Basic Topology

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1 Topological Spaces

Definition 1.1. A **topological space** is an ordered pair (X, τ), where X is a nonempty set and τ is a collection of subsets of X, satisfying the following axioms:

1. The empty set $\emptyset$ and the enitre set X belongs to τ.
2. τ is closed under arbitrary unions: if $U_i \in \tau$ for all i in some arbitrary index set I, then $\bigcup_{i \in I} U_i \in \tau$.
3. τ is closed under finite intersections: if $U_1, \dots, U_n \in \tau$, where $n \in \mathbb{N}$, then $\bigcap_{m=1}^{n} U_m \in \tau$.

The elements of τ are called **open sets** and the collection τ is called a **topology** on X.

Remark.

1. We often just write X instead of (X, τ) to denote a topological space. We also say τ gives X a topology.
2. Typically, one describes a topological space by specifying its open sets.

1.0.1 Comparison of Topologies

Definition 1.2. Let τ and τ' be two topologies on a set X. If $\tau \subseteq \tau'$, then we say τ is a **coarser** (**weaker** or **smaller**) topology than τ. Similarly, if $\tau \subseteq \tau'$, then we say τ' is a **finer** (**stronger** or **larger**) **topology** than τ.

Proposition 1.1. *Let τ and τ' be two topologies on a set X. Suppose that for every $x \in X$ and for every τ-open neighborhood U_x of x, there exists a τ'-open neighborhood U'_x of x such that $U'_x \subseteq U_x$. Then τ' is finer than τ.*

Proof. Let $U \in \tau$. For each $x \in U$, choose a τ'-open neighborhood U'_x of x such that $U'_x \subseteq U$. Then

$$\begin{aligned} U &= \bigcup_{x \in U} U'_x \\ &\in \tau'. \end{aligned}$$

It follows that τ' is finer than τ. □

1.0.2 Subspace Topology

Let (X, τ) be a topological space and let Z be a subset of X. We can give Z a topology by declaring open subsets of Z to be all sets of the form $U \cap Z$, where U is an open subset of X. One easily verifies that the collection of these open sets satisfy the axioms of forming a topology. We call this the **subspace topology induced by** τ.

1.0.3 Generating a Topology from a Collection of Subsets

Proposition 1.2. *Let X be a set and let $\mathscr{C}$ be a nonempty collection of subsets of X. Then there exists a smallest topology on X which contains $\mathscr{C}$. It is called the **topology generated by** $\mathscr{C}$ and is denoted $\tau(\mathscr{C})$. We also call $\mathscr{C}$ a **subbase** for $\tau(\mathscr{C})$.*

Proof. We define $\tau(\mathscr{C})$ to be the collection of all subsets of X obtained by adjoining to $\mathscr{C}$ the set X itself, empty set, and all arbitary unions of finite intersections of members of $\mathscr{C}$. To see that $\tau(\mathscr{C})$ is a topology, note that arbitrary unions of arbitrary unions of finite intersections is an arbitrary union of finite intersections, so it suffices to show that $\tau(\mathscr{C})$ is closed under finite intersections. Let $A, A' \in \tau(\mathscr{C})$. Then A has the form

$$A = \bigcup_{i \in I} (C_{i,1} \cap C_{i,2} \cap \cdots \cap C_{i,n_i})$$

where $C_{i,j} \in \mathscr{C}$ and $n_i \in \mathbb{N}$ for all $i \in I$ and $1 \leq j \leq n_i$. Similarly, A' has the form

$$A' = \bigcup_{i' \in I'} (C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}})$$

where $C'_{i',j'} \in \mathscr{C}$ and $n'_{i'} \in \mathbb{N}$ for all $i' \in I'$ and $1 \leq j' \leq n'_{i'}$. Thus

$$\begin{aligned}
A \cap A' &= A \cap \left(\bigcup_{i' \in I'} (C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}}) \right) \\
&= \bigcup_{i' \in I'} \left(A \cap C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}} \right) \\
&= \bigcup_{i' \in I'} \left(\left(\bigcup_{i \in I} (C_{i,1} \cap C_{i,2} \cap \cdots \cap C_{i,n_i}) \right) \cap C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}} \right) \\
&= \bigcup_{i' \in I'} \left(\bigcup_{i \in I} (C_{i,1} \cap C_{i,2} \cap \cdots \cap C_{i,n_i} \cap C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}}) \right) \\
&= \bigcup_{(i,i') \in I \times I'} (C_{i,1} \cap C_{i,2} \cap \cdots \cap C_{i,n_i} \cap C'_{i',1} \cap C'_{i',2} \cap \cdots \cap C'_{i',n'_{i'}}) \\
&\in \tau(\mathscr{C}).
\end{aligned}$$

Thus $\tau(\mathscr{C})$ is closed under finite intersections. □

Definition 1.3. Let (X, τ) be a topological space. A collection $\mathscr{B}$ of open subsets of X is called a **base** (or **basis**) for τ if

1. $\mathscr{B}$ covers X,
2. For all $U, V \in \mathscr{B}$ and for all points $x \in U \cap V$, there exists $W \in \mathscr{B}$ such that $x \in W \subseteq U \cap V$.

1.0.4 Neighborhoods

Definition 1.4. Let X be a topological space and let $x \in X$. An open subset U of X is called an **open neighborhood of** x if $x \in U$. If U is a basis element in the topology, then we say U is a **basic open neighborhood of** x.

1.1 Continuous Functions

Definition 1.5. Let X and Y be topological spaces. A function $f \colon X \to Y$ is called **continuous** if $f^{-1}(V)$ is an open subset of X whenever V is an open subset of Y. We say f is **continous at a point** $x \in X$ if for any open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$.

Remark. To check continuity of f at $x \in X$, it is enough to shows that for any *basic* open neighborhood V_0 of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V_0$. Indeed, if this property holds, then for any open neighborhood V of $f(x)$, we choose a basic open neighborhood V_0 of $f(x)$ such that $V_0 \subseteq V$ and an open neighborhood U of x such that $f(U) \subseteq V_0 \subseteq V$.

Proposition 1.3. *Let X and Y be topological spaces. A function $f \colon X \to Y$ is continous if and only if it is continuous at every point $x \in X$.*

Proof. First assume that f is continous. Let $x \in X$ and V be an open neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open subset of X since f is continous and, moreover, it contains x. Thus $f^{-1}(V)$ is an open neighborhood of x. Since x was arbitary, f is continous at every point in X.

Conversely, assume that f is continous at every point in X. Let V be an open subset of Y. We need to show that $f^{-1}(V)$ is an open subset of X. For all $x \in f^{-1}(V)$, we can find an open neighborhood U_x of x such that $U_x \subset f^{-1}(V)$ (i.e. $f(U_x) \subseteq V$), since f is continous at every point in X. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

implies that $f^{-1}(V)$ is open. □

Remark. Suppose $f\colon X \to Y$ is continuous at a point $x_0 \in X$. One may suspect that f is continuous in some open neighborhood of x_0, but this is not the case. For a counterexample, consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We claim that f is continuous at 0 but is not continuous anywhere else. To show that f is continuous at 0, let $B_\varepsilon(0)$ be an ε-ball centered at $f(0) = 0$. Then $f\left(B_{\sqrt{\varepsilon}}(0)\right) \subset B_\varepsilon(0)$. Indeed if $x \in B_{\sqrt{\varepsilon}}(0)$ is rational, then $f(x) = 0 \in B_\varepsilon(0)$ and if $x \in B_{\sqrt{\varepsilon}}(0)$ is irrational, then $f(x) = x^2 \in B_\varepsilon(0)$ (since $|x| < \sqrt{\varepsilon}$ and hence $x^2 < \varepsilon$). It is an easy exercise to show that f is not continuous anywhere else.

Proposition 1.4. *Let X, Y be topological spaces, $f\colon X \to Y$ be a continuous function, and let $A \subset X$ be given the subspace topology. Then $f|_A\colon A \to Y$ is continuous.*

Proof. Let V be an open subset of Y. Then $f^{-1}(V)$ is an open subset of X since f is continuous. Therefore $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is an open subset of A. □

1.2 Continuity in Metric Spaces

Definition 1.6. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ which satisfies the following three properties:

1. (Identity of indiscernibles) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
2. (Symmetric) $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A set X together with a choice of a metric d is called a **metric space** and is denoted (X, d), or just denoted X if the metric is understood from context.

Remark. Given the three axioms above, we also have $d(x, y) \geq 0$ (positive-definitene) for all $x, y \in X$. Indeed,

$$\begin{aligned} 0 &= d(x, x) \\ &\leq d(x, y) + d(y, x) \\ &= d(x, y) + d(x, y) \\ &= 2d(x, y). \end{aligned}$$

This implies $d(x, y) \geq 0$.

1.2.1 Open Balls

Definition 1.7. Let (X, d) be a metric space. For $x \in X$ and $\varepsilon > 0$, we define the **open ball centered at x of radius** ε, denoted $B_\varepsilon(x)$, to be the set

$$B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}.$$

Proposition 1.5. *Let (X, d) be a metric space. If $B_\varepsilon(x)$ and $B_{\varepsilon'}(x')$ are two open balls centered at $x \in X$ (resp. $x' \in X$) of radius $\varepsilon > 0$ (resp. $\varepsilon' > 0$) such that $B_\varepsilon(x) \cap B_{\varepsilon'}(x') \neq \emptyset$, then there exists $x'' \in X$ and $\varepsilon'' > 0$ such that*

$$B_{\varepsilon''}(x'') \subseteq B_\varepsilon(x) \cap B_{\varepsilon'}(x).$$

Proof. Pick any $x'' \in B_\varepsilon(x) \cap B_{\varepsilon'}(x')$. Set $\delta = d(x, x'')$ and $\delta' = d(x', x'')$. Without loss of generality, say $\varepsilon - \delta \leq \varepsilon' - \delta'$. Then we set $\varepsilon'' = \varepsilon - \delta$. If $y \in B_{\varepsilon''}(x)$, then

$$\begin{aligned} d(x, y) &\leq d(x, x'') + d(x'', y) \\ &= \delta + d(x'', y) \\ &< \varepsilon \end{aligned}$$

implies $y \in B_\varepsilon(x)$ and

$$\begin{aligned} d(x', y) &\leq d(x', x'') + d(x'', y) \\ &= \delta' + d(x'', y) \\ &< \delta' + \varepsilon - \delta \\ &\leq \varepsilon' \end{aligned}$$

implies $y \in B_{\varepsilon'}(x)$. □

Let (X, d) be a metric space. The proposition above implies that the open balls form the base for a topology of X, making it a topological space. A topological space which can arise in this way from a metric space is called a **metrizable** space.

1.2.2 Epsilon-Delta and Metric Spaces

Let U be an open subset of $\mathbb{R}^n$. Then a function $f\colon U \to \mathbb{R}$ is continuous at a point $p \in U$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|q - p\| < \delta \text{ implies } \|f(q) - f(p)\| < \varepsilon.$$

In terms of open sets, this says for all basic open neighborhoods $B_\varepsilon(f(p))$ of $f(p)$, there is a basic open neighborhood $B_\delta(p)$ of p such that $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$.

Proposition 1.6. *Let (X, d) and (Y, d) be metric spaces. Then $f\colon X \to Y$ is continous at at a point $x \in X$ if and only if for all sequences (x_n) in X such that $x_n \to x$, we have $f(x_n) \to f(x)$.*

Proof. First suppose f is continous at $x \in X$. Let (x_n) be a sequence in X such that $x_n \to x$. We need to show that $f(x_n) \to f(x)$. Let V be an open neighborhood of $f(x)$. Since f is continuous at x, there exists an open neighborhood U of x such that $f(U) \subseteq V$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Then $f(x_n) \in V$ for all $n \geq N$. This shows that $f(x_n) \to f(x)$.

Conversely, suppose that for all sequences (x_n) in X such that $x_n \to x$, we have $f(x_n) \to f(x)$. We need to show that f is continuous. We will prove that f is continuous at x by contradiction: assume that f is not continuous at x. Choose an open neighborhood V of $f(x)$ such that there is no neighborhood U of x where $f(U) \subseteq V$. Let

$$U_n := B_{\frac{1}{n}}(x) := \left\{ x' \in X \mid d(x, x') < \frac{1}{n} \right\}.$$

Choose $x'_n \in U_n$ such that $f(x'_n) \notin V$. Then (x'_n) is a sequence in X such that $x'_n \to x$, but $f(x'_n) \not\to f(x)$ (indeed, $f(x'_n)$ is never in V). Contradiction. □

Example 1.1. Consider the step function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then f is not continuous at $x = 0$ since, for example, the sequence $(1/n) \to 0$ but $(f(1/n)) \to 1 \neq f(0)$. On the other hand, the sequence $(-1/n) \to 0$ and $f(-1/n) = 0 = f(0)$. Thus we really do need f to preserve *all* convergent sequences in order for it to be continuous.

1.3 First-Countable Spaces

Definition 1.8. A topological space X is said to be **first-countable** if each point has a countable neighborhood basis. That is, for each $x \in X$, there exists a sequence (U_n) of open neighborhoods of x such that for any open neighborhood U of x there exists an $n \in \mathbb{N}$ such that $U_n \subseteq U$.

Proposition 1.7. *Let $f\colon X \to Y$ be a function and assume that X is first-countable. Then f is continous at a point $x \in X$ if and only if for all sequences (x_n) in X such that $x_n \to x$, we have $f(x_n) \to f(x)$.*

Proof. First suppose f is continous at $x \in X$. Let (x_n) be a sequence in X such that $x_n \to x$. Let V be an open neighborhood of $f(x)$. Since f is continuous at x, we can choose an open neighborhood U of x such that $f(U) \subseteq V$. Since $x_n \to x$, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$. Then $n \geq N$ implies

$$\begin{aligned} f(x_n) &\in f(U) \\ &\subseteq V. \end{aligned}$$

It follows that $f(x_n) \to f(x)$.

Conversely, suppose that for all sequences (x_n) in X such that $x_n \to x$, we have $f(x_n) \to f(x)$. Assume that f is not continuous at x. Choose an open neighborhood V of $f(x)$ such that there does not exist an open neighborhood U of x with $f(U) \subseteq V$. Now we apply first-countability of X. Choose a neighborhood basis of x, say (U_n). For each $n \in \mathbb{N}$ choose $x_n \in U_n$ such that $f(x_n) \notin V$. Then $x_n \to x$ since for any open neighborhood U of x, we can find an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$x_n \in U_N \subseteq U.$$

On the other hand, $f(x_n) \not\to f(x)$ since $f(x_n) \notin V$ for all $n \in \mathbb{N}$. Contradiction. □

1.4 Discrete Topologies

Definition 1.9. Let X be a set. The **discrete topology** on X is defined by letting every subset of X be open.

Proposition 1.8. *Let X and Y be topological spaces.*

1. *If Y has the discrete topology. Then every continous map $f : X \to Y$ is locally constant*[a]*.*
2. *If X has the discrete topology, then every function $f : X \to Y$ is continous.*

[a]This means for every $x \in X$, there exists an open neighborhood U_x of x such that f is constant on U_x: $f(y) = f(x)$ for all $y \in U_x$.

Proof.

1. Let $f : X \to Y$ be a continous function and $x \in X$. Then $\{f(x)\}$ is open in Y since Y has the discrete topology. Denote U_x to be the inverse image of $\{f(x)\}$ under f:
$$U_x := f^{-1}\{f(x)\}.$$
Then U_x is an open neighborhood x on which f is constant.
2. Let $f : X \to Y$ be a function and let V be an open subset of Y. Since X is discrete, every subset of X is open. In particular, $f^{-1}(V)$ is open.

□

1.4.1 Weakest Topology on Codomain

Let X be a set, $\{X_i\}_{i\in I}$ be a collection of topological spaces, and let $\{f_i : X_i \to X\}$ be a collection of functions. We want to give X a topology such that the maps f_i become continuous. We do this by declaring a subset U of X to be open if and only if $f_i^{-1}(U)$ is open in X_i for all i. That this really is a topology follows from the identities:

$$f_i^{-1}\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) = \bigcup_{\lambda\in\Lambda} f_i^{-1}(U_\lambda) \text{ and } f_i^{-1}\left(\bigcap_{\lambda\in\Lambda} U_\lambda\right) = \bigcap_{\lambda\in\Lambda} f_i^{-1}(U_\lambda).$$

1.4.2 Weakest Topology on Domain

Let X be a set, $\{X_i\}_{i\in I}$ be a collection of topological spaces, and let $\{f_i : X \to X_i\}$ be a collection of functions. We want to give X a topology such that the maps f_i become continuous. If U_i is an open subset of X_i, then certainly we need $f_i^{-1}(U_i)$ to be an open subset of X. We give X the smallest topology that contains all sets of the form $f_i^{-1}(U_i)$, where $i \in I$ and U_i is an open subset of i.

1.5 Gluing

Definition 1.10. Let X be a topological space. An **open covering** of X is a collection $\{U_i\}_{i\in I}$ of open subsets U_i of X such that
$$\bigcup_{i\in I} U_i = X.$$

Let X be a topological space and let $\{X_i\}$ be an open covering, so each X_i gets an induced topology. Note that a subset $U \subseteq X$ is open if and only if $U \cap X_i$ is open in X_i for each i. Indeed, one direction is clear. For the other direction, suppose $U \cap X_i$ is open in X_i for each i. Then for each i, there exists an open subset U_i of X such that $U_i \cap X_i = U \cap X$. Therefore
$$U = \bigcup_{i\in I} U \cap X_i = \bigcup_{i\in I} U_i \cap X_i,$$
shows that U is a union of open subsets of X.

If $f : X \to Y$ is a continouous map, then by restriction to X_i we get continuous maps $f_i : X_i \to Y$ such that
$$f_i \mid_{X_i\cap X_j} = f_j \mid_{X_i\cap X_j} \text{ for all } i \text{ and } j \tag{1}$$
Conversely, if we are given continuous maps $f_i : X_i \to Y$ such that (1) holds, then there is a unique set-theoretic map $f : X \to Y$ satisfying $f \mid_{X_i} = f_i$ for all i, and moreover it is continuous. Indeed, for any open subset V of Y we have $f^{-1}(V)$ is open in X because $f^{-1}(V) \cap X_i = f_i^{-1}(V)$ is open in X_i for every i. Hence, we can view continuous maps $X \to Y$ as collections of continuous maps $X_i \to Y$ that are compatible on the overlaps $X_i \cap X_j$. We want to run this procedure in reverse.

Theorem 1.1. *Let X be a set, and let $\{X_i\}$ be a collection of subsets whose union is X. Suppose on each X_i there is a given topology τ_i and that the τ_i's are compatible in the following sense: $X_i \cap X_j$ is open in each of X_i and X_j, and the induced topologies on $X_i \cap X_j$ from both X_i and X_j coincide. There is a unique topology on X that induces upon each X_i the topology τ_i.*

Remark. We say that the topology in this theorem is obtained by **gluing** the given topologies on the X_i's (We may also say that the topological space (X, τ) is obtained by **gluing** the topological spaces (X_i, τ_i).

Proof. We first prove uniqueness. If τ is a topology on X inducing τ_i for each i and making X_i open in X for each i, then a subset $U \subseteq X$ is open for τ if and only if $U \cap X_i$ is open for the induced topology on X_i for each i (as X_i is τ-open for every i), and hence (by the assumption that the induced topology on X_i is τ_i) if and only if $U \cap X_i$ is τ_i-open in X_i for each i. This final formulation of the openness condition for τ is expressed entirely in terms of the τ_i's and so establishes uniqueness: we have no choice as to what the condition of τ-openness is to be, and it must be the case that the τ-open sets in X are exactly those that meet each X_i in a τ_i-open subset of X_i for each i.

We now run the process in reverse to verify the existence. We *define* τ to be the collection of subsets $U \subseteq X$ such that $U \cap X_i$ is τ_i-open in X_i for each i. This topology is the weakest topology which makes the inclusion maps $X_i \hookrightarrow X$ continuous. Since for each fixed i_0 the overlap $X_{i_0} \cap X_j$ is τ_j-open in X_j for every j, it follows that X_{i_0} is τ-open in X for every i_0. □

2 Compactness

Definition 2.1. Let X be a topological space. We say X is **compact** every open covering of X contains a finite subcovering of X: if $\{U_i\}_{i\in I}$ covers X, then for some $n \in \mathbb{N}$ there exists $U_{i_1}, U_{i_2}, \ldots, U_{i_n} \in \{U_i\}_{i\in I}$ such that $\{U_{i_k}\}_{k=1}^n$ covers X. We say a subset K of X is a **compact subset** of X if K is compact with respect to the subspace topology.

Let $\mathcal{B}$ be a basis for X. To check for compactness for X, it is enough to only consider open coverings $\{U_i\}_{i\in I}$ where the U_i are in $\mathcal{B}$:

Proposition 2.1. *Let X be a topological space and let $\mathcal{B}$ be a basis for X. Then X is compact if and only if every open covering of X consisting of basis elements contains a finite subcovering of X.*

Proof. One direction is clear. For the other direction, assume that every open covering of X consisting of basis elements contains a finite subcovering of X. Let $\{U_i\}_{i\in I}$ be an open covering of X (where the U_i are not necessarily basis elements). For each $i \in I$, let $\{V_{i,j}\}_{j\in J}$ be an open covering of U_i consisting of basis elements (so the $V_{i,j}$ are basis elements). Then $\{V_{i,j}\}_{i\in I, j\in J}$ is an open covering of X consisting of basis elements and so there exists a finite subcovering, say $\{V_{i_\lambda, j_\gamma}\}_{\lambda\in\Lambda, \gamma\in\Gamma}$ where Λ and Γ are finite sets. Then $\{U_{i_\lambda}\}_{\lambda\in\Lambda}$ is a finite subcovering of $\{U_i\}_{i\in I}$. □

Remark. The proposition above is still true if we replace $\mathcal{B}$ with a subbase. However to prove this, we would need to use the Ultrafilter principle.

Lemma 2.1. *Let X be Hausdorff and let K be a compact subset of X. Then K is closed in X.*

Proof. We show that $X \backslash K$ is open. Let $x \in X\backslash K$. For each $y \in K$, choose an open neighborhood U_y of y and an open neighborhood V_y of x such that $U_y \cap V_y = \emptyset$. Since K is compact, the open covering $\{U_y \cap K\}_{y\in K}$ of K contains a finite subcovering of K, say $\{U_{y_i} \cap K\}_{i=1}^n$ where $y_i \in K$ for $i = 1, \ldots, n$. Then

$$V_x := \bigcap_{i=1}^n V_{y_i}$$

is an open neighborhood of x which does not meet K. Therefore

$$X\backslash K = \bigcup_{x\in X\backslash K} V_x,$$

which implies $X\backslash K$ is open, which implies K is closed. □

2.0.1 Image of a Compact Space is Compact

Proposition 2.2. *Let $f\colon X \to Y$ be a continuous function from a compact space X to a topological space Y. Then $f(X)$ is a compact subspace of Y.*

Proof. Let $\{V_j \cap f(X)\}_{j\in J}$ be an open covering of $f(X)$, where the V_j are open subsets of Y. Then $\{f^{-1}(V_j)\}_{j\in J}$ is an open covering of X. Since X is compact, there exists a finite subcover of $\{f^{-1}(V_j)\}_{j\in J}$ wich covers X, say $\{f^{-1}(V_{j_1}),\dots,f^{-1}(V_{j_k})\}$. Then $\{V_{j_1}\cap f(X),\dots,V_{j_k}\cap f(X)\}$ is a finite subcover of $\{V_j\cap f(X)\}_{j\in J}$ which covers $f(X)$. Thus $f(X)$ is compact. □

2.0.2 Finite Intersection Property

There is another way of thinking about compactness.

Definition 2.2. Let X be a topological space. We say that X satisfies the **finite intersection property** (or **FIP**) for closed sets if any collection $\{Z_i\}_{i\in I}$ of closed sets in X with all finite intersections

$$Z_{i_1}\cap\cdots\cap Z_{i_n}\neq\emptyset,$$

the intersection $\bigcap_{i\in I} Z_i$ of all Z_i's is non-empty.

Theorem 2.2. *Let X be a topological space. Then X is compact if and only if it satisfies FIP for closed sets.*

Proof. This is an exercise in linguistics. Suppose first that X is compact. To obtain a contradiction, assume that X does not satisfy FIP for closed sets. Then there exists a collection $\{Z_i\}_{i\in I}$ of closed sets in X with all finite intersections $Z_{i_1}\cap\cdots\cap Z_{i_n}\neq\emptyset$ and with $\bigcap_{i\in I} Z_i=\emptyset$. But this implies $\{X\backslash Z_i\}_{i\in I}$ is an open cover of X with no finite subcover. The converse is proved in exactly the same way. □

2.0.3 When a continuous bijection is a homeomorphism

Lemma 2.3. *Let X be a compact space and let E be a closed subset of X. Then E is also compact.*

Proof. Let $\{U_i\cap E\}_{i\in I}$ be an open cover of E. Then $(X\backslash E)\cup\{U_i\cap E\}_{i\in I}$ is an open cover of X. Since X is compact, there exists a finite subcover in $(X\backslash E)\cup\{U_i\cap E\}_{i\in I}$ of X. In particular, this implies that there exists a finite subcover in $\{U_i\cap E\}_{i\in I}$ of E. □

Lemma 2.4. *Let X be a compact space, Y be any topological space, and let $f: X\to Y$ be continuous surjective map. Then Y is compact.*

Proof. Let $\{V_i\}_{i\in I}$ be an open cover of Y. Since f is continuous, $\{f^{-1}(V_i)\}_{i\in I}$ is an open cover of X. Since X is compact, there exists a finite subcover in $\{f^{-1}(V_i)\}_{i\in I}$ of X, say $\{f^{-1}(V_{i_1}),\dots,f^{-1}(V_{i_n})\}$. But then $\{V_{i_1},\dots,V_{i_n}\}$ is a finite subcover in $\{V_i\}_{i\in I}$ of Y. □

Theorem 2.5. *Let X and Y be topological spaces such that X is compact and Y is Hausdorff, and let $f: X\to Y$ be a continuous bijection. Then f is a homeomorphism.*

Proof. Let $g: Y\to X$ denote the inverse of f. We need to show that g is continuous. We do this by showing that the inverse image of a closed set in X is a closed set in Y: Let E be a closed set in X. Since X is compact, E is compact by Lemma (2.3). Since E is compact, $f(E)$ is compact by Lemma (2.4). Since Y is Hausdorff and $f(E)$ is compact, $f(E)$ is closed by Lemma (2.1). But $f(V)=g^{-1}(V)$, so $g^{-1}(V)$ is closed. □

2.0.4 Closes subspaces of compact spaces are compact

Proposition 2.3. *Let X be a compact space and let A be a closed subspace of X. Then A is compact.*

Proof. Let $\{U_i\cap A\}_{i\in I}$ be an open covering of A. Then $(X\backslash A)\cup\{U_i\}_{i\in I}$ is an open covering of X. Since X is compact, it contains a finite subcovering of X, say $(X\backslash A)\cup\{U_{i_k}\}_{k=1}^n$. But then $\{U_{i_k}\cap A\}_{k=1}^n$ must be a finite subcovering of $\{U_i\cap A\}_{i\in I}$. □

2.1 Heine-Borel Theorem

Definition 2.3. Let S be a subset of a topological space X. We say $x\in X$ is a **limit point** of S if every open neighborhood of x meets S: if U is an open subset of X such that $x\in U$, then $U\cap S\neq\emptyset$.

Theorem 2.6. *Let S be a subset of Euclidean space $\mathbb{R}^n$. Then S is compact if and only if it is closed and bounded.*

Proof. Suppose that S is compact. Since $\mathbb{R}^n$ is Hausdorff, Lemma (2.1) implies S is closed. It remains to show that S is bounded, which we will do by contradiction: assume S is not bounded. For each $x\in S$, let $U_x=B_1(x)$ be the open ball of radius 1 centered at x. Then $\{U_x\}_{x\in S}$ forms an open cover of S. Since S is compact, there eixsts a finite subcover of $\{U_x\}_{x\in S}$, say $\{U_{x_1},\dots,U_{x_n}\}$. Let

$$L_{ij}=\sup\left\{\|a_i-a_j\| \mid a_i\in U_{x_i}\text{ and } a_j\in U_{x_j}\right\}$$

clearly L_{ij} is finite since $L_{ij} \leq \|x_i - x_j\| + 2$. Setting $L = \max_{1\leq i,j\leq n}\{L_{ij}\}$, we see that for all $a, a' \in S$, we must have $\|a - a'\| \leq L$. Thus, S is bounded.

Conversely, suppose that S is closed and bounded. Since S is bounded, it is enclosed within an n-box $T_0 = [-a, a]^n$ where $a > 0$. Since $\mathbb{R}^n$ is Hausdorff, a closed subset of a compact set is compact, and so it suffices to show T_0 is compact. Assume, by way of contradiction, that T_0 is not compact. Then there exists an infinite open cover $\{U_i\}_{i\in I}$ of T_0 that does not admit any finite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into 2^n sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the 2^n sections of T_0 must require an infinite subcover of $\{U_i\}$, otherwise $\{U_i\}$ itself would have a finite subcover, by uniting together the finite covers of the sections. Call this section T_1.

Likewise, the sides of T_1 can be bisected, yielded 2^n sections of T_1, at least one of which must require an infinite subcover of $\{U_i\}$. Continuing in this manner yields a decreasing sequence of nexted n-boxes:

$$T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_k \supset \cdots$$

where the side length of T_k is $(2a)/2^k$, which tends to 0 as k tends to infinity. Let us define a sequence (x_k) such that each x_k is in T_k. This sequence is Cauchy, so it must converge to some limit x. Since each T_k is closed, and for each k the sequence (x_k) is eventually always inside T_k, we see that $x \in T_k$ for each k.

Since $\{U_i\}$ covers T_0, it has some member $U \in \{U_i\}$ such that $x \in U$. Since U is open, there is an n-ball $B_\varepsilon(x) \subset U$ for some $\varepsilon > 0$. For large enough k (for example such that $(2a)/2^k < \varepsilon$), one has

$$T_k \subset B_\varepsilon(x) \subset U,$$

but then the infinite number of members of $\{U_i\}$ needed to cover T_k can be replaced by just one: U, a contradiction. Thus, T_0 is compact. □

Remark. The Heine-Borel theorem does not hold as stated for general metric and topological vector spaces. For instance, at one point in our proof we used completeness, which doesn't hold in a general metric space. A metric space (X, d) is said to have the **Heine-Borel property** if each closed bounded set in X is compact.

2.1.1 Sequential Compactness

A topological space X is said to be **sequentially compact** if every sequence of points in X has a convergent subsequence converging to a point in X. In general, there are compact spaces which are not sequentially compact and there are sequentially compact spaces that are not compact. However, when it comes to metric spaces, these notions are equivalent. We will prove this in the case of the Euclidean space $\mathbb{R}^n$.

Theorem 2.7. *Let S be a subset of Euclidean space $\mathbb{R}^n$. Then S is sequentially compact if and only if it is closed and bounded.*

Proof. We first assume that S is sequentially compact. We will first show that S is closed.

Let (x_n) be a convergent sequence in S, and suppose that $x_n \to x$ as $n \to \infty$. Since S is sequentially compact, we can choose a convergent subsequence (x_{n_k}) of (x_n) which converges to a point in X. Since every convergent subsequence of a convergent sequence converges to the same limit, we have $x_{n_k} \to x$ as $k \to \infty$. This establishes that S is closed.

Now we will show that S is bounded. Assume (for a contradiction) that S is unbounded. Since S is unbounded, there exists a sequence (x_n) in S such that

$$x_n \notin \bigcup_{m=1}^{n-1} B_1(x_m).$$

for all $n \in \mathbb{N}$. Such a sequence has no convergent subsequence since for each $n \in \mathbb{N}$, the neighborhood $B_1(x_n)$ contains only one member in the sequence (namely x_n).

To complete the proof of the theorem, we now assume that S is closed and bounded. We will show that S is sequentially compact. Let (x_n) be a sequence in S. Since S is closed and bounded, it lies in a closed box, say $B_0 = [-a, a]^n$ where $a > 0$. Through bisection of each of the sides of B_0, the box B_0 can be broken up into 2^n sub n-boxes, each of which has diameter equal to half the diameter of B_0. Then at least one of the 2^n sections of B_0 contains infinitely elements in the sequence (x_n). Call this section B_1.

Likewise, the sides of B_1 can be bisected, yielded 2^n sections of B_1, at least one of which must contain infinitely many elements in the sequence (x_n). Continuing in this manner yields a decreasing sequence of nexted n-boxes:

$$B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_k \supset \cdots$$

where the side length of B_k is $(2a)/2^k$, which tends to 0 as k tends to infinity. Now we define a convergent subsequence of (x_n) as follows: For each $k \in \mathbb{N}$, we choose x_{n_k} inductively on k to be a member of the sequence (x_n) which lies in the box B_k and such that $x_{n_k} \neq x_{n_{k-1}}$ for all $k \in \mathbb{N}$. The sequence (x_{n_k}) is is Cauchy, so it must converge to some limit x. Since each T_k is closed, and for each k the sequence (x_k) is eventually always inside T_k, we see that $x \in T_k$ for each k. Finally, since S is closed, we must have $x \in S$. This establishes that S is sequentially compact. □

2.1.2 Extreme Value Theorem

Proposition 2.4. *Let X be a compact space and let $f\colon X \to \mathbb{R}$ be continuous. Then f obtains a maximum value, i.e. there exists $x_0 \in X$ such that $f(x_0) \geq f(x)$ for all $x \in X$.*

Proof. Assume X is nonempty, otherwise it is trivial. Since X is compact, $f(X)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel theorem, $f(X)$ is a closed and bounded subset of $\mathbb{R}$. Since $f(X)$ is nonempty and bounded above, the limit $\sup(f(X))$ exists. Moreover, since $f(X)$ is closed and $\sup f(X)$ is a limit point of $f(X)$, we have $\sup(f(X)) \in f(X)$. Thus $\sup(f(X)) = f(x_0)$ for some $x_0 \in X$, and this is clearly the maximum value. □

3 Closure and Interior

3.1 Closure

Definition 3.1. Let X be a topological space and let A be a subset of X. The **closure** of A in X, denoted $\overline{A}$, is the smallest closed set in X which contains A. It is characterized by the universal property that if E is a closed set in X such that $E \subseteq A$, then $E \subseteq \overline{A}$. Indeed,

$$\overline{A} = \bigcap_{\substack{E \text{ closed} \\ A \subseteq E}} E.$$

3.1.1 Uniqueness of Continuous Extensions of Functions from a Set to its Closure

Lemma 3.1. *Let X be a topological space, A be a subset of X, and let $x \in \overline{A}$. If U is an open neighborhood of x, then U meets A (i.e. $U \cap A \neq \emptyset$).*

Proof. To obtain a contradiction, assume $U \cap A = \emptyset$. Then A is contained in the closed set X/U, which implies $\overline{A}$ is contained in the closed set $X \backslash U$. But this is a contradiction since $x \in \overline{A} \cap U$, whence $x \in \overline{A}$ and $x \notin X \backslash U$. □

Proposition 3.1. *Let X and Y be topological spaces such that Y is Hausdorff. Let A be a subset of X and let $f\colon A \to Y$ be a continuous map. Suppose there exists a continuous extension $\widetilde{f}\colon \overline{A} \to Y$ of f (i.e. $\widetilde{f}$ is continuous and $\widetilde{f}(a) = f(a)$ for all $a \in A$), then $\widetilde{f}$ is unique.*

Proof. To prove uniqueness, suppose that $\widetilde{f}_1\colon \overline{A} \to Y$ and $\widetilde{f}_2\colon \overline{A} \to Y$ are two continuous extensions of f. Then there exists $x \in \overline{A}$ such that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Choose open neighborhoods V_1 and V_2 of $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ respectively such that $V_1 \cap V_2 = \emptyset$ (we can do this since Y is Hausdorff). Then $\widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ is an open neighborhood of $x \in \overline{A}$, and so it must meet A by Lemma (3.1). This is a contradiction though, since $a \in A \cap \widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ implies $V_1 \cap V_2 \neq \emptyset$. Indeed,

$$\begin{aligned} V_1 &\ni \widetilde{f}_1(a) \\ &= f_1(a) \\ &= f_2(a) \\ &= \widetilde{f}_2(a) \in V_2. \end{aligned}$$

□

Proposition 3.2. *Let X be a topological spaces such that Y is Hausdorff. Suppose that for every subset A of X and let $f\colon A \to Y$ be a continuous map. Suppose there exists a continuous extension $\widetilde{f}\colon \overline{A} \to Y$ of f (i.e. $\widetilde{f}$ is continuous and $\widetilde{f}(a) = f(a)$ for all $a \in A$), then $\widetilde{f}$ is unique.*

Proof. To prove uniqueness, suppose that $\widetilde{f}_1\colon \overline{A} \to Y$ and $\widetilde{f}_2\colon \overline{A} \to Y$ are two continuous extensions of f. Then there exists $x \in \overline{A}$ such that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Choose open neighborhoods V_1 and V_2 of $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ respectively such that $V_1 \cap V_2 = \emptyset$ (we can do this since Y is Hausdorff). Then $\widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ is an open neighborhood of $x \in \overline{A}$, and so it must meet A by Lemma (3.1). This is a contradiction though, since $a \in A \cap \widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ implies $V_1 \cap V_2 \neq \emptyset$. Indeed,

$$\begin{aligned} V_1 &\ni \widetilde{f}_1(a) \\ &= f_1(a) \\ &= f_2(a) \\ &= \widetilde{f}_2(a) \in V_2. \end{aligned}$$

□

4 Metric Spaces

Definition 4.1. A **metric** on a set X is a function $d\colon X \times X \to \mathbb{R}$ which satisfies the following three properties:

1. (Identity of Indiscernibles) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$,
2. (Symmetric) $d(x,y) = d(y,x)$ for all $x, y \in X$,
3. (Triangle Inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

A set X together with a choice of a metric d is called a **metric space** and is denoted (X, d), or just denoted X if the metric is understood from context.

Remark. Given the three axioms above, we also have $d(x,y) \geq 0$ (Positive-Definiteness) for all $x, y \in X$. Indeed,

$$\begin{aligned} 0 &= d(x,x) \\ &\leq d(x,y) + d(y,x) \\ &= d(x,y) + d(x,y) \\ &= 2d(x,y). \end{aligned}$$

This implies $d(x,y) \geq 0$.

Example 4.1. On $\mathbb{R}^m$ the **Euclidean metric** is

$$d_E(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}.$$

This is the usual distance used in $\mathbb{R}^m$, and when we speak about $\mathbb{R}^m$ as a metric space without specifying a metric, it's the Euclidean metric that is intended.

To check that d_E is a metric on $\mathbb{R}^m$, the first two conditions in the definition are obvious. The third condition is a consequence of the inequality $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (replace $\mathbf{x}$ with $\mathbf{x} - \mathbf{z}$ and $\mathbf{y}$ with $\mathbf{z} - \mathbf{y}$), and to show this inequality holds we will write $\|\mathbf{x}\|$ in terms of the dot product: $\|\mathbf{x}\|^2 = x_1^2 + \cdots + x_m^2 = \mathbf{x} \cdot \mathbf{x}$, so

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2|\mathbf{x} \cdot \mathbf{y}| + \|\mathbf{y}\|^2. \end{aligned}$$

The famous Cauchy-Schwarz inequality says $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, so

$$\|\mathbf{x} + \mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$$

and now take square roots.

A different metric on $\mathbb{R}^m$ is

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq m} |x_i - y_i|.$$

Again, the first two conditions of being a metric are clear, and to check the triangle inequality we use the fact that it is known for the absolute value. If $\max |x_i - y_i| = |x_k - y_k|$ for a particular k from 1 to m, then $d_\infty(\mathbf{x}, \mathbf{y}) = |x_k - y_k|$, so

$$\begin{aligned} d_\infty(\mathbf{x}, \mathbf{y}) &\leq |x_k - z_k| + |z_k - y_k| \\ &\leq \max_{1 \leq i \leq m} |x_i - z_i| + \max_{1 \leq i \leq m} |z_i - y_i| \\ &= d_\infty(\mathbf{x}, \mathbf{z}) + d_\infty(\mathbf{z}, \mathbf{y}). \end{aligned}$$

While the metrics d_E and d_∞ on $\mathbb{R}^m$ are different, they're not that different from each other since each is bounded by a constant multiple of the other one:

$$d_E(\mathbf{x}, \mathbf{y}) \leq \sqrt{m} d_\infty(\mathbf{x}, \mathbf{y}) \text{ and } d_\infty(\mathbf{x}, \mathbf{y}) \leq d_E(\mathbf{x}, \mathbf{y}).$$

Example 4.2. Let $C[0,1]$ be the space of all continuous functions from $[0,1]$ to $\mathbb{R}$. Two metrics used on $C[0,1]$ are

$$d_\infty(f,g) = \max_{0 \leq x \leq 1} |f(x) - g(x)| \text{ and } d_1(f,g) = \int_0^1 |f(x) - g(x)| dx.$$

Unlike with the two metrics on $\mathbb{R}^m$ while we have $d_1(f,g) \leq d_\infty(f,g)$ there is no constant $A > 0$ that makes $d_\infty(f,g) \leq A d_1(f,g)$ for all f and g. These metrics d_1 and d_∞ on $C[0,1]$ are quite different.

4.1 Metric Space Induced by a Norm

Definition 4.2. Let V be a vector space over a subfield F of the complex numbers. A **norm** on V is a nonnegative-valued scalar function $p : V \to [0, \infty)$ such that for all $a \in F$ and $u, v \in V$, we have

1. (Subadditivity) $p(u + v) \leq p(u) + p(v)$,
2. (Absolutely Homogeneous) $p(av) = |a|p(v)$,
3. (Positive-Definite) $p(v) = 0$ implies $v = 0$.

We call the pair (V, p) a **normed vector space**.

Proposition 4.1. *Let (V, p) be a normed vector space. Define $d : V \times V \to \mathbb{R}$ by $d(u, v) = p(u - v)$ for all $(u, v) \in V \times V$. Then (V, d) is a metric space.*

Proof. Let us first check that d satisfies the identity of indiscernibles property. Since p is positive-definite, $d(u, v) = 0$ implies $p(u - v) = 0$ which implies $u = v$. On the other hand, suppose $u = v$. Then since p is absolutely homogeneous, we have $p(0) = |0|p(0) = 0$, and so $d(u, u) = p(0) = 0$.

Next we check that d is symmetric. For all $(u, v) \in V \times V$, we have

$$\begin{aligned} d(u, v) &= p(u - v) \\ &= p(-1(v - u)) \\ &= |-1|p(v - u) \\ &= p(v - u) \\ &= d(v, u). \end{aligned}$$

Finally, triangle inequality for d follows from subadditivity of p. Indeed, for all $u, v, w \in V$, we have

$$\begin{aligned} d(u, v) + d(v, w) &= p(u - v) + p(v - w) \\ &\geq p(u - w) \\ &= d(u, w). \end{aligned}$$

□

Remark. The metric d induced by a norm p has additional properties that are not true of general metrics. These are

1. (Translation Invariance) $d(u + w, v + w) = d(u, v)$ for all $u, v, w \in V$
2. (Scaling Property) $d(au, av) = |a|d(u, v)$ for all $a \in F$ and $u, v \in V$.

Convserely, if a metric has these properties, then $d(u, 0)$ is a norm.

4.2 Limit of a Sequence in a Metric Space

Definition 4.3. For a sequence (x_n) in a metric space (X, d), we say (x_n) **converges to** $x \in X$, and write $\lim_{n\to\infty} x_n = 0$ or $x_n \to x$, if for every $\varepsilon > 0$ there is an $N = N_\varepsilon \in \mathbb{N}$ such that

$$n \geq N \text{ implies } d(x_n, x) < \varepsilon.$$

If a sequence in (X, d) has a limit, then we say that the sequence is **convergent**.

Theorem 4.1. *If a sequence (x_n) in a metric space (X, d) converges, then $d(x_n, x_{n+1}) \to 0$.*

Proof. Suppose $x_n \to x$. From the triangle inequality, we have

$$d(x_n, x_{n+1}) \leq d(x_n, x) + d(x, x_{n+1}) = d(x_n, x) + d(x_{n+1}, x).$$

The two terms on the right get small when n is large, so $d(x_n, x_{n+1})$ gets small when n is large. To be precise, for $\varepsilon/2 > 0$ there's an $N \geq 1$ such that for all $m \geq N$ we have $d(x_m, x) < \varepsilon/2$. Therefore

$$n \geq N \text{ implies } n + 1 \geq N \text{ implies } d(x_n, x_{n+1}) \leq d(x_n, x) + d(x_{n+1}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 4.2. *Every subsequence of a convergent sequence in a metric space is also convergent, with the same limit.*

Proof. Let $x_n \to x$ in (X,d) and let (x_{n_i}) be a subsequence of (x_n). Set $y_i = x_{n_i}$. We want ot show $y_i \to x$.

For $\varepsilon > 0$ there is an N such that $n \geq N$ implies $d(x_n, x) < \varepsilon$. Since the interegers n_i are increasing, we have $n_i \geq N$ if we go out far enough: there's an I such that $i \geq I$ implies $n_i \geq N$ which imlpies $d(x_{n_i}, x) < \varepsilon$, so $d(y_i, x) < \varepsilon$. Thus $y_i \to x$. □

Theorem 4.3. *In a metric space (X,d), if two sequences (x_n) and (x'_n) converge to the same value, then $d(x_n, x'_n) \to 0$.*

Proof. Suppose $x_n \to x$ and $x'_n \to x$ for some $x \in X$ and let $\varepsilon > 0$. Then there exists some integer $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } d(x_n, x) < \frac{\varepsilon}{2} \text{ and } d(x'_n, x) < \frac{\varepsilon}{2}.$$

In particular,

$$n \geq N \text{ implies } d(x_n, x'_n) \leq d(x_n, x) + d(x, x'_n) < \varepsilon.$$

□

Example 4.3. On $\mathbb{R}^m$, because the metrics d_E and d_∞ are each bounded above by a constant multiple of the other, we have $d_E(x_n, x) \to 0$ if and only if $d_\infty(x_n, x) \to 0$. Indeed, let $\varepsilon/\sqrt{m} > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } d_\infty(x_n, x) < \frac{\varepsilon}{\sqrt{m}},$$

and since $d_E(x_n, x) \leq \sqrt{m} d_\infty(x_n, x)$,

$$n \geq N \text{ implies } d_E(x_n, x) < \varepsilon.$$

The converse is shown the same way. Therefore convergence of sequences in $\mathbb{R}^m$ for both metrics means the same thing (with the same limits).

Example 4.4. In $C[0,1]$ consider the sequence of functions x^n for $n \geq 1$.This sequence converges to 0 in the metric d_1 but not in the metric d_∞:

$$d_1(x^n, 0) = \int_0^1 |x^n| dx = \frac{1}{n+1} \to 0, \qquad d_\infty(x^n, 0) = \max_{0 \leq x \leq 1} |x^n| = 1.$$

In fact the sequence (x^n) in $C[0,1]$ has no limit at all relative to the metric d_∞. To prove (x^n) has no limit in $(C[0,1], d_\infty)$, not just that the constant function 0 is not a limit, we seek a property that all convergent sequences satisfy and the sequence (x^n) in $(C[0,1], d_\infty)$ does not satisfy. This will be provided to us in the next section.

4.3 Cauchy Sequences and Completeness

Recall from Theorem (4.1) that for a sequence (x_n) in a metric space (X,d) to converge, it is necessary that $d(x_n, x_{n+1}) \to 0$. On the other hand, this condition is not sufficient. Indeed, in $C[0,1]$, we have

$$d_\infty(x^n, x^{n+1}) = \max_{0 \leq x \leq 1} |x^n - x^{n+1}| = \max_{0 \leq x \leq 1} (x^n - x^{n+1}),$$

To find the maximal value, we first compute the derivative

$$\frac{d}{dx}\left(x^n - x^{n+1}\right) = nx^{n-1} - (n+1)x^n = x^{n-1}(n - (n+1)x).$$

Setting this equal to 0, we have either $x = 0$ or $x = n/(n+1)$. Since $x^n - x^{n+1} = x^n(1-x)$ is always positive on $[0,1]$, it must be maximized on $[0,1]$ at $x = n/(n+1)$, where the value is

$$\left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \sim \frac{1}{e}\left(\frac{1}{n+1}\right) \to 0.$$

Thus, we do have $d_\infty(x^n, x^{n+1}) \to 0$. However we shall see shortly that this sequence does not converge under the d_∞ metric. Indeed, by the exact same reasoning in the proof of Theorem (4.1), we should also have $d_\infty(x^n, x^{2n}) \to 0$. But

$$d_\infty(x^n, x^{2n}) = \max_{0 \leq x \leq 1} |x^n - x^{2n}| = \max_{0 \leq x \leq 1} (x^n(1 - x^n)),$$

has its maximum value at $x = 1/\sqrt[n]{2}$ where $x^n(1 - x^n) = 1/4$, which is independent of n. This proves that (x^n) has not limit in $(C[0,1], d_\infty)$.

Theorem 4.4. *If (x_n) is a convergent sequence in a metric space (X,d), then the terms of the sequence become "uniformly close": for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$m, n \geq N \textit{ implies } d(x_n, x_m) < \varepsilon.$$

Proof. Letting $x = \lim_{n\to\infty} x_n$, the triangle inequality tells us for all m and n that

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x).$$

We now make an $\varepsilon/2$ argument. For every $\varepsilon > 0$ there's an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n, x) < \varepsilon/2$. Therefore $m, n \geq N$ implies

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Definition 4.4. A sequence (x_n) in a metric space (X, d) is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists an $N = N_\varepsilon \in \mathbb{N}$ such that

$$m, n \geq N \text{ implies } d(x_m, x_n) < \varepsilon.$$

Corollary. *If (X, d) is a metric space and Y is a subset of X given the induced metric $d\mid_Y$, then any sequence in Y that converges in X is a Cauchy sequence in $(Y, d\mid_Y)$.*

Proof. A sequence (y_n) in Y that converges in X is Cauchy in X by Theorem (4.4). Since the metric d on X is the metric we are using on Y, the Cauchy property of (y_n) in X can be viewed as the Cauchy property in Y. □

Example 4.5. Consider the interval $(0, \infty)$ as a metric space using the absolute value metric induced from $\mathbb{R}$. We have $1/n \to 0$ in $\mathbb{R}$, but the sequence $(1/n)$ has no limit in $(0, \infty)$ since $0 \notin (0, \infty)$. The sequence $(1/n)$ is a Cauchy sequence in $(0, \infty)$ by Corollary (4.3) but it is not a convergent sequence in $(0, \infty)$.

Theorem 4.5. *If (x_n) is a sequence in a metric space (X, d) such that $d(x_n, x_{n+1}) \leq ar^n$ for all n, where $a > 0$ and $0 < r < 1$, then (x_n) is a Cauchy sequence.*

Proof. For $1 \leq m < n$, a massive use of the triangle inequality tells us

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq ar^m + ar^{m+1} + \cdots + ar^{n-1} \\ &< \sum_{k=m}^{\infty} ar^k \\ &= \frac{ar^m}{1-r}. \end{aligned}$$

Since $0 < r < 1$, the terms ar^n tend to 0 as $n \to \infty$. Now if we pick an $\varepsilon > 0$, choose N large enough that $ar^N < (1-r)\varepsilon$. For $m, n \geq N$, without loss of generality $m \leq n$ so by our prior calculation

$$\begin{aligned} d(x_m, x_n) &< \frac{ar^m}{1-r} \\ &\leq \frac{ar^N}{1-r} \\ &< \varepsilon. \end{aligned}$$

□

4.4 Complete Metric Spaces

Definition 4.5. A metric space (X, d) is called **complete** if every Cauchy sequence in X converges in X: if (x_n) is Cauchy in X then there's an $x \in X$ such that $x_n \to x$.

4.4.1 Completions exist

Our goal in this subsection is to prove the following theorem:

Theorem 4.6. *Let X be a metric space. Then a completion of X exists.*

Let C_X denote the set of Cauchy sequences in the given metric space X. We say two elements $(x_n), (y_n) \in C_X$ are **equivalent**, written $(x_n) \sim (y_n)$, if $\rho(x_n, y_n) \to 0$ as $n \to \infty$. We claim that this is an equivalence relation on C_X. The only nontrivial part to check is transitivity: Suppose $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$. Then

$$\rho(x_n, z_n) \leq \rho(x_n, y_n) + \rho(y_n, z_n) \to 0 + 0$$

as $n \to \infty$.

We denote by $C_X / \sim$ to be the set of equivalence classes in C_X under $\sim$. Note that there is a natural map of sets

$$\iota_X : X \to C_X / \sim,$$

which assigns to each $x \in X$, the equivalence class of constant sequences $(x) \in C_X$. The map ι_X is injective. Indeed, if $\iota_X(x) = \iota_X(x')$, then $(x) \sim (x')$, and so $\rho(x, x') \to 0$ forces $x = x'$.

We now must enhance the structure of $C_X / \sim$ by giving it a metric (with respect to which we'll easily see ι_X is an isometry with dense image). We first define a "pseudo-metric" on the set C_X. Let's first define what we mean by "pseudo-metric":

Definition 4.6. A **pseudo-metric** on a set S is a function $d : S \times S \to \mathbb{R}$ which satisfies the following three properties:

1. $d(s,s) = 0$ for all $s \in S$,
2. $d(s,t) = d(t,s)$ for all $s, t \in S$,
3. $d(s,u) \le d(s,t) + d(t,u)$ for all $s, t, u \in S$.

Remark.

1. The same proof as in the metric case shows that a pseudo-metric is positive-definite (i.e. $d(s,t) \ge 0$ for all $s, t \in S$).
2. The only difference between a metric and a pseudo-metric is that a pseudo-metric might satisfy $d(s,t) = 0$ for some pair $s, t \in S$ with $s \neq t$.

Lemma 4.7. *If (s_n) and (t_n) are Cauchy sequences in S, then $(\rho(s_n, t_n))$ is a convergent sequence in $\mathbb{R}$.*

Proof. We show that $(\rho(s_n, t_n))$ is a Cauchy sequence in $\mathbb{R}$. Let $\varepsilon > 0$. Since (s_n) and (t_n) are Cauchy sequences, there exists $N \in \mathbb{N}$ such that $\rho(s_n, s_m) < \varepsilon/2$ and $\rho(t_n, t_m) < \varepsilon/2$ for all $n, m \ge N$.

Note that $\rho(s_n, t_n) \le \rho(s_n, s_m) + \rho(s_m, t_m) + \rho(t_m, t_n)$ implies

$$\rho(s_n, t_n) - \rho(s_m, t_m) \le \rho(s_n, s_m) + \rho(t_m, t_n),$$

and implies

$$\rho(s_m, t_m) - \rho(s_n, t_n) \le \rho(s_m, s_n) + \rho(t_n, t_m).$$

Thus, for all $n, m \ge N$, we have

$$|\rho(s_n, t_n) - \rho(s_m, t_m)| \le \rho(s_n, s_m) + \rho(t_m, t_n) < \varepsilon.$$

□

With the lemma established, the following definition makes sense:

Definition 4.7. For $(x_n), (y_n) \in C_X$, we define the **pseudo-distance** between them to be

$$d((x_n), (y_n)) = \lim_{n \to \infty} \rho(x_n, y_n).$$

The pseudo-distance defined is a pseudo-metric since it inherits the properties these from ρ.

Corollary. *For $c_1, c_2, c_1', c_2' \in C_X$ with $c_j \sim c_j'$ we have $d(c_1, c_2) = d(c_1', c_2')$. In other words, the pseudo-metric d on C_X respects the equivalence relation.*

Proof. Note that $c_j \sim c_j'$ if and only if $d(c_j, c_j') = 0$. Thus

$$\begin{aligned} d(c_1, c_2) &\le d(c_1, c_1') + d(c_1', c_2') + d(c_2', c_2) \\ &= d(c_1', c_2'). \end{aligned}$$

Similarly,

$$\begin{aligned} d(c_1', c_2') &\le d(c_1', c_1) + d(c_1, c_2) + d(c_2, c_2') \\ &= d(c_1, c_2). \end{aligned}$$

Thus $d(c_1, c_2) = d(c_1', c_2')$. □

Definition 4.8. Define $X' = C_X / \sim$ and define the function $\rho' : X' \times X' \to \mathbb{R}$ by

$$\rho'(\xi_1, \xi_2) = d(c_1, c_2)$$

where $c_1, c_2 \in C_X$ are respective representative elements for $\xi_1, \xi_2 \in X' = C_X / \sim$.

This definition is well-defined in view of Corollary (4.4.1). In fact, the function $\rho' : X' \times X' \to \mathbb{R}$ is a metric since it inherits the pseudo-metric properties from ρ, and modding out by the equivalence relation ensures that we have identity of indiscernibles. Note that we also have

$$\rho'(\iota_X(x), \iota_X(y)) = \rho(x, y)$$

for all $x, y \in X$. Thus, $\iota_X : X \to X'$ is an isometric embedding. In fact the image of X is dense in X'. Indeed, fix a choice of $\xi \in X'$. Choose a representative Cauchy sequence $(x_n) \in C_X$ for the equivalence class $\xi \in X'$. Then the sequence of elements $\iota_X(x_n) \in \iota_X(X) \subseteq X'$ has limit ξ.

4.5 Open and Closed Subsets

In this section we generalize open and closed intervals in $\mathbb{R}$ to open and closed balls of a metric space. Throughout, let (X, d) be a metric space.

Definition 4.9. For $a \in X$ and $r \geq 0$, the **open ball** with center a and radius r is

$$\mathrm{B}_r(a) = \{x \in \mathrm{X} \mid d(a, x) < r\},$$

and the **closed ball** with center a and radius r is

$$\mathrm{B}_r[a] = \{x \in \mathrm{X} \mid d(a, x) \leq r\}.$$

When $r = 0$, we have $B(a, 0) = \emptyset$ and $\overline{B}(a, 0) = \{a\}$.

Definition 4.10. A subset of X is called **bounded** if it is contained in some ball $B(a, r)$. A subset that is not bounded is called **unbounded**.

Definition 4.11. A subset $U \subset X$ is called **open** if for each $x \in U$ there's an $r > 0$ such that $B(x, r) \subset U$. We also consider the empty subset of X to be an open subset.

5 Pseudometric Spaces

Definition 5.1. A **pseudometric** on a set X is a function $\mathrm{d} \colon X \times X \to \mathbb{R}$ which satisfies the following three properties:

1. (Reflexivity) $\mathrm{d}(x, x) = 0$ for all $x \in X$;
2. (Symmetry) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in X$,
3. (Triangle Inequality) $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$ for all $x, y, z \in X$.

If d is a pseudometric on a set X, then we call the pair (X, d) a **pseudometric space**. If the pseudometric is understood from context, then we often denote a pseudometric space by X instead of (X, d).

Remark. Given the three axioms above, we also have $d(x, y) \geq 0$ for all $x, y \in X$. Indeed,

$$\begin{aligned} 0 &= d(x, x) \\ &\leq d(x, y) + d(y, x) \\ &= d(x, y) + d(x, y) \\ &= 2d(x, y). \end{aligned}$$

This implies $d(x, y) \geq 0$.

5.1 Topology Induced by Pseudometric Space

Proposition 5.1. *Let (X, d) be a pseudometric space. For each $x \in X$ and $r > 0$, define*

$$\mathrm{B}_r^{\mathrm{d}}(x) := \{y \in X \mid \mathrm{d}(x, y) < r\},$$

and let

$$\mathcal{B}^{\mathrm{d}} = \{\mathrm{B}_r(x) \mid x \in X \text{ and } r > 0\}.$$

Finally, let $\tau(\mathcal{B}^{\mathrm{d}})$ be the smallest topology on X which contains $\mathcal{B}^{\mathrm{d}}$. Then $\mathcal{B}^{\mathrm{d}}$ is a basis for $\tau(\mathcal{B}^{\mathrm{d}})$.

Remark. We often remove the d in the superscript in $\mathrm{B}_r^{\mathrm{d}}(x)$ and $\mathcal{B}^{\mathrm{d}}$ whenever context is clear.

Proof. First note that $\mathcal{B}$ covers X. Indeed, for any $r > 0$, we have

$$X \subseteq \bigcup_{x \in X} \mathrm{B}_r(x).$$

Next, let $\mathrm{B}_r(x)$ and $\mathrm{B}_{r'}(x')$ be two members of $\mathcal{B}$ which have nontrivial intersection and let $x'' \in \mathrm{B}_r(x) \cap \mathrm{B}_{r'}(x')$. Set

$$r'' = \min\{r' - \mathrm{d}(x', x''), r - \mathrm{d}(x, x'')\}.$$

We claim that $\mathrm{B}_{r''}(x'') \subseteq \mathrm{B}_r(x) \cap \mathrm{B}_{r'}(x')$. Indeed, assume without loss of generality that $r'' = r - \mathrm{d}(x, x'')$. Let $y \in \mathrm{B}_{r''}(x'')$. Then

$$\begin{aligned} \mathrm{d}(y, x) &\leq \mathrm{d}(y, x'') + \mathrm{d}(x'', x) \\ &< r - \mathrm{d}(x, x'') + \mathrm{d}(x'', x) \\ &= r - \mathrm{d}(x'', x) + \mathrm{d}(x'', x) \\ &= r \end{aligned}$$

implies $y \in \mathrm{B}_r(x)$. Similarly,

$$\begin{aligned} \mathrm{d}(y, x') &\leq \mathrm{d}(y, x'') + \mathrm{d}(x'', x') \\ &< r' - \mathrm{d}(x', x'') + \mathrm{d}(x'', x') \\ &= r' - \mathrm{d}(x'', x') + \mathrm{d}(x'', x') \\ &= r' \end{aligned}$$

implies $y \in \mathrm{B}_{r'}(x')$. Thus $\mathrm{B}_{r''}(x'') \subseteq \mathrm{B}_r(x) \cap \mathrm{B}_{r'}(x')$, and so $\mathcal{B}$ is a basis for $\tau(\mathcal{B})$. □

Definition 5.2. The topology $\tau(\mathcal{B})$ in Proposition (5.1) is called the **topology induced by the pseudometric** d. We also denote this topology by τ_{d}.

5.1.1 Subspace topology agrees with topology induced by pseudometric

Let (X, d) be a pseudometric space and let $A \subseteq X$. Then the pseudometric on X restricts to a pseudometric on A. We denote this restriction by $\mathrm{d}|_A$. Thus there are two natural topologies on A. One is the subspace is the subspace topology given by

$$\tau \cap A := \{U \cap A \mid U \in \tau\}.$$

The other is the topology induced by the pseudometric $\mathrm{d}|_A$ given by

$$\tau_{\mathrm{d}|_A} := \tau(\mathcal{B}^{\mathrm{d}}).$$

The next proposition tells us that these are actually the same.

Proposition 5.2. *Let (X, d) be a pseudometric space and let $A \subseteq X$. Then*

$$\tau_{\mathrm{d}} \cap A = \tau_{\mathrm{d}|_A}.$$

Proof. Let $a \in A$ and $r > 0$. Then

$$\begin{aligned} \mathrm{B}_r^{\mathrm{d}|_A}(a) &= \{b \in A \mid \mathrm{d}|_A(a, b) < r\} \\ &= \{b \in A \mid \mathrm{d}(a, b) < r\} \\ &= A \cap \{x \in X \mid \mathrm{d}(a, x) < r\} \\ &= A \cap \mathrm{B}_r^{\mathrm{d}}(a). \end{aligned}$$

It follows that $\tau_{\mathrm{d}|_A}$ and $\tau_{\mathrm{d}} \cap A$ have the same basis, and hence $\tau_{\mathrm{d}} \cap A = \tau_{\mathrm{d}|_A}$. □

5.1.2 Convergence in (X, d)

Concepts like convergence and completion still make sense in pseudometric spaces. This is because these are purely topological concepts.

Definition 5.3. Let (X, d) be a pseudometric space and let (x_n) be a sequence in X.

1. We say the sequence (x_n) converges to $x \in X$ if for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$[a] such that
$$n \geq N_\varepsilon \text{ implies } \mathrm{d}(x_n, x) < \varepsilon.$$
In this case, we say (x_n) is a **convergent** and that it **converges** to x. We denote this by $x_n \to x$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = x$, or even just $x_n \to x$.

2. We say the sequence (x_n) is **Cauchy** if for all $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that
$$n, m \geq N_\varepsilon \text{ implies } \mathrm{d}(x_n, x_m) < \varepsilon.$$

[a]We write ε in the subscript to remind the reader that N_ε depends on ε. Usually we omit ε in the subscript and just write N.

5.1.3 Completeness in (X, d)

In a metric space, every Cauchy sequence is convergence but the converse may not hold. The same thing is true for pseudometric spaces since the proof is purely topological. Let's go over the proof again:

Proposition 5.3. *Let (x_n) be a sequence in X, let $x \in X$, and suppose $x_n \to x$. Then (x_n) is Cauchy.*

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies
$$\mathrm{d}(x_n, x) < \varepsilon/2.$$
Then $n, m \geq N$ implies
$$\begin{aligned} \mathrm{d}(x_n, x_m) &\leq \mathrm{d}(x_n, x) + \mathrm{d}(x, x_m) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$
This implies (x_n) is Cauchy. □

Thus, the concept of completeness makes sense in a pseudometric space.

Definition 5.4. Let (X, d) be a pseudometric space. We say (X, d) is **complete** if every Cauchy sequence in (X, d) is a convergent.

5.2 Metric Obtained by Pseudometric

Unless otherwise specified, we let (X, d) be a pseudometric space throughout the remainder of this section. There is a natural way to obtain a metric space from (X, d) which we now describe as follows: define a relation $\sim$ on X by
$$x \sim y \text{ if and only if } \mathrm{d}(x, y) = 0.$$
Then $\sim$ is an equivalence relation. Indeed, we have reflexivity of $\sim$ since $\mathrm{d}(x, x) = 0$ for all $x \in X$, we have symmetry of $\sim$ since $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in X$, and we have transitivity of $\sim$ since d satisfies the triangle inequality: if $x \sim y$ and $y \sim z$, then
$$\begin{aligned} \mathrm{d}(x, z) &\leq \mathrm{d}(x, y) + \mathrm{d}(y, z) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$
Thus $\mathrm{d}(x, z) = 0$ which implies $x \sim z$.

Therefore we may consider the quotient space of X with respect to the equivalence relation above. We shall denote this quotient space by $[X] := X/\sim$. A coset in $[X]$ which is represented by $x \in X$ will be written as $[x]$. There is a natural **projection map** $\pi \colon X \to [X]$ that sends $x \in X$ to its equivalence class $[x]$. Since π is surjective, any subset of $[X]$ has the form
$$[A] = \{[a] \in [X] \mid a \in A\}.$$
We are ready to define the metric on $[X]$.

Theorem 5.1. *Define* $[\mathrm{d}]\colon [X] \times [X] \to \mathbb{R}$ *by*

$$[\mathrm{d}]([x],[y]) = \mathrm{d}(x,y) \tag{2}$$

for all $[x],[y] \in [X]$. *Then* $[\mathrm{d}]$ *is a metric on* $[X]$. *It is called the metric* ***induced*** *by the pseudometric.*

Proof. We first show that (2) is well-defined. Indeed, choose different coset representatives of $[x]$ and $[y]$, say x' and y' respectively (so $\mathrm{d}(x,x') = 0$ and $\mathrm{d}(y,y') = 0$). Then

$$\begin{aligned}[\mathrm{d}]([x'],[y']) &= \mathrm{d}(x',y') \\ &\leq \mathrm{d}(x',x) + \mathrm{d}(x,y) + \mathrm{d}(y,y') \\ &= \mathrm{d}(x,y) \\ &= [\mathrm{d}]([x],[y]).\end{aligned}$$

Thus $[\mathrm{d}]$ is well-defined.

Next we show that $[\mathrm{d}]$ is in fact a metric on $[X]$. First we check $[\mathrm{d}]$ is symmetric. Let $[x],[y] \in [X]$. Then

$$\begin{aligned}[\mathrm{d}]([x],[y]) &= \mathrm{d}(x,y) \\ &= \mathrm{d}(y,x) \\ &= [\mathrm{d}]([y],[x]).\end{aligned}$$

Thus $[\mathrm{d}]$ is symmetric. Next we check $[\mathrm{d}]$ satisfies triangle inequality. Let $[x],[y],[z] \in [X]$. Then

$$\begin{aligned}[\mathrm{d}]([x],[z]) &= \mathrm{d}(x,z) \\ &\leq \mathrm{d}(x,y) + \mathrm{d}(y,z) \\ &= [\mathrm{d}]([x],[y]) + [\mathrm{d}]([y],[z]).\end{aligned}$$

Thus $[\mathrm{d}]$ satisfies triangle inequality. Finally we check $[\mathrm{d}]$ satisfies identify of indiscernables. Let $[x],[y] \in [X]$ and suppose $[\mathrm{d}]([x],[y]) = 0$. Then

$$\begin{aligned}0 &= [\mathrm{d}]([x],[y]) \\ &= \mathrm{d}(x,y)\end{aligned}$$

implies $x \sim y$ by definition. Therefore $[x] = [y]$. Thus $[\mathrm{d}]$ satisfies identify of indiscernables. □

5.2.1 Completeness in (X, d) is equivalent to completeness in $([X], [\mathrm{d}])$

As in the case of the pseudometric d, the metric $[\mathrm{d}]$ induces a topology on $[X]$. We denote this topology by $\tau_{[\mathrm{d}]}$.

Proposition 5.4. (X, d) *is complete if and only if* $([X], [\mathrm{d}])$ *is complete.*

Proof. Suppose that (X, d) is complete. Let $([x_n])$ be a Cauchy sequence in $([X], [\mathrm{d}])$. We claim (x_n) is a Cauchy sequence in (X, d). Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$[\mathrm{d}]([x_n],[x_m]) < \varepsilon.$$

Then $m, n \geq N$ implies

$$\begin{aligned}\mathrm{d}(x_n,x_m) &= [\mathrm{d}]([x_n],[x_m]) \\ &< \varepsilon.\end{aligned}$$

This implies (x_n) is a Cauchy sequence in (X, d). Since (X, d) is complete, the sequence converges to a (not necessarily unique) $x \in X$. Then we claim that $[x_n] \to [x]$. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\mathrm{d}(x_n, x) < \varepsilon.$$

Then $n \geq N$ implies

$$\begin{aligned}[\mathrm{d}]([x_n],[x]) &= \mathrm{d}(x_n,x) \\ &< \varepsilon.\end{aligned}$$

This implies $[x_n] \to [x]$. Thus $([X], [\mathrm{d}])$ is complete.

Conversely, suppose $([X], [\mathrm{d}])$ is complete. Let (x_n) be a Cauchy sequence in (X, d). We claim $([x_n])$ is a Cauchy sequence in $([X], [\mathrm{d}])$. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$\mathrm{d}(x_n, x_m) < \varepsilon.$$

Then $m, n \geq N$ implies

$$\begin{aligned}[\mathrm{d}]([x_n],[x_m]) &= \mathrm{d}(x_n, x_m) \\ &< \varepsilon.\end{aligned}$$

This implies (x_n) is a Cauchy sequence in $([X],[\mathrm{d}])$. Since $([X],[\mathrm{d}])$ is complete, the sequence converges to a unique $[x] \in [X]$. We claim that $x_n \to x$ (in fact it converges to any $y \in X$ such that $y \sim x$). Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$[\mathrm{d}]([x_n],[x]) < \varepsilon.$$

Then $n \geq N$ implies

$$\begin{aligned}\mathrm{d}(x_n, x) &= [\mathrm{d}]([x_n],[x]) \\ &< \varepsilon.\end{aligned}$$

This implies $x_n \to x$. Thus (X, d) is complete. □

5.3 Quotient Topology

Recall that we view X as a topological space with toplogy τ_{d}; the topology induced by the pseudometric d. It turns out that there are two natural topologies on $[X]$. One such topology is $\tau_{[\mathrm{d}]}$; the topology induced by the metric $[\mathrm{d}]$. The other topology is called the **quotient topology with respect to** $\sim$, and is denoted by $[\tau_{\mathrm{d}}]$, where $[\tau_{\mathrm{d}}]$ is defined by

$$[\tau_{\mathrm{d}}] = \{[A] \subseteq [X] \mid \pi^{-1}([A]) \in \tau_{\mathrm{d}}\}.$$

In other words, we declare a subset $[A]$ of $[X]$ to be open in $[X]$ if and only if

$$\begin{aligned}\pi^{-1}([A]) &= \{x \in X \mid x \sim a \text{ for some } a \in A\} \\ &= \{x \in X \mid \mathrm{d}(x, a) = 0 \text{ for some } a \in A\}\end{aligned}$$

is open in X. Since $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}([X]) = X$, we see that both $\varnothing$ and $[X]$ are open in $[X]$. Furthermore, since

$$\pi^{-1}\left(\bigcup_{i \in I}[A_i]\right) = \bigcup_{i \in I} \pi^{-1}([A_i]) \text{ and } \pi^{-1}\left(\bigcap_{i \in I}[A_i]\right) = \bigcap_{i \in I} \pi^{-1}([A_i]),$$

we see that the collection of open sets in $[X]$ is closed under arbitrary unions and finite intersections. Therefore $[\tau_{\mathrm{d}}]$ is indeed a topology on $[X]$. Note that $[\tau_{\mathrm{d}}]$ was defined in such a way that it makes the projection map $\pi \colon X \to [X]$ continous.

5.3.1 Universal Mapping Property For Quotient Space

Quotient spaces satisfy the following universal mapping property.

Proposition 5.5. *Let $f \colon X \to Y$ be any continuous function which is constant on each equivalence class. Then there exists a unique continuous function $[f] \colon [X] \to Y$ such that $f = [f] \circ \pi$.*

Proof. We define $[f] \colon [X] \to Y$ by

$$[f]([x]) = f(x) \tag{3}$$

for all $x \in X$. We first show that (**??**) is well-defined. Suppose x and x' are two different representatives of the same coset (so $x \sim x'$). Then $f(x) = f(x')$ as f was assumed to be constant on equivalence classes, and so

$$\begin{aligned}[f]([x']) &= f(x') \\ &= f(x) \\ &= [f]([x]).\end{aligned}$$

Thus (**??**) is well-defined.

Next we want to show that $[f]$ is continuous. Let V be an open set in Y. Then

$$f^{-1}(V) = \pi^{-1}\left([f]^{-1}(V)\right)$$

is open in X. By the definition of quotient topology, this implies $[f]^{-1}(V)$ is open in $[X]$. This implies $[f]$ is continuous.

Finally, we want to show that $f = [f] \circ \pi$ holds. Let $x \in X$. Then we have

$$
\begin{aligned}
([f] \circ \pi)(x) &= [f](\pi(x)) \\
&= [f]([x]) \\
&= f(x).
\end{aligned}
$$

It follows that $[f] \circ \pi = f$. This establishes existence of f.

For uniqueness, assume for a contradiction that $\overline{f} \colon [X] \to Y$ is a continuous function such that $f = \overline{f} \circ \pi$ and such that $\overline{f} \neq [f]$. Choose $[x] \in [X]$ such that $\overline{f}[x] \neq [f][x]$. Then

$$
\begin{aligned}
f(x) &= (\overline{f} \circ \pi)(x) \\
&= \overline{f}(\pi(x)) \\
&= \overline{f}([x]) \\
&\neq [f]([x]) \\
&= f(x),
\end{aligned}
$$

which gives us a contradiction. □

It follows from Proposition (5.5) that we have the following bijection of sets

$$\{f \colon X \to Y \mid f \text{ is continuous and constant on equivalence classes}\} \cong \{\text{continuous functions from } [X] \text{ to } Y\}.$$

In particular, if we want to study continuous functions out of $[X]$, then we just need to study the continuous functions out of X which are constant on equivalence classes.

Proposition 5.6. *Suppose (Y, d_Y) is a metric space and $f \colon (X, \mathrm{d}) \to (Y, \mathrm{d}_Y)$ is continuous. The f is constant on equivalence classes.*

Proof. Let $x, x' \in X$ such that $x \sim x'$. Thus $\mathrm{d}(x, x') = 0$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\mathrm{d}(x, y) < \delta \text{ implies } \mathrm{d}_Y(f(x), f(y)) < \varepsilon.$$

We want to show that $f(x) = f(x')$. □

5.3.2 Open Equivalence Relation

An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection map $\pi \colon X \to [X]$ is open. In other words, the equivalence relation $\sim$ on X is open if and only if for every open set U in X, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point of U is open. The imporance of open equivalence relations is that if $\mathscr{B}$ is a basis for X, then $[\mathscr{B}]$ is a basis for $[X]$.

Lemma 5.2. *Let $x \in X$ and $r > 0$. Then*

$$\mathrm{B}_r(x) = \pi^{-1}([\mathrm{B}_r(x)]).$$

In particular, π is an open mapping.

Proof. We have

$$
\begin{aligned}
\mathrm{B}_r(x) &\subseteq \pi^{-1}\left(\pi\left(\mathrm{B}_r(x)\right)\right) \\
&= \pi^{-1}([\mathrm{B}_r(x)]).
\end{aligned}
$$

For the reverse inclusion, let $y \in \pi^{-1}([\mathrm{B}_r(x)])$. Then $\mathrm{d}(y, z) = 0$ for some $z \in \mathrm{B}_r(x)$. Choose such a $z \in \mathrm{B}_r(x)$. Then

$$
\begin{aligned}
\mathrm{d}(y, x) &\leq \mathrm{d}(y, z) + \mathrm{d}(z, x) \\
&= \mathrm{d}(z, x) \\
&< r
\end{aligned}
$$

implies implies $y \in \mathrm{B}_r(x)$. Therefore

$$\pi^{-1}([\mathrm{B}_r(x)]) \subseteq \mathrm{B}_r(x).$$

Thus each subset in $[X]$ of the form $[\mathrm{B}_r(x)]$ is open in $[X]$.

To see that π is an open mapping, let U be an open set in X. Since the set of all open balls is a basis for τ_{d}, we can cover U by open balls, say

$$U = \bigcup_{i \in I} \mathrm{B}_{r_i}(x_i).$$

Then

$$\begin{aligned}
\pi(U) &= \pi\left(\bigcup_{i \in I} \mathrm{B}_{r_i}(x_i)\right) \\
&= \bigcup_{i \in I} \pi\left(\mathrm{B}_{r_i}(x_i)\right) \\
&= \bigcup_{i \in I} [\mathrm{B}_{r_i}(x_i)] \\
&\in [\tau_{\mathrm{d}}].
\end{aligned}$$

Thus π is an open mapping. □

5.3.3 Quotient Topology Agrees With Metric Topology

Theorem 5.3. *With the notation as above, we have*

$$[\tau_{\mathrm{d}}] = \tau_{[\mathrm{d}]}.$$

Proof. We first note that for each $x \in X$ and $r > 0$, we have

$$\begin{aligned}
[\mathrm{B}_r(x)] &= \{[y] \in [X] \mid y \in \mathrm{B}_r(x)\} \\
&= \{[y] \in [X] \mid \mathrm{d}(y, x) < r\} \\
&= \{[y] \in [X] \mid [\mathrm{d}]([y], [x]) < r\} \\
&= \mathrm{B}_r([x]).
\end{aligned}$$

In particular, $\tau_{[\mathrm{d}]}$ and $[\tau_{\mathrm{d}}]$ share a common basis. Therefore $\tau_{[\mathrm{d}]} = [\tau_{\mathrm{d}}]$. □

6 Quotient Topology

Let (X, τ) be a topological space and let $\sim$ be an equivalence relation on X. We denote $[X] := X/\sim$. A coset in $[X]$ which is represented by $x \in X$ will be written as $[x]$. There is a natural **projection map** $\pi : X \to [X]$ that sends $x \in X$ to its equivalence class $[x]$. Since π is surjective, any subset of $[X]$ has the form

$$[A] = \{[a] \in [X] \mid a \in A\}.$$

With these considerations in mind, we define a topology on $[X]$ called the **quotient topology with respect to the equivalence relation** $\sim$, which we denote by $[\tau]$, by

$$[\tau] = \{[A] \subseteq [X] \mid \pi^{-1}([A]) \in \tau\}.$$

In other words, we declare a subset $[A]$ of $[X]$ to be open in $[X]$ if and only if

$$\pi^{-1}([A]) = \{x \in X \mid x \sim a \text{ for some } a \in A\}$$

is open in X. Since $\pi^{-1}(\varnothing) = \varnothing$ and $\pi^{-1}([X]) = X$, we see that both $\varnothing$ and $[X]$ are open in $[X]$. Furthermore, since

$$\pi^{-1}\left(\bigcup_{i \in I} [A_i]\right) = \bigcup_{i \in I} \pi^{-1}([A_i]) \text{ and } \pi^{-1}\left(\bigcap_{i \in I} [A_i]\right) = \bigcap_{i \in I} \pi^{-1}([A_i]),$$

we see that the collection of open sets in $[X]$ is closed under arbitrary unions and finite intersections. Therefore $[\tau]$ is indeed a topology on $[X]$.

Note that $[\tau]$ was defined in such a way that it makes the projection map $\pi \colon X \to [X]$ continuous. Also, any open set in $[X]$ can be represented by an open set in X. Indeed, suppose $[A]$ is open in $[X]$. Denote $U = \pi^{-1}([A])$. Then $[U] = [A]$.

Proposition 6.1. *Let $\mathscr{B}$ be a basis for X. Then $[\mathscr{B}]$ is a basis for $[X]$.*

Proof. It is clear that $\mathscr{B}$ covers $[X]$. Let $[U]$ and $[V]$ be two elements in $\mathscr{B}$ and assume that U and V are open in X. Then □

6.0.1 Continuity of a Map on a Quotient

Let $f\colon X \to Y$ be a continuous function. If f is constant on each equivalence class, then it induces a map $[f]\colon [X] \to Y$ defined by

$$[f][x] = f(x) \tag{4}$$

for all $x \in X$. To see that (**??**) is well-defined, suppose x and x' are two different representatives of the same coset (so $x \sim x'$). Then $f(x) = f(x')$ as f was assumed to be constant on equivalence classes, and so

$$\begin{aligned}[f][x'] &= f(x') \\ &= f(x) \\ &= [f][x].\end{aligned}$$

Thus (**??**) is well-defined. We also have continuity:

Proposition 6.2. *The induced map $[f]\colon [X] \to Y$ is continuous if and only if the map $f\colon X \to Y$ is continuous.*

Proof. Suppose $[f]$ is continuous. Then f is continuous since $f = [f] \circ \pi$ is a composition of two continuous functions. Conversely, suppose f is continuous. Let V be an open set in Y. Then

$$f^{-1}(V) = \pi^{-1}\left([f]^{-1}(V)\right)$$

is open in X. By the definition of quotient topology, this implies $[f]^{-1}(V)$ is open in $[X]$. This implies $[f]$ is continuous. □

6.0.2 Identification of a Subset to a Point

If A is a subspace of a topological space X, we can define a relation $\sim$ on X by declaring

$$x \sim x \text{ for all } x \in X \text{ and } x \sim y \text{ for all } x, y \in A.$$

This is an equivalence relation on X. We say that the quotient space $X/\sim$ is obtained from X by **identifying** A **to a point**.

Example 6.1. Let I be the unit interval $[0,1]$ and $I/\sim$ be the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1$, given by $f(x) = e^{2\pi i x}$, assumes the same value at 0 and 1, and so induces a function $\overline{f} : I/\sim \to S^1$. Since f is continuous, $\overline{f}$ is continuous. As the continuous image of a compact set I, the quotient $I/\sim$ is compact. Thus $\overline{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. Hence it is a homeomorphism.

6.1 Open Equivalence Relations

An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection map $\pi : X \to X/\sim$ is open. In other words, the equivalence relation $\sim$ on X is open if and only if for every open set U in X, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point of U is open.

Example 6.2. Let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 and let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map. Then π is not an open map. Indeed, let V be the open interval $(-2,0)$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) = (-2,0) \cup \{1\},$$

which is not open in $\mathbb{R}$.

Given an equivalence relation $\sim$ on X, let R be the subset of $X \times X$ that defines the relation

$$R = \{(x,y) \in X \times X \mid x \sim y\}.$$

We call R the **graph** of the equivalence relation $\sim$.

Theorem 6.1. *Suppose $\sim$ is an open equivalence relation on a topological space X. Then the quotient space $X/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $X \times X$.*

Proof. There is a sequence of equivalent statements: R is closed in $X \times X$ iff $(X \times X) \backslash R$ is open in $X \times X$ iff for every $(x, y) \in (X \times X) \backslash R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \emptyset$ iff for every pair $x \not\sim y$ in X, there exist neighborhoods U of x and V of y in X such that no element of U is equivalent to an element of V iff for any two points $[x] \neq [y]$ in $X/\sim$, there exist neighborhoods U of x and V of y in X such that $\pi(U) \cap \pi(V) = \emptyset$ in $X/\sim$.

We now show that this last statement is equivalent to $X/\sim$ being Hausdorff. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $X/\sim$ containing $[x]$ and $[y]$ respectively, so $X/\sim$ is Hausdorff. Conversely, suppose $X/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $X/\sim$. Then there exist disjoint open sets A and B in $X/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$. Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $X/\sim$. □

Theorem 6.2. *Let $\sim$ be an open equivalence relation on a topological space X. If $\mathcal{B} = \{B_\alpha\}$ is a basis for X, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $X/\sim$.*

Proof. Since π is an open map, $\{\pi(B(\alpha)\}$ is a collection of open sets in $X/\sim$. Let W be an open set in $X/\sim$ and $[x] \in W$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that $x \in B \subset \pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B) \subset W$, which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary. *If $\sim$ is an open equivalence relation on a second-countable space X, then the quotient space is second-countable.*

6.2 Quotients by Group Actions

Many important manifolds are constructed as quotients by actions of groups on other manifolds, and this often provides a useful way to understand spaces that may have been constructed by other means. As a basic example, the Klein bottle will be defined as a quotient of $S^1 \times S^1$ by the action of a group of order 2. The circle as defined concretely in $\mathbb{R}^2$ is isomorphic to the quotient of $\mathbb{R}$ by additive translation by $\mathbb{Z}$.

Definition 6.1. Let X be a topological space and G a discrete group. A right action of G on X is **continuous** if for each $g \in G$ the action map $X \to X$ defined by $x \mapsto x.g$ is continuous (and hence a homeomorphism, as the action of g^{-1} gives an inverse). The action is **free** if for each $x \in X$ the stabilizer subgroup $\{g \in G \mid x.g = x\}$ is the trivial subgroup (in other words, $x.g = x$ implies $g = 1$). The action is **properly discontinuous** when it is continuous for the discrete topology on G and each $x \in X$ admits an open neighborhood U_x so that the G-translate $U_x.g$ meets U_x for only finitely many $g \in G$.

Proposition 6.3. *A right action of G on X is continuous if $\pi : X \times G \to X$ is continuous.*

Remark. Here, G has the discrete topology.

Proof. Suppose we have a right action of G on X which is continuous. Let U be an open set in X. For each $g \in G$, let $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) = \bigcup_{g \in G} U_g \times \{g\},$$

which is open. Conversely, suppose π is continuous and let $g \in G$. Let U be open in X and set $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) \cap X \times \{g\} = U_g \times \{g\},$$

which shows that g is continuous since $\pi^{-1}(U)$ and $X \times \{g\}$ are open in $X \times G$. □

Example 6.3. Suppose that X is a locally Hausdorff space, and that G acts on X on the right via a properly discontinuous action. For each $x \in X$, we get an open subset U_x such that U_x meets $U_x.g$ for only finitely many $g \in G$. This property is unaffected by replacing U_x with a smaller open subset around x, so by the locally Hausdorff property we can assume that U_x is Hausdorff. The key is that we can do better: there exists an open set $U'_x \subseteq U_x$ such that U'_x meets $U'_x.g$ if and only if $x = x.g$. Thus, if the action is also free then U'_x is disjoint from $U'_x.g$ for all $g \in G$ with $g \neq 1$.

To find U'_x, let $g_1, \ldots, g_n \in G$ be an enumeration of the finite set of elements $g \in G$ such that U_x meets $U_x.g$. For any open subset $U \subseteq U_x$ we can only have $U \cap U.g \neq \emptyset$ for g equal to one of the g_i's, so it suffices to show that for each i with $x.g_i \in U_x \backslash \{x\}$ there is an open subset $U_i \subseteq U_x$ such that $U_i \cap (U_i).g_i = \emptyset$ (and then we may take U'_x to be the intersection of the U_i's over the finitely many i such that $x.g_i \neq x$). By the Hausdorff property of U_x, when $x.g_i \in U_x \backslash \{x\}$ there exist disjoint opens $V_i, V'_i \subseteq U_x$ around x and $x.g_i$ respectively. By continuity of the action on X by $g_i \in G$ there is an open $W_i \subseteq X$ around x such that $(W_i).g_i \subseteq V'_i$. Thus $U_i = W_i \cap V_i$ is disjoint from V'_i yet satisfies $(U_i).g_i \subseteq V'_i$, so $U_i \cap (U_i).g_i = \emptyset$. This completes the construction of U'_x.

The interest in free and properly discontinuous actions is that for such actions in the locally Hausdorff case we may find an open U_x around each $x \in X$ such that U_x is disjoint from $U_x.g$ whenever $g \neq 1$. Thus, for such actions we may say that in X/G we are identifying points in the same G-orbit with this identification process not "crushing" the space X by identifying points in X that are arbitrarily close to each other. An example where things go horribly wrong is the action of $G = \mathbb{Q}$ on $\mathbb{R}$ via additive translations. This is a continuous action, but the quotient $\mathbb{R}/\mathbb{Q}$ is very bad: any two $\mathbb{Q}$-orbits in $\mathbb{R}$ contain arbitrarily close points!

Here are some examples of free and properly discontinuous actions.

Example 6.4. The antipodal map on S^n, given by $(a_1, \ldots, a_{n+1}) \mapsto (-a_1, \ldots, -a_{n+1})$, viewed as an action of the integers mod 2 is free and properly discontinuous: freeness is clear, as is continuity, and for any $x \in S^n$ the points near x all have their antipodes far away!

Example 6.5. Consider the curve $X := \mathbf{V}(x^3 + y^3 + z^3 - 1) \subseteq \mathbb{C}^3$. Then the action $(a_1, a_2, a_3) \mapsto (\zeta_3 a_1, \zeta_3 a_2, \zeta_3)$, viewed as an action of the integers mod 3 is free and properly discontinuous.

Example 6.6. Let $X = S^1 \times S^1$ be a product of two circles, where the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is viewed as a topological group (using multiplication in $\mathbb{C}$, so both the group law and inversion $z \mapsto 1/z = \overline{z}$ on S^1 are continuous). The visibly continuous map $(z, w) \mapsto (1/z, -w) = (\overline{z}, -w)$ reflects through the x-axis in the first circle and rotates 180-degree in the second circle, and it is its own inverse. Thus, this give an action by the order-2 group G of integers mod 2. The associated quotient X/G will be called the (set-theoretic) **Klein bottle**.

Theorem 6.3. *Let X be a locally Hausdorff topological space with a free and properly discontinuous action by a group G. There is a unique topology on X/G such that the quotient map $\pi : X \to X/G$ is a continuous map that is a local homeomorphism (i.e. each $x \in X$ admits a neighborhood mapping homeomorphically onto an open subset of X/G). Moreover, the quotient map is open.*

A subset $S \subseteq X/G$ is open if and only if its preimage in X is open, and if $U \subseteq X$ is an open set that is disjoint from $U.g$ for all nontrivial $g \in G$ then the map $U \to X/G$ is a homeomorphism onto its open image $\overline{U}$ and the natural map $U \times G \to \pi^{-1}(\overline{U})$ over $\overline{U}$ given by $(u, g) \mapsto u.g$ is a homeomorphism when G is given the discrete topology.

Remark. The topology in this theorem is called the **quotient topology**, and it is locally Hausdorff since $X \to X/G$ is a local homeomorphism.

Proof. Sketch: we show that π is an open map. Let $x \in X$ and pick U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$. We first show that $\pi(U_x)$ is open. The inverse image of $\pi(U_x)$ under π is a disjoint union of open sets $\bigcup_{g \in G} U_x.g$. Therefore $\pi(U_x)$ is open. Now let U be any open subset of X. For each $x \in X$, choose U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$ and $U_x \subset U$. Then

$$\pi(U) = \pi\left(\bigcup_{x \in U} U_x\right) = \bigcup_{x \in U} \pi(U_x)$$

implies $\pi(U)$ is open. □

Example 6.7. (Möbius Strip) Choose $a > 0$. Let $X = (-a, a) \times S^1$, and let the group of order 2 act on it with the non-trivial element acting by $(t, w) \mapsto (-t, -w)$. This is easily checked to be a continuous action for the discrete topology of the group of order 2, and it is free and properly discontinuous. The quotient M_a is the **Möbius strip** of height $2a$.

To check that the Möbius strip M_a is Hausdorff, we use the quotient criterion: the set of points in $X \times X$ with the form $((t, w), (t', w'))$ with $(t', w') = (t, w)$ or $(t', w') = (-t, -w)$ is checked to be closed by using the sequential criterion in $X \times X$: suppose $(t_n, w_n) \sim (t'_n, w'_n)$ are sequences in $X \times X$ which converge (t, w) and (t', w') respectively. Then we need to show that $(t, w) \sim (t', w')$. Assume that $(t, w) \neq (t', w')$. Choose open neighborhoods U of (t, w) and U' of (t', w') respectively such that $U \cap U' = \emptyset$ and such that eventually $(t_n, w_n) \neq (t'_n, w'_n)$ (We can do this because they converge to different limits and our space $X \times X$ is Hausdorff). Thus, eventually we have $(t'_n, w'_n) = (-t_n, -w_n) \to (-t, -w)$.

7 Product Topology

Let Λ be a set and let $(X_\lambda, \tau_\lambda)$ be a topological space for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we denote by $\pi_\lambda \colon \prod_\lambda X_\lambda \to X_\lambda$ to be the λth **projection map** defined by

$$\pi_\lambda((x_\lambda)) = x_\lambda$$

for all $(x_\lambda) \in \prod_\lambda X_\lambda$. We define the **product topology** on $\prod_\lambda X_\lambda$, denoted $\prod_\lambda \tau_\lambda$, to be the topology generated by sets of the form

$$\{\pi_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda \text{ and } U_\lambda \in \tau_\lambda\}.$$

In particular, the product is the *weakest* topology on $\prod_\lambda X_\lambda$ which makes all of the projection maps π_λ continuous. Recall that the topology on X generated by a subcollection $\mathcal{C} \subseteq \mathcal{P}(X)$ is obtained by adjoining X and $\varnothing$ to the entire collection as well as all adjoint all arbitrary unions of finite intersections of members of $\mathcal{C}$ to the entire collection. Note that for each $\mu \in \Lambda$ and $U_\mu \in \tau_\mu$ we have

$$\pi_\mu^{-1}(U_\mu) = U_\mu \times \prod_{\lambda \in \Lambda \backslash \{\mu\}} X_\lambda,$$

and for each distinct $\mu, \kappa \in \Lambda$ and $U_\mu \in \tau_\mu$ and $U_\kappa \in \tau_\kappa$, we have

$$\pi_\mu^{-1}(U_\mu) \cap \pi_\kappa^{-1}(U_\kappa) = U_\mu \times U_\kappa \times \prod_{\lambda \in \Lambda \backslash \{\mu, \kappa\}} X_\lambda.$$

In general, a basis of $\prod_\lambda \tau_\lambda$ consists of sets of the form

$$\prod_{\lambda_0 \in \Lambda_0} U_{\lambda_0} \times \prod_{\lambda \in \Lambda \backslash \Lambda_0} X_\lambda,$$

where Λ_0 is a finite subset of Λ and U_{λ_0} is an open subset of X_{λ_0} for each $\lambda_0 \in \Lambda_0$.

Proposition 7.1. *Let Λ be a set, let $(X_\lambda, \tau_\lambda)$ be a topological space for each $\lambda \in \Lambda$, let Y be a topological space, and let $f\colon Y \to \prod_\lambda X_\lambda$ be a function. Then f is continuous if and only if $\pi_\lambda \circ f\colon Y \to X_\lambda$ is continuous for each $\lambda \in \Lambda$.*

Proof. If f is continous, then each $\pi_\lambda \circ f$ is a composition of continuous functions and is hence continuous. Conversely, suppose that $\pi_\lambda \circ f$ is continuous for each $\lambda \in \Lambda$. To show that f is continous, it suffices to show that the premiage of a subbase element $\pi_\lambda^{-1}(U_\lambda)$ is open. But note that

$$f^{-1}(\pi_\lambda^{-1}(U_\lambda)) = (\pi_\lambda \circ f)^{-1}(U_\lambda)$$

is open since $\pi_\lambda \circ f$ is continuous. Thus f is continuous. □

8 Coproduct Topology

Let Λ be a set and let $(X_\lambda, \tau_\lambda)$ be a topological space for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we denote by $\pi_\lambda\colon \prod_\lambda X_\lambda \to X_\lambda$ to be the λth **projection map** defined by

$$\pi_\lambda((x_\lambda)) = x_\lambda$$

for all $(x_\lambda) \in \prod_\lambda X_\lambda$. We define the **product topology** on $\prod_\lambda X_\lambda$, denoted $\prod_\lambda \tau_\lambda$, to be the topology generated by sets of the form

$$\{\pi_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda \text{ and } U_\lambda \in \tau_\lambda\}.$$

In particular, the product is the *weakest* topology on $\prod_\lambda X_\lambda$ which makes all of the projection maps π_λ continuous. Recall that the topology on X generated by a subcollection $\mathcal{C} \subseteq \mathcal{P}(X)$ is obtained by adjoining X and $\varnothing$ to the entire collection as well as all adjoint all arbitrary unions of finite intersections of members of $\mathcal{C}$ to the entire collection. Note that for each $\mu \in \Lambda$ and $U_\mu \in \tau_\mu$ we have

$$\pi_\mu^{-1}(U_\mu) = U_\mu \times \prod_{\lambda \in \Lambda \backslash \{\mu\}} X_\lambda,$$

and for each distinct $\mu, \kappa \in \Lambda$ and $U_\mu \in \tau_\mu$ and $U_\kappa \in \tau_\kappa$, we have

$$\pi_\mu^{-1}(U_\mu) \cap \pi_\kappa^{-1}(U_\kappa) = U_\mu \times U_\kappa \times \prod_{\lambda \in \Lambda \backslash \{\mu, \kappa\}} X_\lambda.$$

In general, a basis of $\prod_\lambda \tau_\lambda$ consists of sets of the form

$$\prod_{\lambda_0 \in \Lambda_0} U_{\lambda_0} \times \prod_{\lambda \in \Lambda \backslash \Lambda_0} X_\lambda,$$

where Λ_0 is a finite subset of Λ and U_{λ_0} is an open subset of X_{λ_0} for each $\lambda_0 \in \Lambda_0$.

Proposition 8.1. *Let Λ be a set, let $(X_\lambda, \tau_\lambda)$ be a topological space for each $\lambda \in \Lambda$, let Y be a topological space, and let $f\colon Y \to \prod_\lambda X_\lambda$ be a function. Then f is continuous if and only if $\pi_\lambda \circ f\colon Y \to X_\lambda$ is continuous for each $\lambda \in \Lambda$.*

Proof. If f is continous, then each $\pi_\lambda \circ f$ is a composition of continuous functions and is hence continuous. Conversely, suppose that $\pi_\lambda \circ f$ is continuous for each $\lambda \in \Lambda$. To show that f is continous, it suffices to show that the premiage of a subbase element $\pi_\lambda^{-1}(U_\lambda)$ is open. But note that

$$f^{-1}(\pi_\lambda^{-1}(U_\lambda)) = (\pi_\lambda \circ f)^{-1}(U_\lambda)$$

is open since $\pi_\lambda \circ f$ is continuous. Thus f is continuous. $\square$