Homological Algebra

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1 Introduction

Homological Algebra is a subject in Mathematics whose origins can be traced back to Topology. Homological Algebra is a very diverse subject, so we will not attempt to give an all encompassing description of what Homological Algebra is, rather we give a partial description instead:

> Homological is the study of R-complexes and their homology.

Here R is understood to be a commutative ring with identity[1]. Whenever we write, "let M be an R-module" or "let (A,d) be an R-complex", then it is understood that R is a ring.

1.1 Notation and Conventions

Unless otherwise specified, let K be a field and let R be a commutative ring with identity.

1.1.1 Category Theory

In this document, we consider the following categories:

- The category of all sets and functions, denoted **Set**;
- The category of all rings and ring homomorphisms, denoted **Ring**;
- The category of all R-modules and R-linear maps, denoted $\mathbf{Mod}_R$;
- The category of all graded R-modules and graded R-linear maps, denoted $\mathbf{Grad}_R$;
- The category of all R-algebras R-algebra homorphisms, denoted $\mathbf{Alg}_R$;
- The category of all R-complexes and chain maps, denoted $\mathbf{Comp}_R$;
- The category of all R-complexes and homotopy classes of chain maps, denoted $\mathbf{HComp}_R$
- The category of all DG R-algebras DG algebra homomorphisms, denoted $\mathbf{DG}_R$.

2 Graded Rings and Modules

2.1 Graded Rings

Definition 2.1. Let H be an additive semigroup with identity 0. An H-**graded ring** R is a ring together with a direct sum decomposition

$$R = \bigoplus_{h \in H} R_h,$$

where the R_h are abelian groups which satisfy the property that if $r_{h_1} \in R_{h_1}$ and $r_{h_2} \in R_{h_2}$, then $r_{h_1} r_{h_2} \in R_{h_1+h_2}$. The R_h are called **homogeneous components of** R and the elements of R_h are called **homogeneous elements of degree** h. If r is a homogeneous element in R, then unless otherwise specified, we denote the degree of r as $|r|$. When we say "let R be a graded ring", then it is understood that the homogeneous components of R are denoted R_h.

Proposition 2.1. *Let R be an H-graded ring. Then R_0 is a ring.*

Proof. First note that $1 \in R_0$ since if $r \in R_i$, the $1 \cdot r = r \in R_i$. If $r, s \in R_0$, then also $rs \in R_0$. It follows that R_0 is an abelian group equipped with a multiplication map with identity $1 \in R_0$. This multiplication map satisfies all of the properties which are required for R_0 to be a ring since it inherits these properties from R. □

We are mostly interested in the case where $H = \mathbb{N}^n$ or $H = \mathbb{N}$[2]. Whenever we write, "let R be an H-graded ring", then it is understood that H is an additive semigroup with identity 0. If we omit H and simply write "let R be a graded ring", then it is understood that R is an $\mathbb{N}$-graded ring.

It is wrong to think of an H-grading of R as a map $|\cdot| \colon R \backslash \{0\} \to H$ be a map such that

$$|rs| = |r| + |s|$$

[1] Unless otherwise specified, all rings discussed in this document are assumed to be commutative and unital.
[2] Our convention is that $\mathbb{N} = \{0, 1, 2, \dots\}$.

whenever $rs \neq 0$. Indeed, usually there are many nonzero elements $r \in R$ where $|r|$ is not defined. What we can say however is that there exists nonzero elements $r_{h_1}, \dots r_{h_n}$, where $r_{h_k} \in R_{h_k}$ for all $1 \leq k \leq n$ and $h_i \neq h_j$ for all $1 \leq i < j \leq n$, such that r can be expressed *uniquely* as

$$r = r_{h_1} + \cdots + r_{h_n}. \tag{1}$$

The qualifier "uniquely" here means that if we have another expression for r, say

$$r = r_{h'_1} + \cdots + r_{h'_{n'}}$$

where $r_{h'_{k'}} \in R_{h'_{k'}} \backslash \{0\}$ for all $1 \leq k' \leq n'$ and $h'_{i'} \neq h'_{j'}$ for all $1 \leq i' < j' \leq n'$, then we must have $n = n'$ and, after reordering if necessary, we must have $r_{h_k} = r_{h'_k}$ for all $1 \leq k \leq n$. We call (1) the **decomposition of r into its homogeneous parts**.

2.1.1 Examples of Graded Rings

Example 2.1. Let R be any ring, then $R_0 := R$ and $R_i := 0$ for all $i > 0$ defines a trivial structure of a graded ring for R. This grading is called the **trivial grading** and we say R is a **trivially graded ring**. Whenever we introduce a ring without specifying any grading, then we assume R is equipped with the trivial grading unless otherwise specified.

Sometimes we speak of a graded ring as a **ring equipped with an H-grading**. If R is a ring, then it is possible for R to have both an H-grading and an H'-grading. Here is an example of this:

Example 2.2. Let R be a ring and let $\boldsymbol{x} = x_1, \dots, x_n$ be a list of indeterminates. Then $R[\boldsymbol{x}]$ is both an $\mathbb{N}$-graded ring and an $\mathbb{N}^n$-graded ring. The homogeneous component in degree i in the $\mathbb{N}$-grading is given by

$$R[\boldsymbol{x}]_i = \sum_{|\boldsymbol{\alpha}|=i} R\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

The homogeneous component in degree $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ in the $\mathbb{N}^n$-grading is given by

$$R[\boldsymbol{x}]_{\boldsymbol{\alpha}} = R\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

2.2 Graded R-Modules

Let R be an H-graded ring. An **H-graded** R-module M is an R-module together with a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

into abelian groups M_h which satisfies the condition that if $r_{h_1} \in R_{h_1}$ and $u_{h_2} \in M_{h_2}$, then $r_{h_1}u_{h_2} \in M_{h_1+h_2}$ for all $h_1, h_2 \in H$. The u_h are called **homogeneous components** of M and the elements of M_h are called **homogeneous elements** of **degree** h. Whenever we write "let M be an H-graded R-module", then it is assumed that R is an H-graded ring. In the usual case, R will be an $\mathbb{Z}$-graded ring with $R_i = 0$ for all $i < 0$ and M will be a $\mathbb{Z}$-graded R-module. In this case, we will just say "let M be a graded R-module".

Definition 2.2. Let M be an H-graded R-module. For each $h \in H$, we define the **hth twist of** M, denoted $M(h)$, to be the H-graded R-module whose h'th homogeneous component is given by $M(h)_{h'} := M_{h+h'}$ for all $i \in \mathbb{Z}$.

2.2.1 Graded R-Submodules

Lemma 2.1. *Let M be a graded R-module and $N \subset M$ be a submodule. The following conditions are equivalent:*

1. *N is graded R-module whose homogeneous components are $M_i \cap N$.*
2. *N can be generated by homogeneous elements.*

Proof. We first show that 1 implies 2. Let $x \in N$. Since N is graded with homogeneous components $M_i \cap N$, there exists homogeneous elements $x_{i_k} \in M_{i_k} \cap N$ for $1 \leq k \leq n$ such that

$$x = x_{i_1} + \cdots + x_{i_n}.$$

In particular, N can be generated by homogeneous elements.

Now we show that 2 implies 1. Let $\{y_\alpha\}$ be a set of homogeneous generators for N and let $x \in N$. Since $N \subset M$, we can uniquely decompose x as a sum of homogeneous elements, $x = \sum x_i$, where each $x_i \in M$. We need to show that each $x_i \in N$. To do this, note that $x = \sum r_\alpha y_\alpha$ where r_α belongs to R. If we take ith homogeneous components, we find that

$$x_i = \sum (r_\alpha)_{i-\deg y_\alpha} y_\alpha,$$

where $(r_\alpha)_{i-\deg y_\alpha}$ refers to the homogeneous component of y_α concentrated in the degree $i - \deg y_\alpha$. From this it is easy to see that each x_i is a linear combination of the y_α and consequently lies in N. □

Definition 2.3. A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (2.1) is called a **graded** (or **homogeneous**) submodule. A graded submodule of a graded ring is called a **graded** (or **homogeneous**) **ideal**.

Example 2.3. Consider the graded ring $R = k[x, y, z]_{(5,6,15)}$. Then the ideal $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ is a homogeneous ideal in R.

Remark. Let R be a graded ring and let I be a homogeneous ideal in R. Then the quotient ring R/I has an induced structure as a graded ring, where the ith homogeneous component of R/I is

$$(R/I)_i := (R_i + I)/I \cong R_i/(I \cap R_i)$$

Proposition 2.2. *Let $\mathfrak{p} \subset R$ be a homogeneous ideal. In order that $\mathfrak{p}$ be prime, it is necessary and sufficient that whenever x, y are homogeneous elements such that $xy \in \mathfrak{p}$, then at least one of $x, y \in \mathfrak{p}$.*

Proof. Necessity is immediate. For sufficiency, suppose $a, b \in R$ and $ab \in \mathfrak{p}$. We must prove that one of these is $\mathfrak{p}$. Write

$$a = a_{i_1} + \cdots + a_{i_m} \quad \text{and} \quad b = b_{j_1} + \cdots + b_{j_n}$$

as a decomposition into homogeneous components where a_{k_m} and a_{k_n} are nonzero and of the highest degree.

We will prove that one of $a, b \in \mathfrak{p}$ by induction on $m + n$. When $m + n = 2$, then it is just the condition of the lemma. Suppose it is true for smaller values of $m + n$. Then ab has highest homogeneous component $a_{i_m} b_{j_n}$, which must be in $\mathfrak{p}$ by homogeneity. Thus one of a_{i_m}, b_{j_n} belongs to $\mathfrak{p}$, say for definiteness it is a_{i_m}. Then we have

$$(a - a_{i_m})b \equiv ab \equiv 0 \bmod \mathfrak{p}$$

so that $(a - a_{i_m})b \in \mathfrak{p}$. But the resolutions of $a - a_{i_m}$ and b have a smaller $m + n$ value: $a - a_{i_m}$ can be expressed with $m - 1$ terms. By the inductive hypothesis, it follows that one of these is in $\mathfrak{p}$, and since $a_{i_m} \in \mathfrak{p}$, we find that one of $a, b \in \mathfrak{p}$. □

2.3 Homomorphisms of Graded R-Modules

Definition 2.4. Let M and N be graded R-modules. A homomorphism $\varphi \colon M \to N$ is called **homogeneous** (or **graded**) of degree j if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If φ is homogeneous of degree zero then we will simply say φ is **homogeneous**.

Example 2.4. Consider the graded ring $R = k[X, Y, Z, W]$. Then the matrix

$$U := \begin{pmatrix} X + Y + Z & W^2 - X^2 & X^3 \\ 1 & X & XY + Z^2 \end{pmatrix}$$

defines a homomorphism $U \colon R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$ which is graded of degree zero.

Example 2.5. Let R be a graded ring and let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

be an $n \times m$ matrix with entries $a_{ij} \in R_{\pi(i,j)}$ where $\pi(i, j) \in \mathbb{N}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Can we realize $A \colon R^m \to R^n$ as the matrix representation of a graded homomorphism between free R-modules? This answer is no. Indeed, consider the free R-modules F and F' generated by e_1, e_2 and e_1', e_2' respectively. Let $\varphi \colon F \to G$ be the unique R-linear map such that

$$\begin{aligned} \varphi(e_1) &= a_{11} e_1' + a_{21} e_2' \\ \varphi(e_2) &= a_{12} e_1' + a_{22} e_2' \end{aligned}$$

where $a_{11} \in R_1$, $a_{12} \in R_2$, $a_{21} \in R_3$, and $a_{22} \in R_5$. Then φ has matrix representation with respect to these bases as

$$[\varphi] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

but this is not graded. Indeed, the system of equations

$$\begin{aligned}\varphi(e_1) &= a_{11}e'_1 + a_{21}e'_2 \\ \varphi(e_2) &= a_{12}e'_1 + a_{22}e'_2\end{aligned}$$

gives us the system of equations

$$\begin{aligned}\deg(e_1) &= 1 + \deg(e'_1) \\ \deg(e_1) &= 2 + \deg(e'_2) \\ \deg(e_2) &= 3 + \deg(e'_1) \\ \deg(e_2) &= 5 + \deg(e'_2),\end{aligned}$$

but not such solution exists.

Definition 2.5. Let R and S be graded rings. A ring homomorphism $\varphi\colon R \to S$ is said to be **graded** if it respects the grading. Thus if $a \in R_i$, then $\varphi(a) \in S_i$.

Example 2.6. Let $\varphi\colon K[x,y,z]_{(1,2,3)} \to K[x,y,z]$ be the unique ring homomorphism map such that $\varphi(x) = x$, $\varphi(y) = y^2$, and $\varphi(z) = z^3$. Then φ is a graded ring isomorphism onto its image $K[x,y^2,z^3]$. Indeed, the inverse $\psi\colon K[x,y^2,z^3] \to K[x,y,z]_{(1,2,3)}$ is the unique ring homomorphism such that $\psi(x) = x$, $\psi(y^2) = y$, and $\psi(z^3) = z$.

2.4 Category of all Graded R-Modules

2.4.1 Products in the Category of Graded R-Modules

Let Λ be a set and let M_λ be a graded R-module for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ denote the homogeneous component of M_λ in degree i by $M_{\lambda,i}$. If Λ is finite, then

$$\begin{aligned}\prod_{\lambda\in\Lambda} M_\lambda &= \prod_{\lambda\in\Lambda}\bigoplus_{i\in\mathbb{Z}} M_{\lambda,i} \\ &\cong \bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i}.\end{aligned}$$

Therefore, if Λ is finite, we may view $\prod_\lambda M_\lambda$ as a graded R-module whose homogeneous component in degree i is $\prod_\lambda M_{\lambda,i}$. On the other hand, if Λ is infinite, then we only have an injective map

$$\bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i} \to \prod_{\lambda\in\Lambda}\bigoplus_{i\in\mathbb{Z}} M_{\lambda,i}.$$

In particular, $\prod_\lambda M_\lambda$ is not the correct product in $\mathbf{Grad}_R$. The correct product is **graded product**, given by the graded R-module

$$\prod^{\star}_{\lambda\in\Lambda} M_\lambda := \bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i}$$

together with its projection maps $\pi_\lambda\colon \prod^\star_\lambda M_\lambda \to M_\lambda$ for all $\lambda \in \Lambda$. A homogeneous element of degree i in $\prod^\star_\lambda M_\lambda$ is a sequence of the form $(u_{\lambda,i})_\lambda$ where $u_{\lambda,i} \in M_{\lambda,i}$ for all $\lambda \in \Lambda$. Thus any element in $\prod^\star_\lambda M_\lambda$ can be expressed as a finite sum of the form

$$(u_{\lambda,i_1} + u_{\lambda,i_2} + \cdots + u_{\lambda,i_n})$$

where we often assume without loss of generality that $i_1 < i_2 < \cdots < i_n$.

Let us check that this is in fact the correct product in $\mathbf{Grad}_R$. To show that the pair $(\prod^\star_\lambda M_\lambda, \pi_\lambda)$ is the correct product we have to show it satisfies the universal property: for any other such pair (M, ψ_λ), where M is a graded R-module and $\psi_\lambda\colon M \to M_\lambda$ are graded R-linear maps, there is a unique graded R-linear map $\psi\colon M \to \prod^\star_\lambda M_\lambda$ such that $\pi_\lambda\psi = \psi_\lambda$ for all $\lambda \in \Lambda$. So let (M, ψ_λ) be such a pair. We define $\psi\colon M \to \prod^\star_\lambda M_\lambda$ by

$$\psi(u) = (\psi_\lambda(u))$$

for $u \in M_i$. Clearly ψ is a graded R-linear map since ψ_λ is a graded R-linear map for each $\lambda \in \Lambda$. Moreover, for all $u \in M_i$, we have

$$\begin{aligned}(\pi_\lambda\psi)(u) &= \pi_\lambda(\psi(u)) \\ &= \pi_\lambda((\psi_\lambda(u)) \\ &= \psi_\lambda(u).\end{aligned}$$

This implies $\pi_\lambda \psi = \psi_\lambda$. This establishes existence of ψ. For uniqueness, suppose $\widetilde{\psi} \colon M \to \prod_\lambda^\star M_\lambda$ is another such map. Then for all $u \in M_i$, we have

$$\begin{aligned} \widetilde{\psi}(u) = \psi(u) &\iff \pi_\lambda(\widetilde{\psi}(u)) = \pi_\lambda(\psi(u)) \text{ for all } \lambda \in \Lambda \\ &\iff (\pi_\lambda \widetilde{\psi})(u) = (\pi_\lambda \psi)(u) \text{ for all } \lambda \in \Lambda \\ &\iff \psi_\lambda(u) = \psi_\lambda(u) \text{ for all } \lambda \in \Lambda. \end{aligned}$$

It follows that $\widetilde{\psi} = \psi$.

2.4.2 Inverse Systems and Inverse Limits in the Category Graded R-Modules

Definition 2.6. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). An **inverse system** $(M_\lambda, \varphi_{\lambda\mu})$ of graded R-modules and graded R-linear maps over Λ consists of a family of graded R-modules $\{M_\lambda\}$ indexed by Λ and a family of graded R-linear maps $\{\varphi_{\lambda\mu} \colon M_\mu \to M_\lambda\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\lambda\mu}\varphi_{\mu\kappa}.$$

We say the pair (M, ψ_λ) is **compatible** with the inverse system $(M_\lambda, \varphi_{\lambda\mu})$ if

$$\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$$

for all $\lambda \leq \mu$.

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi \colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of inverse systesms consists of a collection of graded R-linear maps $\psi_\lambda \colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\mu = \psi_\lambda \varphi_{\lambda\mu}.$$

Proposition 2.3. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The inverse limit of this system, denoted $\overset{\star}{\varprojlim} M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\overset{\star}{\varprojlim} M_\lambda = \left\{ (u_\lambda) \in \prod_{\lambda \in \Lambda}^{\star} M_\lambda \mid \varphi_{\lambda\mu}(u_\mu) = u_\lambda \text{ for all } \lambda \leq \mu \right\}$$

together with the projection maps

$$\pi_\lambda \colon \overset{\star}{\varprojlim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\overset{\star}{\varprojlim} M_\lambda$ is given by

$$(\overset{\star}{\varprojlim} M_\lambda)_i = \varprojlim M_{\lambda,i}.$$

Remark. We put a $\star$ above $\varprojlim$ to remind ourselves that this is the inverse limit in the category of all graded R-modules. In the category of all R-modules, the inverse limit is denoted by $\varprojlim M_\lambda$. If Λ is finite, then $\lim M_\lambda$ already has a natural interpretation of a graded R-module.

Proof. We need to show that $\overset{\star}{\varprojlim} M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the invserse system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$ for all $\lambda \leq \mu$. By the universal mapping property of the graded product, there exists a unique graded R-linear map $\psi \colon M \to \prod_\lambda^\star M_\lambda$ such that $\pi_\lambda \psi = \psi_\lambda$ for all $\lambda \in \Lambda$. In fact, this map lands in $\overset{\star}{\varprojlim} M_\lambda$ since

$$\begin{aligned} \varphi_{\lambda\mu}\pi_\mu\psi(u) &= \varphi_{\lambda\mu}\psi_\mu(u) \\ &= \psi_\lambda(u) \\ &= \pi_\lambda\psi(u) \end{aligned}$$

for all $u \in M$. This establishes existence and uniqueness, and thus $\overset{\star}{\varprojlim} M_\lambda$ satisfies the universal mapping property. $\square$

2.4.3 Pullbacks in the Category of Graded R-Modules

Here is an interesting example of a limit in the case where Λ is finite. Let $\psi\colon N \to M$ and $\varphi\colon P \to M$ be graded R-linear maps. The **pullback of** $\psi\colon N \to M$ **and** $\varphi\colon P \twoheadrightarrow M$ is defined to be graded R-module

$$N \times_M P = \{(u,v) \in N \times P \mid \psi(u) = \varphi(v)\}$$

endowed with the projection maps

$$\pi_1\colon N \times_M P \to N \quad \text{and} \quad \pi_2\colon N \times_M P \to P.$$

One can check that the pullback satisfies the universal mapping property of the system

$$\begin{tikzcd} & P \arrow[d, "\varphi"] \\ N \arrow[r, "\psi"'] & M \end{tikzcd}$$

Thus there exists a *unique* isomorphism from $N \times_M P$ to the limit of this system which makes everything commute.

2.4.4 Pullbacks Preserves Surjective Maps

Proposition 2.4. *Let $\varphi_{13}\colon M_3 \to M_1$ and $\varphi_{12}\colon M_2 \to M_1$ be graded R-linear maps. Consider their pullback*

$$\begin{tikzcd} M_3 \times_{M_1} M_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\varphi_{12}"] \\ M_3 \arrow[r, "\varphi_{13}"'] & M_1 \end{tikzcd}$$

1. *If both φ_{12} and φ_{13} are injective, then both π_1 and π_2 are injective.*
2. *If φ_{12} is surjective, then π_1 is surjective. Similarly, if φ_{13} is surjective, then π_2 is surjective.*

Proof. 1. Suppose both φ_{12} and φ_{13} are injective. We want to show that π_1 is injective. Let $(u_3, u_2) \in \ker \pi_1$. So $(u_3, u_2) \in M_3 \times_{M_1} M_2$, which means $\varphi_{13}(u_3) = \varphi_{12}(u_2)$, and $\pi_1(u_3, u_2) = 0$, which means $u_3 = 0$. Thus

$$\begin{aligned} \varphi_{12}(u_2) &= \varphi_{13}(u_3) \\ &= \varphi_{13}(0) \\ &= 0. \end{aligned}$$

Since φ_{12} is injective, this implies $u_2 = 0$, which implies $\varphi_{13}(u_3) = 0$. Since φ_{12} is injective, this implies $u_3 = 0$.

2. Suppose φ_{12} is surjective. We want to show that π_1 is surjective. Let $u_3 \in M_3$. Using the fact that φ_{12} is surjective, we choose a lift of $\varphi_{13}(u_3)$ with respect to φ_{12}, say $u_2 \in M_2$. So $\varphi_{12}(u_2) = \varphi_{13}(u_3)$, but this means $(u_3, u_2) \in M_3 \times_{M_1} M_2$, which implies π_1 is surjective since $\pi_1(u_3, u_2) = u_3$. The proof that φ_{13} surjective implies π_2 surjective follows in a similar manner.

□

2.4.5 Coproducts in the Category of Graded R-Modules

Let Λ be a set and let M_λ be a graded R-module for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ denote the homogeneous component of M_λ in degree i by $M_{\lambda,i}$. Then observe that

$$\begin{aligned} \bigoplus_{\lambda \in \Lambda} M_\lambda &= \bigoplus_{\lambda \in \Lambda} \bigoplus_{i \in \mathbb{Z}} M_{\lambda,i} \\ &\cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{\lambda \in \Lambda} M_{\lambda,i}. \end{aligned}$$

Therefore $\bigoplus_\lambda M_\lambda$ has a natural interpretation as a graded R-module with the homogeneous component in degree i being given by $\bigoplus_\lambda M_{\lambda,i}$. One can check that $\bigoplus_\lambda M_\lambda$ together with the inclusion maps $\iota_\lambda\colon M_\lambda \to \bigoplus_\lambda M_\lambda$ is the correct coproduct in $\mathbf{Grad}_R$.

2.4.6 Direct Systems and Direct Limits in the Category of Graded R-Modules

Definition 2.7. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). A **direct system** $(M_\lambda, \varphi_{\lambda\mu})$ of graded R-modules and graded R-linear maps over Λ consists of a family of graded R-modules $\{M_\lambda\}$ indexed by Λ and a family of graded R-linear maps $\{\varphi_{\lambda\mu} \colon M_\lambda \to M_\mu\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\mu\kappa}\varphi_{\lambda\mu}.$$

If $(\Lambda, \leq)$ is also directed set, then we say $(M_\lambda, \varphi_{\lambda\mu})$ is a **directed system**. We say the pair (M, ψ_λ) is **compatible** with the inverse system $(M_\lambda, \varphi_{\lambda\mu})$ if

$$\psi_\mu \varphi_{\lambda\mu} = \psi_\lambda$$

for all $\lambda \leq \mu$.

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi \colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of direct systems consists of a collection of graded R-linear maps $\psi_\lambda \colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\lambda = \psi_\mu \varphi_{\lambda\mu}.$$

The morphism ψ induces a graded R-linear map $\varinjlim \psi_\lambda \colon \varinjlim M_\lambda \to \varinjlim M'_\lambda$ uniquely determined by

$$\varinjlim \psi_\lambda(\overline{u_\lambda}) = \overline{\psi_\lambda(u_\lambda)}$$

for all $u_\lambda \in M_\lambda$ for all $\lambda \in \Lambda$.

Proposition 2.5. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The* ***direct limit*** *of this system, denoted $\varinjlim M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\varinjlim M_\lambda := \bigoplus_{\lambda \in \Lambda} M_\lambda / \langle \{ (\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda) \mid u_\lambda \in M_\lambda \text{ and } \lambda \leq \mu \} \rangle$$

together with the inclusion maps

$$\iota_\lambda \colon M_\lambda \to \varinjlim M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\varinjlim M_\lambda$ is given by

$$(\varinjlim M_\lambda)_i = \varinjlim M_{\lambda,i}.$$

Proof. First observe that the submodule

$$\langle \{ (\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda) \mid u_\lambda \in M_\lambda \text{ and } \lambda \leq \mu \} \rangle$$

of $\bigoplus_\lambda M_\lambda$ is generated by homogeneous elements. Indeed, for any $(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda)$, we express u_λ into its homogeneous parts, say

$$u_\lambda = u_{\lambda,i_1} + \cdots + u_{\lambda,i_n},$$

then since $\iota_\lambda - \iota_\mu \varphi_{\lambda\mu}$ is a graded R-linear map, we have

$$\begin{aligned}(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda) &= (\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_{\lambda,i_1} + \cdots + u_{\lambda,i_n}) \\ &= (\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_{\lambda,i_1}) + (\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_{\lambda,i_n}),\end{aligned}$$

where each $(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_{\lambda,i_m})$ is homogeneous. Thus any such $(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda)$ can be expressed as a sum of finitely many homogeneous terms. It follows that $\varinjlim M_\lambda$ has a natural graded R-module structure.

We need to show that $\varinjlim M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the direct system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\lambda = \psi_\mu$ for all $\lambda \leq \mu$. By the universal mapping property of the coproduct, there exists a unique graded R-linear map $\psi \colon \bigoplus_\lambda M_\lambda \to M$ such that $\psi \iota_\lambda = \psi_\lambda$ for all $\lambda \in \Lambda$. In fact, this map induces a well-defined graded R-linear map $\overline{\psi} \colon \varinjlim M_\lambda \to M$ since

$$\begin{aligned}\psi(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_\lambda) &= \psi\iota_\lambda(u_\lambda) - \psi\iota_\mu \varphi_{\lambda\mu}(u_\lambda) \\ &= \psi_\lambda(u_\lambda) - \psi_\mu \varphi_{\lambda\mu}(u_\lambda) \\ &= \psi_\lambda(u_\lambda) - \psi_\lambda(u_\lambda) \\ &= 0\end{aligned}$$

for all $u_\lambda \in M_\lambda$ and $\lambda \in \Lambda$. $\square$

Proposition 2.6. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a directed system of graded R-modules and graded R-linear maps.*

1. *Each element of $\varinjlim M_\lambda$ has the form $\overline{u_\lambda}$ for some $u_\lambda \in M_\lambda$.*

2. *$\overline{u_\lambda} = 0$ if and only if $\varphi_{\lambda\mu}(u_\lambda) = 0$ for some $\lambda \leq \mu$.*

Proof. 1. An element in $\varinjlim M_\lambda$ has the form $\overline{u_{\lambda_1} + \cdots + u_{\lambda_n}}$, where $\lambda_1, \dots, \lambda_n \in \Lambda$ and $u_{\lambda_i} \in M_{\lambda_i}$ for all $1 \leq i \leq n$. Since Λ is directed, there exists a $\lambda \in \Lambda$ such that $\lambda_i \leq \lambda$ for all $1 \leq i \leq n$. Then we have

$$\begin{aligned}
\overline{u_{\lambda_1} + \cdots + u_{\lambda_n}} &= \overline{u_{\lambda_1}} + \cdots + \overline{u_{\lambda_n}} \\
&= \overline{\varphi_{\lambda_1,\lambda}(u_{\lambda_1})} + \cdots + \overline{\varphi_{\lambda_n,\lambda}(u_{\lambda_n})} \\
&= \overline{\varphi_{\lambda_1,\lambda}(u_{\lambda_1}) + \cdots + \varphi_{\lambda_n,\lambda}(u_{\lambda_n})} \\
&= \overline{u_\lambda},
\end{aligned}$$

where $u_\lambda = \varphi_{\lambda_1,\lambda}(u_{\lambda_1}) + \cdots + \varphi_{\lambda_n,\lambda}(u_{\lambda_n})$. Each $\varphi_{\lambda_i,\lambda}(u_{\lambda_i})$ lands in M_λ, so $u_\lambda \in M_\lambda$.

2. If $\varphi_{\lambda\mu}(u_\lambda) = 0$ for some $\lambda \leq \mu$, then $\overline{u_\lambda} = \overline{\varphi_{\lambda\mu}(u_\lambda)} = 0$. Conversely, suppose $\overline{u_\lambda} = 0$. Then we have

$$u_\lambda = u_{\lambda_1} - \varphi_{\lambda_1\mu_1}(u_{\lambda_1}) + \cdots + u_{\lambda_n} - \varphi_{\lambda_n\mu_n}(u_{\lambda_n})$$

for some $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \Lambda$ and $u_{\lambda_i} \in M_{\lambda_i}$ for all $1 \leq i \leq n$. Choose $\mu \in \Lambda$ such that $\lambda, \lambda_i, \mu_i \leq \mu$ for all $1 \leq i \leq n$. Then

$$\begin{aligned}
\varphi_{\lambda\mu}(u_\lambda) &= \varphi_{\lambda\mu}(u_\lambda) - u_\lambda + u_\lambda \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \varphi_{\lambda_1\mu}u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n} + \varphi_{\lambda_n\mu}u_n - \varphi_{\lambda_n\mu}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}\varphi_{\mu_1\mu}u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n} + \varphi_{\lambda_n\mu_1}\varphi_{\mu_1\mu}u_n - \varphi_{\lambda_n\mu}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \varphi_{\lambda_1\mu_1}(\varphi_{\mu_1\mu}u_{\lambda_1} - u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu}u_{\lambda_n}) + \varphi_{\lambda_n\mu_n}(\varphi_{\mu_n\mu}u_n - u_{\lambda_n}) \\
&= \varphi_{\lambda\mu}(u_\lambda) - u_{\lambda_1} + \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \cdots - u_{\lambda_n} + \varphi_{\lambda_n\mu_n}u_{\lambda_n} + u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}(\varphi_{\mu_1\mu}u_{\lambda_1} - u_{\lambda_1}) + \cdots + u_{\lambda_n} - \varphi_{\lambda_n\mu}u_{\lambda_n} + \varphi_{\lambda_n\mu_n}(\varphi_{\mu_n\mu}u \\
&= \varphi_{\lambda\mu}(u_\lambda) + \cdots - \varphi_{\lambda_1\mu}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}\varphi_{\mu_1\mu}u_{\lambda_1} + \cdots - \varphi_{\lambda_n\mu}u_{\lambda_n} + \varphi_{\lambda_n\mu_n}\varphi_{\mu_n\mu}u_n
\end{aligned}$$

□

2.4.7 Taking Directed Limits is an Exact Functor

Proposition 2.7. *Let*

$$0 \longrightarrow (M_\lambda, \varphi_\lambda) \xrightarrow{\psi} (M'_\lambda, \varphi'_\lambda) \xrightarrow{\psi'} (M''_\lambda, \varphi''_\lambda) \longrightarrow 0$$

be a short exact sequence of directed systems of graded R-modules and graded R-linear maps. Then

$$0 \longrightarrow \varinjlim M_\lambda \xrightarrow{\varinjlim \psi_\lambda} \varinjlim M'_\lambda \xrightarrow{\varinjlim \psi'_\lambda} \varinjlim M_\lambda \longrightarrow 0$$

is a short exact sequence of graded R-modules and graded R-linear maps.

Proof. We first show $\varinjlim \psi_\lambda$ is injective. Let $\overline{u_\lambda} \in \varinjlim M_\lambda$ and suppose $\overline{\psi_\lambda u_\lambda} = 0$. Then there exists $\mu \geq \lambda$ such that $\varphi'_{\lambda\mu}\psi_\lambda u_\lambda = 0$. In other words,

$$\begin{aligned}
0 &= \varphi'_{\lambda\mu}\psi_\lambda u_\lambda \\
&= \psi_\mu \varphi_{\lambda\mu} u_\lambda.
\end{aligned}$$

This implies $\varphi_{\lambda\mu}u_\lambda = 0$ since ψ_μ is injective. Thus

$$\begin{aligned}
\overline{u_\lambda} &= \overline{\varphi_{\lambda\mu}u_\lambda} \\
&= 0.
\end{aligned}$$

So $\varinjlim \psi_\lambda$ is injective. Next we show exactness at $\varinjlim M'_\lambda$. Let $\overline{u'_\lambda} \in \varinjlim M'_\lambda$ and suppose $\overline{\psi'_\lambda u'_\lambda} = 0$. Then there exists $\mu \geq \lambda$ such that $\varphi''_{\lambda\mu}\psi'_\lambda u'_\lambda = 0$. In other words,

$$\begin{aligned} 0 &= \varphi''_{\lambda\mu}\psi'_\lambda u'_\lambda \\ &= \psi'_\mu \varphi'_{\lambda\mu} u'_\lambda. \end{aligned}$$

This implies $\varphi'_{\lambda\mu}u'_\lambda = \psi_\mu u_\mu$ for some $u_\mu \in M_\mu$, by exactness at $(M'_\lambda, \varphi'_\lambda)$. Thus

$$\begin{aligned} \overline{u'_\lambda} &= \overline{\varphi'_{\lambda\mu}u'_\lambda} \\ &= \overline{\psi_\mu u_\mu}. \end{aligned}$$

This implies exactness at $\varinjlim M'_\lambda$. Exactness at $\varinjlim M''_\lambda$ is easy and is left as an exercise. $\square$

2.4.8 Contravariant Hom Converts Direct Limits to Inverse Limits

Proposition 2.8. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-linear module. Then there exists an isomorphism*

2.4.9 Tensor Products

Let M and N be graded R-modules. As R-modules, their tensor product is given by

$$\begin{aligned} M \otimes_R N &= \left(\bigoplus_{i \in \mathbb{Z}} M_i \right) \otimes \left(\bigoplus_{j \in \mathbb{Z}} N_j \right) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} (M_i \otimes N_j) \\ &= \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j} \right). \end{aligned}$$

In particular, $M \otimes_R N$ has a natural interpretation as a graded R-module with the homogeneous component in degree i given by

$$(M \otimes_R N)_i = \bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j}.$$

Indeed, if $x \in M_i$, $y \in N_j$, and $a \in R_k$, then

$$a(x \otimes y) = ax \otimes y = x \otimes ay \in (M \otimes_R N)_{i+j+k}.$$

So the grading is preserved upon R-scaling.

2.4.10 Graded Hom

Unlike the case of tensor products, hom does not have a natural interpretation as a graded R-module Instead we consider the graded version of hom: let M and N be graded R-modules. Their **graded hom**, denoted $\mathrm{Hom}^\star_R(M, N)$, is the graded R-module whose homogeneous component in degree i is

$$\mathrm{Hom}^\star_R(M, N)_i = \{\text{graded homomorphisms } \alpha \colon M \to N \text{ of degree } i\}.$$

Observe that we have a natural inclusion of R-modules

$$\mathrm{Hom}^\star_R(M, N) \subseteq \mathrm{Hom}_R(M, N).$$

In particular, many properties which $\mathrm{Hom}_R(M, N)$ satisfies are inherited by $\mathrm{Hom}^\star_R(M, N)$.

2.4.11 Graded Hom Properties

Proposition 2.9. *Let M be a graded R-module, let Λ be a set, and let N_λ be a graded R-module for each $\lambda \in \Lambda$. Then we have natural isomorphisms*

$$\mathrm{Hom}^\star_R\left(M, \prod^\star_{\lambda \in \Lambda} N_\lambda \right) \cong \prod^\star_{\lambda \in \Lambda} \mathrm{Hom}^\star_R(M, N_\lambda) \quad \textit{and} \quad \mathrm{Hom}^\star_R\left(\bigoplus_{\lambda \in \Lambda} M_\lambda, - \right) \cong \prod^\star_{\lambda \in \Lambda} \mathrm{Hom}^\star_R(M_\lambda, -)$$

Proof. Let $i \in \mathbb{Z}$. Define a map $\Psi\colon \operatorname{Hom}_R^\star\left(M, \prod_{\lambda\in\Lambda} N^\lambda\right)_i \to \prod_{\lambda\in\Lambda} \operatorname{Hom}_R^\star\left(M, N^\lambda\right)_i$ by

$$\Psi(\varphi) = (\pi_\lambda\varphi)_{\lambda\in\Lambda}$$

for all $\varphi \in \operatorname{Hom}_R^\star\left(M, \prod_{\lambda\in\Lambda} N^\lambda\right)_i$, where $\pi_\lambda\colon \prod_{\lambda\in\Lambda} N^\lambda \to N^\lambda$ is the projection to the λth coordinate. We claim that Ψ is a graded isomorphism.

We first check that it is R-linear. Let $a, b \in R$ and $\varphi, \psi \in \operatorname{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}\Psi(a\varphi + b\psi) &= (\pi_i \circ (a\varphi + b\psi)) \\ &= (a(\pi_i \circ \varphi) + b(\pi_i \circ \psi)) \\ &= a(\pi_i \circ \varphi) + b(\pi_i \circ \psi) \\ &= a\Psi(\varphi) + b\Psi(\psi).\end{aligned}$$

Thus Ψ is R-linear. To show that Ψ is an isomorphism, we construct its inverse. Let $(\varphi_i) \in \prod_{i\in I} \operatorname{Hom}_R(M, N_i)$. Define $\Phi((\varphi_i))\colon M \to \prod_{i\in I} N_i$ by

$$\Phi((\varphi_i))(x) := (\varphi_i(x))$$

for all $x \in M$. Then clearly Φ and Ψ are inverse to each other. Indeed, let $\varphi \in \operatorname{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}\Phi(\Psi(\varphi))(x) &= \Phi((\pi_i \circ \varphi))(x) \\ &= ((\pi_i \circ \varphi)(x)) \\ &= \varphi(x)\end{aligned}$$

for all $x \in M$. Thus $\Phi(\Psi(\varphi)) = \varphi$. Conversely, let $(\varphi_i) \in \prod_{i\in I} \operatorname{Hom}_R(M, N_i)$. Then

$$\begin{aligned}\Psi(\Phi(\varphi_i)) &= (\pi_i \circ \Phi(\varphi_i)) \\ &= (\pi_i \circ \varphi)) \\ &= \varphi(x)\end{aligned}$$

Finally, note that Ψ is graded since π_λ is graded of degree 0 for all $\lambda \in \Lambda$. □

In fact we can generalize the above proposition as follows:

Proposition 2.10. *Let $(\Lambda, \leq)$ be a preordered set, let $(M_\lambda, \phi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps over Λ and let $(N_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over Λ. Then we have natural isomorphisms*

$$\operatorname{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda) \cong \overset{\star}{\varprojlim} \operatorname{Hom}_R^\star(M, N_\lambda) \quad \text{and} \quad \operatorname{Hom}_R^\star(\overset{\star}{\varprojlim} M_\lambda, N) \cong \varinjlim \operatorname{Hom}_R^\star(M_\lambda, N)$$

Proof. Let $i \in \mathbb{Z}$. Define a map $\Psi\colon \operatorname{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda)_i \to \overset{\star}{\varprojlim} \operatorname{Hom}_R^\star(M, N_\lambda)_i$ by

$$\Psi(\varphi) = (\pi_\lambda\varphi)$$

for all $\varphi \in \operatorname{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda)_i$, where π_λ is the projection to the λth coordinate. Observe that Ψ lands in $\overset{\star}{\varprojlim} \operatorname{Hom}_R^\star(M, N_\lambda)_i$ since $\pi_\mu\varphi = \varphi_{\lambda\mu}\pi_\lambda\varphi$ for all $\lambda \leq \mu$. We claim that Ψ is a graded isomorphism.

We first check that it is R-linear. Let $a, b \in R$ and $\varphi, \psi \in \operatorname{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}\Psi(a\varphi + b\psi) &= (\pi_i \circ (a\varphi + b\psi)) \\ &= (a(\pi_i \circ \varphi) + b(\pi_i \circ \psi)) \\ &= a(\pi_i \circ \varphi) + b(\pi_i \circ \psi) \\ &= a\Psi(\varphi) + b\Psi(\psi).\end{aligned}$$

Thus Ψ is R-linear. To show that Ψ is an isomorphism, we construct its inverse. Let $(\varphi_i) \in \prod_{i\in I} \operatorname{Hom}_R(M, N_i)$. Define $\Phi((\varphi_i))\colon M \to \prod_{i\in I} N_i$ by

$$\Phi((\varphi_i))(x) := (\varphi_i(x))$$

for all $x \in M$. Then clearly Φ and Ψ are inverse to each other. Indeed, let $\varphi \in \operatorname{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}\Phi(\Psi(\varphi))(x) &= \Phi((\pi_i \circ \varphi))(x) \\ &= ((\pi_i \circ \varphi)(x)) \\ &= \varphi(x)\end{aligned}$$

for all $x \in M$. Thus $\Phi(\Psi(\varphi)) = \varphi$. Conversely, let $(\varphi_i) \in \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$. Then

$$\begin{aligned}\Psi(\Phi(\varphi_i)) &= (\pi_i \circ \Phi(\varphi_i)) \\ &= (\pi_i \circ \varphi)) \\ &= \varphi(x)\end{aligned}$$

Finally, note that Ψ is graded since π_λ is graded of degree 0 for all $\lambda \in \Lambda$. □

2.4.12 Left Exactness of $\mathrm{Hom}^\star_R(M, -)$ and $\mathrm{Hom}^\star_R(-, N)$

Let M and N be graded R-modules. Recall that both $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$ are left exact functors from the category of R-modules to itself. The graded version of these functors are

$$\mathrm{Hom}^\star_R(M, -)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R \quad \text{and} \quad \mathrm{Hom}^\star_R(-, N)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R.$$

We want to check that they are also left exact functors. Let's focus on $\mathrm{Hom}^\star_R(-, N)$ first:

Proposition 2.11. *The sequence of graded R-modules and graded homomorphisms*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{2}$$

is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}^\star_R(M_3, N) \xrightarrow{\varphi_2^*} \mathrm{Hom}^\star_R(M_2, N) \xrightarrow{\varphi_1^*} \mathrm{Hom}^\star_R(M_1, N) \tag{3}$$

is exact.

Proof. Suppose that (28) is exact and let N be any R-module. Exactness at $\mathrm{Hom}^\star_R(M_3, N)$ follows from the fact that φ_2^* is injective (which follows from the fact that $\mathrm{Hom}_R(-, N)$ is left exact). Next we show exactness at $\mathrm{Hom}^\star_R(M_2, N)$. Let $\psi_2\colon M_2 \to N$ be a graded homomorphism of degree i such that $\psi_2\varphi_1 = 0$. By left exactness of $\mathrm{Hom}_R(-, N)$, there exists a $\psi_3 \in \mathrm{Hom}_R(M, N)$ such that $\psi_2 = \psi_3\varphi_2$. Since φ_2 is surjective, ψ_3 is graded of degree i. Thus $\psi_3 \in \mathrm{Hom}^\star_R(M, N)$. Thus we have exactness at $\mathrm{Hom}^\star_R(M_2, N)$. □

2.4.13 Projective Objects and Injective Objects in Grad_R

$\mathrm{Hom}^\star_R(\bigoplus_\lambda P_\lambda, B) \cong \prod_\lambda \mathrm{Hom}^\star_R(P_\lambda, B)$ and $\mathrm{Hom}^\star_R(A, \prod^\star_\lambda E_\lambda) \cong \prod^\star_\lambda \mathrm{Hom}^\star_R(A, E_\lambda)$.

2.5 Noetherian Graded Rings and Modules

2.5.1 The Irrelevant Ideal

Definition 2.8. Let R be a graded ring. The **irrelevant ideal of** R is defined to be

$$R_+ := \bigoplus_{i>0} R_i.$$

It is straightforward to check that R_+ is in fact an ideal of R and that $R/R_+ \cong R_0$.

2.5.2 Noetherian Graded Rings

The following lemma will be used many times without mention.

Lemma 2.2. *Let R be a ring and let $S \subseteq R$. Suppose the ideal $\langle S\rangle$ generated by S is finitely generated. Then we can choose the generators to be in S.*

Proof. Since $\langle S\rangle$ is finitely generated, there are $x_1, \dots, x_n \in \langle S\rangle$ such that $\langle S\rangle = \langle x_1, \dots, x_n\rangle$. In particular we have

$$x_i = \sum_{j=1}^{n_i} r_{ji}s_{ji}$$

where for each $1 \le i \le n$ we have $n_i \in \mathbb{N}$, and for each $1 \le j \le n_i$ we have $r_{ji} \in R$ and $s_{ji} \in S$. In particular, this means

$$\langle S\rangle = \langle s_{ji} \mid 1 \le i \le n \text{ and } 1 \le j \le n_i\rangle.$$

□

Definition 2.9. A **Noetherian** graded ring is a graded ring whose underlying ring is Noetherian.

Proposition 2.12. *Let R be a graded ring. Suppose $R_+ = \langle \{x_\lambda\}_{\lambda \in \Lambda} \rangle$. Then the R_0-algebra map*

$$\varphi \colon R_0[\{X_\lambda\}] \to R$$

given by $\varphi(X_\lambda) = x_\lambda$ for all $\lambda \in \Lambda$ is surjective. In other words, if a subset $S \subset R_+$ generates the irrelevant ideal R_+ as an R-ideal, then it generates R as an R_0-algebra.

Proof. It suffices to show that $R_k \subset \operatorname{im} \varphi$ for all $k \in \mathbb{N}$. We prove this by induction on k. The base case $k = 0$ is trivial. Now suppose it is true for all $i < k$ for some $k > 0$ and let $a \in R_k$. Since $R = R_0 \oplus R_+$, we have a unique decomposition

$$a = a_0 + x$$

where $a_0 \in R_0$ and $x \in R_+$. Since $R_+ = \langle \{x_\lambda\} \rangle$ and $x \in R_+$, there exists $x_{\lambda_1}, \dots, x_{\lambda_n} \in \{x_\lambda\}$ and $a_m \in R_{k - \deg x_{\lambda_m}}$ for all $1 \le m \le n$ such that

$$x = a_1 x_{\lambda_1} + \cdots + a_n x_{\lambda_n}.$$

Choose $A_m \in R_0[\{X_\lambda\}]$ such that $\varphi(A_m) = a_m$ for all $0 \le m \le n$ (we can do this by induction). Then

$$\begin{aligned} a &= a_0 + a_1 x_{\lambda_1} + \cdots + a_n x_{\lambda_n} \\ &= \varphi(A_0) + \varphi(A_1)\varphi(X_{\lambda_1}) + \cdots + \varphi(A_n)\varphi(X_{\lambda_n}) \\ &= \varphi(A_0 + A_1 X_{\lambda_1} + \cdots + A_n X_{\lambda_n}). \end{aligned}$$

This implies $R_k \subset \operatorname{im} \varphi$. Therefore φ is surjective. □

Proposition 2.13. *Let R be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely-generated as an R_0-algebra.*

Proof. Suppose R_0 is Noetherian and R is finitely-generated as an R_0-algebra. Then there exists an $n \ge 0$ and a surjection

$$R_0[X_1, \dots, X_n] \to R.$$

where $R_0[X_1, \dots, X_n]$ is a polynomial algebra over Noetherian ring, and hence Noetherian, which implies that R is Noetherian, as it is a quotient of a Noetherian ring.

Now suppose R is Noetherian. Since $R_0 \cong R/R_+$, we see that R_0 must be Noetherian since it is the quotient of a Noetherian ring. Since R is Noetherian, the irrelevant ideal R_+ is finitely-generated, say by $x_1, \dots, x_n \in R_+$. Since R is graded, we have a surjective R_0-algebra map

$$R_0[X_1, \dots, X_n] \to R$$

sending $X_i \mapsto x_i$ for all $1 \le i \le n$. It follows that R is a finitely-generated R_0-algebra. □

2.6 Localization of Graded Rings

Definition 2.10. If $S \subset R$ is a multiplicative subset of a graded ring R consisting of homogeneous elements, then $S^{-1}R$ is a $\mathbb{Z}$-graded ring: we let the homogeneous elements of degree n be of the form r/s where $r \in R_{n + \deg s}$. We write $R_{(S)}$ for the subring of elements of degree zero; there is thus a map $R_0 \to R_{(S)}$.

If S consists of the powers of a homogeneous element f, we write $R_{(f)}$ for R_S. If $\mathfrak{p}$ is a homogeneous ideal and S is the set of homogeneous elements of R not in $\mathfrak{p}$, we write $R_{(\mathfrak{p})}$ for $R_{(S)}$.

More generally if M is a graded R-module, then we define $M_{(S)}$ to be the submodule of $S^{-1}M$ consisting of elements of degree zero. When S consists of powers of a homogeneous element $f \in R$, then we write $M_{(f)}$ instead of $M_{(S)}$. We similarly define $M_{(\mathfrak{p})}$ for a homogeneous prime ideal $\mathfrak{p}$.

2.7 Graded R-Algebras

An R-algebra A is an R-module equipped with an R-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a + b)c = ab + ac \quad \text{and} \quad a(b + c) = ab + ac.$$

We say the R-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the R-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

Unless otherwise specified, all R-algebras discussed are assumed to be associative and unital, so they are genuinely rings (perhaps not commutative) and being an R-algebra just means they have a little extra structure related to scaling by R. If A is an R-algebra, then can view R as sitting inside A via the map $\varphi \colon R \to A$, given by

$$\varphi(r) = 1 \cdot r$$

for all $r \in R$, though this map need not be injective.

Definition 2.11. An H-**graded** R-**algebra** A is an R-algebra which is also H-graded as a ring. So there is a direct sum decomposition

$$A = \bigoplus_{h \in H} A_h,$$

where the A_h are abelian groups which satisfy the property that if $a_{h_1} \in A_{h_1}$ and $a_{h_2} \in A_{h_2}$, then $a_{h_1}a_{h_2} \in A_{h_1+h_2}$. If R is also an H-graded ring, then we also require A to be an H-graded left R-module. This means that if $r_{h_1} \in R_{h_1}$ and $a_{h_2} \in A_{h_2}$, then $r_{h_1}a_{h_2} \in A_{h_1+h_2}$.

2.7.1 Examples of Graded R-Algebras

Example 2.7. Let R be a graded ring and let $\boldsymbol{x} = x_1, \dots, x_n$. The polynomial ring $R[\boldsymbol{x}]$ over R is both an $\mathbb{N}$-graded R-algebra and an $\mathbb{N}^n$-graded R-algebra. The homogeneous component in degree i with respect to the $\mathbb{N}$-grading is given by

$$R[\boldsymbol{x}]_i = \sum_{\boldsymbol{\alpha}} R_{i-|\boldsymbol{\alpha}|}\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

The homogeneous component in degree $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ with respect to the $\mathbb{N}^n$-grading is given by

More generally, let $\boldsymbol{w} := (w_1, \dots, w_n)$ be an n-tuple of positive integers. We define the **weighted degree of a monomial** of a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, denoted $\deg_w(\boldsymbol{x}^{\boldsymbol{\alpha}})$, by the formula

$$\deg_w(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

The **weighted polynomial ring with respect to the weighted vector** $\boldsymbol{w}$, denoted $R[\boldsymbol{x}]^{\boldsymbol{w}}$, is the polynomial ring $R[\boldsymbol{x}]$ equipped with the **weighted grading**: the homogeneous component in degree i is given by

$$R[\boldsymbol{x}]_i^{\boldsymbol{w}} = \sum_{\boldsymbol{\alpha}} R_{i-\langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle}\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Example 2.8. Let K be a field, let $R = K[x,y]/\langle xy \rangle$, and let $A = R[z,w]$. View R as a graded K-algebra with $|x| = 1$ and $|y| = 2$ and view A as a graded R-algebra with $|z| = 1$ and $|w| = 3$. Then the homogeneous components of A start out as

$$\begin{aligned}
A_0 &= K \\
A_1 &= K\overline{x} + Kz \\
A_2 &= K\overline{x}^2 + K\overline{x}z + K\overline{y} \\
A_3 &= K\overline{x}^3 + K\overline{x}^2 z + K\overline{xy} + K\overline{x}z^2 + K\overline{y}z + Kw \\
&\vdots
\end{aligned}$$

Example 2.9. Let R be a ring and let Q be an ideal in R. The **blowup algebra of** Q **in** R is defined by

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong \bigoplus_{i=0}^{\infty} Q^i.$$

Elements in $B_Q(R)$ have the form

$$t^{i_1}x_{i_1} + \cdots + t^{i_m}x_{i_m}$$

where $0 \leq i_1 < \cdots < i_m$ and $x_{i_\lambda} \in Q^{i_\lambda}$ for all $1 \leq \lambda \leq m$. The t^{i_λ} part keeps track of what degree we are in. We define multiplaction on elements of the form $t^i x$ and $t^j y$ by

$$(t^i x)(t^j y) = t^{i+j}xy,$$

and we extend this to all of $B_Q(R)$ in the obvious way. This gives $B_Q(R)$ the structure of a graded R-algebra.

If Q is finitely generated, say $Q = \langle a_1, \dots, a_n \rangle$, then there is a unique R-algebra homomorphism

$$\varphi \colon R[u_1, \dots, u_n] \to B_Q(R),$$

such that $\varphi(u_\lambda) = ta_\lambda$ for all $1 \leq \lambda \leq n$.

2.7.2 Graded Associative R-Algebras

Let R be a ring and let $\boldsymbol{x} = x_1, \dots, x_n$ be a list of indeterminates. We denote by $R\langle \boldsymbol{x} \rangle$ to be the **free R-algebra generated by $\boldsymbol{x}$**. A basis of $R\langle \boldsymbol{x} \rangle$ as an R-module consists of **words**:

$$\boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k}$$

where $k \in \mathbb{N}$ and $\boldsymbol{\alpha}_j \in \mathbb{N}^n$ for all $1 \leq j \leq k$. For example, in $R\langle x_1, x_2, x_3 \rangle$, we have

$$\boldsymbol{x}^{\boldsymbol{\alpha}_1} \boldsymbol{x}^{\boldsymbol{\alpha}_2} \boldsymbol{x}^{\boldsymbol{\alpha}_3} = x_3^2 x_1^3 x_2 x_3 x_2,$$

where

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (0,0,2) \\ \boldsymbol{\alpha}_2 &= (3,2,1) \\ \boldsymbol{\alpha}_3 &= (0,1,0). \end{aligned}$$

The set of all words is denoted $\mathrm{W}(\boldsymbol{x})$. Words of the form $\boldsymbol{x}^{\boldsymbol{\alpha}}$ are called **standard words** and form a subset of the set of all words. A **standard polynomial** in $R\langle \boldsymbol{x} \rangle$ is a finite linear combination of standard words.

Example 2.10. Let R be a graded ring, let $\boldsymbol{x} = x_1, \dots, x_n$ be a list of indeterminates, and let $\boldsymbol{w} := (w_1, \dots, w_n)$ be an n-tuple of positive integers. We define $R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}}$ to be the graded R-algebra whose homogeneous component in degree i is given by

$$R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} = \sum_{\boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k} \in \mathrm{W}(\boldsymbol{x})} R_{i - \sum_{j=1}^k \langle \boldsymbol{w}, \boldsymbol{\alpha}_j \rangle} \boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k}.$$

2.7.3 Graded Commutative R-Algebras

Definition 2.12. Let A be a $\mathbb{Z}$-graded R-algebra. We say A is **graded-commutative** if for all $a \in A_i$ and $b \in A_j$, we have

$$ab = (-1)^{ij} ba. \tag{4}$$

We say A is **strictly graded-commutative** if, an addition to (4), we also have $a^2 = 0$ for all odd degree elements $a \in A$.

Remark. Cohomology rings are a natural source of graded-commutative rings.

Every finitely-presented R-algebra A is isomorphic to $R\langle \boldsymbol{x} \rangle / I$ where $\boldsymbol{x} = x_1, \dots, x_n$ and where I is a two-sided ideal in $R\langle \boldsymbol{x} \rangle$. For our purposes we will be interested in the following finitely-presented R-algebra.

Definition 2.13. Let R be a ring, let $\boldsymbol{x} = x_1, \dots, x_n$ be indeterminates, and let $\boldsymbol{w} = (w_1, \dots, w_n)$ be their respective weights. Set

$$J = \langle \{ fg - (-1)^{ij} gf \mid f \in R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} \text{ and } g \in R\langle \boldsymbol{x} \rangle_j^{\boldsymbol{w}} \} \cup \{ f^2 \mid f \in R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} \text{ where } i \text{ is odd} \rangle.$$

We define the **free graded-(strictly)-commutative R-algebra generated by $\boldsymbol{x}$ with respect to the weighted vector $\boldsymbol{w}$**, denoted $R\lceil \boldsymbol{x} \rceil_{\boldsymbol{w}}$, to be the graded R-algebra

$$R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}} := R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}} / J.$$

Since $x_\lambda x_\mu - (-1)^{w_\lambda w_\mu} x_\mu x_\lambda \in J$ for all $1 \leq \lambda < \mu \leq n$, we see that every $\overline{f} \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$ can be represented by a standard polynomial $f \in R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}}$. We typically dispense with the overline notation and just write $f \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$. In particular, any $f \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$ can be expressed as

$$f = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$$

where the sum ranges over all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $r_{\boldsymbol{\alpha}} = 0$ for almost all $\boldsymbol{\alpha} \in \mathbb{N}^n$.

2.8 Hilbert Function and Dimension

The Hilbert function of a graded module associates to an integer i the dimension of the ith graded part of the given module. For sufficiently large i, the values of this function are given by a polynomial, the Hilbert polynomial.

Definition 2.14. Let R be a Noetherian graded K-algebra and let M be a finitely-generated graded R-module. The **Hilbert function** $H_M\colon \mathbb{Z} \to \mathbb{Z}$ of M is defined by

$$H_M(i) := \dim_K(M_i)$$

Lemma 2.3. *Let R be a Noetherian graded ring and let $i \in \mathbb{Z}$. Then R_i is a finitely-generated R_0-module.*

Proof. The ideal $\langle R_i \rangle$ is finitely-generated since R is Noetherian. Choose generators in $\langle R_i \rangle$ such that each generator belongs to R_i, say $x_1, \dots, x_n \in R_i$. In particular, $\langle R_i \rangle$ is a graded ideal with $\langle R_i \rangle_0 = R_i$. It follows that

$$R_i = R_0 x_1 + \cdots + R_0 x_n,$$

and so R_i is a finitely-generated R_0-module. □

Corollary. *Let R be a Noetherian graded ring and let M be a finitely-generated graded R-module. Then M_i is a finitely-generated R_0-module for all $i \in \mathbb{Z}$. Moreover, there exists $k \in \mathbb{Z}$ such that $M_j = 0$ for all $j < \mathbb{Z}$.*

Proof. Choose homogeneous generators of M, say $u_1, \dots, u_n$, and let $i \in \mathbb{Z}$. Then

$$M_i = R_{i-\deg(u_1)} u_1 + \cdots + R_{i-\deg(u_n)} u_n.$$

This implies that M_i is a finitely-generated R_0-module since the R_i's are finitely generated R_0-modules by Lemma (2.3).

For the moreover part, let

$$k = \min\{\deg(u_i) \mid 1 \le i \le n\}.$$

Then $M_j = 0$ for all $j < k$ since $R_i = 0$ for all $i < 0$. □

2.9 Semigroup Ordering

Definition 2.15. Let H be an additive semigroup with identity 0. A **semigroup ordering** on H is a partial ordering $>$ on H such that

1. $>$ is a total ordering, i.e. either $h_1 > h_2$ or $h_2 > h_1$ for all $h_1, h_2 \in H$.
2. $>$ is translate invariant, i.e. $h_1 > h_2$ implies $h_1 + h_3 > h_2 + h_3$ for all $h_1, h_2, h_3 \in H$.

If $>$ is a semigroup ordering on H, then we call the pair $(H, >)$ an **additive ordered semigroup**.

Example 2.11. The integers $\mathbb{Z}$ (or the natural numbers $\mathbb{N}$) equipped with the natural order $>$ forms an additive ordered semigroup.

Example 2.12. For $n > 1$, there are many different semigroup orderings we can equip $\mathbb{N}^n$ (or even $\mathbb{Z}^n$). For example, one of them is call **lexicographical ordering**, which is defined as follows: for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, we say $\boldsymbol{\alpha} >_{\text{lex}} \boldsymbol{\beta}$ if for some $1 \le i \le n$ we have

$$\begin{aligned} \alpha_1 &= \beta_1 \\ &\vdots \\ \alpha_{i-1} &= \beta_{i-1} \\ \alpha_i &> \beta_i \end{aligned}$$

Theorem 2.4. *Let $(H, >)$ be an additive ordered semigroup, let R be a Noetherian H-graded ring, and let M be a Noetherian H-graded R-module. Then every associated prime $\mathfrak{p}$ of M is a homogeneous ideal.*

Proof. If $\mathfrak{p}$ is an associated prime of M, it is the annihilator of a nonzero element

$$u = u_{j_1} + \cdots + u_{j_t} \in M,$$

where the u_{j_v} are nonzero homogeneous elements of degrees $j_1 < \cdots < j_t$. Choose u such that t is as small as possible. Suppose that

$$a = a_{i_1} + \cdots + a_{i_s}$$

kills u, where for every ν, a_{i_ν} has degree i_ν, and $i_1 < \cdots < i_s$. We shall show that every a_{i_ν} kills u, which proves that $\mathfrak{p}$ is homogeneous. It suffices to show that a_{i_1} kills u (since $a - a_{i_1}$ kills u and we can proceed by induction). Since $au = 0$, the unique least degree term $a_{i_1}u_{j_1} = 0$. Therefore

$$u' = a_{i_1}u = a_{i_1}u_{j_2} + \cdots + a_{i_1}u_{j_t}.$$

If this element is nonzero, its annihilator is still $\mathfrak{p}$, since $Ru \cong R/\mathfrak{p}$ and every nonzero element has annihilator $\mathfrak{p}$. Since $a_{i_1}u_{j_\nu}$ is homogeneous of degree $i_1 + j_\nu$, or else is 0, u' has fewer nonzero homogeneous components than u does, contradicting our choice of u. □

Corollary. *If I is a homogeneous ideal of a Noetherian ring R graded by a semigroup H equipped with a semigroup ordering $>$, then every minimal prime of I is homogeneous.*

Proof. This is immediate, since the minimal primes of I are among the associated primes of R/I. □

Proposition 2.14. *Let $(H, >)$ be an additive ordered semigroup, let R be a H-graded ring, and let I be a homogeneous ideal. Then $\sqrt{I}$ is homogeneous.*

Proof. Let

$$f_{i_1} + \cdots + f_{i_k} \in \sqrt{I}$$

with $i_1 < \cdots < i_k$ and each f_{i_j} nonzero of degree i_j. We need to show that every $f_{i_j} \in \sqrt{I}$. If any of the components are in $\sqrt{I}$, we may subtract them off, giving a similar sum whose terms are the homogeneous components not in $\sqrt{I}$. Therefore it suffices to show that $f_{i_1} \in \sqrt{I}$. But

$$\left(f_{i_1} + \cdots + f_{i_k}\right)^N \in I$$

for some $N > 0$. When we expand, there is a unique term formally of least degree, namely $f_{i_1}^N$, and therefore this term is in I, since I is homogeneous. But this means that $f_{i_1} \in \sqrt{I}$, as required. □

Corollary. *Let R be a finitely-generated graded K-algebra and let $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$ be the homogeneous maximal ideal of R. Then*

$$\dim R = \operatorname{height} \mathfrak{m} = \dim R_{\mathfrak{m}}.$$

Proof. The dimension of R will be equal to the dimension of $R/\mathfrak{p}$ for one of the minimal primes $\mathfrak{p}$ of R. Since $\mathfrak{p}$ is minimal, it is an associated prime and therefore is homogeneous. Hence, $\mathfrak{p} \subseteq \mathfrak{m}$. The domain $R/\mathfrak{p}$ is finitely-generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathfrak{m}/\mathfrak{p}$. Thus,

$$\begin{aligned}
\dim R &= \dim R/\mathfrak{p} \\
&= \dim(R/\mathfrak{p})_{\mathfrak{m}} \\
&\leq \dim R_{\mathfrak{m}} \\
&\leq \dim R,
\end{aligned}$$

and so equality holds throughout, as required. □

3 Homological Algebra

Throughout this section, let R be a ring (trivially graded).

3.1 R-Complexes

3.1.1 R-Complexes and Chain Maps

Definition 3.1. An R**-complex** (A, d) is a graded R-module A equipped with graded R-linear map $\mathrm{d}\colon A \to A$ of degree -1 such that $\mathrm{d}^2 = 0$. Any such map d which satisfies these properties is called an R**-linear differential**. If we denote the ith homogeneous component of A as A_i and if we denote $\mathrm{d}_i = \mathrm{d}|_{A_i}$, then we may view an R-complex as a sequence of R-modules A_i and R-linear maps $\mathrm{d}_i\colon A_i \to A_{i-1}$ as below

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\mathrm{d}_{i+1}} A_i \xrightarrow{\mathrm{d}_i} A_{i-1} \longrightarrow \cdots \tag{5}$$

such that $\mathrm{d}_i\mathrm{d}_{i+1} = 0$ for all $i \in \mathbb{Z}$. An element in $\ker \mathrm{d}$ is called a **cycle** of (A, d) and an element in $\operatorname{im} \mathrm{d}$ is called a **boundary** of (A, d).

A **chain map** $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ between R-complexes (A, d) and (A', d') is a graded R-linear map $\varphi\colon A \to A'$ of degree 0 which commutes with the differentials:

$$\mathrm{d}'\varphi = \varphi\mathrm{d}.$$

If we denote $\varphi_i = \varphi|_{A_i}$, then we may view φ as a sequence of R-linear maps $\varphi_i\colon A_i \to A'_i$ as below

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "\mathrm{d}_{i+1}"] \arrow[d, "\varphi_{i+1}"] & A_i \arrow[r, "\mathrm{d}_i"] \arrow[d, "\varphi_i"] & A_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \\
\cdots \arrow[r] & A'_{i+1} \arrow[r, "\mathrm{d}'_{i+1}"] & A'_i \arrow[r, "\mathrm{d}'_i"] & A'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $\mathrm{d}'_i\varphi_i = \varphi_{i-1}\mathrm{d}'_i$ for all $i \in \mathbb{Z}$. It is easy to check that the identity map $1_{(A,\mathrm{d})}\colon (A, \mathrm{d}) \to (A, \mathrm{d})$ from an R-complex (A, d) to itself is a chain map. It is also easy to check that the composition of two chain maps is a chain map. We obtain the category $\mathbf{Comp}_R$, whose objects are R-complexes and whose morphisms chain maps.

Remark. To simplify notation, we often write A instead of (A, d) if the differential is understood from context. For instance, we may introduce an R-complex as "(A, d)" but later refer to it as "A", but we also may introduce an R-complex as "A" with the differential understood to be denoted "d_A". In that case, we will denote $\mathrm{d}_{A,i} = (\mathrm{d}_A)|_{A_i}$. Also a chain map is always understood to be a map between R-complexes. For instance, if we write "let $\varphi\colon A \to A'$ be a chain map" without first introducing A or A', then it is understood that A and A' are R-complexes.

3.1.2 Homology

Let (A, d) be an R-complex. The condition $\mathrm{d}^2 = 0$ is equivalent to the condition $\ker \mathrm{d} \supseteq \operatorname{im} \mathrm{d}$. Since d is graded, we see that both $\ker \mathrm{d}$ and $\operatorname{im} \mathrm{d}$ are graded submodules of A. Therefore we have

$$\ker \mathrm{d} = \bigoplus_{i \in \mathbb{Z}} \ker \mathrm{d}_i \quad \text{and} \quad \operatorname{im} \mathrm{d} = \bigoplus_{i \in \mathbb{Z}} \operatorname{im} \mathrm{d}_i,$$

and for each $i \in \mathbb{Z}$, we have $\ker \mathrm{d}_i \supseteq \operatorname{im} \mathrm{d}_{i+1}$. Therefore $\ker \mathrm{d}/\operatorname{im} \mathrm{d}$ is a graded R-module. With this in mind, we are justified in making the following definitions:

Definition 3.2. Let (A, d) be an R-complex.

1. We say A is **exact** if $\ker \mathrm{d} = \operatorname{im} \mathrm{d}$ and we say A is **exact at** A_i if $\ker \mathrm{d}_i = \operatorname{im} \mathrm{d}_i$.
2. The **homology** of A is defined to be the graded R-module

$$\mathrm{H}(A, \mathrm{d}) := \ker \mathrm{d}/\operatorname{im} \mathrm{d}.$$

The ith homogeneous component of $\mathrm{H}(A, \mathrm{d})$ is denoted

$$\mathrm{H}_i(A, \mathrm{d}) := \ker \mathrm{d}_i/\operatorname{im} \mathrm{d}_i.$$

Remark. If the differential d is clear from context, then we will simplify our notation by denoting the homology of A as $\mathrm{H}(A)$ rather than $\mathrm{H}(A, \mathrm{d})$.

3.1.3 Positive, Negative, and Bounded Complexes

Definition 3.3. Let A be an R-complex.

1. We say A is **positive** if $A_i = 0$ for all $i < 0$.
2. We say A is **bounded below** if $A_i = 0$ for $i \ll 0$. In other words, if A_i is eventually 0, that is, if there exists $n \in \mathbb{Z}$ such that $A_i = 0$ for all $i < n$.
3. We say A is **homologically bounded below** if $\mathrm{H}_i(A) = 0$ for $i \ll 0$.

Similarly,

1. We say A is **negative** if $A_i = 0$ for all $i > 0$.
2. We say A is **bounded above** if $A_i = 0$ for $i \gg 0$.
3. We say A is **homologically bounded above** if $\mathrm{H}_i(A) = 0$ for $i \gg 0$.

If A is both bounded below and bounded above, then we will say A is **bounded**. Similarly, if A is both homologically bounded above and homologically bounded below, then we will say A is **homologically bounded**.

3.1.4 Supremum and Infimum

Definition 3.4. Let A be an R-complex. We define its **supremum** to be

$$\sup A := \begin{cases} -\infty & \text{if } A \text{ is exact} \\ \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(A) \neq 0\} & \text{if } A \text{ is not exact and is homologically bounded above} \\ \infty & \text{if } A \text{ is not homologically bounded above.} \end{cases}$$

Similarly, we define its **infimum** to be

$$\inf A := \begin{cases} \infty & \text{if } A \text{ is exact} \\ \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(A) \neq 0\} & \text{if } A \text{ is not exact and is homologically bounded below} \\ -\infty & \text{if } A \text{ is not homologically bounded below.} \end{cases}$$

The **amplitude** of A is defined to be

$$\operatorname{amp} A := \begin{cases} -\infty & \text{if } A \text{ is exact} \\ \infty & \text{if } A \text{ is homologically bounded above but not homologically bounded below} \\ \sup A - \inf A & \text{if } A \text{ is not exact and homologically bounded} \\ \infty & \text{if } A \text{ is homologically bounded below but not homologically bounded above} \\ \infty & \text{if } A \text{ is not homologically bounded above or below.} \end{cases}$$

3.2 Category of R-Complexes

The set of all R-complexes together with the set of all chain maps forms a category, which we denote $\mathbf{Comp}_R$. Similarly, the set of all graded R-modules together with the set of all graded homomorphisms (of degree 0) forms a category, which we denote $\mathbf{Grad}_R$.

3.2.1 Homology Considered as a Functor

We've already seen that if (A, d) is an R-complex, then $\mathrm{H}(A)$ is a graded R-module. We would like to extend this observation to get a functor $\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$. This will follow from the following three propositions:

Proposition 3.1. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map. Then φ induces a graded homomorphism $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$, where*

$$\mathrm{H}(\varphi)(\overline{a}) = \overline{\varphi(a)} \tag{6}$$

for all $\overline{a} \in \mathrm{H}(A)$.

Proof. First let us check that the target of each element in $\mathrm{H}(A)$ under $\mathrm{H}(\varphi)$ lands in $\mathrm{H}(A')$. Let $\overline{a} \in \mathrm{H}(A)$ (so $\mathrm{d}(a) = 0$). Then $\overline{\varphi(a)} \in \mathrm{H}(A')$ since

$$\begin{aligned} \mathrm{d}'(\varphi(a)) &= \varphi(\mathrm{d}(a)) \\ &= 0. \end{aligned}$$

Next let us check that that $\mathrm{H}(\varphi)$ is well-defined. Let $a + \mathrm{d}(b)$ be another representative of the coset class $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a + \mathrm{d}(b)}) &= \overline{\varphi(a + \mathrm{d}(b))} \\ &= \overline{\varphi(a) + \varphi(\mathrm{d}(b))} \\ &= \overline{\varphi(a)} + \overline{\varphi(\mathrm{d}(b))} \\ &= \overline{\varphi(a)} + \overline{\mathrm{d}'(\varphi(b))} \\ &= \overline{\varphi(a)} \\ &= \mathrm{H}(\varphi)(\overline{a}). \end{aligned}$$

Thus $\mathrm{H}(\varphi)$ is well-defined.

So far we have shown that $\mathrm{H}(\varphi)$ is a function. To see that $\mathrm{H}(\varphi)$ is an R-module homomorphism, let $r, s \in R$ and $a, b \in A$. Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{ra + sb}) &= \overline{\varphi(ra + sb)} \\ &= \overline{r\varphi(a) + s\varphi(b)} \\ &= r\overline{\varphi(a)} + s\overline{\varphi(b)} \\ &= r\mathrm{H}(\varphi)(\overline{a}) + s\mathrm{H}(\varphi)(\overline{b}). \end{aligned}$$

Finally, to see that $\mathrm{H}(\varphi)$ is graded, let $\overline{a}_i \in \mathrm{H}_i(A)$ (so $a_i \in A_i$). Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a}_i) &= \overline{\varphi(a_i)} \\ &\in \mathrm{H}_i(A') \end{aligned}$$

since φ is graded. □

Proposition 3.2. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ and $\varphi'\colon (A', \mathrm{d}') \to (A'', \mathrm{d}'')$ be two chain maps. Then*

$$\mathrm{H}(\varphi' \circ \varphi) = \mathrm{H}(\varphi') \circ \mathrm{H}(\varphi).$$

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then we have

$$\begin{aligned} \mathrm{H}(\varphi' \circ \varphi)(\overline{a}) &= \overline{(\varphi' \circ \varphi)(a)} \\ &= \overline{\varphi'(\varphi(a))} \\ &= \mathrm{H}(\varphi')(\overline{\varphi(a)}) \\ &= \mathrm{H}(\varphi')(\mathrm{H}(\varphi)(\overline{a})) \\ &= (H(\varphi') \circ H(\varphi))(\overline{a}). \end{aligned}$$

□

Proposition 3.3. *Let (A, d) be an R-complex. Then we have*

$$\mathrm{H}(\mathrm{id}_{(A,\mathrm{d})}) = \mathrm{id}_{\mathrm{H}(A)}.$$

In particular, if $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is a chain map isomorphism, then $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism between graded R-modules $\mathrm{H}(A)$ and $\mathrm{H}(A')$.

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned} \mathrm{H}(\mathrm{id}_{(A,\mathrm{d})})(\overline{a}) &= \overline{\mathrm{id}_{(A,\mathrm{d})}(a)} \\ &= \overline{a} \\ &= \mathrm{id}_{\mathrm{H}(A)}(\overline{a}). \end{aligned}$$

For the latter statement, let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map isomorphism and let $\psi\colon (A', \mathrm{d}') \to (A, \mathrm{d})$ be its inverse. Then

$$\begin{aligned} \mathrm{id}_{\mathrm{H}(A)} &= \mathrm{H}(\mathrm{id}_{(A,d)}) \\ &= \mathrm{H}(\psi \circ \varphi) \\ &= \mathrm{H}(\psi) \circ \mathrm{H}(\varphi). \end{aligned}$$

A similar computation gives $\mathrm{H}(\varphi) \circ \mathrm{H}(\psi) = \mathrm{id}_{\mathrm{H}(A')}$. □

3.2.2 $\mathbf{Comp}_R$ is an R-linear category

There is more structure on the categories $\mathbf{Comp}_R$ and $\mathbf{Grad}_R$ which we haven't discussed so far. They are examples of R-linear categories[3]. Moreover, homology can be viewed as an additive functor from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$.

Proposition 3.4. *$\mathbf{Comp}_R$ is an R-linear category.*

Proof. Let (A, d) and (A', d') be two R-complexes. We define $\mathcal{C}(A, A')$

$$\mathcal{C}(A, A') := \mathrm{Hom}((A, \mathrm{d}), (A', \mathrm{d}')) := \{\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}') \mid \varphi \text{ is a chain map}\}.$$

Then $\mathcal{C}(A, A')$ has the structure of an R-module. Indeed, if $\varphi, \psi \in \mathcal{C}(A, A')$ and $r \in R$, then we define addition and scalar multiplication by

$$(\varphi + \psi)(a) := \varphi(a) + \psi(a) \quad \text{and} \quad (r\varphi)(a) = \varphi(ra)$$

for all $a \in A$. Since d is an R-linear map, it is clear that $\varphi + \psi$ and $r\varphi$ are chain maps (that is, they are graded R-linear maps which commute with the differentials).

Moreover, let (A'', d'') be another R-complex. We define composition

$$\circ\colon \mathcal{C}(A', A'') \times \mathcal{C}(A, A') \to \mathcal{C}(A, A''),$$

in the usual way: if $(\varphi', \varphi) \in \mathcal{C}(A', A'') \times \mathcal{C}(A, A')$, then we define $\varphi' \circ \varphi \in \mathcal{C}(A, A'')$ by

$$(\varphi' \circ \varphi)(a) = \varphi'(\varphi(a))$$

for all $a \in A$. Again one checks that $\varphi' \circ \varphi$ is indeed a chain map. Observe that composition is an R-bilinear map. For instance, let $\varphi', \psi' \in \mathcal{C}(A', A'')$ and $\varphi \in \mathcal{C}(A, A')$. Then

$$\begin{aligned} ((\varphi' + \psi') \circ \varphi)(a) &= (\varphi' + \psi')(\varphi(a)) \\ &= \varphi'(\varphi(a)) + \psi'(\varphi(a)) \\ &= (\varphi' \circ \varphi)(a) + (\psi' \circ \varphi)(a) \end{aligned}$$

for all $a \in A$. Thus $(\varphi' + \psi') \circ \varphi = \varphi' \circ \varphi + \psi' \circ \varphi$. A similar proof gives the other properties of R-bilinearity. □

Remark. To clean notation, we often drop the $\circ$ symbol when denoting composotin. For instance, we often write $\varphi\psi$ rather than $\varphi \circ \psi$.

3.2.3 The inclusion functor from $\mathbf{Grad}_R$ to $\mathbf{Comp}_R$ is fully faithful

Every graded R-module M can be view as an R-complex with differential $\mathrm{d} = 0$. In fact, we obtain a functor

$$\iota\colon \mathbf{Grad}_R \to \mathbf{Comp}_R,$$

where the graded R-module M is mapped to the trivially R-complex $(M, 0)$, and where graded homomorphisms $\varphi\colon M \to M'$ is mapped to the chain map $\varphi\colon (M, 0) \to (M, 0')$ of trivially R-complexes. Clearly φ is in fact chain map since these are trivial R-complexes. The functor ι is full and faithful. It is left-adjoint to the forgetful functor

$$\rho\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$$

where ρ maps the R-complex (M, d) to the graded R-module M, and where ρ maps the chain map $\varphi\colon (M, \mathrm{d}) \to (M', \mathrm{d}')$ to the graded homomorphism $\varphi\colon M \to M'$. Then ρ is still faithful, but it is not full since there may be many graded homomorphism $M \to M'$ which do not come from forgetting a chain map $(M, \mathrm{d}) \to (M', \mathrm{d}')$.

[3]See Appendix for definition of R-linear categories.

3.2.4 The homology functor from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$

There is another functor which goes from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$ which is called the **homology functor**. It is denoted

$$\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R,$$

and is given by mapping an R-complex (M, d) to the graded R-module $\mathrm{H}(M, \mathrm{d})$, and by mapping the chain map $\varphi\colon (M, \mathrm{d}) \to (M', \mathrm{d}')$ to the graded R-linear map $\mathrm{H}(\varphi)\colon \mathrm{H}(M, \mathrm{d}) \to \mathrm{H}(M', \mathrm{d}')$. Let us show that H is an R-linear functor.

Proposition 3.5. *Let $\varphi, \psi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be two chain maps and let $r, s \in R$. Then*

$$\mathrm{H}(r\varphi + s\psi) = r\mathrm{H}(\varphi) + s\mathrm{H}(\psi)$$

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned}
\mathrm{H}(r\varphi + s\psi)(\overline{a}) &= \overline{(r\varphi + s\psi)(a)} \\
&= \overline{r\varphi(a) + s\psi(a)} \\
&= r\overline{\varphi(a)} + s\overline{\psi(a)} \\
&= r\mathrm{H}(\varphi)(a) + s\mathrm{H}(\psi)(a).
\end{aligned}$$

□

3.2.5 Inverse Systems and Inverse Limits in the Category of R-Complexes

Definition 3.5. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). An **inverse system** $(A_\lambda, \varphi_{\lambda\mu})$ of R-complexes and chains maps over Λ consists of a family of R-complexes $\{(A_\lambda, \mathrm{d}_\lambda)\}$ indexed by Λ and a family of chian maps $\{\varphi_{\lambda\mu}\colon A_\mu \to A_\lambda\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\lambda\mu}\varphi_{\mu\kappa}.$$

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi\colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of inverse systesms consists of a collection of graded R-linear maps $\psi_\lambda\colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\mu = \psi_\lambda\varphi_{\lambda\mu}.$$

Proposition 3.6. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The inverse limit of this system, denoted $\overset{\star}{\varprojlim} M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\overset{\star}{\varprojlim} M_\lambda = \left\{ (u_\lambda) \in \prod_{\lambda\in\Lambda}^{\star} M_\lambda \mid \varphi_{\lambda\mu}(u_\mu) = u_\lambda \textit{ for all } \lambda \leq \mu \right\}$$

together with the projection maps

$$\pi_\lambda\colon \overset{\star}{\varprojlim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\overset{\star}{\varprojlim} M_\lambda$ is given by

$$(\overset{\star}{\varprojlim} M_\lambda)_i = \varprojlim M_{\lambda,i}.$$

Remark. We put a $\star$ above $\varprojlim$ to remind ourselves that this is the inverse limit in the category of all graded R-modules. In the category of all R-modules, the inverse limit is denoted by $\varprojlim M_\lambda$. If Λ is finite, then $\lim M_\lambda$ already has a natural interpretation of a graded R-module.

Proof. We need to show that $\overset{\star}{\varprojlim} M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the invserse system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$ for all $\lambda \leq \mu$. By the universal mapping property of the graded product, there exists a unique graded R-linear map $\psi\colon M \to \prod^{\star}_{\lambda} M_\lambda$ such that $\pi_\lambda\psi = \psi_\lambda$ for all $\lambda \in \Lambda$.

In fact, this map lands in $\overset{\star}{\varprojlim} M_\lambda$ since

$$\begin{aligned}\varphi_{\lambda\mu}\pi_\mu\psi(u) &= \varphi_{\lambda\mu}\psi_\mu(u)\\ &= \psi_\lambda(u)\\ &= \pi_\lambda\psi(u)\end{aligned}$$

for all $u \in M$. □

3.2.6 Homology of Inverse Limit

Proposition 3.7. *Let $(A_\lambda, \varphi_{\lambda\mu})$ be an inverse system of R-complexes and chain maps indexed over a preordered set $(\Lambda, \leq)$. Suppose that each $\varphi_{\lambda\mu}$ is surjective and induces a surjective map $\varphi_{\lambda\mu}|_{\mathrm{ker d}_\mu} \colon \mathrm{ker d}_\mu \to \mathrm{ker d}_\lambda$, and suppose that $\mathrm{H}(A_\lambda) = 0$ for all λ. Then*

$$\mathrm{H}(\varprojlim A_\lambda) = 0.$$

Proof. Let $\overline{(a^n)} \in \mathrm{H}(\varprojlim A^n)$. So $\mathrm{d}^n(a^n) = 0$ and $\varphi_{m,n}(a^n) = a^m$ for all $m \leq n$. To show that $\overline{(a^n)} = 0$, we need to construct a sequence (b^n) in $\prod A^n$ such that $\mathrm{d}^n(b^n) = a^n$,
We want to construct a sequence (b_λ) such that

1. $b_\lambda \in A_\lambda$ for all λ
2. $\mathrm{d}_\lambda(b_\lambda) = a_\lambda$ for all λ
3. $\varphi_{\lambda\mu}(b_\mu) = b_\lambda$ for all λ

We will do this by induction on λ. In the base case $\lambda = 1$, we use the fact that $\mathrm{H}(A_1) = 0$ to get $b_1 \in A_1$ such that $\mathrm{d}^1(b^1) = a^1$. Now suppose that for some $n \in \mathbb{N}$, we have constructed $b^m \in A^m$ for all $m \leq n$ such that $\mathrm{d}^m(b^m) = a^m$ and $\varphi_{lm}(b^m) = b^l$ for all $l \leq m \leq n$. Using the fact that $\varphi_{n,n+1}$ is surjective on kernels, we choose $b^{n+1} \in \ker \mathrm{d}^{n+1}$ such that $\varphi_{n,n+1}(b^{n+1}) = b^n$. Observe that for any $m \leq n$, we have

$$\begin{aligned}\varphi_{m,n+1}(b^{n+1}) &= \varphi_{m,n}\varphi_{n,n+1}(b^{n+1})\\ &= \varphi_{m,n}(b^n)\\ &= b^m,\end{aligned}$$

by induction. Using the fact that $\mathrm{H}^{n+1}(A^{n+1}) = 0$, we choose $c^{n+1} \in A^{n+1}$ such that $\mathrm{d}^{n+1}(c^{n+1}) = b^{n+1}$. □

3.2.7 Homology commutes with coproducts

Proposition 3.8. *Let λ be an index set and let $(A_\lambda, \mathrm{d}_\lambda)$ be an R-complex for each $\lambda \in \Lambda$. Then*

$$\mathrm{H}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{H}(A_\lambda).$$

3.2.8 Homology commutes with graded limits

Proposition 3.9. *Let λ be an index set and let $(A_\lambda, \mathrm{d}_\lambda)$ be an R-complex for each $\lambda \in \Lambda$. Then*

$$\mathrm{H}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{H}(A_\lambda).$$

3.3 Homotopy

Definition 3.6. Let φ and ψ be two chain maps between R-complexes (A, d) and (A', d'). We say φ is **homotopic to** ψ if there exists a graded homomorphism $h \colon A \to A'$ of degree 1 such that

$$\varphi - \psi = \mathrm{d}'h + h\mathrm{d}.$$

We call h a **homotopy from** φ **to** ψ. If $\psi = 0$, then we say φ is **null-homotopic**.

3.3.1 Homotopy is an equivalence relation

Proposition 3.10. *Let $\mathcal{C}(A, A')$ denote the set of all chain maps between R-complexes (A, d) and (A', d'). Homotopy gives an equivalence relation on $\mathcal{C}(A, A')$: for two elements $\varphi, \psi \in \mathcal{C}(A, A')$, write $\varphi \sim \psi$ if φ is homotopic to ψ. Then $\sim$ is an equivalence relation.*

Proof. First we show reflexivity. Let $\varphi \in \mathcal{C}(A, A')$. Then the zero map $h = 0$ gives a homotopy from φ to itself.

Next we show symmetry. Let $\varphi, \psi \in \mathcal{C}(A, A')$ and suppose $\varphi \sim \psi$. Choose a homotopy h from φ to ψ. Then $-h$ is a homotopy from ψ to φ.

Finally we show transitivity. Let $\varphi, \psi, \omega \in \mathcal{C}(A, A')$ and suppose $\varphi \sim \psi$ and $\psi \sim \omega$. Choose a homotopy h from φ to ψ and a homotopy h' from ψ to ω. Then

$$\varphi - \psi = \mathrm{d}'h + h\mathrm{d} \qquad \text{and} \qquad \psi - \omega = \mathrm{d}'h' + h'\mathrm{d}.$$

Adding these together gives us

$$\begin{aligned} \varphi - \omega &= \mathrm{d}'h + h\mathrm{d} + \mathrm{d}'h' + h'\mathrm{d} \\ &= \mathrm{d}'(h + h') + (h + h')\mathrm{d}. \end{aligned}$$

Therefore $h + h'$ is a homotopy from φ to ω. □

3.3.2 Homotopy induces the same map on homology

Proposition 3.11. *Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). If φ is homotopic to ψ, then $\mathrm{H}(\varphi) = \mathrm{H}(\psi)$.*

Proof. Showing $\mathrm{H}(\varphi) = \mathrm{H}(\psi)$ is equivalent to showing $\mathrm{H}(\varphi - \psi) = 0$ since H is additive. Thus, we may assume that φ is null-homotopic and that we are trying to show that $\mathrm{H}(\varphi) = 0$. Let $\overline{a} \in \mathrm{H}(A, d)$. Then $\mathrm{H}(a) = 0$, and so

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a}) &= \overline{\varphi(a)} \\ &= \overline{(\mathrm{d}'h + h\mathrm{d})(a)} \\ &= \overline{\mathrm{d}'(h(a)) + h(\mathrm{d}(a))} \\ &= \overline{\mathrm{d}'(h(a))} \\ &= 0. \end{aligned}$$

□

3.3.3 The Homotopy Category of R-Complexes

Recall that $\mathbf{Comp}_R$ is an R-linear category. In particular, this means that for each pair of R-complexes A and A' we have an R-module structure on the set of all chain maps between them. This R-module is denoted by $\mathcal{C}(A, A')$. Moreover the composition map

$$\circ \colon \mathcal{C}(A', A'') \times \mathcal{C}(A, A') \to \mathcal{C}(A, A'')$$

is R-bilinear. For any two R-complexes A and A' let us denote

$$[\mathcal{C}(A, A')] := \mathcal{C}(A, A')/\sim,$$

where $\sim$ is the homotopy equivalence relation. We shall write $[\varphi]$ for the equivalence class in $[\mathcal{C}(A, A')]$ with $\varphi \in \mathcal{C}(A, A')$ as one of its representatives. We want to show that the R-module structure on $\mathcal{C}(A, A')$ induces an R-module structure on $[\mathcal{C}(A, A')]$ and that the composition map $\circ$ induces an R-bilinear map

$$[\circ] \colon [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')] \to [\mathcal{C}(A, A'')].$$

More generally, we define the **homotopy category** of all R-complexes, denoted $\mathbf{HComp}_R$, to be the category whose objects are R-complexes and whose morphisms are homotopy classes of chain maps. The next theorem will prove that this is in fact a well-defined R-linear category.

Theorem 3.1. $\mathbf{HComp}_R$ *is an R-linear category.*

Proof. Let A and A' be R-complexes. We first show that $[\mathcal{C}(A, A')]$ has an induced R-module structure. Let $[\varphi], [\psi] \in [\mathcal{C}(A, A')]$ and let $r, s \in R$. We set

$$r[\varphi] + s[\psi] := [r\varphi + s\psi]. \tag{7}$$

Let us check that (7) is in fact well-defined. Suppose $\varphi \sim \widetilde{\varphi}$ and $\psi \sim \widetilde{\psi}$. Choose a homotopy σ from φ to φ' and choose a homotopy τ from ψ to ψ'. Thus

$$\varphi - \widetilde{\varphi} = \sigma \mathrm{d} + \mathrm{d}'\sigma \quad \text{and} \quad \psi - \widetilde{\psi} = \tau \mathrm{d} + \mathrm{d}'\tau.$$

We claim that $r\sigma + s\tau$ is a homotopy from $r\varphi + s\psi$ to $r\widetilde{\varphi} + s\widetilde{\psi}$. Indeed, $\sigma + \tau$ is a graded R-linear map of degree 1 from A to A'. Moreover, we have

$$\begin{aligned} r\varphi + s\psi - (r\widetilde{\varphi} + s\widetilde{\psi}) &= r(\varphi - \widetilde{\varphi}) + s(\psi - \widetilde{\psi}) \\ &= r(\sigma\mathrm{d} + \mathrm{d}'\sigma) + s(\tau\mathrm{d} + \mathrm{d}'\tau) \\ &= (r\sigma + s\tau)\mathrm{d} + \mathrm{d}'(r\sigma + s\tau). \end{aligned}$$

Thus (7) is well-defined.

Now we will show that composition in $\mathbf{Comp}_R$ induces a well-defined R-bilinear composition operation in $\mathbf{HComp}_R$. Let A, A', and A'' be R-complexes. Let us check that composition map $\circ$ on chain maps induces an R-bilinear composition map on homotopy classes of chain maps:

$$[\circ] \colon [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')] \to [\mathcal{C}(A, A'')].$$

Let $([\varphi'], [\varphi]) \in [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')]$. We define

$$[\circ]([\varphi'], [\varphi]) = [\varphi'\varphi]. \tag{8}$$

Let us check that (8) is in fact well-defined. Suppose $\varphi \sim \psi$ and $\varphi' \sim \psi'$. Choose a homotopy h from φ to ψ and choose a homotopy h' from φ' to ψ'. Thus

$$\varphi - \psi = h\mathrm{d} + \mathrm{d}'h \quad \text{and} \quad \varphi' - \psi' = h'\mathrm{d}' + \mathrm{d}''h'.$$

We claim that $\varphi'h + h'\psi$ is a homotopy from $\varphi'\varphi$ to $\psi'\psi$. Indeed, $\varphi'h + h'\psi$ is a graded R-linear map of degree 1 from A to A''. Moreover we have

$$\begin{aligned} (\varphi'h + h'\psi)\mathrm{d} + \mathrm{d}''(\varphi'h + h'\psi) &= \varphi'h\mathrm{d} + h'\psi\mathrm{d} + \mathrm{d}''\varphi'h + \mathrm{d}''h'\psi \\ &= \varphi'h\mathrm{d} + h'\mathrm{d}'\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'(\varphi - \psi - \mathrm{d}'h) + (\varphi' - \psi' - \mathrm{d}''h')\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'\varphi - \varphi'\psi - \varphi'\mathrm{d}'h + \varphi'\psi - \psi'\psi - \mathrm{d}''h'\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'\varphi - \psi'\psi. \end{aligned}$$

Therefore $\varphi'\varphi \sim \psi'\psi$, and so (8) is well-defined. Observe that R-bilinearity and associativity of (8) follows trivially from R-bilinearity and associativity of composition in $\mathbf{Comp}_R$. Also for each R-complex A, the homotopy class of the identity map 1_A serves as the identity morphism for A in $\mathbf{HComp}_R$, which is easily seen to satisfy the left and right unity laws since 1_A satisfies the left and right unity laws in $\mathbf{Comp}_R$. □

3.3.4 Homotopy equivalences

Definition 3.7. Let $\varphi \colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map. We say φ is a **homotopy equivalence** if there exists a chain map $\varphi' \colon (A', \mathrm{d}') \to (A, \mathrm{d})$ such that $\varphi'\varphi \sim 1_A$ and $\varphi\varphi' \sim 1_{A'}$. In this case, we call φ' a **homotopy inverse** to φ.

Proposition 3.12. *Let $\varphi \colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be an isomorphism of R-complexes with $\varphi' \colon (A', \mathrm{d}') \to (A, \mathrm{d})$ being its inverse. Then both φ is a homotopy equivalence with φ' being a homotopy inverse.*

Proof. Since φ and φ' are inverse to each other, we see that $\varphi'\varphi = 1_A$ and $\varphi\varphi' = 1_{A'}$. In particular, if we take h to be the zero map, then we have

$$\begin{aligned} h\mathrm{d} + \mathrm{d}'h &= 0 \cdot d + \mathrm{d}' \cdot 0 \\ &= 0 \\ &= \varphi'\varphi - 1_A. \end{aligned}$$

Thus $\varphi'\varphi \sim 1_A$. By a similar argument, we also have $\varphi\varphi' \sim 1_{A'}$. □

Remark. Note that a chain map $\varphi \colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is a homotopy equivalence if and only if $[\varphi]$ is an isomorphism.

3.4 Quasiisomorphisms

Definition 3.8. Let $\varphi\colon A \to A'$ be a chain map. We say φ is a **quasiisomorphism** if the induced map in homology $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism of graded R-modules.

3.4.1 Homotopy equivalence is a quasiisomorphism

Proposition 3.13. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a homotopy equivalence with homotopy inverse $\varphi'\colon (A', \mathrm{d}') \to (A, \mathrm{d})$. Then both φ and φ' are quasiisomorphisms.*

Proof. Since $\varphi'\varphi \sim 1_A$ and since homology takes homotopic maps to equal maps, we see that

$$\begin{aligned} 1_{\mathrm{H}(A)} &= \mathrm{H}(1_A) \\ &= \mathrm{H}(\varphi'\varphi) \\ &= \mathrm{H}(\varphi')\mathrm{H}(\varphi). \end{aligned}$$

A similarl calculation gives us $\mathrm{H}(\varphi')\mathrm{H}(\varphi) = 1_{\mathrm{H}(A')}$. Therefore $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism of graded R-modules with $\mathrm{H}(\varphi')\colon \mathrm{H}(A') \to \mathrm{H}(A)$ being its inverse. □

Remark. The converse is not true. That is, there there are many examples quasiisomorphisms which are not homotopy equivalences.

3.4.2 Quasiisomorphism equivalence relation

Definition 3.9. Let A and A' be R-complexes. We A is **quasiisomorphic** to A', denoted $A \sim_q A'$, if there exists R-complexes $A_0, \dots, A_n$ and $B_1, \dots, B_n$ where $A_0 = A$ and $A_n = A'$, together with quasisomorphisms

$$\sigma_m\colon B_m \to A_{m-1} \quad \text{and} \quad \tau_m\colon B_m \to A_m$$

for each $0 < m \leq n$. In terms of arrows, this looks like

$$\begin{tikzcd}
 & B_1 \arrow[dl, "\sigma_1"'] \arrow[dr, "\tau_1"] & & \cdots \arrow[dl] \arrow[dr] & & B_n \arrow[dl, "\sigma_n"'] \arrow[dr, "\tau_n"] & \\
A_0 & & A_1 & & A_{n-1} & & A_n
\end{tikzcd}$$

One can easily check that being quasiisomorphic is an equivalence relation. It turns out that one can easily simplify this equivalence relation quite a bit. This is described in the following proposition.

Proposition 3.14. *Let A and A' be R-complexes. Then A is quasiisomorphic to A' if and only if there exists a semiprojective R-complex P together with quasiisomorphisms $\pi\colon P \to A$ and $\pi'\colon P \to A$.*

Proof. One direction is clear, so it suffices to prove the other direction. Suppose $A \sim_q A'$. Choose R-complexes $A_0, \dots, A_n$ and $B_1, \dots, B_n$ where $A_0 = A$ and $A_n = A'$, together with quasisomorphisms

$$\sigma_m\colon B_m \to A_{m-1} \quad \text{and} \quad \tau_m\colon B_m \to A_m$$

for each $0 < m \leq n$. Choose a semiprojective resolution $\pi_0\colon P \to A_0$ of A_0. Let $\widetilde{\pi}_0\colon P \to B_1$ be a homotopic lift of π_0 with respect to σ_1 and denote $\pi_1 = \tau_1\widetilde{\pi}_0$. We proceed inductively to construct chain maps $\widetilde{\pi}_{m-1}\colon P_m \to B_m$ and $\pi_m\colon P_m \to A_m$ where $\widetilde{\pi}_{m-1}$ is a homotopic lift of π_{m-1} with respect to σ_m and where $\pi_m = \tau_m\widetilde{\pi}_{m-1}$.

We prove by induction on $1 \leq m \leq n$ that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms. First we consider the base case $m = 1$. Observe that $\sigma_1\widetilde{\pi}_0 \sim \pi_0$ implies $\mathrm{H}(\sigma_1)\mathrm{H}(\widetilde{\pi}_0) = \mathrm{H}(\pi_0)$. Then $\mathrm{H}(\widetilde{\pi}_0)$ is an isomorphism since both $\mathrm{H}(\sigma_1)$ and $\mathrm{H}(\pi_0)$ are isomorphisms. Therefore $\widetilde{\pi}_0$ is a quasiisomorphism. Similarly, π_1 is a quasiisomorphisms since it is a composition of quasiisomorphisms.

Now suppose we have shown that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms for some $m < n$. Observe that $\sigma_m\widetilde{\pi}_{m-1} \sim \pi_m$ implies $\mathrm{H}(\sigma_m)\mathrm{H}(\widetilde{\pi}_{m-1}) = \mathrm{H}(\pi_m)$. Then $\mathrm{H}(\widetilde{\pi}_{m-1})$ is an isomorphism since both $\mathrm{H}(\sigma_m)$ and $\mathrm{H}(\pi_m)$ are isomorphisms. Therefore $\widetilde{\pi}_{m-1}$ is a quasiisomorphism. Similarly, π_{m+1} is a quasiisomorphisms since it is a composition of quasiisomorphisms.

Thus we have shown by induction that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms for all $1 \leq m \leq n$. In particular, $\pi_n\colon P \to A_n$ is a quasiisomorphism. □

3.5 Exact Sequences of R-Complexes

Definition 3.10. Let (A, d), (A', d'), and (A'', d'') be R-complexes and let $\varphi\colon A' \to A$ and $\psi\colon A \to A''$ be chain maps. Then we say that

$$0 \longrightarrow (A', \mathrm{d}') \xrightarrow{\varphi} (A, \mathrm{d}) \xrightarrow{\psi} (A'', \mathrm{d}'') \longrightarrow 0$$

is a **short exact sequence** of R-complexes if it is a short exact sequence when considered as graded R-modules. More specifically, this means that following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathrm{d}'_{i+2}"] & \vdots \arrow[d, "\mathrm{d}_{i+2}"] & \vdots \arrow[d, "\mathrm{d}''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "\mathrm{d}'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "\mathrm{d}_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "\mathrm{d}''_{i+1}"] & 0 \\
0 \arrow[r] & A'_{i} \arrow[r, "\varphi_{i}"] \arrow[d, "\mathrm{d}'_{i}"] & A_{i} \arrow[r, "\psi_{i}"] \arrow[d, "\mathrm{d}_{i}"] & A''_{i} \arrow[r] \arrow[d, "\mathrm{d}''_{i}"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "\mathrm{d}'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "\mathrm{d}_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "\mathrm{d}''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

3.5.1 Long exact sequence in homology

Theorem 3.2. *Let*

$$0 \longrightarrow (A', \mathrm{d}') \xrightarrow{\varphi} (A, \mathrm{d}) \xrightarrow{\psi} (A'', \mathrm{d}'') \longrightarrow 0$$

be a short exact sequence of R-complexes. Then there exists a graded homomorphism $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 such that

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_{i+1}(A'') \arrow[dll, rounded corners, "\eth_{i+1}"] \\
 & \mathrm{H}_i(A') \arrow[r, "\mathrm{H}_i(\varphi)"] & \mathrm{H}_i(A) \arrow[r, "\mathrm{H}_i(\psi)"] & \mathrm{H}_i(A'') \arrow[dll, rounded corners, "\eth_i"] \\
 & \mathrm{H}_{i-1}(A') \arrow[r] & \cdots &
\end{tikzcd} \tag{9}$$

is a long exact sequence of R-modules.

Proof. The proof will consists of three steps. The first step is to construct a graded function $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 (graded here just means $\eth(\mathrm{H}_i(A'')) \subseteq \mathrm{H}_{i-1}(A')$ for all $i \in \mathbb{Z}$). The next step will be to show that $\eth$ is R-linear. The final step will be to show exactness of (17).

Step 1: We construct a graded function $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ as follows: let $[a''] \in \mathrm{H}_i(A'')$. Choose a representative of the coset $[a'']$, say $a'' \in A''_i$ (so $\mathrm{d}''(a'') = 0$), and choose a lift of a'' in A_i with respect to ψ, say $a \in A_i$ (so $\psi(a) = a''$). We can make such a choice since ψ is surjective. Since

$$\begin{aligned}
\psi(\mathrm{d}(a)) &= \mathrm{d}''(\psi(a)) \\
&= \mathrm{d}''(a'') \\
&= 0,
\end{aligned}$$

it follows by exactness of (3.8.3) that there exists a unique $a' \in A'_{i-1}$ such that $\varphi(a') = \mathrm{d}(a)$. Observe that $\mathrm{d}'(a') = 0$ since φ is injective and since

$$\begin{aligned}\varphi(\mathrm{d}'(a')) &= \mathrm{d}(\varphi(a')) \\ &= \varphi(\mathrm{d}(a)) \\ &= 0.\end{aligned}$$

Thus a' represents an element in $\mathrm{H}_{i-1}(A')$. We define $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ by

$$\eth[a''] = [a'].$$

We need to verify that $\eth$ is well-defined. There were two choices that we made in constructing $\eth$. The first choice was the choice of a representative of the coset $[a'']$. Let us consider another choice, say $a'' + \mathrm{d}''(b'')$ where $b'' \in A''_{i+1}$ (every representative of the coset $[a'']$ has this form for some $b'' \in A''_{i+1}$). The second choice that we made was the choice of a lift of a'' in A with respect to ψ. This time we have another coset representative of $[a'']$, so let $a + \varphi(b') + \mathrm{d}(b)$ be another choice of a lift of $a'' + \mathrm{d}''(b'')$ with respect to ψ where $b' \in A'_i$ and $b \in A_{i+1}$ (every such choice has this form for some $b' \in A'_i$ and $b \in A_{i+1}$). Now observe that

$$\begin{aligned}\psi\mathrm{d}(a + \varphi(b') + \mathrm{d}(b)) &= \psi\mathrm{d}(a) + \psi\mathrm{d}\varphi(b')) + \psi\mathrm{dd}(b) \\ &= \psi\mathrm{d}(a) + \psi\mathrm{d}\varphi(b')) \\ &= \psi\mathrm{d}(a) + \psi\varphi\mathrm{d}'(b') \\ &= \psi\mathrm{d}(a) \\ &= \mathrm{d}''\psi(a) \\ &= \mathrm{d}''(a'') \\ &= 0.\end{aligned}$$

Hence there exists a unique element in A'_{i-1} which maps to $\mathrm{d}(a + \varphi(b') + \mathrm{d}(b))$ with respect to φ, and since

$$\begin{aligned}\varphi(a' + \mathrm{d}'(b')) &= \varphi(a') + \varphi\mathrm{d}'(b') \\ &= \mathrm{d}(a) + \mathrm{d}\varphi(b') \\ &= \mathrm{d}(a + \varphi(b') + \mathrm{d}(b)),\end{aligned}$$

this unique element must be $a' + \mathrm{d}'(b')$. Therefore

$$\begin{aligned}\eth[a'' + \mathrm{d}''(b'')] &= [a' + \mathrm{d}'(b')] \\ &= [a'] \\ &= \eth[a''],\end{aligned}$$

which implies $\eth$ is well-defined. Moreover, we see that $\eth(\mathrm{H}(A_i)) \subseteq \mathrm{H}(A_{i-1})$, and is hence graded of degree -1. As usualy, we denote $\eth_i := \eth|_{A_i}$ for all $i \in \mathbb{Z}$.

Step 2: Let $i \in \mathbb{Z}$, let $\overline{a''}, \overline{b''} \in H(A'')$, and let $r, s \in R$. Choose a coset representative $\overline{a''}$ and $\overline{b''}$, say $a'' \in A''_i$ and $b'' \in A''_i$. Then $ra'' + sb''$ is a coset representative of $\overline{ra'' + sb''}$ (by linearity of taking quotients). Next, choose lifts of a'' and b'' in A_i under φ, say $a \in A_i$ and $b \in A_i$ respectively. Then $ra + sb$ is a lift of $ra'' + sb''$ in A_i under φ (by linearity of ψ). Finally, let a' and b' be the unique elements in A'_{i-1} such that $\varphi(a') = d(a)$ and $\varphi(b') = d(b)$. Then $ra' + sb'$ is the unique element in A'_{i-1} such that $\varphi(ra' + sb') = d(ra + sb)$ (by linearity of φ). Thus, we have

$$\begin{aligned}\eth(\overline{ra'' + sb''}) &= \overline{ra' + sb'} \\ &= r\overline{a'} + s\overline{b'} \\ &= r\eth(\overline{a''}) + s\eth(\overline{b''}).\end{aligned}$$

Step 3: To prove exactness of (17), it suffices to show exactness at $H_i(A'')$, $H_i(A)$, and $H_i(A')$. First we prove exactness at $H_i(A)$. Let $\overline{a} \in \mathrm{Ker}(H_i(\psi))$ (so $a \in A_i$, $d(a) = 0$, and $\overline{\psi(a)} = \overline{0}$). Lift $\psi(a) \in A''_i$ to an element $a'' \in A'_{i+1}$ under d'' (we can do this since $\overline{\psi(a)} = \overline{0}$). Lift $a'' \in A''_{i+1}$ to an element $b \in A_{i+1}$ under ψ (we can do this since ψ is surjective). Then

$$\begin{aligned}\psi(d(b) - a) &= \psi(d(b)) - \psi(a) \\ &= d''(a'') - \psi(a) \\ &= \psi(a) - \psi(a) \\ &= 0\end{aligned}$$

implies $d(b) - a \in \mathrm{Ker}(\psi)$. Lift $d(b) - a$ to the unique element $a' \in A'_i$ under φ (we can do this exactness of (3.8.3)). Since φ is injective,

$$
\begin{aligned}
\varphi(d'(a')) &= d(\varphi(a')) \\
&= d(d(b) - a) \\
&= d(d(b)) - d(a)) \\
&= 0
\end{aligned}
$$

implies $d'(a') = 0$. Hence a' represents an element in $H(A')$. Therefore

$$
\begin{aligned}
H_i(\varphi)(a') &= \overline{\varphi(a')} \\
&= \overline{d(b) - a} \\
&= \overline{a}
\end{aligned}
$$

implies $\overline{a} \in \mathrm{Im}(H_i(\varphi))$. Thus we have exactness at $H_i(A)$.

Next we show exactness at $H_i(A')$. Let $\overline{a'} \in \mathrm{Ker}(H_i(\varphi))$ (so $a' \in A'_i$, $d(a') = 0$, and $\overline{\varphi(a')} = \overline{0}$). Lift $\varphi(a') \in A_i$ to an element $a \in A'_{i+1}$ under d (we can do this since $\overline{\varphi(a)} = \overline{0}$). Then

$$
\begin{aligned}
d(\psi(a)) &= \psi(d(a)) \\
&= \psi(\varphi(a')) \\
&= 0.
\end{aligned}
$$

Hence $\psi(a)$ represents an element in $H_{i+1}(A'')$. By construction, we have $\eth(\overline{\psi(a)}) = \overline{a'}$, which implies $\overline{a'} \in \mathrm{Im}(\eth_{i+1})$. Thus we have exactness at $H_i(A')$.

Finally we show exactness at $H_i(A'')$. Let $\overline{a''} \in \mathrm{Ker}(\eth_i)$ (so $a'' \in A''_i$ and $d(a'') = 0$). Lift a'' to an element $a \in A_i$ under ψ. Lift $d(a)$ to the unique element a' in A'_{i-1} under φ. Lift a' to an element $b' \in A'_{i+1}$ under d (we can do this since $0 = \eth(\overline{a''}) = \overline{a'}$). Then

$$
\begin{aligned}
d(a - \varphi(b')) &= d(a) - d(\varphi(b')) \\
&= d(a) - \varphi(d(b')) \\
&= d(a) - \varphi(a') \\
&= 0,
\end{aligned}
$$

and hence $a - \varphi(b')$ represents an element in $H_i(A)$. Moreover, we have

$$
\begin{aligned}
H_i(\psi)(\overline{a - \varphi(b')}) &= \overline{\psi(a - \varphi(b'))} \\
&= \overline{\psi(a) - \psi(\varphi(b'))} \\
&= \overline{\psi(a)} \\
&= \overline{a''},
\end{aligned}
$$

which implies $\overline{a'} \in \mathrm{Im}(H_i(\psi))$. Thus we have exactness at $H_i(A'')$. □

Definition 3.11. Given a short exact sequence of R-complexes as in (3.8.3), we refer to the graded homomorphism $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 as the **induced connecting map**.

3.5.2 When a Graded R-Linear Map is a Chain Map

Proposition 3.15. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\overline{B} = B/\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ and let $\pi\colon B \to \overline{B}$ be the quotient map. Define $\overline{\partial}\colon \overline{B} \to \overline{B}$ by*

$$
\overline{\partial}(\overline{b}) = \overline{\partial(b)}
$$

for all $a \in A$ and $\overline{b} \in \overline{B}$. Then $(\overline{B}, \overline{\partial})$ is an R-complex and $\pi\varphi\colon A \to \overline{B}$ is a chain map. Moreover, if φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$, then we have the following short exact sequence of graded R-modules and graded R-linear maps:

$$
0 \longrightarrow \mathrm{H}(B) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}(\overline{B}) \xrightarrow{\gamma} \mathrm{im}(\partial\varphi - \varphi\mathrm{d})(-1) \longrightarrow 0 \tag{10}
$$

where γ is the connecting map coming from a long exact sequence in homology.

Proof. Observe that $\operatorname{im}(\partial\varphi - \varphi \mathrm{d})$ is a graded R-submodule of B since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on B induces a grading on $\overline{B}$ which makes π into a graded R-linear map. Therefore $\pi\varphi$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that $\overline{\partial}$ is well-defined, that is, we need to check that ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself. Let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Then

$$\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= (\partial\partial\varphi - \partial\varphi\mathrm{d})(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= (-\partial\varphi\mathrm{d}(a) + \varphi\mathrm{d}\mathrm{d})(a) \\
&= (-\partial\varphi + \varphi\mathrm{d})(\mathrm{d}(a)) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d}).
\end{aligned}$$

Thus $\overline{\partial}$ is well-defined. Also $\overline{\partial}$ is an R-linear differential since it inherits these properties from ∂. Therefore $(\overline{B}, \overline{\partial})$ is an R-complex.

Now let us check that $\pi\varphi$ is a chain map. To see this, we just need to show it commutes with the differentials. Let $a \in A$. Then we have

$$\begin{aligned}
\overline{\partial}\pi\varphi(a) &= \overline{\partial}(\overline{\varphi(a)}) \\
&= \overline{\partial\varphi(a)} \\
&= \overline{\partial\varphi(a) - (\partial\varphi - \varphi\mathrm{d})(a)} \\
&= \overline{\partial\varphi(a) - \partial\varphi(a) + \varphi\mathrm{d}(a)} \\
&= \overline{\varphi\mathrm{d}(a)} \\
&= \pi\varphi\mathrm{d}(a).
\end{aligned}$$

Thus $\pi\varphi$ is a chain map.

Since ∂ sends $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ to itself, it restricts to a differential on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. So we have a short exact sequence of R-complexes

$$0 \longrightarrow \operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \xrightarrow{\iota} B \xrightarrow{\pi} \overline{B} \longrightarrow 0 \tag{11}$$

where ι is the inclusion map. The short exact sequence (11) induces the following long exact sequence in homology

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_{i+1}(\overline{B}) \arrow[dlll, rounded corners, "\gamma_{i+1}"'] \\
\mathrm{H}_i(\operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(B) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_i(\overline{B}) \arrow[dll, rounded corners, "\gamma_i"'] & \\
\mathrm{H}_{i-1}(\operatorname{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(B) \arrow[r] & \cdots &
\end{tikzcd} \tag{12}$$

Let us work out the details of the connecting map γ. Let $[\overline{b}] \in \mathrm{H}_i(\overline{B})$, so $\overline{b} \in \overline{B}_i$ is the coset with $b \in B_i$ as a reprsentative and $[\overline{b}] \in \mathrm{H}_i(\overline{B})$ is the coset with $\overline{b} \in \overline{B}_i$ as a representative. In particular, $\overline{\partial}(\overline{b}) = \overline{0}$, which implies

$$\partial(b) = (\partial\varphi - \varphi\mathrm{d})(a) \tag{13}$$

for some $a \in A$. Then (13) implies that $(\partial\varphi - \varphi\mathrm{d})(a)$ is the unique element in $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ which maps to $\partial(b)$ (under the inclusion map). Therefore

$$\gamma_i[\overline{b}] = [(\partial\varphi - \varphi\mathrm{d})(a)].$$

Now suppose φ takes $\operatorname{im}\mathrm{d}$ to $\operatorname{im}\partial$. We claim that ∂ restricts to the zero map on $\operatorname{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \operatorname{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\operatorname{im}\mathrm{d}$ to $\operatorname{im}\partial$, there exists a $b \in B$ such that

$$\varphi\mathrm{d}(a) = \partial(b).$$

Choose such a $b \in B$. Then observe that

$$\begin{aligned}
\partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a) \\
&= -\partial\varphi\mathrm{d}(a) \\
&= -\partial\partial(b) \\
&= 0.
\end{aligned}$$

Thus ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. In particular, $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \cong \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$.

Next we claim that $\mathrm{H}(\iota)$ is the zero map. Indeed, for any $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$, we choose $b \in B$ such that $\varphi\mathrm{d}(a) = \partial(b)$, then we have

$$\begin{aligned}(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\varphi(a) - \varphi\mathrm{d}(a) \\ &= \partial\varphi(a) - \partial b \\ &= \partial(\varphi(a) - b) \\ &\in \mathrm{im}\,\partial.\end{aligned}$$

Therefore $\mathrm{H}(\iota)$ takes the coset in $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}))$ represented by $(\partial\varphi - \varphi\mathrm{d})(a)$ to the coset in $\mathrm{H}(B)$ represented by 0. Thus $\mathrm{H}(\iota)$ is the zero map as claimed.

Combining everything together, we see that the long exact sequence (12) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(B) \xrightarrow{\mathrm{H}_i(\pi)} \mathrm{H}_i(\overline{B}) \xrightarrow{\gamma_i} \mathrm{im}(\partial_{i-1}\varphi_{i-1} - \varphi_{i-2}\mathrm{d}_{i-1}) \longrightarrow 0 \tag{14}$$

for all $i \in \mathbb{Z}$. In other words, (11) is a short exact sequence of graded R-modules.

□

3.6 Operations on R-Complexes

3.6.1 Product of R-complexes

3.6.2 Limits

Definition 3.12. Let $(\Lambda, \leq)$ be a preordered set. A system $(M_\lambda, \varphi_{\lambda\mu})$ of R-complexes and chain maps over Λ consists of a family of a family of R-complexes $\{(M_\lambda, \mathrm{d}_\lambda)\}$ indexed by Λ and a family of chain maps $\{\varphi_{\lambda\mu} \colon M_\lambda \to M_\mu\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\mu\kappa}\varphi_{\lambda\mu}.$$

We say $(M_\lambda, \varphi_{\lambda\mu})$ is a **directed system** if Λ is a directed set.

Proposition 3.16. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a system of R-complexes and chain maps over Λ. The limit of this system, denoted $\lim^\star M_\lambda$, is given by the R-complex $(\lim^\star M_\lambda, \lim^\star \mathrm{d}_\lambda)$ together with together with the projection maps*

$$\pi_\lambda \colon \overset{\star}{\lim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$, where $\lim^\star M_\lambda$ is the graded R-module given by

$$\overset{\star}{\lim} M_\lambda = \left\{ (u_\lambda) \in \overset{\star}{\prod_{\lambda \in \Lambda}} M_\lambda \mid \varphi_{\lambda\kappa}(u_\lambda) = u_\mu \text{ for all } \lambda \leq \mu \right\}$$

and where the differential $\lim^\star \mathrm{d}_\lambda$ is defined pointwise:

$$(\overset{\star}{\lim} \mathrm{d}_\lambda)((u_\lambda)) = (\mathrm{d}_\lambda(u_\lambda))$$

for all $(u_\lambda) \in \lim^\star M_\lambda$.

Proof. We need ot show that $\lim^\star M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the system $(M_\lambda, \varphi_{\lambda\mu})$, so

$$\varphi_{\lambda\mu}\psi_\lambda = \psi_\mu$$

for all $\lambda \leq \mu$. By the universal mapping property of the graded limits, there exists a unique graded R-linear map $\psi \colon M \to \lim^\star M_\lambda$ of graded R-linear maps which commutes with all the arrows. It remains to show that ψ commutes with the differentials. Indeed, we have

$$\begin{aligned}(\overset{\star}{\lim} \mathrm{d}_\lambda \psi)(u) &= \overset{\star}{\lim} \mathrm{d}_\lambda((\psi_\lambda(u))) \\ &= (\mathrm{d}_\lambda(\psi_\lambda(u))) \\ &= (\psi_\lambda(\mathrm{d}(u))) \\ &= \psi(\mathrm{d}(u)) \\ &= (\psi\mathrm{d})(u).\end{aligned}$$

for all $u \in M$.

□

3.6.3 Localization

Let (A, d) be an R-complex and let S be a multiplicatively closed subset of R. The **localization of** (A, d) **with respect to** S is the R_S-complex (A_S, d_S) where A_S is the graded R_S-module whose component in degree i is

$$(A_S)_i = \{a/s \mid a \in A_i \text{ and } s \in S\}.$$

The differential d_S is defined as follows: if $a/s \in (A_S)_i$, then

$$\mathrm{d}_S(a/s) = \mathrm{d}(a)/s.$$

3.6.4 Direct Sum of R-Complexes

Definition 3.13. Let (A, d) and (A', d') be R-complexes. We define their **direct sum** to be the R-complex

$$(A, \mathrm{d}) \oplus_R (A', \mathrm{d}') := (A \oplus A', \mathrm{d} \oplus \mathrm{d}')$$

whose graded R-module $A \oplus A'$ has

$$(A \oplus A')_i = A_i \oplus A'_i$$

as its ith homogeneous component and whose differential $\mathrm{d} \oplus \mathrm{d}'$ is defined by

$$(\mathrm{d} \oplus \mathrm{d}')(a, a') = (\mathrm{d}(a), \mathrm{d}'(a'))$$

for all $(a, a') \in A \oplus A'$.

More generally, suppose $(A_\lambda, \mathrm{d}_\lambda)$ is an R-complex for each λ in some indexing set Λ. We define their **direct sum** to be the R-complex

$$\bigoplus_{\lambda \in \Lambda} (A_\lambda, \mathrm{d}_\lambda) := \left(\bigoplus_{\lambda \in \Lambda} A_\lambda, \bigoplus_{\lambda \in \Lambda} \mathrm{d}_\lambda \right).$$

It is easy to check that

$$\mathrm{H}\left(\bigoplus_{\lambda \in \Lambda} A_\lambda \right) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{H}(A_\lambda).$$

In other words, homology commutes with direct sums.

3.6.5 Shifting an R-complex

Definition 3.14. Let (A, d) be an R-complex. We define the **shift** of (A, d) to be the R-complex

$$\Sigma(A, \mathrm{d}) := (A(-1), -\mathrm{d}).$$

More generally, let $k \in \mathbb{Z}$. We define the kth **shift** of (A, d) to be the R-complex

$$\Sigma^k(A, \mathrm{d}) = (A(-k), (-1)^k \mathrm{d}).$$

Proposition 3.17. *Let A be an R-complex and let $n \in \mathbb{Z}$. Then*

$$\mathrm{H}(\Sigma^n A) = \mathrm{H}(A)(-n).$$

In particular,

$$\mathrm{H}_i(\Sigma^n A) = \mathrm{H}_{i-n}(A)$$

for all $i \in \mathbb{Z}$.

Proof. We have

$$\begin{aligned}
\mathrm{H}(\Sigma^n A) &= \ker\left(\mathrm{d}_{\Sigma^n A}\right) / \operatorname{im}\left(\mathrm{d}_{\Sigma^n A}\right) \\
&= \ker\left((-1)^n \mathrm{d}_{A(-n)}\right) / \operatorname{im}\left((-1)^n \mathrm{d}_{A(-n)}\right) \\
&= \ker\left(\mathrm{d}_{A(-n)}\right) / \operatorname{im}\left(\mathrm{d}_{A(-n)}\right) \\
&= \mathrm{H}(A)(-n).
\end{aligned}$$

□

3.7 The Mapping Cone

Definition 3.15. Let $\varphi\colon A \to B$ be a chain map. The **mapping cone of** φ, denoted $\mathrm{C}(\varphi)$, is the R-complex whose underlying graded R-module is $\mathrm{C}(\varphi) = B \oplus A(-1)$ and whose differential is defined by

$$\mathrm{d}_{\mathrm{C}(\varphi)}(b,a) := (\mathrm{d}_B(b) + \varphi(a), -\mathrm{d}_A(a))$$

for all $(b,a) \in B \oplus A(-1)$.

Remark. To see that we are justified in calling $\mathrm{C}(\varphi)$ an R-complex, let us check that $\mathrm{d}_{\mathrm{C}(\varphi)}\mathrm{d}_{\mathrm{C}(\varphi)} = 0$. Let $(b,a) \in C(\varphi)$. Then we have

$$\begin{aligned}
\mathrm{d}_{\mathrm{C}(\varphi)}\mathrm{d}_{\mathrm{C}(\varphi)}(b,a) &= \mathrm{d}_{\mathrm{C}(\varphi)}(\mathrm{d}_B(b) + \varphi(a), -\mathrm{d}_A(a)) \\
&= (\mathrm{d}_B(\mathrm{d}_B(b) + \varphi(a)) + \varphi(-\mathrm{d}_A(a)), -\mathrm{d}_A\mathrm{d}_A(a)) \\
&= (\mathrm{d}_B\varphi(a) - \varphi\mathrm{d}_A(a), 0) \\
&= (0,0).
\end{aligned}$$

3.7.1 Turning a Chain Map Into a Connecting Map

Theorem 3.3. *Let $\varphi\colon A \to B$ be a chain map. Then we have a short exact sequence of R-complexes*

$$0 \longrightarrow B \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma A \longrightarrow 0 \tag{15}$$

where $\iota\colon B \to \mathrm{C}(\varphi)$ is the inclusion map given by

$$\iota(b) = (b,0)$$

for all $b \in B$, and where $\pi\colon \mathrm{C}(\varphi) \to \Sigma A$ is the projection map given by

$$\pi(b,a) = a$$

for all $(b,a) \in \mathrm{C}(\varphi)$. Moreover the connecting map $\eth\colon \mathrm{H}(\Sigma A) \to \mathrm{H}(B)$ induced by (15) agrees with $\mathrm{H}(\varphi)$.

Proof. It is straightforward to check that (15) is a short exact sequence of R-complexes. Let us show that the connecting map agrees with $\mathrm{H}(\varphi)$. Let $i \in \mathbb{Z}$ and let $\overline{a} \in \mathrm{H}_i(\Sigma A)$. Thus $a \in A_i$ and $\mathrm{d}_A(a) = 0$. Lift $a \in A_i$ to the element $(0,a) \in \mathrm{C}_i(\varphi)$. Now apply $\mathrm{d}_{\mathrm{C}(\varphi)}$ to $(0,a)$ to get $(\varphi(a), 0) \in \mathrm{C}_{i-1}(\varphi)$. Then $\varphi(a)$ is the unique element in B_{i-1} which maps to $(\varphi(a), 0)$ under d_B. Therefore

$$\begin{aligned}
\eth(\overline{a}) &= \overline{\varphi(a)} \\
&= \mathrm{H}(\varphi)(\overline{a}).
\end{aligned}$$

It follows that $\eth$ and $\mathrm{H}(\varphi)$ agree on all of $\mathrm{H}(A)$. □

Remark. In the context of graded R-modules, it would be incorrect to say $\eth = \mathrm{H}(\varphi)$. This is because $\eth$ is graded of degree -1 and $\mathrm{H}(\varphi)$ is graded of degree 0. On the other hand, it would be correct to say $\eth_i = \mathrm{H}_{i-1}(\varphi)$ for all $i \in \mathbb{Z}$.

3.7.2 Quasiisomorphism and Mapping Cone

Corollary. *Let $\varphi\colon A \to B$ be a chain map. Then φ is a quasiisomorphism if and only if $\mathrm{C}(\varphi)$ is an exact complex.*

Proof. Suppose $\mathrm{C}(\varphi)$ is an exact complex, so $\mathrm{H}(\mathrm{C}(\varphi)) \cong 0$. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (15) gives us

$$0 \cong \mathrm{H}_{i+1}(\mathrm{C}(\varphi)) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}_i(A) \xrightarrow{\mathrm{H}(\varphi)} \mathrm{H}_i(B) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}_i(\mathrm{C}(\varphi)) \cong 0$$

which implies $\mathrm{H}_i(A) \cong \mathrm{H}_i(B)$ for all $i \in \mathbb{Z}$.

Conversely, suppose φ is a quaisiisomorphism. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (15) gives us

$$\mathrm{H}_i(A) \cong \mathrm{H}_i(B) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}_i(\mathrm{C}(\varphi)) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}_{i-1}(A) \cong \mathrm{H}_{i-1}(B)$$

which implies $\mathrm{H}_i(\mathrm{C}(\varphi)) \cong 0$ for all $i \in \mathbb{Z}$. □

3.7.3 Translating Mapping Cone With Isomorphisms

Proposition 3.18. *Suppose we have a commutative diagram of R-complexes*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ A' \arrow[r, "\phi'"'] & B' \end{tikzcd}$$

where $\phi\colon A \to B$ and $\phi'\colon A' \to B'$ are isomorphisms. Then we have an isomorphism $\mathrm{C}(\varphi) \cong \mathrm{C}(\psi)$ of R-complexes.

Proof. Define $\phi' \oplus \phi\colon \mathrm{C}(\varphi) \to \mathrm{C}(\psi)$ by

$$(\phi' \oplus \phi)(a', a) = (\phi'(a'), \phi(a))$$

for all $(a', a) \in \mathrm{C}(\varphi)$. Clearly $\phi' \oplus \phi$ is an isomorphism of the underlying graded R-modules. To see that it is an isomorphism of R-complexes, we need to check that it commutes with the differentials. Let $(a', a) \in \mathrm{C}(\varphi)$. We have

$$\begin{aligned} \mathrm{d}_{\mathrm{C}(\psi)}(\phi' \oplus \phi)(a', a) &= \mathrm{d}_{\mathrm{C}(\psi)}(\phi'(a'), \phi(a)) \\ &= (\mathrm{d}_{B'}\phi'(a') + \psi\phi(a), -\mathrm{d}_B\phi(a)) \\ &= (\mathrm{d}_{B'}\phi'(a') + \psi\phi(a), -\mathrm{d}_B\phi(a)) \\ &= (\phi'\mathrm{d}_{A'}(a') + \phi'\varphi(a), -\phi\mathrm{d}_A(a)) \\ &= (\phi' \oplus \phi)(\mathrm{d}_{A'}(a') + \varphi(a), -\mathrm{d}_A(a)) \\ &= (\phi' \oplus \phi)\mathrm{d}_{\mathrm{C}(\varphi)}(a', a). \end{aligned}$$

□

3.7.4 Resolutions by Mapping Cones

Lemma 3.4. *(Lifting Lemma) Let $\varphi\colon M \to M'$ be an R-module homomorphism, let (P, d) be a projective resolution of M, and let (P', d') be a projective resolution of M'. Then there exists a chain map $\varphi\colon (P, d) \to (P', d')$ such that*

$$\begin{tikzcd} H_0(P) \arrow[r, "H_0(\varphi)"] \arrow[d, "\cong"] & H_0(P') \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

Proof. For each $i > 0$, let $M'_i := \mathrm{Im}(d'_i)$ and let $M_i := \mathrm{Im}(d_i)$. We build a chain map $\varphi\colon (P, d) \to (P', d')$ by constructing R-module homomorphism $\varphi_i\colon P_i \to P'_i$ which commute with the differentials using induction on $i \geq 0$.

First consider the base case $i = 0$. Let $\psi_0\colon P_0 \to P'_0/M'_0$ be the composition

$$P_0 \to P_0/M_1 \cong M \to M' \cong P'_0/M'_1.$$

Since P_0 is projective and since $d'_0\colon P'_0 \to P'_0/M_1$ is a surjective homomorphism, we can lift $\psi_0\colon P_0 \to P'_0/M'_0$ along $d'_0 : P'_0 \to P'_0/M_1$ to a homomorphism $\varphi_0\colon P_0 \to P'_0$ such that $d'_0\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructedn an R-module homomorphism $\varphi_i\colon P_i \to P'_i$ such that

$$d'_i\varphi_i = \varphi_{i-1}d_i.$$

We need to construct an R-module homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ such that

$$d'_{i+1}\varphi_{i+1} = \varphi_i d_{i+1}.$$

First, observe that $\mathrm{Im}(\varphi_i d_{i+1}) \subseteq M'_{i+1}$. Indeed, we have

$$\begin{aligned} d'_i\varphi_i d_{i+1} &= \varphi_{i-1}d_i d_{i+1} \\ &= 0, \end{aligned}$$

Thus, since (P', d') is exact for all $i > 0$, we have

$$\begin{aligned} \operatorname{Im}(\varphi_i d_{i+1}) &\subseteq \operatorname{Ker}(d_i') \\ &= \operatorname{Im}(d_{i+1}') \\ &= M_{i+1}'. \end{aligned}$$

Now since P_{i+1} is projective and $d_{i+1}' \colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i d_{i+1} : P_{i+1} \to M_{i+1}'$ along $d_{i+1}' : P_{i+1}' \to M_{i+1}'$ to a homomorphism $\varphi_{i+1} \colon P_{i+1} \to P_{i+1}'$ such that

$$d_{i+1}' \varphi_{i+1} = \varphi_i d_{i+1}.$$

The last part of the lemma, follows from the way φ_0 was constructed.

□

Theorem 3.5. *With the notation as above, the following hold:*

1. *if φ is injective, then $\mathrm{C}(\varphi)$ is a projective resolution of $M' / \operatorname{im} \varphi$.*
2. *if φ is surjective, then $\Sigma\mathrm{C}(\varphi)$ is a projective resolution of* $\ker \varphi$.

Proof. First note that the underlying graded R-module of $\mathrm{C}(\varphi)$ is projective since it is a direct sum of projective modules. Now we first consider the case where φ is injective. The short exact sequence

$$0 \longrightarrow P' \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma P \longrightarrow 0 \tag{16}$$

induces the long exact sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_i(P) \arrow[dll, "\mathrm{H}_i(\varphi)"'] \\ \mathrm{H}_i(P') \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(\mathrm{C}(\varphi)) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_{i-1}(P) \arrow[dll, "\mathrm{H}_{i-1}(\varphi)"'] \\ \mathrm{H}_{i-1}(P') \arrow[r] & \cdots & \end{tikzcd} \tag{17}$$

This gives us $\mathrm{H}_i(\mathrm{C}(\varphi))$ for all $i > 1$ since $\mathrm{H}_i(P') \cong 0 \cong \mathrm{H}_i(P)$ for all $i \geq 1$. For $i = 1$, we get the exact sequence

$$0 \longrightarrow \mathrm{H}_1(\mathrm{C}(\varphi)) \longrightarrow M \xrightarrow{\varphi} M' \tag{18}$$

Then φ being injective implies $\mathrm{H}_1(\mathrm{C}(\varphi)) \cong 0$. Finally, for $i = 0$, we get the short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} M' \longrightarrow \mathrm{H}_0(\mathrm{C}(\varphi)) \longrightarrow 0 \tag{19}$$

This implies $\mathrm{H}_0(\mathrm{C}(\varphi)) \cong M' / \operatorname{im} \varphi$.

Now we consider the case where φ is surjective. We still get $\mathrm{H}_i(\mathrm{C}(\varphi))$ for all $i > 1$ since $\mathrm{H}_i(P') \cong 0 \cong \mathrm{H}_i(P)$ for all $i \geq 1$. For $i = 1$, we get again get the exact sequence (18), but this time we conclude that $\mathrm{H}_1(\mathrm{C}(\varphi)) \cong \ker \varphi$ since φ is surjective. Similarly, for $i = 0$, we again get the short exact sequence (19), but this time we conclude $\mathrm{H}_0(\mathrm{C}(\varphi)) \cong 0$ since φ is surjective. □

Example 3.1. Let $S = K[x_1, \ldots, x_n]$, let $I_{\mathcal{P}}$ be the permutohedron ideal in S, and let $I_{\mathcal{A}}$ be the associahedron ideal in S. Then there are natural free resolution $F_{\mathcal{P}} \xrightarrow{\tau_{\mathcal{P}}} S/I_{\mathcal{P}}$ and $F_{\mathcal{A}} \xrightarrow{\tau_{\mathcal{A}}} S/I_{\mathcal{A}}$ over S where $F_{\mathcal{P}}$ is supported by the permutohedron and $F_{\mathcal{A}}$ is supported by the associahedron. The inclusion of ideals $I_{\mathcal{A}} \subset I_{\mathcal{P}}$ induces a surjective S-linear map $\varphi \colon S/I_{\mathcal{A}} \to S/I_{\mathcal{P}}$ whose kernel is given by $I_{\mathcal{P}}/I_{\mathcal{A}}$. Lift $\varphi\tau_{\mathcal{A}}$ to a chain map $\widetilde{\varphi} \colon F_{\mathcal{A}} \to F_{\mathcal{P}}$ with respect to $\tau_{\mathcal{P}}$, so $\tau_{\mathcal{P}}\widetilde{\varphi} = \varphi\tau_{\mathcal{A}}$. It follows from Theorem (3.5) that $\Sigma\mathrm{C}(\widetilde{\varphi})$ is a free resolution of $I_{\mathcal{P}}/I_{\mathcal{A}}$ over S.

3.8 Tensor Products

3.8.1 Definition of tensor product

Definition 3.16. Let (A, d) and (A', d') be two R-complexes. Their **tensor product** is the R-complex $(A \otimes_R A', \mathrm{d}^{\otimes}_{(A,A')})$, where the graded R-module $A \otimes_R A'$ has

$$(A \otimes_R A')_i = \bigoplus_{j \in \mathbb{Z}} A_j \otimes A'_{j-i}$$

as its ith homogeneous component and whose differential is defined on elementary homogeneous tensors (and extended linearly) by

$$\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') = \mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')$$

for all $a \in A_i$, $a' \in A_j$ and $i, j \in \mathbb{Z}$.

Proposition 3.19. *The map $\mathrm{d}^{\otimes}_{(A,A')}$ is well-defined and is in fact a differential.*

Proof. First we observe that $\mathrm{d}^{\otimes}_{(A,A')}$ is a well-defined R-linear map because the map $A_i \times A'_j \to A_i \otimes_R A'_j$ given by

$$(a, a') \mapsto \mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')$$

for all $(a, a') \in A_i \times A'_j$ is R-bilinear for each $i, j \in \mathbb{Z}$. Next we observe that $\mathrm{d}^{\otimes}_{(A,A')}$ is graded of degree -1. Indeed, if $a \otimes a' \in A_j \otimes_R A'_{i-j}$, then

$$\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a') \in A_{j-1} \otimes_R A'_{i-j} + A_j \otimes_R A_{i-j-1}.$$

Lastly we observe that $\mathrm{d}^{\otimes}_{(A,A')}\mathrm{d}^{\otimes}_{(A,A')} = 0$ since if $a \otimes a' \in (A \otimes_R A')_k$ where $a \in A_i$ and $a' \in A'_j$, then

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(A,A')}\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') &= \mathrm{d}^{\otimes}_{(A,A')}(\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')) \\
&= \mathrm{d}^{\otimes}_{(A,A')}(\mathrm{d}(a) \otimes a') + (-1)^i \mathrm{d}^{\otimes}_{(A,A')}(a \otimes \mathrm{d}'(a')) \\
&= \mathrm{d}\mathrm{d}(a) \otimes a' + (-1)^{i-1}\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i(\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i a \otimes \mathrm{d}'\mathrm{d}'(a')) \\
&= (-1)^{i-1}\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i \mathrm{d}(a) \otimes \mathrm{d}'(a') \\
&= 0.
\end{aligned}$$

□

3.8.2 Commutativity of tensor products

Proposition 3.20. *Let A and B be R-complexes. Then we have an isomorphism of R-complexes*

$$A \otimes_R B \cong B \otimes_R A, \tag{20}$$

which is natural in A and B.

Proof. We define $\tau_{A,B} \colon A \otimes_R B \to B \otimes_R A$ on elementary homogeneous tensors (and extend linearly) by

$$\tau_{A,B}(a \otimes b) = (-1)^{ij} b \otimes a$$

for all $a \otimes b \in A_i \otimes_R B_j$. The map $\tau_{A,B}$ is easily seen to be a well-defined graded R-linear isomorphism. To see that $\tau_{A,B}$ is an isomorphism of R-complexes, we need to show that it commutes with the differentials. That is, we need to show

$$\tau_{A,B}\mathrm{d}^{\otimes}_{(A,B)} = \mathrm{d}^{\otimes}_{(B,A)}\tau_{A,B} \tag{21}$$

It suffices to check (21) on elementary homogeneous tensors, so let $a \otimes b \in A_i \otimes_R B_j$ be such an elementary homogeneous tensor. Then we have

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(B,A)}\tau_{A,B}(a \otimes b) &= (-1)^{ij}\mathrm{d}^{\otimes}_{(B,A)}(b \otimes a) \\
&= (-1)^{ij}\mathrm{d}_B(b) \otimes a + (-1)^{j+ij} b \otimes \mathrm{d}_A(a)) \\
&= (-1)^{i+i(j-1)}\mathrm{d}_B(b) \otimes a + (-1)^{(i-1)j} b \otimes \mathrm{d}_A(a) \\
&= (-1)^{(i-1)j} b \otimes \mathrm{d}_A(a) + (-1)^{i+i(j-1)}\mathrm{d}_B(b) \otimes a \\
&= \tau_{A,B}(\mathrm{d}_A(a) \otimes b + (-1)^i a \otimes \mathrm{d}_B(b)) \\
&= \tau_{A,B}\mathrm{d}^{\otimes}_{(A,B)}(a \otimes b).
\end{aligned}$$

Finally, being natural in A and B means that if $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ are two chain maps, then the following diagram commutes:

$$\begin{tikzcd}
A \otimes_R B \arrow[r, "\varphi\otimes_R B"] \arrow[d, "A\otimes_R\psi"'] & A' \otimes_R B \arrow[d, "A'\otimes_R\psi"] \\
A \otimes_R B' \arrow[r, "\varphi\otimes_R B'"'] & A' \otimes_R B'
\end{tikzcd}$$

We leave it as an exercise for the reader to check that this diagram commutes. □

3.8.3 Associativity of tensor products

Given that the proof of tensor products of R-complexes was nontrivial, we need to be sure that we have associativity of tensor products of R-complexes. The proof in this case turns out to be trivial.

Proposition 3.21. *Let A, A', and A'' be R-complexes. Then we have an isomoprhism of R-complexes*

$$(A \otimes_R A') \otimes_R A'' \cong A \otimes_R (A' \otimes_R A''),$$

which is natural in A, A', and A''.

Proof. Let $\eta_{A,A',A''}\colon (A \otimes_R A') \otimes_R A'' \to A \otimes_R (A' \otimes_R A'')$ to be the unique graded isomorphism such that

$$\eta_{A,A',A''}((a \otimes a') \otimes a'') = a \otimes (a' \otimes a'')$$

for all $a \in A_i$, $a' \in A'_j$, and $a'' \in A''_k$ and for all $i, j, k \in \mathbb{Z}$. To see that $\eta_{A,A',A''}$ is an isomorphism of R-complexes, we need to show that

$$\eta_{A,A',A''}\mathrm{d}^{\otimes}_{((A\otimes_R A'),A'')} = \mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}\eta_{A,A',A''} \tag{22}$$

It suffices to check (22) on elementary homogeneous tensors. Let $(a \otimes a') \otimes a'' \in (A_i \otimes_R A_j) \otimes_R A_k$. To simplify the notation in our calculation, we denote $\eta = \eta_{A,A',A''}$. We have

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}\eta((a \otimes a') \otimes a'') &= \mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}(a \otimes (a' \otimes a'')) \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes \mathrm{d}^{\otimes}_{(A',A'')}(a' \otimes a'') \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (\mathrm{d}_{A'}(a') \otimes a'' + (-1)^j a' \otimes \mathrm{d}_{A''}(a'')) \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (\mathrm{d}_{A'}(a') \otimes a'') + (-1)^{i+j} a \otimes (a' \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta((\mathrm{d}_A(a) \otimes a') \otimes a'') + (-1)^i \eta((a \otimes \mathrm{d}_{A'}(a')) \otimes a'') + (-1)^{i+j}\eta((a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta((\mathrm{d}_A(a) \otimes a') \otimes a'' + (-1)^i (a \otimes \mathrm{d}_{A'}(a')) \otimes a'' + (-1)^{i+j}(a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta(\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \otimes a'' + (-1)^{i+j}(a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta\mathrm{d}^{\otimes}_{((A\otimes_R A'),A'')}((a \otimes a') \otimes a'').
\end{aligned}$$

Therefore (22) holds, and thus $\eta_{A,A',A''}$ is an isomorphism of R-complexes.

Naturality in A, A', and A'' means that if $\varphi\colon A \to B$, $\varphi\colon A' \to B'$, and $\varphi\colon A'' \to B''$ are chains maps, then we have a commutative diagram

$$\begin{tikzcd}
(A \otimes_R A')_R \otimes A'' \arrow[r, "\eta_{A,A',A''}"] \arrow[d, "(\varphi\otimes\varphi')\otimes\varphi''"'] & A \otimes_R (A'_R \otimes A'') \arrow[d, "\varphi\otimes(\varphi'\otimes\varphi'')"] \\
(B \otimes_R B')_R \otimes B'' \arrow[r, "\eta_{B,B',B''}"] & (B \otimes_R B')_R \otimes B''
\end{tikzcd}$$

□

3.8.4 Tensor Commutes with Shifts

Proposition 3.22. *Let $n \in \mathbb{Z}$ and let A and A' be R-complexes. Then*

$$(\Sigma^n A) \otimes_R A' \cong \Sigma^n (A \otimes_R A') \cong A \otimes_R (\Sigma^n A')$$

are isomorphisms of R-complexes.

Proof. We will just show that $(\Sigma^n A) \otimes_R A' \cong \Sigma^n(A \otimes_R A')$. The other isomorphism follows from a similar argument. As graded R-modules, we have

$$\begin{aligned}(\Sigma^n A) \otimes_R A' &= A(-n) \otimes_R A' \\ &= (A \otimes_R A')(-n) \\ &= \Sigma^n(A \otimes_R A').\end{aligned}$$

We define $\Phi\colon (\Sigma^n A) \otimes_R A' \to \Sigma^n(A \otimes_R A')$ by

$$\Phi(a \otimes a') = a \otimes a'$$

for all elementary tensors $a \otimes a' \in \Sigma^n A \otimes_R A'$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $a \otimes a' \in (\Sigma^n A) \otimes_R A'$ with $a \in A_i$ and $a' \in A_j$. Then $a \in (\Sigma^n A)_{i+n}$, and so we have

$$\begin{aligned}(\Sigma^n \mathrm{d}^{\otimes}_{(A,A')}\Phi)(a \otimes a') &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(\Phi(a \otimes a')) \\ &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \\ &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \\ &= (-1)^n(\mathrm{d}_A(a) \otimes a' + (-1)^i a \otimes \mathrm{d}_{A'}(a')) \\ &= (-1)^n \mathrm{d}_A(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a') \\ &= \mathrm{d}_{\Sigma^n A}(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a') \\ &= \Phi(\mathrm{d}_{\Sigma^n A}(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a')) \\ &= \Phi(\mathrm{d}^{\otimes}_{(\Sigma^n A,A')}(a \otimes a')) \\ &= (\Phi \mathrm{d}^{\otimes}_{(\Sigma^n A,A')})(a \otimes a')\end{aligned}$$

□

3.8.5 Tensor Commutes with Mapping Cone

Proposition 3.23. *Let X be an R-complex and let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then*

$$\mathrm{C}(\varphi) \otimes_R X \cong \mathrm{C}(\varphi \otimes_R X)$$

is an isomorphism of R-complexes.

Proof. As graded R-modules, we have

$$\begin{aligned}\mathrm{C}(\varphi) \otimes_R X &= (A' \oplus A(-1)) \otimes_R X \\ &\cong (A' \otimes_R X) \oplus (A(-1) \otimes_R X) \\ &= (A' \otimes_R X) \oplus (A \otimes_R X)(-1) \\ &= \mathrm{C}(\varphi \otimes_R X),\end{aligned}$$

where the graded isomorphism in the second line is given by

$$(a', a) \otimes x \mapsto (a' \otimes x, a \otimes x)$$

for all elementary tensors $(a', a) \otimes x \in (A' \oplus A(-1)) \otimes_R X$.

Let $\Phi\colon \mathrm{C}(\varphi) \otimes_R X \to \mathrm{C}(\varphi \otimes_R X)$ be the unique R-linear map such that

$$\Phi(x \otimes (a', a)) = (x \otimes a', x \otimes a)$$

for all elementary tensors $(a', a) \otimes x \in \mathrm{C}(\varphi) \otimes_R X$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes.

Indeed, let $(a', a) \otimes x \in \mathrm{C}(\varphi) \otimes_R X$ be an elementary tensor with $a' \in A'_i$, $a \in A_{i-1}$, and $x \in X_j$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)} \Phi)((a', a) \otimes x) &= \mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)}(\Phi((a', a) \otimes x)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)}(a' \otimes x, a \otimes x) \\
&= (\mathrm{d}^{\otimes}_{(A', X)}(a' \otimes x) + (\varphi \otimes X)(a \otimes x), -\mathrm{d}^{\otimes}_{(A, X)}(a \otimes x)) \\
&= (\mathrm{d}_{A'}(a') \otimes x + (-1)^i a' \otimes \mathrm{d}_X(x) + \varphi(a) \otimes x, -\mathrm{d}_A(a) \otimes x + (-1)^i a \otimes \mathrm{d}_X(x)) \\
&= ((\mathrm{d}_{A'}(a') \otimes x + \varphi(a) \otimes x + (-1)^i a' \otimes \mathrm{d}_X(x), -\mathrm{d}_A(a) \otimes x + (-1)^i a \otimes \mathrm{d}_X(x)) \\
&= ((\mathrm{d}_{A'}(a') + \varphi(a)) \otimes x, -\mathrm{d}_A(a) \otimes x) + (-1)^i((a' \otimes \mathrm{d}_X(x), a \otimes \mathrm{d}_X(x)) \\
&= \Phi((\mathrm{d}_{A'}(a') + \varphi(a), -\mathrm{d}_A(a)) \otimes x + (-1)^i (a', a) \otimes \mathrm{d}_X(x)) \\
&= \Phi(\mathrm{d}_{\mathrm{C}(\varphi)}(a', a) \otimes x + (-1)^i (a', a) \otimes \mathrm{d}_X(x)) \\
&= \Phi(\mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi), X)}((a', a) \otimes x) \\
&= (\Phi \mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi), X)})((a', a) \otimes x).
\end{aligned}$$

It follows that $\mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)} \Phi = \Phi \mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi), X)}$. Thus Φ gives an isomorphism of R-complexes.

□

Proposition 3.24. *Let A be an R-complex and let $\psi \colon B \to B'$ be a chain map of R-complexes. Then*

$$A \otimes_R \mathrm{C}(\psi) \cong \mathrm{C}(A \otimes_R \psi)$$

is an isomorphism of R-complexes.

Proof. Combining Proposition (3.18) and Proposition (3.23) gives us the isomorphisms

$$\begin{aligned}
A \otimes_R \mathrm{C}(\psi) &\cong \mathrm{C}(\psi) \otimes_R A \\
&\cong \mathrm{C}(\psi \otimes_R A) \\
&\cong \mathrm{C}(A \otimes_R \psi).
\end{aligned}$$

Following these isomorphisms in terms of an elementary homogeneous element $a \otimes (b', b) \in A_i \otimes \mathrm{C}(\psi)_j$, we have

$$\begin{aligned}
a \otimes (b', b) &\mapsto (-1)^{ij}(b', b) \otimes a \\
&\mapsto (-1)^{ij}(b' \otimes a, b \otimes a) \\
&\mapsto (-1)^{ij}((-1)^{ij} a \otimes b', (-1)^{i(j-1)} a \otimes b) \\
&= (a \otimes b', (-1)^{ij + i(j-1)} a \otimes b) \\
&= (a \otimes b', (-1)^i a \otimes b)
\end{aligned}$$

Let us check that this really does commute with the differentials. Define $\Phi \colon A \otimes_R \mathrm{C}(\psi) \to \mathrm{C}(A \otimes_R \psi)$ by

$$\Phi(a \otimes (b', b)) = (a \otimes b', (-1)^i a \otimes b)$$

for all elementary homogeneous tensors $a \otimes (b', b) \in A_i \otimes_R \mathrm{C}(\psi)_j$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(A \otimes_R \psi)} \Phi)(a \otimes (b', b)) &= \mathrm{d}_{\mathrm{C}(A \otimes_R \psi)}(a \otimes b', (-1)^i a \otimes b) \\
&= (\mathrm{d}^{\otimes}_{(A, B')}(a \otimes b') + (-1)^i (A \otimes_R \psi)(a \otimes b), -(-1)^i \mathrm{d}^{\otimes}_{(A, B)}(a \otimes b)) \\
&= (\mathrm{d}_A(a) \otimes b' + (-1)^i a \otimes \mathrm{d}_{B'}(b') + (-1)^i a \otimes \psi(b), -(-1)^i \mathrm{d}_A(a) \otimes b - a \otimes \mathrm{d}_B(b)) \\
&= (\mathrm{d}_A(a) \otimes b', -(-1)^i \mathrm{d}_A(a) \otimes b) + ((-1)^i a \otimes \mathrm{d}_{B'}(b') + (-1)^i a \otimes \psi(b), a \otimes -\mathrm{d}_B(b))) \\
&= \Phi(\mathrm{d}_A(a) \otimes (b', b) + (-1)^i a \otimes (\mathrm{d}_{B'}(b') + \psi(b), -\mathrm{d}_B(b))) \\
&= \Phi(\mathrm{d}_A(a) \otimes (b', b) + (-1)^i a \otimes \mathrm{d}_{\mathrm{C}(\psi)}(b', b)) \\
&= (\Phi \mathrm{d}_{A \otimes_R \mathrm{C}(\psi)})(a \otimes (b', b)).
\end{aligned}$$

□

3.8.6 Tensor Respects Homotopy Equivalences

Proposition 3.25. *Let B be an R-complex, let $\varphi\colon A \to A'$ and $\psi\colon A \to A'$ be two chain maps of R-complexes, and suppose $\varphi \sim \psi$. Then $\varphi \otimes_R B \sim \psi \otimes_R B$.*

Proof. Choose a homotopy $h\colon A \to A'$ from φ to ψ (so $\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A$). We claim that $h \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$ is a homotopy from $\varphi \otimes_R B$ to $\psi \otimes_R B$. Indeed, let $a \otimes b$ be an elementary homogeneous tensor in $A \otimes_R B$. Then we have

$$\begin{aligned}(\mathrm{d}^{\otimes}_{(A',B)}(h\otimes B) + (h\otimes B)\mathrm{d}^{\otimes}_{(A,B)})(a\otimes b) &= \mathrm{d}^{\otimes}_{(A',B)}(h(a)\otimes b) + (h\otimes B)(\mathrm{d}_A(a)\otimes b + (-1)^{|a|}a\otimes \mathrm{d}_B(b))\\ &= \mathrm{d}_{A'}h(a)\otimes b - (-1)^{|a|}h(a)\otimes \mathrm{d}_B(b) + h\mathrm{d}_A(a)\otimes b + (-1)^{|a|}h(a)\otimes \mathrm{d}_B(b)\\ &= \mathrm{d}_{A'}h(a)\otimes b + h\mathrm{d}_A(a)\otimes b\\ &= (\mathrm{d}_{A'}h + h\mathrm{d}_A)(a)\otimes b\\ &= (\varphi-\psi)(a)\otimes b\\ &= \varphi(a)\otimes b - \psi(a)\otimes b\\ &= (\varphi\otimes B - \psi\otimes B)(a\otimes b).\end{aligned}$$

Thus $h \otimes_R B$ is indeed a homotopy from $\varphi \otimes_R B$ to $\psi \otimes_R B$. □

Corollary. *Suppose $\varphi\colon A \to A'$ is a homotopy of equivalence of R-complexes. Then $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$ is a homotopy equivalence of R-complexes.*

Proof. Let $\varphi'\colon A' \to A$ be a homotopy inverse to φ. Thus $\varphi\varphi' \sim 1_{A'}$ and $\varphi'\varphi \sim 1_A$. It follows that

$$\begin{aligned}1_{A'\otimes_R B} &= 1_{A'}\otimes_R B\\ &\sim \varphi\varphi'\otimes_R B\\ &= (\varphi\otimes_R B)(\varphi'\otimes_R B).\end{aligned}$$

Similarly, we have $1_{A\otimes_R B} \sim (\varphi'\otimes_R B)(\varphi\otimes_R B)$. Therefore $\varphi\otimes_R B$ is a homotopy equivalence of R-complexes. □

3.8.7 Twisting the tensor complex with a chain map

Definition 3.17. Let (A, d) be R-complexes and let $\alpha\colon A \to A$ be a chain map. We define an R-complex $A \otimes_R^{\alpha} A$ as follows: as a graded R-module, $A \otimes_R^{\alpha} A$ is just $A \otimes_R A$. We define the differential $\mathrm{d}^{\otimes}_{\alpha}\colon A \otimes_R^{\alpha} A \to A \otimes_R^{\alpha} A$ on elementary tensors $a \otimes b \in A_i \otimes_R A_j$ by

$$\mathrm{d}^{\otimes}_{\alpha}(a\otimes b) = \mathrm{d}(a)\otimes b + (-1)^i\alpha(a)\otimes \mathrm{d}(b) \tag{23}$$

and then we extend $\mathrm{d}^{\otimes}_{\alpha}$ linearly everywhere else. Note that $\mathrm{d}^{\otimes}_{\alpha}$ is a well-defined R-linear map since (23) is R-bilinear in a and b. Also note that $\mathrm{d}^{\otimes}_{\alpha}$ is graded of degree -1 since α is a chain map. Let us show that we have $\mathrm{d}^{\otimes}_{\alpha}\mathrm{d}^{\otimes}_{\alpha} = 0$. Let $a \otimes b \in A_i \otimes_R A_j$. Then we have

$$\begin{aligned}\mathrm{d}^{\otimes}_{\alpha}\mathrm{d}^{\otimes}_{\alpha}(a\otimes b) &= \mathrm{d}^{\otimes}_{\alpha}(\mathrm{d}(a)\otimes b + (-1)^i\alpha(a)\otimes\mathrm{d}(b))\\ &= \mathrm{d}^{\otimes}_{\alpha}(\mathrm{d}(a)\otimes b) + (-1)^i\mathrm{d}^{\otimes}_{\alpha}(\alpha(a)\otimes\mathrm{d}(b))\\ &= \mathrm{d}^2(a)\otimes b + (-1)^{i-1}\alpha\mathrm{d}(a)\otimes\mathrm{d}(b) + (-1)^i\mathrm{d}\alpha(a)\otimes\mathrm{d}(b) + \alpha^2(a)\otimes\mathrm{d}^2(b)\\ &= (-1)^{i-1}\alpha\mathrm{d}(a)\otimes\mathrm{d}(b) + (-1)^i\alpha\mathrm{d}(a)\otimes\mathrm{d}(b)\\ &= 0.\end{aligned}$$

It follows that $\mathrm{d}^{\otimes}_{\alpha}$ is a differential.

If $\alpha\colon A \to A$ is also an R-algebra homomorphism, then observe that

$$\begin{aligned}\mathrm{d}(\alpha(a)(bc) + (ab)\alpha(c)) &= \mathrm{d}(\alpha(a))(bc) + \alpha^2(a)\mathrm{d}(bc) + \mathrm{d}(ab)\alpha(c) + \alpha(ab)\mathrm{d}(\alpha(c))\\ &= \alpha(\mathrm{d}(a))(bc) + \alpha^2(a)(\mathrm{d}(b)c) + \alpha^2(a)(\alpha(b)\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\mathrm{d}(b))\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c))\\ &= \alpha(\mathrm{d}(a))(bc) + (\alpha(a)\mathrm{d}(b))\alpha(c) + (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\mathrm{d}(b))\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c))\\ &= (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c))\\ &= (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + \alpha(ab)\alpha(\mathrm{d}(c))\\ &= 0.\end{aligned}$$

$$\begin{aligned}\mathrm{d}(a(bc) + (ab)c) &= \mathrm{d}(a)(bc) + a\mathrm{d}(bc) + \mathrm{d}(ab)c + (ab)\mathrm{d}(c)\\ &= \mathrm{d}(a)(bc) + a(\mathrm{d}(b)c) + a(b\mathrm{d}(c)) + (\mathrm{d}(a)b)c + (a\mathrm{d}(b))c + (ab)\mathrm{d}(c)\\ &= \mathrm{d}(a)(bc) + (\mathrm{d}(a)b)c + a(\mathrm{d}(b)c) + (a\mathrm{d}(b))c + a(b\mathrm{d}(c)) + (ab)\mathrm{d}(c).\end{aligned}$$

3.9 Hom

Definition 3.18. Let (A, d) and (A', d') be two R-complexes. We define

$$\mathrm{Hom}_R((A,\mathrm{d}),(A',\mathrm{d}')) := (\mathrm{Hom}^\star_R(A,A'), \mathrm{d}^{\mathrm{Hom}^\star_R(A,A')})$$

to be the R-complex whose graded R-module $\mathrm{Hom}^\star_R(A,A')$ has

$$\mathrm{Hom}^\star_R(A,A')_i = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, A'_{n+i})$$

as its ith homogeneous component and whose differential $\mathrm{d}^{\mathrm{Hom}^\star_R(A,A')}$ is defined by

$$\mathrm{d}^{\mathrm{Hom}^\star_R(A,A')}((\varphi^i_n)_{n\in\mathbb{Z}}) = (\mathrm{d}'\varphi^i_n - (-1)^i\varphi^i_{n-1}\mathrm{d})_{n\in\mathbb{Z}} \tag{24}$$

for all $i, n \in \mathbb{Z}$ and $\varphi_{n,i} \in \mathrm{Hom}_R(A_j, A'_{i+j})$.

If context is clear, we will denote $\mathrm{d}^{\mathrm{Hom}^\star_R(A,A')}$ simply as $\mathrm{d}^\star$. We also write (φ^i_n) instead of $(\varphi^i_n)_{n\in\mathbb{Z}}$. The subscript n will clue us in on the fact that (φ^i_n) is a sequence of homomorphisms. Sometimes we will also write $\mathrm{Hom}^\star_R(A,A')$ (rather than the more cumbersome notation $\mathrm{Hom}_R((A,\mathrm{d}),(A',\mathrm{d}'))$) and specify that $\mathrm{Hom}^\star_R(A,A')$ refers to the R-complex hom and not just the graded R-module hom.

Let us check that $\mathrm{d}^\star\mathrm{d}^\star = 0$. Let $(\varphi^i_n) \in \mathrm{Hom}^\star_R(A,A')_i$. Then we have

$$\begin{aligned}
\mathrm{d}^\star\mathrm{d}^\star(\varphi^i_n) &= \mathrm{d}^\star(\mathrm{d}'\varphi^i_n - (-1)^i\varphi^i_{n-1}\mathrm{d}) \\
&= (\mathrm{d}'(\mathrm{d}'\varphi^i_n - (-1)^i\varphi^i_{n-1}\mathrm{d}) - (-1)^{i-1}(\mathrm{d}'\varphi^i_{n-1} - (-1)^i\varphi^i_{n-2}\mathrm{d})\mathrm{d}) \\
&= -(-1)^i\mathrm{d}'\varphi^i_{n-1}\mathrm{d} - (-1)^{i-1}\mathrm{d}'\varphi^i_{n-1}\mathrm{d} \\
&= 0.
\end{aligned}$$

Thus $\mathrm{d}^\star\mathrm{d}^\star = 0$. Note that the sign $-(-1)^i$ in (24) is a little unusual. In the tensor product differential $\mathrm{d}^\otimes$, we had

$$\mathrm{d}^\otimes(a\otimes a') = \mathrm{d}(a)\otimes a' + (-1)^i a\otimes \mathrm{d}'(a')$$

whenever $a \in A_i$ and $a' \in A'_{i'}$. If we replace the sign $-(-1)^i$ with the sign $(-1)^i$ in (24), we would still get $\mathrm{d}^\star\mathrm{d}^\star = 0$. However, for reasons to be clarified later on, we keep the sign $-(-1)^i$.

Note that if A' is just an R-module (so trivially graded with $\mathrm{d}' = 0$), then

$$\mathrm{Hom}^\star_R(A,A')_i \cong \mathrm{Hom}_R(A_{-i}, A').$$

In this case, we have

$$\mathrm{d}^\star(\varphi) = -(-1)^i\varphi\mathrm{d}$$

whenever $\varphi \in \mathrm{Hom}_R(A_{-i}, A')$. Also, if A is just an R-module (so trivially graded with $\mathrm{d} = 0$), then

$$\mathrm{Hom}^\star_R(A,A')_i \cong \mathrm{Hom}_R(A, A'_i).$$

In this case, we have

$$\mathrm{d}^\star(\varphi) = \mathrm{d}'\varphi$$

whenever $\varphi \in \mathrm{Hom}_R(A, A'_i)$.

3.9.1 Reinterpretation of Hom

Definition 3.19. Let A and A' be two R-complexes. We define their **hom complex**, denoted $(\mathrm{Hom}^\star_R(A,A'), \mathrm{d}^\star_{(A,A')})$, to be the R-complex whose underlying graded R-module $\mathrm{Hom}^\star_R(A,A')$ has

$$\mathrm{Hom}^\star_R(A,A')_i = \{\alpha\colon A \to A' \mid \alpha \text{ is graded of degree } i\}$$

as its homogeneous component in degree i, and whose differential is defined by

$$\mathrm{d}^\star_{(A,A')}(\alpha) = \mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_A$$

for all $\alpha \in \mathrm{Hom}^\star_R(A,A')_i$ for all $i \in \mathbb{Z}$.

3.9.2 Homology of Hom

Proposition 3.26. *Let A and A' be two R-complexes. Then*

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(A,A')) = \{\textit{homotopy classes of chain maps } A \to A'\}.$$

Proof. Recall that homotopy gives an equivalence relation $\sim$ on the set of all chain maps $\mathcal{C}(A,A')$ from A to A'. Thus we are saying that

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(A,A')) = \mathcal{C}(A,A')/\sim.$$

Let $\alpha \in \mathrm{Z}_0(\mathrm{Hom}_R^\star(A,A'))$, so $\alpha\colon A \to A'$ be a graded R-linear map of degree 0 such that

$$\begin{aligned} 0 &= \mathrm{d}^\star_{(A,A')}(\alpha) \\ &= \mathrm{d}_{A'}\alpha - \alpha\mathrm{d}_A. \end{aligned}$$

In other words, α is a chain map. It follows that

$$\mathrm{Z}_0(\mathrm{Hom}_R^\star(A,A')) = \mathcal{C}(A,A').$$

Next we observe that elements in $\mathrm{B}_0(\mathrm{Hom}_R^\star(A,A'))$ are of the form

$$\mathrm{d}^\star_{(A,A')}(\beta) = \mathrm{d}_{A'}\beta + \beta\mathrm{d}_A$$

where $\beta\colon A \to A'$ be a graded R-linear map of degree 1. Thus two chain maps α_1 and α_2 represent the same class in homology if and only if they are homotopic to each other. □

Remark. More generally, $\mathrm{H}_i(\mathrm{Hom}_R^\star(A,A'))$ is exact if and only if for all graded R-linear maps $\alpha\colon A \to A'$ of degree i such that

$$\mathrm{d}_{A'}\alpha = (-1)^i\alpha\mathrm{d}_A,$$

there exists a graded R-linear map $\beta\colon A \to A'$ such that

$$\alpha = \mathrm{d}_A\beta + (-1)^i\beta\mathrm{d}_{A'}.$$

3.9.3 Functorial Properties of Hom

Proposition 3.27. *Let (A,d_A), (A',d'_A), (B,d_B), and (B',d'_B) be R-complexes and let $\varphi\colon A \to B$ and $\phi\colon A' \to B'$ be chain maps. Then we get induced chain maps*

$$\phi_*\colon \mathrm{Hom}_R^\star(A,A') \to \mathrm{Hom}_R^\star(A,B') \quad \textit{and} \quad \varphi^*\colon \mathrm{Hom}_R^\star(B,B') \to \mathrm{Hom}_R^\star(A,B')$$

given by

$$\phi_*(\alpha) = \phi\alpha \quad \textit{and} \quad \varphi^*(\beta) = \beta\varphi$$

for all $\alpha \in \mathrm{Hom}_R^\star(A,A')$ and $\beta \in \mathrm{Hom}_R^\star(B,B')$. Furthermore, the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_R^\star(A,A') \arrow[r, "\varphi^*"] \arrow[d, "\phi_*"'] & \mathrm{Hom}_R^\star(B,A') \arrow[d, "\phi_*"] \\ \mathrm{Hom}_R^\star(A,B') \arrow[r, "\varphi^*"'] & \mathrm{Hom}_R^\star(B,B') \end{tikzcd} \tag{25}$$

Proof. First let us check that ϕ_* is a chain map. It is a graded R-linear map since ϕ is a graded R-linear map of degree 0 and composition is R-linear. It remains to show that ϕ_* commutes with the differentials. Let $\alpha \in \mathrm{Hom}_R^\star(A,A')_i$. Then we have

$$\begin{aligned} (\mathrm{d}^\star_{(A,B')}\phi_*)(\alpha) &= \mathrm{d}^\star_{(A,B')}(\phi_*(\alpha)) \\ &= \mathrm{d}^\star_{(A,B')}(\phi\alpha) \\ &= \mathrm{d}_{B'}\phi\alpha - (-1)^i\phi\alpha\mathrm{d}_A \\ &= \phi\mathrm{d}_{A'}\alpha - (-1)^i\phi\alpha\mathrm{d}_A \\ &= \phi_*(\mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_A) \\ &= \phi_*(\mathrm{d}^\star_{(A,A')}(\alpha)) \\ &= (\phi_*\mathrm{d}^\star_{(A,A')})(\alpha). \end{aligned}$$

This implies ϕ_* is a chain map. A similar calculation shows that φ^* is a chain map.

Now we check that the diagram (25) commutes. Let $\alpha \in \mathrm{Hom}_R^\star(A, A')_i$. Then we have

$$\begin{aligned}(\phi_* \varphi^*)(\alpha) &= \phi_*(\varphi^*(\alpha)) \\ &= \phi_*(\alpha\varphi) \\ &= \phi\alpha\varphi \\ &= \varphi^*(\phi\alpha) \\ &= \varphi^*(\phi_*(\alpha)) \\ &= (\varphi^*\phi_*)(\alpha).\end{aligned}$$

This implies the diagram commutes. □

Proposition 3.28. *Let A be an R-complex. Then we obtain functors*

$$\mathrm{Hom}_R^\star(A, -) \colon \mathrm{Comp}_R \to \mathrm{Comp}_R \quad \textit{and} \quad \mathrm{Hom}_R^\star(-, A) \colon \mathrm{Comp}_R \to \mathrm{Comp}_R$$

from the category of R-complexes to itself, where the R-complex B is assigned to the R-complexes

$$\mathrm{Hom}_R^\star(A, B) \quad \textit{and} \quad \mathrm{Hom}_R^\star(B, A)$$

respectively, and where the chain map $\varphi \colon B \to B'$ of R-complexes is assigned to the chain maps

$$\mathrm{Hom}_R^\star(A, \varphi) = \varphi_* \quad \textit{and} \quad \mathrm{Hom}_R^\star(\varphi, A) = \varphi^*$$

respectively.

Proof. We will just show that $\mathrm{Hom}_R^\star(A, -)$ is a functor from the category of R-complexes to itself since a similar argument will show that $\mathrm{Hom}_R^\star(-, A)$ is one too. We need to check that $\mathrm{Hom}_R^\star(A, -)$ preserves compositions and identities. We first check that it preserves compositions. Let $\varphi \colon B \to B'$ and $\varphi' \colon B' \to B''$ be two chain maps and let $\alpha \in \mathrm{Hom}_R^\star(A, B)_i$. Then we have

$$\begin{aligned}(\varphi'\varphi)_*(\alpha) &= \varphi'\varphi\alpha \\ &= \varphi'_*(\varphi\alpha) \\ &= \varphi'_*(\varphi_*(\alpha)) \\ &= (\varphi'_*\varphi_*)(\alpha)\end{aligned}$$

It follows that $(\varphi'\varphi)_* = \varphi'_*\varphi_*$. Hence $\mathrm{Hom}_R^\star(A, -)$ preserves compositions. Next we check that $\mathrm{Hom}_R^\star(A, -)$ preserves identities. Let B be an R-complex and let $\alpha \colon A \to B$ be a chain map. Then we have

$$\begin{aligned}(1_B)_* &= 1_B\alpha \\ &= \alpha \\ &= 1_{\mathrm{Hom}_R^\star(A,B)}(\alpha).\end{aligned}$$

It follows that $(1_B)_* = 1_{\mathrm{Hom}_R^\star(A,-)}$. Hence h_A preserves identities. □

Proposition 3.29. *Let F be a covariant functor from the category of R-complexes to itself. Then F is left exact if and only if it is left exact when viewed as a functor of the underlying graded R-modules.*

Proof. One direction is easy, so we prove the other direction. Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{26}$$

be an exact sequence of R-complexes and chain maps. Then (26) is an exact sequence of graded R-modules and graded homomorphisms. Thus

$$F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2)} F(M_3) \longrightarrow 0 \tag{27}$$

is an exact sequence of graded R-modules and graded homomorphisms. Since the graded homomorphisms in (27) commute with the differentials, we see that (27) is actually an exact sequence of R-complexes and chain maps.

□

Proposition 3.30. *(Yoneda's Lemma) Let A be an R-complex and let $\mathcal{F}\colon \mathbf{Comp}_R \to \mathbf{Set}$ be a functor. Then we have a bijection*

$$\mathrm{Nat}(\mathcal{C}(A,-),\mathcal{F}) \cong \mathcal{F}(A)$$

which is natural in A. In particular, if B is another R-complex, then

$$\mathrm{Nat}(\mathcal{C}(A,-),C(B,-)) \cong \mathcal{C}(B,A)$$

Note that the diagram (25) tells us that each chain map $\varphi\colon A \to B$ gives rise to a natural transformation $\mathrm{h}^-(\varphi)\colon \mathrm{h}_A \to \mathrm{h}_B$. In light of Yoneda's Lemma, we have a map

$$\mathrm{Nat}(C(B,-),\mathcal{C}(A,-)) \to \mathcal{C}(A,B) \to \mathrm{Nat}(\mathrm{h}_A,\mathrm{h}_B).$$

3.9.4 Left Exactness of Contravariant $\mathrm{Hom}^\star_R(-,N)$

Let M and N be R-complexes. We showed earlier that both $\mathrm{Hom}^\star_R(M,-)$ and $\mathrm{Hom}^\star_R(-,N)$ are left exact functors from the category of graded R-modules to itself. In fact, we will see that they The graded version of these functors are

$$\mathrm{Hom}^\star_R(M,-)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R \quad \text{and} \quad \mathrm{Hom}^\star_R(-,N)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R.$$

We want to check that they are also left exact functors. Let's focus on $\mathrm{Hom}^\star_R(-,N)$ first:

Proposition 3.31. *The sequence of graded R-modules and graded homomorphisms*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{28}$$

is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}^\star_R(M_3,N) \xrightarrow{\varphi_2^*} \mathrm{Hom}^\star_R(M_2,N) \xrightarrow{\varphi_1^*} \mathrm{Hom}^\star_R(M_1,N) \tag{29}$$

is exact.

Proof. Suppose that (28) is exact and let N be any R-module. Exactness at $\mathrm{Hom}^\star_R(M_3,N)$ follows from the fact that φ_2^* is injective (which follows from the fact that $\mathrm{Hom}_R(-,N)$ is left exact). Next we show exactness at $\mathrm{Hom}^\star_R(M_2,N)$. Let $\psi_2\colon M_2 \to N$ be a graded homomorphism of degree i such that $\psi_2\varphi_1 = 0$. By left exactness of $\mathrm{Hom}_R(-,N)$, there exists a $\psi_3 \in \mathrm{Hom}_R(M,N)$ such that $\psi_2 = \psi_3\varphi_2$. Since φ_2 is surjective, ψ_3 is graded of degree i. Thus $\psi_3 \in \mathrm{Hom}^\star_R(M,N)$. Thus we have exactness at $\mathrm{Hom}^\star_R(M_2,N)$. □

3.9.5 Tensor-Hom Adjointness

Proposition 3.32. *Let S be an R-algebra, let M_1, M_2 be S-complexes, and let M_3 be an R-complex. Then we have an isomorphism of S-complexes*

$$\mathrm{Hom}^\star_S(M_1,\mathrm{Hom}^\star_R(M_2,M_3)) \cong \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3). \tag{30}$$

Moreover (30) is natural in M_1, M_2, and M_3.

Proof. We define

$$\Psi_{M_1,M_2,M_3}\colon \mathrm{Hom}^\star_S(M_1,\mathrm{Hom}^\star_R(M_2,M_3)) \to \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3)$$

to be the map which sends a $\psi \in \mathrm{Hom}^\star_S(M_1,\mathrm{Hom}^\star_R(M_2,M_3))$ to the map $\Psi(\psi) \in \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3)$ defined by

$$\Psi(\psi)(u_1 \otimes u_2) = (\psi(u_1))(u_2) \tag{31}$$

for all elementary tensors $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Note that $\Psi(\psi)$ is a well-defined R-linear map since the map $M_1 \times M_2 \to M_3$ given by

$$(u_1,u_2) \mapsto (\psi(u_1))(u_2)$$

is R-bilinear. We will show that Ψ is an isomorphism of S-complexes by breaking down the proof into several steps:

Step 1: We show that Ψ is S-linear. Let $s, s' \in S$ and $\psi, \psi' \in \mathrm{Hom}^\star_S(M_1,\mathrm{Hom}^\star_R(M_2,M_3))$. We want to show that

$$\Psi(s\psi + s'\psi') = s\Psi(\psi) + s'\Psi(\psi') \tag{32}$$

We will show (32) holds, by showing that the two maps agree on all elementary tensors in $M_1 \otimes_S M_2$. So let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then

$$\begin{aligned}
\Psi(s\psi + s'\psi')(u_1 \otimes u_2) &= ((s\psi + s'\psi')(u_1))(u_2) \\
&= ((s\psi)(u_1) + (s'\psi')(u_1))(u_2) \\
&= (\psi(su_1) + \psi(s'u_1))(u_2) \\
&= (\psi(su_1))(u_2) + (\psi(s'u_1))(u_2) \\
&= \Psi(\psi)(su_1 \otimes u_2) + \Psi(\psi')(s'u_1 \otimes u_2) \\
&= (s\Psi(\psi))(u_1 \otimes u_2) + (s'\Psi(\psi'))(u_1 \otimes u_2). \\
&= (s\Psi(\psi) + s'\Psi(\psi))(u_1 \otimes u_2)
\end{aligned}$$

It follows that Ψ is S-linear.

Step 2: We show that Ψ is graded. Let ψ be a graded S-linear map from M_1 to $\mathrm{Hom}_R^\star(M_2, M_3)$ of degree n. We want to show that $\Psi(\psi)$ is a graded of degree n too. To see that $\Psi(\psi)$ is graded of degree n, let $u_1 \otimes u_2$ be an elementary tensor in $M_1 \otimes_S M_2$ where u_i has degree i and u_j has degree j. Since ψ is graded of degree n, u_1 is graded of degree i, and u_2 is graded of degree j, we see that $\psi(u_1)$ is graded of degree $i+n$, and hence

$$(\psi(u_1))(u_2) = \Psi(\psi)(u_1 \otimes u_2)$$

is graded of degree $i+j+n$. It follows that $\Psi(\psi)$ is graded of degree n.

Step 3: We show that Ψ commutes with the differentials. In other words, we want to show that

$$\mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)} \Psi = \Psi \mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))} \tag{33}$$

To see that (33) holds, it suffices to show that it holds when we apply to both sides any graded S-linear map of degree n from M_1 to $\mathrm{Hom}_R^\star(M_2, M_3)$. So let ψ be such a map. Then observe on the one hand, we have

$$\begin{aligned}
(\mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)} \Psi)(\psi) &= \mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)}(\Psi(\psi)) \\
&= \mathrm{d}_{M_3}\Psi(\psi) + (-1)^n \Psi(\psi)\mathrm{d}^\otimes_{(M_1, M_2)},
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
(\Psi \mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))})(\psi) &= \Psi(\mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))}(\psi)) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)}\psi + (-1)^n \psi \mathrm{d}_{M_1}) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)}\psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}).
\end{aligned}$$

Thus we are reduced to showing that

$$\mathrm{d}_{M_3}\Psi(\psi) + (-1)^n \Psi(\psi)\mathrm{d}^\otimes_{(M_1, M_2)} = \Psi(\mathrm{d}^\star_{(M_2, M_3)}\psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}) \tag{34}$$

To see that (34) holds, it suffices to show that it holds when we apply any elementary homogeneous tensor in $M_1 \otimes_S M_2$ to both sides. So let $u_1 \otimes u_2 \in M_{1,i} \otimes_R M_{2,j}$ be such an elementary homogeneous tensor, so u_1 is graded of degree i and u_2 is graded of degree j. In the following calculation, we suppress parentheses as much as possible in order to clean notation. We gave

$$\begin{aligned}
(\mathrm{d}_{M_3}\Psi(\psi) + (-1)^n \Psi(\psi)\mathrm{d}^\otimes_{(M_1, M_2)})(u_1 \otimes u_2) &= \mathrm{d}_{M_3}\Psi(\psi)(u_1 \otimes u_2) + (-1)^n \Psi(\psi)\mathrm{d}^\otimes_{(M_1, M_2)}(u_1 \otimes u_2) \\
&= \mathrm{d}_{M_3}\psi(u_1)(u_2) + (-1)^n \Psi(\psi)(\mathrm{d}_{M_1}(u_1) \otimes u_2 + (-1)^i u_1 \otimes \mathrm{d}_{M_2}(u_2)) \\
&= \mathrm{d}_{M_3}\psi(u_1)(u_2) + (-1)^n \Psi(\psi)(\mathrm{d}_{M_1}(u_1) \otimes u_2) + (-1)^{i+n} \Psi(\psi)(u_1 \otimes \mathrm{d}_{M_2}(u_2)) \\
&= \mathrm{d}_{M_3}\psi(u_1)(u_2) + (-1)^n \psi(\mathrm{d}_{M_1}(u_1))(u_2) + (-1)^{i+n} \psi(u_1)(\mathrm{d}_{M_2}(u_2)) \\
&= (\mathrm{d}_{M_3}\psi(u_1) + (-1)^{i+n} \psi(u_1)\mathrm{d}_{M_2})(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= (\mathrm{d}^\star_{(M_2, M_3)}(\psi(u_1))(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= (\mathrm{d}^\star_{(M_2, M_3)}\psi)(u_1)(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)}\psi)(u_1 \otimes u_2) + (-1)^n \Psi(\psi \mathrm{d}_{M_1})(u_1 \otimes u_2) \\
&= (\Psi(\mathrm{d}^\star_{(M_2, M_3)}\psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}))(u_1 \otimes u_2).
\end{aligned}$$

It follows that Ψ commutes with the differentials.

Step 4: We will show that Ψ is a bijection. It will then follows that Ψ gives an isomorphism of S-complexes. We construct its inverse as follows: we define

$$\Phi_{M_1,M_2,M_3} \colon \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3) \to \mathrm{Hom}_S^\star(M_1, \mathrm{Hom}_R^\star(M_2, M_3))$$

to be the map given by

$$(\Phi(\varphi)(u_1))(u_2) = \varphi(u_1 \otimes u_2)$$

for all $\varphi \in \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3)$, $u_1 \in M_1$, and $u_2 \in M_2$. We claim that Ψ and Φ are inverse to each other. Indeed, we have

$$\begin{aligned} \Psi(\Phi(\varphi))(u_1 \otimes u_2) &= (\Phi(\varphi)(u_1))(u_2) \\ &= \varphi(u_1 \otimes u_2) \end{aligned}$$

for all $\varphi \in \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3)$ and $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Thus $\Psi\Phi = 1$. Similarly, we have

$$\begin{aligned} (\Phi(\Psi(\psi))(u_1))(u_2) &= \Psi(\psi)(u_1 \otimes u_2) \\ &= (\psi(u_1))(u_2) \end{aligned}$$

for all $\psi \in \mathrm{Hom}_S^\star(M_1, \mathrm{Hom}_R^\star(M_2, M_3))$ and $u_1 \in M_1$ and $u_2 \in M_2$. Thus $\Phi\Psi = 1$.

Step 5: We show naturality in M_1, M_2, and M_3. Naturality in M_1 means that if $\lambda \colon M_1 \to M_1'$ is an R-module homomorphism, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(M_1', \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1',M_3}"] \arrow[d, "\lambda^*"'] & \mathrm{Hom}_R(M_1' \otimes_S M_2, M_3) \arrow[d, "(\lambda\otimes 1)^*"] \\ \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1,M_3}"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3) \end{tikzcd}$$

Thus we want to show for all $\psi \in \mathrm{Hom}_S^\star(M_1', \mathrm{Hom}_R^\star(M_2, M_3))$, we have

$$(\lambda \otimes 1)^* \left(\Psi_{M_1',M_3}(\psi)\right) = \Psi_{M_1,M_3}(\lambda^*(\psi)) \tag{35}$$

To see that (35) is equal, we apply all elementary tensors to both sides. Let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then we have

$$\begin{aligned} \left((\lambda \otimes 1)^* \left(\Psi_{M_1',M_3}(\psi)\right)\right)(u_1 \otimes u_2) &= (\Psi_{M_1,M_3}(\psi))\,((\lambda \otimes 1)(u_1 \otimes u_2)) \\ &= (\Psi_{M_1,M_3}(\psi))\,(\lambda(u_1) \otimes u_2) \\ &= (\psi(\lambda(u_1))(u_2) \\ &= ((\lambda^*(\psi))(u_1))(u_2) \\ &= (\Psi_{M_1,M_3}(\lambda^*(\psi)))\,(u_1 \otimes u_2) \\ &= (\Psi_{M_1,M_3}(\lambda^*(\psi)))\,(u_1 \otimes u_2). \end{aligned}$$

Similarly, naturality in M_3 means that if $\lambda \colon M_3 \to M_3'$ is an R-module homomorphism, then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1,M_3}"] \arrow[d, "(\lambda_*)_*"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3) \arrow[d, "\lambda_*"] \\ \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3')) \arrow[r, "\Psi_{M_1,M_3'}"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3') \end{tikzcd}$$

Thus we want to show for all $\psi \in \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3))$, we have

$$\lambda_* \left(\Psi_{M_1,M_3}(\psi)\right) = \Psi_{M_1,M_3'}((\lambda_*)_*(\psi)) \tag{36}$$

To see that (36) is equal, we apply all elementary tensors to both sides. Let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then we have

$$
\begin{aligned}
(\lambda_* (\Psi_{M_1,M_3}(\psi)))\,(u_1 \otimes u_2) &= \lambda\left((\Psi_{M_1,M_3}(\psi))\,(u_1 \otimes u_2)\right) \\
&= \lambda\left((\psi(u_1))(u_2)\right) \\
&= (\lambda_*(\psi(u_1)))(u_2) \\
&= ((\lambda_*)_*(\psi))(u_1))(u_2) \\
&= \left(\Psi_{M_1,M_3'}((\lambda_*)_*(\psi))\right)(u_1 \otimes u_2).
\end{aligned}
$$

□

There is another version of Tensor-Hom adjointness which we will state now but not prove.

Proposition 3.33. *Let S be an R-algebra, let M_2, M_3 be S-complexes, and let M_1 be an R-complex. Then we have an isomorphism of S-complexes*

$$
\mathrm{Hom}_R^\star(M_1, \mathrm{Hom}_S^\star(M_2, M_3)) \cong \mathrm{Hom}_S^\star(M_1 \otimes_R M_2, M_3). \tag{37}
$$

Moreover (30) is natural in M_1, M_2, and M_3.

3.9.6 Hom Commutes with Shifts

Proposition 3.34. *Let $n \in \mathbb{Z}$ and let A and A' be R-complexes. Then*

$$
\mathrm{Hom}_R^\star(\Sigma^n A, A') \cong \Sigma^{-n}\mathrm{Hom}_R^\star(A, A') \quad \textit{and} \quad \mathrm{Hom}_R^\star(A, \Sigma^n A') \cong \Sigma^n \mathrm{Hom}_R^\star(A, A')
$$

are isomorphisms of R-complexes.

Remark. Thus the covariant functor $\mathrm{Hom}_R^\star(A, -)$ commutes with shifts and the contravariant functor $\mathrm{Hom}_R^\star(-, A')$ anticommutes with shifts.

Proof. We will first show $\mathrm{Hom}_R^\star(\Sigma^n A, A') \cong \Sigma^{-n}\mathrm{Hom}_R^\star(A, A')$. As graded R-modules, we have

$$
\begin{aligned}
\mathrm{Hom}_R^\star(\Sigma^n A, A') &= \mathrm{Hom}_R^\star(A(-n), A') \\
&= \mathrm{Hom}_R^\star(A, A')(n) \\
&= \Sigma^{-n}\mathrm{Hom}_R^\star(A, A').
\end{aligned}
$$

We define $\Phi\colon \mathrm{Hom}_R^\star(\Sigma^n A, A') \to \Sigma^{-n}\mathrm{Hom}_R^\star(A, A')$ by

$$
\Phi(\alpha) = (-1)^{x_i}\alpha
$$

for all $\alpha \in \mathrm{Hom}_R^\star(\Sigma^n A, A')$ where $x_i \in \mathbb{Z}$ satisfies

$$
x_i = n + x_{i-1}
$$

for all $i \in \mathbb{Z}$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \mathrm{Hom}_R^\star(\Sigma^n A, A')_i$; so $\alpha\colon A \to A'$ is a graded homomorphism of degree $n + i$. Then we have

$$
\begin{aligned}
(\Sigma^{-n}\mathrm{d}^\star_{(A,A')}\Phi)(\alpha) &= (-1)^{-n}\mathrm{d}^\star_{(A,A')}(\Phi(\alpha)) \\
&= (-1)^{-n+x_i}\mathrm{d}^\star_{(A,A')}(\alpha) \\
&= (-1)^{-n+x_i}(\mathrm{d}_{A'}\alpha - (-1)^{n+i}\alpha\mathrm{d}_A) \\
&= (-1)^{-n+x_i}\mathrm{d}_{A'}\alpha - (-1)^{x_i+i}\alpha\mathrm{d}_A) \\
&= (-1)^{x_{i-1}}\mathrm{d}_{A'}\alpha - (-1)^{i+x_{i-1}+n}\alpha\mathrm{d}_A \\
&= (-1)^{x_{i-1}}\mathrm{d}_{A'}\alpha - (-1)^{i+x_{i-1}}\alpha\mathrm{d}_{\Sigma^n A} \\
&= \Phi(\mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_{\Sigma^n A}) \\
&= \Phi(\mathrm{d}^\star_{(\Sigma^n A,A')}(\alpha)) \\
&= (\Phi\mathrm{d}^\star_{(\Sigma^n A,A')})(\alpha)
\end{aligned}
$$

Now we will show $\mathrm{Hom}_R^\star(A, \Sigma^n A') \cong \Sigma^n\mathrm{Hom}_R^\star(A, A')$. As graded R-modules, we have

$$
\begin{aligned}
\mathrm{Hom}_R^\star(A, \Sigma^n A') &= \mathrm{Hom}_R^\star(A, A'(-n)) \\
&= \mathrm{Hom}_R^\star(A, A')(-n) \\
&= \Sigma^n\mathrm{Hom}_R^\star(A, A').
\end{aligned}
$$

We define $\Phi\colon \mathrm{Hom}_R^\star(A,\Sigma^n A') \to \Sigma^n \mathrm{Hom}_R^\star(A,A')$ by

$$\Phi(\alpha) = (-1)^{x_i}\alpha$$

for all $\alpha \in \mathrm{Hom}_R^\star(A,\Sigma^n A')$ where $x_i \in \mathbb{Z}$ satisfies

$$x_i = x_{i-1}$$

for all $i \in \mathbb{Z}$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \mathrm{Hom}_R^\star(A,\Sigma^n A')_i$; so $\alpha\colon A \to A'$ is a graded homomorphism of degree $i-n$. Then we have

$$\begin{aligned}
(\Sigma^n \mathrm{d}^\star_{(A,A')}\Phi)(\alpha) &= (-1)^n \mathrm{d}^\star_{(A,A')}(\Phi(\alpha)) \\
&= (-1)^{n+x_i}\mathrm{d}^\star_{(A,A')}(\alpha) \\
&= (-1)^{n+x_i}(\mathrm{d}_{A'}\alpha - (-1)^{i-n}\alpha \mathrm{d}_A) \\
&= (-1)^{n+x_i}\mathrm{d}_{A'}\alpha - (-1)^{x_i+i}\alpha \mathrm{d}_A) \\
&= (-1)^{x_{i-1}}\mathrm{d}_{\Sigma^n A'}\alpha - (-1)^{x_{i-1}+i}\alpha \mathrm{d}_A \\
&= \Phi(\mathrm{d}_{\Sigma^n A'}\alpha - (-1)^i \alpha \mathrm{d}_A) \\
&= \Phi(\mathrm{d}^\star_{(A,\Sigma^n A')}(\alpha)) \\
&= (\Phi \mathrm{d}^\star_{(A,\Sigma^n A')})(\alpha)
\end{aligned}$$

□

3.9.7 Hom Commutes with Mapping Cone

Proposition 3.35. *Let X and Y be R-complexes and let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then*

$$\mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi)) \cong \mathrm{C}(\mathrm{Hom}_R^\star(X,\varphi)) \quad \text{and} \quad \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi),Y) \cong \mathrm{C}(\mathrm{Hom}_R^\star(\varphi,Y))$$

are isomorphisms of R-complexes.

Proof. We first show $\mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi)) \cong \mathrm{C}(\varphi_*)$. As graded R-modules, we have

$$\begin{aligned}
\mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi)) &= \mathrm{Hom}_R^\star(X,A' \oplus A(-1)) \\
&\cong \mathrm{Hom}_R^\star(X,A') \oplus \mathrm{Hom}_R^\star(X,A(-1)) \\
&= \mathrm{Hom}_R^\star(X,A') \oplus \mathrm{Hom}_R^\star(X,A)(-1) \\
&= \mathrm{C}(\varphi_*),
\end{aligned}$$

where the graded isomorphism in the second line is given by

$$\alpha \mapsto (\pi_1\alpha, \pi_2\alpha)$$

for all $\alpha \in \mathrm{Hom}_R^\star(X,A' \oplus A(-1))$, where

$$\pi_1\colon A' \oplus A(-1) \to A' \quad \text{and} \quad \pi_2\colon A' \oplus A(-1) \to A(-1)$$

are the natural projection maps.

We define $\Phi\colon \mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi)) \to \mathrm{C}(\varphi_*)$ by

$$\Phi(\alpha) = (\pi_1\alpha, \pi_2\alpha)$$

for all $\alpha \in \mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi))$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \mathrm{Hom}_R^\star(X,\mathrm{C}(\varphi))_i$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi_*)}\Phi)(\alpha) &= \mathrm{d}_{\mathrm{C}(\varphi_*)}(\Phi(\alpha)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi_*)}(\pi_1\alpha, \pi_2\alpha) \\
&= (\mathrm{d}^\star_{(X,A')}(\pi_1\alpha) + \varphi_*(\pi_2\alpha), -\mathrm{d}^\star_{(X,A)}(\pi_2\alpha)) \\
&= (\mathrm{d}_{A'}\pi_1\alpha - (-1)^i\pi_1\alpha\mathrm{d}_X + \varphi\pi_2\alpha, -\mathrm{d}_A\pi_2\alpha - (-1)^i\pi_2\alpha\mathrm{d}_X) \\
&= (\pi_1\mathrm{d}_{\mathrm{C}(\varphi)}\alpha - (-1)^i\pi_1\alpha\mathrm{d}_X, \pi_2\mathrm{d}_\varphi\alpha - (-1)^i\pi_2\alpha\mathrm{d}_X) \\
&= \Phi(\mathrm{d}_{\mathrm{C}(\varphi)}\alpha - (-1)^i\alpha\mathrm{d}_X) \\
&= \Phi(\mathrm{d}^\star_{(X,\mathrm{C}(\varphi))}(\alpha)) \\
&= (\Phi\mathrm{d}^\star_{(X,\mathrm{C}(\varphi))})(\alpha)
\end{aligned}$$

where we used the fact that $-\mathrm{d}_A\pi_2 = \pi_2\mathrm{d}_\varphi$ and $\pi_1\mathrm{d}_\varphi = \mathrm{d}_{A'}\pi_1 + \varphi\pi_2$.

Now we show $\Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) \cong \mathrm{C}(\varphi^*)$. As graded R-modules, we have

$$\begin{aligned}
\Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) &= \mathrm{Hom}_R^\star(A' \oplus A(-1), Y)(-1) \\
&\cong \mathrm{Hom}_R^\star(A', Y)(-1) \oplus \mathrm{Hom}_R^\star(A(-1), Y))(-1) \\
&= \mathrm{Hom}_R^\star(A', Y)(-1) \oplus \mathrm{Hom}_R^\star(A, Y)) \\
&\cong \mathrm{Hom}_R^\star(A, Y) \oplus \mathrm{Hom}_R^\star(A', Y)(-1) \\
&= \mathrm{C}(\varphi_*),
\end{aligned}$$

where the graded isomorphism in the second line is given by

$$\alpha \mapsto (\alpha\iota_1, \alpha\iota_2)$$

for all $\alpha \in \mathrm{Hom}_R^\star(X, A' \oplus A(-1))$, where

$$\iota_1 \colon A' \to A' \oplus A(-1) \quad \text{and} \quad \iota_2 \colon A(-1) \to A' \oplus A(-1)$$

are the natural inclusion maps.

We define $\Phi \colon \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) \to \mathrm{C}(\varphi_*)$ by

$$\Phi(\alpha) = (\alpha\iota_2, \alpha\iota_1)$$

for all $\alpha \in \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y)$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y)_i$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi^*)}\Phi)(\alpha) &= \mathrm{d}_{\mathrm{C}(\varphi^*)}(\Phi(\alpha)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi^*)}(\alpha\iota_2, \alpha\iota_1) \\
&= (\mathrm{d}^\star_{(A,Y)}(\alpha\iota_2) + \varphi^*(\alpha\iota_1), -\mathrm{d}^\star_{(A',Y)}(\alpha\iota_1)) \\
&= (\mathrm{d}_Y\alpha\iota_2 + (-1)^i\alpha\iota_2\mathrm{d}_A + \alpha\iota_1\varphi, -\mathrm{d}_Y\alpha\iota_1 + (-1)^i\alpha\iota_1\mathrm{d}_{A'}) \\
&= (-\mathrm{d}_Y\alpha\iota_2 + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}\iota_2, -\mathrm{d}_Y\alpha\iota_1 + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}\iota_1) \\
&= \Phi(-\mathrm{d}_Y\alpha + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}) \\
&= \Phi(-\mathrm{d}^\star_{(\mathrm{C}(\varphi),Y)}(\alpha)) \\
&= (\Phi\Sigma\mathrm{d}^\star_{(\mathrm{C}(\varphi),Y)})(\alpha)
\end{aligned}$$

where we used the fact that $\iota_2\mathrm{d}_A = \iota_1\varphi - \mathrm{d}_{\mathrm{C}(\varphi)}\iota_2$ and $\mathrm{d}_{\mathrm{C}(\varphi)}\iota_1 = \iota_1\mathrm{d}_{A'}$. □

3.9.8 Hom Preserves Homotopy Equivalences

Proposition 3.36. *Let B be an R-complex, let $\varphi \colon A \to A'$ and $\psi \colon A \to A'$ be two chain maps of R-complexes, and suppose $\varphi \sim \psi$. Then $\mathrm{Hom}_R^\star(\varphi, B) \sim \mathrm{Hom}_R^\star(\psi, B)$.*

Proof. Choose a homotopy $h \colon A \to A'$ from φ to ψ (so $\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A$). To ease the notation in the following calculation, we write $\varphi^\star = \mathrm{Hom}_R^\star(\varphi, B)$, $\psi^\star = \mathrm{Hom}_R^\star(\psi, B)$, and $h^\star = \mathrm{Hom}_R^\star(h, B)$. We claim that $h^\star \colon \mathrm{Hom}_R^\star(A', B) \to \mathrm{Hom}_R^\star(A, B)$ is a homotopy from $\varphi^\star$ to $\psi^\star$. Indeed, let $\alpha \colon A' \to B$ be a graded R-linear map of degree i. Then observe that

$$\begin{aligned}
(\mathrm{d}^\star_{(A,B)}h^\star + h^\star\mathrm{d}^\star_{(A',B)})(\alpha) &= (-1)^i\mathrm{d}^\star_{(A,B)}(\alpha h) + h^\star(\mathrm{d}_B\alpha - (-1)^i\alpha\mathrm{d}_{A'}) \\
&= (-1)^i\mathrm{d}_B\alpha h + (-1)^i(-1)^i\alpha h\mathrm{d}_A - (-1)^i\mathrm{d}_B\alpha h - (-1)^i(-1)^{i+1}\alpha\mathrm{d}_{A'}h \\
&= \alpha h\mathrm{d}_A + \alpha\mathrm{d}_{A'}h \\
&= \alpha(h\mathrm{d}_A + \mathrm{d}_{A'}h) \\
&= \alpha(\varphi - \psi) \\
&= (\varphi^\star - \psi^\star)(\alpha)
\end{aligned}$$

Thus $h^\star$ is indeed a homotopy from $\varphi^\star$ to $\psi^\star$. □

Corollary. *Suppose $\varphi \colon A \to A'$ is a homotopy of equivalence of R-complexes. Then $\mathrm{Hom}_R^\star(\varphi, B) \colon \mathrm{Hom}_R^\star(A', B) \to \mathrm{Hom}_R^\star(A, B)$ is a homotopy equivalence of R-complexes.*

Proof. Let $\varphi' \colon A' \to A$ be the homotopy inverse to φ. Thus $\varphi\varphi' \sim 1_{A'}$ and $\varphi'\varphi \sim 1_A$. It follows that

$$\begin{aligned} 1_{\mathrm{Hom}_R^\star(A',B)} &= \mathrm{Hom}_R^\star(1_{A'}, B) \\ &\sim \mathrm{Hom}_R^\star(\varphi\varphi', B) \\ &= \mathrm{Hom}_R^\star(\varphi', B)\mathrm{Hom}_R^\star(\varphi, B). \end{aligned}$$

Similarly, we have $1_{\mathrm{Hom}_R^\star(A,B)} \sim \mathrm{Hom}_R^\star(\varphi, B)\mathrm{Hom}_R^\star(\varphi', B)$. Therefore $\mathrm{Hom}_R^\star(\varphi, B)$ is a homotopy equivalence of R-complexes. □

3.9.9 Twisting the hom complex with a chain map

Definition 3.20. Let (A, d) be an R-complex and let $\alpha \colon A \to A$ be a chain map. We define an R-complex $\mathrm{Hom}_R^{\star_\alpha}(A, A)$ as follows: as a graded R-module, $\mathrm{Hom}_R^{\star_\alpha}(A, A)$ is just $\mathrm{Hom}_R^\star(A, A)$. We define the differential $\mathrm{d}_\alpha^\star \colon \mathrm{Hom}_R^{\star_\alpha}(A, A) \to \mathrm{Hom}_R^{\star_\alpha}(A, A)$ on graded R-linear map $\varphi \colon A \to A$ of degree i by

$$\mathrm{d}_\alpha^\star(\varphi) = \mathrm{d}\varphi + (-1)^i \alpha\varphi\mathrm{d} \tag{38}$$

and then we extend $\mathrm{d}_\alpha^\star$ linearly everywhere else. Note that $\mathrm{d}_\alpha^\star$ is graded of degree -1 since α is a chain map. Let us show that we have $\mathrm{d}_\alpha^\star\mathrm{d}_\alpha^\star = 0$. Let $\varphi \colon A \to A$ be a graded R-linear map of degree i . Then we have

$$\begin{aligned} \mathrm{d}_\alpha^\star\mathrm{d}_\alpha^\star(\varphi) &= \mathrm{d}_\alpha^\star(\mathrm{d}\varphi + (-1)^i\alpha\varphi\mathrm{d}) \\ &= \mathrm{d}\mathrm{d}\varphi + (-1)^{i-1}\alpha\mathrm{d}\varphi\mathrm{d} + (-1)^i\mathrm{d}\alpha\varphi\mathrm{d} + (-1)^{i-1}\alpha\alpha\varphi\mathrm{d}\mathrm{d} \\ &= (-1)^{i-1}\alpha\mathrm{d}\varphi\mathrm{d} + (-1)^i\alpha\mathrm{d}\varphi\mathrm{d} \\ &= 0. \end{aligned}$$

It follows that $\mathrm{d}_\alpha^\star$ is a differential.

4 Ext and Tor

4.1 Projective Resolutions

Definition 4.1. Let M be an R-module. An **augmented projective resolution of** M **over** R is an R-complex (P, d) such that

1. P is a projective R-module. Equivalently, P_i is a projective R-module for all $i \in \mathbb{Z}$;
2. $P_i = 0$ for all $i < 0$;
3. $\mathrm{H}_0(P) \cong M$ and $\mathrm{H}_i(P) = 0$ for all $i > 0$.

Theorem 4.1. *Let (P, d) and (P', d') be two projective resolutions of M over R. Then (P, d) and (P', d') are homotopically equivalent.*

Proof. For each $i \geq 0$, let $M_i' := \mathrm{im}\,\mathrm{d}_i'$ and let $M_i := \mathrm{im}\,\mathrm{d}_i$. We build a chain map $\varphi \colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ by constructing R-module homomorphism $\varphi_i \colon P_i \to P_i'$ which commute with the differentials using induction on $i \geq 0$. First consider the base case $i = 0$. Since $P_0/M_1 \cong P_0'/M_1'$, there exists a homomorphism $\psi_0 \colon P_0 \to P_0'/M_0'$. Then since P_0 is projective and since $\mathrm{d}_0' \colon P_0' \to P_0'/M_1$ is a surjective homomorphism, we can lift $\psi_0 : P_0 \to P_0'/M_0'$ along $\mathrm{d}_0' : P_0' \to P_0'/M_1$ to a homomorphism $\varphi_0 \colon P_0 \to P_0'$ such that $\mathrm{d}_0'\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructed R-module homomorphisms $\varphi_0, \varphi_1, \ldots, \varphi_i$ which commute with the differentials. We need to construct an R-module homomorphism $\varphi_{i+1} \colon P_{i+1} \to P_{i+1}'$ which commutes with the differentials. First, we claim that $\mathrm{im}\,(\varphi_i\mathrm{d}_{i+1}) \subseteq M_{i+1}'$. To see this, note that

$$\begin{aligned} \mathrm{d}_i'\varphi_i\mathrm{d}_{i+1} &= \varphi_{i-1}\mathrm{d}_i\mathrm{d}_{i+1} \\ &= 0. \end{aligned}$$

Thus, since $i > 0$, we have

$$\begin{aligned} \mathrm{im}\,(\varphi_i\mathrm{d}_{i+1}) &\subseteq \ker \mathrm{d}_i \\ &= \mathrm{im}\,\mathrm{d}_{i+1}' \\ &= M_{i+1}'. \end{aligned}$$

Now since P_{i+1} is projective and $\mathrm{d}_{i+1}' \colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i\mathrm{d}_{i+1} : P_{i+1} \to M_{i+1}'$ along $\mathrm{d}_{i+1}' : P_{i+1}' \to M_{i+1}'$ to a homomorphism $\varphi_{i+1} \colon P_{i+1} \to P_{i+1}'$ such that $\mathrm{d}_{i+1}'\varphi_{i+1} = \varphi_i\mathrm{d}_{i+1}$.

By a similar construction as above, we get a chain map $\varphi' \colon (P', \mathrm{d}') \to (P, \mathrm{d})$. Now we claim that $\varphi'\varphi$ is homotopic to id_P and similarly $\varphi\varphi'$ is homotopic to $\mathrm{id}_{P'}$. It suffices to show that $\varphi'\varphi \sim \mathrm{id}_P$ (a similar argument will give $\varphi\varphi' \sim \mathrm{id}_{P'}$). The idea is to build the homotopy $h \colon (P, \mathrm{d}) \to (P, \mathrm{d})$ using induction on $i \geq 0$. The homotopy equation that we need is

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d}, \tag{39}$$

where we write 1 instead of id_P is clean notation. Since P_0 is projective and $\mathrm{d}_1 \colon P_1 \to P_0$ is a surjective morphism, there exists a homomorphism $h_0 \colon P_0 \to P_1$ such that

$$\varphi_0'\varphi_0 - 1 = \mathrm{d}_1 h_0. \tag{40}$$

In homological degree $i = 0$, the equation (39) becomes (40). Thus, we are on the right track.

Now we use induction. Suppose for $i > 0$ we have constructed an R-module homomorphism $h_i \colon P_i \to P_{i+1}$ such that

$$\varphi_i'\varphi_i - 1 = \mathrm{d}_{i+1}h_i + h_{i-1}\mathrm{d}_i. \tag{41}$$

Observe that $\mathrm{Im}(\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i) \subseteq M_{i+1}$. Indeed, note that

$$\begin{aligned}
\mathrm{d}_i(\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i) &= \mathrm{d}_i\varphi_i'\varphi_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \varphi_{i-1}'\mathrm{d}_i'\varphi_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \varphi_{i-1}'\varphi_{i-1}\mathrm{d}_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= (\varphi_{i-1}'\varphi_{i-1} - 1)\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= (\mathrm{d}_i h_{i-1} + h_{i-2}\mathrm{d}_{i-1})\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \mathrm{d}_i h_{i-1}\mathrm{d}_i + h_{i-2}\mathrm{d}_{i-1}\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \mathrm{d}_i h_{i-1}\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= 0.
\end{aligned}$$

Therefore since P_{i+1} is projective and since $\mathrm{d}_{i+2} \colon P_{i+2} \to M_{i+2}$ is a surjective homomorphism, there exists $h_{i+1} \colon P_{i+1} \to P_{i+2}$ such that

$$\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i = \mathrm{d}_{i+2}h_{i+1},$$

which is the homotopy equation in degree $i + 1$. $\square$

4.1.1 Minimal Projective Resolutions over a Noetherian Local Ring

Definition 4.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R-module, and let (P, d) be a projective resolution of M over R. We say (P, d) is **minimal** if $\mathrm{d}(P) \subset \mathfrak{m}P$.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R-module, and let (P, d) and (P', d') be any two minimal projective resolution of M over R. Then for each $i \in \mathbb{Z}$, the ranks of P_i and P_i' are finite and equal to each other. We denote this common rank by $\beta_i(M)$.*

Proof. Choose chain map $\alpha \colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ and $\alpha' \colon (P', \mathrm{d}') \to (P, \mathrm{d})$ together with a homotopy $h \colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ such that

$$\alpha'\alpha - 1 = \mathrm{d}'h + h\mathrm{d}. \tag{42}$$

Since $\mathrm{d}(P) \subset \mathfrak{m}P$ and $\mathrm{d}'(P') \subset \mathfrak{m}P'$, the homotopy equation (42) reduces to

$$\alpha'\alpha - 1 \equiv 0 \bmod \mathfrak{m}.$$

In other words, $\alpha \colon P \to P'$ induces an isomorphism $\overline{\alpha} \colon P/\mathfrak{m}P \to P'/\mathfrak{m}P'$. In particular, for each $i \in \mathbb{Z}$, we have isomorphisms

$$\overline{\alpha}_i \colon P_i/\mathfrak{m}P_i \to P_i'/\mathfrak{m}P_i'$$

of $(R/\mathfrak{m})$-vector spaces. Therefore by Nakayama's Lemma, for all $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathrm{rank}(P_i) &= \dim_{R/\mathfrak{m}}(P_i/\mathfrak{m}P_i) \\
&= \dim_{R/\mathfrak{m}}(P_i'/\mathfrak{m}P_i') \\
&= \mathrm{rank}(P_i').
\end{aligned}$$

$\square$

4.2 Definition of Tor

Definition 4.3. Let M and N be R-modules. We define the **Tor** with respect to M and N as follows: Choose a projective resolution of M, say (P, d), then set

$$\mathrm{Tor}^R(M,N) := \mathrm{H}(P \otimes_R N).$$

We need to check that this definition does not depend on the choice of a projective resolution of M, so suppose (P', d') is another projective resolution of M. By Theorem (4.1), there exists a homotopy equivalence from (P, d) to (P', d'), say $\varphi\colon (P,\mathrm{d}) \to (P, \mathrm{d}')$ and $\varphi'\colon (P', \mathrm{d}') \to (P, \mathrm{d})$ with homotopies $h\colon (P,\mathrm{d}) \to (P,\mathrm{d})$ and $h'\colon (P,\mathrm{d}) \to (P, \mathrm{d}')$ such that

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

We claim that $P \otimes_R N$ is homotopically equivalent to $P' \otimes_R N$ via the pair of maps $\varphi \otimes 1\colon P \otimes_R N \to P' \otimes_R N$ and $\varphi' \otimes 1\colon P' \otimes_R N \to P \otimes_R N$ with homotopies given by $h \otimes 1\colon P \otimes_R N \to P' \otimes_R N$ and $h' \otimes_R 1\colon P' \otimes_R N \to P \otimes_R N$ respectively. Indeed, we have

$$\begin{aligned}
(\varphi' \otimes 1)(\varphi \otimes 1) - 1 \otimes 1 &= \varphi'\varphi \otimes 1 - 1 \otimes 1 \\
&= (\varphi'\varphi - 1) \otimes 1 \\
&= (\mathrm{d}h + h\mathrm{d}) \otimes 1 \\
&= \mathrm{d}h \otimes 1 + h\mathrm{d} \otimes 1 \\
&= \mathrm{d}^{P \otimes_R N}(h \otimes 1) + (h \otimes 1)\mathrm{d}^{P \otimes_R N}.
\end{aligned}$$

A similar calculation shows

$$(\varphi \otimes 1)(\varphi' \otimes 1) = \mathrm{d}^{P' \otimes_R N}(h' \otimes 1) + (h' \otimes 1)\mathrm{d}^{P' \otimes_R N}.$$

Thus $P \otimes_R N$ is homotopically equivalent to $P' \otimes_R N$ and hence

$$\mathrm{H}(P \otimes_R N) = \mathrm{H}(P' \otimes_R N).$$

Therefore the definition of Tor is well-defined.

4.3 Examples of Tor

Example 4.1. Let I and J be ideals in R. We compute $\mathrm{Tor}_1^R(R/I, R/J)$. First we tensor the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

with R/J to get the exact sequence

$$\begin{tikzcd}
 & I/IJ \arrow[r] & R/J \arrow[r] & R/(I+J) \arrow[r] & 0 \\
0 \cong \mathrm{Tor}_1^R(R, R/J) \arrow[r] & \mathrm{Tor}_1^R(R/I, R/J) \arrow[u]
\end{tikzcd}$$

where $\mathrm{Tor}_1^R(R, R/J) \cong 0$ for trivial reasons. From here, it follows that $\mathrm{Tor}_1^R(R/I, R/J)$ is isomorphic to the kernel of the map $I/IJ \to R/J$, which is just $I \cap J/IJ$.

Example 4.2. Let $R = K[x,y,z]$, $I = \langle xy^2z^3, x^2yz^3, x^3yz^2, x^3y^2z, x^2y^3z, xy^3z^2 \rangle$, and $J = \langle x, y \rangle$. We compute $\mathrm{Tor}_i^R(R/I, R/J)$ for all i. An augmented free resolution for R/I comes from the permutohedron of order 3. It is given by

$$0 \longrightarrow R \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R \longrightarrow R/I$$

where

$$\varphi_3 = \begin{pmatrix} xy \\ y^2 \\ yz \\ z^2 \\ xz \\ x^2 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -x & 0 & 0 & 0 & 0 \\ 0 & z & -y & 0 & 0 & 0 \\ 0 & 0 & z & -y & 0 & 0 \\ 0 & 0 & 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 & x & -z \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^3yz^2 & x^3y^2z & x^2y^3z & xy^3z^2 \end{pmatrix}.$$

We now truncate this resolution by replacing the R/I term with 0 and then tensor the truncated resolution with R/J to get:

$$0 \longrightarrow R/J \xrightarrow{\widetilde{\varphi}_3} (R/J)^6 \xrightarrow{\widetilde{\varphi}_2} (R/J)^6 \xrightarrow{\widetilde{\varphi}_1} R/J \longrightarrow 0$$

where $\overline{\varphi}_i$ is given by

$$\overline{\varphi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{z}^2 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{\varphi}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\overline{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\overline{z} \end{pmatrix}, \qquad \overline{\varphi}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we see that

$$\begin{aligned}
\mathrm{Tor}_0^R(R/I, R/J) &\cong R/\langle x, y\rangle \\
\mathrm{Tor}_1^R(R/I, R/J) &\cong (R/\langle x, y\rangle)^2 \oplus (R/\langle x, y, z\rangle)^4 \\
\mathrm{Tor}_2^R(R/I, R/J) &\cong (R/\langle x, y\rangle) \oplus \left(R/\langle x, y, z^2\rangle\right),
\end{aligned}$$

and $\mathrm{Tor}_i^R(R/I, R/J) \cong 0$ for all $i \geq 3$.

4.4 Definition of Ext

Definition 4.4. Let M and N be R-modules. We define the **Ext** with respect to M and N as follows: Choose a projective resolution of M, say (P, d), then set

$$\mathrm{Ext}_R(M, N) := \mathrm{H}(\mathrm{Hom}_R^\star(P, N)).$$

We need to check that this definition does not depend on the choice of a projective resolution of M, so suppose (P', d') is another projective resolution of M. By Theorem (4.1), there exists a homotopy equivalence from (P, d) to (P', d'), say $\varphi\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ and $\varphi'\colon (P', \mathrm{d}') \to (P, \mathrm{d})$ with homotopies $h\colon (P, \mathrm{d}) \to (P, \mathrm{d})$ and $h'\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ such that

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

We claim that $\mathrm{Hom}_R^\star(P, N)$ is homotopically equivalent to $\mathrm{Hom}_R^\star(P', N)$ via the pair of maps $\varphi^\star\colon \mathrm{Hom}_R^\star(P, N) \to \mathrm{Hom}_R^\star(P', N)$ and $\varphi'^\star\colon P' \otimes_R N \to P \otimes_R N$ with homotopies given by $h^\star\colon \mathrm{Hom}_R^\star(P, N) \to \mathrm{Hom}_R^\star(P, N)$ and $h'^\star\colon \mathrm{Hom}_R^\star(P', N) \to \mathrm{Hom}_R^\star(P', N)$ respectively. Indeed, if $\psi \in \mathrm{Hom}_R(P_i, N)$, then we have

$$\begin{aligned}
(\varphi'^\star\varphi^\star - 1^\star)(\psi) &= \psi(\varphi'\varphi - 1) \\
&= \psi(\mathrm{d}h + h\mathrm{d}) \\
&= (\mathrm{d}^\star h^\star + h^\star \mathrm{d}^\star)(\psi).
\end{aligned}$$

It follows that $\varphi'^\star\varphi^\star - 1^\star = \mathrm{d}^\star h^\star + h^\star\mathrm{d}^\star$. A similar calculation shows $\varphi^\star\varphi'^\star - 1^\star = \mathrm{d}^\star h'^\star + h'^\star\mathrm{d}^\star$. Thus $\mathrm{Hom}_R^\star(P, N)$ is homotopically equivalent to $\mathrm{Hom}_R^\star(P', N)$ and hence

$$\mathrm{H}(\mathrm{Hom}_R^\star(P, N)) = \mathrm{H}(\mathrm{Hom}_R^\star(P', N)).$$

Therefore the definition of Ext is well-defined.

4.5 Balance of Ext

We are striving for balance of Ext: the sketch of that proof goes like this: We have

$$\mathrm{Hom}_R(P, N) \underset{\varepsilon_*}{\xrightarrow{\simeq}} \mathrm{Hom}_R(P, E) \underset{\tau^*}{\xleftarrow{\simeq}} \mathrm{Hom}_R(M, E).$$

The quasiisomorphisms are: augment $P \underset{\simeq}{\xrightarrow{\tau}} M$ and $N \underset{\simeq}{\xrightarrow{\varepsilon}} E$. Then $\mathrm{Hom}_R(P, \mathrm{C}(\varepsilon)) \cong \mathrm{C}(\varepsilon_*)$ where $\mathrm{C}(\varepsilon)$ is exact because ε is quasiisomorphism and $\mathrm{Hom}_R(P, \mathrm{C}(\varepsilon))$ is exact because P is bounded below complex of projectives. Therefore $\mathrm{C}(\varepsilon_*)$ is exact, which implies ε_* is a quasiisomorphism.

Lemma 4.2. *Let I be a bounded above complex of injective R-modules. Then $\mathrm{Hom}_R(-, I)$ respects exact complexes. That is, if U is exact, then the complex $\mathrm{Hom}_R(U, I)$ is exact.*

Proposition 4.2. *Let P be a bounded below complex of projective R-modules and let I be a bounded above complex of injective R-modules. Then $\mathrm{Hom}_R(P, -)$ and $\mathrm{Hom}_R(-, I)$ respect quasiisomorphisms. That is, given a quasiisomorphism $\phi\colon U \to V$, the chain maps $\phi_*\colon \mathrm{Hom}_R(P, U) \to \mathrm{Hom}_R(P, V)$ and $\phi^*\colon \mathrm{Hom}_R(V, I) \to \mathrm{Hom}_R(U, I)$ are quasiisomorphisms.*

Proof. We have

$$\begin{aligned} V \xrightarrow[\simeq]{\phi} U &\implies \mathrm{C}(\phi) \text{ is exact} \\ &\implies \mathrm{Hom}_R(\mathrm{C}(\phi), I) \text{ is exact} \\ &\implies \mathrm{C}(\mathrm{Hom}_R(\phi, I)) \text{ is exact} \\ &\implies \mathrm{Hom}(\phi, I) = \phi_* \text{ is quasiisomorphism} \end{aligned}$$

□

Theorem 4.3. *(Balance for Ext) Let P be a projective resolution of an R-module M and let I be an injective resolution of an R-module N. Then*

$$\mathrm{Ext}^i_R(M,N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P,N)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P,I)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M,I)).$$

Proof. Resolution gives us quasiisomorphisms $P \xrightarrow[\simeq]{\tau} M$ and $N \xrightarrow[\simeq]{\varepsilon} I$. Thus

$$\mathrm{Hom}_R(P,N) \xrightarrow[\simeq]{\varepsilon_*} \mathrm{Hom}_R(P,I) \xleftarrow[\simeq]{\tau^*} \mathrm{Hom}_R(M,I).$$

□

4.6 Shift Property of Tor and Ext

Proposition 4.3. *Let A be a ring. Let M and N finitely generated A-modules, and for $i \geq 0$, let M_i and N_i denote there respective nonnegative syzygies. For $j \geq 1$, we have*

$$\begin{aligned} Ext^{j+1}_A(M_i,N) &\cong Ext^j_A(M_{i+1},N) \\ Tor^A_{j+1}(M_i,N) &\cong Tor^A_j(M_{i+1},N) \\ Tor^A_{j+1}(M,N_i) &\cong Tor^A_j(M,N_{i+1}) \end{aligned}$$

Moreover, assume A is Gorenstein, M and N are maximal Cohen-Macaulay, and for $i \leq -1$, let M_i and N_i denote their respective nonnegative syzygies. Then for $j \geq 1$, we have

$$\begin{aligned} Ext^{j+1}_A(M_i,N) &\cong Ext^j_A(M_{i+1},N) \\ Ext^j_A(M,N_i) &\cong Ext^{j+1}_A(M,N_{i+1}) \\ Tor^A_{j+1}(M_i,N) &\cong Tor^A_j(M_{i+1},N) \\ Tor^A_{j+1}(M,N_i) &\cong Tor^A_j(M,N_{i+1}) \end{aligned}$$

5 Differential Graded Algebras

5.1 DG Algebras

Let (A, d) be an R-complex. A **graded-multiplication** on A is a graded R-linear map $\mathrm{m}\colon A \otimes_R A \to A$ of the underlying graded R-modules. The universal mapping property on graded tensor products tells us that there exists a unique graded R-bilinear map $B_{\mathrm{m}}\colon A \times A \to A$ such that

$$B_{\mathrm{m}}(a,b) = \mathrm{m}(a \otimes b)$$

for all $(a,b) \in A \times A$. However since B_{m} is *uniquely* determined by m, we often identify B_{m} with m and simply think of m as a graded R-bilinear map. In fact, we often drop m altogether and simply denote this multiplication map by

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i$$

for all $\sum a_i \otimes b_i \in A \otimes_R A$. At the end of the day, context will make everything clear.

Suppose m is a graded multiplication As the name of the definition suggests, a graded-multiplication on A must respect the grading. In particular, this means that if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. We can also impose other conditions on a graded-multiplication on A.

Definition 5.1. Let (A, d) be an R-complex and let m be a graded-multiplication on A.

1. We say m is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.

2. We say m is **graded-commutative** if
$$ab = (-1)^i ba$$
for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$.

3. We say m is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all $a \in A_i$ for all i odd.

4. We say m is **unital** if there exists an $e \in A$ such that
$$ae = e = ea$$
for all $a \in A$.

5. We say a graded-multiplication satisfies **Leibniz law** if
$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$
for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$. This is equivalent to m being a chain map!

6. We say $(A, \mathrm{m}, \mathrm{d})$ is a **differential graded** R**-algebra** (or **DG** R**-algebra**) if m is a graded-multiplication on A which satisfies conditions 1-5.

Remark. If the differential d and the multiplication map m are understood from context, then we will denote a differential graded R-algebra simply as "A" rather than as a triple "$(A, \mathrm{m}, \mathrm{d})$". We will also often introduce a differential grade R-algebra as "A" without specifying how the differential and multiplication map are to be denoted. In this case, the differential is denoted "d_A" and the multiplication map is denoted "m_A".

Definition 5.2. Let (A, d) and (A', d') be two DG R-algebras. A chain map $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is said to be a **DG-algebra morphism** if it respects multiplication and identity. In other words, we need
$$\varphi(ab) = \varphi(a)\varphi(b)$$
for all $a, b \in A$, and we need
$$\varphi(1) = 1.$$

We obtain a category of DG R-algebras.

5.1.1 Tensor Product of DG Algebras is DG Algebra

Proposition 5.1. *Let A and B be two DG R-algebras. Then $A \otimes_R B$ is is a DG R-algebra.*

Proof. Let $\mathrm{m}_A\colon A \otimes_R A \to A$ be the multiplication map for A and let $\mathrm{m}_B\colon B \otimes_R B \to B$ the multiplication map for B. Then
$$\begin{aligned}(A \otimes_R B) \otimes_R (A \otimes_R B) &\cong A \otimes_R (B \otimes_R (A \otimes_R B)) \\ &\cong A \otimes_R ((B \otimes_R A) \otimes_R B) \\ &\cong A \otimes_R ((A \otimes_R B) \otimes_R B) \\ &\cong \end{aligned}$$
$$A \otimes_R B)$$
□

Proposition 5.2. *Let (A, d) and (A', d') be two DG R-algebras. Then $(A \otimes_R A', \mathrm{d}^{A \otimes_R A'})$ is a DG R-algebra.*

Proof. Throughout this proof, denote $\mathrm{d}^{\otimes} := \mathrm{d}^{A \otimes_R A'}$. We define multiplication on $A \otimes_R A'$ by the formula
$$(a \otimes a')(b \otimes b') = (-1)^{i'j} ab \otimes a'b'. \tag{43}$$

for all $a \otimes a' \in A_i \otimes_R A_{i'}$ and $b \otimes b' \in A_j \otimes_R A_{j'}$. It is easy to check that (43) is associative and unital with with unit being $e_A \otimes e_{A'}$ where e_A is the unit of A and $e_{A'}$ is the unit of A'. Let us check that Leibniz law is satisfied. Let $a \otimes a', b \otimes b' \in A \otimes_R A'$. Then we have

$$\begin{aligned}
\mathrm{d}^{\otimes}((a \otimes a')(b \otimes b')) &= (-1)^{i'j}\mathrm{d}^{\otimes}(ab \otimes a'b') \\
&= (-1)^{i'j}(\mathrm{d}(ab) \otimes a'b' + (-1)^{i+j}ab \otimes \mathrm{d}'(a'b')) \\
&= (-1)^{i'j}((\mathrm{d}(a)b + (-1)^{i}a\mathrm{d}(b)) \otimes a'b' + (-1)^{i+j}ab \otimes (\mathrm{d}'(a')b' + (-1)^{i'}a'\mathrm{d}'(b'))) \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i'j+i}a\mathrm{d}(b) \otimes a'b' + (-1)^{i'j+i+j}ab \otimes \mathrm{d}'(a')b' + (-1)^{i'j+i+j+i'}ab \otimes a'\mathrm{d}'(b') \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i+j(i'+1)}ab \otimes \mathrm{d}'(a')b' + (-1)^{i+i'+i'(j+1)}a\mathrm{d}(b) \otimes a'b' + (-1)^{i+i'+j+i'j}(ab \otimes a'\mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a')(b \otimes b') + (-1)^{i}(a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b') + (-1)^{i+i'+j}(a \otimes a')(b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a' + (-1)^{i}a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b' + (-1)^{j}b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}^{\otimes}(a \otimes a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}^{\otimes}(b \otimes b')).
\end{aligned}$$

Thus $\mathrm{d}^{\otimes}$ satifies Leibniz law with respect to (43).

□

Proposition 5.3. *Let F be an R-complex of free modules and let B be a DG R-algebras. Then $\mathrm{Hom}_R^{\star}(F, B)$ is a DG R-algebra.*

Proof. Let $\{e_\lambda\}$ be a homogeneous basis for F indexed over a set Λ. We define a graded-multiplication on $\mathrm{Hom}_R^{\star}(F, B)$ as follows: let $\varphi \in \mathrm{Hom}_R^{\star}(F, B)_i$ and $\psi \in \mathrm{Hom}_R^{\star}(F, B)_j$, then we define $\varphi \smile \psi \in \mathrm{Hom}_R^{\star}(F, B)_{i+j}$ to be the unique graded R-linear map defined on basis elements $\{e_\lambda\}$ by

$$(\varphi \smile \psi)(e_\lambda) = \varphi(s_-^{n-i}e_\lambda)\psi(s_+^{n-j}e_\lambda)$$

for all $\lambda \in \Lambda$. Note that we are defining $\varphi \smile \psi$ on $\{e_\lambda\}$ and *then* extending R-linearly. Thus $(\varphi \smile \psi)(re_\lambda) = r\varphi(e_\lambda)\psi(e_\lambda)$ (not $r^2\varphi(e_\lambda)\psi(e_\lambda)$)! Similarly, $(\varphi \smile \psi)(e_\lambda + e_\mu) = \varphi(e_\lambda)\psi(e_\lambda) + \varphi(e_\mu)\psi(e_\mu)$ (not $\varphi(e_\lambda)\psi(e_\lambda) + \varphi(e_\mu)\psi(e_\mu) + \varphi(e_\lambda)\psi(e_\mu) + \varphi(e_\mu)\psi(e_\lambda)$)!
for all $a \in A$. Observe that

$$\mathrm{d}(\varphi \cdot \psi) = \mathrm{d}\varphi \cdot \psi + (-1)^{i}\varphi \cdot \mathrm{d}\psi$$

Indeed, we have

$$\begin{aligned}
\mathrm{d}(\varphi \cdot \psi)(a) &= \mathrm{d}(\varphi(a)\psi(a)) \\
&= (\mathrm{d}\varphi(a))\psi(a) + (-1)^{i+n}\varphi(a)(\mathrm{d}\psi(a))
\end{aligned}$$

Now we want to show $\cdot$ induces an R-bilinear map in homology. First let us show that $\mathrm{H}(\varphi \cdot \psi)$ is a graded R-linear map. Let □

5.1.2 Hom of DG Algebras is a Noncommutative DG Algebra

Proposition 5.4. *Let (A, d) be a DG R-algebras. Then $\mathrm{Hom}_R^{\star}(A, A')$ is a noncommutative DG R-algebra.*

Proof. We define multiplication on $\mathrm{Hom}_R^{\star}(A, A)$ via composition of functions. Thus if $\varphi\colon A \to A$ and $\psi\colon A \to A$ are graded homomorphisms of degrees i and j respectively. Then $\varphi\psi\colon A \to A'$ is given by

$$(\varphi\psi)(a) = \varphi(\psi(a))$$

for all $a \in A$. Note that $\varphi\psi$ is a graded R-homomorphism of degree $i + j$. Multiplication is easy seen to satisfy associativity and the identity map $1_A\colon A \to A$ serves as the identity element with respect to this multiplication. Moreover, Leibniz law is satisfied: we have

$$\begin{aligned}
\mathrm{d}^{\star}(\varphi)\psi + (-1)^{i}\varphi\mathrm{d}^{\star}(\psi) &= (\mathrm{d}\varphi - (-1)^{i}\varphi\mathrm{d})\psi + (-1)^{i}\varphi(\mathrm{d}\psi - (-1)^{j}\psi\mathrm{d}) \\
&= \mathrm{d}\varphi\psi - (-1)^{i}\varphi\mathrm{d}\psi + (-1)^{i}\varphi\mathrm{d}\psi - (-1)^{i+j}\varphi\psi\mathrm{d} \\
&= \mathrm{d}\varphi\psi - (-1)^{i+j}\varphi\psi\mathrm{d} \\
&= \mathrm{d}^{\star}(\varphi\psi).
\end{aligned}$$

for all $\varphi \in \mathrm{Hom}_R^{\star}(A, A)_i$ and $\psi \in \mathrm{Hom}_R^{\star}(A, A)_j$. □

5.1.3 DG Algebra Embedding

Proposition 5.5. *Let A be a DG algebra. Define $\varphi\colon A \to \mathrm{Hom}_R^\star(A, A)$ by*

$$\varphi(a) = m_a$$

for all $a \in A$ where $m_a\colon A \to A$ is the homothety map, given by

$$m_a(x) = ax$$

for all $x \in A$. Then φ is an injective DG algebra homomorphism.

Proof. Note that $\varphi\colon A \to \mathrm{Hom}_R^\star(A, A)$ is easily seen to be a graded R-homomorphism. Let us check that it commutes with the differentials so that it is a chain map. Let $a \in A_i$. Observe that

$$\begin{aligned} \mathrm{d}m_a(x) &= \mathrm{d}(ax) \\ &= \mathrm{d}(a)x + (-1)^i a\mathrm{d}(x) \\ &= m_{\mathrm{d}(a)}(x) + (-1)^i m_a(\mathrm{d}(x)) \\ &= (m_{\mathrm{d}(a)} + (-1)^i m_a \mathrm{d})(x) \end{aligned}$$

for all $x \in A$. It follows that

$$\mathrm{d}m_a = m_{\mathrm{d}(a)} + (-1)^i m_a \mathrm{d}.$$

Thus

$$\begin{aligned} (\mathrm{d}^\star\varphi)(a) &= \mathrm{d}^\star(\varphi(a)) \\ &= \mathrm{d}^\star m_a \\ &= \mathrm{d}m_a - (-1)^i m_a \mathrm{d} \\ &= m_{\mathrm{d}(a)} \\ &= \varphi(\mathrm{d}(a)) \\ &= (\varphi\mathrm{d})(a), \end{aligned}$$

and so φ commutes with the differentials. Thus φ is a chain map.

Let us now check that φ is a DG algebra homomorphism. Let $a, b \in A$. Observe that we have

$$\begin{aligned} (m_a m_b)(x) &= m_a(m_b(x)) \\ &= m_a(bx) \\ &= a(bx) \\ &= (ab)x \\ &= m_{ab}(x) \end{aligned}$$

for all $x \in A$. It follows that $m_a m_b = m_{ab}$. Thus

$$\begin{aligned} \varphi(ab) &= m_{ab} \\ &= m_a m_b \\ &= \varphi(a)\varphi(b), \end{aligned}$$

and hence φ respects addition, and also $\varphi(1) = 1_A$, where e is the identity in A and 1_A is the identity in $\mathrm{Hom}_R^\star(A, A)$.

Finally, note that φ is injective. Indeed, suppose $m_a = 0$ for some $a \in A$, then

$$\begin{aligned} 0 &= m_a(1) \\ &= a \cdot 1 \\ &= a \end{aligned}$$

implies $\ker \varphi = 0$. □

Proposition 5.6. *Let R be a ring, let I be an ideal in R, and let (A, d) be a DG algebra resolution of R/I over R. Then I kills $\mathrm{H}(A)$.*

Proof. The embedding of DG Algebras $A \to \mathrm{Hom}_R(A, A)$, given by $a \mapsto m_a$, induces a map in the 0th homology

$$R/I \to \{\text{homotopy classes of chain maps } A \to A\}.$$

In particular, if x is in I, then m_x must be null-homotopic. Hence I kills $\mathrm{H}(A)$. □

Proposition 5.7. *Let R be a ring, let I be an ideal in R, and let (A, d) and (A', d') be two DG algebra resolutions of R/I over R. Then $\mathrm{Hom}^\star_R(A, A)$ is homotopically equivalent to $\mathrm{Hom}^\star_R(A', A')$.*

Proof. Since A and A' are homotopically equivalent, we may choose chain maps $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A$ together with homotopies $h\colon A \to A'$ and $h'\colon A \to A'$ where

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

Define $\gamma\colon \mathrm{Hom}^\star_R(A, A) \to \mathrm{Hom}^\star_R(A', A')$ by

$$\gamma(\alpha) = \varphi\alpha\varphi'$$

for all $\alpha \in \mathrm{Hom}^\star_R(A, A)$. We claim that γ is a chain map. Indeed, it is graded since φ and φ' have degree 0. It is an R-module homomorphism since if $r, s \in R$ and $\alpha, \beta \in \mathrm{Hom}^\star_R(A, A)$, then we have

$$\begin{aligned}
\gamma(r\alpha + s\beta) &= \varphi(r\alpha + s\beta)\varphi' \\
&= \varphi r\alpha\varphi' + \varphi s\phi\varphi' \\
&= r\varphi\alpha\varphi' + s\varphi\beta\varphi' \\
&= r\gamma(\alpha) + s\gamma(\beta).
\end{aligned}$$

It commutes with the differentials since if $\alpha \in \mathrm{Hom}^\star_R(A, A)_i$, then we have

$$\begin{aligned}
(\mathrm{d}^\star_{A'}\gamma)(\alpha) &= \mathrm{d}^\star_{A'}(\gamma(\alpha)) \\
&= \mathrm{d}^\star_{A'}(\varphi\alpha\varphi') \\
&= \mathrm{d}'\varphi\alpha\varphi' + (-1)^i\varphi\alpha\varphi'\mathrm{d}' \\
&= \varphi\mathrm{d}\alpha\varphi' + (-1)^i\varphi\alpha\mathrm{d}\varphi' \\
&= \varphi(\mathrm{d}\alpha + (-1)^i\alpha\mathrm{d})\varphi' \\
&= \gamma(\mathrm{d}\alpha + (-1)^i\alpha\mathrm{d}) \\
&= \gamma(\mathrm{d}^\star_A(\alpha)) \\
&= (\gamma\mathrm{d}^\star_A)(\alpha).
\end{aligned}$$

Similarly, we define $\gamma'\colon \mathrm{Hom}^\star_R(A', A') \to \mathrm{Hom}^\star_R(A, A)$ by

$$\gamma'(\alpha') = \varphi'\alpha'\varphi$$

for all $\alpha' \in \mathrm{Hom}^\star_R(A', A')$. We claim that $\gamma'\gamma \sim 1_{\mathrm{Hom}^\star_R(A,A)}$ and $\gamma'\gamma \sim 1_{\mathrm{Hom}^\star_R(A',A')}$. It suffices to show that $\gamma'\gamma \sim 1_{\mathrm{Hom}^\star_R(A,A)}$ as the other homotopy equivalence will follows by a similar argument. Let $H\colon \mathrm{Hom}^\star_R(A, A) \to \mathrm{Hom}^\star_R(A, A)$ be defined by

$$H(\alpha) = h\alpha\mathrm{d}h + h\alpha h\mathrm{d} + h\alpha + \alpha h$$

for all $\alpha \in \mathrm{Hom}^\star_R(A, A)$. Now let $\alpha \in \mathrm{Hom}^\star_R(A, A)_i$. Then we have

$$\begin{aligned}
(\gamma'\gamma - 1)(\alpha) &= (\gamma'\gamma)(\alpha) - \alpha \\
&= \gamma'(\gamma(\alpha)) - \alpha \\
&= \gamma'(\varphi\alpha\varphi') - \alpha \\
&= \varphi'\varphi\alpha\varphi'\varphi - \alpha \\
&= (\mathrm{d}h + h\mathrm{d} + 1)\alpha(\mathrm{d}h + h\mathrm{d} + 1) - \alpha \\
&= \mathrm{d}h\alpha\mathrm{d}h + \mathrm{d}h\alpha h\mathrm{d} + \mathrm{d}h\alpha + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h + \alpha h\mathrm{d} + \alpha - \alpha \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + (\mathrm{d}h + h\mathrm{d})\alpha + \alpha(\mathrm{d}h + h\mathrm{d}) \\
\\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i h\alpha h\mathrm{d}\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d} - h\alpha\mathrm{d}h)\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d})\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} - (-1)^i h\alpha\mathrm{d}h\mathrm{d} \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d})\mathrm{d} + (h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d}) - (-1)^i(h\alpha\mathrm{d}\mathrm{d}h + h\alpha\mathrm{d}h\mathrm{d}) \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) - (-1)^{i+1} H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha - (-1)^i\alpha\mathrm{d}) \\
&= \mathrm{d}^\star(H(\alpha)) + H(\mathrm{d}^\star(\alpha)) \\
&= (\mathrm{d}^\star H + H\mathrm{d}^\star)(\alpha)
\end{aligned}$$

$\square$

$$
\begin{aligned}
(\gamma'\gamma - 1)(\alpha) &= (\gamma'\gamma)(\alpha) - \alpha \\
&= \gamma'(\gamma(\alpha)) - \alpha \\
&= \gamma'(\varphi\alpha\varphi') - \alpha \\
&= \varphi'\varphi\alpha\varphi'\varphi - \alpha \\
&= (\mathrm{d}h + h\mathrm{d} + 1)\alpha(\mathrm{d}h + h\mathrm{d} + 1) - \alpha \\
&= \mathrm{d}h\alpha\mathrm{d}h + \mathrm{d}h\alpha h\mathrm{d} + \mathrm{d}h\alpha + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h + \alpha h\mathrm{d} + \alpha - \alpha \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + (\mathrm{d}h + h\mathrm{d})\alpha + \alpha(\mathrm{d}h + h\mathrm{d})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{d}h\alpha + \alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h \\
&= \mathrm{d}h\alpha - (-1)^i\mathrm{d}\alpha h + (-1)^i h\alpha\mathrm{d} + \alpha h\mathrm{d} + h\mathrm{d}\alpha + (-1)^i\mathrm{d}\alpha h - (-1)^i h\alpha\mathrm{d} + \alpha\mathrm{d}h \\
&= \mathrm{d}(h\alpha - (-1)^i\alpha h) + (-1)^i(h\alpha - (-1)^i\alpha h)\mathrm{d} + h\mathrm{d}\alpha + (-1)^i\mathrm{d}\alpha h - (-1)^i h\alpha\mathrm{d} + \alpha\mathrm{d}h \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) - (-1)^{i+1} H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha - (-1)^i\alpha\mathrm{d}) \\
&= \mathrm{d}^\star(H(\alpha)) + H(\mathrm{d}^\star(\alpha)) \\
&= (\mathrm{d}^\star H + H\mathrm{d}^\star)(\alpha)
\end{aligned}
$$

5.1.4 Direct Sum of DG Algebras is DG Algebra

Proposition 5.8. *Let (A, d) and (A', d') be two DG R-algebras. Then $(A \oplus_R A', \mathrm{d}^{A \oplus_R A'})$ is a DG R-algebra.*

Proof. Throughout this proof, denote $\mathrm{d}^\oplus := \mathrm{d}^{A \oplus_R A'}$. We define multiplication on $A \oplus_R A'$ by the formula

$$(a, a')(b, b') = (-1)^{i'j}(ab, a'b') \tag{44}$$

for all $a \otimes a' \in A_i \otimes_R A_{i'}$ and $b \otimes b' \in A_j \otimes_R A_{j'}$. It is easy to check that (43) is associative and unital with with unit being $e_A \otimes e_{A'}$ where e_A is the unit of A and $e_{A'}$ is the unit of A'. Let us check that Leibniz law is satisfied. Let $a \otimes a', b \otimes b' \in A \otimes_R A'$. Then we have

$$
\begin{aligned}
\mathrm{d}^\oplus((a, a')(b, b')) &= (-1)^{i'j}\mathrm{d}^\oplus(ab, a'b') \\
&= (-1)^{i'j}\mathrm{d}^\oplus(ab, a'b') \\
&= (-1)^{i'j}((\mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)) \otimes a'b' + (-1)^{i+j} ab \otimes (\mathrm{d}'(a')b' + (-1)^{i'} a'\mathrm{d}'(b'))) \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i'j+i} a\mathrm{d}(b) \otimes a'b' + (-1)^{i'j+i+j} ab \otimes \mathrm{d}'(a')b' + (-1)^{i'j+i+j+i'} ab \otimes a'\mathrm{d}'(b') \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i+j(i'+1)} ab \otimes \mathrm{d}'(a')b' + (-1)^{i+i'+i'(j+1)} a\mathrm{d}(b) \otimes a'b' + (-1)^{i+i'+j+i'j}(ab \otimes a'\mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a')(b \otimes b') + (-1)^i(a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b') + (-1)^{i+i'+j}(a \otimes a')(b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b' + (-1)^j b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}^\otimes(a \otimes a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}^\otimes(b \otimes b')).
\end{aligned}
$$

Thus $\mathrm{d}^\otimes$ satifies Leibniz law with respect to (43). $\square$

5.1.5 Localization of DG-Algebra

Let (A, d) be a DG R-algebra and let S be a multiplicatively-closed subset of A consisting of homogeneous elements of even degree. The **localization of** (A, d) **with respect to** S is the R-complex (A_S, d_S) where A_S is the graded R-module whose component in degree i is

$$(A_S)_i = \{a/s \mid j \in \mathbb{N}, a \in A_{i+j}, \text{ and } s \in A_j\}.$$

The differential d_S is defined as follows: if $a \in A_{i+j}$ and $s \in A_j$, then $a/s \in (A_S)_i$ and

$$\mathrm{d}_S\left(\frac{a}{s}\right) = \frac{\mathrm{d}(a)s - (-1)^{i+j}a\mathrm{d}(s)}{s^2}.$$

To see that this is well-defined, suppose $a/s = a'/s'$ with both $|s|$ and $|s'|$ even, so $as' = a's$ and $|a| = |a'|$. Applying the differential gives us

$$\mathrm{d}(a)s' + (-1)^{|a|}a\mathrm{d}(s') = \mathrm{d}(a')s + (-1)^{|a'|}a'\mathrm{d}(s).$$

We need to show that
$$\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2} = \frac{\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')}{s'^2}.$$

Or in other words, we need to show

$$\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s'^2 = \left(\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')\right)s^2.$$

We have

$$\begin{aligned}
\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s'^2 &= \mathrm{d}(a)ss'^2 - (-1)^{|a|}a\mathrm{d}(s)s'^2 \\
&= \mathrm{d}(a)s'^2 s - (-1)^{|a|}as'^2\mathrm{d}(s) \\
&= (\mathrm{d}(a')s + (-1)^{|a'|}a'\mathrm{d}(s) - (-1)^{|a|}a\mathrm{d}(s'))s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a\mathrm{d}(s')s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a'\mathrm{d}(s')s^2 - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 - (-1)^{|a|}a'\mathrm{d}(s')s^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 - (-1)^{|a'|}a'\mathrm{d}(s')s^2 \\
&= \left(\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')\right)s^2
\end{aligned}$$

Next, we need to check that $\mathrm{d}_S^2 = 0$. We have

$$\begin{aligned}
\mathrm{d}_S^2\left(\frac{a}{s}\right) &= \mathrm{d}_S\left(\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2}\right) \\
&= \frac{\mathrm{d}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s^2 - (-1)^{|a|-1}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)\mathrm{d}(s^2)}{s^4} \\
&= \frac{((-1)^{|a|-1}\mathrm{d}(a)\mathrm{d}(s) - (-1)^{|a|}\mathrm{d}(a)\mathrm{d}(s))s^2 + (-1)^{|a|}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)2s\mathrm{d}(s)}{s^4} \\
&= \frac{(-1)^{|a|-1}2\mathrm{d}(a)\mathrm{d}(s)s^2 + (-1)^{|a|}2\mathrm{d}(a)\mathrm{d}(s)s^2 - 2a\mathrm{d}(s)^2 s}{s^4} \\
&= \frac{0}{s^4} \\
&= 0.
\end{aligned}$$

Next, we need to check that Leibniz law is satisfies. We have

$$
\begin{aligned}
\mathrm{d}_S\left(\frac{aa'}{ss'}\right) &= \frac{\mathrm{d}(aa')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(ss')}{s^2s'^2} \\
&= \frac{\mathrm{d}(aa')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(ss')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)a'ss' + (-1)^{|a|}a\mathrm{d}(a')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(s)s' - (-1)^{|a|+|a'|}aa's\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's' + (-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's' + (-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's'}{s^2s'^2} + \frac{(-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \left(\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2}\right)\frac{a'}{s'} + (-1)^{|a|}\frac{a}{s}\left(\frac{\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')}{s'^2}\right) \\
&= \mathrm{d}_S\left(\frac{a}{s}\right)\frac{a'}{s'} + (-1)^{|a|}\frac{a}{s}\mathrm{d}_S\left(\frac{a'}{s'}\right).
\end{aligned}
$$

5.2 DG Modules

Definition 5.3. Let (A, d_A) be a DG R-algebra. A (right) **differential graded A-module** (or DG A-module for short) is an R-complex (M, d_M) equipped with a chain map

$$\star\colon (M \otimes_R A, \mathrm{d}^{M\otimes_R A}) \to (M, \mathrm{d}_M)$$

denoted $u \otimes a \mapsto \star(u \otimes a)$ (or just ua if context is clear). In other words, M has an A-module structure which behaves well with respect to the Leibniz law:

$$\mathrm{d}_M(ua) = \mathrm{d}_M(u)a + (-1)^i u\mathrm{d}_A(a)$$

for all $u \in M_i$ and $a \in A$. If (I, d_I) is an R-complex with $I \subset A$ and $\star$ being the usual multiplication map, then say (I, d_I) is a **DG ideal** in (A, d_A).

Definition 5.4. Let (A, d) be a DG R-algebra and let (M, d_M) and (N, d_N) be DG A-modules. A chain map $\varphi\colon (M, \mathrm{d}_M) \to (N, \mathrm{d}_N)$ is said to be a **DG-module morphism** if it respects A-scaling. In other words, we need

$$\varphi(ua) = \varphi(u)a$$

for all $u \in M$ and $a \in A$ (so the underlying map $\varphi\colon M \to N$ of A-modules is an A-module homomorphism). The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Obtaining a Differential Graded A-Module from an R-Complex

Example 5.1. Let (A, d_A) be a differential graded R-algebra and let (M, d_M) be an R-complex. Then the R-complex $(M \otimes_R A, \mathrm{d}^{M\otimes_R A})$ is a DG A-module.

5.2.1 Completion of DG Algebra with respect to an Ideal

Let (A, d) be a DG R-algebra and let (I, d) be a DG ideal in (A, d). We define the I-adic DG algebra, denoted $(\widehat{A}_I, \widehat{\mathrm{d}}_I)$, where

$$\widehat{A}_I := \varprojlim A/I^n = \{(\overline{a_n}) \in A/I^n \mid a_n \equiv a_m \bmod I^m \text{ whenever } n \geq m\}$$

and where $\widehat{\mathrm{d}}_I$ is defined pointwise:

$$\widehat{\mathrm{d}}_I((\overline{a_n})) = (\overline{\mathrm{d}(a_n)})$$

for all $(\overline{a_n}) \in \widehat{A}_I$. Note that the ith homogeneous component of $\widehat{A}_I$ is

$$(\widehat{A}_I)_i = \varprojlim_n (A_i/I_i^n) = \{(\overline{a_n}) \in A_i/I_i^n \mid a_n \equiv a_m \bmod I_i^m \text{ whenever } n \geq m\}.$$

In particular, if $(\overline{a_n}) \in (\widehat{A}_I)_i$, then $a_n \in A_i$ for all $i \geq 0$. Suppose $(\overline{a_n}) \in \ker \widehat{\mathrm{d}}_I$. Then $\mathrm{d}(a_n) \in I^n$ for all $n \in \mathbb{N}$.

5.2.2 Blowing up DG Algebra with respect to an Ideal

Let (A, d) be a DG R-algebra and let I be a DG ideal in A. Let

$$\mathrm{N}_I(A) := A \oplus A/I \oplus A/I^2 \oplus \cdots = A + (A/I)t + (A/I^2)t^2 + \cdots$$

and let $\mathrm{d}^{\mathrm{N}_I(A)} \colon \mathrm{N}_I(A) \to \mathrm{N}_I(A)$ be the unique graded linear map such that

$$\mathrm{d}^{\mathrm{N}_I(A)}(\overline{a}t^n) = \overline{\mathrm{d}(a)}t^{n-1},$$

for all $\overline{a}t^n \in (A/I^n)t^n$[4].

Proposition 5.9. *Let (A, d) be a DG R-algebra and let I be a DG ideal in A such that $I \subset A_+$. Then*

$$\mathrm{H}_n(\mathrm{N}_I(A)) = 0 \textit{ for } n \gg 0 \textit{ if and only if } \mathrm{H}(A) = 0.$$

Proof. Suppose first that $\mathrm{H}(A) = 0$ and assume for a contradiction that $\mathrm{H}_n(\mathrm{N}_I(A)) \neq 0$ for $n \gg 0$. Choose a $(\overline{a}$
Suppose $k \in \mathbb{Z}$ such that $\mathrm{H}_i(A) = 0$ for all $i \geq k$. We wish to show that □

Note that

$$\mathrm{H}_n(\mathrm{N}_I(A)) \cong \frac{\mathrm{d}^{-1}(I^{n-1})}{\mathrm{im}\ \mathrm{d} + I^n}.$$

Thus, we want to show that

$$\mathrm{d}^{-1}(I^{n-1}) = \mathrm{im}\ \mathrm{d} + I^n$$

for $n \gg 0$. The theorem would follow at once if we can show that

$$\mathrm{d}^{-1}(I^{n-1}) \subset I^{n-1}$$

for $n \gg 0$. Assume for a contradiction that we can find $a_n \in A \backslash I^n$ such that $\mathrm{d}(a_n) \in I^n$.

We claim that $\mathrm{H}_i(A) \cong \mathrm{H}_i(\mathrm{N}_I(A))$ for all i

5.3 The Koszul Complex

Throughout this subsection, let $\underline{x} = x_1, \ldots, x_n$ be a sequence in R. We will construct a DG R-algebra called the **Koszul complex of** $\underline{x}$. Before doing so, we need to discuss ordered sets.

5.3.1 Ordered Sets

An **ordered set** is a set with a total linear ordering on it. The **ordered set** $[n]$ is the set $\{1, \ldots, n\}$ equipped with the natural ordering $1 < \cdots < n$. Let σ be a subset of $\{1, \ldots, n\}$. Then the natural ordering on $\{1, \ldots, n\}$ induces a natural ordering on σ. If we want to think of σ as a set equipped with this natural ordering, then we will write $[\sigma]$. If $\sigma = \{\lambda_1, \ldots, \lambda_k\}$, where $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$, then we will also write $[\sigma] = [\lambda_1, \ldots, \lambda_k]$. If we write "suppose $[\sigma] = [\lambda_1, \ldots, \lambda_k]$", then it is understood that $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$. For each $i \in \mathbb{Z}$ such that $0 \leq i \leq n$, we denote

$$S_i[n] := \{\sigma \subseteq \{1, \ldots, n\} \mid |\sigma| = i\}.$$

Compliments

Let $\sigma \subseteq [n]$. We denote by $\sigma^\star$ to be the compliment of σ in $[n]$:

$$\sigma^\star := [n] \backslash \sigma.$$

If $[\sigma] = [\lambda_1, \ldots, \lambda_k]$, then we write $\sigma^\star = [\lambda_1^\star, \ldots, \lambda_{n-k}^\star]$.

[4]Here, the $\overline{a}$ is understood to be a coset in A/I^n with represenative $a \in A$.

Signature

Let σ and τ be two disjoint subsets of $\{1, \dots, n\}$. Suppose that

$$[\sigma] = [\lambda_1, \dots, \lambda_k] \quad \text{and} \quad [\sigma'] = [\lambda_{k+1}, \dots, \lambda_{k+m}].$$

Then

$$[\sigma \cup \sigma'] = [\lambda_{\pi(1)}, \dots, \lambda_{\pi(k+m)}],$$

where $\pi \colon S_{k+m} \to S_{k+m}$ is the permutation which puts everything in the correct order. We define

$$\langle \sigma, \tau \rangle := \operatorname{sign}(\pi).$$

Remark. Let $\lambda \in \{1, \dots, n\}$ and let $\sigma \subseteq \{1, \dots, n\}$. To clean notation, we often drop the curly brackets around singleton elements $\{\lambda\}$ in what follows. For instance, we will write $\sigma \backslash \lambda$ instead of $\sigma \backslash \{\lambda\}$ and $\sigma \cup \lambda$ instead of $\sigma \cup \{\lambda\}$. We will also write $\langle \lambda, \sigma \rangle$ (or $\langle \sigma, \lambda \rangle$) instead of $\langle \{\lambda\}, \sigma \rangle$ (respectively $\langle \sigma, \{\lambda\} \rangle$).

Example 5.2. Consider $n = 4$. We perform some computations:

$$\begin{aligned}
\langle 2, \{1,4\} \rangle &= -1 \\
\langle 2, 3 \rangle &= 1 \\
\langle 3, 2 \rangle &= -1 \\
\langle \{1,4\}, 2 \rangle &= -1 \\
\langle 2, \{1,3,4\} \rangle &= -1 \\
\langle \{1,3,4\}, 2 \rangle &= 1 \\
\langle \{1,3\}, \{2,4\} \rangle &= -1 \\
\langle \{2,4\}, \{1,3\} \rangle &= -1
\end{aligned}$$

Signature Identities

Proposition 5.10. *Let σ, τ, and $\{\lambda\}$ be mutually disjoint subsets of $\{1, \dots, n\}$. Then*

$$\langle \lambda, \sigma \cup \tau \rangle = \langle \lambda, \sigma \rangle \langle \lambda, \tau \rangle.$$

Proof. The permutation which places $[\lambda] \cup [\sigma \cup \tau]$ into proper order is a composition of the permutation which places $[\lambda] \cup [\sigma]$ into proper order with the permutation which places $[\lambda] \cup [\tau]$ into proper order. □

Proof. The permutation with puts λ in the proper order in $[\lambda] \cup [\sigma \cup \tau]$ is just a composition of the permutation which puts λ in the proper order in $[\lambda] \cup [\sigma]$ with the permutation which puts λ in the proper order in $[\lambda] \cup [\tau]$. □

Proposition 5.11. *Let σ and τ be two disjoint subsets of $\{1, \dots, n\}$. If $\lambda \in \sigma$, then*

$$\langle \sigma, \tau \rangle = \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle.$$

Similarly, if $\mu \in \tau$, then

$$\langle \sigma, \tau \rangle = \langle \sigma, \mu \rangle \langle \sigma, \tau \backslash \mu \rangle. \tag{45}$$

Proof. Suppose $\lambda \in \sigma$. We can place $[\sigma] \cup [\tau]$ into proper order by moving λ all the way to the left of $[\sigma]$, then place $[\sigma \backslash \lambda] \cup [\tau]$ into proper order, then place $[\lambda] \cup [\sigma \backslash \lambda \cup \tau]$ into proper order. This gives us

$$\begin{aligned}
\langle \sigma, \tau \rangle &= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, (\sigma \backslash \lambda) \cup \tau) \rangle \\
&= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \sigma \backslash \lambda \rangle \langle \lambda, \tau \rangle \\
&= \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle
\end{aligned}$$

An analagous argument gives (45). □

5.3.2 Definition of the Koszul Complex

We are now ready to define the Koszul complex of $\underline{x}$.

Definition 5.5. The **Koszul complex of** $\underline{x}$, denoted $(\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})})$ is the R-complex whose graded R-module $\mathcal{K}(x)$ has

$$\mathcal{K}_i(\underline{x}) := \begin{cases} \bigoplus_{\sigma \in S_i[n]} Re_\sigma & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $\mathrm{d}^{\mathcal{K}(\underline{x})}$ is uniquely determined by

$$\mathrm{d}^{\mathcal{K}(\underline{x})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle x_\lambda e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq \{1, \dots, n\}$.

Exercise 1. Check that $(\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})})$ is an R-complex. In particular, show $\mathrm{d}^{\mathcal{K}(\underline{x})}\mathrm{d}^{\mathcal{K}(\underline{x})} = 0$.

Example 5.3. Here's what the Koszul complex $\mathcal{K}(x_1, x_2, x_3)$ looks like:

$$R \xrightarrow{\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} R$$

$$\begin{aligned}
e_{\{1,2,3\}} &\longmapsto x_1 e_{\{2,3\}} - x_2 e_{\{1,3\}} + x_3 e_{\{1,2\}} \\
e_{\{2,3\}} &\longmapsto x_2 e_{\{3\}} - x_3 e_{\{2\}} \\
e_{\{1,3\}} &\longmapsto x_1 e_{\{3\}} - x_3 e_{\{1\}} \\
e_{\{1,2\}} &\longmapsto x_1 e_{\{2\}} - x_2 e_{\{1\}} \\
e_{\{1\}} &\longmapsto x_1 \\
e_{\{2\}} &\longmapsto x_2 \\
e_{\{3\}} &\longmapsto x_3
\end{aligned}$$

5.3.3 Koszul Complex as Tensor Product

Proposition 5.12. *We have an isomorphism of R-complexes:*

$$(\mathcal{K}(x_1), \mathrm{d}^{\mathcal{K}(x_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(x_n), \mathrm{d}^{\mathcal{K}(x_n)}) \cong (\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})}).$$

Remark. Note that Proposition (3.21) gives an unambiguous interpretation for $(\mathcal{K}(x_1), \mathrm{d}^{\mathcal{K}(x_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(x_n), \mathrm{d}^{\mathcal{K}(x_n)})$.

Proof. For each $1 \leq \lambda \leq n$, write $\mathcal{K}(x_\lambda) = R \oplus Re_\lambda$ (so $\{1\}$ is a basis for $\mathcal{K}(x_\lambda)_0$ and $\{e_\lambda\}$ is a basis for $\mathcal{K}(x_\lambda)_1$). Let

$$\varphi \colon \mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n) \to \mathcal{K}(\underline{x})$$

be the unique graded linear map [5] such that

$$\varphi(1 \otimes \cdots \otimes 1) = 1 \quad \text{and} \quad \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1) = e_{\{\lambda_1, \dots, \lambda_i\}}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism since it restricts to a bijection on basis sets.

For the rest of the proof, denote $\mathrm{d}^{\mathcal{K}} := \mathrm{d}^{\mathcal{K}(\underline{x})}$ and $\mathrm{d}^{\otimes} := \mathrm{d}^{\mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n)}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi \mathrm{d}^{\otimes} = \mathrm{d}^{\mathcal{K}} \varphi. \tag{46}$$

It suffices to check (**??**) on the basis elements. We have

$$\begin{aligned}
\mathrm{d}^{\mathcal{K}} \varphi(1 \otimes \cdots \otimes 1) &= \mathrm{d}^{\mathcal{K}}(1) \\
&= 0 \\
&= \varphi(0) \\
&= \varphi \mathrm{d}^{\otimes}(1 \otimes \cdots \otimes 1),
\end{aligned}$$

[5] We say unique graded linear map here because $\mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n)$ is free with basis elements of the form $1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1$ for $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and φ respects the grading.

and

$$
\begin{aligned}
\mathrm{d}^{\mathcal{K}}\varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1) &= \mathrm{d}^{\mathcal{K}}(e_{\{\lambda_1,\dots,\lambda_i\}})\\
&= \sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}e_{\{\lambda_1,\dots,\lambda_i\}}\\
&= \sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}\varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes\widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1)\\
&= \varphi\sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes\widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1)\\
&= \varphi\mathrm{d}^{\otimes}(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1).
\end{aligned}
$$

for all $1\leq\lambda_1<\cdots<\lambda_i\leq n$. □

5.3.4 Koszul Complex is a DG Algebra

Proposition 5.13. *Let $\underline{x}=x_1,\dots,x_n$ be a sequence of elements in R. The Koszul complex $(\mathcal{K}(\underline{x}),\mathrm{d}^{\mathcal{K}(\underline{x})})$ is a DG algebra, with multiplication being uniquely determined on elementary tensors: for $\sigma,\tau\subseteq[n]$, we map $e_\sigma\otimes e_\tau\mapsto e_\sigma e_\tau$, where*

$$
e_\sigma e_\tau=\begin{cases}\langle\sigma,\tau\rangle e_{\sigma\cup\tau} & \text{if } \sigma\cap\tau=\emptyset\\ 0 & \text{if } \sigma\cap\tau\neq\emptyset\end{cases}\tag{47}
$$

Proof. Throughout this proof, denote $\mathrm{d}:=\mathrm{d}^{\mathcal{K}(\underline{x})}$. We first want to show that $\mathcal{K}(\underline{x})$ is an associative, unital, and strictly graded-commutative R-algebra. Since $\mathcal{K}(\underline{x})$ is a free R-module with $\{e_\sigma\mid\sigma\subseteq[n]\}$ as a basis, it suffices to check associativity and graded-commutativitythese properties on the basis elements. We first note that $e_\emptyset$ serves as the identity for the multiplication rule (**??**). Indeed, let $\sigma\subseteq[n]$. Then since $\sigma\cap\emptyset=\emptyset$, we have

$$
e_\sigma e_\emptyset=e_\sigma=e_\emptyset e_\sigma.
$$

Thus the underlying R-algebra $\mathcal{K}(\underline{x})$ is unital.

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is associative. Let $\sigma,\tau,\kappa\subseteq[n]$. If $\sigma\cap\tau\cap\kappa\neq\emptyset$, then it is clear that

$$
\begin{aligned}
e_\sigma(e_\tau e_\kappa)&=0\\
&=(e_\sigma e_\tau)e_\kappa,
\end{aligned}
$$

so assume $\sigma\cap\tau\cap\kappa=\emptyset$. Then

$$
\begin{aligned}
e_\sigma(e_\tau e_\kappa)&=\langle\tau,\kappa\rangle e_\sigma e_{\tau\cup\kappa}\\
&=\langle\sigma,\tau\cup\kappa\rangle\langle\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&=\langle\sigma,\tau\rangle\langle\sigma,\kappa\rangle\langle\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&=\langle\sigma,\tau\rangle\langle\sigma\cup\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&=\langle\sigma,\tau\rangle e_{\sigma\cup\tau}e_\kappa\\
&=(e_\sigma e_\tau)e_\kappa.
\end{aligned}
$$

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is graded-commutative. Let $\sigma,\tau\subseteq[n]$. If $\sigma\cap\tau\neq\emptyset$, then

$$
\begin{aligned}
e_\sigma e_\tau&=0\\
&=(-1)^{|\sigma||\tau|}e_\tau e_\sigma.
\end{aligned}
$$

Suppose $\sigma\cap\tau=\emptyset$. Then

$$
\begin{aligned}
e_\sigma e_\tau&=\langle\sigma,\tau\rangle e_{\sigma\cup\tau}\\
&=(-1)^{|\sigma||\tau|}\langle\tau,\sigma\rangle e_{\sigma\cup\tau}\\
&=(-1)^{|\sigma||\tau|}e_\tau e_\sigma.
\end{aligned}
$$

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is strictly graded-commutative. Let $\sigma\subseteq[n]$ such that $|\sigma|$ is odd. Then

$$
\begin{aligned}
e_\sigma^2&=e_\sigma e_\sigma\\
&=0
\end{aligned}
$$

since $\sigma \cap \sigma \neq \emptyset$.

Finally, we need to check Leibniz law. First note that multiplication by $e_\emptyset$ and e_σ satisfies Leibniz law:

$$\begin{aligned}
\mathrm{d}(e_\sigma)e_\emptyset - e_\sigma d(e_\emptyset) &= \mathrm{d}(e_\sigma)e_\emptyset \\
&= \mathrm{d}(e_\sigma) \\
&= \mathrm{d}(e_\sigma e_\emptyset),
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathrm{d}(e_\emptyset)e_\sigma + e_\emptyset \mathrm{d}(e_\sigma) &= e_\emptyset \mathrm{d}(e_\sigma) \\
&= \mathrm{d}(e_\sigma) \\
&= \mathrm{d}(e_\emptyset e_\sigma),
\end{aligned}$$

Next, let $\lambda \in [n]$ and let $\tau \subseteq [n]$. If $\lambda \in \tau$, then the pair (e_λ, e_τ) satisfies Leibniz law trivially, so suppose that $\lambda \notin \tau$. Then

$$\begin{aligned}
\mathrm{d}(e_\lambda)e_\tau - e_\lambda \mathrm{d}(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu} \\
&= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\
&= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \rangle \langle \lambda, \mu \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\
&= x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \rangle \langle \mu, \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\
&= x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \cup \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\
&= \langle \lambda, \tau \rangle \langle \lambda, \tau \rangle x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \cup \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\
&= \langle \lambda, \tau \rangle \sum_{\mu \in \tau \cup \lambda} \langle \mu, (\tau \cup \lambda) \backslash \mu \rangle x_\mu e_{(\tau \cup \lambda) \backslash \mu} \\
&= \langle \lambda, \tau \rangle \mathrm{d}(e_{\tau \cup \lambda}) \\
&= \mathrm{d}(e_\lambda e_\tau),
\end{aligned}$$

where we used Proposition (5.11) to get from the second line to the third line. Next suppose $\tau \subseteq [n]$ and $\lambda \in \tau$. Then

$$\begin{aligned}
\mathrm{d}(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu} \\
&= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_\lambda e_{\tau \backslash \mu} \\
&= x_\lambda e_\tau - \langle \lambda, \tau \backslash \lambda \rangle \langle \lambda, \tau \backslash \lambda \rangle x_\lambda e_\tau \\
&= x_\lambda e_\tau - x_\lambda e_\tau \\
&= 0 \\
&= \mathrm{d}(0) \\
&= \mathrm{d}(e_\lambda e_\tau).
\end{aligned}$$

Thus we have shown (**??**) satisfies the Leibniz law for all pairs (λ, τ) where $\lambda \in [n]$ and $\tau \subseteq [n]$. We prove by induction on $|\sigma| = i \geq 1$ that (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$. The base case $i = 1$ was just shown. Now suppose we have shown (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i < n$. Let $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i + 1$. Choose $\lambda \in \sigma$. Then

$$\begin{aligned}
\mathrm{d}(e_\sigma e_\tau) &= \mathrm{d}(e_\lambda e_{\sigma \backslash \lambda} e_\tau) \\
&= x_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda \mathrm{d}(e_{\sigma \backslash \lambda} e_\tau) \\
&= x_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda (\mathrm{d}(e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma|-1} e_{\sigma \backslash \lambda} \mathrm{d}(e_\tau)) \\
&= (x_\lambda e_{\sigma \backslash \lambda} - e_\lambda \mathrm{d}(e_{\sigma \backslash \lambda})) e_\tau + (-1)^{|\sigma|} e_\sigma \mathrm{d}(e_\tau) \\
&= \mathrm{d}(e_\lambda e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma|} e_\sigma \mathrm{d}(e_\tau) \\
&= \mathrm{d}(e_\sigma) e_\tau + (-1)^{|\sigma|+1} e_\sigma \mathrm{d}(e_\tau),
\end{aligned}$$

where we used the base case on the pairs $(e_\lambda, e_{\sigma \backslash \lambda} e_\tau)$[6] and $(e_\lambda, e_{\sigma \backslash \lambda})$ and where we used the induction hypothesis on the pair $(e_{\sigma \backslash \lambda}, e_\tau)$. and where we used the base case on the pair $(e_\lambda, e_{\sigma \backslash \lambda})$. □

[6] If $e_{\sigma \backslash \lambda} e_\tau = 0$, then obviously Leibniz law holds for the pair $(e_\lambda, e_{\sigma \backslash \lambda} e_\tau)$.

5.3.5 The Dual Koszul Complex

We now want to discuss the dual Koszul complex of $\underline{x}$.

Definition 5.6. The **dual Koszul complex of** $\underline{x}$ is the R-complex

$$\mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R),$$

where R is viewed as a trivial R-complex (trivially grading with $\mathrm{d} = 0$). We denote by $\mathcal{K}^\star(\underline{x})$ to be the graded R-module hom $\mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R)$. We also denote by $\mathrm{d}^{\mathcal{K}^\star(\underline{x})}$ to be the corresponding differential. We can describe the dual Koszul complex more explicitly as follows: the graded R-module $\mathcal{K}^\star(\underline{x})$ has

$$\mathcal{K}_i^\star(\underline{x}) := \begin{cases} \bigoplus_{\sigma \in S_{-i}[n]} Re_\sigma^\star & \text{if } -n \leq i \leq 0 \\ 0 & \text{if } i < n \text{ or if } i > 0. \end{cases}$$

as its ith homogeneous component, where $e_\sigma^\star \colon \mathcal{K}(\underline{x}) \to R$ is uniquely determined by

$$e_\sigma^\star(e_{\sigma'}) = \begin{cases} 1 & \sigma = \sigma' \\ 0 & \text{else.} \end{cases}$$

for all $\sigma, \sigma' \subseteq [n]$. The differential $\mathrm{d}^{\mathcal{K}^\star(\underline{x})}$ is uniquely determined by

$$\mathrm{d}^{\mathcal{K}^\star(\underline{x})}(e_\sigma^\star) = (-1)^{|\sigma|+1} \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma \rangle r_\lambda e_{\sigma \cup \lambda^\star}^\star$$

for all $\sigma \subseteq [n]$.

Duality

Theorem 5.1. *There exists an isomorphism of R-complexes*

$$S^n \mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R) \cong \mathcal{K}(\underline{x}).$$

In particular, we have an isomorphism of R-modules

$$\mathrm{H}_i(\mathcal{K}(\underline{x})) \cong \mathrm{H}_{i-n}(\mathcal{K}^\star(\underline{x}))$$

for all $i \in \mathbb{Z}$.

Proof. Let $i \in \mathbb{Z}$. If $i > n$ or $i < 0$, then theorem is obvious, so we may assume that $0 \leq i \leq n$. Let $\varphi \colon S^n(\mathcal{K}^\star(\underline{r}), d^{\mathcal{K}^\star(\underline{r})}) \to (\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ be the unique R-module graded homomorphism such that

$$\varphi(e_\sigma^\star) = \langle \sigma^\star, \sigma \rangle e_{\sigma^\star}.$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism of graded R-modules since it restricts to a bijection of basis sets. To see that φ is an isomorphism of R-complexes, we need to show that it commutes with the differentials. To do this, we first simplify notation by denoting $d^\star := (d^{\mathcal{K}^\star(\underline{r})})^{\Sigma^n}$ and $d := d^{\mathcal{K}(\underline{r})}$. Now we have

$$\begin{aligned}
d\varphi(e^\star_\sigma) &= d(\langle \sigma^\star, \sigma\rangle e_{\sigma^\star}) \\
&= \langle \sigma^\star, \sigma\rangle d(e_{\sigma^\star}) \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \sigma^\star, \sigma\rangle \langle \lambda^\star, \sigma^\star \backslash \lambda^\star\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma^\star \backslash \lambda^\star\rangle \langle \sigma^\star, \sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma^\star \backslash \lambda^\star\rangle \langle \sigma^\star\backslash\lambda^\star, \sigma\rangle \langle \lambda^\star, \sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \sigma^\star\backslash\lambda^\star, \sigma \cup \lambda^\star\rangle \langle \lambda^\star, \sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma\rangle \langle \sigma^\star\backslash\lambda^\star, \sigma \cup \lambda^\star\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma\rangle \langle (\sigma\cup\lambda^\star)^\star, \sigma \cup \lambda^\star\rangle r_{\lambda^\star} e_{(\sigma\cup\lambda^\star)^\star} \\
&= \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma\rangle r_{\lambda^\star} \varphi(e^\star_{\sigma\cup\lambda^\star}) \\
&= \varphi \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma\rangle r_{\lambda^\star} e^\star_{\sigma\cup\lambda^\star} \\
&= \varphi d^\star(e^\star_\sigma)
\end{aligned}$$

where we used the fact that $\sigma^\star\backslash\lambda^\star = (\sigma\cup\lambda^\star)^\star$ and $\langle\sigma^\star,\sigma\rangle = \langle\lambda^\star, \sigma^\star\backslash\lambda^\star\rangle\langle\lambda^\star,\sigma\rangle\langle\sigma^\star\backslash\lambda^\star, \sigma\cup\lambda^\star\rangle$. □

5.3.6 Mapping Cone of Homothety Map as Tensor Product

Proposition 5.14. *Let (A, d) be an R-complex, let $x \in R$, and let $\mu_x \colon (A, \mathrm{d}) \to (A, \mathrm{d})$ be the multiplication by x homothety map. Then*

$$(\mathrm{C}(\mu_x), \mathrm{d}^{\mathrm{C}(\mu_x)}) \cong (\mathcal{K}(x), \mathrm{d}^{\mathcal{K}(x)}) \otimes_R (A, \mathrm{d}).$$

Proof. Let $\mathcal{K}(x) = R \oplus Re$ (so $\{1\}$ is a basis for $\mathcal{K}(x)_0$ and $\{e\}$ is a basis for $\mathcal{K}(x)_1$). Let $\varphi \colon \mathcal{K}(x) \otimes_R A \to \mathrm{C}(\mu_x)$ be defined by

$$\varphi(1 \otimes a + e \otimes b) = (a, b)$$

for all $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Clearly φ is an isomorphism of graded R-modules. To see that φ is an isomorphism of R-complexes, we need to check that

$$\mathrm{d}^{\mathrm{C}(\mu_x)}\varphi = \varphi \mathrm{d}^{\mathcal{K}(x)\otimes_R A} \tag{48}$$

Let $i \in \mathbb{Z}$, $a \in A_i$, and $b \in A_{i-1}$. Then

$$\begin{aligned}
\mathrm{d}^{\mathrm{C}(\mu_x)}\varphi(1\otimes a + e \otimes b) &= \mathrm{d}^{\mathrm{C}(\mu_x)}(a, b) \\
&= (\mathrm{d}(a) + xb, -\mathrm{d}(b)) \\
&= \varphi(1 \otimes (\mathrm{d}(a) + xb) + e \otimes (-\mathrm{d}(b))) \\
&= \varphi(1 \otimes \mathrm{d}(a) + x \otimes b - e \otimes \mathrm{d}(b)) \\
&= \varphi(\mathrm{d}^{\mathcal{K}(x)\otimes_R A}(1\otimes a) + \mathrm{d}^{\mathcal{K}(x)\otimes_R A}(e \otimes b)) \\
&= \varphi \mathrm{d}^{\mathcal{K}(x)\otimes_R A}(1 \otimes a + e \otimes b).
\end{aligned}$$

□

5.3.7 Properties of the Koszul Complex

Proposition 5.15. *Let $\lambda \in [n]$. Then the homothety map*

$$\mu_{x_\lambda} \colon (\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})}) \to (\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})})$$

is null-homotopic. In particular, $x_\lambda \mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$.

Proof. Denote $\mathrm{d} := \mathrm{d}^{\mathcal{K}(\underline{x})}$ and let $h \colon \mathcal{K}(\underline{x}) \to \mathcal{K}(\underline{x})$ be the unique graded homomorphism of degree 1 such that

$$h(e_\sigma) = e_\lambda e_\sigma$$

for all $\sigma \subseteq [n]$. Then

$$\begin{aligned}(h\mathrm{d} + h\mathrm{d})(e_\sigma) &= \mathrm{d}(e_\lambda e_\sigma) + e_\lambda \mathrm{d}(e_\sigma) \\ &= x_\lambda e_\sigma - e_\lambda \mathrm{d}(e_\sigma) + e_\lambda \mathrm{d}(e_\sigma) \\ &= x_\lambda e_\sigma\end{aligned}$$

for all $\sigma \subseteq [n]$. It follows that

$$\mathrm{d}h + h\mathrm{d} = \mu_{x_\lambda}$$

on all of $\mathcal{K}(\underline{x})$. Thus the homothety map μ_{x_λ} is null-homotopic. □

Proposition 5.16. *The following conditions are equivalent.*

1. $\langle \underline{x} \rangle = R$,
2. $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$,
3. $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong 0$.

This follows immediately from Proposition (5.15) and the fact that $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong R/\langle \underline{x} \rangle$, but we will give an alternative proof:

Proof. Throughout this proof, we denote $\mathrm{d} := \mathrm{d}^{\mathcal{K}(\underline{x})}$.

($1 \implies 2$) Since $\langle \underline{x} \rangle = R$, there exists $y_1, \dots, y_n \in R$ such that

$$\sum_{\lambda=1}^{n} x_\lambda y_\lambda = 1.$$

Choose such $y_1, \dots, y_n \in R$ and let $\overline{f} \in \mathrm{H}(\mathcal{K}(\underline{x}))$ (so $f \in \ker \mathrm{d}$ is a representative of the coset $\overline{f}$). Then

$$\begin{aligned}\mathrm{d}\left(\sum_{\lambda=1}^{n} y_\lambda e_\lambda f\right) &= \sum_{\lambda=1}^{n} y_\lambda \mathrm{d}(e_\lambda f) \\ &= \sum_{\lambda=1}^{n} y_\lambda (\mathrm{d}(e_\lambda) f - e_\lambda \mathrm{d}(f)) \\ &= \sum_{\lambda=1}^{n} y_\lambda x_\lambda f \\ &= \left(\sum_{\lambda=1}^{n} y_\lambda x_\lambda\right) f \\ &= f.\end{aligned}$$

Thus, $f \in \operatorname{im} \mathrm{d}$, which implies $\mathrm{H}(\mathcal{K}(\underline{x})) = 0$.

($2 \implies 3$) $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$ if and only if $\mathrm{H}_i(\mathcal{K}(\underline{x})) \cong 0$ for all $i \in \mathbb{Z}$. In particular, $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$ implies $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong 0$.

($3 \implies 1$) We have

$$\begin{aligned}0 &\cong \mathrm{H}(\mathcal{K}(\underline{x})) \\ &= R/\langle \underline{x} \rangle,\end{aligned}$$

which implies $\langle \underline{x} \rangle = R$. □

Denote $\mathcal{K}(\underline{x}; M) := \mathcal{K}(\underline{x}) \otimes_R M$.

6 Advanced Homological Algebra

Definition 6.1. Let

$$0 \longrightarrow A \xrightarrow{\varphi} A' \xrightarrow{\varphi'} A'' \longrightarrow 0 \tag{49}$$

be an exact sequence of R-complexes and chain maps. We say (49) is **degree-wise exact** if it is exact when viewed as a sequence of graded R-modules, that is, if for each $i \in \mathbb{Z}$ the sequence

$$0 \longrightarrow A_i \xrightarrow{\varphi_i} A'_i \xrightarrow{\varphi'_i} A''_i \longrightarrow 0 \tag{50}$$

is exact. Similarly, we say (49) is **degree-wise split exact** if (49) is split exact for each $i \in \mathbb{Z}$.

Proposition 6.1. *Let*

be an exact sequence of R-complexes and chain maps. Assume that for all $p \in \mathbb{Z}$ the sequence $\xi_p = (0 \to A_p \xrightarrow{\alpha_p} B_p \xrightarrow{\beta_p} C_p \to 0)$ is split exact. Then for all R-complexes X, Y the sequences $\xi_* = \mathrm{Hom}_R(X, \xi)$ and $\xi^* = \mathrm{Hom}_R(\xi, Y)$ are short exact.

Proof. Focus on ξ^*. First note that $0 \to C^* \xrightarrow{\beta^*} B \xrightarrow{\alpha^*} A^*$ is exact by left exactness. Need to show α^* is surjective. Note that ξ_p split implies $\gamma_p \colon B_p \to A_p$ such that $\gamma_p \alpha_p = 1_{A_p}$. We have

$$\begin{aligned} \mathrm{Hom}_R(\alpha_p, Y_{p+n}) &= \mathrm{Hom}_R(\gamma_p, Y_{p+n}) \\ &= \mathrm{Hom}_R(\gamma_p \alpha_p, Y_{p+n}) \\ &= \mathrm{Hom}_R(1_{A_p}, Y_{p+n}) \\ &= 1_{\mathrm{Hom}_R(A_p, Y_{p+n})}. \end{aligned}$$

□

Remark. There is a notion of split exactness for sequences of R-complexes and chain maps. Essentially the splitting map has to commute with the differentials.

Definition 6.2. Exact sequence ξ as above is called **degree-wise split exact**

6.1 Resolutions

Definition 6.3. Let M be an R-complex.

1. A **projective resolution of** M is a bounded below R-complex of projective R-modules P equipped with a quasiisomorphism $\tau \colon P \xrightarrow{\simeq} M$. In this case, we say (P, τ) (or just P if context is clear) is a projective resolution of M.

2. An **injective resolution of** M is a bounded above R-complex of injective R-modules E equipped with a quasiisomorphism $\varepsilon \colon M \xrightarrow{\simeq} E$. In this case, we say (E, ε) (or just E if context is clear) is an injective resolution of M.

6.1.1 Existence of projective resolutions

Proposition 6.2. *Let M, N, and P be R-modules, let $\psi\colon N \to M$ be an R-linear map, and let $\varphi\colon P \twoheadrightarrow M$ be a surjective R-linear map. Define the* ***pullback of*** $\psi\colon N \to M$ ***and*** $\varphi\colon P \twoheadrightarrow M$ *to be the R-module*

$$N \times_M P = \{(u,v) \in N \times P \mid \psi(u) = \varphi(v)\}$$

equipped with the R-linear maps $\pi_1\colon N \times_M P \to N$ and $\pi_2\colon N \times_M P \to P$ given by

$$\pi_1(u,v) = u \quad \text{and} \quad \pi_2(u,v) = v$$

for all $(u,v) \in N \times_M P$. Then there exists an isomorphism $\overline{\varphi}\colon P/\pi_1(N \times_M P) \to M/N$ given by

$$\overline{\varphi}(\overline{v}) = \overline{\varphi(v)}$$

for all $\overline{v} \in P/\pi_1(N \times_M P)$. Moreover, the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & \ker\pi_2 \arrow[r] \arrow[d, "\pi_1|_{\ker\pi_2}"] & N \times_M P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & P \arrow[r] \arrow[d, "\varphi"] & P/\pi_1(N \times_M P) \arrow[r] \arrow[d, "\overline{\varphi}"] & 0 \\
0 \arrow[r] & \ker\psi \arrow[r] & N \arrow[r, "\psi"] & M \arrow[r] & M/\psi(N) \arrow[r] & 0
\end{tikzcd}$$

where π_1 induces an isomorphism $\pi_1\colon \ker\pi_2 \to \ker\psi$.

Proof. We first need to check that $\overline{\varphi}$ is well-defined. Suppose $v + v'$ is another representative of $\overline{v}$ where $v' \in \operatorname{im}\pi_2$. Choose $[u',v'] \in N \times_M P$ such that $\pi_2[u',v'] = v'$ (so $\varphi(v') = \psi(u')$). Then

$$\begin{aligned}
\overline{\varphi}(\overline{v+v'}) &= \overline{\varphi(v+v')} \\
&= \overline{\varphi(v) + \varphi(v')} \\
&= \overline{\varphi(v) + \psi(u')} \\
&= \overline{\varphi(v)}.
\end{aligned}$$

Thus $\overline{\varphi}$ is well-defined. Clearly, $\overline{\varphi}$ is a surjective R-linear map since φ is a surjective R-linear map. It remains to show that $\overline{\varphi}$ is injective. Suppose $\overline{v} \in \ker\overline{\varphi}$. Then $\varphi(v) \in \operatorname{im}\psi$. Choose $u \in N$ such that $\psi(u) = \varphi(v)$. Then $[u,v] \in N \times_M P$ and $v = \pi_2[u,v]$. It follows that $\overline{v} = 0$ in $P/\pi_2(N \times_M P)$.

Let us now check that $\pi_1|_{\ker\pi_2}$ lands in $\ker\psi$. Let $u \in \ker\pi_2$. Then

$$\begin{aligned}
\psi\pi_1(u) &= \varphi\pi_2(u) \\
&= \varphi(0) \\
&= 0
\end{aligned}$$

implies $\pi_1(u) \in \ker\psi$. Thus $\pi_1|_{\ker\pi_2}$ lands in $\ker\psi$. Now we check that $\pi_1|_{\ker\pi_2}$ is an R-linear isomorphism. It is clearly an R-linear isomomorphism since it is the restriction of the homomorphism π_1. To see that $\pi_1|_{\ker\pi_2}$ is surjective, let $u \in \ker\psi$. Since

$$\begin{aligned}
\psi(u) &= 0 \\
&= \varphi(0),
\end{aligned}$$

we see that $[u,0] \in N \times_M P$. Moreover we have $\pi_2[u,0] = 0$ and so $[u,0] \in \ker\pi_2$, and since $\pi_1[u,0] = u$, we see that $\pi_1|_{\ker\pi_2}$ is surjetive. To see that $\pi_1|_{\ker\pi_2}$ is injective, suppose $\pi_1[u,v] = 0$ for some $[u,v] \in \ker\pi_2$. Then

$$\begin{aligned}
0 &= \pi_1[u,v] \\
&= u
\end{aligned}$$

implies $u = 0$ and

$$\begin{aligned}
0 &= \pi_2[u,v] \\
&= v
\end{aligned}$$

implies $v = 0$. Thus $[u,v] = [0,0]$, hence $\pi_1|_{\ker\pi_2}$ is injective. □

Theorem 6.1. *Let (M, d) be an R-complex such that $M_i = 0$ for all $i < 0$. Then there exists a projective resolution of (M, d).*

Proof. We construct an R-complex (P,∂) together with a chain map $\tau\colon (P,\partial) \to (M,\mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker\partial}\colon \ker\partial \to \ker\mathrm{d}$$

by induction on homological degree as follows: for the base case $i=0$, we choose a projective R-module P_0 together with a surjective R-linear map $\tau_0\colon P_0 \to M_0$ and we set $\partial_0\colon P_0 \to 0$ to be the zero map. Suppose for some $k>0$, we have constructed R-linear maps $\tau_i\colon P_i \to M_i$ and $\partial_i\colon P_i \to P_{i-1}$ such that

$$\partial_{i-1}\circ\partial_i = 0 \quad \text{and} \quad \tau_{i-1}\circ\partial_i = \mathrm{d}_i\circ\tau_i$$

and such that τ_i restricts to a surjection

$$\tau_i|_{\ker\partial_i}\colon \ker\partial_i \to \ker\mathrm{d}_i$$

for all $0<i<k$. We first construct the pullback:

$$\begin{tikzcd}
P_k \arrow[dr, dashed, "\rho_k"] \arrow[drr, dashed, bend left, "\partial_k"] \arrow[ddr, dashed, bend right, "\tau_k"'] & & \\
& M_k \times_{\ker\mathrm{d}_{k-1}} \ker\partial_{k-1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \ker\partial_{k-1} \arrow[d, "\tau_{k-1}|_{\ker\partial_{k-1}}"] \\
& M_k \arrow[r, "\mathrm{d}_k"] & \ker\mathrm{d}_{k-1}
\end{tikzcd}$$

where the map $\tau_{k-1}|_{\ker\partial_{k-1}}$ lands in $\ker\mathrm{d}_{k-1}$ since the τ_i commute with the differentials. Now we choose a projective R-module P_k together with a surjective R-linear map

$$\rho_k\colon P_k \to M_k \times_{\ker\mathrm{d}_{k-1}} \ker\partial_{k-1}$$

and we set $\partial_k = \pi_2\circ\rho_k$ and $\tau_k = \pi_1\circ\rho_k$. Observe that $\operatorname{im}\partial_k \subset \ker\mathrm{d}_k$ implies $\partial_{k-1}\circ\partial_k = 0$ and observe that

$$\begin{aligned}
\tau_{k-1}\circ\partial_k &= \tau_{k-1}\circ\pi_2\circ\rho_k \\
&= \mathrm{d}_k\circ\pi_1\circ\rho_k \\
&= \mathrm{d}_k\circ\tau_k
\end{aligned}$$

implies $\tau_{k-1}\circ\partial_k = \mathrm{d}_k\circ\tau_k$. Finally, observe that $\tau_k\colon \ker\partial_k \to \ker\mathrm{d}_k$ is surjective since it is a composition of surjective maps

$$\ker\partial_k = \ker(\pi_2\circ\rho_k) \xrightarrow{\rho_k} \ker\pi_2 \xrightarrow[\cong]{\pi_1} \ker\mathrm{d}_k$$

where the isomorphism $\ker\pi_2 \cong \ker\mathrm{d}_k$ follows from Proposition (6.2). This completes the induction step.

Therefore we have an R-complex (P,∂) together with a chain map $\tau\colon (P,\partial) \to (M,\mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker\partial}\colon \ker\partial \to \ker\mathrm{d}.$$

Moreover, Proposition (6.2) implies

$$\begin{aligned}
\mathrm{H}_{k-1}(M) &= \ker\mathrm{d}_{k-1}/\operatorname{im}\mathrm{d}_k \\
&= \ker\mathrm{d}_{k-1}/\mathrm{d}_k(M_k) \\
&\cong \ker\partial_{k-1}/\operatorname{im}\pi_2 \\
&= \ker\partial_{k-1}/\operatorname{im}\partial_k \\
&= \mathrm{H}_{k-1}(P),
\end{aligned}$$

It follows that τ is a quasi-isomorphism. $\square$

6.1.2 Existence of injective resolutions

Lemma 6.2. *Let M, N, and E be R-modules, let $\psi\colon M \to N$ be an R-linear map, and let $\varphi\colon M \to E$ be an injective R-linear map. Define the pushout of $\psi\colon M \to N$ and $\varphi\colon M \to E$ to be the R-module $E +_M N$ given by*

$$E +_M N = E \times N/\{(\varphi(v), 0) - (0, \psi(v)) \mid v \in M\}$$

equipped with the R-linear maps $\iota_1\colon E \to E +_M N$ and $\iota_2\colon N \to E +_M N$ given by

$$\iota_1(u) = [u, 0] \quad \text{and} \quad \iota_2(w) = [0, w]$$

for all $u \in E$ and $w \in N$, where $[u, w]$ denotes the coset class in $E +_M N$ with (u, w) as a representative. Then the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \ker \iota_1 \arrow[r] & E \arrow[r, "\iota_1"] & E +_M N \arrow[r] & E +_M N/E \arrow[r] & 0 \\
0 \arrow[r] & \ker \psi \arrow[r] \arrow[u, "\varphi|_{\ker \varphi}"] & M \arrow[r, "\psi"] \arrow[u, "\varphi"] & N \arrow[r] \arrow[u, "\iota_2"] & N/M \arrow[r] \arrow[u, "\overline{\iota_2}"] & 0
\end{tikzcd}$$

where $\overline{\iota_2}\colon N/M \to E +_M N/E$ is defined by

$$\overline{\iota_2}(\overline{w}) = \overline{[0, w]}$$

for all $\overline{w} \in N/M$ and where $\varphi|_{\ker \psi}\colon \ker \psi \to \ker \iota_1$ is defined by

$$\varphi|_{\ker \varphi}(v) = \varphi(v)$$

for all $v \in \ker \psi$.

Proof. We need to check that $\overline{\iota_2}$ is well-defined. Suppose $w + \psi(v)$ is another representative of $\overline{w}$ where $v \in M$. Then

$$\begin{aligned}
\overline{\iota_2}(\overline{v + \psi(w)}) &= \overline{[0, w + \psi(v)]} \\
&= \overline{[0, w] + [0, \psi(v)]} \\
&= \overline{[0, w] + [\varphi(v), 0]} \\
&= \overline{[0, w]}.
\end{aligned}$$

Thus λ is well-defined. Clearly, λ is a surjective R-linear map since φ is a surjective R-linear map. It remains to show that λ is injective. Suppose $\overline{v} \in P/\pi_2(N \times_M P)$ such that

$$\lambda(\overline{v}) = \overline{\varphi(v)} = \overline{0}.$$

Then $\varphi(v) \in \mathrm{im}(\psi)$. In other words, there exists $u \in N$ such that $\psi(u) = \varphi(v)$. In other words, $(u, v) \in N \times_M P$ and hence

$$\begin{aligned}
v &= \pi_2(u, v) \\
&\in \pi_2(N \times_M P).
\end{aligned}$$

Thus $\overline{v} = \overline{0}$ in $P/\pi_2(N \times_M P)$. □

Theorem 6.3. *Let (M, d) be an R-complex such that $M_i = 0$ for all $i > 0$. Then there exists an injective resolution of (M, d).*

Proof. We construct an R-complex (E, ∂) together with an injective chain map $\varepsilon\colon (M, \mathrm{d}) \to (E, \partial)$ which induces an injective map

$$\overline{\varepsilon}\colon M/\mathrm{im}\,\mathrm{d} \to E/\mathrm{im}\,\partial$$

by induction on homological degree as follows: for $i > 0$, we set $E_i = 0$, $\partial_{i+1} = 0$, and $\varepsilon_i = 0$. For $i = 0$, we choose an injective R-module E_0 together with an injective R-linear map $\varepsilon_0\colon M_0 \to E_0$ and we set $\partial_1\colon E_1 \to E_0$ to be the zero map. Suppose for some $k < 0$, we have constructed R-linear maps $\varepsilon_i\colon M_i \to E_i$ and $\partial_{i+1}\colon E_{i+1} \to E_i$ such that

$$\partial_{i-1}\partial_i = 0 \quad \text{and} \quad \partial_{i+1}\tau_{i+1} = \varepsilon_i \mathrm{d}_{i+1}$$

and such that ε_i induces an injective map

$$\overline{\varepsilon_i}\colon M_i/\mathrm{im}\,\mathrm{d}_{i+1} \to E_i/\mathrm{im}\,\partial_{i+1}$$

for all $i > k$. We first construct the pushout

$$\begin{tikzcd}
E_k/\mathrm{im}\,\partial_{k+1} \arrow[r, "\iota_1"] & \frac{E_k}{\mathrm{im}\,\partial_{k+1}} +_{\frac{M_k}{\mathrm{im}\,\mathrm{d}_{k+1}}} M_{k-1} \\
M_k/\mathrm{im}\,\mathrm{d}_{k+1} \arrow[u, "\overline{\varepsilon_k}"] \arrow[r, "\mathrm{d}_k"] & M_{k-1} \arrow[u, "\iota_2"']
\end{tikzcd}$$

here the map $\overline{\varepsilon_k}$ is well-defined since ε_k commutes with the differentials. Now we choose an injective R-module E_{k-1} together with an injective R-linear map

$$\rho_k \colon \frac{E_k}{\mathrm{im}\,\partial_{k+1}} +_{\frac{M_k}{\mathrm{im}\,\mathrm{d}_{k+1}}} M_{k-1} \to E_{k-1}.$$

and we set $\partial_k = \rho_k \circ \iota_1 \circ \pi$ and $\varepsilon_{k-1} = \rho_k \circ \iota_2$. Observe that $\mathrm{im}\,\partial_k \subset \ker \mathrm{d}_k$ implies $\partial_{k-1} \circ \partial_k = 0$ and observe that

$$\begin{aligned}
\tau_{k-1} \circ \partial_k &= \tau_{k-1} \circ \pi_2 \circ \rho_k \\
&= \mathrm{d}_k \circ \pi_1 \circ \rho_k \\
&= \mathrm{d}_k \circ \tau_k
\end{aligned}$$

implies $\tau_{k-1} \circ \partial_k = \mathrm{d}_k \circ \tau_k$. Finally, observe that $\tau_k \colon \ker \partial_k \to \ker \mathrm{d}_k$ is surjective since it is a composition of surjective maps

$$\ker \partial_k = \ker(\pi_2 \circ \rho_k) \xrightarrow{\rho_k} \ker \pi_2 \xrightarrow[\cong]{\pi_1} \ker \mathrm{d}_k$$

where the isomorphism $\ker \pi_2 \cong \ker \mathrm{d}_k$ follows from Proposition (6.2). This completes the induction step.

Therefore we have an R-complex (P, ∂) together with a chain map $\tau \colon (P, \partial) \to (M, \mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker \partial} \colon \ker \partial \to \ker \mathrm{d}.$$

Moreover, Proposition (6.2) implies

$$\begin{aligned}
\mathrm{H}_{k-1}(M) &= \ker \mathrm{d}_{k-1}/\mathrm{im}\,\mathrm{d}_k \\
&= \ker \mathrm{d}_{k-1}/\mathrm{d}_k(M_k) \\
&\cong \ker \partial_{k-1}/\mathrm{im}\,\pi_2 \\
&= \ker \partial_{k-1}/\mathrm{im}\,\partial_k \\
&= \mathrm{H}_{k-1}(P),
\end{aligned}$$

It follows that τ is a quasi-isomorphism. □

6.1.3 Extra

Let (M, d) be an R-complex. We know wish to show how to construct a projective resolution of (M, d). That is, we will build an R-complex $(P^{-\infty}, \partial^{-\infty})$ together with a quasiisomorphism $\tau^{-\infty} \colon (P^{-\infty}, \partial^{-\infty}) \to (M, \mathrm{d})$. We proceed as follows: for each $n \in \mathbb{Z}$, let (M^n, d^n) be the truncated R-complex where

$$M_i^n = \begin{cases} M_i & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

and where

$$\mathrm{d}_i^0 = \begin{cases} \mathrm{d}_i & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

Next, choose a projective resolution of (M^0, d^0) as in Theorem (6.1), say (P^0, ∂^0). We construct an R-complex (P^{-1}, ∂^{-1}) together with a chain map $\tau^{-1} \colon (P^{-1}, \partial^{-1}) \to (M^{-1}, \mathrm{d}^{-1})$ which restricts to a surjection

$$\tau|_{\ker \partial} \colon \ker \partial \to \ker \mathrm{d}$$

by induction on homological degree as follows: for the base case $i = 0$, we choose a projective R-module P_{-1}^{-1} together with a surjective R-linear map $\tau_{-1}^{-1} \colon P_{-1}^{-1} \to M_{-1}^{-1}$ and we set $\partial_{-1}^{-1} \colon P_{-1} \to 0$ to be the zero map. Suppose for some $k > 0$, we have constructed R-linear maps $\tau_i \colon P_i \to M_i$ and $\partial_i \colon P_i \to P_{i-1}$

6.2 Semiprojective and semiinjective complexes

Definition 6.4. Let P be an R-complex of projective R-modules and let E be an R-complex of injective R-modules.

1. We say P is **semiprojective** if $\mathrm{Hom}_R^\star(P,-)$ respects quasiisomorphisms. If $\tau\colon P \to X$ is a quasiisiomorphism, then we say P is a **semiprojective resolution** of X.

2. We say E is **seminjective** if $\mathrm{Hom}_R^\star(-,E)$ respects quasiisomorphisms. If $\varepsilon\colon X \to E$ is a quasiisiomorphism, then we say E is a **semiinjective resolution** of X.

Proposition 6.3. *Let P be an R-complex of projective modules and let E be an R-complex of injective modules. Then P is semiprojective if and only if $\mathrm{Hom}_R^\star(P,-)$ takes exact complexes to exact complexes. Similarly, E is seminjective if and only if $\mathrm{Hom}_R^\star(-,E)$ takes exact complexes to exact complexes.*

Proof. First suppose that $\mathrm{Hom}_R^\star(P,-)$ is exact. Let $\varphi\colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned}\varphi\colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact}\\ &\implies \mathrm{Hom}_R^\star(P,\mathrm{C}(\varphi)) \text{ is exact}\\ &\implies \mathrm{C}(\mathrm{Hom}_R^\star(P,\varphi)) \text{ is exact}\\ &\implies \mathrm{Hom}_R^\star(P,\varphi) \text{ is a quasiisomorphism.}\end{aligned}$$

Convsersely, suppose P is semiprojective. Let M be an exact R-complex. Then the zero map $M \to 0$ is a quasiisomorphism. Since P is semiprojective, the induced map $\mathrm{Hom}_R^\star(P,M) \to 0$ is a quasiisomorphism. This implies $\mathrm{Hom}_R^\star(P,M)$ is exact. Thus $\mathrm{Hom}_R^\star(P,-)$ is exact. The proof is similar for the injective case. □

6.2.1 Operations on semiprojective R-complexes

Proposition 6.4. *Let P and P' be semiprojective R-complexes.*

1. *ΣP is semiprojective;*

2. *if $\varphi\colon P \to P'$ is a chain map, then $\mathrm{C}(\varphi)$ is semiprojective;*

3. *$P \oplus P'$ is semiprojective;*

4. *if Q is a complex of projective R-modules, then $\mathrm{C}(1_Q)$ is semiprojective.*

5. *$P \otimes_R P'$ is semiprojective.*

Proof. 1. Let M be an exact R-complex. Then

$$\mathrm{Hom}_R^\star(\Sigma P, M) \cong \Sigma^{-1}\mathrm{Hom}_R^\star(P,M)$$

is exact. It follows that ΣP is semiprojective.

2. Let M be an exact R-complex. Observe that the exact sequence

$$0 \longrightarrow P' \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma P \longrightarrow 0$$

is degreewise split exact. Therefore the sequence

$$0 \longrightarrow \mathrm{Hom}_R^\star(\Sigma P, M) \xrightarrow{\pi^*} \mathrm{Hom}_R^\star(\mathrm{C}(\varphi), M) \xrightarrow{\iota^*} \mathrm{Hom}_R^\star(P,M) \longrightarrow 0$$

is exact. It follows from the fact that both $\mathrm{Hom}_R^\star(\Sigma P, M)$ and $\mathrm{Hom}_R^\star(P', M)$ are exact and from the long exact sequence in homology that $\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), M)$ is exact.

3. This follows from 2 and the fact that

$$P \oplus P' \cong \mathrm{C}(\Sigma^{-1}P \xrightarrow{0} P').$$

4. Let M be an exact R-complex. Then

$$\begin{aligned}\mathrm{Hom}_R^\star(\mathrm{C}(1_Q), M) &\cong \Sigma^{-1}\mathrm{C}(\mathrm{Hom}_R^\star(1_Q, M))\\ &= \Sigma^{-1}\mathrm{C}(1_{\mathrm{Hom}_R^\star(Q,M)})\end{aligned}$$

is exact.

5. By hom tensor adjointness, $\mathrm{Hom}_R(P \otimes_R P', -) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(P', -))$ is a composition of two exact functors.

$\square$

Theorem 6.4. *Every R-complex has a semiprojective resolution and a semiinjective resolution.*

6.2.2 A bounded below complex of projective R-modules is semiprojective

Lemma 6.5. *Let (P, ∂) be a bounded below complex of projective R-modules and let (M, d) be an exact R-complex. Then*

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(P, M)) \cong 0. \tag{51}$$

Proof. By reindexing if necessary, we may assume that $P_i = 0$ for all $i < 0$. Recall that

$$\mathrm{Hom}_R^\star(P, M) = \{\text{homotopy classes of chain maps } \varphi\colon P \to M\}.$$

Thus in order to obtain (51), we need to show that any two chain maps from P to M are homotopic to each other. Let $\varphi\colon P \to M$ and $\psi\colon P \to M$ be any two chain maps. The idea is to build the homotopy $h\colon P \to M$ using induction on $i \geq 0$. The homotopy equation that needs to be satisfied is

$$\varphi - \psi = \mathrm{d}h + h\partial, \tag{52}$$

First, for each $i < 0$, we set $h_i\colon P_i \to M_{i+1}$ to be the zero map. Next we observe that $\mathrm{im}(\varphi_0 - \psi_0) \subseteq \mathrm{im}\,\mathrm{d}_1$. Indeed,

$$\begin{aligned} \mathrm{d}_0(\varphi_0 - \psi_0) &= \mathrm{d}_0\varphi_0 - \mathrm{d}_0\psi_0 \\ &= \varphi_{-1}\partial_0 - \psi_{-1}\partial_0 \\ &= (\varphi_{-1} - \psi_{-1})\partial_0 \\ &= (\varphi_{-1} - \psi_{-1}) \circ 0 \\ &= 0 \end{aligned}$$

implies

$$\begin{aligned} \mathrm{im}(\varphi_0 - \psi_0) &\subseteq \ker \mathrm{d}_0 \\ &= \mathrm{im}\,\mathrm{d}_1. \end{aligned}$$

Thus since P_0 is projective, $\mathrm{d}_1\colon M_1 \to \mathrm{im}\,\mathrm{d}_1$ is surjective, and $\varphi_0 - \psi_0\colon P_0 \to M_0$ lands in $\mathrm{im}\,\mathrm{d}_1$, there exists an R-linear map $h_0\colon P_0 \to P_1$ such that

$$\varphi_0 - \psi_0 = \mathrm{d}_1 h_0. \tag{53}$$

In homological degree $i = 0$, the equation (52) becomes (53). Thus, we are on the right track.

Now we use induction. Suppose for some $i > 0$ we have constructed an R-module homomorphism $h_i\colon P_i \to P_{i+1}$ such that

$$\varphi_i - \psi_i = \mathrm{d}_{i+1}h_i + h_{i-1}\partial_i. \tag{54}$$

Observe that $\mathrm{im}\,(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) \subseteq \mathrm{im}\,\mathrm{d}_{i+2}$. Indeed,

$$\begin{aligned} \mathrm{d}_{i+1}(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) &= \mathrm{d}_{i+1}\varphi_{i+1} - \mathrm{d}_{i+1}\psi_{i+1} - \mathrm{d}_{i+1}h_i\partial_{i+1} \\ &= \varphi_i\partial_{i+1} - \psi_i\partial_{i+1} - \mathrm{d}_{i+1}h_i\partial_{i+1} \\ &= (\varphi_i - \psi_i - \mathrm{d}_{i+1}h_i)\partial_{i+1} \\ &= h_{i-1}\partial_i\partial_{i+1} \\ &= h_{i-1} \circ 0 \\ &= 0 \end{aligned}$$

implies

$$\begin{aligned} \mathrm{im}(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) &\subseteq \ker \mathrm{d}_{i+1} \\ &= \mathrm{im}\,\mathrm{d}_{i+2}. \end{aligned}$$

Therefore since P_{i+1} is projective, $\mathrm{d}_{i+2}\colon M_{i+2} \to \mathrm{im}\,\mathrm{d}_{i+2}$ is surjective, and $\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}\colon P_{i+1} \to M_{i+1}$ lands in $\mathrm{im}\,\mathrm{d}_{i+2}$, there exists an R-linear map $h_{i+1}\colon P_{i+1} \to P_{i+2}$ such that

$$\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1} = \mathrm{d}_{i+2}h_{i+1},$$

which is the homotopy equation in degree $i + 1$. $\square$

Corollary. *Let P be a bounded below complex of projective R-modules. Then $\mathrm{Hom}^{\star}_R(P,-)$ respects exact complexes. In particular, this implies P is semiprojective.*

Proof. Let M be an exact R-complex. Observe that $\Sigma^i P$ is a bounded below complex of projective R-modules for each $i \in \mathbb{Z}$. It follows from Lemma (6.5) that for each $i \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{H}_i(\mathrm{Hom}^{\star}_R(P,M)) &= \mathrm{H}_{0-(-i)}(\mathrm{Hom}^{\star}_R(P,M)) \\ &= \mathrm{H}_0(\Sigma^{-i}\mathrm{Hom}^{\star}_R(P,M)) \\ &= \mathrm{H}_0(\mathrm{Hom}^{\star}_R(\Sigma^i P,M)) \\ &= 0. \end{aligned}$$

Therefore $\mathrm{Hom}^{\star}_R(P,-)$ takes exact complexes to exact complexes.

Now we show that this implies $\mathrm{Hom}^{\star}_R(P,-)$ takes quasiisomorphisms to quasiisomorphisms. Let $\varphi\colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned} \varphi\colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact} \\ &\implies \mathrm{Hom}^{\star}_R(P,\mathrm{C}(\varphi)) \text{ is exact} \\ &\implies \mathrm{C}(\mathrm{Hom}^{\star}_R(P,\varphi)) \text{ is exact} \\ &\implies \mathrm{Hom}^{\star}_R(P,\varphi) \text{ is a quasiisomorphism.} \end{aligned}$$

Therefore P is semiprojective. □

6.2.3 Lifting Lemma

Lemma 6.6. *Let P be a semiprojective R-complex, let $\varphi\colon A \to A'$ be a quasiisomorphism, and let $\psi\colon P \to A'$ be a chain map. Then*

1. *Then there exists a chain map $\widetilde{\psi}\colon P \to A$ such that $\varphi\widetilde{\psi} \sim \psi$. Furthermore, if $\widetilde{\psi}'\colon P \to A$ is another such chain map which satisfies $\varphi\widetilde{\psi}' \sim \psi$, then $\widetilde{\psi} \sim \widetilde{\psi}'$. We call $\widetilde{\psi}$ a **homotopic lift of ψ with respect to φ**.*

2. *If in addition φ is surjective, then there exists a chain map $\widetilde{\psi}\colon P \to A$ such that $\varphi\widetilde{\psi} = \psi$.*

Proof. 1. Since $\mathrm{Hom}^{\star}_R(P,-)$ preserves quasiisomorphisms, we see that

$$\varphi_*\colon \mathrm{Hom}^{\star}_R(P,A) \to \mathrm{Hom}^{\star}_R(P,A')$$

is a quasiisomorphism. In particular, φ_* induces an isomorphism in the degree 0 part of homology:

$$\mathrm{H}_0(\varphi_*)\colon \mathrm{H}_0(\mathrm{Hom}^{\star}_R(P,A)) \to \mathrm{H}_0(\mathrm{Hom}^{\star}_R(P,A')).$$

Now ψ represents the the homology class $[\psi]$ in $\mathrm{H}_0(\mathrm{Hom}^{\star}_R(P,A'))$, and since $\mathrm{H}_0(\varphi_*)$ is an isomorphism, there exists a homology class $[\widetilde{\psi}]$ in $\mathrm{H}_0(\mathrm{Hom}^{\star}_R(P,A))$ such that

$$\mathrm{H}_0(\varphi_*)[\widetilde{\psi}] = [\psi].$$

In other words, such that $[\varphi\widetilde{\psi}] = [\psi]$. Since

$$\mathrm{H}_0(\mathrm{Hom}^{\star}_R(P,A')) = \mathcal{C}(A,A')/\sim,$$

we see that $\varphi\widetilde{\psi} \sim \psi$. For the second statement, suppose $\widetilde{\psi}'\colon P \to A$ is another such chain map which satisfies $\varphi\widetilde{\psi}' \sim \psi$. Then $[\widetilde{\psi}'] = [\widetilde{\psi}]$ since $\mathrm{H}_0(\varphi_*)$ is an isomorphism, hence $\widetilde{\psi} \sim \widetilde{\psi}'$.

2. Now suppose that φ is surjective. Choose a homotopic lift of ψ with respect to φ, say $\widetilde{\psi}$. Choose a homotopy from $\varphi\widetilde{\psi}$ to ψ, say $h\colon P \to A'$. So

$$\varphi\widetilde{\psi} - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_P.$$

Using the fact that P is a projective R-module and φ is surjective, we choose a graded lift of h with respect to φ, say $\widetilde{h}\colon P \to A$. So $\widetilde{h}$ is a graded homomorphism of degree 1 such that $\varphi\widetilde{h} = h$. Then note that $\widetilde{\psi} \sim \widetilde{\psi} - \mathrm{d}_A\widetilde{h} - \widetilde{h}\mathrm{d}_P$ and

$$\begin{aligned} \varphi(\widetilde{\psi} - \mathrm{d}_A\widetilde{h} - \widetilde{h}\mathrm{d}_P) &= \varphi\widetilde{\psi} - \varphi\mathrm{d}_A\widetilde{h} - \varphi\widetilde{h}\mathrm{d}_P \\ &= \varphi\widetilde{\psi} - \mathrm{d}_{A'}\varphi\widetilde{h} - \varphi\widetilde{h}\mathrm{d}_P \\ &= \varphi\widetilde{\psi} - \mathrm{d}_{A'}h - h\mathrm{d}_P \\ &= \mathrm{d}_{A'}h + h\mathrm{d}_P + \psi - \mathrm{d}_{A'}h - h\mathrm{d}_P \\ &= \psi. \end{aligned}$$

□

6.3 Ext Functor

Definition 6.5. Let A and B be R-complexes. We define the graded R-module $\mathrm{Ext}_R(A,B)$ as follows: choose a semiprojective resolution $\tau\colon P \to A$. Then

$$\mathrm{Ext}_R(A,B) := \mathrm{H}(\mathrm{Hom}_R^\star(P,B)).$$

The ith homogeneous component of $\mathrm{Ext}_R(A,B)$ is denoted

$$\mathrm{Ext}_R^i(A,B) := \mathrm{H}_{-i}(\mathrm{Hom}_R^\star(P,B))$$

In our definition of $\mathrm{Ext}_R(A,B)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get an isomorphic object. Thus $\mathrm{Ext}_R(A,B)$ is well-defined *up to isomorphism*.

Theorem 6.7. $\mathrm{Ext}_R(A,B)$ *is well-defined up to isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_1\colon P_1 \to P_2$ of τ_1 with respect to τ_2. Similarly, choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1. We claim that $\widetilde{\tau}_1\colon P_1 \to P_2$ is a homotopy equivalence with $\widetilde{\tau}_2\colon P_2 \to P_1$ being its homotopy inverse. Indeed, observe that

$$\begin{aligned}\tau_1\widetilde{\tau}_2\widetilde{\tau}_1 &\sim \tau_2\widetilde{\tau}_1\\ &\sim \tau_1\end{aligned}$$

implies $\widetilde{\tau}_2\widetilde{\tau}_1$ is a homotopic lift of τ_1 with respect to τ_1, but 1_{P_1} is also a homotopic lift of τ_1 with respect to τ_1. Therefore $\widetilde{\tau}_2\widetilde{\tau}_1 \sim 1_{P_1}$. A similar computation gives $\widetilde{\tau}_1\widetilde{\tau}_2 \sim 1_{P_2}$. Now $\mathrm{Hom}_R^\star(-,B)$ preserves homotopy equivalences, and thus $\mathrm{Hom}_R^\star(\widetilde{\tau}_1,B)\colon \mathrm{Hom}_R^\star(P_1,B) \to \mathrm{Hom}_R^\star(P_2,B)$ is a homotopy equivalence. Then since the homology functor takes homotopy equivalences to isomorphisms, we see that

$$\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_1,B))\colon \mathrm{H}(\mathrm{Hom}_R^\star(P_1,B)) \to \mathrm{H}(\mathrm{Hom}_R^\star(P_2,B))$$

is an isomorphism. □

6.3.1 The functor $\mathrm{Ext}_R(A,-)$

Now that we've defined the module $\mathrm{Ext}_R(A,B)$, we want to define the covariant functor

$$\mathrm{Ext}_R(A,-)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Clearly, we want this functor to map an R-complex B to the graded R-module $\mathrm{Ext}_R(A,B)$. Let us show how it should act on chain maps:

Definition 6.6. Let $\psi\colon B \to B'$ be a chain map and let $\tau\colon P \to A$ be a semiprojective resolution of A. We define

$$\mathrm{Ext}_R(A,\psi)\colon \mathrm{Ext}_R(A,B) \to \mathrm{Ext}_R(A,B')$$

by $\mathrm{Ext}_R(A,\psi) := \mathrm{H}(\mathrm{Hom}_R^\star(A,\psi))$.

Again, in our definition of $\mathrm{Ext}_R(A,\psi)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get a *naturally isomorphic* functor. Thus the functor $\mathrm{Ext}_R(A,-)$ is well-defined *up to natural isomorphism*.

Theorem 6.8. $\mathrm{Ext}_R(A,-)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1. Then $\widetilde{\tau}_2$ is a homotopy equivalence, by the same argument as in the proof of Theorem (6.10). Now observe that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R^\star(P_1,B) \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B)"] \arrow[d, "\mathrm{Hom}_R^\star(P_1,\psi)"'] & \mathrm{Hom}_R^\star(P_2,B) \arrow[d, "\mathrm{Hom}_R^\star(P_2,\psi)"] \\
\mathrm{Hom}_R^\star(P_1,B') \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B')"'] & \mathrm{Hom}_R^\star(P_2,B')
\end{tikzcd}$$

is commutative. Therefore we obtain a commutative diagram after apply homology:

$$\begin{tikzcd}
\mathrm{H}(\mathrm{Hom}_R^\star(P_1,B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B))"] \arrow[d, "\mathrm{H}(\mathrm{Hom}_R^\star(P_1,\psi))"'] & \mathrm{H}(\mathrm{Hom}_R^\star(P_2,B)) \arrow[d, "\mathrm{H}(\mathrm{Hom}_R^\star(P_2,\psi))"] \\
\mathrm{H}(\mathrm{Hom}_R^\star(P_1,B')) \arrow[r, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B'))"'] & \mathrm{H}(\mathrm{Hom}_R^\star(P_2,B'))
\end{tikzcd}$$

Since the rows are isomorphisms, we see that $\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_2,-))$ is a natural isomorphism. □

6.3.2 The functor $\mathrm{Ext}_R(-,B)$

Next we want to define the contravariant functor

$$\mathrm{Ext}_R(-,B)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Again, we want this functor to send and an R-complex A to the graded R-module $\mathrm{Ext}_R(A,B)$. This time, the way it acts on chain maps will be a little more involved than in the covariant case.

Definition 6.7. Let $\varphi\colon A \to A'$ be a chain map, let $\tau\colon P \to A$ be a semiprojective resolution of A, let $\tau'\colon P' \to A'$ be a semiprojective resolution of A', and let $\widetilde{\varphi}\colon P \to P'$ be a homotopic lift of $\varphi\tau$ with respect to τ'. We define

$$\mathrm{Ext}_R(\varphi,B)\colon \mathrm{Ext}_R(A',B) \to \mathrm{Ext}_R(A,B).$$

by $\mathrm{Ext}_R(\varphi,B) := \mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\varphi},B))$.

This time our definition of the functor $\mathrm{Ext}_R(-,B)$ involves *three choices*; namely, the semiprojective resolutions $\tau\colon P \to A$ and $\tau'\colon P' \to A'$ as well as the homotopic lift $\widetilde{\varphi}\colon P \to P'$. Even though we made three choices, we shall still see that $\mathrm{Ext}_R(-,B)$ is well-defined up to natural isomorphism.

Theorem 6.9. *$\mathrm{Ext}_R(-,B)$ is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A, suppose $\tau_1'\colon P_1' \to A'$ and $\tau_2'\colon P_2' \to A'$ are two semiprojective resolutions of A', and suppose $\widetilde{\varphi_1}\colon P_1 \to P_1'$ is a homotopic lift of $\varphi\tau_1$ with respect to τ_1' and $\widetilde{\varphi_2}\colon P_2 \to P_2'$ is a homotopic lift of $\varphi\tau_2$ with respect to τ_2'. So altogether we have the diagrams

$$\begin{tikzcd}
P_1 \arrow[r, "\widetilde{\varphi_1}"] \arrow[d, "\tau_1"'] & P_1' \arrow[d, "\tau_1'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
P_2 \arrow[r, "\widetilde{\varphi_2}"] \arrow[d, "\tau_2"'] & P_2' \arrow[d, "\tau_2'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}$$

which commute up to homotopy.

Choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1 and choose a homotopic lift $\widetilde{\tau}_2{}'\colon P_2' \to P_1'$ of τ_2' with respect to τ_1'. Then $\widetilde{\tau}_2$ and $\widetilde{\tau}_2{}'$ are both homotopy equivalences by the same argument as in the proof of Theorem (6.10). Now observe that

$$\begin{aligned}
\tau_1'\widetilde{\tau}_2{}'\widetilde{\varphi_2} &\sim \tau_2'\widetilde{\varphi_2} \\
&\sim \varphi\tau_2 \\
&\sim \varphi\tau_1\widetilde{\tau}_2 \\
&\sim \tau_1'\widetilde{\varphi_1}\widetilde{\tau}_2.
\end{aligned}$$

In particular, both $\widetilde{\tau}_2{}'\widetilde{\varphi_2}\colon P_2 \to P_1'$ and $\widetilde{\varphi_1}\widetilde{\tau}_2\colon P_2 \to P_1'$ are homotopic lifts of $\varphi\tau_2$ with respect to τ_1'. Therefore $\widetilde{\tau}_2{}'\widetilde{\varphi_2} \sim \widetilde{\varphi_1}\widetilde{\tau}_2$, which further implies

$$\begin{aligned}
\mathrm{Hom}_R^\star(\widetilde{\varphi_2},B)\mathrm{Hom}_R^\star(\widetilde{\tau}_2{}',B) &= \mathrm{Hom}_R^\star(\widetilde{\tau}_2{}'\widetilde{\varphi_2},B) \\
&\sim \mathrm{Hom}_R^\star(\widetilde{\varphi_1}\widetilde{\tau}_2,B) \\
&= \mathrm{Hom}_R^\star(\widetilde{\tau}_2,B)\mathrm{Hom}_R^\star(\widetilde{\varphi_1},B)
\end{aligned}$$

since $\mathrm{Hom}_R^\star(-,B)$ respects homotopies. Therefore we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_R^\star(P_1',B) \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2{}',B)"] \arrow[d, "\mathrm{Hom}_R^\star(\widetilde{\varphi_1},B)"'] & \mathrm{Hom}_R^\star(P_2',B) \arrow[d, "\mathrm{Hom}_R^\star(\widetilde{\varphi_2},B)"] \\
\mathrm{Hom}_R^\star(P_1,B') \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B)"'] & \mathrm{Hom}_R^\star(P_2,B)
\end{tikzcd}$$

which commutes up to homotopy. Then since homology takes homotopic maps to equal maps, we see that the diagram

$$\begin{tikzcd} \mathrm{H}(\mathrm{Hom}^\star_R(P_1', B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\tau}_2', B))"] \arrow[d, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\varphi}_1, B))"'] & \mathrm{H}(\mathrm{Hom}^\star_R(P_2', B)) \arrow[d, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\varphi}_2, B))"] \\ \mathrm{H}(\mathrm{Hom}^\star_R(P_1, B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\tau}_2, B))"'] & \mathrm{H}(\mathrm{Hom}^\star_R(P_2, B)) \end{tikzcd}$$

is commutative. Since the rows are isomorphisms, we see that $\mathrm{H}(\mathrm{Hom}^\star_R(-, B))$ is a natural isomorphism. □

6.3.3 Properties of Ext

Proposition 6.5. *Let A, B be R-complexes, let $\{A_\lambda\}$ and $\{B_\lambda\}$ be a collection of R-complexes indexed over a set Λ, and let $S \subseteq R$ be a multiplicatively closed set. Then*

1. $\mathrm{Ext}_R\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) \cong \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R\left(A_\lambda, B\right)$;
2. $\mathrm{Ext}_R\left(A, \prod^\star_{\lambda\in\Lambda} B_\lambda\right) \cong \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A, B_\lambda)$
3. *If A is finitely presented, then* $\mathrm{Ext}_R(A, B)_S \cong \mathrm{Ext}_{R_S}(A_S, B_S)$.

Proof. Choose a semiprojective resolutions $\tau_\lambda \colon P_\lambda \to A_\lambda$ of A_λ for each $\lambda \in \Lambda$. Then $\oplus\tau_\lambda \colon \bigoplus_\lambda P_\lambda \to \bigoplus_\lambda A_\lambda$ is a semiprojective resolution of $\bigoplus_\lambda A_\lambda$. Indeed, the homogeneous piece in degree i of $\bigoplus_\lambda P_\lambda$ is given by $\bigoplus_\lambda P_{\lambda,i}$, where $P_{\lambda,i}$ is the homogeneous piece in degree i of P_λ for all $\lambda \in \Lambda$, and $\bigoplus_\lambda P_{\lambda,i}$ is a projective R-module since each $P_{\lambda,i}$ is a projective R-module. Also, $\oplus\tau_\lambda$ is a quasiisomorphism since each τ_λ is a quasiisomorphism and since homology commutes with direct sums.

Therefore

$$\begin{aligned} \mathrm{Ext}_R\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) &= \mathrm{H}\left(\mathrm{Hom}^\star_R\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right)\right) \\ &= \mathrm{H}\left(\prod^\star_{\lambda\in\Lambda} \mathrm{Hom}^\star_R(A_\lambda, B)\right) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{H}(\mathrm{Hom}^\star_R(A_\lambda, B)) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A_\lambda, B) \end{aligned}$$

Similarly, choose a semiprojective resolution $\tau \colon P \to A$ of A. Then we have

$$\begin{aligned} \mathrm{Ext}_R\left(A, \prod^\star_{\lambda\in\Lambda} B_\lambda\right) &= \mathrm{H}\left(\mathrm{Hom}^\star_R\left(P, \prod^\star_{\lambda\in\Lambda} B_\lambda\right)\right) \\ &= \mathrm{H}\left(\prod^\star_{\lambda\in\Lambda} \mathrm{Hom}^\star_R\left(P, B_\lambda\right)\right) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{H}(\mathrm{Hom}^\star_R(P, B_\lambda)) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A, B_\lambda). \end{aligned}$$

For the final equality, observe that $\tau_S \colon P_S \to A_S$ is a semiprojective resolution of A_S. Thus

$$\begin{aligned} \mathrm{Ext}_{R_S}(A_S, B_S) &= \mathrm{H}\left(\mathrm{Hom}^\star_{R_S}(P_S, B_S)\right) \\ &= \mathrm{H}\left(\mathrm{Hom}^\star_R(P, B)_S\right) \\ &= \mathrm{H}(\mathrm{Hom}^\star_R(P, B))_S \\ &= \mathrm{Ext}_R(A, B)_S. \end{aligned}$$

□

6.4 Semiflat complexes

Definition 6.8. Let M be an R-complex of flat R-modules. We say M is **semiflat** if $-\otimes_R M$ respects quasiisomorphisms. If $\tau\colon M \to X$ is a quasiisomorphism, then we say M is a **semiflat resolution** of X.

Remark. Since $-\otimes_R M$ is naturally isomorphic to $M \otimes_R -$, we see that M is semiflat if and only if $M \otimes_R -$ respects quasiisomorphisms.

Proposition 6.6. *Let M be an R-complex of flat R-modules. Then M is semiflat if and only if $M \otimes_R -$ is exact.*

Proof. First suppose that $-\otimes_R M$ is exact. Let $\varphi\colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned} \varphi\colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact} \\ &\implies \mathrm{C}(\varphi) \otimes_R M \text{ is exact} \\ &\implies \mathrm{C}(\varphi \otimes_R M) \text{ is exact} \\ &\implies \varphi \otimes_R M \text{ is a quasiisomorphism.} \end{aligned}$$

Therefore $-\otimes_R M$ respects quasiisomorphisms.

Conversely, suppose M is semiflat. Let A be an exact R-complex. Then the zero map $M \to 0$ is a quasiisomorphism. Since M is semiflat, the induced map $A \otimes_R M \to 0$ is a quasiisomorphism. This implies $A \otimes_R M$ is exact. Therefore $-\otimes_R M$ is exact. □

6.4.1 Semiprojective complexes are semiflat

Proposition 6.7. *Let P be a semiprojective R-complex. Then P is semiflat.*

Proof. Since projective R-modules are flat, we see that P_i is flat for all $i \in \mathbb{Z}$. Now let A be an exact R-complex and let $\varepsilon\colon P \otimes_R A \to E$ be a semiinjective resolution. Then

$$\begin{aligned} P \otimes_R A \text{ is exact} &\iff \mathrm{Hom}_R^\star(P \otimes_R A, E) \text{ is exact} \\ &\iff \mathrm{Hom}_R^\star(P, \mathrm{Hom}_R^\star(A, E)) \text{ is exact.} \end{aligned}$$

the last line follows from the fact that P is semiprojective and E is semiinjective. □

6.5 Tor Functor

Definition 6.9. Let A and B be R-complexes. We define the graded R-module $\mathrm{Tor}^R(A, B)$ as follows: choose a semiprojective resolution $\tau\colon P \to A$. Then

$$\mathrm{Tor}^R(A, B) := \mathrm{H}(P \otimes_R B).$$

The ith homogeneous component of $\mathrm{Tor}^R(A, B)$ is denoted

$$\mathrm{Tor}_i^R(A, B) := \mathrm{H}_i(P \otimes_R B)$$

In our definition of $\mathrm{Tor}^R(A, B)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get an isomorphic object. Thus $\mathrm{Tor}^R(A, B)$ is well-defined *up to isomorphism*.

Theorem 6.10. $\mathrm{Tor}^R(A, B)$ *is well-defined up to isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_1\colon P_1 \to P_2$ of τ_1 with respect to τ_2. Similarly, choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1. As in the proof of Theorem (6.10), $\widetilde{\tau}_1\colon P_1 \to P_2$ is a homotopy equivalence with $\widetilde{\tau}_2\colon P_2 \to P_1$ being its homotopy inverse. Now $-\otimes_R B$ preserves homotopy equivalences, and thus $\widetilde{\tau}_1 \otimes_R B\colon P_1 \otimes_R B \to P_2 \otimes_R B$ is a homotopy equivalence. Then since the homology functor takes homotopy equivalences to isomorphisms, we see that

$$\mathrm{H}(\widetilde{\tau}_1 \otimes_R B))\colon \mathrm{H}(P_1 \otimes_R B) \to \mathrm{H}(P_2 \otimes_R B)$$

is an isomorphism. This isomorphism is unique in a sense. Indeed, if we had chosen another homotopic lift of τ_1 with respect to τ_2, say $\widetilde{\tau}_1'\colon P_1 \to P_2$, then $\widetilde{\tau}_1 \sim \widetilde{\tau}_1{}'$, which implies $\widetilde{\tau}_1 \otimes_R B \sim \widetilde{\tau}_1{}' \otimes_R B$, which implies $\mathrm{H}(\widetilde{\tau}_1 \otimes_R B)) = \mathrm{H}(\widetilde{\tau}_1{}' \otimes_R B))$. □

6.5.1 The functor $\operatorname{Tor}^R(A,-)$

Now that we've defined the module $\operatorname{Tor}^R(A,B)$, we want to define the covariant functor

$$\operatorname{Tor}^R(A,-)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Clearly, we want this functor to map an R-complex B to the graded R-module $\operatorname{Tor}^R(A,B)$. Let us show how it should act on chain maps:

Definition 6.10. Let $\psi\colon B \to B'$ be a chain map and let $\tau\colon P \to A$ be a semiprojective resolution of A. We define

$$\operatorname{Tor}^R(A,\psi)\colon \operatorname{Tor}^R(A,B) \to \operatorname{Tor}^R(A,B')$$

by $\operatorname{Tor}^R(A,\psi) := \mathrm{H}(A \otimes_R \psi)$.

Again, in our definition of $\operatorname{Tor}^R(A,\psi)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get a *naturally isomorphic* functor. Thus the functor $\operatorname{Tor}^R(A,-)$ is well-defined *up to natural isomorphism*.

Theorem 6.11. $\operatorname{Tor}^R(A,-)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau_1}\colon P_1 \to P_2$ of τ_1 with respect to τ_2. Then $\widetilde{\tau_1}$ is a homotopy equivalence, by the same argument as in the proof of Theorem (6.10). Now observe that the diagram

$$\begin{tikzcd} P_1 \otimes_R B \arrow[r, "\widetilde{\tau_1}\otimes_R B"] \arrow[d, "P_1\otimes_R\psi"'] & P_2 \otimes_R B \arrow[d, "P_2\otimes_R\psi"] \\ P_1 \otimes_R B' \arrow[r, "\widetilde{\tau_2}\otimes_R B'"'] & P_2 \otimes_R B' \end{tikzcd}$$

is commutative where the rows are homotopy equivalences since $-\otimes_R B$ preserves homotopy equivalences. Therefore we obtain a commutative diagram after apply homology

$$\begin{tikzcd} \mathrm{H}(P_1 \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau_1}\otimes_R B)"] \arrow[d, "\mathrm{H}(P_1\otimes_R\psi)"'] & \mathrm{H}(P_2 \otimes_R B) \arrow[d, "\mathrm{H}(P_2\otimes_R\psi)"] \\ \mathrm{H}(P_1 \otimes_R B') \arrow[r, "\mathrm{H}(\widetilde{\tau_2}\otimes_R B')"'] & \mathrm{H}(P_2 \otimes_R B') \end{tikzcd}$$

where the rows are isomorphisms since the $\mathrm{H}(-)$ takes homotopy equivalences to isomorphisms. Since the rows are isomorphisms and the diagram commutes, we see that $\mathrm{H}(\operatorname{Tor}^R(\widetilde{\tau_1},-))$ is a natural isomorphism. $\square$

6.5.2 The functor $\operatorname{Tor}^R(-,B)$

Next we want to define the covariant functor

$$\operatorname{Tor}^R(-,B)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Again, we want this functor to send and an R-complex A to the graded R-module $\operatorname{Tor}^R(A,B)$.

Definition 6.11. Let $\varphi\colon A \to A'$ be a chain map, let $\tau\colon P \to A$ be a semiprojective resolution of A, let $\tau'\colon P' \to A'$ be a semiprojective resolution of A', and let $\widetilde{\varphi}\colon P \to P'$ be a homotopic lift of $\varphi\tau$ with respect to τ'. We define

$$\operatorname{Tor}^R(\varphi,B)\colon \operatorname{Tor}^R(A,B) \to \operatorname{Tor}^R(A',B).$$

by $\operatorname{Tor}^R(\varphi,B) := \mathrm{H}(\widetilde{\varphi} \otimes_R B)$.

This time our definition of the functor $\operatorname{Tor}^R(-,B)$ involves *three choices*; namely, the semiprojective resolutions $\tau\colon P \to A$ and $\tau'\colon P' \to A'$ as well as the homotopic lift $\widetilde{\varphi}\colon P \to P'$. Even though we made three choices, we shall still see that $\operatorname{Tor}^R(-,B)$ is well-defined up to natural isomorphism.

Theorem 6.12. $\operatorname{Tor}^R(-,B)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A, suppose $\tau_1'\colon P_1' \to A'$ and $\tau_2'\colon P_2' \to A'$ are two semiprojective resolutions of A', and suppose $\widetilde{\varphi_1}\colon P_1 \to P_1'$ is a homotopic lift of $\varphi\tau_1$ with respect to τ_1' and $\widetilde{\varphi_2}\colon P_2 \to P_2'$ is a homotopic lift of $\varphi\tau_2$ with respect to τ_2'. So altogether we have the diagrams

$$\begin{tikzcd} P_1 \arrow[r, "\widetilde{\varphi_1}"] \arrow[d, "\tau_1"'] & P_1' \arrow[d, "\tau_1'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd} \qquad \begin{tikzcd} P_2 \arrow[r, "\widetilde{\varphi_2}"] \arrow[d, "\tau_2"'] & P_2' \arrow[d, "\tau_2'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd}$$

which commute up to homotopy.

Choose a homotopic lift $\widetilde{\tau}_1 \colon P_1 \to P_2$ of τ_1 with respect to τ_2 and choose a homotopic lift $\widetilde{\tau}_1{}' \colon P_1' \to P_2'$ of τ_1' with respect to τ_2'. Then $\widetilde{\tau}_1$ and $\widetilde{\tau}_1{}'$ are both homotopy equivalences by the same argument as in the proof of Theorem (6.10). Now observe that

$$\begin{aligned} \tau_2'\widetilde{\varphi_2}\widetilde{\tau}_1 &\sim \varphi\tau_2\widetilde{\tau}_1 \\ &\sim \varphi\tau_1 \\ &\sim \tau_1'\widetilde{\varphi_1} \\ &\sim \tau_2'\widetilde{\tau}_1{}'\widetilde{\varphi_1}. \end{aligned}$$

In particular, both $\widetilde{\varphi_2}\widetilde{\tau}_1 \colon P_1 \to P_2'$ and $\widetilde{\tau}_1{}'\widetilde{\varphi_1} \colon P_1 \to P_2'$ are homotopic lifts of $\varphi\tau_1$ with respect to τ_2'. Therefore

$$\widetilde{\varphi_2}\widetilde{\tau}_1 \sim \widetilde{\tau}_1{}'\widetilde{\varphi_1},$$

and since $- \otimes_R B$ respects homotopies, we have a diagram

$$\begin{tikzcd} P_1 \otimes_R B \arrow[r, "\widetilde{\tau}_1 \otimes_R B"] \arrow[d, "\widetilde{\varphi_1} \otimes_R B"'] & P_2 \otimes_R B \arrow[d, "\widetilde{\varphi_2} \otimes_R B"] \\ P_1' \otimes_R B \arrow[r, "\widetilde{\tau}_1{}' \otimes_R B"] & P_2' \otimes_R B \end{tikzcd}$$

which commutes up to homotopy. Finally, since $\mathrm{H}(-)$ takes homotopic maps to equal maps, we see that the diagram

$$\begin{tikzcd} \mathrm{H}(P_1 \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau}_1 \otimes_R B)"] \arrow[d, "\mathrm{H}(\widetilde{\varphi_1} \otimes_R B)"'] & \mathrm{H}(P_2 \otimes_R B) \arrow[d, "\mathrm{H}(\widetilde{\varphi_2} \otimes_R B)"] \\ \mathrm{H}(P_1' \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau}_1{}' \otimes_R B)"] & \mathrm{H}(P_2' \otimes_R B) \end{tikzcd}$$

which is commutative. Since $\mathrm{H}(-)$ takes homotopy equivalences to isomorphisms, we see that the rows are isomorphisms, and thus $\mathrm{H}(\mathrm{Hom}_R^\star(-, B))$ is a natural isomorphism.

□

6.5.3 Balance of Tor

Proposition 6.8. *Let A and B be R-complexes and let $\sigma \colon P \to A$ and $\tau \colon Q \to B$ be semiprojective resolutions. Then*

$$\mathrm{Tor}^R(A, B) \cong \mathrm{H}(P \otimes_R Q) \cong \mathrm{H}(A \otimes_R Q).$$

Proof. Observe that $P \otimes_R -$ respects quasiisomorphisms since P is semiprojective (and hence semiflat). Therefore $P \otimes_R \tau \colon P \otimes_R Q \to P \otimes_R B$ is a quasiisomoprhism. Thus

$$\mathrm{H}(P \otimes_R \tau) \colon \mathrm{H}(P \otimes_R Q) \to \mathrm{H}(P \otimes_R B)$$

is an isomorphism. Similarly, $- \otimes_R Q$ respects quasiisomorphisms since Q is semiprojective (and hence semiflat). Therefore $\sigma \otimes_R Q \colon P \otimes_R Q \to A \otimes_R Q$ is a quasiisomorphism. Thus

$$\mathrm{H}(\sigma \otimes_R Q) \colon \mathrm{H}(P \otimes_R Q) \to \mathrm{H}(A \otimes_R Q)$$

is an isomorphism. Therefore we have balance of Tor:

$$\begin{aligned} \mathrm{Tor}^R(A, B) &= \mathrm{H}(P \otimes_R B) \\ &\cong \mathrm{H}(P \otimes_R Q) \\ &\cong \mathrm{H}(A \otimes_R Q). \end{aligned}$$

□

6.5.4 Commutativity of Tor

Proposition 6.9. *Let A and B be R-complexes. Then we have an isomorphism of graded R-modules*

$$\mathrm{Tor}^R(A,B) \cong \mathrm{Tor}^R(B,A),$$

which is natural in A and B.

Proof. Let $\sigma\colon P \to A$ be a semiprojective resolution of A and let $\tau\colon Q \to B$ be a semiprojective resolutions of B. We have

$$\begin{aligned}\mathrm{Tor}^R(A,B) &= \mathrm{H}(P \otimes_R B)\\ &\cong \mathrm{H}(P \otimes_R Q)\\ &\cong \mathrm{H}(Q \otimes_R P)\\ &\cong \mathrm{H}(Q \otimes_R A)\\ &= \mathrm{Tor}^R(B,A).\end{aligned}$$

□

6.6 Functors from $\mathbf{Comp}_R$ to $\mathbf{HComp}_R$ and $\mathbf{HComp}_R$ to $\mathbf{HComp}_R$

6.6.1 Semiprojective Version

For every R-complex A we fix a semiprojective resolution $\mathrm{P}_R(A) \xrightarrow{\tau_A} A$ and for every chain map $\varphi\colon A \to B$ we fix a homotopic lift $\mathrm{P}_R(\varphi)\colon \mathrm{P}_R(A) \to \mathrm{P}_R(B)$ of $\varphi\tau_A$ with respect to τ_B. If the ring R is clear from context, then we write $\mathrm{P}(A)$ and $\mathrm{P}(\varphi)$ rather than $\mathrm{P}_R(A)$ and $\mathrm{P}_R(\varphi)$ in order to simplify notation.

Proposition 6.10. *We obtain a well-defined R-linear covariant functor $\mathbb{P}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{P}(A)$ and which takes a chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{P}(\varphi)]$.*

Proof. The well-definedness comes from the fact that we used fixed resolutions and lifts. The functor $\mathbb{P}$ respects identity maps. Indeed, given the identity morphism $1_A\colon A \to A$, we have $\tau_A 1_{\mathrm{P}(A)} = 1_A\tau_A$. In particular, $1_{\mathrm{P}(A)}$ is a homotopic lift of $1_A\tau_A$ with respect to τ_A. Thus $\mathrm{P}(1_A) \sim 1_{\mathrm{P}(A)}$, and thus $[\mathrm{P}(1_A)] = [1_{\mathrm{P}(A)}]$. The functor $\mathbb{P}$ also respects compositions. Indeed, let $\varphi\colon A \to B$ and $\psi\colon B \to C$ be two chain maps. Then

$$\begin{aligned}\tau_C \mathrm{P}(\psi)\mathrm{P}(\varphi) &\sim \psi\tau_B\mathrm{P}(\varphi)\\ &\sim \psi\varphi\tau_A.\end{aligned}$$

Thus $\mathrm{P}(\psi)\mathrm{P}(\varphi)$ is a homotopic lift of $\psi\varphi\tau_A$ with respect to τ_C. Since $\mathrm{P}(\psi\varphi)$ is also a homotopic lift of $\psi\varphi\tau_A$ with respect to τ_C, it follows that $\mathrm{P}(\psi\varphi) \sim \mathrm{P}(\psi)\mathrm{P}(\varphi)$, and thus $[\mathrm{P}(\psi\varphi)] = [\mathrm{P}(\psi)][\mathrm{P}(\varphi)]$.

Now we show that $\mathbb{P}$ is an R-linear functor. Let A and B be R-complexes. We want to show that if $\varphi, \psi \in \mathcal{C}(A,B)$ and $r, s \in R$ then

$$[\mathrm{P}(r\varphi + s\psi)] = [r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)]. \tag{55}$$

To see this, note that $\mathrm{P}(\varphi)$ is a homotopic lift of $\varphi\tau_A$ with respect to τ_B and $\mathrm{P}(\psi)$ is a homotopic lift of $\psi\tau_A$ with respect to τ_B. Now observe that

$$\begin{aligned}\tau_B(r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)) &= r\tau_B\mathrm{P}(\varphi) + s\tau_B\mathrm{P}(\psi)\\ &\sim r\varphi\tau_A + s\psi\tau_A\\ &= (r\varphi + s\psi)\tau_A.\end{aligned}$$

Thus $r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)$ is a homotopic lift of $(r\varphi + s\psi)\tau_A$ with respect to τ_B. Since $\mathrm{P}(r\varphi + s\psi)$ is another homotopic lift of $(r\varphi + s\psi)\tau_A$ with respect to τ_B, it follows that $\mathrm{P}(r\varphi + s\psi) \sim r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)$. In other words, we have (55). □

Definition 6.12. Define $\Omega_R\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ to be functor which sends the R-complex A to the R-complex A and which takes a chain map $\varphi\colon A \to B$ to the homotopy class $[\varphi]$.

Remark. If the ring R is clear from context, then we write Ω rather than Ω_R in order to simplify notation.

Proposition 6.11. *The functor Ω is a well-defined R-linear covariant functor. Moreover it transforms homotopy equivalences to isomorphisms. Furthermore, Ω satisfies the following universal mapping property: for every R-linear covariant functor $F\colon \mathbf{Comp}_R \to \mathcal{C}$ which takes homotopic maps to equal maps, there exists a unique R-linear functor $\widetilde{F}\colon \mathbf{HComp}_R \to \mathcal{C}$ such that $\widetilde{F}\Omega = F$.*

Proof. The first part of the propositions is straightforward. Let us address the universal mapping property. Given such an $F\colon \mathbf{Comp}_R \to \mathcal{C}$, we define $\widetilde{F}\colon \mathbf{HComp}_R \to \mathcal{C}$ to be the functor which takes an R-complex A to the object $F(A)$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$ to the morphism $F(\varphi)\colon F(A) \to F(B)$. Observe that this is well-defined by assumption of F (it takes homotopic chain maps to equal maps). Let us show that $\widetilde{F}$ is a functor. First we check that it respects identity maps. Let $[1_A]$ be the homotopy class of the identity map $1_A\colon A \to A$. Then

$$\begin{aligned}\widetilde{F}[1_A] &= F(1_A)\\ &= 1_{F(A)}.\end{aligned}$$

Thus $\widetilde{F}$ respects identity maps. Next let's check that it respects compositions. Let $[\varphi]$ and $[\psi]$ be the homotopy classes of the chain maps $\varphi\colon A \to B$ and $\psi\colon B \to C$ respectively. Then

$$\begin{aligned}\widetilde{F}[\psi\varphi] &= F(\psi\varphi)\\ &= F(\psi)F(\varphi)\\ &= \widetilde{F}[\psi]\widetilde{F}[\varphi].\end{aligned}$$

Thus $\widetilde{F}$ respects compositions. Now let us check that $\widetilde{F}\Omega = F$. For any R-complex A, we have

$$\begin{aligned}\widetilde{F}\Omega(A) &= \widetilde{F}(A)\\ &= F(A)\end{aligned}$$

and for any chain map $\varphi\colon A \to B$, we have

$$\begin{aligned}\widetilde{F}\Omega(\varphi) &= \widetilde{F}[\mathrm{P}(\varphi)]\\ &= F(\varphi).\end{aligned}$$

Therefore $\widetilde{F}\Omega = F$. Finally, note that uniqueness of $\widetilde{F}$ follows from the fact that we were forced to define $\widetilde{F}$ in this way. Indeed, if $\widetilde{F}'$ was another such functor, then for any R-complex A, we have

$$\begin{aligned}\widetilde{F}'(A) &= \widetilde{F}'\Omega(A)\\ &= F(A)\\ &= \widetilde{F}\Omega(A)\\ &= \widetilde{F}(A),\end{aligned}$$

and for any chain map $\varphi\colon A \to B$, we have

$$\begin{aligned}\widetilde{F}'[\varphi] &= \widetilde{F}'\Omega(\varphi)\\ &= F(\varphi)\\ &= \widetilde{F}\Omega(\varphi)\\ &= \widetilde{F}[\varphi].\end{aligned}$$

□

Remark. One should view Ω as some sort of "localization" functor. Indeed, recall that if S is a multilpicatively closed subset of a commutative ring A and $\rho_S\colon A \to A_S$ is the canonical localization map, then the pair (A_S, ρ_S) satisfies the following universal mapping property: for every ring homomorphism $\varphi\colon A \to B$ such that $\varphi(S) \subseteq B^\times$, there exists a unique ring homomorphism $\widetilde{\varphi}\colon A_S \to B$ such that $\widetilde{\varphi}\rho_S = \varphi$.

Theorem 6.13. *Let $\widetilde{\mathbb{P}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ be the functor which takes an R-complex A to the R-complex $\mathrm{P}(A)$ and which takes a homotopy class $[\varphi]$ of the chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{P}(\varphi)]$ of the chain map $\mathrm{P}(\varphi)\colon \mathrm{P}(A) \to \mathrm{P}(B)$. Then $\widetilde{\mathbb{P}}$ is a well-defined R-linear functor.*

Proof. Note that $\mathbb{P}$ takes homotopic chain maps to equal maps. Thus we may apply Proposition (6.11) to $\mathbb{P}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ (where $\mathcal{C} = \mathbf{HComp}_R$) to get $\widetilde{\mathbb{P}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$. □

6.6.2 Semiinjective Version

For every R-complex A we fix a semiinjective resolution $A \xrightarrow{\varepsilon_A} \mathrm{E}_R(A)$ and for every chain map $\varphi\colon A \to B$ we fix a homotopic lift $\mathrm{E}_R(\varphi)\colon \mathrm{E}_R(A) \to \mathrm{E}_R(B)$ of $\varepsilon_B\varphi$ with respect to ε_A. If the ring R is clear from context, then we write $\mathrm{E}(A)$ and $\mathrm{E}(\varphi)$ rather than $\mathrm{E}_R(A)$ and $\mathrm{E}_R(\varphi)$ in order to simplify notation.

Just like in the semiprojective case, we will denote we obtain a well-defined R-linear covariant functor $\mathbb{E}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{E}(A)$ and which takes a chain map $\varphi\colon A \to B$ to to the homotopy class $[\mathrm{E}(\varphi)]$ of the chain map $\mathrm{E}(\varphi)\colon \mathrm{E}(A) \to \mathrm{E}(B)$. Similarly, we obtain a well-defined R-linear covariant functor $\widetilde{\mathbb{E}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{E}(A)$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{E}(\varphi)]$ of the chain map $\mathrm{E}(\varphi)\colon \mathrm{E}(A) \to \mathrm{E}(B)$.

6.6.3 Covariant Hom

Theorem 6.14. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}^\star_R(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}^\star_R(A,\varphi)]$ *of the chain map* $\mathrm{Hom}^\star_R(A,\varphi)\colon \mathrm{Hom}^\star_R(A,B) \to \mathrm{Hom}^\star_R(A,B')$.

2. $\widetilde{\mathbb{H}}\mathrm{om}^\star_R(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}^\star_R(A,\varphi)]$ *of the chain map* $\mathrm{Hom}^\star_R(A,\varphi)\colon \mathrm{Hom}^\star_R(A,B) \to \mathrm{Hom}^\star_R(A,B')$.

Proof. 1. Observe that $\mathbb{H}\mathrm{om}^\star_R(A,-) = \Omega\mathrm{Hom}^\star_R(A,-)$. The composition of two R-linear covariant functors is a well-defined R-linear covariant functor.

2. Observe that $\mathbb{H}\mathrm{om}^\star_R(A,-)$ takes homotopic maps to equal maps. Indeed, if $\varphi\colon B \to B'$ and $\psi\colon B \to B'$ are two chain maps such that $\varphi \sim \psi$, then $\mathrm{Hom}^\star_R(A,\varphi) \sim \mathrm{Hom}^\star_R(A,\psi)$. Therefore $[\mathrm{Hom}^\star_R(A,\varphi)] = [\mathrm{Hom}^\star_R(A,\psi)]$. Thus we may apply the universal mapping property in Proposition (6.11) to $\mathbb{H}\mathrm{om}^\star_R(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ (where $\mathcal{C} = \mathbf{HComp}_R$) to get $\widetilde{\mathbb{H}}\mathrm{om}^\star_R(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$. □

6.6.4 Contravariant Hom

Theorem 6.15. *Let B be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}^\star_R(-,B)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a chain map $\varphi\colon A \to A'$ to the homotopy class* $[\mathrm{Hom}^\star_R(\varphi,B)]$ *of the chain map* $\mathrm{Hom}^\star_R(\varphi,B)\colon \mathrm{Hom}^\star_R(A',B) \to \mathrm{Hom}^\star_R(A,B)$.

2. $\widetilde{\mathbb{H}}\mathrm{om}^\star_R(-,B)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to A'$ to the homotopy class* $[\mathrm{Hom}^\star_R(\varphi,B)]$ *of the chain map* $\mathrm{Hom}^\star_R(\varphi,B)\colon \mathrm{Hom}^\star_R(A,B) \to \mathrm{Hom}^\star_R(A,B')$.

Proof. Proof is similar to the proof of Theorem (6.18). □

6.6.5 Tensor Product

Theorem 6.16. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $A\underline{\otimes}_R-\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex $A \otimes_R B$ and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class $[A \otimes_R \varphi]$ of the chain map $A \otimes_R \varphi\colon A \otimes_R B \to A \otimes_R B'$.*

2. $A\underline{\widetilde{\otimes}}_R-\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex $A \otimes_R B$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class $[A \otimes_R \varphi]$ of the chain map $A \otimes_R \varphi\colon A \otimes_R B \to A \otimes_R B'$.*

Theorem 6.17. *Let B be an R-complex. Then the following are well-defined R-linear functors*

1. $-\underline{\otimes}_R B\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex $A \otimes_R B$ and which takes a chain map $\varphi\colon A \to A'$ to the homotopy class $[\varphi \otimes_R A]$ of the chain map $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$.*

2. $-\underline{\widetilde{\otimes}}_R B\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex $A \otimes_R B$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to A'$ to the homotopy class $[\varphi \otimes_R B]$ of the chain map $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$.*

Remark. (commutativity) Let A be an R-complex. Then $A\underline{\otimes}_R-$ is naturally isomorphic to $-\underline{\otimes}_R A$. Indeed, we have

$$\begin{aligned} A\underline{\otimes}_R- &= \Omega(A \otimes_R -) \\ &\cong \Omega(- \otimes_R A) \\ &= -\underline{\otimes}_R A, \end{aligned}$$

where the isomorphism at the second line is natural (as shown earlier). Note that this also implies $A\widetilde{\underline{\otimes}}_R-$ is naturally isomorphic to $-\widetilde{\underline{\otimes}}_R A$.

6.6.6 Natural Transformation of Functors

Proposition 6.12. *Let A be an R-complex. The natural chain maps*

$$\mathrm{P}(A) \underset{\simeq}{\xrightarrow{\tau_A}} A \underset{\simeq}{\xrightarrow{\varepsilon_A}} \mathrm{E}(A)$$

induce the following natural transformations

1. $\mathbb{P} \xrightarrow{[\tau]} \Omega \xrightarrow{[\varepsilon]} \mathbb{E}$ *of functors from* $\mathbf{Comp}_R$ *to* $\mathbf{HComp}_R$.

2. $\widetilde{\mathbb{P}} \xrightarrow{[\tau]} \mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$ *of functors from* $\mathbf{HComp}_R$ *to* $\mathbf{HComp}_R$.

Proof. We focus $\Omega \xrightarrow{[\varepsilon]} \mathbb{E}$ and $\mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$ since the proof that the other maps are natural transformations is a similar argument. We first consider $\Omega \xrightarrow{[\varepsilon]} \mathbb{E}$. We need to check that for every chain map $\varphi\colon A \to B$, the following diagram commutes in $\mathbf{HComp}_R$:

$$\begin{tikzcd} A \arrow[r, "{[\varepsilon_A]}"] \arrow[d, "{[\varphi]}"'] & \mathrm{E}(A) \arrow[d, "{[\mathrm{E}(\varphi)]}"] \\ B \arrow[r, "{[\varepsilon_B]}"'] & \mathrm{E}(B) \end{tikzcd}$$

This is clear however since $\mathrm{E}(\varphi)$ is a homotopic lift of $\varepsilon_B\varphi$ with respect to ε_A. Thus $\varepsilon_B\varphi \sim \mathrm{E}(\varphi)\varepsilon_A$, which implies

$$\begin{aligned} [\varepsilon_B][\varphi] &= [\varepsilon_B\varphi] \\ &= [\mathrm{E}(\varphi)\varepsilon_A] \\ &= [\mathrm{E}(\varphi)][\varepsilon_A]. \end{aligned}$$

Now we consider $\mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$. We need to check that for every homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$, the following diagram commutes in $\mathbf{HComp}_R$:

$$\begin{tikzcd} A \arrow[r, "{[\varepsilon_A]}"] \arrow[d, "{[\varphi]}"'] & \mathrm{E}(A) \arrow[d, "{[\mathrm{E}(\varphi)]}"] \\ B \arrow[r, "{[\varepsilon_B]}"'] & \mathrm{E}(B) \end{tikzcd}$$

This was done above. $\square$

Theorem 6.18. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}^\star_R(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}^\star_R(A,\varphi)]$ *of the chain map* $\mathrm{Hom}^\star_R(A,\varphi)\colon \mathrm{Hom}^\star_R(A,B) \to \mathrm{Hom}^\star_R(A,B')$.

2. $\widetilde{\mathbb{H}}\mathrm{om}^\star_R(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}^\star_R(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}^\star_R(A,\varphi)]$ *of the chain map* $\mathrm{Hom}^\star_R(A,\varphi)\colon \mathrm{Hom}^\star_R(A,B) \to \mathrm{Hom}^\star_R(A,B')$.

6.7 Triangulated Categories

Exact sequences are useful for studying modules and complexes, but these are poorly behaved in $\mathbf{HComp}_R$. For instance, the natural chain $0 \xrightarrow{\simeq} \mathcal{K}(1)$ is a quasiisomorphism between semiprojective complexes and so thus must be a homotopy equivalence. Thus $\mathcal{K}(1)$ is isomorphic to 0 in the $\mathbf{HComp}_R$. Now the 0 complex fits into a really silly exact sequence, namely $0 \to 0 \to 0$, but it is not clear whether the sequence $0 \to \mathcal{K}(1) \to 0$ should be exact. To solve this, Grothendieck and Verdier introduced the notion of a **triangulated category**, where instead of considering exact sequences, one considers **distinguished triangles**.

6.7.1 Basic Definitions

Definition 6.13. Let $\mathcal{C}$ be an R-linear category.

1. A **shift functor** (or **translation functor**) on $\mathcal{C}$ is an R-linear functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ with a 2-sided inverse $\Sigma^{-1}\colon \mathcal{C} \to \mathcal{C}$. Sometimes ΣA will be denoted $A[1]$. More generally, $\Sigma^n A = A[n]$. Note that $\Sigma^0 = 1_{\mathcal{C}}$.

2. A **triangle** in $\mathcal{C}$ is a diagram of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A \tag{56}$$

of morphisms in $\mathcal{C}$. Sometimes we call these **pretriangles** or **candidate triangles**. We shall use the short-hand notation $(A, B, C)_{(\alpha,\beta,\gamma)}$ to denote the triangle (**??**).

3. A **morphism** of triangles in $\mathcal{C}$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r, "\gamma"] \arrow[d, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & \Sigma A' \end{tikzcd}$$

Such a morphism is called an **isomorphism** if f, g, h are all isomorphisms, that is, the morphism has a 2-sided inverse.

6.7.2 Triangulated Category

Definition 6.14. A **triangulated R-linear category** is an R-linear category $\mathcal{C}$ equipped with a shift functor Σ and a class of triangles called **distinguished triangles** (or **exact triangles**) such that the following axioms are satisfied.

1. For all objects A in $\mathcal{C}$, the triangle $A \xrightarrow{1_A} A \to 0 \to \Sigma A$ is distinguished.

2. For every morphism $\alpha\colon A \to B$, there exists a distinguished triangle $A \xrightarrow{\alpha} B \to C \to \Sigma A$. In this case we call C a **cone of** α (or a **cofiber** of α).

3. Given an isomorphism of triangles

7 Special Complexes

7.1 Taylor Resolution

Throughout this subsection, let $\underline{m} = m_1, \dots, m_r$ be monomials in $R = K[x_1, \dots, x_n]$. For each subset σ of $\{1, \dots, r\}$ we set $m_\sigma := \operatorname{lcm}(m_\lambda \mid \lambda \in \sigma)$. Let $a_\sigma \in \mathbb{N}^n$ be the exponent vector of m_σ and let $R(-a_\sigma)$ be the free R-module with one generator in multidegree a_σ. The **Taylor resolution** of $R/\langle \underline{m} \rangle$ is the R-complex $(\mathcal{T}(\underline{m}), d^{\mathcal{T}(\underline{m})})$ whose graded R-module $\mathcal{T}(\underline{m})$ has

$$\mathcal{T}_i(\underline{m}) := \begin{cases} \bigoplus_{\sigma \in S_i[n]} Re_\sigma & \text{if } 0 \le i \le n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{T}(\underline{m})}$ is uniquely determined by

$$d^{\mathcal{T}(\underline{m})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma \backslash \lambda}} e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq [n]$.

Remark. We need to check that the differential defined above really is a differential. Denote $d := d^{\mathcal{T}(\underline{m})}$ and let $\sigma \subseteq [n]$. Then

$$\begin{aligned}
d^2(e_\sigma) &= d(d(e_\sigma)) \\
&= d\left(\sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \frac{m_\sigma}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda}\right) \\
&= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \frac{m_\sigma}{m_{\sigma\backslash\lambda}} d(e_{\sigma\backslash\lambda}) \\
&= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \frac{m_\sigma}{m_{\sigma\backslash\lambda}} \sum_{\mu\in\sigma\backslash\lambda} \langle \mu, \sigma\backslash\{\lambda,\mu\}\rangle \frac{m_{\sigma\backslash\lambda}}{m_{\sigma\backslash\{\lambda,\mu\}}} d(e_{\sigma\backslash\{\lambda,\mu\}}) \\
&= \sum_{\substack{\lambda,\mu\in\sigma \\ \lambda\neq\mu}} \langle \lambda, \sigma\backslash\lambda\rangle \langle \mu, \sigma\backslash\{\lambda,\mu\}\rangle \frac{m_\sigma}{m_{\sigma\backslash\{\lambda,\mu\}}} d(e_{\sigma\backslash\{\lambda,\mu\}}) \\
&= 0,
\end{aligned}$$

where the last part follows from symmetry in μ and λ and

$$\begin{aligned}
\langle \lambda, \sigma\backslash\lambda\rangle \langle \mu, \sigma\backslash\{\lambda,\mu\}\rangle &= \langle \lambda, \sigma\backslash\lambda\rangle \langle \mu, \sigma\backslash\{\lambda,\mu\}\rangle \\
&= \langle \lambda, \sigma\backslash\lambda\rangle \langle \mu, \sigma\backslash\lambda\rangle \langle \mu, \lambda\rangle \\
&= -\langle \lambda, \sigma\backslash\lambda\rangle \langle \lambda, \mu\rangle \langle \mu, \sigma\backslash\lambda\rangle \\
&= -\langle \lambda, \sigma\backslash\{\mu,\lambda\}\rangle \langle \mu, \sigma\backslash\lambda\rangle \\
&= -\langle \mu, \sigma\backslash\mu\rangle \langle \lambda, \sigma\backslash\{\mu,\lambda\}\rangle.
\end{aligned}$$

7.1.1 Taylor Resolution as $\mathbb{N}^n$-Graded k-Algebra

The Taylor resolution has an extra graded structure present which is not necessarily shared by the Koszul complex. The underlying graded R-module $\mathcal{T}(\underline{m})$ has an $\mathbb{N}^n$-graded K-module structure. Indeed, for $\mathbf{b} \in \mathbb{N}^n$, the $\mathbf{b}$th homogeneous component of is given by

$$\mathcal{T}_{\mathbf{b}}(\underline{m}) = \bigoplus_{m_\sigma | \mathbf{x}^{\mathbf{b}}} K \cdot \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma.$$

Moreover, the differential is an $\mathbb{N}^n$-graded K-endormorphism (of degree 0): For any $\sigma \subseteq [n]$ such that $m_\sigma | \mathbf{x}^{\mathbf{b}}$, we have

$$\begin{aligned}
d^{\mathcal{T}(\underline{m})}\left(\frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma\right) &= \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} d^{\mathcal{T}(\underline{m})}(e_\sigma) \\
&= \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \frac{m_\sigma}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda} \\
&= \sum_{\lambda\in\sigma} \langle \lambda, \sigma\backslash\lambda\rangle \frac{\mathbf{x}^{\mathbf{b}}}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda} \\
&\in \mathcal{T}_{\mathbf{b}}(\underline{m}).
\end{aligned}$$

In particular, $\ker(d^{\mathcal{T}(\underline{m})})$ and $\operatorname{im}(d^{\mathcal{T}(\underline{m})})$ have induced $\mathbb{N}^n$-graded K-module structures and hence $\mathrm{H}(\mathcal{T}(\underline{m}))$ has an induced $\mathbb{N}^n$-graded K-module structure: For $\mathbf{b} \in \mathbb{N}^n$, the $\mathbf{b}$th homogeneous component of $\mathrm{H}(\mathcal{T}(\underline{m}))$ is

Proposition 7.1. *The Taylor complex is a free resolution of R/I.*

Proof. It suffices to show $\mathrm{H}_{\mathbf{b}}(\mathcal{T}(\underline{m})) \cong 0$ for all $\mathbf{b} \in \mathbb{N}^n \backslash \{0\}$. Observe that the simplicial complex

$$\Delta[\mathbf{x}^{\mathbf{b}}] := \{\sigma \subseteq [n] \mid m_\sigma \text{ divides } \mathbf{x}^{\mathbf{b}}\}$$

□

$$\mathrm{H}_{\mathbf{b}}(\mathcal{T}(\underline{m})) = \frac{\ker_{\mathbf{b}}(d^{\mathcal{T}(\underline{m})})}{\mathrm{im}_{\mathbf{b}}(d^{\mathcal{T}(\underline{m})})}.$$

7.1.2 The K-Complex in Degree $\mathbf{b}$

Let $\mathbf{b} \in \mathbb{N}^n$. The complex $(\mathcal{T}_{\mathbf{b}}(\underline{m}), d^{\mathcal{T}_{\mathbf{b}}(\underline{m})})$ is the K-complex whose underlying graded K-module has

$$\mathcal{T}_{i,\mathbf{b}}(\underline{m}) = \bigoplus_{\substack{m_\sigma | \mathbf{x}^{\mathbf{b}} \\ \sigma \in S_i(n)}} K \cdot \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma$$

as its ith homogeneous and whose differential is the unique differential such that

$$d^{\mathcal{T}_{\mathbf{b}}(\underline{m})} \left(\frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma \right) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{\mathbf{x}^{\mathbf{b}}}{m_{\sigma \backslash \lambda}} e_{\sigma \backslash \lambda}.$$

7.1.3 Taylor Complex is a Free Resolution

In this section, we want to show that the Taylor complex defined above is a free resolution of R/I. We do this by induction on n. The case $n = 1$ is trivial. A

7.1.4 Taylor Complex as a DG Algebra

Proposition 7.2. *Let $I = \langle m_1, \ldots, m_r \rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$. The Taylor resolution $(\mathcal{T}(\underline{m}), d^{\mathcal{T}(\underline{m})})$ is a DG algebra, with multiplication being uniquely determined on elementary tensors: for $\sigma, \tau \subseteq [n]$, we map $e_\sigma \otimes e_\tau \mapsto e_\sigma e_\tau$, where*

$$e_\sigma e_\tau = \begin{cases} \langle \sigma, \tau \rangle \frac{m_\sigma m_\tau}{m_{\sigma \cup \tau}} e_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \varnothing \\ 0 & \text{if } \sigma \cap \tau \neq \varnothing \end{cases} \tag{57}$$

Proof. Throughout this proof, denote $d := d^{\mathcal{T}(\underline{m})}$. We first note that $e_\varnothing$ serves as the identity for the multiplication rule (**??**). Indeed, let $\sigma \subseteq [n]$. Then since $\sigma \cap \varnothing = \varnothing$, we have

$$e_\sigma e_\varnothing = e_\sigma = e_\varnothing e_\sigma.$$

Moreover, multiplication by $e_\varnothing$ and e_σ given in (**??**) satisfies Leibniz law:

$$\begin{aligned} d(e_\sigma) e_\varnothing - e_\sigma d(e_\varnothing) &= d(e_\sigma) e_\varnothing \\ &= d(e_\sigma) \\ &= d(e_\sigma e_\varnothing), \end{aligned}$$

and similarly

$$\begin{aligned} d(e_\varnothing) e_\sigma + e_\varnothing d(e_\sigma) &= e_\varnothing d(e_\sigma) \\ &= d(e_\sigma) \\ &= d(e_\varnothing e_\sigma), \end{aligned}$$

Next, let $\lambda \in [n]$. Suppose $\tau \subseteq [n]$ and $\lambda \notin \tau$. Then

$$\begin{aligned}
d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= m_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \frac{m_\tau}{m_{\tau \backslash \mu}} e_{\tau \backslash \mu} \\
&= m_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \frac{m_\tau}{m_{\tau \backslash \mu}} e_\lambda e_{\tau \backslash \mu} \\
&= m_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \backslash \mu \rangle \frac{m_\tau}{m_{\tau \backslash \mu}} \frac{m_\lambda m_{\tau \backslash \mu}}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= m_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \rangle \langle \lambda, \mu \rangle \frac{m_\lambda m_\tau}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= m_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \rangle \langle \mu, \lambda \rangle \frac{m_\lambda m_\tau}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= m_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \cup \lambda \rangle \frac{m_\lambda m_\tau}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= \langle \lambda, \tau \rangle \left(\langle \lambda, \tau \rangle m_\lambda e_\tau + \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \cup \lambda \rangle \frac{m_\lambda m_\tau}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \right) \\
&= \langle \lambda, \tau \rangle \sum_{\mu \in \tau \cup \lambda} \langle \mu, \tau \backslash \mu \cup \lambda \rangle \frac{m_\lambda m_\tau}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= \langle \lambda, \tau \rangle \frac{m_\lambda m_\tau}{m_{\tau \cup \lambda}} \sum_{\mu \in \tau \cup \lambda} \langle \mu, \tau \backslash \mu \cup \lambda \rangle \frac{m_{\tau \cup \lambda}}{m_{\tau \backslash \mu \cup \lambda}} e_{\tau \backslash \mu \cup \lambda} \\
&= \langle \lambda, \tau \rangle \frac{m_\lambda m_\tau}{m_{\tau \cup \lambda}} d(e_{\tau \cup \lambda}) \\
&= d(e_\lambda e_\tau),
\end{aligned}$$

Next suppose $\tau \subseteq [n]$ and $\lambda \in \tau$. Then

$$\begin{aligned}
d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= m_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \frac{m_\tau}{m_{\tau \backslash \mu}} e_{\tau \backslash \mu} \\
&= m_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \frac{m_\tau}{m_{\tau \backslash \mu}} e_\lambda e_{\tau \backslash \mu} \\
&= m_\lambda e_\tau - \langle \lambda, \tau \backslash \lambda \rangle \langle \lambda, \tau \backslash \lambda \rangle \frac{m_\tau}{m_{\tau \backslash \lambda}} \frac{m_\lambda m_{\tau \backslash \lambda}}{m_\tau} e_\tau \\
&= m_\lambda e_\tau - m_\lambda e_\tau \\
&= 0 \\
&= d(0) \\
&= d(e_\lambda e_\tau).
\end{aligned}$$

Thus we have shown (**??**) satisfies the Leibniz law for all pairs (λ, τ) where $\lambda \in [n]$ and $\tau \subseteq [n]$. We prove by induction on $|\sigma| = i \geq 1$ that (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$. The base case $i = 1$ was just shown. Now suppose we have shown (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i < n$. Let $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i + 1$. Choose $\lambda \in \sigma$. Then

$$\begin{aligned}
\frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} d(e_\sigma e_\tau) &= d\left(\frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} e_\sigma e_\tau \right) \\
&= d(e_\lambda e_{\sigma \backslash \lambda} e_\tau) \\
&= m_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda d(e_{\sigma \backslash \lambda} e_\tau) \\
&= m_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda (d(e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma|-1} e_{\sigma \backslash \lambda} d(e_\tau)) \\
&= (m_\lambda e_{\sigma \backslash \lambda} - e_\lambda d(e_{\sigma \backslash \lambda})) e_\tau + (-1)^{|\sigma|} \frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} e_\sigma d(e_\tau) \\
&= d(e_\lambda e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma|} \frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} e_\sigma d(e_\tau) \\
&= d\left(\frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} e_\sigma \right) e_\tau + (-1)^{|\sigma|+1} \frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} e_\sigma d(e_\tau), \\
&= \frac{m_\lambda m_{\sigma \backslash \lambda}}{m_\sigma} \left(d(e_\sigma) e_\tau + (-1)^{|\sigma|+1} e_\sigma d(e_\tau) \right)
\end{aligned}$$

where we used the base case on the pairs $(e_\lambda, e_{\sigma\setminus\lambda}e_\tau)$[7] and $(e_\lambda, e_{\sigma\setminus\lambda})$ and where we used the induction hypothesis on the pair $(e_{\sigma\setminus\lambda}, e_\tau)$. and where we used the base case on the pair $(e_\lambda, e_{\sigma\setminus\lambda})$. Canceling $\frac{m_\lambda m_{\sigma\setminus\lambda}}{m_\sigma}$ on both sides completes the proof. □

Lemma 7.1. *(DG Algebra Criterion) Let (A, d) be an R-complex such that A is an associative and unital graded R-algebra. Let G be a set of generators for the graded R-algebra A. Suppose the Leibniz law is true for all pairs (a, b) where $a, b \in G$ such that $\deg(a) = 1$. Further suppose that each $a \in G$ is divisible by some $a_1 \in G$ such that $\deg(a_1) = 1$. Then (A, d) is a DG algebra.*

Proof. It suffices to check that the Leibniz law holds for all pairs (a, b) where $a, b \in G$. Indeed, if $x \in A_k$ and $y \in A_l$ and

$$x = \sum_i r_i a_i \quad \text{and} \quad y = \sum_j s_j b_j,$$

then

$$\begin{aligned} d(xy) &= d\left(\sum r_i a_i \sum s_j b_j\right) \\ &= \sum\sum r_i s_j d(a_i b_j) \\ &= \sum\sum r_i s_j (d(a_i)b_j + (-1)^{\deg(a_i)} a_i d(b_j)) \\ &= \sum\sum r_i s_j d(a_i)b_j + \sum\sum r_i s_j (-1)^{\deg(a_i)} a_i d(b_j)) \\ &= d\left(\sum r_i a_i\right)\sum s_j b_j + (-1)^{\deg(x)} \sum r_i a_i d\left(\sum s_j b_j\right) \\ &= d(x)y + (-1)^{\deg(x)} x d(y). \end{aligned}$$

First observe that the Leibniz law is satisfied for all pairs $(1, a)$ where $1 \in A$ is the identity and $a \in A$. Indeed, we have

$$\begin{aligned} d(1)a + 1d(a) &= 0 \cdot a + 1 \cdot d(a) \\ &= d(a) \\ &= d(1 \cdot a). \end{aligned}$$

Similarly, the Leibniz law is satisfied for all pairs $(a, 1)$ where $1 \in A$ is the identity and $a \in A$. Indeed, we have

$$\begin{aligned} d(a) \cdot 1 + (-1)^{\deg(a)} a d(1) &= d(a) + (-1)^{\deg(a)} a \cdot 0 \\ &= d(a) \\ &= d(a \cdot 1). \end{aligned}$$

Now we want to show that the Leibniz law holds for all pairs (a, b) where $a, b \in A$ such that $\deg(a) \geq 1$ by using induction on $\deg(a)$. The base case ($\deg(a) = 1$) is the assumption in the lemma. Now assume that the Leibniz law is satisfied for all pairs (a, b) where $\deg(a) = i \geq 1$. Let $a, b \in A$ such that $\deg(a) = i + 1$. Choose $a_1 \in A_1$ such that $a_1 | a$. Then $a = a_1 a_i$, for some $a_i \in A_i$. Then

$$\begin{aligned} d(ab) &= d(a_1 a_i b) \\ &= d(a_1)a_i b - a_1 d(a_i b) \\ &= d(a_1)a_i b - a_1(d(a_i)b + (-1)^i a_i d(b)) \\ &= d(a_1)a_i b - a_1 d(a_i)b + (-1)^{i+1} a_1 a_i d(b)) \\ &= (d(a_1)a_i - a_1 d(a_i))b + (-1)^{i+1} a_1 a_i d(b)) \\ &= d(a_1 a_i))b + (-1)^{i+1} a_1 a_i d(b)), \\ &= d(a)b + (-1)^{i+1} a d(b). \end{aligned}$$

□

7.1.5 Taylor Complex is a Free Resolution

In this section, we want to show that the Taylor complex defined above is a free resolution of R/I. We do this by induction on r. The case $r = 1$ being trivial. Let $\underline{m}' = m_2, \ldots, m_r$. By induction, $\mathcal{T}(\underline{m})$ is a free resolution of $R/\langle \underline{m}' \rangle$.

[7] If $e_{\sigma\setminus\lambda}e_\tau = 0$, then obviously Leibniz law holds for the pair $(e_\lambda, e_{\sigma\setminus\lambda}e_\tau)$.

7.2 Generalizing Taylor Complex

Let R and S be rings such that $R \subset S$. Let (A, d) be an S-complex. Suppose A is an $\mathbb{N}^n$-graded R-module and d is homogeneous with respect to the $\mathbb{N}^n$-grading. Then for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ we obtain an R-complex $(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}})$ whose graded R-module in degree i is $A_{i,\boldsymbol{\alpha}} := A_i \cap A_{\boldsymbol{\alpha}}$ and whose differential $\mathrm{d}_{\boldsymbol{\alpha}} := \mathrm{d}|_{A_{\boldsymbol{\alpha}}}$ is the restriction of d to A_{α}. Moreover, we have

$$\begin{aligned}
\mathrm{H}(A, \mathrm{d}) &:= \ker \mathrm{d} / \mathrm{im}\, \mathrm{d} \\
&= \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \ker \mathrm{d}_{\boldsymbol{\alpha}} \right) / \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \mathrm{im}\, \mathrm{d}_{\boldsymbol{\alpha}} \right) \\
&\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \ker \mathrm{d}_{\boldsymbol{\alpha}} / \mathrm{im}\, \mathrm{d}_{\boldsymbol{\alpha}} \\
&:= \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \mathrm{H}(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}}) \\
&\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_{i,\boldsymbol{\alpha}}(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}}.)
\end{aligned}$$

8 Some Category Theory

8.1 Preadditive and Additive Categories

8.1.1 Preadditive Categories

Definition 8.1. A category $\mathcal{A}$ is called **preadditive** if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(y, z) \times \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, z)$$

are bilinear. A functor $F \colon \mathcal{A} \to \mathcal{B}$ of preadditive categories is called **additive** if and only if

$$F \colon \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$$

is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

Remark. In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 8.1. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent:*

1. *x is an initial object;*
2. *x is a final object;*
3. $\mathrm{id}_x = 0$ *in* $\mathrm{Mor}(x, x)$.

Definition 8.2. In a preadditive category $\mathcal{A}$, we call **zero object**, and denote it by 0 any final and initial object as in the Lemma above.

Lemma 8.2. *Let $\mathcal{A}$ be a preadditive category and let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case also $x \amalg y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p \colon z \to x$ and $q \colon z \to y$. Denote $i \colon x \to z$ the morphism corresponding to $(1, 0)$. Denote $j \colon y \to z$ the morphism corresponding to $(0, 1)$. Thus we have a commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\
& z \arrow[ur, "p"] \arrow[dr, "q"] & \\
y \arrow[ur, "j"] \arrow[rr, "1"] & & y
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q \colon z \to z$ is the identity since it is a morphism which upon composing p gives p and upon composing q gives q. Suppose given morphisms $a \colon x \to w$ and $b \colon y \to w$. Then we can form the map $a \circ p + b \circ q \colon z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \amalg y$. $\square$

Definition 8.3. Given a pair of objects x, y in a preadditive categore $\mathcal{A}$, the **direct sum** $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma (8.2).

Lemma 8.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. A direct sum z of x and y is characterized by having morphisms $i\colon x \to z$, $j\colon y \to z$, $p\colon z \to x$, and $q\colon z \to y$ such that $p \circ i = 1_x$, $q \circ j = 1_y$, $p \circ j = 0$, $q \circ i = 0$, and $i \circ p + j \circ q = 1_z$. Clearly $F(x)$, $F(y)$, $F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums. $\square$

8.1.2 Additive Category

Definition 8.4. A category $\mathcal{A}$ is called **additive** if it is preadditive and finite products exist. In other words, it has a zero object and direct sums.

Definition 8.5. Let $\mathcal{A}$ be a preadditive category and let $f\colon x \to y$ be a morphism.

1. A **kernel** of f is an equalizer of $f\colon x \to y$ and $0\colon x \to y$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* kernel of f and denote it by $\iota\colon \ker f \to x$. Thus we have $f\iota = 0$ and if $\iota'\colon z \to x$ is an other morphism such that $f\iota' = 0$, then there exists a unique morphism $g\colon z \to \ker f$ such that $\iota' = \iota g$.

2. A **cokernel** of f is a coequalizer of $f\colon x \to y$ and $0\colon x \to y$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* cokernel of f and denote it by $\pi\colon y \to \operatorname{coker} f$. Thus we have $\pi f = 0$ and if $\pi'\colon y \to z$ is an other morphism such that $\pi' f = 0$, then there exists a unique morphism $g\colon \operatorname{coker} f \to z$ such that $\pi' = g\pi$.

3. If a kernel of f exists, then a **coimage** of f is a cokernel of the morphism $\ker f \to x$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* coimage of f and denote it by $x \to \operatorname{coim} f$.

4. If a cokernel of f exists, then a **image** of f is a kernel of the morphism $y \to \operatorname{coker} f$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* image of f and denote it by $\operatorname{im} f \to y$.

Lemma 8.4. *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma (8.2) be a direct sum in $\mathcal{C}$. Then $i\colon x \to x \oplus y$ is a kernel of $q\colon x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f\colon z' \to x \oplus y$ be a morphism such that $qf = 0$. We have to show taht there exists a unique morphism $g\colon z' \to x$ such that $f = ig$. SInce $ip + jq$ is the identity on $x \oplus y$ we see that

$$\begin{aligned} f &= (ip + jq)f \\ &= ipf \end{aligned}$$

and hence $g = pf$ works. Uniqueness holds because pi is the idenity on x. The proof of the second statement is dual. $\square$

Lemma 8.5. *Let $\mathcal{C}$ be a preadditive category. Let $f\colon x \to y$ be a morphism in $\mathcal{C}$.*

1. *If a kernel of f exists, then this kernel is a monomorphism.*

2. *If a cokernel of f exists, then this cokernel is an epimorphism.*

3. *If a kernel and coimage of f exist, then the coimage is an epimorphism.*

4. *If a cokernel and image of f exist, then the image is a monomorphism.*

Lemma 8.6. *Let $f\colon x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image, and coimage all exist. Then f can be factored uniquely as*

$$x \to \operatorname{coim} f \to \operatorname{im} f \to y.$$

Proof. There is a canonical morphism $\operatorname{coim} f \to y$ because $\ker f \to x \to y$ is zero. The composition $\operatorname{coim} f \to y \to \operatorname{coker} f$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \operatorname{coker} f$ which is zero. Hence $\operatorname{coim} f \to y$ factors uniquely through $\operatorname{im} f \to y$, which gives us the desired map. $\square$

8.2 Abelian Category

An abelian category is a category satisfying just enough axioms so the snake lemma holds.

Definition 8.6. A category $\mathcal{A}$ is called **abelian** if

1. it is additive;
2. all kernels and cokernels exist;
3. the natural map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism for all morphisms f in $\mathcal{A}$.

Definition 8.7. Let $f : x \to y$ be a morphism in an abelian category.

1. We say f is **injective** if $\ker f = 0$.
2. We say f is **surjective** if $\operatorname{coker} f = 0$.
3. If $x \to y$ is injective, then we say that x is a **subobject** of y and we use the notation $x \subseteq y$ to denote this. If $x \to y$ is surjective, then we say y is a **quotient** of x.

Lemma 8.7. *Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$. Then*

1. *f is injective if and only if f is a monomorphism.*
2. *f is surjective if and only if f is an epimorphism.*

Lemma 8.8. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

8.3 R-Linear Categories

Definition 8.8. An R-**linear category** $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where $x, y, z \in \operatorname{Ob}(\mathcal{A})$ composition law

$$\operatorname{Hom}_{\mathcal{A}}(y,z) \times \operatorname{Hom}_{\mathcal{A}}(x,y) \to \operatorname{Hom}_{\mathcal{A}}(x,z)$$

is R-bilinear. Thus composition determines an R-linear map

$$\operatorname{Hom}_{\mathcal{A}}(y,z) \otimes_R \operatorname{Hom}_{\mathcal{A}}(x,y) \to \operatorname{Hom}_{\mathcal{A}}(x,z)$$

of R-modules. A functor $F : \mathcal{A} \to \mathcal{B}$ of R-linear categories is called R-**linear** if the map

$$F : \operatorname{Hom}_{\mathcal{A}}(x,y) \to \operatorname{Hom}_{\mathcal{A}}(F(x), F(y))$$

is an R-linear map.

Example 8.1. The category Mod_R of all R-modules and R-linear maps is an R-linear category. Indeed, for each R-module M and N, we have an R-module $\operatorname{Hom}_R(M,N)$. Composition

$$\operatorname{Hom}_R(M_2, M_3) \times \operatorname{Hom}_R(M_1, M_2) \to \operatorname{Hom}_R(M_1, M_3),$$

defined by $(\varphi_2, \varphi_1) \mapsto \varphi_2 \circ \varphi_1$, is easily checked to be R-bilinear.

8.3.1 Additive functor from Graded Modules Induces Functor on Complexes

Proposition 8.1. *Let $\mathcal{F} : \operatorname{Grad}_R \to \operatorname{Grad}_R$ be an additive functor. Then $\mathcal{F}$ induces a functor*

$$\mathcal{F} : \operatorname{Comp}_R \to \operatorname{Comp}_R,$$

where an R-complex (A, d) gets mapped to the R-complex $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$.

Proof. Let (A, d) be an R-complex. We first need to show that $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$ is an R-complex. Indeed, $\mathcal{F}(A)$ is a graded R-module and $\mathcal{F}(\mathrm{d})$ is a graded homomorphism of degree -1. Moreover,

$$\begin{aligned} \mathcal{F}(\mathrm{d})\mathcal{F}(\mathrm{d}) &= \mathcal{F}(\mathrm{dd}) \\ &= \mathcal{F}(0) \\ &= 0. \end{aligned}$$

Thus $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$ is an R-complex.

Next, let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then

$$\begin{aligned}\mathcal{F}(\varphi)\mathcal{F}(\mathrm{d}) &= \mathcal{F}(\varphi\mathrm{d}) \\ &= \mathcal{F}(\mathrm{d}\varphi) \\ &= \mathcal{F}(\mathrm{d})\mathcal{F}(\varphi).\end{aligned}$$

Thus $\mathcal{F}(\varphi)$ is also a chain map. □

8.4 Functors Which Preserve Homotopy

8.4.1 Tensor Product

Proposition 8.2. *Let N be an R-module, let $\varphi\colon M \to M'$ and $\psi\colon M \to M'$ be two chain maps of R-complexes and suppose $\varphi \sim \psi$. Then $\varphi \otimes N \sim \psi \otimes N$.*

Proof. Choose a homotopy $h\colon M \to M'$ from φ to ψ. So

$$\varphi - \psi = \mathrm{d}_{M'}h + h\mathrm{d}_M.$$

We claim that $h \otimes N\colon M \otimes_R N \to M' \otimes_R N$ is a homotopy from $\varphi \otimes N$ to $\psi \otimes N$. Indeed, let $u \otimes v \in M \otimes_R N$ with $u \in M_i$ and $v \in N_j$. Then we have

$$\begin{aligned}(\mathrm{d}^{\otimes}_{(M',N)}(h\otimes N) + (h\otimes N)\mathrm{d}^{\otimes}_{(M,N)})(u\otimes v) &= \mathrm{d}^{\otimes}_{(M',N)}(h(u)\otimes v) + (h\otimes N)(\mathrm{d}_M(u)\otimes v + (-1)^i u\otimes \mathrm{d}_N(v)) \\ &= \mathrm{d}_{M'}h(u)\otimes v - (-1)^i h(u)\otimes \mathrm{d}_N(v) + h\mathrm{d}_M(u)\otimes v + (-1)^i h(u)\otimes \mathrm{d}_N(v)) \\ &= \mathrm{d}_{M'}h(u)\otimes v + h\mathrm{d}_M(u)\otimes v \\ &= (\mathrm{d}_{M'}h(u) + h\mathrm{d}_M(u))\otimes v \\ &= ((\mathrm{d}_{M'}h + h\mathrm{d}_M)(u))\otimes v \\ &= (\varphi-\psi)(u))\otimes v \\ &= \varphi(u)\otimes v - \psi(u)\otimes v \\ &= (\varphi\otimes N)(u\otimes v) - (\psi\otimes N)(u\otimes v) \\ &= (\varphi\otimes N - \psi\otimes N)(u\otimes v).\end{aligned}$$

It follows that

$$\varphi\otimes N - \psi\otimes N = \mathrm{d}^{\otimes}_{(M',N)}(h\otimes N) + (h\otimes N)\mathrm{d}^{\otimes}_{(M,N)}.$$

□

8.4.2 R-linear Functor Preserves Homotopy

Proposition 8.3. *Let $\varphi\colon A \to A'$ and $\psi\colon A \to A'$ be two chain maps of R-complexes which are homotopic to each other, and let $F\colon \mathrm{Comp}_R \to \mathrm{Comp}_R$ be an R-linear functor. Then $F(\varphi)$ is homotopic to $F(\psi)$.*

Proof. Choose a homotopy $h\colon A \to A'$ from φ to ψ. So

$$\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A.$$

We claim that $F(h)\colon F(A) \to F(A')$ is a homotopy from $F(\varphi)$ to $F(\psi)$. Indeed, let $a \in F(A)$ with $a \in F(A)_i$. Then we have

$$(\mathrm{d}_{F(A')}F(h) + F(h)\mathrm{d}_{F(A)})(a)$$

$$= (F(\varphi) - F(\psi))(a).$$

It follows that □

Proposition 8.4. *Let (A, d) and (A', d') be R-complexes and let $F\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$ be an R-linear functor. Suppose A is homotopically equivalent to A'. Then $(F(A), F(\mathrm{d}))$ is homotopically equivalent to $(F(A'), F(\mathrm{d}'))$.*

Proof. Choose chain maps $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A$ together with homotopies $h\colon A \to A'$ and $h'\colon A \to A'$ where

$$\varphi'\varphi - 1_A = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1_{A'} = \mathrm{d}'h' + h'\mathrm{d}'.$$

Then observe that

$$\begin{aligned} F(\varphi')F(\varphi) - 1_{F(A)} &= F(\varphi')F(\varphi) - F(1_A) \\ &= F(\varphi'\varphi - 1_A) \\ &= F(\mathrm{d}h + h\mathrm{d}) \\ &= F(d)F(h) + F(h)F(d). \end{aligned}$$

Thus $\mathcal{F}(\varphi')\mathcal{F}(\varphi) \sim 1_{\mathcal{F}(A)}$. A similar argument shows $\mathcal{F}(\varphi)\mathcal{F}(\varphi) \sim 1_{\mathcal{F}(A')}$. Therefore $\mathcal{F}(A)$ is homotopically equivalent to $\mathcal{F}(A')$. □

8.5 Epimorphisms and Monomorphisms

Definition 8.9. Let $\mathcal{C}$ be a category and let $f\colon x \to y$ be a morphism in $\mathcal{C}$.

1. We say f is an **epimorphism** if it is right-cancellative: $g_1 f = g_2 f$ implies $g_1 = g_2$ for all $g_1\colon y \to z$ and $g_2\colon y \to z$.
2. We say f is a **split epimorphism** if it has a right-sided inverse: there exists $g\colon y \to x$ such that $fg = 1_x$.
3. We say f is a **monomorphism** if it is left-cancellative: $fg_1 = fg_2$ implies $g_1 = g_2$ for all $g_1\colon w \to x$ and $g_2\colon w \to x$.
4. We say f is a **split monomorphism** if it has a left-sided inverse: there exists $g\colon y \to x$ such that $gf = 1_x$.
5. We say f is a **bimorphism** if it is both a monomorphism and an epimorphism.
6. We say f is an **isomorphism** if it is both a split monomomorphism and a split epimorphism.

8.5.1 Epimorphisms and Monomorphisms in $\mathbf{Comp}_R$

Proposition 8.5. *Let $\varphi\colon A \to B$ be a chain map. Then φ is an epimorphism if and only if φ is surjective*

8.6 Adjunctions

Definition 8.10. An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that for all objects x in $\mathcal{C}$ and y in $\mathcal{D}$ we have a bijection

$$\tau_{y,x}\colon \mathrm{Hom}_{\mathcal{C}}(Gy, x) \to \mathrm{Hom}_{\mathcal{D}}(y, Fx)$$

which is natural in x and y. We also say G is **left adjoint to** F and F is **right adjoint to** G.

Proposition 8.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ to left-adjoint to $G\colon \mathcal{D} \to \mathcal{C}$. Then F preserves colimits and G preserves limits.*

Proof. Let us show that F preserves colimits. Let (□

Proposition 8.7. *Let M be a graded R-module. The functor $- \otimes_R M\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$ is left adjoint to the functor $\mathrm{Hom}_R(M, -)\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$. In particular, $- \otimes_R M$ preserves direct limits and $\mathrm{Hom}^\star_R(M, -)$ preserves inverse limits.*

Proof. Let us show that $- \otimes_R M$ being left adjoint to $\mathrm{Hom}^\star_R(M, -)$ implies $- \otimes_R M$ preserves direct limits. Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps indexed over a preordered set $(\Lambda, \leq)$. Since $- \otimes_R M$ is a covariant functor, $(M_\lambda \otimes_R M, \varphi_{\lambda\mu} \otimes 1_M)$ is a direct system of graded R-modules and graded R-linear maps indexed over a preordered set $(\Lambda, \leq)$. Furthermore, □