# Preliminary Material

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## 1 Gröbner Bases

Throughout this section, let $K$ be a field, and let $S$ denote the polynomial ring $K[x_1, \ldots, x_n]$. In this section, we state all of our lemmas, propositions, and theorems without proof. All of the proofs can be found in **[?]** and **[?]**.

### 1.1 Monomials and Polynomials in $S$

A **monomial** $m$ in $S$ is a product in $S$ of the form

$$m = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. Sometimes we will use the notation $x^\alpha$ to denote a monomial, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an $n$-tuple of nonnegative integers. Note that $x^\alpha = 1$ when $\alpha = (0, \ldots, 0)$. If $m = x^\alpha$ is a monomial in $S$ then the **degree** of $m$, denoted $\deg(m)$ or $|x^\alpha|$ , is the sum $\alpha_1 + \cdots + \alpha_n$.

A **polynomial** $f$ in $S$ is a finite linear combination of monomials. We will write a polynomial $f$ in the form

$$f = \sum_\alpha a_\alpha x^\alpha, \quad a_\alpha \in K,$$

where the sum is over a finite number of $n$-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$. We call $a_\alpha$ the **coefficient** of the monomial $x^\alpha$. If $a_\alpha \neq 0$, then we call $a_\alpha x^\alpha$ a **term** of $f$. The **total degree** of $f \neq 0$, denoted $\deg(f)$, is the maximum $|\alpha|$ such that the coefficient $a_\alpha$ is nonzero.

*Remark.* If we replace the field $K$ with a ring $R$, then the same terminology applies to $R[x_1, \ldots, x_n]$. For instance, a **monomial** $m$ in $R[x_1, \ldots, x_n]$ is a product of the form $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and etc...

#### 1.1.1 Monomial Orderings on $S$

A **monomial ordering** on $S$ is a total ordering $>$ on $\mathbb{Z}^n_{\geq 0}$, or equivalently, a total ordering on the set of monomials $x^\alpha$, $\alpha \in \mathbb{Z}^n_{\geq 0}$, satisfying

$$x^\alpha > x^\beta \implies x^\gamma x^\alpha > x^\gamma x^\beta,$$

for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$. We say $>$ is a **global monomial ordering** if $x^\alpha > 1$ for all $\alpha \neq 0$.

*Remark.* By a total ordering, we mean for all distinct pairs of monomials $x^\alpha$ and $x^\beta$, we either have $x^\alpha > x^\beta$ or $x^\beta > x^\alpha$. This property is used in induction arguments.

**Lemma 1.1.** *Let $>$ be a monomial ordering, then the following conditions are equivalent.*

1. *$>$ is a well-ordering, i.e. every nonempty set of monomials has a smallest element, or equivalently, every decreasing sequence*

   $$x^{\alpha(1)} > x^{\alpha(2)} > x^{\alpha(3)} > \cdots$$

   *eventually terminates.*

2. $x_i > 1$ *for* $i = 1, \ldots, n$.

3. *$>$ is global.*

4. *$\alpha \geq_{nat} \beta$ and $\alpha \neq \beta$ implies $x^\alpha > x^\beta$, where $\geq_{nat}$ is a partial order on $\mathbb{Z}^n_{\geq 0}$ defined by*

   $$(\alpha_1, \ldots, \alpha_n) \geq_{nat} (\beta_1, \ldots, \beta_n) \text{ if and only if } \alpha_i \geq \beta_i \text{ for all } i.$$

### 1.1.2 Examples of Monomial Orderings

We now describe some important examples of global monomial orderings: Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$.

1. (Lexicographical ordering): We say $x^\alpha >_{lp} x^\beta$ if
$$\text{there exists } 1 \leq i \leq n \text{ such that } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$
2. (Degree reverse lexicographical ordering) We say $x^\alpha >_{dp} x^\beta$ if
$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and there exists } 1 \leq i \leq n \text{ such that } \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i.$$
3. (Degree lexicographical ordering) We say $x^\alpha >_{Dp} x^\beta$ if
$$|\alpha| = \sum_{i=1}^n \alpha_i > |\beta| = \sum_{i=1}^n \beta_i, \quad \text{or} \quad |\alpha| = |\beta| \text{ and there exists } 1 \leq i \leq n \text{ such that } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

**Example 1.1.** With respect to the lexicographical ordering on $K[x,y,z]$, we have $x^3y^2z >_{lp} x^3yz^3$ and $xy^2z >_{lp} xyz^2$. With respect to the degree reverse lexicographical ordering on $K[x,y,z]$, we have $x^2y^2z^2 >_{dp} x^3yz^3$ and $z^2 >_{dp} x$. With respect to the degree lexicographical ordering on $K[x,y,z]$, we have $x^3yz^3 >_{Dp} x^2y^2z^2$ and $z^2 >_{Dp} x$.

### 1.1.3 Multidegree, Leading Coefficients, Leading Monomials, and Leading Terms

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a nonzero polynomial in $K[x_1, \dots, x_n]$ and let $>$ be a monomial order.

1. The **multidegree** of $f$ is
$$\text{multdeg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \mid c_\alpha \neq 0).$$
2. The **leading coefficient** of $f$ is
$$\text{LC}(f) = c_{\text{multdeg}(f)} \in K.$$
3. The **leading monomial** of $f$ is
$$\text{LM}(f) = x^{\text{multdeg}(f)}.$$
4. The **leading term** of $f$ is
$$\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f).$$

**Example 1.2.** Let $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2$. With respect to lexicographical ordering we have
$$\begin{aligned} \text{multdeg}(f) &= (3,0,0) \\ \text{LC}(f) &= -5 \\ \text{LM}(f) &= x^3 \\ \text{LT}(f) &= -5x^3. \end{aligned}$$
With respect to degree reverse lexicographical ordering we have
$$\begin{aligned} \text{multdeg}(f) &= (2,0,2) \\ \text{LC}(f) &= 7 \\ \text{LM}(f) &= x^2z^2 \\ \text{LT}(f) &= 7x^2z^2. \end{aligned}$$

## 1.2 Monomial Ideals

An ideal $I \subseteq S$ is a called a **monomial ideal** if there is a subset $A \subset \mathbb{Z}^n_{\geq 0}$ (possibly infinite) such that $I$ consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$, where $h_\alpha \in K[x_1, \dots, x_n]$. In this case, we write $I = \langle x^\alpha \mid \alpha \in A \rangle$.

**Example 1.3.** An example of a monomial ideal is given by $I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle \subseteq K[x,y]$. A nontrivial example of a monomial ideal is given by $J = \langle f_1, f_2, f_3, f_4 \rangle = \langle x^2 + x^2y^3, -x^2y^3 + y^3, x^4, y^6 \rangle$. It's easy to see that $J \subset \langle x^2, y^3 \rangle$. For the reverse inclusion, note that
$$\begin{aligned} x^2 &= f_1 - x^2 f_2 - y^3 f_3 \\ y^3 &= f_1 + y^3 f_2 - x^2 f_4. \end{aligned}$$
So $\langle x^2, y^3 \rangle \subset J$. Therefore $J = \langle x^2, y^3 \rangle$.

### 1.2.1 Monomials Ideals are Finitely-Generated

The next theorem tells us that monomials ideals are finitely generated.

**Theorem 1.2.** *(Dickson's Lemma.) Let $I = \langle x^\alpha \mid \alpha \in A \rangle$ be a monomial ideal. Then $I$ can be written as $I = \langle x^{\alpha(1)}, \dots, x^{\alpha(s)} \rangle$ where $\alpha(1), \dots, \alpha(s) \in A$.*

## 1.3 Hilbert Basis Theorem

Throughout the rest of this section, fix a monomial ordering on $S$.

### 1.3.1 Lead Term Ideal

Let $I$ be a nonzero ideal in $S$.

1. We denote by $\mathrm{LT}(I)$ the set of leading terms of nonzero elements of $I$. Thus,

$$\mathrm{LT}(I) = \{cx^\alpha \mid \text{there exists } f \in I \setminus \{0\} \text{ with } \mathrm{LT}(f) = cx^\alpha\}.$$

2. We denote by $\langle \mathrm{LT}(I) \rangle$ be the ideal generated by the elements of $\mathrm{LT}(I)$.

It is easy to see that $\langle \mathrm{LT}(I) \rangle$ is a monomial ideal. Therefore Theorem (1.2) implies that it is finitely-generated. Thus, there are $g_1, \dots, g_t \in I$ such that $\mathrm{LT}(I) = \langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle$. If we are given an arbitrary finite generating set for $I$, say $I = \langle f_1, \dots, f_s \rangle$, then $\langle \mathrm{LT}(f_1), \dots, \mathrm{LT}(f_s) \rangle$ and $\langle \mathrm{LT}(I) \rangle$ may be *different* ideals. To see this, consider the following example.

**Example 1.4.** Let $I = \langle f_1, f_2 \rangle$, where $f_1 = x^3 - 2xy$ and $f_2 = x^2y - 2y^2 + x$, and use grlex ordering on monomials in $K[x, y]$. Then

$$x \cdot (x^2y - 2y^2 + x) - y \cdot (x^3 - 2xy) = x^2,$$

so that $x^2 \in I$. Thus $x^2 = \mathrm{LT}(x^2) \in \langle \mathrm{LT}(I) \rangle$. However $x^2$ is not divisible by $\mathrm{LT}(f_1) = x^3$ or $\mathrm{LT}(f_2) = x^2y$, so that $x^2 \notin \langle \mathrm{LT}(f_1), \mathrm{LT}(f_2) \rangle$.

### 1.3.2 Hilbert Basis Theorem

**Theorem 1.3.** *(Hilbert Basis Theorem). Let $I$ be an ideal in $S$. Then $I$ is finitely-generated.*

## 1.4 Gröbner Bases

Let $I$ be a nonzero ideal in $S$. A finite subset $G = \{g_1, \dots, g_t\}$ is said to be a **reduced Gröbner basis** if

1. $\langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle$
2. $\mathrm{LC}(g) = 1$ for all $g \in G$.
3. For all $g \in G$, no monomial of $g$ lies in $\langle \mathrm{LT}(G \setminus \{g\} \rangle$.

Let $I$ be an ideal in $S$ and let $G = \{g_1, \dots, g_t\}$ be the reduced Gröbner basis for $I$. Then given a polynomial $f$ in $S$, it can be shown that there are unique polynomials $\pi(f)$ and $f^G$ in $S$ such that $f = \pi(f) + f^G$ and no term of $f^G$ is divisible by any of $\mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t)$. We call $f^G$ the **normal form of $f$ with respect to** $G$. It follows from uniqueness of $f^G$ and $\pi(f)$ that taking the normal form of a polynomial is a $K$-linear map:

$$c_1 f_1^G + c_2 f_2^G = (c_1 f_1 + c_2 f_2)^G \tag{1}$$

for all $c_1, c_2 \in K$ and $f_1, f_2 \in S$. We will denote this map as $-^G$. An important property of $-^G$ is that it preserves homogeneity. The details can be found in \cite{GP08} and \cite{CLO15}.

# 2 Graded Rings and Modules

## 2.1 Graded Rings

A **graded ring** $R$ is a ring together with a direct sum decomposition

$$R = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R_i,$$

where the $R_i$ are abelian groups which satisfies the condition that if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$. The $R_i$ are called **homogeneous components** of $R$ and the elements of $R_i$ are called **homogeneous elements** of **degree** $i$. If $r$ is a homogeneous element in $R$, then we denote the degree of $r$ as $\deg(r)$. When we say "Let $R$ be a graded ring", we denote the homogeneous components of $R$ as $R_i$.

*Remark.* The condition that $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$ is equivalent to the condition that $R_i R_j \subset R_{i+j}$.

**Example 2.1.** An important example of a graded ring is a ring $R$ endowed with the **trivial grading**: The homogoneeous components of $R$ being $R_0 := R$ and $R_i := 0$ for all $i > 0$. If $R$ is a field, then will *always* assume that $R$ is a graded ring endowed with the trivial grading.

**Example 2.2.** Let $R$ be a ring and let $Q$ be an ideal in $R$. The **associated graded ring of** $R$ **with respect to** $Q$ is

$$\mathrm{Gr}_Q(R) := \bigoplus_{i=0}^{\infty} Q^i/Q^{i+1}.$$

Multiplication in $\mathrm{Gr}_Q(R)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$.

### 2.1.1 Weighted Polynomial Rings

Let $w := (w_1, \dots, w_n)$ be an $n$-tuple of positive integers. We define the **weighted polynomial ring** $S_w$ with respect to the **weighted vector** $w$ to be the polynomial ring $R[x_1, \dots, x_n]$ endowed with the unique grading such that $\deg(x_\lambda) = \alpha_\lambda$ for all $\lambda = 1, \dots, n$. We define the **weighted degree** of a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in $S_w$, denoted $\deg_w(m)$, to be

$$\deg_w(m) := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

This grading gives $S_w$ the structure of a graded ring, where the homogeneous components are given by

$$(S_w)_i := \mathrm{Span}_R \langle m \in S_w \mid m \text{ is monomial of weighted degree } i \rangle.$$

*Remark.* If $w = (1, \dots, 1)$, then we recover the polynomial ring $R[x_1, \dots, x_n]$ with the usual grading. If the context is clear, we simply use the letter $S$ to denote this graded ring.

**Example 2.3.** Let $K$ be a field and let $S_w$ denote the weighted polynomial ring $K[x, y, z]$ with respect to the weighted vector $w := (1, 2, 3)$. The first few homogeneous components of $S_w$ start out as

$$\begin{aligned}
(S_w)_0 &= K \\
(S_w)_1 &= Kx \\
(S_w)_2 &= Kx^2 + Ky \\
(S_w)_3 &= Kx^3 + Kxy + Kz \\
&\vdots
\end{aligned}$$

## 2.2 Graded $R$-Modules

Let $R$ be a graded ring. An $R$-module $M$, together with a direct sum decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

into abelian groups $M_i$ is called a **graded** $R$**-module** if $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. The $M_i$ are called **homogeneous components** of $M$ and the elements of $M_i$ are called **homogeneous** of **degree** $i$. If $m$ is a homogeneous element in $M$, then we denote the degree of $m$ as $\deg(m)$. When we say "Let $M$ be a graded $R$-module", then the homogeneous components of $M$ are denoted $M_i$.

*Remark.* Unlike in the case of graded rings, we do *not* usually assume that $M_i = 0$ for $i < 0$.

**Example 2.4.** Here's an important example of a graded $R$-module where we do not necessarily have $M_i = 0$ for $i < 0$: If $M$ is a graded $R$-module, then for $j \in \mathbb{Z}$, we define the $j$'**th twist** or the $j$'**th shift** of $M$ to be the graded $R$-module

$$M(j) := \bigoplus_{i \in \mathbb{Z}} M(j)_i$$

where $M(j)_i := M_{i+j}$.

#### 2.2.1 Graded $R$-Submodules

**Lemma 2.1.** *Let $M$ be a graded $R$-module and $N \subset M$ a submodule. The following conditions are equivalent:*

1. *$N$ is graded $R$-module whose homogeneous components are $M_i \cap N$.*
2. *$N$ is generated by homogeneous elements.*
3. *Let $m = \sum m_i$ with $m_i \in M_i$. Then $m \in N$ if and only if $m_i \in N$ for all $i \in \mathbb{Z}$.*

*Proof.* The proof is straightforward and can be found in \cite{GP08}. $\square$

A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (2.1) is called a **graded** (or **homogeneous**) $R$-submodule.

**Example 2.5.** Let $K$ be a field, $S_w$ be the polynoimal ring $K[x, y, z]$ with respect to the weight $w = (5, 6, 15)$, and let $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ be an ideal $S_w$. Then $I$ is a homogeneous ideal in $S_w$.

*Remark.* Let $R$ be a graded ring, and let $I$ be a homogeneous ideal in $R$. Then the quotient $R/I$ has an induced structure as a graded ring, where the homogeneous component of $R/I$ is

$$(R/I)_i := (R_i + I)/I \cong R_i / I \cap R_i$$

#### 2.2.2 Homomorphisms of Graded $R$-Modules

Let $M$ and $N$ be graded $R$-modules. A homomorphism $\varphi : M \to N$ is called **homogeneous** (or **graded**) of degree $j$ if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If $\varphi$ is homogeneous of degree zero then we will simply say $\varphi$ is **homogeneous**.

**Example 2.6.** Let $R$ denote the polynomial ring $K[x, y, z, t]$ with the natural grading. Then the matrix

$$U := \begin{pmatrix} x + y + z & w^2 - x^2 & x^3 \\ 1 & x & xy + z^2 \end{pmatrix}$$

defines a homomorphism $U : R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$ which is graded of degree zero.

### 2.3 Graded $R$-Algebras

Let $R$ be a graded ring and let $A$ be an $R$-algebra. We say $A$ is a **graded $R$-algebra** if $A$ is graded as a ring and $A_0 = R$.

*Remark.* We do not require $A$ to be a *commutative* ring.

**Example 2.7.** Let $Q$ be an ideal in $R$. The **blowup algebra of $Q$ in $R$** is the $R$-algebra

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong R \oplus Q \oplus Q^2 \oplus Q^3 \oplus \cdots .$$

The multiplication in $B_Q(A)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$.

#### 2.3.1 Homomorphisms of Graded $R$-Algebras

Let $A$ and $A'$ be graded $R$-algebras. We say $\varphi : A \to A'$ is an $R$-algebra homomorphism if

1. $\varphi$ is a homomorphism when viewed as a map of $R$-modules. In other words,

   $$\varphi(r_1 a_1 + r_2 a_2) = r_1 \varphi(a_1) + r_2 \varphi(a_2)$$

   for all $r_1, r_2 \in R$ and $a_1, a_2 \in A$.
2. $\varphi$ preserves the algebra structure. In other words

   $$\varphi(ab) = \varphi(a)\varphi(b)$$

   for all $a, b \in A$.

Moreover, we say $\varphi$ is **graded** if $\varphi$ is a graded homomorphism when viewed as a map of graded $R$-modules.

### 2.3.2 Finitely-Generated Graded $R$-Algebras

An graded $R$-algebra $A$ is said to be **finitely-generated** if it is finitely-generated as an $R$-algebra. The next proposition gives a classification of all finitely-generated commutative $R$-algebras.

**Proposition 2.1.** *Every finitely-generated commutative graded $R$-algebra is isomorphic to $S_w/I$, where $S_w$ denotes the polynomial ring $R[x_1,\dots,x_n]$ with respect to the weighted vector $w \in \mathbb{Z}^n_{\geq 0}$ and $I$ is a homogeneous ideal in $S_w$.*

*Proof.* Let $A$ be a finitely-generated commutative $R$-algebra with generators $a_1,\dots,a_n$. Then for each $\lambda = 1,\dots,n$ we have $a_\lambda \in A_{w_\lambda}$, where $w_\lambda \in \mathbb{Z}_{\geq 0}$. Let $\varphi : S_w \to A$ be the unique morphism of graded $R$-algebras such that $\varphi(x_\lambda) = a_\lambda$ for all $\lambda = 1,\dots,n$. Then $A$ is isomorphic to $S_w/\mathrm{Ker}(\varphi)$ as graded $R$-algebras. $\square$

### 2.3.3 Algorithmic Computations in the $R$-algebra $S/I$ using Gröbner Bases

Let $K$ be a field, $S$ denote the polynomials ring $K[x_1,\dots,x_n]$, and $I$ be a homogeneous ideal in $S$. Then $S/I$ is a graded $K$-algebra, where the homogeneous component $S_i$ is the $K$-vector space of all homogeneous polynomials $f \in S$ of degree $i$. Now fix a monomial ordering and let $G$ be the reduced Gröbner basis of $I$ with respect to this ordering. Define

$$S_I := \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle)$$

There is an obvious decompostion of $S_I$ into $K$-vector spaces $(S_I)_i$, where

$$(S_I)_i = \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle \text{ and } \deg(x^\alpha) = i).$$

In fact, $S/I$ and $S_I$ are isomorphic as graded $K$-modules. The isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. Indeed, $K$-linearity follows from (1), and the grading is preserved since $-^G$ preserves homogeneity. This makes $S/I$ isomorphic to $S_I$ as graded $K$-modules. Using this isomorphism, we can carry multiplication from $S/I$ over to $S_I$ to turn $S_I$ into a graded $K$-algebra: For $f_1, f_2 \in S_I$, we define multiplication as

$$f_1 \cdot f_2 = (f_1 f_2)^G. \tag{2}$$

Defining multilpication this way makes $S_I$ isomorphic to $S/I$ as graded $K$-algebras. For computational purposes, it is easier to work with $S_I$ rather than $S/I$.

**Example 2.8.** Consider $S = K[x,y]$ and $I = \langle xy^2 + y^3, x^3 + x^2y \rangle$. Then $G = \{xy^2 + y^3, x^3 + x^2y\}$ is the reduced Gröbner basis with respect to graded reverse lexicographical order. Thus $\mathrm{LT}(I) = \langle xy^2, x^3 \rangle$. Let's do some computations in $S_I$. First, let's write the first few homogeneous terms of $S_I$:

$$\begin{aligned}
(S_I)_0 &= K \\
(S_I)_1 &= Kx + Ky \\
(S_I)_2 &= Kx^2 + Kxy + Ky^2 \\
(S_I)_3 &= Kx^2y + Ky^3 \\
(S_I)_4 &= Ky^4 \\
(S_I)_5 &= Ky^5 \\
&\ \vdots
\end{aligned}$$

Next, we multiply some elements together in $S_I$ in the multiplication table below

| $\cdot$ | $x$ | $y$ | $y^3$ |
|---|---|---|---|
| $x^2y$ | $y^4$ | $y^4$ | $y^6$ |
| $x^2$ | $x^2y$ | $x^2y$ | $y^5$ |
| $x$ | $x^2$ | $xy$ | $y^4$ |

**Example 2.9.** Consider $S = K[x,y]$ and $I = \langle xy + y^2, x^3 \rangle$. We first use Singular to compute a Gröbner basis $G$ of $I$ with respect to graded reverse lexicographical ordering. We obtain $G = \{g_1, g_2, g_3\}$. where $g_1 = xy + y^2$, $g_2 = x^3$, and $g_3 = y^4$. Then the first few homogeneous components of $I$, $S/I$ and $S_I$ are given below

$$\begin{array}{lll}
I_0 = 0 & (S/I)_0 = K \cdot \overline{1} & (S_I)_0 = K \\
I_1 = 0 & (S/I)_1 = K\overline{x} + K\overline{y} & (S_I)_1 = Kx + Ky \\
I_2 = Kg_1 & (S/I)_2 = K\overline{x}^2 + K\overline{y}^2 & (S_I)_2 = Kx^2 + Ky^2 \\
I_3 = Kxg_1 + Kyg_1 + Kg_2 & (S/I)_3 = K\overline{y}^3 & (S_I)_3 = Ky^3 \\
I_4 = S_4 & (S/I)_4 = 0 & (S_I)_4 = 0 \\
\vdots & \vdots & \vdots
\end{array}$$

# 3 Homological Algebra

Throughout this section, let $R$ be a ring.

## 3.1 Chain Complexes over $R$

A **chain complex** $(A, d)$ **over** $R$, or simply a **chain complex** if the base ring $R$ is understood from context, is a sequence of $R$-modules $A_i$ and morphisms $d_i : A_i \to A_{i-1}$

$$(A, d) := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\mathrm{Ker}(d_i) \supset \mathrm{Im}(d_{i+1})$. With this in mind, we define the $i$**th homology of the chain complex** $(A, d)$ to be

$$H_i(A, d) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

Let $(A, d)$ and $(A', d')$ be two chain complexes. A **chain map** $\varphi : (A, d) \to (A', d')$ is a sequence of $R$-module homomoprhisms $\varphi_i : A_i \to A'_i$ such that $d'_i \varphi_i = \varphi_{i-1} d'_i$ for all $i \in \mathbb{Z}$. We can view a chain map visually as illustrated in the diagram below:

$$\begin{tikzcd}
(A, d) := \cdots \arrow[r] & A_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\varphi_{i+1}"] & A_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"] & A_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \\
(A', d') := \cdots \arrow[r] & A'_{i+1} \arrow[r, "d'_{i+1}"] & A'_i \arrow[r, "d'_i"] & A'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

### 3.1.1 Simplifying Notation

To simplify notation in what follows, we think of $R$ as a trivially graded ring. If $(A, d)$ is a chain complex over $R$, then we think of $(A, d)$ as a graded $R$-module $A$ together with a graded endomorphism $d : A \to A$ of degree $-1$ such that $d^2 = 0$. We think of $d_i$ as being the restriction of $d$ to $A_i$ and we often refer to $d$ as the **differential**. An element in $\mathrm{Ker}(d)$ is called a **cycle** of $(A, d)$ and an element in $\mathrm{Im}(d)$ is called a **boundary** of $(A, d)$. We define the **homology** of $(A, d)$ to be

$$H(A, d) := \mathrm{Ker}(d)/\mathrm{Im}(d)$$

Note that $H(A, d) = \bigoplus_{i \in \mathbb{Z}} H_i(A, d)$. We sometimes write $H(A)$ instead of $H(A, d)$ if the differential is understood from context.

Let $(A, d)$ and $(A', d')$ be chain complexes. A chain map $\varphi : (A, d) \to (A', d')$ can be thought of as a homogeneous homomorphism of graded $R$-modules such that $\varphi d = d' \varphi$.

### 3.1.2 Homotopy Equivalence

Let $\varphi$ and $\psi$ be chain maps of chain complexes $(A, d)$ and $(A', d')$. We say $\varphi$ is **homotopic** to $\psi$ if there is a graded homomorphism $h : A \to A'$ of degree 1 such that $\varphi - \psi = d'h + hd$.

**Proposition 3.1.** *Let $\varphi$ and $\psi$ be chain maps of chain complexes $(A, d)$ and $(A', d')$. Then $\varphi$ and $\psi$ induce the same map on homology.*

*Proof.* The proof is straightforward and can be found in [**?**]. □

## 3.2 Exact Sequences of Chain Complexes over $R$

Let $(A, d)$, $(A', d')$, and $(A'', d'')$ be chain complexes and let $\varphi : (A', d') \to (A, d)$ and $\psi : (A, d) \to (A'', d'')$ be chain maps. Then we say that

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

is a **short exact sequence** of chain complexes if the following diagram is commutative with exact rows:

$$
\begin{tikzcd}
 & \vdots \arrow[d, "d'_{i+2}"] & \vdots \arrow[d, "d_{i+2}"] & \vdots \arrow[d, "d''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "d'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "d_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "d''_{i+1}"] & 0 \\
0 \arrow[r] & A'_i \arrow[r, "\varphi_i"] \arrow[d, "d'_i"] & A_i \arrow[r, "\psi_i"] \arrow[d, "d_i"] & A''_i \arrow[r] \arrow[d, "d''_i"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "d'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "d_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "d''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Given such a short exact sequence, we get induced maps $\varphi_i : H_i(A') \to H_i(A)$ and $\psi_i : H_i(A) \to H_i(A'')$, and **connecting homomorphisms** $\gamma_i : H_i(A'') \to H_{i-1}(A')$ which gives rise a long exact sequence in homology:

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{i+1}(A'') \arrow[dll, rounded corners, "\gamma_{i+1}"] \\
H_i(A') \arrow[r, "\varphi_i"] & H_i(A) \arrow[r, "\psi_i"] & H_i(A'') \arrow[dll, rounded corners, "\gamma_i"] \\
H_{i-1}(A') \arrow[r, "\varphi_{i-1}"] & H_{i-1}(A) \arrow[r, "\psi_{i-1}"] & \cdots
\end{tikzcd}
$$

*Remark.* It is a nice exercise in homological algebra to work out the details of the connecting map.

## 3.3 Differential Graded $R$-Algebras

A **differential graded $R$-algebra** is a chain complex $(A, d)$ such that $A$ is a graded $R$-algebra and the differential $d$ satisfies the **Leibniz law** with respect to this algebra structure:

$$d(ab) = d(a)b + (-1)^{\deg(a)} a d(b). \tag{3}$$

for all $a, b \in A$. We say that the differential graded $R$-algebra is **commutative** if $ab = (-1)^{\deg(a)\deg(b)} ba$. We say that the differential graded $R$-algebra is **strictly commutative** if in addition $a^2 = 0$ for $\deg(a)$ odd.

### 3.3.1 Homomorphisms of Differential Graded $R$-Algebras

Let $(A, d)$ and $(A', d')$ be differential graded $R$-algebras. We say $\varphi : (A, d) \to (A', d')$ is **homomorphism of differential graded** $R$-algebras if $\varphi$ is both a chain map and an $R$-algebra homomorphism.

### 3.3.2 Differential Graded $A$-Modules

Let $(A, d)$ be a differential graded $R$-algebra. A **differential graded $A$-module** $(M, d)$ is a chain complex $(M, d)$ over $R$ such that $M$ is an $A$-module and such that the differential $d$ satisfies the **Leibniz law** with respect to the algebra structure in $A$:

$$d(am) = d(a)m + (-1)^{\deg(a)} a d(m). \tag{4}$$

for all $a \in A$ and $m \in M$.

### 3.3.3 Obtaining a Differential Graded $A$-Module from a Chain Complex over $R$

Let $(A, d_A)$ be a differential graded $R$-algebra. If we start with a chain complex over $R$, then we can construct a differential graded $A$-module. Indeed, suppose that $(B, d_B)$ is a chain complex over $R$. Then $A \otimes_R B$ is an $A$-module and a graded $R$-module whose homogeneous component in degree $k$ is

$$(A \otimes_R B)_k := \bigoplus_{i+j=k} A_i \otimes_R B_j.$$

We define a differential $d$ on $A \otimes_R B$ by first defininig it on the elementary tensors as

$$d(a \otimes b) := d_A(a) \otimes b + (-1)^{\deg(a)} a \otimes d_B(b),$$

for all $a \in A$ and $b \in B$, and then extending it $R$-linearly everywhere else. A straightforward calculation shows that $d^2 = 0$ and that the differential satisfies Leibniz law (4). Moreover, if $B$ is a differential graded $R$-algebra, then $A \otimes_R B$ can realized as a differential graded $A$-algebra and a differential graded $B$-algebra. Multiplication in $A \otimes_R B$ is defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'.$$

for all $a, a' \in A$ and $b, b \in B$.

*Remark.* In particular, if $M$ is an $R$-module endowed with the trivial grading, then $(A \otimes_R M, d)$ is a differential graded $A$-module where the homogeneous componenet of degree $k$ in $A \otimes_R M$ is $(A \otimes_R M)_k := A_k \otimes_R M$, and $d$ acts on elementary tensors as $d(a \otimes m) = d(a) \otimes m$.

## 3.4 Exterior Algebras and Koszul Complexes

### 3.4.1 Exterior Algebras

Let $R$ be a ring and $M$ an $R$-module. For $k \geq 2$, the $k$th **exteror power** of $M$, denoted $\Lambda^k(M)$, is the $R$-module $M^{\otimes k}/J_k$ where $J_k$ is the submodule of $M^{\otimes k}$ spanned by all $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for $i \neq j$. For any $m_1, \dots, m_k \in M$, the coset of $m_1 \otimes \cdots \otimes m_k$ in $\Lambda^k(M)$ is denoted $m_1 \wedge \cdots \wedge m_k$. For completeness, we set $\Lambda^0(M) = R$ and $\Lambda^1(M) = M$. A general element in $\Lambda^k(M)$ will be denoted as $\omega$ or $\eta$. Since $M^{\otimes k}$ is spanned by tensors $m_1 \otimes \cdots \otimes m_k$, the quotient module $M^{\otimes k}/J_k = \Lambda^k(M)$ is spanned by their images $m_1 \wedge \cdots \wedge m_k$. That is, any $\omega \in \Lambda^k(M)$ is a finite $R$-linear combination

$$\omega = \sum r_{i_1,\dots,i_k} m_{i_1} \wedge \cdots \wedge m_{i_k},$$

where there coefficients $r_{i_1,\dots,i_k}$ are in $R$ and the $m_i$'s are in $M$. We call $m_1 \wedge \cdots \wedge m_k$ an **elementary wedge product**. Since $r(m_1 \wedge \cdots \wedge m_k) = (rm_1) \wedge \cdots \wedge m_k$, every element of $\Lambda^k(M)$ is a sum (not just a linear combination) of elementary wedge products.

We define the **exterior algebra** of $M$ to be

$$\Lambda(M) := \bigoplus_{k \geq 0} \Lambda^k(M),$$

where the multiplication rule given by the wedge product. The exterior algebra of $M$ is a graded $R$-algebra, where the degree $k$ homogeneous component is $\Lambda^k(M)$. If $R$ does not have characteristic 2, then the exterior algebra of $M$ is **skew commutative**. This means that if $\omega_1$ and $\omega_2$ are homogeneous elements, then

$$\omega_1 \wedge \omega_2 = (-1)^{\deg(\omega_1)\deg(\omega_2)} \omega_2 \wedge \omega_1.$$

The construction of $\Lambda(M)$ is functioral in $M$. This means that if $N$ is another $R$-module and $\varphi : M \to N$ is an $R$-module homomorphism. Then $\varphi$ induces a graded $R$-algebra homomorphism $\wedge\varphi : \Lambda(M) \to \Lambda(N)$, where $\wedge\varphi$ takes the elementary wedge product $m_1 \wedge \cdots \wedge m_k$ in $\Lambda(M)$ and maps it to the wedge product $\varphi(m_1) \wedge \cdots \wedge \varphi(m_k)$ in $\Lambda(N)$. We will write $\wedge^k\varphi$ to be the induced $R$-module homomorphism from $\Lambda^k(M)$ to $\Lambda^k(N)$. In particular, if $N$ is free of rank $n$, then $\Lambda^n(N) \cong R$, and if $\varphi : N \to N$ is an $R$-module homomorphism, then $\wedge^n\varphi$ is multiplication by the determinant of any matrix representing $\varphi$.

**Example 3.1.** Let $R$ be a ring, $M = Rx_1 \oplus Rx_2 \oplus Rx_3 \cong R^3$, and let $\varphi : M \to M$ be the $R$-module homomorphism induced by setting $\varphi(x_\mu) = \sum_{\lambda=1}^{n} a_{\lambda\mu} x_\lambda$ for $1 \leq \lambda, \mu \leq 3$. The matrix representation of $\varphi$ with respect to the ordered basis $\beta_1 = \{x_1, x_2, x_3\}$ is given by

$$[\varphi]_{\beta_1}^{\beta_1} := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

To calculate $\wedge^2\varphi$, we need to see how it acts on the basis vectors $x_\lambda \wedge x_\mu$ where $1 \leq \lambda < \mu \leq 3$:

$$\begin{aligned}
\varphi(x_1) \wedge \varphi(x_2) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \\
&= (a_{11}a_{22} - a_{21}a_{12})x_1 \wedge x_2 + (a_{11}a_{32} - a_{31}a_{12})x_1 \wedge x_3 + (a_{21}a_{32} - a_{31}a_{22})x_2 \wedge x_3 \\
\\
\varphi(x_1) \wedge \varphi(x_3) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{11}a_{23} - a_{21}a_{13})x_1 \wedge x_2 + (a_{11}a_{33} - a_{31}a_{13})x_1 \wedge x_3 + (a_{21}a_{33} - a_{31}a_{23})x_2 \wedge x_3 \\
\\
\varphi(x_2) \wedge \varphi(x_3) &= (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{12}a_{23} - a_{22}a_{13})x_1 \wedge x_2 + (a_{12}a_{33} - a_{32}a_{13})x_1 \wedge x_3 + (a_{22}a_{33} - a_{32}a_{23})x_2 \wedge x_3.
\end{aligned}$$

So the matrix representation of $\wedge^2\varphi$ with respect to the ordered basis $\beta_2 = \{x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3\}$ is

$$[\varphi]_{\beta_2}^{\beta_2} = \begin{pmatrix} a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ a_{11}a_{32} - a_{31}a_{12} & a_{11}a_{33} - a_{31}a_{13} & a_{12}a_{33} - a_{32}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & a_{21}a_{33} - a_{31}a_{23} & a_{22}a_{33} - a_{32}a_{23} \end{pmatrix}$$

To calculate $\wedge^3\varphi$, we need to see how it acts on the basis vector $x_1 \wedge x_2 \wedge x_3$:

$$\begin{aligned}
\varphi(x_1) \wedge \varphi(x_2) \wedge \varphi(x_3) &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3) \wedge (a_{12}x_1 + a_{22}x_2 + a_{32}x_3) \wedge (a_{13}x_1 + a_{23}x_2 + a_{33}x_3) \\
&= (a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13})x_1 \wedge x_2 \wedge x_3 \\
&= \det\left([\varphi]_{\beta_1}^{\beta_1}\right) e_1 \wedge e_2 \wedge e_3.
\end{aligned}$$

### 3.4.2 Koszul Complexes

Let $R$ be a ring, $M$ an $R$-module, and $\varphi : M \to R$ an $R$-module homomorphism. The assignment

$$(m_1, \ldots, m_k) \mapsto \sum_{i=1}^{k} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

defines an alternating $n$-linear map $M^k \to \Lambda^{k-1}(M)$. By the universal property of the $k$th exterior power, there exists an $R$-linear map $d_\varphi^{(k)} : \Lambda^k(M) \to \Lambda^{k-1}(M)$ with

$$d_\varphi^{(k)}(m_1 \wedge \cdots \wedge m_k) = \sum_{i=1}^{n} (-1)^{i+1} \varphi(m_i) m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_k$$

for all $m_1, \ldots, m_k \in L$. The collection of the maps $d_\varphi^{(k)}$ defines a graded $R$-homomorphism

$$d_\varphi : \Lambda(M) \to \Lambda(M)$$

of degree $-1$. A straightforward calculation shows that $d_\varphi$ gives $\Lambda(M)$ the structure of a differential graded $R$-algebra. This differential graded $R$-algebra is called the **Koszul complex** of $\varphi$ and is denoted $\mathcal{K}_\bullet(\varphi)$. The **dual Koszul complex** of $\varphi$, denoted $\mathcal{K}^\bullet(\varphi)$, is the chain complex over $R$ whose underlying graded $R$-module is $\mathrm{Hom}_R(\mathcal{K}_\bullet(\varphi), R)$ and whose differential is $d^\star$, where $d^\star$ is obtained by applying the functor $\mathrm{Hom}_R(-, R)$ to $d$.

**Example 3.2.** Let $R$ be a ring of characteristic 2, $S$ denote the polynomial ring $R[x_1, \ldots, x_n]$, and let $\varphi : S_1 := \bigoplus_{\lambda=1}^n Rx_\lambda \to R$ be the unique $R$-linear map such that $\varphi(x_\lambda) = r_\lambda \in R$ for all $\lambda = 1, \ldots, n$. Then $\Lambda(S_1)$ is isomorphic to $S/\langle x_1^2, \ldots, x_n^2\rangle$ as graded $R$-algebras. Using this isomorphism, we give $S/\langle x_1^2, \ldots, x_n^2\rangle$ the structure of a differential graded $R$-algebra by carrying over the differential $d_\varphi$ for $\Lambda(S_1)$ to a differential $d$ for $S/\langle x_1^2, \ldots, x_n^2\rangle$. A straightforward calculation shows that $d = \sum_{\lambda=1}^n r_\lambda \partial_{x_\lambda}$. We denote this Koszul complex as $\mathcal{K}(r_1, \ldots, r_n)$.

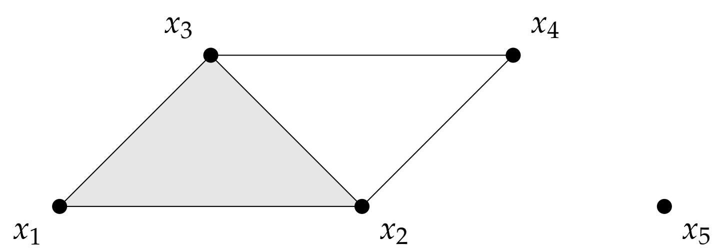
# 4 Simplicial Complexes

A **simplicial complex** $\Delta$ on the set $\{x_1, \ldots, x_n\}$ is a collection of subsets of $\{x_1, \ldots, x_n\}$ such that

1. The simplicial complex $\Delta$ contains all singletons: $\{x_\lambda\} \in \Delta$ for all $\lambda = 1, \ldots, n$.
2. The simplicial complex $\Delta$ is closed under containment: if $\sigma \subseteq \{x_1, \ldots, x_n\}$ and $\tau \supset \sigma$, then $\tau \in \Delta$.

An element of a simplicial complex is called a **face** or **simplex**, and a simplex of $\Delta$ not properly contained in another simplex of $\Delta$ is called a **facet**. A simplex $\sigma \in \Delta$ of cardinality $i+1$ is called an $i$-dimensional face or an $i$-face of $\Delta$. The empty set $\varnothing$, is the unique face of dimension $-1$, as long as $\Delta$ is not the **void complex** $\{\}$ consisting of no subsets of $\{1, \ldots, n\}$. The **dimension** of $\Delta$, denoted $\dim(\Delta)$, is defined to be the maximum of the dimensions of its faces (or $-\infty$ if $\Delta = \{\}$).

**Example 4.1.** The simplicial complex $\Delta$ on $\{x_1, x_2, x_3, x_4, x_5\}$ consisting of all subsets of $\{x_1, x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_4\}$, and $\{x_4\}$ is pictured below




## 4.1 Simplicial Homology

Let $\Delta$ be a simplicial complex on $\{x_1, \ldots, x_n\}$. For $i \in \mathbb{Z}$, let

$$S_i(\Delta) := \operatorname{Span}_K(\sigma \in \Delta \mid \dim(\sigma) = i) \quad \text{and} \quad S(\Delta) := \bigoplus_{i \in \mathbb{Z}} S_i(\Delta).$$

Then $S(\Delta)$ is a graded $K$-module. Let $\partial : S(\Delta) \to S(\Delta)$ be the unique graded endomorphism of degree $-1$ such that

$$\partial(\sigma) = \sum_{\lambda \in \sigma} \sigma \backslash \{\lambda\}.$$

By a direct calculation, we have $\partial^2 = 0$, and so $(S(\Delta), \partial)$ forms a chain complex over $K$; it is called the (**augmented** or **reduced**) **chain complex of** $\Delta$ **over** $K$. The $i$th homology of $(S(\Delta), \partial)$ is called the $i$**th reduced homology** of $\Delta$ over $K$, and is commonly denoted as $\widetilde{H}_i(\Delta, K)$.

**Example 4.2.** For $\Delta$ as in Example (4.1), we have

$$\begin{aligned}
S_2(\Delta) &= \{\{1,2,3\}\} \\
S_1(\Delta) &= \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\} \\
S_0(\Delta) &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \\
S_{-1}(\Delta) &= \{\varnothing\}
\end{aligned}$$

Choosing bases for the $S_i(\Delta)$ as suggested by the ordering of the faces listed above, the chain complex for $\Delta$ becomes

$$0 \longrightarrow K \xrightarrow[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}]{} K^5 \xrightarrow[\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}]{} K^5 \xrightarrow[\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}]{} K \longrightarrow 0$$

For example, $\partial_2(e_{\{1,2,3\}}) = e_{\{2,3\}} + e_{\{1,3\}} + e_{\{1,2\}}$, which we identify with the vector $(1,1,1,0,0)$. The mapping $\partial_1$ has rank 3, so $\widetilde{H}_0(\Delta; K) \cong \widetilde{H}_1(\Delta; K) \cong K$ and the other homology groups are 0. Geometrically, $\widetilde{H}_0(\Delta; K)$ is nontrivial since $\Delta$ is disconnected and $\widetilde{H}_1(\Delta; K)$ is nontrivial since $\Delta$ contains a triangle which is not the boundary of an element of $\Delta$.