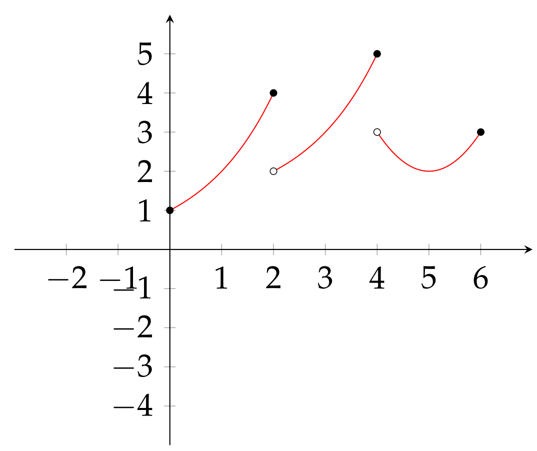
Math 1070 Test 2 Review

Graphs

Exercise 1. Consider the function $f(x)$ defined on the closed interval $[0, 6]$ whose graph is given below



Complete the table below:

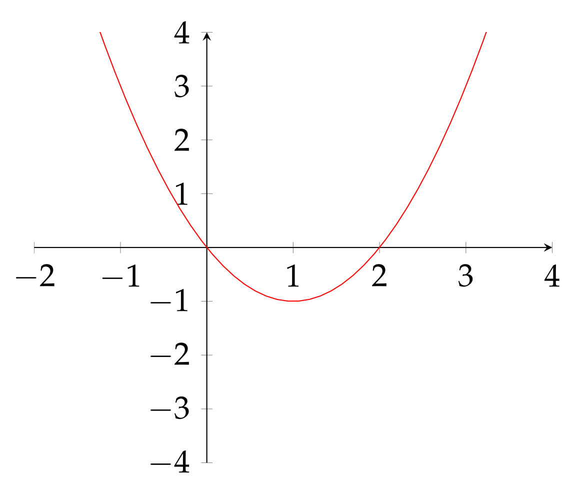
r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f

Keep in mind that r-mins and r-maxes do *not* occur at the endpoints, which in the case are $x = 0$ and $x = 6$.

Solution 1. The completed table is

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f
$x = 2, 4$	$x = 5$	$x = 4$	$x = 0$	$x = 2, 4, 5$

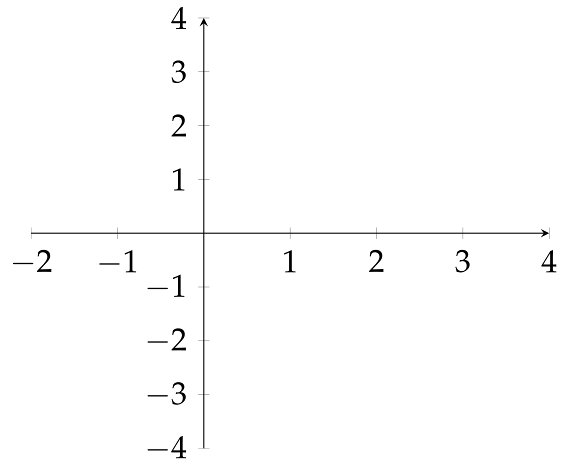
Exercise 2. Consider the function $f(x)$ defined on the whole real line below whose *slope* graph (i.e. the graph of $f'(x)$) is given below



Complete the table below:

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f	inflection points of f

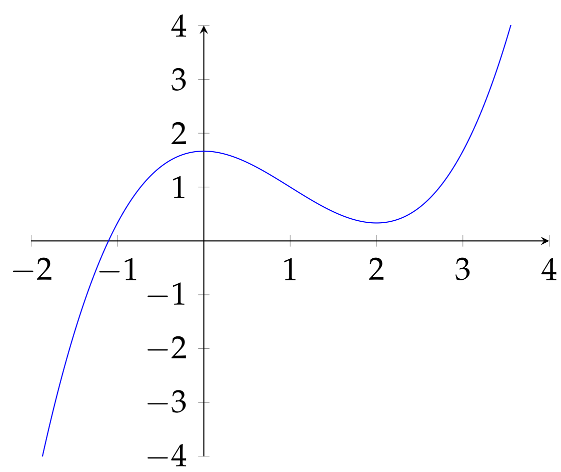
Sketch how the shape of the graph of $f(x)$ should look (there's actually many different functions whose slope graph corresponds to the one above but they all have the same shape):



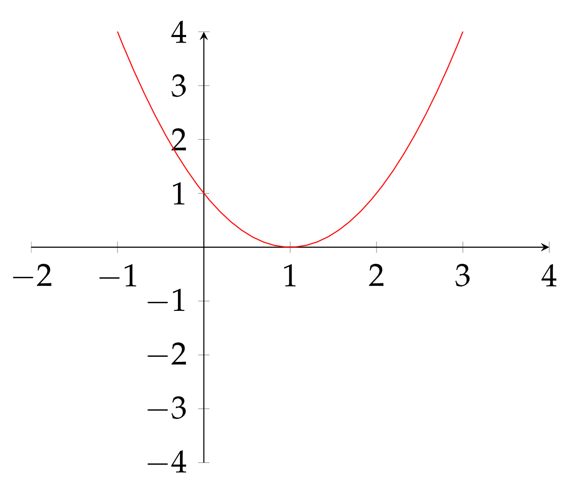
Solution 2. The completed table is

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f	inflection points of f
$x = 0$	$x = 2$	none	none	$x = 0, 2$	at $x = 1$

The sketch of the shape of the graph of $f(x)$ is



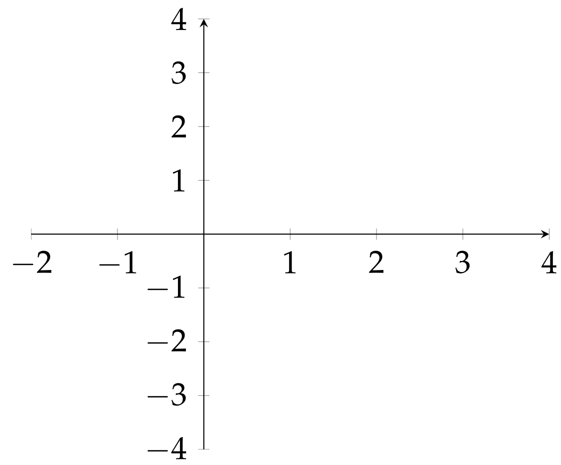
Exercise 3. Consider the function $f(x)$ defined on the whole real line below whose *slope* graph is given below



Complete the table below:

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f	inflection points of f

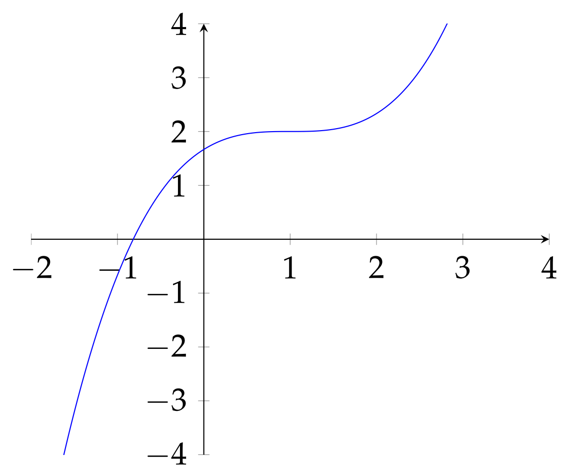
Sketch how the shape of the graph of $f(x)$ should look:



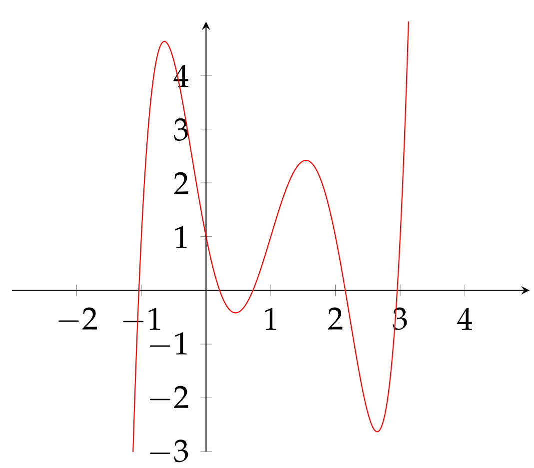
Solution 3. The completed table is

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f	inflection points of f
none	none	none	none	$x = 1$	at $x = 1$

The sketch of the shape of the graph of $f(x)$ is



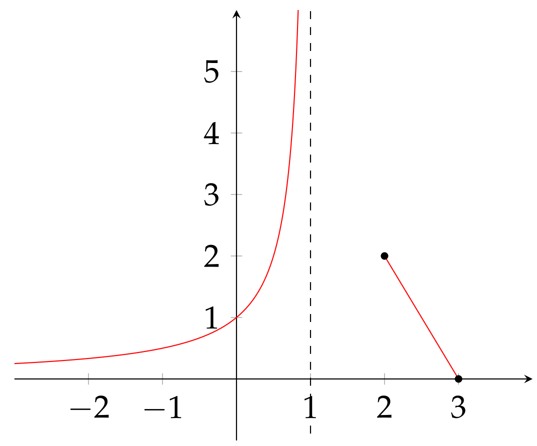
Exercise 4. Consider the function $f(x)$ whose graph is given below



At what x-values are the inflection points of f approximately located at?

Solution 4. They are located at (approximately) $x = 0$, $x = 1$, and $x = 2$. Indeed, there is an inflection point at $x = 0$ because the function changes from being concave down on $(-\infty, 0)$ to being concave up on $(0, 1)$. Similarly, there is an inflection point at $x = 1$ because the function changes from being concave up on $(0, 1)$ to being concave down on $(1, 2)$. Finally there is an inflection point at $x = 2$ because the function changes from being concave down on $(1, 2)$ to being concave up on $(2, \infty)$.

Exercise 5. Consider the function $f(x)$ defined on the interval $(-\infty, 1) \cup [2, 3]$ whose graph is given below



Complete the table below:

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f

Solution 5. The completed table is

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f
none	none	none	$x = 3$

Word Problems

Exercise 6. Consider a square whose side length is given by ℓ. Thus the area of the square is given by $A = \ell^2$. Suppose that ℓ decreases at a rate of 2 inches per hour. Find the rate that the volume is decreasing when $\ell = 4$; make sure to include units!

Solution 6. In these types of problems, you should always think about the variables, like ℓ and A, as secretly being functions of time t – especially when you see the word *rate*! In this problem, we are told that ℓ *decreases* at a *rate* of 2 inches per hour. You should interpret this as saying $\ell' = -2$ (you don't have to think about units here, but you will have to at the end of this problem). From this, we obtain $A' = 2\ell\ell' = -4\ell$. In case, you don't see that, let's do it step-by-step:

$$\begin{aligned}
A' &= A'(t) && \text{(this is just notation)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(A(t)) && \text{(again just notation)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\ell(t)^2\right) \\
&= 2\ell(t)\frac{\mathrm{d}}{\mathrm{d}t}(\ell(t)) \\
&= 2\ell\ell' && \text{(yup, just notation)} \\
&= -4\ell && \text{(since } \ell' = -2\text{)}
\end{aligned}$$

Now that we have A' calculated, let's think about what the problem is asking. We are asking to find the *rate* that the volume is decreasing when $\ell = 4$ [1]. In other words, we are asked to find A' given that $\ell = 4$. This is obtained by plugging in $\ell = 4$ into $A' = -4\ell$: when $\ell = 4$, we find that $A' = -16$. The final part to this problem

[1]There is a slight abuse of notation going on here. When we write $\ell = 4$ – we're *not* saying that the function ℓ is the constant function 4 – rather we are saying when $\ell(t_0) = 4$ for some time t_0. You don't really need to worry about this subtle point on the exam (but you should still be aware of it in general).

is to include units at the end. Recall that

$$
\begin{aligned}
\text{output units of } A' &= \frac{\text{output units of } A}{\text{output units of } t} \\
&= \frac{(\text{output units of } \ell)^2}{\text{output units of } t} \\
&= \frac{\text{in}^2}{\text{hr}}.
\end{aligned}
$$

Thus the area of the square is decreasing at a rate of 16 inches squared per hour.

Exercise 7. Consider a cylinder whose height is increasing by 8 centimeters per second and whose radius is decreasing by 2 centimeters per second. At what rate is the surface area changing at the instant when the height is 3 centimeters and the radius is 5 centimeters?

Solution 7. The first step is to find approriate function names. Let's denote the radius of the cyclinder by r and let's denote the height of the cylinder by h. Remember, r and h are secretly functions of t, so it's okay if you write $r(t)$ and $h(t)$ to denote these functions as well. Finally, let's denote S to be the surface area of the cylinder. Now that we have appropriate function names, we need to see how these functions are related. In particular, note that surface area of the cylinder is determined by the raidus and height of the cylinder, in particular, we have

$$S = 2\pi rh + 2\pi r^2. \tag{1}$$

Now we are asked to find the *rate* of change of the surface area in this problem, so we need to determine what S' is. To do this, we differentiate both sides of (1) with respect to t (remember r, h, and S are all secretly functions of t). We have

$$
\begin{aligned}
S' &= \frac{\mathrm{d}}{\mathrm{d}t}(S) && \text{I'm writing } S \text{ instead of } S(t) \text{ to simplify notation} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(2\pi rh + 2\pi r^2\right) \\
&= 2\pi\frac{\mathrm{d}}{\mathrm{d}t}(rh) + 2\pi\frac{\mathrm{d}}{\mathrm{d}t}(r^2) && \text{I applied linearity of } \frac{\mathrm{d}}{\mathrm{d}t} \\
&= 2\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}(r)h + r\frac{\mathrm{d}}{\mathrm{d}t}(h)\right) + 2\pi\frac{\mathrm{d}}{\mathrm{d}t}(r^2) && \text{product rule} \\
&= 2\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}(r)h + r\frac{\mathrm{d}}{\mathrm{d}t}(h)\right) + 2\pi\cdot 2r\frac{\mathrm{d}}{\mathrm{d}t}(r) && \text{chain rule} \\
&= 2\pi(r'h + rh') + 4\pi rr' && \text{notation simplification}
\end{aligned}
$$

Thus we have S'. Now the problem asks us to find out what S' is, when $h = 3$ and $r = 5$. The problem already tells us that $h' = 8$ (height is increasing by 8 centimeters per minute) and $r' = -2$ (radius is decreasing by 2 centimeters per minute). Thus we plug in all of these numbers to get

$$
\begin{aligned}
S' &= 2\pi(-2\cdot 3 + 5\cdot 8) + 4\pi\cdot 5\cdot(-2) \\
&= 68\pi - 40\pi \\
&= 28\pi.
\end{aligned}
$$

The final step to this problem is to figure out what output the units of S' are:

$$
\begin{aligned}
\text{output units of } S' &= \frac{\text{output units of } S}{\text{output units of } t} \\
&= \frac{\text{cm}^2}{s}.
\end{aligned}
$$

Thus the surface area is increasing by 28π centimeters squared per minute when the height is 3 centimeters and the radius is 5 centimeters.

Exercise 8. A man 6 ft tall walks at a rate of 5 feet per second away from a streetlight that is 16 feet above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 feet from the base of the light?

Solution 8. The first step is to find appropriate function names. Let us denote the distance between the man and the base of the light by m. Next let us denote the the length of the shadow by s. In this problem we are told that $m' = 5$. Indeed, the distance between the man and the base of the light is *increasing* as time goes forward so m' should be positive. In order to proceed further in the problem, we need to determine a relationship between m and s. Here we use the idea of similar triangles:

$$\frac{m+s}{16} = \frac{s}{6}. \tag{2}$$

Now we differentiate both sides of (2) with respect to t to get

$$\frac{m'+s'}{16} = \frac{s'}{6}. \tag{3}$$

Setting $m' = 5$ into (3) and rearranging terms, we find that $s' = 3$. Now let us think about what s' is measuring: it is measuring the rate of change of the length of s. Note that s' is constant (the length of the shadow is constantly changing by 3 feet per second). Thus when the man is 10 feet from the base of the light, the length of the shadow is changing by 3 feet per second.

Finally, to find the rate at which the tip of the shadow is moving, let's introduce one more function: let d denote the distance between the tip of the shadow and the base of the light. Thus

$$d = m + s. \tag{4}$$

By differentiating (4) with respect to t on both sides and setting $m' = 5$ and $s' = 3$, we find that $d' = 8$. Thus the rate at which the tip of the shadow is moving is given by 8 feet per second.

Critical Points

Exercise 9. Find all critical point(s) of $f(x) = \sin(3x + \pi)$ on $\mathbb{R}$, then find all critical points of f on the interval $(-\pi/2, \pi/2)$.

Solution 9. First note that $f'(x) = 3\cos(3x + \pi)$. Note that you should be able to do this derivative in your head at this point, but if you don't see it immediately, don't worry: you just need to more practice calculating derivatives. Here's the step-by-step calculation:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin(3x+\pi)) \\ &= \cos(3x+\pi)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x+\pi) \\ &= \cos(3x+\pi)\cdot\left(3\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\pi)\right) \\ &= \cos(3x+\pi)\cdot(3\cdot 1 + 0) \\ &= 3\cos(3x+\pi). \end{aligned}$$

Okay, now that we have the derivative f', we can find the critical point(s) of f. Recall that a **critical point** of f is a number $c \in \mathbb{R}$ such that either $f'(c) = 0$ or $f'(c)$ does not exist. In this case, $f'(c)$ exists for all $c \in \mathbb{R}$, since $f(x) = \sin(3x + \pi)$ is **differentiable** everywhere with derivative given by $f'(x) = 3\cos(3x + \pi)$. Okay, so the critical points of f are those real numbers $c \in \mathbb{R}$ such that $f'(c) = 0$; for a given $c \in \mathbb{R}$, we have

$$\begin{aligned}
f'(c) = 0 &\iff 3\cos(3c+\pi) = 0 \\
&\iff \cos(3c+\pi) = 0 \\
&\iff 3c + \pi = \frac{\pi}{2} + k\pi \text{ where } k \in \mathbb{Z} \\
&\iff 3c = \frac{\pi}{2} + k\pi - \pi \text{ where } k \in \mathbb{Z} \\
&\iff 3c = \frac{\pi}{2} + \frac{2k\pi}{2} - \frac{2\pi}{2} \text{ where } k \in \mathbb{Z} \\
&\iff 3c = \frac{(2k-1)\pi}{2} \text{ where } k \in \mathbb{Z} \\
&\iff c = \frac{(2k-1)\pi}{6} \text{ where } k \in \mathbb{Z}.
\end{aligned}$$

So for each $k \in \mathbb{Z}$, we obtain a critical point of f, and these exhaust all of them. Let us list the first few below, starting with $\ldots, k=-2, k=-1, k=0, k=1, k=2, \ldots$:

$$\left\{\ldots, \frac{-5\pi}{6}, \frac{-3\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \ldots\right\}. \tag{5}$$

Now let's solve the latter part of the problem: let's find all the critical points of f on the interval $(-\pi/2, \pi/2)$. In this case, we just look at all critical points of f given in (5), and see which ones belong to $(-\pi/2, \pi/2)$. It's clear that only two such numbers belong to $(-\pi/2, \pi/2)$, namely

$$\left\{\frac{-\pi}{6}, \frac{\pi}{6}\right\}.$$

Thus the set of all critical points of f on the interval $(-\pi/2, \pi/2)$ is given by $\{-\pi/6, \pi/6\}$.

Remark 1. On the test, you may see this type of question as an MC question. In this case, you don't actually have to calculate *all* critical points like I did in this example. You could simply test which choices (a,b,c,d) make sense by checking if $f'(c) = 0$ for each choice (a,b,c,d).

Exercise 10. Consider the function $f(x) = \frac{x-1}{x^2+2}$ defined on the interval $[-1, 1]$. Complete the table below:

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f

Solution 10. First we find all of the critical points of f in the interval $(-1, 1)$. Note that f is ratio of two polynomials, and the polynomial in the denominator is never zero on $(-1, 1)$. Thus f is differentiable everywhere on $(-1, 1)$. So the only critical points of f in $(-1, 1)$ are those $c \in (-1, 1)$ such that $f'(c) = 0$. To find these critical points, we first calculate f':

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x-1}{x^2+2}\right) \\
&= \frac{\frac{d}{dx}(x-1)\cdot(x^2+2) - (x-1)\cdot\frac{d}{dx}(x^2+2)}{(x^2+2)^2} \\
&= \frac{(x^2+2) - (x-1)\cdot 2x}{(x^2+2)^2} \\
&= \frac{x^2+2-2x^2+2x}{(x^2+2)^2} \\
&= \frac{-x^2+2x+2}{(x^2+2)^2}.
\end{aligned}$$

Now we can find the critical points of f in $(-1,1)$. We have

$$\begin{aligned}
f'(c) = 0 \text{ and } c \in (-1,1) &\iff \frac{-c^2+2c+2}{(c^2+2)^2} = 0 \text{ and } c \in (-1,1) \\
&\iff -c^2+2c+2 = 0 \text{ and } c \in (-1,1) \\
&\iff c = 1 \pm \sqrt{3} \text{ and } c \in (-1,1) && \text{use quadratic formula to get roots} \\
&\iff c = 1-\sqrt{3} && \text{since } 1-\sqrt{3} \in (-1,1).
\end{aligned}$$

Thus $c = 1-\sqrt{3}$ is the only critical point of f inside $(-1,1)$. Next, note that $f'(-1) = -1/9$ and $f'(1) = 1/3$. In particular $f'(x)$ goes from negative to positive as x increases through the critical point $1-\sqrt{3}$. Thus $f'(x)$ as a relative minimum at $1-\sqrt{3}$. Also note that $f(-1) = -2/3$ and $f(1) = 0$. Since $f'(-1)$ is negative and since $1-\sqrt{3}$ is the only critical point in $(-1,1)$, we see that we must have

$$f(1-\sqrt{3}) < f(-1) < f(1).$$

Thus f in fact has an absolute minimum at $1-\sqrt{3}$. Therefore the completed table is

r-maxes of f	r-mins of f	a-maxes of f	a-mins of f	critical points of f
none	$x = 1-\sqrt{3}$	$x = 1$	$x = 1-\sqrt{3}$	$x = 1-\sqrt{3}$

Exercise 11. Let $f(x)$ be a function defined on $\mathbb{R}$ which is differentiable everywhere with derivative given by $f'(x) = x(x+1)(x-1)$. Use this information to complete the table below:

r-maxes of f	r-mins of f	critical points of f

Solution 11. First we find the critical points of f. We are told that the function is differentiable everywhere, so the only critical points of f are those $c \in \mathbb{R}$ such that $f'(c) = 0$, and since $f(x) = x(x+1)(x-1)$, we see that the critical points are $c = -1$, $c = 0$, and $c = 1$. Now that we've determined what the critical points of f are, we can determine the r-maxes of r-maxes of f. Observe that

- f' is negative on $(-\infty,-1)$ since $f'(-2) = -18$
- f' is positive on $(-1,0)$ since $f'(-1/2) = 3/8$
- f' is negative on $(0,1)$ since $f'(1/2) = -3/8$
- f' is positive on $(1,\infty)$ since $f'(5) = 120$.

Thus we see that f has one r-max at $x = 0$ and it has two r-mins at $x = -1$ and $x = 1$. Therefore the completed table is

r-maxes of f	r-mins of f	critical points of f
$x = 0$	$x = -1$ and $x = 1$	$x = -1$, $x = 0$, and $x = 1$

Exercise 12. Suppose $f(x)$ is a function whose derivative $f'(x)$ and second derivative $f''(x)$ exists everywhere (where everywhere means all of $\mathbb{R}$). Circle true or false in the following statements:

1. If $f'(x) > 0$ everywhere, then $f(x)$ is increasing everywhere.
2. If $f''(x) < 0$ everywhere, then $f(x)$ is concave down everywhere.
3. If $f(x)$ is concave down at $x = a$, then $f''(a) < 0$.

Solution 12. 1 and 2 are true, whereas 3 is false.

Exercise 13. Let $f(x)$ be a function defined on $\mathbb{R}$ which is continuously differentiable everywhere. This means that $f'(x)$ defined on all of $\mathbb{R}$ and is also continuous on all of $\mathbb{R}$. Suppose that

- $f'(x)$ is positive on $(-\infty, -1)$,
- $f'(x)$ is negative on $(-1, 1)$,
- $f'(x)$ is positive on $(1, \infty)$.

Use this information to complete the table below:

r-maxes of f	r-mins of f	critical points of f

Solution 13. The completed table is

r-maxes of f	r-mins of f	critical points of f
$x = -1$	$x = 1$	$x = -1$ and $x = 1$

Exercise 14. Suppose $f(x)$ is a function has derivative $f'(x) = (x+1)(x-1)(x+3)$. Find the intervals where $f(x)$ is increasing.

Solution 14. First we find the critical points of f. In this case, it is easy to see that the critical points are $c = -3$, $c = -1$, and $c = 1$. Now that we've determined what the critical points of f are, we can determine where f is increasing:

- f is decreasing on $(-\infty, -3)$ since $f'(-4) = -15$ is negative
- f is increasing on $(-3, -1)$ since $f'(-2) = 3$ is positive
- f is decreasing on $(-1, 1)$ since $f'(0) = -3$ is negative
- f is increasing on $(1, \infty)$ since $f'(2) = 15$ is positive.

Exercise 15. Find the absolute extrema of the function $f(x) = \frac{x}{1+4x^2}$ on the interval $[-1, 1]$.

Solution 15. Absolute extrema is just another word for absolute mins and absolute maxes. Thus we need to find all of the absolute maxes/mins for f on $[-1, 1]$. There are two places where an absolute max/min of f can be located at:

1. a criticial point of f or
2. a boundary point (namely $x = -1$ or $x = 1$ in this example)

Thus we need to first find all of the critical points of f in the open interval $(-1, 1)$ (note that boundary points are not counted as critical points). To do this, we first calculate

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1+4x^2}\right) \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(x)(1+4x^2) - x\frac{\mathrm{d}}{\mathrm{d}x}(1+4x^2)}{(1+4x^2)^2} \\
&= \frac{(1+4x^2) - x \cdot 8x}{(1+4x^2)^2} \\
&= \frac{1-4x^2}{(1+4x^2)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\{\text{critical points of } f \text{ in } (-1,1)\} &= \{c \in (-1,1) \mid f'(c) = 0\} \\
&= \left\{c \in (-1,1) \mid \frac{1-4c^2}{(1+4c^2)^2} = 0\right\} \\
&= \left\{c \in (-1,1) \mid 1-4c^2 = 0\right\} \\
&= \left\{c \in (-1,1) \mid 1 = 4c^2\right\} \\
&= \left\{c \in (-1,1) \mid \frac{1}{4} = c^2\right\} \\
&= \left\{c \in (-1,1) \mid c = \pm\frac{1}{2}\right\} \\
&= \left\{-\frac{1}{2}, \frac{1}{2}\right\}.
\end{aligned}$$

Thus $c = -1/2$ and $c = 1/2$ are the critical points of f in (-1.1). So to find the absolute mins/maxes of f we calculate

$$\begin{aligned}
f(-1) &= -\frac{1}{5} \\
f\left(-\frac{1}{2}\right) &= -\frac{1}{4} \\
f\left(\frac{1}{2}\right) &= \frac{1}{4} \\
f(1) &= \frac{1}{5}
\end{aligned}$$

Thus f has an absolute min at $x = -1/2$ and it has an absolute max at $x = 1/2$.

Mean Value Theorem Problems

Exercise 16. Let $f(x)$ be a function defined on the interval $[a,b]$. What conditions does f need to satisfy in order for Rolle's Theorem to apply to f? Now suppose that $f(x) = x^2 + x - 1$ and let $I_1 = [0,1]$ and $I_2 = [-1,0]$. Which of these two intervals does Rolle's Theorem apply for f? Finally, state the conclusion of the Mean Value Theorem with respect to both intervals.

Solution 16. There are three conditions that f needs to satisfy:

1. The function f needs to be continuous on $[a,b]$. Recall that this means that for each $x_0 \in [a,b]$, the function is defined at x_0 (so $f(x_0)$ is some real number) and moreover we have

$$\lim_{x \to x_0} f(x) = f(x_0).$$

2. The function f needs to be differentiable on (a,b). Recall that this means that for each $x_0 \in (a,b)$, the limit

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists. When it does exist, we denote that limit by $f'(x_0)$.

3. We have $f(a) = f(b)$.

Now we suppose that $f(x) = x^2 + x - 1$ and let $I_1 = [0,1]$ and $I_2 = [-1,0]$. So f is a polynomial, and these are continous and differentiable on all of $\mathbb{R}$ (let alone on just I_1 or I_2). Furthermore, note that

$$f(-1) = -1 = f(0).$$

In particular, we may apply Rolle's Theorem to f with respect to the interval I_2; which says that there exists a $c \in I_2$ such that $f'(c) = 0$. On the other hand, note that $f(1) = 1$

$$f(0) = -1 \neq 1 = f(1).$$

Thus Rolle's Theorem does not apply to f with respect to the interval I_1. However, since f still satisfies condition 1 and 2 above, we may still apply the Mean Value Theorem to f with respect to the interval I_1; which says that there exists a $c \in I_1$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 2.$$

Exercise 17. Consider the function $\tan|x|$ together with the intervals $[-\pi/2, 0]$, $[-\pi/4, \pi/4]$, and $[0, \pi/4]$. For each interval, determine if the conditions stated in the Mean Value Theorem are satisfied.

Solution 17. First we consider $[-\pi/2, 0]$. The Mean Value Theorem does not apply here because $\tan|x|$ is not continuous on $[-\pi/2, 0]$. Indeed, $\tan|x|$ doesn't even exist at $-\pi/2$ (that is, $-\pi/2$ is not in the domain of f), but even $-\pi/2$ was in the domain of $\tan|x|$, we have another problem:

$$\lim_{x \to -\pi/2} \tan|x| = \infty,$$

so clearly $\tan|x|$ cannot be continuous at $x = -\pi/2$.

Now we consider $[-\pi/4, \pi/4]$. In this case, note that $\tan|x|$ is a composition of continuous functions, namely $\tan x$ and $|x|$; both of which are continuous functions. A composition of continuous functions is always continuous (assuming it is defined everywhere). Since $\tan|x|$ is defined on all of $[-\pi/4, \pi/4]$ and is a composition of continuous functions, we see that $\tan|x|$ is continuous on $[-\pi/4, \pi/4]$. However the Mean Value Theorem does not apply here because $\tan|x|$ is not differentiable on $[-\pi/4, \pi/4]$; indeed $\tan|x|$ is not differentiable at $x = 0$. To see this, note that on the one hand, we have

$$\begin{aligned}
\lim_{h \to 0^+} \frac{\tan|h| - \tan|0|}{h} &= \lim_{h \to 0^+} \frac{\tan|h|}{h} \\
&= \lim_{h \to 0^+} \frac{\tan h}{h} && \text{Since } h > 0, \text{ we have } |h| = h \\
&= 1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\lim_{h \to 0^-} \frac{\tan|h| - \tan|0|}{h} &= \lim_{h \to 0^-} \frac{\tan|h|}{h} \\
&= \lim_{h \to 0^-} \frac{\tan(-h)}{h} && \text{Since } h < 0, \text{ we have } |h| = -h \\
&= \lim_{h \to 0^-} \frac{-\tan h}{h} && \tan x \text{ is an odd function} \\
&= -\lim_{h \to 0^-} \frac{\tan h}{h} && \lim \text{ is a linear operator} \\
&= -1.
\end{aligned}$$

Thus we see that the limit

$$\lim_{h \to 0} \frac{\tan|h| - \tan|0|}{h}$$

does not exist since the left-hand limit and the right-hand limit do not agree.

Finally, we consider $[0, \pi/4]$. In this case, $\tan|x|$ is continuous on $[0, \pi/4]$ by the same reason above. Also $\tan|x|$ is differentiable on $(0, \pi/4)$ (you should check this as an exercise!). Thus the Mean Value Theorem applies here. Notice that it didn't matter that $\tan|x|$ is not differentiable at $x = 0$ since $0 \notin (0, \pi/4)$.

Exercise 18. Consider the polynomial $f(x) = x^3 - 2x^2$. Since f is a polynomial, the conditions of the MVT are satisfied by f with respect to the interval $[0, 1]$. Find all the numbers c that satisfy the conclusion of the MVT.

Solution 18. First we calculate the slope of the secant line from $(0, f(0))$ to $(1, f(1))$:

$$\begin{aligned}\frac{f(1)-f(0)}{1-0} &= \frac{(1-2)-(0-0)}{1} \\ &= \frac{-1}{1} \\ &= -1.\end{aligned}$$

Recall that the MVT tells us that there exists at least one $c \in (0,1)$ such that $f'(c) = -1$. In this problem however, we need to find *all* such $c \in (0,1)$. First, note that $f'(x) = 3x^2 - 4x$. Using this fact, we see that

$$\begin{aligned}f'(c) = -1 \text{ and } c \in (0,1) &\iff 3c^2 - 4c = -1 \text{ and } c \in (0,1) \\ &\iff 3c^2 - 4c + 1 = 0 \text{ and } c \in (0,1) \\ &\iff (3c-1)(c-1) = 0 \text{ and } c \in (0,1) \\ &\iff c = 1/3.\end{aligned}$$

Thus the only c which satisfies the conclusion of MVT is $c = 1/3$.

Exercise 19. Let $f(x) = \frac{x-2}{x^2-4}$. State the domain of f. Then find all (vertical, horizontal, and/or slant asymptotes).

Solution 19. To find the domain of f, we just need to see when the denominator is nonzero. In this case, it is easy to see that the denominator is nonzero if and only $a \neq \pm 2$. Thus

$$\text{domain}(f) = \mathbb{R}\backslash\{-2, 2\}.$$

Next we want to find the vertical asymptotes of f. To do this, we first perform cancellations:

$$\frac{x-2}{x^2-4} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2}.$$

What this tells us is that we have a removable singularity at $a = 2$ and a vertical asymptote at $a = -2$. The line representing this vertical asymptote is given by the equation $x = -2$. Let's justify this using limits. We have

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x-2}{x^2-4} \\ &= \lim_{x\to 2} \frac{1}{x+2} \\ &= \frac{1}{4}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} \frac{x-2}{x^2-4} \\ &= \lim_{x\to -2^+} \frac{1}{x+2} \\ &= \infty.\end{aligned}$$

Finally we want to determine if the function has a horizontal asymptote and/or a slant asymptote. A slant asymptote is found by comparing the leading term in the numerator to the leading term in the denominator. Since the degree of the numerator is less than the degree of the denominator, we see that there is no slant asymptote. On the other hand, a horizontal asymptote exists since the degree of the numerator is less than the degree of the denominator. Indeed, we have

$$\begin{aligned}\lim_{x\to \infty} f(x) &= \lim_{x\to \infty} \frac{x-2}{x^2-4} \\ &= \lim_{x\to \infty} \frac{1}{x+2} \\ &= 0.\end{aligned}$$

This implies the line $y = 0$ represents a horizontal asymptote of $f(x)$. Similarly, we have $\lim_{x\to -\infty} f(x) = 0$, so this is the only horizontal asymptote.

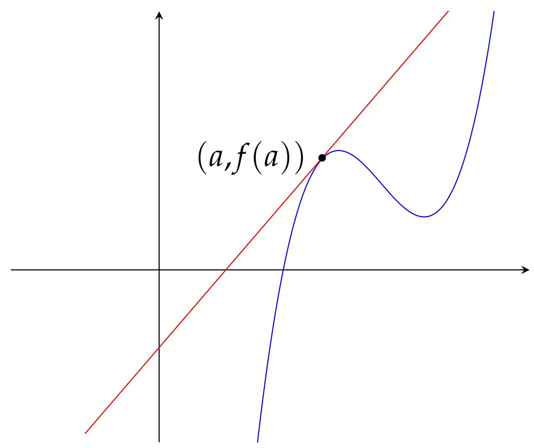
Linearization and Differentials

Exercise 20. Find the linear approximation of e^{x^2-4} at $a = 2$.

Solution 20. In general, the linear approximation (or linearization) of $f(x)$ at $x = a$ is given by

$$L_a(x) = f(a) + f'(a)(x - a). \tag{6}$$

Before I go over this problem, let me explain where the equation (6) comes from. Suppose $f(x)$ has graph below (in blue):



We wish to find the equation of the red tangent line. Recall that in order to do this, we need to find the slope of the red line (which is $f'(a)$) and we need to find a point which lies on this red line (such a point is given by $(a, f(a))$). With this in mind, we find the equation of the tangent line is

$$y - f(a) = f'(a)(x - a).$$

We obtain (6) by rearranging terms and relabeling y by L_a. Thus (6) is just the equation of the tangent line of $f(x)$ at $x = a$. The idea behing L_a is that at $x = a$, we have $f(a) = L_a(a)$, and if $\varepsilon > 0$ is a very small number, then $L_a(a + \varepsilon) \approx f(a + \varepsilon)$. Thus L_a is a nice simple function which approximates $f(x)$ at values close to $x = a$.

With that explained, let's get back to the problem at hand. Here we have $f(x) = e^{x^2-4}$ and we want to figure out the linearization (or linear approximation) of $f(x)$ at $a = 2$. Thus we should first figure out what $f'(x)$ is:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2-4}\right) \\ &= e^{x^2-4} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 4) \\ &= 2xe^{x^2-4}. \end{aligned}$$

Having solved that, we now note that $f(2) = 1$ and $f'(2) = 4$. Thus the linear approximation of $f(x)$ at $a = 2$ is

$$\begin{aligned} L_2(x) &= f(2) + f'(2)(x - 2) \\ &= 1 + 4(x - 2). \end{aligned}$$

Exercise 21. Find the linearization of $f(x) = \ln(x + 2x^2)$ at $a = 1$.

Solution 21. Observe that

$$\begin{aligned} f'(x) &= \frac{4x + 1}{2x^2 + x} \\ f'(1) &= \frac{5}{3} \\ f(1) &= \ln(3). \end{aligned}$$

You should verify that my calculations here are correct. Thus

$$L_1(x) = \ln(3) + \frac{5}{3}(x - 1).$$

Exercise 22. Let $y = e^{1-x}$. Find the differential $\mathrm{d}y$.

Solution 22. We first solve

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(e^{1-x}) \\ &= e^{1-x}\frac{\mathrm{d}}{\mathrm{d}x}(1-x) \\ &= -e^{1-x}.\end{aligned}$$

Multiplying by $\mathrm{d}x$ on both sides gives us $\mathrm{d}y = -e^{1-x}\mathrm{d}x$.

Remark 2. Make sure you leave $\mathrm{d}x$ there! The final answer is *not* $\mathrm{d}y = -e^{1-x}$!

Exercise 23. Use a differential to estimate the change in the volume of a sphere as its radius decreased from 10 inches to 9.5 inches. Be sure to include units.

Solution 23. The first step is to find appropriate names for the functions involved. This is easy: how about r for radius and V for volume. Next, we need to find a relation between these functions. In this case, it is given by

$$V = \frac{4}{3}\pi r^3. \tag{7}$$

Note that you do not need to memorize the formula for the volume a sphere (it'll be given to you on the test). Now for this problem, we need to find the differential $\mathrm{d}V$. Here are two ways to do this: First we could differentiate both sides of (7) with respect to r:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}(V) \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{4}{3}\pi\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3\right) \\ &= 4\pi r^2.\end{aligned}$$

Now multiply both sides by $\mathrm{d}r$ to obtain

$$\mathrm{d}V = 4\pi r^2\mathrm{d}r \tag{8}$$

Alternatively, we could differentiate both sides of (7) with respect to t:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(V) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi r^3\right) \\ &= \frac{4}{3}\pi\frac{\mathrm{d}}{\mathrm{d}t}\left(r^3\right) \\ &= 4\pi r^2\frac{\mathrm{d}}{\mathrm{d}t}(r) \\ &= 4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}.\end{aligned}$$

If we multiply both sides by $\mathrm{d}t$, we obtain we the same differential equation (8). In any case, we use the equation (8) to estimate the change in volume given that the radius decreased from 10 inches to 9.5 inches. In particular, we set $r = 10$ and $\mathrm{d}r = -0.5$ into (8) to obtain

$$\begin{aligned}\mathrm{d}V &= 4\pi(10)^2(-0.5) \\ &= -4\pi \cdot 50 \\ &= -200\pi\end{aligned}$$

The output units of V is inches cubed, so we write: the volume of the sphere decreased approximately by 200π inches cubed when its radius decreased from $r = 10$ inches to $r = 9.5$ inches.

Remark 3. Notice that this is just an approximation. To see how much the volume *actually* decreased, we calculate

$$\begin{aligned}\Delta V &= \frac{4}{3}\pi(9.5)^3 - \frac{4}{3}\pi(10)^3 \\ &\approx -597.426.\end{aligned}$$

Considering that $200\pi \approx 628.319$, we see that our estimation is pretty close to how much the volume actually changed by.