Linear Analysis Homework 4

Michael Nelson

Problem 1

Proposition 0.1. *Let $\mathcal{H}$ be a hilbert space and let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then*

1. $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1\}$;
2. $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} \mid x \in \mathcal{H}\backslash\{0\}\right\}$.

Proof.

1. First note that

$$\begin{aligned}\sup\{\|Tx\| \mid \|x\| = 1\} &\leq \sup\{\|Tx\| \mid \|x\| \leq 1\} \\ &= \|T\|.\end{aligned}$$

We prove the reverse inequality by contradiction. Assume that $\|T\| > \sup\{\|Tx\| \mid \|x\| = 1\}$. Choose $\varepsilon > 0$ such that

$$\|T\| - \varepsilon > \sup\{\|Tx\| \mid \|x\| = 1\} \tag{1}$$

Next, choose $x \in \mathcal{H}$ such that $\|x\| \leq 1$ and $\|Tx\| \geq \|T\| - \varepsilon$. Then since $\|x\| \leq 1$ and $\left\|\frac{x}{\|x\|}\right\| = 1$, we have

$$\begin{aligned}\|T\| &\geq \left\|T\left(\frac{x}{\|x\|}\right)\right\| \\ &= \frac{\|Tx\|}{\|x\|} \\ &\geq \|Tx\| \\ &> \|T\| - \varepsilon,\end{aligned}$$

and this contradicts (1).

2. We have

$$\begin{aligned}\sup\left\{\frac{\|Tx\|}{\|x\|} \mid x \in \mathcal{H}\backslash\{0\}\right\} &= \sup\left\{\left\|T\left(\frac{x}{\|x\|}\right)\right\| \mid x \in \mathcal{H}\backslash\{0\}\right\} \\ &= \sup\left\{\|Ty\| \mid \|y\| = 1\right\} \\ &= \|T\|,\end{aligned}$$

where the last equality follows from 1. □

Problem 2

Proposition 0.2. *Let $k \in C[a,b]$. Then the operator $T\colon C[a,b] \to C[a,b]$ defined by*

$$Tf = kf$$

for all $f \in C[a,b]$ is bounded. It's norm will be explicitly computed in the proof below.

Proof. We first show it is linear. Let $f, g \in C[a,b]$ and let $\lambda, \mu \in \mathbb{C}$. Then we have

$$\begin{aligned}T(\lambda f + \mu g) &= k(\lambda f + \mu g) \\ &= \lambda kf + \mu kg \\ &= \lambda T(f) + \mu T(g).\end{aligned}$$

Thus, T is linear.

Next we show it is bounded. If $k = 0$, then $\|T\| = 0$, so assume $k \neq 0$. Since k is continuous on the compact interval $[a, b]$, there exists $c \in [a, b]$ such that $|k(x)| \leq |k(c)|$ for all $x \in [a, b]$. Choose such a $c \in [a, b]$ and let $f \in C[a, b]$ such that $\|f\| \leq 1$. Then

$$\begin{aligned}
\|Tf\| &= \|kf\| \\
&= \sqrt{\int_a^b |k(x)|^2 |f(x)|^2 dx} \\
&\leq |k(c)| \sqrt{\int_a^b |f(x)|^2 dx} \\
&\leq |k(c)|.
\end{aligned}$$

implies $\|T\| \leq |k(c)|$, and hence T is bounded.

To find the norm of T, let $\varepsilon > 0$ such that $\varepsilon < |k(c)|$. Without loss of generality, assume that $c < b$ (if $c = b$, then we swap the role of b with a in the argument which follows). Choose $c' \in (c, b)$ such that $|k(x)| \geq |k(c)| - \varepsilon$ for all $x \in (c, c')$ (such a c' must exist since k is continuous) and choose f to be a nonzero continuous function in $C[a, b]$ which vanishes outside the interval (c, c'). Then

$$|k(x)||f(x)| \geq (|k(c)| - \varepsilon)|f(x)|$$

for all $x \in (a, b)$. In particular, this implies

$$\begin{aligned}
\|Tf\| &= \|kf\| \\
&= \sqrt{\int_a^b |k(x)f(x)|^2 dx} \\
&\geq \sqrt{\int_a^b (|k(c)| - \varepsilon)|f(x)|^2 dx} \\
&= (|k(c)| - \varepsilon) \sqrt{\int_a^b |f(x)|^2 dx} \\
&= (|k(c)| - \varepsilon)\|f\|.
\end{aligned}$$

Therefore $\|T(f/\|f\|)\| \geq |k(c)| - \varepsilon$, and this implies

$$\|T\| \geq |k(c)| - \varepsilon \tag{2}$$

Since (2) holds for all $\varepsilon > 0$, we must have $\|T\| \geq |k(c)|$. Thus $\|T\| = |k(c)|$. □

Problem 3

Proposition 0.3. *Let $\{x_n \mid n \in \mathbb{N}\}$ be a linearly independent set of vectors in a Hilbert space $\mathcal{H}$. Consider the so called Gram-Schmidt process: set $e_1 = \frac{1}{\|x_1\|} x_1$. Proceed inductively. If $e_1, e_2, \ldots, e_{n-1}$ are computed, compute e_n in two steps by*

$$f_n := x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k, \text{ and then set } e_n := \frac{1}{\|f_n\|} f_n.$$

Then

1. *for every $N \in \mathbb{N}$ we have $\operatorname{span}\{x_1, x_2, \ldots, x_N\} = \operatorname{span}\{e_1, e_2, \ldots, e_N\}$;*
2. *the set $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal set in $\mathcal{H}$;*
3. *if $\overline{\operatorname{span}}\{x_n \mid n \in \mathbb{N}\} = \mathcal{H}$, then $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis for $\mathcal{H}$.*

Proof.
1. Let $N \in \mathbb{N}$. Then for each $1 \leq n \leq N$, we have

$$x_n = \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k + \|f_n\| e_n.$$

This implies $\text{span}\{x_1, x_2, \dots, x_N\} \subseteq \text{span}\{e_1, e_2, \dots, e_N\}$. We show the reverse inclusion by induction on n such that $1 \leq n \leq N$. The base case $n = 1$ being $\text{span}\{x_1\} \supseteq \text{span}\{e_1\}$, which holds since $e_1 = \frac{1}{\|x_1\|} x_1$. Now suppose for some n such that $1 \leq n < N$ we have

$$\text{span}\{x_1, x_2, \dots, x_k\} \supseteq \text{span}\{e_1, e_2, \dots, e_k\} \tag{3}$$

for all $1 \leq k \leq n$. Then

$$e_{n+1} = \frac{1}{\|f_n\|} x_n - \sum_{k=1}^{n} \frac{1}{\|f_n\|} \langle x_n, e_k \rangle e_k \in \text{span}\{x_1, x_2, \dots, x_n\}.$$

where we used the induction step (3) on the e_k's ($1 \leq k \leq n$). Therefore

$$\text{span}\{x_1, x_2, \dots, x_k\} \supseteq \text{span}\{e_1, e_2, \dots, e_k\}$$

for all $1 \leq k \leq n + 1$, and this proves our claim.

2. By construction, we have $\langle e_n, e_n \rangle = 1$ for all $n \in \mathbb{N}$. Thus, it remains to show that $\langle e_m, e_n \rangle = 0$ whenever $m \neq n$. We prove by induction on $n \geq 2$ that $\langle e_n, e_m \rangle = 0$ for all $m < n$. Proving this also give us $\langle e_m, e_n \rangle = 0$ for all $m < n$, since

$$\begin{aligned}
\langle e_m, e_n \rangle &= \overline{\langle e_n, e_m \rangle} \\
&= \overline{0} \\
&= 0.
\end{aligned}$$

The base case is

$$\begin{aligned}
\langle e_2, e_1 \rangle &= \frac{1}{\|x_1\| \|f_2\|} \left\langle \left(x_2 - \frac{\langle x_2, x_1 \rangle}{\langle x_1, x_1 \rangle} x_1 \right), x_1 \right\rangle \\
&= \frac{1}{\|x_1\| \|f_2\|} \left(\langle x_2, x_1 \rangle - \langle x_2, x_1 \rangle \right) \\
&= 0
\end{aligned}$$

Now suppose that $n > 2$ and that $\langle e_n, e_m \rangle = 0$ for all $m < n$. Then

$$\begin{aligned}
\langle e_{n+1}, e_m \rangle &= \frac{1}{\|f_{n+1}\|} \langle x_{n+1} - \sum_{k=1}^{n} \langle x_{n+1}, e_k \rangle e_k, e_m \rangle \\
&= \frac{1}{\|f_{n+1}\|} \left(\langle x_{n+1}, e_m \rangle - \sum_{k=1}^{n} \langle x_{n+1}, e_k \rangle \langle e_k, e_m \rangle \right) \\
&= \frac{1}{\|f_{n+1}\|} \left(\langle x_{n+1}, e_m \rangle - \langle x_{n+1}, e_m \rangle \langle e_m, e_m \rangle \right) \\
&= \frac{1}{\|f_{n+1}\|} \left(\langle x_{n+1}, e_m \rangle - \langle x_{n+1}, e_m \rangle \right) \\
&= 0,
\end{aligned}$$

for all $m < n + 1$, where we used the induction hypothesis to get from the second line to the third line. This proves the induction step, which finishes the proof of part 2 of the proposition.

3. By 2, we know that $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal set. Thus, it suffices to show that $\{e_n \mid n \in \mathbb{N}\}$ is complete. To do this, we use the criterion that the set $\{e_n \mid n \in \mathbb{N}\}$ is complete if and only if the only $x \in \mathcal{H}$ such that $\langle x, e_n \rangle = 0$ for all $n \in \mathbb{N}$ is $x = 0$.

Let $x \in \mathcal{H}$ and suppose $\langle x, e_n \rangle = 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\langle x, x_n \rangle &= \left\langle x, \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k + \|f_n\| e_n \right\rangle \\
&= \sum_{k=1}^{n-1} \langle x_n, e_k \rangle \langle x, e_k \rangle + \|f_n\| \langle x, e_n \rangle \\
&= 0
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{x_n \mid n \in \mathbb{N}\}$ is complete, this implies $x = 0$. Therefore $\{e_n \mid n \in \mathbb{N}\}$ is complete. $\square$

Problem 4

Example 0.1. The first three Legendre polynomials are

$$P_1(x) = 1, \quad P_2(x) = x, \quad , P_3(x) = \frac{1}{2}(3x^2 - 1).$$

We apply Gram-Schmidt process to the polynomials $1, x, x^2$ in the space $C[-1,1]$ to get scalar multiples of the Legendre polynomials above. First we set $f_1(x) = 1$ and then calculate

$$\begin{aligned} \|f_1(x)\| &= \sqrt{\int_{-1}^{1} dx} \\ &= \sqrt{2}. \end{aligned}$$

Thus we set $e_1(x) = 1/\sqrt{2}$. Next we calculate

$$\begin{aligned} f_1(x) &= x - \left\langle x, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} \\ &= x - \frac{1}{2} \int_{-1}^{1} x dx \\ &= x. \end{aligned}$$

Next we calculate

$$\begin{aligned} \|f_1(x)\| &= \sqrt{\int_{-1}^{1} x^2 dx} \\ &= \sqrt{\frac{2}{3}}. \end{aligned}$$

Thus we set $e_2(x) = \sqrt{3/2}x$. Next we calculate

$$\begin{aligned} f_2(x) &= x^2 - \left\langle x^2, \sqrt{\frac{3}{2}}x \right\rangle \sqrt{\frac{3}{2}}x - \left\langle x^2, \sqrt{\frac{1}{2}} \right\rangle \sqrt{\frac{1}{2}} \\ &= x^2 - \frac{3}{2}x \int_{-1}^{1} x^3 dx - \frac{1}{2} \int_{-1}^{1} x^2 dx \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

Then we finally calculate

$$\begin{aligned} \|f_2(x)\| &= \sqrt{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} \\ &= \sqrt{\int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx} \\ &= \sqrt{\int_{-1}^{1} x^4 dx - \frac{2}{3} \int_{-1}^{1} x^2 dx + \frac{1}{9} \int_{-1}^{1} dx} \\ &= \sqrt{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} \\ &= \sqrt{\frac{8}{45}}. \end{aligned}$$

Thus we set $e_3(x) = \sqrt{45/8}(x^2 - 1/3)$. Now observe that

$$\begin{aligned} P_1(x) &= \sqrt{2} e_1(x) \\ P_2(x) &= \sqrt{\frac{2}{3}} e_2(x) \\ P_3(x) &= \sqrt{\frac{2}{5}} e_3(x) \end{aligned}$$

Problem 5

For this problem, we needed to establish some basic results which we proved in the Appendix.

Proposition 0.4. *The expression*

$$\int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx. \tag{4}$$

is minimized in $a, b, c \in \mathbb{C}$ if and only if $a = 0$, $b = 3/5$, and $c = 0$.

Proof. Let

$$\mathcal{H} = \{p(x) \in \mathbb{C}[x] \mid \deg(p(x)) \leq 3\} \quad \text{and} \quad \mathcal{K} = \{p(x) \in \mathbb{C}[x] \mid \deg(p(x)) \leq 2\}.$$

Then $\mathcal{H}$ and $\mathcal{K}$ are subspaces of $C[-1,1]$, Proposition (0.7) implies they are inner-product spaces with the inner-product inherited from $C[-1,1]$. Since $\mathcal{H}$ is finite dimensional, Proposition (0.8) implies $\mathcal{H}$ is a separable Hilbert space. Since $\mathcal{K}$ is a finite dimensional subspace of $\mathcal{H}$, Proposition (0.9) implies $\mathcal{K}$ is closed in $\mathcal{H}$. Let $\{e_1, e_2, e_3\}$ be the orthonormal basis computed in problem 4. A proposition proved in class implies

$$\begin{aligned} P_{\mathcal{K}}(x^3) &= \langle x^3, e_1\rangle e_1 + \langle x^3, e_2\rangle e_2 + \langle x^3, e_3\rangle e_3 \\ &= \frac{1}{2}\int_{-1}^{1} x^3 dx + \frac{3}{2}x \int_{-1}^{1} x^4 dx + \frac{45}{8}\left(x^2 - \frac{1}{3}\right)\int_{-1}^{1} x^3\left(x^2 - \frac{1}{3}\right) dx \\ &= \frac{3}{5}x. \end{aligned}$$

where we used the fact that $x^3(x^2 - 1/3)$ is an odd function to get $\int_{-1}^{1} x^3(x^2 - 1/3)dx = 0$. Therefore

$$\begin{aligned} \int_{-1}^{1} \left| x^3 - \frac{3}{5}x \right|^2 dx &= \|x^3 - P_{\mathcal{K}}(x^3)\|^2 \\ &= \inf\left\{ \|x^3 - (a + bx + cx^2)\|^2 \mid a + bx + cx^2 \in \mathcal{K} \right\} \\ &= \inf\left\{ \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx \mid a, b, c \in \mathbb{C} \right\}. \end{aligned}$$

By uniqueness of $P_{\mathcal{K}}x^3$, (4) is minimized in $a, b, c \in \mathbb{C}$ if and only if $a = 0$, $b = 3/5$, and $c = 0$. □

Problem 6

Proposition 0.5. *$\ell^2(\mathbb{N})$ is a Hilbert space.*

Proof. Let $(a^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\ell^2(\mathbb{N})$.

Step 1: We show that for each $k \in \mathbb{N}$, the sequence of kth coordinates $(a_k^n)_{n\in\mathbb{N}}$ is a Cauchy sequence of complex numbers, and hence must converge (as $\mathbb{C}$ is complete). Let $k \in \mathbb{N}$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|a^n - a^m\| < \varepsilon^2$. Then $n, m \geq N$ implies

$$\begin{aligned} |a_k^n - a_k^m|^2 &\leq \sum_{i=1}^{\infty} |a_i^n - a_i^m|^2 \\ &= \|a^n - a^m\| \\ &< \varepsilon^2, \end{aligned}$$

which implies $|a_k^n - a_k^m| < \varepsilon$. Therefore $(a_k^n)_{n\in\mathbb{N}}$ is a Cauchy sequence of complex numbers. In particular, the sequence $(a_k^n)_{n\in\mathbb{N}}$ converges to some element, say $a_k^n \to a_k$.

Step 2: We show that the sequence $(a_k)_{k\in\mathbb{N}}$ defined in step 1 is square summable. Since (a^n) is a Cauchy sequence of elements in $\ell^2(\mathbb{N})$, there exists an $M > 0$ such that $\|a^n\| < M$ for all $n \in \mathbb{N}$ (see Lemma ((0.1) for a proof of this). Choose such an $M > 0$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$|a_k|^2 < |a_k^N|^2 + \varepsilon/K$$

for all $1 \leq k \leq K$. Then

$$\begin{aligned} \sum_{k=1}^{K} |a_k|^2 &< \sum_{k=1}^{K} |a_k^N|^2 + \varepsilon \\ &\leq \|a^N\| + \varepsilon \\ &\leq M + \varepsilon. \end{aligned}$$

Taking the limit $K \to \infty$, we see that

$$\begin{aligned}
\|a\| &= \sum_{k=1}^{\infty} |a_k|^2 \\
&\leq M + \varepsilon \\
&\leq 0.
\end{aligned}$$

In particular, a is square summable.

Step 3: Let a be the sequence $(a_k)_{k\in\mathbb{N}}$ defined in step 1. We show that $a^n \to a$ in the ℓ^2 norm. Let $\varepsilon > 0$ and let $K \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|a^n - a^m\|^2 < \varepsilon/2$. Then

$$\begin{aligned}
\sum_{k=1}^{K} |a_k^n - a_k^m|^2 &\leq \sum_{k=1}^{\infty} |a_k^n - a_k^m|^2 \\
&= \|a^n - a^m\|^2 \\
&< \varepsilon/2
\end{aligned}$$

for all $n, m \geq N$. Since $a_k^m \to a_k$ as $m \to \infty$ implies

$$\sum_{k=1}^{K} |a_k^n - a_k^m|^2 \to \sum_{k=1}^{K} |a_k^n - a_k|^2$$

as $m \to \infty$, we see that after taking the limit $m \to \infty$, we have

$$\sum_{k=1}^{K} |a_k^n - a_k|^2 \leq \varepsilon/2. \tag{5}$$

for all $n \geq N$. Taking the limit $K \to \infty$ in (5) gives us

$$\|a^n - a\|^2 < \varepsilon$$

for all $n \geq N$. It follows that $a^n \to a$. □

Problem 7

Proposition 0.6. *$C[a, b]$ is not a Hilbert space.*

Proof. For each $n \in \mathbb{N}$, define $f_n \in C[a, b]$ by

$$f_n(x) = \begin{cases} 0 & x \in [a, c - \frac{1}{n}] \\ nx + 1 - nc & x \in [c - \frac{1}{n}, c] \\ 1 & x \in [c, b], \end{cases}$$

where $c = \frac{a+b}{2}$. We will show that the sequence (f_n) is a Cauchy sequence which is not convergent.

Step 1: We first show that the sequence (f_n) is a Cauchy sequence. Let $\varepsilon > 0$ and let $m, n \in \mathbb{N}$ such that $n \geq m$. Then

$$\begin{aligned}
\|f_n - f_m\|^2 &= \int_{c-\frac{1}{m}}^{c-\frac{1}{n}} |mx + 1 - mc|^2 \, dx + \int_{c-\frac{1}{n}}^{c} |nx + 1 - nc - (mx + 1 - mc)|^2 \, dx \\
&= \int_{c-\frac{1}{m}}^{c-\frac{1}{n}} |m(x - c) + 1|^2 \, dx + (n - m)^2 \int_{c-\frac{1}{n}}^{c} |x - c|^2 \, dx \\
&\leq \left(\frac{1}{m} - \frac{1}{n}\right) \left|1 - \frac{m}{n}\right|^2 + \frac{(n - m)^2}{n^3} \\
&\leq \frac{1}{m} - \frac{1}{n} + \frac{(n - m)^2}{n^3}.
\end{aligned}$$

Choose $N \in \mathbb{N}$ such that $n \geq m \geq N$ implies

$$\frac{1}{m} - \frac{1}{n} + \frac{(n - m)^2}{n^3} < \varepsilon^2.$$

Then $n \geq m \geq N$ implies $\|f_n - f_m\| < \varepsilon$. Therefore (f_n) is a Cauchy sequence.

Step 2: We show that the sequence (f_n) is not convergent. Assume for a contradiction that $f_n \to f$ where $f \in C[a,b]$. Then

$$\begin{aligned}\|f_n - f\|^2 &= \int_a^{c-\frac{1}{n}} |f(x)|^2 dx + \int_{c-\frac{1}{n}}^{c} |f(x) - f_n(x)|^2 dx + \int_c^b |f(x) - 1|^2 dx \\ &\leq (c - a - \frac{1}{n}) \sup_{x \in [a, c-\frac{1}{n}]} |f(x)|^2 + \int_{c-\frac{1}{n}}^{c} |f(x) - f_n(x)|^2 dx + (b-c) \sup_{x \in [c, c-\frac{1}{n}]} |f(x) - 1|^2 dx.\end{aligned}$$

Since $\|f_n - f\| \to 0$ as $n \to \infty$, we see that (after taking the limit $n \to \infty$) we must have

$$f(x) = \begin{cases} 0 & \text{if } x \in [a,c] \\ 1 & \text{if } x \in [c,b] \end{cases}$$

but this is not a continuous function. Thus we obtain a contradiction. □

Appendix

Proposition 0.7. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner-product space and let W be a subspace of V. Then $(W, \langle \cdot, \cdot \rangle|_{W \times W}\rangle$ is an inner-product space, where $\langle \cdot, \cdot \rangle|_{W \times W} \colon W \times W \to \mathbb{C}$ is the restriction of $\langle \cdot, \cdot \rangle$ to $W \times W$.*

Proof. All of the required properties for $\langle \cdot, \cdot \rangle|_{W \times W}$ to be an inner-product are *inherited* by $\langle \cdot, \cdot \rangle$ since W is a subset of V. For instance, let $x, y, z \in V$ and let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\langle x + \lambda y, z \rangle|_{W \times W} &= \langle x + \lambda y, z \rangle \\ &= \langle x, z \rangle + \lambda \langle y, z \rangle \\ &= \langle x, z \rangle|_{W \times W} + \lambda \langle y, z \rangle|_{W \times W}\end{aligned}$$

gives us linearity in the first argument. The other properties follow similarly. □

Remark. As long as context is clear, then we denote $\langle \cdot, \cdot \rangle|_{W \times W}$ simply by $\langle \cdot, \cdot \rangle$.

Proposition 0.8. *Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional inner-product space. Then $(V, \langle \cdot, \cdot \rangle)$ is unitarily equivalent to $(\mathbb{C}^n, \langle \cdot, \cdot \rangle_e\rangle$, where $\langle \cdot, \cdot \rangle_e$ is the standard Euclidean inner-product on $\mathbb{C}^n$. In particular, $(V, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space.*

Proof. Let $\{v_1, \dots, v_n\}$ be a basis for V. By applying the Gram-Schmidt process to $\{v_1, \dots, v_n\}$, we can get an orthonormal basis, say $\{u_1, \dots, u_n\}$, of V. Let $\varphi \colon V \to \mathbb{C}^n$ be the unique linear isomorphism such that

$$\varphi(u_i) = e_i$$

where e_i is the standard ith coordinate vector in $\mathbb{C}^n$ for all $1 \leq i \leq n$. Then φ is a unitary equivalence. Indeed, it is an isomorphism since it restricts to a bijection on basis sets. Moreover we have

$$\langle u_i, u_j \rangle = \langle \varphi(u_i), \varphi(u_j) \rangle_{\mathrm{e}} = \langle e_i, e_j \rangle_{\mathrm{e}}$$

for all $1 \leq i, j \leq n$. This implies

$$\langle x, y \rangle = \langle \varphi(x), \varphi(y) \rangle_{\mathrm{e}}$$

for all $x, y \in V$. □

Proposition 0.9. *Let $\mathcal{V}$ be an inner-product space over $\mathbb{C}$ and let $\mathcal{W}$ be a finite dimensional subspace of $\mathcal{V}$. Then $\mathcal{W}$ is a closed.*

Proof. Let $\{w_1, \dots, w_k\}$ be an orthonormal basis for $\mathcal{W}$ and let (x_n) be a sequence of vectors in $\mathcal{W}$ such that $x_n \to x$ where $x \in \mathcal{V}$. For each $n \in \mathbb{N}$, express x_n in terms of the basis $\{w_1, \dots, w_k\}$ say as

$$x_n = \lambda_{1n} w_1 + \cdots + \lambda_{kn} w_k,$$

where $\lambda_{1n}, \dots, \lambda_{kn} \in \mathbb{C}$. Since $x_n \to x$ as $n \to \infty$, the sequence (x_n) is a Cauchy sequence. This implies the sequence $(\lambda_{jn})_{n \in \mathbb{N}}$ of complex numbers is a Cauchy sequence, for each $1 \leq j \leq k$. Indeed, letting $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies $\|x_n - x_m\| < \varepsilon$. Then $n, m \geq N$ implies

$$\begin{aligned}|\lambda_{jn} - \lambda_{jm}| &\leq |\lambda_{1n} - \lambda_{1m}| + \cdots + |\lambda_{kn} - \lambda_{km}| \\ &= \|(\lambda_{1n} - \lambda_{1m}) w_1 + \cdots + (\lambda_{kn} - \lambda_{km}) w_k\| \\ &= \|x_n - x_m\| \\ &< \varepsilon\end{aligned}$$

for each $1 \leq j \leq k$. Now since $\mathbb{C}$ is complete, we must have $\lambda_{jn} \to \lambda_j$ as $n \to \infty$ for some $\lambda_j \in \mathbb{C}$ for all $1 \leq j \leq n$. In particular, we have

$$\begin{aligned} x &= \lim_{n\to\infty} x_n \\ &= \lim_{n\to\infty} (\lambda_{1n} w_1 + \cdots + \lambda_{kn} w_k) \\ &= \lim_{n\to\infty} (\lambda_{1n} w_1) + \cdots + \lim_{n\to\infty} (\lambda_{kn} w_k) \\ &= \lambda_1 w_1 + \cdots + \lambda_k w_k, \end{aligned}$$

and this implies $x \in \mathcal{W}$, which implies $\mathcal{W}$ is closed. □

Lemma 0.1. *Let (x_n) be a Cauchy sequence in $\mathcal{V}$. Then (x_n) is bounded.*

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|x_n - x_m\| < \varepsilon$. Thus, fixing $m \in \mathbb{N}$, we see that $n \geq N$ implies

$$\|x_n\| < \|x_m\| + \varepsilon.$$

Now we let

$$M = \max\{\|x_1\|, \|x_2\|, \dots, \|x_m\| + \varepsilon\}.$$

Then M is a bound for (x_n). □

Proposition 0.10. *Let (x_n) and (y_n) be Cauchy sequences of vectors in $\mathcal{V}$. Then $(\langle x_n, y_n\rangle)$ is a Cauchy sequence of complex numbers.*

Proof. Let $\varepsilon > 0$. Choose M_x and M_y such that $\|x_n\| < M_x$ and $\|y_n\| < M_y$ for all $n \in \mathbb{N}$. We can do this by Lemma (0.1). Next, choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies $\|x_n - x_m\| < \frac{\varepsilon}{2M_y}$ and $\|y_n - y_m\| < \frac{\varepsilon}{2M_x}$. Then $n, m \geq M$ implies

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| &= |\langle x_n, y_n\rangle - \langle x_m, y_n\rangle + \langle x_m, y_n\rangle - \langle x_m, y_m\rangle| \\ &= |\langle x_n - x_m, y_n\rangle + \langle x_m, y_n - y_m\rangle| \\ &\leq |\langle x_n - x_m, y_n\rangle| + |\langle x_m, y_n - y_m\rangle| \\ &\leq \|x_n - x_m\|\|y_n\| + \|x_m\|\|y_n - y_m\| \\ &\leq \|x_n - x_m\| M_y + M_x \|y_n - y_m\| \\ &< \varepsilon. \end{aligned}$$

This implies $(\langle x_n, y_n\rangle)$ is a Cauchy sequence of complex numbers in $\mathbb{C}$. The latter statement in the proposition follows from the fact that $\mathbb{C}$ is complete. □

Homework 2, Problem 5

Proposition 0.11. *Let $\mathcal{V}$ be an inner-product space, let $\mathcal{A}$ be a subspace of $\mathcal{V}$, let $x \in \mathcal{V}$, and let $\lambda \in \mathbb{C}$. Then*

$$d(\lambda x, \mathcal{A}) = |\lambda| d(x, \mathcal{A}).$$

Proof. Choose a sequence (y_n) of elements in $\mathcal{A}$ such that

$$\|x - y_n\| < d(x, \mathcal{A}) + \frac{1}{|\lambda n|}$$

for all $n \in \mathbb{N}$. Then since $\mathcal{A}$ is a subspace, we have

$$\begin{aligned} d(\lambda x, \mathcal{A}) &\leq \|\lambda x - \lambda y_n\| \\ &= |\lambda| \|x - y_n\| \\ &< |\lambda| \left(d(x, \mathcal{A}) + \frac{1}{|\lambda n|} \right) \\ &= |\lambda| d(x, \mathcal{A}) + \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$. In particular, this implies $d(\lambda x, \mathcal{A}) \leq |\lambda| d(x, \mathcal{A})$.

Conversely, choose a sequence (z_n) of elements in $\mathcal{A}$ such that

$$\|\lambda x - z_n\| < d(\lambda x, \mathcal{A}) + \frac{1}{n}$$

Then since $\mathcal{A}$ is a subspace, we have

$$\begin{aligned} |\lambda| d(x, \mathcal{A}) &\leq |\lambda| \|x - z_n / |\lambda|\| \\ &= \|\lambda x - z_n\| \\ &< d(\lambda x, \mathcal{A}) + \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$. In particular, this implies $|\lambda| d(x, \mathcal{A}) \leq d(\lambda x, \mathcal{A})$. $\square$

Proposition 0.12. *Let $\mathcal{V}$ be an inner-product space, let $\mathcal{A}$ be a subspace of $\mathcal{V}$, and let $x, y \in \mathcal{V}$. Then*

$$d(x + y, \mathcal{A}) \leq d(x, \mathcal{A}) + d(y, \mathcal{A}).$$

Proof. Choose a sequences (w_n) and (z_n) of elements in $\mathcal{A}$ such that

$$\|x - w_n\| < d(x, \mathcal{A}) + \frac{1}{2n} \quad \text{and} \quad \|y - z_n\| < d(y, \mathcal{A}) + \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Then since $\mathcal{A}$ is a subspace, we have

$$\begin{aligned} d(x + y, \mathcal{A}) &\leq \|(x + y) - (w_n + z_n)\| \\ &\leq \|x - w_n\| + \|y - z_n\| \\ &< d(x, \mathcal{A}) + \frac{1}{2n} + d(y, \mathcal{A}) + \frac{1}{2n} \\ &= d(x, \mathcal{A}) + d(y, \mathcal{A}) + \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$. In particular, this implies $d(x + y, \mathcal{A}) \leq d(x, \mathcal{A}) + d(y, \mathcal{A})$. $\square$