Measure Theory Homework 6

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Problem 1

Proposition 0.1. *Let $f \in L^1(X, \mathcal{M}, \mu)$ and suppose that $\int_X f 1_E \mathrm{d}\mu = 0$ for every $E \in \mathcal{M}$. Then $f = 0$ almost everywhere.*

Proof. Let $A^+ = \{f^+ \neq 0\}$ and $A^- = \{f^- \neq 0\}$. Then A^+ and A^- are measurable sets since f^+ and f^- are measurable functions. Since f agrees with f^+ on A^+, we have

$$\begin{aligned}\int_X f^+ \mathrm{d}\mu &= \int_X f^+ 1_{A^+} \mathrm{d}\mu \\ &= \int_X f 1_{A^+} \mathrm{d}\mu \\ &= 0.\end{aligned}$$

Similarly, since $-f$ agrees with f^- on A^-, we have

$$\begin{aligned}\int_X f^- \mathrm{d}\mu &= \int_X f^- 1_{A^-} \mathrm{d}\mu \\ &= \int_X -f 1_{A^-} \mathrm{d}\mu \\ &= -\int_X f 1_{A^-} \mathrm{d}\mu \\ &= 0.\end{aligned}$$

It follows that

$$\begin{aligned}\int_X |f| \mathrm{d}\mu &= \int_X (f^+ + f^-) \mathrm{d}\mu \\ &= \int_X f^+ \mathrm{d}\mu + \int_X f^- \mathrm{d}\mu \\ &= 0.\end{aligned}$$

Thus $f = 0$ almost everywhere (by a proposition proved in class). □

Problem 2

Proposition 0.2. *Let $f \colon X \to [0, \infty]$ be a nonnegative measurable function such that $\int_X f \mathrm{d}\mu < \infty$. Then*

1. $\mu(\{f = \infty\}) = 0$;
2. *f does not need to be bounded almost everywhere.*

Proof. 1. Assume for a contradiction that $\mu(\{f = \infty\}) > 0$. Then for any $M \in \mathbb{N}$, we have

$$M 1_{\{f=\infty\}} \leq M f.$$

Therefore

$$\begin{aligned}\infty &> \int_X f \mathrm{d}\mu \\ &\geq \int_X M 1_{\{f=\infty\}} \mathrm{d}\mu \\ &= M\mu(\{f = \infty\}).\end{aligned}$$

Taking $M \to \infty$ gives us a contradiction.

2. To see that f does not need to be bounded, consider $X = [0, 1]$ and $f(x) = x^{-1/2}$. Then

$$\int_0^1 x^{-1/2} \mathrm{d}x = 2,$$

but f is not bounded almost everywhere. Indeed, for any $M \in \mathbb{N}$, the set $[0, 1/M^2]$ has nonzero measure and $f|_{[0,1/M^2]} \geq M$. $\square$

Problem 3

Problem 3.a

Lemma 0.1. *Let (X, d) be a metric space and let (x_n) be a Cauchy sequence in X. Suppose there exists a subsequence $(x_{\pi(n)})$ of the sequence (x_n) such that $x_{\pi(n)} \to x$ for some $x \in X$. Then $x_n \to x$.*

Proof. Let $\varepsilon > 0$. Since $(x_{\pi(n)})$ is convergent, there exists an $N \in \mathbb{N}$ such that $\pi(n) \geq N$ implies

$$\mathrm{d}(x_{\pi(n)}, x) < \frac{\varepsilon}{2}.$$

Since (x_n) is Cauchy, there exists $M \in \mathbb{N}$ such that $m, n \geq M$ implies

$$\mathrm{d}(x_m, x_n) < \frac{\varepsilon}{2}.$$

Choose such M and N and assume without loss of generality that $N \geq M$. Then $n \geq N$ implies

$$\begin{aligned} \mathrm{d}(x_n, x) &\leq \mathrm{d}(x_{\pi(n)}, x_n) + \mathrm{d}(x_{\pi(n)}, x) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

It follows that $x_n \to x$. $\square$

Lemma 0.2. *Let X be a normed linear space. Then $\mathcal{X}$ is a Banach space if and only if every absolutely convergent series in $\mathcal{X}$ is convergent.*

Proof. Suppose first that every absolutely convergent series in $\mathcal{X}$ is convergent. Let (x_n) be a Cauchy sequence in $\mathcal{X}$. To show that (x_n) is convergent, it suffices to show that a subsequence of (x_n) is convergent, by Lemma (0.1). Choose a subsequence $(x_{\pi(n)})$ of (x_n) such that

$$\|x_{\pi(n)} - x_{\pi(n-1)}\| < \frac{1}{2^n}$$

and for all $n \in \mathbb{N}$ (we can do this since (x_n) is Cauchy). Then the series $\sum_{n=1}^{\infty}(x_{\pi(n)} - x_{\pi(n-1)})$ is absolutely convergent since

$$\begin{aligned} \sum_{n=1}^{\infty} \|x_{\pi(n)} - x_{\pi(n-1)}\| &< \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= 1. \end{aligned}$$

Therefore it must be convergent, say $\sum_{n=1}^{\infty}(x_{\pi(n)} - x_{\pi(n-1)}) \to x$. On the other hand, for each $n \in \mathbb{N}$, we have

$$x_{\pi(n)} - x_{\pi(1)} = \sum_{m=1}^{n} (x_{\pi(m)} - x_{\pi(1)}).$$

In particular, $x_{\pi(n)} \to x - x_{\pi(1)}$ as $n \to \infty$. Thus $(x_{\pi(n)})$ is a convergent subsequence of (x_n).

Conversely, suppose $\mathcal{X}$ is a Banach space and suppose $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Let $\varepsilon > 0$ and choose $K \in \mathbb{N}$ such that $N \geq M \geq K$ implies

$$\sum_{n=M}^{N} \|x_n\| < \varepsilon.$$

Then $N \geq M \geq K$ implies

$$\begin{aligned}\left\|\sum_{n=1}^{N} x_n - \sum_{n=1}^{M} x_n\right\| &= \left\|\sum_{n=M}^{N} x_n\right\| \\ &\leq \sum_{n=M}^{N} \|x_n\| \\ &< \varepsilon.\end{aligned}$$

It follows that the sequence of partial sums $(\sum_{n=1}^{N} x_n)_N$ is Cauchy. Since $\mathcal{X}$ is a Banach space, it follows that $\sum_{n=1}^{\infty} x_n$ is convergent. $\square$

Proposition 0.3. *Let $1 < p < \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is a Banach space.*

Proof. By Lemma (0.2), it suffices to show that every absolutely convergent series in $L^p(X, \mathcal{M}, \mu)$ is convergent. Suppose (f_n) is a sequence in $L^p(X, \mathcal{M}, \mu)$ such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. For each $N \in \mathbb{N}$, set $s_N = (\sum_{n=1}^{N} f_n)$. We want to show that (s_N) is convergent in $L^p(X, \mathcal{M}, \mu)$. For each $N \in \mathbb{N}$, define

$$G_N = \sum_{n=1}^{N} |f_n| \quad \text{and} \quad G = \sum_{n=1}^{\infty} |f_n|.$$

Observe that (G_N^p) is increasing sequence of nonnegative measurable (in fact integrable) functions which converges pointwise to G^p. Therefore by MCT it follows that

$$\begin{aligned}\|G\|_p &= \|G^p\|_1^{1/p} \\ &= \lim_{N\to\infty} \|G_N^p\|_1^{1/p} \\ &= \lim_{N\to\infty} \|G_N\|_p.\end{aligned}$$

In particular, since

$$\begin{aligned}\|G_N\|_p &\leq \sum_{n=1}^{N} \|f_n\|_p \\ &\leq \sum_{n=1}^{\infty} \|f_n\|_p\end{aligned}$$

for all N, we have

$$\begin{aligned}\|G\|_p &\leq \sum_{n=1}^{\infty} \|f_n\| \\ &< \infty.\end{aligned}$$

This implies $G \in L^p(X, \mathcal{M}, \mu)$. Since $\|G^p\|_1 = \|G\|_p^p < \infty$, Proposition (0.2) implies $\{G^p = \infty\}$ has measure zero, which implies $\{G = \infty\}$ has measure zero. Define $F \colon X \to \mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } G(x) = \infty. \\ \sum_{n=1}^{\infty} f_n(x) & \text{if } G(x) < \infty. \end{cases}$$

for all $x \in X$. Observe that $F(x)$ lands in $\mathbb{R}$ since if $G(x) < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent (and hence convergent since $\mathbb{R}$ is complete). Since $|F| \leq G$ and $G \in L^p(X, \mathcal{M}, \mu)$, we see that $F \in L^p(X, \mathcal{M}, \mu)$. Finally, observe that

$$\begin{aligned}\lim_{N\to\infty} \|s_N - F\|_p^p &= \lim_{N\to\infty} \int_X |s_N - F|^p \mathrm{d}\mu \\ &= \lim_{N\to\infty} \int_X \left|\sum_{n=N+1}^{\infty} f_n\right|^p \mathrm{d}\mu. \\ &= \int_X \lim_{N\to\infty} \left|\sum_{n=N+1}^{\infty} f_n\right|^p \mathrm{d}\mu \\ &= \int_X 0 \mathrm{d}\mu \\ &= 0.\end{aligned}$$

where we applied DCT to get from the second step to the third step with G^p being the dominating function. $\square$

Problem 3.b

Proposition 0.4. *Let $1 < p < \infty$. Then the set of simple functions in $L^p(X, \mathcal{M}, \mu)$ is a dense subspace of $L^p(X, \mathcal{M}, \mu)$.*

Proof. Let $f \in L^p(X, \mathcal{M}, \mu)$. Decompose f into its positive and negative parts

$$f = f^+ - f^-.$$

There exists an increasing sequence (φ_n) of nonnegative simple functions which converges to f^+ pointwise. Similarly, there exists an increasing sequence (ψ_n) of nonnegative simple functions which converges to f^- pointwise. Then $(\varphi_n + \psi_n)$ is an increasing sequence of nonnegative simple functions which converges pointwise to $|f|$. Also note that $(\varphi_n - \psi_n)$ is a sequence of simple functions which converges pointwise to f. We claim that $\|s_n - f\|_p \to 0$ as $n \to \infty$. Indeed, it suffices to show that $\||s_n - f|^p\|_1 \to 0$ since $\|s_n - f\|_p = \||s_n - f|^p\|_1^{1/p}$ for all n. To this, we'll use DCT. Clearly $(|s_n - f|^p)$ is a sequence of measurable functions which converges pointwise to 0. Also observe that

$$\begin{aligned}
|s_n - f|^p &\leq (|s_n| + |f|)^p \\
&= (|\varphi_n + \psi_n| + |f|)^p \\
&= (\varphi_n + \psi_n + |f|)^p \\
&\leq (|f| + |f|)^p \\
&\leq 2^p |f|^p.
\end{aligned}$$

Thus $2^p|f|^p$ is a dominating function, which means we can apply DCT. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \int_X |s_n - f|^p \mathrm{d}\mu &= \int_X \lim_{n\to\infty} |s_n - f|^p \mathrm{d}\mu \\
&= \int_X 0 \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

□

Problem 4

Problem 4.a

Proposition 0.5. *Assume that $\mu(X) < \infty$. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of integrable functions such that $f_n \to f$ uniformly. Then f is integrable and*

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu. \tag{1}$$

Proof. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f(x) - f_n(x)| < \frac{\varepsilon}{\mu(X)}$$

for all $x \in X$. Then

$$\begin{aligned}
\int_X |f| \mathrm{d}\mu &= \int_X |f_N + f - f_N| \mathrm{d}\mu \\
&\leq \int_X |f_N| \mathrm{d}\mu + \int_X |f - f_n| \mathrm{d}\mu \\
&< \int_X |f_N| \mathrm{d}\mu + \frac{\varepsilon}{\mu(X)} \mu(X) \\
&< \int_X |f_N| \mathrm{d}\mu + \varepsilon \\
&< \infty.
\end{aligned}$$

It follows that f is integrable. Now observe that $n \geq N$ implies

$$\begin{aligned}
\left| \int_X f \mathrm{d}\mu - \int_X f_n \mathrm{d}\mu \right| &= \left| \int_X (f - f_n) \mathrm{d}\mu \right| \\
&\leq \int_X |f - f_n| \mathrm{d}\mu \\
&< \frac{\varepsilon}{\mu(X)} \mu(X). \\
&= \varepsilon.
\end{aligned}$$

This implies (1). □

Problem 4.b

Proposition 0.6. *Assume that $\mu(X) < \infty$. Let $1 \leq p < q < \infty$. Then $L^q(X, \mathcal{M}, \mu) \subseteq L^p(X, \mathcal{M}, \mu)$.*

Proposition 0.7.

Proof. Let $f \in L^q(X, \mathcal{M}, \mu)$. We want to show that $f \in L^p(X, \mathcal{M}, \mu)$. Let

$$A = \{x \in X \mid |f|(x) > 1\}.$$

Then $|f|^p 1_A < |f|^q 1_A$, thus

$$\begin{aligned}
\int_X |f|^p \mathrm{d}\mu &= \int_X (|f|^p 1_A + |f|^p 1_{A^c}) \mathrm{d}\mu \\
&= \int_X |f|^p 1_A \mathrm{d}\mu + \int_X |f|^p 1_{A^c} \mathrm{d}\mu \\
&\leq \int_X |f|^q 1_A \mathrm{d}\mu + \int_X 1_{A^c} \mathrm{d}\mu \\
&\leq \|f\|_q + \mu(A^c) \\
&< \infty.
\end{aligned}$$

It follows that $f \in L^p(X, \mathcal{M}, \mu)$. □

Problem 5

Proposition 0.8. *Let $(f_n \colon X \to \mathbb{R})$ and $(g_n \colon X \to [0, \infty))$ be two sequences of integrable functions which converge almost everywhere to integrable functions $f \colon X \to \mathbb{R}$ and $g \colon X \to \mathbb{R}$ respectively. Suppose $|f_n| \leq g_n$ for all n and $\|g_n\|_1 \to \|g\|_1$. Then*

$$\lim_{n \to \infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

Proof. Observe that $(g_n - f_n)$ is a sequence of nonnegative measurable functions. Thus by Fatou's Lemma, we have

$$\begin{aligned}
\int_X g \mathrm{d}\mu - \int_X f \mathrm{d}\mu &= \int_X (g - f) \mathrm{d}\mu \\
&\leq \liminf_{n \to \infty} \int_X (g_n - f_n) \mathrm{d}\mu \\
&= \int_X g \mathrm{d}\mu - \limsup_{n \to \infty} \int f_n \mathrm{d}\mu,
\end{aligned}$$

where we used the fact that $\|g_n\|_1 \to \|g\|_1$ to get from the second line to the third line. Subtracting $\int_X g \mathrm{d}\mu$ from both sides and canceling the sign gives us

$$\limsup_{n \to \infty} \int_X f_n \mathrm{d}\mu \leq \int_X f \mathrm{d}\mu.$$

Now we apply the same argument with functions $g_n + f_n$ in place of $g_n - f_n$, and we obtain

$$\liminf_{n \to \infty} \int f_n \mathrm{d}\mu \geq \int_X f \mathrm{d}\mu.$$

□

Problem 6

Proposition 0.9. *Let $(f_n \colon X \to \mathbb{R})$ be a sequence of integrable functions that converge almost everywhere to an integrable function $f \colon X \to \mathbb{R}$. Then $\|f_n - f\|_1 \to 0$ if and only if $\|f_n\|_1 \to \|f\|_1$.*

Proof. Suppose $\|f_n - f\|_1 \to 0$. Then

$$\begin{aligned}
\lim_{n \to \infty} |\|f_n\|_1 - \|f\|_1| &\leq \lim_{n \to \infty} \|f_n - f\|_1 \\
&= 0.
\end{aligned}$$

Thus $\|f_n\|_1 \to \|f\|$.

Conversely, suppose $\|f_n\|_1 \to \|f\|_1$. For each $n \in \mathbb{N}$, set $g_n = |f_n| + |f|$, and set $g = 2|f|$. Then $|f_n - f| \leq g_n$, also g_n converges pointwise almost everywhere to g, also

$$\begin{aligned}
\|g_n\|_1 &= \int_X (|f_n| + |f|)\mathrm{d}\mu \\
&= \int_X |f_n|\mathrm{d}\mu + \int_X |f|\mathrm{d}\mu \\
&= \|f_n\|_1 + \|f\|_1 \\
&\to 2\|f\|_1 \\
&= \|g\|,
\end{aligned}$$

and $f_n - f$ converges pointwise almost everywhere to 0. It follows from problem 5 that

$$\begin{aligned}
\|f_n - f\|_1 &\to \|0\|_1 \\
&= 0.
\end{aligned}$$

□

Problem 7

Proposition 0.10. *Let $f : X \to \mathbb{R}$ be an integral function. Then*

$$\lim_{n\to\infty} n\mu(\{|f| > n\}) = 0.$$

Proof. First we consider the case for integrable simple functions, say

$$\varphi = \sum_{i=1}^{n} a_i 1_{A_i}, \tag{2}$$

where (2) is expressed in canonical form. Being integral here means $\mu(A_i) \neq \infty$ for all $1 \leq i \leq n$. In particular, $|\varphi|$ is bounded above by some N. Thus $n \geq N$ implies

$$\begin{aligned}
n\mu(\{\varphi > n\}) &\geq n \cdot 0 \\
&= 0.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} n\mu(\{\varphi > n\}) = 0.$$

Now we prove it for any integral function $f : X \to \mathbb{R}$. First note that since $\mu(\{|f| > n\}) \geq \mu(\{f > n\})$, we may assume that f is nonnegative. Using the fact that the set of all integrable simple functions is dense in $L^1(X, \mathcal{M}, \mu)$, choose a nonnegative integrable simple function φ such that $\varphi \leq f$ and $\|f - \varphi\|_1 < \varepsilon$. Let M be an upper bound for φ. Then we have

$$\begin{aligned}
\lim_{n\to\infty} n\mu(\{f > n\}) &= \lim_{n\to\infty} n\mu(\{\varphi > n\} \cup \{f - \varphi \geq n - \varphi\}) \\
&\leq \lim_{n\to\infty} n\mu(\{\varphi > n\} \cup \{f - \varphi \geq n - M\}) \\
&\leq \lim_{n\to\infty} n\mu(\{\varphi > n\}) + \lim_{n\to\infty} n\mu(\{f - \varphi \geq n - M\}) \\
&= \lim_{n\to\infty} n\mu(\{f - \varphi \geq n - M\}) \\
&\leq \lim_{n\to\infty} \frac{n}{n - M}\|f - \varphi\|_1 \\
&< \lim_{n\to\infty} \frac{n\varepsilon}{n - M} \\
&= \varepsilon.
\end{aligned}$$

Taking $\varepsilon \to 0$ gives us our desired result. □

Problem 8

Exercise 1. Let $x, y \geq 0$ and $0 < \gamma < 1$. Prove that

$$x^{\gamma} y^{1-\gamma} \leq \gamma x + (1 - \gamma)y. \tag{3}$$

Deduce the Young's Inequality.

Solution 1. We may assume that $x, y > 0$ since otherwise it is trivial. Set $t = x/y$ and rewrite (3) as

$$t^\gamma - \gamma t \leq 1 - \gamma. \tag{4}$$

Thus, to show (3) for all $x, y > 0$, we just need to show (4) for all $t > 0$. To see why (4) holds, define $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ by

$$f(t) = t^\gamma - \gamma t$$

for all $t \in \mathbb{R}_{>0}$. Observe that f is a smooth function on $\mathbb{R}_{>0}$, with it's first derivative and second derivative given by

$$f'(t) = \gamma t^{\gamma-1} - \gamma \quad \text{and} \quad f''(t) = \gamma(\gamma-1)t^{\gamma-2}$$

for all $t \in \mathbb{R}_{>0}$. Observe that

$$\begin{aligned} f'(t) = 0 &\iff \gamma t^{\gamma-1} = \gamma \\ &\iff t^{\gamma-1} = 1 \\ &\iff t = 1, \end{aligned}$$

where the last if and only if follows from the fact that t is a positive real number. Also, we clearly have $f''(t) < 0$ for all $t \in \mathbb{R}_{>0}$. Thus, since f is concave down on all of $\mathbb{R}_{>0}$, and $f'(t) = 0$ if and only if $t = 1$, it follows that f has a global maximum at $t = 1$. In particular, we have

$$\begin{aligned} t^\gamma - \gamma t &= f(t) \\ &\leq f(1) \\ &\leq 1^\gamma - \gamma \cdot 1 \\ &= 1 - \gamma \end{aligned}$$

for all $t \in \mathbb{R}_{>0}$.

With (3) established, we now prove Young's Inequality: Let $a, b \geq 0$ and let $1 \leq p, q < \infty$ such that $1/p + 1/q = 1$. We want to show that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Set $\gamma = 1/p$ (so $1 - \gamma = 1/q$), $a = x^\gamma$, and $b = y^{1-\gamma}$. Then Young's Inequality becomes (3), which was proved above.