Free Resolutions

1 Introduction

We want to "understand" mathematical objects using limited technology. For example, given a group G, we obtain a lot of information about G by just knowing its order $|G|$. For instance, if p is a prime such that p divides $|G|$, then the Sylow theorems tells us that there exists a p-Sylow subgroup of G. Another thing we get from knowing the order of a group is whether or not it is isomorphic to another group: if G and H are two groups such that $|G| \neq |H|$, then $G \ncong H$ (or in contrapositive form: if $G \cong H$ then $|G| = |H|$). This means that the order of a group is an invariant. Even though the order of a group is a nice invariant to have, it is not strong enough to determine the group completely: if $|G| = |H|$, then usually it's very difficult to determine whether $G \cong H$ or not.

There are similar tools for understanding rings and modules. -SSW

1.1 Focus of this class

1.1.1 Part I

In part I one of the class, we will focus on understanding the contents of the Hilbert Syzygy Theorem as well as giving a (partial) proof of it. Let us state the Hilbert Syzygy Theorem.

Theorem 1.1. *(The Hilbert Syzygy Theorem) Let $R = k[X_1, \ldots, X_d]$ be the polynomial ring in d indeterminates over a field k and let I be the ideal in R generated by $f_1, \ldots, f_{\beta_1}$. Then there exists an exact sequence of the form*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \tag{1}$$

The exact sequence (8) is called an **augmented free resolution of** R/I **over** R. If we remove R/I from (8), then we get a **free resolution of** R/I **over** R. We view the term R^{β_i} in (8) as sitting in **homological degree** i. The maps ∂_i in (8) are called **differentials**. We often simplify notation by writing (8) as (F, ∂), where we think of F as a graded R-module whose homogeneous component in degree i is R^{β_i} and we think of ∂ as a graded R-homomorphism $\partial \colon F \to F$ of degree -1. Then $\partial_i := \partial|_{F_i}$ where $F_i := R^{\beta_i}$.

1.1.2 Part II

In part II, we will go over a bunch of examples. Here are two to consider now:

Example 1.1. Consider the case where $R = k[X, Y]$ and $I = \langle X^2, XY \rangle$. Then we have the following short exact sequence of R-complexes

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X^2 & XY \end{pmatrix}} R \longrightarrow R/I$$

Example 1.2. (Twisted Cubic) Consider the case where $R = k[X, Y, Z, W]$ and $I = \langle XZ - Y^2, YW - Z^2, XW - YZ \rangle$. A free resolution of R/I is given by

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} W & Z \\ Y & X \\ -Z & -Y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} XZ-Y^2 & YW-Z^2 & WX-YZ \end{pmatrix}} R \longrightarrow R/I$$

Another very important example that we will consider is when $R = k[X_1, \ldots, X_d]$ and $I = \langle X_{i_1}, \ldots, X_{i_{\beta_1}} \rangle$. In this case, we can build a resolution explicitly, called the **Koszul complex of** R/I **over** R, named after Jean Louis Koszul. For example, consider $I = \langle X_1, X_2, X_3 \rangle$. Then the Koszul complex of R/I over R looks like

$$R \xrightarrow{\begin{pmatrix} X_1\\ -X_2\\ X_3\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}0&-X_3&-X_2\\-X_3&0&X_1\\X_2&X_1&0\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}X_1&X_2&X_3\end{pmatrix}} R$$

$$\begin{array}{llllll}
e_{1,2,3} & \longmapsto & X_1e_{2,3}-X_2e_{1,3}+X_3e_{1,2} & & & \\
 & & e_{2,3} & \longmapsto & X_2e_3-X_3e_2 & \\
 & & e_{1,3} & \longmapsto & X_1e_3-X_3e_1 & \\
 & & e_{1,2} & \longmapsto & X_1e_2-X_2e_1 & \\
 & & & & e_1 \longmapsto & X_1 \\
 & & & & e_2 \longmapsto & X_2 \\
 & & & & e_3 \longmapsto & X_3
\end{array}$$

1.1.3 Part III

If all f_j's in Theorem (1.1) are homogeneous, then the resolution (8) can be built minimally and the β_i's are independent of the choice of minimal free resolution. In this case, we call the $\beta_i = \beta_i^R(R/I)$ the ith **Betti number of R/I over R**, named after the Italian mathematician Enrico Betti. If J is another homogeneous ideal of R, and $\beta_i^R(R/I) \neq \beta_i^R(R/J)$ for some i, then $R/I \not\cong R/J$. Thus the Betti numbers give us an invariant of R/I. However, just like in the case of orders in group theory, this invariant is not sufficiently strong enough to determine R/I: if $\beta_i^R(R/I) = \beta_i^R(R/J)$ for all i, then R/I may not be isomorphic to R/J.

In part III of the class, we will try to find finer technology. For instance, it turns out that the Koszul complex is a graded commutative R-algebra with the multiplication rule given by

$$e_Ae_B = \begin{cases} 0 & \text{if } A \cap B = \emptyset \\ e_{A\cup B} & \text{else} \end{cases}$$

subject to the rule

$$e_Ae_B = (-1)^{|A||B|}e_Be_A,$$

where $A, B \subseteq \{1, \dots, m\}$. Moreover, we have the Leibniz rule

$$\partial(e_Ae_B) = \partial(e_A)e_B + (-1)^{|A|}e_A\partial(e_B).$$

This shows that the Koszul complex is a **differential graded algebra (DGA) resolution**.

2 Preliminary Material and Notation

Throughout these notes, let R be a commutative ring with identity. We also let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of a natural numbers together with 0 (SSW: "My class, my rules").

2.1 Linear Algebra

2.1.1 Free Modules

Definition 2.1. Let M be an R-module.

1. A sequence $e_1, \dots, e_n \in M$ is a **(finite) basis** for M if it generates M as an R-module and it is linearly independent over R, i.e. for every $m \in M$ there exists $r_1, \dots, r_n \in R$ such that $m = \sum_{i=1}^n r_ie_i$, and if $a_1, \dots, a_n \in R$ such that $\sum_{i=1}^n a_ie_i = 0$, then $a_i = 0$ for all i.

2. We say M is a **finite rank free R-module** if it has a basis.

Example 2.1. R^n is the **standard free R-module of rank n**. It has as basis the **standard basis elements** e_i where e_i is the vector with 1 in the ith entry and 0 everywhere else.

Example 2.2. If I is a nonzero ideal in R, then R/I is not a free R-module. Indeed, if r is a nonzero element in I, then for all $s \in R$, we have $\overline{r}\overline{s} = \overline{rs} = 0$ in R/I. In other words, "**torsion**" makes linear independence fail for elements of R/I when taking coefficients from R.

2.1.2 Universal Mapping Property of Free R-Modules

The universal mapping property of free R-modules can be stated as follows: Let F be a free R-module with basis $e_1, \dots, e_n \in F$. For all R-modules M and for all $m_1, \dots, m_n \in M$ there exists a unique R-module homomorphism $\varphi \colon F \to M$ such that $\varphi(e_i) = m_i$ for all $i = 1, \dots, n$. In terms of diagrams, this is pictured as follows:

$$\begin{tikzcd} \{e_1, \dots, e_n\} \arrow[r, hook] \arrow[dr, "e_i \mapsto m_i"'] & F \arrow[d, dashed, "\exists! \varphi"] \\ & M \end{tikzcd}$$

Using the universal mapping property of free R-modules, let us prove the following theorem:

Theorem 2.1. *If F and G are finite rank free R-modules with basis $e_1, \dots, e_n$ and $f_1, \dots, f_n$ respectively, then $F \cong G$.*

Proof. By the universal mapping property of free R-modules there exists a unique R-module homomorphism $\varphi \colon F \to G$ such that $\varphi(e_i) = f_i$ for all $i = 1, \dots, n$. Similarly, there exists a unique R-module homomorphism $\psi \colon G \to F$ such that $\psi(f_i) = e_i$ for all $i = 1, \dots, n$. In particular, we see that $\psi \circ \varphi \colon F \to F$ satisfies $(\psi \circ \varphi)(e_i) = e_i$. But we also have $1(e_i) = e_i$ for all $i = 1, \dots, n$, where $1 \colon F \to F$ is the identity map. Therefore by uniqueness of the map in the universal mapping property of free R-modules, we must have $\psi \circ \varphi = 1$. A similar argument shows that $\varphi \circ \psi = 1$. □

Corollary. *Let F be a free R-module with basis $e_1, \dots, e_n \in F$. Then $F \cong R^n$.*

Remark. Note that you can prove Theorem (2.1) without the universal mapping property of free R-modules, but the point is that you'd have to show well-definedness, linearity, etc... of the maps constructed. The point is that all of this is built into the universal mapping property of free R-modules.

2.1.3 Representing R-module Homomorphisms By Matrices

Let $\varphi \colon R^n \to R^m$ be an R-module homomorphism and let $e_1, \dots, e_n$ denote the **canonical basis** of R^n, i.e. $e_i = (0 \dots, 1, \dots 0)^\top$ where 1 is at the ith entry and 0 is in all other entries. Any $x \in R^n$ has a unique linear combination as $x = x_1 e_1 + \cdots + x_n e_n$ where $x_i \in R$. In particular, $\varphi(e_j)$ has a unique representation as

$$\varphi(e_j) = \sum_{i=1}^{n} a_{ij} e_i.$$

Writing x as a column vector $[x]$ and using linearity of φ, we see that

$$\varphi(x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = [\varphi][x],$$

where $[\varphi] = (a_{ij})$ is an $m \times n$ matrix with entries in R. We call $[\varphi]$ the **matrix representation of** φ. Addition, scalar multiplication, and composition of R-module homomorphisms correspond to addition, scalar multiplication, and multiplication of matrices.

2.2 Noetherian Rings and Modules

Definition 2.2. An R-module M is said to be **finitely generated** if there exists a surjective R-module homomorphism from R^n to M for some $n \in \mathbb{N}$.

Theorem 2.2. *The following conditions are equivalent:*

1. *Every ideal of R is finitely generated.*
2. *Ascending chain condition (acc) of ideals in R: given a chain of ideals*
$$I_1 \subseteq I_2 \subseteq \cdots$$
of ideals in R, there exists $N \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that
$$I_N = I_{N+1} = I_{N+2} = \cdots$$
3. *Maximum condition for ideals in R: every non-empty set of ideals of R contains an element maximal with respect to containment.*
4. *For every $n \in \mathbb{N}$ every submodule of R^n is finitely generated.*
5. *For every $n \in \mathbb{N}$, R^n satisfies maximum condition on submodules.*
6. *Acc of on submodules of R^n for every $n \in \mathbb{N}$.*

Definition 2.3. R is said to be **Noetherian** if it satisfies any of the equivalent conditions of Theorem (2.2).

Theorem 2.3. *(Hilbert Basis Theorem) If R is Noetherian then $R[X]$ is also Noetherian.*

Remark. More generally, if I is an ideal of $R[X_1, \dots, X_n]$, then $R[X_1, \dots, X_n]/I$ is Noetherian. In particular, $k[X_1, \dots, X_n]/I$ is Noetherian since k is Noetherian.

2.3 Exact Sequences

Definition 2.4. A sequence of R-module homomorpisms

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is said to be **exact at** B if $\operatorname{Ker}(\psi) = \operatorname{Im}(\varphi)$. A sequence of R-module homomorphisms

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \longrightarrow \cdots$$

is said to be **exact** if it is exact at every A_i. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

2.3.1 Free Resolutions

Theorem 2.4. *Assume that R is Noetherian and let M be a finitely generated R-module. Then there exists an exact sequence of the form*

$$\cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow M \longrightarrow 0 \tag{2}$$

Proof. First note that since M is finitely generated, there exists a surjective R-module homomorphism $\tau_0 \colon R^{\beta_0} \to M$. Denote $M_1 := \operatorname{Ker}(\tau_0)$ the kernel of τ_0 and denote $\iota_1 \colon M_1 \hookrightarrow R^{\beta_0}$ the inclusion map. Since R is Noetherian and M_1 is a submodule of R^{β_0}, we see that M_1 is finitely generated (by Theorem (2.2)). Therefore there exists a surjective homomorphism $\tau_1 \colon R^{\beta_1} \to M_1$. Let $\iota_1 \colon \operatorname{Ker}(\tau_0) \to R^{\beta_0}$ denote the inclusion map and let $\phi_1 := \iota_0 \circ \tau_1$. Continuing in this way, for each $i \in \mathbb{N}$, we construct short exact sequences of the form

$$0 \longrightarrow M_{i+1} \xrightarrow{\iota_{i+1}} R^{\beta_i} \xrightarrow{\tau_i} M_i \longrightarrow 0$$

where we denote $M_0 := M$. A standard argument in homological algebra says that we can connect these short exact sequences together to form the long exact sequence (2) where $\partial_{i+1} := \iota_i \circ \tau_{i+1}$. ☐

Definition 2.5. The exact sequence (2) is called an **augmented free resolution of** M.

In general, free resolutions are hard to compute. However here are two examples to consider.

Example 2.3. Recall that the fundamental theorem of finitely generated abelian groups says that if G is an abelian group, then there exists $r \in \mathbb{Z}_{\geq 0}$ and $d_1, \ldots, d_n \in \mathbb{Z}_{>1}$ such that

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n.$$

This boils down to the fact that there exists a free resolution of the form

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\varphi} \mathbb{Z}^{r+n} \longrightarrow G \longrightarrow 0$$

where the $(n+r) \times n$ matrix representation of φ has the form

$$[\varphi] = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 2.4. If R is an integral domain and r is a nonzero unit in R. Then we have the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/r \longrightarrow 0$$

If R is a PID, then these are all the resolutions.

Example 2.5. Let $R = k[X, Y]/\langle XY \rangle$ and let $M = R/\langle \overline{X} \rangle$. Then we have

$$\cdots \longrightarrow R \xrightarrow{\cdot \overline{X}} R \xrightarrow{\cdot \overline{Y}} R \xrightarrow{\cdot \overline{X}} R \longrightarrow R/r \longrightarrow 0$$

In fact, we can prove that this resolution does not stop. The idea is that straying away from polynomial rings makes this construction go bad.

2.3.2 Hilbert Syzygy Theorem

Theorem 2.5. *(Hilbert Syzygy Theorem) Let $R = k[X_1, \ldots, X_d]$ with k a field and let M be a finitely generated R-module. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow M \longrightarrow 0 \tag{3}$$

with $\beta_i \geq 0$ for all i.

Remark. If $d = 0$, then R is just the field k and M is a k-vector space. If $d = 1$, then R is a PID. Since every submodule of a free R-module is free (when R is a PID), we see that the free resolution terminates at R^{β_1}. More variables require more work.

3 Graded Resolutions

Tracking finer information about certain resolutions. In this section, we assume that k is a field, R is a polynomial ring $R = k[X_1, \ldots, X_d]$.

Definition 3.1. Let I be an ideal in R. We say I is a **homogeneous** (or **graded**) ideal in R if it can be generated by homogeneous polynomials (not necessarily of the same degree).

Example 3.1. Consider the ring $R = k[X, Y]$ and $I = \langle X^2 - XY^2, XY^2 \rangle$. Even though $X^2 - XY^2$ is not a homogeneous polynomial, the ideal I is still a homogeneous ideal. This is because $I = \langle X^2, XY^2 \rangle$.

Theorem 3.1. *(Graded Hilbert Basis Theorem) If $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ where each f_i is homogeneous. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \tag{4}$$

such that each differential ∂_j is represented by a matrix of homogeneous polynomials.

3.1 Examples of Graded Resolutions

Example 3.2. Consider the case where $R = k[X,Y]$ and $I = \langle X^a, Y^b \rangle$ where $a, b \geq 1$. Then we have the following short exact sequence of R-complexes

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y^a \\ X^b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R \longrightarrow R/I$$

This uses unique factorization of the indeterminates.

Example 3.3. Consider the case where $R = k[X,Y]$ and $I = \langle X^a, XY, Y^b \rangle$ where $a, b \geq 2$. We claim that

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -Y & 0 \\ X^{a-1} & Y^{b-1} \\ 0 & X \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} X^a & XY & Y^b \end{pmatrix}} R \longrightarrow R/I$$

is a free resolution of R/I. To see this, we first show $\mathrm{Ker}(\partial_1) = \mathrm{Im}(\partial_2)$. Let $(f, g, h)^\top \in \mathrm{Ker}(\partial_1)$, so we have

$$X^a f + XYg + Y^b h = 0. \tag{5}$$

Then (5) implies $Y | X^a f$ which implies $Y | f$, and so $f = Yf_1$ for some $f_1 \in R$. Similarly, (5) implies $X | Y^b h$ which implies $X | h$, and so $h = Xh_1$ for some $h_1 \in R$. Thus we have

$$\begin{aligned} 0 &= X^a f + XYg + Y^b h \\ &= X^a Y f_1 + XYg + Y^b X h \\ &= XY(X^{a-1} f_1 + g + Y^{b-1} h_1), \end{aligned}$$

and this further implies $X^{a-1} f_1 + g + Y^{b-1} h_1 = 0$. Solving for g, we find that $g = -X^{a-1} f_1 - Y^{b-1} h_1$. Therefore we see that

$$\begin{aligned} \begin{pmatrix} f \\ g \\ h \end{pmatrix} &= \begin{pmatrix} Yf_1 \\ -X^{a-1} f_1 - Y^{b-1} h_1 \\ Xh_1 \end{pmatrix} \\ &= f_1 \begin{pmatrix} Y \\ -X^{a-1} \\ 0 \end{pmatrix} + h_1 \begin{pmatrix} 0 \\ Y^{b-1} \\ X \end{pmatrix} \\ &\in \mathrm{Im}(\partial_2). \end{aligned}$$

In particular, we have $\mathrm{Im}(\partial_2) \supseteq \mathrm{Ker}(\partial_1)$ (and hence $\mathrm{Im}(\partial_2) = \mathrm{Ker}(\partial_1)$ since the reverse inclusion is already known). Finally, since R is an integral domain and since $(-Y, X^{a-1}, 0)^\top$ and $(0, Y^{b-1}, X)$ are linearly independent, we see that ∂_2 is injective.

3.2 Relation on the β_i

Note that the β_i in each of the previous examples satisfy the relations

$$\beta_0 - \beta_1 + \beta_2 = 0.$$

This actually happens in general:

Lemma 3.2. *Let*

$$0 \longrightarrow k^{\beta_d} \longrightarrow \cdots \longrightarrow k^{\beta_i} \xrightarrow{\partial_i} k^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow k^{\beta_1} \xrightarrow{\partial_1} k^{\beta_0} \longrightarrow 0 \tag{6}$$

be an exact sequence of k-vector spaces. Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

Proof. Let $K_i := \mathrm{Ker}(\partial_i)$ for all $0 \leq i \leq d$ and let $K_{-1} = 0$. Then for each $0 \leq i \leq d$, exactness at k^{β_i} in (6) implies exactness of

$$0 \longrightarrow K_i \hookrightarrow k^{\beta_i} \xrightarrow{\partial_i} K_{i-1} \longrightarrow 0.$$

Since the dimension function is additive on short exact sequences, we have $\beta_i = \dim(K_i) + \dim(K_{i-1})$. Therefore we have a telescoping series

$$\begin{aligned}
\sum_{i=0}^{d}(-1)^i\beta_i &= \sum_{i=0}^{d}(-1)^i(\dim(K_i) + \dim(K_{i-1})) \\
&= (-1)^d \dim(K_d) + \dim(K_{-1}) \\
&= 0.
\end{aligned}$$

□

Theorem 3.3. *Let I be a nonzero ideal in R and let*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\partial_0} R/I \longrightarrow 0 \qquad (7)$$

be a free resolutions of R/I over R. Then

$$\sum_{i=0}^{d}(-1)^i\beta_i = 0.$$

Proof. Localizing R at $\langle 0\rangle$ gives us a field $R_{\langle 0\rangle} = k(X_1, \ldots, X_d)$ since R is an integral domain. Since $I \neq 0$, there exists $s \in I\backslash\{0\}$ that is nonzero. We claim then that $(R/I)_{\langle 0\rangle} = 0$. Indeed an element in $(R/I)_{\langle 0\rangle}$ looks like $\overline{r}/t$ where $r \in R$ and $t \in R\backslash 0$. Then

$$\frac{\overline{r}}{t} = \frac{\overline{sr}}{st} = \frac{\overline{0}}{st} = 0.$$

Therefore localizing our resolution at $\langle 0\rangle$ (which is exact) gives us an exact sequence of vector spaces over the field $R_{\langle 0\rangle}$:

$$0 \longrightarrow R_{\mathfrak{p}}^{\beta_d} \longrightarrow \cdots \longrightarrow R_{\mathfrak{p}}^{\beta_i} \xrightarrow{\partial_i} R_{\mathfrak{p}}^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R_{\mathfrak{p}}^{\beta_1} \xrightarrow{\partial_1} R_{\mathfrak{p}}^{\beta_0} \longrightarrow 0 \qquad (8)$$

Now we apply Lemma (3.2) to obtain our desired result. □