Permutativity

December 11, 2019

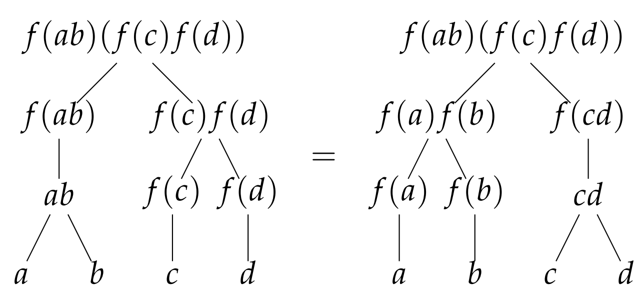
1 Introduction

We introduce a new type of algebraic law which we call the **permutative law** since it corresponds with the permutohedron as the associative law corresponds with associahedron.

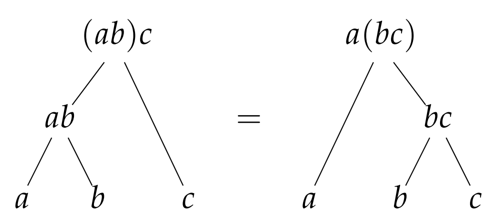
Definition 1.1. Let A be a set equipped with a binary operation $\cdot : A \times A \to A$ and a unary operation $f\colon A \to A$. We say the triple $(A, f, \cdot)$ satisfies the **permutative law** if for all $a, b, c, d \in A$, we have

$$(f(a)f(b))f(cd) = f(ab)(f(c)f(d)) \tag{1}$$

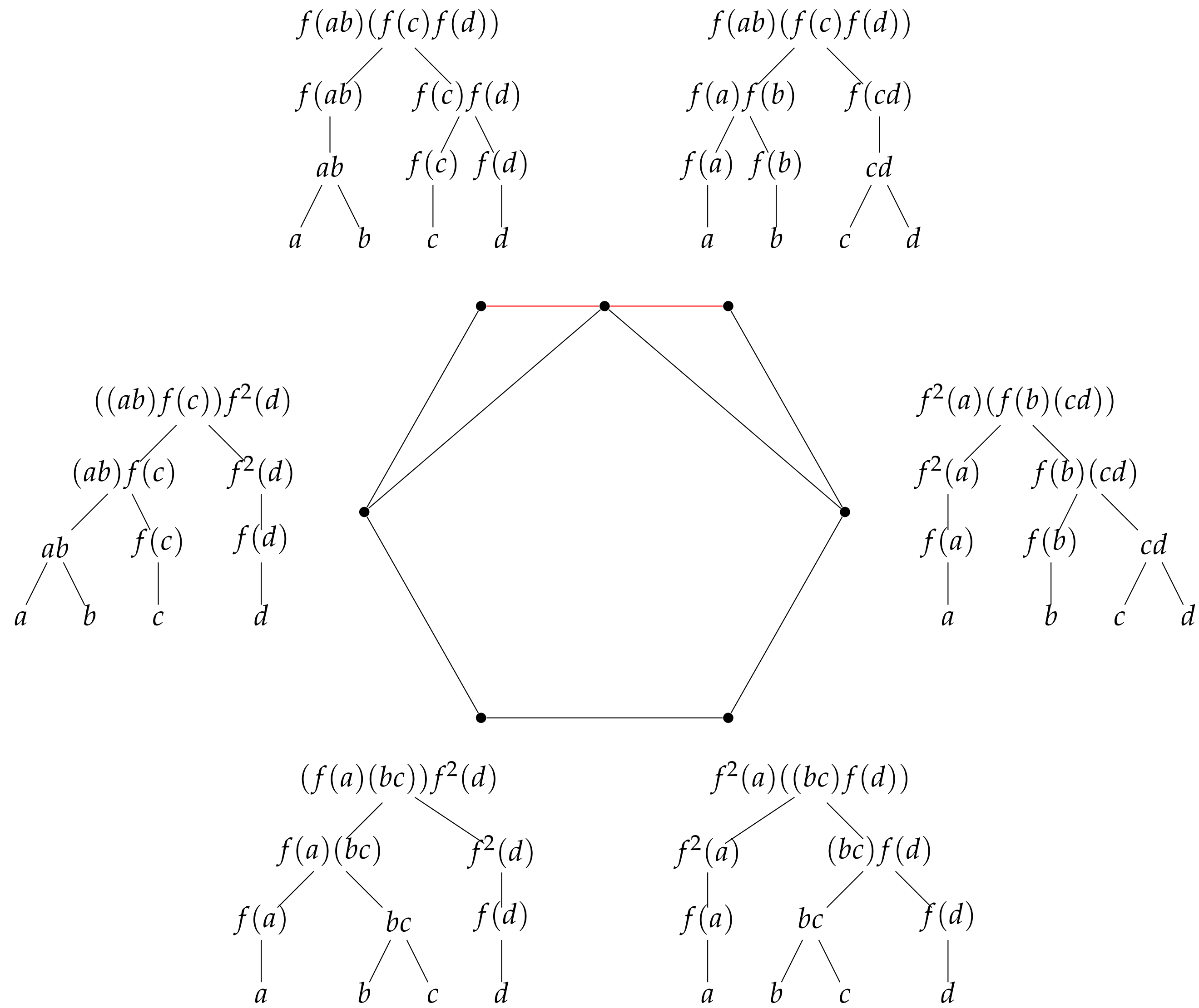
There's a very nice way of capturing visualizing this law in terms of Cayley ordered Bell trees:



This is analagous to how we can express the associative law in terms of binary rooted trees:



The difference between these two types of trees is that the Cayley ordered Bell trees keep track of a unary operator f, whereas the binary rooted trees do not. The Cayley ordered Bell trees can be attached to the vertices of the permutohedron and the ordered binary rooted trees can be attached to the vertices of the associahedron. There is a natural way to map the permutohedron to the associahedron, and it correponds to forgetting the unary operator f. In the image below, we draw the permutohedron P_3 as well as associahedron K_4 inside of it. To each vertex of K_4, we can attach a 4-leaf ordered rooted binary tree, and to each vertex of P_3, we can attach a 4-leaf Cayley ordered Bell tree, which we do here. The map from P_3 to K_4 can be visualized by collapsing the red edge, or in terms of trees, by deleting stretching nodes, or in terms of algebra, by setting f to the identity function.



2 When Do Triples Satisfy The Permutative Law?

In order for a triple $(A, f, \cdot)$ to satisfy the permutative law, we need a mixture of both nice properties for f and nice properties for the binary operation. Let us make a basic assumption on f throughout the rest of this article in order to simplify our results: we will assume that $f\colon A \to A$ is a bijection.

2.1 Groups

In this subsection, we will consider triples $(G, f, \cdot)$ where G is a group and $f\colon G \to G$ is a bijection. Here are two examples:

Example 2.1. Suppose $f\colon G \to G$ is a group homomorphism. Then the triple $(G, f, \cdot)$ satisfies the permutative law. Indeed, for all $a, b, c, d \in G$, we have

$$\begin{aligned} f(ab)(f(c)f(d)) &= (f(ab))f(c)f(d) \\ &= (f(a)f(b))f(cd) \end{aligned}$$

since the binary operation is associative and since f is a group homomorphism.

Example 2.2. Let G be a group with $x \in Z(G)$ and suppose $f(a) = xa$ for all $a \in G$. Then the triple $(G, f, \cdot)$ satisfies the permutative laws permutative. Indeed, for all $a, b, c, d \in G$,

$$\begin{aligned} f(ab)(f(c)f(d)) &= xabxcxd \\ &= xaxbxcd \\ &= (f(a)f(b))(f(cd)) \end{aligned}$$

since the binary operation is associative and since $x \in Z(G)$.

The next proposition tells us that any triples $(G, f, \cdot)$ which satisfies the permutative law essentially comes from one of the two examples above.

Proposition 2.1. *Denote $x = f(e)$. If the triple $(G, f, \cdot)$ satisfies the permutative law, then $x \in Z(G)$ and the map $\ell_x \circ f\colon G \to G$ is a group homomorphism.*

Proof. Since the triple $(G, f, \cdot)$ satisfies the permutative law, we have

$$f(ab)f(c)f(d) = f(a)f(b)f(cd). \tag{2}$$

for all $a, b, c, d \in G$. In particular, setting $a = b = e$ into (2) gives us

$$xf(c)f(d) = x^2 f(cd),$$

and after canceling x on both sides, we obtain

$$f(c)f(d) = xf(cd). \tag{3}$$

Setting $d = e$ into (3) gives us

$$f(c)x = xf(c).$$

Thus $x \in Z(G)$. For the last part, we use (3) to obtain

$$\begin{aligned}(\ell_x \circ f)(cd) &= xf(cd) \\ &= x^2 f(c)f(d) \\ &= (xf(c))(xf(d)) \\ &= (\ell_x \circ f)(c)(\ell_x \circ f)(d)\end{aligned}$$

for all $c, d \in G$. □

2.2 R-Algebras

Let R be a ring. An R-algebra A is an R-module equipped with an R-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a + b)c = ab + ac \quad \text{and} \quad a(b + c) = ab + ac.$$

We say the R-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the R-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

In this case, we call e the **identity** element. We say the R-algebra is **cancellative** if for any element $a \in A$ and any non-zero element $b \in A$ there exists precisely one element $c \in A$ with $a = bc$ and precisely one element $d \in A$ such that $a = db$.

2.2.1 Hom-Associative Algebras

Definition 2.1. A **hom-associative R-algebra**, denoted (A, α) or precisely $(A, \cdot, \alpha)$, is an R-algebra A equipped with an R-linear map $\alpha\colon A \to A$ satisfying the hom-associative law

$$\alpha(a)(bc) = (ab)\alpha(c) \tag{4}$$

for any $a, b, c \in A$. We say (A, α) is **multiplicative** if $\alpha\colon A \to A$ is an R-algebra homomorphism. We say (A, α) is **weakly left unital** if there exists an $e_l \in A$, called a **weak left unit**, such that $e_l a = \alpha(a)$ for all $a \in A$. Similarly, we say (A, α) is **weakly right unital** if there exists an $e_r \in A$, called a **weak right unit**, such that $ae_r = \alpha(a)$ for all $a \in A$. We say (A, α) is **weakly unital** if there exists an $e \in A$, called a **weak unit**, such that e is both a weak left unit and a weak right unit.

Remark. If the R-algebra A is unital with unit e, then hom-associative R-algebra (A, α) is weak unital with weak unit $\alpha(e)$. Indeed, $\alpha(e)$ is weak right unit since

$$\begin{aligned}\alpha(a) &= \alpha(a)(ee) \\ &= (ae)\alpha(e) \\ &= a\alpha(e)\end{aligned}$$

for all $a \in A$. A similar calculation shows that $\alpha(e)$ is a weak left unit as well.

Proposition 2.2. *Let A be a unital, associative R-algebra with unit e, and let $\alpha \colon A \to A$ be an R-algebra endomorphism. Define $* \colon A \times A \to A$ by*

$$a * b = \alpha(ab)$$

for all $a, b \in A$. Then $(A, *, \alpha)$ is a weakly unital hom–associative R-algebra with weak unit e.

2.2.2 From Hom-Associativity to Permutativity

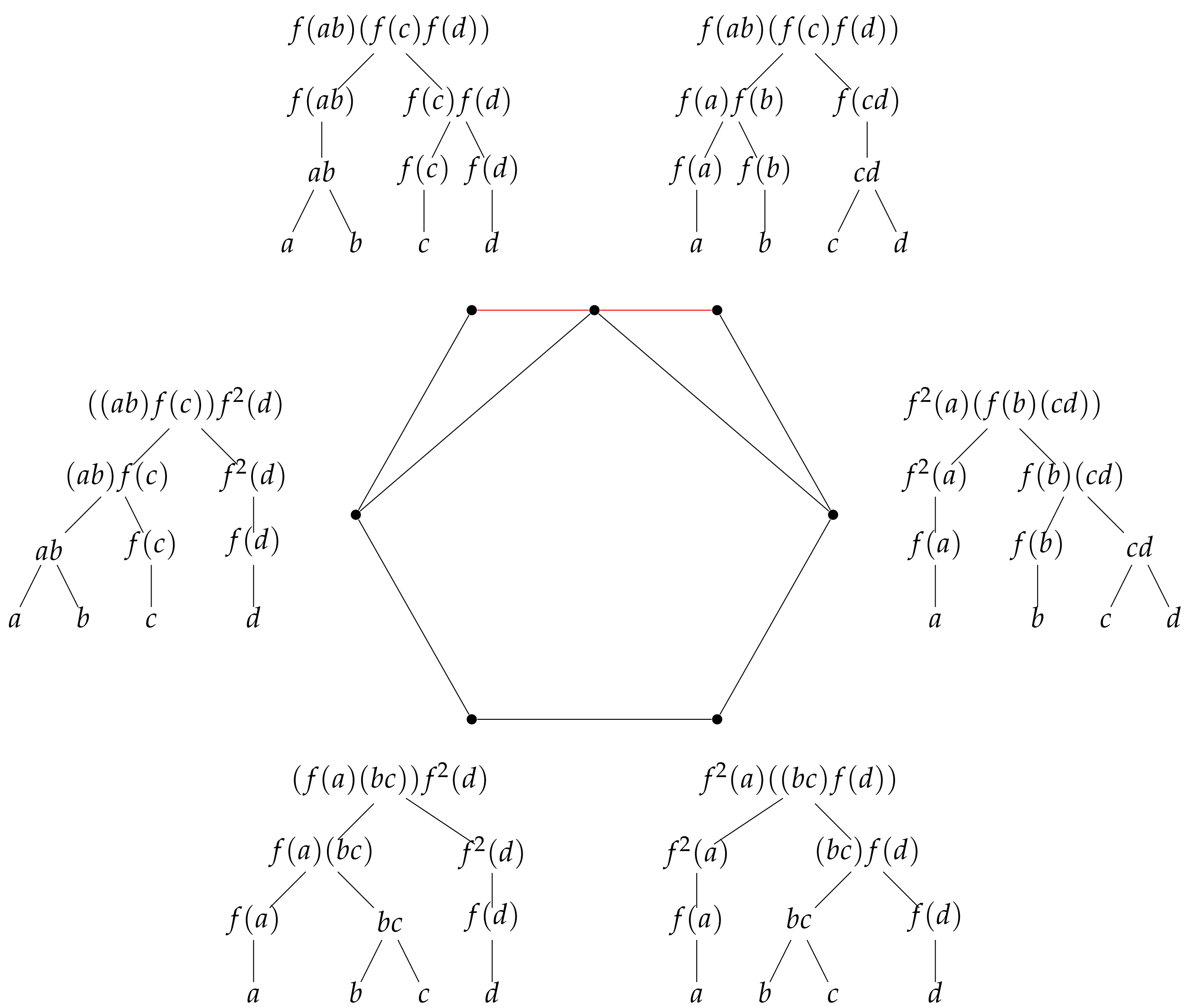
Theorem 2.1. *Every hom-associative R-algebra is a permutative R-algebra.*

Proof. Let A be a hom-associative algebra. Then for all $a, b, c, d \in A$, we have

$$\begin{aligned}f(ab)(f(c)f(d)) &= ((ab)f(c))f^2(d) \\ &= (f(a)(bc))f^2(d) \\ &= f^2(a)((bc)f(d)) \\ &= f^2(a)(f(b)(cd)) \\ &= (f(a)f(b))f(cd).\end{aligned}$$

$\square$

Remark. We can visualize this proof by tracing the edges of the permutohedron:



2.2.3 Hom-Lie Algebra

Definition 2.2. A **hom-Lie R-algebra**, denoted $(A, [\cdot,\cdot], \alpha)$, is an R-algebra $(A, [\cdot,\cdot])$ equipped with an R-linear map $\alpha\colon A \to A$ such that the following properties are satisfied:

1. (skew-symmetry) $[a, b] + [b, a] = 0$ for all $a, b \in A$;

2. (Hom-Jacobi identity) $[\alpha(a), [b, c]] + [\alpha(b), [c, a]] + [\alpha(c), [a, b]]$ for all $a, b, c \in A$.

The R-algebra multiplication map $[\cdot,\cdot]\colon A \otimes_R A \to A$ is called the **hom-Lie bracket** and the R-linear endomorphism $\alpha\colon A \to A$ is called the **twisting map**.

Proposition 2.3. *Let $(M, \cdot, \alpha)$ be a hom-associative R-algebra with commutator $[\cdot,\cdot]$. Then $(M, [\cdot,\cdot], \alpha)$ is a hom-Lie algebra.*

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2.2.4 From Hom-Lie to Permutativity

Theorem 2.2. *Every hom-Lie R-algebra is a permutative R-algebra.*

Proof. Let $(A, [\cdot,\cdot], \alpha)$ be a hom-Lie R-algebra. In the following calculation we denote $[a, b]$ by ab for all $a, b \in A$ in order to make notation look cleaner. Then for all $a, b, c, d \in A$, we have

$$\begin{aligned}
\alpha(ab)(\alpha(c)\alpha(d)) &= -\alpha^2(c)(\alpha(d)(ab)) - \alpha^2(d)((ab)\alpha(c)) \\
&= \alpha^2(c)(\alpha(a)(bd)) + \alpha^2(c)(\alpha(b)(da)) + \alpha^2(d)(\alpha(c)(ab)) \\
&= \alpha^2(c)(\alpha(a)(bd)) + \alpha^2(c)(\alpha(b)(da)) - \alpha^2(d)(\alpha(a)(bc)) - \alpha^2(d)(\alpha(b)(ca)) \\
&= \alpha^2(c)(\alpha(a)(bd)) + \alpha^2(c)(\alpha(b)(da)) - \alpha^2(d)(\alpha(a)(bc)) - \alpha^2(d)(\alpha(b)(ca)) \\
&= -\alpha^2(a)((bd)\alpha(c)) - \alpha(bd)(\alpha(c)\alpha(a)) - \alpha^2(b)((da)\alpha(c)) - \alpha(da)(\alpha(c)\alpha(b)) + \alpha^2(a)((bc)\alpha(d)) + \alpha(bc)(\alpha(d)\alpha(a)) + \alpha^2(b \\
&= -\alpha^2(a)(\alpha(c)(db)) - \alpha^2(a)(\alpha(d)(bc)) + \alpha^2(b)(\alpha(c)(ad)) + \alpha^2(b)(\alpha(a)(dc)) \\
&= -\alpha^2(a)(\alpha(c)(db)) - \alpha^2(a)(\alpha(d)(bc)) - \alpha^2(b)(\alpha(a)(cd)) \\
&= \alpha^2(a)(\alpha(b)(cd)) + \alpha^2(b)((cd)\alpha(a)) \\
&= -\alpha(cd)(\alpha(a)\alpha(b)) \\
&= (\alpha(a)\alpha(b))\alpha(cd),
\end{aligned}$$

where we used the fact that

$$(\alpha(a)\alpha(d))\alpha(bc) - \alpha(ad)(\alpha(b)\alpha(c)) + \alpha(ac)(\alpha(b)\alpha(d)) - (\alpha(a)\alpha(c))\alpha(bd) = 0$$

or in other words

$$(\alpha(a)\alpha(d))\alpha(bc) - (\alpha(a)\alpha(c))\alpha(bd) + \alpha(ac)(\alpha(b)\alpha(d)) - \alpha(ad)(\alpha(b)\alpha(c)) = 0$$

or in other words

$$(\alpha(a)\alpha(d))\alpha(bc) - (\alpha(a)\alpha(c))\alpha(bd) + (\alpha(b)\alpha(c))\alpha(ad) - (\alpha(b)\alpha(d))\alpha(ac) = 0$$

□

And so we have

$$-\alpha(bd)(\alpha(c)\alpha(a)) - \alpha(da)(\alpha(c)\alpha(b)) + \alpha(bc)(\alpha(d)\alpha(a)) + \alpha(ca)(\alpha(d)\alpha(b)) = (\alpha(a)\alpha(d))\alpha(bc) - \alpha(ad)(\alpha(b)\alpha(c)) + \alpha(ac)(\alpha(b)\alpha(d)) - (\alpha(a$$

3 Deformation

For each $1 \leq i_1, i_2, i_3, i_4, k \leq n$, we have

$$\sum_{j_1,j_2,j_3,k_1,k_2} \left(C^{k_1}_{i_1 i_2} C^{k_2}_{j_1 j_2} C^{k}_{j_3 k_2} a_{j_1 i_3} a_{j_2 i_4} a_{j_3 k_1} - C^{k_1}_{i_3 i_4} C^{k_2}_{j_1 j_2} C^{k}_{k_2 j_3} a_{j_1 i_1} a_{j_2 i_2} a_{j_3 k_1} \right) = 0$$

or in other words

$$\sum_{j_1,j_2,j_3,k_1,k_2} C^{k_2}_{j_1 j_2} a_{j_3 k_1} \left(C^{k_1}_{i_1 i_2} C^{k}_{j_3 k_2} a_{j_1 i_3} a_{j_2 i_4} - C^{k_1}_{i_3 i_4} C^{k}_{k_2 j_3} a_{j_1 i_1} a_{j_2 i_2} \right) = 0$$

If the group operation is abelian, then we would have

$$\sum_{j_1,j_2,j_3,k_1,k_2} C^{k_2}_{j_1 j_2} C^{k}_{j_3 j_2} a_{j_3 k_1} \left(C^{k_1}_{i_1 i_2} a_{j_1 i_3} a_{j_2 i_4} - C^{k_1}_{i_3 i_4} a_{j_1 i_1} a_{j_2 i_2} \right) = 0$$

If it is unital, then we would have $C^k_{i1} = a_{ki} = C^k_{1i}$ for all i, k.