Commutative Algebra Homework 6

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I got too caught up in the elections and unfortunately did not finish problem 1 or 3. I apologize for this.

Problem 1

Exercise 1. Let R be an integral domain. Show that R is a Prüfer domain if and only if every overring of R is integrally closed. (Hint: consider $R_{\mathfrak{m}}$ for some maximal ideal and if $x, y \in R_{\mathfrak{m}}$, consider $R_{\mathfrak{m}}[y^2/x^2]$.

Solution 1. Suppose that R is a Prüfer domain and let A be an overring of R. By homework 4, problem 4, we know that A is itself a Prüfer domain. Every Prüfer domain is integrally closed (see Appendix), so A is integrally closed. Since A was arbitrary, it follows that every overring of R is integrally closed.

Conversely, suppose every overring of R is integrally closed and let $\mathfrak{p}$ be a prime ideal of R. We need to show that $R_{\mathfrak{p}}$ is a valuation domain. First note that $R_{\mathfrak{p}}$ is integrally closed since integral closures commute with localization.

Problem 2

Exercise 2. Show that if K is a field then any maximal ideal of $K[T_1, \dots, T_n]$ can be generated by n elements.

Solution 2. By Hilbert's Nullstellensatz, the maximal $\mathfrak{m}$ is in the kernel of a K-algebra homomorphism from $K[T_1, \dots, T_n]$ to L where L/K. is a finite field extension. For each $1 \leq i \leq n$ let α_i be the images of T_i under this homomorphism. We will build a sequence of polynomials $f_1, \dots, f_n$ in $K[T_1, \dots, T_n]$ such that $\mathfrak{m} = \langle f_1, \dots, f_n \rangle$ and such that f_k is a polynomial in $K[T_1, \dots, T_k]$ for all $1 \leq k \leq n$.

First we set $f_1(T_1)$ to be the minimal polynomial of α_1 over K. Next let $\pi_2(X)$ be the minimal polynomial of α_2 over $K(\alpha_1)$. The coefficients of $\pi_2(X)$ can be expressed as polynomials in α_1, and so in particular we can find a polynomial f_2 in $K[T_1, T_2]$ such that $f_2(\alpha_1, X) = \pi_2(X)$. Proceeding inductively, at the kth step, where $1 \leq k \leq n$, we let $\pi_k(X)$ be the minimal polynomial of α_k over $K(\alpha_1, \dots, \alpha_{k-1})$ and we choose a polynomial a polynomials f_k in $K[T_1, \dots, T_k]$ such that

$$f_k(\alpha_1, \dots, \alpha_{k-1}, X) = \pi_k(X).$$

We claim that $\mathfrak{m} = \langle f_1, \dots, f_n \rangle$. Indeed, we have $\mathfrak{m} \supseteq \langle f_1, \dots, f_n \rangle$ since $\langle f_1, \dots, f_n \rangle$ is in the kernel of the K-algebra homomorphism from $K[T_1, \dots, T_n]$ to L. To see this, note that for each $1 \leq k \leq n$ we have

$$f_k(\alpha_1, \dots, \alpha_{k-1}, \alpha_k) = \pi_k(\alpha_k) = 0.$$

We also have $\mathfrak{m} \subseteq \langle f_1, \dots, f_n \rangle$ since $\langle f_1, \dots, f_n \rangle$ is a maximal ideal. Indeed, we prove by induction on n that $K[T_1, \dots, T_n]/\langle f_1, \dots, f_n \rangle \cong K(\alpha_1, \dots, \alpha_n)$. If $n = 1$, then

$$K[T_1]/\langle f_1 \rangle \cong K[X]/\pi_1(X) \cong K(\alpha_1).$$

Now suppose $n > 1$ and we have shown this to be true for all $1 \leq k < n$. Then we have

$$\begin{aligned} K[T_1, \dots, T_{n-1}, T_n]/\langle f_1, \dots, f_{n-1}, f_n \rangle &\cong (K[T_1, \dots, T_{n-1}]/\langle f_1, \dots, f_{n-1} \rangle) [T_n]/\langle f_n(\overline{T_1}, \dots, \overline{T_{n-1}}, T_n) \\ &\cong K(\alpha_1, \dots, \alpha_{n-1})[T_n]/\langle f_n(\alpha_1, \dots, \alpha_{n-1}, T_n) \rangle \\ &\cong K(\alpha_1, \dots, \alpha_{n-1})[X]/\langle \pi_n(X) \rangle \\ &\cong K(\alpha_1, \dots, \alpha_{n-1}, \alpha_n), \end{aligned}$$

where we used the induction step to get from the first line to the second line.

Problem 3

Exercise 3. Let R be an integral domain and let $S \subseteq R$ be a multiplicatively closed subset not containing 0.

1. Show that $R[x]_S = R_S[x]$.
2. Show that $R[[x]]_S \subseteq R_S[[x]]$.
3. Show that equality in 2 holds if and only if for every countable collection (s_n) of elements of S we have $\bigcap_{n\in\mathbb{N}}\langle s_n\rangle \neq 0$.
4. Show that if R is a PID then every $S \subseteq R$ satisfies the above property if and only if R is a field.

Solution 3. 1. Define $\varphi\colon R[x]_S \to R_S[x]$ by

$$\varphi\left(\left(\sum_{i=0}^{n} a_i x^i\right)/s\right) = \sum_{i=0}^{n}(a_i/s)x^i.$$

where $a_i \in R$ and $s \in S$. The map φ is clearly a well-defined injective ring homomorphism. Furthermore, it is surjective. Indeed, if $\sum_{i=0}^{n}(a_i/s_i)x^i \in R_S[x]$, then

$$\begin{aligned}\sum_{i=0}^{n}(a_i/s_i)x^i &= \frac{a_0}{s_0} + \frac{a_1}{s_1}x + \cdots + \frac{a_n}{s_n}x^n \\ &= \frac{a_0 s_1\cdots s_n}{s_0 s_1\cdots s_n} + \frac{s_0 a_1 s_2\cdots s_n}{s_0 s_1\cdots s_n}x + \cdots + \frac{s_0\cdots s_{n-1}a_n}{s_0 s_1\cdots s_n}x^n \\ &= \varphi\left(\frac{a_0 s_1\cdots s_n + s_0 a_1 s_2\cdots s_n x + \cdots + s_0\cdots s_{n-1}a_n x^n}{s_0 s_1\cdots s_n}\right)\end{aligned}$$

Thus $R[x]_S \cong R_S[x]$.

2. Define $\varphi\colon R[[x]]_S \to R_S[[x]]$ by

$$\varphi\left(\left(\sum_{n=0}^{\infty} a_n x^n\right)/s\right) = \sum_{n=0}^{\infty}(a_n/s)x^n \tag{1}$$

for all $\left(\sum_{n=0}^{\infty} a_n x^n\right)/s \in R[[x]]_S$. Let's check that (1) is well-defined. Suppose $\left(\sum_{n=0}^{\infty} a_n x^n\right)/s = \left(\sum_{n=0}^{\infty} a'_n x^n\right)/s'$. Then

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right)/s = \left(\sum_{n=0}^{\infty} a'_n x^n\right)/s' &\iff s'\left(\sum_{n=0}^{\infty} a_n x^n\right) = s\left(\sum_{n=0}^{\infty} a'_n x^n\right) \\ &\iff \sum_{n=0}^{\infty} s' a_n x^n = \sum_{n=0}^{\infty} s a'_n x^n \\ &\iff s' a_n = s a'_n \text{ for each } n \in \mathbb{N} \\ &\iff a_n/s = a'_n/s' \text{ for each } n \in \mathbb{N} \\ &\iff \sum_{n=0}^{\infty}(a_n/s)x^n = \sum_{n=0}^{\infty}(a'_n/s')x^n.\end{aligned}$$

This implies (1) is well-defined. Now we check that φ is injective. Note that

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right)/s \in \ker\varphi &\iff \sum_{n=0}^{\infty}(a_n/s)x^n = 0 \\ &\iff a_n/s = 0 \text{ for all } n \in \mathbb{N} \\ &\iff a_n = 0 \text{ for all } n \in \mathbb{N} \\ &\iff \sum_{n=0}^{\infty} a_n x^n = 0 \\ &\iff \left(\sum_{n=0}^{\infty} a_n x^n\right)/s = 0.\end{aligned}$$

It follows that φ is injective.

3. Keeping the same notation as before, we show φ is surjective if and only if S has the property that for every sequence (s_n) in S we have $\bigcap_{n\in\mathbb{N}}\langle s_n\rangle \neq 0$. Suppose S has the stated property. Let $\sum_{n=0}^{\infty}(a_n/s_n)x^n$ be an element of $R_S[[x]]$. Since $\bigcap_{n\in\mathbb{N}}\langle s_n\rangle \neq 0$, there exists a nonzero $t \in \bigcap_{n\in\mathbb{N}}\langle s_n\rangle$. Write $t = b_n s_n$ for all $n \in \mathbb{N}$ where $b_n \in R$. Note that this implies $b_1 s_1 = b_n s_n$ or $b_1/s_n = b_n/s_1$. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{b_1 a_n}{s_n}x^n &= \sum_{n=0}^{\infty}\frac{b_n a_n}{s_1}x^n \\ &= \left(\sum_{n=0}^{\infty} b_n a_n x^n\right)/s_1\end{aligned}$$

In particular, it follows that

$$\varphi\left(\left(\sum_{n=0}^{\infty} b_n a_n x^n\right)/b_1 s_1\right) = \sum_{n=0}^{\infty}\frac{b_1 a_n}{s_n}x^n,$$

thus φ is surjective.

Problem 4

Definition 0.1. Let R be a commutative ring with identity and let M be an R-module. A prime $\mathfrak{p}$ of R is **weakly associated** to M if there exists an element $u \in M$ such that $\mathfrak{p}$ is minimal among the prime ideals containing the annihilator $0 : u = \{a \in R \mid au = 0\}$. The set of all such primes is denoted $\operatorname{WeakAss} M$.

Proposition 0.1. *Let R be a commutative ring with identity. Then the set of all zerodivisors of R is given by the set*

$$\bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p}.$$

Proof. Suppose $x \in R$ is a zerodivisor. Then $0 : x$ is a proper ideal of R. Choose a minimal prime $\mathfrak{p}$ over $0 : x$. Then $\mathfrak{p}$ is a weakly associated prime to R and $x \in \mathfrak{p}$ implies

$$\{\text{set of zerodivisors of } R\} \subseteq \bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p}.$$

Conversely, suppose $x \in \bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p}$. Then $x \in \mathfrak{p}$ for some prime $\mathfrak{p}$ which is weakly associated to R. Since $\mathfrak{p}$ is weakly associated to R, there exists a $y \in R$ such that $\mathfrak{p}$ is a minimal prime over $0 : y$. Since localization is exact, we see that $\mathfrak{p}_{\mathfrak{p}}$ is a weakly associated prime to $R_{\mathfrak{p}}$, with $\mathfrak{p}_{\mathfrak{p}}$ being minimal over the annihiliator of $y/1$. Since $R_{\mathfrak{p}}$ is local and $\mathfrak{p}_{\mathfrak{p}}$ is minimal over the annihilator $0 : (y/1)$, we have $\operatorname{rad}(0 : (y/1)) = \mathfrak{p}_{\mathfrak{p}}$. In particular, there exists $n \in \mathbb{N}$ and a $z \in R\backslash\mathfrak{p}$ such that $x^n z \in 0 : y$, or in other words, such that $x^n zy = 0$. Note that $zy \neq 0$ as $z \notin \mathfrak{p}$, so if $n = 1$, then $xzy = 0$ implies x is a zerodivisor. Assume $n > 1$. Choose $m \in \mathbb{N}$ such that $m \leq n$ and $x^m zy = 0$ and $x^{m-1}zy \neq 0$. Then $x(x^{m-1}zy) = x^m zy = 0$ implies x is a zerodivisor. Thus

$$\{\text{set of zerodivisors of } R\} \supseteq \bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p}.$$

□

Exercise 4. Let R be a 0-dimensional ring. Then any nonunit of R is a zerodivisor.

Solution 4. We have

$$\begin{aligned}\{\text{set of zerodivisors of } R\} &= \bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p} \\ &= \bigcup_{\mathfrak{p}\in\operatorname{Spec} R}\mathfrak{p} \\ &= \{\text{nonunits of } R\},\end{aligned}$$

where we obtained the second line from the first line from the fact that R is 0-dimensional. Indeed, clearly we have

$$\bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p} \subseteq \bigcup_{\mathfrak{p}\in\operatorname{Spec} R}\mathfrak{p}.$$

Conversely, suppose $\mathfrak{p}$ is a prime ideal of R and choose $x \in \mathfrak{p}$. Then since $x \in \mathfrak{p}$ and $\mathfrak{p}$ is prime we have $\mathfrak{p} \supseteq 0 : x$ and since R is 0-dimensional we see that $\mathfrak{p}$ is minimal over $0 : x$. Thus $\mathfrak{p}$ is a weakly associated prime to R. It follows that

$$\bigcup_{\mathfrak{p}\in\operatorname{WeakAss} R}\mathfrak{p} \supseteq \bigcup_{\mathfrak{p}\in\operatorname{Spec} R}\mathfrak{p}.$$

Appendix

Problem 1

Prüfer domains are integrally closed

Lemma 0.1. *Let R be an integral domain, let K be its quotient field, and let $\overline{R}$ be the integral closure of R in K. Then*

$$\overline{R} \subseteq \bigcap_{R \subseteq A \subseteq K} A$$

where the intersection runs over all valuation overrings A of R.

Proof. This follows from the fact the every valuation ring is integrally closed. Indeed, let A be a valuation overring of R. Then since A is integrally closed and $R \subseteq A$, it follows that $\overline{R} \subseteq A$. Since A was arbitrary, we see that $\overline{R} = \bigcap_{R \subseteq A \subseteq K} A$ where the intersection runs over all valuation overrings A of R. □

Proposition 0.2. *Let R be a Prüfer domain. Then R is integrally closed.*

Proof. Let $\overline{R}$ be the integral closure of R. Observe that

$$\begin{aligned}
R &= \bigcap_{\mathfrak{p} \in \operatorname{Spec} R} R_{\mathfrak{p}} && \text{(Homework 1, Problem 4)} \\
&\supseteq \bigcap_{A \text{ valuation overring of } R} A && \text{(Because } R \text{ is Prüfer)} \\
&\supseteq \overline{R} && \text{(Lemma above)} \\
&\supseteq R.
\end{aligned}$$

It follows that $R = \overline{R}$. Hence R is integrally closed. □