Complex Integration

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Contents

1 Notation

1.1 Open and Closed Balls

Throughout these notes, I denotes the closed interval $[0,1]$ in $\mathbb{R}$. For all $r > 0$ and $z \in \mathbb{C}$, we write $B_r(z)$ (respectively $\overline{B}_r(z)$) to denote the open (respectively closed) ball centered at z and of radius r:

$$B_r(z) := \{w \in \mathbb{C} \mid |z-w| < r\} \quad \text{and} \quad \overline{B}_r(z) := \{w \in \mathbb{C} \mid |z-w| \leq r\}.$$

Similarly, for all $r > 0$ and $z \in \mathbb{C}$, we write $C_r(z)$ to denote the circle centered at z and of radius r:

$$C_r(z) := \{w \in \mathbb{C} \mid |z-w| = r\}.$$

1.2 Complex Numbers

Let z be a nonzero complex number. A **polar representation** of z is given by

$$z := |z|e^{2\pi i(\mathrm{Arg}(z)+n)}$$

for a unique $|z| \in \mathbb{R}_{\geq 0}$, a unique $\mathrm{Arg}(z) \in [-\pi, \pi)$, and a choice of $n \in \mathbb{Z}$. We call $|z|$ the **modulus** of z. We define the **argument** of z to be the set

$$\arg(z) := \{\mathrm{Arg}(z) + n \mid n \in \mathbb{Z}\}$$

and we call $\mathrm{Arg}(z)$ the **main branch** of the argument of z.

2 Paths

Throughout this section, let D be a nonempty subset of $\mathbb{C}$.

2.1 Definition of a Path

Definition 2.1. A **path** is a continuous function of the form $\gamma\colon [a,b] \to \mathbb{C}$ where $[a,b]$ is a closed interval in $\mathbb{R}$. The image set $[\gamma] := \gamma([a,b])$ is said to be the **trace** (or **trajectory**) of γ. If $[\gamma] \subseteq D$, then we say γ is a **path in** D. The space of all paths in D is denoted $\mathcal{P}(D)$. The points $\gamma(a)$ and $\gamma(b)$ are called the **source** and **target** of γ, respectively. We say γ is **simple** if $\gamma(s) = \gamma(t)$ with $s < t$ implies $s = a$ and $t = b$. We say γ is a **loop in** D **based at** z if $\gamma(a) = z = \gamma(b)$. The space of all loops in D is denoted $\mathcal{L}(D)$ and the space of all loops in D based at z is denoted $\mathcal{L}(D,z)$.

2.2 Reparametrization

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Suppose $\varphi\colon [c,d] \to [a,b]$ is a continuous function from the closed interval $[c,d]$ to the closed interval $[a,b]$. Then $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a path and is called a **reparametrization** of γ. More specifically,

1. if $\varphi(c) = a$ and $\varphi(d) = b$, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **positive reparametrization** of γ.
2. if $\varphi(c) = b$ and $\varphi(d) = a$, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **negative reparametrization** of γ.
3. if φ is a linear map, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **linear reparametrization** of γ.
4. if $\gamma \circ \varphi$ is a positive linear reparametrization whose domain is I, then the map $\varphi\colon I \to [a,b]$ is uniquely determined: it is given by the formula $\varphi(t) = a(1-t) + tb$ for all $t \in I$. In this case, we call $\gamma \circ \varphi\colon I \to \mathbb{C}$ *the* **normalized form** of γ. Any path whose domain is I is called a **normal path**.

As it turns out, the choice of the parameter interval $[a,b]$ is not that essential for the definition of a path. Thus we will usually only work with normal paths. Any contstruction we describe which uses normal paths can easily be extended to all paths by taking their normalized forms.

2.3 Standard Examples

Example 2.1. Let $w, z \in D$, $r > 0$, and let $\gamma \colon I \to D$ be a path in D.

1. The **constant path at** z is the path $c_z \colon I \to \mathbb{C}$ defined by $c_z(t) = z$ for all $t \in I$.
2. The **oriented line segment from** w **to** z is the path $[w, z] \colon I \to D$ defined by $[w, z](t) = w(1-t) + tz$ for all $t \in I$.
3. The **standard parametrization of** $C_r(z)$ is the path $\gamma_r(z) \colon I \to \mathbb{C}$ defined by $\gamma_r(z)(t) = z + re^{2\pi i t}$ for all $t \in I$.
4. The **reversed path** (or **negative path**) γ^- of γ is the path $\gamma^- \colon I \to \mathbb{C}$ defined by the formula $\gamma^-(t) = \gamma(1-t)$ for all $t \in I$.

2.4 Concatenation of Paths

let $\gamma_1 \colon I \to \mathbb{C}$ and $\gamma_2 \colon I \to \mathbb{C}$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then their **concatenation** $\gamma_2 \oplus \gamma_1$ is a path $\gamma_2 \oplus \gamma_1 \colon I \to \mathbb{C}$ defined by the formula

$$(\gamma_2 \oplus \gamma_1)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2\left(2\left(t - \frac{1}{2}\right)\right) & \frac{1}{2} < t \leq 1. \end{cases}$$

for all $t \in [0, 1]$. The idea is that we traverse γ_1 and then γ_2 all in one day. The $2t$ in $\gamma_1(2t)$ comes from the fact that we need to traverse γ_1 twice as fast, the $t - \frac{1}{2}$ in $\gamma_2\left(2\left(t - \frac{1}{2}\right)\right)$ comes from the fact that we need to wait half a day before we start traversing γ_2, and the $2\left(t - \frac{1}{2}\right)$ in $\gamma_2\left(2\left(t - \frac{1}{2}\right)\right)$ comes from the fact that we need to traverse γ_2 twice as fast.

2.5 Polygonal Paths and Paraxial Paths

A **polygonal path** is the sum $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_n$ of segments $\gamma_k = [z_{k-1}, z_k]$ A polygonal path is called **paraxial** if each of its segments is parallel to the real or imaginary axis.

3 Homotopy

Throughout this subsection, let D be a nonempty subset of $\mathbb{C}$.

3.1 Homotopy of Paths

Let $\alpha \colon I \to D$ and $\beta \colon I \to D$ be two paths in D. We say that α is **homotopic** to β as paths, denoted $\alpha \sim \beta$, if there exists a continuous function $H \colon I \times I \to D$ such that $H(0, t) = \alpha(t)$ and $H(1, t) = \beta(t)$ for all $t \in I$. The map H is called a **homotopy** joining α to β. It is easy to check that $\sim$ is an equivalence relation in the set $\mathcal{P}(D)$.

3.2 Homotopy of Paths with Fixed Endpoints

Let $\alpha \colon I \to D$ and $\beta \colon I \to D$ be two paths in D with the same source and target, say $\alpha(0) = w = \beta(0)$ and $\alpha(1) = z = \beta(1)$. Then a homotopy H from α to β is called a **homotopy with fixed endpoints** if we additionally have $H(s, 0) = w$ and $H(s, 1) = z$ for all $s \in I$.

3.3 Homotopy of Loops

Let α and β be two loops in D based at z. We say α is homotopic to β as loops based at z, denoted $\alpha \sim_z \beta$, if there exists a homotopy $H \colon I \times I \to D$ with fixed endpoints from α to β. The space of all loops in D based at z is denoted $\mathcal{L}(D, z)$. It is easy to check that $\sim_z$ is an equivalence relation in the set $\mathcal{L}(D, z)$.

3.4 Free Homotopy of Loops

There is a slightly more general notion of homotopy in the context of loops which we will also consider. Let α and β be two loops in D based at z. We say α is **freely homotopic** to β, denoted $\alpha \sim_{\text{free}} \beta$ if there exists a homotopy $H \colon I \times I \to D$ from α to β such that $H(s, 0) = H(s, 1)$ for all $s \in I$. Such a homotopy is called a **free homotopy** from α to β. The added condition "$H(s, 0) = H(s, 1)$ for all $s \in I$" ensures that for each $s \in I$, the function $H(s, -) \colon I \to D$ is a loop. So the base point is allowed to move freely as s goes from 0 to 1.

3.5 The Fundamental Group $\pi(D, z)$

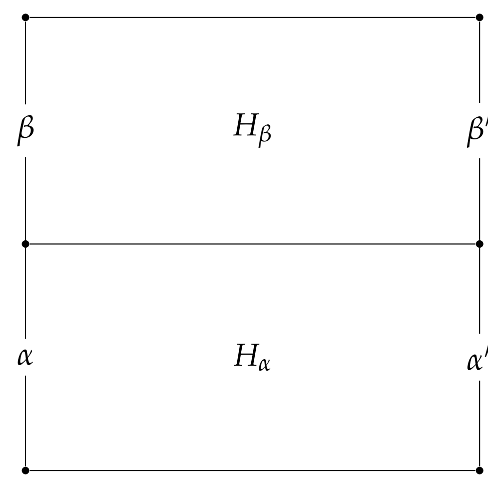
Let $z \in D$. Concatenation serves as a natural binary operation on $\mathcal{L}(D, z)$. However, this binary operation is rather complicated. For example, it isn't associative, it has no identity, and it has no inverses. It turns out however, if we consider loops up to homotopy with fixed endpoints, then we do get these properties. In fact, we will show that $(\mathcal{L}(D, z)/{\sim_z}, \oplus)$ forms a group, called the **fundamental group of** D **based at** z. Let us show this in the following sequence of steps:

3.5.1 Concatenation Passes to Quotient

First we need to show that concatenation passes to the quotient $\mathcal{L}(D, z)/{\sim_z}$, thus giving a well-defined binary operation on $\mathcal{L}(D, z)/{\sim_z}$. Let $\alpha \colon I \to D$ and $\beta \colon I \to D$ be two loops based at z. Suppose α' and β' are two loops based at z such that $\alpha' \sim_z \alpha$ and $\beta' \sim_z \beta$. Let H_α and H_β be their respective homotopies with endpoints. Define $H \colon I \times I \to X$ by the formula

$$H(s, t) = \begin{cases} H_\alpha(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_\beta(s, 2t - 1) & \frac{1}{2} < t \leq 1. \end{cases}$$

for all $(s, t) \in I \times I$. Then H is easily seen to be a homotopy with fixed endpoints from $\beta \oplus \alpha$ to $\beta' \oplus \alpha'$. One may visualize this homotopy as below:

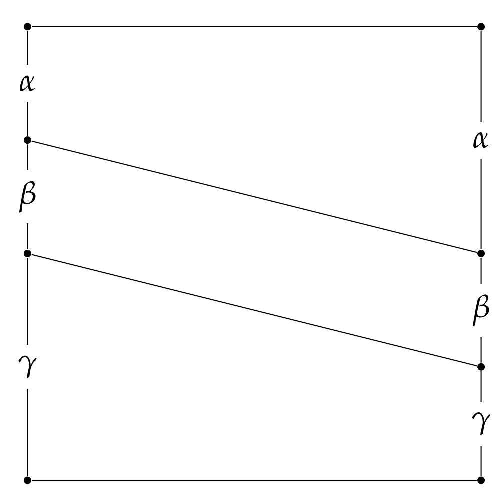


3.5.2 Associativity

Next we show that $(\mathcal{L}(D, z)/{\sim_z}, \oplus)$ is associative. Suppose α, β, and γ are loops based at z. Define $H \colon I \times I \to X$ by the formula

$$H(s, t) = \begin{cases} \gamma\left(\left(\frac{4}{2-s}\right) \cdot t\right) & 0 \leq t \leq \frac{2-s}{4} \\ \beta\left(4 \cdot \left(t - \left(\frac{2-s}{4}\right)\right)\right) & \frac{2-s}{4} < t \leq \frac{3-s}{4} \\ \alpha\left(\left(\frac{4}{1+s}\right) \cdot \left(t - \left(\frac{3-s}{4}\right)\right)\right) & \frac{3-s}{4} < t \leq 1 \end{cases}$$

for all $(s, t) \in I \times I$. Then H is easily seen to be a homotopy with fixed endpoints from $(\alpha \oplus \beta) \oplus \gamma$ to $\alpha \oplus (\beta \oplus \gamma)$. One may visualize this homotopy as below:

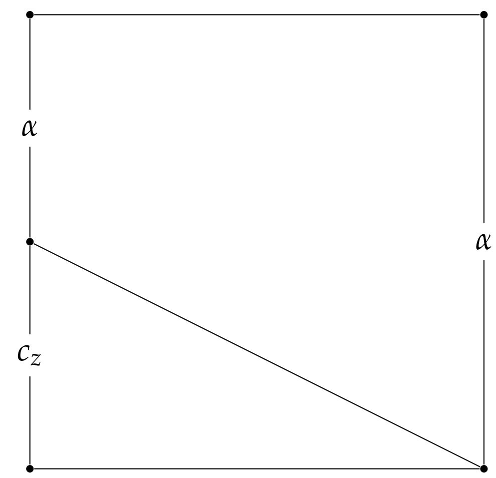


3.5.3 Identity

Next, we want to show that c_z represents the identity element in $(\mathcal{L}(D,z)/{\sim_z}, \oplus)$. Let α be a loop in D based at z. Define $H\colon I \times I \to X$ by the formula

$$H(s,t) = \begin{cases} c_z\left(\left(\frac{2}{1-s}\right)t\right) & 0 \leq t < \frac{1-s}{2} \\ \alpha\left(\left(\frac{2}{1+s}\right)\left(t - \left(\frac{1-s}{2}\right)\right)\right) & \frac{1-s}{2} \leq t \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy fixed at z from $\alpha \oplus c_z$ to c_z. One may visualize this homotopy as below:



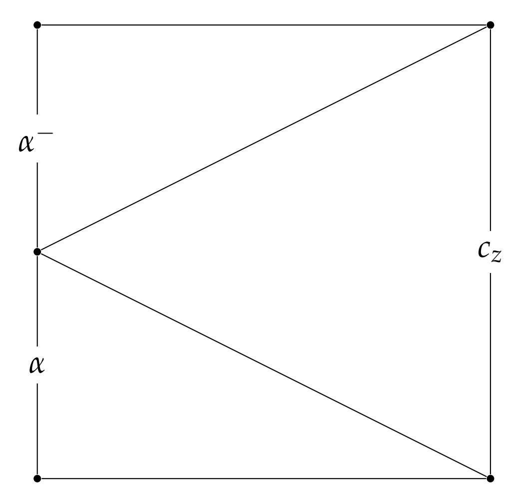
A similar argument gives a homotopy fixed at z from $c_z \oplus \alpha$ to α.

3.5.4 Inverses

Finally, we want to show that $(\mathcal{L}(D,z)/{\sim_z}, \oplus)$ has inverses. Suppose α is a loop based at z. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \alpha\left(\left(\frac{2}{1-s}\right)t\right) & 0 \leq t < \frac{1-s}{2} \\ c_z\left(\left(\frac{1}{s}\right)\left(t - \left(\frac{1-s}{2}\right)\right)\right) & \frac{1-s}{2} \leq t < \frac{1+s}{2} \\ \alpha^-\left(\left(\frac{2}{1-s}\right)\left(t - \frac{2-s}{2}\right)\right) & \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy fixed at z from $\alpha^- \oplus \alpha$ to c_z. One may visualize this homotopy as below:



A similar argument gives a homotopy fixed at z from $\alpha \oplus \alpha^-$ to c_z.

3.5.5 Changing the Base Point

We have shown that $(\mathcal{L}(D,z)/{\sim_z}, \oplus)$ forms a group. This group is called the **fundamental group of** D **based at** z and is denoted as $\pi_1(D,z)$. We now what to consider what happens if we change the base point $z \in D$ to another basepoint, say $w \in D$.

Proposition 3.1. *If D is path connected, then $\pi_1(D,z) \cong \pi_1(D,w)$.*

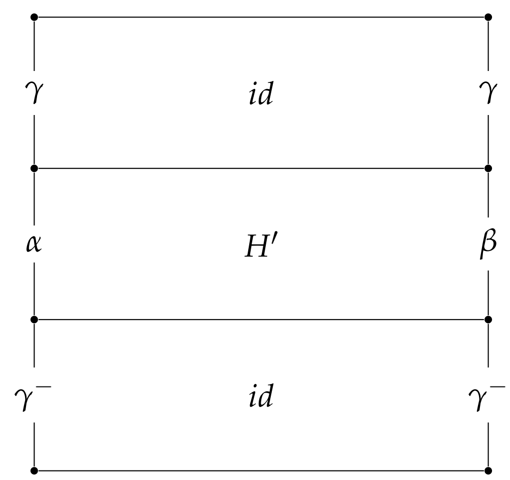
Proof. Choose a path γ from z to w. Define $-^{\gamma}\colon \pi_1(D,z) \to \pi_1(D,w)$ by the formula

$$\overline{\alpha}^{\gamma} = \overline{\gamma \oplus \alpha \oplus \gamma^{-}}$$

for all $\overline{\alpha} \in \pi_1(D,z)$. We need to show that this is a well-defined map, so choose another representative of the equivalence class $\overline{\alpha}$, say β, and let H' be a homotopy with fixed endpoints from α to β. Define $H\colon I \times I \to D$ by the formula

$$H(s,t) = \begin{cases} \gamma^{-}(3t) & 0 \le t < \frac{1}{3} \\ H'(s,t) & \frac{1}{3} \le t < \frac{2}{3} \\ \gamma\,(3t-2) & \frac{2}{3} \le t \le 1 \end{cases}$$

Then H is easily checked to be a homotopy with fixed endpoints from $\gamma \circ \alpha \circ \gamma^{-}$ to $\gamma \circ \beta \circ \gamma^{-}$. Thus, $-^{\gamma}$ is well-defined. One may visualize this homotopy as in the diagram below:



The map $-^{\gamma}$ is easily checked to be a group isomorphism with inverse being given by $-^{\gamma^{-}}$. $\square$

3.5.6 Simply Connected Domains

A domain D is called **simply connected** if for any two paths $\alpha\colon I \to D$ and $\beta\colon I \to D$ which share the same source and target are homotopic with fixed endpoints in D.

3.5.7 Null-Homotopic

A loop which is freely homotopic in D to a constant path is said to be **null-homotopic** in D.

Lemma 3.1. *For any path γ in D, the loop $\gamma \oplus \gamma^{-}$ is null-homotopic in D to its base point $c_{\gamma(0)}$.*

Proof. Define the function $H\colon I \times I \to D$ by the formula

$$H(s,t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1-s}{2} \\ c_{\gamma(1-s)}(t) & \text{if } \frac{1-s}{2} < t \le \frac{1+s}{2} \\ \gamma^{-}(2t-1-s) & \text{if } \frac{1+s}{2} < t \le 1 \end{cases}$$

Then H is easily checked to be a homotopy with fixed endpoints from $\gamma^{-} \oplus \gamma$ to $c_{\gamma(0)}$. $\square$

Lemma 3.2. *If a loop with base point z_0 is null-homotopic in D, then it is also homotopic with fixed endpoints to the constant path c_{z_0}.*

Proof. Let H be a homotopy from the given path γ_0 to a point z_1. We define γ_s and γ_s^{+} by the formulas

$$\gamma_s(t) = H(s,t) \quad \text{and} \quad \gamma_s^{+}(t) = H(st,0)$$

for all $s,t \in I$, and we set $\gamma_s^{-} = (\gamma_s^{+})^{-}$. Then the path γ_s^{+} lies in D and connects z_0 with the moving basepoint $z_s = H(s,0) = H(s,1)$ of the loop γ_s. The family of loops

$$\gamma_s^{*} = \gamma_s^{+} \oplus \gamma_s \oplus \gamma_s^{-}$$

has fixed base point z_0 and all paths in this family are homotopic in D. Now γ_0 is homotopic to γ_0^{*}, γ_0^{*} is homotopic to γ_1^{*}, and γ_1^{*} is homotopic to $\gamma_1^{+} \oplus \gamma_1^{-}$, and the latter is homotopic to the base point z_0. $\square$

4 Smooth Paths

4.1 Definition of Smooth Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. We say that γ is **smooth** if it is continuously differentiable on $[a,b]$ (i.e. $\gamma'(t)$ exists for $t \in [a,b]$ and the function $t \mapsto \gamma'(t)$ is continuous). At the points $t = a$ and $t = b$, the quantities $\gamma'(a)$ and $\gamma'(b)$ are interpreted as one-sided limits

$$\gamma'(a) = \lim_{\substack{h\to 0\\ h>0}} \frac{\gamma(a+h)-\gamma(a)}{h} \quad \text{and} \quad \gamma'(b) = \lim_{\substack{h\to 0\\ h<0}} \frac{\gamma(b+h)-\gamma(b)}{h}.$$

In general, these quantities are called the right-hand derivative of $\gamma(t)$ at a, and the left-hand derivative of $\gamma(t)$ at b, respectively. More generally, we say that γ is **piecewise smooth** if there exists a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0,\dots,n-1$. In particular, the right-hand derivative of γ at t_k may differ from the left-hand derivative of γ at t_k, for $k = 1,\dots,n-1$.

Example 4.1. The paths described in Example (2.1) are all smooth.

4.2 Integrating along a Smooth Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path and suppose f is a continuous function defined on $[\gamma]$. Then the **integral of** f **along** γ is defined by the formula

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If γ is piecewise smooth, then we choose a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0,\dots,n-1$, and we define

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)dt.$$

4.3 Reparametrizing a Smooth Path

Definition 4.1. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path. Suppose $\varphi\colon [c,d] \to [a,b]$ is a continuously differentiable function from the closed interval $[c,d]$ to the closed interval $[a,b]$. Then $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a path and is called a **smooth reparametrization** of γ.

Remark. Note that a linear reparametrization is a smooth reparametrization. Thus the normalized form of a smooth path is also a smooth path.

As in the continuous case, the parameter interval $[a,b]$ is not that essential for the definition of a smooth path. For example, if $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a smooth positive reparametrization of $\gamma\colon [a,b] \to \mathbb{C}$, then the change of variables formula implies

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_c^d f(\gamma(\varphi(s)))\gamma'(\varphi(s))\varphi'(s)ds \\
&= \int_c^d f((\gamma\circ\varphi)(s))(\gamma\circ\varphi)'(s)ds \\
&= \int_{\gamma\circ\varphi} f(z)dz.
\end{aligned}$$

Thus we will usually only work with smooth normal paths. Any construction we describe which uses smooth normal paths can easily be extended to all paths by taking their normalized forms.

4.4 Defining the Length of a Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path. Then the **length** of γ, denoted length(γ), is defined by the formula

$$\text{length}(\gamma) := \int_a^b |\gamma'(t)|\, dt.$$

If γ is piecewise-smooth, then we choose a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \dots, n-1$, and we define

$$\text{length}(\gamma) = \sum_{k=0}^{n-1} \text{length}\left(\gamma|_{[t_k,t_{k+1}]}\right).$$

4.5 Length of a Smooth Path equals the Length of its Normalized Form

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path and let $\varphi\colon I \to [a,b]$ be given by $\varphi(t) = a(1-t) + bt$ for all $t \in I$ (so $\gamma \circ \varphi\colon I \to \mathbb{C}$ is the normalized form of $\gamma\colon [a,b] \to \mathbb{C}$). Then by the change of variables formula, we have

$$\begin{aligned}
\text{length}(\gamma) &= \int_a^b |\gamma'(t)|\, dt. \\
&= \int_0^1 |\gamma'(\varphi(s))|\, \varphi'(s) ds \\
&= \int_0^1 |\gamma'(\varphi(s))|\, |\varphi'(s)|\, ds \\
&= \int_0^1 |\gamma'(\varphi(s))\varphi'(s)|\, ds \\
&= \int_0^1 |(\gamma \circ \varphi)'(s)|\, ds \\
&= \text{length}(\gamma \circ \varphi),
\end{aligned}$$

where we used the fact that for all $s \in I$, we have $\varphi'(s) = b - a > 0$.

4.6 Properties of Integration

Throughout this subsection, let Ω be an open subset of $\mathbb{C}$.

4.6.1 Linearity of Integration

Proposition 4.1. *Let $\gamma\colon I \to \Omega$ be a smooth path in Ω, let f, g be complex-valued functions defined on Ω, and let $\alpha, \beta \in \mathbb{C}$. Then*

$$\int_\gamma (\alpha f + \beta g)(z) dz = \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z) dz.$$

Proof. This follows from linearity of the Riemann integral. Indeed, we have

$$\begin{aligned}
\int_\gamma (\alpha f + \beta g)(z) dz &= \int_0^1 (\alpha f + \beta g)(\gamma(t))\gamma'(t) dt \\
&= \int_0^1 (\alpha f(\gamma(t))\gamma'(t) + \beta g(\gamma(t))\gamma'(t)) dt \\
&= \alpha \int_0^1 f(\gamma(t))\gamma'(t) dt + \beta \int_0^1 g(\gamma(t))\gamma'(t) dt \\
&= \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z) dz.
\end{aligned}$$

$\square$

4.6.2 Additivity of Concatenation of Smooth Paths

Proposition 4.2. *Let $\gamma_1 \colon I \to \Omega$ and $\gamma_2 \colon I \to \Omega$ be smooth paths in Ω such that $\gamma_1(1) = \gamma_2(0)$, and let $f \colon \Omega \to \mathbb{C}$ be a function. Then*

$$\int_{\gamma_2 \oplus \gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz + \int_{\gamma_1} f(z)dz.$$

Proof. This follows from additivity of the Riemann Integral. Indeed, we have

$$\begin{aligned}
\int_{\gamma_2 \oplus \gamma_1} f(z)dz &= \int_0^1 f((\gamma_2 \oplus \gamma_1)(t))(\gamma_2 \oplus \gamma_1)'(t)dt \\
&= 2\int_0^{1/2} f(\gamma_1(2t))\gamma_1'(2t)dt + 2\int_{1/2}^1 f(\gamma_2(2t-1))\gamma_2'(2t-1)dt \\
&= \int_0^1 f(\gamma_1(u))\gamma_1'(u)du + \int_0^1 f(\gamma_2(v))\gamma_2'(v)dv \\
&= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,
\end{aligned}$$

where we used the change of variables $u = 2t$ and $v = 2t - 1$.

□

4.6.3 Negativity of Reverse Orientation

Proposition 4.3. *Let $\gamma \colon I \to \Omega$ be a smooth paths in Ω, and let $f \colon \Omega \to \mathbb{C}$ be a function. Then*

$$\int_{\gamma^-} f(z)dz = -\int_{\gamma} f(z)dz$$

Proof. This follows from a straightforward calculation:

$$\begin{aligned}
\int_{\gamma^-} f(z)dz &= -\int_0^1 f(\gamma(1-t))\gamma'(1-t)dt \\
&= \int_1^0 f(\gamma(s))\gamma'(s)ds \\
&= -\int_0^1 f(\gamma(s))\gamma'(s)ds \\
&= -\int_{\gamma} f(z)dz,
\end{aligned}$$

where we used the change of variable $s = 1 - t$. □

4.6.4 Useful Inequality

Proposition 4.4. *Let $\gamma \colon I \to \Omega$ be a smooth path in Ω, and let $f \colon \Omega \to \mathbb{C}$ be a function. Then one has the inquality*

$$\left| \int_{\gamma} f(z)dz \right| \leq \|f(z)\|_{[\gamma]} \cdot \mathit{length}(\gamma).$$

Proof. This follows from a straightforward calculation:

$$\begin{aligned}
\left| \int_{\gamma} f(z)dz \right| &= \left| \int_0^1 f(\gamma(t))\gamma'(t)dt \right| \\
&\leq \int_0^1 \left| f(\gamma(t))\gamma'(t) \right| dt \\
&\leq \sup_{t \in I} |f(\gamma(t))| \cdot \int_0^1 \left| \gamma'(t) \right| dt \\
&= \sup_{z \in [\gamma]} |f(z)| \cdot \text{length}(\gamma)
\end{aligned}$$

□

4.6.5 Primitives

Definition 4.2. Let $f\colon \Omega \to \mathbb{C}$ be a function. A **primitive** for f on Ω is a function $F\colon \Omega \to \mathbb{C}$ that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 4.1. *Let $\gamma\colon I \to \Omega$ be a piecewise smooth path in Ω from z_1 to z_2, and let $f\colon \Omega \to \mathbb{C}$ be a function. Suppose that $F\colon \Omega \to \mathbb{C}$ is a primitive for f in Ω. Then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

In particular the integral vanishes if γ is a loop.

Proof. We first assume that γ is smooth. Then by the chain rule and the fundamental theorem of calculus, we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\ &= \int_0^1 F'(\gamma(t))\gamma'(t)dt \\ &= \int_0^1 (F\circ\gamma)'(t)dt \\ &= (F\circ\gamma)(b) - (F\circ\gamma)(a) \\ &= F(\gamma(b)) - F(\gamma(a)). \\ &= F(z_2) - F(z_1).\end{aligned}$$

Now we assume γ is piecewise smooth. Choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1,$$

of the interval I such that the restriction $\gamma_k := \gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \ldots, n-1$. Then

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{k=0}^{n-1}\int_{\gamma_k} f(z)dz \\ &= \sum_{k=0}^{n-1} F(\gamma(t_{k+1})) - F(\gamma(t_k)) \\ &= F(\gamma(t_n)) - F(\gamma(t_0)) \\ &= z_2 - z_1.\end{aligned}$$

□

Corollary. *Assume that Ω is path-connected. Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function and suppose that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant on Ω.*

Proof. Fix a point $z_0 \in \Omega$. It suffices to show that $f(z) = f(z_0)$ for all $z \in \Omega$. Let $z \in \Omega$. Since Ω is connected, there exists a path $\gamma\colon I \to \Omega$ from z_0 to z. Since f is clearly a primitive for f', we have

$$\int_\gamma f'(w)dw = f(z) - f(z_0).$$

By assumption, $f' = 0$, so the integral on the left is 0, and we conclude that $f(z) = f(z_0)$ as desired. □

Integral Representation of the Taylor Coefficients

Theorem 4.2. *Let f be the sum of the power series $\sum a_n(z-a)^n$ with radius of convergence R. Then for any $n \geq 0$ and r such that $0 < r < R$, we have*

$$a_m = \frac{1}{r^m}\int_0^1 f(a + re^{2\pi it})e^{-2\pi imt}dt$$

Proof. By uniform convergence of the power series $\sum a_n(z-a)^n$ on $C_r(a)$, we have

$$\begin{aligned}\int_0^1 f(a+re^{2\pi it})e^{-2\pi imt}dt &= \int_0^1 \sum_{n=0}^\infty a_n r^n e^{2\pi i(n-m)t}dt \\ &= \sum_{n=0}^\infty a_n r^n \int_0^1 e^{2\pi i(n-m)t}dt \\ &= a_m r^m.\end{aligned}$$

□

5 More on Paths

5.1 Path Covering Lemma

Let $\gamma\colon I \to \mathbb{C}$ be a path. A **chain of disks covering** γ is a finite sequence $(D_0, D_1, \ldots, D_n)$ of open disks D_k with the following properties

1. There exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval I such that $\gamma(t_k)$ is the center of D_k for $k = 0, 1, \ldots, n$.

2. The section of γ between $\gamma(t_{k-1})$ and $\gamma(t_{k+1})$ is contained in D_k, more precisely,

$$\begin{array}{lll} \gamma(t) \subset D_0, & t_0 \le t \le t_1, & \\ \gamma(t) \subset D_k, & t_{k-1} \le t \le t_{k+1}, & (k = 1, \ldots, n-1) \\ \gamma(t) \subset D_n, & t_{n-1} \le t \le t_n. & \end{array}$$

Lemma 5.1. *(Path Covering Lemma) Let Ω be a nonempty open connected subset of $\mathbb{C}$ and let $\gamma\colon I \to \Omega$ be a path in Ω. Then there exists a chain of disks which is contained in Ω and covers γ. Moreover, the radii of all disks can be chosen to be of the same size and arbitrarily small.*

Proof. Since γ is continuous on the compact interval I, its trace $[\gamma]$ is a compact subset of D. The complement of Ω in $\mathbb{C}$ is closed, and hence the distance d between $[\gamma]$ and $\mathbb{C}\backslash\Omega$ is positive. If $0 < r < d$, then all disks with radius r and centers on $[\gamma]$ are contained in Ω. Because γ is uniformly continuous, there exists a positive number δ such that $s, t \in I$ and $|s - t| < \delta$ imply that $|\gamma(s) - \gamma(t)| < r$. So all requirements are satisfied if the partition $0 = t_0 < t_1 < \cdots < t_n = 1$ is chosen such that $t_k - t_{k-1} < \delta$. □

5.2 Homotopic Paths with Specific Properties

Technically it is of great importance that any path in D can be approximated by homotopic paths with specific properties.

Lemma 5.2. *Let $\gamma\colon I \to D$ be a path in an open set $D \subseteq \mathbb{C}$. Then there exists a smooth path $\widetilde{\gamma}\colon I \to D$ and a paraxial path $\widehat{\gamma}\colon I \to D$ which are homotopic to γ in D. For each $\varepsilon > 0$ both paths can be chosen such that*

$$\|\gamma - \widetilde{\gamma}\|_I < \varepsilon \quad \textit{and} \quad \|\gamma - \widehat{\gamma}\|_I < \varepsilon.$$

Proof. By the path covering lemma, γ can be covered by a sequence of disks D_k with radii less than $\varepsilon/2$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a subdivision of the parameter interval I, and denote by $z_k = \gamma(t_k)$ the centers of the covering disks D_k. Then the restriction γ_k of γ to $[t_{k-1}, t_k]$ is homotopic in D_k (and hence in D) to the line segment $[z_{k-1}, z_k]$ for all $k = 1, \ldots, n$. Indeed, D_k is convex, so the map $H\colon I \times I \to D_k$ given by

$$H(s, t) = \gamma_k(t)(1 - s) + [z_{k-1}, z_k](t)s$$

for all $s, t \in I$ serves as a homotopy from γ_k to $[z_{k-1}, z_k]$.

This induces a homotopy between γ and the polygonal path $\widehat{\gamma} := [z_0, z_1] \oplus \cdots \oplus [z_{n-1}, z_n]$. Smoothing the function $\widehat{\gamma}$ at the points t_k appropriately, we also obtain a smooth path $\widetilde{\gamma}$ which is homotopic to $\widehat{\gamma}$ and hence to γ. Finally, the segments $[z_{k-1}, z_k]$ are homotopic in D_k to the sum $[z_{k-1}, \mathrm{Re}(z_k) + i\mathrm{Im}(z_{k-1})] \oplus [\mathrm{Re}(z_k) + i\mathrm{Im}(z_{k-1}), z_k]$ of two segments which are parallel to the real and imaginary axis, respectively. □

5.3 Winding Numbers

We now introduce a geometric characteristic of loops which describes how many times they "wind around" some point in the plane.

Lemma 5.3. *Let $\gamma\colon I \to \mathbb{C}\backslash\{0\}$ be a path. Then there exist continuous functions $a\colon I \to \mathbb{R}$ and $r\colon I \to \mathbb{R}_+$ such that*

$$\gamma(t) = r(t)e^{ia(t)} \tag{1}$$

for all $t \in I$.

Proof. The function $r(t) := |\gamma(t)|$ is continuous and positive. So the proof amounts to finding an appropriate argument $a(t)$ of $\gamma(t)$ such that $t \mapsto a(t)$ is continuous. For this purpose, we use the path covering lemma with $D := \mathbb{C}\backslash\{0\}$.

At the initial point of γ we choose the principal branch of the argument, $a(0) := \operatorname{Arg}(\gamma(0))$. If $t \in [t_0, t_1]$, all points $\gamma(t)$ lie in the disk D_0. Since D_0 does not contain the origin, it is contained in a sector with vertex at 0 and opening angle less than π. Consequently the argument $a(t) = \arg(\gamma(t))$ can be chosen such that $|a(t) - a(0)| < \pi/2$, which yields a continuous function a on $[0, t_1]$.

Suppose that such a function has already been constructed on some interval $[0, t_k]$. Then it can be prolongated to $[0, t_{k+1}]$ by choosing $a(t) = \arg(\gamma(t))$ on $[t_k, t_{k+1}]$ such that $|a(t) - a(t_k)| < \pi/2$, which is possible since $\gamma(t) \in D_k$ and $0 \notin D_k$. By induction, a can be extended to all of I. $\square$

Any continuous function a satisfying (1) is called a **continuous branch** of the argument along the path γ. The difference of two such functions a_1 and a_2 on I is a constant integral multiple of 2π. If a is continuous branch of the argument along a loop, then $a(1) - a(0)$ is an integral multiple of 2π which does not depend on the special choice of the branch a.

Definition of Winding Numbers

Let γ be a loop in $\mathbb{C}\backslash\{0\}$ and denote by a a continuous branch of the argument along γ. Then the integer

$$\operatorname{wind}(\gamma) := \frac{1}{2\pi}(a(1) - a(0))$$

is called the **winding number** (or **index**) **of** γ. If $z_0 \in \mathbb{C}$ and γ is a loop in $\mathbb{C}\backslash\{z_0\}$, the **winding number of** γ **about** z_0 is defined by

$$\operatorname{wind}(\gamma, z_0) := \operatorname{wind}(\gamma - z_0).$$

Stability of Winding Numbers

Next we prove the intuitive fact that small perturbations of a loop do not change its winding number. Recall that the distance between two paths γ and γ_0 is defined in terms of the uniform norm:

$$\|\gamma - \gamma_0\|_I := \max_{t \in I} |\gamma(t) - \gamma_0(t)|.$$

Lemma 5.4. *Let $\gamma_0 \colon I \to \mathbb{C}\backslash\{0\}$ be a loop, and denote by d the distance of its trace $[\gamma_0]$ from the origin. Then for all loops $\gamma \colon I \to \mathbb{C}$ with $\|\gamma - \gamma_0\|_I < d$,*

$$\mathit{wind}(\gamma) = \mathit{wind}(\gamma_0).$$

Proof. Since $[\gamma_0]$ is a compact subset of $\mathbb{C}\backslash\{0\}$, its distance from the origin is positive. Then $|\gamma(t) - \gamma_0(t)| < d$ implies that $\gamma(t)/\gamma_0(t)$ lies in the right half-plane.

Let a_0 be a continuous branch of the argument along γ_0. If we choose a continuous branch of the argument along γ such that $|a(0) - a_0(0)| < \pi/2$, then $|a(t) - a_0(t)| < \pi/2$ for all $t \in I$. Invoking the triangle inequality we see that

$$|(a(1) - a(0)) - (a_0(1) - a_0(0))| < \pi,$$

and since this number is an integral multiple of 2π, it must be zero. $\square$

Lemma 5.5. *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $z_0 \in D$. Then two loops γ_0 and γ_1 are homotopic in the punctured domain $D\backslash\{z_0\}$ if and only if they have the same winding number about z_0.*

6 Cauchy's Theorem and its Applications

Roughly speaking, Cauchy's Theorem states that if f is holomorphic in an open set Ω and γ is a whose interior is also contained in Ω, then

$$\int_\Gamma f(z)dz = 0.$$

Many results that follow, and in particular the calculus of residues, are related in one way or another to this fact. We first prove this in the special case that our curve Γ is a triangle:

6.1 Goursat's Theorem

Theorem 6.1. *(Goursat's Theorem) If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0,$$

whenever f is holomorphic in Ω.

Proof. We call $T^{(0)}$ our original triangle (with a fixed orientation which we choose to be positive), and let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$, respectively. The first step in our construction consists of bisecting each side of the triangle and connected the midpoints. This creates four new smaller triangles, denote $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, and $T_4^{(1)}$ that are similar to the original triangle. The orientation is chosen to be consistent with that of the original triangle, and so after cancellations arising from integration over the same side in two opposite directions, we have

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz.$$

By triangle inequality, we must have

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4\left|\int_{T_j^{(1)}} f(z)dz\right|$$

for some $j \in \{1,2,3,4\}$. Without loss of generality, assume $j = 1$. Observe that if $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of $T^{(1)}$, respectively, then $d^{(1)} = (1/2)d^{(0)}$ and $p^{(1)} = (1/2)p^{(0)}$. We now repeat this process for the triangle $T^{(1)}$, bisecting into four smaller triangles. Continuing this process, we obtain a sequence of triangles

$$T^{(0)}, T^{(1)}, \ldots, T^{(n)}, \ldots$$

with the properties that

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n\left|\int_{T^{(n)}} f(z)dz\right|$$

and

$$d^{(n)} = 2^{-n}d^{(0)}, \quad p^{(n)} = 2^{-n}p^{(0)},$$

where $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$, respectively. We also denote $\mathcal{T}^{(n)}$ the *solid* closed triangle with boundary $T^{(n)}$, and observe that our construction yields a sequence of nested compact sets

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots \supset \mathcal{T}^{(n)} \supset \cdots$$

whose diameter goes to 0. Thus, by Proposition (**??**), there exists a unique point z_0 that belongs to all the solid triangles $\mathcal{T}^{(n)}$. Since f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$. Since the constant $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, we can integrate the above equality and, using Corollary (**??**), we obtain

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz.$$

Now z_0 belongs to the closure of the triangle $\mathcal{T}^{(n)}$ and z to its boundary, so we must have $|z - z_0| \leq d^{(n)}$, and so we estimate

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n d^{(n)}p^{(n)},$$

where $\varepsilon_n = \|\psi\|_{z\in T^{(n)}} \to 0$ as $n \to \infty$. Therefore

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n 4^{-n}d^{(0)}p^{(0)},$$

which yields our final estimate

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n\left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n d^{(0)}p^{(0)}.$$

Letting $n \to \infty$ concludes the proof since $\varepsilon_n \to 0$. □

Corollary. *If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then*

$$\int_R f(z)dz = 0.$$

Proof. We simply decompose the rectangle into two triangles, and integrate over the two triangles. □

6.2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 6.2. *A holomorphic function in an open disc has a primitive in that disc.*

Proof. After a translation, we may assume without loss of generality that the disc, say D, is centered at the origin. Given a point $z \in D$, consider the piecewise-smooth curve that joins 0 to z first by moving in the horizontal direction form 0 to $\mathrm{Re}(z)$, and then in the vertical direction from $\mathrm{Re}(z)$ to z. We choose the orientation from 0 to z, and denote this polygonal line by Γ_z. Define

$$F(z) = \int_{\Gamma_z} f(w)dw.$$

The choice of Γ_z gives an unambigious definition of the function F. We contend that F is holomorphic in D and that $F'(z) = f(z)$ for all $z \in D$. To prove this, fix $z \in D$ and let $h \in \mathbb{C}$ be sufficiently small so that $z + h$ also belongs to the disc. Now consider the difference

$$F(z+h) - F(z) = \int_{\Gamma_{z+h}} f(w)dw - \int_{\Gamma_z} f(w)dw.$$

The function f is first integrated along Γ_{z+h} with the original orientation, and then along Γ_z with the reverse orientation. Since we integrate f over the line segment starting at the origin in two opposite directions, it cancels. Then we complete the square and triangle, so that after an application of Goursat's theorem for triangles and rectangles, we are left with the line segment from z to $z + h$. Hence, we have

$$F(z+h) - F(z) = \int_\eta f(w)dw,$$

where η is the straight line segment from z to $z + h$. Since f is continuous at z we can write

$$f(w) = f(z) + \psi(w)$$

where $\psi(w) := f(w) - f(z) \to 0$ as $w \to z$. Therefore

$$F(z+h) - F(z) = \int_\eta f(z)dw + \int_\eta \psi(w)dw = f(z)\int_\eta dw + \int_\eta \psi(w)dw.$$

On the one hand, the constant 1 has w as a primitive, so the first integral is simply h. On the other hand, we have

$$\left|\int_\eta \psi(w)dw\right| \leq \sup_{w\in\eta} |\psi(w)| \cdot |h|.$$

Since the supremum above goes to 0 as h tends to 0, we conclude that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

thereby proving that F is a primitive for f on the disc. □

Theorem 6.3. *If f is holomorphic in a disc, then*

$$\int_\Gamma f(z)dz = 0$$

for any closed curve Γ in that disc.

Proof. Since f has a primitive, we can apply Corollary (**??**). □

Example 6.1. We show that if $y \in \mathbb{R}$, then

$$e^{-\pi y^2} = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y} dx.$$

This gives a new proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $y = 0$, then the equality becomes

$$1 = \int_{\mathbb{R}} e^{-\pi x^2} dx.$$

Now suppose that $y > 0$ and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the toy contour Γ_R, where Γ_r consists of a rectangle with vertices R, $R + iy$, $-R + iy$, $-R$ and the positive counterclockwise orientation. By Cauchy's theorem,

$$\int_{\Gamma_R} f(z)dz = 0.$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2} dx,$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$I(R) = \int_0^y f(R+it)idt = \int_0^y e^{-\pi(R^2+2iRt-t^2)} idt.$$

This integral goes to 0 as $R \to \infty$ since y is fixed and we may estimate it by

$$|I(R)| \leq Ce^{-\pi R^2}.$$

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons. Finally, the integral over the horizontal segment on top is

$$\int_R^{-R} e^{-\pi(x+iy)^2} dx = e^{-\pi x^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi ixy} dx.$$

Therefore we find in the limit as $R \to \infty$ that

$$0 = 1 - e^{-\pi x^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi ixy} dx,$$

and our desired formula is established. In the case $y < 0$, we then consider the symmetric rectangle in the lower half-plane.

6.3 Differentiable and Analytic Functions

Lemma 6.4. *Let $f\colon B_R(z_0) \to \mathbb{C}$ be a holomorphic function and let $r > 0$ such that $0 < r < R$. If F is a primitive of f, then for all points $z \in B_r(z_0)$, we have*

$$F(z) = \frac{1}{2\pi i} \int_{\gamma_r(z_0)} \frac{F(w)}{w-z} dw.$$

Proof. 1. We begin with an auxiliary result. Let $z_0 \in B_R(a)$ be fixed. Define a function $\varphi\colon B_R(a - z_0) \to \mathbb{C}$ by the formula

$$\varphi(h) := F(z_0 + h) - F(z_0) - f(z_0)h - \frac{1}{2} f'(z_0)h^2$$

for all $h \in B_R(a - z_0)$. The function φ is differentiable in $B_R(a - z_0)$ and its derivative with respect to h can be computed by the chain rule,

$$\begin{aligned} \varphi'(h) &= F'(z_0 + h) - f(z_0) - f'(z_0)h \\ &= f(z_0 + h) - f(z_0) - f'(z_0)h. \end{aligned}$$

Since f is differentiable at z_0, the right-hand side is of order $o(h)$ as $h \to 0$, that is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varphi'(h)| \leq \varepsilon |h|$ whenever $|h| < \delta$.

The function φ' is continuous, whence the mapping $I \to \mathbb{C}$ given by $t \mapsto \varphi(th)$ is continuously differentiable (with respect to the real variable t), so that $\varphi(h)$ can be represented by the fundamental theorem of calculus,

$$\varphi(h) = \int_0^1 \frac{d}{dt}(\varphi(th))dt = \int_0^1 \varphi'(th)hdt.$$

Using the standard estimate for integrals in combination with $|\varphi'(h)| \leq \varepsilon|h|$, we conclude that $|\varphi(h)| \leq h^2\varepsilon$ for all h with $|h| < \delta$. Since ε can be chosen arbitrarily small, we have

$$\lim_{h \to 0} \varphi(h)/h^2 = 0. \tag{2}$$

2. The function G defined by

$$G(z) := \begin{cases} \frac{F(z)-F(z_0)}{z-z_0} & \text{if } z \in B_R(a) \backslash \{z_0\} \\ f(z_0) & \text{if } z = z_0 \end{cases}$$

is differentiable in $z \in B_R(a)\backslash\{z_0\}$. In order to prove that G is also differentiable at z_0, we consider its difference quotient at z_0. By definition of G and φ, we have

$$\begin{aligned}\frac{G(z)-G(z_0)}{z-z_0} &= \frac{(F(z)-F(z_0)-(z-z_0)f(z_0))}{(z-z_0)^2}\\ &= \frac{1}{2}f'(z_0)+\frac{\varphi(z-z_0)}{(z-z_0)^2},\end{aligned}$$

and using (2), we find $G'(z_0) = (1/2)f'(z_0)$.
3. Because G is differentiable in the disk $B_R(a)$, we can apply Goursat's lemma, which tells us that the integral of G along the closed path $\gamma_r(a)$ vanishes. Hence

$$\begin{aligned}0 &= \int_{\gamma_r(a)} G(z)dz\\ &= \int_{\gamma_r(a)} \frac{F(z)-F(z_0)}{z-z_0}dz\\ &= \int_{\gamma_r(a)} \frac{F(z)}{z-z_0}dz - F(z_0)\int_\gamma \frac{dz}{z-z_0}\\ &= \int_{\gamma_r(a)} \frac{F(z)}{z-z_0}dz - F(z_0)\cdot 2\pi i\cdot \operatorname{wind}(\gamma, z_0).\end{aligned}$$

□

6.3.1 Cauchy Integrals

Definition 6.1. Let γ be a piecewise smooth path in $\mathbb{C}$ and assume that the function $\varphi\colon [\gamma] \to \mathbb{C}$ is continuous. The function $f\colon \mathbb{C}\backslash[\gamma] \to \mathbb{C}$ defined by

$$f(z) := \frac{1}{2\pi i}\int_\gamma \frac{\varphi(w)}{w-z}dw \tag{3}$$

for all $z \in \mathbb{C}\backslash[\gamma]$ is said to be the **Cauchy integral with density φ along γ.**

Theorem 6.5. *Let γ be a piecewise smooth path in $\mathbb{C}$ and assume that $\varphi\colon [\gamma] \to \mathbb{C}$ is continuous. Then the function f defined by the Cauchy integral (3) is analytic on $D := \mathbb{C}\backslash[\gamma]$ and tends to zero as $z \to \infty$. For any disk $D_0 \subset D$ with center z_0, the taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

of f at z_0 converges in D_0 and its coefficients satisfy

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-z_0)^{n+1}}dz.$$

Proof. Fix $z \in D_0$. Invoking a compactness argument, we know of the existence of a constant $q < 1$ such that

$$\left|\frac{z-z_0}{w-z_0}\right| \le q < 1$$

for all $w \in [\gamma]$. Consequently,

$$\begin{aligned}\frac{\varphi(w)}{w-z} &= \frac{\varphi(w)}{(w-z_0)-(z-z_0)}\\ &= \sum_{n=0}^{\infty}\frac{\varphi(w)}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^n.\end{aligned}$$

The function $w \mapsto \varphi(w)/(w-z_0)$ is bounded on the compact set $[\gamma]$, the series converges uniformly with respect to $w \in [\gamma]$ (apply Weierstrass M-test with $M_n = Mq^n$, where M is a bound for the function $w \mapsto \varphi(w)/(w-z_0)$). Interchanging the order of summation and integration, we obtain

$$\begin{aligned}2\pi i f(z) &= \int_\gamma \frac{\varphi(w)}{w-z}dw\\ &= \sum_{n=0}^{\infty}\left(\int_\gamma \frac{\varphi(w)}{(w-z_0)^{n+1}}dw\right)(z-z_0)^n\end{aligned}$$

for all $z \in D_0$, which proves the claim. Finally, the standard integral estimate yields that $f(z) \to 0$ as $z \to \infty$. □

6.4 Cauchy's Integral Formula

Theorem 6.6. *(Cauchy's Integral Formula) Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, let $a \in \Omega$, and let $r > 0$ such that $\overline{B}_r(a) \subset \Omega$. For every $z \in B_r(a)$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-z}dw.$$

To get a feel for how this theorem works, let us assume that f is analytic at a. Then we can choose $\varepsilon > 0$ such that $\overline{B}_\varepsilon(a) \subset B_r(z_0)$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)$$

for all $z \in \overline{B}_\varepsilon(a)$. Then we'd have

$$\begin{aligned}\int_{\gamma_r(z_0)} \frac{f(z)}{z-a}dz &= \int_{\gamma_\varepsilon(a)} \frac{f(z)}{z-a}dz \\ &= \int_{\gamma_\varepsilon(a)} \sum_{n=0}^{\infty} a_n(z-a)^{n-1}dz \\ &= \sum_{n=0}^{\infty} a_n \int_{\gamma_\varepsilon(a)} (z-a)^{n-1}dz \\ &= 2\pi i f(a).\end{aligned}$$

Theorem 6.7. *(Cauchy's Integral Formula) Let Ω be an open subset of $\mathbb{C}$, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, let $a \in \Omega$, and let $r > 0$ such that $C_r(a) \subset \Omega$. For every $z \in B_r(a)$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-z}dw.$$

Proof. Let $z \in B_r(a)$. Define $\gamma\colon I \to \Omega$ to be the path

$$\gamma := \oplus e^{2\pi i \mathrm{Arg}(z-a)}\gamma_r(a)$$

$$\zeta$$

If $a \neq z$, let $\zeta = \min\{w - z \mid w \in C_r(a)\}$, be the and let $\gamma_{\delta,\varepsilon}$ be the loop in Ω be given by

$$\gamma_{\delta,\varepsilon} = -$$

and consider the "keyhole" $\gamma_{\delta,\varepsilon}$ which omits the point z. Here δ is the width of the corridor, and ε is the radius of the small circle centered at z. Since the function $f(w)/(w-z)$ is holomorphic away from the point $w = z$, we have

$$\int_{\Gamma_{\delta,\varepsilon}} \frac{f(w)}{w-z}dw = 0$$

by Cauchy's theorem for the chosen toy contour. Now we make the corridor narrower by letting δ tend to 0, and use the continuity of $f(w)/(w-z)$ to see that in the limit, the integrals over the two sides of the corridor cancel out. The remaining part consists of two curves, the large boundary circle C with the positive orientation, and a small circle C_ε centered at z of radius ε and oriented negatively, that is, clockwise. To see what happens to the integral over the small circle, we write

$$\frac{f(w)}{w-z} = \frac{f(w)-f(z)}{w-z} + \frac{f(z)}{w-z}$$

and note that since f is holomorphic, the first term on the right-hand is bounded so that its integral over C_ε tends to 0 as $\varepsilon \to 0$. To conclude the proof, it suffices to observe that

$$\begin{aligned}\int_{C_\varepsilon} \frac{f(z)}{w-z}dw &= f(z)\int_{C_\varepsilon} \frac{dw}{w-z} \\ &= -f(z)\int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}}dt \\ &= -f(z)2\pi i\end{aligned}$$

so that in the limit we find

$$0 = \int_C \frac{f(w)}{w-z}dw - 2\pi i f(z),$$

as was to be shown. $\square$

Integral Representation of the Taylor Coefficients

Theorem 6.8. *Let f be the sum of the power series $\sum a_n(z-a)^n$ with radius of convergence R. Then for any $n \geq 0$ and r such that $0 < r < R$, we have*

$$a_m = \frac{1}{r^m}\int_0^1 f(a+re^{2\pi it})e^{-2\pi imt}dt$$

Proof. By uniform convergence of the power series $\sum a_n(z-a)^n$ on $C_r(a)$, we have

$$\begin{aligned}\int_0^1 f(a+re^{2\pi it})e^{-2\pi imt}dt &= \int_0^1 \sum_{n=0}^{\infty} a_n r^n e^{2\pi i(n-m)t}dt\\ &= \sum_{n=0}^{\infty} a_n r^n \int_0^1 e^{2\pi i(n-m)t}dt\\ &= a_m r^m.\end{aligned}$$

□

Corollary. *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw$$

for all z in the interior of C.

Proof. Let f be the sum of the power series $\sum a_n(z-a)^n$ with radius of convergence R. Then

$$\begin{aligned}f^{(m)}(a) &= m!a_m\\ &= \frac{m!}{r^m}\int_0^1 f(a+re^{2\pi it})e^{-2\pi imt}dt\\ \\ &= \frac{m!}{2\pi i}\int_0^1 \frac{f(z)}{(re^{2\pi it})^{n+1}}2\pi i re^{2\pi it}dz\\ &= \frac{m!}{2\pi i}\int_{\gamma_r(a)} \frac{f(z)}{(z-a)^{n+1}}dz\end{aligned}$$

□

Corollary. *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw$$

for all z in the interior of C.

Proof. The proof is by induction on n, the case $n = 0$ being simply the Cauchy integral formula. Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^n}dw.$$

Now for small h, the difference quotient for $f^{(n-1)}$ takes the form

$$\begin{aligned}\frac{f^{(n-1)}(z+h)-f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{h}\left(\left(\frac{1}{w-(z+h)}\right)^n - \left(\frac{1}{w-z}\right)^n\right)dw\\ &= \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{h}\left(\frac{1}{(w-(z+h))} - \frac{1}{(w-z)}\right)\left(\sum_{m=0}^{n-1}\left(\frac{1}{w-(z+h)}\right)^{n-m-1}\left(\frac{1}{w-z}\right)^m\right)dw\\ &= \frac{(n-1)!}{2\pi i}\int_C f(w)\left(\frac{1}{(w-(z+h))(w-z)}\right)\left(\sum_{m=0}^{n-1}\left(\frac{1}{w-(z+h)}\right)^{n-m-1}\left(\frac{1}{w-z}\right)^m\right)dw\end{aligned}$$

Now observe that if h is small, then $z+h$ and z stay at a finite distance from the boundary circle C, so in the limit as h tends to 0, we find that the quotient converges to

$$\frac{(n-1)!}{2\pi i}\int_C f(w)\left(\frac{1}{(w-z)^2}\right)\left(\frac{n}{(w-z)^{n-1}}\right)dw = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw,$$

which completes the induction argument and proves the theorem. □

6.4.1 Taylor's Theorem

Theorem 6.9. *Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, and let $a \in \Omega$. Then there exists $r > 0$ and a power series $\sum a_n(z-a)^n$ centered at a such that $\overline{B}_r(a) \subset \Omega$*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in \overline{B}_r(a)$. Furthermore, the coefficients are given by

$$a_n = \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{(w-a)^{n+1}}dw$$

for all $n \geq 0$.

Proof. Let $z \in B_r(a)$. Then

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-a}\left(\frac{1}{1-\left(\frac{z-a}{w-a}\right)}\right)dw \\
&= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n dw \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n. \\
&= \sum_{n=0}^{\infty} a_n(z-a)^n.
\end{aligned}$$

where we are allowed to interchange the integral with the sum since the series $\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$ converges uniformly in $w \in C_r(a)$. □

6.4.2 Limit of Holomorphic Functions Converging Uniformly on Compact Subsets is Holomorphic

Theorem 6.10. *Let Ω be a nonempty open subset of $\mathbb{C}$ and let (f_n) be a sequence of analytic functions on Ω that converges uniformly to f on each compact subset of Ω. Then f is holomorphic and (f_n') converges uniformly to f' on each compact subset.*

Proof. Let $a \in \Omega$. Choose $z_0 \in \Omega$ and $r > 0$ such that $a \in \overline{B}_r(z_0) \subset \Omega$. Then

$$f_n(a) = \frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f_n(z)}{z-a}dz$$

for all $n \in \mathbb{N}$. Since (f_n) converges to f uniformly on $\overline{B}_r(z_0)$, the function f is continuous on $\overline{B}_r(z_0)$, and so $f(z)/(z-a)$ is integrable along $\gamma_r(z_0)$. Thus

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f_n(z)}{z-a}dz - \frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f(z)}{z-a}dz\right| &= \left|\frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f_n(z)-f(z)}{z-a}dz\right| \\
&\leq \frac{1}{2\pi}\|f_n - f\|_{\overline{B}_r(z_0)}\left|\int_{\gamma_r(z_0)} \frac{dz}{z-a}\right| \\
&= \|f_n - f\|_{\overline{B}_r(z_0)},
\end{aligned}$$

which tends to 0 as n tends to ∞. Therefore

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f(z)}{z-a}dz &= \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{f_n(z)}{z-a}dz \\
&= \lim_{n\to\infty} f_n(a) \\
&= f(a).
\end{aligned}$$

This implies f is holomorphic in Ω.

To show $f_n' \to f'$ uniformly on compact subsets of Ω, it suffices to work with closed discs. Let $\overline{D}$ be a closed disc in Ω with radius $r > 0$. Choose a in the interior of $\overline{D}$. Then

$$f_n'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(z)}{(z-a)^2} dz, \quad \text{and} \quad f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2} dz.$$

Therefore

$$\begin{aligned} |f_n'(a) - f'(a)| &\leq \frac{\|f_n - f\|_{\overline{D}}}{r} \\ &\to 0. \end{aligned}$$

□

6.4.3 Cauchy's Inequalities

Corollary. *(Cauchy's inequality) Let f is holomorphic in a given set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \sup_{z \in C} |f(z)| .$$

Proof. Applying Cauchy's Integral Formula for $f^{(n)}(z_0)$, we have

$$\begin{aligned} \left| f^{(n)}(z_0) \right| &= \left| \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| \\ &= \frac{n!}{2\pi} \left| \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{R^n e^{int}} d\zeta \right| \\ &\leq \frac{n!}{R^n} \sup_{z \in C} |f(z)| . \end{aligned}$$

□

6.4.4 Louiville's Theorem

Theorem 6.11. *(Louiville's Theorem) Every bounded entire function must be constant.*

Proof. Let f be a bounded entire function. Suppose $\sum a_n z^n$ is the power series representation of f at 0 and choose a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then for every $r > 0$ and $n \geq 1$, we have

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_{\gamma_r(0)} \frac{f(z)}{z^{n+1}} dz \right| \\ &= \left| \int_0^1 \frac{f(re^{2\pi it})}{r^n e^{2\pi int}} dt \right| \\ &\leq \int_0^1 \left| \frac{f(re^{2\pi it})}{r^n e^{2\pi int}} \right| dt \\ &\leq \frac{M}{r^n}. \end{aligned}$$

This implies $a_n = 0$ for every $n \geq 1$. Thus $f(z) = a_0$, which proves the theorem. □

6.4.5 Fundamental Theorem of Algebra

Corollary. *Every non-constant polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. If $P(z)$ has no roots, then $Q(z) := 1/P(z)$ is a bounded holomorphic function. To see this, we can of course assume that $a_n \neq 0$ and write

$$Q(z) = \frac{1}{a_n z^n + \cdots + a_1 z + a_0} = \left(\frac{1}{z^n} \right) \left(\frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n} \right).$$

As $z \to \infty$, the denominator of the second term in the round brackets converges to $a_n \neq 0$, hence the second term itself goes to $1/a_n$. But the first term tends to zero, hence

$$\lim_{z \to \infty} Q(z) = 0.$$

In particular, $|Q(z)|$ is bounded by 1 outside of some circle $|z| = r$. Inside this circle, $|Q(z)|$ is continuous, hence bounded. Thus $|Q(z)|$, and therefore $Q(z)$ itself is bounded on the whole complex plane. By Liousville's theorem, we then conclude that $Q(z)$ is constant. This contradicts our assumption that $P(z)$ is nonconstant and proves the corollary. $\square$

Theorem 6.12. *(Identity Theorem) Let f, g be holomorphic functions on a connected open set D of $\mathbb{C}$. If $f = g$ on a nonempty open subset of D, then $f = g$ on D.*

Remark. This says that a holomorphic function is completely determined by its values on a (possibly quite small) neighborhood in D. This is not true for real-differentiable functions. In comparison, holomorphy is a much more rigid notion.

Proof. Let S be the set of all $z \in D$ such that $f(z) = g(z)$. We show that S is open and closed, and hence must be D. Since $f - g$ is continuous, and $S = (f - g)^{-1}\{0\}$, we see that S is closed. To show, that S is open, suppose w lies in S. Then, because the Taylor series of f and g at w have non-zero radius of convergence, the open disk $B_r(w)$ also lies in S for some r. $\square$

6.5 Further Applications

6.5.1 Morera's Theorem

Theorem 6.13. *Suppose f is a continuous function in the open disc D such that for any triangle T cotained in D,*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. By the proof of Theorem (6.2), the function f has a primitive F in D that satisfies $F' = f$. By the regularity theorem, we know that F is indefinitely (and hence twice) complex differentiable, and therefore f is holomorphic. $\square$

Theorem 6.14. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.*

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\int_T f_n(z)dz = 0$$

for all n. By assumption, $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\int_T f_n(z)dz \to \int_T f(z)dz.$$

As a result, we find $\int_T f(z)dz = 0$ and by Morera's theorem, we conclude that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω. $\square$

6.5.2 Sequence of Holomorphic Functions

Theorem 6.15. *Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties:*

1. *$F(z, s)$ is holomorphic in z for each s.*
2. *F is continuous on $\Omega \times [0, 1]$.*

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s)ds$$

is holomorphic.

Remark. The second condition says that F is jointly continuous in both arguments. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω, and by Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\int_T \int_0^1 F(z, s)dsdz = 0.$$

Interchanging the order of integration, and using property (1) would then yield the desired result. We can, however, get around the issue of justifying the change in the order of integration by arguing differently. The idea is to interpret the integral as a "uniform" limit of Riemann sums, and then apply the results of the previous section.

Proof. For each $n \geq 1$, we consider the Riemann sum

$$f_n(z) = (1/n) \sum_{k=1}^{n} F(z, k/n).$$

Then f_n is holomorphic in all of Ω by property (1), and we claim that on any disc D whose closure is contained in Ω, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. To see this, we recall that a continuous function on a compact set is uniformly continuous, so if $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon \quad \text{whenever } |s_1 - s_2| < \delta.$$

Then if $n > 1/\delta$, and $z \in D$ we have

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} F(z, k/n) - F(z, s) ds \right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{n} \\ &< \varepsilon. \end{aligned}$$

This proves the claim, and by Theorem (6.14), we conclude that f is holomorphic in D. As a consequence, f is holomorphic in Ω, as was to be shown. $\square$

6.5.3 Scwarz reflection principle

Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, that is $z \in \Omega$ if and only if $\bar{z} \in \Omega$. Let Ω^+ denote the part of Ω that lies in the upper half-plane and Ω^- that part that lies in the lower half-plane. Also, let $I = \Omega \cap \mathbb{R}$ so that I denotes the interior of that part of the boundary of Ω^+ and Ω^- that lies on the real axis. Then we have

$$\Omega^+ \cup I \cup \Omega^- = \Omega$$

and the only interesting case of the next theorem occurs, of course, when I is nonempty.

Theorem 6.16. *If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I and $f^+(x) = f^-(x)$ for all $x \in I$, then the function f defined on Ω by*

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+, \\ f^+(z) = f^-(z) & \text{if } z \in I, \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω.

Proof. One first notes that f is continuous on Ω. The only difficulty is to prove that f is holomorphic at points of I. Suppose D is a disc centered at a point on I and entirely contained in Ω. We prove that f is holomorphic in D by Morera's theorem. Suppose T is a triangle in D. If T does not intersect I, then

$$\int_T f(z) dz = 0$$

since f is holomorphic in the upper and lower half-discs. Suppose now that one side or vertex of T is contained in I, and the rest of$T$ is in, say the upper half-disc. If T_ε is the triangle obained from T by slightly raising the edge or vertex which lies on I, we have

$$\int_{T_\varepsilon} f(z) dz = 0$$

since T_ε is entirely contained in the upper half-disc. We then let $\varepsilon \to 0$, and by continuity we conclude that

$$\int_T f(z) dz = 0.$$

If the interior of T intersects I, we can reduce the siutation to the previous one by writing T as the union of triangles each of which has an edge or vertex on I. By Morera's theorem, we conclude that f is holomorphic in D, as was to be shown. $\square$

Theorem 6.17. *(Schwarz reflection principle) Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that $F = f$ on Ω^+.*

Proof. The idea is simply to define $F(z)$ for $z \in \Omega^-$ by $F(z) = \overline{f(\bar{z})}$. To prove that F is holomorphic in Ω^- we note that if $z, z_0 \in \Omega^-$, then $\bar{z}, \bar{z}_0 \in \Omega^+$ and hence, the power series expansion of f near $\bar{z}_0$ gives

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \left(\bar{z} - \bar{z}_0\right)^n.$$

As a consequence we see that

$$F(z) = \sum_{n=0}^{\infty} \bar{a}_n \left(z - z_0\right)^n$$

and F is holomorphic in Ω^-. Since f is real valued on I, we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence F extends continuously up to I. The proof is complete once we invoke the symmetry principle. □

Theorem 6.18. *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

Proof. Suppose that $z_0 \in \Omega$ is a limit point for the sequence $\{w_k\}_{k=1}^{\infty}$ and that $f(\omega_k) = 0$. First, we show that f is idenitcally zero in a small disc containing z_0. For that, we choose a disc D centered at z_0 and contained in Ω, and consider the power series expansion of f in that disc

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If f is not identically zero, there exists a smallest integer m such that $a_m \neq 0$. But then we can write

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0)),$$

where $g(z - z_0)$ converges to 0 as $z \to z_0$. Taking $z = w_k \neq z_0$ for a sequence of points converging to z_0, we get a contradiction since $a_m (w_k - z_0)^m \neq 0$ and $1 + g(w_k - z_0) \neq 0$, but $f(w_k) = 0$.

We conclude the proof using the fact that Ω is connected. Let U denote the interior of the set of points where $f(z) = 0$. Then U is open by definition and nonempty by the argument just given. The set U is also closed since if $z_n \in U$ and $z_n \to z$, then $f(z) = 0$ by continuity, and f vanishes in a neighborhood of z by the argument above. Hence, $z \in U$. Now if we let V denote the complement of U in Ω, we conclude that U and V are both open, disjoint, and

$$\Omega = U \cup V.$$

Since Ω is connected we conclude that either U or V is empty. Since $z_0 \in U$, we find that $U = \Omega$ and the proof is complete.

□

Corollary. *Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some nonempty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ througout Ω.*

Suppose we are given a pair of functions f and F analytic in regions Ω and Ω', respectively, with $\Omega \subset \Omega'$. If the two functions agree on the smaller set Ω, we say that F is an **analytic continuiation** of f into the region Ω' The corollary then guarantees that there can be only one such analytic continuation, since F is uniquely determined by f.