Category Theory Course

John Baez

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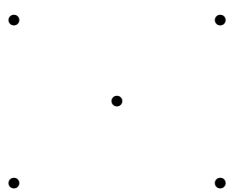
Contents

1 Category Theory:

- Unifies mathematics.

- Studies the mathematics of mathematics (similar to mathematical logic).

- Moves towards higher-dimensional algebra ("homotopifying" mathematics).

Set Theory
0-dimensional
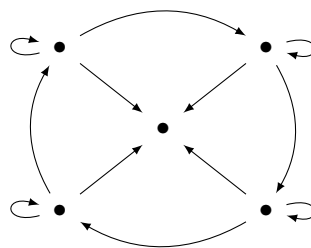
Category Theory
1-dimensional

1.1 Definition of a Category

A *category* **C** consists of:

- A class $Ob(\mathbf{C})$ of *objects*. If $x \in Ob(\mathbf{C})$, we simply write $x \in \mathbf{C}$.

- Given $x, y \in \mathbf{C}$, there's a set $Hom_{\mathbf{C}}(x, y)$, called a *homset*, whose elements are called *morphisms* or *arrows* from x to y. If $f \in Hom_{\mathbf{C}}(x, y)$, we write $f : x \to y$.

- Given $f : x \to y$ and $g : y \to z$, there is a morphism called their *composite* $g \circ f : x \to z$.
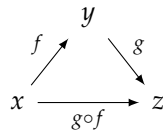
- Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$ if either side is well-defined.
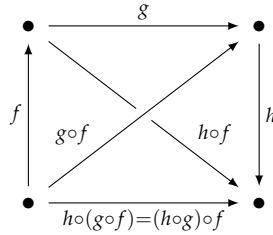
- For any $x \in \mathbf{C}$, there is an *identity morphism* $1_x : x \to x$
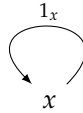
- We have the *left and right unity laws*:

$$1_x \circ f = f \text{ for any } f : x' \to x$$

$$g \circ 1_x = g \text{ for any } g : x \to x'$$

Examples of Categories

1.1.1 Categories of mathematical objects

For any kind of mathematical object, there's a category with objects of that kind and morhpisms being the structure-preserving maps between the objects of that kind.

Example 1.1. **Set** is the category with sets as objects and functions as morphisms.

Example 1.2. Grp is the category with groups as objects and homomorphisms as morphisms.

Example 1.3. For any field k, **Vect**$_k$ is the category with vector spaces over a field k as objects and linear maps as morphisms.

Example 1.4. Ring is the category with rings as objects and ring homomorphisms as morphisms.

These are categories of "algebraic" objects, namely, a set (*stuff*) with operations (*structure*) such that a bunch of equations hold (*properties*), with morphisms being functions that preserve the operations. All this is formalized in "universal algebra", using "algebraic theories". There are also categories of non-algebraic gadgets:

Example 1.5. Top is the category with topological spaces as objects and continuous maps as morphisms.

Example 1.6. Met is the category with metric spaces as objects and continuous maps as morphisms.

Example 1.7. Meas is the category with measurable spaces as objects and measurable maps as morphisms.

1.1.2 Categories as mathematical objects

There are lots of small, managebable categories:

Definition 1.1. A *monoid* is a category with one object.

Remark. Hom$_C(\bullet, \bullet)$ for this object $\bullet$, is a set with associative product and unit.

$$g \circlearrowleft \overset{1_\bullet}{\underset{\bullet}{\curvearrowright}} \circlearrowright f$$

Example 1.8. $1_\bullet \circlearrowleft \bullet \circlearrowright f$

$\circ$	$1_\bullet$	f
$1_\bullet$	$1_\bullet$	f
f	f	$1_\bullet$

The multiplication table above tells us how to compose morphisms. The resulting monoid is usually called $\mathbb{Z}/2\mathbb{Z}$. Now, consider the same diagram but with this multiplication table instead:

$\circ$	$1_\bullet$	f
$1_\bullet$	$1_\bullet$	f
f	f	f

Here we get another famous monoid:

$$
\begin{array}{ccc}
1_\bullet = \text{true} & & 1_\bullet = \text{false} \\
f = \text{false} & \text{or alternatively} & f = \text{true} \\
\circ = \text{or} & & \circ = \text{and}
\end{array}
$$

Definition 1.2. A morphism $f : x \to y$ is an *isomorphism* if it has an *inverse* $g : y \to x$, that is, a morphism with:

$$g \circ f = 1_x$$
$$f \circ g = 1_y$$

If there exists an isomorphism between two objects $x, y \in \mathbf{C}$, we say they're *isomorphic*.

Definition 1.3. A category where all morphisms are isomorphisms is called a *groupoid*.

Example 1.9. "The groupoid of finite sets" is obtained by taking **FinSet**, with finite sets as objects and functions as morphisms, and then throwing out all morphisms except isomorphisms (i.e. bijections).

Definition 1.4. A monoid that is a groupoid is called a *group*.

Remark. the usual "elements" of a group are now the morphisms.

Definition 1.5. A category with only identity morphisms is a *discrete* category.

Remark. So any set is the set of objects of some discrete category in a unique way. So a discrete category is "essentially the same" as a set.
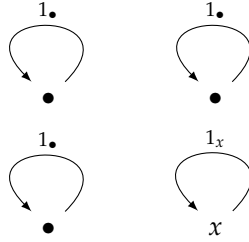
Definition 1.6. A *preorder* is a category with at most one morphism in each homset.

If there is a morphism $f : x \to y$ in a preorder, we say "$x \leq y$"; if not, we say "$x \not\leq y$. For a preorder, the category axioms just say:

- Composition: $x \leq y$ and $y \leq z \implies x \leq z$.

- Associativity is automatic.

- Identities: $x \leq x$ always.

- Left and right unit laws are automatic.

- We're not getting antisymmetry: $x \leq y$ and $y \leq x \implies x = y$.

Definition 1.7. An *equivalence relation* is a preorder that's also a groupoid.

Proposition 1.1. *A preoder is a groupoid if and only if this extra law holds for all* $x, y \in \mathbf{C}$:

$$x \leq y \implies y \leq x$$

Here we have transitivity, reflexivity, and symmetry of "$\leq$". So we usually call this relation $\sim$.

Proposition 1.2. *A preorder is skeletal, i.e. isomorphic objects are equal, if and only if this extra law holds for all $x, y \in \mathbf{C}$:*

$$(x \leq y) \wedge (y \leq x) \implies x = y$$

In this case we say that $\mathbf{C}$ is a *poset*.

Example 1.10. Preorder that is a groupoid but not a poset:

Example 1.11. Preorders that are posets but not groupoids:
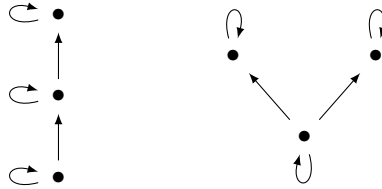
Example 1.12. Preorder that is both a poset and a groupoid:

Since categories can be seen as mathematical objects, we should define maps between them:

Definition 1.8. Given categories $\mathbf{C}$ and $\mathbf{D}$, a *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of:

- a function called F from $Ob(\mathbf{C})$ to $Ob(\mathbf{D})$: if $x \in \mathbf{C}$ then $\mathbf{F}(x) \in \mathbf{D}$.

- functions called F from $Hom_{\mathbf{C}}(x,y)$ to $Hom_{\mathbf{C}}(F(x), F(y))$, for all objects $x, y \in \mathbf{C}$: if $f : x \to y$ then $F(f) : F(x) \to F(y)$

such that:

- $F(g \circ f) = F(g) \circ F(f)$ whenever either side is well defined.

- $F(\mathbf{1}_x) = \mathbf{1}_{F(x)}$ for all $x \in \mathbf{C}$.

So a functor looks like this:
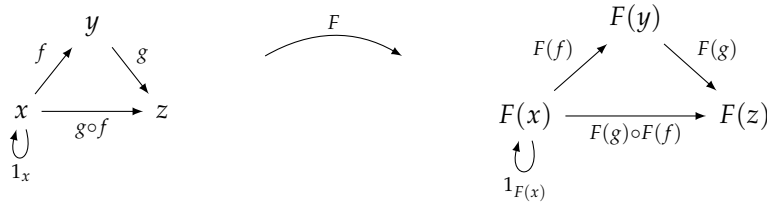
Example 1.13. There's a category called "**1**". It looks like this:

$$1 \underset{\bullet}{\overset{\bullet}{\circlearrowright}}$$

What is a functor $F : \mathbf{1} \to \mathbf{C}$ where $\mathbf{C}$ is any category?

The answer is: "an object in $\mathbf{C}$", since for any object $x \in \mathbf{C}$, there exists a unique functor $F : \mathbf{2} \to \mathbf{C}$ such that $F(\bullet) = x$.

Example 1.14. There's a category called "**2**". It looks like this:

Remark. Also a poset.

$$1_x \circlearrowleft x \xrightarrow{\ f\ } y \circlearrowright 1_y$$

What is a functor $F : \mathbf{2} \to \mathbf{C}$ where $\mathbf{C}$ is any category? It's just a morphism or arrow in $\mathbf{C}$! For any morphism $g : X \to X$ in $\mathbf{C}$, there exists a unique functor $F : \mathbf{2} \to \mathbf{C}$ such that $F(f) = g$.
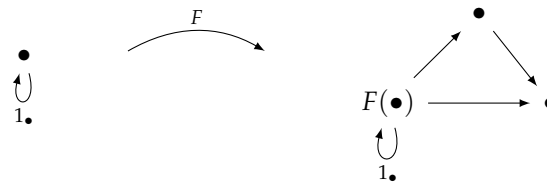
Proposition 1.3. *If $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ are functors, then you can define a functor $G \circ F : \mathbf{C} \to \mathbf{E}$ and $(H \circ G) \circ F = H \circ (G \circ F)$. Also, for any category $\mathbf{C}$ there's an identity functor $\mathbf{1_C} : \mathbf{C} \to \mathbf{C}$ with:*

- $\mathbf{1_C}(x) = x$ for all $x \in \mathbf{C}$
- $\mathbf{1_C}(f) = f$ for all $f : x \to y$ in $\mathbf{C}$
- $F \circ \mathbf{1_C} = F$ for all $F : \mathbf{C} \to \mathbf{D}$
- $\mathbf{1_C} \circ H = H$ for all $H : \mathbf{D} \to \mathbf{C}$

Definition 1.9. **Cat** is the category whose objects are "small" categories and whose morphisms are functors.

Remark. A "small" category is one with a set of objects. For example, **Set** is not a small category because **Set** has a class of objects. **Grp** and **Ring** are also not small categories for the same reason as **Set**. The categories **1** and **2** on the other hand, are small categories.

1.2 Doing Mathematics inside a Category

A lot of math is done inside **Set**, the category of sets and functions. Let's try to generalize all that stuff to other categories by replacing **Set** with a general category **C**.

In **Set**, we have "onto" and "one-to-one" functions. In a category **C**, we generalize these concepts to *epimorphisms* or "*epis*" and *monomorphisms* or "*monos*" respectively.

Definition 1.10. A morhpism $f : X \to Y$ is a *mono* if for all $g, h : Q \to X$ we have:

$$f \circ g = f \circ h \implies g = h$$

$$Q \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

Remark. Also known as being a left-cancellative morhpism

Proposition 1.4. *In* **Set***, a morphism is monic if and only it's a one-to-one function.*

Turning around the arrows in the definition of mono, we get:

Definition 1.11. A morhpism $f : Y \to X$ is a *epi* if for all $g, h : X \to Q$ we have:

$$g \circ f = h \circ f \implies g = h$$

$$Y \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} Q$$

Remark. Also known as being a right-cancellative morhpism

Proposition 1.5. *In* **Set***, a morphism is an epi if and only if it's an onto function.*

Definition 1.12. A morphism $f : X \to Y$ is an *iso* if there exists $f^{-1} : Y \to X$ that's a *left inverse* $f^{-1} \circ f = \mathbf{1}_X$ and a *right inverse* $f \circ f^{-1} = \mathbf{1}_Y$

Proposition 1.6. *In* **Set***,* $f : X \to Y$ *is a mono if and only if it has a left inverse, and an epi if and only if it has a right inverse (using the axiom of choice). Thus, f is an isomorhpism if and only if it is mono and epi.*

Proposition 1.7. *In* **Ring** *(rings and ring homomorphisms)* $f : \mathbb{Z} \to \mathbb{Q}$ $(n \to n)$ *is a mono and an epi, but not an iso. In fact, it has neither a left nor a right inverse.*

Proof. There isn't a ring homorphism $g : \mathbb{Q} \to \mathbb{Z}$, since it would send $\frac{1}{2}$ to some multiplicative inverse of 2. Why is f mono? We need:

$$f \circ g = f \circ h \implies g = h$$

$$R \underset{h}{\overset{g}{\rightrightarrows}} \mathbb{Z} \xrightarrow{f} \mathbb{Q}$$

If $(f \circ g)(r) = (f \circ h)(r) \ \forall r \in R$, since f is one-to-one $g(r) = h(r) \ \forall r$ (as a function), this implies $g = h$. Why is f epi? We need:

$$g \circ f = h \circ f \implies g = h$$

$$\mathbb{Z} \xrightarrow{\ f\ } \mathbb{Q} \underset{h}{\overset{g}{\rightrightarrows}} R$$

The main idea is that any morphism from $\mathbb{Q}$ is completely determined by its values on the integers. We know $g(p) = h(p)$ and $g(q) = h(q)$. So $g(1) = g(\frac{q}{q}) = g(q)g(\frac{1}{q})$, so we can write $g(\frac{1}{q}) = \frac{1}{g(q)}$. So $g(\frac{p}{q}) = g(p)g(\frac{1}{q}) = \frac{g(p)}{g(q)}$. So g (and similarly for h) is determined by its values on the integers; since they agree on $\mathbb{Z}$, they're equal.

$\square$

Puzzle: In **Top**, find $f : X \to Y$ that is epi and mono, but not an iso.

1.3 Limits and Colimits

These are ways of building new objects in a category **C** from diagrams in **C**.

1.3.1 Products

Definition 1.13. Given objects $X, Y \in C$, a *product* of them is an object Z equipped with morphisms, p and q called projections to X and Y.
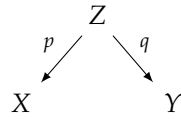
such that for any candidate Q
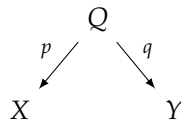
there exists a unique $\psi : Q \to Z$ such that the following diagram commutes

The definition of *coproduct* is just the same but with all arrows reversed.

Proposition 1.8. *In* **Set**, *the product of X and Y, denoted $X \times Y$, is:*

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Proof. Given

$$
\begin{array}{ccc}
 & Q & \\
g \swarrow & & \searrow f \\
X & & Y
\end{array}
$$

Let $\psi : Q \to X \times Y$ be $\psi(q) = (f(q), g(q))$. We indeed get $p \circ \psi = f$, $q \circ \psi = g$, and ψ is the unique map obeying these equations.

$\square$

We could also take as our product any set S that's isomorphic to $X \times Y$, via some iso $\alpha : S \to X \times Y$

$$
\begin{array}{ccc}
 & \alpha & \\
X \times Y & & S \\
p \downarrow & \times & \downarrow q \circ \alpha \\
 & p \circ \alpha \quad q & \\
X & & Y
\end{array}
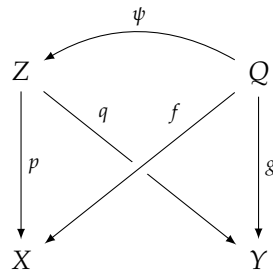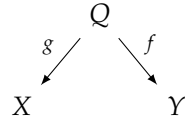$$
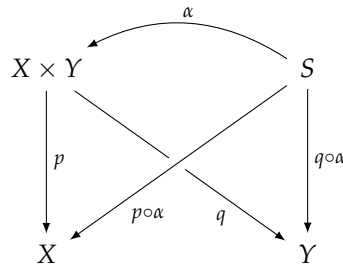
Use $p \circ \alpha$ and $q \circ \alpha$ as projections; then you can check that

$$
\begin{array}{ccc}
 & S & \\
p \circ \alpha \swarrow & & \searrow q \circ \alpha \\
X & & Y
\end{array}
$$
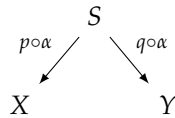
is also a product of X and Y. So "any object isomorphic to a product can also be a product."

Proposition 1.9. *Suppose*

$$
\begin{array}{ccccccc}
 & W & & & & Z & \\
p \swarrow & & \searrow q & and & p \swarrow & & \searrow q \\
X & & Y & & X & & Y
\end{array}
$$
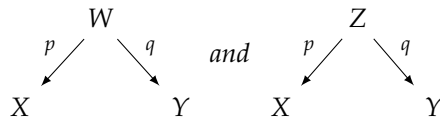
are both a product of X and Y. Then W and Z are isomorphic. That is, products are unique up to isomorphism.

Proof. Since W is a product. There exists a unique $\psi : Z \to W$ making this diagram commute:
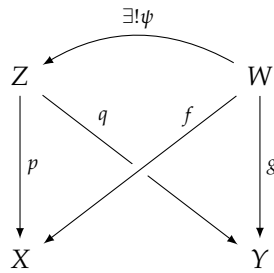
Also, since Z is a product, There exists a unique $\varphi : W \to Z$ making this diagram commute:
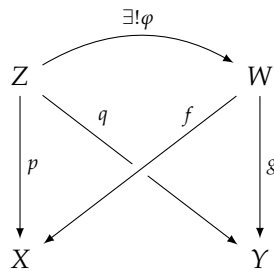
It suffices to show φ and ψ are inverse. Why is $\psi \circ \varphi : W \to W$ the identity? If we can show this, the same argument will show $\varphi \circ \phi = \mathbf{1}_Z$. Since There is a unique arrow making this diagram commute:
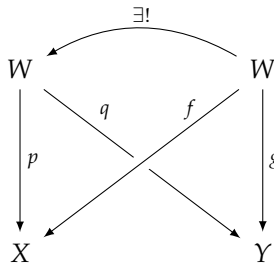
$\mathbf{1}_W : W \to W$ does the job, but so does $\psi \circ \varphi : W \to W$. And so by uniqueness, $\mathbf{1}_W = \psi \circ \varphi$.

$\square$

Proposition 1.10. *If a morphism is an iso, it's both a mono and an epi.*

Remark. We've seen that the converse is false

Proof. If $f : X \to Y$ has a left inverse f^{-1}, it's a mono:

$$f \circ g = f \circ h \implies f^{-1} \circ f \circ g = f^{-1} \circ f \circ h \implies g = h \quad \forall g, h$$

Similarly, If $f : X \to Y$ has a right inverse f^{-1}, it's an epi:

$$g \circ f = h \circ f \implies g \circ f \circ f^{-1} = h \circ f \circ f^{-1} \implies g = h \quad \forall g, h$$

$\square$

Definition 1.14. A morphism with a left inverse is called a *split monomorphism*; a morphism with a right inverse is called a *split epimorphism*.

Remark. In **Set**, every mono (or epi) splits, but we saw that this isn't true in **Ring** or **Top**.

1.3.2 Coproducts

Definition 1.15. Given objects X and Y, a *coproduct* of X and Y is an object Z equipped with morphisms i, j called *inclusions*.

$$
\begin{array}{ccc}
X & & Y \\
& \searrow^{\,i} \;\; \swarrow_{\,j} & \\
& Z &
\end{array}
$$
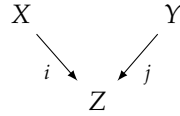
which is *universal*, which means for any diagram of the form:

$$
\begin{array}{ccc}
X & & Y \\
& \searrow^{\,f} \;\; \swarrow_{\,g} & \\
& Q &
\end{array}
$$
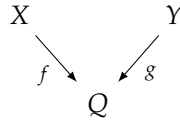
There exists a unique $\psi : Z \to Q$ making the following diagram commute:
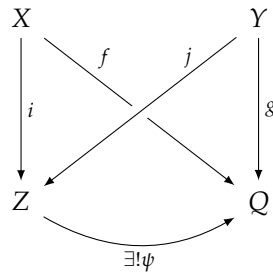
That is, $f = \psi \circ i$ and $g = \psi \circ j$.

Proposition 1.11. *In* **Set**, *a coproduct of X and Y is their disjoint union.*

$$X + Y = X \times \{0\} \sqcup Y \times \{1\}$$

with morphisms:

$$
\begin{array}{ll}
i : X \to X + Y & x \mapsto (x, 0) \\
j : Y \to X + Y & y \to (y, 1)
\end{array}
$$

Category	PRODUCTS $\times$	COPRODUCTS $+$
Set	cartesian product $S \times T$	disjoint union $S \sqcup T$
Top	cartesian product $X \times Y$ with product topology	disjoint union $X \sqcup Y$
Grp	product of groups $G \times H$	free product $G * H$
AbGrp (abelian category)	$A \oplus B$ product of abelian groups	$A \oplus B$
Vect$_k$ (abelian category)	$V \oplus W$ direct sum of vector spaces	$V \oplus W$

The free product $G * H$ consists of equivalence classes of words $x_1 x_2 \dots x_n$ where $x_i \in G \cup H$, with the following relations:

$$x_1 x_2 \dots x_{i-1} 1 x_{i+1} \dots x_n \sim x_1 x_2 \dots x_{i-1} x_{i+1} \dots x_n$$

$$x_1 x_2 \dots x_i x_{i+1} \dots x_n \sim x_1 x_2 \dots x_{i-1} y x_{i+2} \dots x_n$$

where 1 is the identity in G or H, and $x_i, x_{i+1} \in G$ or $x_i, x_{i+1} \in H$, and $y = x_i x_{i+1}$

1.4 General Limits and Colimits

Given any diagram in a category **C**:
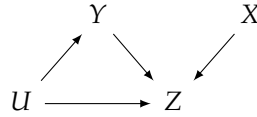
A cone over the diagram is a choice of morphisms from Z to each object in the diagram, such that all newly formed triangles commute:
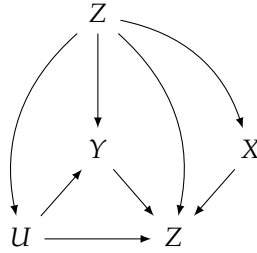
A *limit* of the diagram is a cone that's *universal*, i.e. given any *competitor* Q (another candidate), another cone over the same diagram, there exists a unique $\psi : Q \to Z$ such that all triangles including ψ commute. If U is any object in the diagram and $p : Z \to U$ is the morphism in the universal cone, and $f : Q \to U$ is the morphism in the competitor, then $f = p \circ \psi$
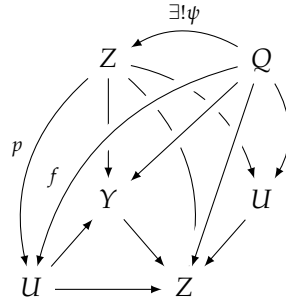
A *cocone* is like a cone but with arrows reversed. A *colimit* is a universal cocone.

Diagrams	LIMITS	COLIMITS
•　　　　•	binary product	binary coproduct
(two arrows between two dots)	equalizer	coequalizer
A　　　B ↓↘ C	pullback	C
C ↙↘ A　　B	C	pushout
•A	A	A
A• ⟶ •B	A	B
	terminal object 1	initial object 0

What's a limit of the empty diagram? It's an object Z such that for all objects Q there exists a unique $\psi : Q \to Z$. This is called a *terminal object*.

- In **Set**, any 1-element set is a terminal object.

- In **Vect**$_k$, any 0-dimensial vector space is a terminal object.

- In **Ring**, the zero ring, which is the unique ring (up to isomorphism) consisting of one element is a terminal object.

Similarly, an *initial object* Z is one such that for any object Q, there exists a unique $\psi : Z \to Q$

- In **Set**, the empty set is an initial object.

- In **Vect**$_k$, any 0-dimensional vector space is an initial object.

- In **Ring**, the ring of integers $\mathbb{Z}$ is an initial object.

In any abelian category, initial objects are terminal and vice-versa.

2   Equalizers, Coequalizers, Pullbacks, and Pushouts (Week 3)

2.1   Equalizers

Definition 2.1. An *equalizer* is a limit of this diagram: $A \underset{g}{\overset{f}{\rightrightarrows}} B$

Proposition 2.1. *In* **Set***, the equalizer of* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *is*

$$q = f \circ p = g \circ p$$

- *with $Z = \{a \in A \mid f(a) = g(a)\}$.*

- *where $p : Z \to A$ has $p(a) = a$ for all $a \in Z$ (It's an inclusion), and q is forced to be $f \circ p = g \circ p$.*

Remark. Since q is determined by p, we usually don't draw it, and write an equalizer like $Z \xrightarrow{p} A \underset{g}{\overset{f}{\rightrightarrows}} B$. Similarly, for lots of other limits and colimits.

Proof. We need to check that this cone is universal, so take a competitor:

$$
\begin{array}{c}
Q \\
\downarrow{\scriptstyle p'} \\
Z \xrightarrow{\;p\;} A \underset{g}{\overset{f}{\rightrightarrows}} B
\end{array}
$$

We want to show there exists a unique $\psi : Q \to Z$ making everything commute: $p \circ \psi = p'$. Since $p(a) = a$ for all $a \in A$, $(p \circ \psi)(q) = \psi(q)$ for all $q \in Q$. Thus, $\psi \circ p = p'$ simply says $\psi(q) = p'(q)$ for all $q \in Q$. Thus, there exists a unique ψ making everything commute, namely $\psi = p'$.

$\square$

Proposition 2.2. *In **Grp**, **AbGrp**, or **Vect**$_k$, the equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is $\ker(f - g)$.*

Remark. $\ker(f - g) = \{a \in A \mid f(a) = g(a)\}$

Proof. The same as before. $\square$

Proposition 2.3. *If $Z \xrightarrow{p} A \underset{g}{\overset{f}{\rightrightarrows}} B$ is an equalizer then p is monic.*

Moral: monics and limits get alone well; epics and colimits do too.

Proof. Assume we have an equalizer. To check that i is monic, we consider:

$$
Y \underset{k}{\overset{h}{\rightrightarrows}} Z \xrightarrow{\;p\;} A \underset{g}{\overset{f}{\rightrightarrows}} B
$$

and show $i \circ h = i \circ k \implies h = k$. Y is a competitor to Z. Since Z is universal, there exists a unique $\psi : Y \to Z$ making everything commute, so $\psi = h = k$.

$\square$

2.2 Coequalizers

Definition 2.2. A *coequalizer* of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a universal cocone over this diagram. i.e.

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{i}{\longrightarrow} Z \qquad \text{(commutes)}$$

s.t. if we have a competitor

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{i}{\longrightarrow} Z$$
$$\downarrow{i'}$$
$$Q$$

there exists a unique $\psi : Z \to Q$ making everything commute.

Proposition 2.4. *In* **Set***, the coequalizer of* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *is* $A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{i}{\longrightarrow} Z$
where $Z = B/\sim$ *where* $\sim$ *is the finest equivalence relation s.t.* $f(a) \sim g(a)$ *for all* $a \in A$ *and i maps b to its equivalence class* $[b]$.

Proof. $i \circ f = i \circ g$ with this definition, so this is a cocone. Why is it universal? Why does there exist a unique $\psi : Z \to Q$ making this diagram commute?

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{i}{\longrightarrow} Z$$
$$\downarrow{i'} \quad \swarrow{\exists!\psi}$$
$$Q$$

To commute, we need:

$$\psi \circ i = i'$$
$$\psi(i(b)) = i'(b) \quad \forall b \in B$$
$$\psi([b]) = i'(b)$$

This shows ψ is unique if it exists; to show it exists, we need to check it's well-defined: If $[b] = [b']$ we need to show $i'(b) = i'(b')$. Since $[b] = [b']$, either $b = b'$, or $f(a) = b$ and $g(a) = b'$ for some $a \in A$. Since $i' \circ f = i' \circ g$ for all $a \in A$, the map is well-defined.

$\square$

Proposition 2.5. *In* **AbGrp** *or* **Vect**$_k$*, the coequalizer of* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *is*
$coker(f - g) = B/im(f - g)$.

Proposition 2.6. *If* $A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{p}{\longrightarrow} Z$ *is a coequalizer, p is epic.*

Proof. Same as proof of the "dual" proposition for equalizers. $\square$

2.3 Pullbacks

Definition 2.3. The limit of this diagram:

$$
\begin{array}{ccc}
 & & B \\
 & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\,f\,} & C
\end{array}
$$

is called a pullback, and denoted:

$$
\begin{array}{ccc}
A \times_C B & \xrightarrow{\,q\,} & B \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\,f\,} & C
\end{array}
$$

The object here, "A times B over C, or the *fibered product*, and we only need to draw its morphisms to A and B called projections. We write:

$$
\begin{array}{ccc}
Z & \longrightarrow & B \\
\downarrow & {\scriptstyle \lrcorner} & \downarrow \\
A & \longrightarrow & C
\end{array}
$$

when Z is a pullback.

Proposition 2.7. *In* **Set***, the pullback of* $A \xrightarrow{\,f\,} C \xleftarrow{\,g\,} B$ *is*

$$
A \times_C B = \{(a,b) \in A \times B \mid f(a) = f(b)\}
$$

with

$$
p : A \times_C B \to A \qquad q : A \times_C B \to B
$$

$$
(a,b) \mapsto a \qquad\qquad (a,b) \mapsto b
$$

Proof. This is clearly a cone: to show it's universal, use the next Prop. □

Proposition 2.8. *Given* $A \xrightarrow{\,f\,} C \xleftarrow{\,g\,} B$ *, if the product $A \times B$ exists and if the equalizer exists:*

$$
\begin{array}{ccc}
Z & & \\
 & \searrow^{\,i} & \\
 & A \times B & \xrightarrow{\,\pi_2\,} & B \\
 & {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\
 & A & \xrightarrow{\,f\,} & C
\end{array}
$$

where $i : Z \to A \times B$ is the equalizer of $A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$ *, then this is a pullback:*

$$
\begin{array}{ccc}
Z & \xrightarrow{\,\pi_2 \circ i\,} & B \\
{\scriptstyle \pi_1 \circ i}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\,f\,} & C
\end{array}
$$

2.4 Pullbacks and Pushouts

Proposition 2.9. *To compute a pullback of* $A \xrightarrow{f} C \xleftarrow{g} B$ *it suffices to take a product of A and B:*

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\pi_2} & B \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{f} & C
\end{array}
$$

and then form the equalizer of: $Z \xrightarrow{i} A \times B \mathrel{\substack{f\circ\pi_1 \\ \longrightarrow \\ \longrightarrow \\ g\circ\pi_2}} C$ *giving the desired pullback:*

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi_2\circ i} & B \\
{\scriptstyle \pi_1\circ i}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{f} & C
\end{array}
$$

Proof. Note the last square commutes since $f \circ \pi_1 \circ i = g \circ \pi_2 \circ i$, so it's a candidate for being the pullback. To show it's universal, consider a competitor:
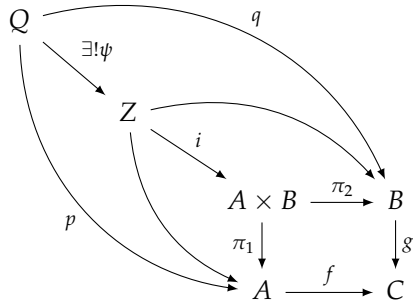
only little square does not commute.

How do we show there exists a unique $\psi : Q \to Z$ making the newly formed triangle commute? By the universal property of the product, we get:
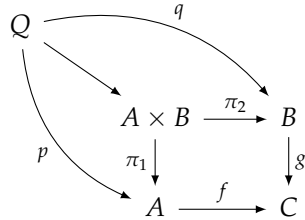
making this commute.

Why is Q a competitor? We need to show $f \circ \pi_1 \circ \psi = g \circ \pi_2 \circ \psi$.

$$
\begin{aligned}
f \circ \pi_1 \circ \psi &= & f \circ p \\
&= & g \circ q \\
&= & g \circ \pi_2 \circ \psi \quad \text{(by various comm. diagrams)}
\end{aligned}
$$

By the universal property of the equalizer, there exists a unique $\psi : Q \to Z$ making this diagram commute:

$$Z \xrightarrow{\ i\ } A \times B \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} C$$

with $\psi : Q \to Z$ and $\varphi : Q \to A \times B$.

In particular, $\varphi = i \circ \psi$. Why does this imply:

1. $\pi_1 \circ i \circ \psi = p$

2. $\pi_2 \circ i \circ \psi = q$

3. a unique ψ making (1) and (2) true.

For (1) and (2), it suffices to show $\pi_1 \circ \psi = p$ and $\pi_2 \circ \varphi = q$, but we already had this by the universal property of the product.

Exercise 1. check (3).

$\square$

"Category theory makes trivial things trivially trivial." - Michael Barr

"I'm content to let them be trivial." - Timothy Gowers

2.5 Limits for all finite diagrams

A category has limits for all finite diagrams if and only if it has:

- products

  $$\bullet \qquad \bullet$$

- equalizers

  $$\bullet \rightrightarrows \bullet$$
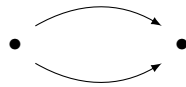
- terminal object 1

Proposition 2.10. *If this is a pullback:*

$$
\begin{array}{ccc}
A \times_C B & \xrightarrow{\ q\ } & B \\
{\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\ f\ } & C
\end{array}
$$

and g is a mono, then p is a mono too.

Proof. Assume g is a mono. Show p is a mono:

$$X \underset{k}{\overset{h}{\rightrightarrows}} A \times_C B \xrightarrow{q} B \qquad \text{Need: } p \circ h = p \circ k \implies h = k$$

with p and g as vertical morphisms, $f: A \to C$.

$$
\begin{aligned}
p \circ h = p \circ k \quad &\implies \quad f \circ p \circ h = f \circ p \circ k \\
&\implies \quad g \circ q \circ h = g \circ q \circ k \qquad \text{(by associativity and commutativity of diagram.)} \\
&\implies \qquad q \circ h = q \circ k \qquad\qquad\qquad \text{(since } g \text{ is mono.)}
\end{aligned}
$$

Note X is a competitor to the pullback:

$$q \circ h = q \circ k$$
$$X \qquad \exists! \psi$$
$$A \times B \xrightarrow{q} B \qquad f \circ p \circ h = g \circ q \circ h = g \circ q \circ k$$
$$p \circ h = p \circ k$$
$$A \xrightarrow{f} C$$

So there exists a unique $\psi : X \to A \times_C B$ making this commute. Both h and k do make it commute, so $h = k$.

$\square$

Proposition 2.11. *Given:*

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & C \\
\downarrow & \mathcal{A} & \downarrow & \mathcal{B} & \downarrow \\
D & \longrightarrow & E & \longrightarrow & F
\end{array}
$$

1. *If $\mathcal{A}$ and $\mathcal{B}$ are pullbacks, so is the combined square $\mathcal{AB}$.*

2. *If $\mathcal{B}$ and $\mathcal{AB}$ are pullbacks, so is $\mathcal{A}$.*

3 Week 4

3.1 Mathematics Between Categories

Recall that given categories $\mathbf{C}$ and $\mathbf{C}$ a *functor* $F : \mathbf{C} \to \mathbf{D}$ is a map sending objects $c \in \mathbf{C}$ to objects $F(c) \in \mathbf{D}$, morphism $f : c \to c'$ in $\mathbf{C}$ to morphism $F(f) : F(c) \to F(c')$ in $\mathbf{D}$ preserving composition $F(f' \circ f) = F(f') \circ F(f)$, and identities $F(1_c) = F(1_{F(c)})$.

There are many "forgetful functor" going from categories of "fancy" mathematical gadgets to categories of less fancy ones, forgetting some extra properties, structure or stuff.
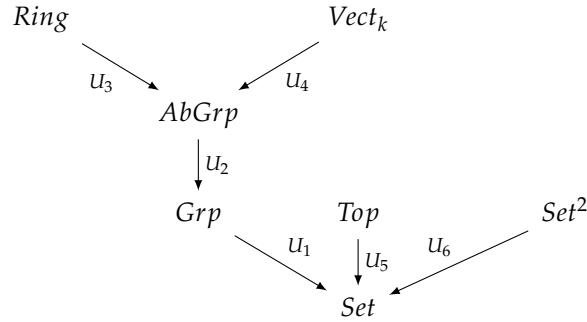
Example 3.1. $U_1 : Grp \to \mathbf{Set}$ sends any group G to its underlying set, and any homomorphism $f : G \to G'$ to its underlying function.

Example 3.2. Given categories $\mathbf{C}$ and $\mathbf{D}$, there's a category $\mathbf{C} \times \mathbf{D}$, where objects are order pairs (c, d) with $c \in \mathbf{C}$, $d \in \mathbf{D}$, and morphism are order pairs (f, g) with f a morphism in $\mathbf{C}$ and g a morphism in $\mathbf{D}$: given $f : c \to c'$ in $\mathbf{C}$ and $g : d \to d'$ in $\mathbf{D}$ then $(f, g) : (c, d) \to (c', d')$. We define $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

In fact $\mathbf{C} \times \mathbf{D}$ is the product of the objects $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, which is the category with

- (small) categories as objects

- functors as morphisms

Among other things this means we have projections
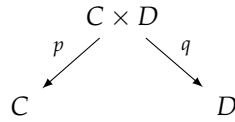
Set is a large category but we can still define $\mathbf{Set}^2 = \mathbf{Set} \times \mathbf{Set}$ with pairs of sets as objects. In the chart, let $U_6 : \mathbf{Set}^2 \to \mathbf{Set}$, $(S, T) \to S$ be the projection onto the first component.

- Functions can be nice in two ways: one-to-one and onto.

- Functors can be nice in three ways:

Definition 3.1. A functor $F : \mathbf{C} \to \mathbf{D}$ is *faithful* if for any $c, c' \in \mathbf{C}$, $F : hom(c, c') \to hom(F(c), F(c'))$ is one-to-one.

Definition 3.2. A functor $F : \mathbf{C} \to \mathbf{D}$ is *full* if for any $c, c' \in \mathbf{C}$, $F : hom(c, c') \to hom(F(c), F(c'))$ is onto.

Definition 3.3. A functor $F : \mathbf{C} \to \mathbf{D}$ is *essentially surjective* if for any $d \in \mathbf{D}$, there exists $c \in \mathbf{C}$ such that $F(c) \cong d$, meaning there exists an isomorphism $g : F(c) \to d$ in $\mathbf{D}$.

Example 3.3. Compare $\mathbf{FinVect}_{\mathbb{R}}$ (finite dimensional vector spaces) to this category $\mathbf{C}$, with

- $\{0\}, \mathbb{R}, \mathbb{R}^2, \dots$ as objects,

- all linear maps between these as morphisms

There is a functor $F : \mathbf{C} \to \mathbf{FinVect}_\mathbb{R}$, defined in objects as

$$\mathbb{R}^n \longmapsto \mathbb{R}^n$$

and similarly for morphisms

$$f : \mathbb{R}^n \to \mathbb{R}^n \longmapsto f : \mathbb{R}^n \to \mathbb{R}^n$$

This is faithfull and full, not surjective on objects, but essentially surjective.

Later we'll define "equivalent" categories and see that if $F : \mathbf{C} \to \mathbf{FinVect}_\mathbb{R}$ is faithfull, full and essentially surjective then $\mathbf{C}$ and $\mathbf{D}$ are equivalent.

Definition 3.4. We say:

- A functor $U : \mathbf{C} \to \mathbf{D}$ *forgets nothing* if it's faithfull, full, and essentially surjective.

- A functor $U : \mathbf{C} \to \mathbf{D}$ *forgets (at most) properties* if it's faithfull and full.

- A functor $U : \mathbf{C} \to \mathbf{D}$ *forgets (at most) structure* if it's faithfull.

- In general we say U *forgets (at most) stuff.*

Example 3.4. $U_1 : \mathbf{Grp} \to \mathbf{Set}$ forgets (at most) structure.
It's faithfull: given $f, f' : G \to G'$ in $\mathbf{Grp}$, $U_1(f) = U_1(f') \Rightarrow f = f'$.
It's not full: there are usually functions $f : U_1(G) \to U_1(G')$ that don't come from group homomorphism, e.g : $f(gh) \neq f(g)f(h)$ or $f(1) \neq 1$.

Example 3.5. $U_2 : \mathbf{AbGrp} \to \mathbf{Grp}$ forgets (at most) properties: the commutative law is forgotten. This is faithfull and also full: if you have any group homomorphism $f : U_2(A) \to U_2(A')$ then $U_2(f') = f$ for some homomorphism of abelian groups $f' : A \to A'$. But it's not esentially surjective, if G is nonabelian, $G \not\cong U_2(A)$ for any $A \in \mathbf{AbGrp}$.

Example 3.6. $U_6 : \mathbf{Set}^2 \to \mathbf{Set}$ forgets stuff: $U_6(S, S') = S$ (it forget the second set in the pair). Technically it is not faithfull: we can have 2 different morphisms $(f, g), (f, g') : (S, S') \to (T, T')$ with $U_6(f, g) = f = U_6(f, g')$.

In our chart, every forgetful functor $U : \mathbf{C} \to \mathbf{D}$ has a "left adjoint" $F : \mathbf{D} \to \mathbf{C}$ which "freely creates" stuff, structure or properties that U forgets.

Example 3.7.
- $F_1 : \mathbf{Set} \to \mathbf{Grp}$ takes a set S and form the free product on S, $F_1(S)$.

- $F_2 : \mathbf{Grp} \to \mathbf{AbGrp}$ *abelianizes* any group G, forming

$$F_2(G) = \frac{G}{< xyx^{-1}y^{-1} >}$$

- $F_6 : \mathbf{Set} \to \mathbf{Set}^2$, $S \to (S, \varnothing)$

To define adjoint functors (and many other things) we need...

3.2 Natural Transformations

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$, we can define a natural transformation $\alpha : F \Rightarrow G$.

$$x \xrightarrow{\ f\ } y \qquad\qquad \overset{F}{\underset{G}{\underset{\Rightarrow}{\alpha}}} \qquad\qquad \begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

Definition 3.5. Given functors $F, G : \mathbf{C} \to \mathbf{D}$ a *transformation* $\alpha : F \Rightarrow G$ is a function sending each object $x \in C$ to a morphism $\alpha_x : F(x) \to G(x)$. We say $\alpha : F \Rightarrow G$ is a *natural transformation* if for each morphism $f : x \to y$ in $\mathbf{C}$ this square commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

Proposition 3.1. *Given categories $\mathbf{C}$ and $\mathbf{D}$ there's a category, the* functor category $\mathbf{D}^{\mathbf{C}}$ *with:*

- *objects being functors $F : \mathbf{C} \to \mathbf{D}$*

- *morphisms being natural transformation $\alpha : F \Rightarrow G$.*

In $\mathbf{D}^{\mathbf{C}}$ we compose $\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$ to get $\beta \circ \alpha : F \Rightarrow H$ as follows: $(\beta \circ \alpha)_x : F(x) \to H(x)$ for all $x \in C$ is given by $\beta_x \circ \alpha_x$.
In $\mathbf{D}^{\mathbf{C}}$ the identity $1_F : F \Rightarrow F$, $(1_F)_x : F(x) \to F(x)$ is given by $1_{F(x)}$.

Proof: We'll check that the compositie $\beta \circ \alpha$ is natural. Given $f : x \to y$ in $\mathbf{C}$, we want the following diagram to commute:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle (\beta\circ\alpha)x}\downarrow & & \downarrow{\scriptstyle (\beta\circ\alpha)_y} \\ H(x) & \xrightarrow[H(f)]{} & H(y) \end{array}$$

We have

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \\ {\scriptstyle \beta_x}\downarrow & & \downarrow{\scriptstyle \beta} \\ H(x) & \xrightarrow{H(f)} & H(y) \end{array}$$

with $(\beta\circ\alpha)_x$ on the left and $(\beta\circ\alpha)_y$ on the right.

Since the top and botton commutes (α and β are natural), the whole diagram commute.∎

Remark. So just as given two sets X and Y, there's a set Y^X of all functions $f : X \to Y$, given two categories, there's a category Y^X of all functors $F : X \Rightarrow Y$.

Given two sets X and Y they have a product:

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

Notice $X \times Y \neq Y \times X$ but if we want to be honest $X \times Y \cong Y \times X$ and there's a specific "good" isomorphism $\alpha_{X,Y} : X \times Y \to Y \times X$, $((x,y) \to (y,x))$.
It's good because it's natural in the sense we just defined.
There are two functors from $\mathbf{Set}^2 \to \mathbf{Set}$,

$$F : (X,Y) \mapsto X \times Y$$

$$G : (X,Y) \mapsto X \times Y$$

and α is a natural transformation from F to G. In fact it's a "natural isomorphism":

Definition 3.6. If $F,G : \mathbf{C} \to \mathbf{D}$ are functors and $\alpha : F \Rightarrow G$ is a natural transformation, we say α is a *natural isomorphism* if $\alpha_x : F(x) \to G(x)$ is an isomorphism for all $x \in C$.

Proposition 3.2. *$\alpha : F \to G$ is a natural isomorphism iff it have and inverse $\alpha^{-1} : G \to F$ in $\mathbf{D}^\mathbf{C}$.*

Proof: Key Idea:$(\alpha^{-1})_x = (\alpha_x)^{-1}$. ∎

Proposition 3.3. *Suppose $\mathbf{C}$ is a category with binary product :any pair of object have a product. Then we can choose, for any pair $x,y \in \mathbf{C}$, a specific product:*

$$X \times Y$$
$$p_{X,Y} \swarrow \qquad \searrow q_{X,Y}$$
$$X \qquad\qquad Y$$

and then there is a functor $X : \mathbf{C}^2 \to C, (X,Y) \mapsto X \times Y$.
In fact there are two functors:

$$X = F : \mathbf{C}^2 \to \mathbf{C}, (X,Y) \mapsto X \times Y$$

$$G : \mathbf{C}^2 \to \mathbf{C}, (X,Y) \mapsto Y \times X$$

and this are naturally isomorphic. We say "products are commutative up to natural isomorphism"

Remark. Also products are associative up to natural isomorphisms.

$$\alpha_{X,Y,Z} : (X \times Y) \times Z \xrightarrow{\sim} (X \times Y) \times Z$$

(Just keep using universal properties of product.)

Definition 3.7. A *cartesian* category is a category with binary products and a terminal object. (I.e. it's a category where any finite set of objects have a product- a finite product category)

One can show that in a cartesian category we have natural isomorphisms.

$$l_X : 1 \times X \xrightarrow{\sim} X.$$

$$r_X : X \times 1 \xrightarrow{\sim} X.$$

All this work similarly in a cat with finite coproducts

$$\beta_{X,Y} : X + Y \xrightarrow{\sim} Y + X.$$

$$\alpha_{X,Y,Z} : (X + Y) + Z \xrightarrow{\sim} X + (Y + Z).$$

$$l_X : 0 + X \xrightarrow{\sim} X.$$

$$r_X : X + 0 \xrightarrow{\sim} X.$$

In case $\mathbf{C} = \mathbf{FinSet}$ (finite sets and functions) this gives laws of arithmetic: $\mathbb{N}$ is the isomorphism clases of objects in $\mathbf{FinSet}$.
Another example:

Example 3.8. A group is a category $\mathbf{G}$ with one object and all morphisms invertible:

$$
\begin{array}{c}
1 = f3 \\
\curvearrowright \\
f2 \,\circlearrowleft\; \bullet \;\circlearrowright f
\end{array}
$$

$$\frac{\mathbb{Z}}{3}$$

What's a functor $F : \mathbf{G} \to \mathbf{Set}$?

$$
\begin{array}{ccc}
\begin{array}{c} 1 = f3 \\ \curvearrowright \\ f2 \,\circlearrowleft\; x \;\circlearrowright f \end{array}
& \xrightarrow{\ F\ }
& \begin{array}{c} F(1) \\ \curvearrowright \\ F(f2) \,\circlearrowleft\; F(x) \;\circlearrowright F(f) \end{array}
\end{array}
$$

$$\qquad\quad \mathbf{G} \qquad\qquad\qquad\qquad\qquad \mathbf{Set}$$

F picks out a set $F(x) = X$ and for each group element f it picks out a function $F(f) : X \to X$ such that $F(ff') = F(f)F(f')$ and $F(1) = 1_X$. So X is a set acted by the group $\mathbf{G}$, or a $\mathbf{G}$-set.

So: a functor $F : \mathbf{G} \to \mathbf{Set}$ is a $\mathbf{G}$-set.

What's a natural transformation between 2 such functors?.

4 Maps Between Categories

4.1 Natural Transformations

4.1.1 Examples of natural transformations

Example 4.1. We saw that a 1-object category $\mathbf{G}$ with all morphisms invertible is a group. We saw that a functor $F : \mathbf{G} \to \mathbf{Set}$ is a $\mathbf{G}$-set:

- a set $F(\bullet)$

- with functions $F(g) : S \to S$ for all $g \in \mathbf{G}$

- such that $F(gg') = F(g) \circ F(g')$ and $F(1_\bullet) = 1_{F(\bullet)}$

Given two functors $F, F' : \mathbf{G} \to \mathbf{Set}$, what is a natural transformation $\alpha : F \Rightarrow F'$? It's called a map of *map of* $\mathbf{G}$-*sets* or $\mathbf{G}$-*equivariant* map, but let's draw one.
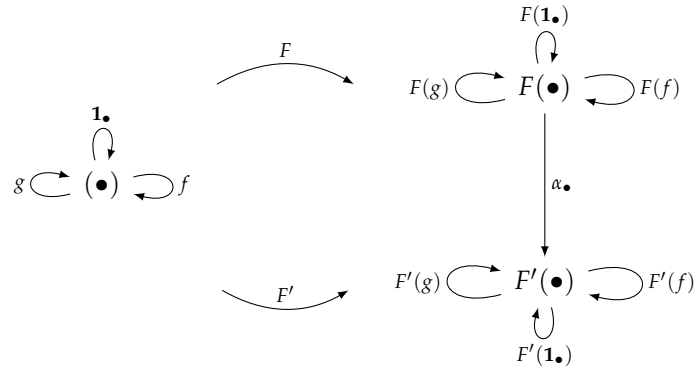
- It's a function $\alpha_\bullet : F(\bullet) \to F'(\bullet)$

- such that for all morhpisms $g \in \mathbf{G}$, we have $F'(g) \circ \alpha_\bullet = \alpha_\bullet \circ F(g)$.

Example 4.2. Two sets are isomorphic if there are functions $F : X \to Y$ and $G : Y \to X$ such that $G \circ F = 1_X$ and $F \circ G = 1_Y$. Given F, when can you find such a G? If and only if F is one-to-one and onto.

4.2 Equivalence of Categories

Definition 4.1. An *equivalence* of categories $\mathbf{C}$ and $\mathbf{D}$ consists of:

- functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$.

- natural transformations $\alpha : G \circ F \Rightarrow \mathbf{1}_X$ and $\beta : F \circ G \Rightarrow \mathbf{1}_X$.

28

We say that F and G are *weak inverses*. We say **C** and **D** are equivalent if there exists an equivalence between them.

Theorem 4.1. *A functor $F : \mathbf{C} \to \mathbf{D}$ is part of an equivalence (F,G,α,β) if and only if F is faithful, full, and essentially surjective. If such a G exists, it may not be unique, but if G' was another one, it's naturally isomorphic to G.*

4.3 Adjunctions

4.3.1 What are adjunctions?

Recall an example:

$U : \mathbf{Grp} \to \mathbf{Set}$ sending each group G to its underlying set $U(G)$.

$F : \mathbf{Set} \to \mathbf{Grp}$ sending each set S to the free group on it $F(S)$.

We say that U is the "right adjoint" of F, or synonymously, F is the "left adjoint" of U. The basic idea is that morphisms from the object $F(S)$ to the object G in **Grp** are in 1-1 correspondence with morphisms from the object S to the object $U(G)$ in **Set**. Given a function $f : S \to U(G)$, we get a homomorphism $\bar{f} : F(S) \to G$, the unique one such that $\bar{f}(s) = f(s)$ for all $s \in S \subseteq F(S)$. And conversely, given a homomorphism $h : F(S) \to G$, we get $\underline{h} : S \to U(G)$ by restricting h to $S \subseteq F(S)$. The usual picture looks like this:

$$
\begin{array}{ccc}
 & S & \\
{\scriptstyle inclusion}\Big\uparrow & \searrow{\scriptstyle f} & \\
F(S) & \xrightarrow[\exists!\bar{f}]{} & G
\end{array}
$$

We prefer to say that there is a bijection $Hom_{\mathbf{Grp}}(F(S), G) \cong Hom_{\mathbf{Set}}(S, U(G))$. Note that F is on the left of $Hom_{\mathbf{Grp}}(F(-), -)$ and G is on the right of $Hom_{\mathbf{Set}}(-, G(-))$. To define adjoint functors, we need to say that this kind of bijection is "natural". What functors give $Hom_{\mathbf{Grp}}(F(S), G)$? They must be two functors from $\mathbf{Set} \times \mathbf{Grp}$ to $\mathbf{Set}$. On objects, these do:

$$(S, G) \to Hom_{\mathbf{Grp}}(F(S), G)$$

$$(S, G) \to Hom_{\mathbf{Set}}(S, U(G))$$

What is the "hom" doing here?

Proposition 4.1. *For any category, there is a functor, called the* hom functor; *Hom* : $\mathbf{C^{op}} \times \mathbf{C} \to \mathbf{Set}$ *which sends each object (X, Y) to the set $Hom_{\mathbf{C}}(X, Y)$*

Remark. Here, $\mathbf{C^{op}}$ is the *opposite* of **C**: the category with one morphism $f^{op} : Y \to X$ for each $f : X \to Y$ in **C**, and $f^{op} \circ g^{op} = (g \circ f)^{op}$ with the same identity morphisms.

Proof. Sketch of proof: We need to define *Hom* : $\mathbf{C^{op}} \times \mathbf{C} \to \mathbf{Set}$ on morphisms. Given a morphism in $\mathbf{C^{op}} \times \mathbf{C}$, $\varphi : (X, Y) \to (X', Y')$. That is, a pair of morphisms: $f^{op} : X \to X'$ in $\mathbf{C^{op}}$ and $g : Y \to Y'$ in **C**. We need to define a morphism, i.e. a function, $Hom(\varphi) : Hom_{\mathbf{C}}(X, Y) \to Hom_{\mathbf{C'}}(X', Y')$ in **Set**.

Given $h \in Hom_{\mathbf{C}}(X, Y)$, what is $Hom(\varphi)(h) \in Hom_{\mathbf{C}'}(X', Y')$? It is $g \circ h \circ f$. Thus, the hom functor $Hom : \mathbf{C^{op}} \times \mathbf{C} \to \mathbf{Set}$ will not only describe hom sets, but also composition in $\mathbf{C}$. Then check it's really a functor: For example, check it preserves composition.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ h\ \ } & Y \\
\uparrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow[g \circ h \circ f = Hom(\varphi)(h)]{} & Y'
\end{array}
$$

$\square$

Given functors $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$, how can we say that the isomorphism $Hom_{\mathbf{D}}(F(X), Y) \cong Hom_{\mathbf{C}}(X, U(Y))$ is natural?

$$
\begin{array}{ccc}
\mathbf{D^{op}} \times \mathbf{D} & \xrightarrow{\ \ Hom\ \ } & \mathbf{Set} \\
\uparrow{\scriptstyle F^{op} \times 1_{\mathbf{D}}} & \overset{\alpha}{\underset{s}{\Longrightarrow}} & \uparrow{\scriptstyle Hom} \\
\mathbf{C^{op}} \times \mathbf{D} & \xrightarrow[1_{\mathbf{C}} \times U]{} & \mathbf{C^{op}} \times \mathbf{C}
\end{array}
$$

4.3.2 Examples of Adjunctions

Let's at first downplay the naturality condition and look at examples focusing on bijections.

Example 4.3. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ sends each group G to its underlying set $U(G)$. The free functor $F : \mathbf{Set} \to \mathbf{Grp}$ sends each set S to the free group on it $F(S)$. Since these two functors form an adjunction between the categories $\mathbf{Grp}$ and $\mathbf{Set}$, we have bijections for every $G \in \mathbf{Grp}$ and $S \in \mathbf{Set}$:

$$
Hom_{\mathbf{Grp}}(F(S), G) \cong Hom_{\mathbf{Set}}(S, U(G))
$$

These bijections let us turn any function $f : S \to U(G)$ into a homomorphism $\bar{f} = \alpha_{S,G}^{-1}(f) : F(S) \to G$. And conversely; any homomorphism $h : F(S) \to G$ comes from a function $\underline{h} = \alpha_{S,G}(h) : S \to U(G)$.

Example 4.4. Does the forgetful functor $U : \mathbf{Vect}_k \to \mathbf{Set}$ sending each vector space V over a field $\mathbb{K}$ to its underlying set $U(V)$ have a left adjoint? Yes, for any set S, there is a vector space $F(S)$ whose basis is S, where the sums are formal expressions:

$$
F(S) = \{ \sum_{s_i \in S} c_i s_i \mid c_i \in \mathbb{K}, \text{ only finitely many nonzero} \}
$$

What does $F : \textbf{Set} \to \textbf{Vect}_k$ do to a morphism $f : S \to T$ in $\textbf{Set}$? It should give a linear map $F(f) : F(S) \to F(T)$. What is it? It is:

$$F(f)(\sum_{s_i \in S} c_i s_i) = \sum_{s_i \in S} c_i f(s_i)$$

Check F is a functor: That is, check that identities $F(g \circ f) = F(g) \circ F(f)$ and $F(1_S) = 1_{F(S)}$ hold.

Exercise 2. Why is the functor F of the last example, left adjoint to U? First, for all $V \in \textbf{Vect}_{\mathbb{K}}$ and $\textbf{Set}$, we need the following bijections to hold (and check they're natural):

$$Hom_{\textbf{Vect}_k}(F(S), V) \cong Hom_{\textbf{Set}}(S, U(V))$$

Given a function $f : S \to U(V)$, we need a linear map $\bar{f} : F(S) \to V$ in some "natural" way. Try $\bar{f}(\sum_{s_i \in S} c_i s_i) = \sum_{s_i \in S} c_i f(s_i)$. Conversely, given a linear map $l : F(S) \to V$, we need a function $\underline{l} : S \to U(V)$. Try $\underline{l}(s) = l(s)$. Check these maps are inverses: $\underline{(\bar{f})} = f$ and $\overline{(\underline{l})} = l$, so that we have a bijection:

$$Hom_{\textbf{Vect}_k}(F(S), V) \cong Hom_{\textbf{Set}}(S, U(V))$$

Example 4.5. To dream up a left adjoint of the forgetful functor $U : \textbf{Top} \to \textbf{Set}$ sending each topological space X to its underlying set $U(X)$, we need to think of ways to turn a set S into a topological space. One way we can do this is to give this set the *discrete toppology*, where you give S as *many* open sets as possible, so every subset is open. Another way we can do this is to give this set the *indiscrete topology*, where you give S as *few* open sets as possible. The left adjoint of $U : \textbf{Top} \to \textbf{Set}$, say $L : \textbf{Set} \to \textbf{Top}$, must have have the following bijections for every $X \in \textbf{Top}$ and $S \in \textbf{Set}$:

$$Hom_{\textbf{Top}}(L(S), X) \cong Hom_{\textbf{Set}}(S, U(X))$$

That is, continuous maps $\bar{f} : L(S) \to X$ are "the same" as functions $f : S \to U(X)$. To make this true, $L(S)$ should have as many open sets as possible, so $L(S)$ is S with the discrete topology. The right adjoint of $U : \textbf{Top} \to \textbf{Set}$, say $R : \textbf{Set} \to \textbf{Top}$, must have have the following bijections for every $X \in \textbf{Top}$ and $S \in \textbf{Set}$:

$$Hom_{\textbf{Set}}(U(X), S) \cong Hom_{\textbf{Top}}(X, R(S))$$

That is, continuous maps $h : X \to R(S)$ are "the same" as functions $\underline{h} : U(X) \to S$. To make this true, $R(S)$ should have as few open sets as possible, so $R(S)$ is S with the indiscrete topology.

4.3.3 Diagonal Functor

Suppose $\textbf{C}$ is any category. There's always a functor $\Delta : \textbf{C} \to \textbf{C} \times \textbf{C}$ called the *diagonal* with:

$$\Delta(X) = (X, X) \text{ for all objects } X \in \textbf{C}$$

$$\Delta(f) = (f, f) : (X, X) \to (Y, Y) \text{ for all objects } X, Y \in \textbf{C}$$

Proposition 4.2. *If* **C** *has binary products, then the functor* $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is the right adjoint of* $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

Remark. In fact, the converse is true: Δ has a right adjoint if and only **C** has binary products, and the right adjoint is $\times$.

Proof. Sketch of proof: For starters, we need bijections for all objects $X, Y, Z \in$ **C**:

$$Hom_{\mathbf{C} \times \mathbf{C}}(\Delta(X), (Y, Z)) \cong Hom_{\mathbf{C}}(X, Y \times Z)$$

since a morphism from (X, X) to (Y, Z) is a pair: $f : X \to Y, g : X \to Z$, for the left side we have:

$$Hom_{\mathbf{C} \times \mathbf{C}}(\Delta(X), (Y, Z)) = Hom_{\mathbf{C} \times \mathbf{C}}((X, X), (Y, Z)) \cong Hom_{\mathbf{C}}(X, Y) \times Hom_{\mathbf{C}}(X, Z)$$

So what we need to show is:

$$Hom_{\mathbf{C}}(X, Y) \times Hom_{\mathbf{C}}(X, Z) \cong Hom_{\mathbf{C}}(X, Y \times Z)$$

Indeed, the universal property of the product says:
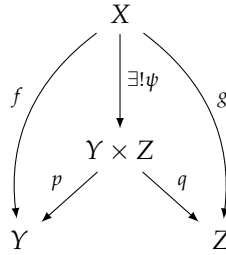
So (f, g) gives ψ and conversely ψ gives $f = p \circ \psi$ and $g = q \circ \psi$, wo we have a bijection:

$$Hom_{\mathbf{C}}(X, Y) \times Hom_{\mathbf{C}}(X, Z) \cong Hom_{\mathbf{C}}(X, Y \times Z)$$

$$(f, g) \longleftrightarrow \psi$$

$\square$

Proposition 4.3. *If* **C** *has binary coproducts, then the functor* $+ : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *is the left adjoint of* $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

Remark. In fact, the converse is true: Δ has a left adjoint if and only **C** has binary coproducts, and the left adjoint is $+$.

Proof. Sketch of proof: For starters, we need bijections for all objects $X, Y, Z \in$ **C**:

$$Hom_{\mathbf{C}}(Y + Z), X) \cong Hom_{\mathbf{C} \times \mathbf{C}}((Y, Z), \Delta(X))$$

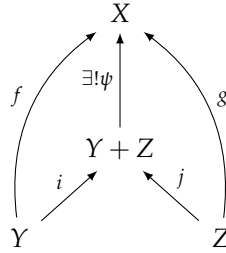since a morphism from (X, X) to (Y, Z) is a pair: $f : X \to Y, g : X \to Z$, for the right side we have:

$$Hom_{\mathbf{C} \times \mathbf{C}}((Y, Z), \Delta(X)) = Hom_{\mathbf{C} \times \mathbf{C}}((Y, Z), (X, X)) \cong Hom_{\mathbf{C}}(Y, X) \times Hom_{\mathbf{C}}(Z, X)$$

So what we need to show is:

$$Hom_{\mathbf{C}}(Y + Z, X) \cong Hom_{\mathbf{C}}(Y, X) \times Hom_{\mathbf{C}}(Z, X)$$

Indeed, the universal property of the coproduct says:

So (f, g) gives ψ and conversely ψ gives $f = i \circ \psi$ and $g = j \circ \psi$, wo we have a bijection:

$$Hom_{\mathbf{C}}(Y + Z, X) \cong Hom_{\mathbf{C}}(Y, X) \times Hom_{\mathbf{C}}(Z, X)$$

$$\psi \longleftrightarrow (f, g)$$

$\square$

A product (an example of a limit) is an example of a right adjoint - it's easy to describe morphisms going into it. A coproduct (an example of a colimit) is an example of a left adjoint - it's easy to describe morphisms going out of it.

5 Diagrams in a Category as Functors

Last time, we saw that if $\mathbf{C}$ has products, the functor $\times : \mathbf{C}^2 \to \mathbf{C}$ is a right adjoint to the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C}^2\ c \mapsto (c, c)$. Similarly, the functor $+ : \mathbf{C}^2 \to \mathbf{C}$, if $\mathbf{C}$ has coproducts, is a left adjoint to Δ. Thus, $\oplus : \mathbf{Vect}_k^2 \to \mathbf{Vect}_k$ is both left and right adjoint to $\Delta : \mathbf{Vect}_{\mathbb{F}}^2 \to \mathbf{Vect}_{\mathbb{F}}$. In fact, if a category has limits, these limits give a right adjoint to some functor:
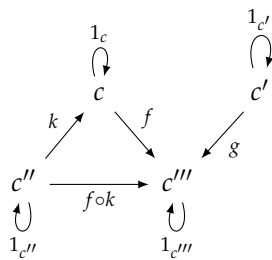
<div align="center">"limits are right adjoints"</div>

<div align="center">"colimits are left adjoints"</div>

We often think about the limit of a diagram in a category $\mathbf{C}$. What's a "diagram in C", really?
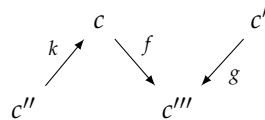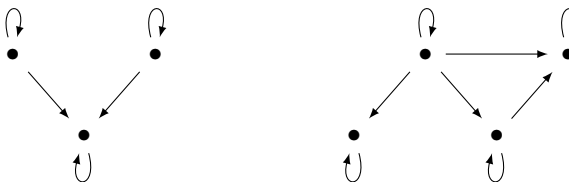

Namely, it's a collection of objects and morphisms between them. We can make it into a subcategory of $\mathbf{C}$:
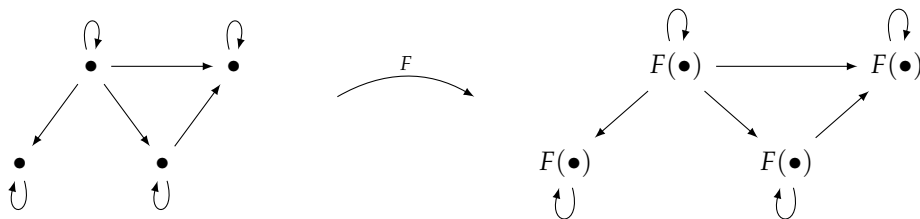
We're often interested in diagrams of some shape, like pullbacks:

These "shapes" can be interpreted as categories:

Let **D** be any category: we'll take this as our "diagram shape". What is a **D**-shaped diagram in some category **C**? It's a functor $F : \mathbf{D} \to \mathbf{C}$:
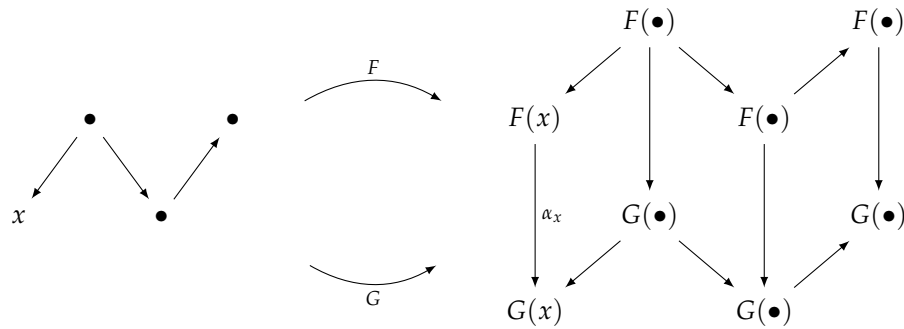

When we take the limit of this diagram, we get an object $limF \in \mathbf{C}$ (defined up to isomorphism). What's the process that takes us from $F : \mathbf{D} \to \mathbf{C}$ to $limF \in \mathbf{C}$? The key is that there is a category $\mathbf{C}^{\mathbf{D}}$ with:

- objects being functors $F : \mathbf{D} \to \mathbf{C}$.

- morphisms being natural transformations α.
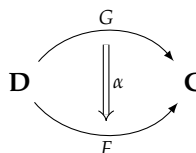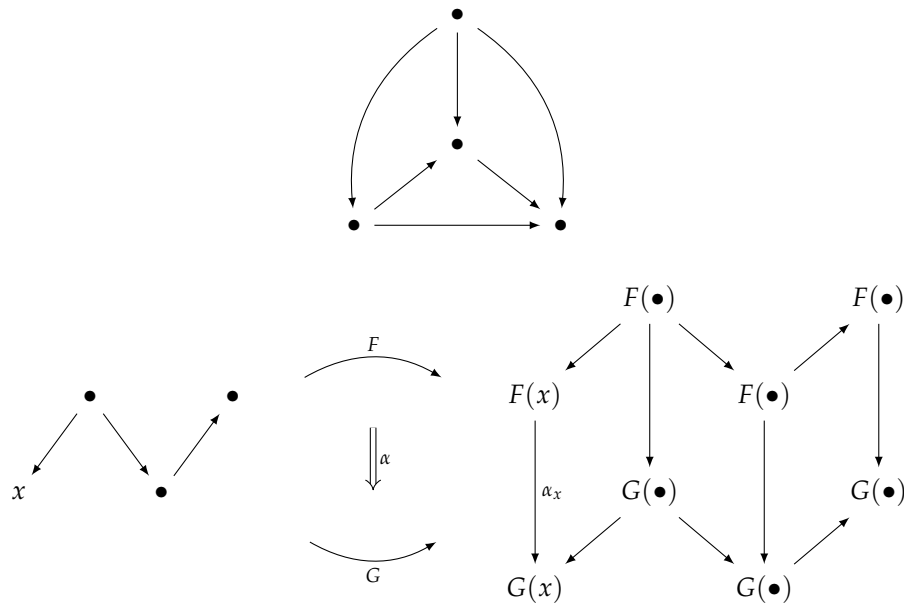

These morphisms look like:

When we take a limit of $F : \mathbf{C} \to \mathbf{D}$, we study dones over F.

Definition 5.1. A *cone* over F is a natural transformation $\alpha : G \to F$ where G sends every object of $\mathbf{D}$ to some object of $\mathbf{C}$, and G sends every morphism of $\mathbf{D}$ to the identity morphism of that object.

Here, $G : \mathbf{D} \to \mathbf{C}$ was determined by the object x via the above recipe. It turns an object $x \in \mathbf{C}$ into an object $G \in \mathbf{C}^{\mathbf{D}}$. So this recipe should be a functor $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$. $\Delta_{\mathbf{C}}(x)$ is the diagram:
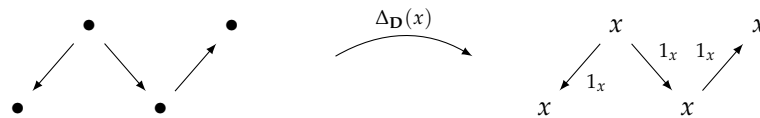
So a cone over F with apex $x \in \mathbf{C}$ is a natural transformation $\alpha : \Delta_{\mathbf{C}}(x) \to F$. What's the limit of a diagram? If $F \in \mathbf{C}^{\mathbf{D}}$, it is a universal cone over that diagram.
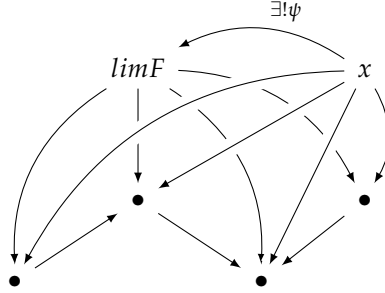
Remember U is the right adjoint of F if:

$$Hom_{\mathbf{D}}(F(x), y) \cong Hom_{\mathbf{C}}(x, U(y))$$

So adjoint functors are about converting one kind of morphisms into another in a bijective way, and that's what we're doing when we're stating the universal property:

- morphisms $\psi : q \to limF$ in $\mathbf{C}$.

- cones over F with apex q, i.e. natural transformations $\alpha : \Delta_{\mathbf{D}}(q) \to F$. (morphisms α from $\Delta_{\mathbf{D}}(q)$ to F in $\mathbf{C^D}$.)

So:

$$Hom_{\mathbf{C^D}}(\Delta_{\mathbf{D}}(q), F) \cong Hom_{\mathbf{C}}(q, limF)$$

So it looks like we have $lim : \mathbf{C^D} \to \mathbf{C}$ which is right adjoint to $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C^D}$. This is true, you need to check that the bijection above is natural to finish the proof of:

Theorem 5.1. *If $\mathbf{C}$ has all limits for $\mathbf{D}$-shaped diagrams, then we have a functor $lim : \mathbf{C^D} \to \mathbf{C}$ which is right adjoint to $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C^D}$. Conversely, if $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C^D}$ has a right adjoint, then this gives limits of $\mathbf{D}$-shaped diagrams in $\mathbf{C}$.*

What choice of $\mathbf{D}$ gives the case of binary products (a special case of limits)?
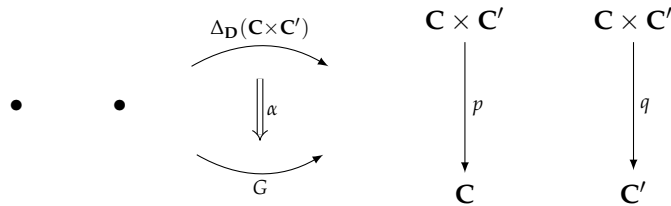
Here, $\mathbf{D}$ has two objects and only identity morphisms, so we could call it **2**, so $\mathbf{C^D} = \mathbf{C^2}$ and $\times : \mathbf{C^2} \to \mathbf{C}$ is right adjoint to $\Delta_{\mathbf{2}} = \Delta : \mathbf{C} \to \mathbf{C^2}$. Similarly,

Theorem 5.2. *If a category $\mathbf{C}$ has colimits of all $\mathbf{D}$-shaped diagrams, there's a functor $colim : \mathbf{C^D} \to \mathbf{C}$ which is left adjoint to $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C^D}$. Conversely, if $\Delta_{\mathbf{D}} : \mathbf{C} \to \mathbf{C^D}$ has a left adjoint, then this gives limits of $\mathbf{D}$-shaped diagrams in $\mathbf{C}$.*

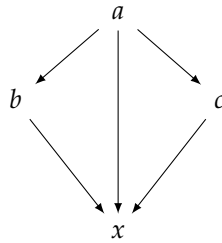Note: $\alpha \in Hom_{\mathbf{C^D}}(F, \Delta_{\mathbf{D}}(q))$ is a cocone:

q
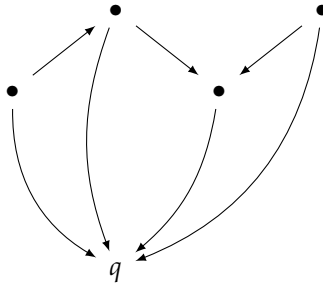
Theorem 5.3. *Left adjoints preserve colimits; right adjoints preserve limits.*

Proof. Sketch of proof:

Let's show that if $F : \mathbf{C} \to \mathbf{D}$ is a left adjoint to $U : \mathbf{D} \to \mathbf{C}$, then F preserves colimits. For concreteness, let's show F preserves pushouts - general case is analogous. So suppose we have a pushout in $\mathbf{C}$:

a

$b \qquad\qquad c$

x

Here, x is the apex of a cocone on the diagram we're taking a colimit of, and the universal property holds. The claim is that applying F to this universal cocone gives a universal cocone in $\mathbf{D}$:

$F(a)$

$F(b) \qquad\qquad F(c)$

$F(x) \xdashrightarrow{\exists!\psi} Q$
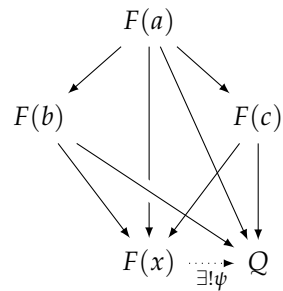
Choose a competitor cocone with apex Q. We need to show $\exists!\psi : F(x) \to Q$ making the newly formed triangle commute. We can look at $U(Q) :\in \mathbf{C}$:
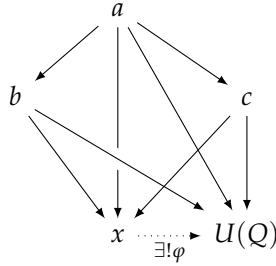
a

$b \qquad\qquad c$

$x \qquad\qquad U(Q)$

37
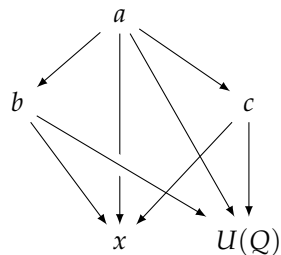
Since F is left adjoint to U, we have:

$$Hom_{\mathbf{D}}(F(x), Q) \cong Hom_{\mathbf{C}}(x, U(Q))$$

So to get $\psi : F(x) \to Q$, let's find $\varphi : x \to U(Q)$. $U(Q)$ becomes a competitor due to the adjointness of F and U, e.g.

$$Hom_{\mathbf{D}}(F(a), Q) \cong Hom_{\mathbf{C}}(a, U(Q))$$

For some reason, the triangles involving $U(Q)$ commute since those involving Q commute. So $U(Q)$ is a competitor. Thus, $\exists! \varphi : x \to U(Q)$ making the newly formed triangles commute.

This gives us $\psi : F(x) \to Q$, check it makes its newly formed triangle commute and is unique (since φ is).

$\square$

Example 5.1. $F : \mathbf{Set} \to \mathbf{Grp}$ preserves colimits, e.g. coproducts, so $F(S + T) \cong F(S) + F(T)$. Here, $S + T$ is the disjoint union of S and T, $F(S + T)$ is the free group with elements of $S + T$ as generators, and $F(S) + F(T) = F(S) * F(T)$ is the "free product" of $F(S)$ and $F(T)$.

Example 5.2. $U : \mathbf{Grp} \to \mathbf{Set}$ preserves limits, e.g. products, so $U(G \times H) \cong U(G) \times U(H)$ where $G \times H$ is the usual product of groups $G \times H$.

Theorem 5.4. *The composite of left adjoints is a left adjoint. The composite of right adjoints is a right adjoint.*

Proof. Suppose we have functors $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{F'} \mathbf{E}$ and F and F' are left adjoint of functors U and U' $\mathbf{C} \xleftarrow{U} \mathbf{D} \xleftarrow{U'} \mathbf{E}$. We'll show that $F' \circ F : \mathbf{C} \to \mathbf{E}$ is the left adjoint of $U \circ U' : \mathbf{E} \to \mathbf{C}$. We want a natural isomorphism:

$$Hom_{\mathbf{E}}(F' \circ F(c)), e) \cong Hom_{\mathbf{C}}(c, U \circ U'(e))$$

Here's how we get it:

$$Hom_{\mathbf{E}}(F' \circ F(c), e) \cong Hom_{\mathbf{D}}(F(c), U'(e)) \quad \text{Since } F' \text{is left adjoint to } U'$$
$$Hom_{\mathbf{D}}(F(c), U'(e)) \cong Hom_{\mathbf{C}}(c, U \circ U'(e)) \quad \text{Since } F \text{ is left adjoint to } U$$

$\square$

Example 5.3. $F' \circ F$ is left adjoint to the forgetful functor $U \circ U'$ from **Ring** to **Set**.
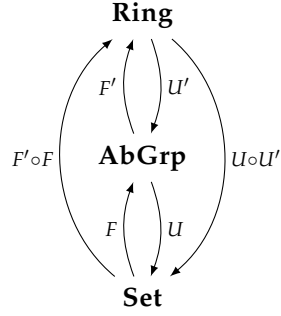
- Starting from the empty set $\varnothing$ (the initial set) we get $F(\varnothing) = \{0\}$ (the trivial abelian group, which is the initial abelian group) and then $F'(F(\varnothing)) = \mathbb{Z}$ (the ring of integers, which is the initial ring).

- Starting from a one-element set $\{x\}$, we get $F(\{x\}) = \{\dots, -x, 0, x, x + x, \dots\} \cong \mathbb{Z}$ and then $F'(F(x)) = \mathbb{Z}[x]$, the ring of polynomials in x with integer coefficients.

5.1 Units and Counits of Adjunctions

Suppose we have $\mathbf{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{D}$ with F left adjoint to U. So that for all $c \in \mathbf{C}$ and $d \in \mathbf{D}$, we have:

$$Hom_{\mathbf{D}}(F(c), d) \cong Hom_{\mathbf{C}}(c, U(d))$$

We can apply this bijection to an identity morphism and get something interesting. We can do this if $d = F(c)$.

$$Hom_{\mathbf{D}}(F(c), F(c)) \xrightarrow[\sim]{\varphi} Hom_{\mathbf{C}}(c, U(F(c)))$$

$$1_{F(c)} \longmapsto \varphi(1_{F(c)})$$

$\varphi(1_{F(c)})$ is called the *unit*, ι_c:

$$\iota_c : c \to U(F(c))$$

We can also apply φ^{-1} to an identity if $c = U(d)$.

$$Hom_{\mathbf{D}}(F(U(d)), d) \xleftarrow[\varphi^{-1}]{\sim} Hom_{\mathbf{C}}(U(d), U(d))$$

$$\varphi^{-1}(1_{F(c)}) \longleftarrow\!\shortmid\ 1_{F(c)}$$

$\varphi^{-1}(1_{U(d)})$ is called the *counit*, ϵ_d:

$$\epsilon_d : F(U(d)) \to d$$

These give various famous morphisms.

Example 5.4.

$$F : \mathbf{Set} \to \mathbf{Grp}$$

$$U : \mathbf{Grp} \to \mathbf{Set}$$

Given any set S, we get a unit:

$$\iota_S : S \to U(F(S))$$

This is the "inclusion of the generators": elements of S are generators of $F(S)$. Given a group G, we get a counit:

$$\epsilon_G : F(U(G)) \to G$$

$$g_1^{\pm 1} * g_2^{\pm 1} * \cdots * g_n^{\pm 1} \qquad \mapsto \qquad g_1^{\pm 1} g_2^{\pm 1} \cdots g_n^{\pm 1}$$

"formal product" in $F(U(G))$.   "actual product" in G.

The counits "convert formal expressions into actual ones".

6   Cartesian Closed Categories

Any category has a *set $Hom(X, Y)$* of morphisms from one object X to another object Y, but in a cartesian closed category (or *ccc*) you also have an object Y^X of morphisms from X to Y.

Example 6.1. If $\mathbf{C} = \mathbf{Cat}$, $Hom_{\mathbf{C}}(\mathbf{X}, \mathbf{Y})$ is the set of functors $F : \mathbf{X} \to \mathbf{Y}$, while $\mathbf{Y}^{\mathbf{X}}$ is the category of functors $F : \mathbf{X} \to \mathbf{Y}$ and natural transformations between them. In general, you can get $Hom_{\mathbf{C}}(\mathbf{X}, \mathbf{Y})$ from $\mathbf{Y}^{\mathbf{X}}$ but not vice versa. We call $Hom_{\mathbf{C}}(\mathbf{X}, \mathbf{Y})$ the homset or *external* hom (it lives outside of $\mathbf{C}$, in $\mathbf{Set}$), and $\mathbf{Y}^{\mathbf{X}}$ the exponential or *internal hom* (since it lives inside $\mathbf{C}$).

Internalization is the process of taking math that lives in $\mathbf{Set}$ and moving it into some category $\mathbf{C}$.

Example 6.2. In $\mathbf{Set}$ you can define a group to be an object $G \in \mathbf{Set}$ with morphisms:

$m : G \times G \to G$ Multiplication
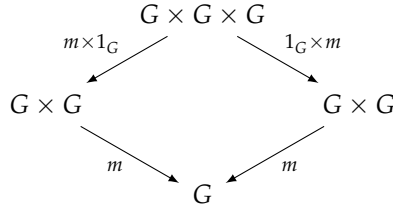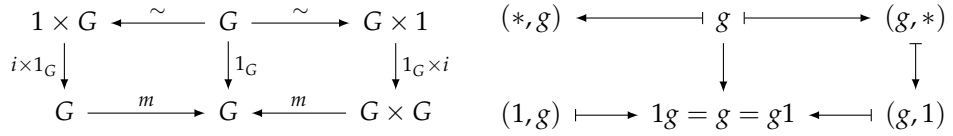
$inv : G \to G$ Inverses

$i : 1 \to G$ The identity-assigning map.
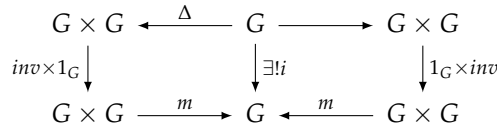It maps the one element of 1 to the identity element in G.

associative law:

$$G \times G \times G$$

$$G \times G \xleftarrow{m \times 1_G} \qquad \xrightarrow{1_G \times m} G \times G$$

$$G \times G \xrightarrow{m} G \xleftarrow{m} G \times G$$

left and right unit laws:

$$1 \times G \xleftarrow{\sim} G \xrightarrow{\sim} G \times 1 \qquad (*,g) \longleftarrow\!\shortmid\; g \;\shortmid\!\longrightarrow (g,*)$$

$$\downarrow{\scriptstyle i \times 1_G} \qquad \downarrow{\scriptstyle 1_G} \qquad \downarrow{\scriptstyle 1_G \times i}$$

$$G \xrightarrow{m} G \xleftarrow{m} G \times G \qquad (1,g) \longmapsto 1g = g = g1 \longleftarrow\!\shortmid (g,1)$$

inverse laws:

$$G \times G \xleftarrow{\Delta} G \longrightarrow G \times G$$

$$\downarrow{\scriptstyle inv \times 1_G} \qquad \downarrow{\scriptstyle \exists! i} \qquad \downarrow{\scriptstyle 1_G \times inv}$$

$$G \times G \xrightarrow{m} G \xleftarrow{m} G \times G$$

All these diagrams make sense in any cartesian category (=category with finite products = category with binary products and terminal object). So we can define a *group internal to* **C** or *group in* **C** using these axioms whenever **C** is cartesian. For example:

- If **C** = **Top**, a group in **C** is called a *topological group*.

- If **C** = **Diff**, a group in **C** is called a *Lie group*.

- If **C** is the category of *algebraic varieties*, a group in **C** is called an *algebraic group*.

Puzzle: If **C** = **Grp**, a group in **C** is a very famous thing. What is it?

6.1 Evaluation and Coevaluation in Cartesian Closed Categories

Recall a cartesian category **C** is a *ccc* if for any $Y \in$ **C**, the functor $- \times$ **C** has a right adjoint:

$$Hom_{\mathbf{C}}(X \times Y, Z) \cong Hom_{\mathbf{C}}(X, Z^Y)$$

Any adjunction $\mathbf{C} \overset{F}{\underset{U}{\rightleftarrows}} \mathbf{D}$ has a unit and counit:

$$\iota_X : X \to UFX \qquad X \in \mathbf{C}$$

$$\epsilon_Y : FUY \to Y \qquad Y \in \mathbf{D}$$

Now we have an adjunction $\mathbf{C} \overset{- \times Y}{\underset{-^Y}{\rightleftarrows}} \mathbf{C}$

$$\iota_X : X \to (X \times Y)^Y$$

$$\epsilon_X : X^Y \times Y \to X$$

$$X \in \mathbf{C}$$

The second one is called *evaluation*: in **Set**

$$\epsilon_X : X^Y \times Y \to X$$

$$(f, y) \mapsto f(y)$$

The first one is called *coevaluation*: in **Set**

$$\iota_X : X \to (X \times Y)^Y$$

$$i_x(x)(y) = (x, y)$$

So we have analogous of these in any ccc.

6.1.1 Internalizing Composition

In any category, we have composition:

$$\circ : Hom(Y, Z) \times Hom(X, Y) \to Hom(X, Z)$$

$$(f, g) \longmapsto f \circ g$$

In a ccc, we can internalize this and define "internal composition":

$$\bullet : Z^Y \times Y^X \to Z^X$$

$$\bullet \in Hom(Z^Y \times Y^X, Z^X) \cong Hom(Z^Y, (Z^X)^{(Y^X)}) \cong Hom(Z^Y \times Y^X \times X, Z)$$

So we get $\bullet$ form a morphism:

$$\tilde{\bullet} : Z^Y \times Y^X \times X \to Z$$

which we indeed have in any ccc:

$$Z^Y \times Y^X \times X \xrightarrow{1_{Z^Y} \times \epsilon} Z^Y \times Y \xrightarrow{\epsilon} Z$$

This is just an internalized way of saying the old definition of composition:
$(f \circ g)(x) = f(g(x))$

Emily Riehl, *Categories in Context*, Dover Pub. free on the web.

6.2 Elements

Sets have elements, but what about objects in other categories? Elements of a set X are in $1 - 1$ correspondence with functions $f : 1 \to X$, where 1 is a terminal object in **Set** (1 = a one element set). So:

Definition 6.1. If **C** is a category with a terminal object, an *element* of an object $X \in \mathbf{C}$ is a morphism $1 \to X$. We define the set $elt(X)$ to be $Hom(1, X)$.

Example 6.3. If $\mathbf{C} = \mathbf{Top}$, elt(X) = {continuous maps $f : \{*\} \to X$, where $\{*\}$ is the one-point space, i.e. the terminal object in **Top**}. In fact, elt(X) is in $1 - 1$ correspondence with the underlying set of X:

Given $x \in X$, $f : \{*\} \to X$ where $* \mapsto x$, and conversely any such $f(*) \in X$.

Example 6.4. If $\mathbf{C} = \mathbf{Grp}$, elt(G) = {homomorphisms $f : 1 \to G$, where 1 is the trivial group, i.e. the terminal object in **Grp**}. So elt(G) has just one element: there's just one homomorphism $f : 1 \to G$, since 1 is also initial!

Example 6.5. If $\mathbf{C} = \mathbf{Cat}$, elt(**D**) = {functors $f : 1 \to \mathbf{D}$, where 1 is the terminal category in **Cat**}. functors $f : 1 \to \mathbf{D}$ are in $1 - 1$ correspondence with the objects of **D**. So elt(**D**) $\cong$ {objects in **D**}
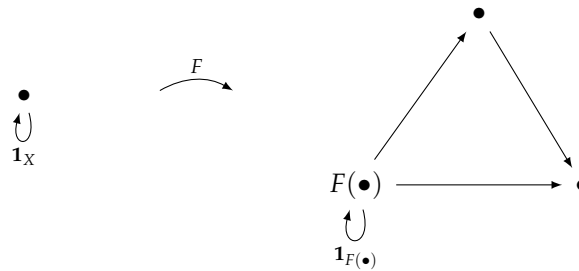
Here, as in the previous example, elt forgets a lot of information:

7 Week 9

Proposition 7.1.

Suppose **C** is a category with terminal object $1 \in \mathbf{C}$. Then there's a functor $elt : \mathbf{C} \to \mathbf{Set}$ with
$$elt(X) = Hom(1, X), \forall X \in \mathbf{C}$$
and given any morphism $g : X \to Y in \mathbf{C}$, $elt(g) : elt(X) \to elt(Y)$ is defined as follows:

Proof: elt preserve composition: given $X \xrightarrow{g} Y \xrightarrow{h} Z$ we need

$$elt(h \circ g) = elt(h) \circ elt(g)$$

$$
\begin{array}{ccc}
1 & \xrightarrow{f} & X \\
& & \downarrow{g} \\
& & Y \\
& & \downarrow{h} \\
& & Z
\end{array}
$$

Given $f \in elt(X)$ we have

$$
\begin{aligned}
elt(h \circ g)f &= (h \circ g) \circ f \\
&= h \circ (g \circ f) \\
&= h \circ (elt(g)f) \\
&= elt(h)(elt(g)(f))
\end{aligned}
$$

Similarly $elt(1_x)f = 1_x \circ f = f$, for all $f \in elt(X)$. So $elt(1_x) = 1_{elt(X)}$. ∎

Example 7.1. $elt : \mathbf{C} \to \mathbf{Set}$ may not be faihtfull, i.e we can have two different morphisms $g, g' : X \to Y$ in $\mathbf{C}$ with $elt(g) = elt(g')$.
If $\mathbf{C} = \mathbf{Grp}$, we saw $elt(G) = 1 \in \mathbf{Set}$ for all G, so any homomorphism $h : G \to G'$ will be get sent to a function $elt(h) : 1 \to 1$, but there's only one of these.

Proposition 7.2. *If $\mathbf{C}$ is a cartesian category* $elt : \mathbf{C} \to \mathbf{Set}$ *preserve finite products.*

Proof: If easy to show elt preserve the terminal object: if $1 \in \mathbf{C}$ then $elt(1) = \{f : 1 \to 1\}$ is one-element set, so it's terminal in $\mathbf{Set}$.
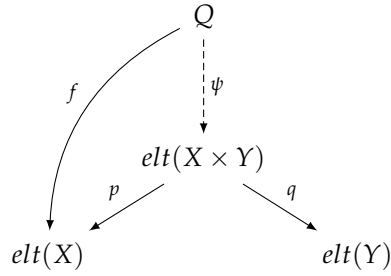
Why does elt preserve binary products?
Suppose $X, Y \in \mathbf{C}$, then their product is a universal cone

$$
\begin{array}{ccc}
& X \times Y & \\
{}^{p}\swarrow & & \searrow^{q} \\
X & & Y
\end{array}
$$

To show elt preserve products, we need this cone is universal in $\mathbf{Set}$:

$$
\begin{array}{ccc}
& elt(X \times Y) & \\
{}^{elt(p)}\swarrow & & \searrow^{elt(q)} \\
elt(X) & & elt(Y)
\end{array}
$$

Choose a competitor:

$$Q$$

Want $\exists!\phi : Q \to elt(X \times Y)$ making the newly formed triangles commute. $f : Q \to elt(X)$ sends any $a \in Q$ to a point $f(a) \in elt(X) = h : 1 \to X$, so $f(a) : 1 \to X$. Similarly $g(a) : 1 \to Y$. We want to define $\psi : Q \to elt(X \times Y)$; this will send any $a \in Q$ to $\psi(a) : 1 \to X \times Y$.

By the universal property of $X \times Y$, for each $a \in Q$ $\exists!\psi(a) : 1 \to X \times Y$ so that this commutes
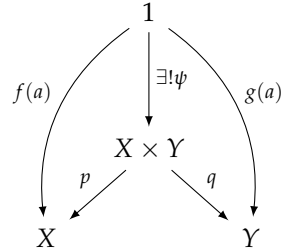
Define ψ this way, check that (*) commutes, and moreover (*) commuting forces us to choose this ψ, so ψ is unique. ∎

What if **C** is a ccc?

then

$$
\begin{aligned}
hom(X, Y) &\cong hom(1 \times X, Y) \\
&\cong hom(1, Y^X) \\
&= elt(Y^X)
\end{aligned}
$$

Since $1 \times X \cong X$ so:

give us a bijection

$$
\begin{aligned}
hom(X \times Y) &\cong hom(1 \times X, Y) \\
f &\longrightarrow f \circ \alpha \\
g \circ \alpha^{-1} &\longleftarrow g
\end{aligned}
$$

The moral: we can convert the hom-object $Y^X \in \mathbf{C}$ into the hom-set $hom(X, Y) \in$ **Set** by taking elements.

Given $f : X \to Y$ in $hom(X, Y)$ we can convert it into an element of Y^X called the *name* of f: $\ulcorner f \urcorner : 1 \to Y^X$.

Conversely, any elemnet of Y^X is the name of a unique morphism $f : X \to Y$. In functional programming, objects are data types, morphisms are programs and any program $f : X \to Y$ have a name $\ulcorner f \urcorner \in elt(Y^X)$.

7.1 Subobjects

Definition 7.1. In a category **C** is an equivalence class of monomorphisms $i : A \to X$, where monos $i : A \to X$, $j : B \to X$ are equivalent if there's an isomorphisms $f : A \to B$ so that this commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & X \\
{\scriptstyle f}\downarrow & \nearrow & \\
B & {\scriptstyle j} &
\end{array}
$$

Example 7.2. If $\mathbf{C} = \mathbf{Set}$, subobjects of $X \in \mathbf{Set}$ corresponds to subsets of X. Given a monomorphism $i : A \to X$ we get a subset $im(i) \subset X$. Any subset $S \subset X$ arise in this way via the inclusion:

$$i : S \to X$$

$$s \to s \in X$$

this has $im(i) = S$.

Finally, given monos $i : A \to X$ and $j : B \to X$ that define the same subset $im(i) = im(j)$, then there exists a bijection $f : A \to B$ so that

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & X \\
{\scriptstyle f}\downarrow & \nearrow & \\
B & {\scriptstyle j} &
\end{array}
$$

coomutes, namely $f = (j|_{im(j)})^{-1} \circ i$.

Example 7.3. In Graph,how many subobjects does this graph?: $v\bullet \overset{e}{\underset{e'}{\rightrightarrows}} \bullet w$ Here
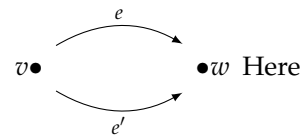
they are
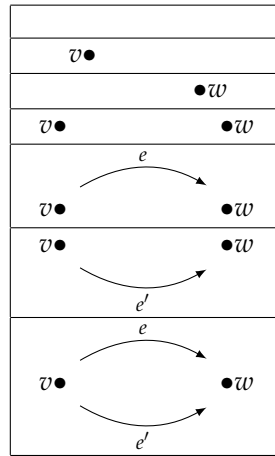
Any object give a subobject of itself:$1_X : X \to X$ is a monomorphism.

(A graph is a pair of functions $E \overset{s}{\underset{t}{\rightrightarrows}} V$

Proposition 7.3. *In* **Set***, subobjects of $S \in$* **Set** *are in 1-1 correspondence with functions $X : S \to 2$ where $2 = \{F, T\}$.*

Proof: Subobjects of S are just subsets $A \subset S$.

Any such subset has a **characteristic function** $X : S \to 2$ given by

$$\chi(s) = \begin{cases} F & s \notin A \\ T & s \in A \end{cases}$$

Conversely, given $\chi : S \to 2$,let $A = \chi^{-1}(T) = \{s \in S : X(s) = T\}$ ∎

Roughly, a "subobject classifier" in a category **C** is an object $\Omega \in$ **C** that plays the role of $2 = \{F, T\}$, in that subobjects of any subset $S \in$ **C** are going to be in 1-1 correspondence with morphisms $\chi : S \to \Omega$.
Set has the "subobject classifier" $2 = \{F, T\}$. What does this really means?.
First,there's a function called true:$t : 1 \to 2$ from $1 = \{*\}$ to 2 given by $t(*) = T \in 2$.
For any set A there's a unique function $!_A : A \to 1$ since 1 is terminal.
I claim that for any monomorphism $i : A \to X$ (that is a 1-1 function), there exists a unique function
$$S : X \to 2$$

called the *characteristic function* of i, such that:

$$\begin{array}{ccc} E & \overset{!A}{\longrightarrow} & 1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\ E & \underset{Xi}{\longrightarrow} & 2 \end{array}$$
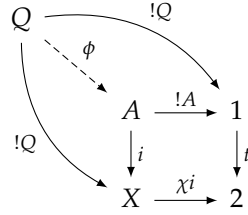
is a pullback.

χ_i,in more familiar terms, will be the characteristic function of the subset $im(i) \subset X$, but we call it the characteristic fucntion of the monomorphism i.

First Let's show that this χ_i :

$$\chi_i(x) = \begin{cases} T & x \in im(i) \\ F & x \notin im(i) \end{cases}$$
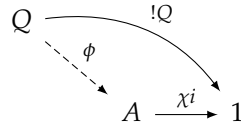
Let Q be a competitor

Then show $\exists!\psi : Q \to A$ making the newly formed triangles commute. Since Q is a competitor:

$$\begin{aligned} \chi_i(f(q)) &= t(!_Q(q)), q \in Q \\ &= t(*) \\ &= T \end{aligned}$$

$\Rightarrow$ (using the definition of χ_i) $f(q) \in im(i)$.

So since i is one-to-one, for each $q \in Q$, $\exists!a \in A$ with $f(q) = i(a)$. So define $\phi : Q \to A$ by $\phi(q) = a$. This makes $f = ß \circ \psi$ and it's the unique $\phi : Q \to A$ that does so (since i is one-to-one).

The other newly formed triangle automatically commutes:

you can also check that $\chi_i : X \to 2$ is the unique morphism from X to 2 that makes the square a pullback.

So generalizing:

Definition 7.2. Given a category **C** with a terminal object, a *subobject classifier* is an object $\Omega \in$ **C** with a morphism $t : 1 \to \Omega$ such that : for any monomorphism $i : A \to X$ there exists a unique $\chi_i : X \to \Omega$ such that this square is a pullback:


Definition 7.3. A *(elementary) topos* is a cartesian closed category with finite limits (limits of finite sized diagrams) and a subobject classifier.

Grothendieck in the 1960's introduced a concept of topos, now **Grothendieck topos**, which is a special case of alementary topos, as part of proving the Weil hypothesis in number theory. Later in the late 60's and early 70's Lawrence and Trerney simplified the concept of topos to define an "elementary topos".

Example 7.4. Examples of elementary topos

1. **Set**: category of sets and functions.

2. **FinSet**: category of finite sets and functions, this doesn't have all limits only finite limits, so topos theory includes finitest mathematics.

3. **Set$'$**: category of sets and functions as defined using ZF=Zermelo-Fraenkel axioms without axiom of choice.
   The axiom of choice is aquivalent to: there exists a monomorphisms $i : A \to X$ so that $p \circ i = 1_A$. If this if true we say the epimorphism splits. In a general topos, not every epimorphisms splits so the axiom of choice need not hold.

4. **Graphs**: The category of graphs:

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$
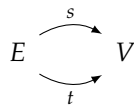
5. Previous example is ana special case of a category **Set$^\mathbf{C}$**, where **C** is any category. These are called *presheaf categories* when we write them as **Set$^{\mathbf{D}^{op}}$** (eg.**D** = **C**^{op} so **D**^{op} = **C**)

If **C** $= 1_x \circlearrowleft x \overset{f}{\underset{g}{\rightrightarrows}} y \circlearrowright 1_y$ then **Set$^\mathbf{C}$** = **Graph**
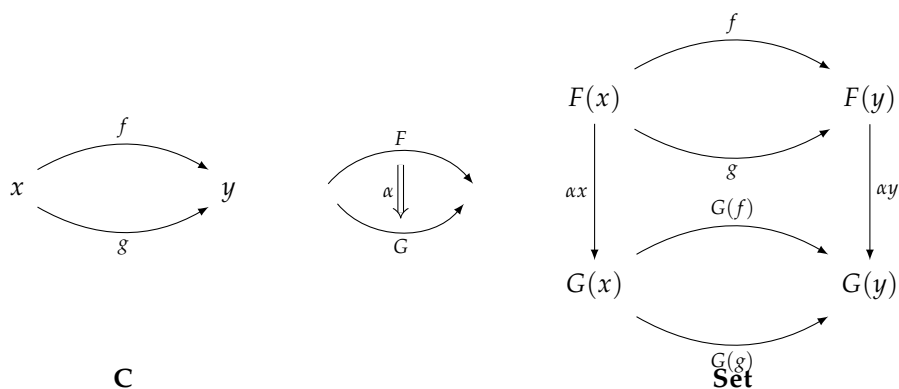
A functor $F : \mathbf{C} \to$ **Set** is a graph with $E = F(x), V = F(y), s = F(f), t = F(g)$. So a graph is an object in **Set$^\mathbf{C}$**. Similarly, a morphism in **Set$^\mathbf{C}$** is a morphism between graphs.

6. Another example of a presheaf category is the category of simplicial sets:
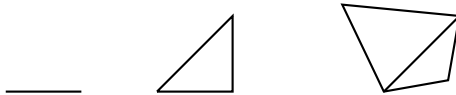
These are fundamental to algebraic topology.

7. Presheaf categories are closely connected to categories of sheaves, which are also topoi. Sheaves are fundamental to algebraic geometry.

8 Symmetric Monoidal Categories

8.1 Guest lecture by Christina Osborne

> A category theorist is sort of like a sociologist. He looks at mathematical objects - he doesn't pry it open and see how it works - but sees how it behaves in relation to all other things.
>
> - Chris Heunen

8.1.1 What is a Monoidal Category?

Definition 8.1. A *monoid* is a nonempty set G together with a binary operation on G which is:

- associative: $(xy)z = x(yz) \quad \forall\, x, y, z \in G$

- and contains a (two-sided) identity element $e \in G$ such that $xe = ex = x$ $\forall\, x, y, z \in G$

Remark. i.e. take the definition of a group and drop the requirement of inverses

Definition 8.2. A *monoidal category* is a category $\mathbf{C}$ which is equipped with:

1. A tensor product functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ where the image of a pair of objects (x, y) is denoted by $x \otimes y$.

2. A *unit object I*.

3. For every $x, y, z \in Ob(\mathbf{C})$, and associativity isomorphism $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$, natural in the objects x, y, and z.

4. For every $x \in Ob(\mathbf{C})$, a left unit isomorphism $\ell_x : I \otimes X \to X$ and a right unit isomorphism $r_x : x \otimes I \to x$, both natural in x.

We further assume the following diagrams commute for any objects w, x, y, and z:

- the **pentagon identity**:

$$((w \otimes x) \otimes y) \otimes z$$

$$a_{w,x,y} \otimes id_z$$

$$a_{w \otimes x, y, z}$$

$$(w \otimes (x \otimes y)) \otimes z \qquad\qquad (w \otimes x) \otimes (y \otimes z)$$

$$a_{w, x \otimes y, z}$$

$$a_{w, x, y \otimes z}$$

$$w \otimes ((x \otimes y) \otimes z) \xrightarrow{\ \ id_z \otimes a_{x,y,z}\ \ } w \otimes (x \otimes (y \otimes z))$$

- the **triangle identity**:

$$(x \otimes I) \otimes y \xrightarrow{\ a_{x,I,y}\ } x \otimes (I \otimes y)$$

$$r_x \otimes id_y \qquad\qquad id_x \otimes \ell_y$$

$$x \otimes y$$

Remark. When we want to emphasize the tensor product and unit, we denote a monoidal category by $(\mathbf{C}, \otimes, I)$.

Example 8.1. $(\mathbf{Set}, \times, \{\bullet\})$

Example 8.2. $(\mathbf{Set}, \coprod, \{\varnothing\})$

Example 8.3. $(\mathbf{Grp}, \times, \{e\})$

Example 8.4. $(\mathbf{Hilb}, \otimes, \mathbb{C})$, where the category **Hilb** has Hilbert spaces as objects and short linear maps (linear maps of norm at most 1) as morphisms.

Why is $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ an isomorphism and not an equality? Let's consider the example $(\mathbf{Set}, \times, \{\bullet\})$:

$$(X \times Y) \times Z = \{(w, z) \mid w \in X \times Y,\ z \in Z\} = \{((x, y), z) \mid x \in X,\ y \in Z,\ z \in Z\}$$

$$X \times (Y \times Z) = \{(x, w) \mid x \in X,\ w \in Y \times Z = \{(x, (y, z)) \mid x \in X,\ y \in Y,\ z \in Z\}$$

These sets are *not* equal - but we can easily construct an isomorphism.

Example 8.5. How can we take a monoid G and construct a monoidal category? First we need a category **C**:

- objects: elements of G.

- morphisms: identity morphisms.

We get a monoidal category $(\mathbf{C}, \bullet, e)$ where $\bullet$ is the binary product of G and e is the identity element of G.

Note: In general:

- If $\mathbf{C}$ has products, we get a monoidal category $(\mathbf{C}, \times, 1)$.

- If $\mathbf{C}$ has coproducts, we get a monoidal category $(\mathbf{C}, +, 0)$.

Definition 8.3. A monoidal category $(\mathbf{C}, \otimes, I)$ is *symmetric* if it additionally is equipped with an isomorphism $s_{x,y} : x \otimes y \to y \otimes x$ for any objects x and y of $\mathbf{C}$, natural in x and y, such that the following diagrams commute for all objects x, y, and z:

$$
\begin{array}{ccc}
(x \otimes y) \otimes z & \xrightarrow{\ s_{x,y} \otimes id_z\ } & (y \otimes x) \otimes z \\
\downarrow{\scriptstyle a_{x,y,z}} & & \downarrow{\scriptstyle a_{y,x,z}} \\
x \otimes (y \otimes z) & & y \otimes (x \otimes z) \\
\downarrow{\scriptstyle s_{x,y\otimes z}} & & \downarrow{\scriptstyle id_y \otimes s_{x,z}} \\
(y \otimes z) \otimes x & \xrightarrow[\ a_{y,z,x}\ ]{} & y \otimes (z \otimes x)
\end{array}
$$

$$
\begin{array}{ccc}
x \otimes I & \xrightarrow{\ s_{x,I}\ } & I \otimes x \\
& {\scriptstyle r_x}\searrow \quad \swarrow{\scriptstyle \ell_x} & \\
& x &
\end{array}
$$

$$
\begin{array}{ccc}
x \otimes y & \xrightarrow{\ s_{x,y}\ } & y \otimes x \\
& {\scriptstyle id_{x\times y}}\searrow \quad \swarrow{\scriptstyle s_{y,x}} & \\
& x \otimes y &
\end{array}
$$

Most of the examples of monoidal categories we have talked about are symmetric. So what's an example of a monoidal category that is not symmetric?

Example 8.6. Let R be a non-commutative ring. The category R-R-bimodules with ${}_R\otimes_R$ as the tensor and R as the unit is an example of a monoidal category that is not symmetric.
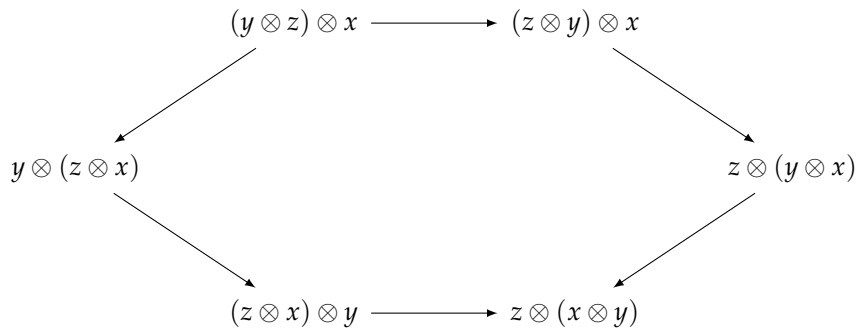
Note: Let $(\mathbf{C}, \bullet, e)$ be the monoidal category given by the monoid G. If G is an abelian group, then $(\mathbf{C}, \bullet, e)$ is symmetric.

8.1.2 Going back to the definition of a symmetric monoidal category...

Q: Why is the hexagon commuting diagram sufficient?

- There are 6 different ways to order 3 elements.

- There are 2 ways of associating 3 elements.

- So there are 12 possibilities (we would expect all of these to be isomorphic).

A: repeat!

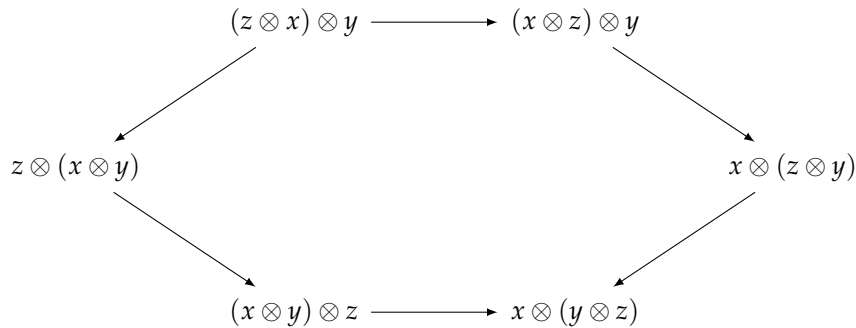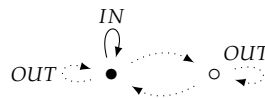$$(z \otimes x) \otimes y \longrightarrow (x \otimes z) \otimes y$$

$$z \otimes (x \otimes y) \qquad\qquad x \otimes (z \otimes y)$$

$$(x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$
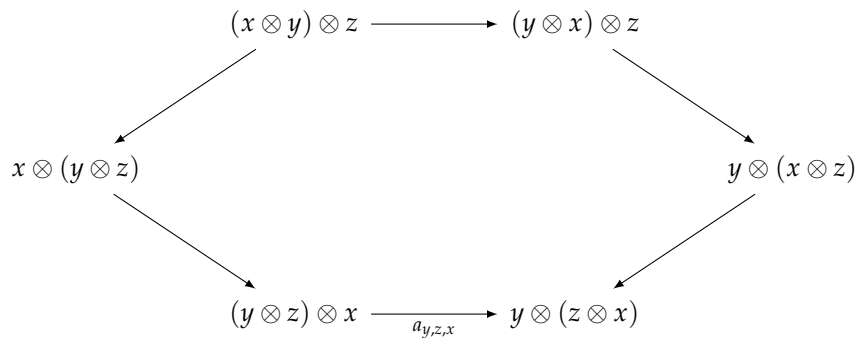
9 Week 10

9.1 The subobject classifier in Graph

This is some graph Ω such that subgraphs A of any graph X corresponds to morphisms of graphs $\chi : X \to \Omega$ in such a way that

$$
\begin{array}{ccc}
A & \xrightarrow{\;!A\;} & 1 \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle t} \\
X & \xrightarrow{\;\chi\;} & \Omega
\end{array}
$$

is a pullback. Ω looks like this:

The terminal graph, "**1**", looks like this:

The purple subgraph of Ω is a copy of **1** (it's isomorphic to **1**). We get this from the morphism $t : 1 \to \Omega$ which you have in any topos. A vertex or edge of X will be mapped to this subgraph of Ω iff it's true that the vertex or edge is in A.

The most important basic properties of topoi:

Proposition 9.1. *A topos has finite colimits, meaning it has colimits of finite-sized.*

Proposition 9.2. *Any morphism $f : X \to Y$ in a topos has an epi-mono factorization i.e. there exists an epimorphism $p : X \to A$ and a mono $i : A \to Y$ making this triangle commute:*

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
& {\scriptstyle p}\searrow \quad \nearrow {\scriptstyle i} & \\
& A &
\end{array}
$$

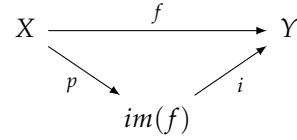**Proposition 9.3.** *In a topos, the epi-mono factorization of any morphism $f : X \to Y$ is unique up to a unique isomorphism. Given two epi-mono factorization:*
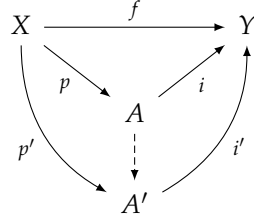
there exists a unique isomorphism $g : A \to A'$ making the resulting diagram commute.

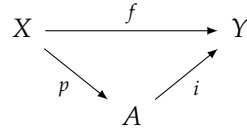**Example 9.1.** In **Set**, we have an epi-mono factorization

where $im(f) = \{y \in Y : y = f(x)$ for some $x \in X\}$, $i : im(f) \to Y$ is the inclusion and $p : X \to im(f)$ is the obvious function $p(x) = f(x) \in im(f)$.

So:

Definition 9.1. Given an epi-mono factorization:

we call A "the" image of f (it's unique up to isomorphism) and denote it as im(f).

Generalize $\subseteq, \cap, \cup$ to any topos, henceforth suppose **C** is a topos.

Definition 9.2. Given $X \in$ **C**, define $Sub(X)$ to be the set of all subobjects of X:equivalence classes of monomorphisms $i : A \to X$, where $i : A \to X$ and $j : A \to X$ are equivalent iff there exists an isomorphism $g : A \to B$ so that:
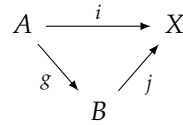
commutes.

Note. $Sub(X) \cong hom(X, \Omega)$ since Ω is the subobject classifier.

Proposition 9.4. *$Sub(X)$ is a poset where we say the equivalence class of $i : A \to X$ is contained in (or $\subseteq$) the equivalence class of $j : B \to X$ if there exists $f : A \to B$ making this commute:*

$$A \xrightarrow{\quad i \quad} X$$

with f going down-left to B and j going up-right from B to X.

(Note: f must be a monomorphism, and it's unique)

Proof. Need to check:

- If $[i] \subseteq [j]$ and $[j] \subseteq [k]$, then $[i] \subseteq [k]$

$$
\begin{array}{ccc}
A \xrightarrow{i} X \\
f \downarrow \quad \nearrow j \\
B \xrightarrow{g} C
\end{array}
\qquad \text{gives} \qquad
\begin{array}{ccc}
A \xrightarrow{i} X \\
g \circ f \downarrow \quad \nearrow k \\
C
\end{array}
$$

- $[i] \subseteq [i]$ - easy

- If $[i] \subseteq [j]$ and $[j] \subseteq [i]$, then $[i] = [j]$

$$
\begin{array}{ccc}
A \xrightarrow{i} X \\
f \downarrow \quad \nearrow j \\
B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \xrightarrow{i} X \\
g \uparrow \quad \nearrow k \\
B
\end{array}
$$

To show $[i] = [j]$, it suffices to show:

$$
\begin{array}{ccc}
A \\
f \downarrow \quad \searrow i \\
B \xrightarrow{j} X \\
g \downarrow \quad \nearrow i \\
A
\end{array}
\qquad\qquad
\begin{array}{ccc}
B \\
g \downarrow \quad \searrow j \\
A \xrightarrow{i} X \\
f \downarrow \quad \nearrow j \\
B
\end{array}
$$

commute, so $i \circ g \circ f = i \circ 1_A$ and $j \circ f \circ g = j \circ 1_B$, and since i and k are monic, they're left cancellable: $g \circ f = 1_A$ and $f \circ g = 1_B$

$\square$

Next time we'll define U for subobjects, and this makes $Sub(X)$, which is a poset (hence a category), into a category with coproducts: $\cup$ is the coproduct in $Sub(X)$. Similarly, $\cap$ is the product in the category $Sub(X)$.

9.2 Set Theory, Topos, and Logic

In **Set**, every subset of $X \in$ **Set** corresponds to a predicate on elements of X:

$$\chi : X \to \{T, F\} \qquad \text{i.e. a characteristic function}$$

χ determines a subset $A \subseteq X$ via:

$$A = \{x \in X \mid \chi(x) = T\}$$

and conversely, any subset $A \subseteq X$ determines $\chi : X \to \{T, F\}$ via:

$$\chi(x) = \begin{cases} T & x \in A \\ F & x \notin A \end{cases}$$

In a topos, we get a similar bijection between $Sub(X)$ and $Hom(X, \Omega)$. The concepts of $\cup$ and $\cap$ for subsets correspond to the operations of $\wedge$ and $\vee$ on predicates.

$$\{x \in X \mid \chi(x) = T\} \cup \{x \in X \mid \varphi(x) = T\} = \{x \in X \mid (\chi \vee \varphi)(x) = T\}$$
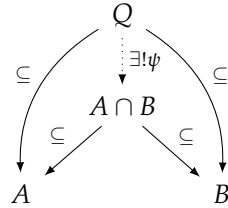
and similarly for $\cap$ and $\wedge$.

Proposition 9.5. *In* **Set***, $Sub(X)$ for $X \in$ **Set** is a poset via $\subseteq$, and thus a category where there exists a unique morphism from A to B if and only if $A \subseteq B$ $(A, B \subseteq X)$. In this category $A \cap B$ is the product of A and B, and $A \cup B$ is the coproduct.*

Proof. We have
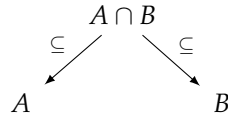
and this cone is universal:

which is true since $Q \subseteq A$, $Q \subseteq B \implies Q \subseteq A \cap B$.

$\square$

In fact, in **Set**, $Sub(X)$ has all finite limits and all finite colimits! A category has all finite limits if an only if it has:

- binary products
- a terminal object
- equalizers

$Sub(X)$ has binary products ($\cap$), a terminal object (X, since $A \subseteq X$ for all $A \in Sub(X)$) and equalizers: $B \underset{g}{\overset{f}{\rightrightarrows}} C$ in any poset is really $B \underset{f}{\overset{f}{\rightrightarrows}} C$, and the equalizer is:

$$B \xrightarrow{\ i\ } B \underset{f}{\overset{f}{\rightrightarrows}} C$$

so equalizers exist in any poset. Similarly in **Set**, $Sub(X)$ has all finite colimits because it has:

- binary coproducts

- an initial object

- coequalizers

The binary coproduct of A and B is $A \cup B$, the initial object is $\varnothing$ (since $\varnothing \subseteq A$ for all $A \in Sub(X)$), and coequalizers (which exist in any poset: just turn arrows around in argument for equalizers).

Definition 9.3. A *lattice* is a poset with all finite limits and colimits.

Remark. This is equivalent to other more popular definitions, though some evil people don't demand the initial and terminal object.

In fact we have:

SET THEORY	LOGIC	CATEGORY THEORY
$\cap$	$\wedge$	binary product
X (the whole set)	T	terminal object
$\cup$	$\vee$	binary coproduct
$\varnothing$	F	initial object
$B \cup A^c$	$Q \vee \neg P$ or "P implies Q"	exponentiation

Note:
$$X = \{x \in X \mid T = T\}$$
$$\varnothing = \{x \in X \mid F = T\}$$

In fact, the poset $Sub(X)$ is cartesian closed. In general, this means:
$$Hom(B \times C, D) \cong Hom(B, D^C)$$

but for $Sub(X)$, being a poset, these sets either have 0 elements or 1 element. Also, the product is the intersection. So this says:
$$B \cap C \subseteq D \quad \text{if and only if} \quad B \subseteq D \cup C^c$$

or in terms of logic:
$$P \wedge Q \implies R \quad \text{if and only if} \quad P \implies R \vee \neg Q$$

Theorem 9.1. *In any topos, for any object X the poset $Sub(X)$ is a* Heyting *algebra: it's a poset that has finite limits, finite colimits, and is cartesian closed.*

Remark. i.e. it's a Cartesian closed lattice

Proof. Given two subobjects of X, $[i]$ and $[j]$, we want to form $[i] \cap [j]$ and $[i] \cup [j]$. Taking the pullback gives us the intersection:

$$
\begin{array}{ccc}
A \cap B & \xrightarrow{\;g\;} & B \\
{\scriptstyle f}\downarrow & {\scriptstyle i \circ f}\searrow\ {\scriptstyle j \circ g} & \downarrow{\scriptstyle j} \\
A & \xrightarrow[\;i\;]{} & X
\end{array}
$$

Since this is a pullback and i,j are monic, then f,g, must be monic too (monics pullback to monics). This implies $i \circ f = j \circ g$ is monic too, so we get a new subobject of X, which is $[i] \cap [j]$. For unions, we start with the product:

$$
\begin{array}{ccc}
A + B & \xleftarrow{\;g\;} & B \\
{\scriptstyle f}\big\uparrow & {\scriptstyle \exists!\psi}\searrow \quad & \big\downarrow{\scriptstyle j} \\
A & \xrightarrow{\;i\;} & X
\end{array}
$$

where we get ψ from the universal property of the coproduct. But ψ need not be monic, so do the epi-mono factorization:

$$
\begin{array}{ccc}
A + B & \xrightarrow{\quad\quad \psi \quad\quad} & X \\
{\scriptstyle p}\searrow & & \nearrow{\scriptstyle k} \\
& im\psi &
\end{array}
$$

where p is epic and k is monic. k gives a new subobject of X, which is $[i] \cup [j]$.

$\square$

9.3   Where does topos theory go from here?

Many directions.... e.g.:

- Using the "Mitchell-Benabov language", we can reason inside any topos:

We can write things like:

$$\{x \in A \cap B \mid \forall y \in Y \; \exists z \in Z \; f(x,z) = y\}$$

and prove things about them using the logic internal to the topos, and "generalized elements".

- There are also maps between topoi:

$$
C \underset{\longleftarrow}{\overset{\longrightarrow}{}} D
$$

consisting of certain nice adjunctions. These maps are called "geometric morphisms". There's a topos called **Th**(**Grp**) - "the theory of groups", and then a geometric morphism from some other topos **C** to **Th**(**Grp**) is the same as a group object in **C**. This idea works for lots of concepts, not just the concept of a group.