Measure Theory Test

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Problem 1

Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \colon X \to \mathbb{R}$ be a measurable function. Prove that $f/(1+f^2)$ is also a measurable function.

Solution 1. By a proposition proved in class (see Appendix for details), the product and sum of two measurable functions is measurable. Therefore both f and $1+f^2$ are measurable. It remains to show that $f/(1+f^2)$ is measurable. To see this, note that for any strictly positive measurable function $h \colon X \to (0,\infty)$, the function $1/h$ is measurable. Indeed, for any $c > 0$, we have

$$\begin{aligned} x \in \left\{ \frac{1}{h} < c \right\} &\iff \frac{1}{h(x)} < c \\ &\iff 1 < ch(x) \\ &\iff x \in \left\{ h > \frac{1}{c} \right\}. \end{aligned}$$

Thus $\{1/h < c\} = \{h > 1/c\} \in \mathcal{M}$. If $c \leq 0$, then we have $\{1/h < c\} = \emptyset \in \mathcal{M}$ since h is strictly positive. In either case, we see that $1/h$ is measurable. In particular, since $1+f^2$ is a strictly positive measurable function, we see that $1/(1+f^2)$ is a measurable function. Therefore the product $f/(1+f^2)$ is a measurable function.

Exercise 2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \colon X \to \mathbb{R}$ be a measurable function. Prove that $\{-1 \leq f \leq 1\}$ is a measurable function.

Solution 2. Since f is measurable, we have

$$\begin{aligned} \{f \leq 1\} &= \bigcap_{n=1}^{\infty} \left\{ f < 1 + \frac{1}{n} \right\} \\ &\in \mathcal{M}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \{f \geq -1\} &= \{f < -1\}^c \\ &\in \mathcal{M}. \end{aligned}$$

Therefore

$$\begin{aligned} \{-1 \leq f \leq 1\} &= \{f \geq -1\} \cap \{f \leq 1\} \\ &\in \mathcal{M}. \end{aligned}$$

Problem 2

Exercise 3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of measurable functions and suppose $f \colon X \to \mathbb{R}$ is another measurable function. Prove that

$$\bigcup_{k=1}^{\infty} \limsup_{n\to\infty} \{|f_n - f| \geq 1/k\} = \{\lim_{n\to\infty} f_n \neq f\} \tag{1}$$

Solution 3. Observe that

$$
\begin{aligned}
x \in \bigcup_{k=1}^{\infty} \limsup_{n\to\infty}\{|f_n - f| \geq 1/k\} &\iff x \in \limsup_{n\to\infty}\{|f_n - f| \geq 1/k\} \text{ for some } k \\
&\iff x \in \{|f_{\pi_k(n)} - f| \geq 1/k\} \text{ for some } k \text{ for some subsequence } (\pi_k(n))_{n\in\mathbb{N}} \text{ of } (n)_{n\in\mathbb{N}} \\
&\iff |f_{\pi_k(n)}(x) - f(x)| \geq 1/k \text{ for some } k \text{ for some subsequence } (\pi_k(n))_{n\in\mathbb{N}} \text{ of } (n)_{n\in\mathbb{N}} \\
&\iff x \in \{\lim_{n\to\infty} f_n \neq f\}
\end{aligned}
$$

where the last if and only if follows from the fact that the distance $|f_n(x) - f(x)|$ is frequently greater than $1/k$, which means $f_n(x) \not\to f(x)$. This gives us (1).

Exercise 4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of measurable functions and suppose $f \colon X \to \mathbb{R}$ is another measurable function. Prove that if $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in X$, then for all $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \mu\{|f_N - f| \geq 1/k\} = 0 \tag{2}$$

Solution 4. Let $k \in \mathbb{N}$. Then observe that

$$
\begin{aligned}
0 &= \mu\left\{\lim_{n\to\infty} f_n \neq f\right\} \\
&= \mu\left\{\bigcup_{m=1}^{\infty} \limsup_{n\to\infty}\{|f_n - f| \geq 1/m\}\right\} \\
&\geq \mu\left\{\limsup\{|f_n - f| \geq 1/k\}\right\} \\
&= \mu\left(\bigcap_{N=1}^{\infty} \bigcup_{n\geq N} \{|f_n - f| \geq 1/k\}\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} \{|f_n - f| \geq 1/k\}\right) \\
&\geq \lim_{N\to\infty} \mu\{|f_N - f| \geq 1/k\}
\end{aligned}
$$

where we used the fact that $\mu(X) < \infty$ to get from the fourth line to the fifth line. This gives us (2).

Exercise 5. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of measurable functions and suppose $f \colon X \to \mathbb{R}$ is another measurable function. Prove that if $f_n \to f$ a.e, then $f_n \to f$ in measure.

Solution 5. Suppose $f_n \to f$ a.e and let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Then by part (b), we have

$$
\begin{aligned}
\lim_{n\to\infty} \mu\{|f_n - f| \geq \varepsilon\} &\leq \lim_{n\to\infty} \mu\{|f_n - f| \geq 1/k\} \\
&= 0.
\end{aligned}
$$

This implies $f_n \to f$ in measure.

Exercise 6. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of measurable functions and suppose $f \colon X \to \mathbb{R}$ is another measurable function. Prove that if $f_n \xrightarrow{L^2} f$, then $f_n \to f$ in measure.

Solution 6. Let $g \in L^2(X, \mathcal{M}, \mu)$. Since $\mu(X) < \infty$, we also have $1_X \in L^2(X, \mathcal{M}, \mu)$. By Hölder's inequality, we have

$$
\begin{aligned}
\|g\|_1 &\leq \|g\|_2 \cdot \|1_X\|_2 \\
&= \sqrt{\mu(X)}\|g\|_2.
\end{aligned}
$$

In particular, $f_n \xrightarrow{L^2} f$ implies $f_n \xrightarrow{L^1} f$ which implies $f_n \to f$ in measure (proved in class).

Problem 3

Exercise 7. Compute the following limit

$$\lim_{n\to\infty}\int_{(0,1)}\frac{1+nx}{(1+x)^n}\mathrm{d}x$$

Solution 7. For each $n\in\mathbb{N}$, let $f_n=(1+nx)(1+x)^{-n}$. Observe that each f_n is measurable since each f_n is continuous (the denominator is never zero for any $x\in\mathbb{R}$). Furthermore, each f_n is nonnegative (since taking squares makes everything nonnegative). We claim that f_n is a decreasing sequence. Indeed

$$\begin{aligned}\frac{f_n}{f_{n+1}}&=\left(\frac{1+nx}{(1+x)^n}\right)\left(\frac{(1+x)^{n+1}}{1+(n+1)x}\right)\\&=\frac{(1+nx)(1+x)}{1+(n+1)x}\\&=\frac{nx^2+(n+1)x+1}{(n+1)x+1}\\&>1\end{aligned}$$

Thus (f_n) is a decreasing sequence which is bounded below, so it must converge pointwise to some function. To see what it converges to, we use L'Hopital's rule:

$$\begin{aligned}\lim_{n\to\infty}f_n&=\lim_{n\to\infty}\frac{1+nx}{(1+x)^n}\\&=\lim_{n\to\infty}\frac{x}{\ln(1+x)(1+x)^n}\\&=0.\end{aligned}$$

Thus (f_n) converges pointwise to 0. Since

$$\begin{aligned}\int_0^1 f_1\mathrm{d}x&=\int_0^1\frac{1+x}{1+x}\mathrm{d}x\\&=\int_0^1\mathrm{d}x\\&=1\\&<\infty,\end{aligned}$$

it follows from (decreasing version of MCT) that

$$\begin{aligned}\lim_{n\to\infty}\frac{1+nx}{(1+x)^n}\mathrm{d}x&=\lim_{n\to\infty}\int_0^1 f_n\mathrm{d}x\\&=\int_0^1 0\mathrm{d}x\\&=0.\end{aligned}$$

Problem 4

Exercise 8. Let $(X,\mathcal{M},\mu)$ be a measure space. Suppose that $f\colon X\to\mathbb{R}$ is a measurable function such that $f(x)>0$ for all $x\in X$. Prove that $\int_X 1_E f\mathrm{d}\mu>0$ for every measurable $E\in\mathcal{M}$ such that $\mu(E)>0$.

Solution 8. Let $E\in\mathcal{M}$ such that $\mu(E)>0$. For each $n\in\mathbb{N}$, define

$$F_n:=\{f\geq 1/n\}.$$

Since $f(x)>0$ for all $x\in X$, we have

$$\begin{aligned}0&<\mu(E)\\&=\mu\left(\bigcup_{n=1}^{\infty}F_n\cap E\right)\\&\leq\sum_{n=1}^{\infty}\mu(F_n\cap E).\end{aligned}$$

The strict inequality implies $\mu(F_n \cap E) > 0$ for some $n \in \mathbb{N}$. Choose such an $n \in \mathbb{N}$, then we have

$$\begin{aligned}
\int_X f 1_E \mathrm{d}\mu &\geq \int_X f 1_{E \cap F_n} \mathrm{d}\mu \\
&\geq \int_X \frac{1}{n} \cdot 1_{E \cap F_n} \mathrm{d}\mu \\
&= \mu(E \cap F_n)/n \\
&> 0.
\end{aligned}$$

Problem 5

Exercise 9. Let $f\colon [0,1] \to [0,\infty)$ be a nonnegative measurable function. Prove that if

$$\lim_{n \to \infty} \int_{[0,1]} f 1_{[0,\frac{n}{n+1}]} \mathrm{d}x \leq 1$$

for all $n \in \mathbb{N}$, then f is integrable and $\int_{[0,1]} f \mathrm{d}x \leq 1$.

Solution 9. Observe that since $(n/n+1)$ is an increasing sequence which converges to 1, the sequence $(f 1_{[0,\frac{n}{n+1}]})$ is an increasing sequence of nonnegative measurable functions which converges pointwise to f. It follows from MCT that

$$\begin{aligned}
\int_{[0,1]} f \mathrm{d}x &= \lim_{n \to \infty} \int_{[0,1]} f 1_{[0,\frac{n}{n+1}]} \mathrm{d}x \\
&\leq 1.
\end{aligned}$$

In particular, f is integrable and $\int_{[0,1]} f \mathrm{d}x \leq 1$.

Problem 6

Exercise 10. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be a nonnegative measurable function such that $\int_X f \mathrm{d}\mu < \infty$. Prove that $\nu\colon \mathcal{M} \to \mathbb{R}$ defined by

$$\nu(E) = \int_X f 1_E \mathrm{d}\mu$$

is a finite measure on $(X, \mathcal{M})$.

Solution 10. This was proved in the homework, but we include it for completeness.

First we prove it for nonnegative simple functions:

Proposition 0.1. *Let $\phi\colon X \to [0,\infty)$ be a nonnegative simple function. Define a function $\nu\colon \mathcal{M} \to [0,\infty]$ by*

$$\nu(E) = \int_X \phi 1_E \mathrm{d}\mu$$

for all $E \in \mathcal{M}$. Then ν is a measure on $(X, \mathcal{M})$.

Proof. First note that

$$\begin{aligned}
\nu(\emptyset) &= \int_X \phi 1_\emptyset \mathrm{d}\mu \\
&= \int_X \phi \cdot 0 \cdot \mathrm{d}\mu \\
&= \int_X 0 \cdot \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

Now we show that ν is finitely additive. Let $(E_n)_{n=1}^N$ be a finite sequence of pairwise disjoint sets in $\mathcal{M}$. Then

$$\begin{aligned}\nu\left(\bigcup_{n=1}^N E_n\right) &= \int_X \phi 1_{\bigcup_{n=1}^N E_n} \mathrm{d}\mu \\ &= \int_X \phi \sum_{n=1}^N 1_{E_n} \mathrm{d}\mu \\ &= \sum_{n=1}^N \int_X \phi 1_{E_n} \mathrm{d}\mu \\ &= \sum_{n=1}^N \nu(E_n),\end{aligned}$$

where we used the fact that each $\phi 1_{E_n}$ is a nonnegative simple function in order to commute the finite sum with the integral. Thus it follows that ν is finitely additive. It remains to show that ν is countably subadditive. Let (E_n) be a sequence of sets in $\mathcal{M}$. We want to show that

$$\int_X \phi 1_{\bigcup_{n=1}^\infty E_n} \mathrm{d}\mu \leq \sum_{n=1}^\infty \int_X \phi 1_{E_n} \mathrm{d}\mu. \tag{3}$$

To do this, we will show that the sum on the righthand side in (3) is greater than or equal to all integrals of the form $\int \varphi \mathrm{d}\mu$ where $\varphi \colon X \to [0, \infty]$ is a simple function such that $\varphi \leq \phi 1_{\bigcup_{n=1}^\infty E_n}$. Then the inequality (3) will follow from the fact that the integral on the lefthand side in (3) is the supremum of this set. So let $\varphi \colon X \to [0, \infty]$ be a simple function such that $\varphi \leq \phi 1_{\bigcup_{n=1}^\infty E_n}$. Write φ and ϕ in terms of their canonical forms, say

$$\varphi = \sum_{i=1}^k a_i 1_{A_i} \quad \text{and} \quad \phi = \sum_{j=1}^m b_j 1_{B_j}.$$

So $a_i \neq a_{i'}$ and $A_i \cap A_{i'} = \emptyset$ whenever $i \neq i'$ and $b_j \neq b_{j'}$ and $B_j \cap B_{j'} = \emptyset$ whenever $j \neq j'$. Observe that the canonical representation of $\phi 1_{\bigcup_{n=1}^\infty E_n}$ is given by

$$\begin{aligned}\phi 1_{\bigcup_{n=1}^\infty E_n} &= \left(\sum_{j=1}^m b_j 1_{B_j}\right) 1_{\bigcup_{n=1}^\infty E_n} \\ &= \sum_{j=1}^m b_j 1_{B_j} 1_{\bigcup_{n=1}^\infty E_n} \\ &= \sum_{j=1}^m b_j 1_{\bigcup_{n=1}^\infty B_j \cap E_n},\end{aligned}$$

where this representation is the canonical representation since $b_j \neq b_{j'}$ and

$$\left(\bigcup_{n=1}^\infty B_j \cap E_n\right) \cap \left(\bigcup_{n=1}^\infty B_{j'} \cap E_n\right) = \emptyset$$

whenever $j \neq j'$ (since $B_j \cap B_{j'} = \emptyset$). Therefore we have

$$\begin{aligned}
\int_X \varphi \mathrm{d}\mu &\leq \int_X \phi 1_{\bigcup_{n=1}^\infty E_n} \mathrm{d}\mu \\
&= \sum_{j=1}^m b_j \mu \left(\bigcup_{n=1}^\infty B_j \cap E_n \right) \\
&\leq \sum_{j=1}^m b_j \sum_{n=1}^\infty \mu(B_j \cap E_n) \\
&= \sum_{n=1}^\infty \sum_{j=1}^m b_j \mu(B_j \cap E_n) \\
&= \sum_{n=1}^\infty \sum_{j=1}^m \int_X b_j 1_{B_j \cap E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^\infty \int_X \sum_{j=1}^m b_j 1_{B_j \cap E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^\infty \int_X \sum_{j=1}^m b_j \left(1_{B_j} 1_{E_n} \right) \mathrm{d}\mu \\
&= \sum_{n=1}^\infty \int_X \left(\sum_{j=1}^m b_j 1_{B_j} \right) 1_{E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^\infty \int_X \phi 1_{E_n} \mathrm{d}\mu,
\end{aligned}$$

where we used monotonicity of integration in the first line and where we used countable subadditivity of μ to get from the second line to the third line.

□

Now we prove it for more general nonnegative measurable functions

Proposition 0.2. *Let $(X, \mathcal{M}, \mu)$ be measure space and let $g\colon X \to [0, \infty)$ be a nonnegative measurable function. Define $\nu_g\colon \mathcal{M} \to [0, \infty]$ by*

$$\nu_g(E) = \int_X g 1_E \mathrm{d}\mu$$

for all $E \in \mathcal{M}$. Then ν is a measure on $(X, \mathcal{M})$. Furthermore, if $\int_X g \mathrm{d}\mu < \infty$, then $(X, \mathcal{M}, \nu)$ is a finite measure space.

Proof. First note that

$$\begin{aligned}
\nu_g(\emptyset) &= \int_X g 1_\emptyset \mathrm{d}\mu \\
&= \int_X g \cdot 0 \cdot \mathrm{d}\mu \\
&= \int_X 0 \cdot \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

Next we show that ν_g is finitely additive. Let $(E_n)_{n=1}^N$ be a finite sequence of pairwise disjoint sets in $\mathcal{M}$. Then

$$\begin{aligned}
\nu_g \left(\bigcup_{n=1}^N E_n \right) &= \int_X g 1_{\bigcup_{n=1}^N E_n} \mathrm{d}\mu \\
&= \int_X g \sum_{n=1}^N 1_{E_n} \mathrm{d}\mu \\
&= \int_X \sum_{n=1}^N g 1_{E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^N \int_X g 1_{E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^N \nu_g(E_n),
\end{aligned}$$

where we used the fact that each $g1_{E_n}$ is a nonnegative measurable function in order to commute the finite sum with the integral. Thus it follows that ν_g is finitely additive.

It remains to show that ν_g is countably subadditive. In problem 3 in HW4, we showed that for any nonnegative simple function $\varphi\colon X \to [0,\infty)$, the function $\nu_\varphi\colon \mathcal{M} \to [0,\infty]$ defined by

$$\nu_\varphi(E) = \int_X \varphi 1_E \mathrm{d}\mu$$

is a measure. We will use this fact in what follows:

Choose an increasing sequence $(\varphi_n\colon X \to [0,\infty])$ of nonnegative simple functions which converges pointwise to g (we can do this since g is a nonnegative measurable function). Then $(\varphi_n 1_E)$ is an increasing sequence of nonnegative simple functions which converges pointwise to $g1_E$ for all $E \in \mathcal{M}$. It follows from the Monotone Convergence Theorem that

$$\begin{aligned}\nu_{\varphi_n}(E) &= \int_X \varphi_n 1_E \mathrm{d}\mu \\ &\to \int_X g1_E \mathrm{d}\mu \\ &= \nu_g(E)\end{aligned}$$

for any $E \in \mathcal{M}$. In particular, for any $E \in \mathcal{M}$ and $\varepsilon > 0$, we can find a $N_{E,\varepsilon} \in \mathbb{N}$ (which depends on E and ε) such that

$$\nu_g(E) < \nu_{\varphi_n}(E) + \varepsilon \tag{4}$$

for all $n \geq N_{E,\varepsilon}$. However we will don't need estimate $\nu_g(E)$ to this level of precision. We just need to know thta for any $\varepsilon > 0$, we can find an $n \in \mathbb{N}$ such that (4) holds.

Now let (E_k) be a sequence of measurable sets and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\nu_g\left(\bigcup_{k=1}^{\infty} E_k\right) < \nu_{\varphi_n}\left(\bigcup_{k=1}^{\infty} E_k\right) + \varepsilon$$

Then we have

$$\begin{aligned}\nu_g\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \nu_{\varphi_n}\left(\bigcup_{k=1}^{\infty} E_k\right) + \varepsilon \\ &\leq \sum_{k=1}^{\infty} \nu_{\varphi_n}(E_k) + \varepsilon \\ &\leq \sum_{k=1}^{\infty} \nu_g(E_k) + \varepsilon\end{aligned}$$

where we obtained the second line from the first line using countable subadditivity of ν_{φ_n}, and where we obtained the third line from the second line from the fact that $\varphi_n 1_{E_k} \leq g1_{E_k}$ for all k and from monotonicity of integration. Taking $\varepsilon \to 0$ gives us countable subadditivity of ν_g.

Finally, for the last part, we note that

$$\begin{aligned}\nu(X) &= \int_X g1_X \mathrm{d}\mu \\ &= \int_X g \mathrm{d}\mu \\ &< \infty.\end{aligned}$$

□

Exercise 11. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be a nonnegative measurable function such that $\int_X f \mathrm{d}\mu < \infty$. Prove that for any sequence (E_n) of measurable sets such that $\sum_{n=1}^{\infty} \nu(E_n) < \infty$, we have

$$\lim_{n\to\infty} 1_{E_n} f = 0$$

for μ a.e. x.

Solution 11. First recall three propositions we proved in the homework:

Proposition 0.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$. Then*

$$\mu\left(\liminf E_n\right) \leq \liminf \mu(E_n)$$

Proof. Note that the sequence

$$\left(\bigcap_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is an ascending sequence in N. Therefore we have

$$\begin{aligned}
\mu\left(\liminf E_n\right) &= \mu\left(\bigcup_{N=1}^{\infty}\left(\bigcap_{n\geq N} E_n\right)\right) \\
&= \liminf \mu\left(\bigcap_{n\geq N} E_n\right) \\
&\leq \lim_{N\to\infty} \inf\left\{\mu(E_n) \mid n \geq N\right\} \\
&= \liminf \mu(E_n),
\end{aligned}$$

where we obtained the third line from the second line since

$$\mu\left(\bigcap_{n\geq N} E_n\right) \leq \mu(E_n)$$

for all $n \geq N$ by monotonicity of μ. $\square$

Proposition 0.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$ such that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$. Then*

$$\mu\left(\limsup E_n\right) \geq \limsup \mu(E_n)$$

Proof. Note that the sequence

$$\left(\bigcup_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is a descending sequence in N. This together with the fact that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$ implies

$$\begin{aligned}
\mu\left(\limsup E_n\right) &= \mu\left(\bigcap_{N=1}^{\infty}\left(\bigcup_{n\geq N} E_n\right)\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} E_n\right) \\
&\geq \lim_{N\to\infty} \sup\left\{\mu(E_n) \mid n \geq N\right\} \\
&= \limsup \mu(E_n),
\end{aligned}$$

where we obtained the third line from the second line since

$$\mu\left(\bigcup_{n\geq N} E_n\right) \geq \mu(E_n)$$

for all $n \geq N$ by monotonicity of μ. $\square$

Proposition 0.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let (E_n) be a sequence in $\mathcal{M}$ such that $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then*

$$\mu\left(\limsup E_n\right) = 0.$$

Proof. Note that the sequence

$$\left(\bigcup_{n\geq N} E_n\right)_{N\in\mathbb{N}}$$

is a descending sequence in N. This together with the fact that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) < \infty$ implies

$$\begin{aligned}
\mu\left(\limsup E_n\right) &= \mu\left(\bigcap_{N=1}^{\infty}\left(\bigcup_{n\geq N} E_n\right)\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} E_n\right) \\
&\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) \\
&= 0,
\end{aligned}$$

where the last equality follows since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. $\square$

Now we prove part (b). Let (E_n) be a sequence of measurable sets such that

$$\sum_{n=1}^{\infty} \nu(E_n) < \infty.$$

Then observe that

$$\begin{aligned}
\int_X \lim_{n\to\infty} f 1_{E_n} \mathrm{d}\mu &= \int_X \liminf f 1_{E_n} \mathrm{d}\mu \\
&\leq \liminf \int_X f 1_{E_n} \mathrm{d}\mu \\
&= \liminf \nu(E_n) \\
&\leq \limsup \nu(E_n) \\
&\leq \nu(\limsup E_n) \\
&= 0.
\end{aligned}$$

where we applied Fatou's Lemma to get the second line from the first line. It follows that $\lim_{n\to\infty} f 1_{E_n} = 0$ almost everywhere (by a proposition proved in class).

Appendix

Problem 1

Proposition 0.6. *Let $f, g\colon X \to \mathbb{R}$ be measurable functions and let $a \in \mathbb{R}$. Then af, $|f|$, f^2, $f+g$, fg, $\max\{f,g\}$, and $\min\{f,g\}$ are all measurable.*

Proof. We first show af is measurable. If $a = 0$, then af is the zero function, which is measurable. So assume $a \neq 0$. Then we have

$$(af)^{-1}(-\infty, c) = \begin{cases} f^{-1}(-\infty, c/a) \in \mathcal{M} & \text{if } a > 0 \\ f^{-1}(c/a, \infty) \in \mathcal{M} & \text{if } a < 0 \end{cases}$$

αf is measurable αf is measurable.
Observe that

$$\begin{aligned}
x \in (f+g)^{-1}(-\infty, c) &\iff f(x) + g(x) < c \\
&\iff \text{there exists an } r \in \mathbb{Q} \text{ such that } f(x) < r \text{ and } r < c - g(x) \\
&\iff \text{there exists an } r \in \mathbb{Q} \text{ such that } x \in f^{-1}(-\infty, r) \cap g^{-1}(-\infty, c-r). \\
&\iff x \in \bigcup_{r\in\mathbb{Q}} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, c-r).
\end{aligned}$$

Therefore

$$(f+g)^{-1}(-\infty, c) = \bigcup_{r\in\mathbb{Q}} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, c-r) \in \mathcal{M}.$$

We first prove f^2 is measurable:

$$(f^2)^{-1}(c, \infty) = \begin{cases} f^{-1}(\sqrt{c}, \infty) \cup f^{-1}(-\infty, -\sqrt{c}) \in \mathcal{M} & c \geq 0 \\ E \in \mathcal{M} & c < 0 \end{cases}$$

Next, note that

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right) \in \mathcal{M}.$$

Finally note that

$$\max\{f, g\} = \frac{1}{2}(|f+g| + |f-g|)$$

and

$$\min\{f, g\} = \frac{1}{2}(|f+g| - |f-g|).$$

□

Proposition 0.7. *Let $(f_n \colon X \to \mathbb{R})$ be a sequence of measurable functions. Then* $\sup f_n$*,* $\inf f_n$*,* $\limsup f_n$*, and* $\liminf f_n$ *are all measurable. In particular, of*

$$\lim_{n\to\infty} f_n(x)$$

exists for all $x \in X$. The corresponding function is also measurable.

Proof. Let $c \in \mathbb{R}$. Then we have

$$(\sup f_n)^{-1}(c, \infty) = \bigcup_n f_n^{-1}(c, \infty) \in \mathcal{M}.$$

Similarly, we have

$$(\inf f_n)^{-1}(-\infty, c) = \bigcup_n f_n^{-1}(-\infty, c) \in \mathcal{M}.$$

Also we have

$$\limsup f_n = \inf_k \sup_{n \geq k} \in \mathcal{M}.$$

Similarly, we have

$$\liminf f_n = \sup_k \inf_{n \geq k} f_n \in \mathcal{M}$$

□

Problem 2

Proposition 0.8. *If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\text{m}} f$.*

Proof. Suppose $f_n \xrightarrow{L^1} f$ and let $\varepsilon, \delta > 0$. Choose $N_{\varepsilon,\delta} \in \mathbb{N}$ such that $n \geq N_{\varepsilon,\delta}$ implies

$$\|f_n - f\|_1 < \varepsilon\delta.$$

Then it follows from Chebyshev's inequality that

$$\begin{aligned}
\mu\left(\{\{x \in X \mid |f_n(x) - f(x)| \geq \varepsilon\}\}\right) &\leq \frac{1}{\varepsilon}\|f_n - f\|_1 \\
&< \frac{1}{\varepsilon}\varepsilon\delta \\
&= \delta.
\end{aligned}$$

Thus $f_n \xrightarrow{\text{m}} f$. □

Problem 3

Proposition 0.9. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(f_n \colon X \to [0, \infty])$ is a decreasing sequence of nonnegative measurable functions which converges pointwise to $f \colon X \to [0, \infty]$. If $\int_X f_1 \mathrm{d}\mu < \infty$, then*

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu. \tag{5}$$

Proof. For each $n \in \mathbb{N}$, set $g_n = f_{n+1} - f_n$. Since (f_n) is a decreasing sequence, each g_n is nonnegative. Furthermore each g_n is measurable since it is a difference of two measurable functions. Set $g = \sum_{n=1}^{\infty} g_n$ and observe that

$$\begin{aligned} g &= \sum_{n=1}^{\infty} g_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} g_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} (f_{n+1} - f_n) \\ &= \lim_{N\to\infty} (f_N - f_1) \\ &= f - f_1. \end{aligned}$$

It follows from problem 4 that

$$\begin{aligned} \int_X f \mathrm{d}\mu - \int_X f_1 \mathrm{d}\mu &= \int_X (f - f_1) \mathrm{d}\mu \\ &= \int_X g \mathrm{d}\mu \\ &= \sum_{n=1}^{\infty} \int_X g_n \mathrm{d}\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X g_n \mathrm{d}\mu \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X (f_{n+1} - f_n) \mathrm{d}\mu \\ &= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} (f_{n+1} - f_n) \mathrm{d}\mu \\ &= \lim_{N\to\infty} \int_X (f_N - f_1) \mathrm{d}\mu \\ &= \lim_{N\to\infty} \int_X f_N \mathrm{d}\mu - \int_X f_1 \mathrm{d}\mu. \end{aligned}$$

Since $\int_X f_1 \mathrm{d}\mu < 0$, we can cancel it from both sides to get (5). □

Problem 6

Proposition 0.10. *If $\int_X |f| \mathrm{d}\mu = 0$, then $\mu(\{f \neq 0\}) = 0$.*

Proof. Note that $\{f \neq 0\} = \{|f| \neq 0\}$. Also $\{|f| \neq 0\} = \bigcup_{n=1}^{\infty} \{|f| \geq 1/n\}$. Thus

$$\begin{aligned} \mu(\{|f| \neq 0\}) &= \mu\left(\bigcup_{n=1}^{\infty} \{|f| \geq 1/n\}\right) \\ &= \lim_{n\to\infty} \mu(\{|f| \geq 1/n\}) \\ \text{(C-M)} &\leq \lim_{n\to\infty} n \int_X |f| \mathrm{d}\mu \\ &= 0. \end{aligned}$$

□