Free Resolutions

Contents

1 Introduction

We want to "understand" mathematical objects using limited technology. For example, given a group G, we obtain a lot of information about G by just knowing its order $|G|$. For instance, if p is a prime such that p divides $|G|$, then the Sylow theorems tells us that there exists a p-Sylow subgroup of G. Another thing we get from knowing the order of a group is whether or not it is isomorphic to another group: if G and H are two groups such that $|G| \neq |H|$, then $G \ncong H$ (or in contrapositive form: if $G \cong H$ then $|G| = |H|$). This means that the order of a group is an invariant. Even though the order of a group is a nice invariant to have, it is not strong enough to determine the group completely: if $|G| = |H|$, then usually it's very difficult to determine whether $G \cong H$ or not.

There are similar tools for understanding rings and modules. -SSW

1.1 Focus of this class

1.1.1 Part I

In part I one of the class, we will focus on understanding the contents of the Hilbert Syzygy Theorem as well as giving a (partial) proof of it. Let us state the Hilbert Syzygy Theorem.

Theorem 1.1. *(The Hilbert Syzygy Theorem) Let $R = k[X_1, \dots, X_d]$ be the polynomial ring in d indeterminates over a field k and let I be the ideal in R generated by $f_1, \dots, f_{\beta_1}$. Then there exists an exact sequence of the form*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \tag{1}$$

The exact sequence (8) is called an **augmented free resolution of** R/I **over** R. If we remove R/I from (8), then we get a **free resolution of** R/I **over** R. We view the term R^{β_i} in (8) as sitting in **homological degree** i. The maps ∂_i in (8) are called **differentials**. We often simplify notation by writing (8) as (F, ∂), where we think of F as a graded R-module whose homogeneous component in degree i is R^{β_i} and we think of ∂ as a graded R-homomorphism $\partial\colon F \to F$ of degree -1. Then $\partial_i := \partial|_{F_i}$ where $F_i := R^{\beta_i}$.

1.1.2 Part II

In part II, we will go over a bunch of examples. Here are two to consider now:

Example 1.1. Consider the case where $R = k[X, Y]$ and $I = \langle X^2, XY \rangle$. Then we have the following short exact sequence of R-complexes

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X^2 & XY \end{pmatrix}} R \longrightarrow R/I$$

Example 1.2. (Twisted Cubic) Consider the case where $R = k[X, Y, Z, W]$ and $I = \langle XZ - Y^2, YW - Z^2, XW - YZ \rangle$. A free resolution of R/I is given by

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} W & Z \\ Y & X \\ -Z & -Y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} XZ-Y^2 & YW-Z^2 & WX-YZ \end{pmatrix}} R \longrightarrow R/I$$

Another very important example that we will consider is when $R = k[X_1, \dots, X_d]$ and $I = \langle X_{i_1}, \dots, X_{i_{\beta_1}} \rangle$. In this case, we can build a resolution explicitly, called the **Koszul complex of** R/I **over** R, named after Jean Louis Koszul. For example, consider $I = \langle X_1, X_2, X_3 \rangle$. Then the Koszul complex of R/I over R looks like

$$R \xrightarrow{\begin{pmatrix}X_1\\-X_2\\X_3\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}0&-X_3&-X_2\\-X_3&0&X_1\\X_2&X_1&0\end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix}X_1&X_2&X_3\end{pmatrix}} R$$

$$\begin{array}{llllll}
e_{1,2,3} & \longmapsto & X_1e_{2,3}-X_2e_{1,3}+X_3e_{1,2} & & & \\
 & & e_{2,3} & \longmapsto & X_2e_3-X_3e_2 & \\
 & & e_{1,3} & \longmapsto & X_1e_3-X_3e_1 & \\
 & & e_{1,2} & \longmapsto & X_1e_2-X_2e_1 & \\
 & & & & e_1 & \longmapsto X_1 \\
 & & & & e_2 & \longmapsto X_2 \\
 & & & & e_3 & \longmapsto X_3
\end{array}$$

1.1.3 Part III

If all f_j's in Theorem (1.1) are homogeneous, then the resolution (8) can be built minimally and the β_i's are independent of the choice of minimal free resolution. In this case, we call the $\beta_i = \beta_i^R(R/I)$ the ith **Betti number of R/I over R**, named after the Italian mathematician Enrico Betti. If J is another homogeneous ideal of R, and $\beta_i^R(R/I) \neq \beta_i^R(R/J)$ for some i, then $R/I \ncong R/J$. Thus the Betti numbers give us an invariant of R/I. However, just like in the case of orders in group theory, this invariant is not sufficiently strong enough to determine R/I: if $\beta_i^R(R/I) = \beta_i^R(R/J)$ for all i, then R/I may not be isomorphic to R/J.

In part III of the class, we will try to find finer technology. For instance, it turns out that the Koszul complex is a graded commutative R-algebra with the multiplication rule given by

$$e_Ae_B = \begin{cases} 0 & \text{if } A \cap B = \emptyset \\ e_{A\cup B} & \text{else} \end{cases}$$

subject to the rule

$$e_Ae_B = (-1)^{|A||B|}e_Be_A,$$

where $A, B \subseteq \{1, \dots, m\}$. Moreover, we have the Leibniz rule

$$\partial(e_Ae_B) = \partial(e_A)e_B + (-1)^{|A|}e_A\partial(e_B).$$

This shows that the Koszul complex is a **differential graded algebra (DGA) resolution**.

2 Preliminary Material and Notation

Throughout these notes, let R be a commutative ring with identity. We also let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of a natural numbers together with 0 (SSW: "My class, my rules").

2.1 Linear Algebra

2.1.1 Free Modules

Definition 2.1. Let M be an R-module.

1. A sequence $e_1, \dots, e_n \in M$ is a **(finite) basis** for M if it generates M as an R-module and it is linearly independent over R, i.e. for every $m \in M$ there exists $r_1, \dots, r_n \in R$ such that $m = \sum_{i=1}^n r_ie_i$, and if $a_1, \dots, a_n \in R$ such that $\sum_{i=1}^n a_ie_i = 0$, then $a_i = 0$ for all i.

2. We say M is a **finite rank free R-module** if it has a basis.

Example 2.1. R^n is the **standard free R-module of rank n**. It has as basis the **standard basis elements** e_i where e_i is the vector with 1 in the ith entry and 0 everywhere else.

Example 2.2. If I is a nonzero ideal in R, then R/I is not a free R-module. Indeed, if r is a nonzero element in I, then for all $s \in R$, we have $r\bar{s} = \overline{rs} = 0$ in R/I. In other words, "**torsion**" makes linear independence fail for elements of R/I when taking coefficients from R.

2.1.2 Universal Mapping Property of Free R-Modules

The universal mapping property of free R-modules can be stated as follows: Let F be a free R-module with basis $e_1, \dots, e_n \in F$. For all R-modules M and for all $m_1, \dots, m_n \in M$ there exists a unique R-module homomorphism $\varphi\colon F \to M$ such that $\varphi(e_i) = m_i$ for all $i = 1, \dots, n$. In terms of diagrams, this is pictured as follows:

$$\begin{tikzcd} \{e_1, \dots, e_n\} \arrow[r, hook] \arrow[dr, "e_i \mapsto m_i"'] & F \arrow[d, dashed, "\exists! \varphi"] \\ & M \end{tikzcd}$$

Using the universal mapping property of free R-modules, let us prove the following theorem:

Theorem 2.1. *If F and G are finite rank free R-modules with basis $e_1, \dots, e_n$ and $f_1, \dots, f_n$ respectively, then $F \cong G$.*

Proof. By the universal mapping property of free R-modules there exists a unique R-module homomorphism $\varphi\colon F \to G$ such that $\varphi(e_i) = f_i$ for all $i = 1, \dots, n$. Similarly, there exists a unique R-module homomorphism $\psi\colon G \to F$ such that $\psi(f_i) = e_i$ for all $i = 1, \dots, n$. In particular, we see that $\psi \circ \varphi\colon F \to F$ satisfies $(\psi \circ \varphi)(e_i) = e_i$. But we also have $1(e_i) = e_i$ for all $i = 1, \dots, n$, where $1\colon F \to F$ is the identity map. Therefore by uniqueness of the map in the universal mapping property of free R-modules, we must have $\psi \circ \varphi = 1$. A similar argument shows that $\varphi \circ \psi = 1$. □

Corollary. *Let F be a free R-module with basis $e_1, \dots, e_n \in F$. Then $F \cong R^n$.*

Remark. Note that you can prove Theorem (2.1) without the universal mapping property of free R-modules, but the point is that you'd have to show well-definedness, linearity, etc... of the maps constructed. The point is that all of this is built into the universal mapping property of free R-modules.

2.1.3 Representing R-module Homomorphisms By Matrices

Let $\varphi\colon R^n \to R^m$ be an R-module homomorphism and let $e_1, \dots, e_n$ denote the **canonical basis** of R^n, i.e. $e_i = (0 \dots, 1, \dots 0)^\top$ where 1 is at the ith entry and 0 is in all other entries. Any $x \in R^n$ has a unique linear combination as $x = x_1 e_1 + \cdots + x_n e_n$ where $x_i \in R$. In particular, $\varphi(e_j)$ has a unique representation as

$$\varphi(e_j) = \sum_{i=1}^{n} a_{ij} e_i.$$

Writing x as a column vector $[x]$ and using linearity of φ, we see that

$$\varphi(x) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = [\varphi][x],$$

where $[\varphi] = (a_{ij})$ is an $m \times n$ matrix with entries in R. We call $[\varphi]$ the **matrix representation of** φ. Addition, scalar multiplication, and composition of R-module homomorphisms correspond to addition, scalar multiplication, and multiplication of matrices.

2.2 Noetherian Rings and Modules

Definition 2.2. An R-module M is said to be **finitely generated** if there exists a surjective R-module homomorphism from R^n to M for some $n \in \mathbb{N}$.

Theorem 2.2. *The following conditions are equivalent:*

1. *Every ideal of R is finitely generated.*
2. *Ascending chain condition (acc) of ideals in R: given a chain of ideals*
$$I_1 \subseteq I_2 \subseteq \cdots$$
of ideals in R, there exists $N \in \mathbb{N} = \{0, 1, 2, \dots\}$ such that
$$I_N = I_{N+1} = I_{N+2} = \cdots$$
3. *Maximum condition for ideals in R: every non-empty set of ideals of R contains an element maximal with respect to containment.*
4. *For every $n \in \mathbb{N}$ every submodule of R^n is finitely generated.*
5. *For every $n \in \mathbb{N}$, R^n satisfies maximum condition on submodules.*
6. *Acc of on submodules of R^n for every $n \in \mathbb{N}$.*

Definition 2.3. R is said to be **Noetherian** if it satisfies any of the equivalent conditions of Theorem (2.2).

Theorem 2.3. *(Hilbert Basis Theorem) If R is Noetherian then $R[X]$ is also Noetherian.*

Remark. More generally, if I is an ideal of $R[X_1, \dots, X_n]$, then $R[X_1, \dots, X_n]/I$ is Noetherian. In particular, $k[X_1, \dots, X_n]/I$ is Noetherian since k is Noetherian.

2.3 Exact Sequences

Definition 2.4. A sequence of R-module homomorpisms

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is said to be **exact at** B if $\mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$. A sequence of R-module homomorphisms

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \longrightarrow \cdots$$

is said to be **exact** if it is exact at every A_i. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

2.3.1 Free Resolutions

Theorem 2.4. *Assume that R is Noetherian and let M be a finitely generated R-module. Then there exists an exact sequence of the form*

$$\cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow M \longrightarrow 0 \tag{2}$$

Proof. First note that since M is finitely generated, there exists a surjective R-module homomorphism $\tau_0 \colon R^{\beta_0} \to M$. Denote $M_1 := \mathrm{Ker}(\tau_0)$ the kernel of τ_0 and denote $\iota_1 \colon M_1 \hookrightarrow R^{\beta_0}$ the inclusion map. Since R is Noetherian and M_1 is a submodule of R^{β_0}, we see that M_1 is finitely generated (by Theorem (2.2)). Therefore there exists a surjective homomorphism $\tau_1 \colon R^{\beta_1} \to M_1$. Let $\iota_1 \colon \mathrm{Ker}(\tau_0) \to R^{\beta_0}$ denote the inclusion map and let $\phi_1 := \iota_0 \circ \tau_1$. Continuing in this way, for each $i \in \mathbb{N}$, we construct short exact sequences of the form

$$0 \longrightarrow M_{i+1} \xrightarrow{\iota_{i+1}} R^{\beta_i} \xrightarrow{\tau_i} M_i \longrightarrow 0$$

where we denote $M_0 := M$. A standard argument in homological algebra says that we can connect these short exact sequences together to form the long exact sequence (2) where $\partial_{i+1} := \iota_i \circ \tau_{i+1}$. □

Definition 2.5. The exact sequence (2) is called an **augmented free resolution of** M.

In general, free resolutions are hard to compute. However here are two examples to consider.

Example 2.3. Recall that the fundamental theorem of finitely generated abelian groups says that if G is an abelian group, then there exists $r \in \mathbb{Z}_{\geq 0}$ and $d_1, \ldots, d_n \in \mathbb{Z}_{>1}$ such that

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n.$$

This boils down to the fact that there exists a free resolution of the form

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{\varphi} \mathbb{Z}^{r+n} \longrightarrow G \longrightarrow 0$$

where the $(n+r) \times n$ matrix representation of φ has the form

$$[\varphi] = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 2.4. If R is an integral domain and r is a nonzero unit in R. Then we have the short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/r \longrightarrow 0$$

If R is a PID, then these are all the resolutions.

Example 2.5. Let $R = k[X,Y]/\langle XY \rangle$ and let $M = R/\langle \overline{X} \rangle$. Then we have

$$\cdots \longrightarrow R \xrightarrow{\cdot \overline{X}} R \xrightarrow{\cdot \overline{Y}} R \xrightarrow{\cdot \overline{X}} R \longrightarrow R/r \longrightarrow 0$$

In fact, we can prove that this resolution does not stop. The idea is that straying away from polynomial rings makes this construction go bad.

2.3.2 Hilbert Syzygy Theorem

Theorem 2.5. *(Hilbert Syzygy Theorem) Let $R = k[X_1, \ldots, X_d]$ with k a field and let M be a finitely generated R-module. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow M \longrightarrow 0 \tag{3}$$

with $\beta_i \geq 0$ for all i.

Remark. If $d = 0$, then R is just the field k and M is a k-vector space. If $d = 1$, then R is a PID. Since every submodule of a free R-module is free (when R is a PID), we see that the free resolution terminates at R^{β_1}. More variables require more work.

3 Graded Resolutions

Tracking finer information about certain resolutions. In this section, we assume that k is a field, R is a polynomial ring $R = k[X_1, \ldots, X_d]$.

Definition 3.1. Let I be an ideal in R. We say I is a **homogeneous** (or **graded**) ideal in R if it can be generated by homogeneous polynomials (not necessarily of the same degree).

Example 3.1. Consider the ring $R = k[X,Y]$ and $I = \langle X^2 - XY^2, XY^2 \rangle$. Even though $X^2 - XY^2$ is not a homogeneous polynomial, the ideal I is still a homogeneous ideal. This is because $I = \langle X^2, XY^2 \rangle$.

Theorem 3.1. *(Graded Hilbert Basis Theorem) If $I = \langle f_1, \ldots, f_{\beta_1} \rangle$ where each f_i is homogeneous. Then there exists a free resolution*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R \longrightarrow R/I \longrightarrow 0 \tag{4}$$

such that each differential ∂_j is represented by a matrix of homogeneous polynomials.

3.1 Examples of Graded Resolutions

Example 3.2. Consider the case where $R = k[X, Y]$ and $I = \langle X^a, Y^b \rangle$ where $a, b \geq 1$. Then we have the following short exact sequence of R-complexes

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -Y^a \\ X^b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} X^a & Y^b \end{pmatrix}} R \longrightarrow R/I$$

This uses unique factorization of the indeterminates.

Example 3.3. Consider the case where $R = k[X, Y]$ and $I = \langle X^a, XY, Y^b \rangle$ where $a, b \geq 2$. We claim that

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -Y & 0 \\ X^{a-1} & Y^{b-1} \\ 0 & X \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} X^a & XY & Y^b \end{pmatrix}} R \longrightarrow R/I$$

is a free resolution of R/I. To see this, we first show $\mathrm{Ker}(\partial_1) = \mathrm{Im}(\partial_2)$. Let $(f, g, h)^\top \in \mathrm{Ker}(\partial_1)$, so we have

$$X^a f + XYg + Y^b h = 0. \tag{5}$$

Then (5) implies $Y | X^a f$ which implies $Y | f$, and so $f = Yf_1$ for some $f_1 \in R$. Similarly, (5) implies $X | Y^b h$ which implies $X | h$, and so $h = Xh_1$ for some $h_1 \in R$. Thus we have

$$\begin{aligned} 0 &= X^a f + XYg + Y^b h \\ &= X^a Y f_1 + XYg + Y^b X h \\ &= XY(X^{a-1} f_1 + g + Y^{b-1} h_1), \end{aligned}$$

and this further implies $X^{a-1} f_1 + g + Y^{b-1} h_1 = 0$. Solving for g, we find that $g = -X^{a-1} f_1 - Y^{b-1} h_1$. Therefore we see that

$$\begin{aligned} \begin{pmatrix} f \\ g \\ h \end{pmatrix} &= \begin{pmatrix} Yf_1 \\ -X^{a-1} f_1 - Y^{b-1} h_1 \\ Xh_1 \end{pmatrix} \\ &= f_1 \begin{pmatrix} Y \\ -X^{a-1} \\ 0 \end{pmatrix} + h_1 \begin{pmatrix} 0 \\ Y^{b-1} \\ X \end{pmatrix} \\ &\in \mathrm{Im}(\partial_2). \end{aligned}$$

In particular, we have $\mathrm{Im}(\partial_2) \supseteq \mathrm{Ker}(\partial_1)$ (and hence $\mathrm{Im}(\partial_2) = \mathrm{Ker}(\partial_1)$ since the reverse inclusion is already known). Finally, since R is an integral domain and since $(-Y, X^{a-1}, 0)^\top$ and $(0, Y^{b-1}, X)$ are linearly independent, we see that ∂_2 is injective.

3.2 Relation on the β_i

Note that the β_i in each of the previous examples satisfy the relations

$$\beta_0 - \beta_1 + \beta_2 = 0.$$

This actually happens in general:

Lemma 3.2. *Let*

$$0 \longrightarrow V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \longrightarrow 0.$$

be a short exact sequence of k-vector spaces.

Lemma 3.3. *Let*

$$0 \longrightarrow k^{\beta_d} \longrightarrow \cdots \longrightarrow k^{\beta_i} \xrightarrow{\partial_i} k^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow k^{\beta_1} \xrightarrow{\partial_1} k^{\beta_0} \longrightarrow 0 \tag{6}$$

be an exact sequence of k-vector spaces. Then

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

Proof. Let $K_i := \mathrm{Ker}(\partial_i)$ for all $0 \le i \le d$ and let $K_{-1} = 0$. Then for each $0 \le i \le d$, exactness at k^{β_i} in (6) implies exactness of

$$0 \longrightarrow K_i \hookrightarrow k^{\beta_i} \xrightarrow{\partial_i} K_{i-1} \longrightarrow 0.$$

Since the dimension function is additive on short exact sequences, we have $\beta_i = \dim(K_i) + \dim(K_{i-1})$. Therefore we have a telescoping series

$$\begin{aligned}\sum_{i=0}^{d}(-1)^i\beta_i &= \sum_{i=0}^{d}(-1)^i(\dim(K_i)+\dim(K_{i-1})) \\ &= (-1)^d\dim(K_d) + \dim(K_{-1}) \\ &= 0.\end{aligned}$$

□

Theorem 3.4. *Let I be a nonzero ideal in R and let*

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_i} \xrightarrow{\partial_i} R^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\partial_1} R^{\beta_0} \xrightarrow{\partial_0} R/I \longrightarrow 0 \tag{7}$$

be a free resolutions of R/I over R. Then

$$\sum_{i=0}^{d}(-1)^i\beta_i = 0.$$

Proof. Localizing R at $\langle 0\rangle$ gives us a field $R_{\langle 0\rangle} = k(X_1,\dots,X_d)$ since R is an integral domain. Since $I \neq 0$, there exists $s \in I\backslash\{0\}$ that is nonzero. We claim then that $(R/I)_{\langle 0\rangle} = 0$. Indeed an element in $(R/I)_{\langle 0\rangle}$ looks like $\overline{r}/t$ where $r \in R$ and $t \in R\backslash 0$. Then

$$\frac{\overline{r}}{t} = \frac{\overline{sr}}{st} = \frac{\overline{0}}{st} = 0.$$

Therefore localizing our resolution at $\langle 0\rangle$ (which is exact) gives us an exact sequence of vector spaces over the field $R_{\langle 0\rangle}$:

$$0 \longrightarrow R_{\mathfrak{p}}^{\beta_d} \longrightarrow \cdots \longrightarrow R_{\mathfrak{p}}^{\beta_i} \xrightarrow{\partial_i} R_{\mathfrak{p}}^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow R_{\mathfrak{p}}^{\beta_1} \xrightarrow{\partial_1} R_{\mathfrak{p}}^{\beta_0} \longrightarrow 0 \tag{8}$$

Now we apply Lemma (3.3) to obtain our desired result. □

4 Graded Rings and Modules

Unless otherwise specified, all rings discussed below are rings with unity.

4.1 Graded Rings

A **graded ring** R is a ring together with a direct sum decomposition

$$R = \bigoplus_{i\in\mathbb{N}} R_i,$$

where the R_i are abelian groups which satisfies the condition that if $r_i \in R_i$ and $r_j \in R_j$, then $r_ir_j \in R_{i+j}$ for all $i, j \in \mathbb{N}$. The R_i are called **homogeneous components** of R and the elements of R_i are called **homogeneous elements** of **degree** i. If r is a homogeneous element in R, then we denote the degree of r as $\deg(r)$. When we say "let R be a graded ring", then it is understood that the homogeneous components of R are denoted R_i. Note that if R is a graded ring, then R_0 must be a ring.

Remark. We sometimes list elements in a graded ring R as $r_{i_1}, \dots, r_{i_n}$. The idea behind this notation is that for $1 \le \lambda \le n$ the subscript i_λ tells us that $\deg(r_{i_\lambda}) = i_\lambda$. Whenever we list elements like this, then unless otherwise stated, we will asumme that for $1 \le \lambda < \mu \le n$ we have $r_{i_\lambda} \neq r_{i_\mu}$ and $i_\lambda \le i_\mu$.

Example 4.1. An important example of a graded ring is a ring R endowed with the **trivial grading**: The homogoneeous components of R being $R_0 := R$ and $R_i := 0$ for all $i > 0$. If we introduce a ring without specfiying a grading, then it is assumed that this ring is endowed with the trivial grading.

4.2 Graded R-Algebras

Let R be a ring and let A be an R-algebra. We say A is a **graded R-algebra** if A is graded as a ring.

Example 4.2. Let R be a ring and let $w := (w_1, \ldots, w_n)$ be an n-tuple of positive integers. The **weighted polynomial ring with respect to the weight** w, denoted $R[u_1, \ldots, u_n]_w$, is the polynomial ring $R[u_1, \ldots, u_n]$ endowed with the unique grading such that $\deg(u_\lambda) = w_\lambda$ for all $\lambda = 1, \ldots, n$. We define the **weighted degree** of a monomial $m = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$, denoted $\deg_w(m)$, by the formula

$$\deg_w(m) := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

This grading gives $R[u_1, \ldots, u_n]_w$ the structure of a graded R-algebra, where the homogeneous components are given by

$$(R[u_1, \ldots, u_n]_w)_i := \mathrm{Span}_R \langle m \in R[u_1, \ldots, u_n]_w \mid \deg_w(m) = i \rangle.$$

For instance, consider the case where $R = k[X, Y, Z]_{(1,2,3)}$. The homogeneous components of R start out as

$$\begin{aligned} R_0 &= k \\ R_1 &= kX \\ R_2 &= kX^2 + kY \\ R_3 &= kX^3 + kXY + kZ \\ &\vdots \end{aligned}$$

Example 4.3. Let R be a ring and let Q be an ideal in R. The **blowup algebra of Q in R** is defined by

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong \bigoplus_{i=0}^{\infty} Q^i.$$

Elements in $B_Q(R)$ have the form

$$t^{i_1} x_{i_1} + \cdots + t^{i_m} x_{i_m}$$

where $0 \le i_1 < \cdots < i_m$ and $x_{i_\lambda} \in Q^{i_\lambda}$ for all $1 \le \lambda \le m$. The t^{i_λ} part keeps track of what degree we are in. We define multiplaction on elements of the form $t^i x$ and $t^j y$ by

$$(t^i x)(t^j y) = t^{i+j} xy,$$

and we extend this to all of $B_Q(R)$ in the obvious way. This gives $B_Q(R)$ the structure of a graded R-algebra.

If Q is finitely generated, say $Q = \langle a_1, \ldots, a_n \rangle$, then there is a unique R-algebra homomorphism

$$\varphi\colon R[u_1, \ldots, u_n] \to B_Q(R),$$

such that $\varphi(u_\lambda) = ta_\lambda$ for all $1 \le \lambda \le n$.

4.3 Graded R-Modules

Let R be a graded ring. A **graded** R-module M is an R-module together with a direct sum decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

into abelian groups M_i which satisfies the condition that if $r_i \in R_i$ and $m_j \in M_j$, then $r_i m_j \in M_{i+j}$ for all $i, j \in \mathbb{Z}$. The M_i are called **homogeneous components** of M and the elements of M_i are called **homogeneous elements** of **degree** i. If m is a homogeneous element in M, then we denote the degree of m as $\deg(m)$. When we say "let M be a graded R-module", then it is understood that the homogeneous components of M are denoted M_i.

Example 4.4. Let M be a graded R-module. Then for each $j \in \mathbb{Z}$, we define the ***j*-th twist** (or ***j*-twist**) of M, denoted $M(-j)$, to be the graded R-module whose i-th homogeneous component is given by $M(j)_i := M_{i+j}$ for all $i \in \mathbb{Z}$. Thus, we have

$$M(j) := \bigoplus_{i \in \mathbb{Z}} M_{i+j}.$$

4.3.1 Graded R-Submodules

Lemma 4.1. *Let M be a graded R-module and $N \subset M$ be a submodule. The following conditions are equivalent:*

1. *N is graded R-module whose homogeneous components are $M_i \cap N$.*
2. *N can be generated by homogeneous elements.*
3. *Let $y \in M$ and suppose that $y = y_{i_1} + \cdots + y_{i_n}$ where $i_1 < \cdots < i_n$ and $y_{i_\lambda} \in M_{i_\lambda}$ for all $1 \leq \lambda \leq n$. Then $y \in N$ if and only if $y_{i_\lambda} \in N$ for all $1 \leq \lambda \leq n$.*

Proof.
(1 implies 2): Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a set of generators for N. For each $\lambda \in \Lambda$, write

$$x_\lambda = x_{\lambda, i_1} + \cdots + x_{\lambda, i_{n(\lambda)}}$$

where $i_1 < \cdots < i_{n(\lambda)}$ and $x_{\lambda, i_\mu} \in M_{i_\mu} \cap N$ for all $1 \leq \mu \leq n(\lambda)$. Then $\{x_{\lambda, i_\mu}\}_{\lambda \in \Lambda, 1 \leq \mu \leq n(\lambda)}$ is a set of homogeneous elements which generated N.

(2 implies 3) Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a set of homogeneous generators for N and let $y \in N$. Write

$$y =$$

If each $y_{i_\lambda} \in N$ for all $1 \leq \lambda \leq n$, then it is clear that $y \in N$. Conversely if $y \in N$, then there is a unique decomposition

Let $y \in N$. Since $y \in M$, we can decompose y into its homogeneous parts in M, say

$$y = y_{i_1} + \cdots + y_{i_n}$$

where $i_1 < \cdots < i_n$ and $y_{i_\lambda} \in M_{i_\lambda}$ for all $1 \leq \lambda \leq n$. Since $y \in N$, we can decompose y into its homogeneous parts in N, say

$$y = y'_{i'_1} + \cdots + y'_{i'_{n'}}$$

where $i'_1 < \cdots < i'_{n'}$ and $y'_{i'_\lambda} \in N_{i'_{\lambda'}}$ for all $1 \leq \lambda \leq n'$. Since the grading on N is just the induced graded of N as a subset of M, we must have $n' = n$ and $y'_{i'_\lambda} = y_{i_\lambda}$ for all $1 \leq \lambda \leq n$.

□

A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (4.1) is called a **graded** (or **homogeneous**) submodule. A graded submodule of a graded ring is called a **graded** (or **homogeneous**) ideal.

Example 4.5. Consider the graded ring $R = k[X, Y, Z]_{(5,6,15)}$. Then the ideal $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ is a homogeneous ideal in R.

Remark. Let R be a graded ring and let I be a homogeneous ideal in R. Then the quotient ring R/I has an induced structure as a graded ring, where the ith homogeneous component of R/I is

$$(R/I)_i := (R_i + I)/I \cong R_i/(I \cap R_i)$$

4.3.2 Homomorphisms of Graded R-Modules

Let M and N be graded R-modules. A homomorphism $\varphi \colon M \to N$ is called **homogeneous** (or **graded**) of degree j if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If φ is homogeneous of degree zero then we will simply say φ is **homogeneous**.

Example 4.6. Consider the graded ring $R = k[X, Y, Z, W]$. Then the matrix

$$U := \begin{pmatrix} X + Y + Z & W^2 - X^2 & X^3 \\ 1 & X & XY + Z^2 \end{pmatrix}$$

defines a homomorphism $U \colon R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$ which is graded of degree zero.

4.4 Operations on Graded R-Modules

4.4.1 Tensor

Let M and N be graded R-modules. As R-modules, their tensor product is given by

$$\begin{aligned} M \otimes_R N &= \left(\bigoplus_{i \in \mathbb{Z}} M_i \right) \otimes \left(\bigoplus_{j \in \mathbb{Z}} N_j \right) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} (M_i \otimes N_j) \\ &= \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j} \right). \end{aligned}$$

This suggests that we can view $M \otimes_R N$ as a graded R-module where the homogeneous component in degree i is

$$(M \otimes_R N)_i = \bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j}.$$

This turns out to be right notion. Indeed, if $x \in M_i$, $y \in N_j$, and $a \in R_k$, then

$$a(x \otimes y) = ax \otimes y = x \otimes ay \in (M \otimes_R N)_{i+j+k}.$$

So the grading is preserved upon R-scaling.

4.4.2 Graded Hom

Unlike the tensor product of two R-modules, hom does not have a natural interpretation as a graded R-module. Instead we consider the graded version. Let M and N be graded R-modules. Then we define

$$\mathrm{Hom}_R^{\star}(M, N) := \bigoplus_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(M_j, N_{i+j}).$$

Observe that if N is just an R-module with the trivial grading, then we get

$$\begin{aligned} \mathrm{Hom}_R^{\star}(M, N) &= \bigoplus_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(M_j, N_{i+j}) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_{-i}, N), \end{aligned}$$

and if M is just an R-module with the trivial grading, then we get

$$\begin{aligned} \mathrm{Hom}_R^{\star}(M, N) &= \bigoplus_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(M_j, N_{i+j}) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_R(M, N_i), \end{aligned}$$

4.5 Noetherian Graded Rings and Modules

4.5.1 The Irrelevant Ideal

Let R be a graded ring. The **irrelevant ideal of** R is defined to be

$$R_+ := \bigoplus_{i > 0} R_i.$$

It is straightforward to check that R_+ is in fact an ideal of R and that $R/R_+ \cong R_0$. If R_+ is finitely generated, say by $x_{i_1}, \dots, x_{i_n} \in R_+$, then there is a surjective R_0-algebra map

$$R_0[X_{i_1}, \dots, X_{i_n}] \to R$$

sending $X_{i_\lambda} \mapsto x_{i_\lambda}$ for all $1 \leq \lambda \leq n$. Thus if R_+ is finitely generated as an ideal in R, then R is a finitely generated as an R_0-algebra.

4.5.2 Noetherian Graded Rings

The following lemma will be used many times without mention.

Lemma 4.2. *Let R be a ring and let $S \subseteq R$. Suppose the ideal $\langle S \rangle$ generated by S is finitely generated. Then we can choose the generators to be in S.*

Proof. Since $\langle S \rangle$ is finitely generated, there are $x_1, \dots, x_n \in \langle S \rangle$ such that $\langle S \rangle = \langle x_1, \dots, x_n \rangle$. In particular we have

$$x_\lambda = \sum_{\mu=1}^{n_\lambda} r_{\mu\lambda} s_{\mu\lambda}$$

where for each $1 \leq \lambda \leq n$ we have $n_\lambda \in \mathbb{N}$, and for each $1 \leq \mu \leq n_\lambda$ we have $r_{\mu\lambda} \in R$ and $s_{\mu\lambda} \in S$. In particular, this mean

$$\langle S \rangle = \langle s_{\mu\lambda} \mid 1 \leq \lambda \leq n \text{ and } 1 \leq \mu \leq n_\lambda \rangle.$$

□

A **Noetherian** graded ring is a graded ring whose underlying ring is Noetherian.

Proposition 4.1. *Let R be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely generated as an R_0-algebra.*

Proof. Suppose R_0 is Noetherian and R is finitely generated as an R_0-algebra. Then there exists an $n \geq 0$ and a surjection

$$R_0[X_1, \dots, X_n] \to R.$$

where $R_0[X_1, \dots, X_n]$ is a polynomial algebra over Noetherian ring, and hence Noetherian, which implies that R is Noetherian, as it is a quotient of a Noetherian ring.

Now suppose R is Noetherian. Then the irrelevant ideal R_+ is finitely generated and we have $R_0 \cong R/R_+$. Thus R_0 is Noetherian since it is a quotient of a Noetherian ring and R is finitely generated as an R_0-algebra since R_+ is finitely generated as an ideal in R. □

Corollary. *Let R be a Noetherian graded ring and let M be a finitely generated graded R-module. Then M_i is a finitely generated R_0-module.*

Proof. Let $m_{i_1}, \dots, m_{i_n}$ be the generators of M as an R-module where $\deg(m_{i_\lambda}) = i_\lambda$ for all $1 \leq \lambda \leq n$. Then

$$M_i = R_{i-i_1} m_{i_1} + \cdots + R_{i-i_n} m_{i_n}.$$

This implies that M_i is a finitely generated R_0-module because the R_i's are finitely generated R_0-modules. □

5 Homological Algebra

Throughout this subsection, let R be a ring (trivially graded).

5.1 Chain Complexes over R

A **chain complex** (A, d) **over** R, or more simply an R**-complex**, is a sequence of R-modules A_i and morphisms $d_i \colon A_i \to A_{i-1}$

$$(A, d) := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\text{Ker}(d_i) \supset \text{Im}(d_{i+1})$. With this in mind, we define the i**th homology of** R**-complex** (A, d) to be

$$H_i(A, d) := \text{Ker}(d_i)/\text{Im}(d_{i+1}).$$

5.1.1 Chain Maps

Let (A, d) and (A', d') be two R-complexes. A **chain map** $\varphi \colon (A, d) \to (A', d')$ is a sequence of R-module homomoprhisms $\varphi_i \colon A_i \to A'_i$ such that $d'_i \varphi_i = \varphi_{i-1} d'_i$ for all $i \in \mathbb{Z}$.

$$\begin{array}{ccccccccc}
(A, d) := \cdots & \longrightarrow & A_{i+1} & \xrightarrow{d_{i+1}} & A_i & \xrightarrow{d_i} & A_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \\
(A', d') := \cdots & \longrightarrow & A'_{i+1} & \xrightarrow{d'_{i+1}} & A'_i & \xrightarrow{d'_i} & A'_{i-1} & \longrightarrow & \cdots
\end{array}$$

5.1.2 Simplifying Notation

There is another description of R-complexes and chains maps which turns out to be useful. Namely, an R-complex (A, d) is a graded R-module A equipped with a graded endomorphism $d\colon A \to A$ of degree -1 such that $d^2 = 0$. The morphisms d_i introduced above are given by $d_i := d|_{A_i}$. We often refer to d as the **differential**. An element in $\mathrm{Ker}(d)$ is called a **cycle** of (A, d) and an element in $\mathrm{Im}(d)$ is called a **boundary** of (A, d). We define the **homology** of (A, d) to be

$$H(A, d) := \mathrm{Ker}(d)/\mathrm{Im}(d)$$

Since the kernel and image of a graded homomorphism (of any degree) are homogeneous, we have

$$H(A, d) = \bigoplus_{i \in \mathbb{Z}} H_i(A, d).$$

Thus $H(A, d)$ can be thought of as a graded R-module.

Remark. We sometimes write A and $H(A)$ instead of (A, d) and $H(A, d)$ if the differential is understood from context.

5.1.3 Chain Maps Revisited

Let us reinterpret what a chain map is in terms of this new notation. Let (A, d) and (A', d') be two R-complexes. A chain map $\varphi\colon (A, d) \to (A', d')$ between them is a graded homomorphism of graded R-modules such that $\varphi d = d'\varphi$.

Example 5.1. Let (A, d) and (A', d') be two R-complexes and let $r \in R$. The **homothety map** $\mu_r\colon (A, d) \to (A', d')$ is defined by

$$\mu_r(a) = ra$$

for all $a \in A$. The homothety map is clearly a graded homomorphism since r is sitting in degree 0. It also commutes with the differential since the differential is R-linear.

5.2 Homology Considered as a Functor

The set of all R-complexes together with the set of all chain maps forms a category, which we denote $\mathbf{Comp}_R$. Similarly, the set of all graded R-modules together with the set of all graded homomorphisms (of degree 0) forms a category, which we denote $\mathbf{Grad}_R$. We've already seen that if (A, d) is an R-complex, then $H(A, d)$ is a graded R-module. We would like to extend this observation to get a functor $H\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$. This will follow from the following three propositions:

Proposition 5.1. *Let $\varphi\colon (A, d) \to (A', d')$ be a chain map between R-complex (A, d) and (A', d'). Then φ induces a graded homomorphism $H(\varphi)\colon H(A, d) \to H(A', d')$, where*

$$H(\varphi)(\overline{a}) := \overline{\varphi(a)} \tag{9}$$

for all $\overline{a} \in H(A, d)$.

Proof. First let us check that the target of each element in $H(A, d)$ under $H(\varphi)$ lands in $H(A', d')$. Let $\overline{a} \in H(A, d)$, so $d(a) = 0$. Then $\overline{\varphi(a)} \in H(A, d)$ since

$$\begin{aligned} d(\varphi(a)) &= \varphi(d(a)) \\ &= 0. \end{aligned}$$

Next let us check that that $H(\varphi)$ is well-defined. Let $a + d(b)$ and a be two representatives of the coset class $\overline{a}$. Then we have

$$\begin{aligned} H(\varphi)(\overline{a + d(b)}) &= \overline{\varphi(a + d(b))} \\ &= \overline{\varphi(a) + \varphi(d(b))} \\ &= \overline{\varphi(a)} + \overline{\varphi(d(b))} \\ &= \overline{\varphi(a)} + \overline{d(\varphi(b))} \\ &= \overline{\varphi(a)}. \end{aligned}$$

So far we have shown that $H(\varphi)$ is a function. To see that $H(\varphi)$ is an R-module homomorphism, let $r, s \in R$ and $a, b \in A$. Then

$$\begin{aligned} H(\varphi)(ra + sb) &= \overline{\varphi(ra + sb)} \\ &= \overline{r\varphi(a) + s\varphi(b)} \\ &= r\overline{\varphi(a)} + s\overline{\varphi(b)} \\ &= rH(\varphi)(a) + sH(\varphi)(b). \end{aligned}$$

Finally, to see that $H(\varphi)$ is graded, let $\overline{a}_i \in H_i(A, d)$, so $a_i \in A_i$. Then

$$H(\varphi)(\overline{a}_i) = \overline{\varphi(a_i)} \in H_i(A', d')$$

since φ is graded. □

Proposition 5.2. *Let $\varphi\colon (A, d) \to (A', d')$ and $\varphi'\colon (A', d') \to (A'', d'')$ be two chain maps between R-complex (A, d) to (A', d') and (A', d') to (A'', d'') respectively. Then*

$$H(\varphi' \circ \varphi) = H(\varphi') \circ H(\varphi).$$

Proof. Let $\overline{a} \in H(A, d)$. Then we have

$$\begin{aligned} H(\varphi' \circ \varphi)(\overline{a}) &= \overline{(\varphi' \circ \varphi)(a)} \\ &= \overline{\varphi'(\varphi(a))} \\ &= H(\varphi')(\overline{\varphi(a)}) \\ &= H(\varphi')(H(\varphi)(a)) \\ &= (H(\varphi') \circ H(\varphi))(a). \end{aligned}$$

□

Proposition 5.3. *Let (A, d) be an R-complex. Then we have $H(\mathrm{id}_{(A,d)}) = \mathrm{id}_{H(A,d)}$, where $\mathrm{id}_{(A,d)}$ is the identity map from (A, d) to (A, d) and $\mathrm{id}_{H(A,d)}$ is the identity map from $H(A)$ to $H(A)$. In particular, if $\varphi\colon (A, d) \to (A', d')$ is an isomorphism between R-complexes (A, d) and (A', d'), then $H(\varphi)\colon H(A) \to H(A')$ is an isomorphism between graded R-modules $H(A)$ and $H(A')$.*

Proof. Let $\overline{a} \in H(A, d)$. Then

$$\begin{aligned} H(\mathrm{id}_{(A,d)})(\overline{a}) &= \overline{\mathrm{id}_{(A,d)}(a)} \\ &= \overline{a} \\ &= \mathrm{id}_{H(A,d)}(\overline{a}). \end{aligned}$$

For the latter statement, let $\varphi\colon (A, d) \to (A', d')$ is an isomorphism between R-complexes (A, d) and (A', d'), and let $\psi\colon (A', d') \to (A, d)$ be its inverse. Then

$$\begin{aligned} \mathrm{id}_{H(A,d)} &= H(\mathrm{id}_{(A,d)}) \\ &= H(\psi \circ \varphi) \\ &= H(\psi) \circ H(\varphi). \end{aligned}$$

A similar computation gives $H(\varphi) \circ H(\psi) = \mathrm{id}_{H(A')}$. □

5.2.1 Viewing Homology as an Additive Functor

There is more structure on the categories $\mathbf{Comp}_R$ and $\mathbf{Grad}_R$ which we haven't discussed so far. Let us fix that now: Let (A, d) and (A', d') be two R-complexes. We define

$$\mathcal{C}(A, A') := \mathrm{Hom}((A, d), (A', d')) = \{\varphi\colon (A, d) \to (A', d') \mid \varphi \text{ is a chain map}\}.$$

Then $\mathcal{C}(A, A')$ has the structure of an R-module. Indeed, if $\varphi, \psi \in \mathcal{C}(A, A')$ and $r \in R$, then we define addition and scalar multiplication by

$$(\varphi + \psi)(a) := \varphi(a) + \psi(a) \quad \text{and} \quad (r\varphi)(a) = \varphi(ra)$$

for all $a \in A$. Since d is an R-linear map, it is clear that $\varphi + \psi$ and $r\varphi$ are chain maps.

Proposition 5.4. *Let $\varphi, \psi\colon (A, d) \to (A', d')$ be two chain maps and let $r, s \in R$. Then*

$$H(r\varphi + s\psi) = rH(\varphi) + sH(\psi)$$

Proof. Let $\overline{a} \in H(A,d)$. Then

$$\begin{aligned} H(r\varphi + s\psi)(\overline{a}) &= \overline{(r\varphi + s\psi)(a)} \\ &= \overline{r\varphi(a) + s\psi(a)} \\ &= r\overline{\varphi(a)} + s\overline{\psi(a)} \\ &= rH(\varphi)(a) + sH(\psi)(a). \end{aligned}$$

□

We can give $\mathbf{Comp}_R$ even more structure as follows:

Proposition 5.5. *Composition* $\circ\colon \mathcal{C}(A,A') \times \mathcal{C}(A',A'') \to \mathcal{C}(A,A'')$ *is an* R*-bilinear map.*

Proof. Let $\varphi, \varphi' \in \mathcal{C}(A,A')$ and $\psi \in \mathcal{C}(A',A'')$. Then

$$\begin{aligned} ((\varphi + \varphi') \circ \psi)(a) &= (\varphi + \varphi')(\psi(a)) \\ &= \varphi(\psi(a)) + \varphi'(\psi(a)) \\ &= (\varphi \circ \psi)(a) + (\varphi' \circ \psi)(a) \end{aligned}$$

for all $a \in A$. Similarly, let $\varphi \in \mathcal{C}(A,A')$ and $\psi, \psi' \in \mathcal{C}(A',A'')$. Then

$$\begin{aligned} (\varphi \circ (\psi + \psi'))(a) &= \varphi((\psi + \psi')(a)) \\ &= \varphi(\psi(a) + \psi'(a)) \\ &= \varphi(\psi(a)) + \varphi(\psi'(a)) \\ &= (\varphi \circ \psi)(a) + (\varphi \circ \psi')(a). \end{aligned}$$

A similar proof works for R-linearity in the first and second argument too. □

5.2.2 Quasiisomorphisms

Definition 5.1. Let $\varphi\colon (A,d) \to (A',d')$ be a chain map. We say φ is a **quasiisomorphism** if $H(\varphi)\colon H(A,d) \to H(A',d')$ is an isomorphism.

Every isomorphism $\varphi\colon (A,d) \to (A',d')$ between R-complexes is a quasiisomorphism since H is a functor.

Example 5.2. Let (A,d) and (A',d') be two R-complexes. The homothety map $\mu^{A,x}\colon A \to A$ induces the homothety map in homology $\mu^{H_i(A),x}\colon H_i(A) \to H_i(A)$, given by $\overline{a} \mapsto x\overline{a}$. In particular, if x is a unit, then multiplication by x is always an isomorphism, and this implies $H_i(\mu^{A,x})$ is an isomorphism for all i, and so $\mu^{A,x}$ is a quasiisomorphism.

Proposition 5.6. *Let* (A,d) *be an* R*-complex. Then we have* $H(\mathrm{id}_{(A,d)}) = \mathrm{id}_{H(A,d)}$*, where* $\mathrm{id}_{(A,d)}$ *is the identity map from* (A,d) *to* (A,d) *and* $\mathrm{id}_{H(A,d)}$ *is the identity map from* $H(A,d)$ *to* $H(A,d)$*. In particular, if* $\varphi\colon (A,d) \to (A',d')$ *is an isomorphism between* R*-complexes* (A,d) *and* (A',d')*, then* $H(\varphi)\colon H(A,d) \to H(A',d')$ *is an isomorphism.*

Proof. Let $\overline{a} \in H(A,d)$. Then

$$\begin{aligned} H(\mathrm{id}_{(A,d)})(\overline{a}) &= \overline{\mathrm{id}_{(A,d)}(a)} \\ &= \overline{a} \\ &= \mathrm{id}_{H(A,d)}. \end{aligned}$$

For the latter statement, let $\varphi\colon (A,d) \to (A',d')$ is an isomorphism between R-complexes (A,d) and (A',d'), and let $\psi\colon (A',d') \to (A,d)$ be its inverse. Then

$$\begin{aligned} \mathrm{id}_{H(A,d)} &= H(\mathrm{id}_{(A,d)}) \\ &= H(\psi \circ \varphi) \\ &= H(\psi) \circ H(\varphi). \end{aligned}$$

A similar computation gives $H(\varphi) \circ H(\psi) = \mathrm{id}_{H(A',d')}$ □

Example 5.3. Let (A,d) and (A',d') be two R-complexes and let $r \in R$. The homothety map $\mu_r\colon (A,d) \to (A',d')$ induces the homothety map in homology $H(\mu_r)\colon H(A,d) \to H(A',d')$. In particular, if r is a unit, then multiplication by r is always an isomorphism, and this implies $H_i(\mu^{A,x})$ is an isomorphism for all i, and so $\mu^{A,x}$ is a quasiisomorphism.

5.2.3 Homotopy Equivalence

Definition 5.2. Let φ and ψ be two chain maps between R-complexes (A,d) and (A',d'). We say φ is **homotopic** to ψ if there is a chain map $h\colon (A,d) \to (A',d')$ such that $\varphi - \psi = d'h + hd$.

Proposition 5.7. *Let φ and ψ be chain maps of chain complexes (A,d) and (A',d'). If φ is homotopic to ψ, then $H(\varphi) = H(\psi)$.*

Proof. Showing $H(\varphi) = H(\psi)$ is equivalent to showing $H(\varphi - \psi) = 0$. Thus, we may assume that φ is homotopic to the 0 map and that we are trying to show that $H(\varphi) = 0$. Let $\overline{a} \in H(A,d)$. Then $d(a) = 0$, and so

$$\begin{aligned} H(\varphi)(\overline{a}) &= \overline{\varphi(a)} \\ &= \overline{(d'h + hd)(a)} \\ &= \overline{d'(h(a)) + h(d(a))} \\ &= \overline{d'(h(a))} \\ &= \overline{0}. \end{aligned}$$

□

5.3 Exact Sequences of R-Complexes

Let (A,d), (A',d'), and (A'',d'') be R-complexes and let $\varphi\colon (A',d') \to (A,d)$ and $\psi\colon (A,d) \to (A'',d'')$ be chain maps. Then we say that

$$0 \longrightarrow (A',d') \xrightarrow{\varphi} (A,d) \xrightarrow{\psi} (A'',d'') \longrightarrow 0$$

is a **short exact sequence** of R-complexes if it is a short exact sequence when considered as graded R-modules. More specifically, this means that following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "d'_{i+2}"] & \vdots \arrow[d, "d_{i+2}"] & \vdots \arrow[d, "d''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "d'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "d_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "d''_{i+1}"] & 0 \\
0 \arrow[r] & A'_{i} \arrow[r, "\varphi_{i}"] \arrow[d, "d'_{i}"] & A_{i} \arrow[r, "\psi_{i}"] \arrow[d, "d_{i}"] & A''_{i} \arrow[r] \arrow[d, "d''_{i}"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "d'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "d_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "d''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Theorem 5.1. *Let*

$$0 \longrightarrow (A', d') \xrightarrow{\varphi} (A, d) \xrightarrow{\psi} (A'', d'') \longrightarrow 0 \tag{10}$$

be a short exact sequence of R-complexes. Then there exists a graded homomorphism $\eth \colon H(A'') \to H(A')$ *of degree* -1 *such that*

$$\begin{tikzcd} & \cdots \arrow[r] & H_{i+1}(A'') \arrow[dll, rounded corners, "\eth_{i+1}"] \\ H_i(A') \arrow[r, "H_i(\varphi)"] & H_i(A) \arrow[r, "H_i(\psi)"] & H_i(A'') \arrow[dll, rounded corners, "\eth_i"] \\ H_{i-1}(A') \arrow[r] & \cdots & \end{tikzcd} \tag{11}$$

is a long exact sequence of R-modules.

Proof. The proof will consists of three steps. The first step is to construct a graded function $\eth \colon H(A'') \to H(A')$ of degree -1 (graded here just means $\eth(H_i(A'')) \subseteq H_{i-1}(A')$ for all $i \in \mathbb{Z}$). The next step will be to show that $\eth$ is R-linear. The final step will be to show exactness of (11).

Step 1: Let $i \in \mathbb{Z}$ and let $\overline{a''} \in H_i(A'')$. Choose a representative of the coset $\overline{a''}$, say $a'' \in A''_i$ (so $d''(a'') = 0$), and choose a lift of a'' in A_i under ψ (This means choose $a \in A_i$ such that $\psi(a) = a''$. We can make such a choice since ψ is surjective). Since

$$\begin{aligned} \psi(d(a)) &= d''(\psi(a)) \\ &= d''(a'') \\ &= 0, \end{aligned}$$

there exists a unique $a' \in A'_{i-1}$ such that $\varphi(a') = d(a)$ (by exactness of (10)). Observe that $d'(a') = 0$ since φ is injective and since

$$\begin{aligned} \varphi(d'(a')) &= d(\varphi(a')) \\ &= d(d(a)) \\ &= 0. \end{aligned}$$

Thus a' represents an element in $H_{i-1}(A')$. We define $\eth \colon H(A'') \to H(A')$ by $\eth(\overline{a''}) = \overline{a'}$.

We need to verify that $\eth$ is well-defined. There were two choices that we made in constructing $\eth$. The first choice was the choice of a representative of the coset $\overline{a''}$. So let $a'' + d''(b'')$ be another choice of this coset where $b'' \in A''_{i+1}$ (every representative of the coset $\overline{a''}$ has this form for some $b'' \in A''_{i+1}$). The second choice that we made was the choice of a lift of a'' in A under ψ. This time we have another coset representative of $\overline{a''}$, so let $a + \varphi(b') + d(b)$ be a lift of $a'' + d''(b'')$ in A_i where $b' \in A'_i$ and $b \in A_{i+1}$ (every such choice has this form for some $b' \in A'_i$ and $b \in A_{i+1}$). We first observe that

$$\begin{aligned} d(a + \varphi(b') + d(b)) &= d(a) + d(\varphi(b')) + d(d(b)) \\ &= d(a) + d(\varphi(b')) \\ &= d(a) + \varphi(d'(b')). \end{aligned}$$

Therefore

$$\begin{aligned} \psi(d(a + \varphi(b') + d(b))) &= \psi(d_i(a) + \varphi(d'(b'))) \\ &= \psi(d(a)) + \psi(\varphi(d'(b'))) \\ &= \psi(d(a)) \\ &= d''(\psi(a)) \\ &= d''(a'') \\ &= 0. \end{aligned}$$

Hence there exists a unique element in A'_{i-1} which maps to $d(a+\varphi(b')+d(b))$ under φ, and since

$$\begin{aligned}\varphi(a'+d'(b')) &= \varphi(a')+\varphi(d'(b'))\\ &= d(a)+\varphi(d'(b'))\\ &= d(a+\varphi(b')+d(b)),\end{aligned}$$

this unique element must be $a'+d'(b')$. Therefore

$$\begin{aligned}\eth(\overline{a''+d''(b'')}) &= \overline{a'+d'(b')}\\ &= \overline{a'}\\ &= \eth(\overline{a''}),\end{aligned}$$

which implies $\eth$ is well-defined. Moreover, we see that $\eth(A_i) \subseteq A_{i-1}$, and is hence graded of degree -1. As usualy, we denote $\eth_i := \eth|_{A_i}$.

Step 2: Let $i \in \mathbb{Z}$, let $\overline{a''}, \overline{b''} \in H(A'')$, and let $r, s \in R$. Choose a coset representative $\overline{a''}$ and $\overline{b''}$, say $a'' \in A''_i$ and $b'' \in A''_i$. Then $ra''+sb''$ is a coset representative of $\overline{ra''+sb''}$ (by linearity of taking quotients). Next, choose lifts of a'' and b'' in A_i under φ, say $a \in A_i$ and $b \in A_i$ respectively. Then $ra+sb$ is a lift of $ra''+sb''$ in A_i under φ (by linearity of ψ). Finally, let a' and b' be the unique elements in A'_{i-1} such that $\varphi(a') = d(a)$ and $\varphi(b') = d(b)$. Then $ra'+sb'$ is the unique element in A'_{i-1} such that $\varphi(ra'+sb') = d(ra+sb)$ (by linearity of φ). Thus, we have

$$\begin{aligned}\eth(\overline{ra''+sb''}) &= \overline{ra'+sb'}\\ &= r\overline{a'}+s\overline{b'}\\ &= r\eth(\overline{a''})+s\eth(\overline{b''}).\end{aligned}$$

Step 3: To prove exactness of (11), it suffices to show exactness at $H_i(A'')$, $H_i(A)$, and $H_i(A')$. First we prove exactness at $H_i(A)$. Let $\overline{a} \in \operatorname{Ker}(H_i(\psi))$ (so $a \in A_i$, $d(a) = 0$, and $\overline{\psi(a)} = \overline{0}$). Lift $\psi(a) \in A''_i$ to an element $a'' \in A'_{i+1}$ under d'' (we can do this since $\overline{\psi(a)} = \overline{0}$). Lift $a'' \in A''_{i+1}$ to an element $b \in A_{i+1}$ under ψ (we can do this since ψ is surjective). Then

$$\begin{aligned}\psi(d(b)-a) &= \psi(d(b))-\psi(a)\\ &= d''(a'')-\psi(a)\\ &= \psi(a)-\psi(a)\\ &= 0\end{aligned}$$

implies $d(b)-a \in \operatorname{Ker}(\psi)$. Lift $d(b)-a$ to the unique element $a' \in A'_i$ under φ (we can do this exactness of (10)). Since φ is injective,

$$\begin{aligned}\varphi(d'(a')) &= d(\varphi(a'))\\ &= d(d(b)-a)\\ &= d(d(b))-d(a))\\ &= 0\end{aligned}$$

implies $d'(a') = 0$. Hence a' represents an element in $H(A')$. Therefore

$$\begin{aligned}H_i(\varphi)(a') &= \overline{\varphi(a')}\\ &= \overline{d(b)-a}\\ &= \overline{a}\end{aligned}$$

implies $\overline{a} \in \operatorname{Im}(H_i(\varphi))$. Thus we have exactness at $H_i(A)$.

Next we show exactness at $H_i(A')$. Let $\overline{a'} \in \operatorname{Ker}(H_i(\varphi))$ (so $a' \in A'_i$, $d(a') = 0$, and $\overline{\varphi(a')} = \overline{0}$). Lift $\varphi(a') \in A_i$ to an element $a \in A'_{i+1}$ under d (we can do this since $\overline{\varphi(a)} = \overline{0}$). Then

$$\begin{aligned}d(\psi(a)) &= \psi(d(a))\\ &= \psi(\varphi(a'))\\ &= 0.\end{aligned}$$

Hence $\psi(a)$ represents an element in $H_{i+1}(A'')$. By construction, we have $\eth(\overline{\psi(a)}) = \overline{a'}$, which implies $\overline{a'} \in \mathrm{Im}(\eth_{i+1})$. Thus we have exactness at $H_i(A')$.

Finally we show exactness at $H_i(A'')$. Let $\overline{a''} \in \mathrm{Ker}(\eth_i)$ (so $a'' \in A''_i$ and $d(a'') = 0$). Lift a'' to an element $a \in A_i$ under ψ. Lift $d(a)$ to the unique element a' in A'_{i-1} under φ. Lift a' to an element $b' \in A'_{i+1}$ under d (we can do this since $0 = \eth(\overline{a''}) = \overline{a'}$). Then

$$\begin{aligned} d(a - \varphi(b')) &= d(a) - d(\varphi(b')) \\ &= d(a) - \varphi(d(b')) \\ &= d(a) - \varphi(a') \\ &= 0, \end{aligned}$$

and hence $a - \varphi(b')$ represents an element in $H_i(A)$. Moreover, we have

$$\begin{aligned} H_i(\psi)(\overline{a - \varphi(b')}) &= \overline{\psi(a - \varphi(b'))} \\ &= \overline{\psi(a) - \psi(\varphi(b'))} \\ &= \overline{\psi(a)} \\ &= \overline{a''}, \end{aligned}$$

which implies $\overline{a'} \in \mathrm{Im}(H_i(\psi))$. Thus we have exactness at $H_i(A'')$. $\square$

Definition 5.3. Given a short exact sequence of R-complexes as in (10), we refer to the graded homomorphism $\eth\colon H(A'') \to H(A')$ of degree -1 as the **induced connecting map**.

6 Operations on R-Complexes

6.1 Direct Sum of R-Complexes

Definition 6.1. Let (A, d) and (A', d') be R-complexes. We define their **direct sum** to be the R-complex

$$(A, d) \oplus (A', d') := (A \oplus A', d \oplus d')$$

whose graded R-module $A \oplus A'$ has

$$(A \oplus A')_i = A_i \oplus A'_i$$

as its ith homogeneous component and whose differential $d \oplus d'$ is defined by

$$(d \oplus d')(a, a') = (d(a), d'(a'))$$

for all $(a, a') \in A \oplus A'$.

6.2 Shifting an R-complex

Definition 6.2. Let (A, d) be an R-complex. We define the **shift** of (A, d) to be the R-complex

$$\Sigma(A, d) := (A(-1), -d).$$

More generally, let $k \in \mathbb{Z}$. We define the kth **shift** of (A, d) to be the R-complex

$$\Sigma^k(A, d) = (\Sigma^k A, d^{\Sigma^k A}).$$

whose graded R-module $\Sigma^k A$ has

$$(\Sigma^k A)_i = A_{i-k}$$

as its ith homogeneous component and whose differential $d^{\Sigma^k A}$ is simply

$$d^{\Sigma^k A} = (-1)^k d$$

6.3 The Mapping Cone

Definition 6.3. Let $\varphi\colon (A,d) \to (A',d')$ be a chain map. The **mapping cone of** φ, denoted $(C(\varphi), d^{C(\varphi)})$, is the R-complex whose graded R-module $C(\varphi)$ has

$$C_i(\varphi) := A'_i \oplus A_{i-1}$$

as its ith homogeneous component and whose differential $d^{C(\varphi)}$ is defined b

$$d^{C(\varphi)}(a,a') := (d'(a') + \varphi(a), -d(a))$$

for all $a' \in A'_i$ and $a \in A_{i-1}$.

Remark. To see that we are justified in calling $(C(\varphi), d^{C(\varphi)})$ an R-complex, let us check that $d^{C(\varphi)}d^{C(\varphi)} = 0$. Let $(a', a) \in C(\varphi)$. Then

$$\begin{aligned} d^{C(\varphi)}d^{C(\varphi)}(a',a) &= d^{C(\varphi)}(d^{C(\varphi)}(a',a)) \\ &= d^{C(\varphi)}(d'(a') + \varphi(a), -d(a)) \\ &= (d'(d'(a')) + d'(\varphi(a)) - \varphi(d(a)), d(d(a))) \\ &= (\varphi(d(a)) - \varphi(d(a)), 0) \\ &= (0,0). \end{aligned}$$

Theorem 6.1. *Let $\varphi\colon (A,d) \to (A',d')$ be a chain map. Then we have a short exact sequence of R-complexes*

$$0 \longrightarrow (A',d') \xrightarrow{\iota} (C(\varphi), d^{C(\varphi)}) \xrightarrow{\pi} (A(-1), -d) \longrightarrow 0 \tag{12}$$

where $\iota\colon (A',d') \to (C(\varphi), d^{C(\varphi)})$ is given by $\iota(a') = (a', 0)$ for all $a \in A'$ and where $\pi\colon (C(\varphi), d^{C(\varphi)}) \to (A(-1), -d)$ is given by $\pi(a', a) = a$ for all $(a', a) \in C(\varphi)$. Moreover the connecting map $\eth\colon H(A(-1)) \to H(A')$ induced by (12) agrees with $H(\varphi)\colon H(A) \to H(A')$.

Proof. It is straightforward to check that (12) is a short exact sequence of R-complexes. Let us show that the connecting map agrees with $H(\varphi)$. Let $i \in \mathbb{Z}$ and let $\overline{a} \in H_i(A(-1))$. Lift $\overline{a}$ to a representative $a \in A(-1)_i = A_{i-1}$ and lift a to the element $(0, a) \in C(\varphi)_i = A'_i \oplus A_{i-1}$. Then $\varphi(a)$ is the unique element in A'_{i-1} such that

$$\iota(\varphi(a)) = (\varphi(a), d(a)) = d^{\varphi}(0, a).$$

Therefore $\eth(\overline{a}) = \overline{\varphi(a)} = H(\varphi)(\overline{a})$. □

Remark. In the context of graded R-modules, it would be incorrect to say $\eth = H(\varphi)$. This is because $\eth$ is graded of degree -1 and $H(\varphi)$ is graded of degree 0. On the other hand, it would be correct to say $\eth_i = H_{i-1}(\varphi)$ for all $i \in \mathbb{Z}$.

Corollary. *Let $\varphi\colon (A,d) \to (A',d')$ be a chain map between R-complexes (A,d) and (A',d'). Then φ is a quasiisomorphism if and only if $H(C(\varphi)) \cong 0$.*

Proof. Suppose $H(C(\varphi)) \cong 0$. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (12) gives us

$$0 \cong H_{i+1}(C(\varphi)) \xrightarrow{H(\pi)} H_i(A) \xrightarrow{H(\varphi)} H_i(A') \xrightarrow{H(\iota)} H_i(C(\varphi)) \cong 0$$

which implies $H_i(A) \cong H_i(A')$ for all $i \in \mathbb{Z}$.

Conversely, suppose φ is a quaisiisomorphism. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (12) gives us

$$H_i(A) \cong H_i(A') \xrightarrow{H(\iota)} H_i(C(\varphi)) \xrightarrow{H(\pi)} H_{i-1}(A) \cong H_{i-1}(A')$$

which implies $H_i(C(\varphi)) \cong 0$ for all $i \in \mathbb{Z}$. □

6.3.1 Resolutions by Mapping Cones

Lemma 6.2. *(Lifting Lemma) Let $\varphi_{-1}\colon M \to M'$ be an R-module homomorphism, let (P, d) be a projective resolution of M, and let (P', d') be a projective resolution of M'. Then there exists a chain map $\varphi\colon (P, d) \to (P', d')$ such that*

$$\begin{tikzcd} H_0(P) \arrow[r, "H_0(\varphi)"] \arrow[d, "\cong"] & H_0(P') \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

Proof. For each $i > 0$, let $M'_i := \mathrm{Im}(d'_i)$ and let $M_i := \mathrm{Im}(d_i)$. We build a chain map $\varphi\colon (P, d) \to (P', d')$ by constructing R-module homomorphism $\varphi_i\colon P_i \to P'_i$ which commute with the differentials using induction on $i \geq 0$.

First consider the base case $i = 0$. Let $\psi_0\colon P_0 \to P'_0/M'_0$ be the composition

$$P_0 \to P_0/M_1 \cong M \to M' \cong P'_0/M'_1.$$

Since P_0 is projective and since $d'_0\colon P'_0 \to P'_0/M_1$ is a surjective homomorphism, we can lift $\psi_0\colon P_0 \to P'_0/M'_0$ along $d'_0 : P'_0 \to P'_0/M_1$ to a homomorphism $\varphi_0\colon P_0 \to P'_0$ such that $d'_0\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructedn an R-module homomorphism $\varphi_i\colon P_i \to P'_i$ such that

$$d'_i\varphi_i = \varphi_{i-1}d_i.$$

We need to construct an R-module homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ such that

$$d'_{i+1}\varphi_{i+1} = \varphi_i d_{i+1}.$$

First, observe that $\mathrm{Im}(\varphi_i d_{i+1}) \subseteq M'_{i+1}$. Indeed, we have

$$\begin{aligned} d'_i\varphi_i d_{i+1} &= \varphi_{i-1}d_i d_{i+1} \\ &= 0, \end{aligned}$$

Thus, since (P', d') is exact for all $i > 0$, we have

$$\begin{aligned} \mathrm{Im}(\varphi_i d_{i+1}) &\subseteq \mathrm{Ker}(d'_i) \\ &= \mathrm{Im}(d'_{i+1}) \\ &= M'_{i+1}. \end{aligned}$$

Now since P_{i+1} is projective and $d'_{i+1}\colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i d_{i+1} : P_{i+1} \to M'_{i+1}$ along $d'_{i+1} : P'_{i+1} \to M'_{i+1}$ to a homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ such that

$$d'_{i+1}\varphi_{i+1} = \varphi_i d_{i+1}.$$

The last part of the lemma, follows from the way φ_0 was constructed.

□

Theorem 6.3. *With the notation as above, if φ_{-1} is injective, then $(C(\varphi), d^{C(\varphi)})$ is a projective resolution of $M'/Im(\varphi_{-1})$.*

Proof. First note that $C(\varphi) \cong P' \oplus P(-1)$ is projective since P' and $P(1)$ are projective. Next, from the short exact

$$0 \longrightarrow (P', d') \xrightarrow{\iota} (C(\varphi), d^{C(\varphi)}) \xrightarrow{\pi} (P(-1), -d) \longrightarrow 0 \tag{13}$$

we obtain

$$0 \cong H_i(P') \longrightarrow H_i(C(\varphi)) \longrightarrow H_{i-1}(P) \cong 0$$

for all $i > 1$. This implies $H_i(C(\varphi)) \cong 0$ for all $i > 1$. For $i = 1$, we use (13) and Lemma (6.2) to obtain

$$\begin{tikzcd} 0 \arrow[r] & H_1(C(\varphi)) \arrow[r] & H_0(P) \arrow[r, "H_0(\varphi)"] \arrow[d, "\cong"] & H_0(P') \arrow[d, "\cong"] \\ & & M \arrow[r, "\varphi_{-1}"] & M' \end{tikzcd} \tag{14}$$

Since φ_{-1} is injective, $H_0(\varphi)$ must be injective too, which implies $H_1(C(\varphi)) \cong 0$ by exactness of the top row in (14). Finally, for $i = 0$, (13) and Lemma (6.2) gives us

$$\begin{array}{ccccccc} H_0(P) & \xrightarrow{H_0(\varphi)} & H_0(P') & \longrightarrow & H_0(C(\varphi)) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & & & \\ M & \xrightarrow{\varphi_{-1}} & M' & \longrightarrow & M'/\mathrm{Im}(\varphi) & \longrightarrow & 0 \end{array} \tag{15}$$

with exact rows. In particular, this implies $H_0(C(\varphi)) \cong M'/\mathrm{Im}(\varphi_{-1})$. □

6.4 Koszul Complex

Definition 6.4. Let $\underline{r} = r_1, \ldots, r_n \in R$. The **Koszul complex** of $\underline{r}$, denoted $(\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ is the R-complex whose graded R-module $\mathcal{K}(\underline{r})$ has

$$\mathcal{K}_i(\underline{r}) := \begin{cases} R & \text{if } i \leq 0 \\ \bigoplus_{1 \leq \lambda_1 < \cdots < \lambda_i \leq n} Re_{\lambda_1 \cdots \lambda_i} & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{K}(\underline{r})}$ is the unique graded endomorphism of degree -1 such that

$$d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \lambda_i}) = \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$, where the hat symbol means omit that subscript.

Remark. We need to justify that $d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})} = 0$ (so that $(\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ really is an R-complex). It suffices to show that $d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})}$ maps all of the basis elements to 0: for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$, we have

$$\begin{aligned} d^{\mathcal{K}(\underline{r})} d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \lambda_i}) &= d^{\mathcal{K}(\underline{r})} \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i} \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i}) \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} \left(\sum_{1 \leq \kappa < \mu} (-1)^{\kappa-1} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\kappa \cdots \widehat{\lambda}_\mu \cdots \lambda_i} + \sum_{\mu < \kappa \leq i} (-1)^{\kappa} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \widehat{\lambda}_\kappa \cdots \lambda_i} \right) \\ &= \sum_{1 \leq \kappa < \mu \leq i} (-1)^{\mu+\kappa-1} r_{\lambda_\mu} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\kappa \cdots \widehat{\lambda}_\mu \cdots \lambda_i} + \sum_{1 \leq \mu < \kappa \leq i} (-1)^{\mu+\kappa} r_{\lambda_\mu} r_{\lambda_\kappa} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \widehat{\lambda}_\kappa \cdots \lambda_i} \\ &= 0, \end{aligned}$$

by symmetry in μ and κ.

6.5 Tensor Products

Definition 6.5. Let (A, d) and (A', d') be two R-complexes. Their **tensor product** is the R-complex

$$(A, d) \otimes_R (A', d') := (A \otimes_R A', d^{A \otimes_R A'}),$$

where the graded R-module $A \otimes_R A'$ has

$$(A \otimes_R A')_i = \bigoplus_{j \in \mathbb{Z}} A_j \otimes A'_{j-i}$$

as its ith component and whose differential is defined on elementary tensors by

$$d^{A \otimes_R A'}(a \otimes a') = d(a) \otimes a' + (-1)^i a \otimes d'(a')$$

for all $i, j \in \mathbb{Z}$, $a \in A_i$ and $a' \in A_j$.

6.5.1 Commutativity of Tensor Products

Proposition 6.1. *Let (A,d) and (A',d') be R-complexes. Then*

$$(A,d) \otimes_R (A',d') \cong (A',d') \otimes_R (A,d).$$

Proof. Let $\varphi\colon A \otimes_R A' \to A' \otimes_R A$ be the unique graded isomorphism[1] such that

$$\varphi(a \otimes a') = (-1)^{ij} a' \otimes a$$

for all $i,j \in \mathbb{Z}$, $a \in A_i$ and $a' \in A'_j$.

For the rest of the proof, denote $d^{\otimes} := d^{A\otimes_R A'}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{\otimes} = d^{\otimes}\varphi \tag{16}$$

It suffices to check (16) on elementary tensors. We have

$$\begin{aligned}
d^{\otimes}\varphi(a \otimes a') &= d^{\otimes}((-1)^{ij} a' \otimes a) \\
&= (-1)^{ij} d'(a') \otimes a + (-1)^{j+ij} a' \otimes d(a)) \\
&= (-1)^{ij} d'(a') \otimes a + (-1)^{j+ij-2j} a' \otimes d(a)) \\
&= (-1)^{ij} d'(a') \otimes a + (-1)^{ij-j} a' \otimes d(a) \\
&= (-1)^{(i-1)j} a' \otimes d(a) + (-1)^{i+i(j-1)} d'(a') \otimes a \\
&= \varphi(d(a) \otimes a' + (-1)^i a \otimes d'(a')) \\
&= \varphi d^{\otimes}(a \otimes a')
\end{aligned}$$

for all $i,j \in \mathbb{Z}$, $a \in A_i$ and $a' \in A_j$. □

6.5.2 Associativity of Tensor Products

Given that the proof of tensor products of R-complexes was nontrivial, we need to be sure that we have associativity of tensor products of R-complexes. The proof in this case turns out to be trivial.

Proposition 6.2. *Let (A,d), (A',d'), and (A'',d'') be R-complexes. Then*

$$((A,d) \otimes_R (A',d') \otimes_R (A'',d'') \cong (A,d) \otimes_R ((A',d') \otimes_R (A',d')).$$

Proof. Let $\varphi\colon (A \otimes_R A') \otimes_R A'' \to A \otimes_R (A' \otimes_R A'')$ be the unique graded isomorphism such that

$$\varphi((a \otimes a') \otimes a'') = a \otimes (a' \otimes a''))$$

for all $i,j,k \in \mathbb{Z}$, $a \in A_i$, $a' \in A'_j$, and $a'' \in A'_k$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{A\otimes(A'\otimes A'')} = d^{(A\otimes A')\otimes A''}\varphi \tag{17}$$

It suffices to check (17) on elementary tensors. We have

$$\begin{aligned}
d^{A\otimes(A'\otimes A'')}\varphi((a \otimes a') \otimes a'') &= d^{A\otimes(A'\otimes A'')}(a \otimes (a' \otimes a'')) \\
&= d(a) \otimes (a' \otimes a'') + (-1)^i a \otimes d^{A'\otimes A''}(a' \otimes a'') \\
&= d(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (d'(a') \otimes a'' + (-1)^j a' \otimes d(a'')) \\
&= d(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (d'(a') \otimes a'') + (-1)^{i+j} a \otimes (a' \otimes d(a'')) \\
&= \varphi(d(a) \otimes a') \otimes a'' + (-1)^i (a \otimes d'(a')) \otimes a'' + (-1)^{i+j} (a \otimes a') \otimes d''(a'')) \\
&= \varphi((d(a) \otimes a' + (-1)^i a \otimes d'(a')) \otimes a'' + (-1)^{i+j} (a \otimes a') \otimes d''(a'')) \\
&= \varphi(d^{A\otimes A'}(a \otimes a') \otimes a'' + (-1)^{i+j} (a \otimes a') \otimes d''(a'')) \\
&= \varphi d^{(A\otimes A')\otimes A''}((a \otimes a') \otimes a'')
\end{aligned}$$

for all $i,j,k \in \mathbb{Z}$, $a \in A_i$, $a' \in A'_j$, and $a'' \in A'_k$. □

[1] The map φ is linear since the map $(a,a') \mapsto a' \otimes a$ is bilinear in a and a'. Also φ is an isomorphism since the map $\psi\colon A' \otimes_R A \to A \otimes_R A'$, defined on elementary tensors by $\psi(a' \otimes a) = (-1)^{ij} a \otimes a'$ is its inverse.

6.5.3 Koszul Complex as Tensor Product

Proposition 6.3. *Let $\underline{r} = r_1, \dots, r_n \in R$. There there exists an isomorphism*

$$(\mathcal{K}(r_1), d^{\mathcal{K}(r_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(r_n), d^{\mathcal{K}(r_n)}) \cong (\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$$

of R-complexes.

Remark. Note that Proposition (6.2) gives an unambiguous interpretation for $(\mathcal{K}(r_1), d^{\mathcal{K}(r_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(r_n), d^{\mathcal{K}(r_n)})$.

Proof. For each $1 \leq \lambda \leq n$, write $\mathcal{K}(r_\lambda) = R \oplus Re_\lambda$ (so $\{1\}$ is a basis for $\mathcal{K}(r_\lambda)_0$ and $\{e_\lambda\}$ is a basis for $\mathcal{K}(r_\lambda)_1$). Le

$$\varphi\colon \mathcal{K}(r_1) \otimes_R \cdots \otimes_R \mathcal{K}(r_n) \to \mathcal{K}(r_1, \dots, r_n)$$

be the unique graded linear map [2] such that

$$\varphi(1 \otimes \cdots \otimes 1) = 1 \quad \text{and} \quad \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1) = e_{\lambda_1 \cdots \lambda_i}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism since it restricts to a bijection on basis sets.

For the rest of the proof, denote $d^{\mathcal{K}} := d^{\mathcal{K}(\underline{r})}$ and $d^{\otimes} := d^{\mathcal{K}(r_1) \otimes \cdots \otimes \mathcal{K}(r_n)}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi d^{\otimes} = d^{\mathcal{K}} \varphi. \tag{18}$$

It suffices to check (**??**) on the basis elements. We have

$$\begin{aligned} d^{\mathcal{K}} \varphi(1 \otimes \cdots \otimes 1) &= d^{\mathcal{K}}(1) \\ &= 0 \\ &= \varphi(0) \\ &= \varphi d^{\otimes}(1 \otimes \cdots \otimes 1), \end{aligned}$$

and

$$\begin{aligned} d^{\mathcal{K}} \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1) &= d^{\mathcal{K}}(e_{\lambda_1 \cdots \lambda_i}) \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i} \\ &= \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes \widehat{e}_{\lambda_\mu} \otimes \cdots \otimes e_{\lambda_i} \otimes \cdots \otimes 1) \\ &= \varphi \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} 1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes \widehat{e}_{\lambda_\mu} \otimes \cdots \otimes e_{\lambda_i} \otimes \cdots \otimes 1) \\ &= \varphi d^{\otimes}(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1). \end{aligned}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. □

6.5.4 Mapping Cone of Homothety Map as Tensor Product

Proposition 6.4. *Let (A, d) be an R-complex, let $x \in R$, and let $\mu_x\colon (A, d) \to (A, d)$ be the multiplication by x homothety map. Then*

$$(C(\mu_x), d^{C(\mu_x)}) \cong (\mathcal{K}(x), d^{\mathcal{K}(x)}) \otimes_R (A, d).$$

Proof. Let $\mathcal{K}(x) = R \oplus Re$ (so $\{1\}$ is a basis for $\mathcal{K}(x)_0$ and $\{e\}$ is a basis for $\mathcal{K}(x)_1$). Let $\varphi\colon \mathcal{K}(x) \otimes_R A \to C(\mu_x)$ be defined by

$$\varphi(1 \otimes a + e \otimes b) = (a, b)$$

for all $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Clearly φ is an isomorphism of graded R-modules. To see that φ is an isomorphism of R-complexes, we need to check that

$$d^{C(\mu_x)} \varphi = \varphi d^{\mathcal{K}(x) \otimes_R A} \tag{19}$$

[2] We say unique graded linear map here because $\mathcal{K}(r_1) \otimes_R \cdots \otimes_R \mathcal{K}(r_n)$ is free with basis elements of the form $1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1$ for $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and φ respects the grading.

Let $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Then

$$\begin{aligned} d^{C(\mu_x)}\varphi(1 \otimes a + e \otimes b) &= d^{C(\mu_x)}(a, b) \\ &= (d(a) + xb, -d(b)) \\ &= \varphi(1 \otimes (d(a) + xb) + e \otimes (-d(b))) \\ &= \varphi(1 \otimes d(a) + x \otimes b - e \otimes d(b)) \\ &= \varphi(d^{\mathcal{K}(x)\otimes A}(1 \otimes a) + d^{\mathcal{K}(x)\otimes A}(e \otimes b)) \\ &= \varphi d^{\mathcal{K}(x)\otimes A}(1 \otimes a + e \otimes b). \end{aligned}$$

□

6.6 Hom

Definition 6.6. Let (A, d) and (A', d') be two R-complexes. We define

$$\mathrm{Hom}_R((A, d), (A', d')) := (\mathrm{Hom}_R^\star(A, A'), d^{\mathrm{Hom}_R^\star(A,A')})$$

to be the R-complex whose graded R-module $\mathrm{Hom}_R^\star(A, A')$ has

$$\mathrm{Hom}_R^\star(A, A')_i = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_R(A_j, A'_{i+j})$$

as its ith homogeneous component and whose differential $d^{\mathrm{Hom}_R^\star(A,A')}$ is defined by

$$d^{\mathrm{Hom}_R^\star(A,A')}(\varphi_{j,i})_{j\in\mathbb{Z}} = (d'\varphi_{j,i} - (-1)^i\varphi_{j-1,i}d)_{j\in\mathbb{Z}}$$

for all $i, j \in \mathbb{Z}$ and $\varphi_{j,i} \in \mathrm{Hom}_R(A_j, A'_{i+j})$.

6.6.1 The Dual Koszul Complex

Definition 6.7. Let $\underline{r} = r_1, \ldots, r_n \in R$. The **dual Koszul complex** on $\underline{r}$, denoted $(\mathcal{K}^\star(\underline{r}), d^{\mathcal{K}^\star(\underline{r})})$, is the R-complex

$$(\mathcal{K}^\star(\underline{r}), d^{\mathcal{K}^\star(\underline{r})}) := \mathrm{Hom}_R((\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})}), (R, 0)).$$

The graded R-module $\mathcal{K}^\star(\underline{r})$ has

$$\mathcal{K}_i^\star(\underline{r}) := \begin{cases} R & \text{if } i \geq 0 \\ \bigoplus_{1\leq\lambda_1<\cdots<\lambda_i\leq n} Re^\star_{\lambda_1\cdots\lambda_i} & \text{if } -1 \leq i \leq -n \\ 0 & \text{if } i < -n. \end{cases}$$

as its ith homogeneous component, where

$$e^\star_{\lambda_1\cdots\lambda_i}(e_{\mu_1\cdots\mu_j}) = \begin{cases} 1 & \text{if } i = j \text{ and } \lambda_\kappa = \mu_\kappa \text{ for all } 1 \leq \kappa \leq i \\ 0 & \text{else.} \end{cases}$$

To describe how the differential $d^{\mathcal{K}^\star(\underline{r})}$ works, we introduce the following notation: Let $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_i \mid 1 \leq \lambda_1 < \cdots < \lambda_i \leq n\}$ be a subset of $\boldsymbol{n} = \{1, 2, \ldots, n\}$. Then the complement of $\boldsymbol{\lambda}$ in $\boldsymbol{n}$ is denoted by

$$\boldsymbol{\lambda}' = \{\lambda'_1, \ldots, \lambda'_{n-i} \mid 1 \leq \lambda'_1 < \cdots < \lambda'_{n-i} \leq n\} = \mathbf{n}\backslash\boldsymbol{\lambda}.$$

With this notation understood, the differential is defined on the dual basis elements by

$$d^{\mathcal{K}^\star(\underline{r})}(e^\star_{\lambda_1\cdots\lambda_i}) = \sum_{\mu=1}^{n-i}(-1)^{\mu-1}r_{\lambda_\mu}e^\star_{\lambda_1\cdots\lambda'_\mu\cdots\lambda_i}.$$

for al $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$.

Definition 6.8. Let $\underline{r} = r_1, \ldots, r_n \in R$. The **dual Koszul complex** on $\underline{r}$, denoted $(\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ is the R-complex whose graded R-module $\mathcal{K}(\underline{r})$ has

$$\mathcal{K}_i(\underline{r}) := \begin{cases} R & \text{if } i \leq 0 \\ \bigoplus_{1 \leq \lambda_1 < \cdots < \lambda_i \leq n} Re_{\lambda_1 \cdots \lambda_i} & \text{if } 1 \leq i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{K}(\underline{r})}$ is the unique graded endomorphism of degree -1 such that

$$d^{\mathcal{K}(\underline{r})}(e_{\lambda_1 \cdots \lambda_i}) = \sum_{\mu=1}^{i} (-1)^{\mu-1} r_{\lambda_\mu} e_{\lambda_1 \cdots \widehat{\lambda}_\mu \cdots \lambda_i}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$, where the hat symbol means omit that subscript.

7 Ext and Tor

7.0.1 Projective Resolutions

Definition 7.1. Let M be an R-module. A **projective resolution of** M **over** R is an R-complex (P, d) such that

1. P is a graded projective R-module. Equivalently, P_i is a projective R-module for all $i \in \mathbb{Z}$;
2. $P_i = 0$ for all $i < 0$;
3. $H_0(P) \cong M$ and $H_i(P) = 0$ for all $i > 0$.

Theorem 7.1. *Let (P, d) and (P', d') be two projective resolutions of M over R. Then (P, d) and (P', d') are homotopically equivalent.*

Proof. For each $i \geq 0$, let $M'_i := \operatorname{Im}(d'_i)$ and let $M_i := \operatorname{Im}(d_i)$. We build a chain map $\varphi\colon (P, d) \to (P', d')$ by constructing R-module homomorphism $\varphi_i\colon P_i \to P'_i$ which commute with the differentials using induction on $i \geq 0$.

First consider the base case $i = 0$. Since $P_0/M_1 \cong P'_0/M'_1$, there exists a homomorphism $\psi_0\colon P_0 \to P'_0/M'_0$. Then since P_0 is projective and since $d'_0\colon P'_0 \to P'_0/M_1$ is a surjective homomorphism, we can lift $\psi_0 : P_0 \to P'_0/M'_0$ along $d'_0 : P'_0 \to P'_0/M_1$ to a homomorphism $\varphi_0\colon P_0 \to P'_0$ such that $d'_0\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructed R-module homomorphisms $\varphi_0, \varphi_1, \ldots, \varphi_i$ which commute with the differentials. We need to construct an R-module homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ which commutes with the differentials. First, we claim that $\operatorname{Im}(\varphi_i d_{i+1}) \subseteq M'_{i+1}$. To see this, note that

$$\begin{aligned} d'_i\varphi_i d_{i+1} &= \varphi_{i-1} d_i d_{i+1} \\ &= 0. \end{aligned}$$

Thus, since $i > 0$, we have

$$\begin{aligned} \operatorname{Im}(\varphi_i d_{i+1}) &\subseteq \operatorname{Ker}(d'_i) \\ &= \operatorname{Im}(d'_{i+1}) \\ &= M'_{i+1}. \end{aligned}$$

Now since P_{i+1} is projective and $d'_{i+1}\colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i d_{i+1} : P_{i+1} \to M'_{i+1}$ along $d'_{i+1} : P'_{i+1} \to M'_{i+1}$ to a homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ such that $d'_{i+1}\varphi_{i+1} = \varphi_i d_{i+1}$.

By a similar construction as above, we get a chain map $\varphi'\colon (P', d') \to (P, d)$. Now we claim that $\varphi'\varphi$ is homotopic to id_P and similarly $\varphi\varphi'$ is homotopic to $\mathrm{id}_{P'}$. It suffices to show that $\varphi'\varphi \sim \mathrm{id}_P$ (a similar argument will give $\varphi \circ \varphi' \sim \mathrm{id}_{P'}$). The idea is to build the homotopy $h\colon (P, d) \to (P, d)$ using induction on $i \geq 0$. The homotopy equation that we need is

$$\varphi'\varphi \sim \mathrm{id}_P - 1 = dh + hd \tag{20}$$

Since P_0 is projective and $d_1\colon P_1 \to P_0$ is a surjective morphism, there exists an $h_0\colon P_0 \to P_1$ such that

$$\varphi'_0\varphi_0 - 1 = d_1 h_0. \tag{21}$$

In degree $i = 0$, the equation (20) becomes (21). Thus, we are on the right track.

Now we use induction. Suppose for $i > 0$ we have constructed an R-module homomorphism $h_i\colon P_i \to P_{i+1}$ such that

$$\varphi'_i\varphi_i - 1 = d_{i+1}h_i + h_{i-1}d_i. \tag{22}$$

Observe that $\operatorname{Im}(\varphi'_i\varphi_i - 1 - h_{i-1}d_i) \subseteq M_{i+1}$. Indeed, note that

$$\begin{aligned}
d_i(\varphi'_i\varphi_i - 1 - h_{i-1}d_i) &= d_i\varphi'_i\varphi_i - d_i - d_ih_{i-1}d_i \\
&= \varphi'_{i-1}d'_i\varphi_i - d_i - d_ih_{i-1}d_i \\
&= \varphi'_{i-1}\varphi_{i-1}d_i - d_i - d_ih_{i-1}d_i \\
&= (\varphi'_{i-1}\varphi_{i-1} - 1)d_i - d_ih_{i-1}d_i \\
&= (d_ih_{i-1} + h_{i-2}d_{i-1})d_i - d_ih_{i-1}d_i \\
&= d_ih_{i-1}d_i + h_{i-2}d_{i-1}d_i - d_ih_{i-1}d_i \\
&= d_ih_{i-1}d_i - d_ih_{i-1}d_i \\
&= 0.
\end{aligned}$$

Therefore since P_{i+1} is projective and since $d_{i+2} \colon P_{i+2} \to M_{i+2}$ is a surjective homomorphism, there exists $h_{i+1} \colon P_{i+1} \to P_{i+2}$ such that

$$\varphi'_i\varphi_i - 1 - h_{i-1}d_i = d_{i+2}h_{i+1},$$

which is the homotopy equation in degree $i+1$. $\square$

Ext-Tor Duality