# Linear Analysis Homework 9

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Throughout this homework, let $\mathcal{H}$ be a separable Hilbert space.

## Problem 1

**Proposition 0.1.** *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a compact positive self-adjoint operator. Then $T = |T|$, and consequently the eigenvalues of $T$ coincide with the singular values of $T$.*

*Proof.* Choose an orthonormal eigenbasis $(e_n)$ of $T$ with $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$ (this exists since $T$ is compact and self-adjoint). Then $(e_n)$ is an orthonormal basis consisting of eigenvectors of $T^2 = T^*T$ with $T^2 e_n = \lambda_n^2 e_n$ for all $n \in \mathbb{N}$. Then since $\lambda_n \geq 0$ for all $n \in \mathbb{N}$ (since $T$ is positive and self-adjoint), we have

$$\begin{aligned} |T|x &= \sum_{n=1}^{\infty} s_n \langle x, e_n \rangle e_n \\ &= \sum_{n=1}^{\infty} \sqrt{\lambda_n^2} \langle x, e_n \rangle e_n \\ &= \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n \\ &= Tx \end{aligned}$$

for all $x \in \mathcal{H}$. It follows that $T = |T|$, and consequently $s_n = \lambda_n$ for all $n \in \mathbb{N}$. □

## Problem 2

**Proposition 0.2.** *Let $(e_n)$ be an orthonormal basis for $\mathcal{H}$. Define $T\colon \mathcal{H} \to \mathcal{H}$ by*

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \langle x, e_n \rangle e_n.$$

*for all $x \in \mathcal{H}$. Then $T\colon \mathcal{H} \to \mathcal{H}$ is compact but not Hilbert-Schmidt.*

*Remark.* For this problem, I decided to prove this in an arbitrary separable Hilbert space than just $\ell^2(\mathbb{N})$.

*Proof.* We first show $T$ is compact. For each $k \in \mathbb{N}$, define $T_k\colon \mathcal{H} \to \mathcal{H}$ by

$$T_k(x) = \sum_{n=1}^{k} \frac{1}{\sqrt{n}} \langle x, e_n \rangle e_n$$

for all $x \in \mathcal{H}$. First note that for each $k \in \mathbb{N}$, the operator $T_k$ is bounded and has finite rank, and hence must be compact. Moreover, we have $\|T - T_k\| \to 0$ as $k \to \infty$. Indeed, let $\varepsilon > 0$ and let $x \in B_1[0]$ (so $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq 1$).

Choose $K \in \mathbb{N}$ such that $1/K < \varepsilon$. Then $k \geq K$ implies

$$\begin{aligned}
\|Tx - T_k x\|^2 &= \left\| \sum_{n=k+1}^{\infty} \frac{1}{\sqrt{n}} \langle x, e_n \rangle e_n \right\|^2 \\
&= \sum_{n=k+1}^{\infty} \left| \frac{\langle x, e_n \rangle}{\sqrt{n}} \right|^2 \\
&= \sum_{n=k+1}^{\infty} \frac{|\langle x, e_n \rangle|^2}{n} \\
&\leq \frac{1}{K} \sum_{n=k+1}^{\infty} |\langle x, e_n \rangle|^2 \\
&\leq \frac{1}{K} \\
&< \varepsilon.
\end{aligned}$$

This implies $\|T - T_k\| \to 0$ as $k \to \infty$. Thus $(T_k)$ is a sequence of compact operators such that $\|T - T_k\| \to 0$ as $k \to \infty$. Therefore $T$ is compact.

To see that $T$ is not Hilbert-Schmidt, observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} \|Te_n\|^2 &= \sum_{n=1}^{\infty} \|\frac{1}{\sqrt{n}} e_n\|^2 \\
&= \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

is the harmonic series which does not converge. □

# Problem 3

## Problem 3.a

**Proposition 0.3.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator and let $\lambda$ be an eigenvalue of $T$. Then $|\lambda| \leq \|T\|$.*

*Proof.* Choose an eigenvector $x$ corresponding to the eigenvalue $\lambda$. By scaling if necessary, we may assume $\|x\| = 1$. Then

$$\begin{aligned}
\|T\| &= \sup\{|\langle Ty, y \rangle| \mid \|y\| \leq 1\} \\
&\geq |\langle Tx, x \rangle| \\
&= |\langle \lambda x, x \rangle| \\
&= |\lambda|.
\end{aligned}$$

□

## Problem 3.b

**Lemma 0.1.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a compact operator. Then $\||T|\| = \|T\|$.*

*Proof.* Combining problem 5 on HW5 and problem 6.b on HW6, we have

$$\begin{aligned}
\||T|\|^2 &= \||T|^2\| \\
&= \|T^* T\| \\
&= \|T\|^2.
\end{aligned}$$

It follows that $\||T|\| = \|T\|$ since the norm of an operator is nonnegative. □

**Proposition 0.4.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a compact operator and let $s$ be a singular value of $T$. Then we have $0 \leq s \leq \|T\|$.*

*Proof.* Clearly we have $s \geq 0$ by definition. Combining Lemma (0.1) and Proposition (0.3) gives us

$$\begin{aligned}
|s| &\leq \||T|\| \\
&= \|T\|.
\end{aligned}$$

□

## Problem 3.c

**Proposition 0.5.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a compact operator. Let $(s_n)$ be the sequence of singular values of $T$. Then $\|T\|_{HS} = \sqrt{\sum_{n=1}^{\infty} s_n^2}$.*

*Proof.* Let $(x_n)$ be an orthonormal basis for $T^*T$. Then

$$\begin{aligned}
\|T\|_{\mathrm{HS}} &= \sqrt{\sum_{n=1}^{\infty} \|Tx_n\|^2} \\
&= \sqrt{\sum_{n=1}^{\infty} \langle Tx_n, Tx_n \rangle} \\
&= \sqrt{\sum_{n=1}^{\infty} \langle T^*Tx_n, x_n \rangle} \\
&= \sqrt{\sum_{n=1}^{\infty} \langle s_n^2 x_n, x_n \rangle} \\
&= \sqrt{\sum_{n=1}^{\infty} s_n^2}.
\end{aligned}$$

□

# Problem 4

**Proposition 0.6.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then $T^2 + T + 1$ cannot be the zero operator.*

*Proof.* Choose an orthonormal eigenbasis $(e_n)$ of $T$ with $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$. Assume for a contradiction that $T^2 + T + 1 = 0$. Then

$$\begin{aligned}
0 &= (T^2 + T + 1)e_n \\
&= \sum_{n=1}^{\infty} (\lambda_n^2 + \lambda_n + 1)\langle e_n, e_n \rangle e_n \\
&= (\lambda_n^2 + \lambda_n + 1)e_n,
\end{aligned}$$

which implies $\lambda_n^2 + \lambda_n + 1 = 0$ for all $n \in \mathbb{N}$. Therefore $\lambda_n = \pm e^{2\pi i/3}$ for all $n \in \mathbb{N}$, but this contradicts the fact that the $\lambda_n$ must be real. □

# Problem 5

**Proposition 0.7.** *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a compact operator. Then there exists a sequence $T_n \colon \mathcal{H} \to \mathcal{H}$ of operators with finite dimensional range such that $\|T_n - T\| \to 0$ as $n \to \infty$.*

*Proof.* Let $T = U|T|$ be the polar decomposition of $T$. Choose a sequence $(S_n)$ of bounded operators with finite dimensional range such that $\|S_n - |T|\| \to 0$ as $n \to \infty$ (such a sequence exists by problem 6 HW8). Then for each $n \in \mathbb{N}$, the operator $T_n := US_n$ has finite dimensional range since $S_n$ has finite dimensional range. Moreover we have $\|T - T_n\| \to 0$ as $n \to \infty$. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $\||T| - S_n\| < \frac{\varepsilon}{\|U\|}$. Then $n \geq N$ implies

$$\begin{aligned}
\|T - T_n\| &= \|U|T| - US_n\| \\
&= \|U(|T| - S_n)\| \\
&= \|U\|\||T| - S_n|\| \\
&< \|U\| \frac{\varepsilon}{\|U\|} \\
&= \varepsilon.
\end{aligned}$$

□