DUE DATE: Wednesday, December 4, in class.

Show full work on each of these problems. Results without explanation will not receive full credit.

Problem 1. Prove that the space of continuous functions $C[a,b]$ equipped with the supremum norm is a Banach space. Don't forget to show first that the supremum norm is indeed a norm.

Problem 2. Prove that a norm in a normed linear space $(\mathcal{X}, \|.\|)$ comes from an inner product if and only if the norm satisfies the parallelogram identity. Hint: Use polarization identity to define a candidate for the inner product (x, y). Show additivity $(x_1+x_2, y) = (x_1, y)+(x_2, y)$ of this candidate by using the parallelogram identity. Then use induction to show $(nx, y) = n(x, y)$ for all $n \in \mathbb{N}$ and then extend this identity to all $n \in \mathbb{Z}$ and eventually all $n \in \mathbb{Q}$. Finally, use continuity to prove the homogeneity property $(\alpha x, y) = \alpha(x, y)$ for all $\alpha \in \mathbb{R}$.

Problem 3. Consider $C[0,1]$ equipped with the supremum norm. Let $T : C[0,1] \to C[0,1]$ be the linear operator defined by $Tf(x) = \int_0^x f(y)dy$. Prove that T is bounded and compute its norm.

Problem 4. Consider the space $C[a,b]$ equipped with the usual $\|.\|_{sup}$ norm. Define a linear functional $l : C[a,b] \to \mathbb{R}$ by

$$l(f) := f(a) - f(b)$$

(a) Prove that l is bounded and compute its norm.

(b) Prove that $\{f \in C[a,b] : f(a) = f(b)\}$ is a closed subspace of $C[a,b]$.

Problem 5. Let $\mathcal{Y}$ be a subset of $(C[-1,1], \|.\|_{sup})$ consisting of all functions $g \in C[-1,1]$ such that

$$\int_{-1}^{0} g(x)dx = \int_{0}^{1} g(x)dx = 0.$$

(a) Prove that $\mathcal{Y}$ is a closed subspace.

(b) Let $h \in C[-1,1]$ be the function $h(x) = 2x$. Prove that there exists NO closest point to h in $\mathcal{Y}$, i.e., there exists NO $f \in \mathcal{Y}$ such that $\|h - f\|_{sup} = d(h, \mathcal{Y})$. Note that, in contrast, such a vector always exists in a Hilbert space case, and it is given by the orthogonal projection of the vector onto the closed subspace.

Problem 6. Let $\mathcal{X}$ be a normed linear space. For a set $A \subseteq \mathcal{X}$ we define $A^{\perp}$ as the subset of $\mathcal{X}^*$ consisting of all $l \in \mathcal{X}^*$ such that $l(a) = 0$ for all $a \in A$. Similarly, for a set $M \subseteq \mathcal{X}^*$ we define $M_{\perp}$ as the subset of $\mathcal{X}$ consisting of all vectors $x \in \mathcal{X}$ such that $l(x) = 0$ for all $l \in M$.

(a) Show that $A^{\perp}$ and $M_{\perp}$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$ respectively.

(b) Prove that $\overline{span}(A) \subseteq (A^{\perp})_{\perp}$ and $\overline{span}(M) \subseteq (M_{\perp})^{\perp}$.

Problem 7. Prove that $(l^1)^* \cong l^\infty$.