Commutative Algebra Homework 7

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Problem 1

Definition 0.1. Let R be a commutative ring with identity and let I be an ideal of R. We say that I is of **strong finite type** (SFT) if there is a finitely generated ideal $\mathfrak{a} \subseteq I$ and an integral $n \in \mathbb{N}$ such that $x^n \in \mathfrak{a}$ for all $x \in I$. We also say that the ring R is SFT if every ideal of R is SFT.

Exercise 1. Let R be a commutative ring with identity.

1. Show that R is SFT if and only if every prime ideal of R is SFT.
2. Show that if R is SFT then R satisfies the ascending chain condition on radical ideals.
3. Given an example of a ring that is SFT but not Noetherian.
4. Given an example of a ring that satisfies the ascending chain condition on radical ideals but is not SFT.

Solution 1. 1. If R is SFT, then every prime ideal of R is SFT by definition. Conversely, suppose every prime ideal of R is SFT and assume for a contradiction that R is not SFT. Let $(\mathcal{F}, \subseteq)$ be the partially ordered set where the underlying set $\mathcal{F}$ is

$$\mathcal{F} = \{\text{ideals } I \text{ of } R \text{ which are not SFT}\},$$

and where the partial order $\subseteq$ is inclusion. Since R is not SFT, the set $\mathcal{F}$ is nonempty. Furthermore, note that if $(I_\lambda)_{\lambda \in \Lambda}$ is a chain in $\mathcal{F}$, then $\bigcup_{\lambda \in \Lambda} I_\lambda \in \mathcal{F}$. Indeed, assume for a contradiction that $\bigcup_{\lambda \in \Lambda} I_\lambda$ is SFT. Then there exists a finitely generated ideal $\mathfrak{a} \subseteq \bigcup_{\lambda \in \Lambda} I_\lambda$ and an $n \in \mathbb{N}$ such that $x^n \in \mathfrak{a}$ for all $x \in \bigcup_{\lambda \in \Lambda} I_\lambda$. Writing $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$, we see that since $\mathfrak{a} \subseteq \bigcup_{\lambda \in \Lambda} I_\lambda$, we must have $x_i \in \bigcup_{\lambda \in \Lambda} I_\lambda$ for each $1 \leq i \leq n$. This means $x_i \in I_{\lambda_i}$ for some $\lambda_i \in \Lambda$ for each $1 \leq i \leq n$. In particular, since $(I_\lambda)_{\lambda \in \Lambda}$ is a chain, there exists a $\lambda \in \Lambda$ such that $x_i \in I_\lambda$ for all $1 \leq i \leq n$. Thus $\mathfrak{a} \subseteq I_\lambda$ and we have $x^n \in \mathfrak{a}$ for all $x \in I_\lambda$ since this is true for all $x \in \bigcup_{\lambda \in \Lambda} I_\lambda$. However this contradicts the fact that I_λ is SFT. Therefore every chain in $(\mathcal{F}, \subseteq)$ has an upper bound in $(\mathcal{F}, \subseteq)$.

Now we apply Zorn's lemma to obtain an ideal I which is maximal in $(\mathcal{F}, \subseteq)$. We claim that I is a prime ideal. Indeed, assume for a contradiction that I is not prime. Choose $x, y \in R \backslash I$ such that $xy \in I$. By maximality of I, both $I + \langle x \rangle$ and $I + \langle y \rangle$ are SFT, so there exists finitely generated ideals $\mathfrak{a} \subseteq I + \langle x \rangle$ and $\mathfrak{b} \subseteq I + \langle y \rangle$ and integers $m, n \in \mathbb{N}$ such that $z^m \in \mathfrak{a}$ for all $z \in I + \langle x \rangle$ and $z^n \in \mathfrak{b}$ for all $z \in I + \langle y \rangle$. Observe that $\mathfrak{ab}$ is a finitely generated ideal, and furthermore we have

$$\begin{aligned}
\mathfrak{ab} &\subseteq (I + \langle x \rangle)(I + \langle y \rangle) \\
&\subseteq I^2 + \langle x \rangle I + I \langle y \rangle + \langle xy \rangle \\
&\subseteq I.
\end{aligned}$$

Moreover, for any $z \in I$, we have $z^{m+n} = z^m z^n \in \mathfrak{ab}$ since $z^m \in \mathfrak{a}$ for all $z \in I + \langle x \rangle \supseteq I$ and $z^n \in \mathfrak{b}$ for all $z \in I + \langle y \rangle \supseteq I$. Thus I is SFT, which is a contradiction. Thus I is a prime ideal. However this contradicts the fact that all prime ideals are assumed to be SFT. Thus $\mathcal{F}$ is empty, which implies R is SFT.

2. Suppose R is SFT. Let (I_k) be an ascending chain of radical ideals of R. Since $\bigcup_{k=1}^\infty I_k$ is SFT, there exists a finitely generated ideal $\mathfrak{a} \subseteq \bigcup_{k=1}^\infty I_k$ and an $n \in \mathbb{N}$ such that $x^n \in \mathfrak{a}$ for all $x \in \bigcup_{k=1}^\infty I_k$. Since $\mathfrak{a}$ is finitely generated, we must have $\mathfrak{a} \subseteq I_N$ for some $N \in \mathbb{N}$ (see argument in proof of part 1 for justification). Since I_N is radical and $x^n \in \mathfrak{a} \subseteq I_N$ for all $x \in \bigcup_{k=1}^\infty I_k$, we must in fact have $I_N = \bigcup_{k=1}^\infty I_k$. Thus R satisfies the ascending chain condition on radical ideals.

3. Let $R = \mathbb{F}_2[\{x_n \mid n \in \mathbb{N}\}]$ and let $\mathfrak{m} = \langle \{x_n \mid n \in \mathbb{N}\} \rangle$ and set $A = R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2$. The prime ideals of A are in bijection with prime ideals $\mathfrak{p}$ of R such that $\mathfrak{m}^2 \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. There is only one such prime, namely $\mathfrak{m}$, so A contains just one prime ideal, namely $\mathfrak{n} = \mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2$. To see that A is SFT, it suffices to show that $\mathfrak{n}$ is SFT, by part 1. However this is clear since the zero ideal is finitely generated and $\gamma^2 = 0$ for all $\gamma \in \mathfrak{n}$. Indeed, suppose $f/g \in R_\mathfrak{m}$ represents γ where $f \in \mathfrak{m}$ and $g \in R \backslash \mathfrak{m}$. Express f in terms of its monomials as

$$f = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}$$

where $a_i \in \mathbb{F}_2$. Here, the x^{α_i} are the monomials of f (see my HW 5, problem 3 for more detail about this notation). Since $f \in \mathfrak{m}$, we may as well assume that $\operatorname{supp} x^{\alpha_i} \neq \emptyset$ for each $1 \leq i \leq n$ (note that $\operatorname{supp} x^{\alpha} = \emptyset$ means $x^{\alpha} = 1$). Now observe that

$$f^2 = a_1 x^{2\alpha_1} + \cdots + a_n x^{2\alpha_n}$$

since we are working over a characteristic 2 ring. In particular, each monomial $x^{2\alpha_i}$ lands in $\mathfrak{m}^2$. In particular, $f^2 \in \mathfrak{m}^2$, and this implies $(f/g)^2 = f^2/g^2$ represents the zero element in $\mathfrak{n}$.

So A is indeed SFT as claimed, however it is not Noetherian as $\mathfrak{n}$ is not finitely-generated. To see why, note that $\mathfrak{n}$ is generated by $\{\overline{x}_n \mid n \in \mathbb{N}\}$

4. Let K be a field, let p be a prime, let $R = \bigcup_{n=0}^{\infty} K[x^{1/p^n}]$, and let $\mathfrak{m} = \bigcup_{n=0}^{\infty} \langle x^{1/p^n} \rangle$. Observe that $\mathfrak{m}$ is a maximal ideal since it is the kernel of the unique ring homomorphism $R \to K$ given by mapping x^{1/p^n} to 0 for all $n \in \mathbb{Z}_{\geq 0}$. Furthermore we have $\operatorname{ht} \mathfrak{m} = 1$. To see this, first note that R is a domain. Indeed, suppose $fg = 0$ for some $f, g \in R$. Since $(K[x^{1/p^n}])$ is an ascending sequence of rings and $R = \bigcup_{n=0}^{\infty} K[x^{1/p^n}]$, we see that $f, g \in K[x^{1/p^N}]$ for some $N \in \mathbb{Z}_{\geq 0}$. Then since $K[x^{1/p^N}]$ is a domain, we must have either $f = 0$ or $g = 0$. Thus we have a chain of prime ideals $0 \subseteq \mathfrak{m}$. Furthermore, suppose that $\mathfrak{p}$ is a nonzero prime ideal of R such that $\mathfrak{p} \subseteq \mathfrak{m}$. Let f be a nonzero element in $\mathfrak{p}$. Again, we must have $f \in K[x^{1/p^N}]$ for some $N \in \mathbb{Z}_{\geq 0}$ and so we can express f as polynoimal in x^{1/p^N} with coefficients in K. Furthermore since $f \in \mathfrak{m}$, the coefficient of f in degree zero must vanish, so if we denote $\gamma = x^{1/p^N}$, then f has the form

$$f = \gamma^m (a_k \gamma^k + a_{k-1} \gamma^{k-1} \cdots + a_1 \gamma + a_0)$$

where $m \geq 1$ and where $a_0, a_1, \dots, a_{k-1}, a_k \in K$, where we may assume without loss of generality that $a_k \neq 0$ since $f \neq 0$.

$$\begin{aligned} f &= a_k x^{k/p^N} + a_{k-1} x^{(k-1)/p^N} \cdots + a_1 x^{1/p^N} \\ &= x^{1/n_1} \left(a_1 + a_2 x^{1/n_2 - 1/n_1} \cdots + a_r x^{1/n_r - 1/n_1} \right) \\ &= x^{1/n_1} \left(a_1 + a_2 x^{(n_1 - n_2)/n_1 n_2} \cdots + a_r x^{(n_1 - n_r)/n_1 n_r} \right). \end{aligned}$$

This impies $x^{1/n_1} \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $\sum_{i=1}^{r} a_i x^{(n_1 - n_i)/n_1 n_i} \notin \mathfrak{m} \supseteq \mathfrak{p}$. Furthermore, for any $n \in \mathbb{N}$, we obtain $x^{1/n} \in \mathfrak{p}$. Indeed, we have $x^{1/nn_1} \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $(x^{1/nn_1})^n = x^{1/n_1} \in \mathfrak{p}$, and thus $x^{1/n} = (x^{1/nn_1})^{n_1} \in \mathfrak{p}$. Therefore $\mathfrak{p} \supseteq \mathfrak{m}$, and since already we have $\mathfrak{p} \subseteq \mathfrak{m}$, we see that $\mathfrak{p} = \mathfrak{m}$.

So by localizing at $\mathfrak{m}$, we see that $R_{\mathfrak{m}}$ has exactly one nonzero prime ideal, and thus easily satisfies the ascending chain condition on radical ideals (all radical ideals are intersection of prime ideals). Note that R is a domain (if $fg = 0$ for some $f, g \in R$, then since $R = \bigcup_{n=1}^{\infty} K[x^{1/n}]$, we have $f, g \in K[x^{1/N}]$ for some $N \in \mathbb{N}$, and since $K[x^{1/N}]$ is a domain, this implies either $f = 0$ or $g = 0$). Thus we may identify $R_{\mathfrak{m}}$ with a subring of the field of fractions of R and we may identify the localization map $\rho \colon R \to R_{\mathfrak{m}}$ with the inclusion map $R \subseteq R_{\mathfrak{m}}$. With this in mind, we will now show that $R_{\mathfrak{m}}$ is not SFT by showing that $\mathfrak{m}_{\mathfrak{m}}$ is not SFT. Assume for a contradiction that there exists a finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{m}_{\mathfrak{m}}$ and an $N \in \mathbb{N}$ such that $\gamma^N \in \mathfrak{a}$ for all $\gamma \in \mathfrak{m}_{\mathfrak{m}}$. In particular, we must have $x^{N/n} \in \mathfrak{a}$ for all $n \in \mathbb{N}$, and by setting $n = Nm$, we see that this implies $x^{1/m} \in \mathfrak{a}$ for all $m \in \mathbb{N}$. This implies $\mathfrak{a} = \mathfrak{m}_{\mathfrak{m}}$, however we have a contradiction here because $\mathfrak{m}_{\mathfrak{m}}$ is not finitely generated. To see this, assume for a contradiction that $\mathfrak{m}_{\mathfrak{m}}$ is finitely generated. Then since $\mathfrak{m}_{\mathfrak{m}}$ is generated by $\{x^{1/n} \mid n \in \mathbb{N}\}$, and $\mathfrak{m}_{\mathfrak{m}}$ is finitely generated, it follows from Lemma (0.1) that $\mathfrak{m}_{\mathfrak{m}}$ can be generated by finitely many of the $x^{1/n}$, say $\mathfrak{m}_{\mathfrak{m}} = \langle x^{1/n_1}, \dots, x^{1/n_r} \rangle$. Choose $N \in \mathbb{N}$ such that $N > \max\{n_1, \dots, n_r\}$. Then there must exists a $g \in R \backslash \mathfrak{m}$ and polynomials $p_1, \dots, p_s \in R$ such that

$$g x^{1/N} = p_1 x^{1/n_1} + \cdots + p_s x^{1/n_s}.$$

Since $g \notin \mathfrak{m}$, one of the monomials in $g x^{1/N}$ will be

$$R_{\mathfrak{m}} = \left\{ \frac{f}{g} \mid f, g \in \bigcup_{n=1}^{\infty} K[x^{1/n}] \text{ and } g \notin \mathfrak{m} \right\}$$

but is not SFT. Indeed, let us first show that is satisfies the ascending chain condition on radical ideals. In fact, we will show that $R_{\mathfrak{m}}$ has exactly one nonzero prime ideal, namely $\mathfrak{m}_{\mathfrak{m}}$. To see this, suppose $\mathfrak{p}_{\mathfrak{m}}$ is a nonzero prime ideal of $R_{\mathfrak{m}}$, where $\mathfrak{p}_{\mathfrak{m}}$ is the localization of the nonzero prime ideal $\mathfrak{p}$ of R where $\mathfrak{p} \subseteq \mathfrak{m}$. Let $f \in \mathfrak{p}_{\mathfrak{m}}$. Since $f \in \mathfrak{m}_{\mathfrak{m}}$, we can express it as

$$f = a_1 x^{1/n_1} + \cdots + a_r x^{1/n_r}$$

for some $a_1, \dots, a_r \in R\backslash\{0\}$. By relabeling if necessary, we may assume that $n_1 = \max\{n_1, \dots, n_r\}$. Then we can express f as

$$\begin{aligned} f &= a_1 x^{1/n_1} + \cdots + a_r x^{1/n_r} \\ &= x^{1/n_1}\left(a_1 + a_2 x^{1/n_2 - 1/n_1} \cdots + a_r x^{1/n_r - 1/n_1}\right) \\ &= x^{1/n_1}\left(a_1 + a_2 x^{(n_1-n_2)/n_1 n_2} \cdots + a_r x^{(n_1 - n_r)/n_1 n_r}\right). \end{aligned}$$

This impies $x^{1/n_1} \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $\sum_{i=1}^r a_i x^{(n_1-n_i)/n_1 n_i} \notin \mathfrak{m} \supseteq \mathfrak{p}$. Furthermore, for any $n \in \mathbb{N}$, we obtain $x^{1/n} \in \mathfrak{p}$. Indeed, we have $x^{1/nn_1} \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $(x^{1/nn_1})^n = x^{1/n_1} \in \mathfrak{p}$, and thus $x^{1/n} = (x^{1/nn_1})^{n_1} \in \mathfrak{p}$. In particular this implies $\mathfrak{p} \supseteq \mathfrak{m}$. Since already we have $\mathfrak{p} \subseteq \mathfrak{m}$, we see that $\mathfrak{p} = \mathfrak{m}$.

Problem 2

Lemma 0.1. *Let R be a ring and let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a collection of elements of R. If the ideal generated by $\{x_\lambda\}_{\lambda\in\Lambda}$ is finitely-generated, then it can be generated by finitely many of the x_λ's*

Proof. Indeed, suppose $\langle \{x_\lambda\}_{\lambda\in\Lambda}\rangle = \langle f_1, \dots, f_r\rangle$ where

$$f_i = a_{i1} x_{\lambda_{i1}} + \cdots + a_{in_i} x_{\lambda_{in_i}}$$

for each $1 \leq i \leq r$ where $a_{ij} \in R$. Then observe that

$$\begin{aligned} \langle \{x_\lambda\}_{\lambda\in\Lambda}\rangle &= \langle f_1, \dots, f_r\rangle \\ &\supseteq \langle \{x_{\lambda_{ij}} \mid 1 \leq i \leq r \text{ and } 1 \leq j \leq n_i\}\rangle \\ &\supseteq \langle \{x_\lambda\}_{\lambda\in\Lambda}\rangle \\ &= \langle f_1, \dots, f_r\rangle. \end{aligned}$$

□

Exercise 2. Let R be a domain with quotient field K with the property that every overring of R is Noetherian. Show that $\dim R \leq 1$.

Solution 2. Assume for a contradiction that $\dim R > 1$. Then there exists nonzero prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R such that $0 \subset \mathfrak{p} \subset \mathfrak{q}$ where the inclusions are proper. Choose a nonzero $x \in \mathfrak{p}$, choose $y \in \mathfrak{q}\backslash\mathfrak{p}$, and let $S = \{x/y^n \mid n \in \mathbb{N}\}$. Since the overring $R[S]$ is Noetherian, we see that the ideal $\langle S\rangle$ of $R[S]$ must be finitely generated, say $\langle S\rangle = \langle x/y^{n_1}, \dots, x/y^{n_r}\rangle$. Here we are using the fact that a finite subset of S can be used as a generating set of $\langle S\rangle$ (see Lemma (0.1). In fact, setting $n = \max\{n_1, \dots, n_r\}$, it is easy to see that $\langle S\rangle = \langle x/y^n\rangle$. In particular, we have

$$\frac{x}{y^{n+1}} = \left(a_0 + a_1\frac{x}{y} + a_2\frac{x}{y^2} + \cdots + a_k\frac{x}{y^k}\right)\frac{x}{y^n} \tag{1}$$

for some $k \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for all $1 \leq i \leq k$. Multiplying both sides of (1) by y^{n+k+1}/x gives us

$$y^k = a_0 y^{k+1} + a_1 x y^k + a_2 x y^{k-1} + \cdots + a_k x y.$$

In particular we see that $y^k(1 - a_0 y) \in \langle x\rangle \subseteq \mathfrak{p}$. Since $y \notin \mathfrak{p}$, it follows that $1 - a_0 y \in \mathfrak{p}$. However since $y \in \mathfrak{q}$, this implies $1 \in \mathfrak{q}$, a contradiction.

Problem 3

Exercise 3. Let A be 1-dimensional Noetherian domain, let $\mathfrak{p}$ be a prime ideal of A, and let B be an overring of A. Then there are only finitely many prime ideals of B which lie over $\mathfrak{p}$.

Solution 3. By a Theorem shown in class, we see that B is Noetherian. Thus $B/\mathfrak{p}B$ is a 0-dimensional Noetherian ring, hence must be Artinian. Since Artinian rings have only finitely many maximal ideals, we see that $B/\mathfrak{p}B$ has only finitely many maximal ideals. In particular, $B/\mathfrak{p}B$ has only finitely many prime ideals since $B/\mathfrak{p}B$ is 0-dimensional. Since the prime ideals in $B/\mathfrak{p}B$ are in bijection with the prime ideals in B which lie over $\mathfrak{p}$, this implies there exists only finitely many prime ideals of B which lie over $\mathfrak{p}$.

Problem 4

Exercise 4. Let R be a commutative ring. We recall that R is von Neumann if for all $x \in R$ there is a $y \in R$ such that $x = xyx$. Suppose that R is 0-dimensional and commutative with no nonzero nilpotent elements. Then R is von Neumann regular.

Solution 4. Let x be a nonzero element of R (clearly $x = 0$ then we just choose $y = 0$ to get $x = xyx$). To show that there exists a $y \in R$ such that $x = xyx = x^2y$, it suffices to show that $\langle x \rangle = \langle x^2 \rangle$. Let $\mathfrak{m}$ be any maximal ideal of R. Note that since R is 0-dimensional, we see that $R_\mathfrak{m}$ is also 0-dimensional, and since R is reduced, we see that $R_\mathfrak{m}$ is reduced. Thus we have $\mathfrak{m}_\mathfrak{m} = \mathrm{N}(R_\mathfrak{m}) = 0$, and in particular, this implies $R_\mathfrak{m}$ is a field. Every nonzero ideal of a field is just the field itself, and thus

$$\langle x \rangle_\mathfrak{m} = R_\mathfrak{m} = \langle x^2 \rangle_\mathfrak{m}.$$

Since $\mathfrak{m}$ was arbitrary, we see that $\langle x \rangle_\mathfrak{m} = \langle x^2 \rangle_\mathfrak{m}$ for all maximal ideals of R. This implies $\langle x \rangle = \langle x^2 \rangle$. Since x was arbitrary, we see that R is von Neumann regular.