Analytic Functions

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1 Definition of an Analytic Function

Definition 1.1. A function $f\colon D \to \widehat{\mathbb{C}}$ is said to be **analytic at the point** z_0 in D if there exists a nonempty disk $B_r(z_0)$ centered at z_0 such that the restriction of f to $B_r(z_0)$ is the sum of a convergent power series with center z_0, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{1}$$

for all $z \in B_r(z_0)$.

1.1 Uniqueness of Representation

In principle, an analytic function could have different representations (1) as power series at z_0. In order to prove that this cannot happen, we investigate to which extent the coefficients of a power series are determined by the values of its sums.

Theorem 1.1. *(Uniqueness Principle, Local Identity Theorem) Let f and g be the sums of two power series with center z_0, and assume that both converge in an open disk $B_r(z_0)$, say*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n. \tag{2}$$

If there exists a sequence $(z_m) \subset B_r(z_0) \backslash \{z_0\}$ such that $z_m \to z_0$ as $m \to \infty$ and $f(z_m) = g(z_m)$ for all $m \in \mathbb{N}$, then $a_n = b_n$ for all $n \in \mathbb{N}$ and $f(z) = g(z)$ for all $z \in B_r(z_0)$.

Proof. The functions f and g are continuous at z_0, and hence

$$\begin{aligned} a_0 &= f(z_0) \\ &= \lim_{m\to\infty} f(z_m) \\ &= \lim_{m\to\infty} g(z_m) \\ &= g(z_0) \\ &= b_0. \end{aligned}$$

Using the arithmetic rules for convergent sequences, we obtain the representations

$$f_1(z) := \frac{f(z) - a_0}{z - z_0} = \sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n \quad \text{and} \quad g_1(z) := \frac{g(z) - b_0}{z - z_0} = \sum_{n=0}^{\infty} b_{n+1}(z - z_0)^n$$

for all $z \in B_r(z_0) \backslash \{z_0\}$. Because of $a_0 = b_0$ we have $f_1(z_m) = g_1(z_m)$ for all $m \in \mathbb{N}$, which implies $a_1 = b_1$, as just been shown. Proceeding inductively, we get $a_n = b_n$ for all n, and finally $f(z) = g(z)$ for all $z \in B_r(z_0)$. □

1.2 Taylor Coefficients

The coefficients a_n of the power series (1) representing a function f analytic at z_0 are referred to as the **Taylor coefficients of f at z_0**. The series (1) itself is said to be the **Taylor series of f at z_0**. So we can say that a function f is analytic at z_0 if it admits a convergent Taylor series at z_0.

Proposition 1.1. *Let Ω be an open set and let $f\colon \Omega \to \mathbb{C}$ be analytic at a point a in Ω. Then f is holomorphic at a. Moreover, if*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

for all $z \in B_r(a)$, then f is holomorphic on $B_r(a)$, and we have $f'(z)$

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z - a)^n$$

for all $z \in B_r(a)$. In particular, f' is analytic at a.

Proof. Let $z \in B_r(a)$. Choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\
&= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.
\end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m \colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned}
|g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \right| \\
&< \left| m a_m r^{m-1} \right|.
\end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. □

Corollary. *Let Ω be an open set, let $f \colon \Omega \to \mathbb{C}$ be a function, let $a \in \Omega$, and let $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $z \in B_r(a)$. Then $f^{(m)}(z)$ exists for all $m \geq 1$ and $z \in B_r(a)$, and moreover

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} (z-a)^n. \tag{3}$$

In particular, we have $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$.

Proof. The first part follows from an easy induction on m, with Proposition (1.1 giving the base case and the induction step. To get $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$, we set $z = a$ in 3). □

1.3 Operations Involving Analytic Functions

In this subsection, our goal is to prove the following theorem.

Theorem 1.2. *If f and g are analytic at z_0, then $f+g$, $f-g$, and fg are analytic at z_0. If, moreover, $g(z_0) \neq 0$, then f/g is analytic at z_0. If f is analytic at z_0 and g is analytic at $w_0 := f(z_0)$, then $g \circ f$ is analytic at z_0.*

Note that the composition $g \circ f$ need not exist on the domain of f, but just in a sufficiently small neighborhood of z_0. The analyticity of $f+g$ and $f-g$ are trivial. The proofs of the remaining assertions are more demanding.

1.3.1 Cauchy Product

The next result is a more sophisticated statement about the analyticity of a product fg, which includes an algorithm for computing the Taylor coefficients of fg from the coefficients of the factors f and g.

Theorem 1.3. *(Cauchy Product) Assume that the power series (2) for f and g converge in an open disk $B_R(z_0)$ centered at z_0 and of radius R, and let*

$$c_n := \sum_{m=0}^{n} a_m b_{n-m}.$$

Then the power series $\sum c_n(z-z_0)^n$ converges in $B_R(z_0)$ to the product $f(z)g(z)$, that is

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n \tag{4}$$

for all $z \in B_R(z_0)$.

Proof. Let f_n, g_n, and p_n be the partial sums of the series in (2) and (4) respetively. Then a rearrangement of the finite sums yields

$$\begin{aligned} f_n(z)g_n(z) &= \sum_{m=0}^{n} a_m(z-z_0)^m \sum_{m=0}^{n} b_m(z-z_0)^m \\ &= \sum_{m=0}^{n}\left(\sum_{k=0}^{m} a_k b_{m-k}\right)(z-z_0)^m + \sum_{m=n+1}^{2n}\left(\sum_{k=m-n}^{n} a_k b_{m-k}\right)(z-z_0)^m \\ &= p_n(z) + \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} a_k b_{m-k}(z-z_0)^m. \end{aligned}$$

Fix $z \in B_R(z_0)$. Choose r such that $|z-z_0| < r < R$ and choose a constant c such that $|a_k| \le cr^{-k}$ and $|b_k| \le cr^{-k}$ for all k. Setting $q := |z-z_0|/r < 1$, we use the triangle inequality to estimate

$$\begin{aligned} |f_n(z)g_n(z) - p_n(z)| &\le \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} |a_k||b_{m-k}||z-z_0|^m \\ &\le \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} c^2 r^{-k}||z-z_0|^k \\ &\le \sum_{m=n+1}^{2n} (2n-m+1)c^2 q^k \\ &\le n^2c^2q^{n+1}. \end{aligned}$$

Since the right-hand side tends to zero as $n \to \infty$, the assertion follows. $\square$

Example 1.1. Let $x, y \in \mathbb{C}$. Computing the Cauchy product of the Taylor series of $\exp x$ and $\exp y$, we obtain the **addition theorem** of the exponential function

$$\begin{aligned} e^x e^y &= \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{j=0}^{\infty}\frac{y^j}{j!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{x^j y^{k-j}}{j!(k-j)!}\right) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}x^j y^{k-j} \\ &= \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!} \\ &= e^{x+y}. \end{aligned}$$

When this identity is applied to $z = x + iy$ with $x, y \in \mathbb{R}$, it yields a representation of the complex exponential function by familiar real functions,

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

which implies that for all $z \in \mathbb{C}$,

$$|e^z| = e^{\mathrm{Re}(z)}, \quad \arg(e^z) = \mathrm{Im}(z), \quad e^{z+2\pi i} = e^z.$$

In particular, the exponential function has no zeros and is **periodic** with purely imaginary period $2\pi i$.

1.3.2 Reciprocal Functions

Theorem 1.4. *If f is analytic at z_0 and $f(z_0) \neq 0$, then $1/f$ is analytic at z_0. The Taylor coefficients b_k of $1/f$ at z_0 can be computed recursively from the Taylor coefficients a_k of f by $b_0 := 1/a_0$ and*

$$b_k := -\frac{1}{a_0}(a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_k b_0) \tag{5}$$

for all $k \in \mathbb{N}$.

Proof. In the first step we *assume* that the function $1/f$ is analytic at z_0. Then the Taylor series

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} b_0 (z - z_0)^n \tag{6}$$

converges in a neighborhood of z_0 and its Cauchy product with the Taylor series of f is the constant function 1. The latter is equivalent to the infinite system of equations

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 \neq 0$, this triangular system can be solved with respect to the coefficients b_k, which yields the recursion (5).

It remains to prove that the series (6), with coefficients b_k given by the recursion (5), indeed has a positive radius of convergence. Choose positive numbers c and r such that $|a_n| \leq c r^{-n}$ for all $n \in \mathbb{N}$. We set $q := 1 + c/|a_0|$ and show that

$$|b_n| \leq \frac{c}{|a_0|^2} \frac{q^{n-1}}{r^n} \tag{7}$$

for all $n \in \mathbb{N}$. For $n = 1$, we have $b_1 = -a_1/a_0^2$ and $|a_1| \leq c/r$, so that indeed

$$\begin{aligned} |b_1| &= \frac{a_1}{a_0^2} \\ &\leq \frac{c}{|a_0|^2} \frac{1}{r}. \end{aligned}$$

Now assume that (7), holds for all $n = 1, 2, \ldots, k-1$ and consider the case where $n = k$. Using $|b_0| = 1/|a_0|$, the recursive definition of b_k, and the triangle inequality, we estimate

$$\begin{aligned} |b_k| &\leq \frac{1}{|a_0|} \left(|a_k b_0| + \sum_{j=1}^{k-1} |a_{k-j}||b_j| \right) \\ &\leq \frac{1}{|a_0|} \left(|a_k b_0| + \sum_{j=1}^{k-1} \frac{c}{r^{k-j}} \frac{c}{r^j |a_0|^2} q^{j-1} \right) \\ &\leq \frac{c}{r^k |a_0|^2} \left(1 + \frac{c}{|a_0|} \sum_{j=0}^{k-2} q^j \right) \\ &\leq \frac{c}{r^k |a_0|^2} \left(1 + \frac{c}{|a_0|} \frac{q^{k-1} - 1}{q - 1} \right) \\ &= \frac{c}{r^k |a_0|^2} q^{k-1}, \end{aligned}$$

which gives (7) for $n = k$ and thus for all n. Consequently, the power series (6) has radius of convergence not less than r/q. □

Example 1.2. Let the function f be defined on the complex plane by $f(z) := (e^z - 1)/z$ if $z \neq 0$ and $f(0) = 1$. Representing e^z by its Taylor series, we obtain the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}$$

which converges in the entire complex plane and attans the correct value $f(0) = 1$ at $z = 0$. SInce $f(0) \neq 0$, the reciprocal function $1/f$ is also analytic at $z_0 = 0$. Writing the Taylor series of $g := 1/f$ in the form

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \tag{8}$$

the numbers B_k are determined by the equations $B_0 = b_0 = 1/a_0 = 1$ and

$$\begin{aligned} 0 &= \sum_{j=0}^{k} a_{k-j} b_j \\ &= \sum_{j=0}^{k} \frac{B_j}{(k-j+1)!j!} \\ &= \frac{1}{(k+1)!} \sum_{j=0}^{k} \binom{k+1}{j} B_j, \end{aligned}$$

for $j \in \mathbb{N}$. Solving this system recursively, we get

$$B_k = -\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j$$

for $j \in \mathbb{N}$. The numbers B_k are called **Bernoulli numbers**. For n odd, all B_n are zero, except B_1 which equals $-1/2$. The first Bernoulli numbers for n ever are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

Note that the series (8) converges for $|z| < 2\pi$.

1.3.3 Composition of Power Series

The final step in proving Theorem (1.2) is concerned with the composition $g \circ f$ of functions given by power series. In order to ensure that the composition makes sense at least locally, we assume that f is analytic at z_0, while g is supposed to be analytic at the image point $w_0 := f(z_0)$. Then, by continuity, f maps a neighborhood of z_0 into the disk of convergence of g. Our goal is to find a convergent power series for $g \circ f$ from the given power series of f and g. The approach is straightforward: we assume that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{and} \quad g(w) = \sum_{k=0}^{\infty} b_k (w - w_0)^k. \tag{9}$$

substitute $w - w_0 = \sum_{n=1}^{\infty} a_n (z - z_0)^n$ in the series for g, rearrange the double sum according to the powers of $z - z_0$, and show that the resulting series converges to $g \circ f$ in a neighborhood of z_0. The details will be worked out next.

For $n \in \mathbb{N}$, the nth power $(f - a_0)^n$ is analytic at z_0 and the n leading terms of its Taylor series at z_0 vanish. Denoting by a_{nk} the Taylor coefficients of this function, we have

$$(f(z) - a_0)^n = \sum_{k=1}^{\infty} a_{nk} (z - z_0)^k = \sum_{k=n}^{\infty} a_{nk} (z - z_0)^k \tag{10}$$

in some neighborhood of z_0. Substituting the term $w - w_0$ in the power series of $g - b_0$ by the power series of $f - a_0$ (recall that $a_0 = f(z_0) = w_0$), we obtain formally

$$\begin{aligned} \sum_{n=0}^{\infty} b_n (w - w_0)^n &= \sum_{n=1}^{\infty} b_n \left(\sum_{k=n}^{\infty} a_{nk} (z - z_0)^k \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} b_n a_{nk} \right) (z - z_0)^k. \end{aligned}$$

Before we justify that changing the order of summation is possible, we state the result

Theorem 1.5. *If f is analytic at z_0 and g is analytic at $w_0 := f(z_0)$, then $g \circ f$ is analytic at z_0. Let f, g, and $(f - a_0)^n$ be represented by the series (9) and (10) respectively. Then the Taylor coefficients c_k of $g \circ f$ at z_0 are given by*

$$c_0 = b_0, \qquad c_k = \sum_{n=1}^{k} b_n a_{nk}$$

for all $k \in \mathbb{N}$.

1.4 Weierstrass Rearrangement Theorem

Theorem 1.6. *(Weierstrass Rearrangement Theorem) The sum of a power series is analytic at any point in its disk of convergence. If f is given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11}$$

for all $z \in B_r(z_0)$, and if $z_1 \in B_r(z_0)$, then

$$f(z) = \sum_{m=0}^{\infty} b_m (z - z_1)^m$$

for all $z \in B_{r_1}(z_1)$, where $r_1 := r - |z_1 - z_0|$ and the coefficients b_k are given by the convergent series

$$b_m = \sum_{n=m}^{\infty} \binom{n}{m} a_n (z_1 - z_0)^{n-m}$$

for all $k \in \mathbb{N}_0$.

Proof. Let $z \in B_{r_1}(z_1)$. Substituting $z - z_0 = (z - z_1) + (z_1 - z_0)$ into (11), we obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n ((z - z_1) + (z_1 - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} (z - z_1)^m (z_1 - z_0)^{n-m} \end{aligned}$$

In order to prove the assertion, it only remains to change the order of summation in the double series. It suffices to show that this series converges absolutely. To this end we remark that

$$\sum_{n=0}^{\infty} |a_n| \sum_{m=0}^{n} \binom{n}{m} |z - z_1|^m |z_1 - z_0|^{n-m} = \sum_{n=0}^{\infty} |a_n| (|z - z_1| + |z_1 - z_0|)^n.$$

The last sum converges because $|z - z_1| + |z_1 - z_0| < r$, so that the power series (11) converges absolutely at the point $z = z_0 + |z - z_1| + |z_1 - z_0|$. □

1.5 Definition of Analytic Function

Definition 1.2. A complex function $f\colon D \subseteq \mathbb{C} \to \mathbb{C}$ is said to be **analytic on** A if A is a subset of D and f is analytic at every point of A. We say that f is **analytic** if it is analytic on its domain set. A function which is analytic on the entire complex plane is called **entire**.

Lemma 1.7. *For any complex function $f\colon D \subseteq \mathbb{C} \to \mathbb{C}$ the set A_f of all points in D at which f is analytic is open.*

Proof. If A_f is empty, there is nothing to prove. If $z_0 \in A_f$, then f has a Taylor expansion at z_0 which converges in a open disk D_0 centered at z_0. By Theorem (1.6), $D_0 \subset A_f$. □

1.5.1 Jacobi Theta Function

An interesting family of entire functions are the **Jacobi Theta functions**, given by the series

$$\vartheta(z; q) := \sum_{n \in \mathbb{Z}} q^{n^2} e^{2\pi i n z}$$

for all $z \in \mathbb{C}$, where q is a complex parameter with modulus less than one. In order to show that ϑ is entire, we consider the power series

$$f(z) := \sum_{n=1}^{\infty} q^{n^2} z^n = qz + q^4 z^2 + q^9 z^3 + \cdots .$$

This series converges for all $z \in \mathbb{C}$ because

$$\operatorname{limsup} \left(\left| q^{n^2} \right|^{1/n} \right) = \operatorname{limsup} \left(|q^n| \right) = 0,$$

and thus the function f is entire. The function g defined by $g(z) := e^{2\pi i z}$ is also entire and has no zeros in $\mathbb{C}$, so that its reciprocal $1/g$ is also entire. Finally, $\vartheta(z) = 1 + 2f(g(z))$

$$\vartheta(z; q) = 1 + f(g(z)) + f(1/g(z))$$

for all $z \in \mathbb{C}$.

The function g, and consequently ϑ, is periodic with period 1. The parameter q is said to be the **nome** of the Theta function. It is often represented as $q = e^{\pi i \tau}$, where τ is a complex number with $\operatorname{Im}(\tau) > 0$.

1.5.2 Local Normal Forms

Theorem 1.8. *(Local Normal Form) Let $f\colon D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic on D. If f is not constant in a neighborhood of $z_0 \in D$, then there exist a positive integer m and an analytic function $g\colon D \subset \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $g(z_0) \neq 0$ such that*

$$f(z) = f(z_0) + (z - z_0)^m g(z) \tag{12}$$

for all $z \in D$. The integer m and the function g are uniquely determined.

Proof. Assume that the Taylor series $f(z) = \sum a_k (z - z_0)^k$ of f at z_0 converges in a disk D_0. Denoting by a_m the first non-zero coefficient among $a_1, a_2, a_3, \dots$, we have

$$f(z) = f(z_0) + (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$$

for all $z \in D_0$. The sum $g_0(z)$ of the series $\sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$ is an analytic function in D_0 with $g_0(z_0) = a_m \neq 0$. The function g defined in D by

$$g(z) := \begin{cases} \frac{f(z) - f(z_0)}{(z - z_0)^m} & \text{if } z \in D \backslash \{z_0\} \\ a_m & \text{if } z = z_0 \end{cases}$$

is analytic on $D \backslash \{z_0\}$. Since it coincides with g_0 in D_0 it is also analytic at z_0.

For proving uniqueness we assume that $(z - z_0)^n g_1(z) = (z - z_0)^m g_2(z)$ with $n > m$ for all $z \in D$. Then $(z - z_0)^{n-m} g_1(z) = g_2(z)$, and the left-hand side vanishes at z_0 while $g_2(z_0) \neq 0$. So $m = n$ and then $g_1 = g_2$ is obvious. □

Definition 1.3. The integer m in the representation (12) is called the **order** of the function f at z_0 and is denoted by $\operatorname{ord}(f, z_0)$. If f is constant in a neighborhood of z_0 we set $\operatorname{ord}(f, z_0) = \infty$. If in particular $f(z_0) = 0$, then m is said to be the **order of the zero** z_0.

As an immediate corollary of Theorem (1.8) we get the following result which shows, in particular, that all zeros of non-constant analytic functions are isolated.

Lemma 1.9. *If f is analytic at z_0 and $a := f(z_0)$, then there exists a disk D_0 with center z_0 such that either $f(z) = a$ for all $z \in D_0$ or $f(z) \neq a$ for all $z \in D_0 \backslash \{z_0\}$.*

1.6 Analytic Functions in Planar Domain

As we have already seen, it is natural to require that the domain set D of an analytic function is open. From now on, we shall also assume that D is a nonempty connected open subset of $\mathbb{C}$, i.e. that D is a **domain**. This assumption is not too strong, since any open set in $\mathbb{C}$ is the disjoint union of domains, but it simplifies life a lot. In particular it is important when local statements about power series will be "lifted" to global results for analytic functions. This will be demonstrated in the proof of the following theorem.

Theorem 1.10. *(Identity Theorem, Uniqueness Principle) Let f and g be analytic functions in a domain D. If there exists a sequence $(z_n) \subset D \backslash \{z_0\}$ such that $z_n \to z_0 \in D$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = g(z)$ for all $z \in D$.*

Proof. The function $h := f - g$ has a sequence of zeros which converge to $z_0 \in D$. Continuity of h implies that $h(z_0) = 0$, so that z_0 is a zero of h which is not isolated. Since h is analytic in D, we infer from Lemma (1.9) that $h(z) = 0$ in some disk D_0 with center z_0.

We pick any point z_1 in D and show that $h(z_1) = 0$. Since D is open and connected, it must be path-connected. So choose a path $\gamma\colon I \to D$ from z_0 to z_1. Then the set

$$S := \{s \in I \mid h(\gamma(t)) = 0 \text{ for all } t \in [0, s]\}$$

is not empty and we denote by s_0 its supremum. Continuity of h implies that $h(\gamma(s_0)) = 0$. Since $h(\gamma(t)) = 0$ for all $t \in [0, s_0]$, Lemma (1.9) tells us that $h(z) = 0$ in a neighborhood of $\gamma(s_0)$. This is only possible if $s_0 = 1$, because otherwise $h(\gamma(t)) = 0$ for all t in an interval $[0, s_1]$ with $s_1 > s_0$. □

1.6.1 Zeros of Analytic Function

The last theorem establishes the surprising fact that a function which is analytic in a domain is completely determined by its values in an arbitrarily small disk. We state another result concerning the zeros of such a function.

Corollary. *If $f \neq 0$ is analytic in a domain D and K is a compact subset of D, then the number of zeros of f in K is finite.*

Proof. If f had infinitely many zeros in K, there would exist a sequence (z_n) of such zeros which converge to a point $z_0 \in K \subset D$. But then $f = 0$ on D by Theorem (1.10). □

Nevertheless an analytic function $f \neq 0$ can have infinitely many zeros in D. If this happens, the zeros must have an accumulation point z_0 on $\widehat{\mathbb{C}}$. Since z_0 cannot lie in D, it must be on the boundary of D (considered as a subset of $\widehat{\mathbb{C}}$). For an entire function, the only possible accumulation point of zeros is the point at infinity.

Example 1.3. The function $\sin(1/z)$ is analytic in $\mathbb{C}\backslash\{0\}$ and has the zeros $z_k = 1/(k\pi)$ with $k = \pm1, \pm2, \ldots$, which accumulate at the origin.

1.6.2 Extremal Values

Theorem 1.11. *(Maximum and Minimum Principle) Let $f\colon D \subset \mathbb{C} \to \mathbb{C}$ be a non-constant analytic function. Then $|f|$ has no local maximum in D, and very local minimum of $|f|$ is a zero of f.*

Proof. Assume that $|f|$ attains a maximum or minimum at $z_0 \in D$. By Theorem (1.10) f is not locally constant, so that we can apply Theorem (1.8) and write

$$f(z) = f(z_0) + (z - z_0)^m g(z),$$

where g is analytic in D and $g(z_0) \neq 0$. □

1.7 Analytic Continuation

1.7.1 Direct Analytic Continuation

Theorem 1.12. *(Direct Analytic Continuation) Let the functions $f_1\colon D_1 \to \mathbb{C}$ and $f_2\colon D_2\colon \to \mathbb{C}$ be analytic in the domains D_1 and D_2, respectively. Assume that the intersection $D_0 := D_1 \cap D_2$ is nonempty and that $f_1 = f_2$ on D_0. Then there is a unique analytic function f on $D := D_1 \cup D_2$ which coincides with f_1 on D_1, namely*

$$f(z) := \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2. \end{cases}$$

Proof. The function f is analytic on D because any point $z \in D$ belongs to D_1 or D_2, so that f coincides with f_1 or f_2 in a neighborhood of z. Since $D_1 \cup D_2$ is a domain, and $D_1 \cap D_2 \neq \emptyset$ is open, uniqueness of f follows from the identity theorem. □

Under the assumptions of Theorem (1.12), the function f is said to be an **analytic continuation of f_1 onto** D. Interchanging the roles of f_1 and f_2, we see that f is also the (unique) analytic extension of f_2 onto D. So direct analytic continuation may extend a function to a larger domain, but this says nothing about how to *find* such an extension. The key to a constructive approach is Weierstrass rearrangement theorem for power series.

1.7.2 Analytic Function Elements

Assume that an analytic function f is given as the sum of a power series which has center z_0 and disk of convergence D_0. It can happen that the rearrangement of that power series to a series centered at a point z_1 in D_0 has a disk of convergence D_1 which protrudes out of D_0. Then by Theorem (1.12), f admits an analytic extension to $D_0 \cup D_1$. In order to explore this further we introduce some notation.

Definition 1.4.

1. An **(analytic) function element** if a pair (f, D) consisting of a disk D and an analytic function $f\colon D \to \mathbb{C}$. The center of the disk D is also referred to as the **center of the function element**.

2. If (f_1, D_1) and (f_2, D_2) are two function elements which satisfy the assumption of Theorem (1.12), we say that (f_2, D_2) is the **direct analytic continuation** of (f_1, D_1) (or vice versa) and write $(f_1, D_1) \bowtie (f_2, D_2)$.

3. A finite sequence $(f_0, D_0), (f_1, D_1), \ldots, (f_n, D_n)$ of funciton elements is said to be a **chain** if any function element (except the first) is the direct analytic continuation of its predecessor,

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_n, D_n). \tag{13}$$

   We then call (f_n, D_n) an **analytic continuation of** (f_0, D_0) **along the chain**.

4. A function element (f_n, D_n) is an **analytic continuation** of (f_0, D_0) if a chain of function elements satisfying (13) exists. We then write $(f_0, D_0) \sim (f_n, D_n)$.

To understand the procedures that follow better it is essential to recognize some subtleties of these definitions. While it is easy to see that $\sim$ is an **equivalence relation**, the relation $\bowtie$ is reflexive and symmetric, but *not transitive*.

Example 1.4. The binomial series

$$f_0(z) = \sum_{n=0}^{\infty} \binom{1/2}{n} (z-1)^n \tag{14}$$

has radius of convergence one and thus defines a function element (f_0, D_0) with $D_0 := B_1(1) = \{z \in \mathbb{C} \mid |z-1| < 1\}$. If z is real and $0 < z < 1$, we have $f_0(z) = \sqrt{z}$. For $k = 0, 1, \ldots, 8$ we denote by $\omega_k = e^{2\pi i k/9}$ the 9th roots of unity and let $D_k := \{z \in \mathbb{C} \mid |z - \omega_k| < 1\}$. All power series

$$f_k(z) := e^{ik\pi/18} \sum_{n=0}^{\infty} e^{-ik\pi/9} \binom{1/2}{n} (z - \omega_k)^n$$

have radius of convergence one and the nine function elements form a chain

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_8, D_8)$$

where neighbors are direct analytic continuations of each other. Consequently any two elements (f_j, D_j) and (f_k, D_k) are **analytic continuations** of each other. Moreover, for $k = 1, 2, 3, 4$, the element (f_k, D_k) is a *direct* analytic continuation of (f_0, D_0), but not for $k = 5, 6, 7, 8$. Since (f_6, D_6) is also a direct analytic continuation of (f_3, D_3), we have

$$(f_0, D_0) \bowtie (f_3, D_3) \bowtie (f_6, D_6) \not\bowtie (f_0, D_0),$$

which again shows that the relation $\bowtie$ is not transitive.

Lemma 1.13. *If $D_1 \cap D_2 \cap D_3 \neq \emptyset$, $(f_1, D_1) \bowtie (f_2, D_2)$ and $(f_2, D_2) \bowtie (f_3, D_3)$, then $(f_1, D_1) \bowtie (f_3, D_3)$.*

Proof. The functions f_1 and f_3 are analytic in the domain $D_1 \cap D_3$ and coincide (with f_2) on its open subset $D_1 \cap D_2 \cap D_3$. Thus $f_1 = f_3$ on $D_1 \cap D_3$. □

1.7.3 Analytic Continuation Along a Path

Definition 1.5. Let $\gamma\colon I \to \mathbb{C}$ be a path. A chain of function elements

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_n, D_n), \tag{15}$$

is said to be a **chain along** γ, if the chain of disks $(D_0, D_1, \ldots, D_n)$ covers γ in the sense of the Path Covering Lemma.

Let (f_0, D_0) and (f, D) be function elements with centers at $\gamma(0)$ and $\gamma(1)$, respectively. We say that (f, D) is an **analytic continuation of** (f_0, D_0) **along** γ, if there exists a chain of function elements (15) along γ such that $(f, D) = (f_n, D_n)$.

It is essential that analytic continuation along a path does not depend on the special choice of the chain of function elements. This statement is made precise in the next lemma.

Lemma 1.14. *Let $(f_0, D_0) \bowtie \cdots \bowtie (f_n, D_n)$ and $(g_0, \widetilde{D}_0) \bowtie \cdots \bowtie (g_m, \widetilde{D}_m)$ be two chains of function elements along a path γ. If $(f_0, D_0) \bowtie (g_0, \widetilde{D}_0)$, then it is also true that $(f_n, D_n) \bowtie (g_m, \widetilde{D}_m)$.*

Proof. Let $\gamma\colon I \to \mathbb{C}$ be a path and let

$$0 = t_0 < t_1 < \cdots < t_n = 1, \qquad 0 = s_0 < s_1 < \cdots < s_m = 1,$$

be partitions of I such that for all $k = 1, \ldots, n$ and $j = 1, \ldots, m$ we have

$$\gamma([t_{k-1}, t_k]) \subset D_k, \qquad \gamma([s_{j-1}, s_j] \subset \widetilde{D}_j.$$

Intuitively, the following procedure can be described as a walk along the path γ, where the left foot is only allowed to step on disks D_k, the right foot is restricted to the disks $\widetilde{D}_j$, and the function elements (f_k, D_k) and $(f_j, \widetilde{D}_j)$ underneath both feet must be direct analytic continuations of each other. We shall show that one can walk step-by-step all the way along γ, following just a simple rule: don't move the foot which is ahead. □

1.7.4 Function Elements and Germs

Though analytic continuation along a path γ is *essentially* independent of the choice of the function elements which cover γ, these elements are by no means uniquely defined. In fact not even the elements at the endpoints of γ are unique, Lemma (1.14) only tells us that the terminal elements of the chain *coincide on some disk* if the initial elements have this property. The redundancy in this process of analytic continuation is sometimes disturbing and makes formulations cumbersome. To eliminate this drawback we utilize the standard technique of forming classes.

Definition 1.6. Two function elements (f_1, D_1) and (f_2, D_2) centered at z_0 are said to be **equivalent** if $f_1(z) = f_2(z)$ in some neighborhood of z_0. A **germ** at z_0 is a class of equivalent function elements centered at z_0. The germ which contains a function element (f, D) is denoted by $f^\star$. We denote by $\mathcal{O}^{\text{an}}_{z_0}$ to be the set of all germs at z_0. One easily checks that $\mathcal{O}^{\text{an}}_{z_0}$ is a $\mathbb{C}$-algebra.

Depending on the situation, one can choose an appropriate **representative** of a germ $f^\star$. The **canonical representative** of a germ $f^\star$ is that function element (f, D) in $f^\star$ which has the disk D of maximal radius (here we allow $D = \mathbb{C}$).

The **value** $f^\star(z_0)$ **of a germ** $f^\star$ at z_0 is the value $f(z_0)$ of any function element (f, D) which represents $f^\star$. Note that the value of a germ is only defined at its center. On the other hand, the germ of a function element (f, D) is *not* determined by the value of f at its center z_0 alone, but by the complete list of its Taylor coefficients. To explain this idea more precisely, let $\mathbb{C}^\infty$ be the set of all sequences $(z_n)_{n\geq 0}$ in $\mathbb{C}$. Then $\mathbb{C}^\infty$ forms a $\mathbb{C}$-algebra, where addition is defined pointwise and where multiplication is defined by the Cauchy product; namely if (a_n) and (b_n) are two sequences in $\mathbb{C}$, then

$$(a_n) + (b_n) = (a_n + b_n) \quad \text{and} \quad (a_n)(b_n) = (c_n),$$

where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$. Finally, let $\varphi \colon \mathcal{O}^{\text{an}}_{z_0} \to \mathbb{C}^\infty$ be the morphism of $\mathbb{C}$-algebras given by sending a function element (f, D) to the Taylor sequence $(f^{(n)}(z_0))$. The identity theorem implies that this morphism is well-defined and injective.

The concept of germs is not restricted to function elements. If the function f is analytic at a point z, it is analytic in a neighborhood of z, and thus it induces a germ at z which we denote by $f^\star_z$.

1.7.5 Analytic Continuation of Germs

Definition 1.7. We say that a germ $f^\star$ at b is an analytic continuation of a germ $f^\star_a$ at a along a path γ from a to b if there exists a chain of function elements

$$(f_0, D_0) \bowtie \cdots \bowtie (f_n, D_n)$$

along γ such that (f_0, D_0) represents $f^\star_a$ and (f_n, D_n) represents $f^\star$, respectively.

Whenever an analytic continuation of a germ along a path γ exists, Lemma (1.14) tells us that the terminal germ is uniquely determined and does not depend on the specific choice of the function element along γ. We thus can speak of *the* analytic continuation $f^\star(\gamma)$ of a germ $f^\star$ along a path γ.

1.7.6 The Monodromy Principle

In the next step we study analytic continuation of a germ along *different paths* with the same endpoints.

Theorem 1.15. *(Monodromy Principle I) Let γ_s, with $s \in I$, be a family of homotopic paths with fixed endpoints. If the germ $f^\star$ admits an analytic continuation $f^\star(\gamma_s)$ along any parth γ_s, then $f^\star(\gamma_0) = f^\star(\gamma_1)$.*

Example 1.5. (The Complex Logarithm) Our starting point is the function element (f_0, D_0) in the disk $D_0 := B_1(1)$ with

$$f_0(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k = \log|z| + i\text{Arg}z. \tag{16}$$

In order to prove that this function element admits an unrestricted analytic continuation in $\mathbb{C}\backslash\{0\}$, we consider any path $\gamma \colon I \to \mathbb{C}\backslash\{0\}$ with initial point $z_0 = 1$ and arbitrary terminal point z_1.

In order to construct function elements of an analytic continuation of (f_0, D_0) along γ, we first pick a point $z_t := \gamma(t)$ on γ and denote by $D_t := B_{|z_t|}(z_t)$ for all $t \in I$ (the largest disk around z_t contained in $\mathbb{C}\backslash\{0\}$). To find an appropriate argument of z_t, we denote by $t \mapsto a(t)$ the continuous branch of the argument along γ which is equal to the principle value $\mathrm{Arg}1 = 0$ at its initial point and set $\arg_\gamma z_t := a(t)$. Finally, we define the function element (f_t, D_t) by

$$f_t(z) := \log|z_t| + i\arg_\gamma z_t + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k z_t^k}(z - z_t)^k$$

for all $z \in D_t$. The series on the right-hand side results from substituting z by z/z_t in (16), so that D_t is indeed its disk of convergence.