MATH 9500
FALL 2020
HOMEWORK 4

Due Wednesday, October 12, 2020

1. (5 pts) Let R be an integral domain. Show that the following conditions are equivalent.
   a) Every $R-$module is free.
   b) Every $R-$module is projective.
   c) Every $R-$module is injective.
   d) R is a field.

2. (5 pts) Show that if P and Q are projective $R-$modules, then so is $P \otimes_R Q$.

3. (5 pts) Prove that every overring of a valuation domain is a localization.

We say that the integral domain R is a *Prüfer* domain if R_P is a valuation domain for all $P \in \operatorname{Spec}(R)$. They are the "global" analog of valuation domains.

4. (5 pts) Show that any overring of a Prüfer domain is a Prüfer domain.

5. (5 pts) Show that if v is a vaulation on K then the set of elements with nonnegative value (and 0) form a valuation domain.

Let $(A, \leq)$ and $(B \preccurlyeq)$ be totally ordered abelian groups. We order the group $A \oplus B$ by declaring

$$(a_1, b_1) \trianglelefteq (a_2, b_2) \text{ if } a_1 < a_2 \text{ or } (a_1 = a_2 \text{ and } b_1 \preccurlyeq b_2).$$

As an exercise, you should convince yourself that this is a total ordering on the group $A \oplus B$ We use these orderings in the following problem.

6. (5 pts) Construct valuation domains with value groups $\mathbb{Z} \oplus \mathbb{R}$ and $\mathbb{R} \oplus \mathbb{Z}$ ordered lexicographically.