# Permutativity

February 7, 2021

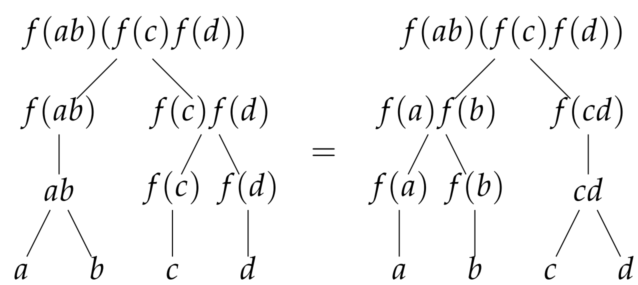
## 1 Introduction

We introduce a new type of algebraic law which we call the **permutative law** since it corresponds with the permutohedron as the associative law corresponds with associahedron.

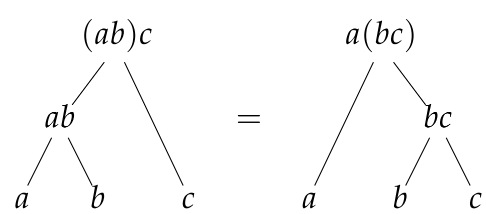
**Definition 1.1.** Let $A$ be a set equipped with a binary operation $\cdot : A \times A \to A$ and a unary operation $f\colon A \to A$. We say the triple $(A, f, \cdot)$ satisfies the **permutative law** if for all $a, b, c, d \in A$, we have

$$(f(a)f(b))f(cd) = f(ab)(f(c)f(d)) \tag{1}$$

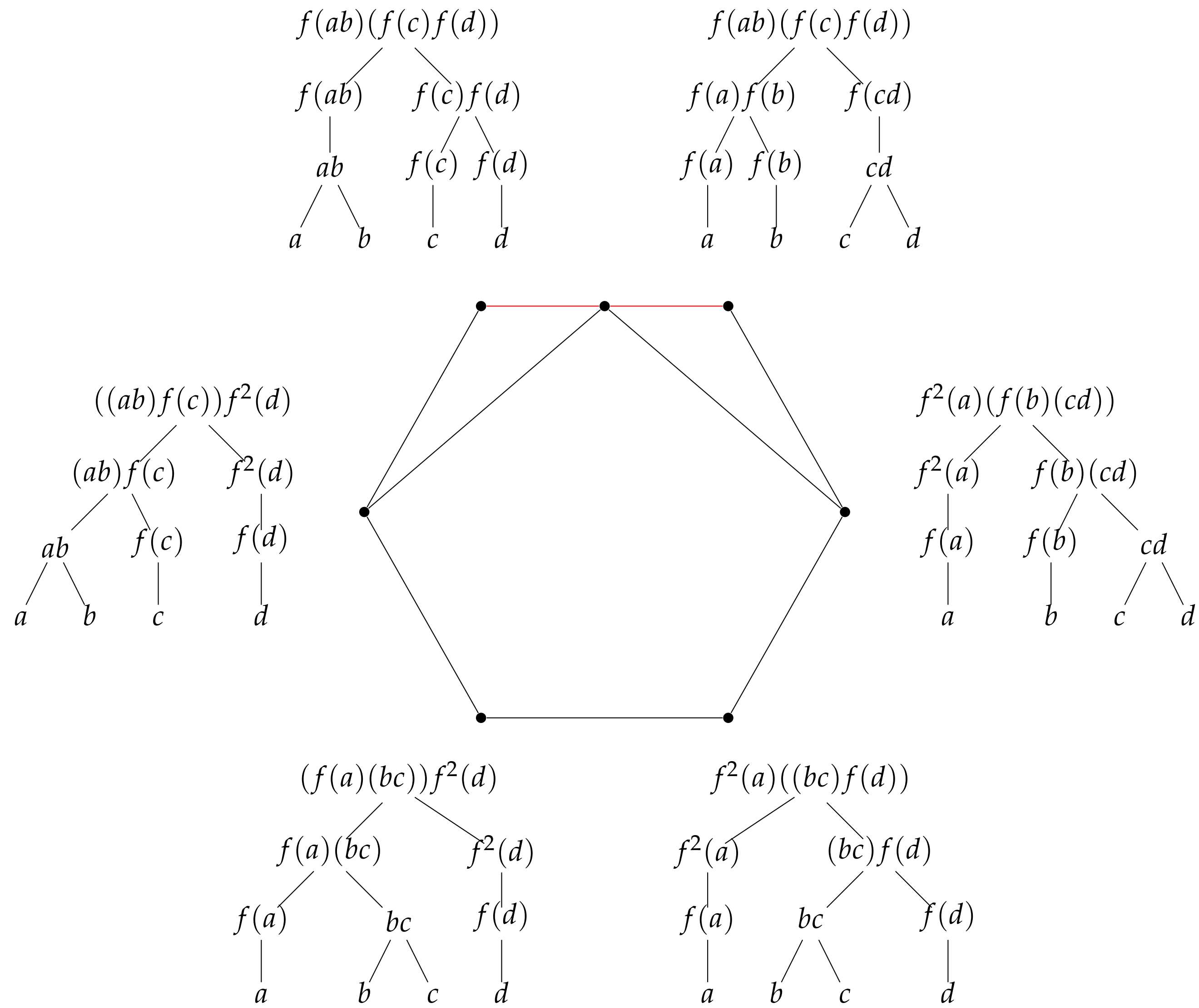
There's a very nice way of capturing visualizing this law in terms of Cayley ordered Bell trees:




This is analagous to how we can express the associative law in terms of binary rooted trees:




The difference between these two types of trees is that the Cayley ordered Bell trees keep track of a unary operator $f$, whereas the binary rooted trees do not. The Cayley ordered Bell trees can be attached to the vertices of the permutohedron and the ordered binary rooted trees can be attached to the vertices of the associahedron. There is a natural way to map the permutohedron to the associahedron, and it correponds to forgetting the unary operator $f$. In the image below, we draw the permutohedron $P_3$ as well as associahedron $K_4$ inside of it. To each vertex of $K_4$, we can attach a 4-leaf ordered rooted binary tree, and to each vertex of $P_3$, we can attach a 4-leaf Cayley ordered Bell tree, which we do here. The map from $P_3$ to $K_4$ can be visualized by collapsing the red edge, or in terms of trees, by deleting stretching nodes, or in terms of algebra, by setting $f$ to the identity function.




## 2 When Do Triples Satisfy The Permutative Law?

In order for a triple $(A, f, \cdot)$ to satisfy the permutative law, we need a mixture of both nice properties for $f$ and nice properties for the binary operation. Let us make a basic assumption on $f$ throughout the rest of this article in order to simplify our results: we will assume that $f\colon A \to A$ is a bijection.

### 2.1 Groups

In this subsection, we will consider triples $(G, f, \cdot)$ where $G$ is a group and $f\colon G \to G$ is a bijection. Here are two examples:

**Example 2.1.** Suppose $f\colon G \to G$ is a group homomorphism. Then the triple $(G, f, \cdot)$ satisfies the permutative law. Indeed, for all $a, b, c, d \in G$, we have

$$\begin{aligned} f(ab)(f(c)f(d)) &= (f(ab))f(c)f(d) \\ &= (f(a)f(b))f(cd) \end{aligned}$$

since the binary operation is associative and since $f$ is a group homomorphism.

**Example 2.2.** Let $G$ be a group with $x \in Z(G)$ and suppose $f(a) = xa$ for all $a \in G$. Then the triple $(G, f, \cdot)$ satisfies the permutative laws permutative. Indeed, for all $a, b, c, d \in G$,

$$\begin{aligned} f(ab)(f(c)f(d)) &= xabxcxd \\ &= xaxbxcd \\ &= (f(a)f(b))(f(cd)) \end{aligned}$$

since the binary operation is associative and since $x \in Z(G)$.

The next proposition tells us that any triples $(G, f, \cdot)$ which satisfies the permutative law essentially comes from one of the two examples above.

**Proposition 2.1.** *Denote $x = f(e)$. If the triple $(G, f, \cdot)$ satisfies the permutative law, then $x \in Z(G)$ and the map $\ell_x \circ f \colon G \to G$ is a group homomorphism.*

*Proof.* Since the triple $(G, f, \cdot)$ satisfies the permutative law, we have

$$f(ab)f(c)f(d) = f(a)f(b)f(cd). \tag{2}$$

for all $a, b, c, d \in G$. In particular, setting $a = b = e$ into (2) gives us

$$xf(c)f(d) = x^2 f(cd),$$

and after canceling $x$ on both sides, we obtain

$$f(c)f(d) = xf(cd). \tag{3}$$

Setting $d = e$ into (3) gives us

$$f(c)x = xf(c).$$

Thus $x \in Z(G)$. For the last part, we use (3) to obtain

$$\begin{aligned}(\ell_x \circ f)(cd) &= xf(cd) \\ &= x^2 f(c)f(d) \\ &= (xf(c))(xf(d)) \\ &= (\ell_x \circ f)(c)(\ell_x \circ f)(d)\end{aligned}$$

for all $c, d \in G$. □

## 2.2 $R$-Algebras

Let $R$ be a ring. An $R$-algebra $A$ is an $R$-module equipped with an $R$-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a + b)c = ab + ac \quad \text{and} \quad a(b + c) = ab + ac.$$

We say the $R$-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the $R$-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

In this case, we call $e$ the **identity** element. We say the $R$-algebra is **cancellative** if for any element $a \in A$ and any non-zero element $b \in A$ there exists precisely one element $c \in A$ with $a = bc$ and precisely one element $d \in A$ such that $a = db$.

### 2.2.1 Hom-Associative Algebras

**Definition 2.1.** A **hom-associative $R$-algebra**, denoted $(A, \alpha)$ or precisely $(A, \cdot, \alpha)$, is an $R$-algebra $A$ equipped with an $R$-linear map $\alpha \colon A \to A$ satisfying the hom-associative law

$$\alpha(a)(bc) = (ab)\alpha(c) \tag{4}$$

for any $a, b, c \in A$. We say $(A, \alpha)$ is **multiplicative** if $\alpha \colon A \to A$ is an $R$-algebra homomorphism. We say $(A, \alpha)$ is **weakly left unital** if there exists an $e_l \in A$, called a **weak left unit**, such that $e_l a = \alpha(a)$ for all $a \in A$. Similarly, we say $(A, \alpha)$ is **weakly right unital** if there exists an $e_r \in A$, called a **weak right unit**, such that $ae_r = \alpha(a)$ for all $a \in A$. We say $(A, \alpha)$ is **weakly unital** if there exists an $e \in A$, called a **weak unit**, such that $e$ is both a weak left unit and a weak right unit.

*Remark.* If the $R$-algebra $A$ is unital with unit $e$, then hom-associative $R$-algebra $(A, \alpha)$ is weak unital with weak unit $\alpha(e)$. Indeed, $\alpha(e)$ is weak right unit since

$$\begin{aligned}\alpha(a) &= \alpha(a)(ee)\\ &= (ae)\alpha(e)\\ &= a\alpha(e)\end{aligned}$$

for all $a \in A$. A similar calculation shows that $\alpha(e)$ is a weak left unit as well.

#### 2.2.2 Obtaining a hom-associative $R$-algebra from an $R$-algebra endomorphism.

**Proposition 2.2.** *Let $A$ be a unital associative $R$-algebra with unit $e$, and let $\alpha \colon A \to A$ be an $R$-algebra endomorphism. Define $\star \colon A \times A \to A$ by*

$$a \star b = \alpha(ab)$$

*for all $a, b \in A$. Then $(A, \star, \alpha)$ is a weakly unital hom–associative $R$-algebra with weak unit $e$.*

*Proof.* Let $a, b, c \in A$. Then we have

$$\begin{aligned}\alpha(a) \star (b \star c) &= \alpha(\alpha(a)(\alpha(bc))\\ &= \alpha(\alpha(a(bc)))\\ &= \alpha(\alpha((ab)c))\\ &= \alpha(\alpha(ab)\alpha(c))\\ &= (a \star b) \star \alpha(c).\end{aligned}$$

Thus $(A, \star, \alpha)$ is hom-associative.

To see that $e$ is a weak unit, let $a \in A$. Then we have

$$\begin{aligned}a \star e &= \alpha(ae)\\ &= \alpha(e)\\ &= \alpha(ea)\\ &= e \star a.\end{aligned}$$

Thus $e$ is a weak unit. □

#### 2.2.3 From Hom-Associativity to Permutativity

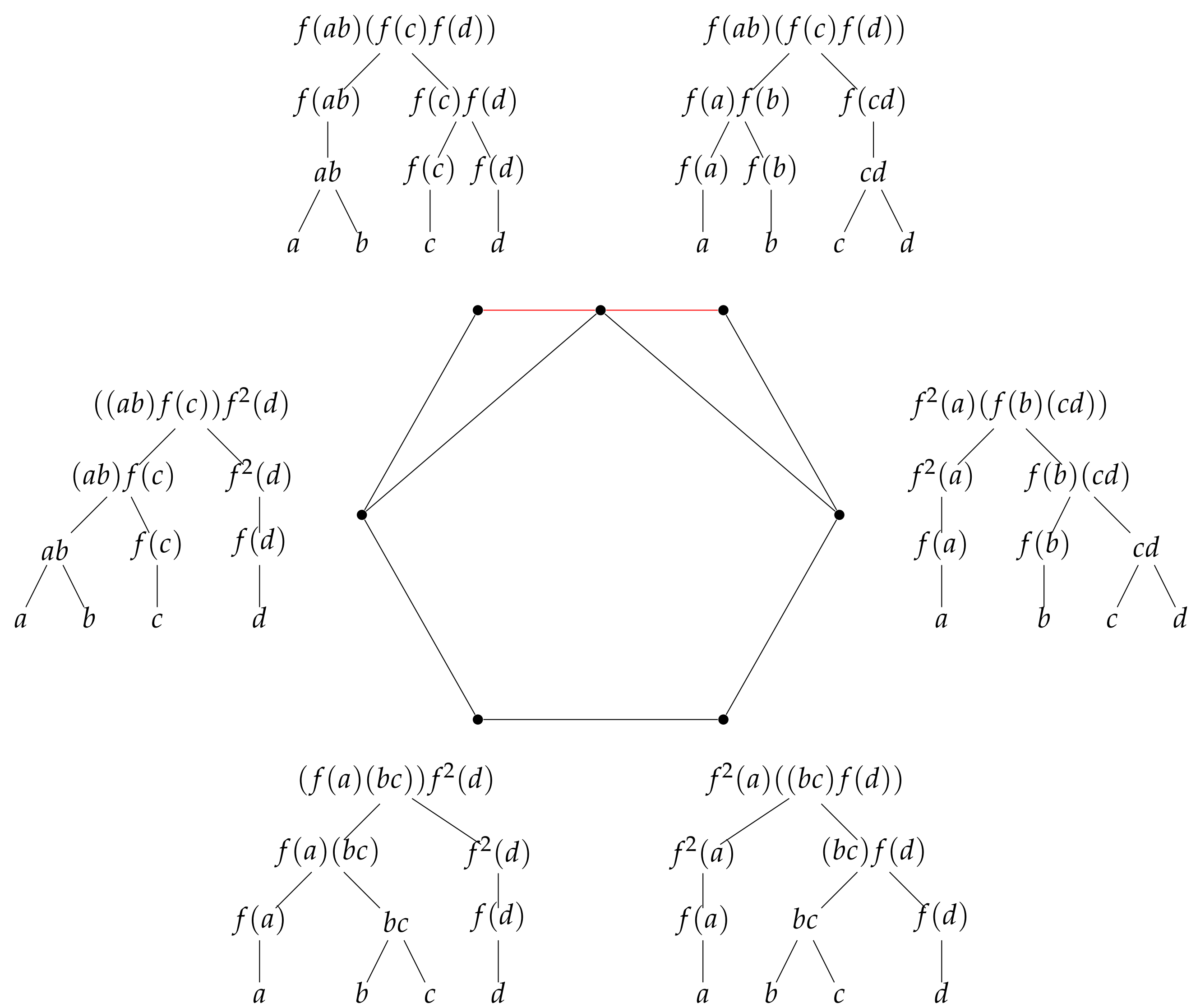
**Theorem 2.1.** *Every hom-associative $R$-algebra is a permutative $R$-algebra.*

*Proof.* Let $(A, f)$ be a hom-associative algebra. Then for all $a, b, c, d \in A$, we have

$$\begin{aligned}f(ab)(f(c)f(d)) &= ((ab)f(c))f^2(d)\\ &= (f(a)(bc))f^2(d)\\ &= f^2(a)((bc)f(d))\\ &= f^2(a)(f(b)(cd))\\ &= (f(a)f(b))f(cd).\end{aligned}$$

□

*Remark.* We can visualize this proof by tracing the edges of the permutohedron:




### 2.2.4 Obtaining a permutative $R$-algebra from an $R$-algebra endomorphism.

**Proposition 2.3.** *Let $A$ be an $R$-algebra and let $\alpha\colon A \to A$ be an $R$-algebra endomorphism. Define an $R$-bilinear map $\star\colon A \times A \to A$ by*

$$a \star b = \alpha(ab)$$

*for all $a, b \in A$. Then $(A, \star, \alpha)$ is a permutative $R$-algebra. Furthermore, if $A$ is unital, then $(A, \star, \alpha)$ is weakly unital.*

*Proof.* Let $a, b, c, d \in A$. Then we have

$$\begin{aligned}(\alpha(a) \star \alpha(b)) \star \alpha(c \star d) &= \alpha(\alpha(a)\alpha(b)) \star \alpha(\alpha(cd)) \\ &= \alpha(\alpha(ab)) \star \alpha(\alpha(c)\alpha(d)) \\ &= \alpha(a \star b) \star (\alpha(c) \star \alpha(d)).\end{aligned}$$

Thus $(A, \star, \alpha)$ is permutative.

Now assume that $A$ is unital with unit $e$. We claim that $\alpha(e)$ is a weak unit in $(A, \star, \alpha)$. Indeed, let $a \in A$. Then we have

$$\begin{aligned}a \star e &= \alpha(ae) \\ &= \alpha(e) \\ &= \alpha(ea) \\ &= e \star a.\end{aligned}$$

Thus $e$ is a weak unit. □

### 2.2.5 Hom-Lie Algebra

**Definition 2.2.** A **hom-Lie $R$-algebra**, denoted $(A, [\cdot,\cdot], \alpha)$, is an $R$-algebra $(A, [\cdot,\cdot])$ equipped with an $R$-linear map $\alpha\colon A \to A$ such that the following properties are satisfied:

1. (skew-symmetry) $[a,b]+[b,a]=0$ for all $a,b\in A$;

2. (Hom-Jacobi identity) $[\alpha(a),[b,c]]+[\alpha(b),[c,a]]+[\alpha(c),[a,b]]$ for all $a,b,c\in A$.

The $R$-algebra multiplication map $[\cdot,\cdot]\colon A\otimes_R A\to A$ is called the **hom-Lie bracket** and the $R$-linear endomorphism $\alpha\colon A\to A$ is called the **twisting map**.

**Proposition 2.4.** *Let $(M,\cdot,\alpha)$ be a hom-associative $R$-algebra with commutator $[\cdot,\cdot]$. Then $(M,[\cdot,\cdot],\alpha)$ is a hom-Lie algebra.*

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#### 2.2.6 From Hom-Lie to Permutativity

**Theorem 2.2.** *Every hom-Lie $R$-algebra is a permutative $R$-algebra.*

*Proof.* Let $(A,[\cdot,\cdot],\alpha)$ be a hom-Lie $R$-algebra. In the following calculation we denote $[a,b]$ by $ab$ for all $a,b\in A$ in order to make notation look cleaner. Then for all $a,b,c,d\in A$, we have

$$\begin{aligned}
\alpha(ab)(\alpha(c)\alpha(d)) &= -\alpha^2(c)(\alpha(d)(ab))-\alpha^2(d)((ab)\alpha(c))\\
&= \alpha^2(c)(\alpha(a)(bd))+\alpha^2(c)(\alpha(b)(da))+\alpha^2(d)(\alpha(c)(ab))\\
&= \alpha^2(c)(\alpha(a)(bd))+\alpha^2(c)(\alpha(b)(da))-\alpha^2(d)(\alpha(a)(bc))-\alpha^2(d)(\alpha(b)(ca))\\
&= \alpha^2(c)(\alpha(a)(bd))+\alpha^2(c)(\alpha(b)(da))-\alpha^2(d)(\alpha(a)(bc))-\alpha^2(d)(\alpha(b)(ca))\\
&= -\alpha^2(a)((bd)\alpha(c))-\alpha(bd)(\alpha(c)\alpha(a))-\alpha^2(b)((da)\alpha(c))-\alpha(da)(\alpha(c)\alpha(b))+\alpha^2(a)((bc)\alpha(d))+\alpha(bc)(\alpha(d)\alpha(a))+\alpha^2(b\\
&= -\alpha^2(a)(\alpha(c)(db))-\alpha^2(a)(\alpha(d)(bc))+\alpha^2(b)(\alpha(c)(ad))+\alpha^2(b)(\alpha(a)(dc))\\
&= -\alpha^2(a)(\alpha(c)(db))-\alpha^2(a)(\alpha(d)(bc))-\alpha^2(b)(\alpha(a)(cd))\\
&= \alpha^2(a)(\alpha(b)(cd))+\alpha^2(b)((cd)\alpha(a))\\
&= -\alpha(cd)(\alpha(a)\alpha(b))\\
&= (\alpha(a)\alpha(b))\alpha(cd),
\end{aligned}$$

where we used the fact that

$$(\alpha(a)\alpha(d))\alpha(bc)-\alpha(ad)(\alpha(b)\alpha(c))+\alpha(ac)(\alpha(b)\alpha(d))-(\alpha(a)\alpha(c))\alpha(bd)=0$$

or in other words

$$(\alpha(a)\alpha(d))\alpha(bc)-(\alpha(a)\alpha(c))\alpha(bd)+\alpha(ac)(\alpha(b)\alpha(d))-\alpha(ad)(\alpha(b)\alpha(c))=0$$

or in other words

$$(\alpha(a)\alpha(d))\alpha(bc)-(\alpha(a)\alpha(c))\alpha(bd)+(\alpha(b)\alpha(c))\alpha(ad)-(\alpha(b)\alpha(d))\alpha(ac)=0$$

$\square$

And so we have

$$-\alpha(bd)(\alpha(c)\alpha(a))-\alpha(da)(\alpha(c)\alpha(b))+\alpha(bc)(\alpha(d)\alpha(a))+\alpha(ca)(\alpha(d)\alpha(b))=(\alpha(a)\alpha(d))\alpha(bc)-\alpha(ad)(\alpha(b)\alpha(c))+\alpha(ac)(\alpha(b)\alpha(d))-(\alpha(a$$

## 3 Permutative Algebra Classification

In this section, we want to classify all finite-dimensional permutative algebras over a field. Let $K$ be a field, let $(A,\alpha)$ be an $n$-dimensional $K$-algebra, and let $\beta:=\{e_1,\dots,e_n\}$ be a basis for $A$ as a $K$-vector space. Write

$$e_ie_j=\sum_{k=1}^{n}c_{i,j}^{k}e_k\quad\text{and}\quad\alpha(e_i)=\sum_{j=1}^{n}a_i^je_j.$$

Suppose $(A,\alpha)$ is hom-associative. Let $1 \le i_1, i_2, i_3 \le n$. Then on the one hand, we have

$$\begin{aligned}\alpha(e_{i_1})(e_{i_2}e_{i_3}) &= \sum_{j_1,k_1} c_{i_2,i_3}^{k_1} a_{i_1}^{j_1}(e_{j_1}e_{k_1}) \\ &= \sum_{j_1,k_1,k_2} c_{i_2,i_3}^{k_1} c_{j_1,k_1}^{k} a_{i_1}^{j_1} e_k.\end{aligned}$$

On the other hand, we gave

$$\begin{aligned}(e_{i_1}e_{i_2})\alpha(e_{i_3}) &= \sum_{j_1,k_1} c_{i_1,i_2}^{k_1} a_{i_3}^{j_1}(e_{k_1}e_{j_1}) \\ &= \sum_{j_1,k_1,k_2} c_{i_1,i_2}^{k_1} c_{k_1,j_1}^{k} a_{i_3}^{j_1} e_k.\end{aligned}$$

The hom-associative law tells us to set these two equal to each other. Thus, for each $1 \le i_1, i_2, i_3, k \le n$, we have

$$0 = \sum_{j_1,k_1} (a_{i_1}^{j_1} c_{i_2,i_3}^{k_1} c_{j_1,k_1}^{k} - a_{i_3}^{j_1} c_{i_1,i_2}^{k_1} c_{k_1,j_1}^{k})$$

Now suppose $(A,\alpha)$ is permutative. Let $1 \le i_1, i_2, i_3, i_4 \le n$. Then on the one hand, we have

$$\begin{aligned}\alpha(e_{i_1}e_{i_2})(\alpha(e_{i_3})\alpha(e_{i_4})) &= \sum_{j_1,j_2,k_1} c_{i_1,i_2}^{k_1} a_{i_3}^{j_1} a_{i_4}^{j_2} \alpha(e_{k_1})(e_{j_1}e_{j_2}) \\ &= \sum_{j_1,j_2,j_3,k_1,k_2} c_{i_1,i_2}^{k_1} a_{i_3}^{j_1} a_{i_4}^{j_2} a_{k_1}^{j_3} c_{j_1,j_2}^{k_2} e_{j_3} e_{k_2} \\ &= \sum_{j_1,j_2,j_3,k_1,k_2,k} c_{i_1,i_2}^{k_1} a_{i_3}^{j_1} a_{i_4}^{j_2} a_{k_1}^{j_3} c_{j_1,j_2}^{k_2} c_{j_3,k_2}^{k} e_k\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha(e_{i_1})\alpha(e_{i_2})(\alpha(e_{i_3}e_{i_4})) &= \sum_{j_1,j_2,k_1} c_{i_3 i_4}^{k_1} a_{i_1}^{j_1} a_{i_2}^{j_2} (e_{j_1}e_{j_2})\alpha(e_{k_1}) \\ &= \sum_{j_1,j_2,j_3,k_1,k_2} c_{i_3 i_4}^{k_1} a_{i_1}^{j_1} a_{i_2}^{j_2} c_{j_1,j_2}^{k_2} a_{k_1}^{j_3} e_{k_2} e_{j_3} \\ &= \sum_{j_1,j_2,j_3,k_1,k_2,k} c_{i_3 i_4}^{k_1} a_{i_1}^{j_1} a_{i_2}^{j_2} c_{j_1,j_2}^{k_2} a_{k_1}^{j_3} c_{k_2,j_3}^{k} e_k\end{aligned}$$

The permutative law tells us to set these two equal to each other. Thus, for each $1 \le i_1, i_2, i_3, i_4, k \le n$, we have

$$\begin{aligned}0 &= \sum_{j_1,j_2,j_3,k_1,k_2} (a_{i_1}^{j_1} a_{i_2}^{j_2} c_{i_3,i_4}^{k_1} c_{j_1,j_2}^{k_2} a_{k_1}^{j_3} c_{k_2,j_3}^{k} - a_{i_3}^{j_1} a_{i_4}^{j_2} c_{i_1,i_2}^{k_1} a_{k_1}^{j_3} c_{j_1,j_2}^{k_2} c_{j_3,k_2}^{k}) \\ &= \sum_{j_1,j_2,j_3,k_1,k_2} (a_{i_1}^{j_1} a_{i_2}^{j_2} c_{i_3,i_4}^{k_1} c_{k_2,j_3}^{k} - a_{i_3}^{j_1} a_{i_4}^{j_2} c_{i_1,i_2}^{k_1} c_{j_3,k_2}^{k}) a_{k_1}^{j_3} c_{j_1,j_2}^{k_2}\end{aligned}$$

We list our summary in the table below

| Algebra | Equation |
|---|---|
| Permutatitive | $0 = \sum_{j_1,j_2,j_3,k_1,k_2} (a_{i_1}^{j_1} a_{i_2}^{j_2} c_{i_3,i_4}^{k_1} c_{k_2,j_3}^{k} - a_{i_3}^{j_1} a_{i_4}^{j_2} c_{i_1,i_2}^{k_1} c_{j_3,k_2}^{k}) a_{k_1}^{j_3} c_{j_1,j_2}^{k_2}$ |
| Permutative + Abelian | $0 = \sum_{j_1,j_2,j_3,k_1,k_2} (a_{i_1}^{j_1} a_{i_2}^{j_2} c_{i_3,i_4}^{k_1} - a_{i_3}^{j_1} a_{i_4}^{j_2} c_{i_1,i_2}^{k_1}) a_{k_1}^{j_3} c_{j_1,j_2}^{k_2} c_{k_2,j_3}^{k}$ |
| Hom-Associative | $0 = \sum_{j_1,k_1} (a_{i_1}^{j_1} c_{i_2,i_3}^{k_1} c_{j_1,k_1}^{k} - a_{i_3}^{j_1} c_{i_1,i_2}^{k_1} c_{k_1,j_1}^{k})$ |
| Hom-Associative + Abelian | $0 = \sum_{j_1,k_1} (a_{i_1}^{j_1} c_{i_2,i_3}^{k_1} - a_{i_3}^{j_1} c_{i_1,i_2}^{k_1}) c_{j_1,k_1}^{k}$ |

$$0 = \sum_{j_1,k_1} (a_{i_1}^{j_1} c_{i_2,i_3}^{k_1} c_{j_1,k_1}^{k} - a_{i_3}^{j_1} c_{i_1,i_2}^{k_1} c_{k_1,j_1}^{k})$$

If we assume that multiplication map is abelian, then we would have

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_1j_2}^{k_2} C_{j_3k_2}^{k_3} a_{j_3k_1} \left( C_{i_1i_2}^{k_1} a_{j_1i_3} a_{j_2i_4} - C_{i_3i_4}^{k_1} a_{j_1i_1} a_{j_2i_2} \right) = 0$$

If it is unital, then we would have $C_{i1}^{k} = a_{ki} = C_{1i}^{k}$ for all $i, k$.

$$\sum_{j_1,k_1} \left( a_{j_1i_1} C_{i_2i_3}^{k_1} C_{j_1k_1}^{k_2} - a_{j_1i_3} C_{i_1i_2}^{k_1} C_{k_1j_1}^{k_2} \right).$$

$$\sum_{j_1,k_1} \left( a_{j_1i_1} C_{i_2i_3}^{k_1} C_{j_1k_1}^{k_2} - a_{j_1i_3} C_{i_1i_2}^{k_1} C_{k_1j_1}^{k_2} \right).$$

Now suppose $\alpha$ is an $R$-algebra homomorphism. Then

$$\begin{aligned}\alpha(e_{i_1}e_{i_2}) &= \sum_{k_1} C_{i_1i_2}^{k_1} \alpha(e_{k_1}) \\ &= \sum_{k_1,m} C_{i_1i_2}^{k_1} a_{mk_1} e_m\end{aligned}$$

and

$$\begin{aligned}\alpha(e_{i_1})\alpha(e_{i_2}) &= \sum_{j_1,j_2} a_{j_1i_1} a_{j_2i_2} e_{j_1} e_{j_2} \\ &= \sum_{j_1,j_2,m} C_{j_1j_2}^{m} a_{j_1i_1} a_{j_2i_2} e_m\end{aligned}$$

Thus we need

$$\sum_{k_1} C_{i_1i_2}^{k_1} a_{mk_1} = \sum_{j_1,j_2} C_{j_1j_2}^{m} a_{j_1i_1} a_{j_2i_2}$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_1j_2}^{k_2} a_{j_3k_1} \left( C_{i_1i_2}^{k_1} C_{j_3k_2}^{k_3} a_{j_1i_3} a_{j_2i_4} - C_{i_3i_4}^{k_1} C_{k_2j_3}^{k_3} a_{j_1i_1} a_{j_2i_2} \right)$$

$$\sum_{j_1,k_1} \left( C_{i_1i_2}^{k_1} C_{k_1j_1}^{k_2} a_{j_1i_3} - C_{i_2i_3}^{k_1} C_{j_1k_1}^{k_2} a_{j_1i_1} \right)$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_1j_2}^{k_2} C_{j_3k_2}^{k_3} a_{j_3k_1} \left( C_{i_1i_2}^{k_1} a_{j_1i_3} a_{j_2i_4} - C_{i_3i_4}^{k_1} a_{j_1i_1} a_{j_2i_2} \right)$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_3k_2}^{k_3} a_{j_3k_1} \left( C_{j_1j_2}^{k_2} C_{i_1i_2}^{k_1} a_{j_1i_3} a_{j_2i_4} - C_{i_3i_4}^{k_1} C_{j_1j_2}^{k_2} a_{j_1i_1} a_{j_2i_2} \right)$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_3k_2}^{k_3} \left( C_{j_1j_2}^{k_2} a_{j_1i_3} a_{j_2i_4} \left( C_{i_1i_2}^{k_1} a_{j_3k_1} \right) - a_{j_3k_1} C_{i_3i_4}^{k_1} \left( C_{j_1j_2}^{k_2} a_{j_1i_1} a_{j_2i_2} \right) \right)$$

## 3.1 Special Case

We consider a special case where $n = 2$ and

$$[\alpha]_\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, (**??**) simplifies as follows: for $(i_1, i_2, i_3, i_4, k) = (1, 1, 1, 1, 1)$, we have

$$\sum_{j_1,j_2,j_3,k_1,k_2} C_{j_1j_2}^{k_2} a_{j_3k_1} \left( C_{11}^{k_1} C_{j_3k_2}^{k_3} a_{j_11} a_{j_21} - C_{11}^{k_1} C_{k_2j_3}^{k_3} a_{j_11} a_{j_21} \right)$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} C^{k_2}_{j_1j_2} a_{j_3k_1} \left( C^{k_1}_{i_1i_2} C^{k_3}_{j_3k_2} a_{j_1i_3} a_{j_2i_4} - C^{k_1}_{i_3i_4} C^{k_3}_{k_2j_3} a_{j_1i_1} a_{j_2i_2} \right)$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} \left( C^{k_1}_{11} C^{k_2}_{j_1j_2} C^{1}_{j_3k_2} a_{j_11} a_{j_21} a_{j_3k_1} - C^{k_1}_{11} C^{k_2}_{j_1j_2} C^{k}_{k_2j_3} a_{j_1i_1} a_{j_2i_2} a_{j_3k_1} \right) = 0$$

$$\sum_{j_1,j_2,j_3,k_1,k_2} \left( C^{k_1}_{i_1i_2} C^{k_2}_{j_1j_2} C^{k}_{j_3k_2} a_{j_1i_3} a_{j_2i_4} a_{j_3k_1} - C^{k_1}_{i_3i_4} C^{k_2}_{j_1j_2} C^{k}_{k_2j_3} a_{j_1i_1} a_{j_2i_2} a_{j_3k_1} \right) = 0$$

Associativity:

$$\begin{aligned}(e_{i_1}e_{i_2})e_{i_3} &= \sum_{j_1} C^{j_1}_{i_1i_2} e_{j_1} e_{i_3} \\ &= \sum_{j_1,j_2} C^{j_1}_{i_1i_2} C^{j_2}_{j_1i_3} e_{j_2}\end{aligned}$$

and

$$\begin{aligned}e_{i_1}(e_{i_2}e_{i_3}) &= \sum C^{j_1}_{i_2i_3} e_{i_1} e_{j_1} \\ &= \sum C^{j_1}_{i_2i_3} C^{j_2}_{i_1j_1} e_{j_2}\end{aligned}$$

so for each $j_2$, we need

$$\sum_{j_1,j_2} C^{j_1}_{i_1i_2} C^{j_2}_{j_1i_3} = \sum_{j_1,j_2} C^{j_1}_{i_2i_3} C^{j_2}_{i_1j_1}$$

## 3.2 Morphisms of Permutative $R$-Algebras

**Definition 3.1.** Let $(A, \star, \alpha)$ and $(A', \star', \alpha')$ be two permutative $R$-algebra. A map $\varphi \colon A \to A'$ is called a **morphism** of permutative $R$-algebras if it satifies the following:

1. $\varphi$ is an $R$-module homomorphism of the underlying $R$-modules;
2. $\varphi$ respects multiplications: $\varphi(a \star b) = \varphi(a) \star' \varphi(b)$ for all $a, b \in A$.
3. $\varphi$ commutes with $\alpha$ and $\alpha'$: $\varphi(\alpha(a)) = \alpha'(\varphi(a))$ for all $a \in A$.

We say $\varphi$ is an **isomorphism** of permutative $R$-algebras if $\varphi$ is an isomorphism of the underlying $R$-modules. Similarly, we say $A$ and $A'$ are **isomorphic** as permutative $R$-algebras if there exists an isomorphism between them.

**Proposition 3.1.** *Let $(A, \star, \alpha)$ be a permutative $R$-algebra and let $\varphi \colon A \to A$ be an $R$-module automorphism. Define $R$-linear maps $\star_\varphi \colon A \otimes_R A \to A$ and $\alpha_\varphi \colon A \to A$ by*

$$a \star_\varphi b = \varphi(\varphi^{-1}(a) \star \varphi^{-1}(b)) \quad \text{and} \quad \alpha_\varphi(a) = \varphi(\alpha(\varphi^{-1}(a)))$$

*for all $a, b \in A$. Then $(A_\varphi, \star_\varphi, \alpha_\varphi)$ is a permutative $R$-algebra. Furthermore, $A$ is isomorphic to $A_\varphi$ as permutative $R$-algebras.*

$$\begin{aligned}[\mathrm{d}(a)\mathrm{d}(b)]\mathrm{d}(cd) &= [\mathrm{d}(a)\mathrm{d}(b)][\mathrm{d}(c)d + c\mathrm{d}(d)] \\ &= [\mathrm{d}(a)\mathrm{d}(b)][\mathrm{d}(c)d] + [\mathrm{d}(a)\mathrm{d}(b)][c\mathrm{d}(d)] \\ &= \mathrm{d}(a)\mathrm{d}(b)\mathrm{d}(c)d + \mathrm{d}(a)\mathrm{d}(b)c\mathrm{d}(d) \\ \\ &= \mathrm{d}(a)b\mathrm{d}(c)\mathrm{d}(d) + a\mathrm{d}(b)\mathrm{d}(c)\mathrm{d}(d) \\ &= [\mathrm{d}(a)b][\mathrm{d}(c)\mathrm{d}(d)] + [a\mathrm{d}(b)][\mathrm{d}(c)\mathrm{d}(d)] \\ &= [\mathrm{d}(a)b + a\mathrm{d}(b)][\mathrm{d}(c)\mathrm{d}(d)] \\ &= \mathrm{d}(ab)[\mathrm{d}(c)\mathrm{d}(d)]\end{aligned}$$

### 3.3 Example

Permutative $R$-algebra $\mathbb{C}^2$ with

$$\alpha = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

for $\lambda \in \mathbb{C}$. For $(i_1, i_2, i_3, i_4, k) = (1,1,1,1,1)$, we have
We list our summary in the table below

| Algebra | Equation | Number of Equations |
|---|---|---|
| Permutatitive | $0 = \sum_{j_1,j_2,j_3,k_1,k_2} (a^{j_1}_{i_1} a^{j_2}_{i_2} c^{k_1}_{i_3,i_4} c^{k}_{k_2,j_3} - a^{j_1}_{i_3} a^{j_2}_{i_4} c^{k_1}_{i_1,i_2} c^{k}_{j_3,k_2}) a^{j_3}_{k_1} c^{k_2}_{j_1,j_2}$ | for each $1 \leq i_1, i_2, i_3, i_4, k \leq n$ |
| Permutative + Abelian | $0 = \sum_{j_1,j_2,j_3,k_1,k_2} (a^{j_1}_{i_1} a^{j_2}_{i_2} c^{k_1}_{i_3,i_4} - a^{j_1}_{i_3} a^{j_2}_{i_4} c^{k_1}_{i_1,i_2}) a^{j_3}_{k_1} c^{k_2}_{j_1,j_2} c^{k}_{k_2,j_3}$ | for each $1 \leq i_1, i_2, i_3, i_4, k \leq n$ |
| Hom-Associative | $0 = \sum_{j_1,k_1} (a^{j_1}_{i_1} c^{k_1}_{i_2,i_3} c^{k}_{j_1,k_1} - a^{j_1}_{i_3} c^{k_1}_{i_1,i_2} c^{k}_{k_1,j_1})$ | for each $1 \leq i_1, i_2, i_3, k \leq n$ |
| Hom-Associative + Abelian | $0 = \sum_{j_1,k_1} (a^{j_1}_{i_1} c^{k_1}_{i_2,i_3} - a^{j_1}_{i_3} c^{k_1}_{i_1,i_2}) c^{k}_{j_1,k_1}$ | for each $1 \leq i_1, i_2, i_3, k \leq n$ |

### 3.4 Permutative $\mathbb{F}_2$-Algebras

**Example 3.1.** Let $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ and let $f = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define multiplication on $A$ by

| $\cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $e_2$ | $e_1$ |
| $e_2$ | $e_2$ | $e_1$ |

Then we have

| $\cdot$ | $f(e_1)$ | $f(e_2)$ |
|---|---|---|
| $f(e_1)$ | $e_1e_1$ | $e_1(e_1+e_2)$ |
| $f(e_2)$ | $(e_1+e_2)e_1$ | $(e_1+e_2)(e_1+e_2)$ |

| $\cdot$ | $f(e_1)$ | $f(e_2)$ |
|---|---|---|
| $f(e_1)$ | $e_2$ | $e_1+e_2$ |
| $f(e_2)$ | $0$ | $e_1$ |

and

| $f \circ \cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $e_1+e_2$ | $e_1$ |
| $e_2$ | $e_1+e_2$ | $e_1$ |

Thus we need

$$\begin{aligned} e_2(e_1+e_2) &= (e_1+e_2)e_2 \\ (e_1+e_2)e_1 &= e_1(e_1+e_2) \\ 0(e_1+e_2) &= (e_1+e_2)0 \\ e_1e_1 &= e_1e_1. \end{aligned}$$

In particular, we just need to show $e_2$ commutes with $e_1+e_2$. We have

$$\begin{aligned} e_2(e_1+e_2) &= e_2e_1 + e_2e_2 \\ &= e_2 + e_1 \\ &\neq 0 \\ &= e_1 + e_1 \\ &= e_1e_2 + e_2e_2 \\ &= (e_1+e_2)e_2 \end{aligned}$$

**Example 3.2.** Let $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ and let $f = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define multiplication on $A$ by

| $\cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_2$ |
| $e_2$ | $e_1$ | $0$ |

Then we have

| $\cdot$ | $f(e_1)$ | $f(e_2)$ |
|---|---|---|
| $f(e_1)$ | $0$ | $e_2$ |
| $f(e_2)$ | $e_1$ | $e_1 + e_2$ |

and

| $f \circ \cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_1 + e_2$ |
| $e_2$ | $e_1$ | $0$ |

Thus we need

$$\begin{aligned} e_2(e_1+e_2) &= (e_1+e_2)e_2 \\ (e_1+e_2)e_1 &= e_1(e_1+e_2) \\ 0(e_1+e_2) &= (e_1+e_2)0 \\ e_1e_1 &= e_1e_1. \end{aligned}$$

In particular, we just need to show $e_2$ commutes with $e_1 + e_2$. We have

$$\begin{aligned} e_2(e_1+e_2) &= e_2e_1 + e_2e_2 \\ &= e_2 + e_1 \\ &\neq 0 \\ &= e_1 + e_1 \\ &= e_1e_2 + e_2e_2 \\ &= (e_1+e_2)e_2 \end{aligned}$$

**Example 3.3.** Let $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ and let $f = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define multiplication on $A$ by

| $\cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_2$ |
| $e_2$ | $e_1 + e_2$ | $0$ |

Then we have

| $\cdot$ | $f(e_1)$ | $f(e_2)$ |
|---|---|---|
| $f(e_1)$ | $0$ | $e_2$ |
| $f(e_2)$ | $e_1 + e_2$ | $e_1$ |

and

| $f \circ \cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_1 + e_2$ |
| $e_2$ | $e_2$ | $0$ |

Thus we need

$$\begin{aligned} e_2(e_1+e_2) &= (e_1+e_2)e_2 \\ (e_1+e_2)e_1 &= e_1(e_1+e_2) \\ 0(e_1+e_2) &= (e_1+e_2)0 \\ e_1e_1 &= e_1e_1. \end{aligned}$$

In particular, we just need to show $e_2$ commutes with $e_1 + e_2$. We have

$$\begin{aligned} e_2(e_1+e_2) &= e_2e_1 + e_2e_2 \\ &= e_2 + e_1 \\ &\neq 0 \\ &= e_1 + e_1 \\ &= e_1e_2 + e_2e_2 \\ &= (e_1+e_2)e_2 \end{aligned}$$

**Example 3.4.** Let $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ and let $f = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define multiplication on $A$ by

| $\cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_1e_2$ |
| $e_2$ | $e_2e_1$ | $0$ |

Then we have

| $\cdot$ | $f(e_1)$ | $f(e_2)$ |
|---|---|---|
| $f(e_1)$ | $0$ | $e_1e_2$ |
| $f(e_2)$ | $e_2e_1$ | $e_1e_2 + e_2e_1$ |

and

| $f \circ \cdot$ | $e_1$ | $e_2$ |
|---|---|---|
| $e_1$ | $0$ | $e_1 + e_2$ |
| $e_2$ | $e_1$ | $0$ |

Thus we need

$$\begin{aligned} e_2(e_1 + e_2) &= (e_1 + e_2)e_2 \\ (e_1 + e_2)e_1 &= e_1(e_1 + e_2) \\ 0(e_1 + e_2) &= (e_1 + e_2)0 \\ e_1e_1 &= e_1e_1. \end{aligned}$$

In particular, we just need to show $e_2$ commutes with $e_1 + e_2$. We have

$$\begin{aligned} e_2(e_1 + e_2) &= e_2e_1 + e_2e_2 \\ &= e_2 + e_1 \\ &\neq 0 \\ &= e_1 + e_1 \\ &= e_1e_2 + e_2e_2 \\ &= (e_1 + e_2)e_2 \end{aligned}$$