Section 3.6

Recall that if $f(x)$ and $g(x)$ are two functions, then the derivative of their product is given by the **product rule:**

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\cdot g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x))\cdot g(x) + f(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(g(x)).$$

Let's calculate some derivatives of product functions.

Example 0.1. Suppose $f(x) = x^{2/3}(x^3 - 5x^2)$. We write $f(x)$ as a product of two functions,

$$g(x) = x^{2/3} \text{ and } h(x) = x^3 - 5x^2.$$

So $f(x) = g(x)h(x)$. We calculate

$$g'(x) = \frac{2}{3}x^{-1/3} \text{ and } h'(x) = 3x^2 - 10x.$$

Therefore

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) \\ &= \left(\frac{2}{3}x^{-1/3}\right)\left(x^3 - 5x^2\right) + x^{2/3}(3x^2 - 10x). \end{aligned}$$

This is the way the book may want you to do it. The way I calculate the derative of $f(x)$ is as follow

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{2/3}(x^3 - 5x^2)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{2/3})(x^3 - 5x^2) + x^{2/3}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x^2) \\ &= \frac{2}{3}x^{-1/3}(x^3 - 5x^2) + x^{2/3}(3x^2 - 10x). \end{aligned}$$

Example 0.2. Let's find the derivative of $e^{2x}\sqrt{x^3 - 5x^2}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\sqrt{x^3 - 5x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{2x})\sqrt{x^3 - 5x^2} + e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^3 - 5x^2\right)^{1/2}\right) \\ &= e^{2x}\frac{\mathrm{d}}{\mathrm{d}x}(2x)\sqrt{x^3 - 5x^2} + e^{2x}\cdot\frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x^2) \\ &= e^{2x}\cdot 2\sqrt{x^3 - 5x^2} + e^{2x}\cdot\frac{1}{2}\left(x^3 - 5x^2\right)^{-1/2}(3x^2 - 10x). \end{aligned}$$

Example 0.3. Let's find the derivative of $(4x^2 - x + 1.5)(2(5^x))$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left((4x^2 - x + 1.5)(2(5^x))\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(4x^2 - x + 1.5)\cdot 2(5^x) + (4x^2 - x + 1.5)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2(5^x)) \\ &= (8x - 1)\cdot 2(5^x) + (4x^2 - x + 1.5)\cdot 2\frac{\mathrm{d}}{\mathrm{d}x}(5^x) \\ &= (8x - 1)\cdot 2(5^x) + (4x^2 - x + 1.5)\cdot 2\ln(5)5^x. \end{aligned}$$

Example 0.4. Let's find the derivative of $\frac{-2(3^x)}{\sqrt{x}}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-2(3^x)}{\sqrt{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-2(3^x)\cdot x^{-1/2}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(-2(3^x)))\cdot x^{-1/2} + -2(3^x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1/2}\right) \\
&= -2\ln(3)3^x x^{-1/2} + -2(3^x)\cdot\frac{-1}{2}x^{-3/2}.
\end{aligned}$$

Example 0.5. Let's find the derivative of $2.5x\sqrt{x^3-x}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(2.5x\sqrt{x^3-x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(2.5x)\cdot\sqrt{x^3-x} + 2.5x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^3-x}\right) \\
&= 2.5\cdot\sqrt{x^3-x} + 2.5x\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left((x^3-x)^{1/2}\right) \\
&= 2.5\cdot\sqrt{x^3-x} + 2.5x\cdot\frac{1}{2}\cdot(x^3-x)^{-1/2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^3-x) \\
&= 2.5\cdot\sqrt{x^3-x} + 2.5x\cdot\frac{1}{2}\cdot(x^3-x)^{-1/2}\cdot(3x^2-1).
\end{aligned}$$

Example 0.6. Let's find the derivative of $(6x-4)^5(2x+1)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((6x-4)^5(2x+1)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}((6x-4)^5)\cdot(2x+1) + (6x-4)^5\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x+1) \\
&= 5\cdot(6x-4)^4\cdot\frac{\mathrm{d}}{\mathrm{d}x}(6x-4)\cdot(2x+1) + (6x-4)^5\cdot 2 \\
&= 5\cdot(6x-4)^4\cdot 6\cdot(2x+1) + (6x-4)^5\cdot 2.
\end{aligned}$$

Example 0.7. Let's find the derivative of $\frac{2x^3+7x}{3x-5}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^3+7x}{3x-5}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((2x^3+7x)(3x-5)^{-1}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(2x^3+7x)\cdot(3x-5)^{-1} + (2x^3+7x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left((3x-5)^{-1}\right) \\
&= (6x^2+7)(3x-5)^{-1} + (2x^3+7x)\cdot(-1)\cdot(3x-5)^{-2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x-5) \\
&= (6x^2+7)(3x-5)^{-1} - (2x^3+7x)(3x-5)^{-2}\cdot 3.
\end{aligned}$$

Example 0.8. Let's find the derivative of $2(5^x)\ln(x)$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(2(5^x)\ln(x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(2(5^x))\cdot\ln(x) + 2(5^x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}(5^x)\cdot\ln(x) + 2(5^x)\cdot\frac{1}{x} \\
&= 2\cdot\ln(5)\cdot 5^x\cdot\ln(x) + 2(5^x)\cdot\frac{1}{x}.
\end{aligned}$$

Example 0.9. To find a rate of change model, we first calculate

$$\begin{aligned}
f'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(110te^{-0.7t}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(110t)\cdot e^{-0.7t} + 110t\cdot\frac{\mathrm{d}}{\mathrm{d}t}(e^{-0.7t}) \\
&= 110e^{-0.7t} + 110t\cdot e^{-0.7t}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(-0.7t) \\
&= 110e^{-0.7t} + 110t\cdot e^{-0.7t}\cdot -0.7 \\
&= (110 + 110t\cdot -0.7)e^{-0.7t} \\
&= (110-77t)e^{-0.7t}.
\end{aligned}$$

Also the units corresponding to $f'(t)$ are ng/mL per hour.

Rate of Change Model: $f'(t) = (110 - 77t)e^{-0.7t}$ ng/mL per hour gives the rate of change in concentration levels of the active ingredient in Ambien in the bloodstream t hours after a single 5 mg dose is taken orally, $0 \leq t \leq 12$