Abstract Algebra Homework 1

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Throughout this homework, let R be a commutative ring.

Problem 1

Proposition 0.1. *Let M be an R-module. Then*

$$\mathrm{Hom}_R(R, M) \cong M.$$

Proof. Define $\Psi \colon \mathrm{Hom}_R(R, M) \to M$ by

$$\Psi(\varphi) = \varphi(1)$$

for all $\varphi \in \mathrm{Hom}_R(R, M)$. We claim that Ψ is an R-module isomorphism. We first check that Ψ is an R-module homomorphism. Let $a, b \in R$ and let $\varphi, \psi \in \mathrm{Hom}_R(R, M)$, then

$$\begin{aligned} \Psi(a\varphi + b\psi) &= (a\varphi + b\psi)(1) \\ &= a\varphi(1) + b\psi(1) \\ &= a\Psi(\varphi) + b\Psi(\psi). \end{aligned}$$

Thus Ψ is an R-module homomorphism.

We next check that Ψ is injective. Suppose $\varphi \in \mathrm{Hom}_R(R, M)$ such that $\Psi(\varphi) = 0$. Then for all $a \in R$, we have

$$\begin{aligned} \varphi(a) &= a\varphi(1) \\ &= a\Psi(\varphi) \\ &= a \cdot 0 \\ &= 0. \end{aligned}$$

Thus $\varphi = 0$. It follows that $\ker \Psi = 0$, which implies Ψ is injective.

We next check that Ψ is surjective. Let $u \in M$. Define $\varphi \colon R \to M$ by setting $\varphi(1) = u$ and extending R-linearly:

$$\begin{aligned} \varphi(a) &= a\varphi(1) \\ &= au \end{aligned}$$

for all $a \in R$. Let us first check that the map φ defined above is indeed an R-module homomorphism. We already have R-scaling by construction, so it suffices to show that φ is additive. Let $a, b \in R$. Then

$$\begin{aligned} \varphi(a + b) &= (a + b)\varphi(1) \\ &= a\varphi(1) + b\varphi(1) \\ &= \varphi(a) + \varphi(b). \end{aligned}$$

Thus $\varphi \in \mathrm{Hom}_R(R, M)$. Finally note that $\Psi(\varphi) = u$, which implies Ψ is surjective. $\square$

Problem 2

Problem 2.a

Proposition 0.2. *Let M be an R-module and let $u \in M$. Define*

$$0 : u = \{a \in R \mid au = 0\}.$$

Then the set $0 : u$ is an ideal in R.

Proof. First note that $0 : u$ is nonempty since $0 \cdot u = 0$ implies $0 \in 0 : u$. Let $x, y \in 0 : u$ and let $a \in R$. Then

$$\begin{aligned}(x + ay)u &= xu + ayu \\ &= 0 + a \cdot 0 \\ &= 0\end{aligned}$$

implies $x + ay \in 0 : u$. This implies $0 : u$ is an ideal in R. $\square$

Problem 2.b

Proposition 0.3. *Suppose R is a domain. Then the set of torsion elements of M forms a submodule of M.*

Proof. Let M_{tor} denote the set of all torsion elements of M. Thus $u \in M_{\text{tor}}$ implies there exists a nonzero $a \in R$ such that $au = 0$. Observe that M_{tor} is nonempty since $0 \in M_{\text{tor}}$ (take $1 \in R$, then $1 \cdot 0 = 0$). Let $u, v \in M_{\text{tor}}$ and let $a \in R$. Choose $c, d \in R\backslash\{0\}$ such that $cu = 0$ and $dv = 0$. Since R is a domain and both c and d are nonzero, we must have cd be nonzero too. Thus

$$\begin{aligned}cd(u + av) &= cdu + cdav \\ &= d(cu) + ac(dv) \\ &= d \cdot 0 + (ac) \cdot 0 \\ &= 0\end{aligned}$$

implies $u + av \in M_{\text{tor}}$. Thus M_{tor} is a submodule of M. $\square$

Remark. If R is not a domain, then it may not be the case that M_{tor} is a submodule of M. Indeed, consider the case where $R = K[x, y]/\langle xy \rangle$ and $M = R$ and K is a field. Note that R is not a domain since $\overline{xy} = \overline{0}$ even though $\overline{x} \neq \overline{0}$ and $\overline{y} \neq \overline{0}$. Also note that R_{tor} is not an ideal of R. Indeed, we have $\overline{x}, \overline{y} \in R_{\text{tor}}$ since $\overline{xy} = \overline{0}$ with $\overline{x}, \overline{y} \neq \overline{0}$, but $\overline{x} + \overline{y} \notin R_{\text{tor}}$. To see that $\overline{x} + \overline{y} \notin R_{\text{tor}}$, suppose we have

$$f(\overline{x}, \overline{y})(\overline{x} + \overline{y}) = \overline{0}. \tag{1}$$

where $f(\overline{x}, \overline{y})$ is the coset in R with $f(x, y) \in K[x, y]$ as a representative. The equation (1) tells us that we can find $g(x, y) \in K[x, y]$ such that

$$f(x, y)(x + y) = g(x, y)xy. \tag{2}$$

Choose such a $g(x, y) \in K[x, y]$. Since $K[x, y]$ is a UFD and $x \nmid (x + y)$ and $y \nmid (x + y)$, we must have $xy \mid f(x, y)$, which implies $f(\overline{x}, \overline{y}) = \overline{0}$ in R.

Problem 3

Proposition 0.4. *Let $\varphi \colon M \to N$ be an R-module homomorphism. Then φ is an isomorphism if and only if there exists an R-module homomorphism $\psi \colon N \to M$ such that $\varphi\psi = \mathrm{id}_N$ and $\psi\varphi = \mathrm{id}_M$.*

Proof. One direction is clear, so suppose that $\varphi \colon N \to M$ is both an R-module homomorphism and a bijection. Let ψ denote the inverse of φ. We want to show that ψ is an R-module homomorphism. Let $a, b \in R$ and $u, v \in N$. Then

$$\begin{aligned}a\psi(u) + b\psi(v) &= \psi\varphi(a\psi(u) + b\psi(v)) \\ &= \psi(a(\varphi(\psi(u))) + b(\varphi(\psi(v)))) \\ &= \psi(au + bv).\end{aligned}$$

Thus ψ is an R-module homomorphism, and so φ is an isomorphism. $\square$

Problem 4

Proposition 0.5. *Let $\varphi\colon M \to M$ be an R-module homomorphism such that $\varphi(\varphi(u)) = \varphi(u)$ for all $u \in M$. Then*

$$M \cong \ker\varphi \oplus \operatorname{im}\varphi.$$

Proof. Define $\Psi\colon M \to \ker\varphi \oplus \operatorname{im}\varphi$ by

$$\Psi(u) = (u - \varphi(u), \varphi(u))$$

for all $u \in M$. Observe that $u - \varphi(u) \in \ker\varphi$ since

$$\begin{aligned}\varphi(u - \varphi(u)) &= \varphi(u) - \varphi(\varphi(u)) \\ &= \varphi(u) - \varphi(u) \\ &= 0.\end{aligned}$$

Thus we really do have $\Psi(u) \in \ker\varphi \oplus \operatorname{im}\varphi$ for all $u \in M$.

Let us check that Ψ is an R-module homomorphism. Let $a, b \in R$ and $u, v \in M$. Then

$$\begin{aligned}\Psi(au + bv) &= ((au + bv) - \varphi(au + bv), \varphi(au + bv)) \\ &= (au + bv - a\varphi(u) - b\varphi(v), a\varphi(u) + b\varphi(v)) \\ &= (a(u - \varphi(u)) + b(v - \varphi(v)), a\varphi(u) + b\varphi(v)) \\ &= (a(u - \varphi(u)), a\varphi(u)) + (b(v - \varphi(v)), b\varphi(v)) \\ &= a(u - \varphi(u), \varphi(u)) + b(v - \varphi(v), \varphi(v)) \\ &= a\Psi(u) + b\Psi(v).\end{aligned}$$

Thus Ψ is an R-module homomorphism.

We now show that Ψ is injective. Let $u \in M$ and suppose $\Psi(u) = (0, 0)$. Then

$$\begin{aligned}(0, 0) &= \Psi(u) \\ &= (u - \varphi(u), \varphi(u))\end{aligned}$$

implies $\varphi(u) = 0$ and $u - \varphi(u) = 0$, which together implies $u = 0$. Thus $\ker\Psi = 0$, and so Ψ is injective.

Finally, we show that Ψ is surjective. Let $(u, \varphi(v)) \in \ker\varphi \oplus \operatorname{im}\varphi$. Then $u + \varphi(v) \in M$, and moreover we have

$$\begin{aligned}\Psi(u + \varphi(v)) &= (u + \varphi(v) - \varphi(u + \varphi(v)), \varphi(u + \varphi(v))) \\ &= (u + \varphi(v) - \varphi(u) - \varphi(v)), \varphi(u) + \varphi(\varphi(v))) \\ &= (u, \varphi(\varphi(v))) \\ &= (u, \varphi(v)).\end{aligned}$$

Thus Ψ is surjective. □

Problem 5

Proposition 0.6. *There is no (unitary) $\mathbb{Q}$-module structure on $\mathbb{Z}$.*

Proof. Suppose $\cdot\colon \mathbb{Q} \times \mathbb{Z} \to \mathbb{Z}$, denoted $(r, m) \mapsto r \cdot m$, gives us a $\mathbb{Q}$-module structure on $\mathbb{Z}$. Set $n = \frac{1}{2} \cdot 1$. Then

$$\begin{aligned}2n &= n + n \\ &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) \cdot 1 \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

implies 2 divides 1, which is a contradiction. □