Combinatorics

Guest Lecture Sills

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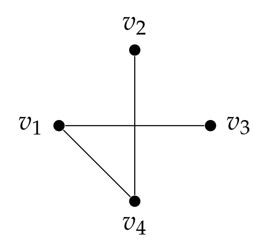
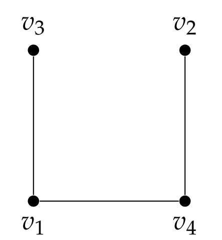
Combinatorics is the art/science of (sophisticated) counting.

Definition 0.1. A **graph** G is a pair (V, E) where V and E are sets and we think of the elements of V as vertices and the set E as a collection of edges that connect various pairs of vertices.

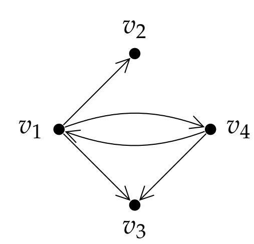
An example of a picture of a graph is like this



The graph G illustrated above is $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}\}$. Another picture of this same G is



A directed graph looks like this



Now, E will be a set of ordered pairs: $E = \{(v_1, v_2), (v_1, v_4), (v_1, v_3), (v_4, v_1), (v_4, v_3)\}$.

Calc II Review

Informally, a **sequence** $\{a_n\}_{n=0}^{\infty}$ is an indexed, infinitely long list of numbers.

$$\{a_n\}_{n=0}^{\infty} = (a_0, a_1, a_2, \dots)$$

Example 0.1. $a_n = 1$ for all $n = 0, 1, 2, 3, \dots$. We can write this as $\{1\}_{n=0}^{\infty}$.

Example 0.2. $a_n = n$ for all $n = 0, 1, 2, 3, \dots$. We can write this as $\{n\}_{n=0}^{\infty}$.

Example 0.3. $a_n = n^2$ for all $n = 0, 1, 2, 3, \dots$. We can write this as $\{n^2\}_{n=0}^{\infty}$.

Alternatively, a **sequence** a_n is a function whose domain is $\mathbb{N}$.

Example 0.4. The **Fibonacci sequence** is defined recursively as follows. Starting with $F_0 = 0$ and $F_1 = 1$, define $F_n = F_{n-1} + F_{n-2}$.

Example 0.5. $a_n = 1$ for all $n = 0, 1, 2, 3, \dots$. We can write this as $\{1\}_{n=0}^{\infty}$.

Definition 0.2. A **partition** of n is a "way" of writing n as a sum of positive integers where the order of the summands is considered irrelevant. The **partition function** $p(n)$ is the number of partitions of n.

$p(n)$ starts out as $1, 1, 2, 3, 5, 7, 11, 15, \dots$. This sequence my seem simple, but consider $p(200) = 3972999029388$.

n	partitions of n	$p(n)$
1	1	1
2	2, $1+1$	2
3	3, $1+2$, $1+1+1$	3
4	4, $1+3$, $2+2$, $1+1+2$, $1+1+1+1$	5

Generating Functions

An extremely useful tool in combinatorics is the generating function of a sequence.

Definition 0.3. The (ordinary power series) **generating function** for the sequence $\{a_n\}_{n=0}^{\infty}$ is the power series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

Example 0.6. The generating function for the sequence $\{1\}_{n=0}^{\infty}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Example 0.7. The generating function for the sequence $\{\frac{1}{n!}\}_{n=0}^{\infty}$ is e^x.

Example 0.8. Let's calculate the generating function for $\{F_n\}_{n=0}^{\infty}$. Let

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} F_n x^n.$$

We have $x^2\mathcal{F}(x) - x\mathcal{F}(x) - \mathcal{F}(x) = -x$. So

$$\mathcal{F}(x) = \frac{x}{1-x-x^2}.$$

A Recurrence Relation for $p(n)$.

In the 1740's, Euler found the infinite product generating function for $p(n)$:

$$\begin{aligned}
\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots &= (1+x^1+x^{1+1}+x^{1+1+1}\cdots)(1+x^2+x^{2+2}+x^{2+2+2}\cdots)(1+x^3+x^{3+3}+x^{3+3+3}+\cdots)\cdots \\
&= 1 + x^1 + (x^{1+1}+x^2) + (x^{1+1+1}+x^{1+2}+x^3) + (x^{1+1+1+1}+x^{1+1+2}+x^{1+3}+x^{2+2}+x^4) + \cdots \\
&= 1 + x + 2x^2 + 3x^3 + 5x^4 \cdots \\
&= \sum_{n=0}^{\infty} p(n)x^n.
\end{aligned}$$

So Euler showed

$$\prod_{n=0}^{\infty}\left(\frac{1}{1-x^n}\right) = \sum_{n=0}^{\infty} p(n)x^n. \tag{1}$$

Later, he studied the reciprocal of the infinite product in (1

$$\begin{aligned}
(1-x)(1-x^2)(1-x^3)\cdots &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \cdots \qquad (2) \\
&= \sum_{j=-\infty}^{\infty} (-1)^j x^{\frac{3}{2}j^2 - \frac{1}{2}j}
\end{aligned}$$

The integers occuring in the exponents on the righthand side of (2) are known as the pentagonal numbers. Euler found a recurrence of the partition numbers as follows:

$$\begin{aligned}
1 &= \left(\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots\right)\Big((1-x)(1-x^2)(1-x^3)\cdots\Big) \\
&= \Big(p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \cdots\Big)\Big(1 - x - x^2 + x^5 + \cdots\Big) \\
&= \sum_{n=0}^{\infty} \left(p(n) - p(n-1) - p(n-2) + p(n-5) + \cdots\right) x^n
\end{aligned}$$

It follows from this that

$$p(n) = p(n-1) + p(n-2) - p(n-5) - \cdots. \tag{3}$$