Math 9853 **Final Exam**: Sample Problems

Part I: All the HW problems and problems from Exam 1.

Part II:

1. Let V and W be vector spaces over a field K and $f\colon V \to W$ any linear map. Suppose f is injective. Describe all linear maps $g\colon W \to V$ so that $g \circ f = I_V$.
2. Let U and V be linear subspaces of a vector space over K. Show that
$$\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V).$$
3. Let $A \in \mathbb{C}^{m\times n}$ be any matrix. Show that $\mathrm{Ker}(A) = \mathrm{Ker}(A^*A)$ (which is a subspace in $\mathbb{C}^n$) where A^* is the conjugate transpose of A.
4. Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian (i.e. $A = A^*$).
   (a) Show that all the eigenvalue of A are real;
   (b) Show that eigenvectors from different eigenvalues of A are orthogonal.
5. Let $v_1, \ldots, v_{n-1} \in K^n$ be independent and $W = \mathrm{Span}(v_1, \ldots, v_{n-1})$ be the subspace of dimension $n-1$ generated by v_i's (also called a hyperplane). For any vector $x \in K^n$, let $A = (v_1, \ldots, v_{n-1}, x)$, an $n \times n$ matrix.
   (a) Show that $x \in W$ iff $\det(A) = 0$.
   (b) Show how to find a nonzero vector that is orthogonal to W (Hint: Expand $\det(A)$ by the column of x and check on the coefficient vector. This vector is often denoted by $v_1 \times v_2 \times \cdots \times v_{n-1}$.)
   (c) Let H be the plane in $\mathbb{R}^3$ containing (or passing through) the three points: $p_1 = (1,1,0)^T, p_2 = (1,0,2)^T, p_3 = (0,1,3)^T$. Apply (b) to the case when $v_1 = p_2 - p_1$ and $v_2 = p_3 - p_1$.
6. Suppose $A \in \mathbb{R}^{4\times 3}$ has SVD: $A = U\Sigma V^T$ where
$$U = 1/2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \quad V^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$
   (a) Find $x \in \mathbb{R}^3$ with $\| x \| = 1$ so that $\| Ax \|$ is minimized.
   (b) Find $y \in \mathbb{R}^4$ with $\| y \| = 1$ so that $\| A^T y \|$ is maximized.
   (c) Describe the eigenvalues and eigenvectors of AA^T.
   (d) Give a basis for the null space of A^T,
7. Find $x \in \mathbb{R}^3$ with $\| x \| = 1$ so that
$$Q(x) = 7x_1^2 + 7x_2^2 + 10x_2^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_4$$

is minimized. (Hint: Find a symmetric matrix A so that $Q(x) = x^T Ax$, and compute the eigenvalues and eigenvectors of A.)

8. Let $T\colon \mathbb{Q}[x]_{<n} \to \mathbb{Q}[x]_{<n}$ be given by $T(f(x)) = f(x) + f'(x)$. You do not need to prove that T is a linear transformation.
   (a) Find a matrix representing T, and compute $\det(T)$ and $\mathrm{Tr}(T)$.
   (b) Is T invertible or not? Justify your answer.
   (c) What are the eigenvalues of T?
   (d) Does $\mathbb{Q}[x]_{<n}$ have a basis of eigenvectors for T? Justify your answer.
9. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric with all positive eigenvalues. Prove that $A + B$ is symmetric with all positive eigenvalues.
10. Let K be a field, and let $A \in k^{n\times n}$ be such that K^n has a basis of eigenvectors of A. Prove that K^n has a basis of eigenvectors of A^T.
11. Let W be the subspace of $\mathbb{R}^5$ spanned by the columns of the matrix

$$A = (a_1, a_2, a_3) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 5 \\ 1 \end{pmatrix}.$$

   (a) Find an orthogonal basis for W.
   (b) Find the projection of b in W.
   (c) What is the minimum value of $\| Ax - b \|$ among all $x \in \mathbb{R}^3$, or the distance from b to the subspace W?
   (d) Find $x \in \mathbb{R}^3$ that is the least square solution to $Ax = b$. (Hint: Use the relationship computed in (a) of the new orthogonal basis to the old basis (a_1, a_2, a_3) of W.)