Ringed Spaces

September 4, 2020

Throughout this article, let R be a commutative ring and let $\alpha \in \widehat{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty, \omega\}$.

1 Ringed Spaces

Ringed spaces formalize the idea of giving a geometric object by specifying its underlying topological space and the "functions" on all open subsets of this space.

Definition 1.1.

1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$.

2. A **locally** R-**ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Remark. Usually we will denote a (locally) R-ringed space $(X, \mathcal{O}_X)$ simply by X.

Our principle example will be sheaves of real-valued C^α functions.

Example 1.1. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by $\mathcal{C}^\alpha_X$ the sheaf of $\mathcal{C}^\alpha$ functions: For all open subsets U of X, we have

$$\mathcal{C}^\alpha_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is } C^\alpha\}.$$

Then $\mathcal{C}^\alpha_X$ is a sheaf of $\mathbb{R}$-algebras. The same argument as for sheaves of continuous functions yields the following observation: For all $x \in X$ the stalk $C^\alpha_{X,x}$ is a local ring. In particular $(X, \mathcal{C}^\alpha_X)$ is a locally $\mathbb{R}$-ringed space.

1.1 Morphisms of (Locally) Ringed Spaces

Definition 1.2. Let $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$ be R-ringed spaces. A **morphism of** R**-ringed spaces** $X \to Y$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a continuous map of the underlying topological spaces and where $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of R-algebras on Y.

The datum of $f^\flat$ is equivalent to the datum of a homomorphism of sheaves of R-algebras $f^\# : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X. Usually we simply write f instead of $(f, f^\#)$ or $(f, f^\flat)$.

Remark. When $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of functions, $f^\flat$ is oftenly defined by the pullback map f^*, where if $g \in \mathcal{O}_Y(U)$, then we set $f^*(g) = g \circ f$. Of course for this to make sense, we need $g \circ f \in \mathcal{O}_X(f^{-1}(U))$.

Morphisms of *locally* ringed spaces have to satisfy an additional property. To state this property, observe that a morphism $f : X \to Y$ of R-ringed spaces induces morphisms on the stalks as follows: let $x \in X$. Using the identification $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$, we get a homomorphism of R-algebras

$$f_x := (f^\#)_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

There is a more explicit description of this homomorphism: for U an open neighborhood of $f(x)$ one has a map

$$\mathcal{O}_Y(U) \xrightarrow{f^\flat_U} \mathcal{O}_X(f^{-1}(U)) \longrightarrow \mathcal{O}_{X,x}$$

These maps induce the map on stalks $f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. In particular, $f_x[(s,V)] = [(f_V^\flat(s), f^{-1}(V))]$.

Now let X and Y be locally R-ringed spaces. We define a **morphism of locally R-ringed spaces** $X \to Y$ to be a morphism $(f, f^\flat)$ of ringed spaces such that the homomorphism of local rings $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is **local** (i.e. $f_x^\#(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$).

In general there exist locally ringed spaces and morphisms of ringed spaces between them that are not morphisms of *locally* ringed spaces. For spaces with functions of C^α functions such as the premanifolds defined below, we will see that every morphism of ringed spaces is automatically a morphism of locally ringed spaces.

Remark. The composition of morphisms of (locally) R-ringed spaces is defined in the obvious way using the compatibility of direct images with composition (i.e. $(g \circ f)_* = g_* \circ f_*$. We obtain the category of (locally) R-ringed spaces.

In general, $f^\flat$ (or $f^\#$) is an additional datum for a morphism. For instance it might happen that f is the identity but $f^\flat$ is not an isomorphism of sheaves. We will usually encounter the simpler case that the structure sheaf is a sheaf of functions on open subsets of X and that $f^\flat$ is given by composition with f. The following special case and its globalization is the main example.

Example 1.2. Let $X \subseteq V$ and $Y \subseteq W$ be open subsets of finite-dimensional $\mathbb{R}$-vector spaces V and W. Every C^α map $f : X \to Y$ defines by composition a morphism of locally $\mathbb{R}$-ringed spaces $(f, f^\flat) : (X, \mathcal{C}_X^\alpha) \to (Y, \mathcal{C}_Y^\alpha)$ by

$$\begin{array}{rcl} f_U^\flat : \mathcal{C}_Y^\alpha(U) & \longrightarrow & f_*(\mathcal{C}_X^\alpha)(U) = \mathcal{C}_X^\alpha(f^{-1}(U)) \\ t & \mapsto & t \circ f \end{array}$$

for $U \subseteq Y$ open.

The induced map on stalks $f_x : \mathcal{C}_{Y,f(x)}^\alpha \to \mathcal{C}_{X,x}^\alpha$ is then also given by composing an $\mathbb{R}$-valued C^α function t, defined in some neighborhood of $f(x)$, with f, which yields an $\mathbb{R}$-valued C^α function $t \circ f$ defined in some neighborhood of x. Conversely, let $(f, f^\flat) : (X, \mathcal{C}_X^\alpha) \to (Y, \mathcal{C}_Y^\alpha)$ be any morphism of $\mathbb{R}$-ringed spaces. We claim:

1. $(f, f^\flat)$ is automatically a morphism of *locally* $\mathbb{R}$-ringed spaces.
2. For all $U \subseteq Y$ open the $\mathbb{R}$-algebra homomorphism $f_U^\flat : \mathcal{C}_Y^\alpha(U) \to \mathcal{C}_X^\alpha(f^{-1}(U))$ is automatically given by the map $t \mapsto t \circ f$. Note that then f is a C^α map (choose a basis of W; considering for t projections to the coordinates shows that each component of f is a C^α map).

To show 1 let $x \in X$. Set $\varphi := f_x^\#$, $B := \mathcal{C}_{X,x}^\alpha$, and $A := \mathcal{C}_{Y,f(x)}^\alpha$. Then $\varphi : A \to B$ is a homomorphism of local $\mathbb{R}$-algebras such that $A/\mathfrak{m}_A = \mathbb{R}$ and $B/\mathfrak{m}_B = \mathbb{R}$. We claim that φ is automatically local, equivalently that $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal of A. Indeed, φ induces an injective homomorphism of $\mathbb{R}$-algebras

$$A/\varphi^{-1}(\mathfrak{m}_B) \hookrightarrow B/\mathfrak{m}_B = \mathbb{R}.$$

As a homomorphism of $\mathbb{R}$-algebras, it is automatically surjective (indeed 1 maps to 1), hence $A/\varphi^{-1}(\mathfrak{m}_B) \cong \mathbb{R}$ is a field and hence $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal.

Let us show 2. Let $U \subseteq Y$ be open and $x \in f^{-1}(U)$. Consider the commutative diagram of $\mathbb{R}$-algebra homomorphisms

$$\begin{tikzcd}
\mathcal{C}_Y^\alpha(U) \arrow[r, "f_U^\flat"] \arrow[d, "t \mapsto t_{f(x)}"'] & \mathcal{C}_X^\alpha(f^{-1}(U)) \arrow[d, "s \mapsto s_x"] \\
\mathcal{C}_{Y,f(x)}^\alpha \arrow[r, "f_x^\#"] \arrow[d, "\mathrm{ev}_{f(x)} : t \mapsto t(f(x))"'] & \mathcal{C}_{X,x}^\alpha \arrow[d, "\mathrm{ev}_x : s \mapsto s(x)"] \\
\mathbb{R} & \mathbb{R}
\end{tikzcd}$$

The evaluation maps are surjective. Hence there exists a homomorphism of $\mathbb{R}$-algebras $\iota : \mathbb{R} \to \mathbb{R}$ making the lower rectangle commutative if and only if one has $f_x^\#(\mathrm{Ker}(\mathrm{ev}_{f(x)})) \subseteq \mathrm{Ker}(\mathrm{ev}_x)$. But this latter condition is satisfied because $f_x^\#$ is local by 1. Moreover, as a homomorphism of $\mathbb{R}$-algebras, one must have $\iota = \mathrm{id}_\mathbb{R}$. Therefore we find $f_U^\flat(t)(x) = t(f(x))$, which shows 2.

1.2 Gluing Ringed Spaces

Let $\{(X_i, \mathcal{F}_i)\}_{i \in I}$ be a collection of ringed spaces.