Measure Theory Extra Problems

Michael Nelson

Problem 3

Proposition 0.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f\colon X \to \mathbb{R}$ be a measurable function such that $\mu(\{f > 1\}) < \infty$, and let $\varepsilon > 0$. Then there exists a bounded measurable function $g\colon X \to \mathbb{R}$ such that $\mu(\{f \neq g\}) < \varepsilon$.*

Proof. We may assume f is nonnegative (otherwise break f up into its positive and negative part $f = f^+ - f^-$, approximate both f^+ and f^- by a bounded measurable function, and then combine these together). For each $n \in \mathbb{N}$, let $E_n = \{f > n\}$. Observe that

$$\bigcap_{n=1}^{\infty} E_n = \varnothing. \tag{1}$$

Indeed, for any $x \in X$, there must exist some $n_x \in \mathbb{N}$ such that $f(x) \leq n_x$ (as $f(x) < \infty$). Thus $x \notin \bigcap_{n=1}^{\infty} E_n$. Since x is arbitrary, this implies (1). Therefore

$$\begin{aligned} 0 &= \mu(\varnothing) \\ &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &= \lim_{n\to\infty} \mu(E_n), \end{aligned}$$

where we obtained the third line from the second line from the fact that $\mu(E_1) < \infty$. In particular, we can choose $N \in \mathbb{N}$ such that

$$\mu(E_N) < \varepsilon.$$

Now we define $g\colon X \to \mathbb{R}$ by $g = \min\{f, N1_{E_N^c}\}$. Note that since f is nonnegative, we have

$$g(x) = \begin{cases} f(x) & \text{if } x \notin E_N \\ N & \text{if } x \in E_N \end{cases}$$

In particular, g is clearly bounded and measurable, and moreover

$$\begin{aligned} \mu(\{g \neq f\}) &\subseteq \mu(E_N) \\ &< \varepsilon. \end{aligned}$$

□

Problem 13

Proposition 0.2. *Let $(X, \mathcal{M}, \mu)$ be measure space and let $g\colon X \to \mathbb{R}$ be a nonnegative measurable function. Recall that the function $\nu\colon \mathcal{M} \to [0, \infty]$ given by*

$$\nu(E) = \int g 1_E \mathrm{d}\mu$$

is a measure on $\mathcal{M}$. For any nonnegative measurable function $f\colon X \to \mathbb{R}$, we have

$$\int_X f \mathrm{d}\nu = \int_X f g \mathrm{d}\mu.$$

Proof. First note that for any nonnegative simple function $\varphi\colon X \to \mathbb{R}$ such that $\varphi \leq f$, we have

$$\int_X \varphi \mathrm{d}\nu = \int_X \varphi g \mathrm{d}\mu.$$

Indeed, write φ in terms of its canonical representation, say $\varphi = \sum_{i=1}^{n} a_i 1_{A_i}$, then

$$\begin{aligned}
\int_X \varphi \mathrm{d}\nu &= \sum_{i=1}^{n} a_i \nu(A_i) \\
&= \sum_{i=1}^{n} a_i \int_X g 1_{A_i} \mathrm{d}\mu \\
&= \int_X \left(\sum_{i=1}^{n} a_i 1_{A_i} \right) g \mathrm{d}\mu \\
&= \int_X g \mathrm{d}\mu.
\end{aligned}$$

Now choose an increasing sequence $(\varphi_n \colon X \to \mathbb{R})$ of nonnegative simple functions which converges pointwise to f. Then $(\varphi_n g)$ is an increasing sequence of nonnegative measurable functions which converges pointwise to fg. It follows from MCT that

$$\begin{aligned}
\int_X f \mathrm{d}\nu &= \lim_{n\to\infty} \int_X \varphi_n \mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_X \varphi_n g \mathrm{d}\mu \\
&= \int_X f g \mathrm{d}\mu.
\end{aligned}$$

□