MATH 8520
SPRING 2019
HOMEWORK 11

Due Monday April 20, 2019

1. (5 pts) Let V be a domain with quotient field K. Show that the following conditions are equivalent.

(1) For all nonzero $a, b \in V$, either $a|b$ or $b|a$.
(2) For all nonzero $\omega \in K$, either ω or ω^{-1} is in V.
(3) There is a valuation v on K such that $V = \{x \in K | v(x) \geq 0\} \bigcup \{0\}$.

3. Let R be a domain with quotient field K and $\overline{R}$ the integral closure of R. V will denote a valuation overring of R.

a) (5 pts) Show that $\overline{R}$ is integrally closed.
b) (5 pts) Show that $\overline{R} \subseteq \bigcap_{R \subseteq V \subseteq K} V$.
c) (5 pts) Now show that $\overline{R} = \bigcap_{R \subseteq V \subseteq K} V$.

4. Let R be a domain with quotient field K. An element $x \in K$ is said to be *almost integral* if there is a nonzero $r \in R$ such that $rx^n \in R$ for all $n \in \mathbb{N}$. We say that a domain is *completely integrally closed* if it contains all of its almost integral elements.

a) (5 pts) Give an example of an element that is almost integral, but not integral.
b) (5 pts) Show that if $x \in K$ is integral over R, then x is almost integral over R..
c) (5 pts) Show that if R is Noetherian, then any almost integral element over R is integral over R.
d) (5 pts) Let V be a valuation domain that is not a field. Show that V is completely integrally closed if and only if V is one-dimensional (that is, every nonzero prime ideal is maximal).