MATH 9850, Free Resolutions
Fall 2019
Exercises 2
Due date: Thu 03 Oct 4:30PM

Let R be a commutative ring with identity.

Let $x_1, \dots, x_n \in R$, and let σ be an element of the symmetric group S_n. The goal of this exercise set is to prove that there is an isomorphism of Koszul complexes

$$K^R(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \cong K^R(x_1, \dots, x_n). \tag{$\dagger$}$$

Note that Exercise 1 can be done with no knowledge of Koszul complexes.

Exercise 1. Let the following commutative diagram of chain maps be given.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & Y \arrow[d, "\gamma"] \\ A' \arrow[r, "\phi'"] & Y' \end{tikzcd}$$

(a) Prove that α and γ induce a well-defined chain map $\Lambda\colon \operatorname{Cone}(\phi) \to \operatorname{Cone}(\phi')$.
(b) Prove that if α and γ are isomorphisms, then so is Λ.

Definition 2. We define the *Koszul complex* $K^R(x_1, \dots, x_n)$ inductively. For $x, y \in R$:

$$\begin{aligned} K^R(x) &= (0 \to R \xrightarrow{x} R \to 0) \\ K^R(x,y) &= \operatorname{Cone}(K^R(y) \xrightarrow{x} K^R(y)) \\ &\cong \left(0 \to R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R \to 0)\right) \\ K^R(x_1, \dots, x_n) &= \operatorname{Cone}(K^R(x_2, \dots, x_n) \xrightarrow{x_1} K^R(x_2, \dots, x_n)) \end{aligned}$$

Exercise 3. Let $x, y \in R$.
(a) Prove that there is an isomorphism between Koszul complexes $K^R(x,y) \cong K^R(y,x)$.
(b) More generally, let A be an R-complex, and set $K^R(x;A) = \operatorname{Cone}(A \xrightarrow{x} A)$ and $K^R(x,y;A) = \operatorname{Cone}(K^R(y;A) \xrightarrow{x} K^R(y;A))$. Define $K^R(y,x;A)$ similarly. Prove that $K^R(x,y;A) \cong K^R(y,x;A)$.

Exercise 4. (a) Prove that if σ is an adjacent transposition $\sigma = (i \;\; i+1)$, then there is an isomorphism ($\dagger$).
(b) Prove that if $\sigma \in S_n$ is arbitrary, then there is an isomorphism ($\dagger$).