Homological Constructions over a Ring of Characteristic 2

March 20, 2019

Throughout this chapter, let R be a ring of characteristic 2.

1 Constructing All Finitely-Generated Differential Graded R-Algebras

Theorem 1.1. *Let S_w denote the weighted polynomial ring $R[x_1, \dots, x_n]$ with respect to the weighted vector $w = (w_1, \dots, w_n)$. Define the map*

$$d := \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda},$$

where f_λ is a nonzero homogeneous polynomial in S_w of weighted degree $w_\lambda - 1$ for all $\lambda = 1, \dots, n$. Then

1. *d is a graded endomorphism $d : S_w \to S_w$ of degree -1 which satisfies Leibniz law.*
2. *Moreover, let $I \subset S_w$ be any d-stable homogeneous ideal such that $d(f_\lambda) \in I$ for all $\lambda = 1, \dots, n$. Then d induces a map $\overline{d} : S_w/I \to S_w/I$, given by $\overline{d}(\overline{f}) = \overline{d(f)}$ for all $\overline{f} \in S_w/I$, and $(S_w/I, \overline{d})$ is a differential graded R-algebra.*

Proof. We first show that d is a graded endomorphism $d : S_w \to S_w$ of degree -1 which satisfies Leibniz law:

- R-linearity: We have

$$\begin{aligned} d(r_1 g_1 + r_2 g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(r_1 g_1 + r_2 g_2) \\ &= \sum_{\lambda=1}^{n} f_\lambda (r_1 \partial_{x_\lambda}(g_1) + r_2 \partial_{x_\lambda}(g_2)) \\ &= r_1 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1) + r_2 \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2)) \\ &= r_1 d(g_1) + r_2 d(g_2), \end{aligned}$$

  for all $r_1, r_2 \in R$ and $g_1, g_2 \in S_w$.

- Leibniz law: We have

$$\begin{aligned} d(g_1 g_2) &= \sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1 g_2) \\ &= \sum_{\lambda=1}^{n} f_\lambda (\partial_{x_\lambda}(g_1) g_2 + g_1 \partial_{x_\lambda}(g_2)) \\ &= \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_1) \right) g_2 + g_1 \left(\sum_{\lambda=1}^{n} f_\lambda \partial_{x_\lambda}(g_2)) \right) \\ &= d(g_1) g_2 + g_1 d(g_2), \end{aligned}$$

  for all $g_1, g_2 \in S_w$.

- Graded of degree -1: By R-linearity, we only need to check this on monomials. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of weighted degree i. A term in $d(x_1^{\alpha_1} \cdots x_n^{\alpha_n})$ has the form $\alpha_\lambda f_\lambda x_1^{\alpha_1} \cdots x_\lambda^{\alpha_\lambda - 1} \cdots x_n^{\alpha_n}$ where $\alpha_\lambda \equiv$

1 mod 3, and

$$\begin{aligned}
\deg_w\left(\alpha_\lambda f_\lambda x_1^{\alpha_1}\cdots x_\lambda^{\alpha_\lambda-1}\cdots x_n^{\alpha_n}\right) &= \deg_w\left(f_\lambda x_1^{\alpha_1}\cdots x_\lambda^{\alpha_\lambda-1}\cdots x_n^{\alpha_n}\right)\\
&= \deg_w(f_\lambda)+\deg_w\left(x_1^{\alpha_1}\cdots x_\lambda^{\alpha_\lambda-1}\cdots x_n^{\alpha_n}\right)\\
&= w_\lambda - 1 + w_1\alpha_1+\cdots+w_\lambda(\alpha_\lambda-1)+\cdots+w_n\alpha_n\\
&= -1+w_1\alpha_1+\cdots+w_\lambda\alpha_\lambda+\cdots+w_n\alpha_n\\
&= -1+i.
\end{aligned}$$

So every term in $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ has weighted degree $-1+i$. This implies that d is graded of degree -1.

Now we will show that $(S_w/I,\overline{d})$ is a differential graded R-algebra. Since I is d-stable, the map $\overline{d}$ is well-defined. The map $\overline{d}$ inherits the properties of being a graded endomorphism of degree -1 which satisfies Leibniz law from d, thus we just need to show that $\overline{d}^2=0$, or in other words, that $d^2(g)\in I$ for all $g\in S_w$. So let $g\in S_w$. Then

$$\begin{aligned}
d^2(g) &= d\left(\sum_{\lambda=1}^n f_\lambda\partial_{x_\lambda}(g)\right)\\
&= \sum_{\lambda=1}^n d(f_\lambda\partial_{x_\lambda}(g))\\
&= \sum_{\lambda=1}^n d(f_\lambda)\partial_{x_\lambda}(g)+f_\lambda d(\partial_{x_\lambda}(g)))\\
&= \sum_{\lambda=1}^n d(f_\lambda)\partial_{x_\lambda}(g)\in I,
\end{aligned}$$

where we used the fact that $\partial_{x_\lambda}^2=0$ and $\partial_{x_\mu}\partial_{x_\lambda}=\partial_{x_\lambda}\partial_{x_\mu}$ to conclude that

$$\begin{aligned}
\sum_{\lambda=1}^n f_\lambda d(\partial_{x_\lambda}(g)) &= \sum_{\lambda=1}^n f_\lambda\sum_{\mu=1}^n f_\mu\partial_{x_\mu}(\partial_{x_\lambda}(g))\\
&= 0.
\end{aligned}$$

□

Remark.

1. We often denote this differential graded R-algebra as $(S_w/I,f_1,\dots f_n)$ instead of $(S_w/I,\overline{d})$.
2. When we write "let $(S_w/I,f_1,\dots f_n)$ be a differential graded R-algebra", it is understood that the conditions in Theorem (1.1) are satisfied. Note that I is a *proper* ideal of S_w.

Proposition 1.1. *Let $(S_w/I,f_1,\dots,f_n)$ be a differential graded R-algebra and let g be a homogeneous polynomial in S of degree j such that $d(g)$ is in I. Then $(S_w/\langle I,g\rangle,f_1,\dots,f_n)$ and $(S/(I:g),f_1,\dots,f_n)$ are differential graded R-algebras.*

Proof. First note that $d(f_\lambda)\in I$ implies $d(f_\lambda)\in\langle I,g\rangle$ and $d(f_\lambda)\in I:g$ for all $\lambda=1,\dots,n$. So we just need to show that $\langle I,g\rangle$ and $I:g$ are d-stable. Since$d(g)$ is in I, Proposition (2.1) implies that $\langle I,g\rangle$ is d-stable. Therefore $S/\langle I,g\rangle$ is a differential graded R-algebra. To prove that $I:g$ is d-stable, let $f\in I:g$. Then since $fg\in I$ and I is d-stable, it follows that $d(fg)=d(f)g+fd(g)\in I$. Which implies $d(f)g\in I$, since $d(g)\in I$. Therefore $d(f)\in I:g$, which implies that $I:g$ is d-stable. □

1.1 Classification of all Finitely-Generated Commutative Differential Graded R-Algebras

Theorem 1.2. *Every finitely-generated commutative differential graded R-algebra is isomorphic to one described in Theorem (1.1).*

Proof. Let (A,d_A) be a finitely generated differential graded R-algebra with generators $a_1,\dots,a_n$. Then for each $\lambda=1,\dots n$, we have $a_\lambda\in A_{w_\lambda}$, where $w_\lambda\in\mathbb{Z}_{\geq 0}$. Let S_w denote the weighted polynomial ring $R[x_1,\dots,x_n]$ with respect to the weighted vector $w=(w_1,\dots,w_n)$, and let $\varphi: S_w\to A$ be the unique morphism of graded R-algebras such that $\varphi(x_\lambda)=a_\lambda$ for all $\lambda=1,\dots,n$. Then A is isomorphic to $S_w/\mathrm{Ker}(\varphi)$ as graded R-algebras. Choose $f_\lambda\in S$ such that $\varphi(f_\lambda)=d_A(a_\lambda)$ and define the map $d: S_w\to S_w$ as

$$d:=\sum_{\lambda=1}^n f_\lambda\partial_{x_\lambda}.$$

Then d is a graded endomorphism of degree -1 which satisfies Leibniz law, by Theorem (1.1). We claim that $\mathrm{Ker}(\varphi)$ is d-stable and that $d(f_\lambda) \in \mathrm{Ker}(\varphi)$ for all $\lambda = 1, \dots, n$. We do this in two steps:

Step 1: We will show that $\varphi d = d_A \varphi$. It suffices to show that for all monomials m, we have $\varphi(d(m)) = d_A(\varphi(m))$. We prove this by induction on $\deg(m)$. For the base case $\deg(m) = 1$, we have $m = x_\lambda$ for some $\lambda \in \{1, \dots, n\}$. Then

$$\begin{aligned} \varphi(d(x_\lambda)) &= \varphi(f_\lambda) \\ &= d_A(a_\lambda) \\ &= d_A(\varphi(x_\lambda)). \end{aligned}$$

Now suppose that $\varphi(d(m)) = d_A(\varphi(m))$ for all monomials m in S of degree less than i, where $i > 1$. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S whose degree is $i+1$. We may assume that $\alpha_1, \alpha_\lambda \geq 1$ for some $\lambda \in \{1, \dots, n\}$. Then using Leibniz law together with induction, we obtain

$$\begin{aligned} \varphi(d(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})) &= \varphi(d(x_1^{\alpha_1}) x_2^{\alpha_2} \cdots x_n^{\alpha_n} + x_1^{\alpha_1} d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\ &= \varphi(d(x_1^{\alpha_1}) \varphi(x_2^{\alpha_2} \cdots x_n^{\alpha_n}) + \varphi(x_1^{\alpha_1}) \varphi(d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\ &= \varphi(d(x_1^{\alpha_1})) a_2^{\alpha_2} \cdots a_n^{\alpha_n} + a_1^{\alpha_1} \varphi(d(x_2^{\alpha_2} \cdots x_n^{\alpha_n})) \\ &= d_A(a_1^{\alpha_1}) a_2^{\alpha_2} \cdots a_n^{\alpha_n} + a_1^{\alpha_1} d_A(a_2^{\alpha_2} \cdots a_n^{\alpha_n}) \\ &= d_A(a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}) \\ &= d_A(\varphi(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n})). \end{aligned}$$

This establishes Step 1.

Step 2: We show that $\mathrm{Ker}(\varphi)$ is d-stable and that $d(f_\lambda) \in \mathrm{Ker}(\varphi)$ for all $\lambda = 1, \dots, n$. Let $g \in \mathrm{Ker}(\varphi)$. Then by Step 1, we have

$$\begin{aligned} \varphi(d(f)) &= d_A(\varphi(f)) \\ &= d_A(0) \\ &= 0. \end{aligned}$$

Thus $d(f) \in \mathrm{Ker}(\varphi)$, which implies $\mathrm{Ker}(\varphi)$ is d-stable. Step 1 also implies

$$\begin{aligned} \varphi(d(f_\lambda)) &= d_A(\varphi(f_\lambda)) \\ &= d_A(d_A(f_\lambda)) \\ &= 0, \end{aligned}$$

for all $\lambda = 1, \dots, n$.

Now Theorem (1.1) implies that $(S_w/\mathrm{Ker}(\varphi), \overline{d})$ is a differential graded R-algebra. Moreover, Step 1 implies $\varphi : (S_w/\mathrm{Ker}(\varphi), \overline{d}) \to (A, d_A)$ is an isomorphism of differential graded R-algebras. □

2 Constructing the Differential Graded R-algebra $(S/I, r_1, \dots, r_n)$

We now want to consider some special cases of Theorem (1.1). In particular, we want to consider the case where the weighted vector is $w = (1, \dots, 1)$. We will write S to denote the polynomial ring $R[x_1, \dots, x_n]$ equipped with this grading. Let $r_1, \dots, r_n$ be nonzero elements in R, and define $d : S \to S$ by

$$d := \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda}.$$

Since $d(r_\lambda) = 0$ for all $\lambda = 1, \dots, n$, it follows from Theorem (1.1) that $(S, r_1, \dots, r_n)$ is a differential graded R-algebra. Moreover, if I is a d-stable ideal, then $(S/I, r_1, \dots, r_n)$ is a differential graded R-algebra. The next proposition gives a necessary and sufficient condition for a finitely generated ideal I to be d-stable.

Proposition 2.1. *Let I be a homogeneous ideal in S. Then I is d-stable if and only if for some generating set $F = \{f_1, \dots, f_r\}$ of I, we have $d(f_\lambda) \in I$ for all $\lambda = 1, \dots, r$.*

Proof. One direction is trivial, so let's prove the other direction. Let $F = \{f_1, \dots, f_r\}$ be a generating set for I such that $d(f_\lambda) \in I$ for all $\lambda = 1, \dots, r$ and let $f \in I$. Since $\{f_1, \dots, f_r\}$ generates I, we can write $f = \sum_{\lambda=1}^r q_\lambda f_\lambda$ for some $q_1, \dots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(f) &= d\left(\sum_{\lambda=1}^r q_\lambda f_\lambda\right) \\ &= \sum_{\lambda=1}^r d(q_\lambda f_\lambda) \\ &= \sum_{\lambda=1}^r (d(q_\lambda) f_\lambda + q_\lambda d(f_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. □

2.1 Koszul Complex

Recall from Example (**??**) that the Koszul complex $\mathcal{K}(r_1, \dots, r_n)$ is a differential graded R-algebra. Indeed, $\mathcal{K}(r_1, \dots, r_n)$ is isomorphic to the differential graded R-algebra $(S/I, r_1, \dots, r_n)$, where I is generated by $\{x_1^2, \dots, x_n^2\}$.C learly I is d-stable since $d(x_\lambda^2) = 0$ for all $\lambda = 1, \dots, n$.

Example 2.1. Let $R = \mathbb{F}_2[x, y]/\langle xy \rangle$ and let $r_1 = x$ and $r_2 = y$. Then $S = R[u, v]$ has a differential graded R-algebra structure with the differential d given by

$$d := x\partial_u + y\partial_v.$$

Using graded lexicographical ordering on the monomials, we can explicitly write S as a chain complex over R using matrices as the linear maps:

$$\cdots \longrightarrow R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & y & 0 \\ 0 & x & 0 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

Now let I be the homogeneous ideal in S generated by $\{x^2, y^2\}$. Then $(S/I, r_1, r_2)$ is isomorphic to the Koszul complex $\mathcal{K}(r_1, r_2)$. Using graded lexicographical ordering on the monomials, we can explicitly write S/I as a chain complex over R using matrices as the linear maps:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0$$

2.2 Blowup algebras

Proposition 2.2. *Let Q be a finitely generated ideal in R with generating set $\{a_1, \dots, a_n\}$. Then the blowup algebra $B_Q(R)$ can be given the structure of differential graded R-algebra.*

Proof. Let $\varphi : S \to B_Q(R)$ be the unique graded R-algebra homomorphism such that $\varphi(x_\lambda) = ta_\lambda$ for all $\lambda = 1, \dots, n$ and let $d := \sum_{\lambda=1}^n a_\lambda \partial_\lambda$. We claim that $\mathrm{Ker}(\varphi)$ is d-stable. Indeed, let $f \in \mathrm{Ker}(\varphi)$. Since $\mathrm{Ker}(\varphi)$ is homogeneous, we may assume that f is homogeneous. Write f and $d(f)$ in terms of the monomial basis:

$$f = \sum_{\lambda=1}^r b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}} \qquad \text{and} \qquad d(f) = \sum_{\substack{1 \le \mu \le n \\ 1 \le \lambda \le r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}.$$

where $b_\lambda \in R$ and $\alpha_{\mu\lambda} \in \mathbb{Z}_{\ge 0}$ for all $\lambda = 1, \dots, r$ and $\mu = 1, \dots n$. Observe that

$$\begin{aligned} 0 &= \varphi(f) \\ &= \varphi\left(\sum_{\lambda=1}^r b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_n^{\alpha_{n\lambda}}\right) \\ &= \sum_{\lambda=1}^r b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_n)^{\alpha_{n\lambda}} \\ &= t^i \left(\sum_{\lambda=1}^r b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n{}^{\alpha_{n\lambda}}\right) \end{aligned}$$

implies that $\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n{}^{\alpha_{n\lambda}} = 0$. Therefore

$$\begin{aligned}
\varphi(d(f)) &= \varphi\left(\sum_{\substack{1\le\mu\le n\\1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda x_1^{\alpha_{1\lambda}} \cdots x_\mu^{\alpha_{\mu\lambda}-1} \cdots x_n^{\alpha_{n\lambda}}\right)\\
&= \sum_{\substack{1\le\mu\le n\\1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda \varphi(x_1)^{\alpha_{1\lambda}} \cdots \varphi(x_\mu)^{\alpha_{\mu\lambda}-1} \cdots \varphi(x_n)^{\alpha_{n\lambda}}\\
&= t^{i-1}\left(\sum_{\substack{1\le\mu\le n\\1\le\lambda\le r}} \alpha_{\mu\lambda} a_\mu b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_\mu^{\alpha_{\mu\lambda}-1} \cdots a_n^{\alpha_{n\lambda}}\right)\\
&= t^{i-1}\left(\left(\sum_{\mu=1}^{n} \alpha_{\mu\lambda}\right)\left(\sum_{\lambda=1}^{r} b_\lambda a_1^{\alpha_{1\lambda}} \cdots a_n{}^{\alpha_{n\lambda}}\right)\right)\\
&= 0.
\end{aligned}$$

Therefore $(S/\mathrm{Ker}(\varphi), a_1, \ldots, a_n)$ is a differential graded R-algebra where $S/\mathrm{Ker}(\varphi) \cong B_Q(R)$. □

Remark. It isn't too difficult to show that this differential graded R-algebra is $(B_Q(R), \partial_t)$, where ∂_t is defined in the obvious way.

Example 2.2. Let $R = \mathbb{F}_2[x,y]/\langle y^2 + x^3 + x^2\rangle$, $\mathfrak{m}$ be the maximal ideal in R generated by $\{\overline{x}, \overline{y}\}$, S denote the polynomial ring $R[u,v]$, and $d = \overline{x}\partial_u + \overline{y}\partial_v$. There is a surjective R-algebra homomorphism from S to the blowup algebra at $\mathfrak{m}$ given by

$$\varphi : S := \mathbb{F}_2[x,y,u,v]/\langle y^2 + x^3 + x^2\rangle \to B_{\mathfrak{m}}(R),$$

where φ is induced by $\varphi(u) = t\overline{x}$ and $v \mapsto t\overline{y}$. Using Singular, we find that the kernel of φ is an ideal which is homogeneous in the variables u, v, and is generated by the set $\{f_1, f_2, f_3\}$, where

$$\begin{aligned}
f_1 &= \overline{x}v + \overline{y}u\\
f_2 &= \overline{x}u^2 + u^2 + v^2\\
f_3 &= \overline{x}^2u + \overline{x}u + \overline{y}v
\end{aligned}$$

Note that $d(f_1) = d(f_2) = d(f_3) \in \mathrm{Ker}(\varphi)$. It follows from Proposition (2.1) that $\mathrm{Ker}(\varphi)$ is d-stable, which we already knew from Proposition (2.2).

2.3 Homology Calculations

Proposition 2.3. *Let $(S/I, r_1, \ldots, r_n)$ be a differential graded R-algebra. Suppose that there are $t_1, \ldots, t_n \in R$ such*

$$\sum_{\lambda=1}^{n} t_\lambda r_\lambda = 1. \tag{1}$$

Then $H(S/I, r_1, \ldots, r_n) = 0$.

Proof. First note that $\sum_{\lambda=1}^{n} t_\lambda x_\lambda \notin I$, otherwise $d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) = 1 \notin I$ would imply that I is not d-stable. Let f be a homogeneous polynomial of degree i such $d(f) \in I$; so f represents a cycle of $(S/I, \overline{d})$. Then

$$\begin{aligned}
d\left(\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f\right) &= d\left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f)\\
&= \left(\sum_{\lambda=1}^{n} t_\lambda r_\lambda\right) f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f)\\
&= f + \left(\sum_{\lambda=1}^{n} t_\lambda x_\lambda\right) d(f)\\
&\equiv f \bmod I.
\end{aligned}$$

thus $\mathrm{Ker}(\overline{d}) = \mathrm{Im}(\overline{d})$, which proves the claim. □

Remark.

1. By setting $I = 0$, we also find that $H(S) = 0$.
2. The condition (1) is equivalent to saying that $\{r_1, \ldots, r_n\}$ generates the unit ideal.

2.3.1 Long Exact Sequence

It is straightforward to check that

$$\begin{tikzcd} 0 \arrow[r] & (S_w(-j)/(I:g),\overline{d}) \arrow[r, "\cdot g"] & (S/I,\overline{d}) \arrow[r] & (S/\langle I,g\rangle,\overline{d}) \arrow[r] & 0 \\ & \overline{f} \arrow[r, mapsto] & \overline{fg} & & \end{tikzcd} \tag{2}$$

is short exact sequence of chain complexes. The short exact sequence (2) gives rise to a long exact sequence in homology:

$$\begin{tikzcd} & \cdots \arrow[r] & H_{i+1}(S_w/\langle I,g\rangle) \arrow[dll, rounded corners, "\lambda"] \\ H_{i-j}(S_w/(I:g)) \arrow[r, "\cdot g"] & H_i(S_w/I) \arrow[r] & H_i(S_w/\langle I,g\rangle) \arrow[dll, rounded corners, "\lambda"] \\ H_{i-j-1}(S_w/(I:g)) \arrow[r, "\cdot g"] & H_{i-1}(S_w/I) \arrow[r] & \cdots \end{tikzcd}$$

Let us work out the details of the connecting map: Let $\overline{f}$ be a homogeneous element in $S_w/\langle I,g\rangle$ which represents a class in $H_i(S_w/\langle I,g\rangle)$. In particular, $f \in S$ and $d(f) \in \langle I,g\rangle$. We lift $\overline{f} \in S_w/\langle I,g\rangle$ to S_w/I and then apply d to get $\overline{d(f)} \in S_w/I$. Since $d(f) \in \langle I,g\rangle$, we can write $d(f) = p + gq$ where $p \in I$. Thus, $\overline{d(f)} = \overline{gq}$, and this pulls back to $\overline{q}$ in $S_w/(I:g)$.

3 Extra

3.1 Classifying d-Stable Ideals

Let $(R[x_1,\dots,x_n]/I, r_1,\dots,r_n)$ be a differential graded R-algebra. Suppose that there are $t_1,\dots,t_m \in R$ such that $\langle r_1,\dots,r_n\rangle = \langle t_1,\dots,t_m\rangle$ and $(R[y_1,\dots,y_m]/I, t_1,\dots,t_m)$ is also a differential graded R-algebra. Then for all $1 \le \lambda \le n$ and $1 \le \mu \le n$, there are $a_{\lambda\mu}$ and $b_{\lambda\mu}$ in R such that

$$r_\lambda = \sum_{\mu=1}^{m} a_{\lambda\mu} t_\mu \text{ and } t_\mu = \sum_{\lambda=1}^{n} b_{\lambda\mu} r_\lambda.$$

Let $\varphi : R[x_1,\dots,x_n] \to R[y_1,\dots,y_m]$ be the unique graded R-algebra homomorphism such that $\varphi(x_\lambda) = \sum_{\mu=1}^{m} a_{\lambda\mu} y_\mu$ for all $\lambda = 1,\dots,n$. Then φ induces a graded R-algebra homomorphism $\overline{\varphi} : R[x_1,\dots,x_n]/I \to R[y_1,\dots,y_m]/\langle\varphi(I)\rangle$ which in turn induces a homomorphism of differential graded R-algebras $\overline{\varphi} : (R[x_1,\dots,x_n]/I, r_1,\dots,r_n) \to (R[y_1,\dots,y_m]/\langle\varphi(I)\rangle, t_1,\dots,t_m)$. Indeed, let us denote the differentials as

$$d_r := \sum_{\lambda=1}^{n} r_\lambda \partial_{x_\lambda} \text{ and } d_t := \sum_{\mu=1}^{m} t_\mu \partial_{y_\mu}.$$

We first show that $\varphi d_r = d_t \varphi$. It is enough to show that $\varphi d_r(x_\lambda) = d_t\varphi(x_\lambda)$ for all $\lambda = 1,\dots,n$. We have

$$\begin{aligned} d_t\varphi(x_\lambda) &= d_t\left(\sum_{\mu=1}^{m} a_{\lambda\mu} y_\mu\right) \\ &= \sum_{\mu=1}^{m} a_{\lambda\mu} t_\mu \\ &= r_\lambda \\ &= d_r(x_\lambda) \\ &= \varphi(d_r(x_\lambda)). \end{aligned}$$

Now we show that $(R[y_1,\dots,y_m]/\langle\varphi(I)\rangle, t_1,\dots,t_m)$ is a differential graded R-algebra. We do this by showing that $\langle\varphi(I)\rangle$ is d_t-stable. Let $\sum_{\kappa=1}^r g_\kappa\varphi(f_\kappa)\in\varphi(I)$. Then

$$\begin{aligned} d_t\left(\sum_{\kappa=1}^r g_\kappa\varphi(f_\kappa)\right) &= \sum_{\kappa=1}^r d_t(g_\kappa)\varphi(f_\kappa) + \sum_{\kappa=1}^r g_\kappa d_t(\varphi(f_\kappa)) \\ &= \sum_{\kappa=1}^r d_t(g_\kappa)\varphi(f_\kappa) + \sum_{\kappa=1}^r g_\kappa \varphi(d_r(f_\kappa)) \in \langle\varphi(I)\rangle. \end{aligned}$$

Similarly, let $\psi: R[y_1,\dots,y_m]\to R[x_1,\dots,x_n]$ be the unique graded R-algebra homomorphism such that $\psi(y_\mu)=\sum_{\lambda=1}^n b_{\lambda\mu}x_\lambda$ for all $\mu=1,\dots,m$. Then φ induces a graded R-algebra homomorphism $\overline{\psi}: R[y_1,\dots,y_m]/\langle\varphi(I)\rangle\to R[x_1,\dots,x_n]/\langle\psi(\varphi(I))\rangle$ which in turn induces a homomorphism of differential graded R-algebras $\overline{\psi}(R[y_1,\dots,y_m]/\langle\varphi(I)\rangle, t_1,\dots,t_m)\to (R[x_1,\dots,x_n]/\langle\psi(\varphi(I))\rangle, r_1,\dots,r_n)$.

3.1.1 Evalutation Map

Let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra such that I is contained in $\langle x_1,\dots,x_n\rangle$. Let $Q=\langle r_1,\dots,r_n\rangle$ and $\mathrm{Ev}_r: S\to R$ be the unique R-algebra homomorphism such that $\mathrm{Ev}_r(x_\lambda)=r_\lambda$ for all $\lambda=1,\dots,n$. We are interested in the ideal $\mathrm{Ev}_r(I)$ in R. Clearly we have $\mathrm{Ev}_r(I)\subset Q$. Suppose $a\in Q\backslash\mathrm{Ev}_r(I)$. Then $a=\sum_{\lambda=1}^n a_\lambda r_\lambda$ for some $a_\lambda\in R$. This implies $x:=\sum_{\lambda=1}^n a_\lambda x_\lambda\notin I$. Now $J=I+\langle x,a\rangle$ is an ideal strictly larger than I such that J is d-stable $\mathrm{Ev}_r(J)$ is strictly larger than $\mathrm{Ev}_r(I)$. This implies that we can find an ideal I such that $\mathrm{Ev}_r(I)=Q$.

Proposition 3.1. *Let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra and let $Q=\langle r_1,\dots,r_n\rangle$ be an ideal in R. Suppose that $Ev_r(I)=0$. Then there exists a unique homomorphism φ which makes the following diagram commute*

$$\begin{tikzcd} & B_Q(R) \arrow[dr, "Ev_1"] & \\ S/I \arrow[ur, dashed, "\varphi"] \arrow[rr, "Ev_r"'] & & R \end{tikzcd}$$

3.1.2 Tensor product of differential graded R-algebras

Let $(R[x_1,\dots,x_n]/I, d_r)$ and $(R[y_1,\dots,y_m]/J, d_t)$ be two differential graded R-algebras, where

$$d_r := \sum_{\lambda=1}^n r_\lambda\partial_{x_\lambda} \text{ and } d_t := \sum_{\mu=1}^m t_\mu\partial_{y_\mu}.$$

for $r_\lambda, t_\mu\in R$ for all $\lambda=1,\dots,n$ and $\mu=1,\dots,m$. Then their tensor product over R is

$$(R[x_1,\dots,x_n]/I, d_r)\otimes_R (R[y_1,\dots,y_m]/J, d_t)\cong (R[x_1,\dots,x_n,y_1,\dots,y_m]/(I+J), d_r+d_t).$$

Example 3.1. The Koszul complex $\mathcal{K}(r_1,\dots,r_n)$ can be realized as a tensor product:

$$\mathcal{K}(r_1,\dots,r_n)\cong\mathcal{K}(r_1)\otimes\cdots\otimes\mathcal{K}(r_n).$$

Let M be an R-module, and let $(S/I, r_1,\dots,r_n)$ be a differential graded R-algebra. Recall that $(M\otimes_R S/I, d)$ is an (S/I)-module.