Section 1.3: Limits and Continuity

For function f defined on an interval containing a constant c (except possibly at c itself), if $f(x)$ approaches the number L_1 as x approaches c from the left, then the **left-hand limit** of f is L_1 and is written $\lim_{x\to c^-} f(x) = L_1$.

Similarly, if $f(x)$ approaches the number L_2 as x approaches c from the right, then the **right-hand limit** of f is L_2 and is written $\lim_{x\to c^+} f(x) = L_2$.

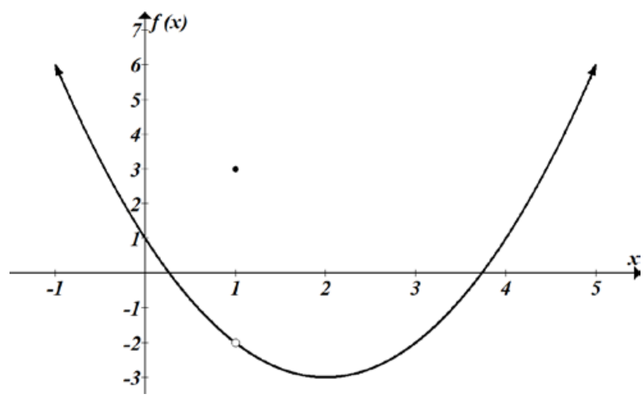
The **limit** of f as x approaches c **exists** if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$. It is written $\lim_{x\to c} f(x) = L$. Otherwise, the limit of f as x approaches c does not exist.

If $\lim_{x\to c^-} f(x) = \infty \; or \; -\infty$ or $\lim_{x\to c^+} f(x) = \infty \; or \; -\infty$, then the graph of f has a **vertical asymptote** at $x = c$.

Example 1:

Use the graph of $f(x) = \begin{cases} x^2 - 4x + 1 \; for \; x \neq 1 \\ 3 \; for \; x = 1 \end{cases}$

shown to the right to find the following.



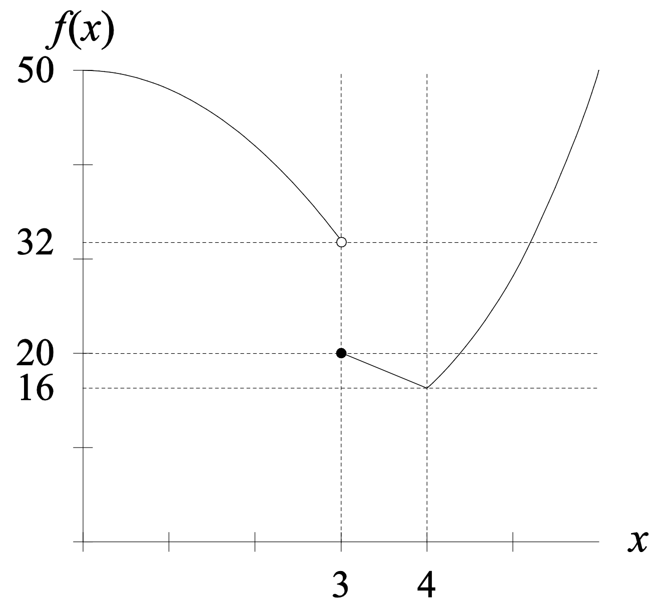
a. $f(1) =$	b. $\lim_{x\to 1^-} f(x) =$	c. $\lim_{x\to 1^+} f(x) =$	d. $\lim_{x\to 1} f(x) =$

$x \to 1^-$	$f(x)$
0.9	-1.79
0.99	-1.9799
0.999	-1.99799
0.9999	-1.99979999

$x \to 1^+$	$f(x)$
1.1	-2.19
1.01	-2.0199
1.001	-2.001999
1.0001	-2.00019999

Example 2: (CC5e pp. 23, 26)

Use the graph of f shown to the right to find the following.



a. $f(3) =$ ________ $\lim_{x\to 3^-} f(x) =$ ________

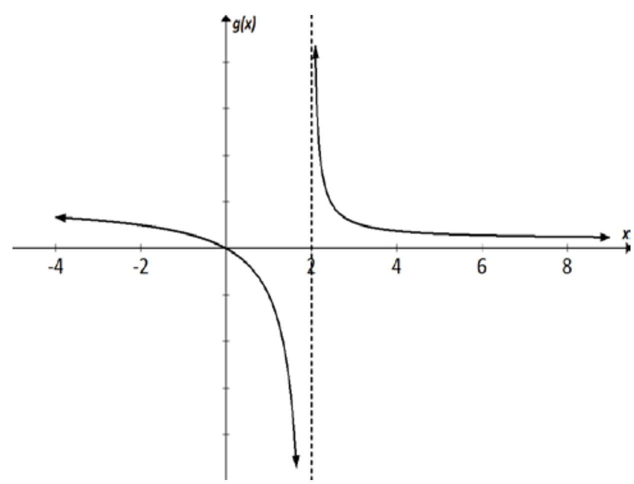
$\lim_{x\to 3^+} f(x) =$ ______ $\lim_{x\to 3} f(x) =$ ________

b. $f(4) =$ _________ $\lim_{x\to 4^-} f(x) =$ ______

$\lim_{x\to 4^+} f(x) =$ _______ $\lim_{x\to 4} f(x) =$ _______

Example 3:

Use the graph of g shown to the right to find the following.



a. $g(2) =$ ________ $\lim_{x\to 2^-} g(x) =$ ________

$\lim_{x\to 2^+} g(x) =$ ______ $\lim_{x\to 2} g(x) =$ ________

b. The graph shows a vertical asymptote. Write its equation.

A function f is **continuous at input *c*** if and only if the following three conditions are satisfied:

(1) $f(c)$ exists
and (2) $\lim_{x \to c} f(x)$ exists
and (3) $\lim_{x \to c} f(x) = f(c)$

A function f is **continuous on an open interval** if all three conditions are met for every input value in the interval.

A function is continuous everywhere if it meets all three conditions for every possible input value. Such a function is called a **continuous function**.

Example 4:

a. Answer the following questions about function f in example 1 to determine whether f is continuous at $x = 1$.

Does $f(1)$ exist?	Does $\lim_{x \to 1} f(x)$ exist?	Does $\lim_{x \to 1} f(x) = f(1)$?
Is f continuous at $x = 1$? Why or why not?		

b. Answer the following questions about function f in example 2 to determine whether f is continuous at $x = 3$.

Does $f(3)$ exist?	Does $\lim_{x \to 3} f(x)$ exist?	
Is f continuous at $x = 3$? Why or why not?		

c. Answer the following questions about function f in example 2 to determine whether f is continuous at $x = 4$.

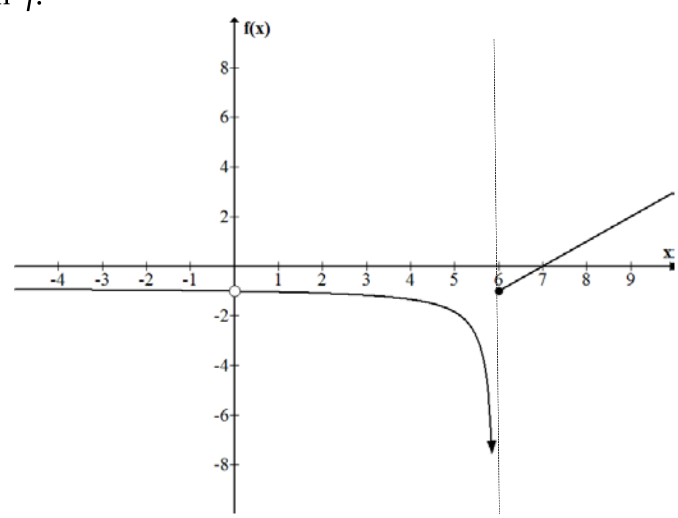
Does $f(4)$ exist?	Does $\lim_{x\to 4} f(x)$ exist?	Does $\lim_{x\to 4} f(x) = f(4)$?
Is f continuous at $x = 4$? Why or why not?		

d. Answer the following questions about function g in example 3 to determine whether g is continuous at $x = 2$.

Does $g(2)$ exist?		
Is g continuous at $x = 2$? Why or why not?		

Example 5: (CC5e p. 30, Activities 5 and 6)

Graphically estimate the values for the function f.



a. $f(6) =$_________ $\lim_{x\to 6^-} f(x) =$_________

$\lim_{x\to 6^+} f(x) =$_______ $\lim_{x\to 6} f(x) =$_______

Explain why f is not continuous at $x = 6$.

b. $f(0) =$_________ $\lim_{x\to 0^-} f(x) =$_______

$\lim_{x\to 0^+} f(x) =$______ $\lim_{x\to 0} f(x) =$______

Explain why f is not continuous at $x = 0$.

Limits – Algebraically (Optional)

Limit Rules

- The limit of a **constant** is that constant.
- The limit of a **sum** is the sum of the limits.
- The limit of a **constant times a function** is the constant times the limit of the function.
- If f is a **polynomial function** and *c* is a real number, then $\lim_{x \to c} f(x) = f(c)$.
- The limit of a **product** is the product of the limits.
- The limit of a **quotient** is the quotient of the limits.
- If f is a **rational** function and *c* is a valid input of f, then $\lim_{x \to c} f(x) = f(c)$.
- If the numerator and denominator of a rational function share a common factor, then the new function obtained by algebraically **cancelling** the common factor has all limits identical to the original function.

Example 6:

Algebraically determine $\lim_{x \to 0} \frac{(5+x)^2 - 5^2}{x}$

Example 7: (CC5e p. 29)

Given $d(x) = \begin{cases} x^2 + 2 & for\ x < 4 \\ -3x + 2 & for\ x \geq 4 \end{cases}$

Use the definition of a continuous function to determine whether d is continuous at $x = 4$.

Example 8: (CC5e p. 29)

Given $f(x) = \begin{cases} x^2 + 2 & for\ \ x < 4 \\ -3x + 30 & for\ \ x \geq 4 \end{cases}$

Use the definition of a continuous function to determine whether f is continuous at $x = 4$.