Computational Algebraic Geometry Homework 3

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Problem 1

I was successful.

Problem 2

Exercise 1. Suppose f_1, f_2, and f_3 are three polynomials in $\mathbb{Q}[x, y, z]$ of degree 3 whose coefficients are all integers of absolute value at most 20. Experiment with Macaulay2 to see how large the coefficients in a Gröbner basis for $\mathcal{V}(f_1, f_2, f_3)$ can be.

Solution 1. The coefficients can become incredibly large. For instance, calculating the Gröbner basis of

I=20x3+19x2y+18xy2+17xyz,17x3+12xy2+14z3,18xyz+20x2y+17z3,20yz2+17xz2+18z3;

gives us

994131220y2z2-815803669z4.

Problem 3

Exercise 2. Find an example of an ideal $I \subseteq \mathbb{R}[x, y]$ such that $\mathcal{V}(I_1) = \mathbb{R}$ but $\overline{\pi_1(\mathcal{V}(I))} = \emptyset$.

Solution 2. Let $I = \langle x^2 + y^2 + 1 \rangle$. Then $I_1 = I \cap \mathbb{R}[y] = 0$, and thus $\mathcal{V}(I_1) = \mathbb{R}$. On the other hand, note that $\mathcal{V}(I) = \emptyset$ since $x^2 + y^2 = -1$ has no solutions in $\mathbb{R}^2$. In particular, this implies $\pi_1(\mathcal{V}(I)) = \emptyset$, which implies $\overline{\pi_1(\mathcal{V}(I))} = \emptyset$.

Problem 4

Exercise 3. Let $f(x, y) \in K[x, y]$ and $V = \mathcal{V}(f)$. Let p be a point on V. We say that p is a **singular point** of $\mathcal{V}(f)$ if both partial derivatives $\partial_x f(p) = 0$ and $\partial_y f(p) = 0$. If at least one of these partial derivatives is nonzero, then p is a **nonsingular point** of $\mathcal{V}(f)$.

Let p be a point on V and let $\ell(t)$ be a line passing through p when the parameter $t = 0$. Without using calculus, we say that ℓ is **tangent** to V at p if $f(\ell(t))$ has a root of multiplicity greater than one as a function of t at $t = 0$.

1. Show that the algebraic variety $\mathcal{V}(x^3 - xy + y^2 - 1)$ has no singular points.
2. Check that the only tangent line to $\mathcal{V}(x^3 - xy + y^2 - 1)$ at the point $(1, 1)$ is the one given by calculus.
3. Construct an algebraic variety whose points correspond to tangent lines to $\mathcal{V}(x^3 - xy + y^2 - 1)$; that is, the set lines which are tangent at some point of $\mathcal{V}(x^3 - xy + y^2 - 1)$.

Solution 3. Throughout this problem, let $f = x^3 - xy + y^2 - 1$ and note that $\partial_x f = 3x^2 - y$ and $\partial_y f = -x + 2y$.

1. Assume for a contradiction that (a, b) is a singular point of $\mathcal{V}(f)$. Then we must have

$$\begin{aligned} 3a^2 - b &= 0 \\ -a + 2b &= 0 \\ a^3 - ab + b^2 - 1 &= 0. \end{aligned}$$

From the first equation above, we see that $b = 3a^2$. From the second equation above, we see that $a = 2b$. Thus we have $b = 12b^2$, which implies either $b = 0$ or $b = 1/12$, and hence either $a = 0$ or $a = 1/6$ (respectively). However neither $(0,0)$ nor $(1/6, 1/12)$ are solutions to the last equation. This is a contradiction.

2. Let $\ell(t)$ be a line passing through $p = (1,1)$ when the parameter $t = 0$. In particular, $\ell(t)$ has the form

$$\ell_{\mathbf{v}}(t) = p + t\mathbf{v} = (1 + tv_1, 1 + tv_2)$$

where $\mathbf{v} = (v_1, v_2)$ is a nonzero vector in K^2. Note that for any nonzero $a \in K$, we have $\ell_{a\mathbf{v}}(t/a) = \ell_{\mathbf{v}}(t)$. In particular, both $\ell_{a\mathbf{v}}(t)$ and $\ell_{\mathbf{v}}(t)$ parametrize the same line (though they are not the same parametrization). Since $\mathbf{v}$ is nonzero, either $v_1 \neq 0$ or $v_2 \neq 0$.

First assume that $v_2 \neq 0$. Since $\ell_{(1/v_2)\mathbf{v}}(t)$ and $\ell_{\mathbf{v}}(t)$ parametrize the same line, we may as well assume that $\mathbf{v} = (v, 1)$ where $v \neq 0$. Then we have

$$\begin{aligned} f(\ell_{\mathbf{v}}(t)) &= (1+tv)^3 - (1+tv)(1+t) + (1+t)^2 - 1 \\ &= v^3t^3 + (3v^2 - v + 1)t^2 + (2v+1)t \\ &= t(v^3t^2 + (3v^2 - v + 1)t + (2v+1)). \end{aligned}$$

This polynomial has a root of multiplicity greater than one as a function of t at $t = 0$ if and only if $v = -1/2$.

Now assume that $v_1 \neq 0$. As noted above, we may as well assume that $\mathbf{v} = (1, v)$ where $v \neq 0$. Then we have

$$\begin{aligned} f(\ell_{\mathbf{v}}(t)) &= (1+t)^3 - (1+t)(1+tv) + (1+tv)^2 - 1 \\ &= v^2t^2 + (-v^2 + v)t + v^3 + 3v^2 + 2v. \end{aligned}$$

This has a root of multiplicity greater than one as a function of t at $t = 0$ if and only if

$$\begin{aligned} 0 &= -v^2 + v = v(1-v) \\ 0 &= v^3 + 3v^2 + 2v = v(v+1)(v+2). \end{aligned}$$

Clearly this implies $v = 0$, which is a contradiction since we assumed $v \neq 0$.

Therefore we see that there is only one line which is tangent to $\mathcal{V}(f)$ at the point $(1,1)$, and it is parametrized by

$$\ell_{(-1,2)}(t) = (1 - t, 1 + 2t).$$

The equation which describes this line is exactly the equation which we obtain from calculus: namel

$$\begin{aligned} \partial_x f(1,1)(x-1) + \partial_y f(1,1)(y-1) &= 2(x-1) + (y-1) \\ &= 2x + y - 3. \end{aligned}$$

Indeed, for all t we have $2(1-t) + (1+2t) - 3 = 0$.

3. We now consider the more general situation where $p = (a,b)$ is a point on $\mathcal{V}(f)$ and $\mathbf{v} = (v_1, v_2)$ is a nonzero vector in K^2. Let $\ell_{p,\mathbf{v}}(t) = p + t\mathbf{v}$. Then observe that

$$\begin{aligned} f(\ell_{p,\mathbf{v}}(t)) &= (a + tv_1)^3 - (a + tv_1)(b + tv_2) + (b + tv_2)^2 - 1 \\ &= (a^3 - ab + b^2 - 1) + (3a^2v_1t + 3av_1^2t^2 + v_1^3t^3 - av_2t - bv_1t - v_1v_2t^2 + 2bv_2t + v_2^2t^2) \\ &= 3a^2v_1t + 3av_1^2t^2 + v_1^3t^3 - av_2t - bv_1t - v_1v_2t^2 + 2bv_2t + v_2^2t^2 \\ &= v_1t^3 + (3av_1^2 - v_1v_2 + v_2^2)t^2 + (3a^2v_1 - av_2 - bv_1 + 2bv_2)t \\ &= t(v_1t^2 + (3av_1^2 - v_1v_2 + v_2^2)t + ((3a^2 - b)v_1 + (2b - a)v_2)). \end{aligned}$$

In particular, $f(\ell_{p,\mathbf{v}}(t))$ has a root of multiplicity greater than one as a function of t at $t = 0$ if and only if

$$(3a^2 - b)v_1 + (2b - a)v_2 = 0.$$

In particular, let

$$X = \left\{ ((a,b), [v_1 : v_2]) \in \mathbb{A}_K^2 \times \mathbb{P}_K^1 \mid f(a,b) = 0 \text{ and } (3a^2 - b)v_1 + (2b - a)v_2 = 0 \right\}$$

The X is an algebraic subvariety of $\mathbb{A}_K^2 \times \mathbb{P}_K^1$ whose points correspond to tangent lines to $\mathcal{V}(f)$.

Problem 5

Exercise 4. Prove that an algebraically closed field is infinite.

Solution 4. We prove the contrapositive: Let K be a finite field and list its elements as $a_1, \ldots, a_n$. Then the polynomial

$$f(x) = 1 + \prod_{i=1}^{n}(x - a_i)$$

has no roots in K since $f(a_i) = 1$ for all $1 \leq i \leq n$. Thus K cannot be algebraically closed.