Scientific Computing Homework 5

Michael Nelson

Problem 1

Exercise 1. Set up the least squares system to find coefficients $\mathbf{x}^\top = (x_1, x_2)$ for fitting the model function

$$f_{\mathbf{x}}(t) = x_1 t + x_2 e^t$$

to the three data points $(1,2)$, $(2,3)$, $(3,5)$. Does this system have a unique solution?

Solution 1. We first attempt to solve the system of equations:

$$\begin{aligned} x_1 + ex_2 &= 2 \\ 2x_1 + e^2 x_2 &= 3 \\ 3x_{1'} + e^3 x_3 &= 5 \end{aligned}$$

If we set

$$A = \begin{pmatrix} 1 & e \\ 2 & e^2 \\ 3 & e^3 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix},$$

then this system is just $A\mathbf{x} = \mathbf{b}$. Note that $\text{rank} A = 2$, and thus $\text{rank}(A^\top A) = \text{rank}\, A = 2$. It follows that $A^\top A$ is nonsingular. Now recall that

$$\begin{aligned} \mathbf{x} \text{ is a critical point of } \|A\mathbf{x} - \mathbf{b}\|_2^2 &\iff \text{the gradient of } \|A\mathbf{x} - \mathbf{b}\|_2^2 \text{ vanishes} \\ &\iff A^\top A\mathbf{x} = A^\top \mathbf{b} \\ &\iff \mathbf{x} = (A^\top A)^{-1} A^\top \mathbf{b} \end{aligned}$$

Since $\|A\mathbf{x} - \mathbf{b}\|_2^2$ is a strictly convex function in $\mathbf{x}$, it follows that a crictical point of $\|A\mathbf{x} - \mathbf{b}\|_2^2$ is a global minimum. Thus the unique solutions is given by

$$\mathbf{x} = (A^\top A)^{-1} A^\top \mathbf{b}$$

Problem 2

Exercise 2. We consider the least squares problem

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

1. Determine the rank of A.
2. What happens if you form the normal equation and try to solve the system in MATLAB? Why?
3. Compute the SVD of A using MATLAB's svd command. Report the singular values. What is the rank if you count the $\sigma_i \neq 0$? That happens to be different than what you found in 1), why?

Solution 2. 1. We determine this using MATLAB using the code below

```
A = [−1 1 0; −1 0 1; 0 −1 1; −2 1 1];
rank(A)

ans = 2
```

Thus rank$A = 2$.

2. Continuing the code above, we now attempt to solve the system using MATLAB:

```
B = transpose(A);
C = B*A;
c = B*b;
C\c

Warning: Matrix is singular to working precision.
```

We obtain an error because $A^\top A$ is singular. This is because $\text{rank}(A^\top A) \neq 3$ since $\text{rank}(A^\top A) = \text{rank}A = 2$.

3. Now we compute the SVD of A:

```
svd(A)

ans = 3.0000      1.7321      0.0000
```

Thus the singular values of A are $\sigma_1 = 3$, $\sigma_2 = 1.7321$, and $\sigma_3 = 0$. Note that the number of singular values which is nonzero is 2, which agrees with what we found in part 1. The reason it does, is because the rank of any square matrix equals the number of nonzero eleigenvalues (with repetitions), so the number of nonzero singular salues of A equals the rank of $\text{rank}(A^\top A)$, but recall that $\text{rank}(A^\top A) = \text{rank}\,A$.

Problem 3

Exercise 3. Write a MATLAB function function x = least_square_svd(A,b,tol) that solves the least square problem $\min \|b - Ax\|_2^2$ using SVD as discussed in class, where the parameter tol is used to determine if a singular value is numerically zero (if $\sigma_i <$ tol). Note that you may use MATLAB's [U,E,V] = svd(A) to compute the SVD. You can test your routine on the system in question 2.

Solution 3.
```
function x = least_square_svd(A,b,tol)

sz = size(A);
m = sz(1);
n = sz(2);

[U,S,V] = svd(A);
s = svd(A);
k = length(s);

% determine value of numerical rank

r = 1;
for i=1:n
        if abs(s(i)) > tol
        r = r + 1;
        end

% calculate minizer

c = U'*b;
for i=r+1:m
        min = min + c(i)^2
        end
```

```
% return minizer
min;
```