PDG Algebras and Modules

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Contents

1 Introduction

1.1 Notation and Conventions

Unless otherwise specified, let K be a field and let $(R, \mathfrak{m})$ be a local Noetherian ring.

1.1.1 Category Theory

In this document, we consider the following categories:

- The category of all sets and functions, denoted **Set**;
- The category of all rings and ring homomorphisms, denoted **Ring**;
- The category of all R-modules and R-linear maps, denoted $\mathbf{Mod}_R$;
- The category of all graded R-modules and graded R-linear maps, denoted $\mathbf{Grad}_R$;
- The category of all R-algebras R-algebra homorphisms, denoted $\mathbf{Alg}_R$;
- The category of all R-complexes and chain maps, denoted $\mathbf{Comp}_R$;
- The category of all R-complexes and homotopy classes of chain maps, denoted $\mathbf{HComp}_R$
- The category of all DG R-algebras DG algebra homomorphisms, denoted $\mathbf{DG}_R$.

2 Basic Definitions

2.1 PDG R-Algebras

Let (A, d) be an R-complex algebra and let $\mu \colon A \otimes_R A \to A$ be a chain map. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i \star_\mu b_i.$$

If μ is understood from context, then we also tend to drop μ from the subscript in $\star_\mu$, or even drop $\star$ altogether and write

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i b_i.$$

Note that μ being a chain map implies it is a **graded-multiplication** which satisfies **Leibniz law**. Being a graded-multiplication means μ is an R-bilinear map which respects the grading. In particular, if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. Satisfying Leibniz law means

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$

for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$. We can also impose other conditions on μ as follows:

1. We say μ is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.
2. We say μ is **graded-commutative** if
$$ab = (-1)^i ba$$
for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$.
3. We say μ is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$aa = 0$$
for all $a \in A_i$ for all i odd.

4. We say μ is **unital** if there exists $1 \in A$ such that

$$a1 = a = 1a$$

for all $a \in A$.

The triple (A, d, μ) is called **differential graded R-algebra** (or **DG R-algebra**) if μ satisfies conditions 1-4. If (A, d, μ) only satisfies conditions 2-4, then it is called a **partial differential graded R-algebra** (or **PDG R-algebra**). To clean notation in what follows, we will often refer to a PDG R-algebra (A, d, μ) via its underlying graded R-module A. In particular, if we write "let A be a PDG R-algebra" without specifying its differential or multiplication, then it will be understood that it's differential is denoted d_A and its multiplication is denoted μ_A.

Given two PDG R-algebras A and A', a **morphism** between them is a chain map $\varphi\colon A \to A'$ which satisfies the following two properties

1. it respects the identity elements, that is, $\varphi(1) = 1$;
2. it respects multiplication, that is, $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

It is straightforward to check that the collection of all PDG R-algebras algebras together with their morphisms forms a category, which we denote by $\mathbf{PDG}_R$.

2.2 PDG A-Modules

Unless otherwise specificed, we fix A to be a PDG R-algebra. A (left) **partial differential graded A-module** (or **PDG A-module**) is a triple (M, d_M, μ_M) where (M, d_M) is an R-complex and where $\mu_M\colon A \otimes_R M \to M$ is a chain map which satisfies $1u = u$ for all $u \in M$. Here again we are using the convention that the image of a tensor $\sum_{i=1}^n a_i \otimes u_i$ in $A \otimes_R M$ under the map μ_M is denoted by

$$\mu_M\left(\sum_{i=1}^n a_i \otimes u_i\right) = \sum_{i=1}^n a_i \star_{\mu_M} u_i = \sum_{i=1}^n a_i u_i$$

where μ_M is understood from context. Also, as before, if we write "let M be a PDG A-module" without specifying its differential or scalar multiplication, then it will be understood that it's differential is denoted d_M and its multiplication is denoted μ_M. Note that μ_M being a chain map implies it is satisfies **Leibniz law**, which in this context says

$$\mathrm{d}_M(au) = \mathrm{d}_A(a)u + (-1)^i a\mathrm{d}_M(u)$$

for all $a \in A_i$ and $u \in M$ for all $i \in \mathbb{Z}$. If in addition μ_M is **associative**, meaning $(ab)u = a(bu)$ for all $a, b \in A$ and $u \in M$, then we say M is a **DG A-module**.

Given two PDG A-modules M and N, an **A-linear map** betweem them is a chain map $\varphi\colon M \to N$ which satisfies $\varphi(au) = a\varphi(u)$ for all $a \in A$ and $u \in M$. The collection of all PDG A-modules together with their A-linear maps forms a category, which we denote by $\mathbf{PMod}_A$.

2.2.1 Submodules

Let M and N be two PDG A-modules. We say M is a PDG **A-submodule** of N if $M \subseteq N$. A PDG A-submodule of A is called an **ideal** of A. Given any collection $\{u_\lambda\}_{\lambda \in \Lambda}$ of elements of M, we denote by $\langle u_\lambda \rangle_{\lambda \in \Lambda}$ to be the smallest PDG A-submodule of M.

Warning: In the category of R-modules, we have the concept of annihilators. In particular, suppose M is an R-module and let $u \in M$. We define the **annihilator** with respect to u to be the subset of R given by

$$0 : u = \{r \in R \mid ru = 0\}.$$

In fact, $0 : u$ is in an ideal of R, but we need the associative law to get this: if $r \in R$ and $x \in 0 : u$, then $(rx)u = r(xu) = 0$ implies $rx \in 0 : u$.

Now let us consider the case where M is a PDG A-module and let $u \in M$. We can define the annihilator $0 : u$ with respect to u as a subset of A as before:

$$0 : u = \{a \in A \mid au = 0\},$$

however as noted above, $0 : u$ need not be a PDG ideal of A. On the other hand, if $u \in \operatorname{Assoc} M$, where

$$\operatorname{Assoc} M = \{u \in M \mid [a, b, u] = 0 \text{ for all } a, b \in A\},$$

then there are no issues with the proof above, so $0 : u$ is an ideal of R in this case.

2.2.2 Hom

Let M and N be two PDG A-modules. We denote by $\mathrm{Hom}_A(M,N)$ to be the set of all A-linear maps from M to N. The set $\mathrm{Hom}_A(M,N)$ as the structure of a PDG A-module via the

2.2.3 Tensor

2.3 Homology of $[M]$

Let A be a PDG R-algebra and let M be a PDG A-module. It is easy to see that μ_M is associative if and only if $[M]=0$. Given that $[M]$ is an R-complex, we have a weaker form of associativity:

Definition 2.1. We say μ_M is **homologically associative** if $\mathrm{H}([M])=0$.

Clearly if μ_M is associative, then μ_M is homologically associative. It turns out that the converse is also true if M bounded below and is **minimal**, that is, if $\mathrm{d}_M(M)\subseteq \mathfrak{m}M$ where $\mathfrak{m}$ is the maximal ideal in the local ring R.

Proposition 2.1. *Let A be a PDG R-algebra and let M and a PDG A-module. Assume that M is minimal and bounded below. Then the following conditions are equivalent*

1. *μ_M is associative.*

2. *μ_M is homologically associative.*

Proof. Clearly 1 implies 2. To show 2 implies 1, we prove the contrapositive: assume μ_M is not associative, so $[M]\neq 0$. Choose $m\in\mathbb{Z}$ minimal so that $[M]_m\neq 0$ and $[M]_{m-1}=0$. By Nakayama's Lemma, we can find a triple (a,b,u) such that $|a|+|b|+|u|=m$ and such that $[a,b,u]\notin \mathfrak{m}[M]_m$. By minimality of m, we have $\mathrm{d}_{[M]}[a,b,u]=0$. Also, since M is minimal, we have $\mathrm{d}_M[M]\subseteq\mathfrak{m}[M]$. Thus $[a,b,u]$ represents a nontrivial element in homology. □

2.4 $\mathbf{PMod}_A$ is an Abelian Category

Throughout the rest of this subsection, we fix a PDG R-algebra A. We would like to talk about the concept of an exact sequence in $\mathbf{PMod}_A$. For this, we just need to check that $\mathbf{PMod}_A$ is abelian category. First let us check that it is a pre-additive category.

2.4.1 Kernels

Proposition 2.2. *Let M and M' be two PDG A-modules and let $\varphi\colon M\to M'$ be an A-linear map. Then $(\ker\varphi,\widetilde{\mathrm{d}},\widetilde{\mu})$ is a PDG A-submodule of M, where $\widetilde{\mathrm{d}}=\mathrm{d}|_{\ker\varphi}$ and $\widetilde{\mu}=\mu|_{\ker\varphi\otimes_R\ker\varphi}$.*

Proof. We just need to check that $\widetilde{\mathrm{d}}$ and $\widetilde{\mu}$ land in $\ker\varphi$. Then it will follow that $(\ker\varphi,\widetilde{\mathrm{d}},\widetilde{\mu})$ is a a PDG A-submodule of M since it will inherit the properties needed to be a PDG A-module from M. First we show $\widetilde{\mathrm{d}}$ lands in $\ker\varphi$. Let $u\in\ker\varphi$. Then

$$\begin{aligned}\varphi\mathrm{d}(u)&=\mathrm{d}\varphi(u)\\&=\mathrm{d}(0)\\&=0\end{aligned}$$

implies $\mathrm{d}(a)\in\ker\varphi$. It follows that $\widetilde{\mathrm{d}}$ lands in $\ker\varphi$. Now we show $\widetilde{\mu}$ lands in $\ker\varphi$. Let $u\otimes v$ be an elementary tensor in $\ker\varphi\otimes_R\ker\varphi$. Then

$$\begin{aligned}\varphi(\mu(u\otimes v))&=\varphi(uv)\\&=\varphi(u)\varphi(v)\\&=0\star 0\\&=0.\end{aligned}$$

It follows that $\widetilde{\mu}$ lands in $\ker\varphi$. □

2.4.2 Images

Proposition 2.3. *Let* $\varphi\colon (A,\mathrm{d},\mu) \to (A',\mathrm{d}',\mu')$ *be a morphism of* R*-complex algebras. Then* $(\operatorname{im}\varphi, \widetilde{\mathrm{d}}', \widetilde{\mu}')$ *is an* R*-complex algebra, where* $\widetilde{\mathrm{d}}' = \mathrm{d}'|_{\ker\varphi}$ *and* $\widetilde{\mu}' = \mu'|_{\operatorname{im}\varphi\otimes_R \operatorname{im}\varphi}$.

Proof. We just need to check that $\widetilde{\mathrm{d}}'$ and $\widetilde{\mu}'$ land in $\ker\varphi$. Then it will follow that $(\ker\varphi, \widetilde{\mathrm{d}}, \widetilde{\mu})$ is an R-complex algebra since it will inherit the properties needed to be an R-complex algebra from (A,d,μ). First we show $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a) \in \operatorname{im}\varphi$. Then

$$\begin{aligned}\mathrm{d}'(\varphi(a)) &= \mathrm{d}'\varphi(a)\\ &= \varphi\mathrm{d}(a)\\ &= \varphi(\mathrm{d}(a)).\end{aligned}$$

It follows that $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Now we show $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a)\otimes\varphi(b)$ be an elementary tensor in $\operatorname{im}\varphi\otimes_R\operatorname{im}\varphi$. Then

$$\begin{aligned}\mu((\varphi(a)\otimes\varphi(b)) &= \varphi(a)\star\varphi(b)\\ &= \varphi(a\star b)\\ &= \varphi(\mu(a\otimes b)).\end{aligned}$$

It follows that $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. □

2.4.3 Cokernels

As we've seen, both kernels and images exist in $\mathbf{CompAlg}_R$. The problem however is that cokernels do not necessarily exist in $\mathbf{CompAlg}_R$. To see what goes wrong, suppose $\varphi\colon (A,\mathrm{d},\mu)\to(A',\mathrm{d}',\mu')$ be a morphism of R-complex algebras. A naive attempt at defining the cokernel of φ would go as follows: first we take the cokernel of the underlying R-complexes, namely $(\overline{A'}, \overline{\mathrm{d}'})$ where $\overline{A'} = A'/\operatorname{im}\varphi$ and $\overline{\mathrm{d}'}$ is defined by $\overline{\mathrm{d}'}(\overline{a'}) = \overline{\mathrm{d}'(a')}$ for all $\overline{a'}\in\overline{A'}$. It is straightforward to check that $\overline{\mathrm{d}'}$ is well-defined and gives $\overline{A'}$ the structure of an R-complex. Next we define multiplication $\overline{\mu'}\colon \overline{A'}\otimes_R\overline{A'}\to\overline{A'}$ by

$$\overline{\mu'}(\overline{a'}\otimes\overline{b'}) = \overline{a'\star_{\mu'} b'} \tag{1}$$

for all elementary tensors $\overline{a'}\otimes\overline{b'}$ in $\overline{A'}\otimes_R\overline{A'}$ and extending $\overline{\mu'}$ everywhere else R-linearly. Unfortunately, upon further inspection, we see that (**??**) is note well-defined. Indeed, if $a'+\varphi(a)$ is another representative of the coset $\overline{a'}$ and $b'+\varphi(b)$ is another representative of the coset $\overline{b'}$, then we have

$$\begin{aligned}\overline{\mu'}(\overline{a'+\varphi(a)}\otimes\overline{b'+\varphi(b)}) &= \overline{(a'+\varphi(a))\star(b'+\varphi(b))}\\ &= \overline{a'\star b'}+\overline{a'\star\varphi(b)}+\overline{\varphi(a)\star b'}+\overline{\varphi(a)\star\varphi(b)}\\ &= \overline{a'\star b'}+\overline{a'\star\varphi(b)}+\overline{\varphi(a)\star b'}+\overline{\varphi(a\star b)}\\ &= \overline{a'\star b'}+\overline{a'\star\varphi(b)}+\overline{\varphi(a)\star b'}.\end{aligned}$$

In particular, (**??**) is well-defined if and only if $\operatorname{im}\varphi$ is an ideal of A'.

2.5 Associator Functor

Let A be a PDG R-algebra and let M be a PDG A-module. Given a triple (a,b,u) where $a,b\in A$ and $u\in M$, its **associator triple** $[a,b,u]$ is defined by

$$[a,b,u] = (ab)u - a(bu). \tag{2}$$

More generally, if $\alpha_{A,A,M}\colon (A\otimes_R A)\otimes_R M\to A\otimes_R(A\otimes_R M)$ denotes the unique chain map defined on elementary tensors by $(a\otimes b)\otimes u\mapsto a\otimes(b\otimes u)$, then we define the **associator** with respect to M to be chain map $[\cdot,\cdot,\cdot]_{\mu_M}\colon (A\otimes_R A)\otimes_R M\to M$ defined by

$$[\cdot,\cdot,\cdot]_{\mu_M} := \mu_M(1\otimes\mu_M)\alpha_{A,A,M} - \mu_M(\mu_A\otimes 1).$$

If μ_M is understood from context, then we will simplify our notation by dropping μ_M from the subscript in $[\cdot,\cdot,\cdot]$. Thus, if $(a\otimes b)\otimes u$ is an elementary tensor in $(A\otimes_R A)\otimes_R M$, then $[\cdot,\cdot,\cdot]((a\otimes b)\otimes u) = [a,b,u]$ as defined above in (2). We denote by $[A,A,M]$ to be the image of $[\cdot,\cdot,\cdot]$. If A is understood from context, then we will simplify our notation even further by writing $[M]$ instead of $[A,A,M]$. Thus

$$[M] = \operatorname{span}_R\{[a,b,u]\mid a,b\in A \text{ and } u\in M\}.$$

Since $[\cdot,\cdot,\cdot]$ is a chain map from $(A \otimes_R A) \otimes_R M$, we see that $[\cdot,\cdot,\cdot]$ is a graded trilinear map satisfies Leibniz law, where Leibniz law in this case is the equation

$$\mathrm{d}_{[M]}[a,b,u] = [\mathrm{d}_A(a),b,u] + (-1)^{|a|}[a,\mathrm{d}_A(b),u] + (-1)^{|a|+|b|}[a,b,\mathrm{d}_M(u)]. \tag{3}$$

for all homogeneous $a, b \in A$ and $u \in M$.

Now suppose M' is another PDG A-module and $\varphi \colon M \to M'$ is an A-linear. We obtain an induced map of R-complexes $[\varphi] \colon [M] \to [M']$, where $[\varphi]$ is the unique chain map which satisfies

$$\begin{aligned}
[\varphi][a,b,u] &= \varphi((ab)u - a(bu)) \\
&= \varphi((ab)u) - \varphi(a(bu)) \\
&= (ab)\varphi(u) - a\varphi(bu) \\
&= (ab)\varphi(u) - a(b\varphi(u)) \\
&= [a,b,\varphi(u)].
\end{aligned}$$

In particular, we have $[\varphi]$ is just the restriction of φ to $[M]$. It is straightforward to check that the assignment $M \mapsto [M]$ and $\varphi \mapsto [\varphi]$ gives rise to a functor

$$\mathcal{A} \colon \mathbf{PMod}_A \to \mathbf{Comp}_R$$

which we call the **associator functor**.

2.5.1 The associator functor need not be exact

The associator functor $\mathcal{A} \colon \mathbf{PMod}_A \to \mathbf{Mod}_R$ need not be exact. To see what goes wrong, let

$$0 \longrightarrow M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{4}$$

be a short exact sequence of PDG A-modules. We obtain an induced sequence of R-complexes

$$0 \longrightarrow [M_1] \xrightarrow{[\varphi_1]} [M_2] \xrightarrow{[\varphi_2]} [M_3] \longrightarrow 0$$

We claim that we have exactness at $[M_1]$ and $[M_3]$. Indeed, this is equivalent to showing $[\varphi_1]$ is injective $[\varphi_3]$ is surjective, and this follows from the fact that $[\varphi_1]$ is restriction of the injective function φ_1 and $[\varphi_3]$ is the restriction of the surjective function φ_3. Let us see what goes wrong when trying to prove exactness at $[M_2]$. Let $\sum_{i=1}^n [a_i, b_i, v_i] \in \ker[\varphi_2]$. In particular, we have $\sum_{i=1}^n [a_i, b_i, v_i] \in \ker \varphi_2$. By exactness of (4), there exists $u \in M_1$ such that $\varphi_1(u) = \sum_{i=1}^n [a_i, b_i, v_i]$. It is not at all clear however that $u \in [M_1]$.

3 Example

Let $R = \mathbb{F}_2[x,y,z,w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2 \rangle$, and let F be the free minimal resolution of R/I over R. The complex F is supported on the simplicial complex drawn below:

Consider the multiplication on F defined as follows: in degree 1 we have the multiplication table

	e_1	e_2	e_3	e_4	e_5
e_1	0	e_{12}	e_{13}	xe_{14}	$yz^2e_{14} + xe_{45}$
e_2	e_{12}	0	we_{23}	e_{24}	$y^2ze_{23} + we_{35}$
e_3	e_{13}	we_{23}	0	e_{34}	ze_{35}
e_4	xe_{14}	e_{24}	e_{34}	0	ye_{45}
e_5	$yz^2e_{14} + xe_{45}$	$y^2ze_{23} + we_{35}$	ze_{35}	ye_{45}	0

in degree 3 we have the multiplication table

	e_{12}	e_{45}	e_3	e_4	e_5
e_1			e_{13}	xe_{14}	$yz^2e_{14} + xe_{45}$
e_2		$yze_{234} + we_{345}$	we_{23}	e_{24}	$y^2ze_{23} + we_{35}$
e_3			0	e_{34}	ze_{35}
e_4			e_{34}	0	ye_{45}
e_5	$y^2ze_{123} + yzwe_{134} + xwe_{345}$		ze_{35}	ye_{45}	0

4 Grobner Basis Computations