Final Project

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Introduction

In this note, we wish to discuss a proof of a Riesz representation theorem due to Garling. Throughout this note, let X be a compact Hausdorff space and let $C(X)$ be the Banach of space of continuous real-valued functions defined on X equipped with the supremum norm.

The Baire σ-algebra

Definition 0.1. Let $\mathcal{C}$ be the collection of all subsets of X of the form

$$\{a \leq f \leq b\} := \{x \in X \mid a \leq f(x) \leq b\},$$

where $a, b \in \mathbb{R}$. The **Baire σ-algebra of** X, denoted by $\mathcal{M}_X$, or just $\mathcal{M}$ if X is understood from context, is the σ-algebra generated $\mathcal{C}$, written $\mathcal{M} = \sigma(\mathcal{C})$. The members of $\mathcal{M}$ are called **Baire sets**. A measure defined on $\mathcal{M}$ whose value on every compact Baire set is called a **Baire measure**. Note that in our case, we are already assuming that X is compact, and thus Baire measures correspond to finite measures defined on $\mathcal{M}$.

The are more general definitions of Baire measures in the case where X is not compact, but we will not pursue this direction since we are always assuming X is compact throughout this document. Note that the Baire σ-algebra of X is the *smallest* σ-algebra which makes every continuous function $f \colon X \to \mathbb{R}$ Baire measurable. Recall that every continuous function $f \colon X \to \mathbb{R}$ is Borel measurable. Thus we certainly have $\mathcal{M} \subseteq \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of X.

The Baire σ-algebra of X is generated by all G_δ-sets

The Baire σ-algebra of X can be generated by another useful collection of special subsets of X, namely the G_δ**-sets**:

Definition 0.2. Let A be a subset of X. We say A is a G_δ**-subset of** X, or just a G_δ**-set** if X is understood from context, if it can be expressed as a countable intersection of open subsets of X.

Proposition 0.1. *Let $\mathcal{D}$ be the collection of all compact G_δ-subsets of X. Then $\mathcal{M} = \sigma(\mathcal{D})$.*

Proof. Let K be a compact G_δ-set and it express it as a countable intersection of open sets, say

$$K = \bigcap_{n=1}^{\infty} U_n.$$

For each $n \in \mathbb{N}$, there exists a continuous function $f_n \colon X \to [0,1]$ such that $f_n|_K = 1$ and $f_n|_{X \backslash U_n} = 0$ by Urysohn's lemma. Clearly the sequence of functions (f_n) converges pointwise to the characteristic function 1_K, and since each f_n is Baire measurable, it follows that 1_K is Baire measurable. In particular, K is a Baire measure since $K = \{1_K = 1\}$. Since $K \in \mathcal{D}$ was arbitrary, it follows that $\sigma(\mathcal{D}) \subseteq \mathcal{M}$.

Conversely, let $f \in C(X)$, let $c \in \mathbb{R}$, and let $A = \{a \leq f \leq b\}$. Then A is a closed subset of X since f is continuous. Note that closed subsets of compact spaces are themselves compact. Indeed, suppose $(U_i)_{i \in I}$ is an open covering of A, that is $A = \bigcup_{i \in I} U_i$. Then $(X \backslash A, U_i)_{i \in I}$ is an open covering of X. Since X is a compact, it can be covered by some finite subcovering, say

$$X = (X \backslash A) \cup U_{i_1} \cup \cdots \cup U_{i_k}.$$

In particular this implies A can be covered by some finite subcovering of (U_i), namely

$$A = U_{i_1} \cup \cdots \cup U_{i_k}.$$

Thus A is a compact subset of X. Moreover, observe that A is a G_δ-set since

$$\begin{aligned} A &= \{a \leq f \leq b\} \\ &= \bigcap_{n=1}^{\infty} \{a + 1/n < f < b + 1/n\}. \end{aligned}$$

Since $f \in C(X)$ and $A \in \mathcal{C}$ were arbitrary, it follows that $\mathcal{M} \subseteq \sigma(\mathcal{D})$. □

The key takeaway from this proposition is that much of this proof involved purely topological arguments. For instance, Urysohn's lemma has nothing to do with measure theory or linear analysis; it's content is only concerned with topological concepts like continuous functions. Essentially it tells us that we have many continous functions $f \colon X \to \mathbb{R}$ to work with.

Riesz representation theorem

Before we state the form of Riesz representation theorem which we will be interested in proving, we consider the following proposition/definition.

Proposition 0.2. *Let μ be a Baire measure. Define $\ell_\mu \colon C(X) \to \mathbb{R}$ by*

$$\ell_\mu(f) = \int_X f \mathrm{d}\mu$$

for all $f \in C(X)$. The map ℓ_μ is a positive linear functional.

Proof. Positivity of ℓ_μ follows from positivity of integration, and linearity of ℓ_μ follows from linearity of integration. To see that ℓ_μ is bounded, note that

$$\begin{aligned} \ell_\mu(f) &= \int_X f \mathrm{d}\mu \\ &\leq \|f\|_\infty \mu(X) \end{aligned}$$

for all $f \in C(X)$. Taking f to be the constant function 1, we see that $\|\ell_\mu\| = \mu(X)$. □

The equality $\|\ell_\mu\| = \mu(X)$ obtained in the proof above seems to suggest that something more is going on than what was stated in the proposition. In fact, there is! Let us denote by $M(X)$ to be the space of signed Baire measures defined on the Baire σ-algebra of X. The sum of two finite signed Baire measures is a finite signed Baire measure, as is the product of a finite signed measure by a real number. Furthermore, the total variation defines a norm, and it turns out that this gives $M(X)$ the structure of a Banach space. The association $\mu \mapsto \ell_\mu$ can be extended to an isomorphism of Banach spaces from $M(X)$ to $C(X)^*$ which is natural in X. For sake of time, we will not pursue this direction too much in this document; however we still wanted to mention it. We are now ready to state the form of the Reisz representation theorem which we will be interested in.

Theorem 0.1. *Let ℓ be a positive linear functional defined on $C(X)$. Then there exists a unique Baire measure μ such that $\ell = \ell_\mu$.*

Extremally disconnected spaces

There way that we will prove Theorem (0.1) is by first proving it in the case where X is extremally disconnected. After we do this, we then proceed to the general case. First, let us recall what extremally disconnected means.

Definition 0.3. We say X is **extremally disconnected** if each open subset of X has open closure.

The condition that X be extremally disconnected turns out to be equivalent to the condition that every pair of disjoint open subsets of X has disjoint closures. Indeed, suppose that X is extremally disconnected and let U and U' be two disjoint open subsets of X. Since both $\overline{U}$ and $\overline{U}'$ are open, their intersection $\overline{U} \cap \overline{U}'$ is also open. If $\overline{U} \cap \overline{U}' \neq \emptyset$, then would have $U \cap U' \neq \emptyset$, which is a contradiction. Thus we must have $\overline{U} \cap \overline{U}' \neq \emptyset$. The converse direction is proved in a similar manner.

Proof of the Riesz representation theorem when X is extremally disconnected

Before we prove Theorem (0.1) in the special case where X is extremally disconnected, we consider the following lemma.

Lemma 0.2. *Suppose X is extremally disconnected compact Hausdorff space. Let $\mathcal{O}$ be the collection of all clopen subsets of X. Then $\mathcal{O}$ forms an algebra. Furthermore, let $\mathcal{V}$ be the space of $\mathcal{O}$-simple functions. Then $\mathcal{V}$ is a uniformly dense subspace of $C(X)$.*

Proof. It is straightforward to check that $\mathcal{O}$ is an algebra, so we will only focus on showing $\mathcal{V}$ is a uniformly dense subspace $C(X)$. First note if $A \in \mathcal{O}$, then the characteristic function $1_A\colon X \to \mathbb{R}$ is continuous since A is clopen. Indeed, suppose U is an open subset of $\mathbb{R}$. Then

$$\{1_A \in U\} = \begin{cases} A & \text{if } 1 \in U \text{ and } 0 \notin U \\ X \backslash A & \text{if } 1 \notin U \text{ and } 0 \in U \\ X & \text{if } 1 \in U \text{ and } 0 \in U \\ \varnothing & \text{if } 1 \notin U \text{ and } 0 \notin U \end{cases}$$

Thus $\mathcal{V}$ really is a subset of $C(X)$. It is straightforward to check that $\mathcal{V}$ is in fact a subspace of $C(X)$. Let us now show that it is uniformly dense in $C(X)$. Let $f \in C(X)$ and let $n \in \mathbb{N}$. For each $k \in \mathbb{Z}$, let

$$A_k = \left\{ \frac{k}{n} < f < \frac{k+1}{n} \right\}.$$

Each A_k is open and are pairwise disjoint from each other. Since X is extremally disconnected, each $\overline{A}_k$ is clopen and pairwise disjoint from each other. Furthermore, since X is compact, only finitely many of the $\overline{A}_k$ are nonempty. Thus the sets $A = \bigcup_{k \in \mathbb{Z}} \overline{A}_k$ and $X \backslash A$ are clopen. Note that $X \backslash A$ is the set of all $x \in X$ such that there exists $k \in \mathbb{Z}$ and an open neighborhood U of x such that $U \subseteq \{f = k/n\}$. For each $k \in \mathbb{Z}$, let

$$B_k = (X \backslash A) \cap \{f = k/n\}.$$

Then each B_k is clopen and pairwise disjoint from each other, and since (B_k) covers the compact space $X \backslash A$, it follows that only finitely many of the B_k are nonempty. Thus we can define the following $\mathcal{O}$-simple function

$$\varphi_n = \sum_{k \in \mathbb{Z}} \frac{k}{n} (1_{\overline{A}_k} + 1_{B_k})$$

which is easily seen to satisfy $|\varphi_n - f| < 1/n$. In other words, the sequence (φ_n) of $\mathcal{O}$-simple functions converges uniformly to f. □

Now we prove Theorem (0.1) in the special case where X is extremally disconnected.

Proposition 0.3. *With the notation as in Theorem (0.1), assume further that X is extremally disconnected. Then $\ell = \ell_\mu$.*

Proof. We first show existence. Define $\mu\colon \mathcal{O} \to [0, \infty)$ by

$$\mu(A) = \ell(1_A)$$

for all $A \in \mathcal{O}$. We claim that $(X, \mathcal{O}, \mu)$ is a finite premeasure space. Indeed, we have

$$\begin{aligned} \mu(\varnothing) &= \ell(1_\varnothing) \\ &= \ell(0) \\ &= 0. \end{aligned}$$

Furthermore, suppose that (A_n) is a disjoint sequence of sets in $\mathcal{O}$ whose union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also in $\mathcal{O}$. Then A is compact since it is a closed subset of a compact space X, and so it follows that only finitely many of the A_n can be nonempty, say $A_{n_1}, \dots, A_{n_k}$. Thus we have

$$\begin{aligned} \nu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \nu\left(\bigcup_{i=1}^{k} A_{n_i}\right) \\ &= \ell\left(1_{\bigcup_{i=1}^{k} A_{n_i}}\right) \\ &= \ell\left(\sum_{i=1}^{k} 1_{A_{n_i}}\right) \\ &= \sum_{i=1}^{k} \ell(1_{A_{n_i}}) \\ &= \sum_{i=1}^{k} \nu(A_{n_i}) \\ &= \sum_{n=1}^{\infty} \nu(A_n). \end{aligned}$$

It follows that $(X, \mathcal{O}, \mu)$ is a finite premeasure space. Therefore by the Caratheodory Extension Theorem, the premeasure μ extends to a unique measure, which we again denote μ, defined on $\sigma(\mathcal{O})$.

We claim that $\sigma(\mathcal{O}) = \mathcal{M}$ so that μ is in fact a Baire measure. Indeed, we have $\sigma(\mathcal{O}) \subseteq \mathcal{M}$ since each clopen subset of X is a compact G_δ-set. To show $\mathcal{M} \subseteq \sigma(\mathcal{O})$, it suffices to show that each $f \in C(X)$ is a $\sigma(\mathcal{O})$-measurable function. Let $f \in C(X)$ and let $c \in \mathbb{R}$. For each $n \in \mathbb{N}$ set $A_n = \{f < c + 1/n\}$ and set $A = \{f \leq c\}$. Since A_n is open, its closure $\overline{A}_n$ is clopen, and since

$$A_n \subseteq \overline{A}_n \subseteq \{f \leq c + 1/n\},$$

we have $A = \bigcap_{n=1}^{\infty} \overline{A}_n$. Thus $A \in \sigma(\mathcal{O})$, which implies f is $\sigma(\mathcal{O})$-measurable. Therefore $\sigma(\mathcal{O}) = \mathcal{M}$ and hence μ is a Baire measure. From Proposition (0.2), we obtain a corresponding positive linear functional $\ell_\mu \in C(X)^*$.

Now we want to show that $\ell = \ell_\mu$ which will establish existence. First observe that $\ell|_{\mathcal{V}} = \ell_\mu|_{\mathcal{V}}$. Indeed, let φ be an $\mathcal{O}$-simple function and express it in canonical form as $\varphi = \sum_{i=1}^{n} a_i 1_{A_i}$. Then we have

$$\begin{aligned}
\ell(\varphi) &= \ell\left(\sum_{i=1}^{n} a_i 1_{A_i}\right) \\
&= \sum_{i=1}^{n} a_i \ell(1_{A_i}) \\
&= \sum_{i=1}^{n} a_i \mu(A_i) \\
&= \int_X \varphi \mathrm{d}\mu. \\
&= \ell_\mu(\varphi).
\end{aligned}$$

Nexxt, observe that since $\mathcal{V}$ is uniformly dense in X, we in fact have $\ell = \ell_\mu$ everywhere. Indeed, given $f \in C(X)$, we choose a sequence (φ_n) of $\mathcal{O}$-simple functions which converges uniformly to f. Then since $\mu(X) < \infty$, we have

$$\begin{aligned}
\ell(f) &= \lim_{n\to\infty} \ell(\varphi_n) \\
&= \lim_{n\to\infty} \ell_\mu(\varphi_n) \\
&= \lim_{n\to\infty} \int_X \varphi_n \mathrm{d}\mu \\
&= \int_X f \mathrm{d}\mu \\
&= \ell_\mu(f).
\end{aligned}$$

This establishes existence.

Now we prove uniqueness. Let μ and ν be two Baire measures such that $\ell_\mu = \ell_\nu$, that is, such that

$$\int_X f \mathrm{d}\mu = \int_X f \mathrm{d}\nu$$

for all $f \in C(X)$. We need to show that $\mu = \nu$. Let K be a compact G_δ-set. Express K as an countable intersection of open sets, say

$$K = \bigcap_{n=1}^{\infty} U_n.$$

For each $n \in \mathbb{N}$, there exists a continuous function $f_n \colon X \to [0,1]$ such that $f_n|_K = 1$ and $f_n|_{X \setminus U_n} = 0$ by Urysohn's lemma. Since both μ and ν are finite measures, the characteristic function 1_K is a nonnegative μ-integrable and ν-integrable function which dominates each f_n. Moreover the sequence (f_n) converges pointwise to 1_K. Thus by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\mu(K) &= \int_X 1_K \mathrm{d}\mu \\
&= \lim_{n\to\infty} \int_X f_n \mathrm{d}\mu \\
&= \lim_{n\to\infty} \int_X f_n \mathrm{d}\nu \\
&= \int_X 1_K \mathrm{d}\nu \\
&= \nu(K).
\end{aligned}$$

Since K is an arbitrary compact G_δ-set and $\mathcal{M}$ is generated by the collection of all of these types of sets, we have $\mu = \nu$. This establishes uniqueness. $\square$

Proof of the Riesz representation theorem in general

In this last section, we will sketch the proof of Theorem (0.1) in the general case where X is not necessarily extremally disconnected. The idea is at follows: let X_0 be the set X equipped with the discrete topology and let $Y = \beta X_0$ be the Stone–Čech compactification with $\iota \colon X_0 \to Y$ denoting the canonical map. Any function out of a discrete space is continuous and so in particular the identity function $1 \colon X_0 \to X$ is continuous. By the universal properties of the Stone–Čech compactification, there exists a unique continuous map $\pi \colon Y \to X$ such that $\pi \circ \iota = 1$. Now the continuous map π induces an isomorphism $\pi_* \colon C(X) \to C(Y)$ of Banach spaces given by

$$\pi_*(f) = \pi \circ f$$

for all $f \in C(X)$. One check that this is in fact an isomorphism using the universal properties of the Stone–Čech compactification. The isomorphism π_* induces another isomorphism $(\pi_*)^* \colon C(Y)^* \to C(X)^*$ of Banach spaces, given by

$$(\pi_*)^*(\psi) = \psi \circ \pi_*$$

for all $\psi \in C(Y)^*$. In particular, there is a uniuqe $\psi \in C(Y)^*$ such that $\psi \circ \pi_* = \ell$. It turns out that the Stone–Čech compactification of a discrete space is an extremally disconnected space, and so one can apply Proposition (0.3) to get a unique Baire measure ν which represents ψ. Finally one shows that $\nu \circ \pi^{-1}$ is the unique Baire measure which represents ℓ.