Uniqueness of Measure Extensions

Uniqueness of Extensions when Target Space is Hausdorff

Proposition 0.1. *Let X be a topological space and let $f\colon A \to Y$ be a continuous function from a dense subspace A of X to a Hausdorff space Y. If there exists a continuous extension of f to all of X, then it must be unique. In other words, suppose $\widetilde{f}_1\colon X \to Y$ and $\widetilde{f}_2\colon X \to Y$ are continuous functions such that*

$$\widetilde{f}_1|_A = f = \widetilde{f}_2|_A.$$

Then $\widetilde{f}_1 = \widetilde{f}_2$.

Proof. To prove uniqueness, assume for a contradiction that $\widetilde{f}_1\colon X \to Y$ and $\widetilde{f}_2\colon X \to Y$ are two continuous extensions of f such that $\widetilde{f}_1 \neq \widetilde{f}_2$. Choose $x \in X$ such that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Since Y is Hausdorff, we may choose open neighborhoods V_1 and V_2 of $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ respectively such that $V_1 \cap V_2 = \emptyset$. Then $\widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ is an open neighborhood of x, and so it must have a nonempty intersection with A. Choose $a \in A \cap \widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$. Then

$$\begin{aligned} f(a) &= \widetilde{f}_1(a) \\ &\in V_1. \end{aligned}$$

Similarly,

$$\begin{aligned} f(a) &= \widetilde{f}_2(a) \\ &\in V_2. \end{aligned}$$

Thus $f(a) \in V_1 \cap V_2$, which is a contradiction since V_1 and V_2 were chosen to disjoint from one another. □

Continuity of Finite Measure

Lemma 0.1. *Let $\mathcal{A}$ be an algebra and let μ be a measure on $\sigma(\mathcal{A})$. Then*

$$(\mu|_{\mathcal{A}})^*(A) \geq \mu(A)$$

for all $A \in \sigma(\mathcal{A})$.

Proof. Let $A \in \sigma(\mathcal{A})$. Then

$$\begin{aligned} (\mu|_{\mathcal{A}})^*(A) &= \inf\left\{ \sum_{n=1}^{\infty} (\mu|_{\mathcal{A}})(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \\ &= \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \\ &\geq \inf\left\{ \mu\left(\bigcup_{n=1}^{\infty} E_n \right) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \\ &\geq \mu(A), \end{aligned}$$

where we used countable subadditivity of μ to get from the second line to the third line, and where we used monotonicity of μ to get from the third line to the fourth line. □

Proposition 0.2. *Let $\mathcal{A}$ be an algebra and let μ be a finite measure on $\sigma(\mathcal{A})$. Then μ is Lipschitz continuous with respect to $\mathrm{d}_{\mu|_{\mathcal{A}}}$.*

Proof. Let $A, B \in \sigma(\mathcal{A})$. Assume without loss of generality that $\mu(A) \geq \mu(B)$. Then

$$\begin{aligned}
\mu(A) - \mu(B) &\leq \mu(A \backslash B) \\
&\leq \mu((A \backslash B) \cup (B \backslash A)) \\
&= \mu(A \Delta B) \\
&\leq (\mu|_{\mathcal{A}})^*(A \Delta B) \\
&= \mathrm{d}_{\mu|_{\mathcal{A}}}(A, B),
\end{aligned}$$

where we used the fact that μ is finite in the first line. $\square$

Uniqueness of Extension for Measures

Proposition 0.3. *Let μ and ν be two finite measures defined on $\sigma(\mathcal{A})$ which coincide on $\mathcal{A}$. Then $\mu = \nu$.*

Proof. We first note that $\mathrm{d}_{\mu|_{\mathcal{A}}} = \mathrm{d}_{\nu|_{\mathcal{A}}}$ since μ and ν agree on $\mathcal{A}$. Indeed, let $A, B \in \sigma(\mathcal{A})$. Then we have

$$\begin{aligned}
\mathrm{d}_{\mu|_{\mathcal{A}}}(A, B) &= (\mu|_{\mathcal{A}})^*(A \Delta B) \\
&= \inf \left\{ \sum_{n=1}^{\infty} (\mu|_{\mathcal{A}})(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A \Delta B \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \\
&= \inf \left\{ \sum_{n=1}^{\infty} (\nu|_{\mathcal{A}})(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A \Delta B \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \\
&= (\nu|_{\mathcal{A}})^*(A \Delta B) \\
&= \mathrm{d}_{\nu|_{\mathcal{A}}}(A, B).
\end{aligned}$$

Therefore $\mathrm{d}_{\mu|_{\mathcal{A}}}$ and $\mathrm{d}_{\nu|_{\mathcal{A}}}$ induce a common topology on $\sigma(\mathcal{A})$. Both $\mu \colon \sigma(\mathcal{A}) \to [0, \infty]$ and $\nu \colon \sigma(\mathcal{A}) \to [0, \infty]$ are continuous extensions of $\mu|_{\mathcal{A}} = \nu|_{\mathcal{A}}$ with respect to this common topology by Proposition (0.2). Since $[0, \infty]$ is Hausdorff and since $\mathcal{A}$ is dense in $\sigma(\mathcal{A})$ with respect to this common topology, it follows from Proposition (0.1) that $\mu = \nu$. $\square$