MATH 9500
FALL 2020
HOMEWORK 7

Due Monday, November 23, 2020

Let R be a commutative ring with identity and $I \subseteq R$ an ideal. We say that I is of *strong finite type* (SFT) if there is a finitely generated ideal $B \subseteq I$ and an integer $n \in \mathbb{N}$ such that $x^n \in B$ for all $x \in I$. We also say that the ring R is SFT if every ideal of R is SFT.

1. Let R be a commutative ring with identity.
   a) (5 pts) Show that R is SFT if and only if every prime ideal of R is SFT.
   b) (5 pts) Show that if R is SFT then R satisfies the ascending chain condition on radical ideals.
   c) (5 pts) Give an example of a ring that is SFT but not Noetherian.
   d) (5 pts) Give an example of a ring that satisfies the ascending chain condition on radical ideals but is not SFT.

2. (5 pts) Let R be a domain with quotient field K with the property that every overring of R is Noetherian. Show that $\dim(R) \leq 1$.

3. (5 pts) Let R be $1-$dimensional and Noetherian. If $\mathfrak{P}$ is a prime ideal of R and T is an overring of R then there are only (at most) finitely many prime ideals in T that lie over $\mathfrak{P}$.

4. (5 pts) Let R be commutative (not necessarily with 1 for this problem). We recall that R is von Neumann regular if for all $x \in R$ there is a $y \in R$ such that $x = xyx$. Suppose that R is $0-$dimensional and commutative with no nonzero nilpotent elements. Show that R is von Neumann regular.