Linear Analysis Homework 10

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Problem 1

Proposition 0.1. *Let* $\|\cdot\|_\infty \colon C[a,b] \times C[a,b] \to \mathbb{R}$ *be given by*

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [a,b]\} \tag{1}$$

for all $f \in C[a,b]$. *Then* $\|\cdot\|_\infty$ *is a norm. Moreover, the pair* $(C[a,b], \|\cdot\|_\infty)$ *forms a Banach space.*

Proof. Let us first show $\|\cdot\|_\infty$ is a norm. First note that the set $\{|f(x)| \mid x \in [a,b]\}$ is non-empty and bounded above (since f is continuous on a compact interval and hence attains a maximum). Therefore the supremum (1) exists.

For positive-definiteness, let $f \in C[a,b]$. Then

$$\begin{aligned}\|f\|_\infty &= \sup\{|f(x)| \mid x \in [a,b]\} \\ &\geq \sup\{0 \mid x \in [a,b]\} \\ &= 0.\end{aligned}$$

We have equality if and only if $|f(x)| = 0$ for all $x \in [a,b]$, and since $|\cdot|$ is positive-definite, this is equivalent to f being the zero function.

For absolute-homogeneity, let $f \in C[a,b]$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}\|\alpha f\|_\infty &= \sup\{|\alpha f(x)| \mid x \in [a,b]\} \\ &= \sup\{|\alpha||f(x)| \mid x \in [a,b]\} \\ &= |\alpha| \sup\{|f(x)| \mid x \in [a,b]\} \\ &= |\alpha|\|f\|_\infty,\end{aligned}$$

where the equality at the third line is justified by Proposition (0.10) (stated and proved in the Appendix).

For subadditivity, let $f, g \in C[a,b]$. Then

$$\begin{aligned}\|f+g\|_\infty &= \sup\{|f(x)+g(x)| \mid x \in [a,b]\} \\ &\leq \sup\{|f(x)| + |g(x)| \mid x \in [a,b]\} \\ &= \sup\{|f(x)| \mid x \in [a,b]\} + \sup\{|g(x)| \mid x \in [a,b]\} \\ &= \|f\|_\infty + \|g\|_\infty,\end{aligned}$$

where the equality at the third line is justified by Proposition (0.11) (stated and proved in the Appendix).

Finally, to show that $(C[a,b], \|\cdot\|_\infty)$ forms a Banach space, we need to show that every Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$ is convegent. Throughout the rest of the proof, we drop the notation $(C[a,b], \|\cdot\|_\infty)$ and simply write $C[a,b]$ instead. Let (f_n) be a Cauchy sequence in $C[a,b]$. We first make the observation that for each $x \in [a,b]$, the sequence $(f_n(x))$ forms a Cauchy sequence of complex numbers. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|f_n - f_m\|_\infty < \varepsilon$. In other words, $m, n \geq N$ implies

$$\sup\{|f_n(x) - f_m(x)| \mid x \in [a,b]\} < \varepsilon.$$

In particular $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| < \varepsilon \tag{2}$$

for all $x \in [a,b]$. This proves our claim.

Since $\mathbb{C}$ is complete, we are justified in defining $f \colon [a,b] \to \mathbb{C}$ by

$$f(x) := \lim_{n\to\infty} f_n(x)$$

for all $x \in [a,b]$. By taking $m \to \infty$ in (2), we see that (f_n) converges *uniformly* to f. In particular, this implies f is continuous (by the usual $\varepsilon/3$ trick). Thus $f \in C[a,b]$. Finally, we note that convergence in $\|\cdot\|_\infty$ is equivalent to uniform convergence. Thus the Cauchy sequence (f_n) converges in the $\|\cdot\|_\infty$ norm to f. $\square$

Problem 2

Proposition 0.2. *Let $(V, \|\cdot\|)$ be a normed linear space over $\mathbb{C}$ which satisfies the parallelogram law. Then the map $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{C}$ defined by*

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2\right) \tag{3}$$

for all $x, y \in V$ is an inner-product. Moreover, the norm induced by this inner-product is precisely $\|\cdot\|$. In other words, we have

$$\langle x, x\rangle = \|x\|^2$$

for all $x \in V$.

Proof. The most difficult part of this proof is showing that (3) is linear in the first argument. Before we do this, let us show that (3) is positive-definite and conjugate-symmetric.

For positive-definiteness, let $x \in V$. Then

$$\begin{aligned}
\langle x, x\rangle &= \frac{1}{4}\left(\|2x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2\right) \\
&= \frac{1}{4}\left(\|2x\|^2 + i((|1+i|^2 - |1-i|^2)\|x\|^2)\right) \\
&= \|x\|^2 \\
&\geq 0,
\end{aligned}$$

with equality if and only if $x = 0$. Note that this also gives us $\langle x, x\rangle = \|x\|^2$ for all $x \in V$.

For conjugate-symmetry, let $x, y \in V$. Then

$$\begin{aligned}
\overline{\langle y, x\rangle} &= \frac{1}{4}\overline{(\|y+x\|^2 + i\|y+ix\|^2 - \|y-x\|^2 - i\|y-ix\|^2)} \\
&= \frac{1}{4}\left(\|y+x\|^2 - i\|y+ix\|^2 - \|y-x\|^2 + i\|y-ix\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 - i\|i(x-iy)\|^2 - \|x-y\|^2 + i\|i(x+iy)\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 - i\|x-iy\|^2 - \|x-y\|^2 + i\|x+iy\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2\right) \\
&= \langle x, y\rangle
\end{aligned}$$

Now we come to the difficult part, namely showing that (3) is linear in the first argument. We do this in several steps:

Step 1: We show that (3) is additive in the first argument (i.e. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ for all $x, y, z \in V$). Let $x, y, z \in V$. First note that by the parallelogram law, we have

$$\begin{aligned}
\|x+z+y\|^2 - \|x+z-y\|^2 &= 2\|x+y\|^2 + 2\|z\|^2 - \|x+y-z\|^2 - 2\|x-y\|^2 - 2\|z\|^2 + \|x-y-z\|^2 \\
&= 2\|x+y\|^2 - 2\|x-y\|^2 - \|z-y-x\|^2 + \|z+y-x\|^2 \\
&= 2\|x+y\|^2 - 2\|x-y\|^2 - 2\|z-y\|^2 - 2\|x\|^2 + \|z-y+x\|^2 + 2\|z+y\|^2 + 2\|x\|^2 - \|z+y+x\|^2 \\
&= 2\|x+y\|^2 - 2\|x-y\|^2 + 2\|z+y\|^2 - 2\|z-y\|^2 + \|x+z-y\|^2 - \|x+z+y\|^2.
\end{aligned}$$

Adding $\|x+z-y\|^2 - \|x+z+y\|^2$ to both sides gives us

$$2(\|x+z+y\|^2 - \|x+z-y\|^2) = 2(\|x+y\|^2 - \|x-y\|^2 + \|z+y\|^2 - \|z-y\|^2),$$

and after cancelling 2 from both sides, we obtain

$$\|x+z+y\|^2 - \|x+z-y\|^2 = \|x+y\|^2 - \|x-y\|^2 + \|z+y\|^2 - \|z-y\|^2.$$

Therefore

$$
\begin{aligned}
\langle x+z,y\rangle &= \frac{1}{4}\left(\|x+z+y\|^2 + i\|x+z+iy\|^2 - \|x+z-y\|^2 - i\|x+z-iy\|^2\right)\\
&= \frac{1}{4}\left(\|x+z+y\|^2 - \|x+z-y\|^2 + i(\|x+z+iy\|^2 - \|x+z-iy\|^2)\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|z+y\|^2 - \|z-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2 + \|z+iy\|^2 - \|z-iy\|^2)\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \|z+y\|^2 - \|z-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2 + \|z+iy\|^2 - \|z-iy\|^2)\right)\\
&= \frac{1}{4}\left(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2 + \|z+y\|^2 + i\|z+iy\|^2 - \|z-y\|^2 - i\|z-iy\|^2\right)\\
&= \langle x,y\rangle + \langle z,y\rangle.
\end{aligned}
$$

Thus we have additivity in the first argument.

Step 2: We show that (3) respects $\mathbb{Z}$-scaling in the first argument (i.e. $m\langle x,y\rangle = \langle mx,y\rangle$ for all integers $m \in \mathbb{Z}$ and for all $x,y \in V$). It suffices to show that (3) respects $\mathbb{N}$-scaling in the first argument since additivity implies

$$
\begin{aligned}
0 &= \langle 0,y\rangle\\
&= \langle x-x,y\rangle\\
&= \langle x,y\rangle + \langle -x,y\rangle,
\end{aligned}
$$

which implies $\langle -x,y\rangle = -\langle x,y\rangle$ for all $x,y \in V$. We prove (3) respects $\mathbb{N}$-scaling in the first argument using induction on $m \geq 2$. The base case $m=2$ follows from Step 1. Now assume that for some $m \geq 2$ and for all $x,y \in V$, we have $\langle mx,y\rangle = m\langle x,y\rangle$. Then we have

$$
\begin{aligned}
\langle (m+1)x,y\rangle &= \langle mx+x,y\rangle\\
&= \langle mx,y\rangle + \langle x,y\rangle\\
&= m\langle x,y\rangle + \langle x,y\rangle\\
&= (m+1)\langle x,y\rangle,
\end{aligned}
$$

where we applied the induction step at the third line.

Step 3: We show that (3) respects $\mathbb{Q}$-scaling in the first argument. Let $\frac{m}{n} \in \mathbb{Q}$ and let $x,y \in V$. Then since (3) is additive in the first argument and since V is a $\mathbb{C}$-vector space, we have

$$
\begin{aligned}
\frac{m}{n}\langle x,y\rangle &= \frac{m}{n}\left\langle \frac{n}{n}x,y\right\rangle\\
&= \frac{mn}{n}\left\langle \frac{1}{n}x,y\right\rangle\\
&= m\left\langle \frac{1}{n}x,y\right\rangle\\
&= \left\langle \frac{m}{n}x,y\right\rangle.
\end{aligned}
$$

Therefore (3) respects $\mathbb{Q}$-scaling in the first argument.

Step 4: We show that (3) respects $\mathbb{R}$-scaling in the first argument. First note that for each $y \in V$, the map $\langle \cdot,y\rangle \colon V \to \mathbb{C}$ is continuous since the norm is continuous. Let $x,y \in V$ and let $r \in \mathbb{R}$. Choose a sequence (r_n) of rational numbers such that $r_n \to r$ (we can do this since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then we have

$$
\begin{aligned}
\langle rx,y\rangle &= \lim_{n\to\infty}\langle r_n x,y\rangle\\
&= \lim_{n\to\infty} r_n\langle x,y\rangle\\
&= r\langle x,y\rangle.
\end{aligned}
$$

Therefore (3) respects $\mathbb{R}$-scaling in the first component.

Step 5: We show that (3) respects $\mathbb{C}$-scaling in the first component. We first show that $\langle ix,y\rangle = i\langle x,y\rangle$ for all $x,y \in V$.

Let $x, y \in V$. Then we have

$$\begin{aligned}
\langle ix, y\rangle &= \frac{1}{4}\left(\|ix+y\|^2 + i\|ix+iy\|^2 - \|ix-y\|^2 - i\|ix-iy\|^2\right)\\
&= \frac{1}{4}\left(\|x-iy\|^2 + i\|x+y\|^2 - \|x+iy\|^2 - i\|x-y\|^2\right)\\
&= \frac{1}{4}\left(i\|x+y\|^2 - \|x+iy\|^2 - i\|x-y\|^2 + \|x-iy\|^2\right)\\
&= \frac{i}{4}\left(\|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2\right)\\
&= i\langle x, y\rangle.
\end{aligned}$$

Now let $\lambda = r + is \in \mathbb{C}$. Then we have

$$\begin{aligned}
\langle \lambda x, y\rangle &= \langle (r+is)x, y\rangle\\
&= \langle rx + isx, y\rangle\\
&= \langle rx, y\rangle + \langle isx, y\rangle\\
&= r\langle x, y\rangle + s\langle ix, y\rangle\\
&= r\langle x, y\rangle + is\langle x, y\rangle\\
&= (r+is)\langle x, y\rangle\\
&= \lambda\langle x, y\rangle
\end{aligned}$$

for all $x, y \in V$. Therefore (3) respects $\mathbb{C}$-scaling in the first component. □

Problem 3

Proposition 0.3. *Consider $C[0,1]$ equipped with the supremum norm. Let $T\colon C[0,1] \to C[0,1]$ be the linear operator defined by*

$$(Tf)(x) = \int_0^x f(y)\mathrm{d}y$$

for all $x \in [0,1]$. Then T is bounded with $\|T\| = 1$.

Proof. Let $f \in C[0,1]$ such that $\|f\|_\infty \leq 1$. Then

$$\begin{aligned}
\|Tf\|_\infty &= \sup\{|(Tf)(x)| \mid x \in [0,1]\}\\
&= \sup\left\{\left|\int_0^x f(y)\mathrm{d}y\right| \mid x \in [0,1]\right\}\\
&\leq \sup\left\{\int_0^x |f(y)|\mathrm{d}y \mid x \in [0,1]\right\}\\
&\leq \sup\left\{\int_0^x \mathrm{d}y \mid x \in [0,1]\right\}\\
&= \sup\{x \mid x \in [0,1]\}\\
&= 1.
\end{aligned}$$

Thus $\|T\| \leq 1$. To see that $\|T\| = 1$, let $f\colon [0,1] \to \mathbb{C}$ be the constant function $f = 1$. Then $\|f\|_\infty = 1$ and

$$\begin{aligned}
\|Tf\|_\infty &= \sup\{|(Tf)(x)| \mid x \in [0,1]\}\\
&= \sup\left\{\left|\int_0^x \mathrm{d}y\right| \mid x \in [0,1]\right\}\\
&= \sup\{|x| \mid x \in [0,1]\}\\
&= \sup\{x \mid x \in [0,1]\}\\
&= 1.
\end{aligned}$$

□

Problem 4

Proposition 0.4. *Consider $C[a,b]$ equipped with the supremum norm. Define a linear functional $\ell\colon C[a,b] \to \mathbb{R}$ by*

$$\ell(f) := f(a) - f(b).$$

for all $f \in C[a,b]$. Then ℓ is bounded. Moreover the set

$$\{f \in C[a,b] \mid f(a) = f(b)\}$$

is a closed subspace of $C[a,b]$.

Proof. Let $f \in C[a,b]$ such that $\|f\|_\infty \leq 1$. Then

$$\begin{aligned} |\ell(f)| &= |f(a) - f(b)| \\ &\leq |f(a)| + |f(b)| \\ &\leq 1 + 1 \\ &= 2. \end{aligned}$$

Thus $\|\ell\| \leq 2$. To see that $\|\ell\| = 2$, let $f\colon [a,b] \to \mathbb{C}$ be given by

$$f(x) = \frac{2}{b-a}(x-a) - 1$$

for all $x \in [a,b]$. So the graph of f is just the line segment from $(a,-1)$ to $(b,1)$. In particular, $\|f\|_\infty = 1$ and

$$\begin{aligned} |\ell(f)| &= |f(a) - f(b)| \\ &= |-1-1| \\ &= 2. \end{aligned}$$

The last part of the proposition follows from

$$\ker \ell = \{f \in C[a,b] \mid f(a) = f(b)\},$$

and $\ker \ell$ is a closed subspace since ℓ is a bounded linear operator. □

Problem 5

Lemma 0.1. *Consider $C[a,b]$ equipped with the supremum norm. Let $[c,d] \subseteq [a,b]$ and define $\ell_{c,d}\colon C[a,b] \to \mathbb{C}$ by*

$$\ell_{c,d}(f) = \int_c^d f(t)\mathrm{d}t$$

for all $f \in C[a,b]$. Then $\ell_{c,d}$ is a bounded linear functional with $\|\ell_{c,d}\| = d - c$.

Proof. Linearity of $\ell_{c,d}$ follows from linearity of integration. So it suffices to check that $\ell_{c,d}$ is bounded. Let $f \in C[a,b]$ such that $\|f\|_\infty \leq 1$. Then

$$\begin{aligned} |\ell_{c,d}(f)| &= \left| \int_c^d f(t)\mathrm{d}t \right| \\ &\leq \int_c^d |f(t)|\mathrm{d}t \\ &\leq \int_c^d \mathrm{d}t \\ &= d - c. \end{aligned}$$

Thus $\|\ell\| \leq d - c$. To see that $\|\ell\| = d - c$, let $f\colon [a,b] \to \mathbb{C}$ be the constant function $f = 1$. Then $\|f\|_\infty = 1$ and

$$\begin{aligned} |\ell_{c,d}(f)| &= \left| \int_c^d f(t)\mathrm{d}t \right| \\ &= \left| \int_c^d \mathrm{d}t \right| \\ &= |d - c| \\ &= d - c. \end{aligned}$$

□

Proposition 0.5. *Consider $C[-1,1]$ equipped with the supremum norm. Let $\mathcal{Y}$ be the subset of $C[-1,1]$ consisting of all functions $g \in C[-1,1]$ such that*

$$\int_{-1}^{0} g(x)\mathrm{d}x = \int_{0}^{1} g(x)\mathrm{d}x = 0.$$

Then $\mathcal{Y}$ is a closed subspace.

Proof. Note that $\mathcal{Y} = \ker \ell_{-1,0} \cap \ker \ell_{0,1}$ is an intersection of two closed subspaces (since $\ell_{-1,0}$ and $\ell_{0,1}$ are bounded linear functionals by Lemma (0.1)). Thus $\mathcal{Y}$ is a closed subspace. □

Proposition 0.6. *With the notation as in Proposition (0.5) above, let $h \in C[-1,1]$ be given by*

$$h(x) = 2x$$

for all $x \in [-1,1]$. Then there does not exist a $g \in \mathcal{Y}$ such that $\|g-h\|_\infty = d(h,\mathcal{Y})$.

Proof.

Step 1: We will first show that $d(h,\mathcal{Y}) = 1$. To prove $d(h,\mathcal{Y}) \geq 1$, assume for a contradiction that $d(h,\mathcal{Y}) < 1$. Choose $\varepsilon > 0$ and $g \in \mathcal{Y}$ such that

$$\|g-h\|_\infty < 1 - \varepsilon.$$

Write g in terms of its real and imaginary parts, say $g = u + iv$. Then

$$\begin{aligned} 0 &= \int_{-1}^{0} g(x)\mathrm{d}x \\ &= \int_{-1}^{0} u(x)\mathrm{d}x + i\int_{-1}^{0} v(x)\mathrm{d}x \end{aligned}$$

implies $\int_{-1}^{0} u(x)\mathrm{d}x = 0$ and $\int_{-1}^{0} v(x)\mathrm{d}x = 0$. Similarly,

$$\begin{aligned} 0 &= \int_{0}^{1} g(x)\mathrm{d}x \\ &= \int_{0}^{1} u(x)\mathrm{d}x + i\int_{0}^{1} v(x)\mathrm{d}x \end{aligned}$$

implies $\int_{0}^{1} u(x)\mathrm{d}x = 0$ and $\int_{0}^{1} v(x)\mathrm{d}x = 0$. Moreover, we have

$$\begin{aligned} 1 - \varepsilon &> \|g-h\|_\infty \\ &= \sup_{x\in[-1,1]} \sqrt{(u(x)-h(x))^2 + v(x)^2} \\ &\geq \sup_{x\in[-1,1]} \sqrt{(u(x)-h(x))^2} \\ &= \|u-h\|_\infty. \end{aligned}$$

Therefore $u \in \mathcal{Y}$, $\|u-h\|_\infty < 1-\varepsilon$, and u is a real-valued function. Since $\|u-h\|_\infty < 1-\varepsilon$, $h(x) = 2x$ for all $x \in [-1,0]$, and both u and h are real-valued functions, we have

$$u(x) \leq 2x + 1 - \varepsilon$$

for all $x \in [-1,0]$. This implies

$$\begin{aligned} 0 &= \int_{-1}^{0} u(x)\mathrm{d}x \\ &\leq \int_{-1}^{0} (2x+1-\varepsilon)\mathrm{d}x \\ &= (x^2 + x - \varepsilon x)|_{-1}^{0} \\ &= \varepsilon \\ &> 0, \end{aligned}$$

which gives us our desired contradiction. Therefore $d(h,\mathcal{Y}) \geq 1$.

Now we will show that $d(h,\mathcal{Y}) \leq 1$. Let $t \in (0,1]$ and define $g_t \colon [-1,0] \to \mathbb{R}$ by the formula

$$g_t(x) = \begin{cases} 2x+1+t & \text{if } -1 \leq x \leq \frac{-2t}{1+t} \\ -\frac{(t-1)^2}{2t}x & \text{if } \frac{-2t}{1+t} \leq x \leq 0. \end{cases}$$

Extend g_t to all of $[-1,1]$ by the formula

$$g_t(x) = g_t(-x)$$

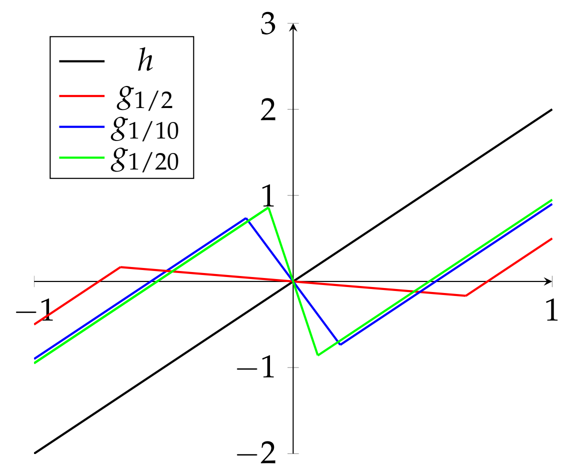
for all $x \in [0,1]$. So g_t is an odd function. Moreover g_t is continous since each segment of g_t is linear and since they agree on their boundaries:

$$\begin{aligned}
2\left(\frac{-2t}{1+t}\right) + 1 + t &= \frac{-4t}{1+t} + \frac{(1+t)^2}{1+t} \\
&= \frac{t^2 - 2t + 1}{1+t} \\
&= \frac{(t-1)^2}{1+t} \\
&= -\frac{(t-1)^2}{2t}\left(\frac{-2t}{1+t}\right)
\end{aligned}$$

and

$$\begin{aligned}
-\frac{(t-1)^2}{2t} \cdot 0 &= 0 \\
&= \frac{(t-1)^2}{2t} \cdot 0.
\end{aligned}$$

The image below gives the graphs for h, $g_{1/2}$, and $g_{1/10}$:



Now observe that

$$\begin{aligned}
\int_{-1}^{0} g_t(x)\mathrm{d}x &= \int_{-1}^{\frac{-2t}{1+t}} (2x+1+t)\mathrm{d}x + \int_{\frac{-2t}{1+t}}^{0} -\frac{(t-1)^2}{2t} x \mathrm{d}x \\
&= (x^2 + x + tx)\Big|_{-1}^{\frac{-2t}{1+t}} + \left(\frac{-(t-1)^2}{4t}x^2\right)\Big|_{\frac{-2t}{1+t}}^{0} \\
&= \left(\frac{2t}{1+t}\right)^2 + \left(\frac{-2t}{1+t}\right) + t\left(\frac{-2t}{1+t}\right) - (1-1-t) + \frac{(t-1)^2}{4t}\left(\frac{2t}{1+t}\right)^2 \\
&= \left(\frac{(t-1)^2}{4t} + 1\right)\left(\frac{2t}{1+t}\right)^2 + (1+t)\left(\frac{-2t}{1+t}\right) + t \\
&= \frac{(t+1)^2}{4t}\frac{4t^2}{(1+t)^2} - t \\
&= t - t \\
&= 0.
\end{aligned}$$

Therefore $g_t \in \mathcal{Y}$ for all $t \in (0,1]$. Moreover, by construction we have

$$\|g_t - h\|_\infty = 1 + t$$

for all $t \in (0,1]$. This implies $d(h, \mathcal{Y}) \le 1$.

Step 2: We claim that there does not exist a $g \in \mathcal{Y}$ such that $\|g - h\|_\infty = 1$. Indeed, assume for a contradiction there does exist a $g \in \mathcal{Y}$ such that $\|g - h\|_\infty = 1$. Choose such a $g \in \mathcal{Y}$. We may assume that g is real-valued: if g is not real-valued, then we pass to its real-valued part u and as argued above we obtain $u \in \mathcal{Y}$ and

$$\begin{aligned} 1 &= \|g - h\|_\infty \\ &= \|u - h\|_\infty \\ &\geq 1. \end{aligned}$$

Since g is real-valued and $\|g - h\|_\infty = 1$, we have

$$2x - 1 \leq g(x) \leq 2x + 1$$

for all $x \in [-1, 1]$. Since g is continous, we cannot have

$$g(x) = \begin{cases} 2x + 1 & \text{for all } x \in (-1, 0) \\ 2x - 1 & \text{for all } x \in (0, 1). \end{cases}$$

Assume $g(x) \neq 2x - 1$ on the interval $(0, 1)$. Choose $c \in (0, 1)$ such that $g(c) \neq 2c - 1$. Since g is continuous and since $g(c) > 2c - 1$, there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$g(x) > 2x - 1 + \varepsilon$$

for all $x \in (c - \delta, c + \delta)$. Choose such ε and δ so that $(c - \delta, c + \delta) \subset (0, 1)$. Then

$$\begin{aligned} 0 &= \int_0^1 g(x)\mathrm{d}x \\ &= \int_0^1 g(x)\mathrm{d}x + \int_{c-\delta}^{c+\delta} g(x)\mathrm{d}x + \int_{c+\delta}^1 g(x)\mathrm{d}x \\ &\geq \int_0^1 (2x - 1 + \varepsilon)\mathrm{d}x + \int_{c-\delta}^{c+\delta} g(x)\mathrm{d}x + \int_{c+\delta}^1 (2x - 1 + \varepsilon)\mathrm{d}x \\ &> \int_0^1 (2x - 1 + \varepsilon)\mathrm{d}x + \int_{c-\delta}^{c+\delta} (2x - 1 + \varepsilon)\mathrm{d}x + \int_{c+\delta}^1 (2x - 1 + \varepsilon)\mathrm{d}x \\ &= \int_0^1 (2x - 1 + \varepsilon)\mathrm{d}x \\ &= (x^2 - x + \varepsilon x)|_0^1 \\ &= \varepsilon \\ &> 0 \end{aligned}$$

gives us a contradiction.

Thus $g(x) \neq 2x + 1$ on the interval $(-1, 0)$. Choose $c \in (-1, 0)$ such that $g(c) \neq 2c + 1$. Then by the same argument as above, we have

$$\begin{aligned} 0 &= \int_{-1}^0 g(x)\mathrm{d}x \\ &< \int_{-1}^0 (2x - 1 + \varepsilon)\mathrm{d}x \\ &= \varepsilon \\ &> 0, \end{aligned}$$

which also gives us a contradiction. Therefore there does not exist a $g \in \mathcal{Y}$ such that $\|g - h\|_\infty = 1$. □

Problem 6

Definition 0.1. Let $\mathcal{X}$ be a normed linear space. For a set $A \subseteq \mathcal{X}$ we define $A^\perp$ to be the subset of $\mathcal{X}^*$ consisting of all $\ell \in \mathcal{X}^*$ such that $\ell(a) = 0$ for all $a \in A$. Similarly, for a set $M \subseteq \mathcal{X}^*$ we define $M_\perp$ to be the subset of $\mathcal{X}$ consisting of all vectors $x \in \mathcal{X}$ such that $\ell(x) = 0$ for all $\ell \in M$.

Proposition 0.7. *Let $\mathcal{X}$ be a normed linear space, let A be a subset of $\mathcal{X}$, and let M be a subset of $\mathcal{X}^*$. Then $A^\perp$ and $M_\perp$ are closed subspaces of $\mathcal{X}^*$ and $\mathcal{X}$ respectively.*

Proof. Let $x \in \mathcal{X}$. Define $\widehat{x}\colon \mathcal{X}^* \to \mathbb{C}$ by
$$\widehat{x}(\ell) = \ell(x)$$
for all $\ell \in \mathcal{X}^*$. We claim that $\widehat{x}$ is a bounded linear functional. To see that $\widehat{x}$ is linear, let $\ell, \ell' \in \mathcal{X}^*$ and let $\lambda, \lambda' \in \mathbb{C}$. Then

$$\begin{aligned}\widehat{x}(\lambda\ell + \lambda'\ell') &= (\lambda\ell + \lambda'\ell')(x) \\ &= \lambda\ell(x) + \lambda'\ell'(x) \\ &= \lambda\widehat{x}(\ell) + \lambda'\widehat{x}(\ell').\end{aligned}$$

To see that $\widehat{x}$ is bounded, let $\ell \in \mathcal{X}^*$. Then

$$\begin{aligned}|\widehat{x}(\ell)| &= |\ell(x)| \\ &\leq \|x\|\|\ell\|.\end{aligned}$$

Therefore $\widehat{x}$ is a bounded linear functional. In particular $\ker\widehat{x}$ is a closed subspace. Thus

$$A^{\perp} = \bigcap_{a \in A} \ker\widehat{a} \quad \text{and} \quad M_{\perp} = \bigcap_{\ell \in M} \ker\ell$$

are closed subspaces since an arbitrary intersection of closed subspaces is a closed subspace. □

Proposition 0.8. *Let $\mathcal{X}$ be a normed linear space, let A be a subset of $\mathcal{X}$, and let M be a subset of $\mathcal{X}^*$. Then $\overline{\text{span}}(A) \subseteq (A^{\perp})_{\perp}$ and $\overline{\text{span}}(M) \subseteq (M_{\perp})^{\perp}$.*

Proof. Proposition (0.7) implies $(A^{\perp})_{\perp}$ and $(M_{\perp})^{\perp}$ are closed subspace. Thus, it suffices to show

$$\text{span}(A) \subseteq (A^{\perp})_{\perp} \quad \text{and} \quad \text{span}(M) \subseteq (M_{\perp})^{\perp}.$$

First we show the former. Let $\lambda_1 a_1 + \cdots + \lambda_n a_n \in \text{span}(A)$ and let $\ell \in A^{\perp}$. Then since $\ell(a) = 0$ for all $a \in A$, we have

$$\begin{aligned}\ell(\lambda_1 a_1 + \cdots + \lambda_n a_n) &= \lambda_1 \ell(a_1) + \cdots + \lambda_n \ell(a_n) \\ &= \lambda_1 \cdot 0 + \cdots + \lambda_n \cdot 0 \\ &= 0.\end{aligned}$$

Since ℓ was arbitrary, this implies $\lambda_1 a_1 + \cdots + \lambda_n a_n \in (A^{\perp})_{\perp}$, and hence $\text{span}(A) \subseteq (A^{\perp})_{\perp}$.

Now we show the latter. Let $\lambda_1 \ell_1 + \cdots + \lambda_n \ell_n \in \text{span}(M)$ and let $x \in M_{\perp}$. Then since $\ell(x) = 0$ for all $\ell \in M$, we have

$$\begin{aligned}(\lambda_1 \ell_1 + \cdots + \lambda_n \ell_n)(x) &= \lambda_1 \ell_1(x) + \cdots + \lambda_n \ell_n(x) \\ &= \lambda_1 \cdot 0 + \cdots + \lambda_n \cdot 0 \\ &= 0.\end{aligned}$$

Since x was arbitrary, this implies $\lambda_1 \ell_1 + \cdots + \lambda_n \ell_n \in (M_{\perp})^{\perp}$, and hence $\text{span}(M) \subseteq (M_{\perp})^{\perp}$. □

Problem 7

Proposition 0.9. *$(\ell^1)^*$ is isometrically isomorphic to ℓ^{∞}.*

Proof. For each $n \in \mathbb{N}$, let e^n denote the sequence with entry 1 in the nth component and entry 0 everywhere else. Define $\Phi\colon (\ell^1)^* \to \ell^{\infty}$ by
$$\Phi(\psi) = (\psi(e^n))$$
for all $\psi \in (\ell^1)^*$. Note that for any $\psi \in (\ell^1)^*$, we have $|(\psi(e^n))| \leq \|\psi\|$, and therefore $(\psi(e^n)) \in \ell^{\infty}$. We claim that $\|\psi\| = \|\Phi(\psi)\|_{\infty}$. Indeed,

$$\begin{aligned}\|\Phi(\psi)\|_{\infty} &= \sup\{|\psi(e^n)| \mid n \in \mathbb{N}\} \\ &\leq \sup\left\{\left|\psi\left(\sum_{n=1}^{\infty} a_n e^n\right)\right| \mid \sum_{n=1}^{\infty} |a_n| \leq 1\right\} \\ &= \|\psi\|.\end{aligned}$$

To prove the reverse inequality assume for a contradiction that $\|\psi\| > \|\Phi(\psi)\|_\infty$. Choose $\varepsilon > 0$ and $\sum_{n=1}^\infty a_n e^n \in \ell^1$ such that $\sum_{n=1}^\infty |a_n| \le 1$ and

$$\left|\psi\left(\sum_{n=1}^\infty a_n e^n\right)\right| > \|\Phi(\psi)\|_\infty + \varepsilon. \tag{4}$$

Choose $N \in \mathbb{N}$ such that $\sum_{n=N}^\infty |a_n| < \varepsilon/\|\psi\|$ (we can find such an N since $\sum_{n=1}^\infty |a_n| < \infty$). Then

$$\begin{aligned}
\left|\psi\left(\sum_{n=1}^\infty a_n e^n\right)\right| &= \left|\psi\left(\sum_{n=1}^N a_n e^n + \sum_{n=N+1}^\infty a_n e^n\right)\right| \\
&= \left|\psi\left(\sum_{n=1}^N a_n e^n\right) + \psi\left(\sum_{n=N+1}^\infty a_n e^n\right)\right| \\
&= \left|\sum_{n=1}^N a_n \psi(e^n) + \psi\left(\sum_{n=N+1}^\infty a_n e^n\right)\right| \\
&\le \left|\sum_{n=1}^N a_n \psi(e^n)\right| + \left|\psi\left(\sum_{n=N+1}^\infty a_n e^n\right)\right| \\
&\le \sum_{n=1}^N |a_n||\psi(e^n)| + \|\psi\| \sum_{n=N+1}^\infty |a_n| \\
&< \|\Phi(\psi)\|_\infty \sum_{n=1}^N |a_n| + \|\psi\| \cdot \frac{\varepsilon}{\|\psi\|} \\
&\le \|\Phi(\psi)\|_\infty + \varepsilon.
\end{aligned}$$

This contradicts (**??**).

Next we show Φ is linear. Let $\varphi, \psi \in (\ell^1)^*$ and let $\lambda, \mu \in \mathbb{C}$. Then

$$\begin{aligned}
\Phi(\lambda\varphi + \mu\psi) &= ((\lambda\varphi + \mu\psi)(e^n)) \\
&= \lambda(\varphi(e^n)) + \mu(\psi(e^n)) \\
&= \lambda\Phi(\varphi) + \mu\Phi(\psi).
\end{aligned}$$

Therefore Φ is an isometric embedding.

Now show that Φ is surjective, and hence an isometric isomorphism. Let $(a_n) \in \ell^\infty$, let $M = \sup\{|a_n|\}$, and let $E = \operatorname{span}\{e^n \mid n \in \mathbb{N}\}$. Define $\varphi\colon E \to \mathbb{C}$ to be the unique linear map such that

$$\varphi(e^n) = a_n$$

for all $n \in \mathbb{N}$. Let $x = x_{n_1}e^{n_1} + \cdots + x_{n_k}e^{n_k} \in E$ such that $|x_{n_1}| + \cdots + |x_{n_k}| \le 1$. Then

$$\begin{aligned}
|\varphi(x_{n_1}e^{n_1} + \cdots + x_{n_k}e^{n_k})| &= |x_{n_1}\varphi(e^{n_1}) + \cdots + x_{n_k}\varphi(e^{n_k})| \\
&= |x_{n_1}a_{n_1} + \cdots + x_{n_k}a_{n_k}| \\
&\le |x_{n_1}||a_{n_1}| + \cdots + |x_{n_k}||a_{n_k}| \\
&\le |x_{n_1}|M + \cdots + |x_{n_k}|M \\
&= (|x_{n_1}| + \cdots + |x_{n_k}|)M \\
&\le M
\end{aligned}$$

It follows that φ is bounded. By the Hahn-Banach Theorem, there exists a bounded linear functional $\widetilde{\varphi}$ defined on all of ℓ^1 such that $\widetilde{\varphi}|_E = \varphi$ and $\|\widetilde{\varphi}\| = \|\varphi\|$. Choose such a $\widetilde{\varphi} \in (\ell^1)^*$. Then clearly $\Phi(\widetilde{\varphi}) = (a_n)$. Therefore Φ is surjective, and hence an isometric isomorphism. □

Appendix

Problem 1

Proposition 0.10. *Let A be a non-emtpy set of real numbers which is bounded above and let λ be any non-negative real number. Then*

$$\sup(\lambda A) = \lambda \sup(A). \tag{5}$$

Proof. If $\lambda = 0$, then (5) is obvious, so assume $\lambda > 0$. Let α denote $\sup(A)$. Choose any element in λA, say λa where $a \in A$. Then since $a \le \alpha$ and λ is non-negative, we have $\lambda a \le \lambda\alpha$. This implies

$$\sup(\lambda A) \le \lambda \sup(A).$$

For the reverse direction, observe that

$$\begin{aligned} \sup(A) &= \sup(\lambda^{-1}\lambda A) \\ &\le \lambda^{-1}\sup(\lambda A), \end{aligned}$$

and this implies

$$\sup(\lambda A) \ge \lambda \sup(A).$$

$\square$

Proposition 0.11. *Let A and B be non-empty sets of non-negative real numbers both of which are bounded above. Then*

$$\sup(A+B) = \sup(A) + \sup(B). \tag{6}$$

Proof. Let α denote $\sup(A)$, let β denote $\sup(B)$, and let $a+b$ be an arbitrary element in $A+B$. Then $a \le \alpha$ and $b \le \beta$ implies $a+b \le \alpha+\beta$. Therefore

$$\sup(A+B) \le \sup(A) + \sup(B). \tag{7}$$

To show the reverse inequality, we assume (for a contradiction) that the inequality (7) is strict

$$\sup(A+B) < \sup(A) + \sup(B).$$

Choose $\varepsilon > 0$ such that

$$\sup(A+B) < \sup(A) + \sup(B) - \varepsilon. \tag{8}$$

Choose $a \in A$ and $b \in B$ such that $a > \alpha - \varepsilon/2$ and $b > \beta - \varepsilon/2$. Then

$$\begin{aligned} a+b &> \alpha - \frac{\varepsilon}{2} + \beta - \frac{\varepsilon}{2} \\ &= \alpha + \beta - \varepsilon. \end{aligned}$$

But this contradicts (8). Therefore

$$\sup(A+B) \ge \sup(A) + \sup(B).$$

$\square$