PDG Algebras and Modules

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Contents

1 Introduction

1.1 Notation and Conventions

Unless otherwise specified, let K be a field and let $(R, \mathfrak{m})$ be a local Noetherian ring.

1.1.1 Category Theory

In this document, we consider the following categories:

- The category of all sets and functions, denoted **Set**;
- The category of all rings and ring homomorphisms, denoted **Ring**;
- The category of all R-modules and R-linear maps, denoted $\mathbf{Mod}_R$;
- The category of all graded R-modules and graded R-linear maps, denoted $\mathbf{Grad}_R$;
- The category of all R-algebras R-algebra homorphisms, denoted $\mathbf{Alg}_R$;
- The category of all R-complexes and chain maps, denoted $\mathbf{Comp}_R$;
- The category of all R-complexes and homotopy classes of chain maps, denoted $\mathbf{HComp}_R$
- The category of all DG R-algebras DG algebra homomorphisms, denoted $\mathbf{DG}_R$.

2 Basic Definitions

2.1 PDG R-Algebras

Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ be a chain map. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i \star_\mu b_i.$$

If μ is understood from context, then we also tend to drop μ from the subscript in $\star_\mu$, or even drop $\star$ altogether and write

$$\mu\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i b_i.$$

Note that μ being a chain map implies it is a **graded-multiplication** which satisfies **Leibniz law**. Being a graded-multiplication means μ is an R-bilinear map which respects the grading. In particular, if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. Satisfying Leibniz law means

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a \mathrm{d}(b)$$

for all homogeneous $a, b \in A$. We can also impose other conditions on μ as follows:

1. We say μ is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.
2. We say μ is **graded-commutative** if
$$ab = (-1)^{|a||b|} ba$$
for all homogeneous $a, b \in A$.
3. We say μ is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$aa = 0$$
for all $a \in A_i$ for all i odd.

4. We say μ is **unital** if there exists $1 \in A$ such that

$$a1 = a = 1a$$

for all $a \in A$.

The triple (A, d, μ) is called a **differential graded R-algebra** (or **DG R-algebra**) if μ satisfies conditions 1-4. If (A, d, μ) only satisfies conditions 2-4, then it is called a **partial differential graded R-algebra** (or **PDG R-algebra**). To clean notation in what follows, we will often refer to a PDG R-algebra (A, d, μ) via its underlying graded R-module A. In particular, if we write "let A be a PDG R-algebra" without specifying its differential or multiplication operations, then it will be understood that it's differential is denoted d_A and its multiplication is denoted μ_A.

Definition 2.1. Let A and A' be two PDG R-algebra. A **morphism** between them is a chain map $\varphi \colon A \to A'$ which satisfies the following two properties

1. it respects the identity elements, that is, $\varphi(1) = 1$;
2. it respects multiplication, that is, $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$.

It is straightforward to check that the collection of all PDG R-algebras algebras together with their morphisms forms a category, which we denote by $\mathbf{PDG}_R$.

2.2 PDG A-Modules

Unless otherwise specificed, we fix A to be a PDG R-algebra.

Definition 2.2. A (left) **partial differential graded A-module** (or **PDG A-module**) is a triple $(X, \mathrm{d}_X, \mu_{A,X})$ where (X, d_X) is an R-complex and where $\mu_{A,X} \colon A \otimes_R X \to X$ is a chain map which satisfies $1x = x$ for all $x \in X$.

Here again we are using the convention that the image of a tensor $\sum_{i=1}^n a_i \otimes x_i$ in $A \otimes_R X$ under the map μ_X is denoted by

$$\mu_{A,X}\left(\sum_{i=1}^n a_i \otimes x_i\right) = \sum_{i=1}^n a_i \star_{\mu_{A,X}} x_i = \sum_{i=1}^n a_i x_i$$

Also, as before, if we write "let X be a PDG A-module" without specifying its differential or scalar-multiplication operations, then it will be understood that it's differential is denoted d_X and its multiplication is denoted $\mu_{A,X}$. In fact, if A is understood from context, then we simplify this notation even further by writing μ_X rather than $\mu_{A,X}$. Note that μ_X being a chain map implies it is satisfies **Leibniz law**, which in this context says

$$\mathrm{d}_X(ax) = \mathrm{d}_A(a)x + (-1)^{|a|} a \mathrm{d}_X(x)$$

for all homogeneous $a \in A$ and $x \in X$. Notice that we do not require μ_X to be associative in order for X to be a PDG A-module, that is, we do not require here the identity

$$(ab)x = a(bx)$$

to hold for all $a, b \in A$ and $x \in X$.

Definition 2.3. Let X and Y be two PDG A-modules. An **A-linear map** betweem them is a chain map $\varphi \colon X \to Y$ which satisfies $\varphi(ax) = a\varphi(x)$ for all $a \in A$ and $x \in X$. The collection of all PDG A-modules together with their A-linear maps forms a category, which we denote by $\mathbf{PMod}_A$.

2.2.1 Submodules

Definition 2.4. Let X and Y be two PDG A-modules. We say X is a **PDG A-submodule** of Y if $X \subseteq Y$. A PDG A-submodule of A is called a **PDG ideal** of A. Given any collection $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of X, we denote by $\langle\langle x_\lambda \rangle\rangle_{\lambda \in \Lambda}$ to be the smallest PDG A-submodule of M which contains $\{x_\lambda\}_{\lambda \in \Lambda}$. We denote by $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ to be the set of all A-linear combinations of $\{x_\lambda\}_{\lambda \in \Lambda}$.

Proposition 2.1. *Let X be a PDG A-module and let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a collection of elements of X. Then*

$$\langle\langle x_\lambda \rangle\rangle_{\lambda \in \Lambda} = \langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle_{\lambda \in \Lambda}$$

Proof. To clean notation in what follows, we drop the "$\lambda \in \Lambda$" from the subscript of our bracket notation. Since $\langle\langle x_\lambda \rangle\rangle$ is the smallest PDG A-submodule of X which contains $\{x_\lambda\}$, we must have $\mathrm{d}_X(x_\lambda) \in \langle\langle x_\lambda \rangle\rangle$ for all $\lambda \in \Lambda$. Furthermore, we must have all A-linear combinations of $\{x_\lambda, \mathrm{d}_X(x_\lambda)\}$ belong to $\langle\langle x_\lambda \rangle\rangle$ as well. Thus

$$\langle x_\lambda, \mathrm{d}_X(x_\lambda) \rangle \subseteq \langle\langle x_\lambda \rangle\rangle.$$

For the reverse direction, notice that Leibniz law ensures that $\langle x_\lambda, \mathrm{d}_X(x_\lambda)\rangle$ is d_X-stable. Indeed, if

$$\sum_{i=1}^{m} a_i x_{\lambda_i} + \sum_{j=1}^{n} b_j \mathrm{d}_X(x_{\lambda_j}),$$

is a finite A-linear combination of elements in $\{x_\lambda, \mathrm{d}_X(x_\lambda)\}$ where each a_i and b_j are homogeneous, then note that

$$\begin{aligned}
\mathrm{d}_X\left(\sum_{i=1}^{m} a_i x_{\lambda_i} + \sum_{j=1}^{n} b_j \mathrm{d}_X(x_{\lambda_j})\right) &= \sum_{i=1}^{m} \mathrm{d}_X(a_i x_{\lambda_i}) + \sum_{j=1}^{n} \mathrm{d}_X(b_j \mathrm{d}(x_{\lambda_j})) \\
&= \sum_{i=1}^{m} \left(\mathrm{d}_A(a_i)x_{\lambda_i} + (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i})\right) + \sum_{j=1}^{n} \left(\mathrm{d}_A(b_j)x_{\lambda_j} + (-1)^{|b_j|} b_j \mathrm{d}_X^2(x_{\lambda_j})\right) \\
&= \sum_{i=1}^{m} \mathrm{d}_A(a_i)x_{\lambda_i} + \sum_{i=1}^{m} (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i}) + \sum_{j=1}^{n} \mathrm{d}_A(b_j)x_{\lambda_j} \\
&\in \langle x_\lambda, \mathrm{d}_X(x_\lambda)\rangle.
\end{aligned}$$

In particular, we see that $\langle x_\lambda, \mathrm{d}_X(x_\lambda)\rangle$ is a PDG A-submodule of X which contains $\{x_\lambda\}$. Since $\langle\langle x_\lambda\rangle\rangle$ is the *smallest* PDG A-submodule of X which contains $\{x_\lambda\}$, it follows that

$$\langle x_\lambda, \mathrm{d}_X(x_\lambda)\rangle \supseteq \langle\langle x_\lambda\rangle\rangle.$$

□

Warning: In the category of R-modules, we have the concept of annihilators. In particular, suppose M is an R-module and let $u \in M$. We define the **annihilator** with respect to u to be the subset of R given by

$$0 : u = \{r \in R \mid ru = 0\}.$$

In fact, $0 : u$ is in an ideal of R, but we need the associative law to get this: if $r \in R$ and $x \in 0 : u$, then $(rx)u = r(xu) = 0$ implies $rx \in 0 : u$.

Now let us consider the case where X is a PDG A-module and let $x \in X$. We can define the annihilator $0 : x$ with respect to x as a subset of A as before:

$$0 : x = \{a \in A \mid ax = 0\},$$

however this time the set $0 : x$ need not be a PDG ideal of A. On the other hand, if $u \in \operatorname{Assoc} M$, where

$$\operatorname{Assoc} M = \{u \in M \mid [a, b, u] = 0 \text{ for all } a, b \in A\},$$

then there are no issues with the proof above, so $0 : u$ is an ideal of R in this case.

2.2.2 Hom

Let M and N be two PDG A-modules. We denote by $\operatorname{Hom}_A(M, N)$ to be the set of all A-linear maps from M to N. The set $\operatorname{Hom}_A(M, N)$ as the structure of an abelian group via pointwise addition of A-linear maps from M to N. On the other hand, suppose we define a scalar "action" on $\operatorname{Hom}_A(M, N)$ by

$$(a \cdot \varphi)(u) = \varphi(au)$$

for all $a \in A$, $\varphi \in \operatorname{Hom}_A(M, N)$, and $u \in M$. Then this "action" does not necessarily give $\operatorname{Hom}_A(M, N)$ the structure of an R-module, since if $a \in A_i$, $b \in A_j$, and $\varphi \in \operatorname{Hom}_A(M, N)$, then

$$\begin{aligned}
((ab) \cdot \varphi)(u) &= \varphi((ab)u) \\
&= \varphi((-1)^{i+j}(ba)u) \\
&= (-1)^{i+j}\varphi((ba)u) \\
&= (-1)^{i+j}\varphi(b(au) + (-1)^{i+j}[b, a, u]) \\
&= (-1)^{i+j}(b \cdot \varphi)(au) + (-1)^{i+j}[b, a, \varphi(u)] \\
&= (-1)^{i+j}(a \cdot (b \cdot \varphi))(u) + (-1)^{i+j}[b, a, \varphi(u)]
\end{aligned}$$

for all $u \in M$. Thus one needs commutativity and associativity in order to conclude that $(ab) \cdot \varphi = a \cdot (b \cdot \varphi)$.

2.3 $\mathbf{PMod}_A$ is an Abelian Category

Throughout the rest of this subsection, we fix a PDG R-algebra A. We would like to talk about the concept of an exact sequence in $\mathbf{PMod}_A$. For this, we just need to check that $\mathbf{PMod}_A$ is abelian category. First let us check that it is a pre-additive category.

2.3.1 Kernels

Proposition 2.2. *Let $\varphi\colon X \to Y$ be a morphism of PDG A-modules and let $K = \ker\varphi$. Then K has the structure of a PDG A-submodule of X, where $\mathrm{d}_K = \mathrm{d}|_K$ and where $\mu_K = \mu_X|_{A\otimes_R K}$.*

Proof. Since both d_K and μ_K are restrictions, we just need to check that d_K and μ_K land in K. Indeed, then all of the properties needed in order for K to be a PDG A-submodule of X will be inherited from X. First we show d_K lands in K. Let $x \in K$. Then

$$\begin{aligned}\varphi\mathrm{d}_K(x) &= \mathrm{d}_K\varphi(x)\\ &= \mathrm{d}_K(0)\\ &= 0\end{aligned}$$

implies $\mathrm{d}_K(x) \in K$. It follows that d_K lands in K. Now we show μ_K lands in K. Let $a \otimes x$ be an elementary tensor in $A \otimes_R K$. Then

$$\begin{aligned}\varphi\mu_K(a\otimes x) &= \varphi(ax)\\ &= a\varphi(x)\\ &= a\cdot 0\\ &= 0.\end{aligned}$$

It follows that μ_K lands in K. $\square$

2.3.2 Images

Proposition 2.3. *Let $\varphi\colon (A, \mathrm{d}, \mu) \to (A', \mathrm{d}', \mu')$ be a morphism of R-complex algebras. Then $(\operatorname{im}\varphi, \widetilde{\mathrm{d}}', \widetilde{\mu}')$ is an R-complex algebra, where $\widetilde{\mathrm{d}}' = \mathrm{d}'|_{\ker\varphi}$ and $\widetilde{\mu}' = \mu'|_{\operatorname{im}\varphi\otimes_R \operatorname{im}\varphi}$.*

Proof. We just need to check that $\widetilde{\mathrm{d}}'$ and $\widetilde{\mu}'$ land in $\ker\varphi$. Then it will follow that $(\ker\varphi, \widetilde{\mathrm{d}}, \widetilde{\mu})$ is an R-complex algebra since it will inherit the properties needed to be an R-complex algebra from (A, d, μ). First we show $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a) \in \operatorname{im}\varphi$. Then

$$\begin{aligned}\mathrm{d}'(\varphi(a)) &= \mathrm{d}'\varphi(a)\\ &= \varphi\mathrm{d}(a)\\ &= \varphi(\mathrm{d}(a)).\end{aligned}$$

It follows that $\widetilde{\mathrm{d}}'$ lands in $\operatorname{im}\varphi$. Now we show $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. Let $\varphi(a) \otimes \varphi(b)$ be an elementary tensor in $\operatorname{im}\varphi \otimes_R \operatorname{im}\varphi$. Then

$$\begin{aligned}\mu((\varphi(a)\otimes\varphi(b)) &= \varphi(a)\star\varphi(b)\\ &= \varphi(a\star b)\\ &= \varphi(\mu(a\otimes b)).\end{aligned}$$

It follows that $\widetilde{\mu}'$ lands in $\operatorname{im}\varphi$. $\square$

2.3.3 Cokernels

As we've seen, both kernels and images exist in $\mathbf{CompAlg}_R$. The problem however is that cokernels do not necessarily exist in $\mathbf{CompAlg}_R$. To see what goes wrong, suppose $\varphi\colon (A, \mathrm{d}, \mu) \to (A', \mathrm{d}', \mu')$ be a morphism of R-complex algebras. A naive attempt at defining the cokernel of φ would go as follows: first we take the cokernel of the underlying R-complexes, namely $(\overline{A'}, \overline{\mathrm{d}'})$ where $\overline{A'} = A'/\operatorname{im}\varphi$ and $\overline{\mathrm{d}'}$ is defined by $\overline{\mathrm{d}'}(\overline{a'}) = \overline{\mathrm{d}'(a')}$ for all $\overline{a'} \in \overline{A'}$. It is straightforward to check that $\overline{\mathrm{d}'}$ is well-defined and gives $\overline{A'}$ the structure of an R-complex. Next we define multiplication $\overline{\mu'}\colon \overline{A'} \otimes_R \overline{A'} \to \overline{A'}$ by

$$\overline{\mu'}(\overline{a'}\otimes\overline{b'}) = \overline{a'\star_{\mu'} b'} \tag{1}$$

for all elementary tensors $\overline{a'} \otimes \overline{b'}$ in $\overline{A'} \otimes_R \overline{A'}$ and extending $\overline{\mu'}$ everywhere else R-linearly. Unfortunately, upon further inspection, we see that (**??**) is note well-defined. Indeed, if $a' + \varphi(a)$ is another representative of the coset $\overline{a'}$ and $b' + \varphi(b)$ is another representative of the coset $\overline{b'}$, then we have

$$\begin{aligned}\overline{\mu'}(\overline{a' + \varphi(a)} \otimes \overline{b' + \varphi(b)}) &= \overline{(a' + \varphi(a)) \star (b' + \varphi(b))} \\ &= \overline{a' \star b'} + \overline{a' \star \varphi(b)} + \overline{\varphi(a) \star b'} + \overline{\varphi(a) \star \varphi(b)} \\ &= \overline{a' \star b'} + \overline{a' \star \varphi(b)} + \overline{\varphi(a) \star b'} + \overline{\varphi(a \star b)} \\ &= \overline{a' \star b'} + \overline{a' \star \varphi(b)} + \overline{\varphi(a) \star b'}.\end{aligned}$$

In particular, (**??**) is well-defined if and only if $\operatorname{im}\varphi$ is an ideal of A'.

2.4 Associator Functor

Let A be a PDG R-algebra and let X be a PDG A-module. Given $a, b \in A$ and $x \in X$, we define the **associator** of the triple (a, b, x), denoted $[a, b, x]$, by the formula

$$[a, b, x] = (ab)x - a(bx). \tag{2}$$

More generally, let $\alpha_{A,A,X} \colon (A \otimes_R A) \otimes_R X \to A \otimes_R (A \otimes_R X)$ denote the unique chain map defined on elementary tensors by

$$(a \otimes b) \otimes x \mapsto a \otimes (b \otimes x).$$

We define the **associator chain map** with respect to μ_X to be chain map $[\cdot,\cdot,\cdot]_{\mu_X} \colon (A \otimes_R A) \otimes_R X \to X$ defined by

$$[\cdot,\cdot,\cdot]_{\mu_X} := \mu_X(1 \otimes \mu_X)\alpha_{A,A,X} - \mu_X(\mu_A \otimes 1).$$

If μ_X is understood from context, then we will simplify our notation by dropping μ_X from the subscript in $[\cdot,\cdot,\cdot]$. Thus, if $(a \otimes b) \otimes x$ is an elementary tensor in $(A \otimes_R A) \otimes_R X$ then the associatior chain map with respect to X maps the elementary tensor $(a \otimes b) \otimes x$ to the associator of the triple (a, b, x):

$$[\cdot,\cdot,\cdot]((a \otimes b) \otimes x) = [a, b, x],$$

where $[a, b, x]$ is as defined above in (2). We define the **associator complex** with respect to μ_X, denoted $[X]_{\mu_X}$, to be the image of $[\cdot,\cdot,\cdot]$, where again we simplify notation by writing $[X]$ instead of $[X]_{\mu_X}$ if μ_X is understood from context. Thus

$$[X] = \mathrm{Span}_R\{[a, b, x] \mid a, b \in A \text{ and } x \in X\}.$$

Since $[\cdot,\cdot,\cdot]$ is a chain map, we see that $[X]$, being the image of $[\cdot,\cdot,\cdot]$, is an R-subcomplex of X. We also see that $[\cdot,\cdot,\cdot]$ is a graded-trilinear map which satisfies Leibniz law, where Leibniz law in this context says

$$\mathrm{d}_X[a, b, x] = [\mathrm{d}_A(a), b, x] + (-1)^{|a|}[a, \mathrm{d}_A(b), x] + (-1)^{|a|+|b|}[a, b, \mathrm{d}_X(x)]. \tag{3}$$

for all homogeneous $a, b \in A$ and $x \in X$.

Now suppose Y is another PDG A-module and $\varphi \colon X \to Y$ is an A-linear map. We obtain an induced chain map of R-complexes $[\varphi] \colon [X] \to [Y]$, where $[\varphi]$ is the unique chain map which satisfies

$$\begin{aligned}[\varphi][a, b, x] &= \varphi((ab)x - a(bx)) \\ &= \varphi((ab)x) - \varphi(a(bx)) \\ &= (ab)\varphi(x) - a\varphi(bx) \\ &= (ab)\varphi(x) - a(b\varphi(x)) \\ &= [a, b, \varphi(x)].\end{aligned}$$

In particular, the map $[\varphi]$ is just the restriction of φ to $[M]$. It is straightforward to check that the assignment $M \mapsto [M]$ and $\varphi \mapsto [\varphi]$ gives rise to a functor from the category of PDG A-modules to the category of R-complex. We call this functor the **associator functor** with respect to A, and we denote this functor by $[\cdot]_{\mu_A} \colon \mathbf{PMod}_A \to \mathbf{Comp}_R$. As usual, we simplify our notation by droping μ_A from $[\cdot]$ when context is clear.

2.4.1 Homology of $[X]$

Let X be a PDG A-module. It is easy to see that μ_X is associative if and only if $[X] = 0$. Given that $[X]$ is an R-complex, we also have a weaker form of associativity:

Definition 2.5. We say μ_X is **homologically associative** if $\mathrm{H}([X]) = 0$.

Clearly if μ_X is associative, then μ_X is homologically associative. It turns out that the converse is also true if $[X]$ is bounded below and **minimal** in the sense that $\mathrm{d}_X([X]) \subseteq \mathfrak{m}[X]$ where $\mathfrak{m}$ is the maximal ideal in the local ring R.

Proposition 2.4. *Let X be a PDG A-module and assume that $[X]$ is bounded below and minimal. Then μ_X is associative if and only if μ_X is homologically associative.*

Proof. Clearly if μ_X is associative, then it is homologically associative. To show the converse, we prove the contrapositive: assume μ_X is not associative, so $[X] \neq 0$. Choose $i \in \mathbb{Z}$ minimal so that $[X]_i \neq 0$ and $[X]_{i-1} = 0$. By Nakayama's Lemma, we can find a triple (a, b, x) such that $|a| + |b| + |x| = i$ and such that $[a, b, x] \notin \mathfrak{m}[X]_i$. By minimality of i, we have $\mathrm{d}_X[a, b, x] = 0$. Also, since X is minimal, we have $\mathrm{d}_X[X] \subseteq \mathfrak{m}[X]$. Thus $[a, b, x]$ represents a nontrivial element in homology. It follows that μ_X is not homologically associative. □

Note that if A and X are both minimal, then the Leibniz law (3) implies $[X]$ is minimal too. Also our assumption in Proposition (2.4) that $[X]$ is bounded below can clearly be weakened since in the proof we just needed to find an $i \in \mathbb{Z}$ such that $[X]_i \neq 0$ and $[X]_{i-1} = 0$. At the same time, the proof of Proposition (2.4) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition

Definition 2.6. Let X be a PDG A-module and assume that $[X]$ is bounded below. We define the **associative index** of μ_X, denoted $\operatorname{index}\mu_X$, is defined to be the smallest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\operatorname{index}\mu_X = \infty$ if μ_X is associative. We extend this definition to case where $[X]$ is not bounded below by setting $\operatorname{index}\mu_X = -\infty$.

With the associative index of μ_X defined, we see, after analyzing the proof of Proposition (2.4), that if we assume μ_X is not associative then

$$\operatorname{index}\mu_X = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i([X]) \neq 0\}$$

In other words, the associative index of μ_X can be measured homologically.

We can also define an associative index of R-complex. Let us record this definition now:

Definition 2.7. Let X be a PDG A-module. We define the **associative index** of X, denoted $\operatorname{index} X$, to be

$$\operatorname{index} X = \sup\{\operatorname{index}\mu \mid \mu \text{ is a multplication on } X\}.$$

Let I be an ideal of R and let F be the minimal free resolution of R/I over R. We define the **associative index** of R/I, denoted $\operatorname{index}(R/I)$, to be the associative index of F.

2.4.2 Stable PDG A-Submodules

The associator functor $[\cdot]\colon \mathbf{PMod}_A \to \mathbf{Mod}_R$ need not be exact. To see what goes wrong, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{4}$$

be a short exact sequence of PDG A-modules. We obtain an induced sequence of R-complexes

$$0 \longrightarrow [X] \xrightarrow{[\varphi]} [Y] \xrightarrow{[\psi]} [Z] \longrightarrow 0 \tag{5}$$

We claim that we have exactness at $[X]$ and $[Z]$. Indeed, this is equivalent to showing $[\varphi]$ is injective and $[\psi]$ is surjective, which follows from the fact that $[\varphi]$ (respectively $[\psi]$) is the restriction of the injective map φ (respectively the surjective map ψ). Let us see what goes wrong when trying to prove exactness at $[Y]$. Let $\sum_{i=1}^n [a_i, b_i, y_i] \in \ker[\psi]$. Then $\sum_{i=1}^n [a_i, b_i, y_i] \in \ker \varphi_2$, and so by exactness of (4), there exists $x \in X$ such that $\varphi(x) = \sum_{i=1}^n [a_i, b_i, y_i]$. It is not at all clear however that $x \in [X]$. This leads us to consider the following definition:

Definition 2.8. Let X be a PDG A-submodule of Y. We say X is a **stable** PDG A-submodule of Y if it satisfies $[X] = X \cap [Y]$.

Now it is easy to check that (5) is a short exact sequence of R-complexes if and only if $\varphi(X)$ is a stable PDG A-submodule of Y. Thus if $\varphi(X)$ is a stable PDG A-submodule of Y, then the short exact sequence (5) of R-complexes induces a long exact sequence in homology

$$\begin{tikzcd}
& & \cdots \arrow[r] & \mathrm{H}_{i+1}([Z]) \arrow[dll, rounded corners, to path={}] \\
& \mathrm{H}_i([X]) \arrow[r] & \mathrm{H}_i([Y]) \arrow[r] & \mathrm{H}_i([Z]) \\
& \mathrm{H}_{i-1}([X]) \arrow[r] & \cdots &
\end{tikzcd} \tag{6}$$

From this, one concludes immediately the following theorem:

Theorem 2.1. *Suppose X is a PDG A-submodule of Y. Then μ_Y is homologically associative if and only if μ_X and $\mu_{Y/X}$ are homologically associative.*

3 Invariant

Let I be an ideal of R and let F be the minimal free resolution of R/I over R. The multilpication map $R/I \otimes_R R/I \to R/I$ can be lifted to a multiplication map $\mu_F \colon F \otimes_R F \to F$, which in general is associative and graded-commutative only up to homotopy. Moreover μ_F is unique only up to homotopy. It is known that μ_F can be chosen to be graded-commutative "on the nose", but in general it is not possible to choose μ_F such that it is associative. Choose such a μ_F throughout the rest of this section.

Now suppose $r \in \mathfrak{m}$ is an (R/I)-regular element. Then the mapping cone $\mathrm{C}(r)$ is the minimal free resolution of $R/\langle I, x\rangle$ over R. The multiplication μ_F on F induces a multilpication $\mu_{\mathrm{C}(r)}$ on $\mathrm{C}(r)$ as follows: First note that $F \oplus F(-1)$ is the underlying graded R-module of $\mathrm{C}(r)$. Express this graded R-module in the form $F + Fe$ where e is a generator of degree -1 and where $\{1, e\}$ is an F-linearly independent set. Thus an element in $F + Fe$ can be expressed in the form $\alpha + \beta e$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i, then α and β are homogeneous of degrees i and $i-1$ respectively. With this understood, the multilpication $\mu_{\mathrm{C}(r)}$ is defined on homogeneous elements $\alpha, \beta, \gamma, \delta \in F$ by

$$(\alpha + \beta e)(\gamma + \delta e) = \alpha\gamma + (\alpha\delta + (-1)^{|\gamma|}\beta\gamma)e$$

and extended R-linearly everywhere else. The mapping cone $\mathrm{C}(r)$ inherits a natural PDG F-module structure via restriction of scalars. Now there are two associator complexes to consider. The first is the associator complex with respect to $\mu_{F,\mathrm{C}(r)}$, given by

$$[\mathrm{C}(r)]_{\mu_{F,\mathrm{C}(r)}} = \mathrm{Span}_R\{[\alpha, \beta, \gamma + \delta e] \mid \alpha, \beta, \gamma, \delta \in F\}.$$

The second is the associator complex with respect to $\mu_{\mathrm{C}(r)}$, given by

$$[\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}} = \mathrm{Span}_R\{[\alpha + \beta e, \gamma + \delta e, \varepsilon + \zeta e] \mid \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F\}.$$

It turns out that these two associator complexes are the same. Indeed, clearly we have

$$[\mathrm{C}(r)]_{\mu_{F,\mathrm{C}(r)}} \subseteq [\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}}.$$

Conversely, a calculation gives us

$$\begin{aligned}
[\alpha, \beta, \gamma + \delta e] &= [\alpha, \beta, \gamma] + [\alpha, \beta, \delta]e \\
[\alpha, \beta + \gamma e, \delta] &= [\alpha, \beta, \gamma] + (-1)^{|\delta|}[\alpha, \gamma, \delta]e \\
[\alpha + \beta e, \gamma, \delta] &= [\alpha, \gamma, \delta] + (-1)^{|\gamma|+|\delta|}[\beta, \gamma, \delta]e
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in F$. Using these identities together with the fact that $e^2 = 0$ and the identity 1 associates with everything, we obtain

$$\begin{aligned}
[\alpha + \beta e, \gamma + \delta e, \varepsilon + \zeta e] &= [\alpha, \gamma, \varepsilon] + [\alpha, \gamma, \zeta]e + (-1)^{|\varepsilon|}[\alpha, \delta, \varepsilon]e + (-1)^{|\gamma|+|\varepsilon|}[\beta, \gamma, \varepsilon]e \\
&= [\alpha, \gamma, \varepsilon + \zeta e] + (-1)^{|\varepsilon|}[\alpha, \delta, \varepsilon]e + (-1)^{|\gamma|+|\varepsilon|}[\beta, \gamma, \varepsilon]e \\
&= [\alpha, \gamma, \varepsilon + \zeta e] + (-1)^{|\varepsilon|}[\alpha, \delta, 1 + \varepsilon e] + (-1)^{|\gamma|+|\varepsilon|}[\beta, \gamma, 1 + \varepsilon e],
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F$. It follows that

$$[\mathrm{C}(r)]_{\mu_{F,\mathrm{C}(r)}} \supseteq [\mathrm{C}(r)]_{\mu_{\mathrm{C}(r)}}.$$

Thus we are justified in simplifying our notation by dropping either $\mu_{F,\mathrm{C}(r)}$ and $\mu_{\mathrm{C}(r)}$ from the subscript and just writing $[\mathrm{C}(r)]$ to denote the common R-complex.

Now the homothety map $F \xrightarrow{r} F$ gives rise to a short exact sequence of R-complexes

$$0 \longrightarrow F \xrightarrow{\iota} \mathrm{C}(r) \xrightarrow{\pi} \Sigma F \longrightarrow 0 \tag{7}$$

where $\iota\colon F \to \mathrm{C}(r)$ is the inclusion map and where $\pi\colon \mathrm{C}(r) \to \Sigma F$ is the projection map given by

$$\pi(\alpha + \beta e) = \beta$$

for all $\alpha, \beta \in F$. In fact, both ι and π are A-linear maps, and so (7) is a short exact sequence of PDG F-modules. In fact, it is a stable short exact sequence of PDG F-modules, as the next proposition shows

Proposition 3.1. *With the notation above, F is a stable PDG F-submodule of $\mathrm{C}(r)$.*

Proof. We must check that $[\mathrm{C}(r)] \cap F \subseteq [F]$ since the reverse inclusion is trivial. Suppose $\sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i + \delta_i e] \in [\mathrm{C}(r)] \cap F$ where $r_i \in R$ abd $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ for each $1 \leq i \leq m$. Observe that

$$\begin{aligned} \sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i + \delta_i e] &= \sum_{i=1}^m r_i([\alpha_i, \beta_i, \gamma_i] + ([\alpha_i, \beta_i, \delta_i]e) \\ &= \sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i] + \sum_{i=1}^m r_i[\alpha_i, \beta_i, \delta_i]e \end{aligned}$$

Since $\sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i + \delta_i e] \in F$, it follows that $\sum_{i=1}^m r_i[\alpha_i, \beta_i, \delta_i] = 0$. Thus

$$\sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i + \delta_i e] = \sum_{i=1}^m r_i[\alpha_i, \beta_i, \gamma_i] \in [F].$$

Therefore $[\mathrm{C}(r)] \cap F \subseteq [F]$. □

Since (7) is a stable short exact sequence of PDG F-modules, we obtain a long exact sequence in homology

$$\cdots \longrightarrow \mathrm{H}_i([F]) \xrightarrow{r} \mathrm{H}_i([F]) \longrightarrow \mathrm{H}_i([\mathrm{C}(r)]) \longrightarrow \mathrm{H}_{i-1}([F]) \xrightarrow{r} \mathrm{H}_{i-1}([F]) \longrightarrow \cdots \tag{8}$$

Using Nakayama's lemma, we obtain $\mathrm{index}(\mu_F) = \mathrm{index}(\mu_{\mathrm{C}(r)})$. In particular, this implies

$$\mathrm{index}(R/\langle I, r\rangle) \geq \mathrm{index}(R/I).$$

4 Example

Let $R = \mathbb{F}_2[x, y, z, w]$, let $I = \langle x^2, w^2, zw, xy, y^2z^2\rangle$, and let F be the free minimal resolution of R/I over R. The complex F is supported on the simplicial complex drawn below:

Consider the multiplication on F defined as follows: in degree 1 we have the multiplication table

	e_1	e_2	e_3	e_4	e_5
e_1	0	e_{12}	e_{13}	xe_{14}	$yz^2e_{14} + xe_{45}$
e_2	e_{12}	0	we_{23}	e_{24}	$y^2ze_{23} + we_{35}$
e_3	e_{13}	we_{23}	0	e_{34}	ze_{35}
e_4	xe_{14}	e_{24}	e_{34}	0	ye_{45}
e_5	$yz^2e_{14} + xe_{45}$	$y^2ze_{23} + we_{35}$	ze_{35}	ye_{45}	0

in degree 3 we have the multiplication table

	e_{12}	e_{45}	e_3	e_4	e_5
e_1			e_{13}	xe_{14}	$yz^2e_{14} + xe_{45}$
e_2		$yze_{234} + we_{345}$	we_{23}	e_{24}	$y^2ze_{23} + we_{35}$
e_3			0	e_{34}	ze_{35}
e_4			e_{34}	0	ye_{45}
e_5	$y^2ze_{123} + yzwe_{134} + xwe_{345}$		ze_{35}	ye_{45}	0

5 Grobner Basis Computations