Free Resolutions Homework 1

Michael Nelson

Troughout this homework assignment, let k be a field.

Exercise 1

Proposition 0.1. *Let*

$$0 \longrightarrow k^{\beta_d} \longrightarrow \cdots \longrightarrow k^{\beta_i} \xrightarrow{\partial_i} k^{\beta_{i-1}} \longrightarrow \cdots \longrightarrow k^{\beta_1} \xrightarrow{\partial_1} k^{\beta_0} \longrightarrow 0 \tag{1}$$

be an exact sequence of k-vector spaces. Then

$$\sum_{i=0}^{d}(-1)^i \beta_i = 0.$$

Proof. Let $K_i := \mathrm{Ker}(\partial_i)$ for all $0 \leq i \leq d$ and let $K_{-1} = 0$. Then for each $0 \leq i \leq d$, exactness at k^{β_i} in (1) implies exactness of

$$0 \longrightarrow K_i \hookrightarrow k^{\beta_i} \xrightarrow{\partial_i} K_{i-1} \longrightarrow 0.$$

Since the dimension function is additive on short exact sequences, we have $\beta_i = \dim(K_i) + \dim(K_{i-1})$. Therefore we have a telescoping series

$$\begin{aligned}
\sum_{i=0}^{d}(-1)^i \beta_i &= \sum_{i=0}^{d}(-1)^i(\dim(K_i) + \dim(K_{i-1})) \\
&= (-1)^d \dim(K_d) + \dim(K_{-1}) \\
&= 0.
\end{aligned}$$

□

Exercise 2

Proposition 0.2. *Let* $R = k[X, Y, Z]$ *and let* $I = \langle XY, XZ, YZ \rangle$*. Then*

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -Z & -Z \\ Y & 0 \\ 0 & X \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} XY & XZ & YZ \end{pmatrix}} R \longrightarrow R/I \longrightarrow 0$$

is an augmented free resolution of R/I *over* R*.*

Proof. Exactness at homological degree -1 follows from the fact that the quotient map $R \to R/I$ is surjective. Exactness at homological degree 0 follows from the fact that I is generated by $\mathrm{Im}(\partial_1)$.

To prove exactness at homological degree 1, let $(f, g, h)^\top \in \mathrm{Ker}(\partial_1)$, so

$$fXY + gXZ + hYZ = 0. \tag{2}$$

Then (2) implies $X|h$ which implies $h = h_1 X$ for some $h_1 \in R$. Similarly, (2) implies $Y|g$ and $Z|f$, which implies $g = g_1 Y$ and $f = f_1 X$ for some $g_1, f_1 \in R$. Substituting this in to (2), we obtain

$$XYZ(f_1 + g_1 + h_1) = 0, \tag{3}$$

and (3) implies $f_1 = -g_1 - h_1$ since R is an integral domain. Therefore

$$\begin{aligned}\begin{pmatrix} f \\ g \\ h \end{pmatrix} &= \begin{pmatrix} f_1 Z \\ g_1 Y \\ h_1 X \end{pmatrix} \\ &= \begin{pmatrix} (-g_1 - h_1) Z \\ g_1 Y \\ h_1 X \end{pmatrix} \\ &= g_1 \begin{pmatrix} -Z \\ Y \\ 0 \end{pmatrix} + h_1 \begin{pmatrix} -Z \\ 0 \\ X \end{pmatrix} \\ &\in \operatorname{Im}(\partial_2),\end{aligned}$$

which implies exactness at homological degree 1.

For exactness in homological degree 2, we just need to show that ∂_2 is injective. Let $(f,g)^\top \in R^2$ such that $\partial_2(f,g)^\top = 0$, so we obtain the system of equations

$$\begin{aligned} -fZ - Zg &= 0 \\ fY &= 0 \\ gX &= 0. \end{aligned}$$

Then $gX = 0$ implies $g = 0$ (since R is an integral domain) and $fY = 0$ implies $f = 0$ (since R is an integral domain), and thus $(f,g)^\top = (0,0)^\top$. This implies ∂_2 is injective which implies exactness at homological degree 2. □

Exercise 3

Proposition 0.3. *Let R be a commutative ring with identity and let F be a free R-module. Then F is projective.*

Proof. Let $\varphi\colon M \to N$ be a surjective R-module homomorphism and let $\psi\colon F \to N$ be an R-module homomorphism. We need to show that there exists an R-module homomorphism $\widetilde{\psi}\colon F \to M$ such that $\psi = \varphi \circ \widetilde{\psi}$.

Suppose that B is an R-module basis for F and let $b \in B$. Choose $m_b \in M$ such that $\varphi(m_b) = \psi(b)$ (we can do this since φ is surjective). Define $\widetilde{\psi}(b) = m_b$. By the universal mapping property of free R-modules, there exists a unique R-module homomorphism $\widetilde{\psi}\colon F \to M$ such that $\widetilde{\psi}(b) = m_b$ for all $b \in B$. By construction, we have

$$\begin{aligned} \varphi(\widetilde{\psi}(b)) &= \varphi(m_b) \\ &= \psi(b) \end{aligned}$$

for all $b \in B$. We invoke the universal mapping property of free R-modules again to conclude that $\varphi \circ \widetilde{\psi} = \psi$, which completes the proof. □

Proposition 0.4. *Let R be a commutative ring with identity, and consider the following diagram of R-module homomorphisms where the rows are exact:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & P \arrow[r, "\tau"] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\tau'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Assume that P is a projective R-module. Then there exists R-module homomorphisms g and h making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "h"] & P \arrow[r, "\tau"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\tau'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Proof. Since P is a projective R-module and $\tau'\colon B' \to C'$ is a surjective R-module homomorphism, there exists a morphism $g\colon P \to B'$ such that $f \circ \tau = \tau' \circ g$ (by definition of what it means to be a projective R-module). This takes care of g.

We now define $h\colon A \to A'$ as follows: Let $a \in A$. Then by commutativity of the right square and exactness of the top row, we have

$$\begin{aligned} \tau'(g(\alpha(a))) &= f(\tau(\alpha(a))) \\ &= f(0) \\ &= 0. \end{aligned}$$

This implies $g(\alpha(a)) \in \mathrm{Ker}(\alpha')$. Since the bottom row is exact and since α' is injective, there exists a unique $a' \in A'$ such that $\alpha'(a') = g(\alpha(a))$. We set $h(a) = a'$.

Since a' is uniquely determined by a, this map is well-defined. We now want to show that this map is an R-module homomorphism. To do this, let $r_1, r_2 \in R$ and $a_1, a_2 \in A$. Then

$$\begin{aligned} \alpha'((r_1a_1 + r_2a_2)') &= g(\alpha(r_1a_1 + r_2a_2)) \\ &= g(r_1\alpha(a_1) + r_2\alpha(a_2)) \\ &= r_1g(\alpha(a_1)) + r_2g(\alpha(a_2)) \\ &= r_1\alpha'(a_1') + r_2\alpha'(a_2') \\ &= \alpha'(r_1a_1' + r_2a_2'), \end{aligned}$$

and since α' is injective, this implies $(r_1a_1 + r_2a_2)' = r_1a_1' + r_2a_2'$. Therefore h is indeed an R-module homomorphism.

Finally, we need to show that the left square commutes. Let $a \in A$. Then we have

$$\begin{aligned} g(\alpha(a)) &= \alpha'(a') \\ &= \alpha'(h(a)) \end{aligned}$$

by definition of h. We are done. $\square$