Section 3.4: Rates of Change of Composite Functions

A composite function $f(x)=g(h(x))$ has an *outside function* g and an *inside function* h.
Its derivative is given by the Second Form of the Chain Rule: $f'(x)=g'\left(h(x)\right)\cdot h'(x)$
The Second Form of the Chain Rule says that the derivative of a composition function is found
by multiplying: $\left(\begin{array}{c}\text{the } \textit{derivative of the outside} \\ \text{function evaluated at the} \\ \text{inside function}\end{array}\right)\bullet\left(\begin{array}{c}\text{the } \textit{derivative of} \\ \textit{the inside} \text{ function}\end{array}\right)$

Example 1:

$f(x)=(x^3-5x^2)^{2/3}$ is a composite function $f(x)=g(h(x))$.

a. Note that $f(x)$ takes the form $f(x)=(\textit{inside function})^{2/3}$. The outside function is $g(h)=h^{2/3}$. Circle the "inside function" in $f(x)=(x^3-5x^2)^{2/3}$.

Outside function: $g(h)=h^{2/3}$	Inside function: $h(x)=x^3-5x^2$

b. Write the derivative of the outside function and the inside function in the table below. Note that the derivative of the outside function uses the Power Rule.

Derivative of outside function: $g'(h)=\frac{2}{3}h^{-1/3}$	Derivative of inside function: $h'(x)=3x^2-10x$

c. Use the Second Form of the Chain Rule to find the derivative of the composition:

$$f'(x)=g'\left(h(x)\right)\cdot h'(x)=\frac{2}{3}(\textit{inside function})^{-1/3}\cdot(\textit{derivative of the inside function})$$

$$=\frac{2}{3}(x^3-5x^2)^{-1/3}\cdot(3x^2-10x)$$

Example 2:

$f(x) = \ln(x^3 - 5x^2)$ is a composite function $f(x) = g(h(x))$.

a. Note that $f(x)$ takes the form $f(x) = \ln(\textit{inside function})$. The outside function is $g(h) = \ln(h)$. Circle the "inside function" in $f(x) = \ln(x^3 - 5x^2)$.

Outside function:	Inside function:

b. Write the derivative of the outside function and the inside function in the table below. Note that the derivative of the outside function uses the Natural Logarithm Rule.

Derivative of outside function:	Derivative of inside function:

c. Use the Second Form of the Chain Rule to find the derivative of the composition:

$$f'(x) = g'\left(h(x)\right) \cdot h'(x) = \frac{1}{\textit{inside function}} \cdot (\textit{derivative of the inside function})$$

$$= \frac{1}{x^3 - 5x^2} \cdot (3x^2 - 10x)$$

Example 3:

$f(x) = e^{(x^3 - 5x^2)}$ is a composite function $f(x) = g(h(x))$.

a. Note that $f(x)$ takes the form $f(x) = e^{(\textit{inside function})}$. The outside function is $g(h) = e^h$. Circle the "inside function" in $f(x) = e^{(x^3 - 5x^2)}$.

Outside function:	Inside function:

b. Write the derivative of the outside function and the inside function in the table below. Note that the derivative of the outside function uses the e^x Rule.

Derivative of outside function:	Derivative of inside function:

c. Use the Second Form of the Chain Rule to find the derivative of the composition:

$$f'(x) = g'\left(h(x)\right) \cdot h'(x) = e^{(\textit{inside function})} \cdot (\textit{derivative of the inside function})$$

$$= e^{(x^3 - 5x^2)} \cdot (3x^2 - 10x)$$

Example 4:

$f(x) = 2^{(x^3-5x^2)}$ is a composite function $f(x) = g(h(x))$.

a. Note that $f(x)$ takes the form $f(x) = 2^{(inside\ function)}$. The outside function is $g(h) = 2^h$.
Circle the "inside function" in $f(x) = 2^{(x^3-5x^2)}$.

Outside function:	Inside function:

b. Write the derivative of the outside function and the inside function in the table below. Note that the derivative of the outside function uses the Exponential Rule.

Derivative of outside function:	Derivative of inside function:

c. Use the Second Form of the Chain Rule to find the derivative of the composition:

$$f'(x) = g'\left(h(x)\right) \cdot h'(x) = \ln(2) \cdot 2^{(inside\ function)} \cdot (derivative\ of\ the\ inside\ function)$$
$$= \ln(2) \cdot 2^{(x^3-5x^2)} \cdot (3x^2 - 10x)$$

Example 5:

Use the Second Form of the Chain Rule to find the derivative of each function. Verify that each function is a composite function by identifying the outside function *g* and the inside function *h*. Identify the Rate-of-Change Rule used in finding the derivative of the outside function. Use proper notation.

a. $f(x) = (-3x^2 + 2x - 5)^{-2}$

$f'(x) =$

b. $f(x) = \sqrt{-3x^2 + 2x - 5}$

$f'(x) =$

c. $f(x) = (\ln(x))^{-2}$

d. $f(x) = 3\sqrt{\ln(x)}$

e. $f(x) = \ln(-3x^2 + 2x - 5)$

f. $f(x) = 2\ln(\sqrt{x})$

g. $f(x) = 0.5e^{(-3x^2+2x-5)}$

h. $f(x) = 2^{(-3x^2+2x-5)}$

Example 6:

Use Addition and Subtraction Rules to find the derivative of the function. Apply the Chain Rule for terms that require it.

$f(x) = e^{(2x^3)} + 2\ln(x) - 5\sqrt{x} - \pi^2$

$f'(x) =$

The Chain Rule can be used multiple times for a function of the form $f(x) = g(h(k(x)))$. The derivative is: $f'(x) = g'\left(h(k(x))\right) \cdot h'(k(x)) \cdot k'(x)$.

Example 7:

Find the derivative of $f(x) = e^{(\sqrt{-3x^2+2x-5}\,)}$.

Example 8:

Find the derivative of each function. Use proper notation.

a. $f(x) = 5 - 3\ln\left(x^2 + 1\right)$

b. $s(t) = 2e^{(3t^2)}$

c. $f(x) = 12e^{(0.5x)} - \pi x + e^2$

d. $f(x) = 6\sqrt[3]{x^2 + 5x}$

e. $f(x) = \dfrac{5}{7\left(x^3 - x\right)^2}$ Hint: Before taking the derivative, rewrite the function using a negative exponent for the denominator: $f(x) = \dfrac{5}{7}\left(x^3 - x\right)^{-2}$

f. $f(x) = \dfrac{15.2}{1 + 2.4e^{(0.3x)}}$ Hint: Before taking the derivative, rewrite the function using a negative exponent for the denominator: $f(x) = 15.2\left(1 + 2.4e^{(0.3x)}\right)^{-1}$

g. $f(x) = 2^{\ln(x)}$

h. $f(x) = 3\sqrt{x^2 + 1}$

Example 9:

Write a completely defined rate of change model for each of the models given below.

a. A model for the average number of chirps each minute by a cricket is $f(t) = 7.8e^{(0.0407t)}$ chirps when the temperature is t degrees Fahrenheit.

b. The temperature on an average late-summer evening in south-central Michigan can be modeled as $t(h) = \dfrac{24}{1+0.04e^{(0.6h+0.02)}} + 52$ degrees Fahrenheit, h hours after sunset.