Matrix Representation of a Linear Map

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In this note, let K be a field, let V be a K-vector space with basis $\beta = \{\beta_1, \dots, \beta_m\}$, and let W be a K-vector space with basis $\gamma = \{\gamma_1, \dots, \gamma_n\}$.

1 Introduction

On a first encounter in linear algebra, one typically studies *concrete* vector spaces like $\mathbb{R}^2$ and *concrete* matrices like $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathbb{R}^2 \to \mathbb{R}^2$. In a more abstract setting, one studies *abstract* vectors spaces like V, W and *abstract* linear maps between them like $T \colon V \to W$. However, this abstract setting is not as abstract as it may first seem. Indeed, it turns out that we can translate everything in the abstract setting to the more concrete setting. We will describe this translation in this note.

2 From the Abstract Setting to the Concrete Setting

2.1 Column Representation of a Vector

Let $v \in V$. Then for each $1 \le i \le m$, there exists unique $a_i \in K$ such that

$$v = \sum_{i=1}^{m} a_i \beta_i.$$

Since the a_i are uniquely determined, we are justified in making the following definition:

Definition 2.1. The **column representation of v with respect to the basis** β, denoted $[v]_\beta$, is defined by

$$[v]_\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}.$$

Proposition 2.1. *Let $[\cdot]_\beta \colon V \to K^m$ be given by*

$$[\cdot]_\beta(v) = [v]_\beta$$

for all $v \in V$. Then $[\cdot]_\beta$ is an isomorphism.

Proof. We first show that $[\cdot]_\beta$ is linear. Let $v_1, v_2 \in V$ and $c_1, c_2 \in K$. Then for each $1 \le i \le m$, there exists unique $a_{i1}, a_{i2} \in K$ such that

$$v_1 = \sum_{i=1}^{m} a_{i1}\beta_i \quad \text{and} \quad v_2 = \sum_{i=1}^{m} a_{i2}\beta_i.$$

Therefore we have

$$\begin{aligned} a_1 v_1 + a_2 v_2 &= a_1 \sum_{i=1}^{m} a_{i1}\beta_i + a_2 \sum_{i=1}^{m} a_{i2}\beta_i \\ &= \sum_{i=1}^{m} (a_1 a_{i1} + a_2 a_{i2})\beta_i. \end{aligned}$$

This implies

$$\begin{aligned} [a_1 v_1 + a_2 v_2]_\beta &= \begin{pmatrix} a_1 a_{11} + a_2 a_{12} \\ \vdots \\ a_1 a_{m1} + a_2 a_{m2} \end{pmatrix} \\ &= a_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + a_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} \\ &= a_1 [v_1]_\beta + a_2 [v_2]_\beta. \end{aligned}$$

Therefore $[\cdot]_\beta$ is linear. To see that $[\cdot]_\beta$ is an isomorphism, note that $[\beta_i] = e_i$, where e_i is the column vector in K^n whose i-th entry is 1 and whose entry everywhere else is 0. Thus, $[\cdot]_\beta$ restricts to a bijection on basis sets

$$[\cdot]_\beta \colon \{\beta_1, \dots, \beta_m\} \to \{e_1, \dots, e_n\},$$

and so it must be an isomorphism. ☐

2.2 Matrix Representation of a Linear Map

Let T be a linear map from V to W. Then for each $1 \leq i \leq m$ and $1 \leq j \leq n$, there exists unique elements $a_{ji} \in K$ such that

$$T(\beta_i) = \sum_{j=1}^{n} a_{ji}\gamma_j \tag{1}$$

for all $1 \leq i \leq m$. Since the a_{ji} are uniquely determined, we are justified in making the following definition:

Definition 2.2. The **matrix representation of** T **with respect to the bases** β **and** γ, denoted $[T]_{\beta}^{\gamma}$, is defined to be the $n \times m$ matrix

$$[T]_{\beta}^{\gamma} := \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}.$$

Proposition 2.2. *Let T be a linear map from V to W. Then*

$$[T]_{\beta}^{\gamma}[v]_{\beta} = [T(v)]_{\gamma}$$

for all $v \in V$.

Remark. In terms of diagrams, this proposition says that the following diagram is commutative

$$\begin{tikzcd} K^m \arrow[r, "{[T]_{\beta}^{\gamma}}"] & K^n \\ V \arrow[u, "{[\cdot]_{\beta}}"] \arrow[r, "T"'] & W \arrow[u, "{[\cdot]_{\gamma}}"'] \end{tikzcd}$$

Proof. Let $v \in V$ and let $a_i, a_{ji} \in K$ be the unique elements such that

$$v = \sum_{i=1}^{m} a_i\beta_i \quad \text{and} \quad T(\beta_i) = \sum_{j=1}^{n} a_{ji}\gamma_j$$

for all $1 \leq i \leq m$. Then

$$\begin{aligned} [T]_{\beta}^{\gamma}[v]_{\beta} &= \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{m} a_{1i}a_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni}a_i \end{pmatrix} \\ &= [T(v)]_{\gamma}. \end{aligned}$$

Where the last equality follows from

$$\begin{aligned} T(v) &= T\left(\sum_{i=1}^{m} a_i\beta_i\right) \\ &= \sum_{i=1}^{m} a_i T(\beta_i) \\ &= \sum_{i=1}^{m} a_i \sum_{j=1}^{n} a_{ji}\gamma_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ji}a_i\right)\gamma_j. \end{aligned}$$

□

Theorem 2.1. *Let V, V', and V'' be K-vector spaces with bases β, β', and β'' respectively and let $T\colon V \to V'$ and $T'\colon V' \to V''$ be two K-linear maps. Then*

$$[T' \circ T]_{\beta}^{\beta''} = [T']_{\beta'}^{\beta''}[T]_{\beta}^{\beta'}.$$

Proof. Let $[v]_\beta \in K^n$. Then we have

$$\begin{aligned}
[T' \circ T]_\beta^{\beta''}[v]_\beta &= [(T' \circ T)(v)]_{\beta''} \\
&= [T'(T(v))]_{\beta''} \\
&= [T']_{\beta'}^{\beta''}[T(v)]_{\beta'} \\
&= [T']_{\beta'}^{\beta''}[T]_\beta^{\beta'}[v]_\beta.
\end{aligned}$$

Therefore $[T' \circ T]_\beta^{\beta''} = [T']_{\beta'}^{\beta''}[T]_\beta^{\beta'}$. □

2.3 Change of Basis Matrix

In this subsection, let α be another basis for V and let δ be another basis for W.

Definition 2.3. Let $1_V : V \to V$ denote the identity map. The **change of basis matrix from** β **to** α is defined to be the matrix $[1_V]_\alpha^\beta$.

Remark.

1. The reason why we say from β to α and not from α to β is because we want to express the new basis α in terms of the old basis β.

2. Observe that the change of basis matrix from β to α is invertible, with inverse being $[1_V]_\beta^\alpha$. Indeed, we have

$$\begin{aligned}
[1_V]_\alpha^\beta[1_V]_\beta^\alpha &= [1_V \circ 1_V]_\beta^\beta \\
&= [1_V]_\beta^\beta \\
&= \mathrm{I}_m,
\end{aligned}$$

   where I_m is the $m \times m$ identity matrix.

In applications, we often describe a change of basis from β to α as a concrete matrix like

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mm} \end{pmatrix}.$$

Let us show how to work with C in terms of our notation.

Proposition 2.3. *Let C be the change of basis matrix from β to α. Then*

$$C[v]_\alpha = [v]_\beta$$

for all $v \in V$.

Proof. Let $v \in V$. Then

$$\begin{aligned}
C[v]_\alpha &= [1_V]_\alpha^\beta[v]_\alpha \\
&= [1_V(v)]_\beta \\
&= [v]_\beta.
\end{aligned}$$

□

Proposition 2.4. *Let $T : V \to W$ be a linear map, let C be the change of basis matrix from β to α, and let D be the change of basis matrix from γ to δ. Then*

$$[T]_\alpha^\delta = D^{-1}[T]_\beta^\gamma C.$$

In particular, if $U : V \to V$ is an endomorphism, then

$$[U]_\alpha^\alpha = C^{-1}[U]_\beta^\beta C.$$

Proof. We have

$$\begin{aligned}
[T]_\alpha^\delta &= [1_W \circ T \circ 1_V]_\alpha^\delta \\
&= [1_W]_\gamma^\delta[T]_\beta^\gamma[1_V]_\alpha^\beta \\
&= D^{-1}[T]_\beta^\gamma C.
\end{aligned}$$

$\square$

2.4 Matrix Notation

Let $T: V \to W$ be a linear. A useful way to keep track of (1) for each i is to write it using matrix notation:

$$(T(\beta_1), \ldots, T(\beta_m)) = (\gamma_1, \cdots, \gamma_n)[T]_\beta^\gamma.$$

Using matrix notation, we obtain another proof of Proposition (2.4):

Proof. As matrix equations, we have

$$(\beta_1, \cdots, \beta_m)C = (\alpha_1, \ldots, \alpha_m) \quad \text{and} \quad (\gamma_1, \cdots, \gamma_n)D = (\delta_1, \ldots, \delta_n).$$

Thus, we have

$$\begin{aligned} (T(\beta_1), \ldots, T(\beta_m)) &= (\gamma_1, \cdots, \gamma_n)[T]_\beta^\gamma \\ (T(\beta_1), \ldots, T(\beta_m))C \cdot C^{-1} &= (\gamma_1, \cdots, \gamma_n)D \cdot D^{-1}[T]_\beta^\gamma \\ (T(\alpha_1), \ldots, T(\alpha_m)) &= (\delta_1, \ldots, \delta_n)D^{-1}[T]_\beta^\gamma C, \end{aligned}$$

where $(T(\beta_1), \ldots, T(\beta_m))C = (T(\alpha_1), \ldots, T(\alpha_m))$ follows from linearity of T. It follows that

$$[T]_\alpha^\delta = D^{-1}[T]_\beta^\gamma C.$$

$\square$

Example 2.1. Suppose V and W are 3-dimensional K-vector spaces with basis $\beta = \{\beta_1, \beta_2, \beta_3\}$ for V and basis $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ for W. Suppose $T: V \to W$ is a linear transformation such that the matrix representation of T with respect to β and γ is

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So $T(\beta_1) = \gamma_1$, $T(\beta_2) = \gamma_1 + \gamma_3$, and $T(\beta_3) = \gamma_2$. We summarize in the table below how to convert this matrix into a diagonal matrix using elementary row and column operations. We also show what effect each operation has on the basis elements.

Basis for V	Basis for W	Matrix Representation
$\{\beta_1, \beta_2, \beta_3\}$	$\{\gamma_1, \gamma_2, \gamma_3\}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\{\beta_1, \beta_2 - \beta_1, \beta_3\}$	$\{\gamma_1, \gamma_2, \gamma_3\}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} e_{12}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\{\beta_1, \beta_2 - \beta_1 + \beta_3, \beta_3\}$	$\{\gamma_1, \gamma_2, \gamma_3\}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} e_{32}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$\{\beta_1, \beta_2 - \beta_1 + \beta_3, \beta_1 - \beta_2\}$	$\{\gamma_1, \gamma_2, \gamma_3\}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} e_{23}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
$\{\beta_1, \beta_2 - \beta_1 + \beta_3, \beta_1 - \beta_2\}$	$\{\gamma_1, \gamma_2 + \gamma_3, \gamma_3\}$	$e_{32}(-1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

2.5 Linear Isomorphism from $\mathbf{Hom}_K(V, W)$ to $\mathbf{M}_{n \times m}(K)$

So far, we have shown how to obtain a column vector $[v]_\beta$ from an abstract vector v, and we have shown how to obtain a matrix $[T]_\beta^\gamma$ from an abstract linear map $T: V \to W$. We've also shown that the column represenation map $[\cdot]_\beta: V \to K^m$ is a *linear* map. This means, for example, that $[v_1 + v_2]_\beta = [v_1]_\beta + [v_2]_\beta$ for any two vectors $v_1, v_2 \in V$. Can we view the matrix representation map $[\cdot]_\beta^\gamma$ as a linear map? Indeed we can. To see how this works, we first need to describe the domain of $[\cdot]_\beta^\gamma$.

We denote by $\mathrm{Hom}_K(V,W)$ to be the set of all K-linear maps from V to W. We give $\mathrm{Hom}_K(V,W)$ the structure of a K-vector space as follows: If $T,U \in \mathrm{Hom}_K(V,W)$ and $a \in K$, then we define addition of T and U, denoted $T+U$, and scalar multiplication of a with T, denoted aT, by

$$(T+U)(v) = T(v) + U(v) \quad \text{and} \quad (aT)(v) = T(av)$$

for all $v \in V$.

Exercise 1. Check that the addition and scalar multiplication as defined above gives $\mathrm{Hom}_K(V,W)$ the structure of a K-vector space.

Exercise 2. For each $1 \le i \le m$ and $1 \le j \le n$, let $T_{ji}\colon V \to W$ be unique the linear map such that

$$T_{ji}(\beta_k) = \begin{cases} \gamma_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

for all $1 \le k \le m$. Check that the set $\{T_{ji} \mid 1 \le i \le m \text{ and } 1 \le j \le n\}$ is a basis for $\mathscr{L}(V,W)$.

Theorem 2.2. *Let V and W be K-vector spaces with basis $\beta = \{\beta_1, \dots, \beta_m\}$ for V and basis $\gamma = \{\gamma_1, \dots \gamma_n\}$ for W. Then we have an isomorphism of K-vector spaces*

$$[\cdot]_\beta^\gamma \colon Hom_K(V,W) \cong M_{n\times m}(K)$$

where the map $[\cdot]_\beta^\gamma$ is defined by

$$[\cdot]_\beta^\gamma(T) = [T]_\beta^\gamma$$

for all $T \in Hom_K(V,W)$.

Proof. We first show that the map $[\cdot]_\beta^\gamma$ is linear. Let $T,U \in \mathrm{Hom}_K(V,W)$ and let $a,b \in K$. Then it follows from Proposition (2.2) and Proposition (2.1) that

$$\begin{aligned} [aT+bU]_\beta^\gamma [v]_\beta &= [(aT+bU)(v)]_\gamma \\ &= [aT(v) + bU(v)]_\gamma \\ &= a[T(v)]_\gamma + b[U(v)]_\gamma \\ &= a[T]_\gamma [v]_\beta + b[U]_\gamma [v]_\beta. \end{aligned}$$

Therefore $[\cdot]_\beta^\gamma$ is a linear map. To see that $[\cdot]_\beta^\gamma$ is an isomorphism, note that $[T_{ji}]_\beta^\gamma = E_{ji}$, where E_{ji} is the matrix in K^n whose (j,i)-th entry is 1 and whose entry everywhere else is 0. Thus, $[\cdot]_\beta^\gamma$ restricts to a bijection on basis sets

$$[\cdot]_\beta^\gamma \colon \{T_{ji} \mid 1 \le i \le m \text{ and } 1 \le j \le n\} \to \{E_{ji} \mid 1 \le i \le m \text{ and } 1 \le j \le n\},$$

and so it must be an isomorphism. □

2.6 K-Algebra Isomorphism from End(V) to M$_n(K)$

We write $\mathrm{End}_K(V)$ instead of $\mathrm{Hom}_K(V,V)$ to denote the set of all K-linear maps from V to itself. Simmilarly we write $\mathrm{M}_n(K)$ instead of $\mathrm{M}_{n\times n}(K)$ to denote the set of all $n \times n$ matrices. There is extra structure present in $\mathrm{End}_K(V)$ and $\mathrm{M}_n(K)$ that is not necessarily present in $\mathrm{Hom}_K(V,W)$ and $\mathrm{M}_{n\times m}(K)$; namely, $\mathrm{End}_K(V)$ and $\mathrm{M}_n(K)$ have K-algebra structures. Composition gives $\mathrm{End}_K(V)$ a K-algebra structure and matrix multiplication gives $\mathrm{M}_n(K)$ a K-algebra structure. It's reasonable to suspect that the matrix representation map $[\cdot]_\beta^\beta$ is a K-algebra isomorphism. In fact, this is indeed the case: Theorem (2.2) tells us that the matrix representation map $[\cdot]_\beta^\beta$ can be viewed as an isomorphism from $\mathrm{End}_K(V)$ to $\mathrm{M}_n(K)$ as K-vector spaces, and Theorem (2.1) tells us that the matrix representation map preserves the K-algebra structures (it takes composition to matrix multiplication). Combining these two theorems together tells us that the matrix representation map $[\cdot]_\beta^\beta$ can be viewed as an isomorphism from $\mathrm{End}_K(V)$ to $\mathrm{M}_n(K)$ as K-algebras.

2.7 Duality

Definition 2.4. The **dual** of V is defined to be the K-vector space

$$V^\star := \{\varphi\colon V \to K \mid \varphi \text{ is linear}\}.$$

where addition and scalar multiplication are defined by

$$(\varphi + \psi)(v) = \varphi(v) + \psi(v) \quad \text{and} \quad (\lambda\varphi)(v) = \varphi(\lambda v)$$

for all $\varphi, \psi \in V^\star$, $\lambda \in \mathbb{C}$, and $v \in V$. The **dual** of β is defined to be the basis of $V^\star$ given by $\beta^\star := \{\beta_1^\star, \dots, \beta_m^\star\}$, where each $\beta_i^\star$ is uniquely determined by

$$\beta_i^\star(\beta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Exercise 3. Check that $V^\star$ is indeed a K-vector space and that $\beta^\star$ is indeed a basis for $V^\star$.

Definition 2.5. Let $T\colon V \to W$ be a linear map. The **dual** of T is defined to be the map $T^\star\colon W^\star \to V^\star$ given by

$$T^\star(\varphi) = \varphi \circ T$$

for all $\varphi \in W^\star$.

Proposition 2.5. *The map $T^\star$ defined above is linear.*

Proof. Let $\varphi, \psi \in W^\star$ and let $a, b \in K$. Then

$$\begin{aligned} T^\star(a\varphi + b\psi)(v) &= (a\varphi + b\psi)(T(v)) \\ &= a\varphi(T(v)) + b\psi(T(v)) \\ &= aT^\star(\varphi)(v) + bT^\star(\psi)(v) \end{aligned}$$

for all $v \in V$. Thus $T^\star(a\varphi + b\psi)$ and $aT^\star(\varphi) + bT^\star(\psi)$ agree on all of V, and so they must be equal. □

Remark. An important remark here is that to determine whether two linear maps out of V are equal, we do *not* need to check that they agree on all of V as we did in the proof above. In fact, we just need to show that they agree on the basis β.

2.8 Matrix Representation of the Dual of a Linear Map

Proposition 2.6. *Let $T\colon V \to W$ be a linear map. Then*

$$[T^\star]_{\gamma^\star}^{\beta^\star} = ([T]_\beta^\gamma)^\top,$$

where $([T]_\beta^\gamma)^\top$ is the transpose of $[T]_\beta^\gamma$.

Proof. Suppose that

$$T(\beta_i) = \sum_{j=1}^n a_{ji}\gamma_j \tag{2}$$

for all $1 \leq i \leq m$. So a_{ji} lands in the jth row and ith column in $[T]_\beta^\gamma$ since we are summing over j in (2).

Let $1 \leq j \leq n$. We compute

$$\begin{aligned} T^\star(\gamma_j^\star)(\beta_i) &= \gamma_j^\star(T(\beta_i)) \\ &= \gamma_k^\star\left(\sum_{k=1}^n a_{ki}\gamma_k\right) \\ &= \sum_{j=1}^n a_{ki}\gamma_j^\star(\gamma_k) \\ &= a_{ki} \end{aligned}$$

for all $1 \leq i \leq m$. In particular, this implies

$$T^\star(\gamma_j^\star) = \sum_{i=1}^m a_{ji}\beta_i^\star \tag{3}$$

since both sides of (3) agree on β. So a_{ji} lands in the ith row and jth column in $[T^\star]_{\gamma^\star}^{\beta^\star}$ since we are summing over i in (3). Therefore the transpose of $[T]_\beta^\gamma$ is $[T^\star]_{\gamma^\star}^{\beta^\star}$. □

2.9 Bilinear Forms

Definition 2.6. A **bilinear form** on V is a function $B : V \times V \to K$ which satisfies the following properties

1. It is linear in the first variable when the second variable is fixed: for fixed $w \in V$, we have $B(av + a'v', w) = aB(v, w) + a'B(v', w)$ for all $a, a' \in K$ and $v, v' \in V$.
2. It is linear in the second variable when the first variable is fixed: for fixed $v \in V$, we have $B(v, bw + b'w') = bB(v, w) + b'B(v, w')$ for all $b, b' \in K$ and $w, w' \in V$.

Moreover, we say

- B is **symmetric** if $B(v, w) = B(w, v)$ for all $v, w \in V$,
- B is **skew-symmetric** if $B(v, w) = -B(w, v)$ for all $v, w \in V$,
- B is **alternating** if $B(v, v) = 0$ for all $v \in V$.

Let B be a bilinear form on V. Pick v and w in V and express them in the basis β:

$$v = \sum_{i=1}^m a_i \beta_i \quad \text{and} \quad w = \sum_{j=1}^m b_j \beta_j.$$

Then bilinearity of B gives us

$$\begin{aligned} B(v, w) &= B\left(\sum_{i=1}^m a_i \beta_i, \sum_{j=1}^m b_j \beta_j\right) \\ &= \sum_{1 \le i,j \le m} a_i b_j B(\beta_i, \beta_j) \\ &= \begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix} \begin{pmatrix} B(\beta_1, \beta_1) & \cdots & B(\beta_1, \beta_m) \\ \vdots & \ddots & \vdots \\ B(\beta_m, \beta_1) & \cdots & B(\beta_m, \beta_m) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ &= [v]_\beta^\top [B]_\beta [w]_\beta. \end{aligned}$$

where $\cdot$ denoted the dot product and $[B]_\beta = (B(\beta_i, \beta_j))$. We call $[B]_\beta$ the **matrix representation of B with respect to the basis β**.

Bilinear forms are not linear maps, but each bilinear form B on V can be interpreted as a linear map $V \to V^*$ in two ways, as L_B and R_B, where $L_B(v) = B(v, \cdot)$ and $R_B(v) = B(\cdot, v)$ for all $v \in V$.

Theorem 2.3. *Let B be a bilinear form on V and let $[B]_\beta = (a_{ij})$ be the matrix representation of B with respect to the basis β. Then*

$$M = [R_B]_\beta^{\beta^*}.$$

Proof. For each $1 \le i, j \le m$, we have

$$B(\beta_j, \beta_i) = a_{ji}.$$

Therefore

$$R_B(\beta_i) = B(\cdot, \beta_i) = \sum_{j=1}^m a_{ji} \beta_j^*$$

for all $1 \le i \le m$. It follows that

$$[R_B]_\beta^{\beta^*} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} = [B]_\beta.$$

□

Remark. That the matrix associated to B is the matrix of R_B rather than L_B is related to our *convention* that we view bilinear forms concretely using $[v]_\beta^\top M[w]_\beta$ instead of $(M[v]_\beta)^\top [w]_\beta$. If we adopted the latter convention, then the matrix associated to B would equal the matrix for L_B.

Proposition 2.7. *Let α be another basis of V, let C be a change of basis matrix from β to α, and let B be a bilinear form on V. Then*

$$[B]_\alpha = C^\top [B]_\beta C.$$

Proof. We have

$$\begin{aligned} [B]_\alpha &= [R_B]_\alpha^{\alpha^*} \\ &= [1_{V^*} \circ R_B \circ 1_V]_\alpha^{\alpha^*} \\ &= [1_{V^*}]_{\beta^*}^{\alpha^*} [R_B]_\beta^{\beta^*} [1_V]_\alpha^\beta \\ &= C^\top [B]_\beta C. \end{aligned}$$

□

Definition 2.7. Two bilinear forms B_1 and B_2 on the respective vector spaces V_1 and V_2 are called **equivalent** if there is a vector space isomorphism $A : V_1 \to V_2$ such that

$$B_2(Av, Aw) = B_1(v, w)$$

for all v and w in V_1.

Although all matrix representations of a linear transformation $T \colon V \to V$ have the same determinant, the matrix representations of a bilinear form B on V have the same determinant only up to a nonzero square factor since $\det(C^\top MC) = \det(C)^2 \det(M)$. This provides a sufficient (although far from necessary) condition to show two bilinear forms are inequivalent.

Example 2.2. Let d be a squarefree positive integer. On $\mathbb{Q}^2$, the bilinear form $B_d(v, w) = v^\top \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) w$ has a matrix with determinant d, so different (squarefree) d's give inequivalent bilinear forms on $\mathbb{Q}^2$. As bilinear forms on $\mathbb{R}^2$, however, these B_d's are equivalent. Indeed, we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) = C^\top I_2 C$ for $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{smallmatrix}\right)$. Another way of framing that is that, relative to coordinates in the basis $\{(1, 0), (0, 1/\sqrt{d})\}$ of $\mathbb{R}^2$, B_d looks like the dot product B_1.