Homological Constructions over a Field of Characteristic 2

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Throughout this section, let K be a field of characteristic 2, S denote the polynomial ring $K[x_1, \dots, x_n]$, I be a homogeneous ideal in S, and let G be the reduced Gröbner basis for I with respect to some fixed monomial ordering.

1 Constructing the Chain Complexes (S, d), (S_I, d), and $(I, \underline{d})$.

1.1 Construction of (S, d)

Let $d : S \to S$ be the graded K-linear map of degree -1 given by $d := \sum_{j=1}^{n} \partial_{x_j}$. Since K has characteristic 2, we have $d^2 = 0$. Indeed, it suffices to show that $d^2(m) = 0$ for all monomials m in S. So let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. Then

$$\begin{aligned} d^2(m) &= \left(\sum_{k=1}^{n} \partial_{x_k} \right)^2 \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) \\ &= \left(\sum_{k=1}^{n} \partial_{x_k}^2 \right) \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) \\ &= \sum_{k=1}^{\infty} \alpha_k(\alpha_k - 1) x_1^{\alpha_k - 2} \\ &= 0. \end{aligned}$$

Thus the differential d gives the graded K-module S the structure of a chain complex over K.

1.2 Construction of (S_I, d)

Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree i in S. We denote

$$[m]_o = \{1 \leq \lambda \leq n \mid \alpha_\lambda \text{ is odd}\} \qquad \text{and} \qquad [m]_e = \{1 \leq \mu \leq n \mid \alpha_\mu \text{ is even}\}$$

Using this notation, we can express the differential in another way:

$$d(m) = \sum_{\lambda \in [m]_o} x_\lambda^{-1} m.$$

This makes it clear that the differential d maps S_I into S_I. Indeed, if m is not in $\mathrm{LT}(I)$, then every term $x_\lambda^{-1} m$ of $d(m)$ is not in $\mathrm{LT}(I)$ either. Thus the differential d gives the graded K-module S_I the structure of a chain complex over K.

1.3 Construction of $(S/I, \overline{d})$

Definition 1.1. We say I is d-**stable** if d maps I into I.

Suppose I is d-stable. Then the differential $d : S \to S$ induces a graded linear map of degree -1, denoted $\overline{d} : S/I \to S/I$, where

$$\overline{d}(\overline{f}) = \overline{d(f)} \text{ for all } f \in S.$$

Indeed, the map $\overline{d}$ is well-defined since d is I-stable. To see why, let $f+g$ and f, where $g \in I$ and $f \in S$, be two different representatives of a class in S/I, i.e. $\overline{f+g} = \overline{f} \in S/I$. Then

$$\begin{aligned}\overline{d}\left(\overline{f+g}\right) &= \overline{d(f+g)} \\ &= \overline{d(f)+d(g)} \\ &= \overline{d(f)} \\ &= \overline{d}(\overline{f}).\end{aligned}$$

where $\overline{d(f)+d(g)} = \overline{d(f)}$ since $d(g) \in I$.

Moreover, the differential $\overline{d}$ gives S/I the structure of a differential graded K-algebra. Indeed, $\overline{d}$ is a graded linear map of degree -1 such that $\overline{d}^2 = 0$ and such that $\overline{d}$ satisfies Leibniz law. This is because $\overline{d}$ inherits all of the properties from d. For instance, to see that $\overline{d}$ satisfies Leibniz law, let $\overline{f}_1$ and $\overline{f}_2$ be in S/I. Then

$$\begin{aligned}\overline{d}(\overline{f_1 f_2}) &= \overline{d(f_1 f_2)} \\ &= \overline{d(f_1)f_2 + f_1 d(f_2)} \\ &= \overline{d(f_1)f_2} + \overline{f_1 d(f_2)} \\ &= \overline{d}(\overline{f_1})\overline{f_2} + \overline{f_1}\overline{d}(\overline{f_2}).\end{aligned}$$

Thus if I is d-stable, then the differential $\overline{d}$ gives the graded K-algebra S/I the structure of a differential graded K-algebra.

1.4 Construction of $(I, \underline{d})$

Our final construction involves the graded K-module I. Let $\underline{d} : I \to I$ be the graded K-linear map of degree -1 given by

$$\underline{d}(f) := d(f) + d(f)^G = \pi(d(f))$$

for all $f \in I$. Then $\underline{d}^2 = 0$. Indeed, for all $f \in I$, we have

$$\begin{aligned}\underline{d}(\underline{d}(f)) &= \underline{d}(d(f) + d(f)^G) \\ &= d(d(f) + d(f)^G) + d(d(f) + d(f)^G)^G \\ &= d(d(f)^G) + d(d(f)^G)^G \\ &= d(d(f)^G) + d(d(f)^G) \\ &= 0,\end{aligned}$$

where $d(d(f)^G)^G = d(d(f)^G)$ since every term in $d(d(f)^G)$ is not in I. Thus the differential $\underline{d}$ gives the graded K-module I the structure of a chain complex over K.

Remark. Let J be a homogeneous ideal in S such that $I \supset J$. If J is $\underline{d}$-stable, then the differential $\underline{d}$ gives the graded K-module I/J the structure of a chain complex over K, which we denote by $(I/J, \underline{d})$. If I and J are both monomial ideals, then J is always $\underline{d}$-stable.

2 Differential Graded K-Algebras

Since d is defined in terms of partial derivatives, it is clear that d satisfies Leibniz law. Thus (S, d) is more than just a chain complex over K; it is a differential graded K-algebra. Since S_I is a graded K-algebra, it is natural wonder if (S_I, d) is also a differential graded K-algebra. A quick counterexample shows that this is not necessarily the case:

Example 2.1. Consider $S = K[x]$ and $I = \langle x^5 \rangle$. Then

$$\begin{aligned}d(x \cdot x^4) &= d((x^5)^G) \\ &= d(0) \\ &= 0,\end{aligned}$$

but

$$\begin{aligned}d(x) \cdot x^4 + x \cdot d(x^4) &= 1 \cdot x^4 + x \cdot 0 \\ &= (x^4)^G + 0^G \\ &= x^4,\end{aligned}$$

so $d(x \cdot x^5) \neq d(x) \cdot x^4 + x \cdot d(x^4)$.

2.1 When (S_I, d) Has the Structure of a Differential Graded K-Algebra

The next theorem tells us precisely when (S_I, d) is a differential graded K-algebra.

Theorem 2.1. *(S_I, d) is a differential graded K-algebra if and only if $d(g) = 0$ for all $g \in G$.*

Proof. Assume that $d(g) = 0$ for all $g \in G$. We first prove that $d(f^G) = d(f)^G$ for all $f \in S$. Let $f \in S$. From the division algorithm, we have $f = g_1 q_1 + \cdots + g_r q_r + f^G$ for some $q_1, \ldots, q_r \in S$. Thus

$$\begin{aligned} d(f) &= d(g_1 q_1 + \cdots + g_r q_r + f^G) \\ &= d(g_1 q_1) + \cdots + d(g_r q_r) + d(f^G) \\ &= g_1 d(q_1) + \cdots + g_r d(q_r) + d(f^G). \end{aligned}$$

Since $g_1 d(h_1) + \cdots + g_r d(h_r) \in I$ and no term of $d(f^G)$ is divisible by any element of $\mathrm{LT}(I)$, it follows from uniqueness of normal forms that $d(f^G) = d(f)^G$.

Now we show that this implies that (S_I, d) is a differential graded K-algebra. Let $f_1, f_2 \in S_I$. Then

$$\begin{aligned} d(f_1 \cdot f_2) &= d((f_1 f_2)^G) \\ &= (d(f_1 f_2))^G \\ &= (d(f_1) f_2 + f_1 d(f_2))^G \\ &= (d(f_1) f_2)^G + (f_1 d(f_2))^G \\ &= d(f_1) \cdot f_2 + f_1 \cdot d(f_2). \end{aligned}$$

Therefore (S_I, d) is a differential graded K-algebra.

Now we prove the converse. Assume (S_I, d) is a differential graded K-algebra. Let $g \in G$ and let m be the lead term of g. We may assume g is not a constant (otherwise we'd clearly have $d(g) = 0$). Thus, there exists some x_λ such that x_λ divides m. Then on the one hand, we have

$$\begin{aligned} d(x_\lambda \cdot x_\lambda^{-1} m) &= d(m^G) \\ &= d(g + m) \\ &= d(g) + d(m), \end{aligned}$$

since $m^G = g + m$. On the other hand, we have

$$\begin{aligned} d(x_\lambda) \cdot x_\lambda^{-1} m + x_\lambda \cdot d(x_\lambda^{-1} m) &= (x_\lambda^{-1} m)^G + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda d(x_\lambda^{-1} m))^G \\ &= x_\lambda^{-1} m + (x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m + d(m))^G \\ &= x_\lambda^{-1} m + (x_\lambda^{-1} m)^G + d(m)^G \\ &= x_\lambda^{-1} m + x_\lambda^{-1} m + d(m)^G \\ &= d(m), \end{aligned}$$

since $(x_\lambda^{-1} m)^G = x_\lambda^{-1} m$ and $d(m)^G = d(m)$ (every term of $d(m)$ does not lie in $\langle \mathrm{LT}(G) \rangle$). Since (S_I, d) is a differential graded K-algebra, we must have $d(g) = 0$. This establishes this theorem. $\square$

Remark. We should note that the identity $x_\lambda d(x_\lambda^{-1} m) = x_\lambda (x_\lambda^{-2} m + x_\lambda^{-1} d(m)) = x_\lambda^{-1} m + d(m)$ follows since d satisfies Leibniz law not just in S, but also in $S[x_1^{-1}, \ldots, x_n^{-1}]$. Again, this is because d is defined in terms of partial derivatives.

Example 2.2. Going back to Example (**??**), where $S = K[x, y]$, $I = \langle xy^2 + y^3, x^3 + x^2 y \rangle$, and $G = \{xy^2 + y^3, x^3 + x^2 y\}$. We have $d(xy^2 + y^3) = d(x^3 + x^2 y) = 0$. Therefore Proposition (**??**) implies (S_I, d) is a differential graded K-algebra.

Now we want to show that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra, and moreover, they are isomorphic to each other.

Lemma 2.2. *Let I be a homogeneous ideal in the polynomial ring S, and let $G = \{g_1, g_2, \ldots, g_r\}$ be the reduced Gröbner basis for I. Then $d(g) = d(g)^G$ for all $g \in G$.*

Proof. Let $g \in G$. If $d(g) = 0$, then clearly we have $d(g) = d(g)^G$, so assume $d(g) \neq 0$. We need to prove that $d(g) = d(g)^G$. This is equivalent to saying that no term of $d(g)$ belongs to $\langle \mathrm{LT}(G) \rangle := \langle \mathrm{LT}(g_1), \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_r) \rangle$, since G is a Gröbner basis. Every term in of $d(g)$ has the form $x_\lambda^{-1} m$ where m is some term of g. It is easy to see that this term cannot belong to $\langle \mathrm{LT}(G) \rangle$. Indeed, if $x_\lambda^{-1} m \in \langle \mathrm{LT}(G) \rangle$, then $m \in \langle \mathrm{LT}(g_2), \ldots, \mathrm{LT}(g_r) \rangle$, and this contradicts the fact that G is a *reduced* Gröbner basis. $\square$

Proposition 2.1. *I is d-stable if and only if $d(g) \in I$ for all $g \in G$.*

Proof. One direction is trivial, so let's prove the other direction. Suppose $d(g) \in I$ for all $g \in G$ and let $f \in I$. Since G generates I, we can write $f = \sum_{\lambda=1}^r q_\lambda g_\lambda$ for some $q_1, \ldots, q_r \in S$. Thus, by Leibniz law, we have

$$\begin{aligned} d(f) &= d\left(\sum_{\lambda=1}^r q_\lambda g_\lambda\right) \\ &= \sum_{\lambda=1}^r d(q_\lambda g_\lambda) \\ &= \sum_{\lambda=1}^r (d(q_\lambda) g_\lambda + q_\lambda d(g_\lambda)) \in I. \end{aligned}$$

Thus, I is d-stable. $\square$

Combining Lemma (2.2) and Proposition (2.1), we find that that $d(g) = 0$ for all $g \in G$ if and only if $d(g) \in I$ for all $g \in G$ if and only if I is d-stable. Combining this with Theorem (2.1), we find that (S_I, d) is a differential graded K-algebra if and only if $(S/I, \overline{d})$ is a differential graded K-algebra. Now we will show that they are in fact isomorphic to each other.

Theorem 2.3. *Suppose I is d-stable. Then (S_I, d) is isomorphic to $(S/I, \overline{d})$ as differential graded K-algebras.*

Proof. Recall that S/I is isomorphic to S_I as graded K-algebras, where the isomorphism is given by mapping $\overline{f} \in S/I$ to $f^G \in S_I$. It remains to show that this isomorphism respects the differential graded algebra structure. In particular, we need to show that $d(f^G) = d(f)^G$ for all $f \in S$. This was already proven in Theorem (2.1). $\square$

2.2 More Differential Graded K-algebras

Proposition 2.2. *Suppose I is d-stable and let g be a homogeneous polynomial such that $d(g) = 0$. Then $(S_{\langle I, g \rangle}, d)$ and $(S_{I:g}, d)$ are differential graded K-algebras.*

Proof. We just need to show that $\langle I, g \rangle$ and $I : g$ are both d-stable. Since $d(g) = 0$, it follows that $\langle I, g \rangle$ is d-stable. To prove that $I : g$ is d-stable, let $f \in I : g$. Then since $fg \in I$, $d(g) = 0$, and I is d-stable, it follows that

$$d(f)g = d(f)g + fd(g) = d(fg) \in I$$

Therefore $d(f) \in I : g$, which implies that $I : g$ is d-stable. $\square$

Example 2.3. Consider $S = K[x, y, z]$, $g = x^2 y + x^2 z$, and $I = \langle f_1, f_2, f_3 \rangle$ where

$$\begin{aligned} f_1 &= xy + xz + yz \\ f_2 &= x^4 y + x^5 \\ f_3 &= y^3 + y^2 z \end{aligned}$$

Then $d(f_1) = d(f_2) = d(f_3) = 0$ implies that (S_I, d) is a differential graded K-algebra. The reduced Gröbner basis for I with respect to graded lexicographical ordering is $G = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$\begin{aligned} g_1 &= xy + xz + yz \\ g_2 &= y^3 + y^2 z \\ g_3 &= y^2 z^2 \\ g_4 &= xz^4 + yz^4 \\ g_5 &= x^5 + x^4 z + x^3 z^2 + x^2 z^3 \\ g_6 &= x^4 z^2 \end{aligned}$$

Since $d(g) = 0$, we know that $(S_{\langle I,g\rangle}, d)$ and $(S_{I:g}, d)$ are also differential graded K-algebras. The reduced Gröbner basis for $I : g$ with respect to graded lexicographical ordering is $G'' = \{g_1'', g_2'', g_3''\}$, where

$$\begin{aligned} g_1'' &= y + z \\ g_2'' &= z^2 \\ g_3'' &= x^3 + x^2 z \end{aligned}$$

and the reduced Gröbner basis for $\langle I, g\rangle$ with respect to graded lexicographical ordering is $G' = \{g_1', g_2', g_3', g_4', g_5'\}$, where

$$\begin{aligned} g_1' &= xy + xz + yz \\ g_2' &= y^3 + y^2 z \\ g_3' &= xz^2 + yz^2 \\ g_4' &= y^2 z^2 \\ g_5' &= x^5 + x^4 z + x^3 z^2 + x^2 z^3 \end{aligned}$$

3 Homology Calculations

3.1 $H(S/I, \overline{d}) \cong 0$

Proposition 3.1. *Suppose I is d-stable. Then $H(S_I) = 0$.*

Proof. Let f be a homogeneous polynomial in S_I such that $d(f) = 0$. Then for any $x_\lambda \in (S_I)_1$, we have

$$d(x_\lambda f) = d(x_\lambda) f + x_\lambda d(f) = f.$$

Therefore $\mathrm{Ker}(d) = \mathrm{Im}(d)$, hence $H(S_I) = 0$. $\square$

Remark. Taking $I = 0$ shows that $H(S) = 0$.

3.2 $H_i(I) \cong H_{i-1}(S_I)$

Proposition 3.2. *The differential d induces isomorphisms $H_i(I) \cong H_{i-1}(S_I)$ for all $i > 0$.*

Proof. First we show that $\underline{d}\pi = \pi d$. For all $f \in S$, we have

$$\begin{aligned} \underline{d}(\pi(f)) &= \underline{d}(f + f^G) \\ &= d(f + f^G) + d(f + f^G)^G \\ &= d(f) + d(f^G) + d(f)^G + d(f^G)^G \\ &= d(f) + d(f^G) + d(f)^G + d(f^G) \\ &= d(f) + d(f)^G \\ &= \pi(d(f)), \end{aligned}$$

where $d(f^G)^G = d(f^G)$ because no term in $d(f^G)$ lies in $\mathrm{LT}(I)$.

Therefore we have a short exact sequence of chain complexes over K:

$$0 \longrightarrow (S_I, d) \longrightarrow (S, d) \xrightarrow{\pi} (I, \underline{d}) \longrightarrow 0.$$

From this, we obtain a long exact sequence in homology, which gives for each $i > 0$, the following short exact sequences:

$$0 = H_i(S) \longrightarrow H_i(I) \xrightarrow{d} H_{i-1}(S_I) \longrightarrow H_{i-1}(S) = 0.$$

where d is obtained from the connecting map. In more detail, d maps the element $[f] \in H_i(I)$ to the element $[d(f)] \in H_{i-1}(S_I)$. $\square$

3.3 Decomposing $H_i(S_I)$

Let g be a homogeneous polynomial of degree j and let G' be the reduced Gröbner basis for $\langle I, g\rangle$ with respect to our fixed monomial ordering. In Commutative Algebra, we learn about the following short exact sequence of graded S-modules

$$\begin{tikzcd}
0 \arrow[r] & (S/(I:g))(-j) \arrow[r, "g"] & S/I \arrow[r] & S/\langle I,g\rangle \arrow[r] & 0 \\
& \overline{f} \arrow[r, mapsto] & \overline{fg} & &
\end{tikzcd}$$

We want to use this short exact sequence to our advantage. First, using the isomorphisms $S_{I:g} \cong S/(I:g)$, $S_I \cong S/I$, and $S_{\langle I,g\rangle} \cong S/\langle I,g\rangle$, we get, for each i, a short exact sequence of K-vector spaces

$$\begin{tikzcd}
0 \arrow[r] & (S_{I:g})_{j-i} \arrow[r, "\cdot g"] & (S_I)_i \arrow[r, "_^{G'}"] & (S_{\langle I,g\rangle})_i \arrow[r] & 0 \\
& f \arrow[r, mapsto] & (fg)^G & & \\
& & f \arrow[r, mapsto] & f^{G'} &
\end{tikzcd}$$

or in other words, a short exact sequence of graded K-vector spaces

$$\begin{tikzcd}
0 \arrow[r] & (S_{I:g})(-j) \arrow[r, "\cdot g"] & S_I \arrow[r, "_^{G'}"] & S_{\langle I,g\rangle} \arrow[r] & 0
\end{tikzcd}$$

We want to know under what conditions this becomes a short exact sequence of chain complexes over K, that is, when does the following diagram commute?

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & (S_{I:g})_{j-i} \arrow[r, "\cdot g"] \arrow[d, "d"'] & (S_I)_i \arrow[r, "_^{G'}"] \arrow[d, "d"'] & (S_{\langle I,g\rangle})_i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & (S_{I:g})_{j-i-1} \arrow[r, "\cdot g"] \arrow[d] & (S_I)_{i-1} \arrow[r, "_^{G'}"] \arrow[d] & (S_{\langle I,g\rangle})_{i-1} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

After some thought, we find that the conditions which need to be satisfied are the following:

$$(gd(m))^G = d((gm)^G) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I:g) \tag{1}$$

$$d(m)^{G'} = d(m^{G'}) \text{ for all monomials } m \text{ which are not in } \mathrm{LT}(I) \tag{2}$$

For the moment, let's assume that these conditions are satisfied so that we have a short exact sequence of chain complexes. Then by the usual argument, the short exact sequence of chain complexes gives rise to a long exact sequence in homology:

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{i+1}(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\
H_{i-j}(S_{I:g}) \arrow[r, "\cdot g"] & H_i(S_I) \arrow[r, "_^{G'}"] & H_i(S_{\langle I,g\rangle}) \arrow[dll, rounded corners, "\lambda"] \\
H_{i-j-1}(S_{I:g}) \arrow[r, "\cdot g"] & H_{i-1}(S_I) \arrow[r, "_^{G'}"] & \cdots
\end{tikzcd}$$

It's easy to see that the connecting maps λ all induce the zero map. So in fact, we get for each i, the short exact sequence of K-vector spaces:

$$0 \longrightarrow H_{i-j}(S_{I:g}) \xrightarrow{\cdot g} H_i(S_I) \xrightarrow{-^{G'}} H_i(S_{\langle I,g\rangle}) \longrightarrow 0,$$

and since the inclusion map $S_{\langle I,g\rangle} \hookrightarrow S_I$ splits the map $-^{G'}$, we obtain the following isomorphism

$$H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle}) \cong H_i(S_I) \tag{3}$$

where we map the representative (f_1, f_2) in $H_{i-j}(S_{I:g}) \oplus H_i(S_{\langle I,g\rangle})$ to the representative $gf_1 + f_2$ in $H_i(S_I)$.

3.3.1 Decomposing $H_i(S_I)$ in a Special Case and an Example

We will now discuss a special case of when the conditions in Theorem (**??**) are satisfied. Consider the case where I is a monomial ideal and g is a monomial of degree j which is not in I. Then condition (1) is satisfied since if m is not in $I : g$, then gm is not in I, and so $(gm)^G = gm$ which implies $(gd(m))^G = gd(m)$.

For condition (2) first assume that m is not in $\langle I, g\rangle$. Then then $m^{G'} = m$, which implies $d(m)^{G'} = d(m) = d(m^{G'})$. Thus condition (2) is satisfied in this case. Now assume that $m = g$. Then $m^{G'} = 0$, which implies $d(m^{G'}) = 0$. Thus, we must have $d(g) = 0$ in order for condition (2) to be satisfied in this case. So assume $d(g) = 0$ and consider the final case where $m = m_1 g$. Since $d(g) = 0$, we obtain $d(m)^{G'} = (d(m_1)g)^{G'} = 0$, and thus (2) is satisfied in this case as well.

In the next example, we show how we can apply Theorem (**??**) recursively. In what follows, we frequently use the notation I, g to mean $\langle I, g\rangle$ and $I : g$ to mean $I : \langle g\rangle$. For example, $I, g_1 : g_2 = \langle I, g_1\rangle : \langle g_2\rangle$, and $I : g_1, g_2 = \langle (I : g_1), \langle g_2\rangle\rangle$, and so on. We also note that $I : g_1 : g_2 = I : g_1 g_2$.

Example 3.1. Consider $S = K[x, y, z]$ and $I = \langle x^3 y, yz^3\rangle$. Then $d(x^2) = d(z^2) = 0$, and so

$$\begin{aligned}
H_i(S_I) &= x^2 H_{i-2}(S_{I:x^2}) \oplus H_i(S_{I,x^2}) \\
&= x^2(z^2 H_{i-4}(S_{I:x^2z^2}) \oplus H_{i-2}(S_{I:x^2,z^2})) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \\
&= x^2 z^2 H_{i-4}(S_{I:x^2z^2}) \oplus x^2 H_{i-2}(S_{I:x^2,z^2}) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2})
\end{aligned}$$

We calculate

$$\begin{aligned}
I : x^2 z^2 &= \langle xy, yz\rangle \\
I, x^2 : z^2 &= \langle x^2, yz\rangle \\
I : x^2, z^2 &= \langle xy, z^2\rangle \\
I, x^2, z^2 &= \langle x^2, z^2\rangle
\end{aligned}$$

The only part which has nontrivial homology is $S_{I:x^2z^2}$. Thus, $H_5(S_I) = [d(x^3yz^2)]K$ and $H_i(S_I) = 0$ for all $i \neq 5$.

Example 3.2. Consider $S = K[x, y, z]$ and $I = \langle x^7 y^3 z^2, x^3 y z^3, x^2 y^2 z^5\rangle$. We calculate $I : x^2 z^2, y^2 = \langle xyz, y^2\rangle$. Thus, $d(x^3 y z^3)$ represents a nontrivial element in $H_6(S_I)$.

$$\begin{aligned}
H_i(S_I) &= x^2 H_{i-2}(S_{I:x^2}) \oplus H_i(S_{I,x^2}) \\
&= x^2(z^2 H_{i-4}(S_{I:x^2z^2}) \oplus H_{i-2}(S_{I:x^2,z^2})) \oplus z^2 H_{i-2}(S_{I,x^2:z^2}) \oplus H_i(S_{I,x^2,z^2}) \\
&= x^2 z^2 H_{i-4}(S_{I:x^2z^2}) \oplus x^2 H_{i-2}(S_{I:x^2,z^2}) \oplus z^2 H_{i-2}(S_{I:z^2,x^2}) \oplus H_i(S_{I,x^2,z^2}) \\
&= x^2 z^2 H_{i-4}(S_{I:x^2z^2}) \oplus x^2 H_{i-2}(S_{I:x^2,z^2}) \oplus z^2 H_{i-2}(S_{I:z^2,x^2}) \oplus H_i(S_{I,x^2,z^2})
\end{aligned}$$

$$H_i(S_I) = \sum\sum x^2_{\sigma(1)} x^2_{\sigma(2)} x^2_{\sigma(3)} H(S_{I:x^2_{\sigma(1)} x^2_{\sigma(2)} x^2_{\sigma(3)}}) + x^2_{\sigma(1)} x^2_{\sigma(2)} x^2_{\sigma(3)} H(S_{I:x^2_{\sigma(1)} x^2_{\sigma(2)}, x^2_{\sigma(3)}})$$

4 Constructing the Cochain Complexes (S, δ), (S_I, δ), and (I, δ)

For a K-vector space V, let $V^\star := \mathrm{Hom}_K(V, K)$. We refer to $V^\star$ as the **dual** of V. If $\varphi : V \to W$ is a K-linear map from the vector space V to the vector space W, then we denote the K-linear map $\mathrm{Hom}_K(\varphi, K) : \mathrm{Hom}_K(W, K) \to \mathrm{Hom}_K(V, K)$ simply as $\varphi^\star$ and call it the **dual** of φ.

4.1 Construction of (S, δ) and (I, δ)

The duals of S_I, S, and I are all graded K-modules, where the homogeneous components are simply the duals of $(S_I)_i$, S_i, and I_i respectively. In fact, S_I, S/I, S, and I are all isomorphic as graded K-modules to their duals: To get an isomorphism from S_i to $S_i^\star$, we map the monomial $x^\alpha \in S_i$ to the element $\underline{x}^\alpha \in S_i^\star$, where $\underline{x}^\alpha$ is defined on the monomial $x^\beta \in S_i$ as

$$\underline{x}^\alpha(x^\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

and is extended linearly everywhere else. The isomorphisms $(S_I)_i^\star \cong (S_I)_i$ and $I_i^\star \cong I_i$ are induced from this isomorphism.

We can describe $d^\star$ as follows: Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree $i-1$ in S. Then

$$d^\star(\underline{m}) = \sum_{\mu \in [m]_e} \underline{x^\mu m}.$$

Using the isomorphism from S to $S^\star$ described above, we pull $d^\star$ back to a map on S and we denote this map as δ and call it the **codifferential**. Thus, for each monomial $m \in S$, we have

$$\delta(m) = \sum_{\mu \in [m]_e} x_\mu m. \tag{4}$$

It is clear that δ is a graded endomorphism of S of degree 1 such that $\delta^2 = 0$, and thus gives S the structure of a cochain complex over K.

Remark. Note that (S, δ) is not a differential graded K-algebra with respect to the usual multiplication maps. For instance, consider $S = K[x, y]$. Then $\delta(xy) = 0$ but $\delta(x)y + x\delta(y) = x^2y + xy^2$. Later on we will introduce a product, called **cup product**, which will give (S, δ) the structure of a differential graded K-algebra with respect to this product.

4.2 Construction of (I, δ)

Using the description of δ in (4) and the fact that I is an ideal, we see that δ restricts to a map $\delta : I \to I$, which is also a graded endomorphism of S of degree 1 such that $\delta^2 = 0$, and thus gives I the structure of a cochain complex over K.

4.2.1 Kronecker Pairing

When we identify monomials $\underline{x}^\alpha$ in $S^\star$ with monomials x^α in S_i, we are forgetting the way monomials in $S_i^\star$ act on monomials in S_i. To make up for this, we introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on S_i called the **Kronecker pairing**: For monomials x^α and x^β in S_i, we set

$$\langle x^\alpha, x^\beta \rangle = \underline{x}^\alpha(x^\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}.$$

Then we extend this linearly to $\langle \cdot, \cdot \rangle : S_i \times S_i \to K$. From the way we constructed δ, we have for all $f_1, f_2 \in S_i$, we have $\langle \delta(f_1), f_2 \rangle = \langle f_1, d(f_2) \rangle$.

4.3 Construction of (S_I, δ)

Let $\underline{\delta} : S_I \to S_I$ be the graded K-linear map of degree 1 given by

$$\underline{\delta}(f) := \delta(f)^G$$

for all $f \in S_I$. The map $\underline{\delta}$ gives the graded K-module S_I the structure of a cochain complex over K.

4.4 Cup and Cap Product

We introduce some notation. Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S. If $\alpha_\lambda > 0$, then we say x_λ is in the **support** of m. We denote by $\text{supp}(m)$ to be the set of all x_λ in the support of m. We will assume that $x_1 > x_2 > \cdots > x_n$. We say x_λ is the **last nonzero coordinate** of m if x_λ is the smallest element in $\text{supp}(m)$. We say x_λ is the **first nonzero coordinate** of m if x_λ is the largest element in $\text{supp}(m)$.

4.4.1 Cup Product

Definition 4.1. Let m_1 and m_2 be monomials S_i and S_j respectively. The **cup product** of m_1 and m_2 is

$$m_1 \smile m_2 = \begin{cases} \frac{m_1 m_2}{x_\lambda} & \text{if } x_\lambda \text{ is the last nonzero coordinate of } m_1 \text{ and the first nonzero coordinate of } m_2 \\ 0 & \text{otherwise} \end{cases}$$

This extends to a linear map $\smile : S_i \times S_j \to S_{i+j-1}$ which we call the cup product.

Example 4.1. Let $S = K[x, y, z]$. Then

$$\begin{aligned}(x^2y + xy^2) \smile (x^5 + y^4z) &= x^2y \smile x^5 + xy^2 \smile x^5 + x^2y \smile y^4z + xy^2 \smile y^4z \\ &= x^2y \smile y^4z + xy^2 \smile y^4z \\ &= x^2y^4z + xy^5z.\end{aligned}$$

Proposition 4.1. *Let m_1 and m_2 be two monomials in S. Then*

$$\delta(m_1 \smile m_2) = \delta(m_1) \smile m_2 + m_1 \smile \delta(m_2). \tag{5}$$

Proof. Let x_{λ_1} be the last nonzero coordinate of m_1 and let x_{λ_2} be the first nonzero coordinate of m_2. First assume that $x_{\lambda_1} > x_{\lambda_2}$. Then $m_1 \smile m_2 = 0$, and this implies $\delta(m_1 \smile m_2) = 0$. Also, the last nonzero coordinate of every monomial in $\delta(m_1)$ will be greater than or equal to x_{λ_1} which is strictly greater than x_{λ_2}. Therefore $\delta(m_1) \smile m_2 = 0$. Similarly, the first nonzero in $\delta(m_2)$ will be smaller than or equal to x_{λ_2} which is strictly smaller than x_{λ_1}. Therefore $m_1 \smile \delta(m_2) = 0$. So we trivially have (5) in this case.

Now we assume $x_{\lambda_2} > x_{\lambda_1}$. Then $m_1 \smile m_2 = 0$, and this implies $\delta(m_1 \smile m_2) = 0$. Also, since $x_{\lambda_1} \in [m_2]_e$ and $x_{\lambda_2} \in [m_1]_e$, we will have $\delta(m_1) \smile m_2 = m_1m_2$ and $m_1 \smile \delta(m_2) = m_1m_2$. Adding everything together, we get

$$\delta(m_1 \smile m_2) = 0 = \delta(m_1) \smile m_2 + m_1 \smile \delta(m_2).$$

Finally, assume $x_{\lambda_1} = x_{\lambda_2}$. Let's denote this common variable as x_μ. On the one hand, we have

$$\delta(m_1 \smile m_2) = \delta\left(\frac{m_1m_2}{x_\mu}\right) = \sum_{\substack{x_\lambda \in [m_1]_e \\ x_\lambda \leq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu} + \sum_{\substack{x_\lambda \in [m_2]_e \\ x_\lambda \geq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu}$$

On the other hand, we have

$$\delta(m_1) \smile m_2 = \sum_{\substack{x_\lambda \in [m_1]_e \\ x_\lambda \leq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu} \qquad \text{and} \qquad m_1 \smile \delta(m_2) = \sum_{\substack{x_\lambda \in [m_2]_e \\ x_\lambda \geq x_\mu}} \frac{x_\lambda m_1 m_2}{x_\mu}.$$

Combining these together gives the desired result. □

4.4.2 Cap Product

Definition 4.2. Let m_1 and m_2 be monomials S_i and S_j respectively. The **cap product** of m_1 and m_2 is

$$m_1 \frown m_2 = \begin{cases} x_\lambda \frac{m_2}{m_1} & \text{if } m_1 \mid m_2,\ x_\lambda \text{ is the last nonzero coordinate of } m_1, \text{ and } x_\lambda \text{ is the first nonzero coordinate of } \frac{x_\lambda m_2}{m_1}. \\ 0 & \text{otherwise} \end{cases}$$

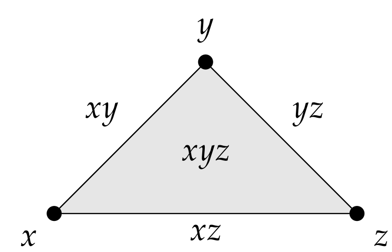
The cap product extends linearly to a map $\frown : S_i \times S_j \to S_{i-j+1}$.

5 Topological Interpretation of $H(S_I)$

In this section, we give a topological interpretation of $H(S_I)$. Since $H(S_{\mathrm{LT}(I)}) \cong H(S_I)$, we only need to consider the case where I is a monomial ideal. Moreover, by using Theorem (**??**), we can specialize further to the case that I is a squarefree monomial ideal before tackling the more general case. Thus we will assume that I is a squarefree monomial ideal in this section. Our goal is to show that $H(S_I)$ is isomorphic to the simplicial homology of a simplicial complex which is associated with I.

5.1 Reinterpreting Simplicial Complexes

We want to reinterpret the theory simplicial complexes using the language of monomials. There is a bijection between the set of subsets of $\{x_1, \ldots, x_n\}$ and the set of squarefree monomials in the variables $x_1, \ldots, x_n$. Indeed, if m is a squarefree monomial, then the corresponding subset of $\{x_1, \ldots, x_n\}$ is $\text{supp}(m)$. Moreover, if m and m' are squarefree monomials, then m divides m' if and only if $\text{supp}(m) \subseteq \text{supp}(m')$. Here's how we think of the squarefree monomials in x, y, z sit on the 2-simplex:



5.2 Stanley-Reisner Rings

5.2.1 Stanley-Reisner Ring

Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. We denote by I_Δ to be the ideal of nonfaces of Δ, that is, I_Δ is generated by the squarefree monomials m in S which are not in Δ. We define the **Stanley-Reisner ring** $K[\Delta]$ of the simplicial complex Δ to be the K-algebra $K[\Delta] := S/I_\Delta$. We will also denote by I_Δ^{sq} to mean $I_\Delta^{\text{sq}} := \langle I_\Delta, x_1^2, \ldots, x_n^2 \rangle$. Conversely, if I is a squarefree monomial ideal in S. Then we denote by Δ_I the simplicial complex on $\{x_1, \ldots, x_n\}$ whose ideal of nonfaces is I, that is, Δ_I consists of all squarefree monomials which do not belong to I.

Lemma 5.1. *Let I be a monomial ideal. If $H(S_I) = 0$, then $H(S_{I,x_\lambda^2}) = H(S_{I:x_\lambda^2}) = 0$.*

Proof. From Theorem (**??**), we have a decomposition

$$H_i(S_I) \cong x_\lambda^2 H_{i-2}(S_{I:x_\lambda^2}) \oplus H_i(S_{I,x_\lambda^2})$$

for all $i \in \mathbb{Z}$. Thus, $H(S_I) = 0$ implies $H(S_{I,x_\lambda^2}) = H(S_{I:x_\lambda^2}) = 0$. □

Lemma 5.2. *Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. Then*

$$H(S_{I_\Delta, x_1^2, \ldots, x_{\lambda-1}^2 : x_\lambda^2}) = 0$$

for all $\lambda = 1, \ldots, n$.

Proof. We prove this by induction. For the base case, let f represent an element in $H(S_{I_\Delta : x_1^2})$ and write f in terms of its monomial basis as

$$f = \sum_{\lambda=1}^{s} a_\lambda m_\lambda,$$

where $a_\lambda \in K$ for all $\lambda = 1, \ldots, s$. Since I_Δ is a squarefree monomial ideal, we have $I_\Delta : x_1^2 = I_\Delta : x_1$. We claim that $x_1 f \in S_{I_\Delta : x_1^2}$. Indeed, for all $\lambda = 1, \ldots, s$, we have $x_1 m_\lambda \in S_{I_\Delta : x_1} = S_{I_\Delta : x_1^2}$. This implies our claim. Now since $x_1 f \in S_{I_\Delta : x_1^2}$ and $d(x_1 f) = f$, it follows that f represents the zero element in $H(S_{I_\Delta : x_1^2})$.

Now assume that $H(S_{I_\Delta, x_1^2, \ldots, x_{\lambda-1}^2 : x_\lambda^2}) = 0$ for some $1 \leq \lambda < n$. We prove that this implies $H(S_{I_\Delta, x_1^2, \ldots, x_\lambda^2 : x_{\lambda+1}^2}) = 0$. First note that

$$H(S_{I_\Delta, x_1^2, \ldots, x_\lambda^2 : x_{\lambda+1}^2}) = H(S_{I_\Delta : x_{\lambda+1}^2, x_1^2, \ldots, x_\lambda^2}) = 0,$$

The same argument in the paragraph above implies $H(S_{I_\Delta : x_{\lambda+1}^2}) = 0$. Now we inductively apply Lemma (5.1) to get

$$H(S_{I_\Delta : x_{\lambda+1}^2, x_1^2, \ldots, x_\lambda^2}) = 0.$$

□

Theorem 5.3. *Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. Then*

$$H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{\text{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$$

for all $i \in \mathbb{Z}$.

Proof. Let us first show that $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$. The map $\varphi : S_{I_\Delta^{\mathrm{sq}}} \to S(\Delta)$, given by $\varphi(m) = [m]_o$ for all monomials $m \in S_{I_\Delta^{\mathrm{sq}}}$, is a graded isomorphism of degree -1. Moreover, it is easy to check that $\varphi d = \partial \varphi$. Thus φ induces an isomorphism $H_i(S_{I_\Delta^{\mathrm{sq}}}) \cong \widetilde{H}_{i-1}(\Delta; K)$.

Now we will prove that $H_i(S_{I_\Delta}) \cong H_i(S_{I_\Delta^{\mathrm{sq}}})$. We do this by combining Theorem (**??**) and Lemma (5.2). We have

$$\begin{aligned} H_i(S_{I_\Delta}) &\cong x_1^2 H_{i-2}(S_{I_\Delta : x_1^2}) \oplus H_i(S_{I_\Delta, x_1^2}) \\ &\cong H_i(S_{I_\Delta, x_1^2}) \\ &\cong x_2^2 H_i(S_{I_\Delta, x_1^2 : x_2^2}) \oplus H_i(S_{I_\Delta, x_1^2, x_2^2}) \\ &\cong H_i(S_{I_\Delta, x_1^2, x_2^2}) \\ &\vdots \\ &\cong H_i(S_{I_\Delta^{\mathrm{sq}}}) \end{aligned}$$

for all $i \in \mathbb{Z}$. $\square$

Example 5.1. Consider $S = K[x_1, x_2, x_3, x_4, x_5]$ and $I_\Delta = \langle x_1x_4, x_1x_5, x_2x_5, x_2x_3x_4, x_3x_5, x_4x_5 \rangle$. Then S/I_Δ is the Stanley-Reisner ring of the simplex Δ given in Example (**??**). Let's write down each homogeneous piece side by side:

$$\begin{aligned} S_2(\Delta) &= K\{1,2,3\} & (S_{I_\Delta^{\mathrm{sq}}})_3 &= Kx_1x_2x_3 \\ S_1(\Delta) &= K\{1,3\} + K\{1,2\} + K\{2,3\} + K\{2,4\} + K\{3,4\} & (S_{I_\Delta^{\mathrm{sq}}})_2 &= Kx_1x_3 + Kx_1x_2 + Kx_2x_3 + Kx_2x_4 + Kx_3x_4 \\ S_0(\Delta) &= K\{1\} + K\{2\} + K\{3\} + K\{4\} + K\{5\} & (S_{I_\Delta^{\mathrm{sq}}})_1 &= Kx_1 + Kx_2 + Kx_3 + Kx_4 + Kx_5 \\ S_{-1}(\Delta) &= K \cdot \emptyset & (S_{I_\Delta^{\mathrm{sq}}})_0 &= K \cdot 1 \end{aligned}$$