Test 1 Review Solutions

Problem 1

Exercise 1. It is very important to be able to work with interval notation. Recall that for real numbers $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$, we define

$$\begin{aligned}(a,b) &= \{x \in \mathbb{R} \mid a < x < b\}\\ (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\}\\ [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\}\\ [a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\}.\end{aligned}$$

Recall that $\cap$ denotes the operation of taking intersections of sets and $\cup$ denotes operation of taking unions of sets. Thus for example, we have

$$(2,4) \cap (3,7) = \{x \in \mathbb{R} \mid 2 < x < 4 \text{ and } 3 < x < 7\} = (3,4)$$

$$(2,4) \cup (3,7) = \{x \in \mathbb{R} \mid 2 < x < 4 \text{ or } 3 < x < 7\} = (2,7)$$

If possible, try to simplify the following sets. If you cannot simplify them, then just write "already in simplified form".

$$\begin{aligned}(2,3] \cap (3,4) &=\\ (2,3) \cap (3,4) &=\\ (-\infty,3) \cup (-3,2) &=\\ [4,8) \cap (4,8] &=\\ (-3,3] \cup (3,8) &=\\ (-3,3] \cup [3,8) &=\end{aligned}$$

Solution 1. We have

$$\begin{aligned}(2,3] \cap (3,4) &= \emptyset\\ (2,3) \cap (3,4) &= \emptyset\\ (-\infty,3) \cup (-3,2) &= (-\infty,3)\\ [4,8) \cap (4,8] &= (4,8)\\ (-3,3] \cup (3,8) &= (-3,8)\\ (-3,3] \cup [3,8) &= (-3,8)\end{aligned}$$

Problem 2

Exercise 2. Consider the function $f(x) = 1/\sqrt{x-1}$. State the domain and range of $f(x)$.

Solution 2. Let us first find the domain of $f(x)$. There are two issues to consider. First, we need to make sure the denominator is not zero. This is the case if and only if $\sqrt{x-1} \neq 0$ which is the case if and only if $x \neq 1$. The second issue to consider is that we need to make sure that we have a nonnegative number under the radical. This is the case if and only if $x - 1 \geq 0$ which is the case if and only if $x \geq 1$. Thus the domain of $f(x)$ is

$$\text{Domain}(f(x)) = \{x \in \mathbb{R} \mid x \geq 1 \text{ and } x \neq 1\} = (1, \infty).$$

To find the range of $f(x)$, we note that $f(x)$ is always positive since taking square roots is always positive. Also we can obtain every positive number: if you plug in $x = 1.001$ into $f(x)$, you'll get

$$\begin{aligned} f(1.001) &= \frac{1}{\sqrt{1.001 - 1}} \\ &= \frac{1}{\sqrt{0.001}} \\ &= \frac{1}{\text{small number}} \\ &= \text{big number,} \end{aligned}$$

and if you plug in $x = 1000$ into $f(x)$, you'll get

$$\begin{aligned} f(1000) &= \frac{1}{\sqrt{1000 - 1}} \\ &= \frac{1}{\sqrt{999}} \\ &= \frac{1}{\text{big number}} \\ &= \text{small number.} \end{aligned}$$

It should be clear that the range of $f(x)$ is given by

$$\text{Range}(f(x)) = (0, \infty).$$

Note that $0 \notin \text{Range}(f(x))$ because there is no $x \in \text{Domain}(f(x)) = (1, \infty)$ such that $f(x) = 0$.

Problem 3

Exercise 3. Recall the trig identities

$$\begin{aligned} \cos(a + b) &= \cos a \cos b - \sin a \sin b \\ \sin(a + b) &= \sin a \cos b + \cos a \sin b \end{aligned}$$

By setting $a = b$, use the identities above to derive the double angle formulas

$$\begin{aligned} \cos(2a) &= \\ \sin(2a) &= \end{aligned}$$

Note that you should use the formula

$$\cos^2 a + \sin^2 a = 1$$

to simplify what you get in $\cos(2a)$. Now use the double angle formulas to derive the half angle formulas

$$\begin{aligned} \cos(a/2) &= \\ \sin(a/2) &= \end{aligned}$$

Use the half angle formula for $\sin x$ to find the value of $\sin(\pi/12)$. Then use the double angle formula to find the value of $\sin(7\pi/12)$ (hint write $7\pi/12 = \pi/12 + \pi/2$).

Solution 3. You need to derive the double and half angle formulas on your own. In this solution, I will just find the value of $\sin(7\pi/12)$. There are many ways you can do this. Here's the easiest way: we write $7\pi/12 = 3\pi/12 + 4\pi/12$ and we use the addition formula for sin. We have

$$\begin{aligned} \sin(7\pi/12) &= \sin(3\pi/12)\cos(4\pi/12) + \sin(4\pi/12)\cos(3\pi/12) \\ &= \sin(\pi/4)\cos(\pi/3) + \sin(\pi/3)\cos(\pi/4) \\ &= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{3}\sqrt{2}}{4} \\ &= \frac{\sqrt{2} + \sqrt{3}\sqrt{2}}{4} \\ &= \frac{\sqrt{2}(1 + \sqrt{3})}{4}. \end{aligned}$$

Another way to solve this is to use $7\pi/12 = \pi/12 + \pi/2$. You should do this one for practice. Note that you'll need to figure out what $\sin(\pi/12)$ and $\cos(\pi/12)$ are. To do that, you need to use the half angle formulas.

Problem 4

Exercise 4. Given that $\sin\theta = 5/13$ and θ is in the second quadrant $(\pi/2 < \theta < \pi)$, find the exact value of $\tan\theta$.

Solution 4. Recall that $\tan\theta = \sin\theta/\cos\theta$. So to find $\tan\theta$, we first to figure out what $\cos\theta$ is. To do this, we use the identity

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

By rearranging (1), we obtain

$$\begin{aligned}
\cos^2\theta &= 1 - \sin^2\theta \\
&= 1 - \frac{5^2}{13^2} \\
&= \frac{13^2}{13^2} - \frac{5^2}{13^2} \\
&= \frac{13^2 - 5^2}{13^2} \\
&= \frac{12^2}{13^2} \\
&= \left(\frac{12}{13}\right)^2.
\end{aligned}$$

By taking square roots on both sides, we see that $\cos\theta = \pm 12/13$. Since θ is in the second quadrant, we see that $\cos\theta$ must be negative! Thus $\cos\theta = -12/13$. Finally, we can find $\tan\theta$:

$$\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
&= \frac{5/13}{-12/13} \\
&= -\frac{5}{13}\frac{13}{12} \\
&= -\frac{5}{12}.
\end{aligned}$$

Problem 5

Exercise 5. Find all solutions to the system of equations given by

$$\begin{aligned}
x^2 + y^2 &= 1 \\
-3x^2 + y &= -2
\end{aligned}$$

Hint: plug in $y = 3x^2 - 2$ into the first equation, and you should get a quartic polynomial of the form

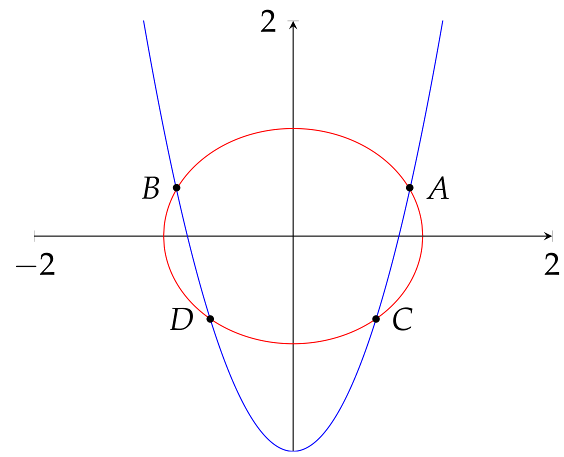
$$ax^4 + bx^2 + c = 0, \tag{2}$$

where you need to figure out what a, b, c. Once you have that, rewrite (2) as

$$au^2 + bu + c = 0$$

where $u = x^2$. Now you can use the *quadratic formula* to determine what u needs to be. Then you can determine what $x = \pm\sqrt{u}$ needs to be.

Solution 5. Let's put this problem into context. Consider the curves below:



Here the red curve is the solution set to $x^2 + y^2 = 1$ and the blue curve is the solution set to $-3x^2 + y = -2$. The points A, B, C, D which lie on the intersection of these two curves are precisely the solution set to $x^2 + y^2 = 1$ *and* $-3x^2 + y = -2$. So solving the system of equations

$$\begin{aligned} x^2 + y^2 &= 1 \\ -3x^2 + y &= -2 \end{aligned}$$

corresponds to determining what the intersection points A, B, C, D are.

Now let's solve these system of equations as follows: First we plug in $y = 3x^2 - 2$ into the first equation. We obtain

$$\begin{aligned} x^2 + (3x^2 - 2)^2 = 1 &\iff x^2 + (3x^2 - 2)^2 - 1 = 0 \\ &\iff x^2 + (3x^2 - 2)^2 - 1 = 0 \\ &\iff x^2 + 9x^4 - 12x^2 + 4 - 1 = 0 \\ &\iff 9x^4 - 11x^2 + 3 = 0. \end{aligned}$$

Now to solve

$$9x^4 - 11x^2 + 3 = 0, \tag{3}$$

we first make the change of variable $u = x^2$, so that (3) becomes the quadratic equation

$$9u^2 - 11u + 3 = 0. \tag{4}$$

The solutions to (4) are given by the quadratic formula:

$$\begin{aligned} u &= \frac{11 \pm \sqrt{121 - 108}}{18} \\ &= \frac{11 \pm \sqrt{13}}{18}. \end{aligned}$$

Finally since $u = x^2$, we see that the solutions to (3) are

$$x = \pm\sqrt{\frac{11 \pm \sqrt{13}}{18}}.$$

In other words, the are four solutions, given by

$$\begin{aligned} x &= \sqrt{\frac{11 + \sqrt{13}}{18}} \\ x &= -\sqrt{\frac{11 + \sqrt{13}}{18}} \\ x &= \sqrt{\frac{11 - \sqrt{13}}{18}} \\ x &= -\sqrt{\frac{11 - \sqrt{13}}{18}}. \end{aligned}$$

These four solutions give us the x-coordinates of A, B, C, and D. We still need to find the y-coordinates of A, B, C, and D. This is easy though, since we know that $y = 3x^2 - 2$. Thus the y-coordinates are given by

$$\begin{aligned} y &= 3\left(\pm\sqrt{\frac{11 \pm \sqrt{13}}{18}}\right)^2 - 2 \\ &= 3\left(\frac{11 \pm \sqrt{13}}{18}\right) - 2 \\ &= \frac{11 \pm \sqrt{13}}{6} - 2 \\ &= \frac{11 \pm \sqrt{13}}{6} - \frac{12}{6} \\ &= \frac{23 \pm \sqrt{13}}{6} \end{aligned}$$

In other words, the y-coordinates are

$$\begin{aligned} y &= \frac{23 + \sqrt{13}}{6} \\ y &= \frac{23 - \sqrt{13}}{6}. \end{aligned}$$

Putting everything together, we see that

$$\begin{aligned} A &= \left(\sqrt{\frac{11 + \sqrt{13}}{18}}, \frac{23 + \sqrt{13}}{6}\right) \\ B &= \left(-\sqrt{\frac{11 + \sqrt{13}}{18}}, \frac{23 + \sqrt{13}}{6}\right) \\ C &= \left(\sqrt{\frac{11 - \sqrt{13}}{18}}, \frac{23 - \sqrt{13}}{6}\right) \\ D &= \left(-\sqrt{\frac{11 - \sqrt{13}}{18}}, \frac{23 - \sqrt{13}}{6}\right) \end{aligned}$$

Problem 6

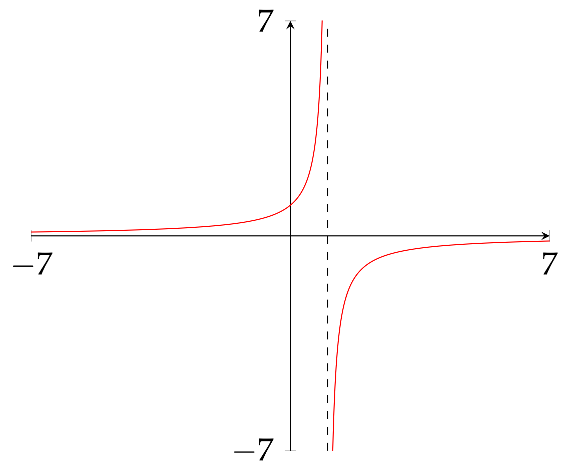
Exercise 6. Factor the polynomial $x^4 + 2x^2 + 1$ as much as you can.

Solution 6. We have

$$x^4 + 2x^2 + 1 = (x^2 + 1)^2.$$

Problem 7

Exercise 7. Which of the following is a correct equation for the graph below?



a) $y = 1/x$

b) $y = 1/(1-x)$

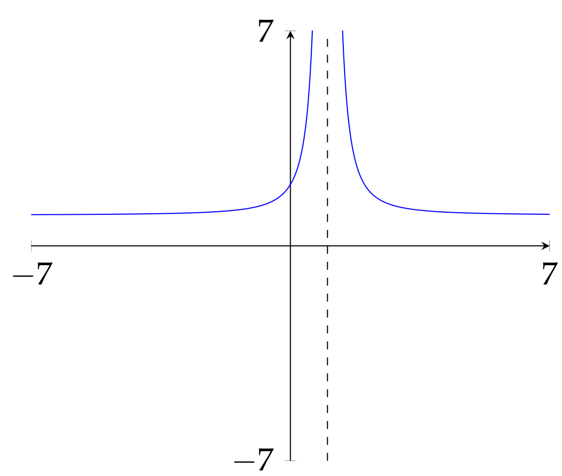
c) $y = 1/x^2$

d) $y = 1/(1-x)^2$

Solution 7. The answer is b). To see this, note that the function whose graph is given above does not seem to be defined at $x = 1$. If I plug in $x = 1$ into the function $1/x$, then I'll get 1, so $1/x$ is defined at $x = 1$ which means a) can't be the solution. Similarly, $1/x^2$ is defined at $x = 1$, so c) can't be the solution either. The functions $1/(1-x)$ and $1/(1-x)^2$ are both not defined at $x = 1$, so they are both candidates. However note that $1/(1-x)^2$ is *always* nonnegative, whereas the function whose graph is given above clearly takes negative values. Thus d) can't be the solution either. This leads us to b) being the solution.

Problem 8

Exercise 8. Which of the following is a correct equation for the graph below?



a) $y = x^{-1} + 1$

b) $y = x^{-2} + 1$

c) $y = (2-x)^{-1}$

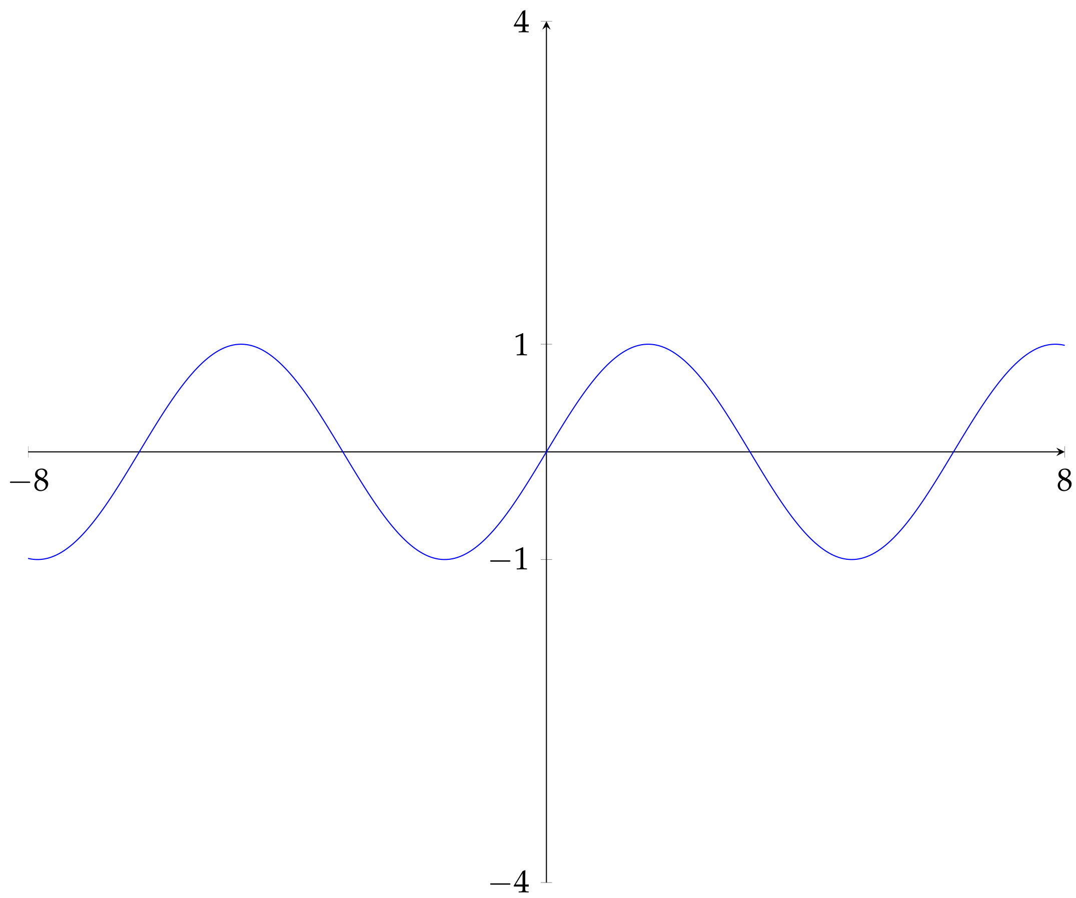
d) $y = (1-x)^{-1} + 1$

e) $y = (1-x)^{-2} + 1$

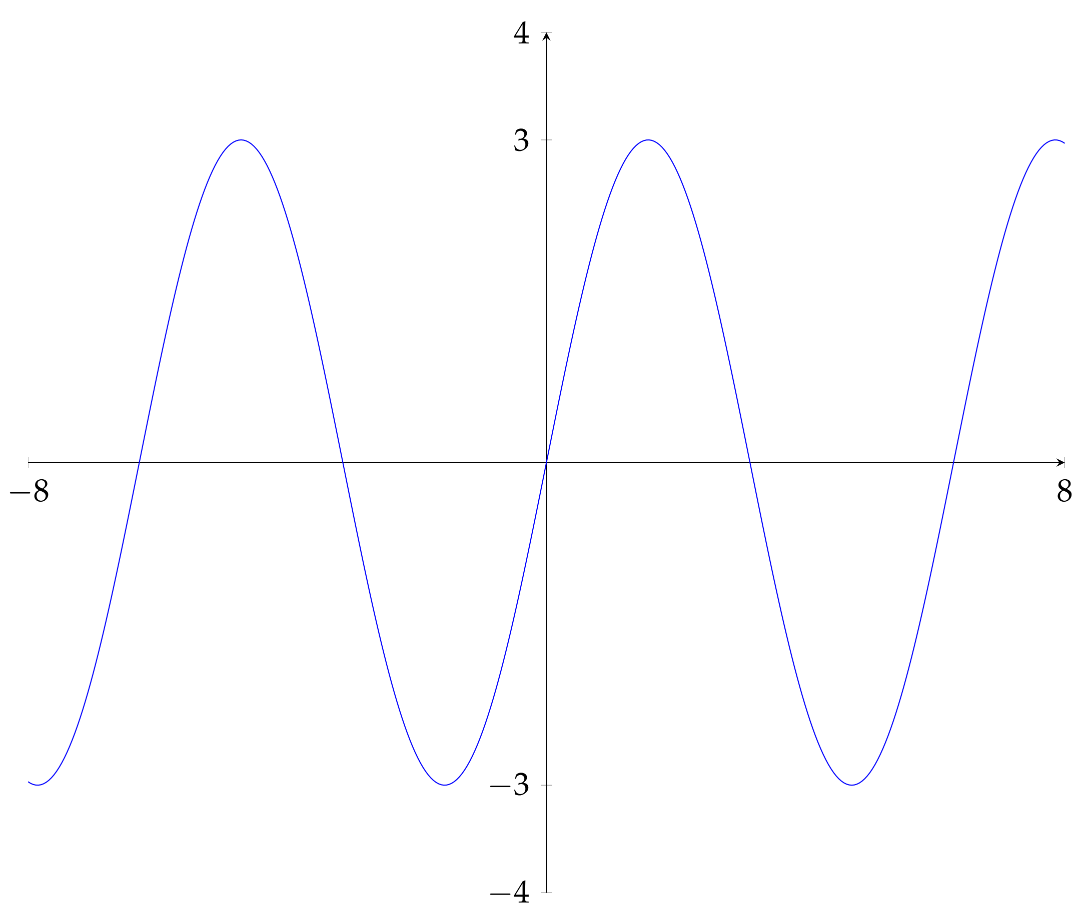
Solution 8. The answer is e). The reason is similar to problem 7.

Problem 8

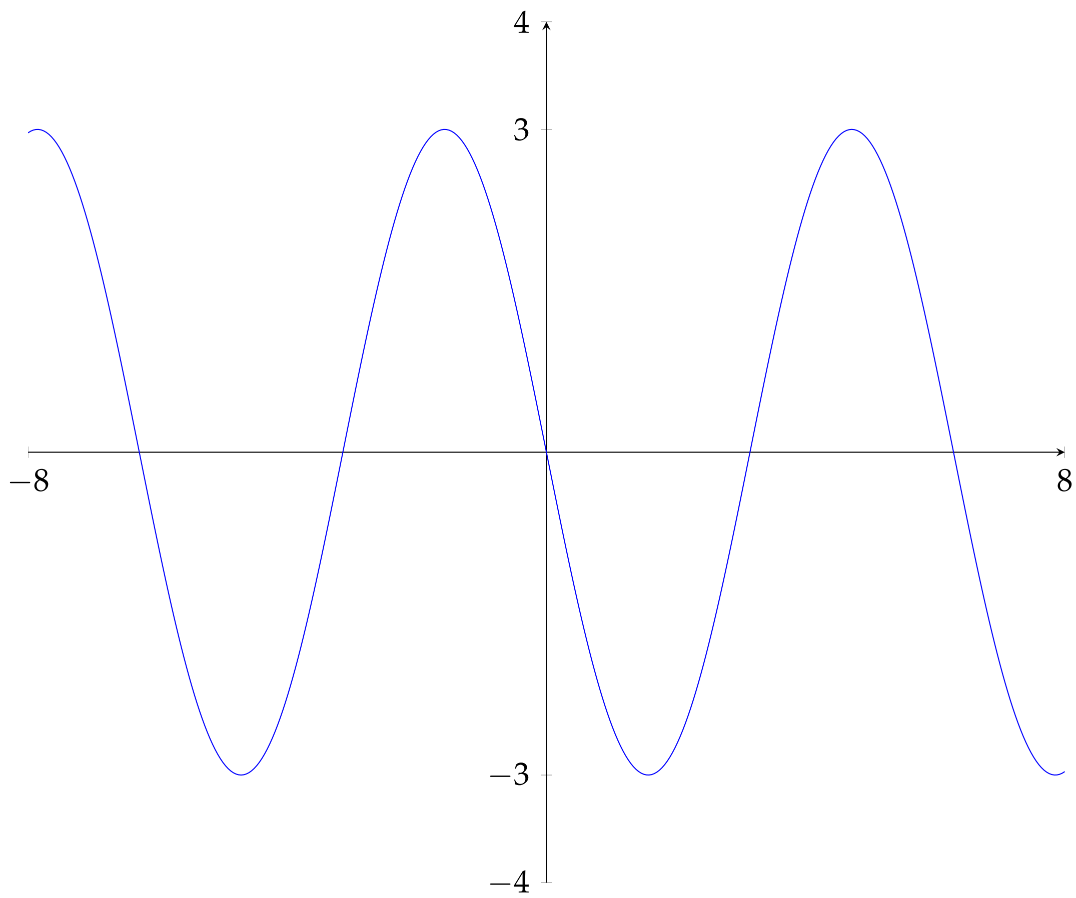
Solution 9. You should definitinely know what the graphs of $\sin x$, $\cos x$, and $\tan x$ look like. You should also know what $\csc x$, $\sec x$, and $\cot x$ look like as well. For this problem, let's focus on $\sin x$. First graph $\sin x$ below



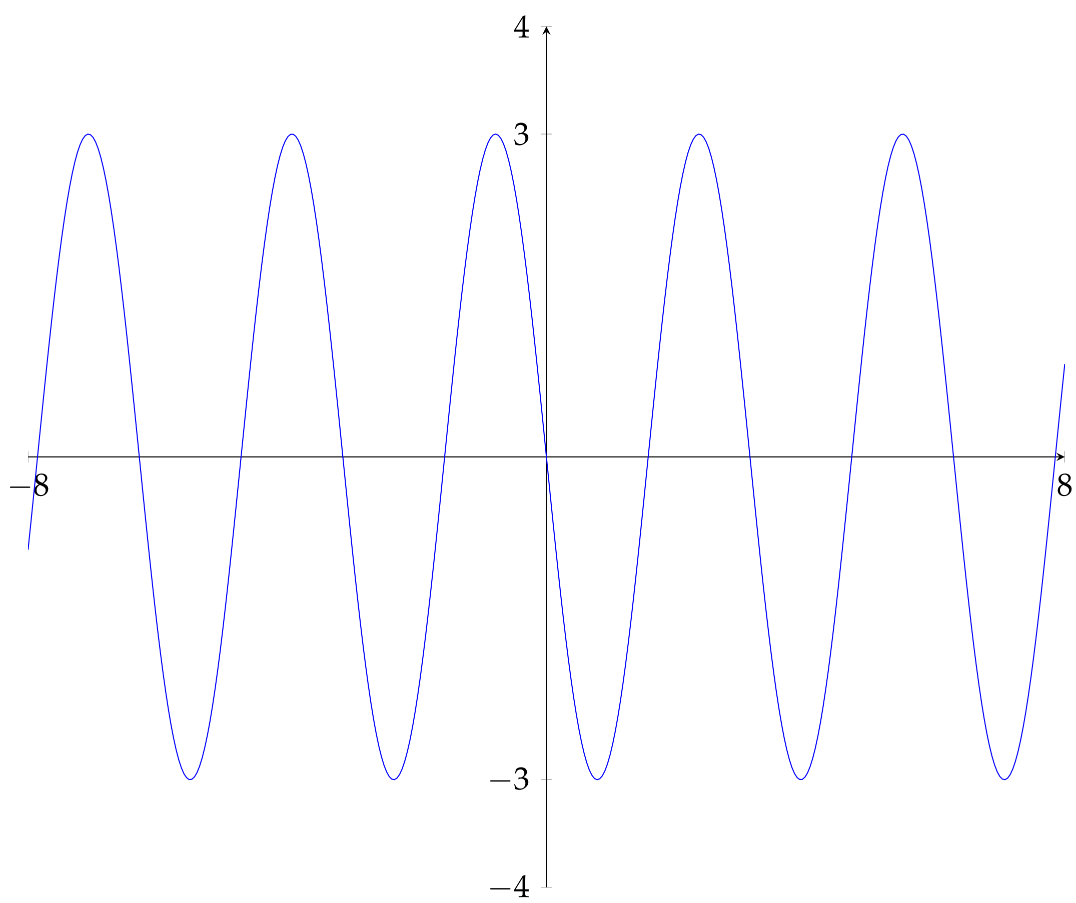
Next graph $3\sin x$ below



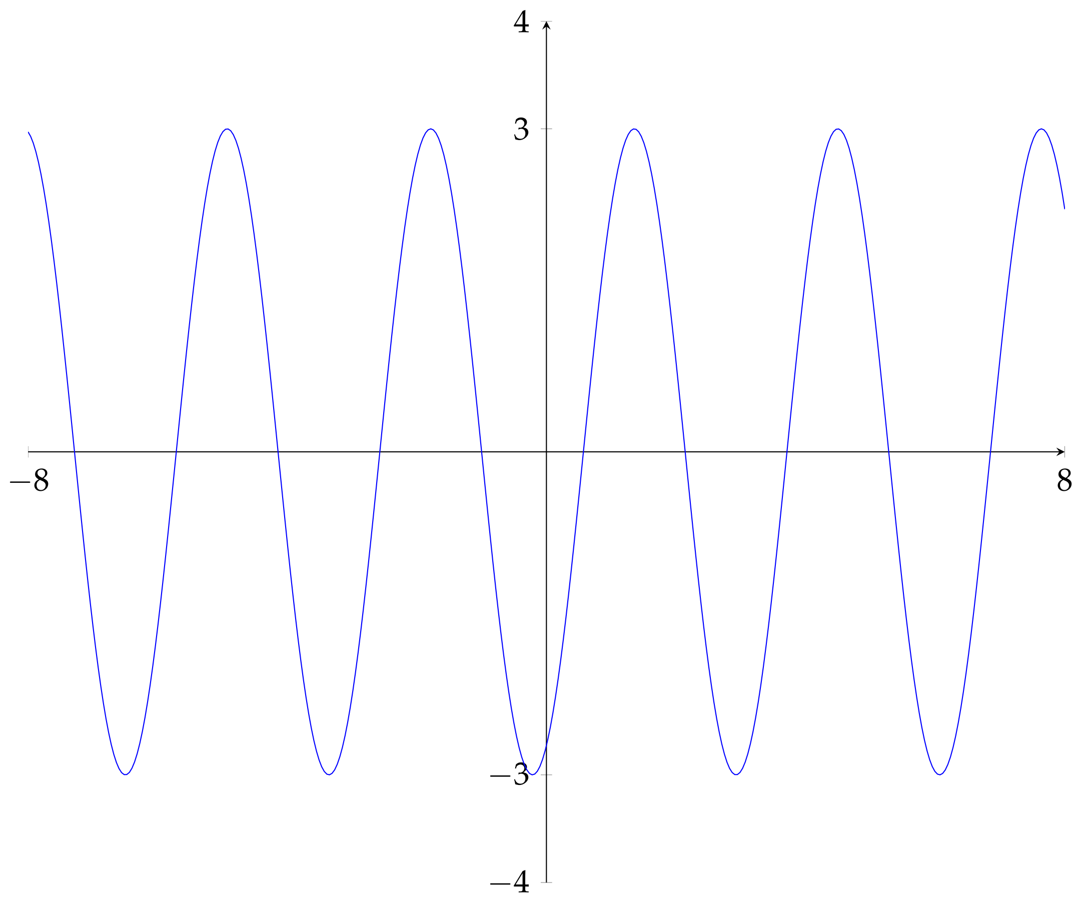
Next graph $-3\sin x$ below



Next graph $-3\sin(2x)$ below



Finally, graph $-3\sin(2(x+1))$ below



What is the amplitude, period, and phase shift of $-3\sin(2(x+1))$. **Solution:** The amplitude is 3, the period is $2\pi/2 = \pi$, and the phase shift is 1. Note that I don't think you need to know what "phase shift" is for the test. However you definitely need to know what the amplitude and period are.

Problem 9

Exercise 9. Express the function $f(x) = |x-2|$ as a piecewise function

Solution 10. We have

$$|x-2| = \begin{cases} x-2 & \text{if } x \geq 2 \\ -(x-2) & \text{if } x < 2 \end{cases}$$

You should check that this is indeed the piecewise description of $|x-2|$ by plugging in various values for x and seeing that they work.

Problem 10

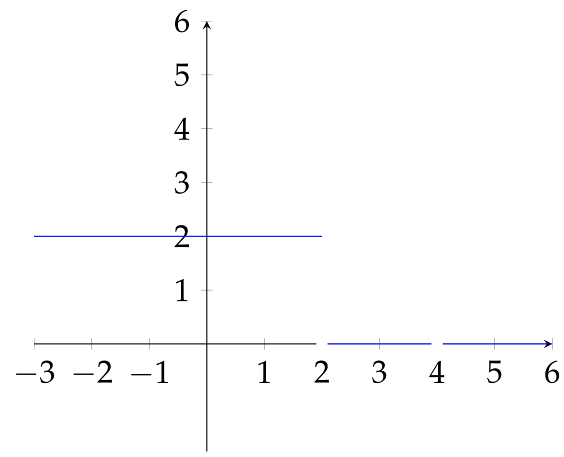
Exercise 10. Convert 310° to radians (just multiply 310 by $\pi/180$). Convert $2\pi/7$ radians to degrees (plug in $\pi = 180$).

Problem 11

Exercise 11. Simplify $\sqrt{x^4 + 2x^2}$.

Problem 12

Exercise 12. Let $f(x)$ be the function whose graph is given below.



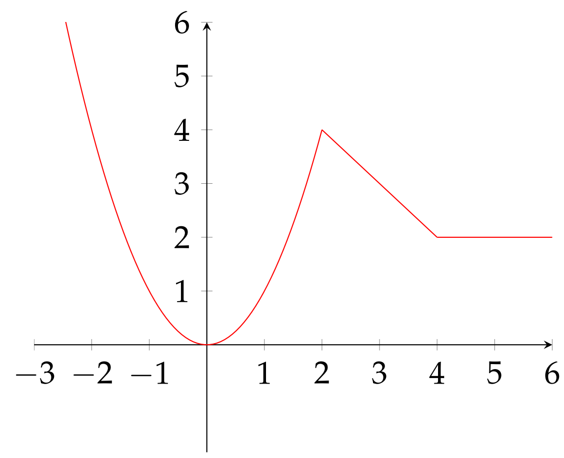
Express the function $f(x)$ as a peicewise function. Note that $f(x)$ is not defined at $x = 2$ nor $x = 4$!

Solution 11. We have

$$f(x) = \begin{cases} 2 & \text{if } x \in (-\infty, 2) \\ 0 & \text{if } x \in (2, 4) \\ 0 & \text{if } x \in (4, \infty) \end{cases}$$

Problem 13

Exercise 13. Let $f(x)$ be the function whose graph is given below.



Express the function $f(x)$ as a peicewise function

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 2) \\ & \text{if } x \in \\ & \text{if } x \in \end{cases}$$

Solution 12. This function consists of three parts: a quadratic part, a linear part, and a constant. We are told that the quadratic part is given by $f(x) = x^2$ whenever $x \in (-\infty, 2]$. Also clearly the constant part is given by $f(x) = 2$ whenever $x \in [4, \infty)$. We need to figure out what the linear part is. We first calculate the slope of that line:

$$\text{slope} = \frac{2-4}{4-2} = -2.$$

Next, note that the line segment passes through the point $(2, 4)$. Thus the equation of this line segment is given by

$$y - 4 = -2(x - 2).$$

In other words, $f(x) = -2(x-2) + 4$ whenever $x \in [2, 4]$. Thus we have our piecewise description of our function:

$$f(x) = \begin{cases} x^2 & \text{if } x \in (-\infty, 2) \\ -2(x-2)+4 & \text{if } x \in [2, 4] \\ 2 & \text{if } x \in [2, \infty) \end{cases}$$

Problem 14

Exercise 14. Suppose $f(x) = x^2 + 2x + y^2 - 3y + 5$. Recall that the graph of $f(x) = 0$ forms a circle. What is the center and radius of this circle? Hint: write $f(x) = (x-a)^2 + (y-b)^2 - r^2$ where $a, b \in \mathbb{R}$ and $r > 0$ are to be determined.

Problem 15

Exercise 15. Find the equation that is perpendicular to the line $8x + 3y = 2$ which passes through the point $(1, 1)$.

Solution 13. In general, if $y = ax + b$ is a line whose slope is a, then a line which is perpendicular to $y = ax + b$ has slope given by $-1/a$ (remember this!). We'll use this fact to solve this problem. First we rewrite $8x + 3y = 2$ into the standard form:

$$\begin{aligned} 8x + 3y = 2 &\iff 3y = -8x + 2 \\ &\iff y = -(8/3)x + 2/3 \end{aligned}$$

So the slope of the line which is perpendicular to $y = -(8/3)x + 2/3$ is given by $3/8$! Finally, we need this perpendicular line to pass through the point $(1, 1)$. Thus the equation of the line which is perpendicular to $y = -(8/3)x + 2/3$ is given by

$$y - 1 = (3/8)(x - 1).$$

Problem 16

Exercise 16. Simplify the following expressions

$$\frac{x^2 - x}{\sqrt{x^4 - x^2}} =$$

$$\frac{1}{1-x} + \frac{1}{1+x} =$$

$$\frac{3x^2 + 2}{xyz} - \frac{2x - 1}{x^2 y} =$$