

Commutative Algebra Homework 1

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Problem 1

Exercise 1. Given an example of a commutative ring (necessarily without identity) that does not have a maximal proper ideal.

Solution 1. Let A be any divisible group (for instance $A = \mathbb{Q}$). So $A = nA$ for every $n \in \mathbb{Z} \setminus \{0\}$. Then observe that A has no maximal proper subgroups. Indeed, assume for a contradiction that B is a maximal proper subgroup of A . Then B must have finite index in A (otherwise we can find a nonzero proper subgroup B'/B of A/B and pull this back to a proper subgroup B' of A which contains B), say $[A : B] = m$. Then we have

$$\begin{aligned} A &= mA \\ &\subseteq B \\ &\subseteq A, \end{aligned}$$

which forces $A = B$ which gives us a contradiction.

Now we turn A into a ring in a rather trivial way, namely we define multiplication on A by

$$a \cdot a' = 0$$

for all $a, a' \in A$. Clearly multiplication defined in this way gives A the structure of a commutative ring (but without an identity element). Moreover since A has no maximal proper subgroups, we see that A has no maximal ideals as a ring.

Problem 2

Exercise 2. Let R be a commutative ring with identity and let $I \subset R$ be a proper ideal of R . We denote by $\text{rad } I$ to be the radical of I and we denote by $N(R)$ to be the set of nilpotents of R .

1. Show that $\text{rad } I$ is contained in the intersection of all prime ideals that contain I .
2. Show the other containment.
3. Show that $N(R)$ is the intersection of all prime ideals of R .

Solution 2. 1. Let $x \in \text{rad } I$ and let \mathfrak{p} be a prime ideal in R which contains I . Choose $n \in \mathbb{N}$ such that $x^n \in I$. Then since $I \subseteq \mathfrak{p}$, we have $x^n \in \mathfrak{p}$. It follows that $x \in \mathfrak{p}$ since \mathfrak{p} is prime. Since x and \mathfrak{p} were arbitrary, it follows that $\text{rad } I$ is contained in all prime ideals which contains I . Thus $\text{rad } I$ is contained in the intersection of all prime ideals which contains I .

2. Assume for a contradiction that

$$\text{rad } I \not\subseteq \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p}.$$

Choose $x \in \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ such that $x \notin \text{rad } I$. Thus $x \in \mathfrak{p}$ for all prime ideals \mathfrak{p} which contain I and $x^n \notin I$ for all $n \in \mathbb{N}$. We will find a prime ideal in R which contains I but does not contain x , which will give us a contradiction.

Consider the ring obtained by localizing R at the multiplicative set $\{x^n \mid n \in \mathbb{N}\}$:

$$R_x = \{a/x^n \mid a \in R \text{ and } n \in \mathbb{N}\},$$

and let $\rho: R \rightarrow R_x$ be the corresponding localization map, given by

$$\rho(a) = a/1$$

for all $a \in R$. Since $x^n \neq 0$ for all $n \in \mathbb{N}$, we see that $I_x = \rho(I)R_x$ is a proper ideal of R_x . In particular, there exists a prime ideal \mathfrak{q} in R_x which contains I_x . Then $\rho^{-1}(\mathfrak{q})$ is a prime ideal in R which contains I but does not contain x . Indeed, if $\rho^{-1}(\mathfrak{q})$ contained x , then \mathfrak{q} would contain a unit, namely $x/1$, and hence would not be prime.

3. By parts 1 and 2, we have

$$\text{rad } I \neq \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p}$$

for *all* ideals I of R . In particular, since $N(R) = \text{rad } \langle 0 \rangle$, we have

$$N(R) = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Problem 3

Exercise 3. Let R be a commutative ring with identity. Denote the Jacobson radical of R by $J(R)$. Then $x \in J(R)$ if and only if $1 + ax$ is a unit for all $a \in R$.

Solution 3. Suppose $x \in J(R)$ and assume for a contradiction that $1 + ax$ is not a unit for some $a \in R$. Choose a maximal ideal in R which contains $1 + ax$, say \mathfrak{m} . Since $x \in J(R)$, we see that in particular $x \in \mathfrak{m}$. Since $1 + ax$ and ax belong to \mathfrak{m} , their difference also belongs to \mathfrak{m} . In other words, $1 \in \mathfrak{m}$. This contradicts the fact that \mathfrak{m} is a proper ideal of R . Thus our original assumption was wrong, which means that $1 + ax$ is a unit for all $a \in R$.

Conversely, suppose $1 + ax$ is a unit for all $a \in R$ and assume for a contradiction that $x \notin J(R)$. Choose a maximal ideal in R which does not contain x , say \mathfrak{m} . Then $Rx + \mathfrak{m} = R$ since \mathfrak{m} is maximal. Thus there exists $a \in R$ and $y \in \mathfrak{m}$ such that $ax + y = 1$, or in other words,

$$1 - ax = y.$$

By assumption, this implies y is a unit. This contradicts the fact that $y \in \mathfrak{m}$ and \mathfrak{m} is a proper ideal.

Problem 4

Exercise 4. Let R be an integral domain. Then

$$R = \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$

Solution 4. Since R is an integral domain, it has no zerodivisors. Thus all of the localization maps $\rho_{\mathfrak{m}}: R \rightarrow R_{\mathfrak{m}}$ are injective. In fact, they are just inclusion maps since we are identifying R and its localizations $R_{\mathfrak{m}}$ with subrings of the fraction field K of R . Thus we have

$$R \subseteq \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$

For the reverse inclusion, let $\gamma \in R_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} in R . Consider the set

$$R : \gamma = \{a \in R \mid a\gamma \in R\}.$$

Note that since $\gamma \in K$, we can express it as $\gamma = x/y$ where $x \in R$ and $y \neq 0$. Then it's easy to see that $y \in R : \gamma$. So $R : \gamma$ can be thought of as "the set of all denominators of γ ". It is easy to see that $R : \gamma$ is an ideal in R . We claim that $R : \gamma = R$. Indeed, assume for a contradiction that $R : \gamma$ is proper ideal of R . Then $R : \gamma$ is contained in a maximal ideal, say \mathfrak{m} . However this means that $\gamma \notin R_{\mathfrak{m}}$: if $\gamma \in R_{\mathfrak{m}}$, then we could express it as $\gamma = x/y$ where $x \in R$ and $y \notin \mathfrak{m}$. Then $y \in R : \gamma \subseteq \mathfrak{m}$ which is a contradiction. So we've found a maximal ideal \mathfrak{m} such that $\gamma \notin R_{\mathfrak{m}}$ which gives us a contradiction. Thus $R : \gamma = R$. In that case, we see that $1 \in R : \gamma$, so $\gamma = 1 \cdot \gamma \in R$. Thus we have

$$R \supseteq \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$