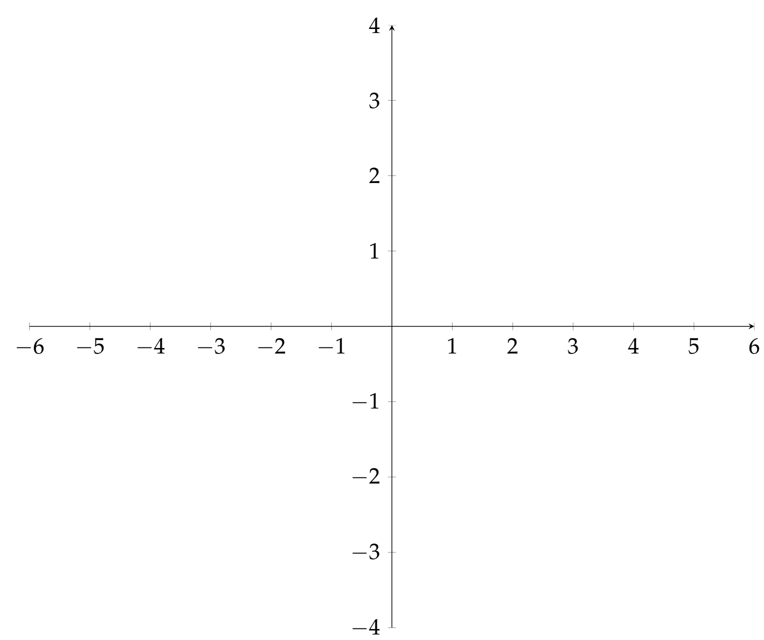
Final Exam Study Guide

This study guide is deisgned to help you be prepared for the final exam. I've separated this study guide into two parts. The first part consists of exercises which involve graphs, whereas the second part consists of exercises which do not invove graphs. I will go over the solution to these exercises in a review session, but please make sure that you're able to do these on your own, and please try to solve these exercises *before* I give you my solutions. As I've said many times before, learning Mathematics is a lot like learning any other activity. For instance, you cannot learn how to ride a bike by watching someone else ride one; you need to ride one yourselves! Similarly you cannot learn how to compute derivatives by watching me do it; you need to compute them yourselves! I put a lot of time and effort into making this study guide, so I hope you'll do me the same favor and put a lot of time and effort into it as well. Remember that we will replace your lowest test grade with your final exam grade, unless your final exam grade is the worst out of the three test grades. At the end of the day, I really want you all to do well on this final.

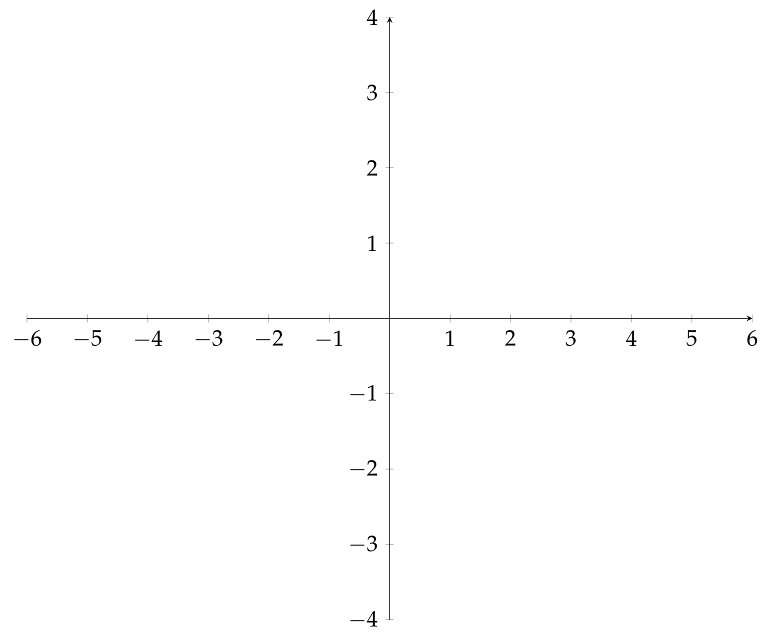
Questions Involving Graphs

Note that some of the exercises below may seem similar to ones I've given you before, however you should know that they are not the same (well except for one problem, but it's kinda important so I added it to this study guide as well). So please make sure you attempt all of them, even if you've already worked on similar ones in the past.

Exercise 1. You should definitely know what the graphs of $\sin x$, $\cos x$, and $\tan x$ look like. You should also know what the graphs of $\csc x$, $\sec x$, and $\cot x$ look like as well. For this problem, let's focus on the graph of $\cos x$. First graph $\cos x$ below

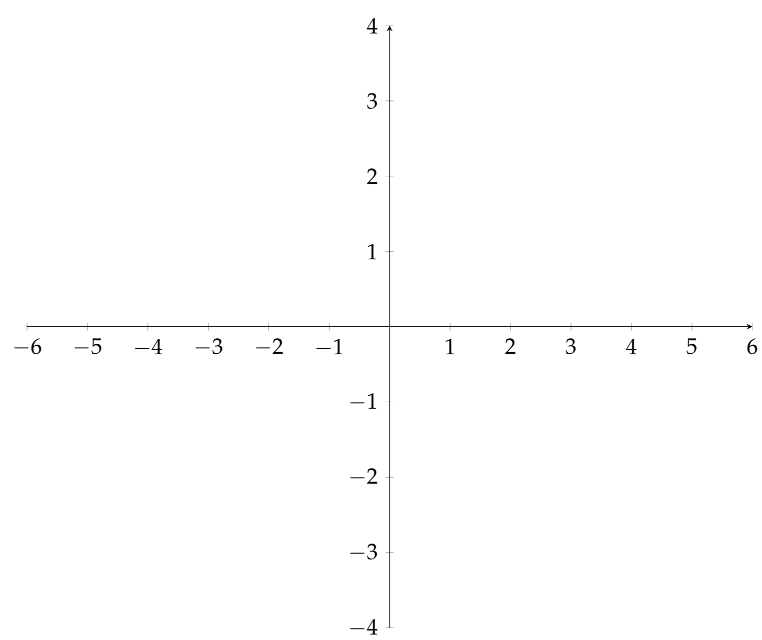


Next graph $3\cos x$ below



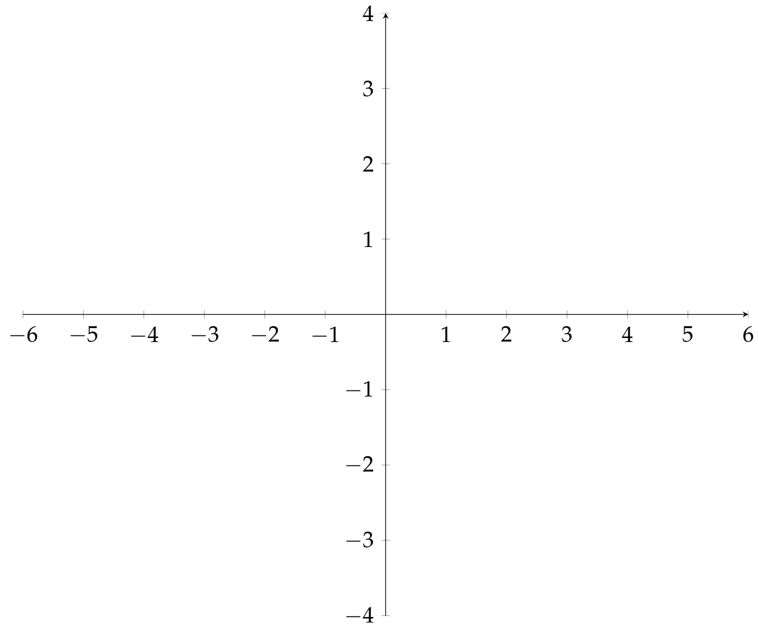
How does multiplying the function by 3 change the graph of $\cos x$?

Next graph $3\cos(x - \pi)$ below

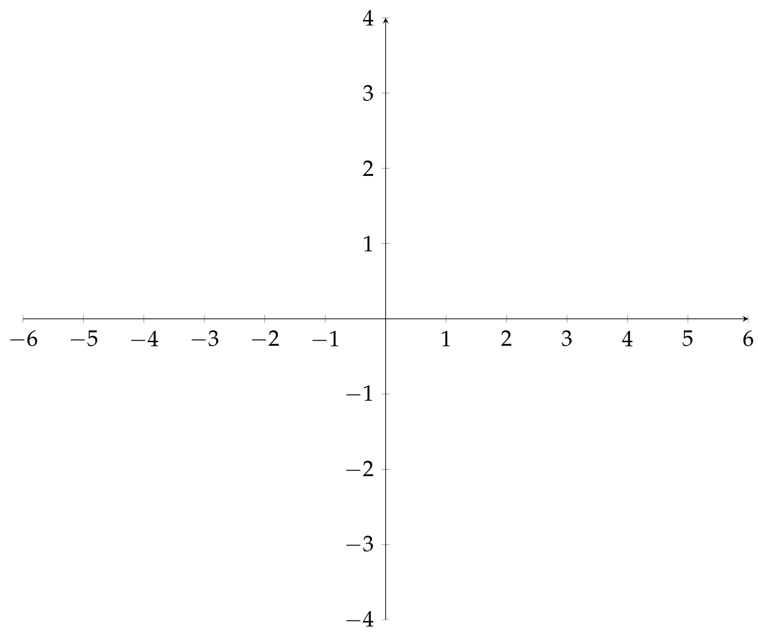


How does replacing x with $x - \pi$ change the graph of $3\cos x$?

Exercise 2. Another function which you should know how to graph is the function $\frac{1}{x^2}$. Graph this function below

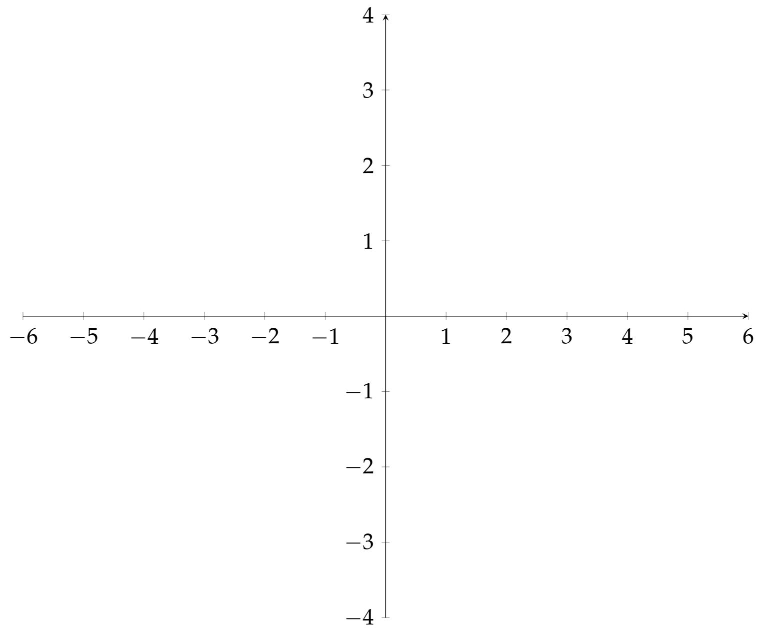


Next graph $\frac{1}{(x+2)^2}$ below



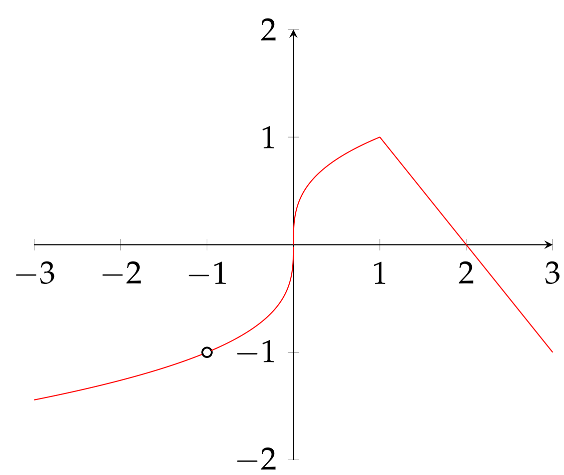
How does replacing x with $x+2$ change the graph of $\frac{1}{x^2}$?

Next graph $\frac{1}{(x+2)^2} + 3$ below



How does adding 3 to the function change the graph of $\frac{1}{(x+2)^2}$?

Exercise 3. Consider the function $f(x)$ whose graph is given below

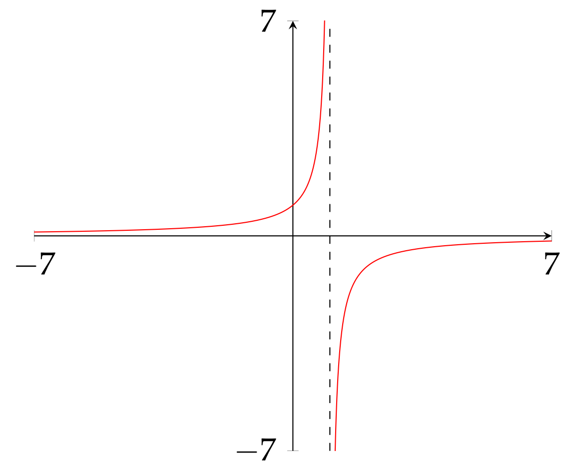


State all x-values where this function is not differentiable. State all x-values where this function is not continuous.

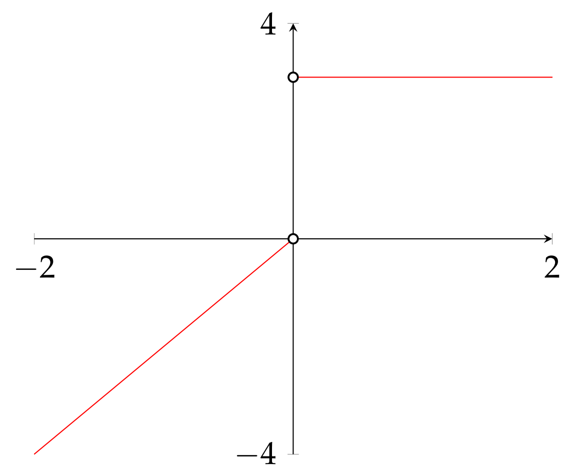
It is not differentiable at:

It is not continuous at:

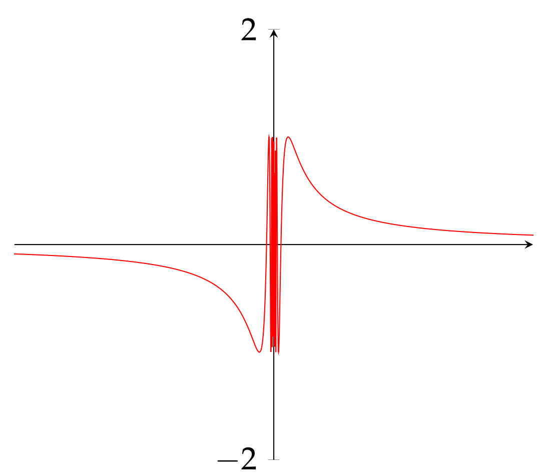
Exercise 4. Make sure you know the different types of discontinuities a function can have, namely jump, infinite, removable, and oscilatting discontinuities. What type of discontinuity is presented in the graph below?



What type of discontinuity is presented in the graph below?



What type of discontinuity is presented in the graph below?



What type of discontinuity is given by the function below?

$$f(x) = \frac{(x-3)(x+2)}{x+2}.$$

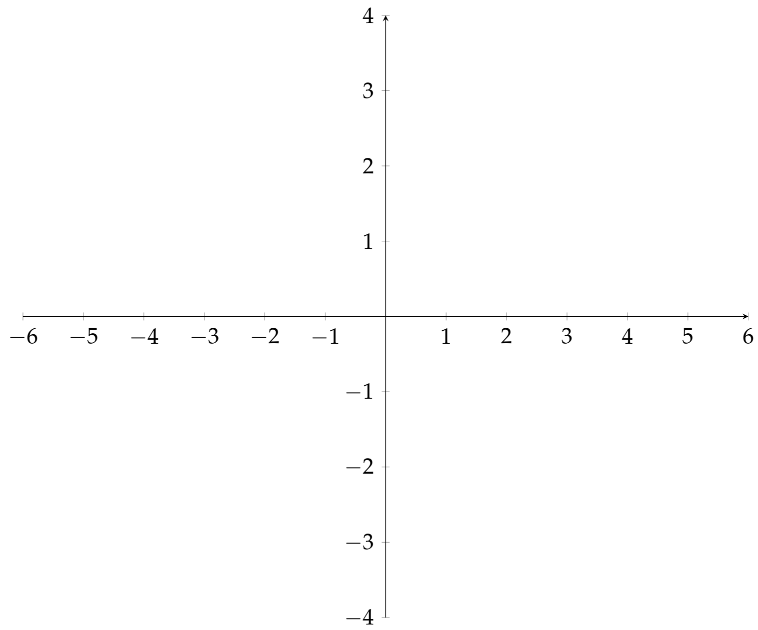
Exercise 5. In this problem we focus on the function

$$f(x) = \frac{(x-3)(x+2)}{(x-1)(x+2)}.$$

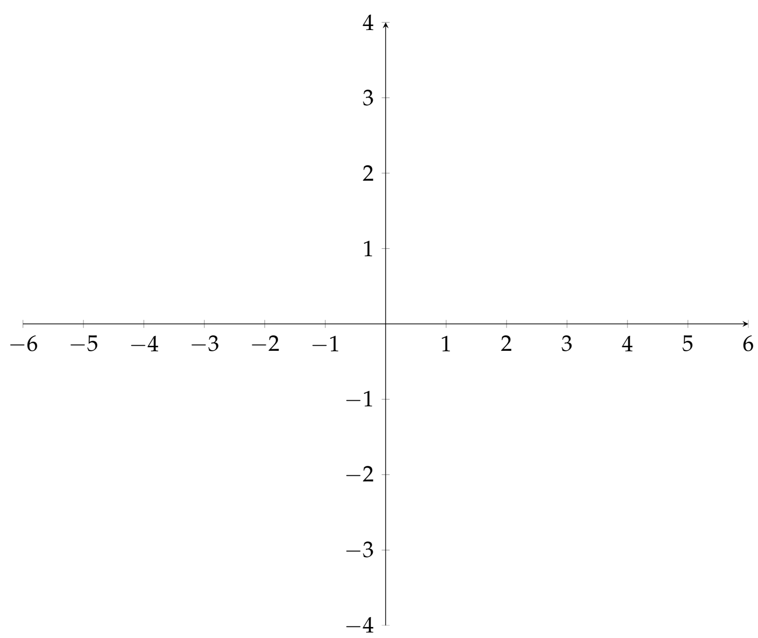
Note that this function is *not* equal to the function

$$g(x) = \frac{x-3}{x-1}.$$

However the functions $f(x)$ and $g(x)$ are *very close* to being equal to each other! In fact, $f(x)$ and $g(x)$ agree everywhere *except* at $x = -2$. To see this, first graph $f(x)$ below



Next graph $g(x)$ below



If you graphed everything correctly, you'll notice that $f(x)$ is essentially equal to $g(x)$ except at $x = -2$ where the graph of $f(x)$ should have a hole and the graph of $g(x)$ has this hole filled in. In particular, $f(x)$ has a *discontinuity* at $x = -2$. What kind of discontinuity is this?

Next, compute the following limits

$$\begin{aligned} \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} \frac{(x-3)(x+2)}{(x-1)(x+2)} \\ &= \end{aligned}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{(x-3)(x+2)}{(x-1)(x+2)}$$
$$=$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{(x-3)(x+2)}{(x-1)(x+2)}$$
$$=$$

If I had replaced the function $f(x)$ with the function $g(x)$ in the limits above, would you get the same answer?

Finally, give the equations of the vertical and horizontal asymptotes of $g(x)$.

Vertical Asymptotes:

Horizontal Asymptotes:

Questions Which Do Not Involve Graphs

Exercise 6. It is very important to be able to work with interval notation. Recall that for real numbers $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$, we define

$$\begin{aligned}(a,b) &= \{x \in \mathbb{R} \mid a < x < b\}\\ (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\}\\ [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\}\\ [a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\}.\end{aligned}$$

where $\cap$ denotes the operation of taking intersections of sets and where $\cup$ denotes operation of taking unions of sets. Thus for example, we have

$$(2,4) \cap (3,7) = \{x \in \mathbb{R} \mid 2 < x < 4 \text{ and } 3 < x < 7\} = (3,4)$$

$$(2,4) \cup (3,7) = \{x \in \mathbb{R} \mid 2 < x < 4 \text{ or } 3 < x < 7\} = (2,7)$$

Now consider the function

$$f(x) = \frac{\sqrt{x+4}}{x^2-1}.$$

Using the interval notation above, state the domain and range of $f(x)$.

Domain:

Range:

Exercise 7. For this problem we consider the function $f(x) = 4x^2 - 2x + 1$. First, find the average rate of change of $f(x)$ on the interval $[-1, 2]$.

Next, what is the slope of the tangent line of $f(x)$ at $x = 1$?

Finally, at what x-values does $f(x)$ have a tangent line with slope equal to 6?

Exercise 8. At what x-value is the tangent line of $f(x) = e^x - 2x$ at that x-value parallel to the line $y = -2x + 1$?

Exercise 9. Find the derivative of $f(x) = -4x^2 + 3x - 2$ using the *limit* definition. Note that you'll see this type of question on the test. **Make sure you do this right and do not skip any steps!** I'll show you how to do it when it comes time to review this question in class, but you need to be able to write it on your own *exactly* the way I do it.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Exercise 10. Find the derivative of the following functions (I'll do the first one for you).

$$p(x) = x^3 + 3x^2 \qquad \begin{aligned} p'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(p(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + \frac{\mathrm{d}}{\mathrm{d}x}(3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \\ &= 3x^{3-1} + 3 \cdot 2x^{2-1} \\ &= 3x^2 + 6x \end{aligned}$$

You don't have to do every step as I did above. If you feel comfortable, you can skip some steps like this:

$$p(x) = x^3 + 3x^2 \qquad \begin{aligned} p'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(p(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3x^2) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(x^3) + 3\frac{\mathrm{d}}{\mathrm{d}x}(x^2) \\ &= 3x^2 + 6x. \end{aligned}$$

Now you do the rest

$$h(t) = e^{\sqrt{t}} - t^e + 5e \qquad \begin{aligned} h'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(h(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\sqrt{t}} - t^e + 5e\right) \end{aligned}$$

$$f(x) = \tan(2e^{3x+1}) \qquad f'(x) =$$

$$g(x) = \sqrt[5]{2x}\sin(x^4) + \sec^3(x+1) \qquad\qquad f'(x) =$$

Exercise 11. In this problem we focus on the function $f(x) = x^3 - 2x - 1$. First, factor this polynomial completely using the fact that $x = -1$ is a root of $f(x)$.

Next, find the x-values at which the tangent line of $f(x)$ at that value is horizontal.

Exercise 12. Evaluate the following limits. Do *not* use L'Hospital's rule! Note that some of these limits are secretly derivatives of simple functions in disguise.

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} =$$

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3} =$$

$$\lim_{x\to -2} \frac{-2 - x}{1 - \sqrt{x+3}} =$$

$$\lim_{x\to 0} \frac{\frac{1}{x+1} - 1}{x} =$$

Exercise 13. Solve for x in the following equations

$$2^x = 4^{x^2}$$

$$\frac{d^2}{dx^2}(e^{2x}) = 4e$$

$$\log_{125}(x+1) = \frac{1}{3}$$

$$3^{2x} - 3^{x+1} + 1 = 0$$

Exercise 14. Let $f(x)$ and $g(x)$ be two functions defined on the whole real line both of which are differentiable everywhere. Suppose that

$$f(4) = 3$$
$$g(1) = 4$$
$$f'(4) = 2$$
$$g'(1) = 7$$

Using the information above, find $h'(1)$ where $h(x)$ is the composite function $h(x) = f(g(x))$.