Abstract Algebra Homework 11

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Problem 1

Proposition 0.1. *Let A be a domain and let K be its quotient field. The following conditions are equivalent*

1. *For all nonzero $a, b \in A$, either $a \mid b$ or $b \mid a$;*
2. *For all nonzero $x \in K$, either x or x^{-1} is in A;*
3. *There is a valuation v on K such that $A = \{x \in K \mid v(x) \geq 0\} \cup \{0\}$.*

Proof. ($1 \implies 2$): Let $x \in K^\times$. Write $x = a/b$ where $a, b \in A \backslash \{0\}$. Then either $a \mid b$ or $b \mid a$. If $b \mid a$, then we can write $a = bc$ for some nonzero $c \in A$. In this case, we have

$$\begin{aligned} x &= a/b \\ &= bc/b \\ &= c, \end{aligned}$$

and hence $x \in A$. On the other hand, if $a \mid b$, then we can write $b = ad$ for some nonzero $d \in A$. In this case, we have

$$\begin{aligned} x^{-1} &= b/a \\ &= ad/a \\ &= d, \end{aligned}$$

and hence $x^{-1} \in A$.

($2 \implies 3$): Let $\Gamma = K^\times / A^\times$. We define a total ordering on Γ as follows: Let $\overline{x}, \overline{y} \in \Gamma$. We say

$$\overline{x} \geq \overline{y} \text{ if and only if } xy^{-1} \in A. \tag{1}$$

Let us check that (1) is well-defined. Suppose xa and yb are two different representatives of the cosets $\overline{x}$ and $\overline{y}$ respectively, where $a, b \in A^\times$. Then

$$\begin{aligned} (xa)(yb)^{-1} &= (xa)(b^{-1}y^{-1}) \\ &= (xy^{-1})(ab^{-1}) \\ &\in A \end{aligned}$$

implies $\overline{xa} \geq \overline{yb}$. Thus (1) is well-defined. Next, observe that the relation given in (1) is antisymmetric: if $\overline{x} \geq \overline{y}$ and $\overline{y} \geq \overline{x}$, then $xy^{-1} \in A$ and $yx^{-1} \in A$, which implies $xy^{-1} \in A^\times$, and hence

$$\begin{aligned} \overline{x} &= \overline{x(yy^{-1})} \\ &= \overline{(xy^{-1})y} \\ &= \overline{y}. \end{aligned}$$

It is also transitive: if $\overline{x} \geq \overline{y}$ and $\overline{y} \geq \overline{z}$, then

$$\begin{aligned} xz^{-1} &= x(y^{-1}y)z^{-1} \\ &= (xy^{-1})(yz^{-1}) \\ &\in A, \end{aligned}$$

which implies $\overline{x} \geq \overline{z}$. It is also a total relation since either $\overline{x} \geq \overline{y}$ or $\overline{y} \geq \overline{x}$ (since either $xy^{-1} \in A$ or $yx^{-1} \in A$ by our assumption). Thus (1) gives us a total ordering on Γ.

Now we define $v \colon K^\times \to \Gamma$ to be the natural quotient map. Clearly v is a surjective homomorphism. We also have

$$v(x+y) \geq \min\{v(x), v(y)\} \text{ with equality if } v(x) \neq v(y).$$

Indeed, assume without loss of generality that $v(y) \geq v(x)$, so $v(x) = \min\{v(x), v(y)\}$. Then $(x+y)x^{-1} = 1 + yx^{-1} \in A$ implies $v(x+y) \geq v(x)$. Now assume $v(x) \neq v(y)$, so $yx^{-1} \notin A$. Then $x^{-1}(x+y) = 1 + yx^{-1} \notin A$. This implies $x(x+y)^{-1} \in A$ (by our assumption). Thus $v(x) \geq v(x+y)$, which implies $v(x) = v(x+y)$ by antisymmetry of $\geq$. Finally, we observe that

$$A^\times = \{x \in K \mid v(x) = 0\}$$

by construction. Moreover, we have

$$A = \{x \in K \mid v(x) \geq 0\} \cup \{0\},$$

since $v(x) \geq 0$ if and only if $v(x) \geq v(1)$ if and only if $x \in A$.

($3 \implies 1$): Let $(\Gamma, \geq)$ be a totally ordered abelian group and let $v \colon K^\times \to \Gamma$ be such a valuation. Suppose $a, b \in A \backslash \{0\}$, and without loss of generality, assume that $v(b) \geq v(a)$. Then

$$\begin{aligned} v(ba^{-1}) &= v(b) - v(a) \\ &\geq 0 \end{aligned}$$

implies $ba^{-1} \in A$. In particular, this implies $a \mid b$. □

Problem 3

Proposition 0.2. *Let A be an integral domain, let K be its quotient field, and let $\overline{A}$ be the integral closure of A in K. Then*

1. *$\overline{A}$ is integrally closed in K.*
2. *$\overline{A} \subseteq \bigcap_{A \subseteq B \subseteq K} B$ where the intersection runs over all valuation overrings B of A.*
3. *$\overline{A} = \bigcap_{A \subseteq B \subseteq K} B$ where the intersection runs over all valuation overrings B of A.*

Proof. 1. This follows from transitivity of integral extensions (see Appendix for proof of this). Indeed, let $x \in K$ be integral over $\overline{A}$. Then since $\overline{A}[x]$ is integral over $\overline{A}$ and since $\overline{A}$ is integral over A, we see that $\overline{A}[x]$ is integral over A. In particular, x is integral over A. This implies $x \in \overline{A}$ (by definition of integral closure). Thus $\overline{A}$ is integrally closed in K.

2. This follows from the fact the every valuation ring is integrally closed (see Appendix for proof of this). Indeed, let B be a valuation overring of A. Then since B is integrally closed and $A \subseteq B$, it follows that $\overline{A} \subseteq B$. Since B was arbitrary, we see that $\overline{A} = \bigcap_{A \subseteq B \subseteq K} B$ where the intersection runs over all valuation overrings B of A.

3. Let $x \in \bigcap_{A \subseteq B \subseteq K} B$ and assume for a contradiction that x is not integral over A. Observe that $x^{-1}A[x^{-1}]$ is a proper ideal in $A[x]$. Indeed, if $x^{-1}A[x^{-1}] = A[x^{-1}]$, then there exists $n \geq 0$ and $a_1, \dots, a_{n-1}, a_n \in A$ such that

$$a_n x^{-n} + a_{n-1} x^{-n+1} + \cdots + a_1 x^{-1} = 1. \tag{2}$$

Multiplying both sides of (2) by x^n and rearranging terms gives us

$$x^n - a_1 x^{n-1} - \cdots - a_{n-1} x - a_n = 0,$$

which contradicts the fact that x is not integral over A. Thus $x^{-1}A[x^{-1}]$ is a proper ideal in $A[x^{-1}]$. In particular, it is contained some maximal ideal, say $\mathfrak{m}$. Then there is a valuation ring $(B, \mathfrak{n})$ that dominates $(A[x^{-1}]_\mathfrak{m}, \mathfrak{m}A[x^{-1}]_\mathfrak{m})$ (see Appendix for proof of this). Since $x^{-1} \in \mathfrak{m} \subseteq \mathfrak{n}$, we see that $x \notin B$ (we can't have $x \in B$ *and* $x^{-1} \in \mathfrak{n}$ since $\mathfrak{n}$ does not contain any units). This contradicts our assumption that $x \in \bigcap_{A \subseteq B \subseteq K} B$. □

Problem 4

Exercise 1. Let A be a domain and let K be its fraction field. An element $x \in K$ is said to be **almost integral** if there is a nonzero $a \in A$ such that $ax^n \in A$ for all $n \in \mathbb{N}$. We say that a domain is **completely integrally closed** if it contains all of its almost integral elements.

1. Give an example of an element that is almost integral, but not integral.
2. Show that if $x \in K$ is integral over A, then x is almost integral over A;
3. Show that if A is Noetherian, then any almost integral element over A is integral over A;
4. Let A be a valuation domain that is not a field. Show that A is completely integrally closed if and only if A is one-dimensional (that is, every nonzero prime ideal is maximal).

Solution 1. 1. Consider ring $A = K[y, \{x/y^n \mid n \in \mathbb{N}\}]$. We have a strict inclusion of rings

$$K[x,y] \subset A \subset K[x,y,1/y].$$

In particular, A is a domain with fraction field $K(x,y)$. Note that $1/y \in K(x,y)$ is almost integral over A since $1/y \notin A$ and $x/y^n \in A$ for all $n \in \mathbb{N}$. On the other hand, $1/y$ is not integral over A. Indeed, if it were, then there would exists $m \in \mathbb{N}$ and $f_0, \dots, f_{m-1} \in A$ such that

$$\frac{1}{y^m} = \frac{f_{m-1}}{y^{m-1}} + \cdots + \frac{f_1}{y} + f_0. \tag{3}$$

Multilpying y^m on both sides of (3) gives us

$$1 = (f_{m-1} + \cdots + f_1 y^{m-2} + f_0 y^{m-1})y. \tag{4}$$

Evaluating $x = 0$ to both sides of (4) gives us

$$1 = (\widetilde{f}_{m-1} + \cdots + \widetilde{f}_1 y^{m-2} + \widetilde{f}_0 y^{m-1})y. \tag{5}$$

where $\widetilde{f}_0, \widetilde{f}_1, \dots, \widetilde{f}_{m-1}$ are polynomials over K in the variable y. Evaluating $y = 0$ to both sides of (5) gives us $1 = 0$, which is a contradiction.

2. Let $x \in K$ be integral over A. Write $x = a/b$ and choose $n \geq 1$ minimal and $a_0, a_1, \dots, a_{n-1} \in A$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{6}$$

We claim that for any for any $k \geq 0$, we have $b^n x^k \in A$. Indeed, first note that if $k > n$, then we can use the fact that x is integral (so $A[x] = \sum_{i=0}^{n-1} Ax^i$) to write

$$x^k = a_{n-1,k}x^{n-1} + \cdots + a_{1,k}x + a_{0,k}$$

for some $a_{0,k}, a_{1,k}, \dots, a_{n-1,k} \in A$. So it suffices to show that $b^n x^k \in A$ when $k \leq n$. This is clear though since

$$\begin{aligned} b^n x^k &= b^n \frac{a^k}{b^k} \\ &= b^{n-k}a^k \\ &\in A. \end{aligned}$$

It follows that x is almost integral over A.

3. Suppose A is a Noetherian domain and let $x \in K$ be almost integral over A. Choose $a \in A$ such that $ax^n \in A$ for all $n \in \mathbb{N}$. Consider the ascending chain of ideals, given by

$$\begin{aligned} I_0 &= \langle a \rangle \\ I_1 &= \langle a, ax \rangle \\ &\vdots \\ I_n &= \langle a, ax, \dots, ax^n \rangle \end{aligned}$$

for all $n \in \mathbb{N}$. The ascending chain of ideals (I_n) must terminate since A is Noetherian, say at $m \in \mathbb{N}$. It follows that $ax^{m+1} \in I_m$, which implies

$$ax^{m+1} = a_m a x^m + \cdots + a_1 a x + a_0 a \tag{7}$$

for some $a_0, a_1, \ldots, a_m \in A$. Canceling a from both sides of (7) (we can do this since A is a domain) and rearranging terms gives us

$$x^{m+1} - a_m x^m - \cdots - a_1 x - a_0 = 0.$$

This implies x is integral over A.

4. First suppose $(A, \mathfrak{m})$ is one-dimensional valuation domain. Let $x \in K$ be almost integral over A and assume for a contradiction that $x \notin A$. Then $x^{-1} \in A$ since A is a valuation domain. Choose a nonzero $a \in A$ such that $ax^n \in A$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose $a_n \in A$ such that $ax^n = a_n$. If a is a unit in A, then clearly $x \in A$, which is a contradiction, thus a is not a unit in A. Similarly, if x^{-1} is a unit in A, then again $x \in A$, which is a contradiction. Thus x^{-1} is also not a unit in A. We claim that $a \mid x^{-n}$ for some $n \in \mathbb{N}$. To see this, suppose that $a \nmid x^{-n}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} x^{-1} &\notin \operatorname{rad}\langle a \rangle \\ &= \bigcap_{\substack{a \in \mathfrak{p} \\ \mathfrak{p} \text{ prime}}} \mathfrak{p} \\ &= \mathfrak{m}, \end{aligned}$$

where the last equality follows from the fact that $(A, \mathfrak{m})$ is one-dimensional local ring. Thus $x^{-1} \notin \mathfrak{m}$ which implies x^{-1} is a unit in A, a contradiction. Thus $a \mid x^{-n}$ for some $n \in \mathbb{N}$. Choose such an $n \in \mathbb{N}$ and choose $b \in A$ such that $ab = x^{-n}$. Then

$$\begin{aligned} a &= a_n x^{-n} \\ &= a_n b a, \end{aligned}$$

which implies $a_n b = 1$. That is, a_n is a unit in A, but this implies $ax^n a_n^{-1} = 1$, which impilies a is unit in A, a contradiciton.

Conversely, suppose $(A, \mathfrak{m})$ is completely integrally closed valuation domain and let $\mathfrak{p}$ be a prime ideal in A. We will show that $\mathfrak{p}$ must be the maximal ideal in A. Choose $a \in A \backslash \mathfrak{p}$. Then observe that since $\mathfrak{p}$ is a prime ideal, we must have $a^n \notin \mathfrak{p}$ for all $n \in \mathbb{N}$. Furthemore, since A is a valuation domain and since $a^n \notin \mathfrak{p}$ for all $n \in \mathbb{N}$, we see that $a^n \mid b$ for all $b \in \mathfrak{p}$ for all $n \in \mathbb{N}$. In particular, we have $\langle a^n \rangle \supset \mathfrak{p}$ for all $n \in \mathbb{N}$. In other words, we have $A \supset a^{-n}\mathfrak{p}$ for all $n \in \mathbb{N}$. So for any $b \in \mathfrak{p}$, we have $a^{-n} b \in A$ for all $n \in \mathbb{N}$. Thus a^{-1} is almost integral over A. Since A is integrally closed, we see that $a^{-1} \in A$. Thus a is a unit in A, which implies $A \backslash \mathfrak{p}$ consists of units of A. Thus $\mathfrak{p}$ must be the maximal ideal $\mathfrak{m}$.

Appendix

Problem 3

Transitivity of Integral Extensions

Proposition 0.3. *Let $A \subseteq B$ be a finite extension of rings. Then $A \subseteq B$ is an integral extension of rings.*

Proof. Let $b \in B$, let $\mathrm{m}_b \colon B \to B$ be the "multiplication by b" map, given by $\mathrm{m}_b(x) = bx$ for all $x \in B$, and suppose $b_1, \ldots, b_n$ are generators for B as an A-module. Then for each $1 \le i \le n$, there exists (not necessarily unique) $a_{ji} \in A$ for all $1 \le j \le n$, such that

$$bb_i = \sum_{j=1}^{n} a_{ji} b_j.$$

Let $[\mathrm{m}_b] = (a_{ij})$ be there corresponding matrix representation. By the Cayley-Hamiltonian Theorem (over any commutative ring), the matrix $[\mathrm{m}_b]$ satisfies it's own characteristic polynomial, which is a monic polynomial $\chi_{[\mathrm{m}_b]}(T) \in A[T]$. In particular, this implies $\chi_{[\mathrm{m}_b]}(\mathrm{m}_b) = 0$. Note that the map $\mathrm{m}_{(-)} \colon B \to \mathrm{End}_A(B)$, given by $\mathrm{m}_{(-)}(b) = \mathrm{m}_b$ for all $b \in B$, is an injective A-algebra homomoprhism. Thus $\chi_{[\mathrm{m}_b]}(\mathrm{m}_b) = 0$ implies $\chi_{[\mathrm{m}_b]}(b) = 0$. Hence b integral, and since b was arbitary, this implies $A \subseteq B$ is an integral extension. $\square$

Corollary 1. *Let $A \subset B$ be a ring extension. Then an element $b \in B$ is integral over A if and only if $A[b]$ is a finitely generated A-module.*

Proof. If b is integral over A, then there is a monic polynomial $f(T) \in A[T]$ satisfying $f(b) = 0$. Then $A[b] \cong A[T]/\langle f(T) \rangle$ as A-modules, and $A[T]/\langle f(T) \rangle$ is generated by $\overline{1}, \overline{T}, \dots, \overline{T}^{n-1}$ as an A-module, where $n = \deg f$. The converse direction follows from Proposition (0.3) □

Corollary 2. *(Transitivity of Integral Extensions) Let $A \subseteq B$ and $B \subseteq C$ be integral extensions. Then $A \subseteq C$ is an integral extension.*

Proof. Let $c \in C$. Since c is integral over B, there exists $b_0, \dots, b_{n-1} \in B$ such that

$$c^n + b_{n-1}c^{n-1} + \cdots + b_0 = 0.$$

Then

$$A \subset A[b_0, \dots, b_{n-1}] \subset A[b_0, \dots, b_{n-1}][c]$$

is a composition of finite extensions. Thus, $A \subset A[b_0, \dots, b_{n-1}, c]$ is a finite extension, and hence an integral extension by Proposition (**??**). Therefore c is integral over A, which implies $A \subseteq C$ is an integral extension since c was arbitrary. □

Every Valuation Ring is Integrally Closed

Proposition 0.4. *Every Valuation Ring is Integrally Closed.*

Proof. Let A be a valuation ring with fraction field K and let $x \in K$ be integral over A. Then there exists $n \geq 1$ and $a_{n-1}, \dots, a_0 \in A$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

If $x \in A$ we are done, so assume $x \notin A$. Then $x^{-1} \in A$, since A is a valuation ring. Multiplying the equation above by $x^{-(n-1)} \in A$ and moving all but the first term on the lefthand side to the righthand side yields

$$x = -a_{n-1} - \cdots - a_0 x^{-(n-1)} \in A,$$

contradicting our assumption that $x \notin A$. It follows that $x \in A$, and hence A is integrally closed. □

Domination

Definition 0.1. Let K be a field. We define a preordered set $(\mathcal{D}_K, \geq_{\mathrm{d}})$ as follows: the underlying set is defined to be

$$\mathcal{D}_K := \{A \mid A \text{ is a local domain such that } A \subseteq K\}.$$

The preorder $\leq_{\mathrm{d}}$ is defined as follows: let $A, B \in \mathcal{D}_K$. We write $B \geq_{\mathrm{d}} A$ if $B \supseteq A$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$. In this case, we also say B **dominates** A. More generally, if R is a subring of K (so necessarily a domain), then we define a preordered set $(\mathcal{D}_{K/R}, \geq_{\mathrm{d}})$ as follows: the underlying set is defined to be

$$\mathcal{D}_{K/R} := \{A \mid A \text{ is a local domain such that } R \subseteq A \subseteq K\}.$$

The preorder $\leq_{\mathrm{d}}$ is defined as above. If $A \in \mathcal{D}_{K/R}$, then we say A is **centered** on R.

Proposition 0.5. *Let K be a field and let $A \in \mathcal{D}_K$. A maximal element in $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$ exists. Furthemore, any such maximal element is a valuation ring with K as its fraction field.*

Proof. We appeal to Zorn's Lemma. First note that $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$ is nonempty since $A \in (\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a totally ordered collection of local subrings of K (so $A_\mu \geq_{\mathrm{d}} A_\lambda$ for each $\mu \geq \lambda$, which means $A_\mu \supseteq A_\lambda$ and $\mathfrak{m}_\lambda = A_\lambda \cap \mathfrak{m}_\mu$ for each $\mu \geq \lambda$). Then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a local subring of K which dominates all of the A_λ. Indeed, it is straightforward to check that $\bigcup_{\lambda \in \Lambda} A_\lambda$ is a subring of K and $\bigcup_{\lambda \in \Lambda} \mathfrak{m}_\lambda$ is an ideal in $\bigcup_{\lambda \in \Lambda} A_\lambda$. To see that $\bigcup_{\lambda \in \Lambda} \mathfrak{m}_\lambda$ is the unique maximal ideal in $\bigcup_{\lambda \in \Lambda} A_\lambda$, we will show that its complement consists of units. Let $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$ and suppose $x \notin \bigcup_{\lambda \in \Lambda} \mathfrak{m}_\lambda$. Since $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$, there exists some λ such that $x \in A_\lambda$. Since $x \notin \bigcup_{\lambda \in \Lambda} \mathfrak{m}_\lambda$, we see that $x \notin \mathfrak{m}_\lambda$. Thus x is a unit in A_λ since $(A_\lambda, \mathfrak{m}_\lambda)$ is a local ring. It follows that x is a unit in $\bigcup_{\lambda \in \Lambda} A_\lambda$ since $A_\lambda \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$. Thus $\bigcup_{\lambda \in \Lambda} \mathfrak{m}_\lambda$ is the unique maximal ideal in $\bigcup_{\lambda \in \Lambda} A_\lambda$. Thus every totally ordered subset of $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$ has an upper bound. It follows from Zorn's Lemma that $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$ has a maximal element.

Now we prove the latter part of the proposition. Let $(B, \mathfrak{m})$ be a maximal element in $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$. First we show B has K as its fraction field. Assume for a contradiction that K is not the fraction field of B. Choose $x \in K$ which is not in the fraction field of B. If x is transcendental over B, then $B[x]_{(x,\mathfrak{m})} \in (\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$, which contradicts maximality of B. If x is algebraic over B, then for some $b \in B$, the element bx is integral over B. In this case, the subring $B' \subseteq K$ generated by B and bx is finite over B. In particular, there exists a prime ideal $\mathfrak{m}' \subseteq B'$ lying over $\mathfrak{m}$. Then $B'_{\mathfrak{m}'}$ dominates B. In particular, this implies $B = B'_{\mathfrak{m}'}$ by maximality of B, and then x is in the fraction field of B which is a contradiction.

Finally, we show that B is a valuation ring. Let $x \in K$ and assume that $x \notin B$. Let B' denote the subring of K generated by B and x. Since B is maximal in $(\mathcal{D}_{K/A}, \geq_{\mathrm{d}})$, there is no prime of B' lying over $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal we see that $\mathrm{V}(\mathfrak{m}B') = \emptyset$. Then $\mathfrak{m}B' = B'$, hence we can write

$$1 = \sum_{i=0}^{d} t_i x^i$$

with $t_i \in \mathfrak{m}$. This implies

$$(1 - t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0.$$

In particular we see that x^{-1} is integral over B. Thus the subring B'' of K generated by B and x^{-1} is finite over B and we see that there exists a prime ideal $\mathfrak{m}'' \subseteq B''$ lying over $\mathfrak{m}$. By maximality of B, we conclude that $B = (B'')_{\mathfrak{m}''}$, and hence $x^{-1} \in B$. $\square$