Matrix Analysis Homework 1

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(1.a) $-z = -\sqrt{3} - i$ and $z^{-1} = (1/4)(\sqrt{3} - i)$.

(1.b) We have

$$\begin{aligned} z^2 &= (\sqrt{3}+i)(\sqrt{3}+i) \\ &= 3 + 2\sqrt{3}i - 1 \\ &= 2 + 2\sqrt{3}i \end{aligned}$$

and

$$\begin{aligned} z^3 &= zz^2 \\ &= (\sqrt{3}+i)(2+2\sqrt{3}i) \\ &= 2\sqrt{3} + 6i + 2i - 2\sqrt{3} \\ &= 8i. \end{aligned}$$

(1.c) We have $r = 2$ and $\theta = \pi/6$.

(1.d) We have

$$\begin{aligned} z^2 &= (2e^{i(\pi/6)})^2 \\ &= 4e^{i(\pi/3)} \end{aligned}$$

and

$$\begin{aligned} z^3 &= (2e^{i(\pi/6)})^3 \\ &= 8e^{i(\pi/2)} \end{aligned}$$

and

$$\begin{aligned} z^{-1} &= (2e^{i(\pi/6)})^{-1} \\ &= (1/2)e^{-i(\pi/6)}. \end{aligned}$$

2. We show that $\mathbb{Q}[i]$ is a field by showing that every nonzero element in $\mathbb{Q}[i]$ has an inverse. Let $a+bi$ be a nonzero element in $\mathbb{Q}[i]$. Then the inverse of $a+bi$ is given by

$$(a+bi)^{-1} = \frac{a-bi}{\sqrt{a^2+b^2}}. \tag{1}$$

This is well-defined since $a+bi$ being nonzero implies the denominator in (1) is nonzero.

(3.a) We first write the addition table

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Next we write the multiplication table

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

(3.b) $-2 \equiv 5 \bmod 7$ and $2^{-1} \equiv 4 \bmod 7$.

(3.c) We have $2^2 \equiv 4 \bmod 7$ and $2^2 \cdot 2 \equiv 1 \bmod 7$. Therefore

$$\begin{aligned} 2^{-1} &\equiv 2^2 \bmod 7 \\ &\equiv 4 \bmod 7. \end{aligned}$$

(3.d) Consider the sequence $a^1, a^2, \ldots, a^p$ of p elements. Since each $a^i \in \mathbb{F}_p^\times$ and $\mathbb{F}_p^\times$ has only $p-1$ elements, there must be a positive integer $k \leq p$ such that $a^k = 1$ (pidgeonhole principle). In particular, this implies $a^{k-1} = a^{-1}$, where $k - 1 < p$.