An Interesting Complex and its Homology

Michael Nelson

November 30, 2018

Contents

1 Notations and Preliminary Material

1.1 Homological Algebra

Throughout this article, R be a ring and let K be a field of characteristic 2. Recall that a **chain complex** $A = (A_\bullet, d_\bullet)$ **over** R is a sequence of R-modules A_i and morphisms $d_i : A_i \to A_{i-1}$

$$A := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots \tag{1}$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\mathrm{Ker}(d_i) \supset \mathrm{Im}(d_{i+1})$. With this in mind, we define the i**th homology of the chain complex** to be

$$H_i(A) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

A **chain map** between two chain complexes $(A_\bullet, d_\bullet)$ and $(A'_\bullet, d'_\bullet)$ over R is a sequence $\varphi_\bullet$ of R-module homomorphisms $\varphi_i : A_i \to B_i$ such that $d_i\varphi_{i-1} = \varphi_i d'_{i-1}$ for all i. It then follows that a chain map gives rise to map an induced map on homology $H_i(A) \to H_i(A')$.

The dual notion to a chain complex is a **cochain complex** $A = (A^\bullet, d^\bullet)$ over R. It is just a chain complex, except we label things differently. Namely, we replace subscripts with superscripts. We will be studying objects which have compatible chain and cochain complex structures.

To simplify notation, we think of R as a trivially graded ring, that is, the degree equals 0 part is R and all the other homogeneous components are 0. We think of the complex (1) as a graded R-module A (where the degree i homogoneous component is A_i) together with an endomorphism d of degree -1 such that $d^2 = 0$. We write $A[j]$ for the graded module obtained from A by the rule $A[j]_i = A_{i+j}$.

1.2 Gröbner Basis

Throughout this article, let $S = K[x_1, \dots, x_n]$, I be a homogeneous ideal in the polynomial ring $K[x_1, \dots, x_n]$, and let $G = \{g_1, g_2 \dots, g_r\}$ be the reduced gröbner basis for I with respect to a fixed monomial ordering (see the book "Ideals, Varieties, and Algorithms: An Introduciton to Computation Algebraic Geometry and Commutative Algebra" for details). Recall that $f \in K[x_1, \dots, x_n]$ can be written in the form $f = g + r$, where $g \in I$ and no term of r is divisible by any element of $\mathrm{LT}(I)$, and, moreover, g and r are uniquely determined. We use the notation $f^G := r$ and call this the **normal form of** f **with respect to** I (or simply the **normal form of** f if the there is no confusion of the ideal I). It follows from uniqueness of f^G and $f - f^G$ that taking the normal form of a polynomial is a K-linear map:

$$\alpha f_1^G + \alpha_1 f_2^G = (\alpha_1 f_1 + \alpha_2 f_2)^G \qquad \text{for all } \alpha_1, \alpha_2 \in K \text{ and } f_1, f_2 \in S. \tag{2}$$

Recall that S is a graded K-algebra, where the homogeneous component S_i is the K-vector space of all homogeneous polynomials $f \in S$ of degree i. Since I is a homogeneous ideal, the ring S/I also inherits the structure of a graded algebra over K, where the homogeneous component $(S/I)_i$ is $(S/I)_i := S_i/(I \cap S_i)$. In fact, S/I is isomorphic as a graded K-vector space to $S_I := \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle)$, where S_I has homogeneous components $(S_I)_i = \mathrm{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle$ and $\deg(x^\alpha) = i\rangle$. The isomorphism is given by mapping $\overline{f} \in (S/I)_i$ to $f^G \in (S_I)_i$, where K-linearity follows from (2). Call this map φ. The inverse of this map is given by mapping $f \in (S_I)_i$ to $\overline{f} \in (S/I)_i$. Composing this map with the natural inclusion map $(S_I)_i \subset S_i$, we obtain an injective map $(S/I)_i \to S_i$, call this map φ.

Similarly, I has a graded K-module structure, where the homogeneous component I_i is $I_i := I \cap S_i$, and the map from S_i to I_i sending f to $f - f^G$ is K-linear (since we are working over a characteristic 2 field, we can just write $f + f^G$). Call this map ψ. Since $f^G = 0$ if and only if $f \in I$, the natural inclusion map $I_i \subset S_i$ splits ψ. Combining everything together, we obtain the following short exact sequences of K-vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (S/I)_i & \xrightarrow{\varphi} & S_i & \xrightarrow{\psi} & I_i & \longrightarrow & 0 \\ & & \overline{f} & \longmapsto & f^G & & & & \\ & & & & f & \longmapsto & f + f^G & & \end{array} \tag{3}$$

Example 1.1. Consider $S = K[x,y]$ and $I = \langle xy + y^2, x^3 \rangle$. We first use Singular to compute a Gröbner basis of G of I with respect to graded reverse lex order. We obtain $G = \{g_1, g_2, g_3\}$. where $g_1 = xy + y^2$, $g_2 = x^3$, and $g_3 = y^4$. Then

$$\begin{aligned}
&I_0 = 0 && (S/I)_0 = K \cdot \overline{1} \\
&I_1 = 0 && (S/I)_1 = K\overline{x} + K\overline{y} \\
&I_2 = Kg_1 && (S/I)_2 = K\overline{x}^2 + K\overline{y}^2 \\
&I_3 = Kxg_1 + Kyg_1 + Kg_2 && (S/I)_3 = K\overline{y}^3 \\
&I_4 = S_4 && (S/I)_4 = 0 \\
&\quad\vdots && \qquad\vdots
\end{aligned}$$

2 $\mathbf{A}(I)$, $\mathbf{A}(S)$, and $\mathbf{A}(S/I)$

2.1 Construction of $\mathbf{A}_\bullet(S)$

Let $d_i : S_i \to S_{i-1}$ be the map given by $d_i := \sum_{j=1}^n \partial_{x_j}$. This map is clearly K-linear, and since we are working over a field of characteristic 2, we have $d^2 = 0$: Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in S_i, then

$$\begin{aligned}
d^2\left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) &= \left(\sum_{k=1}^n \partial_{x_k}\right)^2 \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) \\
&= \left(\sum_{k=1}^n \partial_{x_k}^2\right) \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) \\
&= \sum_{k=1}^{\infty} \alpha_k(\alpha_k - 1) x_1^{\alpha_k - 2} \\
&= 0.
\end{aligned}$$

Thus, d gives the graded K-algebra S the structure of a chain complex over K, which we call $\mathbf{A}_\bullet(S) := (S_\bullet, d_\bullet)$.

2.2 Construction of $\mathbf{A}_\bullet(S/I)$

Since we are working over a field of characteristic 2. We describe d in a simpler manner: Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree i in S. We can decompose m as $m = uv$, where

$$u = \prod_{\substack{1 \le j \le n \\ \alpha_j \text{ is odd}}} x_j^{\alpha_j} \qquad \text{and} \qquad v = \prod_{\substack{1 \le \ell \le n \\ \alpha_\ell \text{ is even}}} x_\ell^{\alpha_\ell}.$$

We denote $[m]_o = \{1 \le j \le n \mid \alpha_j \text{ is odd}\}$ and $[m]_e = \{1 \le \ell \le n \mid \alpha_\ell \text{ is even}\}$. From this decomposition and the definition of d_i, we can describe d_i another way:

$$d_i(m) = \sum_{x_j | u} x_j^{-1} m.$$

This makes it clear, for example, that d_i restricts to a map $d_i : (S_I)_i \to (S_I)_{i-1}$: If m is not in $\mathrm{LT}(I)$, then every term $x_\lambda^{-1} m$ of $d(m)$ is not in $\mathrm{LT}(I)$ either. Thus, using the isomorphism $(S/I)_i \to (S_I)_i$, we can construct a map $\underline{d}_i : (S/I)_i \to (S/I)_{i-1}$, given by $\underline{d}_i(\overline{f}) = \overline{d_i(f^G)}$ for all $\overline{f} \in (S/I)_i$. It's easy to see now that the map $\underline{d}$ gives the graded K-algebra S/I the structure of a chain complex over K, which we call $\mathbf{A}_\bullet(S/I) := ((S/I)_\bullet, \underline{d}_\bullet)$.

2.3 Construction of $\mathbf{A}_\bullet(I)$

Our final construction involves the graded K-module I. Using the fact that the inclusion map $I_i \subset S_i$ splits ψ, we can construct a map $\overline{d}_i : I_i \to I_{i-1}$, where $\overline{d}_i(f) = d_i(f) + d_i(f)^G$ for all $f \in I_i$. The map $\overline{d}$ gives the graded K-module I the structure of a chain complex over K, which we call $\mathbf{A}_\bullet(I) := ((I)_\bullet, \overline{d}_\bullet)$.

2.4 Summary and Example

Summarizing everything up. We have the following chain complexes over K:

Chain Complex	Homogeneous Components	Differential
$\mathbf{A}_\bullet(S) := (S_\bullet, d_\bullet)$	$\mathbf{A}_\bullet(S)_i := S_i$	$d_i : S_i \to S_{i-1}$
$\mathbf{A}_\bullet(I) := ((I)_\bullet, \overline{d}_\bullet)$	$\mathbf{A}_\bullet(I)_i := I_i$	$\overline{d}_i : I_i \to I_{i-1}$
$\mathbf{A}_\bullet(S/I) := ((S/I)_\bullet, \underline{d}_\bullet)$	$\mathbf{A}_\bullet(S/I)_i := (S/I)_i$	$\underline{d}_i : (S/I)_i \to (S/I)_{i-1}$

We now bring these ideas together in the following example.

Example 2.1. Consider the case where $S = \mathbb{F}_2[x,y,z]$ and $I = \langle xy, x^3, y^2z, xz^3, z^4, y^5\rangle$, where $G = \{xy, x^3, y^2z, xz^3, z^4, y^5\}$ is the minimal basis for I. Let's write down the first fiew graded pieces of $\mathbf{A}_\bullet(S/I)$:

$$\begin{aligned}
\mathbf{A}_\bullet(S/I)_0 &= \mathbb{F}_2 \\
\mathbf{A}_\bullet(S/I)_1 &= \mathbb{F}_2\overline{x} + \mathbb{F}_2\overline{y} + \mathbb{F}_2\overline{z} \\
\mathbf{A}_\bullet(S/I)_2 &= \mathbb{F}_2\overline{x}^2 + \mathbb{F}_2\overline{xz} + \mathbb{F}_2\overline{y}^2 + \mathbb{F}_2\overline{yz} + \mathbb{F}_2\overline{z}^2 \\
\mathbf{A}_\bullet(S/I)_3 &= \mathbb{F}_2\overline{x}^2\overline{z} + \mathbb{F}_2\overline{xz}^2 + \mathbb{F}_2\overline{y}^3 + \mathbb{F}_2\overline{yz}^2 + \mathbb{F}_2\overline{z}^3 \\
\mathbf{A}_\bullet(S/I)_4 &= \mathbb{F}_2\overline{x}^2\overline{z}^2 + \mathbb{F}_2\overline{y}^4 + \mathbb{F}_2\overline{yz}^3 \\
\mathbf{A}_\bullet(S/I)_5 &= 0 \\
&\vdots
\end{aligned}$$

The homology of this chain complex is not trivial. For instance, we will see that $H_1(\mathbf{A}_\bullet(S\backslash I)) = \mathbb{F}_2\overline{d(xy)}$. Now let's write down the first few graded pieces of $\mathbf{A}_\bullet(I)$:

$$\begin{aligned}
\mathbf{A}_\bullet(I)_1 &= 0 \\
\mathbf{A}_\bullet(I)_2 &= \mathbb{F}_2 xy \\
\mathbf{A}_\bullet(I)_3 &= \mathbb{F}_2x^3 + \mathbb{F}_2x^2y + \mathbb{F}_2xy^2 + \mathbb{F}_2xyz + \mathbb{F}_2y^2z \\
&\vdots
\end{aligned}$$

3 Differential Graded Algebras

3.1 Differential Graded Algebra Structure on $\mathbf{A}_\bullet(S)$

It turns out that $\mathbf{A}_\bullet(S)$ is more than just a chain complex over K, in fact it also has the structure of a differential graded algebra over $\mathbb{F}_2$. Before we discuss this, let's define what a differential graded algebra over a ring is.

Definition 3.1. A **differential graded algebra over** R is a chain complex $A = (A_\bullet, d_\bullet)$ over R together with R-bilinear maps $A_i \times A_j \to A_{i+j}$, denoted $(a,b) \mapsto ab$, such that the **Leibniz law** holds:

$$d_{i+j}(ab) = d_i(a)b + (-1)^i a d_j(b). \tag{4}$$

for all $a, b \in A$.

Remark. To ease notation, we usually drop the subscripts in (3.1) as long as everything is clear. Combining this with the fact that we are working over a field of characteristic 2, we can simplify (3.1) to

$$d(ab) = d(a)b + ad(b). \tag{5}$$

Since S is already a graded K-algebra, we already have bilinear maps $S_i \times S_j \to S_{i+j}$. Also, the Leibniz law () follows since d is defined in terms of partial derivatives.

3.2 Almost Differential Graded Algebra Structure on $\mathbf{A}_\bullet(S/I)$.

We can try making $\mathbf{A}_\bullet(S/I)$ into a differential graded algebra over K by using the multiplication maps from the K-algebra S/I as the bilinear maps $(S/I)_i \times (S/I)_j \to (S/I)_{i+j}$, but this does not work. The reason is because there are pairs $(\overline{f}, \overline{g}) \in (S/I) \times (S/I)$, which fail the Leibniz law:

$$\underline{d}(\overline{f}\overline{g}) \neq \underline{d}(\overline{f})\overline{g} + \overline{f}\underline{d}(\overline{g}).$$

For instance, consider $I = \langle x^5 \rangle$ in $\mathbb{F}_2[x]$. Then $\underline{d}(\overline{x} \cdot \overline{x}^4) = \underline{d}(\overline{0}) = \overline{0}$, but $\underline{d}(\overline{x})\overline{x}^4 + \overline{x}\underline{d}(\overline{x}^4) = \overline{x}^4$.

3.3 Measuring The Failure of $\mathbf{A}_\bullet(S/I)$ to be a Differential Graded Algebra over K

Our task now is to measure the failure of $\mathbf{A}_\bullet(S/I)$ to be a differential graded algebra. To do this, we introduce the following notation. Let $\underline{d}(-,-) : (S/I)_i \times (S/I)_j \to (S/I)_{i+j}$ be the map given by

$$\underline{d}(\overline{f},\overline{g}) = \underline{d}(\overline{f}\overline{g}) + \underline{d}(f)\overline{g} + \overline{f}\underline{d}(\overline{g}).$$

It is easy to see that bilinearity of $\underline{d}(-,-)$ follows from bilinearity of multiplication and linearity of $\underline{d}$. It is also easy to see that $\underline{d}(\overline{f},\overline{g}) = 0$ if and only if the pair $(\overline{f},\overline{g}) \in (S/I)_i \times (S/I)_j$ satisfies Leibniz law.

From the bilinearity of $\underline{d}(-,-)$, we see that $\underline{d}(-,-)$ is completely determined by where is maps pairs of monomials $(\overline{m}_1,\overline{m}_2) \in (S/I)_i \times (S/I)_j$.

Proposition 3.1. *Suppose $\overline{m}_1\overline{m}_2 \neq 0$, then $\underline{d}(\overline{m}_1,\overline{m}_2) = \overline{0}$.*

Proof. Assume m_1 and m_2 are already in reduced form, that is, $m_1^G = m_1$ and $m_2^G = m_2$. Then $m_1m_2 \notin I$ and $m_1m_2 = m_1^G m_2^G = (m_1m_2)^G$. Therefore

$$\begin{aligned}
\underline{d}(\overline{m}_1\overline{m}_2) &= \overline{d(m_1m_2)} \\
&= \overline{d(m_1)m_2 + m_1d(m_2)} \\
&= \overline{d(m_1)}\overline{m}_2 + \overline{m}_1\overline{d(m_2)} \\
&= \underline{d}(\overline{m}_1)\overline{m}_2 + \overline{m}_1\underline{d}(\overline{m}_2).
\end{aligned}$$

□

Remark. Actually, $\underline{d}(-,-)$ is completely determined by where it maps the pairs $(\overline{x}_\lambda,\overline{m}) \in (S/I)_1 \times (S/I)_i$, where m is a monomial. To see this, suppose $x_\lambda \mid m_1$ and assume $\underline{d}(\overline{x}_\lambda,\overline{m}) = 0$ for every monomial m. Then setting $\overline{m}_1' = x_\lambda^{-1}\overline{m}_1$, and using an inductive argument, we obtain

$$\begin{aligned}
\underline{d}(\overline{m}_1\overline{m}_2) &= \underline{d}(x_\lambda\overline{m}_1'\overline{m}_2) \\
&= \overline{m}_1'\overline{m}_2 + \overline{x}_\lambda\underline{d}(\overline{m}_1'\overline{m}_2) \\
&= \overline{m}_1'\overline{m}_2 + \overline{x}_\lambda d(\overline{m}_1')\overline{m}_2 + \overline{x}_\lambda\overline{m}_1'\underline{d}(\overline{m}_2) \\
&= \underline{d}(\overline{x}_\lambda\overline{m}_1')\overline{m}_2 + \overline{x}_\lambda\overline{m}_1'\underline{d}(\overline{m}_2) \\
&= \underline{d}(\overline{m}_1)\overline{m}_2 + \overline{m}_1\underline{d}(\overline{m}_2)
\end{aligned}$$

Proposition 3.2. *Let $(x_\lambda, m) \in S_1 \times S_i$ be a pair where x_λ is not in I, m is a monomial not in I, and $x_\lambda m$ is in I. Then*

$$\underline{d}\,(\overline{x}_\lambda,\overline{m}) = \overline{d(x_\lambda m)}.$$

Proof. We may assume $x_\lambda = x_\lambda^G$ and $m = m^G$, so that $\underline{d}(\overline{x}_\lambda) = \overline{d(x_\lambda)}$ and $\underline{d}(\overline{m}) = \overline{d(m)}$. Then

$$\begin{aligned}
\underline{d}\,(\overline{x}_\lambda,\overline{m}) &= \underline{d}(\overline{x_\lambda m}) + \underline{d}(\overline{x}_\lambda)\overline{m} + \overline{x}_\lambda\underline{d}\,(\overline{m}) \\
&= \underline{d}(\overline{0}) + \underline{d}(\overline{x}_\lambda)\overline{m} + \overline{x}_\lambda\underline{d}\,(\overline{m}) \\
&= \underline{d}(\overline{x}_\lambda)\overline{m} + \overline{x}_\lambda\underline{d}\,(\overline{m}) \\
&= \overline{d(x_\lambda)}\overline{m} + \overline{x}_\lambda\overline{d(m)} \\
&= \overline{d(x_\lambda m)}
\end{aligned}$$

□

4 Calculating the Homologies $H_i(\mathbf{A}_\bullet(I))$, $H_i(\mathbf{A}_\bullet(S))$, and $H_i(\mathbf{A}_\bullet(S/I))$

We first start with what is essentially a consequence of $\mathbf{A}_\bullet(S)$ being a differential graded algebra.

Proposition 4.1. *$H_i(\mathbf{A}(S)) = 0$ for all $i \geq 0$.*

Proof. Let f be a homogeneous polynomial of degree i such that $d(f) = 0$. Then for any $x_\lambda \in S_1$, we have

$$d(x_\lambda f) = d(x_\lambda)f + x_\lambda d(f) = f.$$

Therefore, $\mathrm{Ker}(d) = \mathrm{Im}(d)$, which proves the claim. □

Proposition 4.2. *The differential d induces isomorphisms $H_i(\mathbf{A}_\bullet(I)) \cong H_{i-1}(\mathbf{A}_\bullet(S/I))$ for all $i > 0$.*

Proof. The short exact sequence (3) gives rise to a long exact sequence of chain complexes over K:

$$0 \longrightarrow \mathbf{A}(S/I) \longrightarrow \mathbf{A}(S) \longrightarrow \mathbf{A}(I) \longrightarrow 0.$$

From this short exact sequence of chain complexes over K, we obtain, for all $i > 0$, the exact sequences

$$0 = H_i(\mathbf{A}(S)) \longrightarrow H_i(\mathbf{A}(I)) \xrightarrow{d} H_{i-1}(\mathbf{A}(S/I)) \longrightarrow H_{i-1}(\mathbf{A}(S)) = 0,$$

where d is obtained by working out the details of the connecting map. □

Proposition 4.3. *Let m be a monomial such that $x_\mu m \in I$ for all $\mu \in [m]_e$. Fix $\mu_0 \in [m]_e$. If $x_\lambda^{-1} x_{\mu_0} m \notin I$ for every $\lambda \in [x_{\mu_0} m]_o$, then $\overline{d(x_{\mu_0} m)}$ represents a nontrivial element in $H(\mathbf{A}_\bullet(S/I))$.*

Proof. Saying $x_\lambda^{-1} x_{\mu_0} m \notin I$ for every $\lambda \in [x_{\mu_0} m]_o$, is equivalent to saying $d(x_{\mu_0} m)^G = d(x_{\mu_0} m)$. Thus,

$$\begin{aligned} \underline{d}(\overline{d(x_{\mu_0} m)}) &= d(d(x_{\mu_0} m)^G) \\ &= d(d(x_{\mu_0} m)) \\ &= 0 \end{aligned}$$

implies $\overline{d(x_{\mu_0} m)}$ represents an element in $H(\mathbf{A}_\bullet(S/I))$. To see that this element is nontrivial, note that the condition $x_\mu m \in I$ for all $\mu \in [m]_e$ implies $\overline{m} \notin \mathrm{Im}(\underline{d})$. Since $\overline{m}$ is a term in $\overline{d(x_{\mu_0} m)}$, this implies $\overline{d(x_{\mu_0} m)} \notin \mathrm{Im}(d)$ either.

□

5 Duality

For a K-vector space V, let $V^\star := \mathrm{Hom}_K(V, K)$. Then $(S/I)^\star := \mathrm{Hom}_K(S/I, K)$, $S^\star := \mathrm{Hom}_K(S, K)$, and $I^\star := \mathrm{Hom}_K(I, K)$ are all graded K-modules, where the homogeneous components are $(S/I)_i^\star$, $S_i^\star$, and $I_i^\star$ respectively. To get an isomorphism from S_i to $S_i^\star$, we map the monomial $x^\alpha \in S_i$ to the element $\varphi_{x^\alpha} \in S_i^\star$, where

$$\varphi_{x^\alpha}(x^\beta) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

for all monomials $x^\beta \in S_i$. The isomorphisms $(S/I)_i^\star \cong (S/I)_i$ and $I_i^\star \cong I_i$ are then induced from the $S_i^\star \cong S_i$ given above.

Recall that $\mathrm{Hom}_K(-, K)$ is a contravariant functor, and so we get a map $d^\star : S_{i-1}^\star \to S_i^\star$ which comes from applying $\mathrm{Hom}_K(-, K)$ to $d : S_i \to S_{i-1}$. Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial of degree i in S. Then it easy to see that

$$d^\star(\varphi_m) = \sum_{\mu \in [m]_e} \varphi_{x^\mu m}.$$

This fits nicely with how d acts on m:

$$d(m) = \sum_{\lambda \in [m]_o} x_\lambda^{-1} m.$$

As long as there is no potential for confusion, we shall identify m with φ_m and relabel $d^\star$ as δ and call it the **codifferential**. Then we have two maps

$$\delta(m) = \sum_{\mu \in [m]_e} x_\mu m \qquad \text{and} \qquad d(m) = \sum_{\lambda \in [m]_o} x_\lambda^{-1} m$$

which gives S the structure of a cochain and chain complex over K. The cochain complex corresponding to δ will be denoted $\mathbf{A}^\bullet(S)$, and as before, the chain complex corresponding to d is denoted $\mathbf{A}_\bullet(S)$. Similarly, we denote $\mathbf{A}^\bullet(S/I)$ and $\mathbf{A}^\bullet(I)$ denote the cochain complex over K induced by δ.

Theorem 5.1. *We have isomorphisms*

$$H(\mathbf{A}^\bullet(S/I)) \cong H(\mathbf{A}^\bullet(I)) \cong H(\mathbf{A}_\bullet(S/I)) \cong H(\mathbf{A}_\bullet(I)),$$

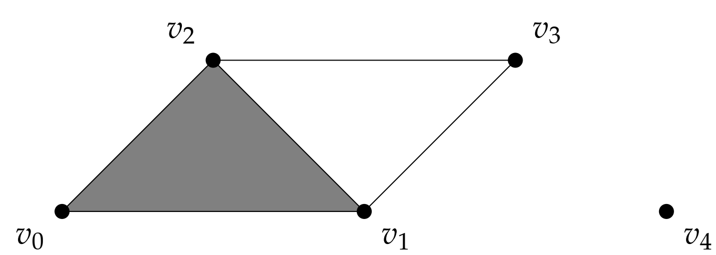
where the elements are given by pairs (x_μ, m) where m is a monomial not in I and $\mu \in [m]_e$, with

$$\begin{aligned} [x_\mu m] &\in H(\mathbf{A}_\bullet(I)) \\ [d(x_\mu m)] &\in H(\mathbf{A}_\bullet(S/I)) \\ [\delta(m)] &\in H(\mathbf{A}^\bullet(I)) \\ [m] &\in H(\mathbf{A}^\bullet(S/I)) \end{aligned}$$

6 Squarefree Monomials and Stanley-Reisner Ring

A **squarefree monomial** in S is a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in S such that for all $1 \leq j \leq n$, either $\alpha_j = 0$ or $\alpha_j = 1$. Thus, a squarefree monomial in S looks like $x_{j_1} \cdots x_{j_k}$ where $1 \leq j_1 < \cdots < j_k \leq n$. There is a standard way of assigning a simplex to a squarefree monomial. Given a square free monomial $x_{j_1} \cdots x_{j_k}$, we form the $(k-1)$-simplex whose vertices are labeled x_{j_s} where $1 \leq s \leq k$. An edges whose boundary vertices are x_{j_s} and x_{j_t} is labeled $x_{j_s} x_{j_t}$ here $1 \leq s < t \leq k$, and so on. Under this correspondence, the usual boundary map defined on simplices matches the differential d acting on monomials.

Example 6.1. Consider $S = K[v_0, v_1, v_2, v_3, v_4]$ and $I = \langle v_0 v_3, v_0 v_4, v_1 v_2 v_3, v_1 v_4, v_2 v_4, v_3 v_4 \rangle$. Then S/I is the **Stanley-Reisner ring** of the simplex Δ given below:



One can show that

$$\begin{aligned} H_2(\mathbf{A}(S/I)) &= \left[\overline{d(v_1 v_2 v_3)}\right] K \\ H_1(\mathbf{A}(S/I)) &= \left[\overline{d(v_1 v_4)}\right] K + \left[\overline{d(v_2 v_4)}\right] K. \end{aligned}$$

In fact, one can show that $H_{i+1}(\mathbf{A}(S/I)) \cong \widetilde{H}_i(\Delta; K)$, where $\widetilde{H}_i(\Delta; K)$ is the i**th reduced simplicial homology of** Δ **over** K.

7 3-Chain Complexes

Recall that a chain complex $\mathcal{A} = (A_\bullet, d_\bullet)$ is a sequence of R-modules A_i and morphisms $d_i : A_i \to A_{i-1}$

$$\mathcal{A} := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\mathrm{Ker}(d_i) \supset \mathrm{Im}(d_{i+1})$. With this in mind, we define the ith homology of the chain complex $\mathcal{A}$ to be $H_i(\mathcal{A}) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$. Chain complexes have been studied quite intensively and are very well understood. The next definition provides a generalization of a chain complex:

Definition 7.1. A 3**-chain complex** $\mathcal{A} := (A_\bullet, d_\bullet)$ is a sequence of R-modules A_i and morphisms $d_i : A_i \to A_{i-1}$

$$\mathcal{A} := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

such that $d_{i-1} \circ d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$.

Since $d_i \circ d_{i+1} = 0$ implies $d_{i-1} \circ d_i \circ d_{i+1}$ for all $i \in \mathbb{Z}$, every chain complex is a 3-chain complex. On the other hand, there are interesting examples of 3-chain complexes which are not chain complexes, as illustrated in the next proposition.

Proposition 7.1. *Let I be a monomial ideal in the polynomial ring $\mathbb{F}_3[x_1, \ldots, x_n]$. For $i \geq 0$, let A_i denote the $\mathbb{F}_3$-vector space generated by monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = i$ and $x^\alpha \notin I$. For $i < 0$, let $A_i = 0$. Let d denote the $\mathbb{F}_3$-linear operator $d := \sum_{k=1}^n \partial_{x_k}$. Then*

$$\mathcal{A} := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

is a 3-chain complex of $\mathbb{F}_3$-vector spaces.

Proof. We just need to check that $d^3\left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) = 0$ for any monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in $\mathbb{F}_3[x_1, \ldots, x_n]$. This follows since we are working mod 3:

$$\begin{aligned} d^3\left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) &= \left(\sum_{k=1}^n \partial_{x_k}\right)^3 \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) \\ &= \left(\sum_{k=1}^n \partial_{x_k}^3\right) \left(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) \\ &= \sum_{k=1}^{\infty} \alpha_k(\alpha_k - 1)(\alpha_k - 2) x_1^{\alpha_k - 2} \\ &= 0. \end{aligned}$$

□

Turning a 3-Chain Complex into a Chain Complex

In this section, we desribe how we can obtain a chain complex from a 3-complex. Let $\mathcal{A} := (A_\bullet, d_\bullet)$ be a 3-chain complex.

$$\mathcal{A} := \cdots \longrightarrow A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \longrightarrow \cdots$$

We **collapse** $\mathcal{A}$ into a chain complex $\mathcal{A}_\star$ as follows:

$$\mathcal{A}_\star := \cdots A_5 \xrightarrow{d_4 d_5} A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_1 d_2} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-2} d_{-1}} A_{-3} \xrightarrow{d_{-3}} A_{-4} \longrightarrow \cdots .$$

More formally, $\mathcal{A}_\star = (\widetilde{A}_\bullet, \widetilde{d}_\bullet)$, where

$$\widetilde{A}_i = A_{\frac{6i+1+(-1)^{i+1}}{4}} \qquad \widetilde{d}_i = \begin{cases} d_{\frac{6i+1+(-1)^{i+1}}{4}} & \text{if } |i| \text{ is even} \\ d_{\frac{6i-3+(-1)^{i+1}}{4}} d_{\frac{6i+1+(-1)^{i+1}}{4}} & \text{if } |i| \text{ is odd} \end{cases}$$

For all $j \in \mathbb{Z}$, let $\mathcal{A}[j] = (A[j]_\bullet, d[i]_\bullet)$ be the sequence of R-modules $A[j]_i$ are morphisms $d[j]_{i+j}$ where $A[j]_i = A_{i+j}$ and $d[j] = d_{i+j}$. It is straightforward to verify that $\mathcal{A}[j]$ is also a 3-Chain Complex. We can also define $\mathcal{A}[1]_\star$ and $\mathcal{A}[2]_\star$ in a similar way as $\mathcal{A}_\star$:

$$\mathcal{A}_\star := \cdots A_5 \xrightarrow{d_4 d_5} A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_1 d_2} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-2} d_{-1}} A_{-3} \xrightarrow{d_{-3}} A_{-4} \longrightarrow \cdots$$

$$\mathcal{A}[1]_\star := \cdots A_6 \xrightarrow{d_5 d_6} A_4 \xrightarrow{d_4} A_3 \xrightarrow{d_2 d_3} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_{-1} d_0} A_{-2} \xrightarrow{d_{-2}} A_{-3} \longrightarrow \cdots$$

$$\mathcal{A}[2]_\star := \cdots A_7 \xrightarrow{d_6 d_5} A_5 \xrightarrow{d_5} A_4 \xrightarrow{d_3 d_4} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_0 d_1} A_{-1} \xrightarrow{d_{-1}} A_{-2} \longrightarrow \cdots$$

Theorem 7.1. *With the notation above, there is a long exact sequence in homology of the form*

$$\begin{tikzcd}
H_{i-1}(\mathcal{A}_\star) \arrow[r] & \cdots & \\
H_i(\mathcal{A}_\star) \arrow[r, "d_{i+1}"] & H_{i-1}(\mathcal{A}[1]_\star) \arrow[r] & H_{i-2}(\mathcal{A}[2]_\star) \arrow[ull, "d_i"'] \\
H_{i+1}(\mathcal{A}_\star) \arrow[r] & H_i(\mathcal{A}[1]_\star) \arrow[r, "d_{i+2}"] & H_{i-1}(\mathcal{A}[2]_\star) \arrow[ull] \\
H_{i+2}(\mathcal{A}_\star) \arrow[r, "d_{i+4}"] & H_{i+1}(\mathcal{A}[1]_\star) \arrow[r] & H_i(\mathcal{A}[2]_\star) \arrow[ull, "d_{i+3}"'] \\
 & \cdots \arrow[r] & H_{i+1}(\mathcal{A}[2]_\star) \arrow[ull]
\end{tikzcd}$$

Proof. Let $K_i = \mathrm{Ker}(d_i)/\mathrm{Ker}(d_i) \cap \mathrm{Im}(d_{i+1})$ for all $i \in \mathbb{Z}$. For each $i \in 3\mathbb{Z}$, we have the following short exact sequences

$$0 \longrightarrow K_{i+1} \longrightarrow H_i(\mathcal{A}_\star) \xrightarrow{d_{i+1}} H_{i-1}(\mathcal{A}[1]_\star) \longrightarrow K_i \longrightarrow 0$$

$$0 \longrightarrow K_{i+2} \longrightarrow H_i(\mathcal{A}[1]_\star) \xrightarrow{d_{i+2}} H_{i-1}(\mathcal{A}[2]_\star) \longrightarrow K_{i+1} \longrightarrow 0$$

$$0 \longrightarrow K_{i+3} \longrightarrow H_i(\mathcal{A}[2]_\star) \xrightarrow{d_{i+3}} H_{i-1}(\mathcal{A}_\star) \longrightarrow K_{i+2} \longrightarrow 0$$

Connecting these together gives us our desired result. $\square$