Sheaves and Locally Ringed Spaces

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1 Presheaves and Sheaves

1.1 Presheaves

A **presheaf** $\mathcal{F}$ on a topological space X assigns to each open set U in X a set $\mathcal{F}(U)$, and to every pair of nested open subsets $U \subset V$ of X, a function $\text{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$, called the **restriction map**, such that

1. $\mathcal{F}(\varnothing) = 0$,
2. res_U^U is the identity map for all open sets U in X,
3. $\text{res}_U^V \circ \text{res}_V^W = \text{res}_U^W$ for all open sets $U \subset V \subset W$ in X.

The elements $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U; elements of $\mathcal{F}(X)$ are called **global sections**. The restriction maps res_U^V are written as $f \mapsto f|_U$. Very often we will also write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$.

1.1.1 Morphism of Presheaves

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be presheaves on X. A **morphism of presheaves** $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a family of maps $\varphi_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ for all open sets U in X such that for all pairs of open sets $U \subseteq V$ in X, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_1(V) \arrow[r, "\varphi_V"] \arrow[d, "\text{res}_U^V"'] & \mathcal{F}_2(V) \arrow[d, "\text{res}_U^V"] \\
\mathcal{F}_2 \arrow[r, "\varphi_U"'] & \mathcal{F}_2
\end{tikzcd}$$

Composition of morphisms φ and ψ of presheaves is defined in the obvious way: $\varphi \circ \psi := (\varphi_U \circ \psi_U)_U$. We obtain the category $\mathbf{Psh}(X)$ of presheaves on X.

1.1.2 Category Theory

Using the language of category theory, we can define presheaves in a very concise way. Let X be a topological space. We denote by $\mathbf{O}(X)$ to be the category whose objects are open subsets of X and whose morphisms are inclusion maps. Then a presheaf $\mathcal{F}$ is just a contravariant functor from $\mathbf{O}(X)$ to **Set**, and morphisms of presheaves are natural transformations between functors.

We can also replace the category **Set** with any other category **C** to obtain the notion of a presheaf $\mathcal{F}$ with values in **C**. This signifies that $\mathcal{F}(U)$ is an object in **C** for every open subset U of X and that the restriction maps are morphisms in **C**.

1.2 Sheaves

Presheaves on a topological space X are top-down constructions; we can restrict information from larger to smaller sets. However, many objects in mathematics are bottom-up constructions; they are defined locally, which we then piece together to obtain something global. Presheaves do not provide the means to deduce global properties from the properties we find locally in the open sets of X. This is where the idea of sheaves come in.

Definition 1.1. A **sheaf** on X is a presheaf $\mathcal{F}$ on X which satisfies the following **sheaf axiom**:

- Suppose $\{U_i\}_{i \in I}$ is an open covering of an open subset U and suppose that for each $i \in I$ a section $s_i \in \mathcal{F}(U_i)$ is given such that for each pair $U_{i_1}, U_{i_2} \in \{U_i\}_{i \in I}$ we have $s_{i_1}|_{U_{i_1} \cap U_{i_2}} = s_{i_2}|_{U_{i_1} \cap U_{i_2}}$. Then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A **morphism of sheaves** is a morphism of presheaves. We denote by $\mathbf{Sh}(X)$ to be the full subcategory of $\mathbf{Psh}(X)$, whose objects are sheaves on X and morphisms being natural transformations.

Remark. Let X be a topological space, $\mathcal{F}$ a sheaf on X, $U \subseteq X$ be open, and let $\{U_i\}_{i \in I}$ be an open covering of U. Suppose for all $i \in I$, we have $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$. By the sheaf condition, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Conversely, if $s, s' \in \mathcal{F}(U)$ such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$, then $s = s'$. In particular, we can think of sections $s \in \mathcal{F}(U)$ as being a collection of compatible sections $s_i \in \mathcal{F}(U_i)$ (compatible meaning $s_i|_{U_{ij}} = s_j|_{U_{ij}}$). More formally, we have

$$\mathcal{F}(U) = \left\{ (s_i)_i \in \prod_i \mathcal{F}(U_i) \mid s_i|_{U_{ij}} = s_j|_{U_{ij}} \text{ for all } i, j \in I. \right\}$$

Proposition 1.1. *The sheaf axioms imply that any sheaf has exactly one section of the empty set.*

Proof. The empty set $\emptyset$ can be written as the union of an empty family (that is, the indexing set I is $\emptyset$). The condition given for the sheaf property is vacuously true. So there must exist a unique section in $F(\emptyset)$. $\square$

Example 1.1. Let E be a set and let $\mathcal{F}$ be a presheaf of functions on a topological space X with values in E. The only thing preventing $\mathcal{F}$ from being a sheaf is the *existence* of global functions since *uniqueness* is already guarenteed. Indeed, suppose $\{U_i\}_{i\in I}$ is an open covering of an open subset U of X, and suppose that for all $i \in I$ we have $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all $i, j \in I$. Then if $f, g \in \mathcal{F}(U)$ satisfy $f|_{U_i} = f_i = g|_{U_i}$ for all $i \in I$, then we must have $f = g$. This is because $f = g$ if and only if $f(x) = g(x)$ for all $x \in U$, and this is true since $x \in U_{i(x)}$ for some $i(x) \in I$ (depending on x), hence $f(x) = f_{i(x)}(x) = g(x)$.

1.2.1 Reformulating the sheaf axiom

We give a reformulation of the sheaf axiom in terms of arrows. Let $\mathcal{F}$ be a presheaf on a topological space X, let U be an open set in X and let $\{U_i\}_{i\in I}$ be an open covering of U. We define maps

$$
\begin{aligned}
\rho : \mathcal{F}(U) &\to \prod_{i\in I}\mathcal{F}(U_i), && s \mapsto (s|_{U_i})_i \\
\sigma : \prod_{i\in I}\mathcal{F}(U_i) &\to \prod_{(i,j)\in I\times I}\mathcal{F}(U_i\cap U_j), && (s_i)_i \mapsto (s_i|_{U_i\cap U_j})_{(i,j)} \\
\sigma' : \prod_{i\in I}\mathcal{F}(U_i) &\to \prod_{(i,j)\in I\times I}\mathcal{F}(U_i\cap U_j), && (s_i)_i \mapsto (s_j|_{U_i\cap U_j})_{(i,j)}
\end{aligned}
$$

The presheaf $\mathcal{F}$ is a sheaf, if it satisfies for all U and all open coverings $\{U_i\}_{i\in I}$ the following condition: The diagram

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\rho"] & \prod_{i\in I}\mathcal{F}(U_i) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\sigma'"'] & \prod_{(i,j)\in I\times I}\mathcal{F}(U_i\cap U_j)
\end{tikzcd}
$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_i \in \prod_{i\in I}\mathcal{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

For presheaves of abelian groups (or with values in any abelian category) we can reformulate the definition of a sheaf as follows: A presheaf $\mathcal{F}$ is a sheaf if and only if for all open subsets U and all coverings $\{U_i\}$ of U the sequence of abelian groups

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \prod_i \mathcal{F}(U_i) \arrow[r] & \prod_{i,j}\mathcal{F}(U_i\cap U_j) \\
 & s \arrow[r, mapsto] & (s|_{U_i})_i & \\
 & & (s_i)_i \arrow[r, mapsto] & (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}
\end{tikzcd}
$$

is exact.

1.3 Examples of Sheaves

1.3.1 Sheaf of Continuous Functions

Example 1.2. Let X and Y be topological spaces. For each open subset U of X, we define

$$\mathcal{C}_{X;Y}(U) := \{f : U \to Y \mid f \text{ is continuous}\}.$$

Then $\mathcal{C}_{X;Y}$ is a presheaf of Y-valued functions on X. In fact, more is true: $\mathcal{C}_{X;Y}$ is a sheaf. Indeed, let $\{U_i\}$ be an open covering of U. If $f : U \to Y$ is a continuous function, then by restriction to U_i, we get continuous maps $f_i : U_i \to Y$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i and j. Conversely, if we are given continuous maps $f_i : U_i \to Y$ that agree on the overlaps (that is, $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i and j) then there is a unique set-theoretic map $f : X \to Y$ satisfying $f|_{U_i} = f_i$ for all i and it is continuous. Indeed, for any open $V \subseteq Y$ we have that $f^{-1}(V)$ is open in U because $f^{-1}(V) \cap U_i = f_i^{-1}(V)$ is open in U_i for every i.

1.3.2 Sheaf of C^α Functions

Example 1.3. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces and let X be an open subspace of V. Let $\alpha \in \widehat{\mathbb{N}}_0$. For each open subset U of X, we define

$$\mathcal{C}^\alpha_{X;W}(U) := \{f : U \to W \mid f \text{ is } C^\alpha \text{ map}\}.$$

Then $\mathcal{C}^\alpha_{X;W}$ is a sheaf of functions on X. It is a sheaf of $\mathbb{R}$-vector spaces. If $W = \mathbb{R}$, then we simply write $\mathcal{C}^\alpha_X$.

Example 1.4. Let $\alpha \in \widehat{\mathbb{N}}$. For $X = \mathbb{R}$ and $U \subseteq \mathbb{R}$ open let $d_U : \mathcal{C}^\alpha_{\mathbb{R}}(U) \to \mathcal{C}^{\alpha-1}_{\mathbb{R}}(U)$ be the derivative $f \mapsto f'$. Then $(d_U)_U$ is a morphism of sheaves of $\mathbb{R}$-vector spaces (but not of $\mathbb{R}$-algebras because usually $(fg)' \neq f'g'$).

1.3.3 Sheaf of Holomorphic Functions

Example 1.5. Let V and W be finite-dimensional $\mathbb{C}$-vector spaces and let X be an open subspace of V. For each open subset U of X, we define

$$\mathcal{O}_{X;W}(U) := \mathcal{O}^{\text{hol}}_{X;W}(U) := \{f \colon U \to W \mid f \text{ is holomorphic}\}.$$

Then $\mathcal{O}_{X;W}$ (with the usual restriction maps) is a sheaf of $\mathbb{C}$-vector spaces.

1.3.4 Constant Sheaf

Example 1.6. Let X be a topological space and let E be a set. For each open subset U of X, we define $\mathcal{F}(U) := E$. Then $\mathcal{F}$ is a sheaf (whose restriction maps being the identity map) called the **constant sheaf with value** E.

1.4 Sheaves are determined by their values on a basis

Let X be a topological space and let $\mathcal{F}$ be a sheaf on X. Let $\mathcal{B}$ be a basis of the topology on X. If we know the value $\mathcal{F}(U)$ of a sheaf on every element U of $\mathcal{B}$, then we can use the sheaf property to determine $\mathcal{F}(V)$ on an arbitrary open subset V of X. We simply cover V by elements of $\mathcal{B}$. Here is a more systematic way of saying this:

$$\begin{aligned}\mathcal{F}(V) &= \left\{ (s_U)_U \in \prod_{\substack{U \in \mathcal{B} \\ U \subseteq V}} \mathcal{F}(U) \mid \text{for } U' \subseteq U \text{ both in } \mathcal{B} \text{ we have } s_U|_{U'} = s_{U'} \right\} \\ &= \lim_{\substack{U \in \mathcal{B} \\ U \subseteq V}} \mathcal{F}(U). \end{aligned} \tag{1}$$

Using this observation, we see that it suffices to define a sheaf on a basis $\mathcal{B}$ of open sets of the topology of a topological space X: Consider $\mathcal{B}$ as a full subcategory of $\mathbf{O}(X)$, then a presheaf on $\mathcal{B}$ is a contravariant functor $\mathcal{F} : \mathcal{B} \to \mathbf{Set}$. Every such presheaf $\mathcal{F}$ on $\mathcal{B}$ can be extended to a presheaf $\mathcal{F}'$ on X by using (1) as a definition.

Example 1.7. Let $\mathcal{F}$ be the presheaf of bounded continuous functions on $\mathbb{R}$ with values in $\mathbb{R}$. Then $\mathcal{F}$ is not a sheaf. Indeed, for each $i \in \mathbb{Z}$ let $U_i = (i, i+2)$ and $f_i = x|_{U_i}$. Then $\{U_i\}$ is a covering of $\mathbb{R}$ and there is no bounded continuous function f on $\mathbb{R}$ such that $f|_{U_i} = f_i$ for all i. The sheafification of $\mathcal{F}$ is isomorphic to $\mathcal{C}_{\mathbb{R};\mathbb{R}}$.

1.5 Gluing Sheaves

Proposition 1.2. *Let X be a topological space and let $\{U_i\}_{i \in I}$ be an open covering of X. For all $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i. Assume that for each pair (i,j) of indices we are given isomorphisms $\varphi_{ij} : \mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$ satisfying for all $i, j, k \in I$ the "cocycle condition" $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ on U_{ijk}. Then there exists a sheaf $\mathcal{F}$ on X and for all $i \in I$ isomorphisms $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ such that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i, j \in I$. Moreover, $\mathcal{F}$ and ψ_i are uniquely determined up to unique isomorphism by these conditions.*

Proof. Let U be an open subset of X. We define $\mathcal{F}(U)$ to be the set of collections of sections which are locally compatible:

$$\mathcal{F}(U) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}_i(U_i \cap U) \mid s_i|_{U_{ij} \cap U} = \varphi_{ij}(s_j)|_{U_{ij} \cap U} \text{ for all } i, j \in I. \right\}$$

The restriction maps are defined pointwise (i.e. if V and an open subset of U, then $(s_i)_{i \in I}|_V = (s_i|_{U_i \cap V})_{i \in I}$). The cocycle ensures that this definition makes sense. Let us verify that this is indeed a sheaf.

Let U be an open subset of X. By relabeling if neccessary, we may assume that $U_i = U_i \cap U$. Let $\{U_{i'}\}_{i' \in I'}$ be an open cover of U and let $(s_{ii'})_{i \in I} \in \mathcal{F}(U_{i'})$ such that $(s_{ii'})_{i \in I}|_{U_{i'j'}} = (s_{ij'})_{i \in I}|_{U_{ij'i'}}$ for all $i', j' \in I'$ (so $s_{ii'} \in \mathcal{F}_i(U_{ii'})$ for all $i \in I$ and $i' \in I'$, $s_{ii'}|_{U_{ii'j}} = \varphi_{ij}(s_{ji'})|_{U_{ji'i}}$ for all $i' \in I'$ and $i, j \in I$, and $s_{ii'}|_{U_{ii'j'}} = s_{ij'}|_{U_{ij'i'}}$ for all $i \in I$ and $i', j' \in I'$).

For each $i \in I$, since $\mathcal{F}_i$ is a sheaf, $\{U_{ii'}\}_{i' \in I'}$ is a cover of U_i, and $s_{ii'}|_{U_{ii'j'}} = s_{ij'}|_{U_{ij'i'}}$ for all $i', j' \in I'$, there exists a unique element $s_i \in \mathcal{F}_i(U_i)$ such that $s_i|_{U_{ii'}} = s_{ii'}$ for all $i' \in I$. We claim that $(s_i)_{i \in I} \in \mathcal{F}(U)$. It suffices to show that $s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}}$ for all $i, j \in I$. Indeed, note that $\{U_{iji'}\}_{i' \in I'}$ is a cover of U_{ij}, and

$$
\begin{aligned}
s_i|_{U_{iji'}} &= s_{ii'}|_{U_{iji'}} \\
&= \varphi_{ij}(s_{ji'})|_{U_{iji'}} \\
&= \varphi_{ij}(s_{ji'}|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j)|_{U_{iji'}}
\end{aligned}
$$

for all $i' \in I'$. Thus, we must have $s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}}$ for all $i, j \in I$.

Now fix $i \in I$. We define the map $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. Let U be an open subset of U_i. Then for $s \in \mathcal{F}_i(U)$, we set $\psi_i(s) = (\varphi_{ji}(s|_{U_j \cap U}))_{j \in I}$. Conversely, if $(s_j)_{j \in I} \in \mathcal{F}|_{U_i}(U)$, then we set $\psi_i^{-1}((s_j)_{j \in I}) = s_i$. It is clear that ψ_i is a bijection with inverse ψ_i^{-1}. Furthermore, if V is an open subset of U, then $\psi_i(s|_V) = \psi_i(s)|_V$. Thus, ψ_i is an isomorphism of sheaves $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. We repeat this construction for all $i \in I$ to get an isomorpism $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ for all $i \in I$. Finally, that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i, j \in I$ follows from a direct calculation: for $s \in \mathcal{F}_j(U_{ij})$, we have

$$
\begin{aligned}
(\psi_i \circ \varphi_{ij})(s) &= \psi_i(\varphi_{ij}(s)) \\
&= (\varphi_{ki}(\varphi_{ij}(s)|_{U_{ijk}}))_{k \in I} \\
&= (\varphi_{ki}(\varphi_{ij}(s))|_{U_{ijk}})_{k \in I} \\
&= (\varphi_{kj}(s)|_{U_{ijk}})_{k \in I} \\
&= (\varphi_{kj}(s|_{U_{ijk}}))_{k \in I} \\
&= \psi_j(s).
\end{aligned}
$$

□

1.6 Stalks

Let $\mathcal{F}$ be a presheaf on a topological space X. Suppose that for each $x \in X$, there exists a smallest neighborhood containing x, say U_x. Then we can determine the sheaf completely by computing the values of the sheaf on these open sets. The problem of course is that the limit of the diagram which consists of all open neighborhoods of x may not exist, i.e. we may not have a smallest open neighborhood of x. However, colimits do exist in **Set**, and there's nothing stopping us from looking at colimits in the diagram of $\mathcal{F}$-images of neighborhoods of x.

Definition 1.2. Let $\mathcal{F}$ be a presheaf on a topological space X, considered as a contravariant functor $\mathcal{F} : \mathbf{O}(X) \to$ **Set**, and let $x \in X$ be a point. Let

$$\mathbf{U}(x) := \{U \subseteq X \mid U \text{ is an open neighborhood of } x\}$$

be the set of open neighborhoods of x, ordered by inclusion. We consider $\mathbf{U}(x)$ as a full subcategory of $\mathbf{O}(x)$. By restricting $\mathcal{F}$ to $\mathbf{O}(x)$, we obtain a contravariant functor $\mathcal{F} : \mathbf{U}(x) \to$ **Set**. Note that the category $\mathbf{U}(x)$ is filtered: for any two neighborhoods U_1 and U_2 of x there exists a neighborhood V of x with $V \subseteq U_1 \cap U_2$.

1. The colimit
$$\mathcal{F}_x := \operatorname{colim}_{\mathbf{U}(x)} \mathcal{F}$$
is called the **stalk** of $\mathcal{F}$ at x. More concretely, one has
$$\mathcal{F}_x = \{(U, s) \mid U \text{ is an open neighborhood of } x \text{ and } s \in \mathcal{F}(U)\} \,/ \sim,$$
where two pairs (U_1, s_1) and (U_2, s_2) are equivalent if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. We often write $[U, s]_x$ (or simply s_x as the open set U often doesn't matter) to denote the equivalence class of (U, s).

2. For each open neighborhood U of x we have a canonical map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}_x, \qquad s \mapsto s_x,$$

which sends $s \in \mathcal{F}(U)$ to the class of (U, s) in $\mathcal{F}_x$. We call s_x the **germ of** s **at** x.

3. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X, then we have an induced map

$$\varphi_x := \mathrm{colim}_{\mathbf{U}(x)} \varphi_U : \mathcal{F}_x \to \mathcal{G}_x$$

of the stalks at x. It sends the equivalence class of (U, s) in $\mathcal{F}_x$ to the class of $(U, \varphi_U(s))$ in $\mathcal{G}_x$ (this is well-defined since φ commutes with restriction maps). We obtain a functor $\mathcal{F} \to \mathcal{F}_x$ from the category of presheaves to the category of sets.

Remark. If $\mathcal{F}$ is a presheaf of functions, one should think of the stalk $\mathcal{F}_x$ as the set of functions defined in some unspecified open neighborhood of x.

Remark. If $\mathcal{F}$ is a presheaf on X with values in $\mathbf{C}$, where $\mathbf{C}$ is any category in which filtered colimits exist (for instance the category of groups, of rings, of R-modules, or R-algebras, etc...), then the stalk $\mathcal{F}_x$ is an object in $\mathbf{C}$ and we obtain a functor $\mathcal{F} \mapsto \mathcal{F}_x$ from the category of presheaves on X with values in $\mathbf{C}$ to the category $\mathbf{C}$.

Let us make this more precise for a sheaf $\mathcal{G}$ of groups. The group law of $\mathcal{G}_x$ is defined as follows: Let $g, h \in \mathcal{G}$ be represented by (U, s) and (V, t). Choose an open neighborhood W of x with $W \subseteq U \cap V$. Then $(U, s) \sim (W, s|_W)$ and $(V, t) \sim (W, t_W)$ and the product gh is the equivalence class of $(W, (s|_W)(t|_W))$. In the same way addition and multiplication is defined on the stalk for a sheaf of rings.

1.6.1 Examples of Stalks

Example 1.8. Let $\mathcal{O}_{\mathbb{R}^n}$ be the sheaf of real analytic functions on $\mathbb{R}^n$ and let $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Let s be a germ in $\mathcal{O}_{\mathbb{R}^n, p}$ and let (U, f) be a representative of s. Since f is analytic at p, there exists an open neighborhood $V \subseteq U$ such that $f|_V$ is equal to its Taylor series at p:

$$f(x) = f(p) + \sum_i \partial_{x_i} f(p)(x_i - p_i) + \cdots + \frac{1}{k!} \sum_{i_1, \dots, i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f(p)(x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) + \cdots$$

for all $x \in V$. Two real analytic functions f_1 and f_2 defined in open neighborhoods U_1 and U_2, respectively, of p agree on some open neighborhood $V \subseteq U_1 \cap U_2$ if and only if they have the same Taylor expansion around p. So we have a well-defined map $\mathcal{O}_{\mathbb{R}^n, p} \to$

Example 1.9. (Stalk of the sheaf of continuous functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U, f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$\mathrm{ev}_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(\mathrm{ev}_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. Let $s \in \mathcal{C}_{X,x} \backslash \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X \backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$. Therefore the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. This shows that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

Example 1.10. (Stalk of the sheaf of C^α functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U, f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$e_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(e_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. We claim that this is the unique maximal ideal of $\mathcal{C}_{X,x}$, i.e. that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

To prove this, we need to show that the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. Let $s \in \mathcal{C}_{X,x}\backslash\mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X\backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$.

Example 1.11. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let X be an open subspace of V, and let $\mathcal{O}$ denote the sheaf $\mathcal{C}^{\alpha}_{X;W}$. We claim that $\mathcal{O}_x$ is a local ring. Indeed, let $s \in \mathcal{O}_x$ be a germ and let (f, U) be a representative of s. By the very same argument as in the example above, we may assume that f does not vanish on U so that $1/f$ exists on U. It remains to show that $1/f$ is C^{α} on X. This follows from the stability of the C^{α} property under composition and the fact that $x \mapsto 1/x$ is a C^{α} map from $\mathbb{R}^{\times}$ to $\mathbb{R}^{\times}$.

1.6.2 Working With Stalks

The following result will be used very often.

Proposition 1.3. *Let X be a topological space, $\mathcal{F}$ and $\mathcal{G}$ presheaves on X, and let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves.*

1. *Assume that $\mathcal{F}$ is a sheaf. Then the induced map on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ are injective for all $x \in X$ if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets U of X.*

2. *If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, the maps φ_x are bijective for all $x \in X$ if and only if φ_U is bijective for all open subsets U of X.*

3. *If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.*

We give two proofs:

Proof.

1. Suppose that $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets U of X. Let $x \in X$ and suppose $\varphi_x([U, s]_x) = \varphi_x([U', s']_x)$ where $[U, s]_x, [U', s']_x \in \mathcal{F}_x$. This implies that there exists an open neighborhood U'' of x such that $U'' \subseteq U \cap U'$ and $\varphi_U(s)|_{U''} = \varphi_{U'}(s')|_{U''}$, or in other words $\varphi_{U''}(s|_{U''}) = \varphi_{U''}(s'|_{U''})$. Since $\varphi_{U''}$ injective, we must have $s|_{U''} = s'|_{U''}$, which implies $[U, s]_x = [U', s']_x$. Thus, φ_x is injective. Since x was arbitrary, $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Conversely, suppose $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Let U be an open subset of X and suppose $\varphi_U(s) = \varphi_U(s')$ where $s, s' \in \mathcal{F}(U)$. Therefore $[U, \varphi_U(s)]_x = [U, \varphi_U(s')]_x$ for all $x \in U$, i.e. $\varphi_x([U, s]_x) = \varphi_x([U, s']_x)$ for all $x \in U$. Since φ_x is injective for all $x \in X$, we must have $[U, s]_x = [U, s']_x$ for all $x \in X$. Therefore there exists an open neighborhood U^x of x such that $U^x \subseteq U$ and $s|_{U^x} = s'|_{U^x}$ for all $x \in U$. This implies $s = s'$ since $\mathcal{F}$ is a sheaf. Therefore φ_U is injective, and since U was arbitrary, $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets U of X.

2. Suppose that $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective for all open subsets U of X . By 1, $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$, thus it remains to show that φ_x is surjective for all $x \in X$. Choose $x \in X$ and let $[U, t]_x \in \mathcal{G}_x$. Since φ_U is surjective, choose $s \in \mathcal{F}(U)$ such that $\varphi_U(s) = t$. Then $\varphi_x[U, s] = [U, t]_x$ implies φ_x is surjective. Since x was arbitrary, φ_x is surjective for all $x \in X$. Conversely, suppose that φ_x is surjective for all $x \in X$. Let U be an open subset of X. By 1, $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective, thus it remains to show that φ_U is surjective. Let $t \in \mathcal{G}(U)$. By surjectivity of φ_x, we can choose $[s^x, U^x]_x$ such that $\varphi_x([s^x, U^x]_x) = [t, U]_x$, where we may assume that $U^x \subseteq U$ and $\varphi_{U^x}(s^x) = t|_{U^x}$, for all $x \in U$. Now observe that

$$\begin{aligned}\varphi_{U^{xy}}(s^x|_{U^{xy}}) &= \varphi_{U^x}(s^x)|_{U^{xy}}\\ &= t|_{U^{xy}}\\ &= \varphi_{U^x}(s^y)|_{U^{xy}}\\ &= \varphi_{U^{xy}}(s^y|_{U^{xy}}),\end{aligned}$$

for all $x, y \in U$. Since $\varphi_{U^{xy}}$ is injective, we must have $s^x|_{U^{xy}} = s^y|_{U^{xy}}$ for all $x, y \in U$. Since $\{U^x\}_{x\in U}$ forms an open covering of U and since $\mathcal{F}$ is sheaf, this implies that there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U^x} = s^x$ for all $x \in U$. Then $\varphi_U(s) = t$ since

$$\varphi_U(s)|_{U^x} = \varphi_{U^x}(s|_{U^x}) = t|_{U^x}$$

for all $x \in U$ and since $\mathcal{G}$ is a sheaf.

3. Suppose $\varphi = \psi$. Then it is clear that $\varphi_x = \psi_x$ for all $x \in X$. Conversely, suppose $\varphi_x = \psi_x$ for all $x \in X$. Assume for a contradiction that $\varphi \neq \psi$. Then there exists an open subset U of X and an $s \in \mathcal{F}(U)$ such that $\varphi_U(s) \neq \psi_U(s)$. On the other hand,

$$\begin{aligned} [U, \varphi_U(s)]_x &= \varphi_x([U,s]_x) \\ &= \psi_x([U,s]_x) \\ &= [U, \psi_U(s)]_x \end{aligned}$$

for all $x \in U$. This means that there exists there exists an open neighborhood U^x of x such that $U^x \subseteq U$ and $\varphi_U(s)|_{U^x} = \psi_U(s)|_{U^x}$ for all $x \in U$. But $\mathcal{G}$ is a sheaf and $\{U^x\}_{x \in U}$ is an open covering of U, and so $\varphi_U(s) = \psi_U(s)$. Contradiction.

□

Proof. For $U \subseteq X$ open consider the map

$$\mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x, \qquad s \mapsto (s_x)_{x \in U}.$$

We claim that this map is injective if $\mathcal{F}$ is a sheaf. Indeed, let $s, t \in \mathcal{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood U^x of x such that $U^x \subseteq U$ and $s|_{U^x} = t|_{U^x}$. Clearly, $U = \bigcup_{x \in U} U^x$ and therefore $s = t$ by the sheaf axiom.

Using the commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d] & \prod_{x \in U} \mathcal{F}_x \arrow[d] \\ \mathcal{G}(U) \arrow[r] & \prod_{x \in U} \mathcal{G}_x \end{tikzcd}$$

we see that 3 and the necessity of the condition in 1 are implied by the above claim. Moreover, a filtered colimit of injective maps is always injective again. Indeed, this follows either from abstract nonsense or we can argue directly: let $s_0, t_0 \in \mathcal{F}_x$ such that $\varphi_x(s_0) = \varphi_x(t_0)$. Let s_0 be represented by (s, U) and t_0 by (t, V). By shrinking U and V, we may assume $U = V$. As

$$\begin{aligned} \varphi_U(s)_x &= \varphi_x(s_0) \\ &= \varphi_x(t_0) \\ &= \varphi_U(t)_x, \end{aligned}$$

there exists an open neighborhood $W \subseteq U$ of x such that

$$\begin{aligned} \varphi_W(s|_W) &= \varphi_U(s)|_W \\ &= \varphi_U(t)|_W \\ &= \varphi_W(t|_W). \end{aligned}$$

As φ_W is injective, we find $s|_W = t|_W$ and hence

$$\begin{aligned} s_0 &= s_x \\ &= t_x \\ &= t_0. \end{aligned}$$

Therefore the condition in 1 is also sufficient.

Hence we are done if we show that the bijectivity of φ_x for all $x \in U$ implies the surjectivity of φ_U. Let $t \in \mathcal{G}(U)$. For all $x \in U$ we choose an open neighborhood U^x of x such that $U^x \subseteq U$ and we choose an element $s^x \in \mathcal{F}(U^x)$ such that $\varphi_{U^x}(s^x) = t|_{U^x}$. Note that we can do this because φ_x is surjective. Then $\{U^x\}_{x \in U}$ is an open covering of U and for $x, y \in U$, we have

$$\begin{aligned} \varphi_{U^{xy}}(s^x|_{U^{xy}}) &= \varphi_{U^x}(s^x)|_{U^{xy}} \\ &= t|_{U^{xy}} \\ &= \varphi_{U^y}(s^y)|_{U^{xy}} \\ &= \varphi_{U^{xy}}(s^y|_{U^{xy}}). \end{aligned}$$

As we already know that $\varphi_{U^{xy}}$ is injective, this shows $s^x|_{U^{xy}} = s^y|_{U^{xy}}$ and the sheaf condition ensures that we find $s \in \mathcal{F}(U)$ such that $s|_{U^x} = s^x$ for all $x \in U$. Clearly, we have $\varphi_U(s)_x = t_x$ for all $x \in U$ and hence $\varphi_U(s) = t$. □

Definition 1.3. We call a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves **injective** (respectively **surjective**, respectively **bijective**) if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (respectively surjective, respectively bijective) for all $x \in X$.

Remark. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.

1. Then φ is injective (respectively bijective) if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (respectively bijective) for all open subsets U of X.

2. The morphism φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathcal{G}(U)$ there exist an open covering $\{U_i\}_{i \in I}$ of U (depending on t) and sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$ for all $i \in I$. In other words, if locally we can find a preimage of t. But the surjectivity of φ does not imply that $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open sets U of X. Indeed, in the proof above we needed injectivity of $\varphi_{U^{xy}}$ in order to patch up the various local sections.

Similary, as we defined the notions of injectivity or surjectivity "stalkwise", we also define the notion of an exact sequence of sheaves of groups "stalkwise".

Definition 1.4. A sequence

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

of homomorphisms of sheaves of groups is called **exact** if for all $x \in X$ the induced sequence of stalks

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is an exact sequence of groups.

Example 1.12. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on an open subset X of $\mathbb{C}$. For every open subspace $U \subseteq X$ and $f \in \mathcal{O}_X(U)$, we let $D_U(f) := f'$ be the derivative. We obtain a morphism $D : \mathcal{O}_X \to \mathcal{O}_X$ of sheaves of $\mathbb{C}$-vector spaces. Then D is surjective because locally every holomorphic function has a primitive.

But there exist open subsets U of X and functions f on U that have no primitive. For instance $U = B_r(z_0) \backslash \{z_0\}$ contained in X and $f : z \mapsto 1/(z - z_0)$. More precisely, by complex analysis we know that D_U is surjective if and only if every connected component of U is simply connected. The sufficiency of this condition will also be an immediate application of cohomologial methods developed later.

We obtain an exact sequence of sheaves of $\mathbb{C}$-vector spaces

$$0 \longrightarrow \mathbb{C}_X \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{D} \mathcal{O}_X \longrightarrow 0$$

where $\mathbb{C}_X$ denotes the sheaf of locally constant $\mathbb{C}$-valued functions on X and where ι_U is the inclusion for all $U \subseteq X$ open.

Remark. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X and let $\{U_i\}_{i \in I}$ be an open covering of X. A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective (respectively surjective, respectively bijective) if and only if its restriction $\varphi|_{U_i} : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ to morphisms of sheaves on U_i is injective (respectively surjective, respectively bijective) for all $i \in I$ because these notions are defined via the stalks. But note that the existence of the morphism φ is crucial. There exists sheaves $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{G}|_{U_i}$ for all i and such that $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic.

1.7 Sheafification

There is a functorial way to attach to a presheaf a sheaf:

Proposition 1.4. *Let $\mathcal{F}$ be a presheaf on a topological space X. Then there exists a pair $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ where $\widetilde{\mathcal{F}}$ is a sheaf on X and $\iota_{\mathcal{F}} : \mathcal{F} \to \widetilde{\mathcal{F}}$ is a morphism of presheaves, such that the following holds: If $\mathcal{G}$ is a sheaf on X and $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ with $\widetilde{\varphi} \circ \iota_{\mathcal{F}} = \varphi$. The pair $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is unique up to unique isomorphism. Moreover, the following properties hold:*

1. *For all $x_0 \in X$, the map of stalks $\iota_{\mathcal{F},x_0} : \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is bijective.*

2. *For every presheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ making the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\iota_{\mathcal{F}}"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{F}} \arrow[d, "\widetilde{\varphi}"] \\ \mathcal{G} \arrow[r, "\iota_{\mathcal{G}}"] & \widetilde{\mathcal{F}} \end{tikzcd}$$

commutative. In particular, $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ is a functor from the category of presheaves on X to the category of sheaves on X.

*The sheaf $\widetilde{\mathcal{F}}$ is called the sheaf associated to $\mathcal{F}$ or the **sheafification** of $\mathcal{F}$.*

Proof. For $U \subseteq X$ open, elements of $\widetilde{\mathcal{F}}(U)$ are by definition families of elements in the stalks of $\mathcal{F}$, which locally give rise to sections of $\mathcal{F}$. More precisely, we define

$$\widetilde{\mathcal{F}}(U) := \left\{ (s_x^x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \mid \forall x_0 \in U,\ \exists \text{ open neighborhood } U_0 \subseteq U \text{ of } x_0 \text{ and } s \in \mathcal{F}(U_0) \text{ such that } s_x^x = s_x, \forall x \in U_0. \right\}.$$

Here we use the notation s_x^x for a germ at x with representative (U^x, s^x), where U^x is a sufficiently small neighborhood of x and $s^x \in \mathcal{F}(U^x)$. For $U \subseteq V$ the restriction map $\widetilde{\mathcal{F}}(V) \to \widetilde{\mathcal{F}}(U)$ is induced by the natural projection $\prod_{x \in V} \mathcal{F}_x \to \prod_{x \in U} \mathcal{F}_x$ (i.e. $(s_x^x)_{x \in V}|_U = (s_x^x)_{x \in U}$). To see that $\widetilde{\mathcal{F}}$ is a sheaf, let $\{U_i\}_{i \in I}$ be a covering of an open set U and for each $i \in I$, let $(s_x^x)_{x \in U_i} \in \widetilde{\mathcal{F}}(U_i)$ such that $(s_x^x)_{x \in U_i}|_{U_{ij}} = (s_x^x)_{x \in U_{ij}} = (s_x^x)_{x \in U_j}|_{U_{ij}}$. Then it is easy to check that $(s_x^x)_{x \in U}$ is the unique section such that $(s_x^x)_{x \in U}|_{U_i} = (s_x^x)_{x \in U_i}$ for all $i \in I$. Now we prove the two properties stated in the proposition:

1. For $U \subseteq X$ open, we define $\iota_{\mathcal{F},U} : \mathcal{F}(U) \to \widetilde{\mathcal{F}}(U)$ by $s \mapsto (s_x)_{x \in U}$. Let us check that $\iota_{\mathcal{F},x_0} : \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is bijective. First we will show that $\iota_{\mathcal{F},x_0}$ is injective: let s_{x_0} and t_{x_0} be two germs at x, represented by (s, U) and (t, U) respectively, and suppose that $\iota_{\mathcal{F},x_0}(s_{x_0}) = \iota_{\mathcal{F},x_0}(t_{x_0})$. Then $(s_x)_{x \in U} = (t_x)_{x \in U}$, and hence $s_{x_0} = t_{x_0}$ since $x_0 \in U$. Thus $\iota_{\mathcal{F},x}$ is injective. Now let us show that $\iota_{\mathcal{F},x_0}$ is surjective. Let s_0 be a germ in $\widetilde{\mathcal{F}}_{x_0}$ and suppose $(U, (s_x^x)_{x \in U})$ is a representative of s_0. Then in some open neighborhood U_0 of x_0, there exists $s \in \mathcal{F}(U_0)$ such that $s_x^x = s_x$ for all $x \in U_0$. In particular, this means $(U_0, (s_x)_{x \in U_0})$ is a representative of s_0. Therefore $\iota_{\mathcal{F},x}(s_x) = s_0$, where $s_x \in \mathcal{F}_x$ is the germ at x with representative (U_0, s).

2. Now let $\mathcal{G}$ be a presheaf on X and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism. We define $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ by defining, for each open subset U of X, the map $\widetilde{\varphi}_U : \widetilde{\mathcal{F}}(U) \to \widetilde{\mathcal{G}}(U)$ by $\widetilde{\varphi}\left((s_x^x)_{x \in U}\right) = (\varphi_x(s_x^x))_{x \in U}$ (in terms of representatives at x_0, we are mapping (U^{x_0}, s^{x_0}) to $(U^{x_0}, \varphi_{U^{x_0}}(s^{x_0}))$). We claim that $\widetilde{\varphi}$ is the unique morphism making the diagram commutative. Indeed, if $\widetilde{\psi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ is another morphism making the diagram commute, then $\widetilde{\psi} \circ \iota_{\mathcal{F}} = \iota_{\mathcal{G}} \circ \varphi = \widetilde{\varphi} \circ \iota_{\mathcal{F}}$ implies $\widetilde{\psi}_x \circ \iota_{\mathcal{F},x} = \widetilde{\varphi}_x \circ \iota_{\mathcal{F},x}$ for all $x \in X$. Since $\iota_{\mathcal{F},x}$ is a bijection, it follows that $\widetilde{\varphi}_x = \widetilde{\psi}_x$ for all $x \in X$. Therefore $\widetilde{\varphi} = \widetilde{\psi}$ by Proposition (1.3).

If we assume in addition that $\mathcal{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathcal{G}} : \mathcal{G} \to \widetilde{\mathcal{G}}$, which is bijective on stalks, is an isomorphism by Proposition (1.3). Composing the morphism $\widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ with $\iota_{\mathcal{G}}^{-1}$, we obtain the morphism $\iota_{\mathcal{G}}^{-1} \circ \widetilde{\varphi} : \widetilde{\mathcal{F}} \to \mathcal{G}$. Finally, the uniqueness of $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is a formal consequence. □

Lemma 1.1. *Let X be a topological space, $\mathcal{F}$ a presheaf on X, and $\mathcal{G}$ a sheaf on X. Then there is a bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}}, \mathcal{G}) \quad \longleftrightarrow \quad \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{F}, \mathcal{G}),$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. If $\varphi : \mathcal{F} \to \mathcal{G}$, then there exists a unique morphism $\widetilde{\varphi} : \widetilde{F} \to \mathcal{G}$ such that $\widetilde{\varphi} \circ \iota_{\mathcal{F}} = \varphi$. Conversely, if $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \mathcal{G}$, then we *define* $\varphi : \mathcal{F} \to \mathcal{G}$ by $\varphi := \widetilde{\varphi} \circ \iota_{\mathcal{F}}$. Functoriality in $\mathcal{F}$ and $\mathcal{G}$ is an easy exercise. □

1.7.1 Sheafification of Presheaf of Functions

Example 1.13. Let E be a set and let $\mathcal{F}$ be a presheaf of functions with values in E. Then

$$\widetilde{\mathcal{F}}(U) \cong \left\{ g : U \to E \mid \exists \text{ open covering } \{V_i\} \text{ of } U \text{ such that } g|_{V_i} \in \mathcal{F}(V_i) \text{ for all } i \right\}.$$

Indeed, let $(f_x^x)_{x \in U} \in \widetilde{\mathcal{F}}(U)$. For all $x \in U$, choose an open neighborhood V^x of x together with $g^x \in \mathcal{F}(V^x)$ such that $V^x \subseteq U$ and $g_y^x = f_y^x$ for all $y \in V^x$. Now define a function $g : U \to E$ as follows: for all $x \in U$, we set

$g(x) := g^x(x)$. We have to be sure that this is well-defined (i.e. does not depend on the choice of g^x), so suppose $x \in V^y$ for some $y \in U$. Then

$$\begin{aligned} g(x) &= g^y(x) \\ &= g^y_x(x) \\ &= g^x_x(x) \\ &= g^x(x), \end{aligned}$$

which shows that this function is well-defined. The point here is that g^x and g^y agree on some neighborhood of x. Indeed, (V^x, g^x) can be used as a representative for all germs f^z_z where $z \in V^x$. Similarly, (V^y, g^y) can be used as a representative for all germs f^z_z where $z \in V^y$. Thus, both (V^x, g^x) and (V^y, g^y) are representatives for the germ f^x_x since $x \in V^x \cap V^y$. In particular, this means that there is a sufficiently small neighborhood V of x such that $f^x|_V = g^x|_V = g^y|_V$.

It is easy to see that $g|_{V^x} = g^x \in \mathcal{F}(V^x)$ for all $x \in X$. Thus, we have constructed a well-defined map given by $(f^x_x)_{x \in U} \mapsto g$. Conversely, if we start with a $g : U \to E$ such that there exists an open covering $\{V_i\}_{i \in I}$ of U where $g|_{V_i} \in \mathcal{F}(V_i)$ for all $i \in I$, then it's easy to see that $(g_x)_{x \in U} \in \widetilde{\mathcal{F}}(U)$.

Example 1.14. Let E be a set and denote by E_X the sheaf of locally constant functions, i.e.

$$E_X(U) := \{f : U \to E \mid f \text{ is locally constant}\}.$$

This is the sheafification of the presheaf of constant functions with values in E. The sheaf E_X is called the **constant sheaf with value** E.

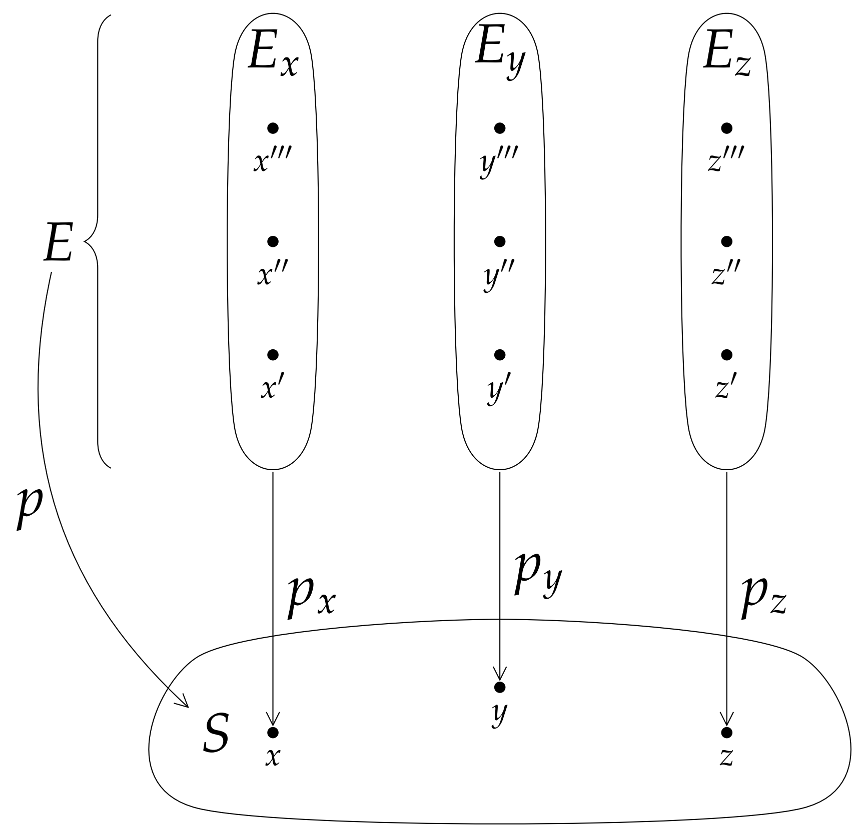
1.8 Sheaves and Etale Spaces

1.8.1 Bundles

Definition 1.5. The **slice category** $\mathbf{C}/c$ of a category $\mathbf{C}$ over an object $c \in \mathbf{C}$ is the category whose objects are morphisms $f : d \to c$ and whose morphisms from $f : d \to c$ to $f' : d' \to c$ are the morphisms $g : d \to d'$ such that $f' \circ g = f$:

Objects	**Morphisms**
$d \xrightarrow{f} c$	$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$
object (d, f)	morphism g

Example 1.15. Let S be a set. An object $(E, p) \in \mathbf{Set}/S$ can be pictured like this:



Let's take a moment to reflect on this image, because it will serve as a nice visualization tool for other categories. For each element $s \in S$, there is an associated set E_s, which is just the inverse image of s under p, i.e. $p^{-1}(s) = E_s$. E_s is called the **fiber** of p over s. Notice that for any distinct $s, s' \in S$, $E_s \cap E_{s'} = \emptyset$, and that $\bigcup_s E_s = E$. We also have functions p_s, which is just the restriction of p to E_s. The commutativity condition for morphisms in the slice category tells us that a morphism $f : (E, p) \to (E', p')$ satisfies $f(E_s) \subseteq E'_s$ for all $s \in S$. The whole structure is called a **bundle** of sets over the **base space** S, with E being called the **total space** and p being called the **projection**.

1.8.2 Etale Spaces

Definition 1.6. Let E and X be topological spaces. A **local homeomorphism** is a continuous map $\pi : E \to X$ with the additional property that for each point $e \in E$ there exists an open neighborhood U_e in E such that $\pi(U_e)$ is open in X, and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to \pi(U_e)$.

Intuitively, a local homeomorphism preserves "local structure". For example, B is locally compact if and only if $\pi(B)$ is.

Proposition 1.5. *Let $\pi : E \to X$ be a local homeomorphism, then π is an open map.*

Proof. Let U be open in E, we need to show that $\pi(U)$ is open in X. For each $e \in U$, choose an open neighborhood U_e of e such that $\pi(U_e)$ is open in X and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to p(U_e)$. Then $U \cap U_e$ is open in U_e, and since homeomorphisms are open maps, $\pi|_{U_e}(U \cap U_e)$ is open in $\pi(U_e)$ in the subspace topology. Since $\pi(U_e)$ is open in X, $\pi(U \cap U_e)$ is open in X too. Finally, since

$$\bigcup_{e \in U} \pi(U \cap U_e) = \pi(U),$$

$\pi(U)$ is open in X.

□

Example 1.16. If X is a topological space and Y is a discrete space, then the projection $Y \times X \to X$ is a local homeomorphism. On the other hand, the projection map $\mathbb{R} \times X \to X$ is never a local homeomorphism, because no product neighborhood is projected homeomorphically into X. For much the same reason, a nontrivial vector bundle is never a locally homeomorphism either.

Definition 1.7. An **etale space** over X is an object $(E, \pi) \in$ **Top**$/X$ such that π is a local homeomorphism. We denote by **Etale**(X) to be the full subcategory of **Top**$/X$ whose objects are etale spaces over X and whose morphisms being the same as in **Top**$/X$.

1.9 An equivalence of categories

We start with the main theorem:

Theorem 1.2. *For any topological space X there is a pair of adjoint functors*

$$\mathbf{Top}/X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set}^{O(X)^{op}}$$

where Γ assigns to each object in $(E, \pi) \in \mathbf{Top}/X$, the sheaf of all sections $\mathcal{F}_\pi$ of π, while its left adjoint Λ assigns to each preasheaf $\mathcal{F}$, the etale space $(E_\mathcal{F}, \pi_\mathcal{F})$. There are natural transformations

$$\eta_\mathcal{F} : \mathcal{F} \to \Gamma\Lambda\mathcal{F} \qquad \epsilon_E : \Lambda\Gamma E \to E$$

for a presheaf $\mathcal{F}$ and an object $(E, \pi) \in \mathbf{Top}/X$, which are unit and counit making Λ a left adjoint for Γ. If $\mathcal{F}$ is a sheaf, $\eta_\mathcal{F}$ is an isomorphism, while if (E, π) is etale, ϵ_E is an isomorphism.

1.9.1 From $\mathbf{Top}/X$ to $\mathbf{Sh}(X)$

Given an object $(E, \pi) \in \mathbf{Top}/X$, we can associate a sheaf $\mathcal{F}_\pi$ as follows: for all open subsets U of X, we define

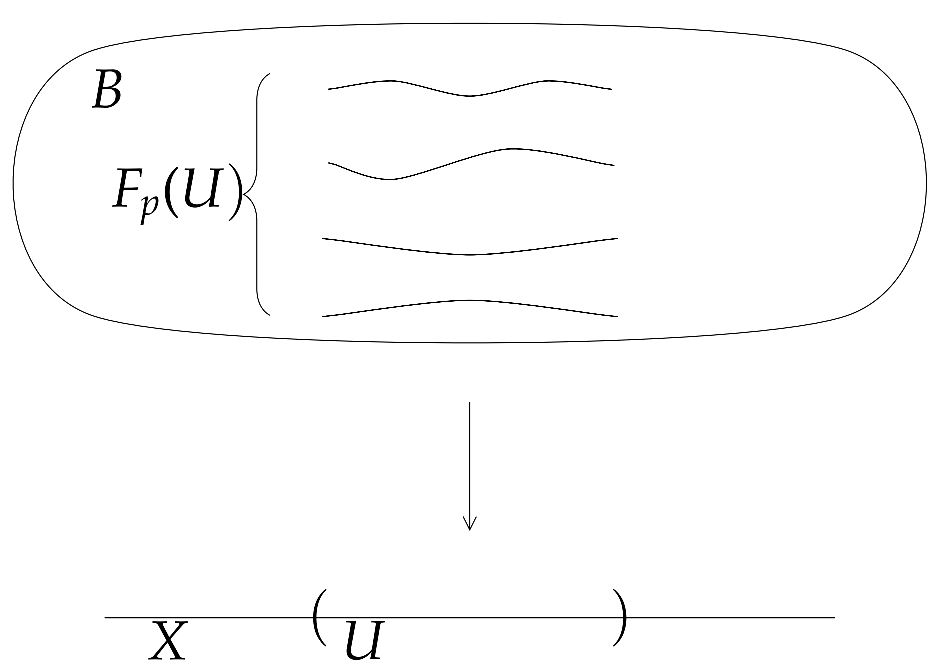
$$\mathcal{F}_\pi(U) := \{s : U \to E \mid s \text{ is continuous and } \pi|_U \circ s = \mathrm{id}_U\}.$$

For all inclusions of open sets $U \subseteq V$, we use the obvious restriction maps: if $s \in \mathcal{F}_\pi(V)$ then $s|_U \in \mathcal{F}_\pi(U)$. We claim that $\mathcal{F}_\pi$ is a sheaf (and not just a presheaf).

Indeed, let $\{U_i\}_{i \in I}$ be an open covering of an open subset U of X, and let $s_i \in \mathcal{F}_\pi(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all $i, j \in I$. We can construct an $s \in \mathcal{F}_\pi(U)$ such that $s|_{U_i} = s_i$ as follows: if $x \in U$, choose some U_i that has $x \in U_i$, and set $s(x) = s_i(x)$. We need to check that this is well-defined (i.e. independent of the choice of neighborhood of x). Suppose $x \in U_j$ for some $j \neq i$. Then because $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, we have $s(x) = s_j(x) = s_i(x)$. Thus, this construction is well-defined. Moreover, s is continous since if V is an open subset of E, then

$$s^{-1}(V) = \bigcup_{i \in I} s_i^{-1}(V)$$

is open in X. Finally, uniqueness of s is guaranteed since $\mathcal{F}_\pi$ is a presheaf of functions. We call $\mathcal{F}_\pi$ the **sheaf of sections of** π.



Let $f : (E, \pi) \to (E', \pi')$ be a morphism in $\mathbf{Top}/X$. Then for each open subset U of X, we define $f_U : \mathcal{F}_\pi(U) \to \mathcal{F}_{\pi'}(U)$ to be the function that maps a section $s \in \mathcal{F}_\pi(U)$ to $f \circ s$. The maps f_U are the components of a natural transformation from $\mathcal{F}_\pi \to \mathcal{F}_{\pi'}$. Thus, we have constructed a functor $\Gamma : \mathbf{Top}/X \to \mathbf{Sh}(X)$.

1.9.2 From $\mathbf{Psh}(X)$ to $\mathbf{Etale}(X)$

Let $\mathcal{F}$ be a presheaf on X. Define

$$E_{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x = \bigcup_{x \in X} \{(x, s_0) \mid s_0 \in \mathcal{F}_x\}.$$

and let $\pi_{\mathcal{F}} : E_{\mathcal{F}} \to X$ be the obvious projection map (i.e. $(x,s) \mapsto x$). For each open subset U of X and section $s \in \mathcal{F}(U)$, let $[U,s] = \{(x, s_x) \mid x \in U\}$. Let τ be the topology on $E_{\mathcal{F}}$ with the collection of all $[U,s]$ as a subbasis. We claim that the collection of all $[U,s]$ is actually a basis for this topology. Indeed, let $[U,s],[V,t] \in \mathcal{B}$ and suppose $(x_0, s_{x_0}) \in [U,s] \cap [V,t]$. Then $x_0 \in U \cap V$ and $s_{x_0} = t_{x_0}$. This implies that there exists a neighborhood U_0 of x such that $U_0 \subseteq U \cap V$ and $s|_{U_0} = t|_{U_0}$. Hence $s_x = t_x$ for all $x \in U_0$. In particular, $[U_0, s|_{U_0}] \subseteq [U,s] \cap [V,t]$.

Finally, we want to show that $\pi_{\mathcal{F}} : E_{\mathcal{F}} \to X$ a local homeomorphism with respect to this topology. To see why, note that $\pi_{\mathcal{F}}$ maps basis elements to basis elements (i.e. $[U,s] \mapsto U$). Thus, it must be an open mapping. Also, if $(x, s_0) \in E_{\mathcal{F}}$, then after choosing a representative of s_0, say (U,s), we see that $(x,s_0) \in [U,s]$ and $\pi|_{[U,s]} : [U,s] \to U$ is a homeomorphism. Indeed, $\pi|_{[U,s]}$ is an open mapping and a bijection, hence its inverse must be continuous.

1.9.3 co-unit

Let $p : B \to X$ be any local homeomorphism, F_p its sheaf of sections, and $p_{F_p} : B_{F_p} \to X$ the associated sheaf of germs. Define a map $k : B \to B_{F_p}$ as follow: If $b \in B$, there exist a local section s of p through b, defined on an open set V, i.e. $b \in s(V)$ (we proved this earlier). Let $k(b) = (f(b), [s]_{f(b)})$ be the germ of s at $f(b)$. The definition of $k(b)$ does not depend on which section through b is chosen (we proved this earlier too). This gives us the following commutative diagram:

$$\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "p"'] & & B_{F_p} \arrow[dl, "p_{F_p}"] \\ & X & \end{tikzcd}$$

And k is a $\mathbf{Etale}(X)$-arrow from p to p_{F_p}, in fact, it is an iso.

1.9.4 unit

Define $\tau_U : F(U) \to F_{p_F}(U)$ by putting, for $s \in F(U)$, $\tau_U(s) = s_U$, where $s_U : U \to A_F$ is defined by putting $s_U(x) = (x, [s]_x)$ for all $x \in U$...

1.10 Direct and Inverse Images of Sheaves

In this section $f : X \to Y$ denotes a continuous map of topological spaces. We will now see how to use f in order to attach a sheaf on X a sheaf on Y (direct image) and to a sheaf on Y a sheaf on X (inverse image). We start with the direct image.

1.10.1 Direct Image

Definition 1.8. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a presheaf on X. We define a presheaf $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets V of Y. The restriction maps are given by the restriction maps for $\mathcal{F}$. We call $f_*\mathcal{F}$ the **direct image of $\mathcal{F}$ under** f. Whenever $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathcal{F}_1 \to f_*\mathcal{F}_2$. Therefore we obtain a functor f_* from the category of presheaves on X to the category of presheaves on Y.

Example 1.17. Let $p : X \to Y$ be a continuous map, E be a set, and let E_X and E_Y be the sheaf of locally constant E-valued functions on X and Y respectively. For $V \subseteq Y$ open and for locally constant map $g : V \to E$ the composition $g \circ p : p^{-1}(V) \to E$ is locally constant. Hence we obtain a morphism of sheaves

$$\varphi : E_Y \to p_* E_X.$$

Now suppose that p is surjective, that Y has the quotient topology of X and that p has connected fibers. For $V \subseteq Y$ open, a locally constant map $h : p^{-1}(V) \to E$ is the same as a continuous map if we endow E with the discrete topology. The restriction of h to the fibers of p is constant and hence by the universal property of the quotient topology there exists a unique continuous map $g : V \to E$ such that $g \circ p = h$. Hence we see that φ is an isomorphism in this case.

Remark. If $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is a sheaf on Y. Indeed, let $\{V_i\}_{i\in I}$ be an open covering of an open subset V of Y and let $s_i \in f_*\mathcal{F}(V_i) = \mathcal{F}(f^{-1}(V_i))$ such that $s_i|_{f^{-1}(V_{ij})} = s_j|_{f^{-1}(V_{ij})}$ for all i, j. Then $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of $f^{-1}(V)$, and so by the sheaf property of $\mathcal{F}$, there exists a unique $s \in \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V)$ such that $s|_{f^{-1}(V_i)} = s_i$. Thus, f_* defines a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Proposition 1.6. *Suppose $f : X \to Y$ is a homeomorphism and let $f(x) \in Y$ and $\mathcal{F}$ be a presheaf on X. Then $(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x$.*

Proof. Let $\pi_{\mathcal{F},x} : (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ be the map given by $\pi_{\mathcal{F},x}\left([V,s]_{f(x)}\right) = [f^{-1}(V), s]_x$, where V is an open neighborhood of $f(x)$ and $s \in \mathcal{F}(f^{-1}(V))$. We need to show that this map is well-defined. Let (V', s') be another representative of the equivalence class $[V,s]_{f(x)}$. Then there exists an open neighborhood V'' of $f(x)$ such that $V'' \subseteq V \cap V'$ and $s|_{f^{-1}(V'')} = s'|_{f^{-1}(V'')}$. Since $f^{-1}(V'')$ is an open neighborhood of x such that $f^{-1}(V'') \subseteq f^{-1}(V) \cap f^{-1}(V')$, we have

$$\begin{aligned}\pi_{\mathcal{F},x}\left([V',s']_{f(x)}\right) &= [f^{-1}(V'), s']_x \\ &= [f^{-1}(V), s]_x.\end{aligned}$$

Thus this map is well-defined.

To show that $\pi_{\mathcal{F},x}$ is bijective, we simply describe the inverse map: Let $\pi^{-1}_{\mathcal{F},x} : \mathcal{F}_x \to (f_*\mathcal{F})_{f(x)}$ be the map given by $\pi^{-1}_{\mathcal{F},x}([U,s]_x) = [f(U), s]_{f(x)}$. Note that we need f to be an injective open map for this it to be well-defined (It is not enough that f is an open mapping. We also need $s \in \mathcal{F}(f^{-1}(f(U)))$, so we must have $f^{-1}(f(U)) = U$. However in general we only have $f^{-1}(f(U)) \supseteq U$). Clearly $\pi_{\mathcal{F},x}$ and $\pi^{-1}_{\mathcal{F},x}$ are inverse to each other. □

1.10.2 Inverse Image

Definition 1.9. Let $f : X \to Y$ be a continuous map and let $\mathcal{G}$ be a presheaf on Y. Define a presheaf $f^+\mathcal{G}$ on X by

$$f^+\mathcal{G}(U) = \varinjlim_{\substack{V \supseteq f(U) \\ V \subseteq Y \text{ open}}} \mathcal{G}(V).$$

the restriction maps being induced by the restriction maps on $\mathcal{G}$. Let $f^{-1}\mathcal{G}$ be the sheafification of $f^+\mathcal{G}$. We call $f^{-1}\mathcal{G}$ the **inverse image of $\mathcal{G}$ under** f.

More concretely,

$$f^+\mathcal{G}(U) = \{(V,t) \mid V \text{ is open subset of } Y \text{ such that } f(U) \subseteq V, \text{ and } t \in \mathcal{G}(V)\} / \sim,$$

where two pairs (V,t) and (V',t') are equivalent if there exists an open neighborhood V'' of x with $f(U) \subseteq V'' \subseteq V \cap V'$ such that $t|_{V''} = t'|_{V''}$. We shall write $[V,t]_{f(U)}$ (or simply $[V,t]$ to ease notation) for the equivalence class of (V,t). If U' is an open subset of X such that $U' \subseteq U$, then the restriction map is defined as $[V,t]|_{U'} = [V,t]_{f(U')}$.

Let's see why $f^+\mathcal{G}$ can fail to be a sheaf. Let $\{U_i\}_{i\in I}$ be an open cover of U and let $[V_i, t_i] \in f^+\mathcal{G}(U_i)$ such that $[V_i,t_i]|_{U_{ij}} = [V_j,t_j]|_{U_{ij}}$. In particular, this means there exists an open subset $V'_{i,j}$ of Y such that $f(U_{ij}) \subseteq V'_{i,j} \subseteq V_{ij}$ and $t_i|_{V'_{i,j}} = t_j|_{V'_{i,j}}$ for all $i,j \in I$.

If there existed $[V,t] \in f^+\mathcal{G}(U)$ such that $[V,t]|_{U_i} = [V_i,t_i]_{f(U_i)}$, then for all $i \in I$ there would exist an open subset V'_i of Y such that $f(U_i) \subseteq V'_i \subseteq V_i$ and $t|_{V'_i} = t_i|_{V'_i}$. By replacing $[V_i,t_i]$ with $[V'_i, t_i|_{V'_i}]$ for all $i \in I$, we may assume that $V'_{i,j} \subseteq V'_{ij}$.

Remark.

1. If f is the inclusion of a subspace X of Y, we also write $\mathcal{G}|_X$ instead of $f^{-1}\mathcal{G}$ and we write $\mathcal{G}(X) := (f^{-1}(\mathcal{G}))(X)$.
2. The construction of $f^+\mathcal{G}$ and hence of $f^{-1}\mathcal{G}$ is functorial in $\mathcal{G}$. Therefore we obtain a functor

$$f^{-1} : \mathbf{PSh}(Y) \to \mathbf{Sh}(X)$$

3. If $f : X \to Y$ is an open continuous map, then for $U \subseteq X$ open one has $f^+\mathcal{G}(U) \cong \mathcal{G}(f(U))$. Indeed, if $[V,t] \in f^+\mathcal{G}(U)$, then $[V,t] = [f(U), t|_{f(U)}]$. Thus, the map sending the element $[V,t] \in f^+\mathcal{G}(U)$ to the element $t|_{f(U)} \in \mathcal{G}(f(U))$ is well-defined and gives rise to an isomorphism of presheaves. Moreover, if $\mathcal{G}$ is a sheaf, then $f^+\mathcal{G}$ is a sheaf and hence $f^+\mathcal{G} = f^{-1}\mathcal{G}$. In particular, if f is the inclusion of an open subspace $U = X$ of Y, then for every sheaf $\mathcal{G}$ on Y and $U' \subseteq U$ open, we have

$$\mathcal{G}|_U(U') = \mathcal{G}(U').$$

Proposition 1.7. *Let $x \in X$ and let $\mathcal{G}$ be a presheaf on Y. Then $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$.*

Proof. It suffices to show that $f^+\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ by Proposition (1.4). Recall that

$$(f^+\mathcal{G})_x = \left\{(U, [V,t]) \mid U \text{ is an open neighborhood of } x \text{ and } [V,t] \in f^+\mathcal{G}(U)\right\} / \sim,$$

where $[V,t]$ denotes the equivalence class of $(V,t) \in f^+\mathcal{G}(U)$ and $(U,[V,t]) \sim (U',[V',t'])$ if there exists an open neighborhood U'' of x such that $U'' \subseteq U \cap U'$ and $[V,t]|_{U''} = [V',t']|_{U''}$, i.e. and there exists an open neighborhood V'' of $f(U'')$ such that $V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. Recall that $[U,[V,t]]_x$ denotes the equivalence class of $(U,[V,t]) \in f^+\mathcal{G}_x$.

Let $\lambda_{\mathcal{G},x} : f^+\mathcal{G}_x \to \mathcal{G}_{f(x)}$ be given by $\lambda_{\mathcal{G},x}([U,[V,t]]_x) = [V,t]_{f(x)}$. We need to show that this map is well-defined. Let $(U',[V',t'])$ be another representative of the equivalence class $[U,[V,t]]_x$. Then as described above, there exists an open neighborhood U'' of x such that $U'' \subseteq U \cap U'$ and there exists an open neighborhood V'' of $f(U'')$ such that $f(U'') \subseteq V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. As V'' is also an open neighborhood of $f(x)$, this implies

$$\begin{aligned} \lambda_{\mathcal{G},x}\left([U',[V',t']]_x\right) &= [V',t']_{f(x)} \\ &= [V,t]_{f(x)}. \end{aligned}$$

Thus this map is well-defined.

Next we show that $\lambda_{\mathcal{G},x}$ is injective. Suppose

$$\begin{aligned} \lambda_{\mathcal{G},x}\left([U,[V,t]]_x\right) &= [V,t]_{f(x)} \\ &= [V',t']_{f(x)} \\ &= \lambda_{\mathcal{G},x}\left([U',[V',t']]_x\right). \end{aligned}$$

Then there exists an open neighborhood V'' of $f(x)$ such that $V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. Setting $U'' = f^{-1}(V'') \cap U \cap U'$, we see that U'' is an open neighborhood of x such that $U'' \subseteq U \cap U'$, V'' is an open neighborhood of $f(U'')$ such that $V'' \subseteq V \cap V'$, and $t|_{V''} = t|_{V''}$. Therefore $[U,[V,t]]_x = [U',[V',t']]_x$.

Finally, we that $\lambda_{\mathcal{G},x}$ is is onto (which will imply that $\lambda_{\mathcal{G},x}$ is a bijection since we've already shown it is injective). Let $[V,t]_{f(x)} \in \mathcal{G}_{f(x)}$. We just need to find an open neighborhood U of x such that $f(U) \subseteq V$. Then $\lambda_{\mathcal{G},x}([U,[V,t]]_x) = [V,t]_{f(x)}$. In fact $U = f^{-1}(V)$ does the trick, and we are done. □

1.10.3 Adjunction between direct image and inverse image

Direct image and inverse image are functors that are adjoint to each other. More precisely:

Proposition 1.8. *Let $\mathcal{F}$ be a presheaf on X, and let $\mathcal{G}$ be a presheaf on Y. Then there is a bijection*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Psh}(X)}(f^+\mathcal{G},\mathcal{F}) & \longleftrightarrow & \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \varphi & \mapsto & \varphi^\flat \\ \psi^\# & \leftarrow\!\shortmid & \psi, \end{array}$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Moreover, if $\mathcal{F}$ is a sheaf, then there is a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G},\mathcal{F}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{G}, f_*\mathcal{F}),$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Moreover if $\mathcal{G}$ is also a sheaf then there is a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G},\mathcal{F}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}),$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\varphi : f^+\mathcal{G} \to \mathcal{F}$ be a morphism of presheaves on X. We define $\varphi^\flat : \mathcal{G} \to f_*\mathcal{F}$ as follows: let V be open in Y and let $t \in \mathcal{G}(V)$. Then we set $\varphi^\flat_V(t) = \varphi_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right)$. Similarly, let $\psi : \mathcal{G} \to f_*\mathcal{F}$ be a morphism of presheaves on Y. We define $\psi^\# : f^+\mathcal{G} \to \mathcal{F}$ as follows: let U be open in X and let $[V,t]_{f(U)} \in f^+\mathcal{G}(U)$. Then we set $\psi^\#_U\left([V,t]_{f(U)}\right) = \psi_V(t)|_U$.

Let us check that these two maps are inverse to each other, i.e. $(\varphi^\flat)^\# = \varphi$ and $(\psi^\#)^\flat = \psi$. First we show $(\varphi^\flat)^\# = \varphi$: let U be open in X and let $[V,t]_{f(U)} \in f^+\mathcal{G}(U)$. Then

$$\begin{aligned}
(\varphi^\flat)^\#_U\left([V,t]_{f(U)}\right) &= \varphi^\flat_V(t)|_U \\
&= \varphi_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right)|_U \\
&= \varphi_U\left([V,t]_{f\left(f^{-1}(V)\right)}|_U\right) \\
&= \varphi_U\left([V,t]_{f(U)}\right),
\end{aligned}$$

where we used the fact that $U \subseteq f^{-1}(V)$ since $f(U) \subseteq V$.

Next we show $(\psi^\#)^\flat = \psi$: let V be open in Y and let $t \in \mathcal{G}(V)$. Then

$$\begin{aligned}
(\psi^\#)^\flat_V(t) &= \psi^\#_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right) \\
&= \psi_V(t)|_{f^{-1}(V)} \\
&= \psi_V(t).
\end{aligned}$$

Therefore these two maps are inverse to each other. Moreover, it is straightforward - albeit quite cumbersome - to check that the constructed maps are functorial in $\mathcal{F}$ and $\mathcal{G}$. The last part of the proposition follows from Lemma (1.1). □

Remark. We will almost never use the concrete description of $f^{-1}\mathcal{G}$ in the sequel. Very often we are given f, $\mathcal{F}$, and $\mathcal{G}$, and a morphism of shaves $f^\flat : \mathcal{G} \to f_*\mathcal{F}$. Then usually it is sufficient to understand for each $x \in X$ the map

$$f^\#_x : \mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x \cong \mathcal{F}_x,$$

induced by $f^\# : f^{-1}\mathcal{G} \to \mathcal{F}$ on stalks.

2 Ringed Spaces

Throughout the rest of this article, let R be a commutative ring and let $\alpha \in \widehat{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty, \omega\}$. Ringed spaces formalize the idea of giving a geometric object by specifying its underlying topological space and the "functions" on all open subsets of this space.

Definition 2.1.

1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$.
2. A **locally** R**-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Remark. Usually we will denote a (locally) R-ringed space $(X, \mathcal{O}_X)$ simply by X.

Our principle example will be sheaves of real-valued C^α functions.

Example 2.1. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by $\mathcal{C}^\alpha_X$ the sheaf of $\mathcal{C}^\alpha$ functions: For all open subsets U of X, we have

$$\mathcal{C}^\alpha_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is } C^\alpha\}.$$

Then $\mathcal{C}^\alpha_X$ is a sheaf of $\mathbb{R}$-algebras. The same argument as for sheaves of continuous functions yields the following observation: For all $x \in X$ the stalk $C^\alpha_{X,x}$ is a local ring. In particular $(X, \mathcal{C}^\alpha_X)$ is a locally $\mathbb{R}$-ringed space.

Another example comes from Algebraic Geometry:

Example 2.2. Let k be an algebraically closed field and let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. The space X is equipped with the Zariski topology. Recall that a function $\varphi : U \to k$ from an open subset U of X to the field k is called **regular** at the point $x_0 \in X$ if there exists an open neighborhood U_0 of x_0 such that $U_0 \subseteq U$ and there are polynomials $f, g \in k[T_1, \ldots, T_n]$ with $g(x) \neq 0$ and $\varphi(x) = \frac{f(x)}{g(x)}$ for all $x \in X$. With this in mind, we define the structure sheaf $\mathcal{O}_X$ of X as follows: for all open subsets U of X, we define

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

2.1 Morphisms of (Locally) Ringed Spaces

Definition 2.2. Let $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$ be R-ringed spaces. A **morphism of R-ringed spaces** $X \to Y$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a continuous map of the underlying topological spaces and where $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of R-algebras on Y.

The datum of $f^\flat$ is equivalent to the datum of a homomorphism of sheaves of R-algebras $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X by Proposition (1.8). Usually we simply write f instead of $(f, f^\#)$ or $(f, f^\flat)$.

Morphisms of *locally* ringed spaces have to satisfy an additional property. To state this property, let $f : X \to Y$ be a morphism of R-ringed spaces and for each $x \in X$ let f_x be the composition of induced homomorphisms on the stalks

$$\mathcal{O}_{Y,f(x)} \xrightarrow{\lambda_{\mathcal{O}_Y,x}} f^{-1}(\mathcal{O}_Y)_x \xrightarrow{f^\#_x} \mathcal{O}_{X,x}$$

i.e. $f_x := f^\#{}_x \circ \lambda_{\mathcal{O}_Y,x}$, where $\lambda_{\mathcal{O}_Y,x}$ is the isomorphism constructed in Proposition (1.7). We can better understand f_x by following an element $[V,t]_{f(x)} \in \mathcal{O}_{Y,f(x)}$ under the composition:

$$\begin{aligned}
[V,t]_{f(x)} &\mapsto [f^{-1}(V), [V,t]_{f(f^{-1}(V))}]_x \\
&\mapsto [f^{-1}(V), f^\#_{f^{-1}(V)}([V,t]_{f(f^{-1}(V))})]_x \\
&= [f^{-1}(V), f^\flat_V(t)]_x,
\end{aligned}$$

where the last equality follows from Proposition (1.8).

Definition 2.3. Let X and Y be locally R-ringed spaces. We define a **morphism of locally R-ringed spaces** $X \to Y$ to be a morphism $(f, f^\flat)$ of R-ringed spaces such that for all $x \in X$, the homomorphisms of local rings $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is **local** (i.e. $f_x(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$).

Remark. In the case where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of functions, the condition that f_x is local is equivalent to the condition that if $[V,t]_{f(x)} \in \mathcal{O}_{Y,f(x)}$ such that $t(x) = 0$, then $f^\flat_V(t)(x) = 0$.

In general there exist locally ringed spaces and morphisms of ringed spaces between them that are not morphisms of *locally* ringed spaces. For spaces with functions of C^α functions such as the premanifolds defined below, we will see that every morphism of ringed spaces is automatically a morphism of locally ringed spaces.

Remark. The composition of morphisms of (locally) R-ringed spaces is defined in the obvious way using the compatibility of direct images with composition (i.e. $(g \circ f)_* = g_* \circ f_*$. We obtain the category of (locally) R-ringed spaces.

In general, $f^\flat$ (or $f^\#$) is an additional datum for a morphism. For instance it might happen that f is the identity but $f^\flat$ is not an isomorphism of sheaves. We will usually encounter the simpler case that the structure sheaf is a sheaf of functions on open subsets of X and that $f^\flat$ is given by composition with f. The following special case and its globalization is the main example.

Example 2.3. Let $X \subseteq V$ and $Y \subseteq W$ be open subsets of finite-dimensional $\mathbb{R}$-vector spaces V and W. Every C^α map $f : X \to Y$ defines by composition a morphism of locally $\mathbb{R}$-ringed spaces $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ by

$$\begin{array}{rcl}
f^\flat_U : \mathcal{C}^\alpha_Y(U) & \longrightarrow & f_*(\mathcal{C}^\alpha_X)(U) = \mathcal{C}^\alpha_X(f^{-1}(U)) \\
t & \mapsto & t \circ f
\end{array}$$

for $U \subseteq Y$ open.

The induced map on stalks $f_x : \mathcal{C}^\alpha_{Y,f(x)} \to \mathcal{C}^\alpha_{X,x}$ is then also given by composing an $\mathbb{R}$-valued C^α function t, defined in some neighborhood of $f(x)$, with f, which yields an $\mathbb{R}$-valued C^α function $t \circ f$ defined in some neighborhood of x. Conversely, let $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ be any morphism of $\mathbb{R}$-ringed spaces. We claim:

1. $(f, f^\flat)$ is automatically a morphism of *locally* $\mathbb{R}$-ringed spaces.
2. For all $U \subseteq Y$ open the $\mathbb{R}$-algebra homomorphism $f^\flat_U : \mathcal{C}^\alpha_Y(U) \to \mathcal{C}^\alpha_X(f^{-1}(U))$ is automatically given by the map $t \mapsto t \circ f$. Note that then f is a C^α map (choose a basis of W; considering for t projections to the coordinates shows that each component of f is a C^α map).

To show 1 let $x \in X$. Set $\varphi := f_x$, $B := \mathcal{C}^{\alpha}_{X,x}$, and $A := \mathcal{C}^{\alpha}_{Y,f(x)}$. Then $\varphi : A \to B$ is a homomorphism of local $\mathbb{R}$-algebras such that $A/\mathfrak{m}_A = \mathbb{R}$ and $B/\mathfrak{m}_B = \mathbb{R}$. We claim that φ is automatically local, equivalently that $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal of A. Indeed, φ induces an injective homomorphism of $\mathbb{R}$-algebras

$$A/\varphi^{-1}(\mathfrak{m}_B) \hookrightarrow B/\mathfrak{m}_B = \mathbb{R}.$$

As a homomorphism of $\mathbb{R}$-algebras, it is automatically surjective (indeed 1 maps to 1), hence $A/\varphi^{-1}(\mathfrak{m}_B) \cong \mathbb{R}$ is a field and hence $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal.

Let us show 2. Let $V \subseteq Y$ be open and $x \in f^{-1}(V)$. Consider the commutative diagram of $\mathbb{R}$-algebra homomorphisms

$$\begin{tikzcd}
\mathcal{C}^{\alpha}_{Y}(V) \arrow[rrr, "f^{\flat}_V"] \arrow[ddd] & & & \mathcal{C}^{\alpha}_{X}(f^{-1}(V)) \arrow[ddd] \\
& t \arrow[r, mapsto] \arrow[d, mapsto] & f^{\flat}_V(t) \arrow[d, mapsto] & \\
& {[V,t]_{f(x)}} \arrow[r, mapsto] & {[f^{-1}(V), f^{\flat}_V(t)]_x} & \\
\mathcal{C}^{\alpha}_{Y,f(x)} \arrow[rrr, "f_x"] \arrow[ddd] & & & \mathcal{C}^{\alpha}_{X,x} \arrow[ddd] \\
& {[V,t]_{f(x)}} \arrow[r, mapsto] \arrow[d, mapsto] & {[f^{-1}(V), f^{\flat}_V(t)]_x} \arrow[d, mapsto] & \\
& t(f(x)) & f^{\flat}_V(t)(x) & \\
\mathbb{R} & & & \mathbb{R}
\end{tikzcd}$$

The evaluation maps are surjective. Hence there exists a homomorphism of $\mathbb{R}$-algebras $\iota : \mathbb{R} \to \mathbb{R}$ making the lower rectangle commutative if and only if one has $f_x(\mathrm{Ker}(\mathrm{ev}_{f(x)})) \subseteq \mathrm{Ker}(\mathrm{ev}_x)$. But this latter condition is satisfied because f_x is local by 1. Moreover, as a homomorphism of $\mathbb{R}$-algebras, one must have $\iota = \mathrm{id}_{\mathbb{R}}$. Therefore we find $f^{\flat}_V(t)(x) = t(f(x))$, which shows 2.

Remark. A morphism $f : X \to Y$ of R-ringed spaces is an isomorphism in the category of R-ringed spaces if and only if f is a homeomorphism and $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism of R-algebras for all $x \in X$. Indeed, $(f, f^{\flat})$ is an isomorphism if and only if f is a homeomorphism and $f^{\flat}$ is an isomorphism of sheaves of rings. We claim that if f is a homeomorphism, then $f^{\flat}$ is an isomorphism if and only if f_x is an isomorphism for all $x \in X$. To see this, note that since f is a homeomorphism, we have $f_x = \pi_{\mathcal{O}_X,x} \circ f^{\flat}_x$, where $\pi_{\mathcal{O}_X,x}$ is the isomorphism constructed in Proposition (1.6).

2.1.1 Open embedding

Let X be a locally R-ringed space and let $U \subseteq X$ be open. Then $(U, \mathcal{O}_{X|U})$ is a locally R-ringed space, which we usually denote simply by U. Such a locally ringed R-space is called an **open subspace** of X. There is an **inclusion morphism** $i : U \to X$ of locally R-ringed spaces, where the continuous map $i : U \to X$ is the inclusion of the underlying topological spaces and where $i^{\flat}$ is given by the restriction $\mathcal{O}_X(U') \to i_*(\mathcal{O}_{X|U})(U') = \mathcal{O}_X(U \cap U')$ for all $U' \subseteq X$ open. Then $i^{\#} : i^{-1}\mathcal{O}_X \to \mathcal{O}_{X|U}$ is the identity. In particular i_x is the identity for all $x \in U$.

For a morphism $f : X \to Y$ of locally R-ringed spaces, we denote by $f|_U : U \to Y$ the composition $f \circ i$ of morphisms of locally R-ringed spaces.

Finally, a morphism $j : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ of locally R-ringed spaces is called an **open embedding** if $U := j(Z)$ is open in X and j induces an isomorphism $(Z, \mathcal{O}_Z) \cong (U, \mathcal{O}_{X|U})$.

Definition 2.4. A morphisms $f : X \to Y$ of locally R-ringed spaces is called a **local isomorphism** if there exists an open covering $\{U_i\}_{i \in I}$ of X such that $f|_{U_i} : U_i \to Y$ is an open embedding for all $i \in I$.

In other words, a morphism f is a local isomorphism if there exists an open covering $\{U_i\}_{i \in I}$ of X and for all $i \in I$ an open subspace V_i of Y such that f induces an isomorpism $U_i \cong V_i$ of locally R-ringed spaces for all $i \in I$.

Remark. A morphism $f : X \to Y$ of R-ringed spaces is a local isomorphism if and only if f is a local homeomorphism and $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism.

2.2 Gluing Ringed Spaces

Let $\{(X_i, \mathcal{F}_i)\}_{i\in I}$ be a collection of ringed spaces, $X_{i,j}$ be open subsets of X_i, and let $\varphi_{i,j} : X_{i,j} \to X_{j,i}$ be homeomorphisms for all $i, j \in I$. Let

$$X := \coprod_{i\in I} X_i / \sim$$

where if $x \in X_{i,j}$, then $x \sim \varphi_{i,j}(x)$ for all $i, j \in I$. Now $\{X_i\}_{i\in I}$ forms an open cover of X, and the $\mathcal{F}_i$ glue to a unique sheaf $\mathcal{F}$ on X.

Example 2.4. Let $(X_1, \mathcal{O}_{X_1})$ and $(X_2, \mathcal{O}_{X_2})$ be locally ringed spaces, $X_{1,2} \subset X_1$ and $X_{2,1} \subset X_2$ be non-empty open subsets, and let $f : (X_{1,2}, \mathcal{O}_{X_1|X_{1,2}}) \to (X_{2,1}, \mathcal{O}_{X_2|X_{2,1}})$ be an isomorphism of locally ringed spaces. Then we can define a locally ringed space $(X, \mathcal{O}_X)$, obtained by **gluing** X_1 and X_2 along $X_{1,2}$ and $X_{2,1}$ via the isomorphism f:

- As a set, the space X is just the disjoint union $X_1 \coprod X_2$ modulo the equivalence relation $x \sim f(x)$ for all $x \in X_{1,2}$.
- As a topological space, we endow X with the so-called **quotient topology** induced by the above equivalence relation, i.e. we say that a subset $U \subset X$ is open if and only if $U \cap U_1 \subseteq U_1$ and $U \cap U_2 \subseteq U_2$ are both open, where $U_1 = i_1(X_1)$ and $U_2 = i_2(X_2)$ with $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ being the obvious inclusion maps.
- As a ringed space, we define the structure sheaf $\mathcal{O}_X$ by

$$\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2), \text{ and } f^*(s_2)|_{U\cap X_{2,1}} = s_1|_{U\cap X_{1,2}})\}$$

Example 2.5. Let $X_1 = X_2 = \mathbb{A}^1$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $x \mapsto \frac{1}{x}$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $K = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$.
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X.

Example 2.6. Let X be the complex affine curve

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2\widetilde{x}^4 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x, y) \mapsto (\widetilde{x}, \widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right)$$

$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x}, \widetilde{y}) \mapsto (x, y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^n}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.