Convergence of Sequences of Functions and Power Series

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Contents

1 Series

1.1 Basic Definitions

Let (a_n) be a sequence of complex numbers in $\mathbb{C}$. The **series with nth term a_n**, denoted $\sum a_n$, is defined to be the sequence of the partial sums

$$\sum a_n := \left(\sum_{m=1}^{n} a_n\right)$$

If the limit of $\sum a_n$ exists, then this limit is called the **sum** of the series $\sum a_n$ and is denoted

$$\sum_{n=1}^{\infty} a_n.$$

If the limit of $\sum a_n$ exists and is a complex number, then we say $\sum a_n$ is **convergent**. Otherwise, we will say $\sum a_n$ is **divergent**.

1.2 Absolute Convergence of a Series

Let (a_n) be a sequence of complex numbers in $\mathbb{C}$. If the limit of $\sum |a_n|$ converges, then we say $\sum a_n$ is **absolutely convergent**.

Lemma 1.1. *Let (a_n) be a sequence of positive real numbers and assume the series $\sum a_n$ converges, say to a. For every permutaiton π of the index set, the series $\sum a_{\pi(n)}$ also converges to a.*

Proof. Let $\varepsilon > 0$ and let π be a permutation of the index set. Choose $M \in \mathbb{N}$ such that $N \geq M$ implies

$$a - \varepsilon \leq \sum_{n=1}^{N} a_n \leq a + \varepsilon.$$

The permutation π takes on all values $1, 2, \ldots, M$ among some initial segment of the positive integers, say

$$\{1, 2, \ldots, M\} \subset \{\pi(1), \pi(2), \ldots, \pi(K)\}$$

for some K. In particular, for $N \geq K$, the set $\{a_{\pi(1)}, \ldots, a_{\pi(N)}\}$ contains $\{a_1, \ldots, a_M\}$. Let J be the maximal value of $\pi(n)$ for $n \leq N$. So for $N \geq K$,

$$a - \varepsilon \leq \sum_{n=1}^{M} a_n \leq \sum_{n=1}^{N} a_{\pi(n)} \leq \sum_{n=1}^{J} a_n \leq a + \varepsilon. \tag{1}$$

So for every ε, $\sum_{n=1}^{N} a_{\pi(n)}$ is within ε of a for all large N. Therefore $\sum_{n=1}^{\infty} a_{\pi(n)} = a$. □

Theorem 1.2. *If $f(z) = \sum c_n z^n$ converges at the point z_0, then $f(z_0)$ is the limit of $f(z)$ as $z \to z_0$ along a radial path from the origin. In particular, if $\sum c_n$ converges, then*

$$\lim_{x \to 1^-} \sum c_n x^n = \sum c_n$$

Proof. The case of a series at z_0 is easily reduced to the case $z_0 = 1$ by a scaling and a rotation. So we assume $z_0 = 1$. Since $\sum c_n z^n$ converges at $z = 1$, the series converges on the open unit disc. Let $b_n = c_0 + \cdots + c_n$, $b = \lim_{n\to\infty} b_n$, $0 < x < 1$. Then

$$\sum_{n=0}^{N} c_n x^n = \sum_{n=0}^{N} b_n x^n - x \sum_{n=0}^{N-1} b_n x^n = (1 - x) \sum_{n=0}^{N-1} b_n x^n + b_N x^N.$$

Let $N \to \infty$. Since $b_N \to b$ and $x^N \to 0$, we get

$$\sum_{n \geq 0} c_n x^n = (1 - x) \sum_{n \geq 0} b_n x^n.$$

Since $\sum c_n x^n - b = (1 - x) \sum (b_n - b) x^n$, we choose $\varepsilon > 0$ and then M so that $|b_n - b| \leq \varepsilon$ for $n > M$. Then

$$\left| \sum_{n \geq 0} c_n x^n - b \right| \leq (1 - x) \sum_{n=0}^{M} |b_n - b| \, x^n + \varepsilon \leqq (1 - x) \sum_{n=0}^{M} |b_n - b| + \varepsilon.$$

For $|x - 1|$ small enough, the first term on the right side can be made $\leq \varepsilon$. □

2 Convergence of Sequences of Functions

Definition 2.1. Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})_{n\in\mathbb{N}}$ be a sequence of functions. Then

1. The sequence (f_n) converges **pointwise** on D to a function f if for all $z \in D$ and for all $\varepsilon > 0$ there exists $N_z \in \mathbb{N}$ (which depends on $z \in D$) such that
$$n \geq N_z \text{ implies } |f_n(z) - f(z)| < \varepsilon$$
2. The sequence (f_n) converges **uniformly** on D to a function f if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (which does *not* depend a particular $z \in D$) such that
$$n \geq N \text{ implies } |f_n(z) - f(z)| < \varepsilon$$
for all $z \in D$. We say (f_n) converges **locally uniformly** on D if every $z_0 \in D$ has an open neighborhood U for which (f_n) converges uniformly on $U \cap S$.
3. The sequence (f_n) is **uniformly Cauchy** on D if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$m, n \geq N \text{ implies } |f_m(z) - f_n(z)| < \varepsilon$$
for all $z \in D$.
4. The series $\sum f_n$ converges **pointwise** on D to a function f if the sequence of partial sums $(\sum_{m=1}^{n} f_m)_{n\in\mathbb{N}}$ converges uniformly on D to f.
5. The series $\sum f_n$ converges **uniformly** on D to a function f if the sequence of partial sums $(\sum_{m=1}^{n} f_m)_{n\in\mathbb{N}}$ converges uniformly on D to f.

The main advantage in determining whether or not a sequence of functions (f_n) is uniformly Cauchy is that we do not need to know what (f_n) converges to. In contrast, the definition of (f_n) converging uniformly assumes that we already know what it converges to from the outset. Fortunately, since $\mathbb{C}$ is complete, we only need to know that (f_n) is uniformly Cauchy to determine whether it converges uniformly or not.

Because $\mathbb{C}$ is locally compact, locally uniform convergence is the same thing as compact convergence:

Proposition 2.1. *Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})_{n\in\mathbb{N}}$ be a sequence of functions. Then (f_n) converges locally uniformly to f on D if and only if (f_n) converges uniformly to f on every compact subset of D.*

Proof. Suppose that (f_n) converges locally uniformly to f on D. Let K be a compact subset of D. For each $x \in K$, choose an open neighborhood U_x of x such that (f_n) converges uniformly on $U_x \cap D$. Choose $x_1, \dots, x_n \in K$ such that $\{U_{x_i}\}_{i=1}^{n}$ covers K (that such a choice exists follows from compactness of K). Then since (f_n) converges uniformly on $U_{x_i} \cap D$ for each $i = 1, \dots, n$, it also converges uniformly on the *finite* union $\bigcup_{i=1}^{n} U_{x_i} \cap D \supseteq K$.

Now suppose that (f_n) converges uniformly to f on every compact subset of D. Let $x \in D$. Since D is locally compact, there exists a compact neighborhood of x. Choose an open U of D and a compact subset K of D such that

□

Theorem 2.1. *Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})_{n\in\mathbb{N}}$ be a sequence of functions.*

1. *The sequence (f_n) converges uniformly on D to a function $f \colon D \to \mathbb{C}$ if and only if (f_n) is uniformly cauchy on D.*
2. *(Weierstrass M-test) Suppose that for each $n \in \mathbb{N}$ there exists $M_n \in [0, \infty)$ such that $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(z)| \leq M_n$ for all $z \in D$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on D.*

Proof.

1. First we assume that (f_n) is uniformly cauchy on D. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that
$$m, n \geq N \text{ implies } |f_m(z) - f_n(z)| < \varepsilon \tag{2}$$

for all $z \in D$. Then for each $z \in D$, the sequence $(f_n(z))$ is a Cauchy sequence in $\mathbb{C}$, and by completeness of $\mathbb{C}$, it must converge to a limit. Let $f(z)$ denote this limit. As we vary $z \in D$, we obtain a function $f \colon D \to \mathbb{C}$, given by
$$f(z) := \lim_{n\to\infty} f_n(z).$$

Clearly (f_n) converges pointwise to $f\colon D \to \mathbb{C}$. To see that it converges *uniformly* to f, we fix $m \in \mathbb{N}$ and let $n \to \infty$ in (2) and we see that

$$m \geq N \text{ implies } |f_m(z) - f(z)| \leq \varepsilon$$

for all $z \in D$.

Now we assume that (f_n) converges uniformly on D to a function $f\colon D \to \mathbb{C}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } |f_n(z) - f(z)| < \frac{\varepsilon}{2}$$

for all $z \in D$. Then for all $m, n \geq N$, we have

$$\begin{aligned}
|f_m(z) - f_n(z)| &\leq |f_m(z) - f(z)| + |f_n(z) - f(z)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

for all $z \in D$. Thus, (f_n) is uniformly cauchy.

2. By 1, it suffices to show that the sequence $(\sum_{m=1}^{n} f_m)_{n\in\mathbb{N}}$ of partial sums is uniformly Cauchy on D. Let $\varepsilon > 0$. Since the series $\sum_{n=1}^{\infty} M_n$ converges, the sequence $(\sum_{k=1}^{n} M_k)_{k\in\mathbb{N}}$ of partial sums is necessarily a Cauchy sequence. Therefore, there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \text{ implies } \left|\sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k\right| < \varepsilon.$$

In particular, $m, n \geq N$ implies

$$\begin{aligned}
\left|\sum_{k=1}^{m} f_k(z) - \sum_{k=1}^{n} f_k(z)\right| &= \left|\sum_{k=m+1}^{n} f_k(z)\right| \\
&\leq \sum_{k=m+1}^{n} |f_k(z)| \\
&\leq \sum_{k=m+1}^{n} M_k \\
&= \left|\sum_{k=m+1}^{n} M_k\right| \\
&= \left|\sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k\right| \\
&< \varepsilon.
\end{aligned}$$

for all $z \in D$. □

2.1 Uniform Norm

Proposition 2.2. *Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n\colon D \to \mathbb{C})$ be a sequence of continuous functions. If (f_n) converges to f uniformly on D, then f is continuous on D.*

Proof. Choose any $a \in D$. We will show that f is continuous at a. Let $\varepsilon > 0$. Since (f_n) converges to f uniformly, there exists $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon/3$ for all $n \geq N$ and $z \in D$. Since f_N is continuous at a, there exists $\delta > 0$ such that $|z - a| < \delta$ implies $|f_N(z) - f_N(a)| < \varepsilon/3$. Combining these together, we see that $|z - a| < \delta$ implies

$$\begin{aligned}
|f(z) - f(a)| &\leq |f(z) - f_N(z))| + |f_N(z) - f_N(a)| + |f_N(a) - f(a)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

□

Examining the proof in Proposition (2.2) reveals that we can weaken the hypothesis under certain conditions. Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{B}(K, \mathbb{C})$ be the $\mathbb{C}$-vector space set of all bounded functions from D to $\mathbb{C}$. We define the **uniform norm** on $\mathcal{B}(K, \mathbb{C})$ by

$$\|f\|_K = \sup \{|f(x)| \mid x \in K\}$$

for all $f \in \mathcal{B}(K, \mathbb{C})$. The pair $(\mathcal{B}(K, \mathbb{C}), \|\cdot\|_K)$ is easily checked to be a normed vector space. This normed vector space gives rise to a metric space in the usual way. Namely, we define the metric $d_K \colon \mathcal{B}(K, \mathbb{C}) \times \mathcal{B}(K, \mathbb{C}) \to \mathbb{R}$ by

$$d_K(f, g) = \|f - g\|_K$$

for all $f, g \in \mathcal{B}(K, \mathbb{C})$. A sequence $(f_n \colon K \to \mathbb{C})_{n \in \mathbb{N}}$ of bounded functions converges *uniformly* to a function f (which must necessarily be bounded) if and only if it converges to f with respect to d_K. This is where the name *uniform* norm comes from.

Proposition 2.3. *Let K be a nonempty compact subset of $\mathbb{C}$ and let $(f_n \colon K \to \mathbb{C})_{n \in \mathbb{N}}$ be of continuous functions in $(\mathcal{B}(K, \mathbb{C}), d_K)$. If f is a limit point of (f_n), then f is continuous on D.*

3 Power Series

Let a be a complex number. A **power series centered at** a is a series of the form $\sum a_n(z-a)^n$, where z is a complex variable, a is a given complex number, and (a_n) is a sequence of complex numbers. In this section, we will show that a power series $\sum a_n(z-a)^n$ has a radius of convergence $R \geq 0$, and for any $r \geq 0$ such that $r < R$, we will see that the power series $\sum a_n(z-a)^n$ converges *absolutely* and *uniformly* for all $z \in B_r(a)$.

3.1 Limit Supremum

To study the convergence of a power series, we study the notion of the limit supremum of a positive real-valued sequence. Let (a_n) be a sequence of positive real numbers. We define the **limit supremum** of (a_n), denoted $\text{limsup}(a_n)$, to be

$$\text{limsup}(a_n) := \lim_{m \to \infty} (\sup\{a_n \mid n \geq m\}).$$

Since $\sup\{a_n \mid n \geq m\}$ is a non-increasing function of m, the limit always exists or equals $+\infty$.

Properties of Limit Supremum

Proposition 3.1. *Let (a_n) be a sequence of positive real-valued numbers.*

1. *If $\text{limsup}(a_n) = A$, then for each $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists some $n \geq N$ such that $a_n > A - \varepsilon$.*
2. *If $\text{limsup}(a_n) = A$, then for each $\varepsilon > 0$, there exists $N \in N$ such that $a_n < A + \varepsilon$ for all $n \geq N$.*
3. *Conversely, if $A \in \mathbb{R}_{\geq 0}$ satisfies 1 and 2, then $\text{limsup}(a_n) = A$.*

Proof.

1. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$. To obtain a contradiction, assume that there does not exist an $n \geq N$ such that $a_n > A - \varepsilon$. Then $A - \varepsilon > a_n$ for all $n > N$. This implies $\sup\{a_n \mid n \geq N\} < A$. This is a contradiction since $\sup\{a_n \mid n \geq m\}$ is a non-increasing function of m.
2. Let $\varepsilon > 0$. To obtain a contradiction, assume that there does not exist an $N \in \mathbb{N}$ such that $a_n < A + \varepsilon$ for all $n \geq N$. Then $\sup\{a_n \mid n \geq N\} \geq A + \varepsilon$ for all $N \in \mathbb{N}$. This implies $\text{limsup}(a_n) \geq A + \varepsilon$, which is a contradiction.
3. Let $A' = \text{limsup}(a_n)$. Assume that $A < A'$. Let $\varepsilon = A' - A$. Then by 2, there exists $N \in \mathbb{N}$ such that $a_n < A + \varepsilon/2$ for all $n \geq N$. So we choose such an $N \in \mathbb{N}$. On the other hand, by 1, there must exist an $n \geq N$ such that $a_n > A' - \varepsilon/2 = A + \varepsilon/2$. Contradiction. An analagous argument gives a contradiction when we assume $A > A'$. Therefore $A = A'$.

□

Lemma 3.1. *Let (a_n) and (b_n) be two sequences of positive real numbers such that $\text{limsup}(a_n) = A$ and $\lim(b_n) = B$. Then*

1. *$\text{limsup}(a_n b_n) = AB$*

2. *limsup*$(a_n + b_n) = A + B$

Proof.

1. Let $\nu > 0$ and let $\delta > 0$ such that $\delta A + \delta B + \delta^2 < \nu$. Choose $N \in \mathbb{N}$ such that $a_n < A + \delta$ and $b_n < B + \delta$ for all $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned} a_n b_n &< (A+\delta)(B+\delta) \\ &= AB + \delta A + \varepsilon B + \delta^2 \\ &< AB + \nu. \end{aligned}$$

Next, let $\varepsilon > 0$, let $N \in \mathbb{N}$, and set $\varepsilon' = \varepsilon/(A+B)$. Choose $n \geq N$ such that $a_n > A - \varepsilon'$ and $b_n > B - \varepsilon'$. Then

$$\begin{aligned} a_n b_n &> (A-\varepsilon')(B-\varepsilon') \\ &= AB - \varepsilon' A - \varepsilon' B + \varepsilon'^2 \\ &> AB - \varepsilon' A - \varepsilon' B \\ &= AB - \varepsilon'(A+B) \\ &= AB - \varepsilon. \end{aligned}$$

Therefore, we must have

$$\text{limsup}(a_n b_n) = AB.$$

2. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $a_n < A + \varepsilon/2$ and $b_n < B + \varepsilon/2$ for all $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned} a_n + b_n &< A + \varepsilon/2 + B + \varepsilon/2 \\ &= A + B + \varepsilon. \end{aligned}$$

Next, let $\varepsilon > 0$ and let $N \in \mathbb{N}$. Choose $n \geq N$ such that $a_n > A - \varepsilon/2$ and $b_n > B - \varepsilon/2$. Then

$$\begin{aligned} a_n + b_n &> A - \varepsilon/2 + B - \varepsilon/2 \\ &= A + B - \varepsilon \end{aligned}$$

Therefore, we must have

$$\text{limsup}(a_n + b_n) = A + B.$$

□

Example 3.1. Let (a_n) be a sequence of complex numbers and let $a \in \mathbb{C}$. Suppose that $\text{limsup}(|a_n|^{1/n}) = A$. Then since $\lim(n^{1/n}) = 1$, we have $\text{limsup}(|na_n|^{1/n}) = A$.

Limit Supremum Test of Convergence of Power Series

Theorem 3.2. *(Cauchy-Hadamard) Let (a_n) be a sequence of complex numbers and let $a \in \mathbb{C}$. Suppose that limsup$(|a_n|^{1/n}) = L$.*

1. *If $L = 0$, then the power series $\sum a_n(z-a)^n$ centered at a converges for all $z \in \mathbb{C}$.*
2. *If $L = \infty$, then the the power series $\sum a_n(z-a)^n$ centered at a converges for $z = 0$ only.*
3. *If $0 < L < \infty$, set $R = 1/L$. For any r with $0 < r < R$ the series $\sum a_n(z-a)^n$ converges absolutely and uniformly on the closed disk $\overline{B}_r(a)$ and diverges for $z \notin \overline{B}_R(a)$. In this case, R is called the **radius of convergence** of the power series.*

Proof. We only prove 3, leaving 1 and 2 as easy exercises. Choose r such that $0 < r < R$. Let $\varepsilon = (R-r)/2rR$ (so $r = 1/(L+2\varepsilon)$). Choose $N \in \mathbb{N}$ such that $|a_n|^{1/n} < L + \varepsilon$ for all $n \geq N$. Then

$$|a_n|^{1/n}|z-a| < \frac{L+\varepsilon}{L+2\varepsilon} \tag{3}$$

for all $z \in \overline{B}_r(a)$. Therefore, letting $M = (L+\varepsilon)/(L+2\varepsilon)$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n(z-a)^n| &= \sum_{n=1}^{N} |a_n(z-a)^n| + \sum_{n=N+1}^{\infty} |a_n(z-a)^n| \\ &\le \sum_{n=1}^{N} |a_n(z-a)^n| + \sum_{n=N+1}^{\infty} M^n \\ &\le \sum_{n=1}^{N} |a_n(z-a)^n| + \frac{1}{1-M}.\end{aligned}$$

for all $z \in \overline{B}_r(a)$. Thus, the series converges absolutely in $\overline{B}_r(a)$. The series also converges uniformly in $\overline{B}_r(a)$, by Weierstrass M-test, with $M_n = M^n$.

On the other hand, if $z \notin \overline{B}_R(a)$, then

$$\operatorname{limsup}\left(|a_n|^{1/n}|z-a|\right) > 1,$$

so that for infinitely many values of n, $a_n(z-a)^n$ has absolute value greater than 1 and thus $\sum a_n(z-a)^n$ diverges. □

Alternative Characterizations of R

Analyzing the proof of the Cauchy-Hadamard theorem we obtain an alternative characterization of the radius of convergence which avoids the limit superior: R is the supremum of all $r \geq 0$ for which the sequence $(r^n|a_n|)$ is bounded. The following refomulation of this statement is even more convenient in applications.

Lemma 3.3. *Let R be the radius of convergence of the power series $\sum a_n(z-a)^n$.*

1. *If $0 \le r < R$, then there exists a constant c such that for all $n \in \mathbb{N}$,*

$$r^n|a_n| \le c. \tag{4}$$

2. *If there exist positive numbers r and c such that (4) holds for all sufficiently large $n \in \mathbb{N}$, then $R \geq r$.*

Proof.

1. Setting $z = a + r$ in (3), we get the desired estimate with $c = 1$ for all sufficiently large n. By increasing c, if necessary, we can also capture the finite number of remaining a_n.

2. To prove the second result we remark that the estimate (4) implies

$$|a_n|^{1/n} \le \frac{|c|^{1/n}}{r} \to \frac{1}{r},$$

and apply Cauchy-Hadamard formula.

□

Power Series Examples

Example 3.2.

1. The power series $\sum_{n=1}^{\infty} nz^n$ centered at 0 has radius of convergence 1 since $\operatorname{limsup}(n^{1/n}) = 1$.

2. The power series $\sum_{n=0}^{\infty} z^{n^2}$ centered at 0 has radius of convergence 1 since $\operatorname{limsup}(a_n^{1/n}) = 1$, where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

3. The generating function for the Catalan numbers C_n is given by

$$f(z) = (z^2+z)^2+z)^2+z)^2+\cdots) = z + 2z^2 + 5z^3 + 14z^4 + \cdots.$$

Since $\operatorname{limsup}(C_n^{1/n}) = 4$, we see that $\sum C_n z^n$ has radius of convergence $1/4$.

Properties of Sums

Lemma 3.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series centered at a and suppose R is its radius of convergence. Then for all r such that $0 < r < R$, we have the estimate*

$$|a_n| \leq r^{-n} \|f(z)\|_{C_r(a)}.$$

for all $n \geq 0$.

Proof. The partial sum f_n of the power series is a polynomial of degree at most n, and hence the coefficient formula tells us that

$$r^n|a_n| \leq \frac{1}{n+1}\sum_{m=0}^{n} |f_n(r\omega^m)| \leq \sup_{|z-a|=r} |f_n(z)|.$$

Now the assertion follows since f_n converges uniformly to f on the disk $|z| \leq r$. □

Write $f_n(z) = \sum_{m=0}^{n} a_m(z-a)^m$. Then

$$f_n(z) = a_n z^n + g_n(z),$$

where $\deg(g_n) < m$. In particular, this implies

$$\begin{aligned} |f_n(z)| &= |a_n z^n + g_n(z)| \\ &\leq |a_n| r^n + |g_n(z)| \end{aligned}$$

$$|f_n(z)| \leq$$

Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$.

3.2 Functions Representable by a Power Series

A function f defined on an open set Ω is said to be **representable by a power series in** Ω if, whenever $a \in \Omega$ and $r > 0$ and the disk $B_r(a)$ is included in Ω, there exists a sequence (a_n) of complex numbers such that the equation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

holds for every $z \in B_r(a)$.

Proposition 3.2. *Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a function, and let $a \in \Omega$ and $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in B_r(a)$. Then $f'(z)$ exists for all $z \in B_r(a)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

Proof. Let $z \in B_r(a)$. Choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n\right) \\ &= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m\right) \\ &= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\ &= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\ &= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}. \end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m \colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned} |g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \right| \\ &< \left| m a_m r^{m-1} \right|. \end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. □

Corollary. *Let Ω be an open set, let $f \colon \Omega \to \mathbb{C}$ be a function, let $a \in \Omega$, and let $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $z \in B_r(a)$. Then $f^{(m)}(z)$ exists for all $m \geq 1$ and $z \in B_r(a)$, and moreover

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} (z-a)^n. \tag{5}$$

In particular, we have $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$.

Proof. The first part follows from an easy induction on m, with Proposition (3.2 giving the base case and the induction step. To get $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$, we set $z = a$ in 5). □