Commutative Algebra Homework 1

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Problem 1

Exercise 1. Given an example of a commutative ring (necessarily without identity) that does not have a maximal proper ideal.

Solution 1. Let A be any divisible group (for instance $A = \mathbb{Q}$). So $A = nA$ for every $n \in \mathbb{Z} \backslash \{0\}$. Then observe that A has no maximal proper subgroups. Indeed, assume for a contradiction that B is a maximal proper subgroup of A. Then B must have finite prime index in A (otherwise we can find a nonzero proper subgroup B'/B of A/B and pull this back to a proper subgroup B' of A which contains B), say $[A : B] = p$. Then we have

$$\begin{aligned} A &= pA \\ &\subseteq B \\ &\subseteq A, \end{aligned}$$

which forces $A = B$. This gives us a contradiction.

Now we turn A into a ring in a trivial way, namely we define multiplication on A by

$$a \cdot a' = 0$$

for all $a, a' \in A$. Clearly multiplication defined in this way gives A the structure of a commutative ring (but without an identity). Moroever since A has no maximal proper subgroups, we see that A has no maximal ideals as a ring.

Problem 2

Exercise 2. Let R be a commutative ring with identity and let $I \subset R$ be a proper ideal of R. We denote by $\operatorname{rad} I$ to be the radical of I and we denote by $\mathrm{N}(R)$ to be the set of nilpotents of R.

1. Show that $\operatorname{rad} I$ is contained in the intersection of all prime ideals that contain I.
2. Show the other containment.
3. Show that $\mathrm{N}(R)$ is the intersection of all prime ideals of R.

Solution 2. 1. Let $x \in \operatorname{rad} I$ and let $\mathfrak{p}$ be a prime ideal in R which contains I. Choose $n \in \mathbb{N}$ such that $x^n \in I$. Then since $I \subseteq \mathfrak{p}$, we have $x^n \in \mathfrak{p}$. It follows that $x \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Since and x and $\mathfrak{p}$ were arbitrary, it follows that $\operatorname{rad} I$ is contained in all prime ideals which contains I. Thus $\operatorname{rad} I$ is contained in the intersection of all prime ideals which contains I.

2. Assume for a contradiction that

$$\operatorname{rad} I \not\supseteq \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p}.$$

Choose $x \in \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ such that $x \notin \operatorname{rad} I$. Thus $x \in \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ which contain I and $x^n \notin I$ for all $n \in \mathbb{N}$. We will find a prime ideal in R which contains I but does not contain x, which will give us a contradiction.

Consider the ring obtained by localizing R at the multiplicative set $\{x^n \mid n \in \mathbb{N}\}$:

$$R_x = \{a/x^n \mid a \in \mathbb{R} \text{ and } n \in \mathbb{N}\},$$

and let $\rho \colon R \to R_x$ be the corresponding localization map, given by

$$\rho(a) = a/1$$

for all $a \in R$. Since $x^n \neq 0$ for all $n \in \mathbb{N}$, we see that $I_x = \rho(I)R_x$ is a proper ideal of R_x. In particular, there exists a prime ideal $\mathfrak{q}$ in R_x which contains I_x. Then $\rho^{-1}(\mathfrak{q})$ is a prime ideal in R which contains I but does not contain x. Indeed, if $\rho^{-1}(\mathfrak{q})$ contained x, then $\mathfrak{q}$ would contain a unit, namely $x/1$, and hence would not be prime.

3. By parts 1 and 2, we have

$$\operatorname{rad} I \neq \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \supseteq I}} \mathfrak{p}$$

for *all* ideals I of R. In particular, since $\mathrm{N}(R) = \operatorname{rad}\langle 0 \rangle$, we have

$$\mathrm{N}(R) = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Problem 3

Exercise 3. Let R be a commutative ring with identity. Denote the Jacobson radical of R by $\mathrm{J}(R)$. Then $x \in \mathrm{J}(R)$ if and only if $1 + ax$ is a unit for all $a \in R$.

Solution 3. Suppose $x \in \mathrm{J}(R)$ and assume for a contradiction that $1 + ax$ is not a unit for some $a \in R$. Choose a maximal ideal in R which contains $1 + ax$, say $\mathfrak{m}$. Since $x \in \mathrm{J}(R)$, we see that in particular $x \in \mathrm{m}$. Since $1 + ax$ and ax belong to $\mathfrak{m}$, their difference also belongs to $\mathfrak{m}$. In other words, $1 \in \mathfrak{m}$. This contradicts the fact that $\mathfrak{m}$ is a proper ideal of R. Thus our original assumption was wrong, which means that $1 + ax$ is a unit for all $a \in R$.

Conversely, suppose $1 + ax$ is a unit for all $a \in R$ and assume for a contradiction that $x \notin \mathrm{J}(R)$. Choose a maximal ideal in R which does not contain x, say $\mathfrak{m}$. Then $Rx + \mathfrak{m} = R$ since $\mathfrak{m}$ is maximal. Thus there exists $a \in R$ and $y \in \mathfrak{m}$ such that $ax + y = 1$, or in other words,

$$1 - ax = y.$$

By assumption, this implies y is a unit. This contradicts the fact that $y \in \mathfrak{m}$ and $\mathfrak{m}$ is a proper ideal.

Problem 4

Exercise 4. Let R be an integral domain. Then

$$R = \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$

Solution 4. Since R is an integral domain, it has no zerodivisors. Thus all of the localization maps $\rho_{\mathfrak{m}} \colon R \to R_{\mathfrak{m}}$ are injective. In fact, they are just inclusion maps since we are identifying R and its localizations $R_{\mathfrak{m}}$ with subrings of the fraction field K of R. Thus we have

$$R \subseteq \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$

For the reverse direction, let $x/y \in R_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R. In particular, this means $x, y \in R$ and $y \notin \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R. In particular, y must be a unit in R! Indeed, if y is not a unit, then there would exist a maximal ideal which contains y, but y does not belong to *any* maximal ideal of R. Thus y is a unit of R, and this implies $x/y \in R$. Thus we have the reverse direction

$$R \supseteq \bigcap_{\mathfrak{m} \text{ maximal}} R_{\mathfrak{m}}.$$