PDG Algebras and Modules

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January 12, 2021

Contents

1 Introduction

1.1 Notation and Conventions

Unless otherwise specified, let K be a field and let $(R, \mathfrak{m})$ be a local Noetherian ring.

1.1.1 Category Theory

In this document, we consider the following categories:

- The category of all sets and functions, denoted **Set**;
- The category of all rings and ring homomorphisms, denoted **Ring**;
- The category of all R-modules and R-linear maps, denoted $\mathbf{Mod}_R$;
- The category of all graded R-modules and graded R-linear maps, denoted $\mathbf{Grad}_R$;
- The category of all R-algebras R-algebra homorphisms, denoted $\mathbf{Alg}_R$;
- The category of all R-complexes and chain maps, denoted $\mathbf{Comp}_R$;
- The category of all R-complexes and homotopy classes of chain maps, denoted $\mathbf{HComp}_R$
- The category of all DG R-algebras DG algebra homomorphisms, denoted $\mathbf{DG}_R$.

2 Basic Definitions

2.1 PDG R-Algebras

Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ be a chain map. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i \star_\mu b_i.$$

If μ is understood from context, then we also tend to drop μ from the subscript in $\star_\mu$, or even drop $\star$ altogether and simply write

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i b_i.$$

Note that μ being a chain map implies it is a **graded-multiplication** which satisfies **Leibniz law**. Being a graded-multiplication means μ is an R-bilinear map which respects the grading. In particular, if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. Satisfying Leibniz law here means

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|}a\mathrm{d}(b)$$

for all $a, b \in A$ with a homogeneous We can also impose other conditions on μ as follows:

1. We say μ is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.
2. We say μ is **graded-commutative** if
$$ab = (-1)^{|a||b|}ba$$
for all homogeneous $a, b \in A$.
3. We say μ is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all homogeneous $a \in A$ where $|a|$ is odd.

4. We say μ is **unital** if there exists $1 \in A$ such that

$$a1 = a = 1a$$

for all $a \in A$.

The triple (A, d, μ) is called a **differential graded R-algebra** (or **DG R-algebra**) if μ satisfies conditions 1-4. If (A, d, μ) only satisfies conditions 2-4, then it is called a **partial differential graded R-algebra** (or **PDG R-algebra**). To clean notation in what follows, we will often refer to a PDG R-algebra (A, d, μ) via its underlying graded R-module A. In particular, if we write "let A be a PDG R-algebra" without specifying its differential or multiplication operations, then it will be understood that it's differential is denoted d_A and its multiplication is denoted μ_A.

2.1.1 Morphisms of PDG R-algebras

Definition 2.1. Let A and B be two PDG R-algebra and let $f\colon A \to B$ be a function. We say f is a **morphism** of PDG R-algebras (or simply morphism if context is clear) if f is a chain map which satisfies the following two properties

1. it respects the identity element, that is, $f(1) = 1$;
2. it respects multiplication, that is, $f(a_1 a_2) = f(a_1) f(a_2)$ for all $a_1, a_2 \in A$.

Note that property 2 is equivalent to the following diagram being commutative

$$\begin{tikzcd} A \otimes_R A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes_R B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Property 1 can be interpretted in terms of a diagram as well. The collection of all PDG R-algebras algebras together with their morphisms forms a category, which we denote by $\mathbf{PDG}_R$.

2.2 PDG A-Modules

Definition 2.2. Let A be a PDG R-algebra.

1. A **left partial differential graded A-module** (or **left PDG A-module**) is a triple $(X, \mathrm{d}_X, \mu_{A,X})$ where (X, d_X) is an R-complex and where $\mu_{A,X}\colon A \otimes_R X \to X$ is a chain map, called the **left scalar-multiplication** map, which satisfies $1x = x$ for all $x \in X$.

2. A **right partial differential graded A-module** (or **right PDG A-module**) is a triple $(X, \mathrm{d}_X, \mu_{X,A})$ where (X, d_X) is an R-complex and where $\mu_{X,A}\colon X \otimes_R A \to X$ is a chain map called the **right scalar-multiplication** map which satisfies $x1 = x$ for all $x \in X$.

3. A **two-sided partial differential graded A-module** (or **two-sided PDG A-module**) is a quadruple $(X, \mathrm{d}_X, \mu_{A,X}, \mu_{X,A})$ where $(X, \mathrm{d}_X, \mu_{A,X})$ is a left partial differential graded A-module and where $(X, \mathrm{d}_X, \mu_{X,A})$ is a right partial differential graded A-module. In other words, it is an R-complex (X, d_X) which is equppied with both a left and right scalar-multiplication map.

Here again we are using the convention that the image of a tensor $\sum_{i=1}^n a_i \otimes x_i$ in $A \otimes_R X$ under the scalar-multiplication map $\mu_{A,X}$ is denoted by

$$\mu_{A,X}\left(\sum_{i=1}^n a_i \otimes x_i \right) = \sum_{i=1}^n a_i \star_{\mu_{A,X}} x_i = \sum_{i=1}^n a_i x_i.$$

Unless otherwise specificed, we fix A to be a PDG R-algebra. The theory of left/right PDG A-modules and the theory of two-sided PDG A-modules are completely analagous for the most part, though there are some notable differences. When these differences arise, we will explicitly mention them, but for now we will mostly focus on the theory of left PDG A-modules. Thus, if we write "let X be a PDG A-module", then it will be understood that A is a PDG R-algebra, that X is a *left* PDG A-module, that it's differential is denoted d_X, and its scalar-multiplication is denoted $\mu_{A,X}$. In fact, if A is understood from context, then we simplify this notation even further by writing μ_X rather than $\mu_{A,X}$.

Note that μ_X being a chain map implies it is satisfies **Leibniz law**, which in this context says

$$\mathrm{d}_X(ax) = \mathrm{d}_A(a)x + (-1)^{|a|} a \mathrm{d}_X(x)$$

for all homogeneous $a \in A$ and $x \in X$. Note also that we do not require μ_X to be associative in order for X to be a PDG A-module, that is, we do not require here the identity

$$(ab)x = a(bx)$$

to hold for all $a, b \in A$ and $x \in X$.

2.2.1 A-linear maps of PDG A-modules

Definition 2.3. Let X and Y be two PDG A-modules and let $\varphi \colon X \to Y$ be a function. We say φ is an **A-linear map** between them if it is a chain map which satisfies

$$\varphi(ax) = a\varphi(x)$$

for all $a \in A$ and $x \in X$. The collection of all PDG A-modules together with their A-linear maps forms a category, which we denote by $\mathbf{PMod}_A$.

2.2.2 Submodules

Definition 2.4. Let X and Y be two PDG A-modules. We say X is a **PDG A-submodule** of Y if $X \subseteq Y$. A PDG A-submodule of A is called a **PDG ideal** of A. Given any collection $\{x_\lambda\}_{\lambda\in\Lambda}$ of elements of X, we denote by $\langle\langle x_\lambda \rangle\rangle_{\lambda\in\Lambda}$ to be the smallest PDG A-submodule of M which contains $\{x_\lambda\}_{\lambda\in\Lambda}$. We denote by $\langle x_\lambda \rangle_{\lambda\in\Lambda}$ to be the set of all A-linear combinations of $\{x_\lambda\}_{\lambda\in\Lambda}$.

Proposition 2.1. *Let X be a PDG A-module and let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of X. Then*

$$\langle\langle x_\lambda \rangle\rangle_{\lambda\in\Lambda} = \langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle_{\lambda\in\Lambda}$$

Proof. To clean notation in what follows, we drop the "$\lambda \in \Lambda$" from the subscript of our bracket notation. Since $\langle\langle x_\lambda \rangle\rangle$ is the smallest PDG A-submodule of X which contains $\{x_\lambda\}$, we must have $\mathrm{d}_X(x_\lambda) \in \langle\langle x_\lambda \rangle\rangle$ for all $\lambda \in \Lambda$. Furthermore, we must have all A-linear combinations of $\{x_\lambda, \mathrm{d}_X(x_\lambda)\}$ belong to $\langle\langle x_\lambda \rangle\rangle$ as well. Thus

$$\langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle \subseteq \langle\langle x_\lambda \rangle\rangle.$$

For the reverse direction, notice that Leibniz law ensures that $\langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle$ is d_X-stable. Indeed, if

$$\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j \mathrm{d}_X(x_{\lambda_j}),$$

is a finite A-linear combination of elements in $\{x_\lambda, \mathrm{d}_X(x_\lambda)\}$ where each a_i and b_j are homogeneous, then note that

$$\begin{aligned}
\mathrm{d}_X\left(\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j \mathrm{d}_X(x_{\lambda_j})\right) &= \sum_{i=1}^m \mathrm{d}_X(a_i x_{\lambda_i}) + \sum_{j=1}^n \mathrm{d}_X(b_j \mathrm{d}(x_{\lambda_j})) \\
&= \sum_{i=1}^m \left(\mathrm{d}_A(a_i)x_{\lambda_i} + (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i})\right) + \sum_{j=1}^n \left(\mathrm{d}_A(b_j)x_{\lambda_j} + (-1)^{|b_j|} b_j \mathrm{d}_X^2(x_{\lambda_j})\right) \\
&= \sum_{i=1}^m \mathrm{d}_A(a_i)x_{\lambda_i} + \sum_{i=1}^m (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i}) + \sum_{j=1}^n \mathrm{d}_A(b_j)x_{\lambda_j} \\
&\in \langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle.
\end{aligned}$$

In particular, we see that $\langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle$ is a PDG A-submodule of X which contains $\{x_\lambda\}$. Since $\langle\langle x_\lambda \rangle\rangle$ is the *smallest* PDG A-submodule of X which contains $\{x_\lambda\}$, it follows that

$$\langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle \supseteq \langle\langle x_\lambda \rangle\rangle.$$

□

Warning: In the category of R-modules, we have the concept of annihilators. In particular, suppose M is an R-module and let $u \in M$. We define the **annihilator** with respect to u to be the subset of R given by

$$0 : u = \{r \in R \mid ru = 0\}.$$

In fact, $0 : u$ is in an ideal of R, but we need the associative law to get this: if $r \in R$ and $x \in 0 : u$, then $(rx)u = r(xu) = 0$ implies $rx \in 0 : u$.

Now let us consider the case where X is a PDG A-module and let $x \in X$. We can define the annihilator $0 : x$ with respect to x as a subset of A as before:

$$0 : x = \{a \in A \mid ax = 0\},$$

however this time the set $0 : x$ need not be a PDG ideal of A. On the other hand, if $u \in \operatorname{Assoc} M$, where

$$\operatorname{Assoc} M = \{u \in M \mid [a, b, u] = 0 \text{ for all } a, b \in A\},$$

then there are no issues with the proof above, so $0 : u$ is an ideal of R in this case.

2.2.3 Hom

Let M and N be two PDG A-modules. We denote by $\operatorname{Hom}_A(M, N)$ to be the set of all A-linear maps from M to N. The set $\operatorname{Hom}_A(M, N)$ as the structure of an abelian group via pointwise addition of A-linear maps from M to N. On the other hand, suppose we define a scalar "action" on $\operatorname{Hom}_A(M, N)$ by

$$(a \cdot \varphi)(u) = \varphi(au)$$

for all $a \in A$, $\varphi \in \operatorname{Hom}_A(M, N)$, and $u \in M$. Then this "action" does not necessarily give $\operatorname{Hom}_A(M, N)$ the structure of an R-module, since if $a \in A_i$, $b \in A_j$, and $\varphi \in \operatorname{Hom}_A(M, N)$, then

$$\begin{aligned}
((ab) \cdot \varphi)(u) &= \varphi((ab)u) \\
&= \varphi((-1)^{i+j}(ba)u) \\
&= (-1)^{i+j}\varphi((ba)u) \\
&= (-1)^{i+j}\varphi(b(au) + (-1)^{i+j}[b, a, u]) \\
&= (-1)^{i+j}(b \cdot \varphi)(au) + (-1)^{i+j}[b, a, \varphi(u)] \\
&= (-1)^{i+j}(a \cdot (b \cdot \varphi))(u) + (-1)^{i+j}[b, a, \varphi(u)]
\end{aligned}$$

for all $u \in M$. Thus one needs commutativity and associativity in order to conclude that $(ab) \cdot \varphi = a \cdot (b \cdot \varphi)$.

2.3 $\mathbf{PMod}_A$ is an Abelian Category

Throughout the rest of this subsection, we fix a PDG R-algebra A. We would like to talk about the concept of an exact sequence in $\mathbf{PMod}_A$. For this, we just need to check that $\mathbf{PMod}_A$ is abelian category. First let us check that it is a pre-additive category.

2.3.1 Kernels

Proposition 2.2. *Let $\varphi\colon X \to Y$ be a morphism of PDG A-modules and let $K = \ker\varphi$. Then K has the structure of a PDG A-submodule of X, where $\mathrm{d}_K = \mathrm{d}|_K$ and where $\mu_K = \mu_X|_{A\otimes_R K}$.*

Proof. Since both d_K and μ_K are restrictions, we just need to check that d_K and μ_K land in K. Indeed, then all of the properties needed in order for K to be a PDG A-submodule of X will be inherited from X. First we show d_K lands in K. Let $x \in K$. Then

$$\begin{aligned}
\varphi\mathrm{d}_K(x) &= \mathrm{d}_K\varphi(x) \\
&= \mathrm{d}_K(0) \\
&= 0
\end{aligned}$$

implies $\mathrm{d}_K(x) \in K$. It follows that d_K lands in K. Now we show μ_K lands in K. Let $a \otimes x$ be an elementary tensor in $A \otimes_R K$. Then

$$\begin{aligned}
\varphi\mu_K(a \otimes x) &= \varphi(ax) \\
&= a\varphi(x) \\
&= a \cdot 0 \\
&= 0.
\end{aligned}$$

It follows that μ_K lands in K. □

2.3.2 Images

Proposition 2.3. *Let* $\varphi\colon (A, \mathrm{d}, \mu) \to (A', \mathrm{d}', \mu')$ *be a morphism of R-complex algebras. Then* $(\operatorname{im}\varphi, \widetilde{\mathrm{d}}', \widetilde{\mu}')$ *is an R-complex algebra, where* $\widetilde{\mathrm{d}}' = \mathrm{d}'|_{\ker\varphi}$ *and* $\widetilde{\mu}' = \mu'|_{\operatorname{im}\varphi\otimes_R \operatorname{im}\varphi}$.

Proof. We just need to check that $\widetilde{\mathrm{d}}'$ and $\widetilde{\mu}'$ land in $\ker\varphi$. Then it will follow that $(\ker\varphi, \widetilde{\mathrm{d}}, \widetilde{\mu})$ is an R-complex algebra since it will inherit the properties needed to be an R-complex algebra from (A, d, μ). First we show $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a) \in \operatorname{im}\varphi$. Then

$$\begin{aligned} \mathrm{d}'(\varphi(a)) &= \mathrm{d}'\varphi(a) \\ &= \varphi\mathrm{d}(a) \\ &= \varphi(\mathrm{d}(a)). \end{aligned}$$

It follows that $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Now we show $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a)\otimes\varphi(b)$ be an elementary tensor in $\operatorname{im}\varphi\otimes_R\operatorname{im}\varphi$. Then

$$\begin{aligned} \mu((\varphi(a)\otimes\varphi(b)) &= \varphi(a)\star\varphi(b) \\ &= \varphi(a\star b) \\ &= \varphi(\mu(a\otimes b)). \end{aligned}$$

It follows that $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. □

2.3.3 Cokernels

As we've seen, both kernels and images exist in $\mathbf{CompAlg}_R$. The problem however is that cokernels do not necessarily exist in $\mathbf{CompAlg}_R$. To see what goes wrong, suppose $\varphi\colon (A, \mathrm{d}, \mu) \to (A', \mathrm{d}', \mu')$ be a morphism of R-complex algebras. A naive attempt at defining the cokernel of φ would go as follows: first we take the cokernel of the underlying R-complexes, namely $(\overline{A'}, \overline{\mathrm{d}'})$ where $\overline{A'} = A'/\operatorname{im}\varphi$ and $\overline{\mathrm{d}'}$ is defined by $\overline{\mathrm{d}'}(\overline{a'}) = \overline{\mathrm{d}'(a')}$ for all $\overline{a'} \in \overline{A'}$. It is straightforward to check that $\overline{\mathrm{d}'}$ is well-defined and gives $\overline{A'}$ the structure of an R-complex. Next we define multiplication $\overline{\mu'}\colon \overline{A'}\otimes_R\overline{A'}\to\overline{A'}$ by

$$\overline{\mu'}(\overline{a'}\otimes\overline{b'}) = \overline{a'\star_{\mu'} b'} \tag{1}$$

for all elementary tensors $\overline{a'}\otimes\overline{b'}$ in $\overline{A'}\otimes_R\overline{A'}$ and extending $\overline{\mu'}$ everywhere else R-linearly. Unfortunately, upon further inspection, we see that (**??**) is note well-defined. Indeed, if $a' + \varphi(a)$ is another representative of the coset $\overline{a'}$ and $b' + \varphi(b)$ is another representative of the coset $\overline{b'}$, then we have

$$\begin{aligned} \overline{\mu'}(\overline{a'+\varphi(a)}\otimes\overline{b'+\varphi(b)}) &= \overline{(a'+\varphi(a))\star(b'+\varphi(b))} \\ &= \overline{a'\star b'} + \overline{a'\star\varphi(b)} + \overline{\varphi(a)\star b'} + \overline{\varphi(a)\star\varphi(b)} \\ &= \overline{a'\star b'} + \overline{a'\star\varphi(b)} + \overline{\varphi(a)\star b'} + \overline{\varphi(a\star b)} \\ &= \overline{a'\star b'} + \overline{a'\star\varphi(b)} + \overline{\varphi(a)\star b'}. \end{aligned}$$

In particular, (**??**) is well-defined if and only if $\operatorname{im}\varphi$ is an ideal of A'.

2.4 Associator Functor

Let X be a PDG A-module. Given $a, b \in A$ and $x \in X$, we define the **associator** of the triple (a, b, x), denoted $[a, b, x]$, by the formula

$$[a, b, x] = (ab)x - a(bx). \tag{2}$$

More generally, let $\alpha_{A,A,X}\colon (A\otimes_R A)\otimes_R X \to A\otimes_R(A\otimes_R X)$ denote the unique chain map defined on elementary tensors by

$$(a\otimes b)\otimes x \mapsto a\otimes(b\otimes x).$$

We define the **associator chain map** with respect to μ_X to be chain map $[\cdot,\cdot,\cdot]_{\mu_X}\colon (A\otimes_R A)\otimes_R X\to X$ defined by

$$[\cdot,\cdot,\cdot]_{\mu_X} := \mu_X(1\otimes\mu_X)\alpha_{A,A,X} - \mu_X(\mu_A\otimes 1).$$

If μ_X is understood from context, then we will simplify our notation by dropping μ_X from the subscript in $[\cdot,\cdot,\cdot]$. Thus, if $(a\otimes b)\otimes x$ is an elementary tensor in $(A\otimes_R A)\otimes_R X$ then the associatior chain map with respect to X maps the elementary tensor $(a\otimes b)\otimes x$ to the associator of the triple (a, b, x):

$$[\cdot,\cdot,\cdot]((a\otimes b)\otimes x) = [a, b, x],$$

where $[a,b,x]$ is as defined above in (2). We define the **associator complex** with respect to μ_X, denoted $[X]_{\mu_X}$, to be the image of $[\cdot,\cdot,\cdot]$, where again we simplify notation by writing $[X]$ instead of $[X]_{\mu_X}$ if μ_X is understood from context. Thus

$$[X] = \mathrm{Span}_R\{[a,b,x] \mid a,b \in A \text{ and } x \in X\}.$$

Since $[\cdot,\cdot,\cdot]$ is a chain map, we see that $[X]$, being the image of $[\cdot,\cdot,\cdot]$, is an R-subcomplex of X. We also see that $[\cdot,\cdot,\cdot]$ is a graded-trilinear map which satisfies Leibniz law, where Leibniz law in this context says

$$\mathrm{d}_X[a,b,x] = [\mathrm{d}_A(a),b,x] + (-1)^{|a|}[a,\mathrm{d}_A(b),x] + (-1)^{|a|+|b|}[a,b,\mathrm{d}_X(x)]. \tag{3}$$

for all homogeneous $a,b \in A$ and $x \in X$.

Now suppose Y is another PDG A-module and $\varphi\colon X \to Y$ is an A-linear map. We obtain an induced chain map of R-complexes $[\varphi]\colon [X] \to [Y]$, where $[\varphi]$ is the unique chain map which satisfies

$$\begin{aligned}[\varphi][a,b,x] &= \varphi((ab)x - a(bx)) \\ &= \varphi((ab)x) - \varphi(a(bx)) \\ &= (ab)\varphi(x) - a\varphi(bx) \\ &= (ab)\varphi(x) - a(b\varphi(x)) \\ &= [a,b,\varphi(x)].\end{aligned}$$

In particular, the map $[\varphi]$ is just the restriction of φ to $[M]$. It is straightforward to check that the assignment $M \mapsto [M]$ and $\varphi \mapsto [\varphi]$ gives rise to a functor from the category of PDG A-modules to the category of R-complex. We call this functor the **associator functor** with respect to A, and we denote this functor by $[\cdot]_{\mu_A}\colon \mathbf{PMod}_A \to \mathbf{Comp}_R$. As usual, we simplify our notation by droping μ_A from $[\cdot]$ when context is clear.

2.4.1 Homology of $[X]$

Let X be a PDG A-module. It is easy to see that μ_X is associative if and only if $[X] = 0$. Given that $[X]$ is an R-complex, we also have a weaker form of associativity:

Definition 2.5. We say μ_X is **homologically associative** if $\mathrm{H}([X]) = 0$. We also say μ_X is **homologically associative** in degree i if $\mathrm{H}_i([X]) = 0$.

Clearly if μ_X is associative, then μ_X is homologically associative. It turns out that the converse is also true if $[X]$ is bounded below and **minimal** in the sense that $\mathrm{d}_X([X]) \subseteq \mathfrak{m}[X]$ where $\mathfrak{m}$ is the maximal ideal in the local ring R.

Proposition 2.4. *Let X be a PDG A-module. Assume that X is minimal and that μ_X is associative in degree i. Then μ_X is associative in degree $i+1$ if and only if μ_X is homologically associative in degree $i+1$. In particular, if $[X]$ is bounded below and minimal. Then μ_X is associative if and only if μ_X is homologically associative.*

Proof. Clearly if μ_X is associative in degree i, then it is homologically associative in degree i. To show the converse, assume for a contradiction that μ_X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume $\mathrm{H}_{i+1}([X]) = 0$ and $[X]_{i+1} \neq 0$. By Nakayama's Lemma, we can find a triple (a,b,x) such that $|a|+|b|+|x| = i+1$ and such that $[a,b,x] \notin \mathfrak{m}[X]_{i+1}$. Since $[X]_i = 0$ by assumption, we have $\mathrm{d}_X[a,b,x] = 0$. Also, since X is minimal, we have $\mathrm{d}_X[X] \subseteq \mathfrak{m}[X]$. Thus $[a,b,x]$ represents a nontrivial element in homology in degree $i+1$. It follows that μ_X is not homologically associative in degree $i+1$, which is a contradiction. □

Note that if A and X are both minimal, then the Leibniz law (3) implies $[X]$ is minimal too. Also our assumption in Proposition (2.4) that $[X]$ is bounded below can clearly be weakened since in the proof we just needed to find an $i \in \mathbb{Z}$ such that $[X]_i \neq 0$ and $[X]_{i-1} = 0$. At the same time, the proof of Proposition (2.4) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition

Definition 2.6. Let X be a PDG A-module and assume that $[X]$ is bounded below. We define the **associative index** of μ_X, denoted $\operatorname{index}\mu_X$, is defined to be the smallest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\operatorname{index}\mu_X = \infty$ if μ_X is associative. We extend this definition to case where $[X]$ is not bounded below by setting $\operatorname{index}\mu_X = -\infty$.

With the associative index of μ_X defined, we see, after analyzing the proof of Proposition (2.4), that if we assume μ_X is not associative then

$$\operatorname{index}\mu_X = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i([X]) \neq 0\}$$

In other words, the associative index of μ_X can be measured homologically.

We can also define an associative index of R-complex. Let us record this definition now:

Definition 2.7. Let X be a PDG A-module. We define the **associative index** of X, denoted index X, to be

$$\operatorname{index} X = \sup\{\operatorname{index} \mu \mid \mu \text{ is a multplication on } X\}.$$

Let I be an ideal of R and let F be the minimal free resolution of R/I over R. We define the **associative index** of R/I, denoted index(R/I), to be the associative index of F.

2.4.2 Stable PDG A-Submodules

The associator functor $[\cdot]\colon \mathbf{PMod}_A \to \mathbf{Mod}_R$ need not be exact. To see what goes wrong, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{4}$$

be a short exact sequence of PDG A-modules. We obtain an induced sequence of R-complexes

$$0 \longrightarrow [X] \xrightarrow{[\varphi]} [Y] \xrightarrow{[\psi]} [Z] \longrightarrow 0 \tag{5}$$

We claim that we have exactness at $[X]$ and $[Z]$. Indeed, this is equivalent to showing $[\varphi]$ is injective and $[\psi]$ is surjective, which follows from the fact that $[\varphi]$ (respectively $[\psi]$) is the restriction of the injective map φ (respectively the surjective map ψ). Let us see what goes wrong when trying to prove exactness at $[Y]$. Let

$$\sum_{i=1}^{n} [a_i, b_i, y_i] \in \ker[\psi] = \ker \psi.$$

Then by exactness of (4), there exists $x \in X$ such that $\varphi(x) = \sum_{i=1}^{n} [a_i, b_i, y_i]$. It is not at all clear however that $x \in [X]$. Indeed, we will see a counterexample to this later on. This leads us to consider the following definition:

Definition 2.8. Let X be a PDG A-submodule of Y. We say X is a **stable** PDG A-submodule of Y if it satisfies $[X] = X \cap [Y]$.

Now it is easy to check that (5) is a short exact sequence of R-complexes if and only if $\varphi(X)$ is a stable PDG A-submodule of Y. Thus if $\varphi(X)$ is a stable PDG A-submodule of Y, then the short exact sequence (5) of R-complexes induces a long exact sequence in homology

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}([Z]) \arrow[dll, rounded corners] \\ & \mathrm{H}_i([X]) \arrow[r] & \mathrm{H}_i([Y]) \arrow[r] & \mathrm{H}_i([Z]) \arrow[dll, rounded corners] \\ & \mathrm{H}_{i-1}([X]) \arrow[r] & \cdots & \end{tikzcd} \tag{6}$$

From this, one concludes immediately the following theorem:

Theorem 2.1. *Suppose X is a PDG A-submodule of Y. Then μ_Y is homologically associative if and only if μ_X and $\mu_{Y/X}$ are homologically associative.*

3 Invariant

Let I be an ideal of R and let F be a free resolution of R/I over R. Choose a multiplication μ_F on F which lifts the multiplication map $R/I \otimes_R R/I \to R$ and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $\mathrm{C}(r)$ is a free resolution of $R/\langle I, x\rangle$ over R. The multiplication μ_F on F induces a multiplication $\mu_{\mathrm{C}(r)}$ on $\mathrm{C}(r)$ as follows: first note that $F \oplus F(-1)$ is the underlying graded R-module of $\mathrm{C}(r)$. Express this graded R-module in the form $F + eF$ where e is an exterior variable of degree -1 so that $\{1, e\}$ is an F-linearly independent set. Thus an element in $F + eF$ can be expressed in the form $\alpha + e\beta$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i, then α and β are homogeneous of degrees i and $i-1$ respectively. With this understood, the multilpication $\mu_{\mathrm{C}(r)}$ is defined on homogeneous elements $\alpha, \beta, \gamma, \delta \in F$ by

$$(\alpha + e\beta)(\gamma + e\delta) = \alpha\gamma + e(\beta\gamma + (-1)^{|\alpha|}\alpha\delta)$$

and extended R-linearly everywhere else. The mapping cone $\mathrm{C}(r)$ inherits a natural *right* PDG F-module structure via restriction of scalars. The reason why it is more naturally viewed as a right PDG F-module rather than a left PDG F-module can be seen in the way the mapping cone differential acts on elements: given $\alpha, \beta \in F$ where α is homogeneous, we have

$$\mathrm{d}_{\mathrm{C}(r)}(\alpha + e\beta) = \mathrm{d}_F(\alpha) + r\beta - e\mathrm{d}_F(\beta).$$

Thus the mapping cone differential behaves as if e is an exterior variable of degree -1.

3.0.1 Associator Complex Corresponding to Mapping Cone

There are two associator complexes to consider. The first one is the associator complex with respect to $\mu_{\mathrm{C}(r),F}$, given by

$$[\mathrm{C}(r)]_{\mu_{\mathrm{C}(r),F}} = \mathrm{Span}_R\{[\alpha + e\beta, \gamma, \delta] \mid \alpha, \beta, \gamma, \delta \in F\}.$$

The second is the associator complex with respect to $\mu_{\mathrm{C}(r)}$, given by

$$[\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}} = \mathrm{Span}_R\{[\alpha + e\beta, \gamma + e\delta, \varepsilon + e\zeta] \mid \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F\}.$$

It turns out that these two associator complexes are the same. Indeed, clearly we have

$$[\mathrm{C}(r)]_{\mu_{\mathrm{C}(r),F}} \subseteq [\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}}.$$

Conversely, first note that a quick calculation gives us

$$\begin{aligned}
[\alpha, \beta, \gamma + e\delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|+|\beta|}e[\alpha, \beta, \delta] \\
[\alpha, \beta + e\gamma, \delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|}e[\alpha, \gamma, \delta] \\
[\alpha + e\beta, \gamma, \delta] &= [\alpha, \gamma, \delta] + e[\beta, \gamma, \delta]
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in F$. Using these identities together with the fact that $e^2 = 0$ and the fact that identity 1 associates with everything, we obtain

$$\begin{aligned}
[\alpha + e\beta, \gamma + e\delta, \varepsilon + e\zeta] &= [\alpha, \gamma, \varepsilon] + e[\beta, \gamma, \varepsilon] + (-1)^{|\alpha||}e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||}e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||}[1 + e\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}[1 + e\alpha, \gamma, \zeta]
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F$. It follows that

$$[\mathrm{C}(r)]_{\mu_{\mathrm{C}(r),F}} \supseteq [\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}}.$$

Thus we are justified in simplifying our notation by dropping either $\mu_{\mathrm{C}(r),F}$ and $\mu_{\mathrm{C}(r)}$ from the subscript and just writing $[\mathrm{C}(r)]$ to denote the common R-complex.

3.0.2 Homethety Map

The homothety map $F \xrightarrow{r} F$ gives rise to a short exact sequence of R-complexes

$$0 \longrightarrow F \xrightarrow{\iota} \mathrm{C}(r) \xrightarrow{\pi} \Sigma F \longrightarrow 0 \tag{7}$$

where $\iota\colon F \to \mathrm{C}(r)$ is the inclusion map and where $\pi\colon \mathrm{C}(r) \to \Sigma F$ is the projection map given by

$$\pi(\alpha + e\beta) = \alpha$$

for all $\alpha, \beta \in F$. In fact, both ι and π are A-linear maps, and so (7) is a short exact sequence of right PDG F-modules. In fact, it is a stable short exact sequence of right PDG F-modules, as the next proposition shows

Proposition 3.1. *With the notation above, F is a stable PDG F-submodule of* $\mathrm{C}(r)$.

Proof. We must check that $[\mathrm{C}(r)] \cap F \subseteq [F]$ since the reverse inclusion is trivial. Suppose $\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in [\mathrm{C}(r)] \cap F$ where $r_i \in R$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ for each $1 \leq i \leq m$. Observe that

$$\begin{aligned}
\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] &= \sum_{i=1}^m r_i([\alpha_i, \gamma_i, \delta_i] + e[\beta_i, \gamma_i, \delta_i]) \\
&= \sum_{i=1}^m r_i[\alpha_i, \gamma_i, \delta_i] + e\sum_{i=1}^m r_i[\beta_i, \gamma_i, \delta_i].
\end{aligned}$$

Since $\sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in F$, it follows that $\sum_{i=1}^{m} r_i[\beta_i, \gamma_i, \delta_i] = 0$. Thus

$$\sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] = \sum_{i=1}^{m} r_i[\alpha_i, \gamma_i, \delta_i] \in [F].$$

Therefore $[\mathrm{C}(r)] \cap F \subseteq [F]$. $\square$

3.0.3 Long Exact Sequence in Homology

Since (7) is a stable short exact sequence of PDG F-modules, we obtain a long exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_i([F]) \xrightarrow{r} \mathrm{H}_i([F]) \longrightarrow \mathrm{H}_i([\mathrm{C}(r)]) \longrightarrow \mathrm{H}_{i-1}([F]) \xrightarrow{r} \mathrm{H}_{i-1}([F]) \longrightarrow \cdots \tag{8}$$

In particular, if $\mathrm{H}_i([\mathrm{C}(r)]) = 0$, then Nakayama's lemma implies $\mathrm{H}_i([F]) = 0$. Thus, we have

Theorem 3.1. *With the notation above, if $\mu_{\mathrm{C}(r)}$ is homologically associative in degree i, then μ_F is homologically associative in degree i. Moreover, we have*

$$\mathrm{index}(\mu_F) = \mathrm{index}(\mu_{\mathrm{C}(r)}).$$

Corollary. *With the notation above, we have*

$$\mathrm{index}(R/\langle I, r\rangle) \geq \mathrm{index}(R/I).$$

4 Example

Let $R = \mathbb{F}_2[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2\rangle$, and let F be the free minimal resolution of R/I over R. The complex F is supported on the simplicial complex drawn below:

Define a multiplication μ on F as follows: in degree 2 we have the multiplication table

$\mu(e_i, e_j)$	e_1	e_2	e_3	e_4	e_5
e_1	0	e_{12}	e_{13}	xe_{14}	$yz^2e_{14} + xe_{45}$
e_2	e_{12}	0	we_{23}	e_{24}	$y^2ze_{23} + we_{35}$
e_3	e_{13}	we_{23}	0	e_{34}	ze_{35}
e_4	xe_{14}	e_{24}	e_{34}	0	ye_{45}
e_5	$yz^2e_{14} + xe_{45}$	$y^2ze_{23} + we_{35}$	ze_{35}	ye_{45}	0

In degree 3 we have the multiplication table

$\mu(e_i, e_{jk})$	e_{12}	e_{13}	e_{14}	e_{23}	e_{24}	e_{34}	e_{35}	e_{45}
e_1	0	0	0	e_{123}	xe_{124}	xe_{134}	$yze_{134} + xe_{345}$	0
e_2	0	we_{123}	e_{124}	0	0	we_{234}	0	$yze_{234} + we_{345}$
e_3	we_{123}	0	e_{134}	0	we_{234}	0	0	ze_{345}
e_4	xe_{124}	xe_{134}	0	e_{234}	0	0	ye_{345}	0
e_5	$y^2ze_{123} + yzwe_{134} + xwe_{345}$	$yz^2e_{134} + xze_{345}$	0	0	$y^2ze_{234} + ywe_{345}$	ze_{345}	0	0

Notice that μ is already not associative since

$$\begin{aligned}
[e_1, e_5, e_2] &= (e_1e_5)e_2 + e_1(e_5e_2) \\
&= (yz^2e_{14} + xe_{45})e_2 + e_1(y^2ze_{23} + we_{35}) \\
&= yz^2e_{124} + xyze_{234} + xwe_{345} + y^2ze_{123} + yzwe_{134} + xwe_{345} \\
&= yz^2e_{124} + xyze_{234} + y^2ze_{123} + yzwe_{134} \\
&= yz\mathrm{d}(e_{1234}).
\end{aligned}$$

In degree 4 we have the multiplication table

$\mu(e_i, e_{jkl})$	e_{123}	e_{124}	e_{134}	e_{234}	e_{345}
e_1	0	0	0	xe_{1234}	0
e_2	0	0	we_{1234}	0	0
e_3	0	we_{1234}	0	0	0
e_4	xe_{1234}	0	0	0	0
e_5	0	y^2ze_{1234}	0	0	0

Now observe that

$$\begin{aligned}
\mathrm{d}[e_1, e_{35}, e_2] &= [x^2, e_{35}, e_2] + [e_1, y^2ze_3 + we_5, e_2] + [e_1, e_{35}, w^2] \\
&= [e_1, y^2ze_3 + we_5, e_2] \\
&= [e_1, y^2ze_3, e_2] + [e_1, we_5, e_2] \\
&= w[e_1, e_5, e_2],
\end{aligned}$$

and hence $[e_1, e_{35}, e_2] = yzwe_{1234}$. A similar calculation give us $[e_1, e_{45}, e_2] = x[e_1, e_5, e_2]$, and hence $[e_1, e_{35}, e_2] = xyze_{1234}$. One can show that

$$\begin{aligned}
[F]_3 &= R[e_1, e_5, e_2] \\
&= \langle yz \rangle \mathrm{d}(e_{1234})
\end{aligned}$$

Similarly

$$\begin{aligned}
[F]_4 &= R[e_1, e_{35}, e_2] + R[e_1, e_{45}, e_2] \\
&= \langle yzw, xyz \rangle e_{1234}
\end{aligned}$$

Let T_1 be the Taylor subalgebra of F generated by x^2, w^2, zw, xy, and let T_2 be the Taylor subalgebra of F generated by zw, xy, y^2z^2. If we view F as a left PDG T_1-module in the natural way, then T_1 is not a stable PDG T_1-submodule of F since $[e_1, e_5, e_2] \in T_1 \cap [F]$ and $[e_1, e_5, e_2] \notin [T_1]$. On the other hand, if we view F as a left PDG T_2-module in the natural way, then T_2 is a stable PDG T_2-submodule of F since

$$T_2 \cap [F] = 0 = [T_2].$$

We also have

$$\begin{aligned}
[e_1, e_5, e_2] &= (e_1e_5)e_2 + e_1(e_5e_2) \\
&= (yz^2e_{14} + xe_{45})e_2 + e_1(y^2ze_{23} + we_{35}) \\
&= yz^2e_{124} + xyze_{234} + xwe_{345} + y^2ze_{123} + yzwe_{134} + xwe_{345} \\
&= yz^2e_{124} + xyze_{234} + y^2ze_{123} + yzwe_{134} \\
&= yz\mathrm{d}(e_{1234}).
\end{aligned}$$

5 Grobner Basis Computations