Almost Prime Counting Functions

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1 Introduction

Let $\pi\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ be the prime counting function. Much is known about the prime counting function. For example, the prime number theorem states that the prime counting function satisfies the asymptotic

$$\pi(x) \sim \frac{x}{\log x}.$$

In this paper, we study an analog of the prime counting function, called the k-**almost prime counting function**. To see what this analog is, let us fix some notation.

Let $m \in \mathbb{N}$ and suppose $m = p_1^{e_1} \cdots p_s^{e_s}$ is the prime factorization of m. The **degree of** m, denoted $\deg m$, is given by

$$\deg m := \sum_{r=1}^{s} e_r.$$

To be complete, we also require $\deg 1 = 0$. We say m is k**-almost prime** if $\deg m = k$. For each prime number p, we also define

$$\mathrm{ord}_p(m) = \begin{cases} e_r & \text{if } p = p_r \text{ for some } r = 1, \dots, s \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for each $x \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{N}$, and p prime, we define the sets

$$S_k(x) := \{m \in \mathbb{N} \mid m \leq x \text{ and } \deg m = k\} \quad \text{and} \quad S_{k,p}(x) := \left\{m \in \mathbb{N} \mid m \leq x,\ \deg m = k, \text{ and } \mathrm{ord}_p(m) = 0\right\}$$

Similarly, for each $k \in \mathbb{N}$ and p prime, we define the functions $\pi_k\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ and $\pi_{k,p}\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ by the formulas

$$\pi_k(x) = \#S_k(x) \quad \text{and} \quad \pi_{k,p}(x) = \#S_{k,p}(x)$$

for all $x \in \mathbb{R}_{\geq 0}$. The function π_k is called the k**-almost prime counting function**.

1.1 Some Motivation

To give some motivation for what follows, we list the values $\pi_k(2^n)$ for small values of k and n in the table below:

$\pi_k(2^n)$	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
π_0	1	1	1	1	1	1	1	1	1	1	1
π_1	0	1	2	4	6	11	18	31	54	97	172
π_2	0	0	1	2	6	10	22	42	82	157	304
π_3	0	0	0	1	2	7	13	30	60	125	256
π_4	0	0	0	0	1	2	7	14	34	71	152
π_5	0	0	0	0	0	1	2	7	15	36	77
π_6	0	0	0	0	0	0	1	2	7	15	37
π_7	0	0	0	0	0	0	0	1	2	7	15
π_8	0	0	0	0	0	0	0	0	1	2	7
π_9	0	0	0	0	0	0	0	0	0	1	2
π_{10}	0	0	0	0	0	0	0	0	0	0	1

For example, we have entry 6 in the π_2-row and 2^4-column since

$$\pi_2(2^4) = \#S_2(2^4) = \#\{4, 6, 9, 10, 14, 15\}.$$

As we study the table in detail, we notice an interesting pattern. For fixed k, the value $\pi_{n-k}(2^n)$ stabilizes when n is large enough. For example,

$$7 = \pi_3(2^5) = \pi_4(2^6) = \pi_5(2^7) \quad \text{and} \quad 15 = \pi_5(2^8) = \pi_6(2^9) = \pi_7(2^{10}).$$

In the next section, we give a precise statement for this and prove it. In doing so, we solve a conjecture which was stated by Robert G. Wilson as a comment in https://oeis.org/A126281.

2 Main Theorem

We begin with the following recursive relation.

Theorem 2.1. *Let $k, n \in \mathbb{N}$ and let p be a prime. Then*

$$\pi_k(p^n) = \pi_{k-1}(p^{n-1}) + \pi_{k,p}(p^n). \tag{1}$$

Proof. We first note that $\pi_{k-1}(p^{n-1}) = \#(S_k(p^n)\backslash S_{k,p}(p^n))$. Indeed, let

$$\varphi\colon S_{k-1}(p^{n-1}) \to S_k(p^n)\backslash S_{k,p}(p^n)$$

be the function given by $\varphi(m) = pm$ for all $m \in S_{k-1}(p^{n-1})$. The function φ is well-defined since $m \leq p^{n-1}$ if and only if $pm \leq p^n$. Moreover, the function φ is easily checked to be a bijection. Now the theorem follows from the fact that the collection

$$\{S_k(p^n)\backslash S_{k,p}(p^n), S_{k,p}(p^n)\}$$

forms a partition of the set $S_k(p^n)$. $\square$

By iterating (1), we obtain the following corollary.

Corollary. *Let $k, n \in \mathbb{N}$ and let p be a prime. Then*

$$\pi_k(p^n) = \sum_{i=0}^{k} \pi_{k-i,p}(p^{n-i}).$$

2.1 Stabilization of $\pi_k(2^n)$

We now specialize to the case where $p = 2$.

Theorem 2.2. *Let $k, n \in \mathbb{N}$ such that $\lceil n\ln(2)/\ln(3)\rceil \leq k$. Then $\pi_{k,2}(2^n) = 0$. In particular, we have*

$$\pi_k(2^n) = \pi_{k-1}(2^{n-1}).$$

Before we give a proof, we first explain where the number $\lceil n\ln(2)/\ln(3)\rceil$ comes from. For a given $n \in \mathbb{N}$, we wish to find the least $k \in \mathbb{N}$ such that $2^n < 3^k$. Clearly 3^k grows much faster as a function in k than 2^n grow as function in n, so we just need to solve for x in the equation $2^y = 3^x$, where $x, y \in \mathbb{R}_{\geq 0}$. A straightforward calculation gives us $x = y\ln(2)/\ln(3)$. In particular, since $\ln(2)/\ln(3)$ is never rational, $\lceil n\ln(2)/\ln(3)\rceil \leq k$ implies $2^n < 3^k$.

Proof. We prove that $S_{k,2}(2^n)$ is empty by contradiction. Assume that $S_{k,2}(2^n)$ is nonempty. Choose $m \in S_{k,2}(2^n)$. Thus $m \leq 2^n$, $\deg(m) = k$, and $\operatorname{ord}_2(m) = 0$. In particular, this implies $3^k \leq m \leq 2^n$. But this is a contradiction since $\lceil n\ln(2)/\ln(3)\rceil \leq k$ implies $2^n < 3^k$. $\square$

3 Conclusion

We end with some generalizations.

3.1 Twisting the Almost Prime Counting Functions by a Multiplicative Function

Definition 3.1. A function $\varphi\colon \mathbb{N} \to \mathbb{C}$ is said to be **completely multiplicative** if $\varphi(mn) = \varphi(m)\varphi(n)$ for all $m, n \in \mathbb{N}$.

Let $\varphi\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function, let $k \in \mathbb{N}$, and let p be a prime. We define the functions $\pi_k^{\varphi}\colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ and $\pi_{k,p}^{\varphi}\colon \mathbb{R}_{\geq 0} \to \mathbb{C}$ by the formulas

$$\pi_k^{\varphi}(x) = \sum_{m \in S_k(x)} \varphi(m) \quad \text{and} \quad \pi_{k,p}^{\varphi}(x) = \sum_{m \in S_{k,p}(x)} \varphi(m)$$

for all $x \in \mathbb{R}_{\geq 0}$. Note that we recover the functions π_k and $\pi_{k,p}$ by setting φ to be the identity function.

Theorem 3.1. *Let $\varphi\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function, let $k, n \in \mathbb{N}$, and let p be a prime. Then*

$$\pi_k^{\varphi}(p^n) = \varphi(p)\pi_{k-1}^{\varphi}(p^{n-1}) + \pi_{k,p}^{\varphi}(p^n) \tag{2}$$

Proof. Indeed, we have

$$\begin{aligned}
\pi_k^{\varphi}(p^n) &= \sum_{m \in S_k(p^n)} \varphi(m) \\
&= \sum_{m \in S_{k-1}(p^{n-1})} \varphi(pm) + \sum_{m \in S_{k,p}(p^n)} \varphi(m) \\
&= \varphi(p) \sum_{m \in S_{k-1}(p^{n-1})} \varphi(m) + \sum_{m \in S_{k,p}(p^n)} \varphi(m) \\
&= \varphi(p)\pi_{k-1}^{\varphi}(p^{n-1}) + \pi_{k,p}^{\varphi}(p^n).
\end{aligned}$$

□

By iterating (2), we obtain the following corollary.

Corollary. *Let $\varphi\colon \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function, let $k, n \in \mathbb{N}$, and let p be a prime. Then*

$$\pi_k^{\varphi}(p^n) = \sum_{i=0}^{k} \varphi(p^i)\pi_{k-i,p}^{\varphi}(p^{n-i}).$$

3.2 Number Fields

Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_K$ and suppose $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}$ is the prime factorization of $\mathfrak{a}$. The **degree of** $\mathfrak{a}$, denoted $\deg \mathfrak{a}$, is given by

$$\deg \mathfrak{a} := \sum_{r=1}^{s} f_r e_r,$$

where $f_r = [\mathcal{O}_K/\mathfrak{p}_r : \mathbb{Z}/(\mathfrak{p}_r \cap \mathbb{Z})]$ for each $r = 1, \ldots, s$. In particular, if $\mathfrak{p}$ is a prime ideal, then the degree of $\mathfrak{p}$ is the degree of the field extension $\mathbb{Z}/p \hookrightarrow \mathcal{O}_K/\mathfrak{p}$ where $p = \mathfrak{p} \cap \mathbb{Z}$. This explains why we chose our terminology in the way that we did. To be complete, we also require $\deg\langle 1 \rangle = 0$.

For each prime ideal $\mathfrak{p}$, we also define

$$\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) = \begin{cases} e_r & \text{if } \mathfrak{p} = \mathfrak{p}_r \text{ for some } r = 1, \ldots, s \\ 0 & \text{otherwise.} \end{cases}$$

The **norm** of $\mathfrak{a}$ is given by $\mathrm{N}(\mathfrak{a}) := \#\mathcal{O}_K/\mathfrak{a}$. Note that the norm is a multiplicative function, and so in particular we have

$$\begin{aligned}
\mathrm{N}(\mathfrak{a}) &= \#\mathcal{O}_K/\mathfrak{a} \\
&= (\#\mathcal{O}_K/\mathfrak{p}_1^{e_1}) \cdots (\#\mathcal{O}_K/\mathfrak{p}_s^{e_s}) \\
&= p_1^{f_1 e_1} \cdots p_s^{f_s e_s},
\end{aligned}$$

where p_r denotes the prime $\mathfrak{p}_r \cap \mathbb{Z}$ for each $r = 1, \ldots, s$.

For each $x \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{N}$, and $\mathfrak{p}$ prime ideal in $\mathcal{O}_K$, we define the sets

$$S_k(x) := \{\mathfrak{a} \in \mathrm{Ideal}(\mathcal{O}_K) \mid \mathrm{N}(\mathfrak{a}) \leq x \text{ and } \deg \mathfrak{a} = k\} \quad \text{and} \quad S_{k,\mathfrak{p}}(x) := \{\mathfrak{a} \in \mathrm{Ideal}(\mathcal{O}_K) \mid \mathrm{N}(\mathfrak{a}) \leq x,\ \deg \mathfrak{a} = k, \text{ and } \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}) = 0\}.$$

Similarly, for each $k \in \mathbb{N}$ and $\mathfrak{p}$ prime, we define the functions $\pi_k\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ and $\pi_{k,\mathfrak{p}}\colon \mathbb{R}_{\geq 0} \to \mathbb{N}$ by

$$\pi_k(x) = |S_k(x)| \quad \text{and} \quad \pi_{k,\mathfrak{p}}(x) = |S_{k,\mathfrak{p}}(x)|$$

for all $x \in \mathbb{R}_{\geq 0}$.

Theorem 3.2. *Let $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$ and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$. Then*

$$\pi_k(N(\mathfrak{p})^n) = \pi_{k-f_p}(N(\mathfrak{p})^{n-1}) + \pi_{k,\mathfrak{p}}(N(\mathfrak{p})^n)$$