Homework #5 Solutions

(1) $f(x) = x^3 - 5.85x^2 + 5.27x + 13.56$ hundred dollars gives the stock price of Company A, x hours after 9:30 am on a given day, on the closed interval $0 \leq x \leq 5$.

(1.a) At what x value(s) does the function have a relative maximum?

Solution: $x \approx 0.52$

(1.b) At what x value(s) does the function have a relative minimum?

Solution: $x \approx 3.38$

(1.c) At what x value(s) does the function have an inflection point?

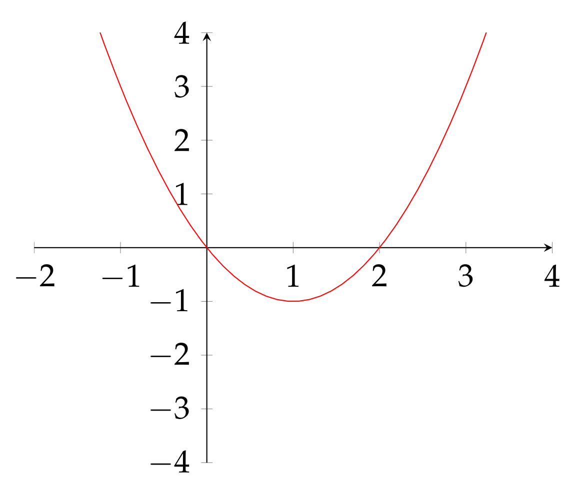
Solution: $x \approx 1.95$

(1.d) On the closed interval $0 \leq x \leq 5$, the stock price was *highest* 5 hours after 9:30 am, at which the price of the stock was 18.66 hundred dollars.

(1.e) On the closed interval $0 \leq x \leq 5$, the stock price was *lowest* 3.38 hours after 9:30 am, at which the price of the stock was 3.154 hundred dollars

(1.f) On the closed interval $0 \leq x \leq 5$, the stock price was *decreasing most rapidly* 1.95 hours after 9:30 am, at a rate of 6.137 hundred dollars per hour.

(2) Consider the function $f(x)$ defined on the whole real line below whose *slope* graph (i.e. the graph of $f'(x)$) is given below



(2.a) Where are the relative maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 0$

(2.b) Where are the absolute maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: There are none.

(2.c) Where are the relative mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 2$

(2.d) Where are the absolute mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: There are none.

(2.e) Where are the critical points of $f(x)$ located? (If there are none, just say "there are none")

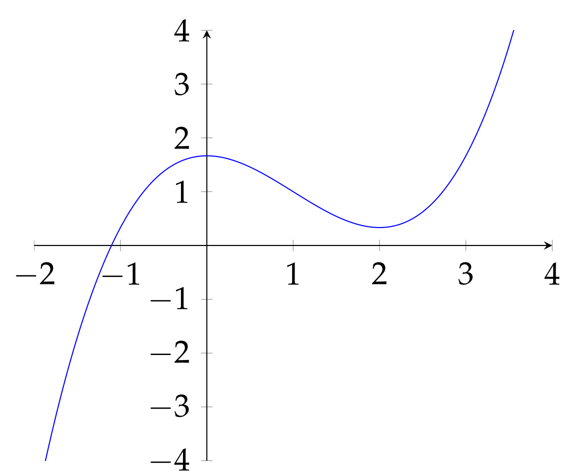
Solution: $x = 0$ and $x = 2$

(2.f) Where are the inflection points of $f(x)$ located? (If there are none, just say "there are none")

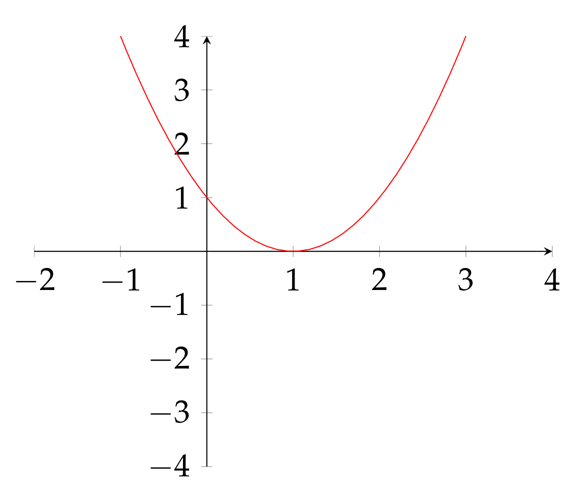
Solution: $x = 1$.

(2.g) Sketch how the shape of the graph of $f(x)$ should look (there's actually many different functions whose slope graph corresponds to the one above but they all have the same shape):

Solution:



(3) Consider the function $f(x)$ defined on the whole real line below whose *slope* graph is given below



(3.a) Where are the relative maxes of $f(x)$ located? (If there are none, just say “there are none”)

Solution: There are none.

(3.b) Where are the absolute maxes of $f(x)$ located? (If there are none, just say “there are none”)

Solution: There are none.

(3.c) Where are the relative mins of $f(x)$ located? (If there are none, just say “there are none”)

Solution: There are none.

(3.d) Where are the absolute mins of $f(x)$ located? (If there are none, just say “there are none”)

Solution: There are none.

(3.e) Where are the critical points of $f(x)$ located? (If there are none, just say “there are none”)

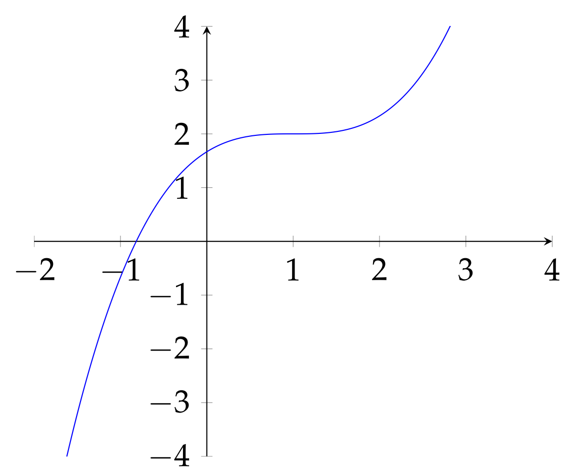
Solution: $x = 1$

(3.f) Where are the inflection points of $f(x)$ located? (If there are none, just say “there are none”)

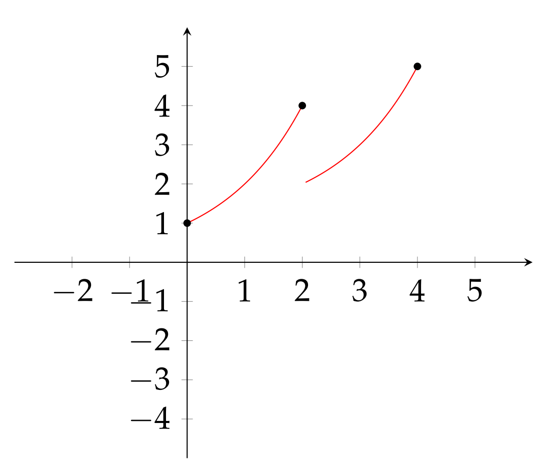
Solution: $x = 1$

(3.g) Sketch how the shape of the graph of $f(x)$ should look:

Solution:



(4) Consider the function $f(x)$ defined on the closed interval $[0, 4]$ whose graph is given below



(4.a) Where are the relative maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 2$

(4.b) Where are the absolute maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 4$

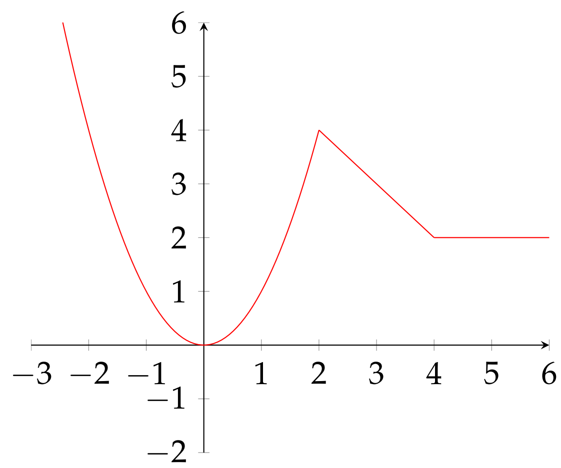
(4.c) Where are the relative mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: There are none.

(4.d) Where are the absolute mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 0$

(5) Consider the function $f(x)$ defined on the whole real line whose graph is given below



(5.a) Where are the critical points of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 0$ and $x = 2$ and $x \in [4, \infty)$[1]

(5.b) Where are the relative maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 2$ and $x \in (4, \infty)$

(5.c) Where are the absolute maxes of $f(x)$ located? (If there are none, just say "there are none")

Solution: There are none

(5.d) Where are the relative mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 0$ and $x \in [4, \infty)$

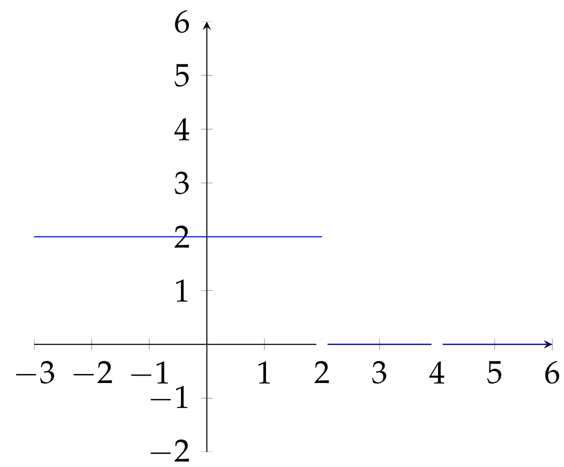
(5.e) Where are the absolute mins of $f(x)$ located? (If there are none, just say "there are none")

Solution: $x = 0$

(5.f) Sketch the graph of $f''(x)$ below (hint: keep in mind that there are two places where $f''(x)$ is not defined):

[1] This is technically true since $f'(x) = 0$ for all $x \in (4, \infty)$ and $f'(4)$ does not exist. Thus every $x \in [4, \infty)$ is a critical point. On the test, you won't have to worry about this though. This is because there will be only finitely many critical points to find on the test (and not a whole interval of them like in this example).

Solution:



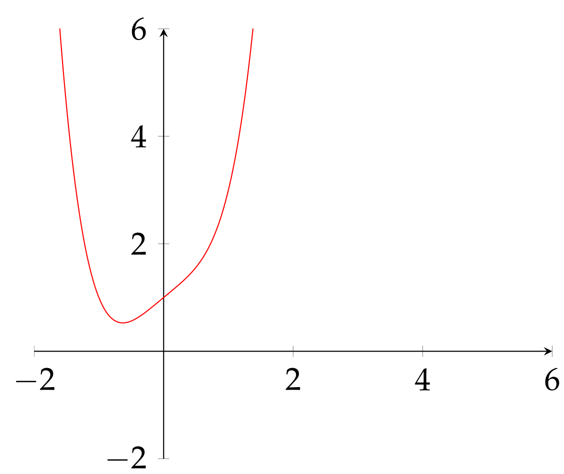
(6) Let $f(x)$ a continuous differentiable function on the whole real line and let a be a real number.

(6.a) True or False: If $f''(a) \neq 0$, then the function may have an inflection point at $x = a$.

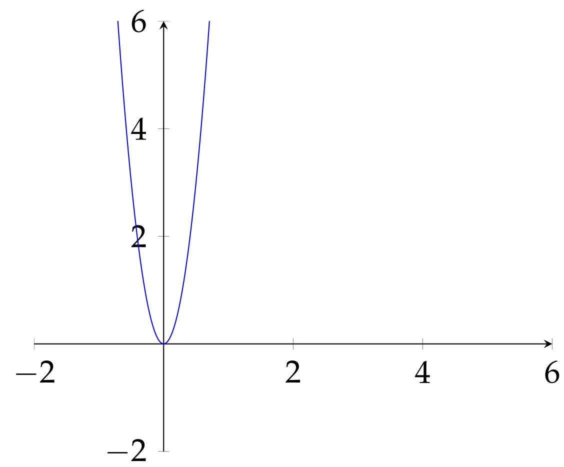
Solution: False

(6.b) True or False: If $f''(a) = 0$, then the function has an inflection point at $x = a$. (this one is a little more subtle than you think)

Solution: False. For example, the function $f(x) = x^4 + x + 1$ has $f''(0) = 0$ but the function $f(x)$ does not have an inflection point at $x = 0$. Indeed, the graph of $f(x)$ looks like:



As you can see, the function $f(x)$ does *not* change concavity at $x = 0$ (the function $f(x)$ is concave up everywhere!). For another reason, look at the graph of $f''(x)$:



See how the function $f''(x)$ just touches the x-axis at $x = 0$ and doesn't actually go through it?

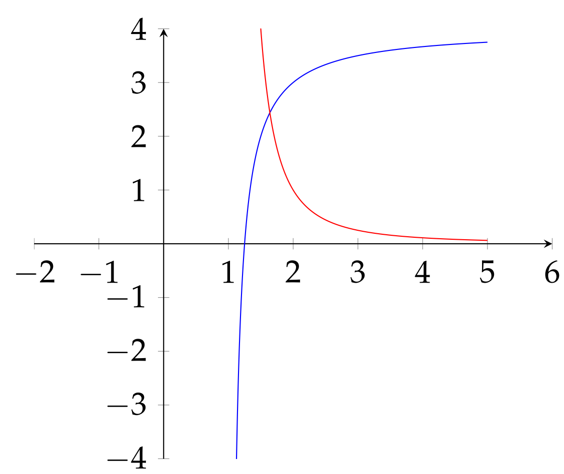
(6.c) True or False: If $f''(a) > 0$, then the function is concave up at $x = a$.

Solution: True.

(6.d) True or False: If $f'(a) = 0$ and $f''(a) > 0$, then the function has a relative max at $x = a$.

Solution: False! If $f'(a) = 0$ and $f''(a) > 0$, then the function has a relative *min* at $x = a$.

(7) Consider the graphs of the two functions below:



One of them is $f(x)$ and one of them is $f'(x)$. Which is which?

Solution: The graph of $f(x)$ is the blue graph and the graph of $f'(x)$ is the red graph.

(8.a) Let $h(x) = 11(1.8^x)$ and $g(h) = h^3$. Find the derivative of $g(h(x))$.

Solution: Let's first figure out what $g(h(x))$ is: we substitute $h(x) = 11(1.8^x)$ for h in $g(h) = h^3$ to get $g(h(x)) = (11(1.8^x))^3$. So

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(g(h(x))) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((11(1.8^x))^3\right) \\
&= 3(11(1.8^x))^2 \frac{\mathrm{d}}{\mathrm{d}x}(11(1.8^x)) \\
&= 3(11(1.8^x))^2 \cdot 11 \frac{\mathrm{d}}{\mathrm{d}x}(1.8^x) \\
&= 3(11(1.8^x))^2 \cdot 11 \ln(1.8) 1.8^x
\end{aligned}$$

(8.b) Write $f(x) = e^{\sqrt{x^2+1}}$ as a composition $g(h(x))$.

Solution:

$$\begin{aligned} g(h) &= e^h \\ h(x) &= \sqrt{x^2+1} \end{aligned}$$

(8.c) Write $f(x) = \sqrt{x^3+1}$ as a composition $g(h(x))$.

Solution:

$$\begin{aligned} g(h) &= \sqrt{h} \\ h(x) &= x^3+1 \end{aligned}$$

(8.d) Find the derivative of $f(x) = (x^2+2)e^{3x^2+1}$.

Solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left((x^2+2)e^{3x^2+1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2+2)e^{3x^2+1} + (x^2+2)\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x^2+1}\right) \\ &= 2xe^{3x^2+1} + (x^2+2)e^{3x^2+1}\frac{\mathrm{d}}{\mathrm{d}x}(3x^2+1) \\ &= 2xe^{3x^2+1} + (x^2+2)e^{3x^2+1}6x \end{aligned}$$

(8.e) Find the derivative of $f(x) = \frac{2+x}{3+\ln x}$.

Solution:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2+x}{3+\ln x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((2+x)(3+\ln x)^{-1}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(2+x)\cdot(3+\ln x)^{-1} + (2+x)\frac{\mathrm{d}}{\mathrm{d}x}((3+\ln x)^{-1}) \\ &= (3+\ln x)^{-1} + (2+x)\cdot -1(3+\ln x)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(3+\ln x) \\ &= (3+\ln x)^{-1} - (2+x)(3+\ln x)^{-2}\cdot\frac{1}{x} \\ &= \frac{1}{3+\ln x} - \frac{2+x}{(3+\ln x)^2 x} \\ &= \frac{(3+\ln x)x}{(3+\ln x)^2 x} - \frac{2+x}{(3+\ln x)^2 x} \\ &= \frac{(3+\ln x)x - 2 - x}{(3+\ln x)^2 x}. \end{aligned}$$

(8.f) Find the *second* derivative of $\ln(3x+1)$.

Solution: First let's find the derivative of $\ln(3x+1)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\ln(3x+1)) &= \frac{1}{3x+1}\frac{\mathrm{d}}{\mathrm{d}x}(3x+1) \\ &= \frac{3}{3x+1}. \end{aligned}$$

Now let's find the derivative of $\frac{3}{3x+1}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{3x+1}\right) &= 3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3x+1}\right)\\ &= 3\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{-1}\right)\\ &= 3\cdot -1\cdot(3x+1)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(3x+1)\\ &= 3\cdot -1\cdot(3x+1)^{-2}\cdot 3\\ &= \frac{-9}{(3x+1)^2}.\end{aligned}$$

Therefore the second derivative of $\ln(3x+1)$ is $\frac{-9}{(3x+1)^2}$.

(8.g) Find the derivative of $2^{(x^3-1)}e^{2x}$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{(x^3-1)}e^{2x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2^{(x^3-1)}\right)e^{2x} + 2^{(x^3-1)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\right)\\ &= \ln(2)2^{(x^3-1)}\frac{\mathrm{d}}{\mathrm{d}x}(x^3-1)e^{2x} + 2^{(x^3-1)}e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}(2x)\\ &= \ln(2)2^{(x^3-1)}\cdot 3x^2\cdot e^{2x} + 2^{(x^3-1)}e^{2x}\cdot 2\\ &= 2^{(x^3-1)}e^{2x}(3\ln(2)x^2+2).\end{aligned}$$

(9) Let f be a differentiable function and suppose that $f(3)=5$ and $f'(3)=-2$.

(9.a) Find the linearization of f at $a=3$.

Solution: In general, linearization is given by $f_L(x)=f(a)+f'(a)(x-a)$. In this case we have

$$f_L(x)=5-2(x-3).$$

(9.b) Estimate the change between $x=3$ and $x=4$.

Solution: In general, an estimate of change between $(a, f(a))$ and a nearby point $(a+h, f(a+h))$ is given by $f'(a)h$. In this case we have

$$f'(3)\cdot(4-3)=-2.$$

(9.c) Estimate $f(4)$. If $f(x)$ is concave up, then is this an overestimate or an underestimate?

Solution: We estimate $f(4)$ to be

$$f_L(4)=3.$$

Since $f(x)$ is concave up, the tangent line lies below the graph, and so this is an underestimate.

(9.d) $p(t)$ cents is the average retail price of a pound of salted, graded A butter, t years since 1990. Suppose that in 1998, the average retail price of salted, graded A butter was 296 cents and was increasing by 54 cents per year. Use a linear estimation to find the average retail price of salted, graded A butter in 1999.

Solution: The linear estimation for $p(t)$ at $t=a$ is given by

$$\begin{aligned}p_L(t) &= p(8)+p'(8)(t-8)\\ &= 296+54(t-8).\end{aligned}$$

So we estimate the average retail price of salted, graded A butter in 1999 to be

$$p_L(9)=350 \text{ cents}$$