Premanifolds

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Let $0 \le k \le \infty$ and let X be a topological premanifold. We want to explain in a precise sense how the concepts of C^k-structure on X and "maximal" C^k-atlas on X are equivalent notions.

1 Definitions

Let $\mathcal{A} = \{(\phi_i, U_i)\}$ and $\mathcal{A}' = \{(\phi'_{i'}, U'_{i'})\}$ be two C^k-atlases on X, so $\phi_i : U_i \to V_i$ and $\phi'_{i'} : U'_{i'} \to V'_{i'}$ are homeomorphisms onto non-empty open subsets of finite-dimensional $\mathbb{R}$-vector spaces, and the resulting homeomorphisms

$$\phi_{i_1} \circ \phi_{i_2}^{-1} : \phi_{i_2}(U_{i_1} \cap U_{i_2}) \to \phi_{i_1}(U_{i_1} \cap U_{i_2}) \text{ and } \phi'_{i'_1} \circ {\phi'_{i'_2}}^{-1} : \phi'_{i'_2}(U'_{i'_1} \cap U'_{i'_2}) \to \phi'_{i'_1}(U'_{i'_1} \cap U'_{i'_2})$$

between open domains in vector spaces are C^k isomorphisms in the usual sense.

2 Ringed Spaces

Definition 2.1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally R-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

2.1 From C^k Structures to Maximal C^k Atlases

Let $\mathcal{O}$ be a C^k structure on X. Let $\mathcal{A}$ be the set of all pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : (U, \mathcal{O}|_U) \to \mathbb{R}^n$ is a C^k isomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$ (with $\mathbb{R}^n$ given its usual C^k structure). The collection $\mathcal{A}$ is a C^k atlas because of two facts: a composite of C^p maps is C^p, and for maps between opens in finite-dimensional $\mathbb{R}$-vector spaces the "old" notion of C^p is the same as the "new" notion (in terms of structured $\mathbb{R}$-spaces). It is obvious that $\mathcal{A}$ is standardized. We want to prove that the standardized C^p-atlas $\mathcal{A}$ is maximal.

2.2 From Maximal C^p-Atlases to C^p-Structures

Let $\mathcal{A}$ be a maximal standarized C^p-atlas on X. For any non-empty open set $U_0 \subseteq X$, we define $\mathcal{O}(U_0)$ to be the set of functions $f : U_0 \to \mathbb{R}$ such that for all $(U, \phi) \in \mathcal{A}$, the composite map

$$f \circ \phi^{-1} : \phi(U \cap U_0) \to \mathbb{R}$$

is a C^p function on the open subset $\phi(U \cap U_0)$ in the Euclidean space $\mathbb{R}^n$ that is the target of ϕ. Also define $\mathcal{O}(\emptyset) = \{0\}$.

Lemma 2.1. *The correspondence $U_0 \mapsto \mathcal{O}(U_0)$ is an $\mathbb{R}$-space structure on X. For any $(U, \phi) \in \mathcal{A}$ and open $U_0 \subseteq U$, $\mathcal{O}(U_0)$ is the set of $f : U_0 \to \mathbb{R}$ such that $f \circ \phi^{-1} : \phi(U_0) \to \mathbb{R}$ is a C^p function on the open domain $\phi(U_0)$ in a Euclidean space.*

Proof. The usual notion of C^p function on an open set in a Euclidean space is preserved under restirction to smaller opens and can be checked by working on an open covering. Thus, the first claim in the lemma follows easily from the definition of $\mathcal{O}$. $\square$

3 Premanifolds and Manifolds

Definition 3.1.

1. A locally $\mathbb{R}$-ringed space $(M, \mathcal{O}_M)$ is called a C^α **premanifold** if there exists an open covering $\{U_i\}_{i \in I}$ of M such that for all $i \in I$ there exists $m \in \mathbb{N}_0$, an open subspace Y of $\mathbb{R}^m$, and an isomorphism of locally $\mathbb{R}$-ringed spaces $(U_i, \mathcal{O}_{M|U_i}) \to (Y, \mathcal{C}^\alpha_Y)$ (called a **chart**). Here m and Y may depend on i. In this case the structure sheaf is denoted by $\mathcal{C}^\alpha_M$ and is called the **sheaf of** $\mathcal{C}^\alpha$ **functions on** M.

2. A **morphism of** C^α **premanifolds** $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is defined as a morphism of locally $\mathbb{R}$-ringed spaces. Such a morphism is also called a C^α **map**. If $\alpha = \infty$, then we say the map is **smooth**. If $\alpha = \omega$, then we say the map is **analytic**.

We obtain the category of C^α premanifolds. A (local) isomorphism in the category of C^α premanifolds is called a (local) C^α-diffeomorphism.

Definition 3.2. 1. A locally $\mathbb{C}$-ringed space $(M, \mathcal{O}_M)$ is called a **complex premanifold** if there exists an open covering $\{U_i\}_{i \in I}$ of M such that for all $i \in I$ there exists $m \in \mathbb{N}_0$, an open subspace Y of $\mathbb{C}^m$ (both dependent on i), and an isomorphism of locally $\mathbb{C}$-ringed spaces $(U_i, \mathcal{O}_{M|U_i}) \to (Y, \mathcal{O}^{\mathrm{hol}}_Y)$ (again called a **chart**).

2. A **morphism of complex premanifolds** $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is defined as a morphism of locally $\mathbb{C}$-ringed spaces. Such a morphism is also called a **holomorphic map**.

Again we obtain the category of complex premanifolds. A (local) isomorphism in the category of complex premanifolds is called a (locally) **biholomorphic map**.

Definition 3.3. A (**real**) C^α **manifold** (respectively a **complex manifold**) is a C^α premanifold (respectively a complex premanifold) whose underlying topological space is Hausdorff and second countable. A **morphism of manifolds** is a morphism of premanifolds.

To ease the handling of the different types of (pre)manifolds, we will use the following terminology. A (pre)manifold is

- either a real C^α (pre)manifold where α will be always an element in $\widehat{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty, \omega\}$; in this case we set $\mathbb{K} := \mathbb{R}$. This will be called the **real case** or that the (pre)manifold is of **real type**;
- or a complex (pre)manifold. In this case we set $\mathbb{K} := \mathbb{C}$. This will be called the **complex case** or that the (pre)manifold is of **complex type**.