Abstract Algebra Homework 7

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Problem 1

Exercise 1. Consider the field extension $\mathbb{Q} \subseteq \mathbb{Q}(x)$. Show that $\mathbb{Q}(x^2)$ is a closed intermediate extension but $\mathbb{Q}(x^3)$ is not.

Solution 1. First we show $\mathbb{Q}(x^2)$ is a closed intermediate extension. Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^2))$. Then σ is completely determined by where it sends x since

$$\sigma \cdot (a_n x^n + \cdots + a_1 x + a_0) = a_n \sigma(x)^n + \cdots + a_1 \sigma(x) + a_0$$

for any $a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Q}[x]$ (so $\sigma \cdot (f(x)/g(x)) = f(\sigma \cdot x)/g(\sigma \cdot x)$ for any $f/g \in \mathbb{Q}(x)$). Since σ fixes x^2, we see that $\sigma(x)$ must be a root of the monic

$$T^2 - x^2 = (T - x)(T + x).$$

In particular, either $\sigma(x) = x$ or $\sigma(x) = -x$. In particular, does not fix $\mathbb{Q}(x)$. Since there are no intermediate fields between $\mathbb{Q}(x^2)$ and $\mathbb{Q}(x)$ (as $[\mathbb{Q}(x) : \mathbb{Q}(x^2)] = 2$ is prime), we see that the fixed field of $\mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^2))$ is $\mathbb{Q}(x^2)$. Thus $\mathbb{Q}(x^2)$ is a closed intermediate extension.

Now we show $\mathbb{Q}(x^3)$ is not a closed intermediate extension. Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^3))$. As seen above, σ is completely determined by where it sends x. Since σ fixes x^3, we see that $\sigma(x)$ must be a root of the monic

$$T^3 - x^3 = (T - x)(T - \zeta_3 x)(T - \zeta_3^2 x).$$

Since $\zeta_3 \notin \mathbb{Q}$, we see that the only possible choice is $\sigma(x) = x$. Thus the fixed field of $\mathrm{Aut}(\mathbb{Q}(x)/\mathbb{Q}(x^3))$ is $\mathbb{Q}(x)$ (and not $\mathbb{Q}(x^3)$). Thus $\mathbb{Q}(x^3)$ is not a closed intermediate extension.

Problem 2

Problem 2.a

Proposition 0.1. *Let F/K be a field extension such that $[F : K] = 2$. Suppose that $\mathrm{char}\, K \neq 2$. Then L is Galois over K.*

Proof. It suffices to show that F is a splitting field of a separable polynomial over K. Let $\alpha \in L \backslash K$ and let $\pi_\alpha(T)$ be the minimal polynomial of α over K. Then $\pi_\alpha(T)$ must have degree 2 (it can't have degree 1 this would imply $\alpha \in K$ and it can't have degree > 2 since this would imply $[F : K] > 2$). Since α is a root of $\pi_\alpha(T)$, we see that $\pi_\alpha(T)$ factors as

$$\pi_\alpha(T) = (T - \alpha)p(T)$$

where $p(T)$ has degree 1 since $\pi_\alpha(T)$ has degree 2. Since $\mathrm{char}\, K \neq 2$, we have $\pi'_\alpha(T) \neq 0$ (since the lead term of $\pi'_\alpha(T)$ is $2T \neq 0$). Thus $\pi_\alpha(T)$ is a separable polynomial over K. Since $p(T)$ has degree 1, it obviously has a root in F. Thus $\pi_\alpha(T)$ splits completely in F. In particular F is the splitting field of $\pi_\alpha(T)$ since $[F : K] = 2 = \deg \pi_\alpha$. $\square$

Problem 2.b

Exercise 2. Give an example of a field extension F/K such that $[F : K] = 2$ and $\mathrm{char}\, K = 2$ but F/K is not Galois.

Solution 2. Let $K = \mathbb{F}_2(t)$ and let $F = K(\sqrt{t})$. Then L/K is an inseparable extension. Indeed, the minimal polynomial of $\sqrt{t}$ over K is $X^2 + t$, which factors over F as

$$X^2 + t = (X + \sqrt{t})^2.$$

This has a multiple root, which implies $\sqrt{t}$ in inseparable over K. Thus L/K is an inseparable extension, and hence is not Galois.

Problem 3.c

Exercise 3. Give an example of a field extension F/K such that $[F : K] = 2$ and $\operatorname{char} K = 2$ with F/K being Galois.

Solution 3. Let $K = \mathbb{F}_2$ and $F = \mathbb{F}_2[T]/\langle f(T) \rangle$ where $f(T) = T^2 + T + 1$. The minimal polynomial of $\overline{T} \in F$ is given by

$$f(X) = X^2 + X + 1,$$

indeed, observe $f(X)$ is irreducible over $\mathbb{F}_2$ by a brute force calculation:

$$\begin{aligned} XX &= X^2 \\ X(X+1) &= X^2 + X \\ (X+1)(X+1) &= X^2 + 1. \end{aligned}$$

Furthermore, $f(X)$ is separable over $\mathbb{F}_2$ since $f(X)$ is irreducible and $f'(X) = 1 \neq 0$. Finally, note that

$$\begin{aligned} (X + \overline{T})(X + \overline{T+1}) &= X^2 + (\overline{T+1} + \overline{T})X + \overline{T}(\overline{T+1}) \\ &= X^2 + X + \overline{T}^2 + \overline{T} \\ &= X^2 + X + 1. \end{aligned}$$

Thus $f(X)$ splits in F. In particular, F is a splitting field of the separable polynomial $f(X)$ (again for degree reasons).

Problem 3

Exercise 4. Let E/K and F/E be Galois extensions. Then is F/K a Galois extension?

Solution 4. No. Consider the following tower of field extensions

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}).$$

Observe that $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ are Galois extensions since they are field extensions of degree 2 and since we are working over characteristic 0 fields. However $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not Galois since $\sqrt[4]{2}$ is the root of the polynomial $T^4 - 2$, but this polynomial factors over $\mathbb{Q}(\sqrt[4]{2}, i)$ as

$$T^4 - 2 = (T - \sqrt[4]{2})(T - i\sqrt[4]{2})(T + \sqrt[4]{2})(T + i\sqrt[4]{2}).$$

In particular, $T^4 - 2$ only has two roots in $\mathbb{Q}(\sqrt[4]{2})$ (the other roots are imaginary numbers whereas $\mathbb{Q}(\sqrt[4]{2})$ consists of real numbers).