# Commutative Algebra Homework 3

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## Problem 1

**Definition 0.1.** Let $R$ be a commutative ring (maybe without identity). We say $R$ is **von Neumann regular** if for every $x \in R$ there exists $y \in R$ such that $x = xyx$.

**Exercise 1.** Show that any direct product or direct sum of fields is von Neumann regular.

**Solution 1.** Let $\{K_\lambda\}_{\lambda \in \Lambda}$ be a collection of fields indexed over a set $\Lambda$. First let us show that $\prod_\lambda K_\lambda$ is von Neumann regular. Let $(x_\lambda)$ be an arbitrary element in $\prod K_\lambda$. For each $\lambda \in \Lambda$, note that $K_\lambda$ is von Neumann regular. Indeed, $K_\lambda$ is a field, so if $x_\lambda \neq 0$, we can choose $y_\lambda = x_\lambda^{-1}$, and if $x_\lambda = 0$, we can choose $y_\lambda = 0$. In any case, we see that $(y_\lambda) \in \prod K_\lambda$ satisfies

$$\begin{aligned}(x_\lambda)(y_\lambda)(x_\lambda) &= (x_\lambda y_\lambda x_\lambda) \\ &= (x_\lambda).\end{aligned}$$

Thus $\prod_\lambda K_\lambda$ is von Neumann regular.

The same proof also shows $\bigoplus_\lambda K_\lambda$ is von Neumann regular. Indeed, we view $\bigoplus_\lambda K_\lambda$ as a subring of $\prod_\lambda K_\lambda$ given by the set of all sequences $(x_\lambda) \in \prod_\lambda K_\lambda$ such that there exists a finite subset $\Lambda_0$ of $\Lambda$ where $x_\lambda = 0$ for all $\lambda \in \Lambda \backslash \Lambda_0$. In this case, for each $\lambda_0 \in \Lambda_0$, we choose $y_{\lambda_0} \in K_{\lambda_0}$ such that $x_{\lambda_0} y_{\lambda_0} x_{\lambda_0} = x_{\lambda_0}$ as before, and for each $\lambda \in \Lambda \backslash \Lambda_0$, we simply set $y_\lambda = 0$. Then clearly $(y_\lambda) \in \bigoplus_\lambda K_\lambda$ satisfies

$$\begin{aligned}(x_\lambda)(y_\lambda)(x_\lambda) &= (x_\lambda y_\lambda x_\lambda) \\ &= (x_\lambda).\end{aligned}$$

Thus $\bigoplus_\lambda K_\lambda$ is von Neumann regular.

## Problem 2

**Exercise 2.** Let $R$ be a commutative ring with identity. Suppose $R$ is von Neumann regular. Then $R$ is 0-dimensional.

**Solution 2.** Assume for a contradiction that $R$ is not 0-dimensional. Choose primes $\mathfrak{p}, \mathfrak{q} \in R$ such that $\mathfrak{p} \subset \mathfrak{q}$ where the inclusion is strict. Clearly $R/\mathfrak{p}$ is von Neumann, so by passing the to quotient $R/\mathfrak{p}$ if necessary, we may as well assume that $R$ is an integral domain, that $\mathfrak{p} = 0$, and that $\mathfrak{q}$ is a nonzero ideal in $R$. Choose a nonzero element $x \in \mathfrak{q}$. Since $R$ is von Neumann, there exists $y \in R$ such that $xyx = x$. This implies

$$x(yx - 1) = 0.$$

Since $x \neq 0$ and $R$ is a domain, we see that $yx = 1$. So $x$ is a unit. This contradicts the fact that $x \in \mathfrak{q}$ (prime ideals do not contain units!). Thus our assumption that $R$ is not 0-dimensional leads to a contradiction, so $R$ must be 0-dimensional.

## Problem 3

**Exercise 3.** Let $R$ be a commutative ring with identity. Suppose $R$ is von Neumann regular and let $\mathfrak{p}$ be a prime ideal in $R$. Then $R_\mathfrak{p} \cong R/\mathfrak{p}$.

**Solution 3.** Note that since $R$ is 0-dimensional (by problem 2) we see that $\mathfrak{p}$ is a maximal ideal, and thus $R/\mathfrak{p}$ is a field. In particular, it follows that $R/\mathfrak{p} \cong (R/\mathfrak{p})_{\mathfrak{p}/\mathfrak{p}} = (R/\mathfrak{p})_\mathfrak{p}$. We claim that $R_\mathfrak{p}$ is also a field. Indeed, to this, it suffices to show that the maximal ideal $\mathfrak{p}R_\mathfrak{p} = 0$. Let $x/s \in \mathfrak{p}R_\mathfrak{p}$ where $x \in \mathfrak{p}$ and $s \notin \mathfrak{p}$. Choose $y \in R$ such that $xyx = x$. In other words, we have have $(xy-1)x = 0$. Note that $xy - 1 \notin \mathfrak{p}$ since $xy \in \mathfrak{p}$ and $1 \notin \mathfrak{p}$. It follows that $x/s = 0$ in $R_\mathfrak{p}$. Therefore $\mathfrak{p}R_\mathfrak{p} = 0$. In particular, it follows that $R_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Finally, since localization is exact, we already have $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p}$. Thus

$$R_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p} \cong R/\mathfrak{p}.$$

## Problem 4

**Exercise 4.** Let $R$ be an integral domain. Then $R$ is a unique factorization domain if and only if $R[X]$ is a unique factorization domain.

We give two solutions.

**Solution 4.** First suppose $R[X]$ is a unique factorization domain. Let $a$ be a nonzero nonunit in $R$. Then viewing $a$ as a constant polynomial in $R[X]$ we see that $a$ has an irreducible factorization, say

$$a = p_1(X) \cdots p_m(X) \tag{1}$$

where $p_1, \ldots, p_m$ are irreducible polynomials in $R[X]$. By taking degrees on both sides of (1), we obtain

$$0 = \deg(p_1 \cdots p_m) = \deg p_1 + \cdots + \deg p_m, \tag{2}$$

where we used the fact that $R$ is a domain to get the equality on the right in (2). In particular, $\deg p_i = 0$ for all $1 \leq i \leq m$. Thus each $p_i$ is a constant polynomial. Irreducible constant polynomials in $R[X]$ are precisely the irreducible elements in $R$, so (1) is an irreducible factorization in $R$. Furthermore, the factorization (1) is unique since $R[X]$ is a unique factorization domain.

Now suppose $R$ is a unique factorization domain. Let $f(X)$ be a nonzero nonunit in $R[X]$ and let $K$ be the fraction field of $R$. First note that $R[X]$ is Noetherian, and thus $f$ has an irreducible factorization (see Appendix for proof of this). Suppose

$$p_1(X) \cdots p_m(X) = f(T) = q_1(X) \cdots q_n(X)$$

are two irreducible factorizations of $f$ in $R[X]$. By Gauss' Lemma, each $p_i$ and $q_j$ is irreducible in $K[X]$. Since $K[X]$ is a unique factorization domain, we see that $m = n$ and (perhaps after relabeling) $p_i \sim q_i$ in $K[X]$. In particular, $p_i = x_i q_i$ for some $x_i \in K[X]^\times = K^\times$. Note that since $p_i, q_i \in R[X]$, we must have $x_i \in R \backslash \{0\}$. Therefore

$$\begin{aligned} 0 &= p_1(X) \cdots p_m(X) - q_1(X) \cdots q_m(X) \\ &= p_1(X) \cdots p_m(X) - x_1 \cdots x_m p_1(X) \cdots p_m(X) \\ &= p_1(X) \cdots p_m(X)(1 - x_1 \cdots x_m) \\ &= f(X)(1 - x_1 \cdots x_m), \end{aligned}$$

and since $f \neq 0$ and $R[X]$ is a domain, this implies $1 = x_1 \cdots x_n$, which implies each $x_i$ is a unit in $R$. Thus $p_i \sim q_i$ in $R[X]$. It follows that $R[X]$ is a unique factorization domain.

**Solution 5.** By the same proof as in the solution above, we see that if $R[X]$ is a unique factorization domain, then $R$ is a unique factorization domain. We want to give an alternative proof for the converse direction. Suppose $R$ is a unique factorization domain. Let $\mathfrak{q}$ be a prime ideal in $R[X]$. Then $\mathfrak{q} \cap R$ is a prime ideal in $R$. Since $R$ is a unique factorization domain, there exists a prime element of $R$ which is contained in $\mathfrak{q} \cap R$, say $a \in \mathfrak{q} \cap R$. Then observe that $a$ is a prime element of $R[X]$ which is contained in $\mathfrak{q}$. Indeed, suppose $a = fg$ where $f, g \in R[X]$. By taking degrees on both sides, we see that $\deg f = 0 = \deg g$. Thus $f, g \in R$, which means either $f \in \langle a \rangle$ or $g \in \langle a \rangle$. Hence $a$ is a prime element of $R[X]$. It follows that every prime ideal in $R[X]$ contains a prime element, thus $R[X]$ is a unique factorization domain.

## Problem 5

**Exercise 5.** Let $R$ be a commutative ring with identity. Characterize $(R[X])^\times$.

**Solution 6.** Let $f(X) \in R[X]$ and it express it as

$$f(X) = a_m X^m + \cdots + a_1 X + a_0$$

where $a_0, a_1, \ldots, a_m \in R$. We claim that $f$ is a unit in $R[X]$ if and only if $a_0$ is a unit in $R$ and $a_i$ is nilpotent for all $1 \leq i \leq m$.

To see this, first suppose $a_0$ is a unit in $R$ and $a_i$ is nilpotent for all $1 \leq i \leq m$. Then each $a_i X^i$ is also nilpotent, and since the sum of two nilpotent elements is nilpotent, we see that $\sum_{i=1}^m a_i X^i$ is nilpotent. Also since $a_0$ is a unit in $R$, it is also a unit in $R[X]$. So $f$ is the sum of a unit plus a nilpotent element. This implies $f$ is a unit since the sum of a unit plus a nilpotent element is always a unit (if $u$ is a unit with $uv = 1$, and $\varepsilon$ is a nilpotent element with $\varepsilon^m = 0$, then $(u + \varepsilon) \sum_{i=1}^m v^i \varepsilon^{i-1} = 1$). This establishes one direction.

For the reverse direction, suppose $f$ is a unit in $R[X]$. We consider two steps:

**Step 1:** Assume that $R$ is a domain. In this case, we want to show that $a_0$ is a unit in $R$ and $a_i = 0$ for all $1 \leq i \leq m$. To see this, first we assume for a contradiction that $a_i \neq 0$ for some $1 \leq i \leq m$. By relabeling if necessary, we may in fact that assume $a_m \neq 0$ where $a_m$ is the lead coefficient of $f$. Now let $g(X) \in R[X]$ such that $fg = 1$ and it express it as

$$g(X) = b_n X^n + \cdots + b_1 X + b_0$$

where $b_0, b_1, \ldots, b_n \in R$ and $b_n \neq 0$. Then the lead term of $fg$ is just $a_m b_n X^{m+n}$ since $a_m \neq 0$ and $b_n \neq 0$ and $R$ is a domain. This is a contradiction since $fg = 1$ and $m + n \geq 1$. Thus we must have $a_i = 0$ for all $1 \leq i \leq m$. By the same proof, we must also have $b_j = 0$ for all $1 \leq j \leq n$. Thus $f(X) = a_0$ and $g(X) = b_0$, and $fg = 1$ implies $a_0 b_0 = 1$ which implies $a_0$ is a unit.

**Step 2:** Now we consider the more general case where $R$ may not be a domain. First, to see why $a_0$ is a unit, note that $a_0$ is in the image of the unit $f$ under the evaluation map $\mathrm{ev}_0 \colon R[X] \to R$, where $\mathrm{ev}_0$ is given by $\mathrm{ev}_0(h) = h(0)$ for all $h(X) \in R[X]$. Thus $a_0 = \mathrm{ev}_0(f)$ is a unit since $f$ is a unit and $\mathrm{ev}_0$ is a ring homomorphism (which preserves the identity element). Next, to see why $a_i$ is nilpotent for all $1 \leq i \leq m$, first note that for any prime ideal $\mathfrak{p}$ in $R$, the quotient $R/\mathfrak{p}$ is a domain. Since $f$ is a unit in $R[X]$, its image $\overline{f}$ is a unit in $(R/\mathfrak{p})[X]$. Since $\overline{f}$ is obtained from $f$ by reducing coefficients modulo $\mathfrak{p}$, we see from step 1 above that $a_i \in \mathfrak{p}$ for all $1 \leq i \leq m$. Since $\mathfrak{p}$ was arbitrary, we see that

$$a_i \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, R} \mathfrak{p} = \mathrm{N}(R)$$

where $\mathrm{N}(R)$ is the set of all nilpotents in $R$ (see homework 1 for why $\bigcap \mathfrak{p} = \mathrm{N}(R)$).

## Problem 6

**Definition 0.2.** Let $R$ be a commutative ring with identity and let $(I_\lambda)_{\lambda \in \Lambda}$ be a chain of ideals between the ideals $I \subseteq J$. We say $(I_\lambda)$ is **maximal** if any ideal $\mathfrak{a} \subseteq R$ that is comparable to every ideal in $(I_\lambda)$, must in fact belong to $(I_\lambda)$.

**Exercise 6.** Show that for any ideals $I \subseteq J$, there is a maximal chain of ideals between $I$ and $J$ (inclusive of $I$ and $J$).

**Solution 7.** If $I = J$, then clearly $(I, J)$ is a maximal chain, so assume $I \subset J$ is a proper inclusion. Let $\mathcal{F}$ be the family of all chains of ideals between $I$ and $J$ which include $I$ and $J$. Thus $(I_\lambda)_{\lambda \in \Lambda} \in \mathcal{F}$ means the following:

- $\Lambda$ is a totally ordered set with a minimal and maximal element. To each $\lambda \in \Lambda$ we have an ideal $I_\lambda$ such that if $\lambda < \mu$[1], then $I_\lambda \subset I_\mu$, where the inclusion is proper. If $\lambda_0$ is the minimal element of $\Lambda$ and $\lambda_1$ is the maximal element of $\Lambda$, then $I = I_{\lambda_0}$ and $J = I_{\lambda_1}$,

We give $\mathcal{F}$ the structure of a partially ordered set via set inclusion. In particular, if this means that if $(I_\lambda)_{\lambda \in \Lambda}$ and $(I_{\lambda'})_{\lambda' \in \Lambda'}$ are two members of $\mathcal{F}$, then we say $(I_\lambda)_{\lambda \in \Lambda} \subseteq (I_{\lambda'})_{\lambda' \in \Lambda'}$ if $\Gamma \subseteq \Lambda$, or in other words, if every member of $(I_\lambda)_{\lambda \in \Lambda}$ is also a member of $(I_{\lambda'})_{\lambda' \in \Lambda'}$. We say the chain $(I_{\lambda'})_{\lambda' \in \Lambda'}$ is **larger** than the chain $(I_\lambda)_{\lambda \in \Lambda}$ if $(I_\lambda)_{\lambda \in \Lambda} \subseteq (I_{\lambda'})_{\lambda' \in \Lambda'}$ and $(I_{\lambda'})_{\lambda' \in \Lambda'} \not\subseteq (I_\lambda)_{\lambda \in \Lambda}$.

Note that $\mathcal{F}$ is nonempty since $(I, J) \in \mathcal{F}$. We claim that every totally ordered subset of $\mathcal{F}$ has an upper bound. Indeed, let

$$((I_\lambda)_{\lambda \in \Lambda(\alpha)})_{\alpha \in A} \tag{3}$$

be a totally ordered subset of $\mathcal{F}$. In detail, this means:

- $A$ is a totally ordered set. To each $\alpha \in A$, we have a chain of ideals $(I_\lambda)_{\lambda \in \Lambda(\alpha)}$ such that if $\alpha < \beta$, then $\Lambda(\alpha) \subset \Lambda(\beta)$ where this inclusion is strict.

Clearly an upper bound of (3) is given by

$$(I_\lambda)_{\lambda \in \bigcup_{\alpha \in A} \Lambda(\alpha)}.$$

Thus $\mathcal{F}$ is nonempty and every totally ordered subset of $\mathcal{F}$ has an upper bound. It follows from Zorn's Lemma that $\mathcal{F}$ has a maximal element, say $(I_\lambda)_{\lambda \in \Lambda}$. In fact, $(I_\lambda)_{\lambda \in \Lambda}$ is maximal in the sense of Definition (0.2). To see this, assume for a contradiction that $(I_\lambda)_{\lambda \in \Lambda}$ is note maximal in the sense of Definition (0.2). Then there exists an ideal $\mathfrak{a}$ in $R$ such that $\mathfrak{a}$ is suppose $\mathfrak{a}$ is comparable to every ideal in $(I_\lambda)_{\lambda \in \Lambda}$ and $\mathfrak{a} \neq I_\lambda$ for any $\lambda \in \Lambda$. Define $\widetilde{\Lambda} = \Lambda \cup \{\widetilde{\lambda}\}$ and set $I_{\widetilde{\lambda}} = \mathfrak{a}$ . Then observe that chain $(I_\lambda)_{\lambda \in \widetilde{\Lambda}}$ is larger than $(I_\lambda)_{\lambda \in \Lambda}$, contradicting maximality of $(I_\lambda)_{\lambda \in \Lambda}$. Thus $(I_\lambda)_{\lambda \in \Lambda}$ is maximal in the sense of Definition (0.2). Furthermore, the chain $(I_\lambda)_{\lambda \in \Lambda}$ contains $I$ and $J$ by definition of $\mathcal{F}$, so we are done.

[1]Note by $\lambda < \mu$ we mean $\lambda \leq \mu$ and $\lambda \neq \mu$

# Appendix

## Nonzero Nonunits in Noetherian Domains have Irreducible Factorizations

**Proposition 0.1.** *Let $R$ be a Noetherian domain and let $a$ be a nonzero nonunit in $R$. Then $a$ has an irreducible factorization.*

*Proof.* If $a$ is irreducible, then we are done, so assume that $a$ is reducible. We assume for a contradiction that $a$ cannot be factored into irreducible. Since $a$ is reducible, there is a factorization of $a$ into nonzero nonunits, say

$$a = a_1 b_1.$$

If both $a_1$ and $b_1$ can be factored into irreducibles, then so can $a$, so at least one of them cannot be factored into irreducible elements, say $a_1$. In particular, $a_1$ is reducible, and thus there is factorization of $a_1$ into nonzero nonunits, say

$$a_1 = a_2 b_2.$$

By the same reasoning above, we may assume that $a_2$ cannot be factored into irreducibles. Proceeding inductively, we construct sequences $(a_n)$ and $(b_n)$ in $R$ where each $a_n$ is reducible and each $b_n$ is a nonzero nonunit, furthermore we have the factorization

$$a_n = a_{n+1} b_{n+1}$$

for all $n \in \mathbb{N}$. In particular, we have an ascending chain of ideals $(\langle a_n \rangle)$. Indeed, $\langle a_n \rangle \subseteq \langle a_{n+1} \rangle$ because $a_n = a_{n+1} b_{n+1}$. Since $R$ is Noetherian, this ascending chain must terminate, say at $N \in \mathbb{N}$. In particular, we have $\langle a_N \rangle = \langle a_{N+1} \rangle$. This implies there exists $c_N \in R$ such that

$$a_N c_N = a_{N+1}.$$

Thus we have

$$\begin{aligned} 0 &= a_N - a_{N+1} b_{N+1} \\ &= a_N - a_N c_N b_{N+1} \\ &= a_N (1 - c_N b_{N+1}). \end{aligned}$$

Since $R$ is an integral domain, this implies $b_{N+1} c_N = 1$ (as $a_N \neq 0$), which implies $b_{N+1}$ is a unit. This is a contradiction. $\square$