Goldbach Rings

Throughout these notes, let K be a field.

1 Goldbach Rings

Definition 1.1. Let C be an abelian group, and let $A, B \subseteq C$ such that $A \subseteq B$. We define the **Goldbach ring with respect to the triple** (A, B, C), denoted $G_{(A,B,C)}$, to be the K-algebra

$$G_{(A,B,C)} := R_{(B,C)}/I_{(A,C)}.$$

where

$$R_{(B,C)} := K[X_b \mid b \in B] \quad \text{and} \quad I_{(A,C)} := \langle \{X_{a_1}X_{a_2} - X_{a_1+a_2} \mid a_1, a_2 \in A,\ a_1 + a_2 \in B\}\rangle.$$

We also give R_B the unique C-graded ring structure defined by $\deg_C X_b = b$ for all $b \in B$. In this case, it is easy to see that I_A is homogeneous with respect to this grading, and hence $G_{(A,B)}$ gets an induced C-graded ring structure as well.

Given the data above, we also fix $F^{(A,B,C)}$ to be a C-graded free resolution of $G_{(A,B,C)}$ over $R_{(B,C)}$. Here, we write (A, B, C) in the superscript of $F^{(A,B,C)}$ since we reserve the subscript for homological degree and C-graded degree; namely

$$F^{(A,B,C)} = \bigoplus_{i \in \mathbb{Z},\, c \in C} F_{i,c}^{(A,B,C)},$$

where i ranges over the homological degree and where c ranges over C-graded degree. Furthermore, if B is finite, then we choose $F^{(A,B,C)}$ to be a minimal free resolution (which is unique up to isomorphism) of $G_{(A,B,C)}$.

Remark 1. Before proceeding further let us make a few remarks.

1. If the abelian group C is understood from context, then we simplify notation by writing

$$\begin{aligned} F^{(A,B)} &= F^{(A,B,C)} \\ G_{(A,B)} &= G_{(A,B,C)} \\ R_B &= R_{(B,C)} \\ I_A &= I_{(A,C).} \end{aligned}$$

   Also if B is understood from context, then we can simplify notation further by suppressing B from our notation too.

2. Whenever we write "let G denote the Goldbach ring with respect to the triple (A, B, C)", then it is understood that the triple (A, B, C) satisfies the conditions laid out in Definition (1.1). Also if we say "let G be a Goldbach ring in C", then it is understood that G has the form $G = G_{(A,B,C)}$ for some (A, B) which again satisfies the conditions laid out in Definition (1.1).

3. We use capital letters to denote the indeterminates in $R_{(B,C)}$ and we use lowercase letters are to denote their cosets in $G_{(A,B,C)}$. Thus the indeterminate X_b in $R_{(B,C)}$ is a representative of the coset x_b in $G_{(A,B,C)}$. More generally, if $f(\{X_b\})$ is a polynomial in $R_{(B,C)}$, then its induced coset is written $f(\{x_b\})$.

1.1 Goldbach Rings in $\mathbb{Z}$

There are many Goldbach rings in $\mathbb{Z}$, however we will pay special attention to the ones of the form $G_{(A,B,\mathbb{Z})}$ and $G_{(A_{2k},B_{2k},\mathbb{Z})}$ where

$$\begin{aligned}
B &= \{p, 2j \mid p \text{ odd prime, } j \in \mathbb{Z}_{\geq 3}\} \\
A &= \{p \mid p \text{ odd prime}\} \\
B_{2k} &= \{p, 2j \mid p \text{ odd prime, } j \in \mathbb{Z}_{\geq 3},\ p, 2j \leq 2k\} && k \in \mathbb{Z}_{\geq 3} \\
A_{2k} &= \{p \mid p \text{ odd prime, } p \leq 2k\} && k \in \mathbb{Z}_{\geq 3}
\end{aligned}$$

In fact, these rings are where the source of the name "Goldbach ring" comes from. Thus we give them a special notation: we denote

$$\begin{aligned}
\mathrm{F} &= F^{(A,B,\mathbb{Z})} \\
\mathrm{G} &= G_{(A,B,\mathbb{Z})} \\
\mathrm{R} &= R_{(B,\mathbb{Z})} \\
\mathrm{I} &= I_{(A,\mathbb{Z})},
\end{aligned}$$

and similarly for each $k \in \mathbb{Z}_{\geq 3}$ we denote

$$\begin{aligned}
\mathrm{F}^{2k} &= F^{(A_{2k},B_{2k},\mathbb{Z})} \\
\mathrm{G}_{2k} &= G_{(A_{2k},B_{2k},\mathbb{Z})} \\
\mathrm{R}_{2k} &= R_{(B_{2k},\mathbb{Z})} \\
\mathrm{I}_{2k} &= I_{(A_{2k},\mathbb{Z})},
\end{aligned}$$

Note that F^{2k} is unique up to isomorphism, but it is not at all clear if F is also unique up to isomorphism. In fact, we will see later on that it is in fact unique up to isomorphism. We will also see that

$$\mathrm{F} = \bigcup_{k \geq 3} \mathrm{F}^{2k}.$$

Of course, Goldbach's conjecture is one of the oldest unsolved problems in number theory. It states

Conjecture 1. *Every even integer greater than* 2 *can be expressed as the sum of two primes.*

In terms of our notation above, Goldbach's conjecture is equivalent to $A + A = B$. In this article we use tools and methods from commutative algebra in order to study this and related conjectures.

1.2 Goldbach Rings in a Finite Abelian Group