Measure Theory Homework 7

Michael Nelson

Problem 2

Proposition 0.1. *Let $(X, \mathcal{M}, \mathcal{N})$ and $(Y, \mathcal{N}, \nu)$ be two finite measure spaces, and let $A \subseteq X$ and $B \subseteq Y$ be nonempty subsets of X and Y respectively. Then $A \times B \in \mathcal{M} \otimes \mathcal{N}$ if and only if $A \in \mathcal{M}$ and $B \in \mathcal{N}$.*

Proof. One direction is clear since $\mathcal{M} \otimes \mathcal{N}$ contains all measurable rectangles. For the reverse direction, suppose $A \times B \in \mathcal{M} \otimes \mathcal{N}$. Choose $x \in A$ and $y \in B$. Then observe that $B = (A \times B)_x$. Moreover, since

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \in A^c \end{cases}$$

we see that $B = (A \times B)_x \in \mathcal{N}$. By a similar argument, we have $A = (A \times B)^y \in \mathcal{M}$. □

Problem 3

Proposition 0.2. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two finite measure spaces. Let $f \colon X \times Y \to \mathbb{R}$ be a μ-integrable functions and let $g \colon X \times Y \to \mathbb{R}$ be a ν-integrable function. Define $h \colon X \times Y \to \mathbb{R}$ by*

$$h(x, y) = f(x)g(y)$$

for all $(x, y) \in X \times Y$. Then h is a $(\mu \otimes \nu)$-integrable function. Moreover we have

$$\int_{X \times Y} h \mathrm{d}(\mu \otimes \nu) = \left(\int_X f \mathrm{d}\mu \right) \left(\int_Y g \mathrm{d}\nu \right).$$

Proof. Define $\widetilde{f} \colon X \times Y \to \mathbb{R}$ and $\widetilde{g} \colon X \times Y \to \mathbb{R}$ by

$$\widetilde{f}(x, y) = f(x) \quad \text{and} \quad \widetilde{g}(x, y) = g(y)$$

for all $(x, y) \in X \times Y$. Observe that both $\widetilde{f}$ and $\widetilde{g}$ are $(\mathcal{M} \otimes \mathcal{N})$-measurable. Indeed, if $c \in \mathbb{R}$, then we have

$$\{\widetilde{f} < c\} = \{f < c\} \times Y \quad \text{and} \quad \{\widetilde{g} < c\} = X \times \{g < c\}.$$

It follows that $h = \widetilde{f}\widetilde{g}$ is $(\mathcal{M} \otimes \mathcal{N})$-measurable since it is the product of two $(\mathcal{M} \otimes \mathcal{N})$-measurable functions. Now observe that for each $x \in X$, we have

$$\begin{aligned} \int_Y h_x \mathrm{d}\nu &= \int_Y f(x) g \mathrm{d}\nu \\ &= f(x) \int_Y g \mathrm{d}\nu, \end{aligned}$$

where we were allowed to pull the constant $f(x)$ out of the integral from the fact that g is ν-integrable. Therefore as functions on X, we have $\int_Y h_{(-)} \mathrm{d}\nu = f \int_Y g \mathrm{d}\nu$. By a similar calculation, we also have $\int_Y |h_{(-)}| \mathrm{d}\nu = |f| \int_Y |g| \mathrm{d}\nu$. It follows from Tonelli's theorem that

$$\begin{aligned} \int_{X \times Y} |h| \mathrm{d}(\mu \otimes \nu) &= \int_X \left(\int_Y |h|_{(-)} \mathrm{d}\nu \right) \mathrm{d}\mu \\ &= \int_X \left(|f| \int_Y |g| \mathrm{d}\nu \right) \mathrm{d}\mu \\ &= \left(\int_X |f| \mathrm{d}\mu \right) \left(\int_Y |g| \mathrm{d}\nu \right) \\ &< \infty. \end{aligned}$$

Thus h is $(\mu \otimes \nu)$-integrable. Therefore by Fubini's theorem, we have

$$\begin{aligned}\int_{X \times Y} h \mathrm{d}(\mu \otimes \nu) &= \int_X \left(\int_Y h_{(-)} \mathrm{d}\nu \right) \mathrm{d}\mu \\ &= \int_X \left(f \int_Y g \mathrm{d}\nu \right) \mathrm{d}\mu \\ &= \left(\int_X f \mathrm{d}\mu \right) \left(\int_Y g \mathrm{d}\nu \right),\end{aligned}$$

where we were allowed to pull the constant $\int_Y g \mathrm{d}\nu$ out of the integral from the fact that f is μ-integrable. □

Problem 5

Proposition 0.3. *Let $\mathcal{B}$ be the σ-algebra of all Borel measurable subset of $[0,1]$ and let m be the Lebesgue measure. Define the function $f\colon [0,1] \times [0,1] \to \mathbb{R}$ by*

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

for all $(x,y) \in [0,1] \times [0,1]$. Then f is not $(\mathrm{m} \otimes \mathrm{m})$-integrable.

Proof. Assume for a contradiction that f is $(\mathrm{m} \otimes \mathrm{m})$-integrable. Then by Fubini's theorem, we must have

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \mathrm{d}x \right) \mathrm{d}y.$$

However, note that

$$\frac{x^2 - y^2}{(x^2 + y^2)^2} = -\partial_x \partial_y \arctan\left(\frac{y}{x} \right).$$

Thus, on the one hand, we have

$$\begin{aligned}\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \mathrm{d}y \right) \mathrm{d}x &= \int_0^1 \left(-\partial_x \arctan\left(\frac{y}{x} \right) \Big|_0^1 \right) \mathrm{d}x \\ &= -\int_0^1 \partial_x \arctan\left(\frac{1}{x} \right) \mathrm{d}x \\ &= -\arctan\left(\frac{1}{x} \right) \Big|_0^1 \\ &= -\frac{\pi}{4} + \frac{\pi}{2} \\ &= \frac{\pi}{4}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \mathrm{d}x \right) \mathrm{d}y &= \int_0^1 \left(-\partial_y \arctan\left(\frac{y}{x} \right) \Big|_0^1 \right) \mathrm{d}y \\ &= -\int_0^1 \partial_y \arctan y \mathrm{d}y \\ &= -\arctan y \Big|_0^1 \\ &= -\frac{\pi}{4}.\end{aligned}$$

This is a contradiction. □

Problem 6

Proposition 0.4. *Let $\mathcal{B}$ be the Borel σ-algebra of all Borel measurable subsets of $I = [0,1]$ equipped with the Lebesgue measure, and let $\mathcal{P}(\mathbb{N})$ denote the power set of $\mathbb{N}$ equipped with the counting measure μ. Then we have the following:*

1. *(Tonelli) Suppose $(f_n \colon I \to [0,\infty])$ is a sequence of nonnegative Borel-measurable functions. Then the function $f \colon I \to [0,\infty]$, defined by*
$$f(t) = \sum_{n=1}^{\infty} f_n(t)$$
for all $t \in I$, is Borel-measurable. Furthermore, we have
$$\int_0^1 f\mathrm{d}t = \sum_{n=1}^{\infty} \int_0^1 f_n\mathrm{d}t.$$

2. *(Fubini) Suppose $(f_n \colon I \to \mathbb{R})$ is a sequence of Borel-integrable functions. Then for m almost all $t \in [0,1]$, the series $\sum_{n=1}^{\infty} f_n(t)$ is absolutely convergent. Also, the series $\sum_{n=1}^{\infty} \int_0^1 f_n\mathrm{d}t$ is absolutely convergent. Furthermore, we have*
$$\int_0^1 \sum_{n=1}^{\infty} f_n\mathrm{d}t = \sum_{n=1}^{\infty} \int_0^1 f_n\mathrm{d}t, \tag{1}$$

Remark 1. We want to explain in a little more detail how to interpret the integral on the left-hand side in (1). The integrand $\sum_{n=1}^{\infty} f_n$ in (1) is only a partially defined function. However, it is defined almost everywhere. Indeed, let
$$E = \left\{ t \in [0,1] \mid \sum_{n=1}^{\infty} |f_n(t)| < \infty \right\}.$$
Then the function $\sum_{n=1}^{\infty} f_n$ is defined on all of E and $\mathrm{m}(E^c) = 0$. In order to compute integral on the left-hand side in (1), we choose any function $f \colon I \to \mathbb{R}$ such that $f|_E = \sum_{n=1}^{\infty} f_n$, for instance, say
$$f(t) = \begin{cases} \sum_{n=1}^{\infty} f_n(t) & \text{if } \sum_{n=1}^{\infty} |f_n(t)| < \infty \\ 0 & \text{else} \end{cases}$$
Then we define
$$\int_0^1 \sum_{n=1}^{\infty} f_n\mathrm{d}t = \int_0^1 f\mathrm{d}t. \tag{2}$$
where the integral on the right-hand side in (2) is computed via Lebesgue integration. Note that (2) is well-defined for two reasons. First, the Lebesgue measure m on the collection of all Borel-measurable subsets of I extends uniquely to a measure on the collection of all Lebesgue-measurable subsets of I. Second, if $g \colon I \to \mathbb{R}$ is another choice of a function such that $g|_E = \sum_{n=1}^{\infty} f_n$, then $\mathrm{m}\{g - f \neq 0\} \subseteq \mathrm{m}(E^c) = 0$. Therefore $\int_0^1 f\mathrm{d}t = \int_0^1 g\mathrm{d}t$, and hence (2) is well-defined.

Problem 7

Proposition 0.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a μ-integrable function. Define $\nu \colon \mathcal{M} \to \mathbb{R}$ by*
$$\nu(E) = \int_X f 1_E \mathrm{d}\mu$$
for all $E \in \mathcal{M}$. Then ν is a finite signed measure.

Proof. Write $f = f^+ - f^-$ in terms of of its positive and negative parts. We define $\nu^+ \colon \mathcal{M} \to \mathbb{R}$ and $\nu^- \colon \mathcal{M} \to \mathbb{R}$ by
$$\nu^+(E) = \int_X f^+ 1_E \mathrm{d}\mu \quad \text{and} \quad \nu^-(E) = \int_X f^- 1_E \mathrm{d}\mu$$
for all $E \in \mathcal{M}$. By a previous HW, it was shown that ν^+ and ν^- are measures on $\mathcal{M}$. In fact, they are finite

measures on $\mathcal{M}$ since f is μ-integrable. Now observe that

$$\begin{aligned}
\nu(E) &= \int_X f 1_E \mathrm{d}\mu \\
&= \int_X (f 1_E)^+ \mathrm{d}\mu - \int_X (f 1_E)^- \mathrm{d}\mu \\
&= \int_X f^+ 1_E \mathrm{d}\mu - \int_X f^- 1_E \mathrm{d}\mu \\
&= \nu^+(E) - \nu^-(E)
\end{aligned}$$

for all $E \in \mathcal{M}$. It follows that ν is a signed measure since it is a difference of two finite measures: clearly $\nu(\emptyset) = 0$ and $\nu(X) = \nu^+(X) - \nu^-(X) < \infty$, also if (E_n) is a pairwise disjoint sequence of members in $\mathcal{M}$, then

$$\begin{aligned}
\nu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \nu^+\left(\bigcup_{n=1}^{\infty} E_n\right) - \nu^-\left(\bigcup_{n=1}^{\infty} E_n\right) \\
&= \sum_{n=1}^{\infty} \nu^+(E_n) - \sum_{n=1}^{\infty} \nu^-(E_n) \\
&= \sum_{n=1}^{\infty} \nu^+(E_n) - \nu^-(E_n) \\
&= \sum_{n=1}^{\infty} \nu(E_n).
\end{aligned}$$

Finally, note that (ν^+, ν^-) is a Jordan decomposition of ν. Indeed, let $A = \{f \geq 0\}$ and $B = \{f < 0\}$. Then clearly $A \cup B = X$ and $A \cap B = \emptyset$. Furthermore, if $E \subseteq A$, then we have

$$\begin{aligned}
\nu^-(E) &= \int_X f^- 1_E \mathrm{d}\mu \\
&\leq \int_X f^- 1_A \mathrm{d}\mu \\
&= \int_X 0 \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

Similarly, if $E \subseteq B$, then we have

$$\begin{aligned}
\nu^+(E) &= \int_X f^+ 1_E \mathrm{d}\mu \\
&\leq \int_X f^+ 1_B \mathrm{d}\mu \\
&= \int_X 0 \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

Therefore $\nu^+ \perp \nu^-$. Thus (ν^+, ν^-) gives the Jordan decomposition of ν with (A, B) being a Hahn decomposition ν. □

Problem 9

For problems 9 and 13.i, we use the following notation:

Definition 0.1. Let $(X, \mathcal{M})$ be a measurable space, let μ and ν be two (possibly signed) measures on $\mathcal{M}$, and let $A, B \subseteq X$. We say (A, B) is a **singular pair** for μ, ν if $A \cup B = X$, $A \cap B = \emptyset$, A is a ν-null set, and B is a μ-null set.

Proposition 0.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let ν be a (finite) signed measure on $\mathcal{M}$. Then $\nu \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. Suppose $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Let (A, B) be a singular pair for (ν^+, μ) and let (C, D) be a singular pair for (ν^-, μ). We claim that $(A \cup C, B \cap D)$ is a singular pair for (ν, μ). Indeed, clearly we have $(A \cup C) \cup (B \cap D) = X$ and $(A \cup C) \cap (B \cap D) = \emptyset$. Also, if $E \subseteq B \cap D$, then we have

$$\begin{aligned}
\nu(E) &= \nu^+(E) - \nu^-(E) \\
&= 0 - 0 \\
&= 0.
\end{aligned}$$

Similarly, if $E \subseteq A \cup C$, then we have

$$\begin{aligned}
\mu(E) &= \mu((A \cup C) \cap E)) \\
&= \mu((A \cap E) \cup (C \cap E)) \\
&= \mu(A \cap E) + \mu(C \cap E) \\
&= 0 + 0 \\
&= 0.
\end{aligned}$$

This proves our claim, and thus $\nu \perp \mu$.

Now suppose $\nu \perp \mu$. Let (A, B) be a singular pair for (ν, μ) and let (C, D) be a singular pair for (ν^+, ν^-). We claim that $(A \cap C, B \cup D)$ is a singular pair for (ν^+, μ). Indeed, clearly we have $(A \cap C) \cup (B \cup D) = X$ and $(A \cap C) \cap (B \cup D) = \emptyset$. Also, if $E \subseteq A \cap C$, then we have $\mu(E) = 0$. Similarly, if $E \subseteq B \cup D$, then we have

$$\begin{aligned}
\nu^-(E) &= \nu^-((E \cap B) \cup (E \cap D)) \\
&= \nu^-(E \cap B) + \nu^-(E \cap D) \\
&= \nu^+(E \cap B) + \nu^-(E \cap D) \\
&= \nu^+(E \cap B) + \nu^-(E \cap D) + \nu^-(E \cap C) \\
&= \nu^+(E \cap B) + \nu^-((E \cap D) \cup (E \cap C)) \\
&= \nu^+(E \cap B) + \nu^-(E) \\
&= \nu^+(E \cap B) + \nu^+(E \cap D) + \nu^-(E). \\
&= \nu^+((E \cap B) \cup (E \cap D)) + \nu^-(E) \\
&= \nu^+(E) + \nu^-(E)
\end{aligned}$$

Canceling $\nu^-(E)$ from both sides gives us $\nu^+(E) = 0$. This proves our claim, and thus $\nu^+ \perp \mu$. A similar computation shows that $(A \cap D, B \cup C)$ is a singular pair for (ν^-, μ). Thus $\nu^- \perp \mu$. □

Problem 10

Proposition 0.7. *$(X, \mathcal{M})$ be a measurable space and let ν be a signed measure on $\mathcal{M}$ and let f be a $|\nu|$-integrable function. Then we have*

$$\left| \int_X f \mathrm{d}\nu \right| \leq \int_X |f| \mathrm{d}|\nu|$$

Proof. We have

$$\begin{aligned}
\left| \int_X f \mathrm{d}\nu \right| &= \left| \int_X f \mathrm{d}\nu^+ - \int_X f \mathrm{d}\nu^- \right| \\
&\leq \left| \int_X f \mathrm{d}\nu^+ \right| + \left| \int_X f \mathrm{d}\nu^- \right| \\
&\leq \int_X |f| \mathrm{d}\nu^+ + \int_X |f| \mathrm{d}\nu^- \\
&= \int_X |f| \mathrm{d}|\nu|.
\end{aligned}$$

□

Problem 13(i)

Proposition 0.8. *Let $(X, \mathcal{M})$ be a measure space and let ν be a signed measure no $\mathcal{M}$. Then*

$$\nu^+(E) = \sup\{\nu(F) \mid F \in \mathcal{M} \text{ and } F \subseteq E\} \tag{3}$$

for all $E \in \mathcal{M}$. Similarly,

$$-\nu^-(E) = \inf\{\nu(F) \mid F \in \mathcal{M} \text{ and } F \subseteq E\} \tag{4}$$

for all $E \in \mathcal{M}$.

Proof. Let (A, B) be a singular pair with respect to $\nu^+ \perp \nu^-$ and let $E \in \mathcal{M}$. We will first show (3). Let $F \in \mathcal{M}$ such that $F \subseteq E$. Then

$$\begin{aligned}
\nu(F) &= \nu^+(F) - \nu^-(F) \\
&\leq \nu^+(F) \\
&\leq \nu^+(E).
\end{aligned}$$

This implies $\geq$ in (3). Conversely, we have

$$\begin{aligned}
\nu(A \cap E) &= \nu^+(A \cap E) - \nu^-(A \cap E) \\
&= \nu^+(A \cap E) \\
&= \nu^+(A \cap E) + \nu^+(B \cap E) \\
&= \nu^+((A \cap E) \cup (B \cap E)) \\
&= \nu^+(E).
\end{aligned}$$

This implies $\leq$ in (3).

Now we will show (4). Let $F \in \mathcal{M}$ such that $F \subseteq E$. Then

$$\begin{aligned}
\nu(F) &= \nu^+(F) - \nu^-(F) \\
&\geq -\nu^-(F) \\
&\geq -\nu^-(E).
\end{aligned}$$

This implies $\leq$ in (4). Conversely, we have

$$\begin{aligned}
\nu(B \cap E) &= \nu^+(B \cap E) - \nu^-(B \cap E) \\
&= -\nu^-(B \cap E) \\
&= -\nu^-(A \cap E) - \nu^-(B \cap E) \\
&= -\nu^-((A \cap E) \cup (B \cap E)) \\
&= -\nu^-(E).
\end{aligned}$$

This implies $\geq$ in (4). $\square$