Goldbach Conjecture

Partitions

Definition 0.1. We have the following definitions:

1. A **partition** is an ordered tuble $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ where each $\lambda_i \in \mathbb{N}_{\geq 1}$ $\lambda_j \geq \lambda_i$ for all $1 \leq i \leq j \leq k$. In this case, we call k the **length** of the partition $\boldsymbol{\lambda}$, and we sometimes denote this by $|\boldsymbol{\lambda}|$.

2. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two partitions. We say they are **disjoint** from each other, denoted $\boldsymbol{\lambda} \perp \boldsymbol{\mu}$, if $\lambda_i \neq \mu_j$ for all $1 \leq i \leq |\boldsymbol{\lambda}|$ and $1 \leq j \leq |\boldsymbol{\mu}|$.

3. Let N be a natural number. A **partition** of N is a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = N$. We denote by $\boldsymbol{\lambda} \vdash N$ to means $\boldsymbol{\lambda}$ is a partition of N. The collection of all partitions of N will be denoted by $\mathcal{P}_N$. The partition $(1, \ldots, 1)$ of N will be denoted by 1_N.

Partition Homomorphism

Proposition 0.1. *Let $\varphi\colon \mathbb{Q}[\{x_n\}] \to \mathbb{Q}(e)$ be the ring homomorphism given by*

$$\varphi(x_n) = e^n$$

for all $n \in \mathbb{N}$. Then as a $\mathbb{Q}$-vector space, we have

$$\ker \varphi = \operatorname{Span}_{\mathbb{Q}}\{\underline{x}^{\boldsymbol{\lambda}} - \underline{x}^{\boldsymbol{\mu}} \mid N \in \mathbb{N} \text{ and } \boldsymbol{\lambda}, \boldsymbol{\mu} \vdash N\}.$$

As a $\mathbb{Q}[\{x_n\}]$-ideal, we have

$$\ker \varphi = \langle \{\underline{x}^{\lambda} - \underline{x}^{\mu} \mid N \in \mathbb{N} \text{ and } \lambda, \mu \vdash N \text{ and } \lambda \perp \mu\} \rangle$$

Proof. Suppose $a_1 \underline{x}^{\lambda_1} + \cdots + a_k \underline{x}^{\lambda_k} \in \ker \varphi$. Since e is transcendental over $\mathbb{Q}$, we may assume that $\lambda_1, \ldots, \lambda_n \vdash N$ for some $N \in \mathbb{N}$. Then observe that

$$\begin{aligned} 0 &= \varphi(a_1 \underline{x}^{\lambda_1} + \cdots + a_k \underline{x}^{\lambda_k}) \\ &= (a_1 + \cdots + a_k) e^N \end{aligned}$$

implies $a_1 + \cdots + a_k = 0$. Therefore, we have

$$\begin{aligned} a_1 \underline{x}^{\lambda_1} + \cdots + a_k \underline{x}^{\lambda_k} &= a_1(\underline{x}^{\lambda_1} - \underline{x}^{\lambda_2}) + (a_1 + a_2)(\underline{x}^{\lambda_2} - \underline{x}^{\lambda_3}) + \cdots + (a_1 + \cdots + a_{k-1})(\underline{x}^{\lambda_{k-1}} - \underline{x}^{\lambda_k}) + (a_1 + \cdots + a_k)(\underline{x}^{\lambda_k}) \\ &= a_1(\underline{x}^{\lambda_1} - \underline{x}^{\lambda_2}) + (a_1 + a_2)(\underline{x}^{\lambda_2} - \underline{x}^{\lambda_3}) + \cdots + (a_1 + \cdots + a_{k-1})(\underline{x}^{\lambda_{k-1}} - \underline{x}^{\lambda_k}) \\ &\in \langle \{\underline{x}^{\lambda} - \underline{x}^{\mu}\} \mid N \in \mathbb{N} \text{ and } \lambda, \mu \vdash N\} \rangle. \end{aligned}$$

□

Partition Ideal

Definition 0.2. Let $\mathbb{Q}[\{x_n\}]$ be the polynomial ring over $\mathbb{Q}$ whose indeterminates are indexed over the natural numbers. For each $N \in \mathbb{N}$, we define the Nth **partition** ideal

$$I_N = \langle \{\underline{x}^{1_N} - \underline{x}^{\lambda}\} \mid \lambda \vdash N \rangle.$$