. In this case, we will have $|L(\boldsymbol{p}+\boldsymbol{h}) - L(\boldsymbol{p})| = 94$, and thus taking the limit in (2) will give us $\widehat{\kappa}(\boldsymbol{p}) = \infty$.

3 Problem 3

Exercise 3. Let

$$p_{24}(x) = (x-1)(x-2)\cdots(x-24) = x^{24} + a_{23}x^{23} + \cdots + a_1 x + a_0,$$

where

$$\begin{aligned} a_{14} &= 9.2447 \times 10^{16} \\ a_{15} &= -5.7006 \times 10^{15} \\ a_{16} &\approx 2.9089 \times 10^{14} \\ a_{17} &\approx -1.2191 \times 10^{13} \\ a_{18} &\approx 4.1491 \times 10^{11}. \end{aligned}$$

Evaluate the relative condition number of the kth root $x_k = k$ subject to the perturbation of a_k for $k = 14, 15, \ldots, 18$ and find the root that is most sensitive to perturbation of the corresponding coefficient. Use the attached MATLAB data file wilk24mc.mat containing the coefficients $a_{24}, a_{23}, \ldots, a_1, a_0$ and use MATLAB's roots function to find the roots. Compare with the true roots and comment on what you see.

Solution 3. Recall from class that the relative condition number of the root x_k subject to the perturbation of a_k is given by

$$\kappa_k := \left| \frac{a_k x_k^{k-1}}{p'(x_k)} \right|.$$

In particular, we have

$$\begin{aligned} \kappa_{14} &= \frac{9.2447 \cdot 10^{16} \cdot 14^{13}}{13!10!} \approx 3.2472 \cdot 10^{15} \\ \kappa_{15} &= \frac{5.7006 \cdot 10^{15} \cdot 15^{14}}{14!9!} \approx 5.2608 \cdot 10^{15} \\ \kappa_{16} &\approx \frac{2.9089 \cdot 10^{14} \cdot 16^{15}}{15!8!} \approx 6.3608 \cdot 10^{15} \\ \kappa_{17} &\approx \frac{1.2191 \cdot 10^{13} \cdot 17^{16}}{16!7!} \approx 5.6256 \cdot 10^{15} \\ \kappa_{18} &\approx \frac{4.1491 \cdot 10^{11} \cdot 18^{17}}{17!6!} \approx 3.5415 \cdot 10^{15}. \end{aligned}$$

Thus the root that is most sensitive to the corresponding coefficient is $x_{16} = 16$.

Next we use variable containing the coefficients $a_{24}, a_{23}, \ldots, a_1, a_0$ and we use MATLAB's roots function to find the roots:

```
>> r = roots(wilkinson24_monomial_coeffs);
>> r

r =

 24.139509745437273 + 0.000000000000000i
 23.055621068264180 + 0.909065814514705i
 23.055621068264180 - 0.909065814514705i
 21.063652983597962 + 1.921738964419476i
 21.063652983597962 - 1.921738964419476i
 18.655051341123166 + 2.383055588891501i
 18.655051341123166 - 2.383055588891501i
 16.260559651064732 + 2.274095501070119i
 16.260559651064732 - 2.274095501070119i
 14.098908262310534 + 1.849084567628511i
 14.098908262310534 - 1.849084567628511i
 12.099341571155342 + 1.288025678365456i
 12.099341571155342 - 1.288025678365456i
 10.245284640997927 + 0.570821949140902i
 10.245284640997927 - 0.570821949140902i
  8.882242501021073 + 0.000000000000000i
  8.023850969323298 + 0.000000000000000i
  6.997354850545262 + 0.000000000000000i
  6.000213454514602 + 0.000000000000000i
  4.999989129325873 + 0.000000000000000i
  4.000000317378681 + 0.000000000000000i
  2.999999995400954 + 0.000000000000000i
  2.000000000025441 + 0.000000000000000i
  1.000000000000035 + 0.000000000000000i
```

We find that the roots start splitting into conjugate pairs starting at the 10th and 11th root $x_{10} = 10$ and $x_{11} = 11$. For instance, the pair (x_{10}, x_{11}) becomes

$$(\widehat{x}_{10}, \widehat{x}_{11}) \approx (10.2453 - 0.5708i, 10.2453 + 0.5708i).$$

Similarly, the pair (x_{16}, x_{17}) which becomes

$$(\widehat{x}_{16}, \widehat{x}_{17}) \approx (16.2606 - 2.2741i, 16.2606 + 2.2741i),$$

and so on. The point is that the relative conditions numbers $\kappa_{10}, \kappa_{11}, \ldots, \kappa_{24}$ are very large and thus the roots $x_{10}, x_{11}, \ldots, x_{24}$ are sensitive to small perturbations in the a_i.