Free Resolutions Homework 2

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Troughout this homework assignment, let R be a commutative ring with identity.

Exercise 1

Proposition 0.1. *Let the following commutative diagram of chain maps be given.*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\alpha"'] & Y \arrow[d, "\gamma"] \\ A' \arrow[r, "\phi'"] & Y' \end{tikzcd} \tag{1}$$

1. *Prove that α and γ induce a well-defined chain map $\Lambda \colon C(\phi) \to C(\phi')$.*
2. *Prove that if α and γ are isomorpisms, then so is Λ.*

Proof.

1. Let $i \in \mathbb{Z}$ and define $\Lambda_i \colon C(\phi)_i \to C(\phi')_i$ be given by

$$\Lambda_i((a,y)) = (\alpha(a), \gamma(y))$$

for all $a \in A_{i-1}$ and $y \in Y_i$. This map is well-defined since α and γ are well-defined and since every element in $C(\phi)_i$ can be uniquely expressed as (a,y) for some $a \in A_{i-1}$ and $y \in Y_i$. Let us check that this is an R-module homomorphism: Let $r, r' \in R$, $a, a' \in A_{i-1}$, and $y, y' \in Y_i$. Then

$$\begin{aligned} \Lambda_i((r(a,y) + r'(a',y')) &= \Lambda_i((ra + r'a', ry + r'y')) \\ &= (\alpha(ra + r'a'), \gamma(ry + r'y')) \\ &= (r\alpha(a) + r'\alpha(a'), r\gamma(y) + r'\gamma(y')) \\ &= r(\alpha(a), \gamma(y)) + r'(\alpha(a'), \gamma(y')) \\ &= \Lambda_i(r(a,y)) + r'\Lambda_i((a',y')). \end{aligned}$$

Finally, we check that $\Lambda := \bigoplus_i \Lambda_i$ is a chain map: Let $(a,y) \in C(\phi)_i$. Then

$$\begin{aligned} \Lambda\partial^{C(\phi)}(a,y) &= \Lambda(-\partial^A(a), \phi(a) + \partial^Y(y)) \\ &= (\alpha(-\partial^A(a)), \gamma(\phi(a) + \partial^Y(y))) \\ &= (-\partial^A(\alpha(a)), \gamma(\phi(a) + \partial^Y(y))) \\ &= (-\partial^A(\alpha(a)), \phi'(\alpha(a)) + \partial^Y(\gamma(y))) \\ &= \partial^{C(\phi')}(\alpha(a), \gamma(y)) \\ &= \partial^{C(\phi')}\Lambda(a,y). \end{aligned}$$

2. Suppose α and γ are isomorphisms and let $\alpha' \colon A' \to A$ and $\gamma' \colon Y' \to Y$ denote their inverses respectively. Then by 1, the morphisms α' and γ' induce a well-defined chain map $\Lambda' \colon C(\phi') \to C(\phi)$, given by

$$\Lambda'(a,y) = (\alpha'(a'), \gamma'(y'))$$

for all $i \in \mathbb{Z}$, $a' \in A'_{i-1}$, and $y' \in Y'_i$. Moreover, we have

$$\begin{aligned} \Lambda'(\Lambda(a,y)) &= \Lambda'(\alpha(a), \gamma(y)) \\ &= (\alpha'(\alpha(a)), \gamma'(\gamma(y)) \\ &= (a,y), \end{aligned}$$

for all $i \in \mathbb{Z}$, $a \in A_{i-1}$, and $y \in Y_i$. Similarly, we have

$$\begin{aligned}\Lambda(\Lambda'(a',y')) &= \Lambda(\alpha'(a'),\gamma'(y')) \\ &= (\alpha(\alpha'(a')),\gamma(\gamma'(y'))) \\ &= (a',y'),\end{aligned}$$

for all $i \in \mathbb{Z}$, $a' \in A'_{i-1}$, and $y' \in Y'_i$. Thus, Λ and Λ' are inverses, which implies $\Lambda\colon C(\phi) \to C(\phi')$ is an isomorphism. □

Exercises 2 and 3

Througout the rest of this homework, let $\underline{r} = r_1, \dots, r_n \in R$. We begin with a concrete definition of the Koszul complex. Then we will develop some theory of tensor products of R-complexes and show how the Koszul complex can be constructed via tensor products. We will also show how the mapping cone of the homothety map can be realized as a tensor product. After all of this, we will finally be in a position to solve exercises 2 and 3.

Definition of Koszul Complex

Definition 0.1. Let $\underline{r} = r_1, \dots, r_n \in R$. The **Koszul complex** of $\underline{r}$, denoted $(\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ is the R-complex whose graded R-module $\mathcal{K}(\underline{r})$ has

$$\mathcal{K}_i(\underline{r}) := \begin{cases} R & \text{if } i \leq 0 \\ \bigoplus_{1 \leq \lambda_1 < \cdots < \lambda_i \leq n} Re_{\lambda_1 \cdots \lambda_i} & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{K}(\underline{r})}$ is the unique graded endomorphism of degree -1 such that

$$d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \lambda_i}) = \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$, where the hat symbol means omit that subscript.

Remark. We need to justify that $d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})} = 0$ (so that $(\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ really is an R-complex). It suffices to show that $d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})}$ maps all of the basis elements to 0: for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$, we have

$$\begin{aligned} d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \lambda_i}) &= d^{\mathcal{K}(\underline{r})} \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i} \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i}) \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} \left(\sum_{1 \leq \kappa < \mu} (-1)^{\kappa-1} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\kappa \cdots \widehat{\lambda}_\mu \cdots \lambda_i} + \sum_{\mu < \kappa \leq i} (-1)^{\kappa} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \widehat{\lambda}_\kappa \cdots \lambda_i} \right) \\ &= \sum_{1 \leq \kappa < \mu \leq i} (-1)^{\mu+\kappa-1} r_{\lambda_\mu} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\kappa \cdots \widehat{\lambda}_\mu \cdots \lambda_i} + \sum_{1 \leq \mu < \kappa \leq i} (-1)^{\mu+\kappa} r_{\lambda_\mu} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \widehat{\lambda}_\kappa \cdots \lambda_i} \\ &= 0, \end{aligned}$$

by symmetry in μ and κ.

Tensor Products

Definition 0.2. Let (A, d) and (A', d') be two R-complexes. Their **tensor product** is the R-complex

$$(A, d) \otimes_R (A', d') := (A \otimes_R A', d^{A \otimes_R A'}),$$

where the graded R-module $A \otimes_R A'$ has

$$(A \otimes_R A')_i = \bigoplus_{j \in \mathbb{Z}} A_j \otimes A'_{j-i}$$

as its ith component and whose differential is defined on elementary tensors by

$$d^{A\otimes_R A'}(a\otimes a') = d(a)\otimes a' + (-1)^i a\otimes d'(a')$$

for all $i,j\in\mathbb{Z}$, $a\in A_i$ and $a'\in A_j$.

Remark. I'm stating the definition of a tensor product of R-complexes so that you are familiar with my notation. Since you mentioned earlier that I don't need to check all of the details (like whether $d^{A\otimes_R A'}$ is a differential), I won't bother proving them here.

Commutativity of Tensor Products

Proposition 0.2. *Let (A,d) and (A',d') be R-complexes. Then*

$$(A,d)\otimes_R(A',d')\cong(A',d')\otimes_R(A,d).$$

Proof. Let $\varphi\colon A\otimes_R A'\to A'\otimes_R A$ be the unique graded isomorphism[1] such that

$$\varphi(a\otimes a') = (-1)^{ij}a'\otimes a$$

for all $i,j\in\mathbb{Z}$, $a\in A_i$ and $a'\in A'_j$.

For the rest of the proof, denote $d^{\otimes} := d^{A\otimes_R A'}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{\otimes} = d^{\otimes}\varphi \tag{2}$$

It suffices to check (2) on elementary tensors. We have

$$\begin{aligned}
d^{\otimes}\varphi(a\otimes a') &= d^{\otimes}((-1)^{ij}a'\otimes a)\\
&= (-1)^{ij}d'(a')\otimes a + (-1)^{j+ij}a'\otimes d(a))\\
&= (-1)^{ij}d'(a')\otimes a + (-1)^{j+ij-2j}a'\otimes d(a))\\
&= (-1)^{ij}d'(a')\otimes a + (-1)^{ij-j}a'\otimes d(a)\\
&= (-1)^{(i-1)j}a'\otimes d(a) + (-1)^{i+i(j-1)}d'(a')\otimes a\\
&= \varphi(d(a)\otimes a' + (-1)^i a\otimes d'(a'))\\
&= \varphi d^{\otimes}(a\otimes a')
\end{aligned}$$

for all $i,j\in\mathbb{Z}$, $a\in A_i$ and $a'\in A_j$. □

Associativity of Tensor Products

Given that the proof of tensor products of R-complexes was nontrivial, we need to be sure that we have associativity of tensor products of R-complexes. The proof in this case turns out to be trivial.

Proposition 0.3. *Let (A,d), (A',d'), and (A'',d'') be R-complexes. Then*

$$((A,d)\otimes_R(A',d')\otimes_R(A'',d'')\cong(A,d)\otimes_R((A',d')\otimes_R(A',d')).$$

Proof. Let $\varphi\colon(A\otimes_R A')\otimes_R A''\to A\otimes_R(A'\otimes_R A'')$ be the unique graded isomorphism such that

$$\varphi((a\otimes a')\otimes a'') = a\otimes(a'\otimes a''))$$

for all $i,j,k\in\mathbb{Z}$, $a\in A_i$, $a'\in A'_j$, and $a''\in A'_k$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{A\otimes(A'\otimes A'')} = d^{(A\otimes A')\otimes A''}\varphi \tag{3}$$

It suffices to check (3) on elementary tensors. We have

[1] The map φ is linear since the map $(a,a')\mapsto a'\otimes a$ is bilinear in a and a'. Also φ is an isomorphism since the map $\psi\colon A'\otimes_R A\to A\otimes_R A'$, defined on elementary tensors by $\psi(a'\otimes a) = (-1)^{ij}a\otimes a'$ is its inverse.

$$
\begin{aligned}
d^{A\otimes(A'\otimes A'')}\varphi((a\otimes a')\otimes a'') &= d^{A\otimes(A'\otimes A'')}(a\otimes(a'\otimes a'')) \\
&= d(a)\otimes(a'\otimes a'') + (-1)^i a\otimes d^{A'\otimes A''}(a'\otimes a'') \\
&= d(a)\otimes(a'\otimes a'') + (-1)^i a\otimes(d'(a')\otimes a'' + (-1)^j a'\otimes d(a'')) \\
&= d(a)\otimes(a'\otimes a'') + (-1)^i a\otimes(d'(a')\otimes a'') + (-1)^{i+j} a\otimes(a'\otimes d(a'')) \\
&= \varphi(d(a)\otimes a')\otimes a'' + (-1)^i(a\otimes d'(a'))\otimes a'' + (-1)^{i+j}(a\otimes a')\otimes d''(a'')) \\
&= \varphi((d(a)\otimes a' + (-1)^i a\otimes d'(a'))\otimes a'' + (-1)^{i+j}(a\otimes a')\otimes d''(a'')) \\
&= \varphi(d^{A\otimes A'}(a\otimes a')\otimes a'' + (-1)^{i+j}(a\otimes a')\otimes d''(a'')) \\
&= \varphi d^{(A\otimes A')\otimes A''}((a\otimes a')\otimes a'')
\end{aligned}
$$

for all $i, j, k \in \mathbb{Z}$, $a \in A_i$, $a' \in A'_j$, and $a'' \in A'_k$. □

Koszul Complex as Tensor Product

Proposition 0.4. *We have an isomorphism of R-complexes*

$$(\mathcal{K}(r_1), d^{\mathcal{K}(r_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(r_n), d^{\mathcal{K}(r_n)}) \cong (\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})}).$$

Remark. Note that Proposition (0.3) gives an unambiguous interpretation for $(\mathcal{K}(r_1), d^{\mathcal{K}(r_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(r_n), d^{\mathcal{K}(r_n)})$.

Proof. For each $1 \leq \lambda \leq n$, write $\mathcal{K}(r_\lambda) = R \oplus Re_\lambda$ (so $\{1\}$ is a basis for $\mathcal{K}(r_\lambda)_0$ and $\{e_\lambda\}$ is a basis for $\mathcal{K}(r_\lambda)_1$). Le

$$\varphi\colon \mathcal{K}(r_1) \otimes_R \cdots \otimes_R \mathcal{K}(r_n) \to \mathcal{K}(r_1, \ldots, r_n)$$

be the unique graded linear map [2] such that

$$\varphi(1 \otimes \cdots \otimes 1) = 1 \quad \text{and} \quad \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1) = e_{\lambda_1 \cdots \lambda_i}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism since it restricts to a bijection on basis sets.

For the rest of the proof, denote $d^{\mathcal{K}} := d^{\mathcal{K}(\underline{r})}$ and $d^{\otimes} := d^{\mathcal{K}(r_1)\otimes\cdots\otimes\mathcal{K}(r_n)}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{\otimes} = d^{\mathcal{K}} \varphi. \tag{4}$$

It suffices to check (4) on the basis elements. We have

$$
\begin{aligned}
d^{\mathcal{K}}\varphi(1\otimes\cdots\otimes 1) &= d^{\mathcal{K}}(1) \\
&= 0 \\
&= \varphi(0) \\
&= \varphi d^{\otimes}(1\otimes\cdots\otimes 1),
\end{aligned}
$$

and

$$
\begin{aligned}
d^{\mathcal{K}}\varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1) &= d^{\mathcal{K}}(e_{\lambda_1\cdots\lambda_i}) \\
&= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1\cdots\widehat{\lambda}_\mu\cdots\lambda_i} \\
&= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} \varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes \widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1) \\
&= \varphi \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} 1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes \widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1) \\
&= \varphi d^{\otimes}(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1).
\end{aligned}
$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. □

[2] We say unique graded linear map here because $\mathcal{K}(r_1) \otimes_R \cdots \otimes_R \mathcal{K}(r_n)$ is free with basis elements of the form $1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1$ for $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and φ respects the grading.

Mapping Cone

Definition 0.3. Let $\varphi\colon (A,d) \to (A',d')$ be a chain map. The **mapping cone of** φ, denoted $(C(\varphi), d^{C(\varphi)})$, is the R-complex whose graded R-module $C(\varphi)$ has

$$C_i(\varphi) := A'_i \oplus A_{i-1}$$

as its ith homogeneous component and whose differential $d^{C(\varphi)}$ is defined b

$$d^{C(\varphi)}(a,a') := (d'(a') + \varphi(a), -d(a))$$

for all $a' \in A'_i$ and $a \in A_{i-1}$.

Mapping Cone of Homothety Map as Tensor Product

Proposition 0.5. *Let (A,d) be an R-complex, let $x \in R$, and let $\mu_x\colon (A,d) \to (A,d)$ be the multiplication by x homothety map. Then*

$$(C(\mu_x), d^{C(\mu_x)}) \cong (\mathcal{K}(x), d^{\mathcal{K}(x)}) \otimes_R (A,d).$$

Proof. Let $\mathcal{K}(x) = R \oplus Re$ (so $\{1\}$ is a basis for $\mathcal{K}(x)_0$ and $\{e\}$ is a basis for $\mathcal{K}(x)_1$). Let $\varphi\colon \mathcal{K}(x) \otimes_R A \to C(\mu_x)$ be defined by

$$\varphi(1 \otimes a + e \otimes b) = (a,b)$$

for all $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Clearly φ is an isomorphism of graded R-modules. To see that φ is an isomorphism of R-complexes, we need to check that

$$d^{C(\mu_x)}\varphi = \varphi d^{\mathcal{K}(x)\otimes_R A} \tag{5}$$

Let $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Then

$$\begin{aligned}
d^{C(\mu_x)}\varphi(1 \otimes a + e \otimes b) &= d^{C(\mu_x)}(a,b) \\
&= (d(a) + xb, -d(b)) \\
&= \varphi(1 \otimes (d(a) + xb) + e \otimes (-d(b))) \\
&= \varphi(1 \otimes d(a) + x \otimes b - e \otimes d(b)) \\
&= \varphi(d^{\mathcal{K}(x)\otimes A}(1 \otimes a) + d^{\mathcal{K}(x)\otimes A}(e \otimes b)) \\
&= \varphi d^{\mathcal{K}(x)\otimes A}(1 \otimes a + e \otimes b).
\end{aligned}$$

□

Exercise Solutions

Exercise 2.a: Follows from Proposition (0.4) and Proposition (0.2).
Exercise 2.b: Follows from Proposition (0.5) and Proposition (0.2).
Exercise 3.a: Follows from Proposition (0.4) and Proposition (0.2)
Exercise 3.b: Follows from Proposition (0.4) and Proposition (0.2)