Problem 6

Proposition 0.1. *Let $\mathcal{H}$ be a separable Hilbert space and let $T\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then there exists a sequence T_m of operators with finite dimensional range such that $\|T - T_m\| \to 0$ and $m \to \infty$.*

Proof. Choose an orthonormal basis (e_n) consisting of eigenvectors of T and let (λ_n) be the corresponding sequence of eigenvalues. By reindexing if necessary, we may assume that $|\lambda_n| \geq |\lambda_{n+1}|$ for all $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, we define $T_m\colon \mathcal{H} \to \mathcal{H}$ by

$$T_m x = \sum_{n=1}^{m} \lambda_n \langle x, e_n \rangle e_n$$

for all $x \in \mathcal{H}$. Observe that $\operatorname{im}(T_m) = \operatorname{span}(\{e_1, \dots, e_m\})$ is finite dimensional. We claim that $\|T - T_m\| \to 0$ and $m \to \infty$. Indeed, let $\varepsilon > 0$ and let Λ denote the set of all eigenvalues of T. If Λ is finite, then the claim is clear by the spectral theorem for compact self-adjoint operators, so assume Λ is infinite. Then 0 must be an accumulation point of Λ. In particular, $|\lambda_n| \to 0$ as $n \to \infty$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Then for all $x \in B_1[0]$, we have

$$\begin{aligned}
\|Tx - T_m x\|^2 &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n - \sum_{n=1}^{m} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \\
&= \left\| \sum_{n=m+1}^{\infty} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \\
&= \sum_{n=m+1}^{\infty} |\lambda_n \langle x, e_n \rangle|^2 \\
&\leq |\lambda_N|^2 \sum_{n=m+1}^{\infty} |\langle x, e_n \rangle|^2 \\
&\leq |\lambda_N|^2 \|x\|^2 \\
&< \varepsilon^2.
\end{aligned}$$

This implies $\|T - T_m\| \to 0$ and $m \to \infty$. $\square$