Measure Theory Homework 3

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Throughout this homework, let X be a set and let $\mathcal{P}(X)$ denote the power set of X.

Problem 1

Proposition 0.1. *Let (A_n) be a sequence in $\mathcal{P}(X)$. Then*

1. $(\liminf A_n)^c = \limsup A_n^c$;
2. $\liminf A_n = \{x \in X \mid \sum_{n=1}^{\infty} 1_{A_n^c}(x) < \infty\} = \{x \in X \mid x \in A_n$ *for all but finitely many* $n\}$.
3. $\limsup A_n = \{x \in X \mid \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty\} = \{x \in X \mid x \in A_{\pi(n)}$ *for all* n *some subsequence* $(A_{\pi(n)})$ *of* $(A_n)\}$.
4. $\liminf A_n \subseteq \limsup A_n$;
5. $1_{\liminf A_n} = \liminf 1_{A_n}$ *and* $1_{\limsup A_n} = \limsup A_n$.

Proof. 1. We have

$$
\begin{aligned}
(\liminf A_n)^c &= \left(\bigcup_{N=1}^{\infty} \left(\bigcap_{n \geq N} A_n \right) \right)^c \\
&= \bigcap_{N=1}^{\infty} \left(\left(\bigcap_{n \geq N} A_n \right)^c \right) \\
&= \bigcap_{N=1}^{\infty} \left(\bigcup_{n \geq N} A_n^c \right) \\
&= \limsup A_n^c.
\end{aligned}
$$

2. First note that

$$
\begin{aligned}
x \in \liminf A_n &\iff x \in \bigcup_{N=1}^{\infty} \left(\bigcap_{n \geq N} A_n \right) \\
&\iff x \in \bigcap_{n \geq N} A_n \text{ for some } N \in \mathbb{N} \\
&\iff x \in A_n \text{ for all } n \geq N \text{ for some } N \in \mathbb{N}.
\end{aligned}
$$

Now if $x \in A_n$ for all $n \geq N$ for some $N \in \mathbb{N}$, then clearly $x \in A_n$ for all but finitely many n. Conversely, let $x \in A_n$ for all but finitely many n. Set $N = \max\{n \mid x \notin A_n\}$. Then $x \in A_n$ for all $n \geq N$. Thus

$$
\liminf A_n = \{x \in X \mid x \in A_n \text{ for all but finitely many } n\}.
$$

Similarly, if $x \in X$ such that

$$
\sum_{n=1}^{\infty} 1_{A_n^c}(x) < \infty,
$$

then $1_{A_n^c}(x) = 1$ for only finitely many n. In other words, $x \in A_n$ for all but finitely many n. Convsersely, if $x \in A_n$ for all but finitely many n, then $x \in A_n^c$ for only finitely many n, and thus

$$
\sum_{n=1}^{\infty} 1_{A_n^c}(x) < \infty.
$$

Therefore
$$\left\{x \in X \mid \sum_{n=1}^{\infty} 1_{A_n^c}(x) < \infty\right\} = \{x \in X \mid x \in A_n \text{ for all but finitely many } n\}.$$

3. First note that

$$\begin{aligned} x \in \limsup A_n &\iff x \in \bigcap_{N=1}^{\infty} \left(\bigcup_{n \geq N} A_n \right) \\ &\iff x \in \bigcup_{n \geq N} A_n \text{ for all } N \in \mathbb{N} \\ &\iff x \in A_n \text{ for some } n \geq N \text{ for all } N \in \mathbb{N}. \end{aligned}$$

In other words, $x \in \limsup A_n$ if and only if for each $n \in \mathbb{N}$ we can find a $\pi(n) \geq n$ such that $x \in A_{\pi(n)}$, or equivalently, if and only if $x \in A_{\pi(n)}$ for all $n \in \mathbb{N}$ where $(A_{\pi(n)})$ is a subsequence of (A_n). Thus

$$\limsup A_n = \{x \in X \mid x \in A_{\pi(n)} \text{ for some subsequence } (A_{\pi(n)}) \text{ of } (A_n)\}.$$

Similarly, suppose $x \in A_{\pi(n)}$ for all $n \in \mathbb{N}$ where $(A_{\pi(n)})$ is a subsequence of (A_n). Then

$$\begin{aligned} \sum_{n=1}^{\infty} 1_{A_n}(x) &\geq \sum_{n=1}^{\infty} 1_{A_{\pi(n)}}(x) \\ &= \infty. \end{aligned}$$

Conversely, if

$$\sum_{n=1}^{\infty} 1_{A_n}(x) = \infty,$$

then $x \in A_n$ for infinitely many n. Thus there is a subsequence $(A_{\pi(n)})$ of (A_n) such that $x \in A_{\pi(n)}$ for all $n \in \mathbb{N}$. Therefore

$$\left\{x \in X \mid \sum_{n=1}^{\infty} 1_{A_n}(x) = \infty\right\} = \{x \in X \mid x \in A_{\pi(n)} \text{ for some subsequence } (A_{\pi(n)}) \text{ of } (A_n)\}.$$

4. We have

$$\begin{aligned} x \in \liminf A_n &\iff x \in A_n \text{ for all } n \geq N \text{ for some } N \\ &\implies x \in A_n \text{ for infinitely many } n \\ &\iff x \in \limsup A_n. \end{aligned}$$

Thus

$$\liminf A_n \subseteq \limsup A_n.$$

5. We first show $1_{\liminf A_n} = \liminf 1_{A_n}$. Let $x \in X$. First assume that $x \in \liminf A_n$. Then $x \in A_n$ for all $n \geq N$ for some $N \in \mathbb{N}$. Then

$$\begin{aligned} 1 &\geq \liminf(1_{A_n}(x)) \\ &= \lim_{M \to \infty} \inf\{1_{A_m}(x) \mid m \geq M\} \\ &\geq \inf\{1_{A_n}(x) \mid n \geq N\} \\ &= \inf\{1 \mid n \geq N\} \\ &= 1 \end{aligned}$$

implies

$$\begin{aligned} 1_{\liminf A_n}(x) &= 1 \\ &= \liminf(1_{A_n}(x)) \\ &= (\liminf 1_{A_n})(x). \end{aligned}$$

Now assume that $x \notin \liminf A_n$. Then $x \notin A_n$ for infinitely many n. In particular, for each $N \in \mathbb{N}$, there exists a $\pi(N) \geq N$ such that $x \notin A_{\pi(N)}$. Then

$$\begin{aligned} 0 &\leq \liminf(1_{A_n}(x)) \\ &= \lim_{N\to\infty} \inf\{1_{A_n}(x) \mid n \geq N\} \\ &= \lim_{N\to\infty} 0 \\ &= 0 \end{aligned}$$

implies

$$\begin{aligned} 1_{\liminf A_n}(x) &= 0 \\ &= \liminf(1_{A_n}(x)) \\ &= (\liminf 1_{A_n})(x). \end{aligned}$$

Thus all cases we have $1_{\liminf A_n}(x) = (\liminf 1_{A_n})(x)$, and therefore

$$1_{\liminf A_n} = \liminf 1_{A_n}.$$

Now we will show $1_{\limsup A_n} = \limsup 1_{A_n}$. Let $x \in X$. First assume that $x \notin \limsup A_n$. Then $x \notin A_n$ for all $n \geq N$ for some $N \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\leq \limsup(1_{A_n}(x)) \\ &= \lim_{M\to\infty} \sup\{1_{A_m}(x) \mid m \geq M\} \\ &\leq \sup\{1_{A_n}(x) \mid n \geq N\} \\ &= \sup\{0 \mid n \geq N\} \\ &= 0 \end{aligned}$$

implies

$$\begin{aligned} 1_{\limsup A_n}(x) &= 0 \\ &= \limsup(1_{A_n}(x)) \\ &= (\limsup 1_{A_n})(x). \end{aligned}$$

Now assume that $x \in \limsup A_n$. Then $x \in A_n$ for infinitely many n. In particular, for each $N \in \mathbb{N}$, there exists a $\pi(N) \geq N$ such that $x \in A_{\pi(N)}$. Then

$$\begin{aligned} 1 &\geq \limsup(1_{A_n}(x)) \\ &= \lim_{N\to\infty} \sup\{1_{A_n}(x) \mid n \geq N\} \\ &\geq \lim_{N\to\infty} 1 \\ &= 1 \end{aligned}$$

implies

$$\begin{aligned} 1_{\limsup A_n}(x) &= 0 \\ &= \limsup(1_{A_n}(x)) \\ &= (\limsup 1_{A_n})(x). \end{aligned}$$

Thus all cases we have $1_{\limsup A_n}(x) = (\limsup 1_{A_n})(x)$, and therefore

$$1_{\limsup A_n} = \limsup 1_{A_n}.$$

$\square$

Problem 2

Problem 2.a

Proposition 0.2. *Let (A_n) be a sequence in $\mathcal{P}(X)$. Then*

$$\bigcup_{n=1}^{\infty}(A_n \Delta A_{n+1}) = \bigcup_{n=1}^{\infty} A_n \backslash \bigcap_{m=1}^{\infty} A_m.$$

Proof. Suppose $x \in \bigcup_{n=1}^{\infty}(A_n \Delta A_{n+1})$. Choose $n \in \mathbb{N}$ such that $x \in A_n \Delta A_{n+1}$. Thus either $x \in A_n \backslash A_{n+1}$ or $x \in A_{n+1} \backslash A_n$. Without loss of generality, say $x \in A_n \backslash A_{n+1}$. Then since $x \in A_n$, we see that $x \in \bigcup_{n=1}^{\infty} A_n$ and since $x \notin A_{n+1}$, we see that $x \notin \bigcap_{n=1}^{\infty} A_n$. Therefore $x \in \bigcup_{n=1}^{\infty} A_n \backslash \bigcap_{m=1}^{\infty} A_m$. This implies

$$\bigcup_{n=1}^{\infty}(A_n \Delta A_{n+1}) \subseteq \bigcup_{n=1}^{\infty} A_n \backslash \bigcap_{m=1}^{\infty} A_m.$$

Conversely, suppose $x \in \bigcup_{n=1}^{\infty} A_n \backslash \bigcap_{m=1}^{\infty} A_m$. Since $x \in \bigcup_{n=1}^{\infty} A_n$, there exists some $n \in \mathbb{N}$ such that $x \in A_n$. Since $x \notin \bigcap_{m=1}^{\infty} A_m$, there exists some $k \in \mathbb{N}$ such that $x \notin A_k$. Assume without loss of generality that $k < n$. Choose m to be the least natural number number such that $x \in A_m$, $x \notin A_{m-1}$, and $k < m \leq n$. Clearly this number exists since $x \notin A_k$ and $x \in A_n$. Then $x \in A_m \Delta A_{m-1}$, which implies $x \in \bigcup_{n=1}^{\infty}(A_n \Delta A_{n+1})$. Thus

$$\bigcup_{n=1}^{\infty}(A_n \Delta A_{n+1}) \supseteq \bigcup_{n=1}^{\infty} A_n \backslash \bigcap_{m=1}^{\infty} A_m.$$

□

Problem 2.b

Proposition 0.3. *Let (A_n) be a sequence in $\mathcal{P}(X)$. Then*

$$\limsup A_n \backslash \liminf A_n = \limsup (A_n \Delta A_{n+1}).$$

Proof. Suppose $x \in \limsup A_n \backslash \liminf A_n$. Then the sets

$$\{n \in \mathbb{N} \mid x \in A_n\} \quad \text{and} \quad \{n \in \mathbb{N} \mid x \notin A_n\}$$

are both infinite. We claim this implies that the set

$$\begin{aligned}\{n \in \mathbb{N} \mid x \in A_n \Delta A_{n+1}\} &= \{n \in \mathbb{N} \mid x \in (A_n \backslash A_{n+1}) \cup (A_{n+1} \backslash A_n)\} \\ &= \{n \in \mathbb{N} \mid x \in A_n \backslash A_{n+1}\} \cup \{n \in \mathbb{N} \mid x \in A_{n+1} \backslash A_n\}\end{aligned}$$

is infinite. To see this, we first assume without loss of generality that $x \in A_1$. Choose the least $\pi(1) > 1$ such that $x \notin A_{\pi(1)}$ and $x \in A_{\pi(1)-1}$. Observe that $\pi(1)$ exists since otherwise $\{n \in \mathbb{N} \mid x \notin A_n\}$ would be finite. Next, choose $\pi(2) > \pi(1)$ such that $x \in A_{\pi(2)}$ and $x \notin A_{\pi(2)-1}$. We again observe that $\pi(2)$ exists since otherwise $\{n \in \mathbb{N} \mid x \in A_n\}$ would be finite. Continuing in this manner, we obtain a strictly increasing sequence $(\pi(n))$ of natural numbers with

$$x \in A_{\pi(2n)} \backslash A_{\pi(2n)-1} \quad \text{and} \quad x \in A_{\pi(2n-1)-1} \backslash A_{\pi(2n-1)}$$

for all $n \geq 1$. In particular, both sets

$$\{n \in \mathbb{N} \mid x \in A_n \backslash A_{n+1}\} \quad \text{and} \quad \{n \in \mathbb{N} \mid x \in A_{n+1} \backslash A_n\}$$

are infinite. Thus $\{n \in \mathbb{N} \mid x \in A_n \Delta A_{n+1}\}$ is infinite, which implies $x \in \limsup(A_n \Delta A_{n+1})$. Therefore

$$\limsup A_n \backslash \liminf A_n \subseteq \limsup (A_n \Delta A_{n+1}).$$

Conversely, suppose $x \in \limsup(A_n \Delta A_{n+1})$. Then the set

$$\begin{aligned}\{n \in \mathbb{N} \mid x \in A_n \Delta A_{n+1}\} &= \{n \in \mathbb{N} \mid x \in (A_n \backslash A_{n+1}) \cup (A_{n+1} \backslash A_n)\} \\ &= \{n \in \mathbb{N} \mid x \in A_n \backslash A_{n+1}\} \cup \{n \in \mathbb{N} \mid x \in A_{n+1} \backslash A_n\}\end{aligned}$$

is infinite. This implies one of

$$\{n \in \mathbb{N} \mid x \in A_n \backslash A_{n+1}\} \quad \text{or} \quad \{n \in \mathbb{N} \mid x \in A_{n+1} \backslash A_n\}$$

is infinite. Without loss of generality, suppose $\{n \in \mathbb{N} \mid x \in A_n \backslash A_{n+1}\}$ is infinite. Thus there exists a strictly increasing sequence $(\pi(n))$ of natural numbers with $x \in A_{\pi(n)}$ and $x \notin A_{\pi(n)+1}$. In particular, both sets

$$\{n \in \mathbb{N} \mid x \in A_n\} \quad \text{and} \quad \{n \in \mathbb{N} \mid x \notin A_n\}$$

are infinite. Equivalently, we have $x \in \limsup A_n \backslash \liminf A_n$. Therefore

$$\limsup A_n \backslash \liminf A_n \supseteq \limsup (A_n \Delta A_{n+1}).$$

□

Problem 3

Problem 3.a

Proposition 0.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a descending sequence in $\mathcal{M}$ such that $\mu(E_1) < \infty$. Then*

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n) \tag{1}$$

Proof. The sequence $(E_1 \backslash E_n)_{n\in\mathbb{N}}$ is an ascending sequence in $\mathcal{M}$, hence

$$\begin{aligned}
\mu(E_1) - \lim_{n\to\infty} \mu(E_n) &= \lim_{n\to\infty} \mu(E_1) - \lim_{n\to\infty} \mu(E_n) \\
&= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) \\
&= \lim_{n\to\infty} \mu(E_1 \backslash E_n) \\
&= \mu\left(\bigcup_{n=1}^{\infty} (E_1 \backslash E_n)\right) \\
&= \mu\left(E_1 \backslash \left(\bigcap_{n=1}^{\infty} E_n\right)\right) \\
&= \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right),
\end{aligned}$$

where we used the fact that $\mu(E_1) < \infty$ to get from the second line to the third line and also from fifth line to the sixth line. Also since $\mu(E_1) < \infty$, we can subtract $\mu(E_1)$ from both sides to obtain (1). □

Problem 3.b

Proposition 0.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$. Then*

$$\mu(\liminf E_n) \leq \liminf \mu(E_n)$$

Proof. Note that the sequence

$$\left(\bigcap_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is an ascending sequence in N. Therefore we have

$$\begin{aligned}
\mu(\liminf E_n) &= \mu\left(\bigcup_{N=1}^{\infty}\left(\bigcap_{n\geq N} E_n\right)\right) \\
&= \liminf \mu\left(\bigcap_{n\geq N} E_n\right) \\
&\leq \lim_{N\to\infty} \inf\{\mu(E_n) \mid n \geq N\} \\
&= \liminf \mu(E_n),
\end{aligned}$$

where we obtained the third line from the second line since

$$\mu\left(\bigcap_{n\geq N} E_n\right) \leq \mu(E_n)$$

for all $n \geq N$ by monotonicity of μ. □

Problem 3.c

Proposition 0.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$ such that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$. Then*

$$\mu\left(\limsup E_n\right) \geq \limsup \mu(E_n)$$

Proof. Note that the sequence

$$\left(\bigcup_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is a descending sequence in N. This together with the fact that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$ implies

$$\begin{aligned}
\mu\left(\limsup E_n\right) &= \mu\left(\bigcap_{N=1}^{\infty}\left(\bigcup_{n\geq N} E_n\right)\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} E_n\right) \\
&\geq \lim_{N\to\infty} \sup\left\{\mu(E_n) \mid n \geq N\right\} \\
&= \limsup \mu(E_n),
\end{aligned}$$

where we obtained the third line from the second line since

$$\mu\left(\bigcup_{n\geq N} E_n\right) \geq \mu(E_n)$$

for all $n \geq N$ by monotonicity of μ. □

Problem 3.d

Proposition 0.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$ such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then*

$$\mu\left(\limsup E_n\right) = 0.$$

Proof. Note that the sequence

$$\left(\bigcup_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is a descending sequence in N. This together with the fact that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$ implies

$$\begin{aligned}
\mu\left(\limsup E_n\right) &= \mu\left(\bigcap_{N=1}^{\infty}\left(\bigcup_{n\geq N} E_n\right)\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} E_n\right) \\
&\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) \\
&= 0,
\end{aligned}$$

where the last equality follows since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. □

Problem 4

Let $\mathcal{A}$ be an algebra of subsets of X and let μ be a finite measure on $\mathcal{A}$. Let μ^* be the outer measure on X induced by μ. Define a relation $\sim$ on $\mathcal{P}(X)$ as follows: if $A, B \in \mathcal{P}(X)$, then

$$A \sim B \text{ if and only if } \mu^*(A\Delta B) = 0.$$

We also define the pseudometric d_μ on $\mathcal{P}(X)$ by

$$\mathrm{d}_\mu(A, B) = \mu^*(A\Delta B)$$

for all $A, B \in \mathcal{P}(X)$.

Problem 4.a

Proposition 0.8. *The relation $\sim$ is an equivalence relation.*

Proof. We first check reflexivity. Let $A \in \mathcal{P}(X)$. Then

$$\begin{aligned}\mu^*(A\Delta A) &= \mu^*(\emptyset)\\ &= 0\end{aligned}$$

implies $A \sim A$. Next we check symmetry. Let $A, B \in \mathcal{P}(X)$ and suppose $A \sim B$. Then

$$\begin{aligned}\mu^*(B\Delta A) &= \mu^*(A\Delta B)\\ &= 0\end{aligned}$$

implies $B \sim A$. Finally we check transitivity. Let $A, B, C \in \mathcal{P}(X)$ and suppose $A \sim B$ and $B \sim C$. Then

$$\begin{aligned}\mu^*(A\Delta C) &= \mu^*(A\Delta B\Delta B\Delta C)\\ &\leq \mu^*((A\Delta B) \cup (B\Delta C))\\ &\leq \mu^*(A\Delta B) + \mu^*(B\Delta C)\\ &= 0 + 0\\ &= 0\end{aligned}$$

implies $A \sim C$. □

Problem 4.b

Proposition 0.9. *Let $A, B \in \mathcal{P}(X)$. If $A \sim B$, then $\mu^*(A) = \mu^*(B)$. The converse need not be true.*

Proof. Suppose that $A \sim B$. Then $\mu^*(A\Delta B) = 0$ implies

$$\begin{aligned}\mu^*(A) &= \mu^*(A) + \mu^*(A\Delta B)\\ &\geq \mu^*(A \cup (A\Delta B))\\ &\geq \mu^*(A\Delta A\Delta B)\\ &= \mu^*(B).\end{aligned}$$

Similarly,

$$\begin{aligned}\mu^*(B) &= \mu^*(B) + \mu^*(B\Delta A)\\ &\geq \mu^*(B \cup (B\Delta A))\\ &\geq \mu^*(B\Delta B\Delta A)\\ &= \mu^*(A).\end{aligned}$$

Thus $\mu^*(A) = \mu^*(B)$.

To see that the converse does not hold, consider the case where $X = \{a, b\}$ and μ is counting measure on this set. Then on the one hand, we have

$$\mu(\{a\}) = 1 = \mu(\{b\}),$$

but on the other hand, we have

$$\begin{aligned}\mu(\{a\}\Delta\{b\}) &= \mu(\{a,b\}) \\ &= 2 \\ &\neq 0.\end{aligned}$$

□

Problem 6

Let $\mathcal{A}$ be an algebra of subsets of X and let μ be a finite measure on $\mathcal{A}$. Let μ^* be the outer measure on X induced by μ. A set E is said to be μ^*-measurable if

$$\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S \backslash E)$$

for all $S \in \mathcal{P}(X)$. Note that by countable subadditivity of μ^*, this implies

$$\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \backslash E).$$

Denote by $\mathcal{M}$ to be the collection of all μ^*-measurable sets.

Problem 6.a

Proposition 0.10. *Let $A \in \mathcal{A}$. Then A is μ^*-measurable.*

Proof. Let $S \in \mathcal{P}(X)$. Assume for a contradiction that

$$\mu^*(S) < \mu^*(S \cap A) + \mu^*(S \backslash A).$$

Choose $\varepsilon > 0$ such that

$$\mu^*(S) < \mu^*(S \cap A) + \mu^*(S \backslash A) - \varepsilon.$$

Choose $B \in \mathcal{A}$ such that $S \subseteq B$ and

$$\mu(B) \leq \mu^*(S) + \varepsilon.$$

Then

$$\begin{aligned}\mu^*(S) &\geq \mu(B) - \varepsilon \\ &= \mu\left((B \cap A) \cup (B \backslash A)\right) - \varepsilon \\ &= \mu(B \cap A) + \mu(B \backslash A) - \varepsilon \\ &\geq \mu^*(S \cap A) + \mu^*(S \backslash A) - \varepsilon.\end{aligned}$$

This is a contradiction. □

Problem 6.b

Proposition 0.11. *$\mathcal{M}$ is a σ-algebra.*

Proof. We prove this in several steps:

Step 1: We first show $\mathcal{M}$ is an algebra. First we show it is closed under finite unions. Let $A, B \in \mathcal{M}$ and let $S \in \mathcal{P}(X)$. Then

$$\begin{aligned}\mu^*(S) &= \mu^*\left(S \cap A\right) + \mu^*\left(S \backslash A\right) \\ &= \mu^*\left(S \cap A\right) + \mu^*\left((S \backslash A) \cap B\right) + \mu^*\left((S \backslash A) \backslash B\right) \\ &\geq \mu^*\left((S \cap A) \cup ((S \backslash A) \cap B)\right) + \mu^*\left((S \backslash A) \backslash B\right) \\ &= \mu^*\left(S \cap (A \cup B)\right) + \mu^*\left(S \backslash (A \cup B)\right)\end{aligned}$$

Therefore $A \cap B \in \mathcal{M}$.

Next we shows it is closed under complements. Let $A \in \mathcal{M}$ and let $S \in \mathcal{P}(X)$. Then

$$\begin{aligned} \mu^*(S) &\geq \mu^*(S \cap A) + \mu^*(S \backslash A) \\ &= \mu^*\left(S \backslash (X \backslash A)\right) + \mu^*(S \backslash A) \\ &= \mu^*\left(S \backslash (X \backslash A)\right) + \mu^*\left(S \cap (X \backslash A)\right). \end{aligned}$$

Therefore $X \backslash A \in \mathcal{M}$.

Step 2: We show μ^* is finitely additive on $\mathcal{M}$. In fact, we claim that for any $S \in \mathcal{P}(X)$ and pairwise disjoint $A_1, \ldots, A_n \in \mathcal{M}$, we have

$$\mu^*\left(S \cap \left(\bigcup_{m=1}^{n} A_m\right)\right) = \sum_{m=1}^{n} \mu^*\left(S \cap A_m\right). \tag{2}$$

We prove (2) by induction on n. The equality holds trivially for $n = 1$. For the induction step, assume that it holds for some $n \geq 1$. Let S be a subset of X and let$A_1, \ldots, A_{n+1}$ be a finite sequence of members in $\mathcal{M}$. Then

$$\begin{aligned} \mu^*\left(S \cap \left(\bigcup_{m=1}^{n+1} A_m\right)\right) &\geq \mu^*\left(S \cap \left(\bigcup_{m=1}^{n+1} A_m\right) \cap A_{n+1}\right) + \mu^*\left(S \cap \left(\bigcup_{m=1}^{n+1} A_m\right) \cap (X \backslash A_{n+1})\right) \\ &= \mu^*\left(S \cap A_{n+1}\right) + \mu^*\left(S \cap \left(\bigcup_{m=1}^{n} A_m\right)\right) \\ &= \mu^*\left(S \cap A_{n+1}\right) + \sum_{m=1}^{n} \mu^*(S \cap A_m) \\ &= \sum_{m=1}^{n+1} \mu^*(S \cap A_m). \end{aligned}$$

This establishes (2). Setting $S = X$ in (2) gives us finite additivity of μ^* on $\mathcal{M}$.

Step 3: We prove that $\mathcal{M}$ is a σ-algebra. Since $\mathcal{M}$ was already shown to be an algebra, it suffices to show that $\mathcal{M}$ is closed under countable unions. Let (A_n) be a sequence in $\mathcal{M}$. Disjointify the sequence (A_n) to the sequence (D_n): set $D_1 = A_1$ and $D_n = A_n \backslash \bigcup_{m=1}^{n-1} A_m$ for all $n > 1$. Note that (D_n) is a sequence in $\mathcal{M}$ since $\mathcal{M}$ is algebra. Let $S \in \mathcal{P}(X)$ and $n \in \mathbb{N}$. Observe that

$$\begin{aligned} \mu^*(S) &\geq \mu^*\left(S \cap \left(\bigcup_{m=1}^{n} D_m\right)\right) + \mu^*\left(S \backslash \left(\bigcup_{m=1}^{n} D_m\right)\right) \\ &\geq \mu^*\left(S \cap \left(\bigcup_{m=1}^{n} D_m\right)\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right) \\ &= \sum_{m=1}^{n} \mu^*\left(S \cap D_m\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right), \end{aligned}$$

where we applied finite-additivity of μ^* to the first term on the right-hand side and we applied monotonicity of μ^* to the second term on the right-hand side. Taking the limit as $n \to \infty$. We obtain

$$\begin{aligned} \mu^*(S) &\geq \sum_{m=1}^{\infty} \mu^*\left(S \cap D_m\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right) \\ &\geq \mu^*\left(\bigcup_{n \in \mathbb{N}} \left(S \cap D_m\right)\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right) \\ &= \mu^*\left(S \cap \bigcup_{n \in \mathbb{N}} D_m\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right) \\ &= \mu^*\left(S \cap \left(\bigcup_{n \in \mathbb{N}} A_m\right)\right) + \mu^*\left(S \backslash \left(\bigcup_{n \in \mathbb{N}} A_n\right)\right), \end{aligned}$$

where we applied countable subadditivity of μ^* to the first expression on the right-hand side. Thus $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M}$.

$\square$

Problem 6.c

Proposition 0.12. *We have $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.*

Proof. By problem 6.a and 6.b, we see that $\mathcal{M}$ is a σ-algebra which contains $\mathcal{A}$. Since $\sigma(\mathcal{A})$ is the *smallest* σ-algebra which contains $\mathcal{A}$, we must have $\sigma(\mathcal{A}) \subseteq \mathcal{M}$. $\square$

Problem 6.d

Proposition 0.13. *The outer measure μ^* restricted to $\mathcal{M}$ is a measure.*

Proof. In Proposition (0.11), we showed that μ^* is finitely additive on $\mathcal{M}$. We already know that μ^* is already countably subadditive on $\mathcal{M}$. Therefore μ^* is countably additive on $\mathcal{M}$ since

$$\text{finte additivity} + \text{countable subadditivity} \ = \ \text{countable additivity}.$$

To see this, let (A_n) be a sequence of pairwise disjoint members of $\mathcal{M}$. By countable subadditivity of μ^*, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

For the reverse inequality, notat that for each $N \in \mathbb{N}$, finite additivity of μ^* imlpies

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\geq \mu^*\left(\bigcup_{n=1}^{N} A_n\right) \\ &= \sum_{n=1}^{N} \mu^*(A_n).\end{aligned}$$

Taking $N \to \infty$ gives us

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \sum_{n=1}^{\infty} \mu^*(A_n).$$

$\square$

Problem 6.e

Proposition 0.14. *Let $E \in \mathcal{M}$ such that $\mu^*(E) = 0$, and let $F \in \mathcal{P}(X)$ such that $F \subseteq E$. Then $F \in \mathcal{M}$.*

Proof. Let $S \in \mathcal{P}(X)$. Then

$$\begin{aligned}\mu^*(S) &\geq \mu^*(S \backslash F) \\ &= \mu^*(S \cap F) + \mu^*(S \backslash F),\end{aligned}$$

where we used the fact that $\mu^*(S \cap F) = 0$ since $S \cap F \subseteq E$ and $\mu^*(E) = 0$. $\square$

More generally:

Proposition 0.15. *Let $E \in \mathcal{P}(X)$ such that $\mu^*(E) = 0$. Then $E \in \mathcal{M}$.*

Proof. Let $S \in \mathcal{P}(X)$. First note that

$$\begin{aligned}0 &= \mu^*(E) \\ &\geq \mu^*(S \cap E)\end{aligned}$$

implies $\mu^*(S \cap E) = 0$ by monotonicity of μ^*. Therefore

$$\begin{aligned}\mu^*(S) &\geq \mu^*(S \backslash E) \\ &= \mu^*(S \cap E) + \mu^*(S \backslash E).\end{aligned}$$

This implies $E \in \mathcal{M}$. $\square$