Permutativity

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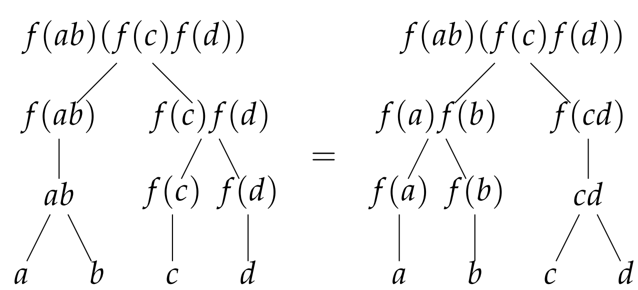
1 Introduction

We introduce a new type of algebraic law which we call the **permutative law** since it corresponds with the permutohedron as the associative law corresponds with associahedron.

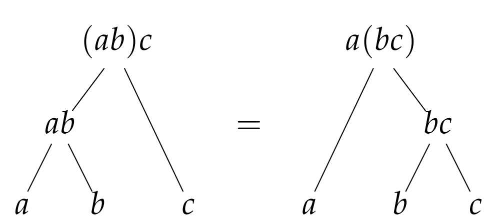
Definition 1.1. Let A be a set equipped with a binary operation $\cdot : A \times A \to A$ and a unary operation $f\colon A \to A$. We say the triple $(A, f, \cdot)$ satisfies the **permutative law** if for all $a, b, c, d \in A$, we have

$$(f(a)f(b))f(cd) = f(ab)(f(c)f(d)) \tag{1}$$

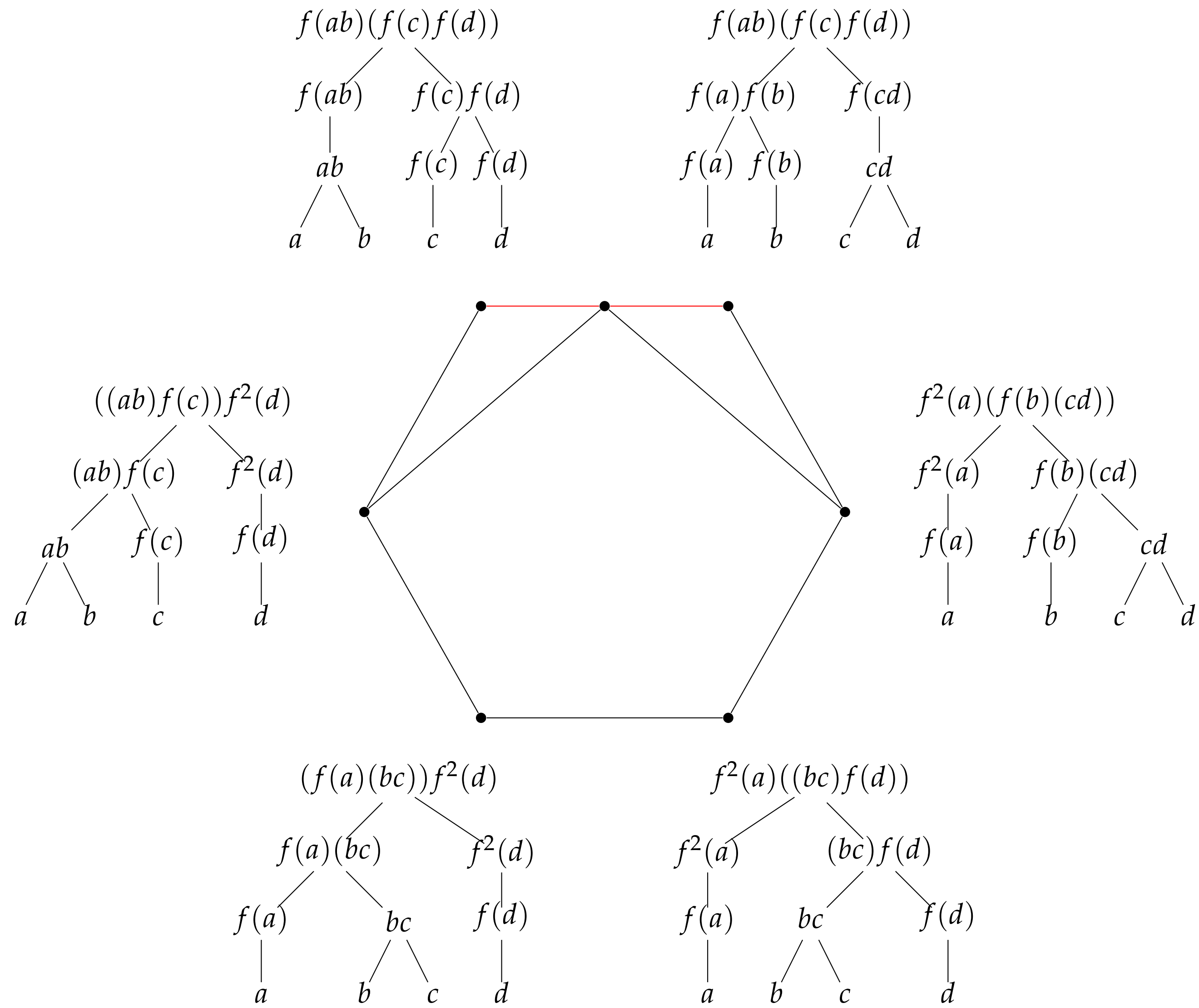
There's a very nice way of capturing visualizing this law in terms of Cayley ordered Bell trees:



This is analagous to how we can express the associative law in terms of binary rooted trees:



The difference between these two types of trees is that the Cayley ordered Bell trees keep track of a unary operator f, whereas the binary rooted trees do not. The Cayley ordered Bell trees can be attached to the vertices of the permutohedron and the ordered binary rooted trees can be attached to the vertices of the associahedron. There is a natural way to map the permutohedron to the associahedron, and it correponds to forgetting the unary operator f. In the image below, we draw the permutohedron P_3 as well as associahedron K_4 inside of it. To each vertex of K_4, we can attach a 4-leaf ordered rooted binary tree, and to each vertex of P_3, we can attach a 4-leaf Cayley ordered Bell tree, which we do here. The map from P_3 to K_4 can be visualized by collapsing the red edge, or in terms of trees, by deleting stretching nodes, or in terms of algebra, by setting f to the identity function.



2 When Do Triples Satisfy The Permutative Law?

In order for a triple $(A, f, \cdot)$ to satisfy the permutative law, we need a mixture of both nice properties for f and nice properties for the binary operation. Let us make a basic assumption on f throughout the rest of this article in order to simplify our results: we will assume that $f\colon A \to A$ is a bijection.

2.1 Groups

In this subsection, we will consider triples $(G, f, \cdot)$ where G is a group and $f\colon G \to G$ is a bijection. Here are two examples:

Example 2.1. Suppose $f\colon G \to G$ is a group homomorphism. Then the triple $(G, f, \cdot)$ satisfies the permutative law. Indeed, for all $a, b, c, d \in G$, we have

$$\begin{aligned} f(ab)(f(c)f(d)) &= (f(ab))f(c)f(d) \\ &= (f(a)f(b))f(cd) \end{aligned}$$

since the binary operation is associative and since f is a group homomorphism.

Example 2.2. Let G be a group with $x \in Z(G)$ and suppose $f(a) = xa$ for all $a \in G$. Then the triple $(G, f, \cdot)$ satisfies the permutative laws permutative. Indeed, for all $a, b, c, d \in G$,

$$\begin{aligned} f(ab)(f(c)f(d)) &= xabxcxd \\ &= xaxbxcd \\ &= (f(a)f(b))(f(cd)) \end{aligned}$$

since the binary operation is associative and since $x \in Z(G)$.

The next proposition tells us that any triples $(G, f, \cdot)$ which satisfies the permutative law essentially comes from one of the two examples above.

Proposition 2.1. *Denote $x = f(e)$. If the triple $(G, f, \cdot)$ satisfies the permutative law, then $x \in Z(G)$ and the map $\ell_x \circ f : G \to G$ is a group homomorphism.*

Proof. Since the triple $(G, f, \cdot)$ satisfies the permutative law, we have

$$f(ab)f(c)f(d) = f(a)f(b)f(cd). \tag{2}$$

for all $a, b, c, d \in G$. In particular, setting $a = b = e$ into (2) gives us

$$xf(c)f(d) = x^2 f(cd),$$

and after canceling x on both sides, we obtain

$$f(c)f(d) = xf(cd). \tag{3}$$

Setting $d = e$ into (3) gives us

$$f(c)x = xf(c).$$

Thus $x \in Z(G)$. For the last part, we use (3) to obtain

$$\begin{aligned}(\ell_x \circ f)(cd) &= xf(cd) \\ &= x^2 f(c)f(d) \\ &= (xf(c))(xf(d)) \\ &= (\ell_x \circ f)(c)(\ell_x \circ f)(d)\end{aligned}$$

for all $c, d \in G$. □

2.2 R-Algebras

Let R be a ring. An R-algebra A is an R-module equipped with an R-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a + b)c = ab + ac \quad \text{and} \quad a(b + c) = ab + ac.$$

We say the R-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the R-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

In this case, we call e the **identity** element. We say the R-algebra is **cancellative** if for any element $a \in A$ and any non-zero element $b \in A$ there exists precisely one element $c \in A$ with $a = bc$ and precisely one element $d \in A$ such that $a = db$.

2.2.1 Hom-Associative Algebras

Definition 2.1. A **hom-associative R-algebra**, denoted (A, f), is an R-algebra A equipped with an R-algebra homomorphism $f : A \to A$ satisfying the hom-associative law

$$f(a)(bc) = (ab)f(c) \tag{4}$$

for any $a, b, c \in A$. Moreover, we say (A, f) is **unital** if there exists an $e \in A$ such that $\alpha(e) = e$ and

$$ae = \alpha(a) = ea$$

for all $a \in A$.

Remark. If the R-algebra A is unital with unit e, then (A, α) is also unital with unit e. Indeed, we have $\alpha(e) = e$ since α is an R-algebra homomorphism. Moreover, we have

$$\begin{aligned}\alpha(a) &= \alpha(a)(ee)\\ &= (ae)\alpha(e)\\ &= a.\end{aligned}$$

Remark. Hom-associative algebras generalize associative algebras in the sense that any associative algebra A is a hom-associative algebra with f being the identity map.

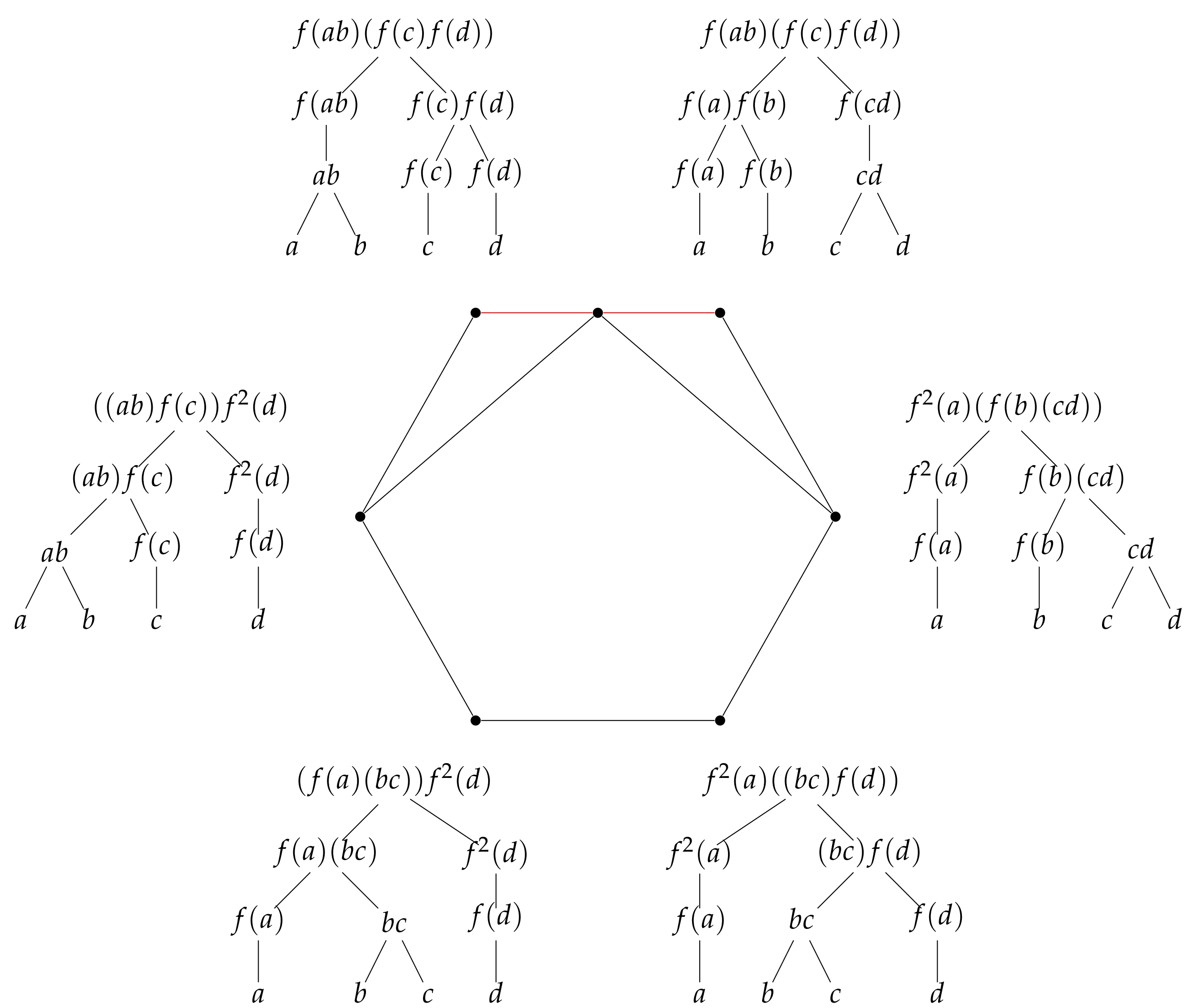
Theorem 2.1. *Every hom-associative R-algebra is a permutative R-algebra.*

Proof. Let A be a hom-associative algebra. Then for all $a, b, c, d \in M$, we have

$$\begin{aligned}f(ab)(f(c)f(d)) &= ((ab)f(c))f^2(d)\\ &= (f(a)(bc))f^2(d)\\ &= f^2(a)((bc)f(d))\\ &= f^2(a)(f(b)(cd))\\ &= (f(a)f(b))f(cd).\end{aligned}$$

□

Remark. We can visualize this proof by tracing the edges of the permutohedron:



2.2.2 Permutative R-Algebra

Definition 2.2. A **permutative R-algebra** is an R-algebra A equipped with an R-linear map $f\colon A \to A$ such that the the triple $(A, f, \cdot)$ satisfes the permutative law.

Theorem 2.2. *Let $(A, f, \cdot)$ be a permutative R-algebra with identity $e \in A$ and denote $x := f(e)$. Furthermore, suppose that A is cancellative. Then for all $a, b \in A$, we have*

1. $x^2 f(a) = x(xf(a))$
2. $xf(a) = f(a)x$
3. $(f(a)x)f(b) = f(a)((xf(b))$
4. $f(ab) = x(f(a)f(b))$
5. $x^2 = e$

Proof. Since the triple $(A, f, \cdot)$ satisfies the permutative law, we have

$$f(ab)(f(c)f(d)) = (f(a)f(b))f(cd). \tag{5}$$

for all $a, b, c, d \in A$. Setting $a = b = e$ into (5) gives us

$$x(f(c)f(d)) = x^2 f(cd).$$

Setting $a = c = e$ into (5) gives us

$$f(b)(xf(d)) = (xf(b))f(d)$$

Setting $a = d = e$ into (5) gives us

$$f(b)(f(c)x) = (xf(b))f(c)$$

Setting $b = c = e$ into (5) gives us

$$f(a)(xf(d)) = (f(a)x)f(d)$$

Setting $b = d = e$ into (5) gives us

$$f(a)(f(c)x) = (f(a)x)f(c)$$

Setting $a = b = c = e$ into (5) gives us

$$x(xf(d)) = x^2 f(d)$$

Setting $a = b = d = e$ into (5) gives us

$$x(f(c)x) = x^2 f(c)$$

Setting $a = c = d = e$ into (5) gives us

$$(xf(b))x = f(b)x^2$$

Setting $b = c = d = e$ into (5) gives us

$$(f(a)x)x = f(a)x^2.$$

.

In particular, setting $a = b = c = e$ into (5) gives us $x(xf(d)) = x^2 f(d)$. Therefore

$$x(xa) = x^2 a$$

for all $a \in A$ since f is a bijection.
(1) : This is obtained by setting $a = b = c = e$ in Equation (1).
(2) : Set $a = b = d = e$ in Equation (1) to get

$$x(f(c)x) = (f(e)f(e))f(c) = f(e)(f(e)f(c))$$

Then cancel x on both sides to obtain $f(e)f(c) = f(c)f(e)$. This implies $f(e)f(a) = f(a)f(e)$ for all $a \in A$.
(3) : Set $a = c = e$ in Equation (1) to get

$$f(b) \star (f(e) \star f(d)) = (f(e) \star f(b)) \star f(d) = (f(b) \star f(e)) \star f(d)$$

This implies $(f(a) \star f(e)) \star f(b) = f(a) \star ((f(e) \star f(b))$ for all $a, b \in A$.
(4) : Set $a = b = e$ in Equation (1) to get

$$f(e) \star (f(c) \star f(d)) = (f(e) \star f(e)) \star f(c \star d) = f(e) \star (f(e) \star f(c \star d))$$

Then cancel $f(e)$ on both sides to obtain $f(c) \star f(d) = f(e) \star f(c \star d)$. This implies $f(a \star b) = f(e) \star (f(a) \star f(b))$ for all $a, b \in A$.
(5) : Write

$$f(e) = f(e \star e) = f(e) \star (f(e) \star f(e))$$

Then cancel $f(e)$ on both sides to obtain $f(e)^2 = e$. $\square$

Theorem 2.3. *Let (A, μ, f) be a permutative R-algebra such that the R-algebra (A, μ) is unital with identity $e \in A$. Then for all $a, b \in A$, we have*

1. $(f(e)f(e))f(a) = f(e)(f(e)f(a))$ *and* $f(a)(f(e)f(e)) = (f(a)f(e))f(e)$ *(set $a = b = c = e$ and $b = c = d = e$)*
2. $(f(e)f(a))f(e) = f(a)(f(e)f(e))$ *and* $f(e)(f(a)f(e)) = (f(e)f(e))f(a)$ *(set $a = c = d = e$ and $a = b = d = e$)*
3. $f(e)(f(a)f(e)) = f(e)(f(e)f(a))$ *(use $f(e)(f(a)f(e)) = (f(e)f(e))f(a) = f(e)(f(e)f(a))$)*
4. $(f(e)f(a))f(e) = (f(a)f(e))f(e)$ *(use $(f(e)f(a))f(e) = f(a)(f(e)f(e)) = (f(a)f(e))f(e)$)*
5. *Don't need* 1 *through* 4*.*
6. $f(e)^2 f(ab) = f(e)(f(a)f(b))$ *(set $a = b = e$)*
7. $(f(a)f(b))f(e) = f(ab)f(e)^2$ *(set $c = d = e$)*
8. $(f(e)f(a))f(b) = f(a)(f(e)f(b))$ *(set $a = c = e$)*
9. $(f(e)f(a))f(b) = f(a)(f(b)f(e))$ *(set $a = d = e$)*
10. $(f(a)f(e))f(b) = f(a)(f(b)f(e))$ *(set $b = d = e$)*
11. $(f(a)f(e))f(b) = f(a)(f(e)f(b))$ *(set $b = c = e$) Important*
12. $(f(a)f(e))f(b) = (f(e)f(a))f(b)$ *and* $f(a)(f(e)f(b)) = f(a)(f(b)f(e))$
13. 11 *and* 12 *imply* 8 *through* 10 *and* 1 *through* 4*.*
14. $(f(e)f(b))(f(cd)) = f(b)(f(c)f(d))$ *set $a = e$.*
15. $f(a)(f(e)f(bc)) = f(a)(f(b)f(c))$*. If identity belongs to the image of f, then $f(e)f(bc) = f(b)f(c)$! Setting $c = e$ implies $f(e)f(b) = f(b)f(e)$. In a homassociative algebra, this would imply $f^2(ab) = f(a)f(b)$. In particular set $b = e$ we get $f^2(c) = f(e)f(c) = f(e)^2 c$ which is already known.*
16. $(f(a)f(b))f(c) = (f(ab)f(e))f(c)$
17. *With homassociativity, we get $f(a)f^2(bc) = f(a)(f(b)f(c))$ or $f(a)f^2(b) = f(a)f(b)^2$.*

Examples of Permutative R-Algebras

Example 2.3. Let $(A, \cdot)$ be an associative R-algebra and $f : A \to A$ be the idenity map. Then $(A, \cdot, f)$ is a permutative R-algebra.

We now show how to construct permutative R-algebras using nonassociative R-algebras.

Definition 2.3. A **homassociative algebra** is a triple (A, μ, f) consisting of an R-algebra (A, μ) equipped with an R-linear map $f : A \to A$ satisfying the homassociativity condition

$$f(a) \star (b \star c) = (a \star b) \star f(c) \tag{6}$$

for any $a, b, c \in A$.

Remark. Homassociative algebras generalize associative algebras in the sense that any associative algebra A is a homassociative algebra with f being the identity map.

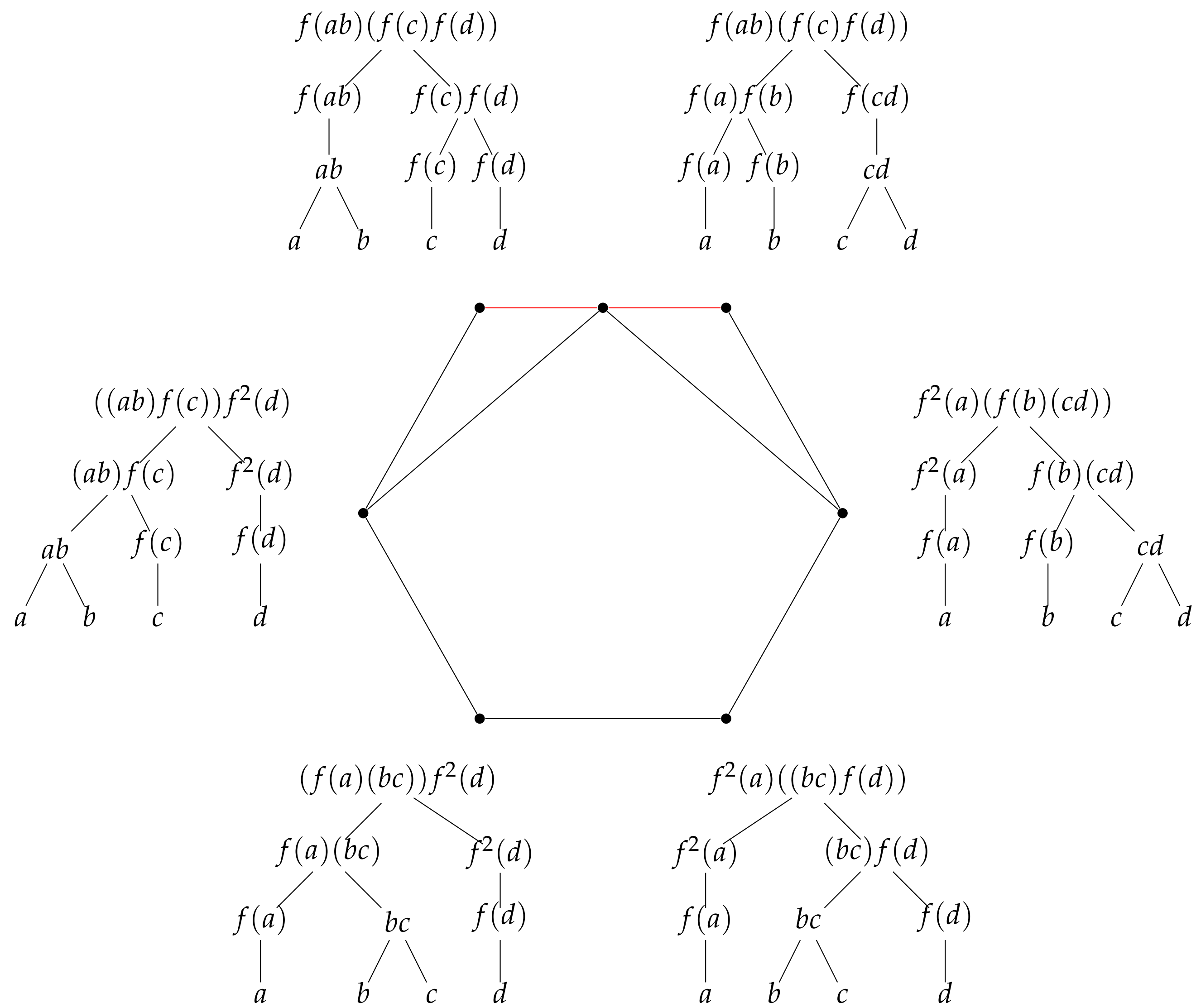
Theorem 2.4. *Every homassociative algebra is a permutative algebra.*

Proof. Let (A, μ, f) be a hom-associative algebra. Then for all $a, b, c, d \in M$, we have

$$\begin{aligned} f(a \star b) \star (f(c) \star f(d)) &= ((a \star b) \star f(c)) \star f^2(d) \\ &= (f(a) \star (b \star c)) \star f^2(d) \\ &= f^2(a) \star ((b \star c) \star f(d)) \\ &= f^2(a) \star (f(b) \star (c \star d)) \\ &= (f(a) \star f(b)) \star f(c \star d). \end{aligned}$$

□

Remark. The way to see this is by starting at the top left vertex of the hexagon in the image below and tracing the edges of the hexagon via the homassociative law to get to the top right edge.



Theorem 2.5. *Let (A, μ, f) be a homassociative R-algebra such that the R-algebra (A, μ) is unital with identity $e \in A$ and cancellative. Then we have*

$$f^2(a) = a.$$

for all $a \in A$.

Proof. Since (A, μ, f) is a homassociative R-algebra, it follows from the homassociativity condition we have

$$a \star f(b) = f(a) \star b = f(a \star b),$$

for all $a, b \in A$. In particular, when $a = e$, we have $f(b) = f(e) \star b$. Since (A, μ, f) is also a permutative R-algebra, it follows from the permutativity law together with the homassociativity condition that we have

$$f(a \star b) = f(e) \star (f(a) \star f(b)) = f(a) \star f^2(b)$$

In particular, when $a = e$, we have $f^2(b) = f(e) \star f(b)$ for all $b \in A$. Since $f(e)^2 = e$ in any permutative R-algebra with identity e, it then follows that we have

$$\begin{aligned} f^2(b) &= f(e) \star f(b) \\ &= f(e) \star (f(e) \star b) \\ &= (f(e) \star f(e)) \star b \\ &= b \end{aligned}$$

$\square$

Idea

If A is not cancellative, then we have
$$f(f(a)f(b)) = f(f(e)f(ab))$$
This follows since
Now
$$f(a(bc)) = f(1)(f(a)f(bc)) = f(1)(f(a)bf(c)) = f(1)(f(ab)f(c)) = f((ab)c)$$
Therefore if f is injective, then $\cdot$ is associative. One way we can create a permutative algebra from a hom-associative algebra is by twisting it by a 3-cocyle α:
$$(ab)f(c) = \alpha_{a,b,c}f(a)(bc)$$
Then it follows
$$\begin{aligned}
f(ab)(f(c)f(d)) &= \alpha^{-1}_{ab,f(c),f(d)}((ab)f(c))f^2(d) \\
&= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}(f(a)(bc))f^2(d) \\
&= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}f^2(a)((bc)f(d)) \\
&= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}f^2(a)(f(b)(cd)) \\
&= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}\alpha^{-1}_{f(a),f(b),cd}(f(a)f(b))f(cd).
\end{aligned}$$
Now set $\beta_{a,b,c,d} = \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}\alpha^{-1}_{f(a),f(b),cd}$. Then
$$f(ab)(f(c)f(d)) = (f(ab)f(c))f(d) =$$
$$(d\beta)_{a,b,c,d,e} = \beta_{b,c,d,e}\beta^{-1}_{ab,c,d,e}\beta_{a,bc,d,e}\beta^{-1}_{a,b,cd,e}\beta_{a,b,c,de}\beta^{-1}_{a,b,c,d}$$
This is
$$\beta_{b,c,d,e} = \alpha^{-1}_{bc,f(d),f(e)}\alpha_{b,c,d}\alpha_{f(b),cd,f(e)}\alpha_{c,d,e}\alpha^{-1}_{f(b),f(c),de}$$
$$\beta^{-1}_{ab,c,d,e} = \alpha_{abc,f(d),f(e)}\alpha^{-1}_{ab,c,d}\alpha^{-1}_{f(ab),cd,f(e)}\alpha^{-1}_{c,d,e}\alpha_{f(ab),f(c),de}$$
$$\beta_{a,bc,d,e} = \alpha^{-1}_{abc,f(d),f(e)}\alpha_{a,bc,d}\alpha_{f(a),bcd,f(e)}\alpha_{bc,d,e}\alpha^{-1}_{f(a),f(bc),de}$$
$$\beta^{-1}_{a,b,cd,e} = \alpha_{ab,f(cd),f(e)}\alpha^{-1}_{a,b,cd}\alpha^{-1}_{f(a),bcd,f(e)}\alpha^{-1}_{b,cd,e}\alpha_{f(a),f(b),cde}$$
$$\beta_{a,b,c,de} = \alpha^{-1}_{ab,f(c),f(de)}\alpha_{a,b,c}\alpha_{f(a),bc,f(de)}\alpha_{b,c,de}\alpha^{-1}_{f(a),f(b),cde}$$
$$\beta^{-1}_{a,b,c,d} = \alpha_{ab,f(c),f(d)}\alpha^{-1}_{a,b,c}\alpha^{-1}_{f(a),bc,f(d)}\alpha^{-1}_{b,c,d}\alpha_{f(a),f(b),cd}$$
So
$$(d\beta)_{a,b,c,d,e} = \alpha^{-1}_{a,b,c}\alpha_{c,d,e}\alpha_{abc,f(d),f(e)}\alpha_{f(a),bcd,f(e)}\alpha_{f(a),f(b),cde}$$

Are The Octonions Permutative?

The **octonions** $\mathbb{O}$ are a normed division algebra over the real numbers. They are not commutative nor associative, but satisfy a weaker form of associativity, namely they are alternative. This means for all $x, y \in \mathbb{O}$, we have
$$x(xy) = (xx)y \quad \text{and} \quad (yx)x = y(xx)$$
Thus if they were to be f-associative for some $f : \mathbb{O} \to \mathbb{O}$, we must have
$$f(x)(xf^{-1}(y)) = x(xy)$$
So if $f(x) = x$ for any $x \in \mathbb{O}$, then $f(x) = x$ for all $x \in \mathbb{O}$.

New Section

Formula $a \star b = f^{-1}(f(e) \star f(a) \star f(b))$.

$$a \star b = -(-a \star -b)$$

Quaternions

$\star$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
$k = ij$	k	j	$-i$	-1

$(\mathbb{Z}_4, \star)$

$\star$	0	1	2	3
0	0	1	2	3
1	1	a	b	c
2	2	d	e	$-d$
3	3	$-c$	$-b$	$-a$

$1 \star 2 = -(3 \star 2)$
$2 \star 2 = -(2 \star 2)$
$1 \star 3 = -(3 \star 1)$
$2 \star 1 = -(2 \star 3)$
$1 \star 1 = -(3 \star 3)$
Now assume $(a \star b) \star c = -a \star (b \star -c)$
$1 \star 3 = (1 \star 0) \star 3 = 3 \star (0 \star 1) = 3 \star 1$
$2 \star 3 = (2 \star 0) \star 3 = 2 \star (0 \star 1) = 2 \star 1$
$1 \star 2 = (1 \star 0) \star 2 = 3 \star (0 \star 2) = 3 \star 2$
$1 \star 1 = (1 \star 0) \star 1 = 3 \star (0 \star 3) = 3 \star 3$
$a \star b = (a \star 0) \star b = f(a) \star (0 \star f^{-1}(b)) = f(a) \star f^{-1}(b)$. So if $f(a) = b$, then $a \star b = b \star a$.
$(a \star b) \star c = f(a) \star (b \star f^{-1}(c))$. So if $f(a) = c$, then $(a \star b) \star c = c \star (b \star a)$.
$a \star b = f^{-1}(f(a) \star f(b))$.
$a \star 1 = (1 \star 1) \star 1 = 3 \star (1 \star 3) = 3 \star c$
$b \star 3 = (1 \star 2) \star 3 = 3 \star (2 \star 1) = 3 \star d$

$\star$	0	1	2	3
0	0	1	2	3
1	1	a	b	c
2	2	b	e	b
3	3	c	b	a

Next

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	o	p	q	r	s	t	u
4	4	v	w	x	y	$-x$	$-w$	$-v$
5	5	$-u$	$-t$	$-s$	$-r$	$-q$	$-p$	$-o$
6	6	$-n$	$-m$	$-l$	$-k$	$-j$	$-i$	$-h$
7	7	$-g$	$-f$	$-e$	$-d$	$-c$	$-b$	$-a$

Now assume $(a \star b) \star c = -a \star (b \star -c)$
$1 \star 7 = 7 \star 1$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	o	p	q	r	s	t	u
4	4	v	w	x	y	x	w	v
5	5	u	t	s	r	q	p	o
6	6	n	m	l	k	j	i	h
7	7	g	f	e	d	c	b	a

Or $a \star b = 3 \star (3a \star 3b)$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	$3c$	$3f$	$3a$	$3d$	$3g$	$3b$	$3e$
4	4	v	w	$3v$	$\{0,4\}$	x	$3w$	$3x$
5	5	o	p	q	r	s	t	u
6	6	$3l$	$3j$	$3h$	$3m$	$3k$	$3i$	$3n$
7	7	$3q$	$3t$	$3o$	$3r$	$3u$	$3p$	$3s$

Now assume $a \star b = (a \star 0) \star b = 3a \star 3b$

$(t_4)_{g,h}$	e	(12)	(23)	(34)	(123)	$(12)(34)$	$(13)(24)$	$(14)(23)$	...
e	1	1	1	1	1	1	1	1	
(12)	1	1	1	1	1				
(23)	1	1	1	1	1				
(34)	1	-1	1	1	1				
$(12)(34)$	1	-1	-1	1	1	-1	1	1	
$(13)(24)$	1					-1	-1	1	
$(14)(23)$	1					1	-1	-1	
...									

References

https://arxiv.org/pdf/1504.03019.pdf

Twisting Associativity

Definition 2.4. A **magma** $(M, \cdot)$ is a set M equipped with a binary operation.

Remark. This sequence of equalities can be seen as tracing the edges of P_2.

Definition 2.5. A **quasigroup** $(Q, \cdot)$ is a magma such that for every $a, b \in Q$, there exist unique elements $x, y \in Q$ such that

$$ax = b \quad \text{and} \quad ya = b.$$

The unique solutions to these equations are written $x = a \backslash b$ and $y = b/a$. The operations $\backslash$ and $/$ are called **left** and **right division** respectively.

Remark. This property ensures that the each element of Q occurs exactly once in each row and exactly once in each column of the quasigroup's multiplication table. The uniqueness requirement can be replaced by the requirement that the magma be cancellative: (Suppose $ax = ay$. Then $x = a \lhd ay = y$).

Definition 2.6. A **pique** $(Q, \cdot)$ is a quasigroup with an idempotent element, that is, an element $e \in Q$ such that $e^2 = e$.

Definition 2.7. A **loop** $(Q, \cdot)$ is a quasigroup with an identity element $e \in Q$, i.e. for all $a \in Q$, we have

$$ea = a = ae$$

Remark. It follows that the identity element e is unique and that every element of Q has a unique left and right inverse.

Theorem 2.6. *Let $(M, f, \cdot)$ be a magma equipped with a unary operation $f : M \to M$ such that for all $a, b, c \in M$,*

$$(ab)f(c) = f(a)(bc)$$

Then the triple $(M, f, \cdot)$ is permutative.

Remark. If f is the identity map, then the binary operation is associative.

Proof. For all $a, b, c, d \in M$, we have

$$\begin{aligned} f(ab)(f(c)f(d)) &= ((ab)f(c))f^2(d) \\ &= (f(a)(bc))f^2(d) \\ &= f^2(a)((bc)f(d)) \\ &= f^2(a)(f(b)(cd)) \\ &= (f(a)f(b))f(cd) \end{aligned}$$

□

In a loop $(Q, \cdot)$, there is a unique element $\alpha_{a,b,c} \in Q$ such that

$$\alpha_{a,b,c}(ab)c = a(bc)$$

This leads us to to the following defintion. The **associator** is a map $\alpha : Q \times Q \times Q \to Q$ given by $(a, b, c) \mapsto \alpha_{a,b,c}$. The associator is a measure of nonassociativity of Q. The associator satisfies the 3-cocycle equation

$$\alpha_{b,c,d}(\alpha_{ab,c,d})^{-1}\alpha_{a,bc,d}(\alpha_{a,b,cd})^{-1}\alpha_{b,c,d} = 1$$

Also in a loop $(Q, \cdot)$, there is a unique element $\alpha^f_{a,b,c} \in Q$ such that

$$\alpha^f_{a,b,c}(ab)c = f(a)(bf^{-1}(c))$$

We define the f-**associator** to be the map $\alpha^f : Q \times Q \times Q \to Q$ given by $(a, b, c) \mapsto \alpha^f_{a,b,c}$. The f-associator also satisfies the 3-cocycle equation

$$\alpha^f_{b,c,d}(\alpha^f_{ab,c,d})^{-1}\alpha^f_{a,bc,d}(\alpha^f_{a,b,cd})^{-1}\alpha^f_{b,c,d} = 1$$

Test

On the one hand

$$\begin{aligned} e_g(e_h e_k) &= e_g(\alpha_{h,k} e_{hk}) \\ &= \beta_g^{-1}\beta_{hk}(e_g \alpha_{h,k})e_{hk} \\ &= \beta_g^{-1}\beta_{hk}(\alpha_{h,k} e_g)e_{hk} \\ &= \beta_g^{-1}\alpha_{h,k}(\alpha_{g,hk} e_{ghk}) \end{aligned}$$

On the other hand

$$\begin{aligned} \beta_g^{-1}\beta_k(e_g e_h)e_k &= \beta_g^{-1}\beta_k(\alpha_{g,h} e_{gh})e_k \\ &= \beta_g^{-1}\alpha_{g,h}(e_{gh} e_k) \\ &= \beta_g^{-1}\alpha_{g,h}(\alpha_{gh,k} e_{ghk}) \end{aligned}$$

Equating the two, we obtain

$$\beta_g^{-1}\alpha_{g,h}(\alpha_{gh,k}e_{ghk}) = \beta_g^{-1}\alpha_{h,k}(\alpha_{g,hk}e_{ghk})$$

Or

$$\alpha_{g,h}(\alpha_{gh,k}e_{ghk}) = \alpha_{h,k}(\alpha_{g,hk}e_{ghk})$$

Or

$$\alpha_{g,h}\alpha_{gh,k} = \alpha_{h,k}\alpha_{g,hk}$$

Theorem 2.7. *Let $(Q,\cdot)$ be a loop with unit $e \in Q$ equipped with a unary operation $f : Q \to Q$ such that $(Q, f, \cdot)$ satisfies the permutative law. Then for all $a, b \in Q$ and for idempotents $e, e' \in Q$,*

1. $f(e)f(a) = f(a)f(e)$
2. $(f(a)f(e))f(b) = f(a)((f(e)f(b))$
3. $f(ab) = f(e)(f(a)f(b))$
4. $f(e)^2 = e$

Proof.
(1) : Set $a = b = c = e$ in Equation (1) to get

$$(f(e)f(e))f(d) = f(e)(f(e)f(d)). \tag{7}$$

Then set $a = b = c = e$ in Equation (1) to get

$$(f(e)f(e))f(c) = f(e)(f(c)f(e)). \tag{8}$$

Equations (7) and (8) imply $f(e)(f(e)f(a)) = f(e)(f(a)f(e))$ for all $a \in Q$. Since $(Q,\cdot)$ is cancellative, this implies

$$f(e)f(a) = f(a)f(e) \tag{9}$$

for all $a \in Q$.
(2) : Set $a = c = e$ in Equation (1) to get

$$(f(e)f(b))f(d) = f(b)(f(e)f(d)).$$

This together with Equation (9) implies $(f(a)f(e))f(b) = f(a)((f(e)f(b))$ for all $a, b \in Q$.
(3) : Choose any idempotent $e \in Q$, then set $a = b = e$ in Equation (1) to get

$$(f(e)f(e))f(cd) = f(e)(f(c)f(d)).$$

This implies $f(e)f(ab) = f(a)f(b)$ for all $a, b \in Q$, and any idempotent $e \in Q$, since $(f(e)f(e))f(cd) = f(e)(f(e)f(cd))$ and $(Q,\cdot)$ is cancellative.
(4) : Given two idempotents e and e' in Q, we have

$$\begin{aligned} f(e')f(ab) &= f(a)f(b) \\ &= f(e)f(ab) \end{aligned}$$

This implies $f(e) = f(e')$, since $(Q,\cdot)$ is cancellative.
(5) : $f(e) = f(ee) = f(e)^3$.

□

Corollary. *Let $(Q,\cdot)$ be a loop with identity e. Then $f(e)^2 = e$.*

Proof. We have

$$\begin{aligned} f(e)^3 &= f(e) \\ &= f(e)e. \end{aligned}$$

Since $(Q,\cdot)$ is cancellative, $f(e)^2 = e$. □

3 Hom-Associative Algebras

Let R be a ring. An R-algebra A is an R-module equipped with an R-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a+b)c = ab + ac \quad \text{and} \quad a(b+c) = ab + ac.$$

We say the R-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the R-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

In this case, we call e the **identity** element. We say the R-algebra is **cancellative** if for any element $a \in A$ and any non-zero element $b \in A$ there exists precisely one element $c \in A$ with $a = bc$ and precisely one element $d \in A$ such that $a = db$.

3.1 Hom-associative Algebras

Definition 3.1. A **hom-associative R-algebra** is an R-algebra A equipped with an R-linear map $f\colon A \to A$ satisfying the homassociativity condition

$$f(a)(bc) = (ab)f(c) \tag{10}$$

for any $a, b, c \in A$.

Remark. Hom-associative algebras generalize associative algebras in the sense that any associative algebra A is a homassociative algebra with f being the identity map.

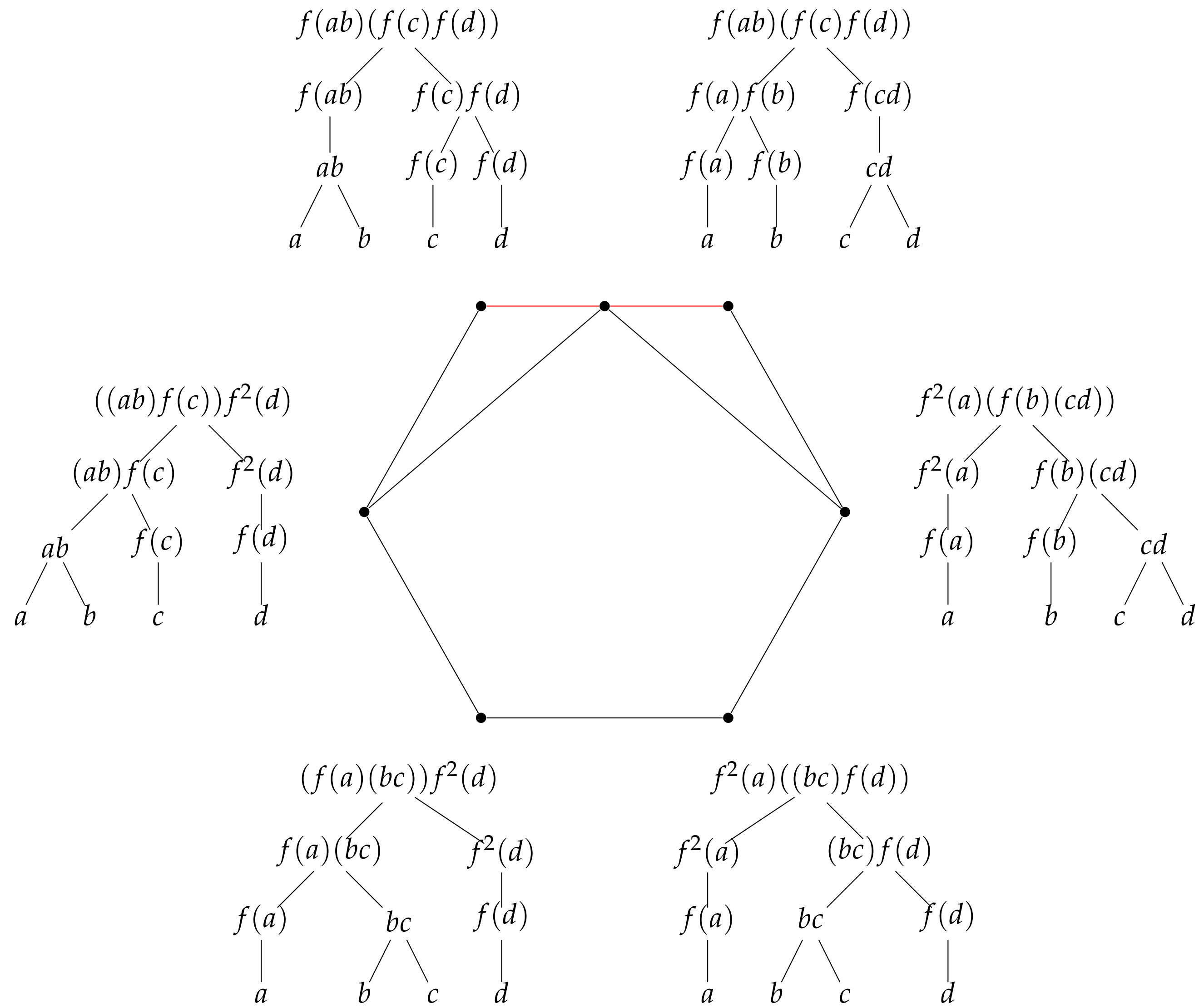
Theorem 3.1. *Every hom-associative R-algebra is a permutative R-algebra.*

Proof. Let A be a hom-associative algebra. Then for all $a, b, c, d \in M$, we have

$$\begin{aligned} f(ab)(f(c)f(d)) &= ((ab)f(c))f^2(d) \\ &= (f(a)(bc))f^2(d) \\ &= f^2(a)((bc)f(d)) \\ &= f^2(a)(f(b)(cd)) \\ &= (f(a)f(b))f(cd). \end{aligned}$$

□

Remark. We can visualize this proof by tracing the edges of the permutohedron:



Definition 3.2. A **permutative R-algebra** is a triple (A, μ, f) consisting of an R-algebra (A, μ) equipped with an R-linear map $f : A \to A$ such that the permutative law is satisfied. (1).

Theorem 3.2. *Let (A, μ, f) be a permutative R-algebra such that the R-algebra (A, μ) is unital with identity $e \in A$ and cancellative. Then for all $a, b \in A$, we have*

1. $f(e)^2 f(a) = f(e)(f(e)f(a))$
2. $f(e)f(a) = f(a)f(e)$
3. $(f(a)f(e))f(b) = f(a)((f(e)f(b))$
4. $f(ab) = f(e)(f(a)f(b))$
5. $f(e)^2 = e$

Proof.
(1) : This is obtained by setting $a = b = c = e$ in Equation (1).
(2) : Set $a = b = d = e$ in Equation (1) to get

$$f(e)(f(c)f(e)) = (f(e)f(e))f(c) = f(e)(f(e)f(c))$$

Then cancel $f(e)$ on both sides to obtain $f(e)f(c) = f(c)f(e)$. This implies $f(e)f(a) = f(a)f(e)$ for all $a \in A$.
(3) : Set $a = c = e$ in Equation (1) to get

$$f(b) \star (f(e) \star f(d)) = (f(e) \star f(b)) \star f(d) = (f(b) \star f(e)) \star f(d)$$

This implies $(f(a) \star f(e)) \star f(b) = f(a) \star ((f(e) \star f(b))$ for all $a, b \in A$.
(4) : Set $a = b = e$ in Equation (1) to get

$$f(e) \star (f(c) \star f(d)) = (f(e) \star f(e)) \star f(c \star d) = f(e) \star (f(e) \star f(c \star d))$$

Then cancel $f(e)$ on both sides to obtain $f(c) \star f(d) = f(e) \star f(c \star d)$. This implies $f(a \star b) = f(e) \star (f(a) \star f(b))$ for all $a, b \in A$.
(5) : Write

$$f(e) = f(e \star e) = f(e) \star (f(e) \star f(e))$$

Then cancel $f(e)$ on both sides to obtain $f(e)^2 = e$. □

Theorem 3.3. *Let (A, μ, f) be a permutative R-algebra such that the R-algebra (A, μ) is unital with identity $e \in A$. Then for all $a, b \in A$, we have*

1. $(f(e)f(e))f(a) = f(e)(f(e)f(a))$ *and* $f(a)(f(e)f(e)) = (f(a)f(e))f(e)$ *(set $a = b = c = e$ and $b = c = d = e$)*
2. $(f(e)f(a))f(e) = f(a)(f(e)f(e))$ *and* $f(e)(f(a)f(e)) = (f(e)f(e))f(a)$ *(set $a = c = d = e$ and $a = b = d = e$)*
3. $f(e)(f(a)f(e)) = f(e)(f(e)f(a))$ *(use $f(e)(f(a)f(e)) = (f(e)f(e))f(a) = f(e)(f(e)f(a))$)*
4. $(f(e)f(a))f(e) = (f(a)f(e))f(e)$ *(use $(f(e)f(a))f(e) = f(a)(f(e)f(e)) = (f(a)f(e))f(e)$)*
5. *Don't need* 1 *through* 4.
6. $f(e)^2 f(ab) = f(e)(f(a)f(b))$ *(set $a = b = e$)*
7. $(f(a)f(b))f(e) = f(ab)f(e)^2$ *(set $c = d = e$)*
8. $(f(e)f(a))f(b) = f(a)(f(e)f(b))$ *(set $a = c = e$)*
9. $(f(e)f(a))f(b) = f(a)(f(b)f(e))$ *(set $a = d = e$)*
10. $(f(a)f(e))f(b) = f(a)(f(b)f(e))$ *(set $b = d = e$)*
11. $(f(a)f(e))f(b) = f(a)(f(e)f(b))$ *(set $b = c = e$) Important*
12. $(f(a)f(e))f(b) = (f(e)f(a))f(b)$ *and* $f(a)(f(e)f(b)) = f(a)(f(b)f(e))$
13. 11 *and* 12 *imply* 8 *through* 10 *and* 1 *through* 4.
14. $(f(e)f(b))(f(cd)) = f(b)(f(c)f(d))$ *set $a = e$.*
15. $f(a)(f(e)f(bc)) = f(a)(f(b)f(c))$. *If identity belongs to the image of f, then $f(e)f(bc) = f(b)f(c)$! Setting $c = e$ implies $f(e)f(b) = f(b)f(e)$. In a homassociative algebra, this would imply $f^2(ab) = f(a)f(b)$. In particular set $b = e$ we get $f^2(c) = f(e)f(c) = f(e)^2c$ which is already known.*
16. $(f(a)f(b))f(c) = (f(ab)f(e))f(c)$
17. *With homassociativity, we get $f(a)f^2(bc) = f(a)(f(b)f(c))$ or $f(a)f^2(b) = f(a)f(b)^2$.*

Examples of Permutative R-Algebras

Example 3.1. Let $(A, \cdot)$ be an associative R-algebra and $f : A \to A$ be the idenity map. Then $(A, \cdot, f)$ is a permutative R-algebra.

We now show how to construct permutative R-algebras using nonassociative R-algebras.

Definition 3.3. A **homassociative algebra** is a triple (A, μ, f) consisting of an R-algebra (A, μ) equipped with an R-linear map $f : A \to A$ satisfying the homassociativity condition

$$f(a) \star (b \star c) = (a \star b) \star f(c) \tag{11}$$

for any $a, b, c \in A$.

Remark. Homassociative algebras generalize associative algebras in the sense that any associative algebra A is a homassociative algebra with f being the identity map.

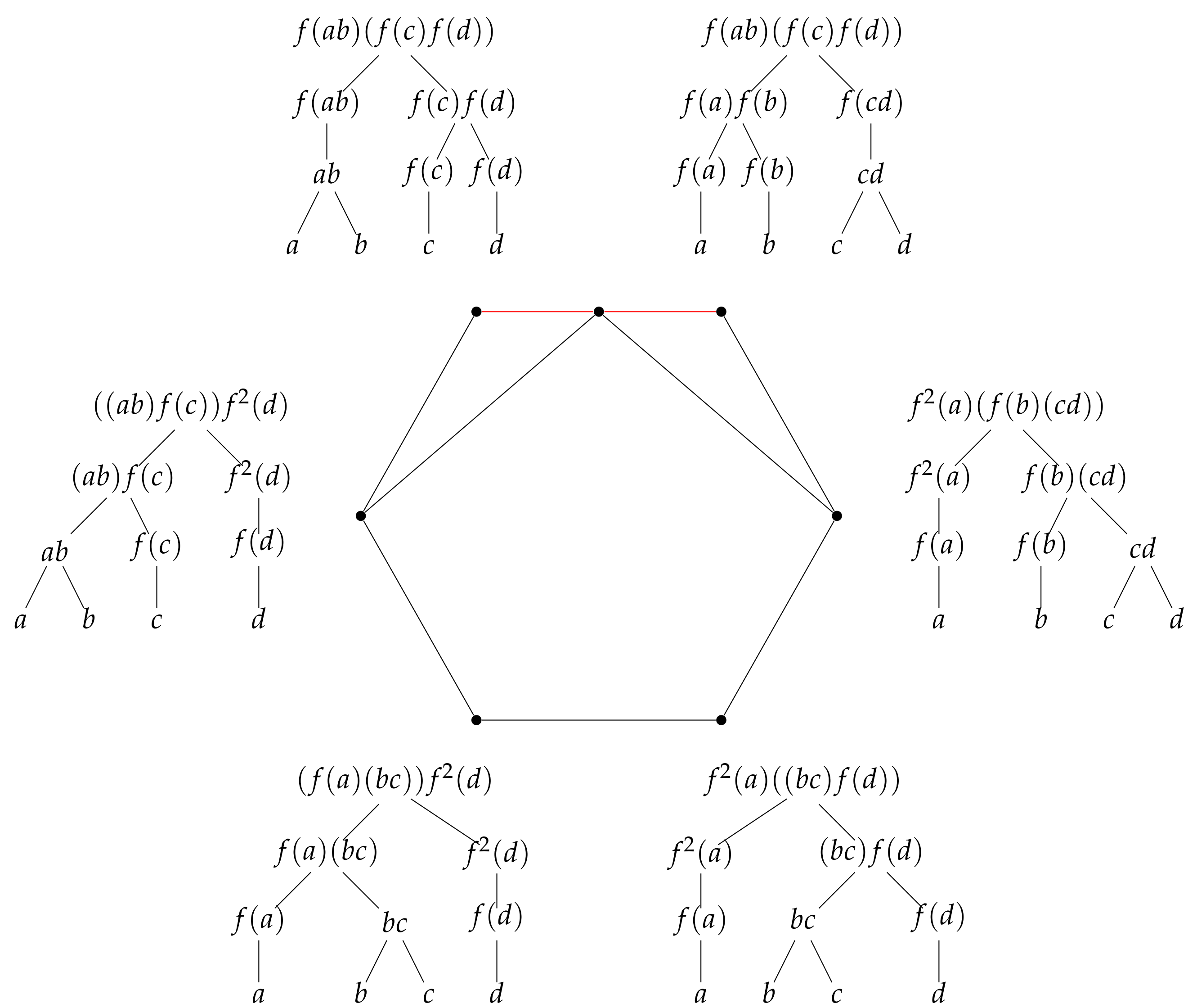
Theorem 3.4. *Every homassociative algebra is a permutative algebra.*

Proof. Let (A, μ, f) be a hom-associative algebra. Then for all $a, b, c, d \in M$, we have

$$\begin{aligned} f(a \star b) \star (f(c) \star f(d)) &= ((a \star b) \star f(c)) \star f^2(d) \\ &= (f(a) \star (b \star c)) \star f^2(d) \\ &= f^2(a) \star ((b \star c) \star f(d)) \\ &= f^2(a) \star (f(b) \star (c \star d)) \\ &= (f(a) \star f(b)) \star f(c \star d). \end{aligned}$$

□

Remark. The way to see this is by starting at the top left vertex of the hexagon in the image below and tracing the edges of the hexagon via the homassociative law to get to the top right edge.



Theorem 3.5. *Let (A, μ, f) be a homassociative R-algebra such that the R-algebra (A, μ) is unital with identity $e \in A$ and cancellative. Then we have*

$$f^2(a) = a.$$

for all $a \in A$.

Proof. Since (A, μ, f) is a homassociative R-algebra, it follows from the homassociativity condition we have

$$a \star f(b) = f(a) \star b = f(a \star b),$$

for all $a, b \in A$. In particular, when $a = e$, we have $f(b) = f(e) \star b$. Since (A, μ, f) is also a permutative R-algebra, it follows from the permutativity law together with the homassociativity condition that we have

$$f(a \star b) = f(e) \star (f(a) \star f(b)) = f(a) \star f^2(b)$$

In particular, when $a = e$, we have $f^2(b) = f(e) \star f(b)$ for all $b \in A$. Since $f(e)^2 = e$ in any permutative R-algebra with identity e, it then follows that we have

$$\begin{aligned} f^2(b) &= f(e) \star f(b) \\ &= f(e) \star (f(e) \star b) \\ &= (f(e) \star f(e)) \star b \\ &= b \end{aligned}$$

□

Idea

If A is not cancellative, then we have

$$f(f(a)f(b)) = f(f(e)f(ab))$$

This follows since
Now

$$f(a(bc)) = f(1)(f(a)f(bc)) = f(1)(f(a)bf(c)) = f(1)(f(ab)f(c)) = f((ab)c)$$

Therefore if f is injective, then $\cdot$ is associative. One way we can create a permutative algebra from a hom-associative algebra is by twisting it by a 3-cocyle α:

$$(ab)f(c) = \alpha_{a,b,c} f(a)(bc)$$

Then it follows

$$\begin{aligned} f(ab)(f(c)f(d)) &= \alpha^{-1}_{ab,f(c),f(d)}((ab)f(c))f^2(d) \\ &= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}(f(a)(bc))f^2(d) \\ &= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}f^2(a)((bc)f(d)) \\ &= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}f^2(a)(f(b)(cd)) \\ &= \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}\alpha^{-1}_{f(a),f(b),cd}(f(a)f(b))f(cd). \end{aligned}$$

Now set $\beta_{a,b,c,d} = \alpha^{-1}_{ab,f(c),f(d)}\alpha_{a,b,c}\alpha_{f(a),bc,f(d)}\alpha_{b,c,d}\alpha^{-1}_{f(a),f(b),cd}$. Then

$$f(ab)(f(c)f(d)) = (f(ab)f(c))f(d) =$$

$$(d\beta)_{a,b,c,d,e} = \beta_{b,c,d,e}\beta^{-1}_{ab,c,d,e}\beta_{a,bc,d,e}\beta^{-1}_{a,b,cd,e}\beta_{a,b,c,de}\beta^{-1}_{a,b,c,d}$$

This is

$$\beta_{b,c,d,e} = \alpha^{-1}_{bc,f(d),f(e)}\alpha_{b,c,d}\alpha_{f(b),cd,f(e)}\alpha_{c,d,e}\alpha^{-1}_{f(b),f(c),de}$$

$$\beta^{-1}_{ab,c,d,e} = \alpha_{abc,f(d),f(e)}\alpha^{-1}_{ab,c,d}\alpha^{-1}_{f(ab),cd,f(e)}\alpha^{-1}_{c,d,e}\alpha_{f(ab),f(c),de}$$

$$\beta_{a,bc,d,e} = \alpha^{-1}_{abc,f(d),f(e)}\alpha_{a,bc,d}\alpha_{f(a),bcd,f(e)}\alpha_{bc,d,e}\alpha^{-1}_{f(a),f(bc),de}$$

$$\beta^{-1}_{a,b,cd,e} = \alpha_{ab,f(cd),f(e)}\alpha^{-1}_{a,b,cd}\alpha^{-1}_{f(a),bcd,f(e)}\alpha^{-1}_{b,cd,e}\alpha_{f(a),f(b),cde}$$

$$\beta_{a,b,c,de} = \alpha^{-1}_{ab,f(c),f(de)}\alpha_{a,b,c}\alpha_{f(a),bc,f(de)}\alpha_{b,c,de}\alpha^{-1}_{f(a),f(b),cde}$$

$$\beta^{-1}_{a,b,c,d} = \alpha_{ab,f(c),f(d)}\alpha^{-1}_{a,b,c}\alpha^{-1}_{f(a),bc,f(d)}\alpha^{-1}_{b,c,d}\alpha_{f(a),f(b),cd}$$

So

$$(d\beta)_{a,b,c,d,e} = \alpha^{-1}_{a,b,c}\alpha_{c,d,e}\alpha_{abc,f(d),f(e)}\alpha_{f(a),bcd,f(e)}\alpha_{f(a),f(b),cde}$$

Are The Octonions Permutative?

The **octonions** $\mathbb{O}$ are a normed division algebra over the real numbers. They are not commutative nor associative, but satisfy a weaker form of associativity, namely they are alternative. This means for all $x, y \in \mathbb{O}$, we have

$$x(xy) = (xx)y \quad \text{and} \quad (yx)x = y(xx)$$

Thus if they were to be f-associative for some $f : \mathbb{O} \to \mathbb{O}$, we must have

$$f(x)(xf^{-1}(y)) = x(xy)$$

So if $f(x) = x$ for any $x \in \mathbb{O}$, then $f(x) = x$ for all $x \in \mathbb{O}$.

New Section

Formula $a \star b = f^{-1}(f(e) \star f(a) \star f(b))$.

$$a \star b = -(-a \star -b)$$

Quaternions

$\star$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
$k = ij$	k	j	$-i$	-1

$(\mathbb{Z}_4, \star)$

$\star$	0	1	2	3
0	0	1	2	3
1	1	a	b	c
2	2	d	e	$-d$
3	3	$-c$	$-b$	$-a$

$1 \star 2 = -(3 \star 2)$
$2 \star 2 = -(2 \star 2)$
$1 \star 3 = -(3 \star 1)$
$2 \star 1 = -(2 \star 3)$
$1 \star 1 = -(3 \star 3)$
Now assume $(a \star b) \star c = -a \star (b \star -c)$
$1 \star 3 = (1 \star 0) \star 3 = 3 \star (0 \star 1) = 3 \star 1$
$2 \star 3 = (2 \star 0) \star 3 = 2 \star (0 \star 1) = 2 \star 1$
$1 \star 2 = (1 \star 0) \star 2 = 3 \star (0 \star 2) = 3 \star 2$
$1 \star 1 = (1 \star 0) \star 1 = 3 \star (0 \star 3) = 3 \star 3$
$a \star b = (a \star 0) \star b = f(a) \star (0 \star f^{-1}(b)) = f(a) \star f^{-1}(b)$. So if $f(a) = b$, then $a \star b = b \star a$.
$(a \star b) \star c = f(a) \star (b \star f^{-1}(c))$. So if $f(a) = c$, then $(a \star b) \star c = c \star (b \star a)$.
$a \star b = f^{-1}(f(a) \star f(b))$.
$a \star 1 = (1 \star 1) \star 1 = 3 \star (1 \star 3) = 3 \star c$
$b \star 3 = (1 \star 2) \star 3 = 3 \star (2 \star 1) = 3 \star d$

$\star$	0	1	2	3
0	0	1	2	3
1	1	a	b	c
2	2	b	e	b
3	3	c	b	a

Next

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	o	p	q	r	s	t	u
4	4	v	w	x	y	$-x$	$-w$	$-v$
5	5	$-u$	$-t$	$-s$	$-r$	$-q$	$-p$	$-o$
6	6	$-n$	$-m$	$-l$	$-k$	$-j$	$-i$	$-h$
7	7	$-g$	$-f$	$-e$	$-d$	$-c$	$-b$	$-a$

Now assume $(a \star b) \star c = -a \star (b \star -c)$
$1 \star 7 = 7 \star 1$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	o	p	q	r	s	t	u
4	4	v	w	x	y	x	w	v
5	5	u	t	s	r	q	p	o
6	6	n	m	l	k	j	i	h
7	7	g	f	e	d	c	b	a

Or $a \star b = 3 \star (3a \star 3b)$

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	a	b	c	d	e	f	g
2	2	h	i	j	k	l	m	n
3	3	$3c$	$3f$	$3a$	$3d$	$3g$	$3b$	$3e$
4	4	v	w	$3v$	$\{0,4\}$	x	$3w$	$3x$
5	5	o	p	q	r	s	t	u
6	6	$3l$	$3j$	$3h$	$3m$	$3k$	$3i$	$3n$
7	7	$3q$	$3t$	$3o$	$3r$	$3u$	$3p$	$3s$

Now assume $a \star b = (a \star 0) \star b = 3a \star 3b$

$(t_4)_{g,h}$	e	(12)	(23)	(34)	(123)	(12)(34)	(13)(24)	(14)(23)	...
e	1	1	1	1	1	1	1	1	
(12)	1	1	1	1	1				
(23)	1	1	1	1	1				
(34)	1	-1	1	1	1				
(12)(34)	1	-1	-1	1	1	-1	1	1	
(13)(24)	1					-1	-1	1	
(14)(23)	1					1	-1	-1	
...									

References

https://arxiv.org/pdf/1504.03019.pdf