When a Graded Map is a Chain Map

Let (A, d) and (B, ∂) be R-complexes and let $\psi\colon \mathrm{H}(A) \to \mathrm{H}(B)$ be a graded R-linear map. Suppose that we could lift ψ to a graded R-linear map $\widetilde{\psi}\colon A \to B$ of the underlying graded R-modules. So $\widetilde{\psi}$ takes $\ker \mathrm{d}$ to $\ker \partial$ and it takes $\operatorname{im} \mathrm{d}$ to $\operatorname{im} \partial$ and $\mathrm{H}(\widetilde{\psi}) = \psi$. It's easy to see that $\widetilde{\psi}$ is a chain map if and only if $\operatorname{im}(\partial\widetilde{\psi} - \widetilde{\psi}\mathrm{d}) = 0$. If $\widetilde{\psi}$ is not a chain map, then can we adjust our R-complexes in a way so that it *induces* a chain map? It turns out that the answer is yes, and knowing that $\widetilde{\psi}$ induces $\psi\colon \mathrm{H}(A) \to \mathrm{H}(B)$ gives us a little more information about this induced chain map.

Proposition 0.1. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\overline{B} = B/\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ and let $\pi\colon B \to \overline{B}$ be the quotient map. Define $\overline{\partial}\colon \overline{B} \to \overline{B}$ by*

$$\overline{\partial}(\overline{b}) = \overline{\partial(b)}$$

for all $a \in A$ and $\overline{b} \in \overline{B}$. Then $(\overline{B}, \overline{\partial})$ is an R-complex and $\pi\varphi\colon A \to \overline{B}$ is a chain map. Moreover, if φ takes $\operatorname{im} \mathrm{d}$ to $\operatorname{im} \partial$, then we have the following short exact sequence of graded R-modules and graded R-linear maps:

$$0 \longrightarrow \mathrm{H}(B) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}(\overline{B}) \xrightarrow{\gamma} \operatorname{im}(\partial\varphi - \varphi\mathrm{d})(-1) \longrightarrow 0 \tag{1}$$

where γ is the connecting map coming from a long exact sequence in homology.

Proof. Observe that $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ is a graded R-submodule of B since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on B induces a grading on $\overline{B}$ which makes π into a graded R-linear map. Therefore $\pi\varphi$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that $\overline{\partial}$ is well-defined, that is, we need to check that ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself. Let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Then

$$\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= (\partial\partial\varphi - \partial\varphi\mathrm{d})(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= (-\partial\varphi\mathrm{d}(a) + \varphi\mathrm{d}\mathrm{d})(a) \\
&= (-\partial\varphi + \varphi\mathrm{d})(\mathrm{d}(a)) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d}).
\end{aligned}$$

Thus $\overline{\partial}$ is well-defined. Also $\overline{\partial}$ is an R-linear differential since it inherits these properties from ∂. Therefore $(\overline{B}, \overline{\partial})$ is an R-complex.

Now let us check that $\pi\varphi$ is a chain map. To see this, we just need to show it commutes with the differentials. Let $a \in A$. Then we have

$$\begin{aligned}
\overline{\partial}\pi\varphi(a) &= \overline{\partial}(\overline{\varphi(a)}) \\
&= \overline{\partial\varphi(a)} \\
&= \overline{\partial\varphi(a) - (\partial\varphi - \varphi\mathrm{d})(a)} \\
&= \overline{\partial\varphi(a) - \partial\varphi(a) + \varphi\mathrm{d}(a)} \\
&= \overline{\varphi\mathrm{d}(a)} \\
&= \pi\varphi\mathrm{d}(a).
\end{aligned}$$

Thus $\pi\varphi$ is a chain map.

Since ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself, it restricts to a differential on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. So we have a short exact sequence of R-complexes

$$0 \longrightarrow \operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \xrightarrow{\iota} B \xrightarrow{\pi} \overline{B} \longrightarrow 0 \tag{2}$$

where ι is the inclusion map. The short exact sequence (9) induces the following long exact sequence in homology

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_{i+1}(\overline{B}) \arrow[dll, rounded corners, "\gamma_{i+1}"'] \\
 & \mathrm{H}_i(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(B) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_i(\overline{B}) \arrow[dll, rounded corners, "\gamma_i"'] \\
 & \mathrm{H}_{i-1}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(B) \arrow[r] & \cdots
\end{tikzcd}
\tag{3}
$$

Let us work out the details of the connecting map γ. Let $[\overline{b}] \in \mathrm{H}_i(\overline{B})$, so $\overline{b} \in \overline{B}_i$ is the coset with $b \in B_i$ as a reprsentative and $[\overline{b}] \in \mathrm{H}_i(\overline{B})$ is the coset with $\overline{b} \in \overline{B}_i$ as a representative. In particular, $\overline{\partial}(\overline{b}) = \overline{0}$, which implies

$$\partial(b) = (\partial\varphi - \varphi\mathrm{d})(a) \tag{4}$$

for some $a \in A$. Then (4) implies that $(\partial\varphi - \varphi\mathrm{d})(a)$ is the unique element in $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ which maps to $\partial(b)$ (under the inclusion map). Therefore

$$\gamma_i[\overline{b}] = [(\partial\varphi - \varphi\mathrm{d})(a)].$$

Now suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. We claim that ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$, there exists a $b \in B$ such that

$$\varphi\mathrm{d}(a) = \partial(b).$$

Choose such a $b \in B$. Then observe that

$$
\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= -\partial\partial(b) \\
&= 0.
\end{aligned}
$$

Thus ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. In particular, $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \cong \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$.

Next we claim that $\mathrm{H}(\iota)$ is the zero map. Indeed, for any $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$, we choose $b \in B$ such that $\varphi\mathrm{d}(a) = \partial(b)$, then we have

$$
\begin{aligned}
(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\varphi(a) - \varphi\mathrm{d}(a) \\
&= \partial\varphi(a) - \partial b \\
&= \partial(\varphi(a) - b) \\
&\in \mathrm{im}\,\partial.
\end{aligned}
$$

Therefore $\mathrm{H}(\iota)$ takes the coset in $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}))$ represented by $(\partial\varphi - \varphi\mathrm{d})(a)$ to the coset in $\mathrm{H}(B)$ represented by 0. Thus $\mathrm{H}(\iota)$ is the zero map as claimed.

Combining everything together, we see that the long exact sequence (3) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(B) \xrightarrow{\mathrm{H}_i(\pi)} \mathrm{H}_i(\overline{B}) \xrightarrow{\gamma_i} \mathrm{im}(\partial_{i-1}\varphi_{i-1} - \varphi_{i-2}\mathrm{d}_{i-1}) \longrightarrow 0 \tag{5}$$

for all $i \in \mathbb{Z}$. In other words, (6) is a short exact sequence of graded R-modules. □

Corollary. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. Then φ is a chain map if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$.*

Corollary. *Indeed, φ is a chain map if and only if $\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) = 0$ if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$ by (6).*

Corollary. *Let (P, d) and (B, ∂) be R-complexes and let $\varphi\colon P \to B$ be a graded R-linear map of the underlying graded modules. Suppose P is a semiprojective R-complex and suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. Then φ is a chain map if and only if $\mathrm{H}(\overline{B}) \cong \mathrm{H}(B)$.*

Proof. Indeed, φ is a chain map if and only if $\pi\colon B \to \overline{B}$ is a quasiisomorphism. Since π is surjective and P is semiprojective, there exists a chain map $\phi\colon P \to B$ such that $\pi\phi = \pi\varphi$. □

There is a dual version to Proposition (0.2). Let us state it now.

Proposition 0.2. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\widetilde{A} = \ker(\partial\varphi - \varphi\mathrm{d})$ and let $\iota\colon \widetilde{A} \to A$ be the inclusion map. Then d restricts to a differential $\mathrm{d}\colon \widetilde{A} \to \widetilde{A}$. Furthermore, $(\widetilde{A}, \mathrm{d})$ is an R-complex and $\varphi\iota\colon \widetilde{A} \to B$ is a chain map.*
Then the differential d restricts to a differential $\mathrm{d}\colon A_0 \to A_0$. Furthermore, $(A_0, \mathrm{d})$is an R-complex and $\pi\varphi\colon A \to \overline{B}$ is a chain map. Moreover, if φ takes $\operatorname{im}\mathrm{d}$ to $\operatorname{im}\partial$, then we have the following short exact sequence of graded R-modules and graded R-linear maps:

$$0 \longrightarrow \mathrm{H}(B) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}(\overline{B}) \xrightarrow{\gamma} \operatorname{im}(\partial\varphi - \varphi\mathrm{d})(-1) \longrightarrow 0 \tag{6}$$

where γ is the connecting map coming from a long exact sequence in homology.

Proof. Observe that $\widetilde{A}$ is a graded R-submodule of A since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on A induces a grading on $\widetilde{A}$ which makes ι into a graded R-linear map. Therefore $\varphi\iota$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that d restricted to $\widetilde{A}$ lands in $\widetilde{A}$. Suppose $a \in \widetilde{A}$. Thus $a \in A$ and $\partial\varphi(a) = \varphi\mathrm{d}(a)$. Then

$$\begin{aligned}
(\partial\varphi - \varphi\mathrm{d})\mathrm{d}(a) &= \partial\varphi\mathrm{d}(a) - \varphi\mathrm{d}\mathrm{d}(a) \\
&= \partial\varphi\mathrm{d}(a) \\
&= \partial\partial\varphi(a) \\
&= 0.
\end{aligned}$$

This implies $\mathrm{d}(a) \in \widetilde{A}$. Thus d restricted to $\widetilde{A}$ lands in $\widetilde{A}$. Clearly d is an R-linear differential. Therefore $(\widetilde{A}, \mathrm{d})$ is an R-complex.

Now let us check that $\varphi\iota$ is a chain map. To see this, we just need to show it commutes with the differentials. Let $a \in \widetilde{A}$. Thus $a \in A$ and $\partial\varphi(a) = \varphi\mathrm{d}(a)$. Then we have

$$\begin{aligned}
\partial\varphi\iota(a) &= \partial\varphi(a) \\
&= \varphi\mathrm{d}(a)
\end{aligned}$$

Thus $\varphi\iota$ is a chain map.

We have a short exact sequence of R-complexes

$$0 \longrightarrow \widetilde{A} \xrightarrow{\iota} A \xrightarrow{\partial\varphi - \varphi\mathrm{d}} \Sigma\operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \longrightarrow 0 \tag{7}$$

where ι is the inclusion map. The short exact sequence (9) induces the following long exact sequence in homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}_i(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \arrow[dll, "\lambda_i"] \\
\mathrm{H}_i(\widetilde{A}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(A) \arrow[r, "\mathrm{H}_i(\partial\varphi - \varphi\mathrm{d})"] & \mathrm{H}_{i-1}(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \arrow[dll, "\lambda_{i-1}"] \\
\mathrm{H}_{i-1}(\widetilde{A}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(A) \arrow[r] & \cdots
\end{tikzcd} \tag{8}$$

Now suppose φ takes $\ker\mathrm{d}$ to $\ker\partial$. We claim that ∂ restricts to the zero map on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\operatorname{im}\mathrm{d}$ to $\operatorname{im}\partial$, there exists a $b \in B$ such that

$$\varphi\mathrm{d}(a) = \partial(b).$$

Choose such a $b \in B$. Then observe that

$$\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= -\partial\partial(b) \\
&= 0.
\end{aligned}$$

Thus ∂ restricts to the zero map on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. In particular, $\mathrm{H}(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \cong \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$.

Next we claim that $\mathrm{H}(\partial\varphi - \varphi\mathrm{d})$ is the zero map. Indeed, let $[a] \in \mathrm{H}(A)$, so $a \in A$ and $\mathrm{d}(a) = 0$. Since φ takes $\ker \mathrm{d}$ to $\ker \partial$, we see that $\partial\varphi(a) = 0$. Therefore

$$\begin{aligned} \mathrm{H}(\partial\varphi - \varphi\mathrm{d})[a] &= [(\partial\varphi - \varphi\mathrm{d})(a)] \\ &= [\partial\varphi(a) - \varphi\mathrm{d}(a)] \\ &= [0]. \end{aligned}$$

Thus $\mathrm{H}(\partial\varphi - \varphi\mathrm{d})$ is the zero map as claimed.

Combining everything together, we see that the long exact sequence (8) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(\operatorname{im}(\partial\varphi - \varphi\mathrm{d})) \xrightarrow{\lambda_i} \mathrm{H}_i(\widetilde{A}) \xrightarrow{\iota_i} \mathrm{H}(A) \longrightarrow 0 \tag{9}$$

for all $i \in \mathbb{Z}$. In other words, (6) is a short exact sequence of graded R-modules. $\square$

Applications

Example 0.1. Let $\underline{x} = x_1, \dots, x_n$ be a sequence of elements in R, let $\mathcal{K}(\underline{x})$ be the Koszul complex with respect to that sequence, and let π be a permutation of $[n]$. For any subset $\sigma \subseteq [n]$, we write $\sigma = \{\lambda_1, \dots, \lambda_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and we define $\pi \cdot \sigma$ to be the subset in $[n]$ defined by

$$\pi \cdot \sigma = \{\pi(\lambda_1), \dots, \pi(\lambda_k)\}.$$

We also define $\operatorname{sign}(\pi|_\sigma)$ to be the sign of the permutation which puts $(\pi(\lambda_1), \dots, \pi(\lambda_k))$ into the correct order. Then π induces a graded R-linear map $\pi\colon \mathcal{K}(\underline{x}) \to \mathcal{K}(\underline{x})$, uniquely determined by

$$\pi(e_\sigma) = (-1)^{\operatorname{sign}(\pi|_\sigma)} e_{\pi\cdot\sigma} \tag{10}$$

for all $\sigma \subseteq [n]$. If $\underline{x} = \underline{1}$, then (10) is a chain map. Indeed, we have

$$\begin{aligned} \mathrm{d}_{\mathcal{K}(\underline{x})}\pi(e_\sigma) &= (-1)^{\operatorname{sign}(\pi|_\sigma)} \mathrm{d}_{\mathcal{K}(\underline{x})}(e_{\pi\cdot\sigma}) \\ &= \sum_{\pi(\lambda)\in\pi\cdot\sigma} (-1)^{\operatorname{sign}(\pi|_\sigma)} \langle \pi(\lambda), \sigma \backslash \pi(\lambda)\rangle e_{(\pi\cdot\sigma)\backslash\pi(\lambda)} \\ &= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle (-1)^{\operatorname{sign}(\pi|_{\sigma\backslash\lambda})} e_{\pi\cdot(\sigma\backslash\lambda)} \\ &= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \pi e_{\sigma\backslash\lambda} \\ &= \pi \mathrm{d}_{\mathcal{K}(\underline{x})}(e_\sigma) \end{aligned}$$

for all $\sigma \subseteq [n]$.

Example 0.2. Let $R = K[x, y, z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi = (12)$. We first calculate

$$\begin{aligned} (\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) - \pi(f_1) \\ &= f_2 - f_1. \end{aligned}$$

Next we calculate

$$\begin{aligned} (\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_2) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_2) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) - \pi(f_2) \\ &= f_1 - f_2. \end{aligned}$$

Next we calculate

$$\begin{aligned} (\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_3) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_3) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) - \pi(f_3) \\ &= f_3 - f_3 \\ &= 0. \end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{12}) - \pi(f_1e_2 - f_2e_1) \\
&= -f_1e_2 + f_2e_1 - f_1e_1 + f_2e_2 \\
&= (f_2 - f_1)(e_1 + e_2).
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1e_3 - f_3e_1) \\
&= f_2e_3 - f_3e_2 - f_1e_3 + f_3e_2 \\
&= (f_2 - f_1)e_3.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) - \pi(f_2e_3 - f_3e_2) \\
&= f_1e_3 - f_3e_1 - f_2e_3 + f_3e_1 \\
&= (f_1 - f_2)e_3.
\end{aligned}$$

Finally we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123}) - \pi(f_1e_{23} - f_2e_{13} + f_3e_{12}) \\
&= -f_1e_{23} + f_2e_{13} - f_3e_{12} - f_1e_{13} + f_2e_{23} + f_3e_{12} \\
&= (f_2 - f_1)(e_{23} + e_{13}).
\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}) = \langle f_2 - f_1 \rangle$$

Example 0.3. Let $R = K[x, y, z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi = (12)$. We first calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) - \pi(f_1) \\
&= f_2 - f_1.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_2) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_2) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_2) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) - \pi(f_2) \\
&= f_1 - f_2.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_3) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_3) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_3) - \pi(f_3) \\
&= f_3 - f_3 \\
&= 0.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{12}) - \pi(f_1e_2 - f_2e_1) \\
&= -f_1e_2 + f_2e_1 - f_1e_1 + f_2e_2 \\
&= (f_2 - f_1)(e_1 + e_2).
\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1e_3 - f_3e_1) \\ &= f_2e_3 - f_3e_2 - f_1e_3 + f_3e_2 \\ &= (f_2 - f_1)e_3.\end{aligned}$$

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) - \pi(f_2e_3 - f_3e_2) \\ &= f_1e_3 - f_3e_1 - f_2e_3 + f_3e_1 \\ &= (f_1 - f_2)e_3.\end{aligned}$$

Finally we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123}) - \pi(f_1e_{23} - f_2e_{13} + f_3e_{12}) \\ &= -f_1e_{23} + f_2e_{13} - f_3e_{12} - f_1e_{13} + f_2e_{23} + f_3e_{12} \\ &= (f_2 - f_1)(e_{23} + e_{13}).\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}) = \langle f_2 - f_1\rangle$$

Example 0.4. Let $R = K[x,y,z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi\colon \mathcal{K}(\underline{f}) \to \mathcal{K}(\underline{f})$ be a graded R-linear map. For each $1 \leq i < j \leq 3$, we have

$$\begin{aligned}\pi(1) &= f_0^0 \\ \pi(e_i) &= f_i^1e_1 + f_i^2e_2 + f_i^3e_3 \\ \pi(e_{ij}) &= f_{ij}^{12}e_{12} + f_{ij}^{13}e_{13} + f_{ij}^{23}e_{23} \\ \pi(e_{ijk}) &= f_{123}^{123}e_{123}\end{aligned}$$

where the f_i^k's and f_{ij}^{kl}'s are in R. Then we have

$$\begin{aligned}&(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(1) = f_0^0 \\ &(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_i) = f_i^1f_1 + f_i^2f_2 + f_i^3f_3 - f_0^0f_i \\ &(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{ij}) = (f_jf_i^1 - f_if_j^1 - f_{ij}^{13}f_3 - f_{ij}^{12}f_2)e_1 + (f_jf_i^2 - f_if_j^2 - f_{ij}^{23}f_3 + f_{ij}^{12}f_1)e_2 + (f_jf_i^3 - f_if_j^3 + f_{ij}^{23}f_j + f_{ij}^{13})e_3\end{aligned}$$

One can calculate more generally that

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) = f_1^1f_1 + f_1^2f_2 + f_1^3f_3 - f_1f_0$$

We first calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_1^1e_1 + f_1^2e_2 + f_1^3e_3) - \pi(f_1) \\ &= f_1^1f_1 + f_1^2f_2 + f_1^3f_3 - f_1f_0\end{aligned}$$

More generally we have

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_i) = f_i^1f_1 + f_i^2f_2 + f_i^3f_3 - f_if_0$$

for $i = 1, 2, 3$.

Next we calculate

$$\begin{aligned}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\ &= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_{12}^{12}e_{12} + f_{12}^{13}e_{13} + f_{12}^{23}e_{23}) - \pi(f_1e_2 - f_2e_1) \\ &= f_{12}^{12}(f_1e_2 - f_2e_1) + f_{12}^{13}(f_1e_3 - f_3e_1) + f_{12}^{23}(f_2e_3 - f_3e_2) - f_1(f_2^1e_1 + f_2^2e_2 + f_2^3e_3) + f_2(f_1^1e_1 + f_1^2e_2 + f_1^3e_3) \\ &= (f_2f_1^1 - f_1f_2^1 - f_{12}^{13}f_3 - f_{12}^{12}f_2)e_1 + (f_2f_1^2 - f_1f_2^2 - f_{12}^{23}f_3 + f_{12}^{12}f_1)e_2 + (f_2f_1^3 - f_1f_2^3 + f_{12}^{23}f_2 + f_{12}^{13})e_3\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1 e_3 - f_3 e_1) \\
&= f_2 e_3 - f_3 e_2 - f_1 e_3 + f_3 e_2 \\
&= (f_2 - f_1) e_3.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) - \pi(f_2 e_3 - f_3 e_2) \\
&= f_1 e_3 - f_3 e_1 - f_2 e_3 + f_3 e_1 \\
&= (f_1 - f_2) e_3.
\end{aligned}$$

Finally we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123}) - \pi(f_1 e_{23} - f_2 e_{13} + f_3 e_{12}) \\
&= -f_1 e_{23} + f_2 e_{13} - f_3 e_{12} - f_1 e_{13} + f_2 e_{23} + f_3 e_{12} \\
&= (f_2 - f_1)(e_{23} + e_{13}).
\end{aligned}$$

Thus, we have.

$$\operatorname{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}) = \langle f_2 - f_1 \rangle$$

Example 0.5. Let $R = K[x, y, z]$, let $\underline{f} = f_1, f_2, f_3$ be a sequence of elements in R, and let $\pi = (12)$. We first calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_1) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_1) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_1) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_1^1 e_1 + f_1^2 e_2 + f_1^3 e_3) - \pi(f_1) \\
&= f_1^1 f_1 + f_1^2 f_2 + f_1^3 f_3 - f_1 f_0
\end{aligned}$$

More generally we have

$$(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_i) = f_i^1 f_1 + f_i^2 f_2 + f_i^3 f_3 - f_i f_0$$

for $i = 1, 2, 3$.

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{12}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{12}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{12}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(f_{12}^{12} e_{12} + f_{12}^{13} e_{13} + f_{12}^{23} e_{23}) - \pi(f_1 e_2 - f_2 e_1) \\
&= f_{12}^{12}(f_1 e_2 - f_2 e_1) + f_{12}^{13}(f_1 e_3 - f_3 e_1) + f_{12}^{23}(f_2 e_3 - f_3 e_2) - f_1(f_2^1 e_1 + f_2^2 e_2 + f_2^3 e_3) + f_2(f_1^1 e_1 + f_1^2 e_2 + f_1^3 e_3) \\
&= (f_2 f_1^1 - f_1 f_2^1 - f_{12}^{13} f_3 - f_{12}^{12} f_2) e_1 + (f_2 f_1^2 - f_1 f_2^2 - f_{12}^{23} f_3 + f_{12}^{12} f_1) e_2 + (f_2 f_1^3 - f_1 f_2^3 + f_{12}^{23} f_2 + f_{12}^{13}) e_3
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{13}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{13}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) - \pi(f_1 e_3 - f_3 e_1) \\
&= f_2 e_3 - f_3 e_2 - f_1 e_3 + f_3 e_2 \\
&= (f_2 - f_1) e_3.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi \mathrm{d}_{\mathcal{K}(\underline{f})})(e_{23}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{23}) - \pi \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{23}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(e_{13}) - \pi(f_2 e_3 - f_3 e_2) \\
&= f_1 e_3 - f_3 e_1 - f_2 e_3 + f_3 e_1 \\
&= (f_1 - f_2) e_3.
\end{aligned}$$

Finally we calculate

$$\begin{aligned}
(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})})(e_{123}) &= \mathrm{d}_{\mathcal{K}(\underline{f})}\pi(e_{123}) - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}(e_{123}) \\
&= \mathrm{d}_{\mathcal{K}(\underline{f})}(-e_{123}) - \pi(f_1e_{23} - f_2e_{13} + f_3e_{12}) \\
&= -f_1e_{23} + f_2e_{13} - f_3e_{12} - f_1e_{13} + f_2e_{23} + f_3e_{12} \\
&= (f_2 - f_1)(e_{23} + e_{13}).
\end{aligned}$$

Thus, we have.

$$\mathrm{im}(\mathrm{d}_{\mathcal{K}(\underline{f})}\pi - \pi\mathrm{d}_{\mathcal{K}(\underline{f})}) = \langle f_2 - f_1 \rangle$$

DG Algebras

Let (A, d) be an R-complex. A **graded-multiplication** on A is a graded R-linear map $\mathrm{m}\colon A \otimes_R A \to A$ of the underlying graded R-modules. The universal mapping property on graded tensor products tells us that there exists a unique graded R-bilinear map $B_{\mathrm{m}}\colon A \times A \to A$ such that

$$B_{\mathrm{m}}(a, b) = \mathrm{m}(a \otimes b)$$

for all $(a, b) \in A \times A$. However since B_{m} is *uniquely* determined by m, we often identify B_{m} with m and simply think of m as a graded R-bilinear map. In fact, we often drop m altogether and simply denote this multiplication map by

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i$$

for all $\sum a_i \otimes b_i \in A \otimes_R A$. At the end of the day, context will make everything clear.

Suppose m is a graded multiplication As the name of the definition suggests, a graded-multiplication on A must respect the grading. In particular, this means that if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. We can also impose other conditions on a graded-multiplication on A.

Definition 0.1. Let (A, d) be an R-complex and let m be a graded-multiplication on A.

1. We say a m is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.

2. We say m is **graded-commutative** if
$$ab = (-1)^i ba$$
for all $a \in A_i$ and $b \in B_j$ for all $i, j \in \mathbb{Z}$.

3. We say m is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all $a \in A_i$ for all i odd.

4. We say m is **unital** if there exists an $e \in A$ such that
$$ae = e = ea$$
for all $a \in A$.

5. We say a graded-multiplication satisfies **Leibniz law** if
$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$
for all $a \in A_i$ and $b \in B_j$ for all $i, j \in \mathbb{Z}$. This is equivalent to m being a chain map!

6. We say $(A, \mathrm{m}, \mathrm{d})$ is a **differential graded R-algebra** (or **DG R-algebra**) if m is a graded-multiplication on A which satisfies conditions 1-5.

We are often presented with the following scenario: we are given a graded-multiplication m on an R-complex (A, d) and would like to know if $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra. For instance, we may know that m satisfies the conditions 2-5 in Definition (0.1), which would reduce the question of whether $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra to the question of whether m is associative. On the other hand, we may know that m satisfies the conditions 1-4 in Definition (0.1), which would reduce the question of whether $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra to the question of whether m is chain map. Proposition (0.2) gives us some insight on how to proceed in this direction.

Proposition 0.3. *Let (A, d) be an R-complex and let m be a graded-multiplication on A which satisfies conditions 1-4 in Definition (0.1). Furthermore, suppose that*

$$\mathrm{d}(a)b + (-1)^i a\mathrm{d}(b) \in \operatorname{im} \mathrm{d} \tag{11}$$

for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$. Then $(A, \mathrm{m}, \mathrm{d})$ is a DG R-algebra if and only if $\pi \colon A \to \overline{A}$ is a quasiisomorphism where

$$\overline{A} = A / \langle \{ \mathrm{d}(ab) - \mathrm{d}(a)b - (-1)^i a\mathrm{d}(b) \} \rangle.$$

Proof. The condition (11) is equivalent to the condition that m takes $\operatorname{im} \mathrm{d}_A^{\otimes}$ to $\operatorname{im} \mathrm{d}$. □