Goldbach Rings

1 Introduction

Goldbach's conjecture is one of the oldest unsolved problems in number theory. It states

Conjecture 1. *Every even integer greater than* 2 *can be expressed as the sum of two primes.*

In this article we use tools and methods from commutative algebra in order to study this and related conjectures.

2 Preliminary Material

3 The Goldbach Ring

Throughout this article, let K be a field. We are going to consider polynomial rings over K with infinitely variables. In particular, let

$$\begin{aligned} R &= K[\{x_p, x_{2k} \mid p \text{ odd prime and } k \in \mathbb{Z}_{\geq 3}\}] \\ I &= \langle \{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\} \rangle \\ G &= R/I. \end{aligned}$$

We will refer to G as the **Goldbach ring**. We simplify our notation by writing $\{x_p, x_{2k}\}$ to denote the set $\{x_p, x_{2k} \mid p \text{ odd prime and } k \in \mathbb{Z}_{\geq 3}\}$. Similarly, we write $\{x_p x_q - x_{p+q}\}$ to denote the set $\{x_p x_q - x_{p+q} \mid p, q \text{ odd primes}\}$.

3.1 Representing Monomials

We will denote by $\mathcal{M}$ to be the set of all monomials in R. There are two ways we can represent monomials in R. The first way is as a finite product of the indeterminates $\{x_p, x_{2k}\}$, namely, a monomial can be expressed in the form

$$x_{p_1} \cdots x_{p_r} x_{2k_1} \cdots x_{2k_s}$$

where $p_1, \dots, p_r$ are odd primes (not necessarily distinct) and $2k_1, \dots, 2k_s$ are even integers greater than or equal to 6 (not necessarily distinct). We will use the way of representing monomials to give R a graded structure.

The second way of representing monomials is described as follows: given a function $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$, we define its **support**, denoted $\operatorname{supp} \alpha$, to be the set

$$\operatorname{supp} \alpha = \{m \in \mathbb{N} \mid \alpha(m) \neq 0\}.$$

We denote by $\mathcal{F}$ to be the set

$$\mathcal{F} = \{\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0} \mid \operatorname{supp} \alpha \text{ is finite and contained in } \{x_p, x_{2k}\}\}.$$

Thus if $\alpha \in \mathcal{F}$, then α takes value 0 zero almost everywhere, and the only places where it is nonzero is at an odd prime or an even greater than or equal to 6. Then there is a bijection from $\mathcal{F}$ to $\mathcal{M}$ given by assigning $\alpha \in \mathcal{F}$ to the monomial

$$x^\alpha := \prod_{m \in \mathbb{N}} x_m^{\alpha(m)} = \prod_{m \in \operatorname{supp} \alpha} x_m^{\alpha(m)}.$$

For instance, suppose $\alpha \colon \mathbb{N} \to \mathbb{Z}_{\geq 0}$ is defined by

$$\alpha(m) = \begin{cases} 3 & \text{if } m = 2 \\ 2 & \text{if } m = 6 \\ 4 & \text{if } m = 11 \\ 0 & \text{if } m \in \mathbb{N} \backslash \{2, 6, 11\} \end{cases}$$

Then $x^\alpha = x_3^3 x_6^2 x_{11}^4$ and $\operatorname{supp} x^\alpha = \{2, 6, 11\}$. This second way of expressing monimals gives us a cleaner way of expressing nonzero polynomials in R, namely, every nonzero polynomial $f \in R$ can be expressed in the form

$$f = a_1 x^{\alpha_1} + \cdots + a_n x^{\alpha_n}$$

for unique $a_1, \ldots a_n \in K$ and for unique $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$. We often pass back and forth between functions $\alpha \in \mathcal{F}$ and monimals $x^\alpha \in \mathcal{M}$. For instance, given a monimal $x^\alpha \in \mathcal{M}$, we define its **support**, denoted $\operatorname{supp} x^\alpha$, to be $\operatorname{supp} x^\alpha = \operatorname{supp} \alpha$.

3.1.1 Graded Ring Structure on R

We describe a graded structure on R using this representation of monoimals, namely, we define $\deg_1 \colon \mathcal{M} \to \mathbb{N}$ and $\deg_2 \colon \mathcal{M} \to \mathbb{N}$ by

$$\deg_1(x_{p_1} \cdots x_{p_r} x_{2k_1} \cdots x_{2k_s}) = \sum_{i=1}^{r} p_r + 2 \sum_{j=1}^{s} k_j \qquad \text{and} \qquad \deg_2(x_{p_1} \cdots x_{p_r} x_{2k_1} \cdots x_{2k_s}) = r + 2s.$$

In particular, we have

$$\begin{aligned} \deg_1(x_p) &= p \\ \deg_1(x_{2k}) &= 2k \\ \deg_2(x_p) &= 1 \\ \deg_2(x_{2k}) &= 2 \end{aligned}$$

In terms of our other notation, we have

$$\deg_1 x^\alpha = \sum_{m \in \mathbb{N}} m\alpha(m) \qquad \text{and} \qquad \deg_2 x^\alpha = \sum_{p \text{ odd prime}} \alpha(p) + 2 \sum_{m \text{ even } \geq 6} \alpha(m).$$

Next, for each $n \in \mathbb{N}$, we set

$$R_n = \operatorname{span}_K \{x^\alpha \in \mathcal{M} \mid \deg_1 x^\alpha = n\}.$$

Also for each $n, d \in \mathbb{N}$, we set

$$R_{n,d} = \operatorname{span}_K \{x^\alpha \in \mathcal{M} \mid \deg_1 x^\alpha = n \text{ and } \deg_2 x^\alpha = d\}.$$

For instance, we have

$$\begin{aligned} R_{14} &= Kx_3x_3x_3x_5 + Kx_6x_8 + Kx_3x_5x_6 + Kx_7x_7 + Kx_3x_{11} \\ R_{14,4} &= Kx_3x_3x_3x_5 + Kx_6x_8 + Kx_3x_5x_6 \\ R_{14,2} &= Kx_7x_7 + Kx_3x_{11} \end{aligned}$$

Observe that $R_n R_{n'} \subseteq R_{n+n'}$, so R has the structure of a graded ring

$$R = \bigoplus_{n \in \mathbb{N}} R_n,$$

moreover we also have $R_{n,d} R_{n',d'} \subseteq R_{n+n',d+d'}$, so we can refine this grading

$$R = \bigoplus_{n,d \in \mathbb{N}} R_{n,d}.$$

Note that the ideal I is homogeneous with respect to this grading, and thus G inherits both of these gradings. In particular, we have

$$G = \bigoplus_{n \in \mathbb{N}} G_n \quad \text{and} \quad G = \bigoplus_{n,d \in \mathbb{N}} G_{n,d},$$

where $G_n = R_n / I \cap R_n$ and $G_{n,d} = R_{n,d} / I \cap R_{n,d}$. We will be interested in describing the dimensions of the K-vector spaces R_n, G_n, $R_{n,d}$, and $G_{n,d}$.

4 Conjecture

Let us rephrase Goldbach's conjecture to a statement about the graded ring R.

Conjecture 2. *For each $k \geq 3$, we have $\dim_K R_{2k,2} > 1$.*

We now make the following conjecture about the graded ring G.

Conjecture 3. *For each*