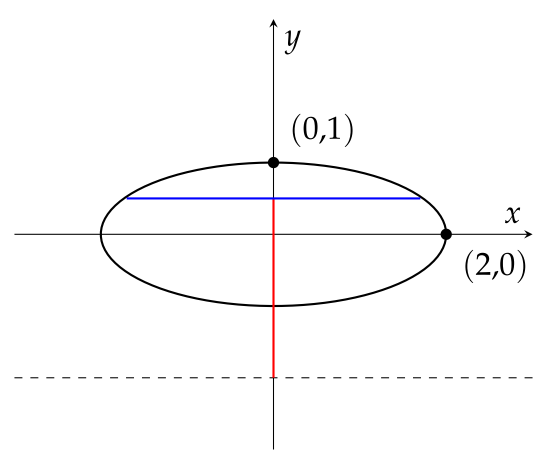
Volume of a Torus

Consider the ellipse E defined by the set of all points (x, y) in the plane such that $\frac{x^2}{4} + y^2 = 1$. By rotating E around the $y = -2$ line, we obtain an elliptic torus $\widetilde{E}$. We want to calculate the volume of $\widetilde{E}$. We will do this using the shell method. As y ranges from -1 to 1, let $S_r(y)$ be the shell radius and let $S_h(y)$ be the shell height. In the image below, the curve E is drawn using a thick black line. The axis or rotation is drawn using a dashed line. The length of the red line is given by $S_r(1/2) = 3/2$ and the length of the blue line is given by $S_h(1/2) = 2\sqrt{3}$.



An easy calculation shows that $S_r(y) = 2 + y$ and $S_h(y) = 4\sqrt{1-y^2}$. Now, let V be the volume of $\widetilde{E}$. Then by the shell method, we have

$$\begin{aligned}
V &= \int_{-1}^{1} 2\pi S_r(y) S_h(y) dy \\
&= 8\pi \int_{-1}^{1} (2+y)\sqrt{1-y^2} dy \\
&= 16\pi \int_{-1}^{1} \sqrt{1-y^2} dy + 8\pi \int_{-1}^{1} y\sqrt{1-y^2} dy \\
&= 16\pi \int_{-1}^{1} \sqrt{1-y^2} dy \\
&= 32\pi \int_{0}^{1} \sqrt{1-y^2} dy
\end{aligned}$$

Here, we have $\int_{-1}^{1} y\sqrt{1-y^2} dy = 0$ because $y\sqrt{1-y^2}$ is an odd function[1] and $16\pi \int_{-1}^{1} \sqrt{1-y^2} dy = 32\pi \int_0^1 \sqrt{1-y^2} dy$ because $\sqrt{1-y^2}$ is an even function [2]. To solve $32\pi \int_0^1 \sqrt{1-y^2} dy$, we use the trig substitution $y = \sin\theta$:

$$\begin{aligned}
32\pi \int_0^1 \sqrt{1-y^2} dy &= 32\pi \int_0^{\pi/2} \cos^2\theta d\theta \\
&= 32\pi \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2} d\theta \\
&= 16\pi \int_0^{\pi/2} d\theta + 16\pi \int_0^{\pi/2} \cos(2\theta) d\theta \\
&= 16\pi \int_0^{\pi/2} d\theta \\
&= 8\pi^2.
\end{aligned}$$

Here, we have $\int_0^{\pi/2} \cos(2\theta) d\theta = 0$ because $\cos(2\theta)$ is antisymmetric across the $y = \pi/4$ line [3] .

[1] A function $f : [-1,1] \to \mathbb{R}$ is called an **odd function** if $f(-x) = -f(x)$ for all $x \in [-1,1]$.
[2] A function $f : [-1,1] \to \mathbb{R}$ is called an **even function** if $f(-x) = f(x)$ for all $x \in [-1,1]$.
[3] A function $f : [0,1] \to \mathbb{R}$ is **antisymmetric across the** $y = \pi/4$ **line** if $f(\pi/2 - x) = -f(x)$ for all $x \in [0,1]$.