Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, g : X \to \mathbb{R}$ be two measurable functions. Prove that the sets $\{x \in X : f(x) = g(x)\}$ and $\{x \in X : f(x) < g(x)\}$ are both measurable sets.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $A, B \subseteq X$ are two measurable sets such that $X = A \cup B$. Prove that $f : X \to \mathbb{R}$ is measurable if and only if its restrictions $f : A \to \mathbb{R}$ and $f : B \to \mathbb{R}$ are both measurable.

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Prove that for any $\epsilon > 0$ there exists a bounded measurable function $g : X \to \mathbb{R}$ such that $\mu(\{f \neq g\}) < \epsilon$.

Problem 4. Let $(X, \mathcal{M}, \mu)$ be a measure space. A set E is said to be locally measurable if $E \cap A$ is measurable for each $A \in \mathcal{M}$ with finite measure. Recall that the collection $\mathcal{L}$ of all locally measurable sets is a σ-algebra. A function $f : X \to \mathbb{R}$ is said to be locally measurable if $f 1_A$ is measurable for every $A \in \mathcal{M}$ with finite measure. Prove that $f : X \to \mathbb{R}$ is locally measurable if and only if it is measurable with respect to the σ-algebra $\mathcal{L}$ of locally measurable sets.

Problem 5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n X \to \mathbb{R}$ be a sequence of measurable functions. Prove that the set $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists } \}$ is a measurable set.

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be a measurable function. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the function $h \circ f : X \to \mathbb{R}$ is also measurable.

Problem 7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{A_r\}_{r\in\mathbb{Q}}$ be a collection of measurable sets $A_r \in \mathcal{M}$ such that $A_r \subseteq A_s$ whenever $r < s$. Assume also that $\cup_{r\in\mathbb{Q}} A_r = X$ and $\cap_{r\in\mathbb{Q}} A_r = \emptyset$. Prove that there exists a unique measurable function $f : X \to \mathbb{R}$ such that $A_r \subseteq \{f \leq r\}$ and $\{f \geq r\} \subseteq A_r^c$.

For the next few problems the concept of a complete measure space is needed. This was introduced and discussed in the lecture posted on the e-learning day in February. You can still find it posted in Canvas in the Files folder.

Problem 8. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and let $f : X \to \mathbb{R}$ be a measurable function. Prove that if $f = g$ a.e., then g is also measurable.

Problem 9. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and let $\{f_n\}$ be a sequence of measurable functions which converges to a function f a.e.. Prove that f is also measurable.

Problem 10. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \bar{\mathcal{M}}, \bar{\mu})$ be its completion. A function $f : X \to \mathbb{R}$ is $\bar{\mathcal{M}}$-measurable if and only if there exists $g : X \to \mathbb{R}$ which is $\mathcal{M}$-measurable and such that $f = g$ everywhere except on a set $E \in \mathcal{M}$ with $\mu(E) = 0$.

Problem 11. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Let $\{B_r\}_{r\in\mathbb{Q}}$ be a collection of measurable sets $B_r \in \mathcal{M}$ such that $\mu(B_r \setminus B_s) = 0$ whenever $r < s$. Assume also that $\cup_{r\in\mathbb{Q}} B_r = X$ and $\cap_{r\in\mathbb{Q}} B_r = \emptyset$. Prove that there exists a measurable function $f : X \to \mathbb{R}$ such that $\mu(B_r \setminus \{f \leq r\}) = 0$ and $\mu(\{f \geq r\} \setminus B_r^c) = 0$.

Problem 12. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0, \infty)$ be a non-negative integrable function. Show that for every $\epsilon > 0$ there exists a measurable set $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_X 1_E f d\mu > \int_X f d\mu - \epsilon$.

Problem 13. Let $(X, \mathcal{M}, \mu)$ be a measure space and $g : X \to \mathbb{R}$ be a nonnegative measurable function. In HW5 you proved that the set function $\nu : \mathcal{M} \to [0, \infty]$ defined by $\nu(E) = \int g 1_E d\mu$ is a measure on $\mathcal{M}$. Prove that for any nonnegative measurable function $f : X \to \mathbb{R}$ the following identity holds $\int f d\nu = \int f g d\mu$.

Problem 14. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be an integrable function. Prove that $\left| \int_X f d\mu \right| < \int_X |f| d\mu$.

Problem 15. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be an integrable function. Suppose $\{E_n\}$ is a sequence of measurable sets such that $\mu(E_n) \to 0$, as $n \to \infty$. Prove that $\int_{E_n} f d\mu = 0$.

Problem 16. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be an integrable function. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that for each measurable set E with $\mu(E) < \delta$ we have $\left| \int f d\mu \right| < \epsilon$.

Problem 17. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function. Prove that f is integrable if and only if

$$\sum_{k=1}^{\infty} k\mu\{k \leq |f| < k+1\} < \infty.$$

Problem 18. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ be a nonnegative measurable function. Prove that f is integrable if and only if

$$\sum_{k=1}^{\infty} \mu\{f \geq k\} < \infty.$$

Problem 19. Prove that if the underlying measure space is complete then the pointwise convergence can be replaced with the pointwise almost everywhere convergence in

(a) the Monotone Convergence Theorem;

(b) Fatou's Lemma;

(c) the Dominated Convergence Theorem.

Problem 20. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n : X \to [0, \infty)$ be a sequence of non-negative measurable functions such that $f_n \to f$ pointwise. Prove that if $f_n(x) \leq f(x)$ for every $x \in X$ and all $n \in \mathbb{N}$ then

$$\int_X f_n d\mu \to \int_X f d\mu.$$

Problem 21. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. Let (f_n) be a sequence of L^p functions such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for almost every $x \in X$ and in particular $\lim_{n\to\infty} f_n(x) = 0$ for a.e. $x \in X$. Prove also that the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is in L^p.

Problem 22. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n : X \to \mathbb{R}$ is a sequence of integrable functions such that $\int_{\mathbb{R}} |f_n| d\mu < \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} f_n(x) = 0$ for a.e. $x \in X$.

Problem 23. Let $(X, \mathcal{M}, \mu)$ be a measure space. We proved in class that $f_n \to f$ in L^1 sense (i.e. $\|f_n - f\|_1 \to 0$) doesn't in general imply that $f_n \to f$ pointwise a.e.. Prove that if for some $\delta > 0$ we have $\|f_n - f\|_1 < \frac{1}{n^{1+\delta}}$ for all $n \in \mathbb{N}$, then $f_n \to f$ pointwise a.e.. Prove also that $\|f_n - f\|_1 < \frac{1}{n}$ for all $n \in \mathbb{N}$ does not in general imply $f_n \to f$ pointwise a.e..

Problem 24. Prove that the pointwise convergence can be replaced by convergence in measure in

(a) Monotone Convergence Theorem;

(b) Fatou's Lemma;

(c) Dominated Convergence Theorem.

Problem 25. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that:

(i) If $f_n \to f$ a.e., then $h \circ f_n \to h \circ f$ a.e..

(ii) If $f_n \to f$ in measure, then $h \circ f_n \to h \circ f$ in measure.

(iii) Prove that if $h : \mathbb{R} \to \mathbb{R}$ is uniformly continuous then (ii) continues to hold even when $\mu(X) = \infty$.

Problem 26. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_n \to f$ in measure and $g_n \to g$ in measure. Prove that $f_n + g_n \to f + g$ in measure. If $\mu(X) < \infty$ prove that $f_n g_n \to fg$ in measure.

Problem 27. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Prove that if a sequence $\{f_n\}$ of measurable functions converges in measure to both f and g, then $f = g$ a.e..

Problem 28. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. If $\{f_n\}$ is a monotone sequence of measurable functions such that $f_n \to f$ in measure, then $f_n \to f$ a.e..

Problem 29. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that $f_n \to f$ in measure if and only if each subsequence of $\{f_n\}$ has a further subsequence that converges to f a.e..

Problem 30. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that $f_n \to f$ pointwise implies $f_n \to f$ in measure. Prove that the converse is not true in general.

Problem 31. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that if $f_n \to f$ in measure if and only if

$$\int \frac{|f_n - f|}{1 + |f_n - f|} d\mu \to 0.$$

Problem 32. We know that $f_n \to f$ in L^1 implies $f_n \to f$ in measure. Prove that the converse is not true in general.

Problem 33. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that if $\{f_n\}$ converges to f almost uniformly, then there exists a subsequence $\{f_{n_k}\}$ which converges a.e. to f.

Problem 34. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that if $\{f_n\}$ converges to f almost uniformly, then $f_n \to f$ pointwise a.e..

Problem 35. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that if $\{f_n\}$ converges to f almost uniformly, then $f_n \to f$ in measure.

Problem 36. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f_n : X \to [0, \infty)$ be a sequence of measurable functions such that $f_n \to f$ pointwise. Prove that there exist measurable sets $F, E_1, E_2, E_3, \dots \subseteq X$, such that $\mu(F) = 0, X = \cup_{n=1} E_n \cup F$, and $f_n \to f$ uniformly on every $E_n, n \in \mathbb{N}$.

In the following problems the measure space is $X = \mathbb{R}$ equipped with the Lebesgue σ-algebra and the Lebesgue measure m. As usual we will denote $dm(x)$ by dx.

Problem 37. Using the fact that Lebesgue measure m on $\mathbb{R}$ is translation invariant, that is, for every measurable set E and $a \in \mathbb{R}$, $m(E) = m(a + E)$, prove that for every $f \in L^1(\mathbb{R})$ and every $a \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f(x + a) dx.$$

Problem 38. Let $f : \mathbb{R} \to [0, \infty)$ be a nonnegative measurable function on R. Prove that if the series $\sum_{n=1}^{\infty} f(x + n)$ converges a.e., and $g(x) := \sum_{n=1}^{\infty} f(x + n)$ is integrable on $\mathbb{R}$, then $f = 0$ a.e..

Problem 39. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous, integrable function. Prove that $\lim_{x \to \infty} f(x) = 0$. Does this still hold if f is continuous, but not uniformly continuous? Prove it or give a counterexample.

Problem 40. Suppose $f : \mathbb{R} \to [0, \infty)$ is a non-negative measurable function. Prove that

$$\lim_{n \to \infty} \int_{-n}^{n} f(x) dx = \int_{\mathbb{R}} f(x) dx.$$

Problem 41. Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x + h) - f(x)| dx = 0.$$

Problem 42. Suppose $f : \mathbb{R} \to [0, \infty)$ is a non-negative measurable function such that the series $\sum_{n=1}^{\infty} \int_{\mathbb{R}} f^n$ converges. Prove that $f < 1$ a.e., and that $\frac{f}{1-f}$ is integrable.

Problem 43. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an integrable function. Prove that for any $\delta > 0$, $\lim_{n\to\infty} \frac{f(nx)}{n^\delta} = 0$ for a.e. $x \in \mathbb{R}$.

In the remaining problems the measure space is $X = [0,1]$ equipped with the Lebesgue σ-algebra and the Lebesgue measure m. As usual we will denote $dm(x)$ by dx.

Problem 44. Compute the following limits:

(a)

$$\lim_{n\to\infty} \int_0^1 \frac{n \sin(\frac{x}{n})}{x} dx.$$

(b)

$$\lim_{n\to\infty} \int_0^1 \frac{n^3 x^{3/4}}{1 + n^4 x^2} dx.$$

(c)

$$\lim_{n\to\infty} \int_0^1 \frac{nx \sin(nx)}{1 + n^2 x^2} dx.$$

(d)

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} \frac{n}{1 + n^2 x^2 + n^4 x^6} dx.$$

Problem 45. Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x) dx = 0$.

Problem 46. Let $f : [0,1] \to \mathbb{R}$ be a non-negative, integrable function. Prove that $\lim_{n\to\infty} \int_0^1 f(x)^{1/n} dx = m(\{f > 0\})$.

Problem 47. Let $f : [0,1] \to \mathbb{R}$ be a non-negative, integrable function. Suppose that f is bounded above by 1, and $\int_0^1 f(x) dx = 1$. Prove that $f(x) = 1$ a.e..

Problem 48. Let $f : [0,1] \to \mathbb{R}$ be a measurable function. Prove that there exists a number h such that $m\{f \geq h\} \geq 1/2$ and for all $k > h$, $m\{f \geq k\} < 1/2$.

Problem 49. Let $f : [0,1] \to \mathbb{R}$ be a measurable function. Prove that for every $\epsilon > 0$ there exists a closed set F such that f is continuous at every point of F and $m([0,1] \setminus F) < \epsilon$.

Problem 50. Let $f : [0,1] \to \mathbb{R}$ be a function such that for every $\epsilon > 0$ there exists a closed set F such that f is continuous at every point of F and $m([0,1] \setminus F) < \epsilon$. Prove that f must be a measurable function.

Problem 51. Prove that for any measurable function $f : [0,1] \to \mathbb{R}$ there exists a sequence of continuous functions $\{f_n\}$ which converges to f a.e. on $[0,1]$.