Homological Algebra

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1 Introduction

Homological Algebra is a subject in Mathematics whose origins can be traced back to Topology. Homological Algebra is a very diverse subject, so we will not attempt to give an all encompassing description of what Homological Algebra is, rather we give a partial description instead:

> Homological is the study of R-complexes and their homology.

Here R is understood to be a commutative ring with identity[1]. Whenever we write, "let M be an R-module" or "let (A, d) be an R-complex", then it is understood that R is a ring.

1.1 Notation and Conventions

Unless otherwise specified, let K be a field and let R be a commutative ring with identity.

1.1.1 Category Theory

In this document, we consider the following categories:

- The category of all sets and functions, denoted **Set**;
- The category of all rings and ring homomorphisms, denoted **Ring**;
- The category of all R-modules and R-linear maps, denoted $\mathbf{Mod}_R$;
- The category of all graded R-modules and graded R-linear maps, denoted $\mathbf{Grad}_R$;
- The category of all R-algebras R-algebra homorphisms, denoted $\mathbf{Alg}_R$;
- The category of all R-complexes and chain maps, denoted $\mathbf{Comp}_R$;
- The category of all R-complexes and homotopy classes of chain maps, denoted $\mathbf{HComp}_R$
- The category of all DG R-algebras DG algebra homomorphisms, denoted $\mathbf{DG}_R$.

2 Graded Rings and Modules

2.1 Graded Rings

Definition 2.1. Let H be an additive semigroup with identity 0. An H-**graded ring** R is a ring together with a direct sum decomposition

$$R = \bigoplus_{h \in H} R_h,$$

where the R_h are abelian groups which satisfy the property that if $r_{h_1} \in R_{h_1}$ and $r_{h_2} \in R_{h_2}$, then $r_{h_1} r_{h_2} \in R_{h_1+h_2}$. The R_h are called **homogeneous components of** R and the elements of R_h are called **homogeneous elements of degree** h. If r is a homogeneous element in R, then unless otherwise specified, we denote the degree of r by $\deg r$. When we say "let R be a graded ring", then it is understood that the homogeneous components of R are denoted R_h.

Proposition 2.1. *Let R be an H-graded ring. Then R_0 is a ring.*

Proof. First note that $1 \in R_0$ since if $r \in R_i$, the $1 \cdot r = r \in R_i$. If $r, s \in R_0$, then also $rs \in R_0$. It follows that R_0 is an abelian group equipped with a multiplication map with identity $1 \in R_0$. This multiplication map satisfies all of the properties which are required for R_0 to be a ring since it inherits these properties from R. □

We are mostly interested in the case where $H = \mathbb{N}^n$ or $H = \mathbb{N}$[2]. Whenever we write, "let R be an H-graded ring", then it is understood that H is an additive semigroup with identity 0. If we omit H and simply write "let R be a graded ring", then it is understood that R is an $\mathbb{N}$-graded ring.

It is wrong to think of an H-grading of R as a map $|\cdot| \colon R \backslash \{0\} \to H$ be a map such that

$$|rs| = |r| + |s|$$

[1]Unless otherwise specified, all rings discussed in this document are assumed to be commutative and unital.
[2]Our convention is that $\mathbb{N} = \{0, 1, 2, \dots\}$.

whenever $rs \neq 0$. Indeed, usually there are many nonzero elements $r \in R$ where $|r|$ is not defined. What we can say however is that for each $r \in R$ there exists nonzero elements $r_{h_1}, \dots r_{h_n}$, where $r_{h_k} \in R_{h_k}$ for all $1 \leq k \leq n$ and $h_i \neq h_j$ for all $1 \leq i < j \leq n$, such that r can be expressed *uniquely* as

$$r = r_{h_1} + \cdots + r_{h_n}. \tag{1}$$

The qualifier "uniquely" here means that if we have another expression for r, say

$$r = r_{h'_1} + \cdots + r_{h'_{n'}}$$

where $r_{h'_{k'}} \in R_{h'_{k'}} \backslash \{0\}$ for all $1 \leq k' \leq n'$ and $h'_{i'} \neq h'_{j'}$ for all $1 \leq i' < j' \leq n'$, then we must have $n = n'$ and, after reordering if necessary, we must have $r_{h_k} = r_{h'_k}$ for all $1 \leq k \leq n$. We call (1) the **decomposition of r into its homogeneous parts**.

2.1.1 Trivially Graded Ring

Example 2.1. Let R be any ring, then $R_0 := R$ and $R_i := 0$ for all $i > 0$ defines a trivial structure of a graded ring for R. This grading is called the **trivial grading** and we say R is a **trivially graded ring**. Whenever we introduce a ring without specifying any grading, then we assume R is equipped with the trivial grading unless otherwise specified.

2.1.2 A Ring Equipped with Two Gradings

Sometimes we speak of a graded ring as a **ring equipped with an H-grading**. If R is a ring, then it may possible to equip R with two gradings. Here is an example of this:

Example 2.2. Let R be a ring and let $\boldsymbol{x} = x_1, \dots, x_n$ be a list of indeterminates. Then $R[\boldsymbol{x}]$ is both an $\mathbb{N}$-graded ring and an $\mathbb{N}^n$-graded ring. The homogeneous component in degree i in the $\mathbb{N}$-grading is given by

$$R[\boldsymbol{x}]_i = \sum_{|\boldsymbol{\alpha}|=i} R\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

The homogeneous component in degree $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ in the $\mathbb{N}^n$-grading is given by

$$R[\boldsymbol{x}]_{\boldsymbol{\alpha}} = R\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

2.2 Graded R-Modules

Let R be an H-graded ring. An H**-graded** R-module M is an R-module together with a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

into abelian groups M_h which satisfies the condition that if $r_{h_1} \in R_{h_1}$ and $u_{h_2} \in M_{h_2}$, then $r_{h_1}u_{h_2} \in M_{h_1+h_2}$ for all $h_1, h_2 \in H$. The u_h are called **homogeneous components** of M and the elements of M_h are called **homogeneous elements** of **degree** h. If u is a homogeneous element in M, then unless otherwise specified, we denote the degree of u by $\deg u$. Whenever we write "let M be an H-graded R-module", then it is assumed that R is an H-graded ring. In the usual case, R will be an $\mathbb{N}$-graded ring and M will be a $\mathbb{Z}$-graded R-module. In this case, we will just say "let M be a graded R-module".

2.2.1 Twist of Graded Module

Definition 2.2. Let M be an H-graded R-module. For each $h \in H$, we define the h**th twist of** M, denoted $M(h)$, to be the H-graded R-module whose h'th homogeneous component is given by $M(h)_{h'} := M_{h+h'}$ for all $i \in \mathbb{Z}$.

2.3 Graded R-Submodules

Lemma 2.1. *Let M be a graded R-module and $N \subset M$ be a submodule. The following conditions are equivalent:*

1. *N is graded R-module whose homogeneous components are $M_i \cap N$.*
2. *N can be generated by homogeneous elements.*

Proof. We first show that 1 implies 2. Let $x \in N$. Since N is graded with homogeneous components $M_i \cap N$, there exists homogeneous elements $x_{i_k} \in M_{i_k} \cap N$ for $1 \leq k \leq n$ such that

$$x = x_{i_1} + \cdots + x_{i_n}.$$

In particular, N can be generated by homogeneous elements.

Now we show that 2 implies 1. Let $\{y_\alpha\}$ be a set of homogeneous generators for N and let $x \in N$. Since $N \subset M$, we can uniquely decompose x as a sum of homogeneous elements, $x = \sum x_i$, where each $x_i \in M$. We need to show that each $x_i \in N$. To do this, note that $x = \sum r_\alpha y_\alpha$ where r_α belongs to R. If we take ith homogeneous components, we find that

$$x_i = \sum (r_\alpha)_{i - \deg y_\alpha} y_\alpha,$$

where $(r_\alpha)_{i-\deg y_\alpha}$ refers to the homogeneous component of r_α concentrated in the degree $i - \deg y_\alpha$. From this it is easy to see that each x_i is a linear combination of the y_α and consequently lies in N. $\square$

Definition 2.3. A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (2.1) is called a **graded submodule**. A graded submodule of a graded ring is called a **homogeneous ideal**.

Example 2.3. Consider the graded ring $R = k[x, y, z]_{(5,6,15)}$. Then the ideal $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$ is a homogeneous ideal in R.

Remark. Let R be a graded ring and let I be a homogeneous ideal in R. Then the quotient ring R/I has an induced structure as a graded ring, where the ith homogeneous component of R/I is

$$(R/I)_i := (R_i + I)/I \cong R_i/(I \cap R_i)$$

2.3.1 Criterion for Homogoneous Ideal to be Prime

Proposition 2.2. *Let $\mathfrak{p} \subset R$ be a homogeneous ideal. In order that $\mathfrak{p}$ be prime, it is necessary and sufficient that whenever x, y are homogeneous elements such that $xy \in \mathfrak{p}$, then at least one of $x, y \in \mathfrak{p}$.*

Proof. Necessity is immediate. For sufficiency, suppose $a, b \in R$ and $ab \in \mathfrak{p}$. We must prove that one of these is $\mathfrak{p}$. Write

$$a = a_{i_1} + \cdots + a_{i_m} \quad \text{and} \quad b = b_{j_1} + \cdots + b_{j_n}$$

as a decomposition into homogeneous components where a_{k_m} and a_{k_n} are nonzero and of the highest degree.

We will prove that one of $a, b \in \mathfrak{p}$ by induction on $m + n$. When $m + n = 2$, then it is just the condition of the lemma. Suppose it is true for smaller values of $m + n$. Then ab has highest homogeneous component $a_{i_m} b_{j_n}$, which must be in $\mathfrak{p}$ by homogeneity. Thus one of a_{i_m},b_{j_n} belongs to $\mathfrak{p}$, say for definiteness it is a_{i_m}. Then we have

$$(a - a_{i_m})b \equiv ab \equiv 0 \bmod \mathfrak{p}$$

so that $(a - a_{i_m})b \in \mathfrak{p}$. But the resolutions of $a - a_{i_m}$ and b have a smaller $m + n$ value: $a - a_{i_m}$ can be expressed with $m - 1$ terms. By the inductive hypothesis, it follows that one of these is in $\mathfrak{p}$, and since $a_{i_m} \in \mathfrak{p}$, we find that one of $a, b \in \mathfrak{p}$. $\square$

2.4 Homomorphisms of Graded R-Modules

Definition 2.4. Let M and N be graded R-modules. A homomorphism $\varphi \colon M \to N$ is called **graded of degree** j if $\varphi(M_i) \subset N_{i+j}$ for all $i \in \mathbb{Z}$. If φ is graded of degree zero then we will simply say φ is **graded**.

Example 2.4. Consider the graded ring $R = k[X, Y, Z, W]$. Then the matrix

$$U := \begin{pmatrix} X + Y + Z & W^2 - X^2 & X^3 \\ 1 & X & XY + Z^2 \end{pmatrix}$$

defines a graded homomorphism $U \colon R(-1) \oplus R(-2) \oplus R(-3) \to R \oplus R(-1)$.

Example 2.5. Let R be a graded ring and let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

be an $n \times m$ matrix with entries $a_{ij} \in R_{\pi(i,j)}$ where $\pi(i, j) \in \mathbb{N}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Can we realize $A \colon R^m \to R^n$ as the matrix representation of a graded homomorphism between free R-modules? This answer is no. Indeed, consider the free R-modules F and F' generated by e_1, e_2 and e'_1, e'_2 respectively. Let $\varphi\colon F \to G$ be the unique R-linear map such that

$$\begin{aligned}\varphi(e_1) &= a_{11}e'_1 + a_{21}e'_2\\ \varphi(e_2) &= a_{12}e'_1 + a_{22}e'_2\end{aligned}$$

where $a_{11} \in R_1$, $a_{12} \in R_2$, $a_{21} \in R_3$, and $a_{22} \in R_5$. Then φ has matrix representation with respect to these bases as

$$[\varphi] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

but this is not graded. Indeed, the system of equations

$$\begin{aligned}\varphi(e_1) &= a_{11}e'_1 + a_{21}e'_2\\ \varphi(e_2) &= a_{12}e'_1 + a_{22}e'_2\end{aligned}$$

gives us the system of equations

$$\begin{aligned}\deg(e_1) &= 1 + \deg(e'_1)\\ \deg(e_1) &= 2 + \deg(e'_2)\\ \deg(e_2) &= 3 + \deg(e'_1)\\ \deg(e_2) &= 5 + \deg(e'_2),\end{aligned}$$

but not such solution exists.

Definition 2.5. Let R and S be graded rings. A ring homomorphism $\varphi\colon R \to S$ is said to be **graded** if it respects the grading. Thus if $a \in R_i$, then $\varphi(a) \in S_i$.

Example 2.6. Let $\varphi\colon K[x,y,z]_{(1,2,3)} \to K[x,y,z]$ be the unique ring homomorphism map such that $\varphi(x) = x$, $\varphi(y) = y^2$, and $\varphi(z) = z^3$. Then φ is a graded ring isomorphism onto its image $K[x,y^2,z^3]$. Indeed, the inverse $\psi\colon K[x,y^2,z^3] \to K[x,y,z]_{(1,2,3)}$ is the unique ring homomorphism such that $\psi(x) = x$, $\psi(y^2) = y$, and $\psi(z^3) = z$.

2.5 Category of all Graded R-Modules

2.5.1 Products in the Category of Graded R-Modules

Let Λ be a set and let M_λ be a graded R-module for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ denote the homogeneous component of M_λ in degree i by $M_{\lambda,i}$. If Λ is finite, then

$$\begin{aligned}\prod_{\lambda\in\Lambda} M_\lambda &= \prod_{\lambda\in\Lambda}\bigoplus_{i\in\mathbb{Z}} M_{\lambda,i}\\ &\cong \bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i}.\end{aligned}$$

Therefore, if Λ is finite, we may view $\prod_\lambda M_\lambda$ as a graded R-module whose homogeneous component in degree i is $\prod_\lambda M_{\lambda,i}$. On the other hand, if Λ is infinite, then we only have an injective map

$$\bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i} \to \prod_{\lambda\in\Lambda}\bigoplus_{i\in\mathbb{Z}} M_{\lambda,i}.$$

In particular, $\prod_\lambda M_\lambda$ is not the correct product in $\mathbf{Grad}_R$. The correct product is **graded product**, given by the graded R-module

$$\prod_{\lambda\in\Lambda}^{\star} M_\lambda := \bigoplus_{i\in\mathbb{Z}}\prod_{\lambda\in\Lambda} M_{\lambda,i}$$

together with its projection maps $\pi_\lambda\colon \prod^\star_\lambda M_\lambda \to M_\lambda$ for all $\lambda \in \Lambda$. A homogeneous element of degree i in $\prod^\star_\lambda M_\lambda$ is a sequence of the form $(u_{\lambda,i})_\lambda$ where $u_{\lambda,i} \in M_{\lambda,i}$ for all $\lambda \in \Lambda$. Thus any element in $\prod^\star_\lambda M_\lambda$ can be expressed as a finite sum of the form

$$(u_{\lambda,i_1} + u_{\lambda,i_2} + \cdots + u_{\lambda,i_n})$$

where we often assume without loss of generality that $i_1 < i_2 < \cdots < i_n$.

Let us check that this is in fact the correct product in $\mathbf{Grad}_R$. To show that the pair $(\prod^\star_\lambda M_\lambda, \pi_\lambda)$ is the correct product we have to show it satisfies the universal property: for any other such pair (M, ψ_λ), where M is a graded

R-module and $\psi_\lambda\colon M \to M_\lambda$ are graded R-linear maps, there is a unique graded R-linear map $\psi\colon M \to \prod^\star_\lambda M_\lambda$ such that $\pi_\lambda \psi = \psi_\lambda$ for all $\lambda \in \Lambda$. So let (M, ψ_λ) be such a pair. We define $\psi\colon M \to \prod^\star_\lambda M_\lambda$ by

$$\psi(u) = (\psi_\lambda(u))$$

for $u \in M_i$. Clearly ψ is a graded R-linear map since ψ_λ is a graded R-linear map for each $\lambda \in \Lambda$. Moreover, for all $u \in M_i$, we have

$$\begin{aligned}(\pi_\lambda \psi)(u) &= \pi_\lambda(\psi(u)) \\ &= \pi_\lambda((\psi_\lambda(u)) \\ &= \psi_\lambda(u).\end{aligned}$$

This implies $\pi_\lambda \psi = \psi_\lambda$. This establishes existence of ψ. For uniqueness, suppose $\widetilde{\psi}\colon M \to \prod^\star_\lambda M_\lambda$ is another such map. Then for all $u \in M_i$, we have

$$\begin{aligned}\widetilde{\psi}(u) = \psi(u) &\iff \pi_\lambda(\widetilde{\psi}(u)) = \pi_\lambda(\psi(u)) \text{ for all } \lambda \in \Lambda \\ &\iff (\pi_\lambda \widetilde{\psi})(u) = (\pi_\lambda \psi)(u) \text{ for all } \lambda \in \Lambda \\ &\iff \psi_\lambda(u) = \psi_\lambda(u) \text{ for all } \lambda \in \Lambda.\end{aligned}$$

It follows that $\widetilde{\psi} = \psi$.

2.5.2 Inverse Systems and Inverse Limits in the Category Graded R-Modules

Definition 2.6. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). An **inverse system** $(M_\lambda, \varphi_{\lambda\mu})$ of graded R-modules and graded R-linear maps over Λ consists of a family of graded R-modules $\{M_\lambda\}$ indexed by Λ and a family of graded R-linear maps $\{\varphi_{\lambda\mu}\colon M_\mu \to M_\lambda\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\lambda\mu}\varphi_{\mu\kappa}.$$

We say the pair (M, ψ_λ) is **compatible** with the inverse system $(M_\lambda, \varphi_{\lambda\mu})$ if

$$\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$$

for all $\lambda \leq \mu$.

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi\colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of inverse systesms consists of a collection of graded R-linear maps $\psi_\lambda\colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\mu = \psi_\lambda \varphi_{\lambda\mu}.$$

Proposition 2.3. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The inverse limit of this system, denoted $\overset{\star}{\varprojlim} M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\overset{\star}{\varprojlim} M_\lambda = \left\{ (u_\lambda) \in \overset{\star}{\prod_{\lambda \in \Lambda}} M_\lambda \mid \varphi_{\lambda\mu}(u_\mu) = u_\lambda \text{ for all } \lambda \leq \mu \right\}$$

together with the projection maps

$$\pi_\lambda\colon \overset{\star}{\varprojlim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\overset{\star}{\varprojlim} M_\lambda$ is given by

$$(\overset{\star}{\varprojlim} M_\lambda)_i = \varprojlim M_{\lambda,i}.$$

Remark. We put a $\star$ above $\varprojlim$ to remind ourselves that this is the inverse limit in the category of all graded R-modules. In the category of all R-modules, the inverse limit is denoted by $\varprojlim M_\lambda$. If Λ is finite, then $\lim M_\lambda$ already has a natural interpretation of a graded R-module.

Proof. We need to show that $\overset{\star}{\varprojlim} M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the invserse system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$ for all $\lambda \leq \mu$. By the universal mapping property of the graded product, there exists a unique graded R-linear map $\psi\colon M \to \prod_\lambda^\star M_\lambda$ such that $\pi_\lambda \psi = \psi_\lambda$ for all $\lambda \in \Lambda$. In fact, this map lands in $\varprojlim^\star M_\lambda$ since

$$\begin{aligned}\varphi_{\lambda\mu}\pi_\mu\psi(u) &= \varphi_{\lambda\mu}\psi_\mu(u)\\ &= \psi_\lambda(u)\\ &= \pi_\lambda\psi(u)\end{aligned}$$

for all $u \in M$. This establishes existence and uniqueness, and thus $\varprojlim^\star M_\lambda$ satisfies the universal mapping property. □

2.5.3 Pullbacks in the Category of Graded R-Modules

Here is an interesting example of a limit in the case where Λ is finite. Let $\psi\colon N \to M$ and $\varphi\colon P \to M$ be graded R-linear maps. The **pullback of** $\psi\colon N \to M$ **and** $\varphi\colon P \twoheadrightarrow M$ is defined to be graded R-module

$$N \times_M P = \{(u,v) \in N \times P \mid \psi(u) = \varphi(v)\}$$

endowed with the projection maps

$$\pi_1\colon N \times_M P \to N \quad \text{and} \quad \pi_2\colon N \times_M P \to P.$$

One can check that the pullback satisfies the universal mapping property of the system

$$\begin{tikzcd} & P \arrow[d, "\varphi"] \\ N \arrow[r, "\psi"'] & M \end{tikzcd}$$

Thus there exists a *unique* isomorphism from $N \times_M P$ to the limit of this system which makes everything commute.

2.5.4 Pullbacks Preserves Surjective Maps

Proposition 2.4. *Let $\varphi_{13}\colon M_3 \to M_1$ and $\varphi_{12}\colon M_2 \to M_1$ be graded R-linear maps. Consider their pullback*

$$\begin{tikzcd} M_3 \times_{M_1} M_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\varphi_{12}"] \\ M_3 \arrow[r, "\varphi_{13}"'] & M_1 \end{tikzcd}$$

1. *If both φ_{12} and φ_{13} are injective, then both π_1 and π_2 are injective.*

2. *If φ_{12} is surjective, then π_1 is surjective. Similarly, if φ_{13} is surjective, then π_2 is surjective.*

Proof. 1. Suppose both φ_{12} and φ_{13} are injective. We want to show that π_1 is injective. Let $(u_3,u_2) \in \ker \pi_1$. So $(u_3,u_2) \in M_3 \times_{M_1} M_2$, which means $\varphi_{13}(u_3) = \varphi_{12}(u_2)$, and $\pi_1(u_3,u_2) = 0$, which means $u_3 = 0$. Thus

$$\begin{aligned}\varphi_{12}(u_2) &= \varphi_{13}(u_3)\\ &= \varphi_{13}(0)\\ &= 0.\end{aligned}$$

Since φ_{12} is injective, this implies $u_2 = 0$, which implies $\varphi_{13}(u_3) = 0$. Since φ_{12} is injective, this implies $u_3 = 0$.

2. Suppose φ_{12} is surjective. We want to show that π_1 is surjective. Let $u_3 \in M_3$. Using the fact that φ_{12} is surjective, we choose a lift of $\varphi_{13}(u_3)$ with respect to φ_{12}, say $u_2 \in M_2$. So $\varphi_{12}(u_2) = \varphi_{13}(u_3)$, but this means $(u_3,u_2) \in M_3 \times_{M_1} M_2$, which implies π_1 is surjective since $\pi_1(u_3,u_2) = u_3$. The proof that φ_{13} surjective implies π_2 surjective follows in a similar manner.

□

2.5.5 Coproducts in the Category of Graded R-Modules

Let Λ be a set and let M_λ be a graded R-module for all $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ denote the homogeneous component of M_λ in degree i by $M_{\lambda,i}$. Then observe that

$$\begin{aligned} \bigoplus_{\lambda\in\Lambda} M_\lambda &= \bigoplus_{\lambda\in\Lambda}\bigoplus_{i\in\mathbb{Z}} M_{\lambda,i} \\ &\cong \bigoplus_{i\in\mathbb{Z}}\bigoplus_{\lambda\in\Lambda} M_{\lambda,i}. \end{aligned}$$

Therefore $\bigoplus_\lambda M_\lambda$ has a natural interpretation as a graded R-module with the homogeneous component in degree i being given by $\bigoplus_\lambda M_{\lambda,i}$. One can check that $\bigoplus_\lambda M_\lambda$ together with the inclusion maps $\iota_\lambda \colon M_\lambda \to \bigoplus_\lambda M_\lambda$ is the correct coproduct in $\mathbf{Grad}_R$.

2.5.6 Direct Systems and Direct Limits in the Category of Graded R-Modules

Definition 2.7. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). A **direct system** $(M_\lambda, \varphi_{\lambda\mu})$ of graded R-modules and graded R-linear maps over Λ consists of a family of graded R-modules $\{M_\lambda\}$ indexed by Λ and a family of graded R-linear maps $\{\varphi_{\lambda\mu}\colon M_\lambda \to M_\mu\}_{\lambda\leq\mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\mu\kappa}\varphi_{\lambda\mu}.$$

If $(\Lambda, \leq)$ is also directed set, then we say $(M_\lambda, \varphi_{\lambda\mu})$ is a **directed system**. We say the pair (M, ψ_λ) is **compatible** with the inverse system $(M_\lambda, \varphi_{\lambda\mu})$ if

$$\psi_\mu\varphi_{\lambda\mu} = \psi_\lambda$$

for all $\lambda \leq \mu$.

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi\colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of direct systems consists of a collection of graded R-linear maps $\psi_\lambda\colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\lambda = \psi_\mu\varphi_{\lambda\mu}.$$

The morphism ψ induces a graded R-linear map $\varinjlim \psi_\lambda\colon \varinjlim M_\lambda \to \varinjlim M'_\lambda$ uniquely determined by

$$\varinjlim \psi_\lambda(\overline{u_\lambda}) = \overline{\psi_\lambda(u_\lambda)}$$

for all $u_\lambda \in M_\lambda$ for all $\lambda \in \Lambda$.

Proposition 2.5. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The **direct limit** of this system, denoted $\varinjlim M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\varinjlim M_\lambda := \bigoplus_{\lambda\in\Lambda} M_\lambda / \langle\{(\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda) \mid u_\lambda \in M_\lambda \text{ and } \lambda \leq \mu\}\rangle$$

together with the inclusion maps

$$\iota_\lambda\colon M_\lambda \to \varinjlim M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\varinjlim M_\lambda$ is given by

$$(\varinjlim M_\lambda)_i = \varinjlim M_{\lambda,i}.$$

Proof. First observe that the submodule

$$\langle\{(\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda) \mid u_\lambda \in M_\lambda \text{ and } \lambda \leq \mu\}\rangle$$

of $\bigoplus_\lambda M_\lambda$ is generated by homogeneous elements. Indeed, for any $(\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda)$, we express u_λ into its homogeneous parts, say

$$u_\lambda = u_{\lambda,i_1} + \cdots + u_{\lambda,i_n},$$

then since $\iota_\lambda - \iota_\mu\varphi_{\lambda\mu}$ is a graded R-linear map, we have

$$\begin{aligned} (\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda) &= (\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_{\lambda,i_1} + \cdots + u_{\lambda,i_n}) \\ &= (\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_{\lambda,i_1}) + (\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_{\lambda,i_n}), \end{aligned}$$

where each $(\iota_\lambda - \iota_\mu \varphi_{\lambda\mu})(u_{\lambda,i_m})$ is homogeneous. Thus any such $(\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda)$ can be expressed as a sum of finitely many homogeneous terms. It follows that $\varinjlim M_\lambda$ has a natural graded R-module structure.

We need to show that $\varinjlim M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the direct system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\lambda = \psi_\mu$ for all $\lambda \leq \mu$. By the universal mapping property of the coproduct, there exists a unique graded R-linear map $\psi \colon \bigoplus_\lambda M_\lambda \to M$ such that $\psi\iota_\lambda = \psi_\lambda$ for all $\lambda \in \Lambda$. In fact, this map induces a well-defined graded R-linear map $\overline{\psi} \colon \varinjlim M_\lambda \to M$ since

$$\begin{aligned}
\psi(\iota_\lambda - \iota_\mu\varphi_{\lambda\mu})(u_\lambda) &= \psi\iota_\lambda(u_\lambda) - \psi\iota_\mu\varphi_{\lambda\mu}(u_\lambda) \\
&= \psi_\lambda(u_\lambda) - \psi_\mu\varphi_{\lambda\mu}(u_\lambda) \\
&= \psi_\lambda(u_\lambda) - \psi_\lambda(u_\lambda) \\
&= 0
\end{aligned}$$

for all $u_\lambda \in M_\lambda$ and $\lambda \in \Lambda$. □

Proposition 2.6. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a directed system of graded R-modules and graded R-linear maps.*

1. *Each element of $\varinjlim M_\lambda$ has the form $\overline{u_\lambda}$ for some $u_\lambda \in M_\lambda$.*
2. *$\overline{u_\lambda} = 0$ if and only if $\varphi_{\lambda\mu}(u_\lambda) = 0$ for some $\lambda \leq \mu$.*

Proof. 1. An element in $\varinjlim M_\lambda$ has the form $\overline{u_{\lambda_1} + \cdots + u_{\lambda_n}}$, where $\lambda_1, \dots, \lambda_n \in \Lambda$ and $u_{\lambda_i} \in M_{\lambda_i}$ for all $1 \leq i \leq n$. Since Λ is directed, there exists a $\lambda \in \Lambda$ such that $\lambda_i \leq \lambda$ for all $1 \leq i \leq n$. Then we have

$$\begin{aligned}
\overline{u_{\lambda_1} + \cdots + u_{\lambda_n}} &= \overline{u_{\lambda_1}} + \cdots + \overline{u_{\lambda_n}} \\
&= \overline{\varphi_{\lambda_1,\lambda}(u_{\lambda_1})} + \cdots + \overline{\varphi_{\lambda_n,\lambda}(u_{\lambda_n})} \\
&= \overline{\varphi_{\lambda_1,\lambda}(u_{\lambda_1}) + \cdots + \varphi_{\lambda_n,\lambda}(u_{\lambda_n})} \\
&= \overline{u_\lambda},
\end{aligned}$$

where $u_\lambda = \varphi_{\lambda_1,\lambda}(u_{\lambda_1}) + \cdots + \varphi_{\lambda_n,\lambda}(u_{\lambda_n})$. Each $\varphi_{\lambda_i,\lambda}(u_{\lambda_i})$ lands in M_λ, so $u_\lambda \in M_\lambda$.

2. If $\varphi_{\lambda\mu}(u_\lambda) = 0$ for some $\lambda \leq \mu$, then $\overline{u_\lambda} = \overline{\varphi_{\lambda\mu}(u_\lambda)} = 0$. Conversely, suppose $\overline{u_\lambda} = 0$. Then we have

$$u_\lambda = u_{\lambda_1} - \varphi_{\lambda_1\mu_1}(u_{\lambda_1}) + \cdots + u_{\lambda_n} - \varphi_{\lambda_n\mu_n}(u_{\lambda_n})$$

for some $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \Lambda$ and $u_{\lambda_i} \in M_{\lambda_i}$ for all $1 \leq i \leq n$. Choose $\mu \in \Lambda$ such that $\lambda, \lambda_i, \mu_i \leq \mu$ for all $1 \leq i \leq n$. Then

$$\begin{aligned}
\varphi_{\lambda\mu}(u_\lambda) &= \varphi_{\lambda\mu}(u_\lambda) - u_\lambda + u_\lambda \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \varphi_{\lambda_1\mu}u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n} + \varphi_{\lambda_n\mu}u_n - \varphi_{\lambda_n\mu}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}\varphi_{\mu_1\mu}u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu_n}u_{\lambda_n} + \varphi_{\lambda_n\mu_1}\varphi_{\mu_1\mu}u_n - \varphi_{\lambda_n\mu}u_{\lambda_n}) \\
&= (\varphi_{\lambda\mu}(u_\lambda) - u_\lambda) + (u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1}) + \varphi_{\lambda_1\mu_1}(\varphi_{\mu_1\mu}u_{\lambda_1} - u_{\lambda_1}) + \cdots + (u_{\lambda_n} - \varphi_{\lambda_n\mu}u_{\lambda_n}) + \varphi_{\lambda_n\mu_n}(\varphi_{\mu_n\mu}u_n - u_{\lambda_n}) \\
&= \varphi_{\lambda\mu}(u_\lambda) - u_{\lambda_1} + \varphi_{\lambda_1\mu_1}u_{\lambda_1} + \cdots - u_{\lambda_n} + \varphi_{\lambda_n\mu_n}u_{\lambda_n} + u_{\lambda_1} - \varphi_{\lambda_1\mu}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}(\varphi_{\mu_1\mu}u_{\lambda_1} - u_{\lambda_1}) + \cdots + u_{\lambda_n} - \varphi_{\lambda_n\mu}u_{\lambda_n} + \varphi_{\lambda_n\mu_n}(\varphi_{\mu_n\mu}u \\
&= \varphi_{\lambda\mu}(u_\lambda) + \cdots - \varphi_{\lambda_1\mu}u_{\lambda_1} + \varphi_{\lambda_1\mu_1}\varphi_{\mu_1\mu}u_{\lambda_1} + \cdots - \varphi_{\lambda_n\mu}u_{\lambda_n} + \varphi_{\lambda_n\mu_n}\varphi_{\mu_n\mu}u_n
\end{aligned}$$

□

2.5.7 Taking Directed Limits is an Exact Functor

Proposition 2.7. *Let*

$$0 \longrightarrow (M_\lambda, \varphi_\lambda) \xrightarrow{\psi} (M'_\lambda, \varphi'_\lambda) \xrightarrow{\psi'} (M''_\lambda, \varphi''_\lambda) \longrightarrow 0$$

be a short exact sequence of directed systems of graded R-modules and graded R-linear maps. Then

$$0 \longrightarrow \varinjlim M_\lambda \xrightarrow{\varinjlim \psi_\lambda} \varinjlim M'_\lambda \xrightarrow{\varinjlim \psi'_\lambda} \varinjlim M_\lambda \longrightarrow 0$$

is a short exact sequence of graded R-modules and graded R-linear maps.

Proof. We first show $\varinjlim \psi_\lambda$ is injective. Let $\overline{u_\lambda} \in \varinjlim M_\lambda$ and suppose $\overline{\psi_\lambda u_\lambda} = 0$. Then there exists $\mu \geq \lambda$ such that $\varphi'_{\lambda\mu}\psi_\lambda u_\lambda = 0$. In other words,

$$\begin{aligned} 0 &= \varphi'_{\lambda\mu}\psi_\lambda u_\lambda \\ &= \psi_\mu \varphi_{\lambda\mu} u_\lambda. \end{aligned}$$

This implies $\varphi_{\lambda\mu}u_\lambda = 0$ since ψ_μ is injective. Thus

$$\begin{aligned} \overline{u_\lambda} &= \overline{\varphi_{\lambda\mu}u_\lambda} \\ &= 0. \end{aligned}$$

So $\varinjlim \psi_\lambda$ is injective. Next we show exactness at $\varinjlim M'_\lambda$. Let $\overline{u'_\lambda} \in \varinjlim M'_\lambda$ and suppose $\overline{\psi'_\lambda u'_\lambda} = 0$. Then there exists $\mu \geq \lambda$ such that $\varphi''_{\lambda\mu}\psi'_\lambda u'_\lambda = 0$. In other words,

$$\begin{aligned} 0 &= \varphi''_{\lambda\mu}\psi'_\lambda u'_\lambda \\ &= \psi'_\mu \varphi'_{\lambda\mu} u'_\lambda. \end{aligned}$$

This implies $\varphi'_{\lambda\mu}u'_\lambda = \psi_\mu u_\mu$ for some $u_\mu \in M_\mu$, by exactness at $(M'_\lambda, \varphi'_\lambda)$. Thus

$$\begin{aligned} \overline{u'_\lambda} &= \overline{\varphi'_{\lambda\mu}u'_\lambda} \\ &= \overline{\psi_\mu u_\mu}. \end{aligned}$$

This implies exactness at $\varinjlim M'_\lambda$. Exactness at $\varinjlim M''_\lambda$ is easy and is left as an exercise. □

2.5.8 Contravariant Hom Converts Direct Limits to Inverse Limits

Proposition 2.8. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-linear module. Then there exists an isomorphism*

2.5.9 Tensor Products

Let M and N be graded R-modules. As R-modules, their tensor product is given by

$$\begin{aligned} M \otimes_R N &= \left(\bigoplus_{i \in \mathbb{Z}} M_i \right) \otimes \left(\bigoplus_{j \in \mathbb{Z}} N_j \right) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j \in \mathbb{Z}} (M_i \otimes N_j) \\ &= \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j} \right). \end{aligned}$$

In particular, $M \otimes_R N$ has a natural interpretation as a graded R-module with the homogeneous component in degree i given by

$$(M \otimes_R N)_i = \bigoplus_{j \in \mathbb{Z}} M_j \otimes N_{i-j}.$$

Indeed, if $x \in M_i$, $y \in N_j$, and $a \in R_k$, then

$$a(x \otimes y) = ax \otimes y = x \otimes ay \in (M \otimes_R N)_{i+j+k}.$$

So the grading is preserved upon R-scaling.

2.5.10 Graded Hom

Unlike the case of tensor products, hom does not have a natural interpretation as a graded R-module Instead we consider the graded version of hom: let M and N be graded R-modules. Their **graded hom**, denoted $\mathrm{Hom}^\star_R(M, N)$, is the graded R-module whose homogeneous component in degree i is

$$\mathrm{Hom}^\star_R(M, N)_i = \{\text{graded homomorphisms } \alpha \colon M \to N \text{ of degree } i\}.$$

Observe that we have a natural inclusion of R-modules

$$\mathrm{Hom}^\star_R(M, N) \subseteq \mathrm{Hom}_R(M, N).$$

In particular, many properties which $\mathrm{Hom}_R(M, N)$ satisfies are inherited by $\mathrm{Hom}^\star_R(M, N)$.

2.5.11 Graded Hom Properties

Proposition 2.9. *Let M be a graded R-module, let Λ be a set, and let N_λ be a graded R-module for each $\lambda \in \Lambda$. Then we have natural isomorphisms*

$$\mathrm{Hom}_R^\star\left(M, \prod_{\lambda\in\Lambda}^\star N_\lambda\right) \cong \prod_{\lambda\in\Lambda}^\star \mathrm{Hom}_R^\star(M, N_\lambda) \quad \textit{and} \quad \mathrm{Hom}_R^\star\left(\bigoplus_{\lambda\in\Lambda} M_\lambda, -\right) \cong \prod_{\lambda\in\Lambda}^\star \mathrm{Hom}_R^\star(M_\lambda, -)$$

Proof. Let $i \in \mathbb{Z}$. Define a map $\Psi\colon \mathrm{Hom}_R^\star\left(M, \prod_{\lambda\in\Lambda} N^\lambda\right)_i \to \prod_{\lambda\in\Lambda} \mathrm{Hom}_R^\star\left(M, N^\lambda\right)_i$ by

$$\Psi(\varphi) = (\pi_\lambda\varphi)_{\lambda\in\Lambda}$$

for all $\varphi \in \mathrm{Hom}_R^\star\left(M, \prod_{\lambda\in\Lambda} N^\lambda\right)_i$, where $\pi_\lambda\colon \prod_{\lambda\in\Lambda} N^\lambda \to N^\lambda$ is the projection to the λth coordinate. We claim that Ψ is a graded isomorphism.

We first check that it is R-linear. Let $a, b \in R$ and $\varphi, \psi \in \mathrm{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}
\Psi(a\varphi + b\psi) &= (\pi_i \circ (a\varphi + b\psi)) \\
&= (a(\pi_i \circ \varphi) + b(\pi_i \circ \psi)) \\
&= a(\pi_i \circ \varphi) + b(\pi_i \circ \psi) \\
&= a\Psi(\varphi) + b\Psi(\psi).
\end{aligned}$$

Thus Ψ is R-linear. To show that Ψ is an isomorphism, we construct its inverse. Let $(\varphi_i) \in \prod_{i\in I} \mathrm{Hom}_R\left(M, N_i\right)$. Define $\Phi((\varphi_i))\colon M \to \prod_{i\in I} N_i$ by

$$\Phi((\varphi_i))(x) := (\varphi_i(x))$$

for all $x \in M$. Then clearly Φ and Ψ are inverse to each other. Indeed, let $\varphi \in \mathrm{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}
\Phi(\Psi(\varphi))(x) &= \Phi((\pi_i \circ \varphi))(x) \\
&= ((\pi_i \circ \varphi)(x)) \\
&= \varphi(x)
\end{aligned}$$

for all $x \in M$. Thus $\Phi(\Psi(\varphi)) = \varphi$. Conversely, let $(\varphi_i) \in \prod_{i\in I} \mathrm{Hom}_R\left(M, N_i\right)$. Then

$$\begin{aligned}
\Psi(\Phi(\varphi_i)) &= (\pi_i \circ \Phi(\varphi_i)) \\
&= (\pi_i \circ \varphi)) \\
&= \varphi(x)
\end{aligned}$$

Finally, note that Ψ is graded since π_λ is graded of degree 0 for all $\lambda \in \Lambda$. $\square$

In fact we can generalize the above proposition as follows:

Proposition 2.10. *Let $(\Lambda, \leq)$ be a preordered set, let $(M_\lambda, \phi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps over Λ and let $(N_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over Λ. Then we have natural isomorphisms*

$$\mathrm{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda) \cong \overset{\star}{\varprojlim} \mathrm{Hom}_R^\star(M, N_\lambda) \quad \textit{and} \quad \mathrm{Hom}_R^\star(\overset{\star}{\varprojlim} M_\lambda, N) \cong \varinjlim \mathrm{Hom}_R^\star(M_\lambda, N)$$

Proof. Let $i \in \mathbb{Z}$. Define a map $\Psi\colon \mathrm{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda)_i \to \overset{\star}{\varprojlim} \mathrm{Hom}_R^\star(M, N_\lambda)_i$ by

$$\Psi(\varphi) = (\pi_\lambda\varphi)$$

for all $\varphi \in \mathrm{Hom}_R^\star(M, \overset{\star}{\varprojlim} N_\lambda)_i$, where π_λ is the projection to the λth coordinate. Observe that Ψ lands in $\overset{\star}{\varprojlim} \mathrm{Hom}_R^\star(M, N_\lambda)_i$ since $\pi_\mu\varphi = \varphi_{\lambda\mu}\pi_\lambda\varphi$ for all $\lambda \leq \mu$. We claim that Ψ is a graded isomorphism.

We first check that it is R-linear. Let $a, b \in R$ and $\varphi, \psi \in \mathrm{Hom}_R\left(M, \prod_{i\in I} N_i\right)$. Then

$$\begin{aligned}
\Psi(a\varphi + b\psi) &= (\pi_i \circ (a\varphi + b\psi)) \\
&= (a(\pi_i \circ \varphi) + b(\pi_i \circ \psi)) \\
&= a(\pi_i \circ \varphi) + b(\pi_i \circ \psi) \\
&= a\Psi(\varphi) + b\Psi(\psi).
\end{aligned}$$

Thus Ψ is R-linear. To show that Ψ is an isomorphism, we construct its inverse. Let $(\varphi_i) \in \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$. Define $\Phi((\varphi_i))\colon M \to \prod_{i\in I} N_i$ by

$$\Phi((\varphi_i))(x) := (\varphi_i(x))$$

for all $x \in M$. Then clearly Φ and Ψ are inverse to each other. Indeed, let $\varphi \in \mathrm{Hom}_R(M, \prod_{i\in I} N_i)$. Then

$$\begin{aligned}\Phi(\Psi(\varphi))(x) &= \Phi((\pi_i \circ \varphi))(x)\\ &= ((\pi_i \circ \varphi)(x))\\ &= \varphi(x)\end{aligned}$$

for all $x \in M$. Thus $\Phi(\Psi(\varphi)) = \varphi$. Conversely, let $(\varphi_i) \in \prod_{i\in I} \mathrm{Hom}_R(M, N_i)$. Then

$$\begin{aligned}\Psi(\Phi(\varphi_i)) &= (\pi_i \circ \Phi(\varphi_i))\\ &= (\pi_i \circ \varphi))\\ &= \varphi(x)\end{aligned}$$

Finally, note that Ψ is graded since π_λ is graded of degree 0 for all $\lambda \in \Lambda$. □

2.5.12 Left Exactness of $\mathrm{Hom}_R^\star(M, -)$ and $\mathrm{Hom}_R^\star(-, N)$

Let M and N be graded R-modules. Recall that both $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, N)$ are left exact functors from the category of R-modules to itself. The graded version of these functors are

$$\mathrm{Hom}_R^\star(M, -)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R \quad \text{and} \quad \mathrm{Hom}_R^\star(-, N)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R.$$

We want to check that they are also left exact functors. Let's focus on $\mathrm{Hom}_R^\star(-, N)$ first:

Proposition 2.11. *The sequence of graded R-modules and graded homomorphisms*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{2}$$

is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R^\star(M_3, N) \xrightarrow{\varphi_2^*} \mathrm{Hom}_R^\star(M_2, N) \xrightarrow{\varphi_1^*} \mathrm{Hom}_R^\star(M_1, N) \tag{3}$$

is exact.

Proof. Suppose that (28) is exact and let N be any R-module. Exactness at $\mathrm{Hom}_R^\star(M_3, N)$ follows from the fact that φ_2^* is injective (which follows from the fact that $\mathrm{Hom}_R(-, N)$ is left exact). Next we show exactness at $\mathrm{Hom}_R^\star(M_2, N)$. Let $\psi_2\colon M_2 \to N$ be a graded homomorphism of degree i such that $\psi_2\varphi_1 = 0$. By left exactness of $\mathrm{Hom}_R(-, N)$, there exists a $\psi_3 \in \mathrm{Hom}_R(M, N)$ such that $\psi_2 = \psi_3\varphi_2$. Since φ_2 is surjective, ψ_3 is graded of degree i. Thus $\psi_3 \in \mathrm{Hom}_R^\star(M, N)$. Thus we have exactness at $\mathrm{Hom}_R^\star(M_2, N)$. □

2.5.13 Projective Objects and Injective Objects in $\mathbf{Grad}_R$

$\mathrm{Hom}_R^\star(\bigoplus_\lambda P_\lambda, B) \cong \prod_\lambda \mathrm{Hom}_R^\star(P_\lambda, B)$ and $\mathrm{Hom}_R^\star(A, \prod_\lambda^\star E_\lambda) \cong \prod_\lambda^\star \mathrm{Hom}_R^\star(A, E_\lambda)$.

2.6 Noetherian Graded Rings and Modules

2.6.1 The Irrelevant Ideal

Definition 2.8. Let R be a graded ring. The **irrelevant ideal of** R is defined to be

$$R_+ := \bigoplus_{i>0} R_i.$$

It is straightforward to check that R_+ is in fact an ideal of R and that $R/R_+ \cong R_0$.

2.6.2 Noetherian Graded Rings

The following lemma will be used many times without mention.

Lemma 2.2. *Let R be a ring and let $S \subseteq R$. Suppose the ideal $\langle S\rangle$ generated by S is finitely generated. Then we can choose the generators to be in S.*

Proof. Since $\langle S \rangle$ is finitely generated, there are $x_1, \dots, x_n \in \langle S \rangle$ such that $\langle S \rangle = \langle x_1, \dots, x_n \rangle$. In particular we have

$$x_i = \sum_{j=1}^{n_i} r_{ji} s_{ji}$$

where for each $1 \leq i \leq n$ we have $n_i \in \mathbb{N}$, and for each $1 \leq j \leq n_i$ we have $r_{ji} \in R$ and $s_{ji} \in S$. In particular, this means

$$\langle S \rangle = \langle s_{ji} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i \rangle.$$

□

Definition 2.9. A **Noetherian** graded ring is a graded ring whose underlying ring is Noetherian.

Proposition 2.12. *Let R be a graded ring. Suppose $R_+ = \langle \{x_\lambda\}_{\lambda \in \Lambda} \rangle$. Then the R_0-algebra map*

$$\varphi\colon R_0[\{X_\lambda\}] \to R$$

given by $\varphi(X_\lambda) = x_\lambda$ for all $\lambda \in \Lambda$ is surjective. In other words, if a subset $S \subset R_+$ generates the irrelevant ideal R_+ as an R-ideal, then it generates R as an R_0-algebra.

Proof. It suffices to show that $R_k \subset \operatorname{im} \varphi$ for all $k \in \mathbb{N}$. We prove this by induction on k. The base case $k = 0$ is trivial. Now suppose it is true for all $i < k$ for some $k > 0$ and let $a \in R_k$. Since $R = R_0 \oplus R_+$, we have a unique decomposition

$$a = a_0 + x$$

where $a_0 \in R_0$ and $x \in R_+$. Since $R_+ = \langle \{x_\lambda\} \rangle$ and $x \in R_+$, there exists $x_{\lambda_1}, \dots, x_{\lambda_n} \in \{x_\lambda\}$ and $a_m \in R_{k - \deg x_{\lambda_m}}$ for all $1 \leq m \leq n$ such that

$$x = a_1 x_{\lambda_1} + \cdots + a_n x_{\lambda_n}.$$

Choose $A_m \in R_0[\{X_\lambda\}]$ such that $\varphi(A_m) = a_m$ for all $0 \leq m \leq n$ (we can do this by induction). Then

$$\begin{aligned} a &= a_0 + a_1 x_{\lambda_1} + \cdots + a_n x_{\lambda_n} \\ &= \varphi(A_0) + \varphi(A_1)\varphi(X_{\lambda_1}) + \cdots + \varphi(A_n)\varphi(X_{\lambda_n}) \\ &= \varphi(A_0 + A_1 X_{\lambda_1} + \cdots + A_n X_{\lambda_n}). \end{aligned}$$

This implies $R_k \subset \operatorname{im} \varphi$. Therefore φ is surjective. □

Proposition 2.13. *Let R be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely-generated as an R_0-algebra.*

Proof. Suppose R_0 is Noetherian and R is finitely-generated as an R_0-algebra. Then there exists an $n \geq 0$ and a surjection

$$R_0[X_1, \dots, X_n] \to R.$$

where $R_0[X_1, \dots, X_n]$ is a polynomial algebra over Noetherian ring, and hence Noetherian, which implies that R is Noetherian, as it is a quotient of a Noetherian ring.

Now suppose R is Noetherian. Since $R_0 \cong R/R_+$, we see that R_0 must be Noetherian since it is the quotient of a Noetherian ring. Since R is Noetherian, the irrelevant ideal R_+ is finitely-generated, say by $x_1, \dots, x_n \in R_+$. Since R is graded, we have a surjective R_0-algebra map

$$R_0[X_1, \dots, X_n] \to R$$

sending $X_i \mapsto x_i$ for all $1 \leq i \leq n$. It follows that R is a finitely-generated R_0-algebra. □

2.7 Localization of Graded Rings

Definition 2.10. If $S \subset R$ is a multiplicative subset of a graded ring R consisting of homogeneous elements, then $S^{-1}R$ is a $\mathbb{Z}$-graded ring: we let the homogeneous elements of degree n be of the form r/s where $r \in R_{n + \deg s}$. We write $R_{(S)}$ for the subring of elements of degree zero; there is thus a map $R_0 \to R_{(S)}$.

If S consists of the powers of a homogeneous element f, we write $R_{(f)}$ for R_S. If $\mathfrak{p}$ is a homogeneous ideal and S is the set of homogeneous elements of R not in $\mathfrak{p}$, we write $R_{(\mathfrak{p})}$ for $R_{(S)}$.

More generally if M is a graded R-module, then we define $M_{(S)}$ to be the submodule of $S^{-1}M$ consisting of elements of degree zero. When S consists of powers of a homogeneous element $f \in R$, then we write $M_{(f)}$ instead of $M_{(S)}$. We similarly define $M_{(\mathfrak{p})}$ for a homogeneous prime ideal $\mathfrak{p}$.

2.8 Graded R-Algebras

An R-algebra A is an R-module equipped with an R-linear map $A \otimes_R A \to A$, denoted $a \otimes b \mapsto ab$. This means that for all $r \in R$ and $a, b \in A$, we have

$$r(ab) = (ra)b = a(rb),$$

and for all $a, b, c \in A$, we have

$$(a+b)c = ab + ac \quad \text{and} \quad a(b+c) = ab + ac.$$

We say the R-algebra is **associative** when for all $a, b, c \in A$, we have

$$(ab)c = a(bc).$$

We say the R-algebra is **unital** when there exists an element $e \in A$ such that for all $a \in A$, we have

$$ae = a = ea.$$

Unless otherwise specified, all R-algebras discussed are assumed to be associative and unital, so they are genuinely rings (perhaps not commutative) and being an R-algebra just means they have a little extra structure related to scaling by R. If A is an R-algebra, then can view R as sitting inside A via the map $\varphi \colon R \to A$, given by

$$\varphi(r) = 1 \cdot r$$

for all $r \in R$, though this map need not be injective.

Definition 2.11. An H**-graded** R**-algebra** A is an R-algebra which is also H-graded as a ring. So there is a direct sum decomposition

$$A = \bigoplus_{h \in H} A_h,$$

where the A_h are abelian groups which satisfy the property that if $a_{h_1} \in A_{h_1}$ and $a_{h_2} \in A_{h_2}$, then $a_{h_1} a_{h_2} \in A_{h_1+h_2}$. If R is also an H-graded ring, then we also require A to be an H-graded left R-module. This means that if $r_{h_1} \in R_{h_1}$ and $a_{h_2} \in A_{h_2}$, then $r_{h_1} a_{h_2} \in A_{h_1+h_2}$.

2.8.1 Examples of Graded R-Algebras

Example 2.7. Let R be a graded ring and let $\boldsymbol{x} = x_1, \dots, x_n$. The polynomial ring $R[\boldsymbol{x}]$ over R is both an $\mathbb{N}$-graded R-algebra and an $\mathbb{N}^n$-graded R-algebra. The homogeneous component in degree i with respect to the $\mathbb{N}$-grading is given by

$$R[\boldsymbol{x}]_i = \sum_{\boldsymbol{\alpha}} R_{i-|\boldsymbol{\alpha}|} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

The homogeneous component in degree $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ with respect to the $\mathbb{N}^n$-grading is given by

More generally, let $\boldsymbol{w} := (w_1, \dots, w_n)$ be an n-tuple of positive integers. We define the **weighted degree of a monomial** of a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, denoted $\deg_w(\boldsymbol{x}^{\boldsymbol{\alpha}})$, by the formula

$$\deg_w(\boldsymbol{x}^{\boldsymbol{\alpha}}) := \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle := \sum_{\lambda=1}^{n} w_\lambda \alpha_\lambda.$$

The **weighted polynomial ring with respect to the weighted vector** $\boldsymbol{w}$, denoted $R[\boldsymbol{x}]^{\boldsymbol{w}}$, is the polynomial ring $R[\boldsymbol{x}]$ equipped with the **weighted grading**: the homogeneous component in degree i is given by

$$R[\boldsymbol{x}]_i^{\boldsymbol{w}} = \sum_{\boldsymbol{\alpha}} R_{i-\langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle} \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Example 2.8. Let K be a field, let $R = K[x,y]/\langle xy \rangle$, and let $A = R[z,w]$. View R as a graded K-algebra with $|x| = 1$ and $|y| = 2$ and view A as a graded R-algebra with $|z| = 1$ and $|w| = 3$. Then the homogeneous components of A start out as

$$\begin{aligned}
A_0 &= K \\
A_1 &= K\overline{x} + Kz \\
A_2 &= K\overline{x}^2 + K\overline{x}z + K\overline{y} \\
A_3 &= K\overline{x}^3 + K\overline{x}^2 z + K\overline{xy} + K\overline{x}z^2 + K\overline{y}z + Kw \\
&\vdots
\end{aligned}$$

Example 2.9. Let R be a ring and let Q be an ideal in R. The **blowup algebra of** Q **in** R is defined by

$$B_Q(R) := R + tQ + t^2Q^2 + t^3Q^3 + \cdots \cong \bigoplus_{i=0}^{\infty} Q^i.$$

Elements in $B_Q(R)$ have the form

$$t^{i_1}x_{i_1} + \cdots + t^{i_m}x_{i_m}$$

where $0 \leq i_1 < \cdots < i_m$ and $x_{i_\lambda} \in Q^{i_\lambda}$ for all $1 \leq \lambda \leq m$. The t^{i_λ} part keeps track of what degree we are in. We define multiplaction on elements of the form $t^i x$ and $t^j y$ by

$$(t^i x)(t^j y) = t^{i+j}xy,$$

and we extend this to all of $B_Q(R)$ in the obvious way. This gives $B_Q(R)$ the structure of a graded R-algebra.
If Q is finitely generated, say $Q = \langle a_1, \ldots, a_n \rangle$, then there is a unique R-algebra homomorphism

$$\varphi\colon R[u_1, \ldots, u_n] \to B_Q(R),$$

such that $\varphi(u_\lambda) = ta_\lambda$ for all $1 \leq \lambda \leq n$.

2.8.2 Graded Associative R-Algebras

Let R be a ring and let $\boldsymbol{x} = x_1, \ldots, x_n$ be a list of indeterminates. We denote by $R\langle \boldsymbol{x} \rangle$ to be the **free** R**-algebra generated by** $\boldsymbol{x}$. A basis of $R\langle \boldsymbol{x} \rangle$ as an R-module consists of **words**:

$$\boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k}$$

where $k \in \mathbb{N}$ and $\boldsymbol{\alpha}_j \in \mathbb{N}^n$ for all $1 \leq j \leq k$. For example, in $R\langle x_1, x_2, x_3 \rangle$, we have

$$\boldsymbol{x}^{\boldsymbol{\alpha}_1}\boldsymbol{x}^{\boldsymbol{\alpha}_2}\boldsymbol{x}^{\boldsymbol{\alpha}_3} = x_3^2 x_1^3 x_2 x_3 x_2,$$

where

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (0, 0, 2) \\ \boldsymbol{\alpha}_2 &= (3, 2, 1) \\ \boldsymbol{\alpha}_3 &= (0, 1, 0). \end{aligned}$$

The set of all words is denoted $\mathrm{W}(\boldsymbol{x})$. Words of the form $\boldsymbol{x}^{\boldsymbol{\alpha}}$ are called **standard words** and form a subset of the set of all words. A **standard polynomial** in $R\langle \boldsymbol{x} \rangle$ is a finite linear combination of standard words.

Example 2.10. Let R be a graded ring, let $\boldsymbol{x} = x_1, \ldots, x_n$ be a list of indeterminates, and let $\boldsymbol{w} := (w_1, \ldots, w_n)$ be an n-tuple of positive integers. We define $R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}}$ to be the graded R-algebra whose homogeneous component in degree i is given by

$$R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} = \sum_{\boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k} \in \mathrm{W}(\boldsymbol{x})} R_{i - \sum_{j=1}^k \langle \boldsymbol{w}, \boldsymbol{\alpha}_j \rangle} \boldsymbol{x}^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{x}^{\boldsymbol{\alpha}_k}.$$

2.8.3 Graded Commutative R-Algebras

Definition 2.12. Let A be a $\mathbb{Z}$-graded R-algebra. We say A is **graded-commutative** if for all $a \in A_i$ and $b \in A_j$, we have

$$ab = (-1)^{ij}ba. \tag{4}$$

We say A is **strictly graded-commutative** if, an addition to (4), we also have $a^2 = 0$ for all odd degree elements $a \in A$.

Remark. Cohomology rings are a natural source of graded-commutative rings.

Every finitely-presented R-algebra A is isomorphic to $R\langle \boldsymbol{x} \rangle / I$ where $\boldsymbol{x} = x_1, \ldots, x_n$ and where I is a two-sided ideal in $R\langle \boldsymbol{x} \rangle$. For our purposes we will be interested in the following finitely-presented R-algebra.

Definition 2.13. Let R be a ring, let $\boldsymbol{x} = x_1, \ldots, x_n$ be indeterminates, and let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be their respective weights. Set

$$J = \langle \{fg - (-1)^{ij} gf \mid f \in R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} \text{ and } g \in R\langle \boldsymbol{x} \rangle_j^{\boldsymbol{w}}\} \cup \{f^2 \mid f \in R\langle \boldsymbol{x} \rangle_i^{\boldsymbol{w}} \text{ where } i \text{ is odd} \rangle.$$

We define the **free graded-(strictly)-commutative R-algebra generated by $\boldsymbol{x}$ with respect to the weighted vector $\boldsymbol{w}$**, denoted $R\lceil \boldsymbol{x} \rceil_{\boldsymbol{w}}$, to be the graded R-algebra

$$R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}} := R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}} / J.$$

Since $x_\lambda x_\mu - (-1)^{w_\lambda w_\mu} x_\mu x_\lambda \in J$ for all $1 \leq \lambda < \mu \leq n$, we see that every $\overline{f} \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$ can be represented by a standard polynomial $f \in R\langle \boldsymbol{x} \rangle^{\boldsymbol{w}}$. We typically dispense with the overline notation and just write $f \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$. In particular, any $f \in R\lceil \boldsymbol{x} \rceil^{\boldsymbol{w}}$ can be expressed as

$$f = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$$

where the sum ranges over all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $r_{\boldsymbol{\alpha}} = 0$ for almost all $\boldsymbol{\alpha} \in \mathbb{N}^n$.

2.9 Hilbert Function and Dimension

The Hilbert function of a graded module associates to an integer i the dimension of the ith graded part of the given module. For sufficiently large i, the values of this function are given by a polynomial, the Hilbert polynomial.

Definition 2.14. Let R be a Noetherian graded K-algebra and let M be a finitely-generated graded R-module. The **Hilbert function** $H_M \colon \mathbb{Z} \to \mathbb{Z}$ of M is defined by

$$H_M(i) := \dim_K(M_i)$$

Lemma 2.3. *Let R be a Noetherian graded ring and let $i \in \mathbb{Z}$. Then R_i is a finitely-generated R_0-module.*

Proof. The ideal $\langle R_i \rangle$ is finitely-generated since R is Noetherian. Choose generators in $\langle R_i \rangle$ such that each generator belongs to R_i, say $x_1, \ldots, x_n \in R_i$. In particular, $\langle R_i \rangle$ is a graded ideal with $\langle R_i \rangle_0 = R_i$. It follows that

$$R_i = R_0 x_1 + \cdots + R_0 x_n,$$

and so R_i is a finitely-generated R_0-module. □

Corollary. *Let R be a Noetherian graded ring and let M be a finitely-generated graded R-module. Then M_i is a finitely-generated R_0-module for all $i \in \mathbb{Z}$. Moreover, there exists $k \in \mathbb{Z}$ such that $M_j = 0$ for all $j < \mathbb{Z}$.*

Proof. Choose homogeneous generators of M, say $u_1, \ldots, u_n$, and let $i \in \mathbb{Z}$. Then

$$M_i = R_{i - \deg(u_1)} u_1 + \cdots + R_{i - \deg(u_n)} u_n.$$

This implies that M_i is a finitely-generated R_0-module since the R_i's are finitely generated R_0-modules by Lemma (2.3).

For the moreover part, let

$$k = \min\{\deg(u_i) \mid 1 \leq i \leq n\}.$$

Then $M_j = 0$ for all $j < k$ since $R_i = 0$ for all $i < 0$. □

2.10 Semigroup Ordering

Definition 2.15. Let H be an additive semigroup with identity 0. A **semigroup ordering** on H is a partial ordering $>$ on H such that

1. $>$ is a total ordering, i.e. either $h_1 > h_2$ or $h_2 > h_1$ for all $h_1, h_2 \in H$.
2. $>$ is translate invariant, i.e. $h_1 > h_2$ implies $h_1 + h_3 > h_2 + h_3$ for all $h_1, h_2, h_3 \in H$.

If $>$ is a semigroup ordering on H, then we call the pair $(H, >)$ an **additive ordered semigroup**.

Example 2.11. The integers $\mathbb{Z}$ (or the natural numbers $\mathbb{N}$) equipped with the natural order $>$ forms an additive ordered semigroup.

Example 2.12. For $n > 1$, there are many different semigroup orderings we can equip $\mathbb{N}^n$ (or even $\mathbb{Z}^n$). For example, one of them is call **lexicographical ordering**, which is defined as follows: for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, we say $\boldsymbol{\alpha} >_{\text{lex}} \boldsymbol{\beta}$ if for some $1 \leq i \leq n$ we have

$$\begin{gathered} \alpha_1 = \beta_1 \\ \vdots \\ \alpha_{i-1} = \beta_{i-1} \\ \alpha_i > \beta_i \end{gathered}$$

Theorem 2.4. *Let $(H, >)$ be an additive ordered semigroup, let R be a Noetherian H-graded ring, and let M be a Noetherian H-graded R-module. Then every associated prime $\mathfrak{p}$ of M is a homogeneous ideal.*

Proof. If $\mathfrak{p}$ is an associated prime of M, it is the annihilator of a nonzero element

$$u = u_{j_1} + \cdots + u_{j_t} \in M,$$

where the u_{j_ν} are nonzero homogeneous elements of degrees $j_1 < \cdots < j_t$. Choose u such that t is as small as possible. Suppose that

$$a = a_{i_1} + \cdots + a_{i_s}$$

kills u, where for every ν, a_{i_ν} has degree i_ν, and $i_1 < \cdots < i_s$. We shall show that every a_{i_ν} kills u, which proves that $\mathfrak{p}$ is homogeneous. It suffices to show that a_{i_1} kills u (since $a - a_{i_1}$ kills u and we can proceed by induction). Since $au = 0$, the unique least degree term $a_{i_1} u_{j_1} = 0$. Therefore

$$u' = a_{i_1} u = a_{i_1} u_{j_2} + \cdots + a_{i_1} u_{j_t}.$$

If this element is nonzero, its annihilator is still $\mathfrak{p}$, since $Ru \cong R/\mathfrak{p}$ and every nonzero element has annihilator $\mathfrak{p}$. Since $a_{i_1} u_{j_\nu}$ is homogeneous of degree $i_1 + j_\nu$, or else is 0, u' has fewer nonzero homogeneous components than u does, contradicting our choice of u. □

Corollary. *If I is a homogeneous ideal of a Noetherian ring R graded by a semigroup H equipped with a semigroup ordering $>$, then every minimal prime of I is homogeneous.*

Proof. This is immediate, since the minimal primes of I are among the associated primes of R/I. □

Proposition 2.14. *Let $(H, >)$ be an additive ordered semigroup, let R be a H-graded ring, and let I be a homogeneous ideal. Then $\sqrt{I}$ is homogeneous.*

Proof. Let

$$f_{i_1} + \cdots + f_{i_k} \in \sqrt{I}$$

with $i_1 < \cdots < i_k$ and each f_{i_j} nonzero of degree i_j. We need to show that every $f_{i_j} \in \sqrt{I}$. If any of the components are in $\sqrt{I}$, we may subtract them off, giving a similar sum whose terms are the homogeneous components not in $\sqrt{I}$. Therefore it suffices to show that $f_{i_1} \in \sqrt{I}$. But

$$\left(f_{i_1} + \cdots + f_{i_k}\right)^N \in I$$

for some $N > 0$. When we expand, there is a unique term formally of least degree, namely $f_{i_1}^N$, and therefore this term is in I, since I is homogeneous. But this means that $f_{i_1} \in \sqrt{I}$, as required. □

Corollary. *Let R be a finitely-generated graded K-algebra and let $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$ be the homogeneous maximal ideal of R. Then*

$$\dim R = \text{height}\, \mathfrak{m} = \dim R_{\mathfrak{m}}.$$

Proof. The dimension of R will be equal to the dimension of $R/\mathfrak{p}$ for one of the minimal primes $\mathfrak{p}$ of R. Since $\mathfrak{p}$ is minimal, it is an associated prime and therefore is homogeneous. Hence, $\mathfrak{p} \subseteq \mathfrak{m}$. The domain $R/\mathfrak{p}$ is finitely-generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathfrak{m}/\mathfrak{p}$. Thus,

$$\begin{aligned} \dim R &= \dim R/\mathfrak{p} \\ &= \dim (R/\mathfrak{p})_{\mathfrak{m}} \\ &\leq \dim R_{\mathfrak{m}} \\ &\leq \dim R, \end{aligned}$$

and so equality holds throughout, as required. □

3 Homological Algebra

Throughout this section, let R be a ring (trivially graded).

3.1 R-Complexes

3.1.1 R-Complexes and Chain Maps

Definition 3.1. An R**-complex** (A, d) is a graded R-module A equipped with graded R-linear map $\mathrm{d}\colon A \to A$ of degree -1 such that $\mathrm{d}^2 = 0$. Any such map d which satisfies these properties is called an R**-linear differential**. If we denote the ith homogeneous component of A as A_i and if we denote $\mathrm{d}_i = \mathrm{d}|_{A_i}$, then we may view an R-complex as a sequence of R-modules A_i and R-linear maps $\mathrm{d}_i\colon A_i \to A_{i-1}$ as below

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\mathrm{d}_{i+1}} A_i \xrightarrow{\mathrm{d}_i} A_{i-1} \longrightarrow \cdots \tag{5}$$

such that $\mathrm{d}_i\mathrm{d}_{i+1} = 0$ for all $i \in \mathbb{Z}$. An element in $\ker \mathrm{d}$ is called a **cycle** of (A, d) and an element in $\operatorname{im} \mathrm{d}$ is called a **boundary** of (A, d).

A **chain map** $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ between R-complexes (A, d) and (A', d') is a graded R-linear map $\varphi\colon A \to A'$ of degree 0 which commutes with the differentials:

$$\mathrm{d}'\varphi = \varphi\mathrm{d}.$$

If we denote $\varphi_i = \varphi|_{A_i}$, then we may view φ as a sequence of R-linear maps $\varphi_i\colon A_i \to A'_i$ as below

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "\mathrm{d}_{i+1}"] \arrow[d, "\varphi_{i+1}"] & A_i \arrow[r, "\mathrm{d}_i"] \arrow[d, "\varphi_i"] & A_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \\
\cdots \arrow[r] & A'_{i+1} \arrow[r, "\mathrm{d}'_{i+1}"] & A'_i \arrow[r, "\mathrm{d}'_i"] & A'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $\mathrm{d}'_i\varphi_i = \varphi_{i-1}\mathrm{d}'_i$ for all $i \in \mathbb{Z}$. It is easy to check that the identity map $1_{(A,\mathrm{d})}\colon (A, \mathrm{d}) \to (A, \mathrm{d})$ from an R-complex (A, d) to itself is a chain map. It is also easy to check that the composition of two chain maps is a chain map. We obtain the category $\mathbf{Comp}_R$, whose objects are R-complexes and whose morphisms chain maps.

Remark. To simplify notation, we often write A instead of (A, d) if the differential is understood from context. For instance, we may introduce an R-complex as "(A, d)" but later refer to it as "A", but we also may introduce an R-complex as "A" with the differential understood to be denoted "d_A". In that case, we will denote $\mathrm{d}_{A,i} = (\mathrm{d}_A)|_{A_i}$. Also a chain map is always understood to be a map between R-complexes. For instance, if we write "let $\varphi\colon A \to A'$ be a chain map" without first introducing A or A', then it is understood that A and A' are R-complexes.

3.1.2 Homology

Let (A, d) be an R-complex. The condition $\mathrm{d}^2 = 0$ is equivalent to the condition $\ker \mathrm{d} \supseteq \operatorname{im} \mathrm{d}$. Since d is graded, we see that both $\ker \mathrm{d}$ and $\operatorname{im} \mathrm{d}$ are graded submodules of A. Therefore we have

$$\ker \mathrm{d} = \bigoplus_{i \in \mathbb{Z}} \ker \mathrm{d}_i \quad \text{and} \quad \operatorname{im} \mathrm{d} = \bigoplus_{i \in \mathbb{Z}} \operatorname{im} \mathrm{d}_i,$$

and for each $i \in \mathbb{Z}$, we have $\ker \mathrm{d}_i \supseteq \operatorname{im} \mathrm{d}_{i+1}$. Therefore $\ker \mathrm{d}/\operatorname{im} \mathrm{d}$ is a graded R-module. With this in mind, we are justified in making the following definitions:

Definition 3.2. Let (A, d) be an R-complex.

1. We say A is **exact** if $\ker \mathrm{d} = \operatorname{im} \mathrm{d}$ and we say A is **exact at** A_i if $\ker \mathrm{d}_i = \operatorname{im} \mathrm{d}_i$.
2. The **homology** of A is defined to be the graded R-module

$$\mathrm{H}(A, \mathrm{d}) := \ker \mathrm{d}/\operatorname{im} \mathrm{d}.$$

   The ith homogeneous component of $\mathrm{H}(A, \mathrm{d})$ is denoted

$$\mathrm{H}_i(A, \mathrm{d}) := \ker \mathrm{d}_i/\operatorname{im} \mathrm{d}_i.$$

Remark. If the differential d is clear from context, then we will simplify our notation by denoting the homology of A as $\mathrm{H}(A)$ rather than $\mathrm{H}(A, \mathrm{d})$.

3.1.3 Positive, Negative, and Bounded Complexes

Definition 3.3. Let A be an R-complex.

1. We say A is **positive** if $A_i = 0$ for all $i < 0$.
2. We say A is **bounded below** if $A_i = 0$ for $i \ll 0$. In other words, if A_i is eventually 0, that is, if there exists $n \in \mathbb{Z}$ such that $A_i = 0$ for all $i < n$.
3. We say A is **homologically bounded below** if $\mathrm{H}_i(A) = 0$ for $i \ll 0$.

Similarly,

1. We say A is **negative** if $A_i = 0$ for all $i > 0$.
2. We say A is **bounded above** if $A_i = 0$ for $i \gg 0$.
3. We say A is **homologically bounded above** if $\mathrm{H}_i(A) = 0$ for $i \gg 0$.

If A is both bounded below and bounded above, then we will say A is **bounded**. Similarly, if A is both homologically bounded above and homologically bounded below, then we will say A is **homologically bounded**.

3.1.4 Supremum and Infimum

Definition 3.4. Let A be an R-complex. We define its **supremum** to be

$$\sup A := \begin{cases} -\infty & \text{if } A \text{ is exact} \\ \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(A) \neq 0\} & \text{if } A \text{ is not exact and is homologically bounded above} \\ \infty & \text{if } A \text{ is not homologically bounded above.} \end{cases}$$

Similarly, we define its **infimum** to be

$$\inf A := \begin{cases} \infty & \text{if } A \text{ is exact} \\ \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(A) \neq 0\} & \text{if } A \text{ is not exact and is homologically bounded below} \\ -\infty & \text{if } A \text{ is not homologically bounded below.} \end{cases}$$

The **amplitude** of A is defined to be

$$\operatorname{amp} A := \begin{cases} -\infty & \text{if } A \text{ is exact} \\ \infty & \text{if } A \text{ is homologically bounded above but not homologically bounded below} \\ \sup A - \inf A & \text{if } A \text{ is not exact and homologically bounded} \\ \infty & \text{if } A \text{ is homologically bounded below but not homologically bounded above} \\ \infty & \text{if } A \text{ is not homologically bounded above or below.} \end{cases}$$

3.2 Category of R-Complexes

The set of all R-complexes together with the set of all chain maps forms a category, which we denote $\mathbf{Comp}_R$. Similarly, the set of all graded R-modules together with the set of all graded homomorphisms (of degree 0) forms a category, which we denote $\mathbf{Grad}_R$.

3.2.1 Homology Considered as a Functor

We've already seen that if (A, d) is an R-complex, then $\mathrm{H}(A)$ is a graded R-module. We would like to extend this observation to get a functor $\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$. This will follow from the following three propositions:

Proposition 3.1. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map. Then φ induces a graded homomorphism $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$, where*

$$\mathrm{H}(\varphi)(\overline{a}) = \overline{\varphi(a)} \tag{6}$$

for all $\overline{a} \in \mathrm{H}(A)$.

Proof. First let us check that the target of each element in $\mathrm{H}(A)$ under $\mathrm{H}(\varphi)$ lands in $\mathrm{H}(A')$. Let $\overline{a} \in \mathrm{H}(A)$ (so $\mathrm{d}(a) = 0$). Then $\overline{\varphi(a)} \in \mathrm{H}(A')$ since

$$\begin{aligned} \mathrm{d}'(\varphi(a)) &= \varphi(\mathrm{d}(a)) \\ &= 0. \end{aligned}$$

Next let us check that that $\mathrm{H}(\varphi)$ is well-defined. Let $a + \mathrm{d}(b)$ be another representative of the coset class $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a + \mathrm{d}(b)}) &= \overline{\varphi(a + \mathrm{d}(b))} \\ &= \overline{\varphi(a) + \varphi(\mathrm{d}(b))} \\ &= \overline{\varphi(a)} + \overline{\varphi(\mathrm{d}(b))} \\ &= \overline{\varphi(a)} + \overline{\mathrm{d}'(\varphi(b))} \\ &= \overline{\varphi(a)} \\ &= \mathrm{H}(\varphi)(\overline{a}). \end{aligned}$$

Thus $\mathrm{H}(\varphi)$ is well-defined.

So far we have shown that $\mathrm{H}(\varphi)$ is a function. To see that $\mathrm{H}(\varphi)$ is an R-module homomorphism, let $r, s \in R$ and $a, b \in A$. Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{ra + sb}) &= \overline{\varphi(ra + sb)} \\ &= \overline{r\varphi(a) + s\varphi(b)} \\ &= r\overline{\varphi(a)} + s\overline{\varphi(b)} \\ &= r\mathrm{H}(\varphi)(\overline{a}) + s\mathrm{H}(\varphi)(\overline{b}). \end{aligned}$$

Finally, to see that $\mathrm{H}(\varphi)$ is graded, let $\overline{a}_i \in \mathrm{H}_i(A)$ (so $a_i \in A_i$). Then

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a}_i) &= \overline{\varphi(a_i)} \\ &\in \mathrm{H}_i(A') \end{aligned}$$

since φ is graded. □

Proposition 3.2. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ and $\varphi'\colon (A', \mathrm{d}') \to (A'', \mathrm{d}'')$ be two chain maps. Then*

$$\mathrm{H}(\varphi' \circ \varphi) = \mathrm{H}(\varphi') \circ \mathrm{H}(\varphi).$$

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then we have

$$\begin{aligned} \mathrm{H}(\varphi' \circ \varphi)(\overline{a}) &= \overline{(\varphi' \circ \varphi)(a)} \\ &= \overline{\varphi'(\varphi(a))} \\ &= \mathrm{H}(\varphi')(\overline{\varphi(a)}) \\ &= \mathrm{H}(\varphi')(\mathrm{H}(\varphi)(\overline{a})) \\ &= (H(\varphi') \circ H(\varphi))(\overline{a}). \end{aligned}$$

□

Proposition 3.3. *Let (A, d) be an R-complex. Then we have*

$$\mathrm{H}(\mathrm{id}_{(A,\mathrm{d})}) = \mathrm{id}_{\mathrm{H}(A)}.$$

In particular, if $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is a chain map isomorphism, then $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism between graded R-modules $\mathrm{H}(A)$ and $\mathrm{H}(A')$.

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned} \mathrm{H}(\mathrm{id}_{(A,\mathrm{d})})(\overline{a}) &= \overline{\mathrm{id}_{(A,\mathrm{d})}(a)} \\ &= \overline{a} \\ &= \mathrm{id}_{\mathrm{H}(A)}(\overline{a}). \end{aligned}$$

For the latter statement, let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map isomorphism and let $\psi\colon (A', \mathrm{d}') \to (A, \mathrm{d})$ be its inverse. Then

$$\begin{aligned}\mathrm{id}_{\mathrm{H}(A)} &= \mathrm{H}(\mathrm{id}_{(A,d)}) \\ &= \mathrm{H}(\psi \circ \varphi) \\ &= \mathrm{H}(\psi) \circ \mathrm{H}(\varphi).\end{aligned}$$

A similar computation gives $\mathrm{H}(\varphi) \circ \mathrm{H}(\psi) = \mathrm{id}_{\mathrm{H}(A')}$. □

3.2.2 $\mathbf{Comp}_R$ is an R-linear category

There is more structure on the categories $\mathbf{Comp}_R$ and $\mathbf{Grad}_R$ which we haven't discussed so far. They are examples of R-linear categories[3]. Moreover, homology can be viewed as an additive functor from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$.

Proposition 3.4. $\mathbf{Comp}_R$ *is an R-linear category.*

Proof. Let (A, d) and (A', d') be two R-complexes. We define $\mathcal{C}(A, A')$

$$\mathcal{C}(A, A') := \mathrm{Hom}((A, \mathrm{d}), (A', \mathrm{d}')) := \{\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}') \mid \varphi \text{ is a chain map}\}.$$

Then $\mathcal{C}(A, A')$ has the structure of an R-module. Indeed, if $\varphi, \psi \in \mathcal{C}(A, A')$ and $r \in R$, then we define addition and scalar multiplication by

$$(\varphi + \psi)(a) := \varphi(a) + \psi(a) \quad \text{and} \quad (r\varphi)(a) = \varphi(ra)$$

for all $a \in A$. Since d is an R-linear map, it is clear that $\varphi + \psi$ and $r\varphi$ are chain maps (that is, they are graded R-linear maps which commute with the differentials).

Moreover, let (A'', d'') be another R-complex. We define composition

$$\circ\colon \mathcal{C}(A', A'') \times \mathcal{C}(A, A') \to \mathcal{C}(A, A''),$$

in the usual way: if $(\varphi', \varphi) \in \mathcal{C}(A', A'') \times \mathcal{C}(A, A')$, then we define $\varphi' \circ \varphi \in \mathcal{C}(A, A'')$ by

$$(\varphi' \circ \varphi)(a) = \varphi'(\varphi(a))$$

for all $a \in A$. Again one checks that $\varphi' \circ \varphi$ is indeed a chain map. Observe that composition is an R-bilinear map. For instance, let $\varphi', \psi' \in \mathcal{C}(A', A'')$ and $\varphi \in \mathcal{C}(A, A')$. Then

$$\begin{aligned}((\varphi' + \psi') \circ \varphi)(a) &= (\varphi' + \psi')(\varphi(a)) \\ &= \varphi'(\varphi(a)) + \psi'(\varphi(a)) \\ &= (\varphi' \circ \varphi)(a) + (\psi' \circ \varphi)(a)\end{aligned}$$

for all $a \in A$. Thus $(\varphi' + \psi') \circ \varphi = \varphi' \circ \varphi + \psi' \circ \varphi$. A similar proof gives the other properties of R-bilinearity. □

Remark. To clean notation, we often drop the $\circ$ symbol when denoting composotin. For instance, we often write $\varphi\psi$ rather than $\varphi \circ \psi$.

3.2.3 The inclusion functor from $\mathbf{Grad}_R$ to $\mathbf{Comp}_R$ is fully faithful

Every graded R-module M can be view as an R-complex with differential $\mathrm{d} = 0$. In fact, we obtain a functor

$$\iota\colon \mathbf{Grad}_R \to \mathbf{Comp}_R,$$

where the graded R-module M is mapped to the trivially R-complex $(M, 0)$, and where graded homomorphisms $\varphi\colon M \to M'$ is mapped to the chain map $\varphi\colon (M, 0) \to (M, 0')$ of trivially R-complexes. Clearly φ is in fact chain map since these are trivial R-complexes. The functor ι is full and faithful. It is left-adjoint to the forgetful functor

$$\rho\colon \mathbf{Comp}_R \to \mathbf{Grad}_R$$

where ρ maps the R-complex (M, d) to the graded R-module M, and where ρ maps the chain map $\varphi\colon (M, \mathrm{d}) \to (M', \mathrm{d}')$ to the graded homomorphism $\varphi\colon M \to M'$. Then ρ is still faithful, but it is not full since there may be many graded homomorphism $M \to M'$ which do not come from forgetting a chain map $(M, \mathrm{d}) \to (M', \mathrm{d}')$.

[3] See Appendix for definition of R-linear categories.

3.2.4 The homology functor from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$

There is another functor which goes from $\mathbf{Comp}_R$ to $\mathbf{Grad}_R$ which is called the **homology functor**. It is denoted

$$\mathrm{H}\colon \mathbf{Comp}_R \to \mathbf{Grad}_R,$$

and is given by mapping an R-complex (M, d) to the graded R-module $\mathrm{H}(M, \mathrm{d})$, and by mapping the chain map $\varphi\colon (M, \mathrm{d}) \to (M', \mathrm{d}')$ to the graded R-linear map $\mathrm{H}(\varphi)\colon \mathrm{H}(M, \mathrm{d}) \to \mathrm{H}(M', \mathrm{d}')$. Let us show that H is an R-linear functor.

Proposition 3.5. *Let $\varphi, \psi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be two chain maps and let $r, s \in R$. Then*

$$\mathrm{H}(r\varphi + s\psi) = r\mathrm{H}(\varphi) + s\mathrm{H}(\psi)$$

Proof. Let $\overline{a} \in \mathrm{H}(A)$. Then

$$\begin{aligned} \mathrm{H}(r\varphi + s\psi)(\overline{a}) &= \overline{(r\varphi + s\psi)(a)} \\ &= \overline{r\varphi(a) + s\psi(a)} \\ &= r\overline{\varphi(a)} + s\overline{\psi(a)} \\ &= r\mathrm{H}(\varphi)(a) + s\mathrm{H}(\psi)(a). \end{aligned}$$

□

3.2.5 Inverse Systems and Inverse Limits in the Category of R-Complexes

Definition 3.5. Let $(\Lambda, \leq)$ be a preordered set (i.e. $\leq$ is reflexive and transitive). An **inverse system** $(A_\lambda, \varphi_{\lambda\mu})$ of R-complexes and chains maps over Λ consists of a family of R-complexes $\{(A_\lambda, \mathrm{d}_\lambda)\}$ indexed by Λ and a family of chian maps $\{\varphi_{\lambda\mu}\colon A_\mu \to A_\lambda\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\lambda\mu}\varphi_{\mu\kappa}.$$

Suppose $(M_\lambda, \varphi_{\lambda\mu})$ and $(M'_\lambda, \varphi'_{\lambda\mu})$ are two direct systems over a partially ordered set $(\Lambda, \leq)$. A **morphism** $\psi\colon (M_\lambda, \varphi_{\lambda\mu}) \to (M'_\lambda, \varphi'_{\lambda\mu})$ of inverse systesms consists of a collection of graded R-linear maps $\psi_\lambda\colon M_\lambda \to M'_\lambda$ indexed by Λ such that for all $\lambda \leq \mu$ we have

$$\varphi'_{\lambda\mu}\psi_\mu = \psi_\lambda\varphi_{\lambda\mu}.$$

Proposition 3.6. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be an inverse system of graded R-modules and graded R-linear maps over a preordered set $(\Lambda, \leq)$. The inverse limit of this system, denoted $\overset{\star}{\varprojlim} M_\lambda$, is (up to unique isomorphism) given by the graded R-module*

$$\overset{\star}{\varprojlim} M_\lambda = \left\{ (u_\lambda) \in \prod_{\lambda\in\Lambda}^{\star} M_\lambda \mid \varphi_{\lambda\mu}(u_\mu) = u_\lambda \text{ for all } \lambda \leq \mu \right\}$$

together with the projection maps

$$\pi_\lambda\colon \overset{\star}{\varprojlim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$. In particular, the homogeneous component of degree i in $\overset{\star}{\varprojlim} M_\lambda$ is given by

$$(\overset{\star}{\varprojlim} M_\lambda)_i = \varprojlim M_{\lambda,i}.$$

Remark. We put a $\star$ above $\varprojlim$ to remind ourselves that this is the inverse limit in the category of all graded R-modules. In the category of all R-modules, the inverse limit is denoted by $\varprojlim M_\lambda$. If Λ is finite, then $\lim M_\lambda$ already has a natural interpretation of a graded R-module.

Proof. We need to show that $\overset{\star}{\varprojlim} M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the invserse system $(M_\lambda, \varphi_{\lambda\mu})$, so $\varphi_{\lambda\mu}\psi_\mu = \psi_\lambda$ for all $\lambda \leq \mu$. By the universal mapping property of the graded product, there exists a unique graded R-linear map $\psi\colon M \to \prod_\lambda^\star M_\lambda$ such that $\pi_\lambda\psi = \psi_\lambda$ for all $\lambda \in \Lambda$.

In fact, this map lands in $\overset{\star}{\varprojlim} M_\lambda$ since

$$\begin{aligned} \varphi_{\lambda\mu}\pi_\mu\psi(u) &= \varphi_{\lambda\mu}\psi_\mu(u) \\ &= \psi_\lambda(u) \\ &= \pi_\lambda\psi(u) \end{aligned}$$

for all $u \in M$. $\square$

3.2.6 Homology of Inverse Limit

Proposition 3.7. *Let $(A_\lambda, \varphi_{\lambda\mu})$ be an inverse system of R-complexes and chain maps indexed over a preordered set $(\Lambda, \leq)$. Suppose that each $\varphi_{\lambda\mu}$ is surjective and induces a surjective map $\varphi_{\lambda\mu}|_{\mathrm{kerd}_\mu} \colon \mathrm{kerd}_\mu \to \mathrm{kerd}_\lambda$, and suppose that $\mathrm{H}(A_\lambda) = 0$ for all λ. Then*

$$\mathrm{H}(\varprojlim A_\lambda) = 0.$$

Proof. Let $\overline{(a^n)} \in \mathrm{H}(\varprojlim A^n)$. So $\mathrm{d}^n(a^n) = 0$ and $\varphi_{m,n}(a^n) = a^m$ for all $m \leq n$. To show that $\overline{(a^n)} = 0$, we need to construct a sequence (b^n) in $\prod A^n$ such that $\mathrm{d}^n(b^n) = a^n$,
We want to construct a sequence (b_λ) such that

1. $b_\lambda \in A_\lambda$ for all λ
2. $\mathrm{d}_\lambda(b_\lambda) = a_\lambda$ for all λ
3. $\varphi_{\lambda\mu}(b_\mu) = b_\lambda$ for all λ

We will do this by induction on λ. In the base case $\lambda = 1$, we use the fact that $\mathrm{H}(A_1) = 0$ to get $b_1 \in A_1$ such that $\mathrm{d}^1(b^1) = a^1$. Now suppose that for some $n \in \mathbb{N}$, we have constructed $b^m \in A^m$ for all $m \leq n$ such that $\mathrm{d}^m(b^m) = a^m$ and $\varphi_{lm}(b^m) = b^l$ for all $l \leq m \leq n$. Using the fact that $\varphi_{n,n+1}$ is surjective on kernels, we choose $b^{n+1} \in \ker \mathrm{d}^{n+1}$ such that $\varphi_{n,n+1}(b^{n+1}) = b^n$. Observe that for any $m \leq n$, we have

$$\begin{aligned} \varphi_{m,n+1}(b^{n+1}) &= \varphi_{m,n}\varphi_{n,n+1}(b^{n+1}) \\ &= \varphi_{m,n}(b^n) \\ &= b^m, \end{aligned}$$

by induction. Using the fact that $\mathrm{H}^{n+1}(A^{n+1}) = 0$, we choose $c^{n+1} \in A^{n+1}$ such that $\mathrm{d}^{n+1}(c^{n+1}) = b^{n+1}$. $\square$

3.2.7 Homology commutes with coproducts

Proposition 3.8. *Let λ be an index set and let $(A_\lambda, \mathrm{d}_\lambda)$ be an R-complex for each $\lambda \in \Lambda$. Then*

$$\mathrm{H}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{H}(A_\lambda).$$

3.2.8 Homology commutes with graded limits

Proposition 3.9. *Let λ be an index set and let $(A_\lambda, \mathrm{d}_\lambda)$ be an R-complex for each $\lambda \in \Lambda$. Then*

$$\mathrm{H}\left(\bigoplus_{\lambda\in\Lambda} A_\lambda\right) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{H}(A_\lambda).$$

3.3 Homotopy

Definition 3.6. Let φ and ψ be two chain maps between R-complexes (A, d) and (A', d'). We say φ is **homotopic to** ψ if there exists a graded homomorphism $h \colon A \to A'$ of degree 1 such that

$$\varphi - \psi = \mathrm{d}'h + h\mathrm{d}.$$

We call h a **homotopy from** φ **to** ψ. If $\psi = 0$, then we say φ is **null-homotopic**.

3.3.1 Homotopy is an equivalence relation

Proposition 3.10. *Let $\mathcal{C}(A, A')$ denote the set of all chain maps between R-complexes (A, d) and (A', d'). Homotopy gives an equivalence relation on $\mathcal{C}(A, A')$: for two elements $\varphi, \psi \in \mathcal{C}(A, A')$, write $\varphi \sim \psi$ if φ is homotopic to ψ. Then $\sim$ is an equivalence relation.*

Proof. First we show reflexivity. Let $\varphi \in \mathcal{C}(A, A')$. Then the zero map $h = 0$ gives a homotopy from φ to itself.

Next we show symmetry. Let $\varphi, \psi \in \mathcal{C}(A, A')$ and suppose $\varphi \sim \psi$. Choose a homotopy h from φ to ψ. Then $-h$ is a homotopy from ψ to φ.

Finally we show transitivity. Let $\varphi, \psi, \omega \in \mathcal{C}(A, A')$ and suppose $\varphi \sim \psi$ and $\psi \sim \omega$. Choose a homotopy h from φ to ψ and a homotopy h' from ψ to ω. Then

$$\varphi - \psi = \mathrm{d}'h + h\mathrm{d} \qquad \text{and} \qquad \psi - \omega = \mathrm{d}'h' + h'\mathrm{d}.$$

Adding these together gives us

$$\begin{aligned} \varphi - \omega &= \mathrm{d}'h + h\mathrm{d} + \mathrm{d}'h' + h'\mathrm{d} \\ &= \mathrm{d}'(h + h') + (h + h')\mathrm{d}. \end{aligned}$$

Therefore $h + h'$ is a homotopy from φ to ω. □

3.3.2 Homotopy induces the same map on homology

Proposition 3.11. *Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). If φ is homotopic to ψ, then $\mathrm{H}(\varphi) = \mathrm{H}(\psi)$.*

Proof. Showing $\mathrm{H}(\varphi) = \mathrm{H}(\psi)$ is equivalent to showing $\mathrm{H}(\varphi - \psi) = 0$ since H is additive. Thus, we may assume that φ is null-homotopic and that we are trying to show that $\mathrm{H}(\varphi) = 0$. Let $\overline{a} \in \mathrm{H}(A, d)$. Then $\mathrm{H}(a) = 0$, and so

$$\begin{aligned} \mathrm{H}(\varphi)(\overline{a}) &= \overline{\varphi(a)} \\ &= \overline{(\mathrm{d}'h + h\mathrm{d})(a)} \\ &= \overline{\mathrm{d}'(h(a)) + h(\mathrm{d}(a))} \\ &= \overline{\mathrm{d}'(h(a))} \\ &= 0. \end{aligned}$$

□

3.3.3 The Homotopy Category of R-Complexes

Recall that $\mathbf{Comp}_R$ is an R-linear category. In particular, this means that for each pair of R-complexes A and A' we have an R-module structure on the set of all chain maps between them. This R-module is denoted by $\mathcal{C}(A, A')$. Moreover the composition map

$$\circ \colon \mathcal{C}(A', A'') \times \mathcal{C}(A, A') \to \mathcal{C}(A, A'')$$

is R-bilinear. For any two R-complexes A and A' let us denote

$$[\mathcal{C}(A, A')] := \mathcal{C}(A, A')/\sim,$$

where $\sim$ is the homotopy equivalence relation. We shall write $[\varphi]$ for the equivalence class in $[\mathcal{C}(A, A')]$ with $\varphi \in \mathcal{C}(A, A')$ as one of its representatives. We want to show that the R-module structure on $\mathcal{C}(A, A')$ induces an R-module structure on $[\mathcal{C}(A, A')]$ and that the composition map $\circ$ induces an R-bilinear map

$$[\circ] \colon [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')] \to [\mathcal{C}(A, A'')].$$

More generally, we define the **homotopy category** of all R-complexes, denoted $\mathbf{HComp}_R$, to be the category whose objects are R-complexes and whose morphisms are homotopy classes of chain maps. The next theorem will prove that this is in fact a well-defined R-linear category.

Theorem 3.1. $\mathbf{HComp}_R$ *is an R-linear category.*

Proof. Let A and A' be R-complexes. We first show that $[\mathcal{C}(A, A')]$ has an induced R-module structure. Let $[\varphi], [\psi] \in [\mathcal{C}(A, A')]$ and let $r, s \in R$. We set

$$r[\varphi] + s[\psi] := [r\varphi + s\psi]. \tag{7}$$

Let us check that (7) is in fact well-defined. Suppose $\varphi \sim \widetilde{\varphi}$ and $\psi \sim \widetilde{\psi}$. Choose a homotopy σ from φ to φ' and choose a homotopy τ from ψ to ψ'. Thus

$$\varphi - \widetilde{\varphi} = \sigma \mathrm{d} + \mathrm{d}'\sigma \quad \text{and} \quad \psi - \widetilde{\psi} = \tau \mathrm{d} + \mathrm{d}'\tau.$$

We claim that $r\sigma + s\tau$ is a homotopy from $r\varphi + s\psi$ to $r\widetilde{\varphi} + s\widetilde{\psi}$. Indeed, $\sigma + \tau$ is a graded R-linear map of degree 1 from A to A'. Moreover, we have

$$\begin{aligned} r\varphi + s\psi - (r\widetilde{\varphi} + s\widetilde{\psi}) &= r(\varphi - \widetilde{\varphi}) + s(\psi - \widetilde{\psi}) \\ &= r(\sigma \mathrm{d} + \mathrm{d}'\sigma) + s(\tau \mathrm{d} + \mathrm{d}'\tau) \\ &= (r\sigma + s\tau)\mathrm{d} + \mathrm{d}'(r\sigma + s\tau). \end{aligned}$$

Thus (7) is well-defined.

Now we will show that composition in $\mathbf{Comp}_R$ induces a well-defined R-bilinear composition operation in $\mathbf{HComp}_R$. Let A, A', and A'' be R-complexes. Let us check that composition map $\circ$ on chain maps induces an R-bilinear composition map on homotopy classes of chain maps:

$$[\circ]\colon [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')] \to [\mathcal{C}(A, A'')].$$

Let $([\varphi'], [\varphi]) \in [\mathcal{C}(A', A'')] \times [\mathcal{C}(A, A')]$. We define

$$[\circ]([\varphi'], [\varphi]) = [\varphi'\varphi]. \tag{8}$$

Let us check that (8) is in fact well-defined. Suppose $\varphi \sim \psi$ and $\varphi' \sim \psi'$. Choose a homotopy h from φ to ψ and choose a homotopy h' from φ' to ψ'. Thus

$$\varphi - \psi = h\mathrm{d} + \mathrm{d}'h \quad \text{and} \quad \varphi' - \psi' = h'\mathrm{d}' + \mathrm{d}''h'.$$

We claim that $\varphi'h + h'\psi$ is a homotopy from $\varphi'\varphi$ to $\psi'\psi$. Indeed, $\varphi'h + h'\psi$ is a graded R-linear map of degree 1 from A to A''. Moreover we have

$$\begin{aligned} (\varphi'h + h'\psi)\mathrm{d} + \mathrm{d}''(\varphi'h + h'\psi) &= \varphi'h\mathrm{d} + h'\psi\mathrm{d} + \mathrm{d}''\varphi'h + \mathrm{d}''h'\psi \\ &= \varphi'h\mathrm{d} + h'\mathrm{d}'\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'(\varphi - \psi - \mathrm{d}'h) + (\varphi' - \psi' - \mathrm{d}''h')\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'\varphi - \varphi'\psi - \varphi'\mathrm{d}'h + \varphi'\psi - \psi'\psi - \mathrm{d}''h'\psi + \varphi'\mathrm{d}'h + \mathrm{d}''h'\psi \\ &= \varphi'\varphi - \psi'\psi. \end{aligned}$$

Therefore $\varphi'\varphi \sim \psi'\psi$, and so (8) is well-defined. Observe that R-bilinearity and associativity of (8) follows trivially from R-bilinearity and associativity of composition in $\mathbf{Comp}_R$. Also for each R-complex A, the homotopy class of the identity map 1_A serves as the identity morphism for A in $\mathbf{HComp}_R$, which is easily seen to satisfy the left and right unity laws since 1_A satisfies the left and right unity laws in $\mathbf{Comp}_R$. □

3.3.4 Homotopy equivalences

Definition 3.7. Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a chain map. We say φ is a **homotopy equivalence** if there exists a chain map $\varphi'\colon (A', \mathrm{d}') \to (A, \mathrm{d})$ such that $\varphi'\varphi \sim 1_A$ and $\varphi\varphi' \sim 1_{A'}$. In this case, we call φ' a **homotopy inverse** to φ.

Proposition 3.12. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be an isomorphism of R-complexes with $\varphi'\colon (A', \mathrm{d}') \to (A, \mathrm{d})$ being its inverse. Then both φ is a homotopy equivalence with φ' being a homotopy inverse.*

Proof. Since φ and φ' are inverse to each other, we see that $\varphi'\varphi = 1_A$ and $\varphi\varphi' = 1_{A'}$. In particular, if we take h to be the zero map, then we have

$$\begin{aligned} h\mathrm{d} + \mathrm{d}'h &= 0 \cdot d + \mathrm{d}' \cdot 0 \\ &= 0 \\ &= \varphi'\varphi - 1_A. \end{aligned}$$

Thus $\varphi'\varphi \sim 1_A$. By a similar argument, we also have $\varphi\varphi' \sim 1_{A'}$. □

Remark. Note that a chain map $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is a homotopy equivalence if and only if $[\varphi]$ is an isomorphism.

3.4 Quasiisomorphisms

Definition 3.8. Let $\varphi\colon A \to A'$ be a chain map. We say φ is a **quasiisomorphism** if the induced map in homology $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism of graded R-modules.

3.4.1 Homotopy equivalence is a quasiisomorphism

Proposition 3.13. *Let $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a homotopy equivalence with homotopy inverse $\varphi'\colon (A', \mathrm{d}') \to (A, \mathrm{d})$. Then both φ and φ' are quasiisomorphisms.*

Proof. Since $\varphi'\varphi \sim 1_A$ and since homology takes homotopic maps to equal maps, we see that

$$\begin{aligned} 1_{\mathrm{H}(A)} &= \mathrm{H}(1_A) \\ &= \mathrm{H}(\varphi'\varphi) \\ &= \mathrm{H}(\varphi')\mathrm{H}(\varphi). \end{aligned}$$

A similarl calculation gives us $\mathrm{H}(\varphi')\mathrm{H}(\varphi) = 1_{\mathrm{H}(A')}$. Therefore $\mathrm{H}(\varphi)\colon \mathrm{H}(A) \to \mathrm{H}(A')$ is an isomorphism of graded R-modules with $\mathrm{H}(\varphi')\colon \mathrm{H}(A') \to \mathrm{H}(A)$ being its inverse. □

Remark. The converse is not true. That is, there there are many examples quasiisomorphisms which are not homotopy equivalences.

3.4.2 Quasiisomorphism equivalence relation

Definition 3.9. Let A and A' be R-complexes. We A is **quasiisomorphic** to A', denoted $A \sim_q A'$, if there exists R-complexes $A_0, \dots, A_n$ and $B_1, \dots, B_n$ where $A_0 = A$ and $A_n = A'$, together with quasisomorphisms

$$\sigma_m\colon B_m \to A_{m-1} \quad \text{and} \quad \tau_m\colon B_m \to A_m$$

for each $0 < m \leq n$. In terms of arrows, this looks like

$$\begin{tikzcd} & B_1 \arrow[dl, "\sigma_1"'] \arrow[dr, "\tau_1"] & & \cdots \arrow[dl] \arrow[dr] & & B_n \arrow[dl, "\sigma_n"'] \arrow[dr, "\tau_n"] & \\ A_0 & & A_1 & & A_{n-1} & & A_n \end{tikzcd}$$

One can easily check that being quasiisomorphic is an equivalence relation. It turns out that one can easily simplify this equivalence relation quite a bit. This is described in the following proposition.

Proposition 3.14. *Let A and A' be R-complexes. Then A is quasiisomorphic to A' if and only if there exists a semiprojective R-complex P together with quasiisomorphisms $\pi\colon P \to A$ and $\pi'\colon P \to A$.*

Proof. One direction is clear, so it suffices to prove the other direction. Suppose $A \sim_q A'$. Choose R-complexes $A_0, \dots, A_n$ and $B_1, \dots, B_n$ where $A_0 = A$ and $A_n = A'$, together with quasisomorphisms

$$\sigma_m\colon B_m \to A_{m-1} \quad \text{and} \quad \tau_m\colon B_m \to A_m$$

for each $0 < m \leq n$. Choose a semiprojective resolution $\pi_0\colon P \to A_0$ of A_0. Let $\widetilde{\pi}_0\colon P \to B_1$ be a homotopic lift of π_0 with respect to σ_1 and denote $\pi_1 = \tau_1\widetilde{\pi}_0$. We proceed inductively to construct chain maps $\widetilde{\pi}_{m-1}\colon P_m \to B_m$ and $\pi_m\colon P_m \to A_m$ where $\widetilde{\pi}_{m-1}$ is a homotopic lift of π_{m-1} with respect to σ_m and where $\pi_m = \tau_m\widetilde{\pi}_{m-1}$.

We prove by induction on $1 \leq m \leq n$ that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms. First we consider the base case $m = 1$. Observe that $\sigma_1\widetilde{\pi}_0 \sim \pi_0$ implies $\mathrm{H}(\sigma_1)\mathrm{H}(\widetilde{\pi}_0) = \mathrm{H}(\pi_0)$. Then $\mathrm{H}(\widetilde{\pi}_0)$ is an isomorphism since both $\mathrm{H}(\sigma_1)$ and $\mathrm{H}(\pi_0)$ are isomorphisms. Therefore $\widetilde{\pi}_0$ is a quasiisomorphism. Similarly, π_1 is a quasiisomorphisms since it is a composition of quasiisomorphisms.

Now suppose we have shown that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms for some $m < n$. Observe that $\sigma_m\widetilde{\pi}_{m-1} \sim \pi_m$ implies $\mathrm{H}(\sigma_m)\mathrm{H}(\widetilde{\pi}_{m-1}) = \mathrm{H}(\pi_m)$. Then $\mathrm{H}(\widetilde{\pi}_{m-1})$ is an isomorphism since both $\mathrm{H}(\sigma_m)$ and $\mathrm{H}(\pi_m)$ are isomorphisms. Therefore $\widetilde{\pi}_{m-1}$ is a quasiisomorphism. Similarly, π_{m+1} is a quasiisomorphisms since it is a composition of quasiisomorphisms.

Thus we have shown by induction that π_m and $\widetilde{\pi}_{m-1}$ are quasiisomorphisms for all $1 \leq m \leq n$. In particular, $\pi_n\colon P \to A_n$ is a quasiisomorphism. □

3.5 Exact Sequences of R-Complexes

Definition 3.10. Let (A, d), (A', d'), and (A'', d'') be R-complexes and let $\varphi\colon A' \to A$ and $\psi\colon A \to A''$ be chain maps. Then we say that

$$\begin{tikzcd} 0 \arrow[r] & (A', \mathrm{d}') \arrow[r, "\varphi"] & (A, \mathrm{d}) \arrow[r, "\psi"] & (A'', \mathrm{d}'') \arrow[r] & 0 \end{tikzcd}$$

is a **short exact sequence** of R-complexes if it is a short exact sequence when considered as graded R-modules. More specifically, this means that following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathrm{d}'_{i+2}"] & \vdots \arrow[d, "\mathrm{d}_{i+2}"] & \vdots \arrow[d, "\mathrm{d}''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "\mathrm{d}'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "\mathrm{d}_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "\mathrm{d}''_{i+1}"] & 0 \\
0 \arrow[r] & A'_{i} \arrow[r, "\varphi_{i}"] \arrow[d, "\mathrm{d}'_{i}"] & A_{i} \arrow[r, "\psi_{i}"] \arrow[d, "\mathrm{d}_{i}"] & A''_{i} \arrow[r] \arrow[d, "\mathrm{d}''_{i}"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "\mathrm{d}'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "\mathrm{d}_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "\mathrm{d}''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

3.5.1 Long exact sequence in homology

Theorem 3.2. *Let*

$$\begin{tikzcd} 0 \arrow[r] & (A', \mathrm{d}') \arrow[r, "\varphi"] & (A, \mathrm{d}) \arrow[r, "\psi"] & (A'', \mathrm{d}'') \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence of R-complexes. Then there exists a graded homomorphism $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 such that

$$\begin{tikzcd}
 & & \cdots \arrow[r] & \mathrm{H}_{i+1}(A'') \arrow[dll, rounded corners, "\eth_{i+1}"'] \\
\mathrm{H}_i(A') \arrow[r, "\mathrm{H}_i(\varphi)"] & \mathrm{H}_i(A) \arrow[r, "\mathrm{H}_i(\psi)"] & \mathrm{H}_i(A'') \arrow[dll, rounded corners, "\eth_i"'] & \\
\mathrm{H}_{i-1}(A') \arrow[r] & \cdots & &
\end{tikzcd} \tag{9}$$

is a long exact sequence of R-modules.

Proof. The proof will consists of three steps. The first step is to construct a graded function $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 (graded here just means $\eth(\mathrm{H}_i(A'')) \subseteq \mathrm{H}_{i-1}(A')$ for all $i \in \mathbb{Z}$). The next step will be to show that $\eth$ is R-linear. The final step will be to show exactness of (17).

Step 1: We construct a graded function $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ as follows: let $[a''] \in \mathrm{H}_i(A'')$. Choose a representative of the coset $[a'']$, say $a'' \in A''_i$ (so $\mathrm{d}''(a'') = 0$), and choose a lift of a'' in A_i with respect to ψ, say $a \in A_i$ (so $\psi(a) = a''$). We can make such a choice since ψ is surjective. Since

$$\begin{aligned} \psi(\mathrm{d}(a)) &= \mathrm{d}''(\psi(a)) \\ &= \mathrm{d}''(a'') \\ &= 0, \end{aligned}$$

it follows by exactness of (3.8.3) that there exists a unique $a' \in A'_{i-1}$ such that $\varphi(a') = \mathrm{d}(a)$. Observe that $\mathrm{d}'(a') = 0$ since φ is injective and since

$$\begin{aligned}\varphi(\mathrm{d}'(a')) &= \mathrm{d}(\varphi(a')) \\ &= \varphi(\mathrm{d}(a)) \\ &= 0.\end{aligned}$$

Thus a' represents an element in $\mathrm{H}_{i-1}(A')$. We define $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ by

$$\eth[a''] = [a'].$$

We need to verify that $\eth$ is well-defined. There were two choices that we made in constructing $\eth$. The first choice was the choice of a representative of the coset $[a'']$. Let us consider another choice, say $a'' + \mathrm{d}''(b'')$ where $b'' \in A''_{i+1}$ (every representative of the coset $[a'']$ has this form for some $b'' \in A''_{i+1}$). The second choice that we made was the choice of a lift of a'' in A with respect to ψ. This time we have another coset representative of $[a'']$, so let $a + \varphi(b') + \mathrm{d}(b)$ be another choice of a lift of $a'' + \mathrm{d}''(b'')$ with respect to ψ where $b' \in A'_i$ and $b \in A_{i+1}$ (every such choice has this form for some $b' \in A'_i$ and $b \in A_{i+1}$). Now observe that

$$\begin{aligned}\psi\mathrm{d}(a + \varphi(b') + \mathrm{d}(b)) &= \psi\mathrm{d}(a) + \psi\mathrm{d}\varphi(b')) + \psi\mathrm{dd}(b) \\ &= \psi\mathrm{d}(a) + \psi\mathrm{d}\varphi(b')) \\ &= \psi\mathrm{d}(a) + \psi\varphi\mathrm{d}'(b') \\ &= \psi\mathrm{d}(a) \\ &= \mathrm{d}''\psi(a) \\ &= \mathrm{d}''(a'') \\ &= 0.\end{aligned}$$

Hence there exists a unique element in A'_{i-1} which maps to $\mathrm{d}(a + \varphi(b') + \mathrm{d}(b))$ with respect to φ, and since

$$\begin{aligned}\varphi(a' + \mathrm{d}'(b')) &= \varphi(a') + \varphi\mathrm{d}'(b') \\ &= \mathrm{d}(a) + \mathrm{d}\varphi(b') \\ &= \mathrm{d}(a + \varphi(b') + \mathrm{d}(b)),\end{aligned}$$

this unique element must be $a' + \mathrm{d}'(b')$. Therefore

$$\begin{aligned}\eth[a'' + \mathrm{d}''(b'')] &= [a' + \mathrm{d}'(b')] \\ &= [a'] \\ &= \eth[a''],\end{aligned}$$

which implies $\eth$ is well-defined. Moreover, we see that $\eth(\mathrm{H}(A_i)) \subseteq \mathrm{H}(A_{i-1})$, and is hence graded of degree -1. As usualy, we denote $\eth_i := \eth|_{A_i}$ for all $i \in \mathbb{Z}$.

Step 2: Let $i \in \mathbb{Z}$, let $\overline{a''}, \overline{b''} \in H(A'')$, and let $r, s \in R$. Choose a coset representative $\overline{a''}$ and $\overline{b''}$, say $a'' \in A''_i$ and $b'' \in A''_i$. Then $ra'' + sb''$ is a coset representative of $\overline{ra'' + sb''}$ (by linearity of taking quotients). Next, choose lifts of a'' and b'' in A_i under φ, say $a \in A_i$ and $b \in A_i$ respectively. Then $ra + sb$ is a lift of $ra'' + sb''$ in A_i under φ (by linearity of ψ). Finally, let a' and b' be the unique elements in A'_{i-1} such that $\varphi(a') = d(a)$ and $\varphi(b') = d(b)$. Then $ra' + sb'$ is the unique element in A'_{i-1} such that $\varphi(ra' + sb') = d(ra + sb)$ (by linearity of φ). Thus, we have

$$\begin{aligned}\eth(\overline{ra'' + sb''}) &= \overline{ra' + sb'} \\ &= r\overline{a'} + s\overline{b'} \\ &= r\eth(\overline{a''}) + s\eth(\overline{b''}).\end{aligned}$$

Step 3: To prove exactness of (17), it suffices to show exactness at $H_i(A'')$, $H_i(A)$, and $H_i(A')$. First we prove exactness at $H_i(A)$. Let $\overline{a} \in \mathrm{Ker}(H_i(\psi))$ (so $a \in A_i$, $d(a) = 0$, and $\overline{\psi(a)} = \overline{0}$). Lift $\psi(a) \in A''_i$ to an element $a'' \in A'_{i+1}$ under d'' (we can do this since $\overline{\psi(a)} = \overline{0}$). Lift $a'' \in A''_{i+1}$ to an element $b \in A_{i+1}$ under ψ (we can do this since ψ is surjective). Then

$$\begin{aligned}\psi(d(b) - a) &= \psi(d(b)) - \psi(a) \\ &= d''(a'') - \psi(a) \\ &= \psi(a) - \psi(a) \\ &= 0\end{aligned}$$

implies $d(b) - a \in \mathrm{Ker}(\psi)$. Lift $d(b) - a$ to the unique element $a' \in A'_i$ under φ (we can do this exactness of (3.8.3)). Since φ is injective,

$$\begin{aligned}\varphi(d'(a')) &= d(\varphi(a')) \\ &= d(d(b) - a) \\ &= d(d(b)) - d(a)) \\ &= 0\end{aligned}$$

implies $d'(a') = 0$. Hence a' represents an element in $H(A')$. Therefore

$$\begin{aligned}H_i(\varphi)(a') &= \overline{\varphi(a')} \\ &= \overline{d(b) - a} \\ &= \overline{a}\end{aligned}$$

implies $\overline{a} \in \mathrm{Im}(H_i(\varphi))$. Thus we have exactness at $H_i(A)$.

Next we show exactness at $H_i(A')$. Let $\overline{a'} \in \mathrm{Ker}(H_i(\varphi))$ (so $a' \in A'_i$, $d(a') = 0$, and $\overline{\varphi(a')} = \overline{0}$). Lift $\varphi(a') \in A_i$ to an element $a \in A'_{i+1}$ under d (we can do this since $\overline{\varphi(a)} = \overline{0}$). Then

$$\begin{aligned}d(\psi(a)) &= \psi(d(a)) \\ &= \psi(\varphi(a')) \\ &= 0.\end{aligned}$$

Hence $\psi(a)$ represents an element in $H_{i+1}(A'')$. By construction, we have $\eth(\overline{\psi(a)}) = \overline{a'}$, which implies $\overline{a'} \in \mathrm{Im}(\eth_{i+1})$. Thus we have exactness at $H_i(A')$.

Finally we show exactness at $H_i(A'')$. Let $\overline{a''} \in \mathrm{Ker}(\eth_i)$ (so $a'' \in A''_i$ and $d(a'') = 0$). Lift a'' to an element $a \in A_i$ under ψ. Lift $d(a)$ to the unique element a' in A'_{i-1} under φ. Lift a' to an element $b' \in A'_{i+1}$ under d (we can do this since $0 = \eth(\overline{a''}) = \overline{a'}$). Then

$$\begin{aligned}d(a - \varphi(b')) &= d(a) - d(\varphi(b')) \\ &= d(a) - \varphi(d(b')) \\ &= d(a) - \varphi(a') \\ &= 0,\end{aligned}$$

and hence $a - \varphi(b')$ represents an element in $H_i(A)$. Moreover, we have

$$\begin{aligned}H_i(\psi)(\overline{a - \varphi(b')}) &= \overline{\psi(a - \varphi(b'))} \\ &= \overline{\psi(a) - \psi(\varphi(b'))} \\ &= \overline{\psi(a)} \\ &= \overline{a''},\end{aligned}$$

which implies $\overline{a'} \in \mathrm{Im}(H_i(\psi))$. Thus we have exactness at $H_i(A'')$. □

Definition 3.11. Given a short exact sequence of R-complexes as in (3.8.3), we refer to the graded homomorphism $\eth\colon \mathrm{H}(A'') \to \mathrm{H}(A')$ of degree -1 as the **induced connecting map**.

3.5.2 When a Graded R-Linear Map is a Chain Map

Proposition 3.15. *Let (A, d) and (B, ∂) be R-complexes and let $\varphi\colon A \to B$ be a graded R-linear map of the underlying graded modules. Let $\overline{B} = B/\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ and let $\pi\colon B \to \overline{B}$ be the quotient map. Define $\overline{\partial}\colon \overline{B} \to \overline{B}$ by*

$$\overline{\partial}(\overline{b}) = \overline{\partial(b)}$$

for all $a \in A$ and $\overline{b} \in \overline{B}$. Then $(\overline{B}, \overline{\partial})$ is an R-complex and $\pi\varphi\colon A \to \overline{B}$ is a chain map. Moreover, if φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$, then we have the following short exact sequence of graded R-modules and graded R-linear maps:

$$0 \longrightarrow \mathrm{H}(B) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}(\overline{B}) \xrightarrow{\gamma} \mathrm{im}(\partial\varphi - \varphi\mathrm{d})(-1) \longrightarrow 0 \tag{10}$$

where γ is the connecting map coming from a long exact sequence in homology.

Proof. Observe that $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ is a graded R-submodule of B since $\partial\varphi - \varphi\mathrm{d}$ is a graded R-linear map of degree -1, therefore the grading on B induces a grading on $\overline{B}$ which makes π into a graded R-linear map. Therefore $\pi\varphi$, being a composite of two graded R-linear maps, is a graded R-linear map. We need to check that $\overline{\partial}$ is well-defined, that is, we need to check that ∂ sends $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ to itself. Let $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Then

$$\begin{aligned} \partial(\partial\varphi - \varphi\mathrm{d})(a) &= (\partial\partial\varphi - \partial\varphi\mathrm{d})(a) \\ &= -\partial\varphi\mathrm{d}(a) \\ &= (-\partial\varphi\mathrm{d}(a) + \varphi\mathrm{d}\mathrm{d})(a) \\ &= (-\partial\varphi + \varphi\mathrm{d})(\mathrm{d}(a)) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d}). \end{aligned}$$

Thus $\overline{\partial}$ is well-defined. Also $\overline{\partial}$ is an R-linear differential since it inherits these properties from ∂. Therefore $(\overline{B}, \overline{\partial})$ is an R-complex.

Now let us check that $\pi\varphi$ is a chain map. To see this, we just need to show it commutes with the differentials. Let $a \in A$. Then we have

$$\begin{aligned} \overline{\partial}\pi\varphi(a) &= \overline{\partial}(\overline{\varphi(a)}) \\ &= \overline{\partial\varphi(a)} \\ &= \overline{\partial\varphi(a) - (\partial\varphi - \varphi\mathrm{d})(a)} \\ &= \overline{\partial\varphi(a) - \partial\varphi(a) + \varphi\mathrm{d}(a)} \\ &= \overline{\varphi\mathrm{d}(a)} \\ &= \pi\varphi\mathrm{d}(a). \end{aligned}$$

Thus $\pi\varphi$ is a chain map.

Since ∂ sends $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ to itself, it restricts to a differential on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. So we have a short exact sequence of R-complexes

$$0 \longrightarrow \mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \xrightarrow{\iota} B \xrightarrow{\pi} \overline{B} \longrightarrow 0 \tag{11}$$

where ι is the inclusion map. The short exact sequence (11) induces the following long exact sequence in homology

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_{i+1}(\overline{B}) \arrow[dll, "\gamma_{i+1}"'] \\ & \mathrm{H}_i(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(B) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_i(\overline{B}) \arrow[dll, "\gamma_i"'] \\ & \mathrm{H}_{i-1}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}) \arrow[r, "\mathrm{H}_{i-1}(\iota)"] & \mathrm{H}_{i-1}(B) \arrow[r] & \cdots \end{tikzcd} \tag{12}$$

Let us work out the details of the connecting map γ. Let $[\overline{b}] \in \mathrm{H}_i(\overline{B})$, so $\overline{b} \in \overline{B}_i$ is the coset with $b \in B_i$ as a reprsentative and $[\overline{b}] \in \mathrm{H}_i(\overline{B})$ is the coset with $\overline{b} \in \overline{B}_i$ as a representative. In particular, $\overline{\partial}(\overline{b}) = \overline{0}$, which implies

$$\partial(b) = (\partial\varphi - \varphi\mathrm{d})(a) \tag{13}$$

for some $a \in A$. Then (13) implies that $(\partial\varphi - \varphi\mathrm{d})(a)$ is the unique element in $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ which maps to $\partial(b)$ (under the inclusion map). Therefore

$$\gamma_i[\overline{b}] = [(\partial\varphi - \varphi\mathrm{d})(a)].$$

Now suppose φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$. We claim that ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. Indeed, let $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$. Since φ takes $\mathrm{im}\,\mathrm{d}$ to $\mathrm{im}\,\partial$, there exists a $b \in B$ such that

$$\varphi\mathrm{d}(a) = \partial(b).$$

Choose such a $b \in B$. Then observe that

$$\begin{aligned} \partial(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\partial\varphi - \partial\varphi\mathrm{d}(a) \\ &= -\partial\varphi\mathrm{d}(a) \\ &= -\partial\partial(b) \\ &= 0. \end{aligned}$$

Thus ∂ restricts to the zero map on $\mathrm{im}(\partial\varphi - \varphi\mathrm{d})$. In particular, $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d})) \cong \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$.

Next we claim that $\mathrm{H}(\iota)$ is the zero map. Indeed, for any $(\partial\varphi - \varphi\mathrm{d})(a) \in \mathrm{im}(\partial\varphi - \varphi\mathrm{d})$ where $a \in A$, we choose $b \in B$ such that $\varphi\mathrm{d}(a) = \partial(b)$, then we have

$$\begin{aligned}(\partial\varphi - \varphi\mathrm{d})(a) &= \partial\varphi(a) - \varphi\mathrm{d}(a) \\ &= \partial\varphi(a) - \partial b \\ &= \partial(\varphi(a) - b) \\ &\in \mathrm{im}\,\partial.\end{aligned}$$

Therefore $\mathrm{H}(\iota)$ takes the coset in $\mathrm{H}(\mathrm{im}(\partial\varphi - \varphi\mathrm{d}))$ represented by $(\partial\varphi - \varphi\mathrm{d})(a)$ to the coset in $\mathrm{H}(B)$ represented by 0. Thus $\mathrm{H}(\iota)$ is the zero map as claimed.

Combining everything together, we see that the long exact sequence (12) breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(B) \xrightarrow{\mathrm{H}_i(\pi)} \mathrm{H}_i(\overline{B}) \xrightarrow{\gamma_i} \mathrm{im}(\partial_{i-1}\varphi_{i-1} - \varphi_{i-2}\mathrm{d}_{i-1}) \longrightarrow 0 \tag{14}$$

for all $i \in \mathbb{Z}$. In other words, (11) is a short exact sequence of graded R-modules.

□

3.6 Operations on R-Complexes

3.6.1 Product of R-complexes

3.6.2 Limits

Definition 3.12. Let $(\Lambda, \leq)$ be a preordered set. A system $(M_\lambda, \varphi_{\lambda\mu})$ of R-complexes and chain maps over Λ consists of a family of a family of R-complexes $\{(M_\lambda, \mathrm{d}_\lambda)\}$ indexed by Λ and a family of chain maps $\{\varphi_{\lambda\mu} \colon M_\lambda \to M_\mu\}_{\lambda \leq \mu}$ such that for all $\lambda \leq \mu \leq \kappa$,

$$\varphi_{\lambda\lambda} = 1_{M_\lambda} \quad \text{and} \quad \varphi_{\lambda\kappa} = \varphi_{\mu\kappa}\varphi_{\lambda\mu}.$$

We say $(M_\lambda, \varphi_{\lambda\mu})$ is a **directed system** if Λ is a directed set.

Proposition 3.16. *Let $(M_\lambda, \varphi_{\lambda\mu})$ be a system of R-complexes and chain maps over Λ. The limit of this system, denoted $\lim^\star M_\lambda$, is given by the R-complex $(\lim^\star M_\lambda, \lim^\star \mathrm{d}_\lambda)$ together with together with the projection maps*

$$\pi_\lambda \colon \overset{\star}{\lim} M_\lambda \to M_\lambda$$

for all $\lambda \in \Lambda$, where $\lim^\star M_\lambda$ is the graded R-module given by

$$\overset{\star}{\lim} M_\lambda = \left\{ (u_\lambda) \in \prod_{\lambda \in \Lambda}^{\star} M_\lambda \mid \varphi_{\lambda\kappa}(u_\lambda) = u_\mu \text{ for all } \lambda \leq \mu \right\}$$

and where the differential $\lim^\star \mathrm{d}_\lambda$ is defined pointwise:

$$(\overset{\star}{\lim} \mathrm{d}_\lambda)((u_\lambda)) = (\mathrm{d}_\lambda(u_\lambda))$$

for all $(u_\lambda) \in \lim^\star M_\lambda$.

Proof. We need ot show that $\lim^\star M_\lambda$ satisfies the universal mapping property. Let (M, ψ_λ) be compatible with respect to the system $(M_\lambda, \varphi_{\lambda\mu})$, so

$$\varphi_{\lambda\mu}\psi_\lambda = \psi_\mu$$

for all $\lambda \leq \mu$. By the universal mapping property of the graded limits, there exists a unique graded R-linear map $\psi \colon M \to \lim^\star M_\lambda$ of graded R-linear maps which commutes with all the arrows. It remains to show that ψ commutes with the differentials. Indeed, we have

$$\begin{aligned}(\overset{\star}{\lim} \mathrm{d}_\lambda\psi)(u) &= \overset{\star}{\lim} \mathrm{d}_\lambda((\psi_\lambda(u))) \\ &= (\mathrm{d}_\lambda(\psi_\lambda(u))) \\ &= (\psi_\lambda(\mathrm{d}(u))) \\ &= \psi(\mathrm{d}(u)) \\ &= (\psi\mathrm{d})(u).\end{aligned}$$

for all $u \in M$.

□

3.6.3 Localization

Let (A, d) be an R-complex and let S be a multiplicatively closed subset of R. The **localization of** (A, d) **with respect to** S is the R_S-complex (A_S, d_S) where A_S is the graded R_S-module whose component in degree i is

$$(A_S)_i = \{a/s \mid a \in A_i \text{ and } s \in S\}.$$

The differential d_S is defined as follows: if $a/s \in (A_S)_i$, then

$$\mathrm{d}_S(a/s) = \mathrm{d}(a)/s.$$

3.6.4 Direct Sum of R-Complexes

Definition 3.13. Let (A, d) and (A', d') be R-complexes. We define their **direct sum** to be the R-complex

$$(A, \mathrm{d}) \oplus_R (A', \mathrm{d}') := (A \oplus A', \mathrm{d} \oplus \mathrm{d}')$$

whose graded R-module $A \oplus A'$ has

$$(A \oplus A')_i = A_i \oplus A'_i$$

as its ith homogeneous component and whose differential $\mathrm{d} \oplus \mathrm{d}'$ is defined by

$$(\mathrm{d} \oplus \mathrm{d}')(a, a') = (\mathrm{d}(a), \mathrm{d}'(a'))$$

for all $(a, a') \in A \oplus A'$.

More generally, suppose $(A_\lambda, \mathrm{d}_\lambda)$ is an R-complex for each λ in some indexing set Λ. We define their **direct sum** to be the R-complex

$$\bigoplus_{\lambda \in \Lambda} (A_\lambda, \mathrm{d}_\lambda) := \left(\bigoplus_{\lambda \in \Lambda} A_\lambda, \bigoplus_{\lambda \in \Lambda} \mathrm{d}_\lambda \right).$$

It is easy to check that

$$\mathrm{H}\left(\bigoplus_{\lambda \in \Lambda} A_\lambda \right) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{H}(A_\lambda).$$

In other words, homology commutes with direct sums.

3.6.5 Shifting an R-complex

Definition 3.14. Let (A, d) be an R-complex. We define the **shift** of (A, d) to be the R-complex

$$\Sigma(A, \mathrm{d}) := (A(-1), -\mathrm{d}).$$

More generally, let $k \in \mathbb{Z}$. We define the kth **shift** of (A, d) to be the R-complex

$$\Sigma^k(A, \mathrm{d}) = (A(-k), (-1)^k \mathrm{d}).$$

Proposition 3.17. *Let A be an R-complex and let $n \in \mathbb{Z}$. Then*

$$\mathrm{H}(\Sigma^n A) = \mathrm{H}(A)(-n).$$

In particular,

$$\mathrm{H}_i(\Sigma^n A) = \mathrm{H}_{i-n}(A)$$

for all $i \in \mathbb{Z}$.

Proof. We have

$$\begin{aligned}
\mathrm{H}(\Sigma^n A) &= \ker(\mathrm{d}_{\Sigma^n A}) / \operatorname{im}(\mathrm{d}_{\Sigma^n A}) \\
&= \ker\left((-1)^n \mathrm{d}_{A(-n)}\right) / \operatorname{im}\left((-1)^n \mathrm{d}_{A(-n)}\right) \\
&= \ker\left(\mathrm{d}_{A(-n)}\right) / \operatorname{im}\left(\mathrm{d}_{A(-n)}\right) \\
&= \mathrm{H}(A)(-n).
\end{aligned}$$

□

3.7 The Mapping Cone

Definition 3.15. Let $\varphi\colon A \to B$ be a chain map. The **mapping cone of** φ, denoted $\mathrm{C}(\varphi)$, is the R-complex whose underlying graded R-module is $\mathrm{C}(\varphi) = B \oplus A(-1)$ and whose differential is defined by

$$\mathrm{d}_{\mathrm{C}(\varphi)}(b, a) := (\mathrm{d}_B(b) + \varphi(a), -\mathrm{d}_A(a))$$

for all $(b, a) \in B \oplus A(-1)$.

Remark. To see that we are justified in calling $\mathrm{C}(\varphi)$ an R-complex, let us check that $\mathrm{d}_{\mathrm{C}(\varphi)}\mathrm{d}_{\mathrm{C}(\varphi)} = 0$. Let $(b, a) \in C(\varphi)$. Then we have

$$\begin{aligned}
\mathrm{d}_{\mathrm{C}(\varphi)}\mathrm{d}_{\mathrm{C}(\varphi)}(b, a) &= \mathrm{d}_{\mathrm{C}(\varphi)}(\mathrm{d}_B(b) + \varphi(a), -\mathrm{d}_A(a)) \\
&= (\mathrm{d}_B(\mathrm{d}_B(b) + \varphi(a)) + \varphi(-\mathrm{d}_A(a)), -\mathrm{d}_A\mathrm{d}_A(a)) \\
&= (\mathrm{d}_B\varphi(a) - \varphi\mathrm{d}_A(a), 0) \\
&= (0, 0).
\end{aligned}$$

3.7.1 Turning a Chain Map Into a Connecting Map

Theorem 3.3. *Let $\varphi\colon A \to B$ be a chain map. Then we have a short exact sequence of R-complexes*

$$0 \longrightarrow B \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma A \longrightarrow 0 \tag{15}$$

where $\iota\colon B \to \mathrm{C}(\varphi)$ is the inclusion map given by

$$\iota(b) = (b, 0)$$

for all $b \in B$, and where $\pi\colon \mathrm{C}(\varphi) \to \Sigma A$ is the projection map given by

$$\pi(b, a) = a$$

for all $(b, a) \in \mathrm{C}(\varphi)$. Moreover the connecting map $\eth\colon \mathrm{H}(\Sigma A) \to \mathrm{H}(B)$ induced by (15) agrees with $\mathrm{H}(\varphi)$.

Proof. It is straightforward to check that (15) is a short exact sequence of R-complexes. Let us show that the connecting map agrees with $\mathrm{H}(\varphi)$. Let $i \in \mathbb{Z}$ and let $\overline{a} \in \mathrm{H}_i(\Sigma A)$. Thus $a \in A_i$ and $\mathrm{d}_A(a) = 0$. Lift $a \in A_i$ to the element $(0, a) \in \mathrm{C}_i(\varphi)$. Now apply $\mathrm{d}_{\mathrm{C}(\varphi)}$ to $(0, a)$ to get $(\varphi(a), 0) \in \mathrm{C}_{i-1}(\varphi)$. Then $\varphi(a)$ is the unique element in B_{i-1} which maps to $(\varphi(a), 0)$ under d_B. Therefore

$$\begin{aligned}
\eth(\overline{a}) &= \overline{\varphi(a)} \\
&= \mathrm{H}(\varphi)(\overline{a}).
\end{aligned}$$

It follows that $\eth$ and $\mathrm{H}(\varphi)$ agree on all of $\mathrm{H}(A)$. □

Remark. In the context of graded R-modules, it would be incorrect to say $\eth = \mathrm{H}(\varphi)$. This is because $\eth$ is graded of degree -1 and $\mathrm{H}(\varphi)$ is graded of degree 0. On the other hand, it would be correct to say $\eth_i = \mathrm{H}_{i-1}(\varphi)$ for all $i \in \mathbb{Z}$.

3.7.2 Quasiisomorphism and Mapping Cone

Corollary. *Let $\varphi\colon A \to B$ be a chain map. Then φ is a quasiisomorphism if and only if $\mathrm{C}(\varphi)$ is an exact complex.*

Proof. Suppose $\mathrm{C}(\varphi)$ is an exact complex, so $\mathrm{H}(\mathrm{C}(\varphi)) \cong 0$. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (15) gives us

$$0 \cong \mathrm{H}_{i+1}(\mathrm{C}(\varphi)) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}_i(A) \xrightarrow{\mathrm{H}(\varphi)} \mathrm{H}_i(B) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}_i(\mathrm{C}(\varphi)) \cong 0$$

which implies $\mathrm{H}_i(A) \cong \mathrm{H}_i(B)$ for all $i \in \mathbb{Z}$.

Conversely, suppose φ is a quaisiisomorphism. Then for each $i \in \mathbb{Z}$, the long exact sequence induced by (15) gives us

$$\mathrm{H}_i(A) \cong \mathrm{H}_i(B) \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}_i(\mathrm{C}(\varphi)) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}_{i-1}(A) \cong \mathrm{H}_{i-1}(B)$$

which implies $\mathrm{H}_i(\mathrm{C}(\varphi)) \cong 0$ for all $i \in \mathbb{Z}$. □

3.7.3 Translating Mapping Cone With Isomorphisms

Proposition 3.18. *Suppose we have a commutative diagram of R-complexes*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"] \\ A' \arrow[r, "\phi'"'] & B' \end{tikzcd}$$

where $\phi\colon A \to B$ and $\phi'\colon A' \to B'$ are isomorphisms. Then we have an isomorphism $\mathrm{C}(\varphi) \cong \mathrm{C}(\psi)$ of R-complexes.

Proof. Define $\phi' \oplus \phi\colon \mathrm{C}(\varphi) \to \mathrm{C}(\psi)$ by

$$(\phi' \oplus \phi)(a', a) = (\phi'(a'), \phi(a))$$

for all $(a', a) \in \mathrm{C}(\varphi)$. Clearly $\phi' \oplus \phi$ is an isomorphism of the underlying graded R-modules. To see that it is an isomorphism of R-complexes, we need to check that it commutes with the differentials. Let $(a', a) \in \mathrm{C}(\varphi)$. We have

$$\begin{aligned} \mathrm{d}_{\mathrm{C}(\psi)}(\phi' \oplus \phi)(a', a) &= \mathrm{d}_{\mathrm{C}(\psi)}(\phi'(a'), \phi(a)) \\ &= (\mathrm{d}_{B'}\phi'(a') + \psi\phi(a), -\mathrm{d}_B\phi(a)) \\ &= (\mathrm{d}_{B'}\phi'(a') + \psi\phi(a), -\mathrm{d}_B\phi(a)) \\ &= (\phi'\mathrm{d}_{A'}(a') + \phi'\varphi(a), -\phi\mathrm{d}_A(a)) \\ &= (\phi' \oplus \phi)(\mathrm{d}_{A'}(a') + \varphi(a), -\mathrm{d}_A(a)) \\ &= (\phi' \oplus \phi)\mathrm{d}_{\mathrm{C}(\varphi)}(a', a). \end{aligned}$$

□

3.7.4 Resolutions by Mapping Cones

Lemma 3.4. *(Lifting Lemma) Let $\varphi\colon M \to M'$ be an R-module homomorphism, let (P, d) be a projective resolution of M, and let (P', d') be a projective resolution of M'. Then there exists a chain map $\varphi\colon (P, d) \to (P', d')$ such that*

$$\begin{tikzcd} H_0(P) \arrow[r, "H_0(\varphi)"] \arrow[d, "\cong"] & H_0(P') \arrow[d, "\cong"] \\ M \arrow[r, "\varphi"] & M' \end{tikzcd}$$

Proof. For each $i > 0$, let $M_i' := \mathrm{Im}(d_i')$ and let $M_i := \mathrm{Im}(d_i)$. We build a chain map $\varphi\colon (P, d) \to (P', d')$ by constructing R-module homomorphism $\varphi_i\colon P_i \to P_i'$ which commute with the differentials using induction on $i \geq 0$.

First consider the base case $i = 0$. Let $\psi_0\colon P_0 \to P_0'/M_0'$ be the composition

$$P_0 \to P_0/M_1 \cong M \to M' \cong P_0'/M_1'.$$

Since P_0 is projective and since $d_0'\colon P_0' \to P_0'/M_1$ is a surjective homomorphism, we can lift $\psi_0\colon P_0 \to P_0'/M_0'$ along $d_0'\colon P_0' \to P_0'/M_1$ to a homomorphism $\varphi_0\colon P_0 \to P_0'$ such that $d_0'\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructedn an R-module homomorphism $\varphi_i\colon P_i \to P_i'$ such that

$$d_i'\varphi_i = \varphi_{i-1}d_i.$$

We need to construct an R-module homomorphism $\varphi_{i+1}\colon P_{i+1} \to P_{i+1}'$ such that

$$d_{i+1}'\varphi_{i+1} = \varphi_i d_{i+1}.$$

First, observe that $\mathrm{Im}(\varphi_i d_{i+1}) \subseteq M_{i+1}'$. Indeed, we have

$$\begin{aligned} d_i'\varphi_i d_{i+1} &= \varphi_{i-1}d_i d_{i+1} \\ &= 0, \end{aligned}$$

Thus, since (P', d') is exact for all $i > 0$, we have

$$\begin{aligned}\mathrm{Im}(\varphi_i d_{i+1}) &\subseteq \mathrm{Ker}(d'_i)\\ &= \mathrm{Im}(d'_{i+1})\\ &= M'_{i+1}.\end{aligned}$$

Now since P_{i+1} is projective and $d'_{i+1}\colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i d_{i+1} : P_{i+1} \to M'_{i+1}$ along $d'_{i+1} : P'_{i+1} \to M'_{i+1}$ to a homomorphism $\varphi_{i+1}\colon P_{i+1} \to P'_{i+1}$ such that

$$d'_{i+1}\varphi_{i+1} = \varphi_i d_{i+1}.$$

The last part of the lemma, follows from the way φ_0 was constructed.

□

Theorem 3.5. *With the notation as above, the following hold:*

1. *if φ is injective, then $\mathrm{C}(\varphi)$ is a projective resolution of $M'/\operatorname{im}\varphi$.*
2. *if φ is surjective, then $\Sigma\mathrm{C}(\varphi)$ is a projective resolution of* $\ker\varphi$.

Proof. First note that the underlying graded R-module of $\mathrm{C}(\varphi)$ is projective since it is a direct sum of projective modules. Now we first consider the case where φ is injective. The short exact sequence

$$0 \longrightarrow P' \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma P \longrightarrow 0 \tag{16}$$

induces the long exact sequence

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_i(P) \arrow[dll, "\mathrm{H}_i(\varphi)"] \\ \mathrm{H}_i(P') \arrow[r, "\mathrm{H}_i(\iota)"] & \mathrm{H}_i(\mathrm{C}(\varphi)) \arrow[r, "\mathrm{H}_i(\pi)"] & \mathrm{H}_{i-1}(P) \arrow[dll, "\mathrm{H}_{i-1}(\varphi)"] \\ \mathrm{H}_{i-1}(P') \arrow[r] & \cdots & \end{tikzcd} \tag{17}$$

This gives us $\mathrm{H}_i(\mathrm{C}(\varphi))$ for all $i > 1$ since $\mathrm{H}_i(P') \cong 0 \cong \mathrm{H}_i(P)$ for all $i \geq 1$. For $i = 1$, we get the exact sequence

$$0 \longrightarrow \mathrm{H}_1(\mathrm{C}(\varphi)) \longrightarrow M \xrightarrow{\varphi} M' \tag{18}$$

Then φ being injective implies $\mathrm{H}_1(\mathrm{C}(\varphi)) \cong 0$. Finally, for $i = 0$, we get the short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} M' \longrightarrow \mathrm{H}_0(\mathrm{C}(\varphi)) \longrightarrow 0 \tag{19}$$

This implies $\mathrm{H}_0(\mathrm{C}(\varphi)) \cong M'/\operatorname{im}\varphi$.

Now we consider the case where φ is surjective. We still get $\mathrm{H}_i(\mathrm{C}(\varphi))$ for all $i > 1$ since $\mathrm{H}_i(P') \cong 0 \cong \mathrm{H}_i(P)$ for all $i \geq 1$. For $i = 1$, we get again get the exact sequence (18), but this time we conclude that $\mathrm{H}_1(\mathrm{C}(\varphi)) \cong \ker\varphi$ since φ is surjective. Similarly, for $i = 0$, we again get the short exact sequence (19), but this time we conclude $\mathrm{H}_0(\mathrm{C}(\varphi)) \cong 0$ since φ is surjective. □

Example 3.1. Let $S = K[x_1, \ldots, x_n]$, let $I_{\mathcal{P}}$ be the permutohedron ideal in S, and let $I_{\mathcal{A}}$ be the associahedron ideal in S. Then there are natural free resolution $F_{\mathcal{P}} \xrightarrow{\tau_{\mathcal{P}}} S/I_{\mathcal{P}}$ and $F_{\mathcal{A}} \xrightarrow{\tau_{\mathcal{A}}} S/I_{\mathcal{A}}$ over S where $F_{\mathcal{P}}$ is supported by the permutohedron and $F_{\mathcal{A}}$ is supported by the associahedron. The inclusion of ideals $I_{\mathcal{A}} \subset I_{\mathcal{P}}$ induces a surjective S-linear map $\varphi\colon S/I_{\mathcal{A}} \to S/I_{\mathcal{P}}$ whose kernel is given by $I_{\mathcal{P}}/I_{\mathcal{A}}$. Lift $\varphi\tau_{\mathcal{A}}$ to a chain map $\widetilde{\varphi}\colon F_{\mathcal{A}} \to F_{\mathcal{P}}$ with respect to $\tau_{\mathcal{P}}$, so $\tau_{\mathcal{P}}\widetilde{\varphi} = \varphi\tau_{\mathcal{A}}$. It follows from Theorem (3.5) that $\Sigma\mathrm{C}(\widetilde{\varphi})$ is a free resolution of $I_{\mathcal{P}}/I_{\mathcal{A}}$ over S.

3.8 Tensor Products

3.8.1 Definition of tensor product

Definition 3.16. Let (A, d) and (A', d') be two R-complexes. Their **tensor product** is the R-complex $(A \otimes_R A', \mathrm{d}^{\otimes}_{(A,A')})$, where the graded R-module $A \otimes_R A'$ has

$$(A \otimes_R A')_i = \bigoplus_{j \in \mathbb{Z}} A_j \otimes A'_{j-i}$$

as its ith homogeneous component and whose differential is defined on elementary homogeneous tensors (and extended linearly) by

$$\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') = \mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')$$

for all $a \in A_i$, $a' \in A_j$ and $i, j \in \mathbb{Z}$.

Proposition 3.19. *The map $\mathrm{d}^{\otimes}_{(A,A')}$ is well-defined and is in fact a differential.*

Proof. First we observe that $\mathrm{d}^{\otimes}_{(A,A')}$ is a well-defined R-linear map because the map $A_i \times A'_j \to A_i \otimes_R A'_j$ given by

$$(a, a') \mapsto \mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')$$

for all $(a, a') \in A_i \times A'_j$ is R-bilinear for each $i, j \in \mathbb{Z}$. Next we observe that $\mathrm{d}^{\otimes}_{(A,A')}$ is graded of degree -1. Indeed, if $a \otimes a' \in A_j \otimes_R A'_{i-j}$, then

$$\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a') \in A_{j-1} \otimes_R A'_{i-j} + A_j \otimes_R A_{i-j-1}.$$

Lastly we observe that $\mathrm{d}^{\otimes}_{(A,A')}\mathrm{d}^{\otimes}_{(A,A')} = 0$ since if $a \otimes a' \in (A \otimes_R A')_k$ where $a \in A_i$ and $a' \in A'_j$, then

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(A,A')}\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') &= \mathrm{d}^{\otimes}_{(A,A')}(\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')) \\
&= \mathrm{d}^{\otimes}_{(A,A')}(\mathrm{d}(a) \otimes a') + (-1)^i \mathrm{d}^{\otimes}_{(A,A')}(a \otimes \mathrm{d}'(a')) \\
&= \mathrm{d}\mathrm{d}(a) \otimes a' + (-1)^{i-1}\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i(\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i a \otimes \mathrm{d}'\mathrm{d}'(a')) \\
&= (-1)^{i-1}\mathrm{d}(a) \otimes \mathrm{d}'(a') + (-1)^i \mathrm{d}(a) \otimes \mathrm{d}'(a') \\
&= 0.
\end{aligned}$$

□

3.8.2 Commutativity of tensor products

Proposition 3.20. *Let A and B be R-complexes. Then we have an isomorphism of R-complexes*

$$A \otimes_R B \cong B \otimes_R A, \tag{20}$$

which is natural in A and B.

Proof. We define $\tau_{A,B} \colon A \otimes_R B \to B \otimes_R A$ on elementary homogeneous tensors (and extend linearly) by

$$\tau_{A,B}(a \otimes b) = (-1)^{ij} b \otimes a$$

for all $a \otimes b \in A_i \otimes_R B_j$. The map $\tau_{A,B}$ is easily seen to be a well-defined graded R-linear isomorphism. To see that $\tau_{A,B}$ is an isomorphism of R-complexes, we need to show that it commutes with the differentials. That is, we need to show

$$\tau_{A,B}\mathrm{d}^{\otimes}_{(A,B)} = \mathrm{d}^{\otimes}_{(B,A)}\tau_{A,B} \tag{21}$$

It suffices to check (21) on elementary homogeneous tensors, so let $a \otimes b \in A_i \otimes_R B_j$ be such an elementary homogeneous tensor. Then we have

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(B,A)}\tau_{A,B}(a \otimes b) &= (-1)^{ij}\mathrm{d}^{\otimes}_{(B,A)}(b \otimes a) \\
&= (-1)^{ij}\mathrm{d}_B(b) \otimes a + (-1)^{j+ij} b \otimes \mathrm{d}_A(a)) \\
&= (-1)^{i+i(j-1)}\mathrm{d}_B(b) \otimes a + (-1)^{(i-1)j} b \otimes \mathrm{d}_A(a) \\
&= (-1)^{(i-1)j} b \otimes \mathrm{d}_A(a) + (-1)^{i+i(j-1)}\mathrm{d}_B(b) \otimes a \\
&= \tau_{A,B}(\mathrm{d}_A(a) \otimes b + (-1)^i a \otimes \mathrm{d}_B(b)) \\
&= \tau_{A,B}\mathrm{d}^{\otimes}_{(A,B)}(a \otimes b).
\end{aligned}$$

Finally, being natural in A and B means that if $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ are two chain maps, then the following diagram commutes:

$$\begin{tikzcd} A \otimes_R B \arrow[r, "\varphi\otimes_R B"] \arrow[d, "A\otimes_R\psi"'] & A' \otimes_R B \arrow[d, "A'\otimes_R\psi"] \\ A \otimes_R B' \arrow[r, "\varphi\otimes_R B'"'] & A' \otimes_R B' \end{tikzcd}$$

We leave it as an exercise for the reader to check that this diagram commutes. □

3.8.3 Associativity of tensor products

Given that the proof of tensor products of R-complexes was nontrivial, we need to be sure that we have associativity of tensor products of R-complexes. The proof in this case turns out to be trivial.

Proposition 3.21. *Let A, A', and A'' be R-complexes. Then we have an isomoprhism of R-complexes*

$$(A \otimes_R A') \otimes_R A'' \cong A \otimes_R (A' \otimes_R A''),$$

which is natural in A, A', and A''.

Proof. Let $\eta_{A,A',A''}\colon (A \otimes_R A') \otimes_R A'' \to A \otimes_R (A' \otimes_R A'')$ to be the unique graded isomorphism such that

$$\eta_{A,A',A''}((a \otimes a') \otimes a'') = a \otimes (a' \otimes a'')$$

for all $a \in A_i$, $a' \in A'_j$, and $a'' \in A''_k$ and for all $i, j, k \in \mathbb{Z}$. To see that $\eta_{A,A',A''}$ is an isomorphism of R-complexes, we need to show that

$$\eta_{A,A',A''}\mathrm{d}^{\otimes}_{((A\otimes_R A'),A'')} = \mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}\eta_{A,A',A''} \tag{22}$$

It suffices to check (22) on elementary homogeneous tensors. Let $(a \otimes a') \otimes a'' \in (A_i \otimes_R A_j) \otimes_R A_k$. To simplify the notation in our calculation, we denote $\eta = \eta_{A,A',A''}$. We have

$$\begin{aligned}
\mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}\eta((a \otimes a') \otimes a'') &= \mathrm{d}^{\otimes}_{(A,(A'\otimes_R A''))}(a \otimes (a' \otimes a'')) \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes \mathrm{d}^{\otimes}_{(A',A'')}(a' \otimes a'') \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (\mathrm{d}_{A'}(a') \otimes a'' + (-1)^j a' \otimes \mathrm{d}_{A''}(a'')) \\
&= \mathrm{d}_A(a) \otimes (a' \otimes a'') + (-1)^i a \otimes (\mathrm{d}_{A'}(a') \otimes a'') + (-1)^{i+j} a \otimes (a' \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta((\mathrm{d}_A(a) \otimes a') \otimes a'') + (-1)^i \eta((a \otimes \mathrm{d}_{A'}(a')) \otimes a'') + (-1)^{i+j}\eta((a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta((\mathrm{d}_A(a) \otimes a') \otimes a'' + (-1)^i (a \otimes \mathrm{d}_{A'}(a')) \otimes a'' + (-1)^{i+j}(a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta(\mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \otimes a'' + (-1)^{i+j}(a \otimes a') \otimes \mathrm{d}_{A''}(a'')) \\
&= \eta\mathrm{d}^{\otimes}_{((A\otimes_R A'),A'')}((a \otimes a') \otimes a'').
\end{aligned}$$

Therefore (22) holds, and thus $\eta_{A,A',A''}$ is an isomorphism of R-complexes.

Naturality in A, A', and A'' means that if $\varphi\colon A \to B$, $\varphi\colon A' \to B'$, and $\varphi\colon A'' \to B''$ are chains maps, then we have a commutative diagram

$$\begin{tikzcd} (A \otimes_R A')_R \otimes A'' \arrow[r, "\eta_{A,A',A''}"] \arrow[d, "(\varphi\otimes\varphi')\otimes\varphi''"'] & A \otimes_R (A'_R \otimes A'') \arrow[d, "\varphi\otimes(\varphi'\otimes\varphi'')"] \\ (B \otimes_R B')_R \otimes B'' \arrow[r, "\eta_{B,B',B''}"] & (B \otimes_R B')_R \otimes B'' \end{tikzcd}$$

□

3.8.4 Tensor Commutes with Shifts

Proposition 3.22. *Let $n \in \mathbb{Z}$ and let A and A' be R-complexes. Then*

$$(\Sigma^n A) \otimes_R A' \cong \Sigma^n(A \otimes_R A') \cong A \otimes_R (\Sigma^n A')$$

are isomorphisms of R-complexes.

Proof. We will just show that $(\Sigma^n A) \otimes_R A' \cong \Sigma^n(A \otimes_R A')$. The other isomorphism follows from a similar argument. As graded R-modules, we have

$$\begin{aligned}(\Sigma^n A) \otimes_R A' &= A(-n) \otimes_R A' \\ &= (A \otimes_R A')(-n) \\ &= \Sigma^n(A \otimes_R A').\end{aligned}$$

We define $\Phi\colon (\Sigma^n A) \otimes_R A' \to \Sigma^n(A \otimes_R A')$ by

$$\Phi(a \otimes a') = a \otimes a'$$

for all elementary tensors $a \otimes a' \in \Sigma^n A \otimes_R A'$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $a \otimes a' \in (\Sigma^n A) \otimes_R A'$ with $a \in A_i$ and $a' \in A_j$. Then $a \in (\Sigma^n A)_{i+n}$, and so we have

$$\begin{aligned}(\Sigma^n \mathrm{d}^{\otimes}_{(A,A')}\Phi)(a \otimes a') &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(\Phi(a \otimes a')) \\ &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \\ &= (-1)^n \mathrm{d}^{\otimes}_{(A,A')}(a \otimes a') \\ &= (-1)^n(\mathrm{d}_A(a) \otimes a' + (-1)^i a \otimes \mathrm{d}_{A'}(a')) \\ &= (-1)^n \mathrm{d}_A(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a') \\ &= \mathrm{d}_{\Sigma^n A}(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a') \\ &= \Phi(\mathrm{d}_{\Sigma^n A}(a) \otimes a' + (-1)^{i+n} a \otimes \mathrm{d}_{A'}(a')) \\ &= \Phi(\mathrm{d}^{\otimes}_{(\Sigma^n A,A')}(a \otimes a')) \\ &= (\Phi \mathrm{d}^{\otimes}_{(\Sigma^n A,A')})(a \otimes a')\end{aligned}$$

□

3.8.5 Tensor Commutes with Mapping Cone

Proposition 3.23. *Let X be an R-complex and let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then*

$$\mathrm{C}(\varphi) \otimes_R X \cong \mathrm{C}(\varphi \otimes_R X)$$

is an isomorphism of R-complexes.

Proof. As graded R-modules, we have

$$\begin{aligned}\mathrm{C}(\varphi) \otimes_R X &= (A' \oplus A(-1)) \otimes_R X \\ &\cong (A' \otimes_R X) \oplus (A(-1) \otimes_R X) \\ &= (A' \otimes_R X) \oplus (A \otimes_R X)(-1) \\ &= \mathrm{C}(\varphi \otimes_R X),\end{aligned}$$

where the graded isomorphism in the second line is given by

$$(a', a) \otimes x \mapsto (a' \otimes x, a \otimes x)$$

for all elementary tensors $(a', a) \otimes x \in (A' \oplus A(-1)) \otimes_R X$.

Let $\Phi\colon \mathrm{C}(\varphi) \otimes_R X \to \mathrm{C}(\varphi \otimes_R X)$ be the unique R-linear map such that

$$\Phi(x \otimes (a', a)) = (x \otimes a', x \otimes a)$$

for all elementary tensors $(a', a) \otimes x \in \mathrm{C}(\varphi) \otimes_R X$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes.

Indeed, let $(a', a) \otimes x \in \mathrm{C}(\varphi) \otimes_R X$ be an elementary tensor with $a' \in A'_i$, $a \in A_{i-1}$, and $x \in X_j$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)} \Phi)((a', a) \otimes x) &= \mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)}(\Phi((a', a) \otimes x)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)}(a' \otimes x, a \otimes x) \\
&= (\mathrm{d}^{\otimes}_{(A',X)}(a' \otimes x) + (\varphi \otimes X)(a \otimes x), -\mathrm{d}^{\otimes}_{(A,X)}(a \otimes x)) \\
&= (\mathrm{d}_{A'}(a') \otimes x + (-1)^i a' \otimes \mathrm{d}_X(x) + \varphi(a) \otimes x, -\mathrm{d}_A(a) \otimes x + (-1)^i a \otimes \mathrm{d}_X(x)) \\
&= ((\mathrm{d}_{A'}(a') \otimes x + \varphi(a) \otimes x + (-1)^i a' \otimes \mathrm{d}_X(x), -\mathrm{d}_A(a) \otimes x + (-1)^i a \otimes \mathrm{d}_X(x)) \\
&= ((\mathrm{d}_{A'}(a') + \varphi(a)) \otimes x, -\mathrm{d}_A(a) \otimes x) + (-1)^i((a' \otimes \mathrm{d}_X(x), a \otimes \mathrm{d}_X(x)) \\
&= \Phi((\mathrm{d}_{A'}(a') + \varphi(a), -\mathrm{d}_A(a)) \otimes x + (-1)^i (a', a) \otimes \mathrm{d}_X(x)) \\
&= \Phi(\mathrm{d}_{\mathrm{C}(\varphi)}(a', a) \otimes x + (-1)^i (a', a) \otimes \mathrm{d}_X(x)) \\
&= \Phi(\mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi),X)}((a', a) \otimes x) \\
&= (\Phi \mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi),X)})((a', a) \otimes x).
\end{aligned}$$

It follows that $\mathrm{d}_{\mathrm{C}(\varphi \otimes_R X)} \Phi = \Phi \mathrm{d}^{\otimes}_{(\mathrm{C}(\varphi),X)}$. Thus Φ gives an isomorphism of R-complexes.

□

Proposition 3.24. *Let A be an R-complex and let $\psi\colon B \to B'$ be a chain map of R-complexes. Then*

$$A \otimes_R \mathrm{C}(\psi) \cong \mathrm{C}(A \otimes_R \psi)$$

is an isomorphism of R-complexes.

Proof. Combining Proposition (3.18) and Proposition (3.23) gives us the isomorphisms

$$\begin{aligned}
A \otimes_R \mathrm{C}(\psi) &\cong \mathrm{C}(\psi) \otimes_R A \\
&\cong \mathrm{C}(\psi \otimes_R A) \\
&\cong \mathrm{C}(A \otimes_R \psi).
\end{aligned}$$

Following these isomorphisms in terms of an elementary homogeneous element $a \otimes (b', b) \in A_i \otimes \mathrm{C}(\psi)_j$, we have

$$\begin{aligned}
a \otimes (b', b) &\mapsto (-1)^{ij}(b', b) \otimes a \\
&\mapsto (-1)^{ij}(b' \otimes a, b \otimes a) \\
&\mapsto (-1)^{ij}((-1)^{ij} a \otimes b', (-1)^{i(j-1)} a \otimes b) \\
&= (a \otimes b', (-1)^{ij+i(j-1)} a \otimes b) \\
&= (a \otimes b', (-1)^i a \otimes b)
\end{aligned}$$

Let us check that this really does commute with the differentials. Define $\Phi\colon A \otimes_R \mathrm{C}(\psi) \to \mathrm{C}(A \otimes_R \psi)$ by

$$\Phi(a \otimes (b', b)) = (a \otimes b', (-1)^i a \otimes b)$$

for all elementary homogeneous tensors $a \otimes (b', b) \in A_i \otimes_R \mathrm{C}(\psi)_j$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(A \otimes_R \psi)} \Phi)(a \otimes (b', b)) &= \mathrm{d}_{\mathrm{C}(A \otimes_R \psi)}(a \otimes b', (-1)^i a \otimes b) \\
&= (\mathrm{d}^{\otimes}_{(A,B')}(a \otimes b') + (-1)^i (A \otimes_R \psi)(a \otimes b), -(-1)^i \mathrm{d}^{\otimes}_{(A,B)}(a \otimes b)) \\
&= (\mathrm{d}_A(a) \otimes b' + (-1)^i a \otimes \mathrm{d}_{B'}(b') + (-1)^i a \otimes \psi(b), -(-1)^i \mathrm{d}_A(a) \otimes b - a \otimes \mathrm{d}_B(b)) \\
&= (\mathrm{d}_A(a) \otimes b', -(-1)^i \mathrm{d}_A(a) \otimes b) + ((-1)^i a \otimes \mathrm{d}_{B'}(b') + (-1)^i a \otimes \psi(b), a \otimes -\mathrm{d}_B(b))) \\
&= \Phi(\mathrm{d}_A(a) \otimes (b', b) + (-1)^i a \otimes (\mathrm{d}_{B'}(b') + \psi(b), -\mathrm{d}_B(b))) \\
&= \Phi(\mathrm{d}_A(a) \otimes (b', b) + (-1)^i a \otimes \mathrm{d}_{\mathrm{C}(\psi)}(b', b)) \\
&= (\Phi \mathrm{d}_{A \otimes_R \mathrm{C}(\psi)})(a \otimes (b', b)).
\end{aligned}$$

□

3.8.6 Tensor Respects Homotopy Equivalences

Proposition 3.25. *Let B be an R-complex, let $\varphi\colon A \to A'$ and $\psi\colon A \to A'$ be two chain maps of R-complexes, and suppose $\varphi \sim \psi$. Then $\varphi \otimes_R B \sim \psi \otimes_R B$.*

Proof. Choose a homotopy $h\colon A \to A'$ from φ to ψ (so $\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A$). We claim that $h \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$ is a homotopy from $\varphi \otimes_R B$ to $\psi \otimes_R B$. Indeed, let $a \otimes b$ be an elementary homogeneous tensor in $A \otimes_R B$. Then we have

$$\begin{aligned}
(\mathrm{d}^{\otimes}_{(A',B)}(h \otimes B) + (h \otimes B)\mathrm{d}^{\otimes}_{(A,B)})(a \otimes b) &= \mathrm{d}^{\otimes}_{(A',B)}(h(a) \otimes b) + (h \otimes B)(\mathrm{d}_A(a) \otimes b + (-1)^{|a|} a \otimes \mathrm{d}_B(b)) \\
&= \mathrm{d}_{A'}h(a) \otimes b - (-1)^{|a|} h(a) \otimes \mathrm{d}_B(b) + h\mathrm{d}_A(a) \otimes b + (-1)^{|a|} h(a) \otimes \mathrm{d}_B(b) \\
&= \mathrm{d}_{A'}h(a) \otimes b + h\mathrm{d}_A(a) \otimes b \\
&= (\mathrm{d}_{A'}h + h\mathrm{d}_A)(a) \otimes b \\
&= (\varphi - \psi)(a) \otimes b \\
&= \varphi(a) \otimes b - \psi(a) \otimes b \\
&= (\varphi \otimes B - \psi \otimes B)(a \otimes b).
\end{aligned}$$

Thus $h \otimes_R B$ is indeed a homotopy from $\varphi \otimes_R B$ to $\psi \otimes_R B$. □

Corollary. *Suppose $\varphi\colon A \to A'$ is a homotopy of equivalence of R-complexes. Then $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$ is a homotopy equivalence of R-complexes.*

Proof. Let $\varphi'\colon A' \to A$ be a homotopy inverse to φ. Thus $\varphi\varphi' \sim 1_{A'}$ and $\varphi'\varphi \sim 1_A$. It follows that

$$\begin{aligned}
1_{A' \otimes_R B} &= 1_{A'} \otimes_R B \\
&\sim \varphi\varphi' \otimes_R B \\
&= (\varphi \otimes_R B)(\varphi' \otimes_R B).
\end{aligned}$$

Similarly, we have $1_{A \otimes_R B} \sim (\varphi' \otimes_R B)(\varphi \otimes_R B)$. Therefore $\varphi \otimes_R B$ is a homotopy equivalence of R-complexes. □

3.8.7 Twisting the tensor complex with a chain map

Definition 3.17. Let (A, d) be R-complexes and let $\alpha\colon A \to A$ be a chain map. We define an R-complex $A \otimes_R^{\alpha} A$ as follows: as a graded R-module, $A \otimes_R^{\alpha} A$ is just $A \otimes_R A$. We define the differential $\mathrm{d}^{\otimes}_{\alpha}\colon A \otimes_R^{\alpha} A \to A \otimes_R^{\alpha} A$ on elementary tensors $a \otimes b \in A_i \otimes_R A_j$ by

$$\mathrm{d}^{\otimes}_{\alpha}(a \otimes b) = \mathrm{d}(a) \otimes b + (-1)^i \alpha(a) \otimes \mathrm{d}(b) \tag{23}$$

and then we extend $\mathrm{d}^{\otimes}_{\alpha}$ linearly everywhere else. Note that $\mathrm{d}^{\otimes}_{\alpha}$ is a well-defined R-linear map since (23) is R-bilinear in a and b. Also note that $\mathrm{d}^{\otimes}_{\alpha}$ is graded of degree -1 since α is a chain map. Let us show that we have $\mathrm{d}^{\otimes}_{\alpha}\mathrm{d}^{\otimes}_{\alpha} = 0$. Let $a \otimes b \in A_i \otimes_R A_j$. Then we have

$$\begin{aligned}
\mathrm{d}^{\otimes}_{\alpha}\mathrm{d}^{\otimes}_{\alpha}(a \otimes b) &= \mathrm{d}^{\otimes}_{\alpha}(\mathrm{d}(a) \otimes b + (-1)^i \alpha(a) \otimes \mathrm{d}(b)) \\
&= \mathrm{d}^{\otimes}_{\alpha}(\mathrm{d}(a) \otimes b) + (-1)^i \mathrm{d}^{\otimes}_{\alpha}(\alpha(a) \otimes \mathrm{d}(b)) \\
&= \mathrm{d}^2(a) \otimes b + (-1)^{i-1} \alpha\mathrm{d}(a) \otimes \mathrm{d}(b) + (-1)^i \mathrm{d}\alpha(a) \otimes \mathrm{d}(b) + \alpha^2(a) \otimes \mathrm{d}^2(b) \\
&= (-1)^{i-1} \alpha\mathrm{d}(a) \otimes \mathrm{d}(b) + (-1)^i \alpha\mathrm{d}(a) \otimes \mathrm{d}(b) \\
&= 0.
\end{aligned}$$

It follows that $\mathrm{d}^{\otimes}_{\alpha}$ is a differential.

If $\alpha\colon A \to A$ is also an R-algebra homomorphism, then observe that

$$\begin{aligned}
\mathrm{d}(\alpha(a)(bc) + (ab)\alpha(c)) &= \mathrm{d}(\alpha(a))(bc) + \alpha^2(a)\mathrm{d}(bc) + \mathrm{d}(ab)\alpha(c) + \alpha(ab)\mathrm{d}(\alpha(c)) \\
&= \alpha(\mathrm{d}(a))(bc) + \alpha^2(a)(\mathrm{d}(b)c) + \alpha^2(a)(\alpha(b)\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\mathrm{d}(b))\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c)) \\
&= \alpha(\mathrm{d}(a))(bc) + (\alpha(a)\mathrm{d}(b))\alpha(c) + (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\mathrm{d}(b))\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c)) \\
&= (\mathrm{d}(a)b)\alpha(c) + (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + (\mathrm{d}(a)b)\alpha(c) + \alpha(ab)\alpha(\mathrm{d}(c)) \\
&= (\alpha(a)\alpha(b))(\alpha(\mathrm{d}(c)) + \alpha(ab)\alpha(\mathrm{d}(c)) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}(a(bc) + (ab)c) &= \mathrm{d}(a)(bc) + a\mathrm{d}(bc) + \mathrm{d}(ab)c + (ab)\mathrm{d}(c) \\
&= \mathrm{d}(a)(bc) + a(\mathrm{d}(b)c) + a(b\mathrm{d}(c)) + (\mathrm{d}(a)b)c + (a\mathrm{d}(b))c + (ab)\mathrm{d}(c) \\
&= \mathrm{d}(a)(bc) + (\mathrm{d}(a)b)c + a(\mathrm{d}(b)c) + (a\mathrm{d}(b))c + a(b\mathrm{d}(c)) + (ab)\mathrm{d}(c).
\end{aligned}$$

3.9 Hom

Definition 3.18. Let (A, d) and (A', d') be two R-complexes. We define

$$\mathrm{Hom}_R((A, \mathrm{d}), (A', \mathrm{d}')) := (\mathrm{Hom}_R^\star(A, A'), \mathrm{d}^{\mathrm{Hom}_R^\star(A,A')})$$

to be the R-complex whose graded R-module $\mathrm{Hom}_R^\star(A, A')$ has

$$\mathrm{Hom}_R^\star(A, A')_i = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(A_n, A'_{n+i})$$

as its ith homogeneous component and whose differential $\mathrm{d}^{\mathrm{Hom}_R^\star(A,A')}$ is defined by

$$\mathrm{d}^{\mathrm{Hom}_R^\star(A,A')}((\varphi_n^i)_{n\in\mathbb{Z}}) = (\mathrm{d}'\varphi_n^i - (-1)^i \varphi_{n-1}^i \mathrm{d})_{n\in\mathbb{Z}} \tag{24}$$

for all $i, n \in \mathbb{Z}$ and $\varphi_{n,i} \in \mathrm{Hom}_R(A_j, A'_{i+j})$.

If context is clear, we will denote $\mathrm{d}^{\mathrm{Hom}_R^\star(A,A')}$ simply as $\mathrm{d}^\star$. We also write (φ_n^i) instead of $(\varphi_n^i)_{n\in\mathbb{Z}}$. The subscript n will clue us in on the fact that (φ_n^i) is a sequence of homomorphisms. Sometimes we will also write $\mathrm{Hom}_R^\star(A, A')$ (rather than the more cumbersome notation $\mathrm{Hom}_R((A, \mathrm{d}), (A', \mathrm{d}'))$) and specify that $\mathrm{Hom}_R^\star(A, A')$ refers to the R-complex hom and not just the graded R-module hom.

Let us check that $\mathrm{d}^\star\mathrm{d}^\star = 0$. Let $(\varphi_n^i) \in \mathrm{Hom}_R^\star(A, A')_i$. Then we have

$$\begin{aligned}
\mathrm{d}^\star\mathrm{d}^\star(\varphi_n^i) &= \mathrm{d}^\star(\mathrm{d}'\varphi_n^i - (-1)^i\varphi_{n-1}^i\mathrm{d}) \\
&= (\mathrm{d}'(\mathrm{d}'\varphi_n^i - (-1)^i\varphi_{n-1}^i\mathrm{d}) - (-1)^{i-1}(\mathrm{d}'\varphi_{n-1}^i - (-1)^i\varphi_{n-2}^i\mathrm{d})\mathrm{d}) \\
&= -(-1)^i\mathrm{d}'\varphi_{n-1}^i\mathrm{d} - (-1)^{i-1}\mathrm{d}'\varphi_{n-1}^i\mathrm{d} \\
&= 0.
\end{aligned}$$

Thus $\mathrm{d}^\star\mathrm{d}^\star = 0$. Note that the sign $-(-1)^i$ in (24) is a little unusual. In the tensor product differential $\mathrm{d}^\otimes$, we had

$$\mathrm{d}^\otimes(a \otimes a') = \mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a')$$

whenever $a \in A_i$ and $a' \in A'_{i'}$. If we replace the sign $-(-1)^i$ with the sign $(-1)^i$ in (24), we would still get $\mathrm{d}^\star\mathrm{d}^\star = 0$. However, for reasons to be clarified later on, we keep the sign $-(-1)^i$.

Note that if A' is just an R-module (so trivially graded with $\mathrm{d}' = 0$), then

$$\mathrm{Hom}_R^\star(A, A')_i \cong \mathrm{Hom}_R(A_{-i}, A').$$

In this case, we have

$$\mathrm{d}^\star(\varphi) = -(-1)^i\varphi\mathrm{d}$$

whenever $\varphi \in \mathrm{Hom}_R(A_{-i}, A')$. Also, if A is just an R-module (so trivially graded with $\mathrm{d} = 0$), then

$$\mathrm{Hom}_R^\star(A, A')_i \cong \mathrm{Hom}_R(A, A'_i).$$

In this case, we have

$$\mathrm{d}^\star(\varphi) = \mathrm{d}'\varphi$$

whenever $\varphi \in \mathrm{Hom}_R(A, A'_i)$.

3.9.1 Reinterpretation of Hom

Definition 3.19. Let A and A' be two R-complexes. We define their **hom complex**, denoted $(\mathrm{Hom}_R^\star(A, A'), \mathrm{d}_{(A,A')}^\star)$, to be the R-complex whose underlying graded R-module $\mathrm{Hom}_R^\star(A, A')$ has

$$\mathrm{Hom}_R^\star(A, A')_i = \{\alpha\colon A \to A' \mid \alpha \text{ is graded of degree } i\}$$

as its homogeneous component in degree i, and whose differential is defined by

$$\mathrm{d}_{(A,A')}^\star(\alpha) = \mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_A$$

for all $\alpha \in \mathrm{Hom}_R^\star(A, A')_i$ for all $i \in \mathbb{Z}$.

3.9.2 Homology of Hom

Proposition 3.26. *Let A and A' be two R-complexes. Then*

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(A,A')) = \{\textit{homotopy classes of chain maps } A \to A'\}.$$

Proof. Recall that homotopy gives an equivalence relation $\sim$ on the set of all chain maps $\mathcal{C}(A,A')$ from A to A'. Thus we are saying that

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(A,A')) = \mathcal{C}(A,A')/\sim.$$

Let $\alpha \in \mathrm{Z}_0(\mathrm{Hom}_R^\star(A,A'))$, so $\alpha\colon A \to A'$ be a graded R-linear map of degree 0 such that

$$\begin{aligned} 0 &= \mathrm{d}^\star_{(A,A')}(\alpha) \\ &= \mathrm{d}_{A'}\alpha - \alpha\mathrm{d}_A. \end{aligned}$$

In other words, α is a chain map. It follows that

$$\mathrm{Z}_0(\mathrm{Hom}_R^\star(A,A')) = \mathcal{C}(A,A').$$

Next we observe that elements in $\mathrm{B}_0(\mathrm{Hom}_R^\star(A,A'))$ are of the form

$$\mathrm{d}^\star_{(A,A')}(\beta) = \mathrm{d}_{A'}\beta + \beta\mathrm{d}_A$$

where $\beta\colon A \to A'$ be a graded R-linear map of degree 1. Thus two chain maps α_1 and α_2 represent the same class in homology if and only if they are homotopic to each other. □

Remark. More generally, $\mathrm{H}_i(\mathrm{Hom}_R^\star(A,A'))$ is exact if and only if for all graded R-linear maps $\alpha\colon A \to A'$ of degree i such that

$$\mathrm{d}_{A'}\alpha = (-1)^i\alpha\mathrm{d}_A,$$

there exists a graded R-linear map $\beta\colon A \to A'$ such that

$$\alpha = \mathrm{d}_A\beta + (-1)^i\beta\mathrm{d}_{A'}.$$

3.9.3 Functorial Properties of Hom

Proposition 3.27. *Let (A,d_A), (A',d'_A), (B,d_B), and (B',d'_B) be R-complexes and let $\varphi\colon A \to B$ and $\phi\colon A' \to B'$ be chain maps. Then we get induced chain maps*

$$\phi_*\colon \mathrm{Hom}_R^\star(A,A') \to \mathrm{Hom}_R^\star(A,B') \quad \textit{and} \quad \varphi^*\colon \mathrm{Hom}_R^\star(B,B') \to \mathrm{Hom}_R^\star(A,B')$$

given by

$$\phi_*(\alpha) = \phi\alpha \quad \textit{and} \quad \varphi^*(\beta) = \beta\varphi$$

for all $\alpha \in \mathrm{Hom}_R^\star(A,A')$ and $\beta \in \mathrm{Hom}_R^\star(B,B')$. Furthermore, the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_R^\star(A,A') \arrow[r, "\varphi^*"] \arrow[d, "\phi_*"'] & \mathrm{Hom}_R^\star(B,A') \arrow[d, "\phi_*"] \\ \mathrm{Hom}_R^\star(A,B') \arrow[r, "\varphi^*"'] & \mathrm{Hom}_R^\star(B,B') \end{tikzcd} \tag{25}$$

Proof. First let us check that ϕ_* is a chain map. It is a graded R-linear map since ϕ is a graded R-linear map of degree 0 and composition is R-linear. It remains to show that ϕ_* commutes with the differentials. Let $\alpha \in \mathrm{Hom}_R^\star(A,A')_i$. Then we have

$$\begin{aligned} (\mathrm{d}^\star_{(A,B')}\phi_*)(\alpha) &= \mathrm{d}^\star_{(A,B')}(\phi_*(\alpha)) \\ &= \mathrm{d}^\star_{(A,B')}(\phi\alpha) \\ &= \mathrm{d}_{B'}\phi\alpha - (-1)^i\phi\alpha\mathrm{d}_A \\ &= \phi\mathrm{d}_{A'}\alpha - (-1)^i\phi\alpha\mathrm{d}_A \\ &= \phi_*(\mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_A) \\ &= \phi_*(\mathrm{d}^\star_{(A,A')}(\alpha)) \\ &= (\phi_*\mathrm{d}^\star_{(A,A')})(\alpha). \end{aligned}$$

This implies ϕ_* is a chain map. A similar calculation shows that φ^* is a chain map.

Now we check that the diagram (25) commutes. Let $\alpha \in \mathrm{Hom}_R^\star(A, A')_i$. Then we have

$$\begin{aligned}(\phi_*\varphi^*)(\alpha) &= \phi_*(\varphi^*(\alpha)) \\ &= \phi_*(\alpha\varphi) \\ &= \phi\alpha\varphi \\ &= \varphi^*(\phi\alpha) \\ &= \varphi^*(\phi_*(\alpha)) \\ &= (\varphi^*\phi_*)(\alpha).\end{aligned}$$

This implies the diagram commutes. □

Proposition 3.28. *Let A be an R-complex. Then we obtain functors*

$$\mathrm{Hom}_R^\star(A, -)\colon \mathrm{Comp}_R \to \mathrm{Comp}_R \quad \text{and} \quad \mathrm{Hom}_R^\star(-, A)\colon \mathrm{Comp}_R \to \mathrm{Comp}_R$$

from the category of R-complexes to itself, where the R-complex B is assigned to the R-complexes

$$\mathrm{Hom}_R^\star(A, B) \quad \text{and} \quad \mathrm{Hom}_R^\star(B, A)$$

respectively, and where the chain map $\varphi\colon B \to B'$ of R-complexes is assigned to the chain maps

$$\mathrm{Hom}_R^\star(A, \varphi) = \varphi_* \quad \text{and} \quad \mathrm{Hom}_R^\star(\varphi, A) = \varphi^*$$

respectively.

Proof. We will just show that $\mathrm{Hom}_R^\star(A, -)$ is a functor from the category of R-complexes to itself since a similar argument will show that $\mathrm{Hom}_R^\star(-, A)$ is one too. We need to check that $\mathrm{Hom}_R^\star(A, -)$ preserves compositions and identities. We first check that it preserves compositions. Let $\varphi\colon B \to B'$ and $\varphi'\colon B' \to B''$ be two chain maps and let $\alpha \in \mathrm{Hom}_R^\star(A, B)_i$. Then we have

$$\begin{aligned}(\varphi'\varphi)_*(\alpha) &= \varphi'\varphi\alpha \\ &= \varphi'_*(\varphi\alpha) \\ &= \varphi'_*(\varphi_*(\alpha)) \\ &= (\varphi'_*\varphi_*)(\alpha)\end{aligned}$$

It follows that $(\varphi'\varphi)_* = \varphi'_*\varphi_*$. Hence $\mathrm{Hom}_R^\star(A, -)$ preserves compositions. Next we check that $\mathrm{Hom}_R^\star(A, -)$ preserves identities. Let B be an R-complex and let $\alpha\colon A \to B$ be a chain map. Then we have

$$\begin{aligned}(1_B)_* &= 1_B\alpha \\ &= \alpha \\ &= 1_{\mathrm{Hom}_R^\star(A,B)}(\alpha).\end{aligned}$$

It follows that $(1_B)_* = 1_{\mathrm{Hom}_R^\star(A,-)}$. Hence h_A preserves identities. □

Proposition 3.29. *Let F be a covariant functor from the category of R-complexes to itself. Then F is left exact if and only if it is left exact when viewed as a functor of the underlying graded R-modules.*

Proof. One direction is easy, so we prove the other direction. Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{26}$$

be an exact sequence of R-complexes and chain maps. Then (26) is an exact sequence of graded R-modules and graded homomorphisms. Thus

$$F(M_1) \xrightarrow{F(\varphi_1)} F(M_2) \xrightarrow{F(\varphi_2)} F(M_3) \longrightarrow 0 \tag{27}$$

is an exact sequence of graded R-modules and graded homomorphisms. Since the graded homomorphisms in (27) commute with the differentials, we see that (27) is actually an exact sequence of R-complexes and chain maps.

□

Proposition 3.30. *(Yoneda's Lemma) Let A be an R-complex and let $\mathcal{F}\colon \mathbf{Comp}_R \to \mathbf{Set}$ be a functor. Then we have a bijection*

$$\mathrm{Nat}(\mathcal{C}(A,-),\mathcal{F}) \cong \mathcal{F}(A)$$

which is natural in A. In particular, if B is another R-complex, then

$$\mathrm{Nat}(\mathcal{C}(A,-),C(B,-)) \cong \mathcal{C}(B,A)$$

Note that the diagram (25) tells us that each chain map $\varphi\colon A \to B$ gives rise to a natural transformation $\mathrm{h}^-(\varphi)\colon \mathrm{h}_A \to \mathrm{h}_B$. In light of Yoneda's Lemma, we have a map

$$\mathrm{Nat}(C(B,-),\mathcal{C}(A,-)) \to \mathcal{C}(A,B) \to \mathrm{Nat}(\mathrm{h}_A,\mathrm{h}_B).$$

3.9.4 Left Exactness of Contravariant $\mathrm{Hom}_R^\star(-,N)$

Let M and N be R-complexes. We showed earlier that both $\mathrm{Hom}_R^\star(M,-)$ and $\mathrm{Hom}_R^\star(-,N)$ are left exact functors from the category of graded R-modules to itself. In fact, we will see that they The graded version of these functors are

$$\mathrm{Hom}_R^\star(M,-)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R \quad \text{and} \quad \mathrm{Hom}_R^\star(-,N)\colon \mathrm{Grad}_R \to \mathrm{Grad}_R.$$

We want to check that they are also left exact functors. Let's focus on $\mathrm{Hom}_R^\star(-,N)$ first:

Proposition 3.31. *The sequence of graded R-modules and graded homomorphisms*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{28}$$

is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R^\star(M_3,N) \xrightarrow{\varphi_2^*} \mathrm{Hom}_R^\star(M_2,N) \xrightarrow{\varphi_1^*} \mathrm{Hom}_R^\star(M_1,N) \tag{29}$$

is exact.

Proof. Suppose that (28) is exact and let N be any R-module. Exactness at $\mathrm{Hom}_R^\star(M_3,N)$ follows from the fact that φ_2^* is injective (which follows from the fact that $\mathrm{Hom}_R(-,N)$ is left exact). Next we show exactness at $\mathrm{Hom}_R^\star(M_2,N)$. Let $\psi_2\colon M_2 \to N$ be a graded homomorphism of degree i such that $\psi_2\varphi_1 = 0$. By left exactness of $\mathrm{Hom}_R(-,N)$, there exists a $\psi_3 \in \mathrm{Hom}_R(M,N)$ such that $\psi_2 = \psi_3\varphi_2$. Since φ_2 is surjective, ψ_3 is graded of degree i. Thus $\psi_3 \in \mathrm{Hom}_R^\star(M,N)$. Thus we have exactness at $\mathrm{Hom}_R^\star(M_2,N)$. □

3.9.5 Tensor-Hom Adjointness

Proposition 3.32. *Let S be an R-algebra, let M_1, M_2 be S-complexes, and let M_3 be an R-complex. Then we have an isomorphism of S-complexes*

$$\mathrm{Hom}_S^\star(M_1,\mathrm{Hom}_R^\star(M_2,M_3)) \cong \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3). \tag{30}$$

Moreover (30) is natural in M_1, M_2, and M_3.

Proof. We define

$$\Psi_{M_1,M_2,M_3}\colon \mathrm{Hom}_S^\star(M_1,\mathrm{Hom}_R^\star(M_2,M_3)) \to \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3)$$

to be the map which sends a $\psi \in \mathrm{Hom}_S^\star(M_1,\mathrm{Hom}_R^\star(M_2,M_3))$ to the map $\Psi(\psi) \in \mathrm{Hom}_R^\star(M_1 \otimes_S M_2, M_3)$ defined by

$$\Psi(\psi)(u_1 \otimes u_2) = (\psi(u_1))(u_2) \tag{31}$$

for all elementary tensors $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Note that $\Psi(\psi)$ is a well-defined R-linear map since the map $M_1 \times M_2 \to M_3$ given by

$$(u_1,u_2) \mapsto (\psi(u_1))(u_2)$$

is R-bilinear. We will show that Ψ is an isomorphism of S-complexes by breaking down the proof into several steps:

Step 1: We show that Ψ is S-linear. Let $s,s' \in S$ and $\psi,\psi' \in \mathrm{Hom}_S^\star(M_1,\mathrm{Hom}_R^\star(M_2,M_3))$. We want to show that

$$\Psi(s\psi + s'\psi') = s\Psi(\psi) + s'\Psi(\psi') \tag{32}$$

We will show (32) holds, by showing that the two maps agree on all elementary tensors in $M_1 \otimes_S M_2$. So let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then

$$\begin{aligned}
\Psi(s\psi + s'\psi')(u_1 \otimes u_2) &= ((s\psi + s'\psi')(u_1))(u_2) \\
&= ((s\psi)(u_1) + (s'\psi')(u_1))(u_2) \\
&= (\psi(su_1) + \psi(s'u_1))(u_2) \\
&= (\psi(su_1))(u_2) + (\psi(s'u_1))(u_2) \\
&= \Psi(\psi)(su_1 \otimes u_2) + \Psi(\psi')(s'u_1 \otimes u_2) \\
&= (s\Psi(\psi))(u_1 \otimes u_2) + (s'\Psi(\psi'))(u_1 \otimes u_2). \\
&= (s\Psi(\psi) + s'\Psi(\psi))(u_1 \otimes u_2)
\end{aligned}$$

It follows that Ψ is S-linear.

Step 2: We show that Ψ is graded. Let ψ be a graded S-linear map from M_1 to $\mathrm{Hom}_R^\star(M_2, M_3)$ of degree n. We want to show that $\Psi(\psi)$ is a graded of degree n too. To see that $\Psi(\psi)$ is graded of degree n, let $u_1 \otimes u_2$ be an elementary tensor in $M_1 \otimes_S M_2$ where u_i has degree i and u_j has degree j. Since ψ is graded of degree n, u_1 is graded of degree i, and u_2 is graded of degree j, we see that $\psi(u_1)$ is graded of degree $i+n$, and hence

$$(\psi(u_1))(u_2) = \Psi(\psi)(u_1 \otimes u_2)$$

is graded of degree $i + j + n$. It follows that $\Psi(\psi)$ is graded of degree n.

Step 3: We show that Ψ commutes with the differentials. In other words, we want to show that

$$\mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)} \Psi = \Psi \mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))} \tag{33}$$

To see that (33) holds, it suffices to show that it holds when we apply to both sides any graded S-linear map of degree n from M_1 to $\mathrm{Hom}_R^\star(M_2, M_3)$. So let ψ be such a map. Then observe on the one hand, we have

$$\begin{aligned}
(\mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)} \Psi)(\psi) &= \mathrm{d}^\star_{(M_1 \otimes_S M_2, M_3)}(\Psi(\psi)) \\
&= \mathrm{d}_{M_3} \Psi(\psi) + (-1)^n \Psi(\psi) \mathrm{d}^\otimes_{(M_1, M_2)},
\end{aligned}$$

and on the other hand, we have

$$\begin{aligned}
(\Psi \mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))})(\psi) &= \Psi(\mathrm{d}^\star_{(M_1, \mathrm{Hom}_R^\star(M_2, M_3))}(\psi)) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)} \psi + (-1)^n \psi \mathrm{d}_{M_1}) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)} \psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}).
\end{aligned}$$

Thus we are reduced to showing that

$$\mathrm{d}_{M_3} \Psi(\psi) + (-1)^n \Psi(\psi) \mathrm{d}^\otimes_{(M_1, M_2)} = \Psi(\mathrm{d}^\star_{(M_2, M_3)} \psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}) \tag{34}$$

To see that (34) holds, it suffices to show that it holds when we apply any elementary homogeneous tensor in $M_1 \otimes_S M_2$ to both sides. So let $u_1 \otimes u_2 \in M_{1,i} \otimes_R M_{2,j}$ be such an elementary homogeneous tensor, so u_1 is graded of degree i and u_2 is graded of degree j. In the following calculation, we suppress parentheses as much as possible in order to clean notation. We gave

$$\begin{aligned}
(\mathrm{d}_{M_3} \Psi(\psi) + (-1)^n \Psi(\psi) \mathrm{d}^\otimes_{(M_1, M_2)})(u_1 \otimes u_2) &= \mathrm{d}_{M_3} \Psi(\psi)(u_1 \otimes u_2) + (-1)^n \Psi(\psi) \mathrm{d}^\otimes_{(M_1, M_2)}(u_1 \otimes u_2) \\
&= \mathrm{d}_{M_3} \psi(u_1)(u_2) + (-1)^n \Psi(\psi)(\mathrm{d}_{M_1}(u_1) \otimes u_2 + (-1)^i u_1 \otimes \mathrm{d}_{M_2}(u_2)) \\
&= \mathrm{d}_{M_3} \psi(u_1)(u_2) + (-1)^n \Psi(\psi)(\mathrm{d}_{M_1}(u_1) \otimes u_2) + (-1)^{i+n} \Psi(\psi)(u_1 \otimes \mathrm{d}_{M_2}(u_2)) \\
&= \mathrm{d}_{M_3} \psi(u_1)(u_2) + (-1)^n \psi(\mathrm{d}_{M_1}(u_1))(u_2) + (-1)^{i+n} \psi(u_1)(\mathrm{d}_{M_2}(u_2)) \\
&= (\mathrm{d}_{M_3} \psi(u_1) + (-1)^{i+n} \psi(u_1) \mathrm{d}_{M_2})(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= (\mathrm{d}^\star_{(M_2, M_3)}(\psi(u_1))(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= (\mathrm{d}^\star_{(M_2, M_3)} \psi)(u_1)(u_2) + (-1)^n (\psi \mathrm{d}_{M_1})(u_1)(u_2) \\
&= \Psi(\mathrm{d}^\star_{(M_2, M_3)} \psi)(u_1 \otimes u_2) + (-1)^n \Psi(\psi \mathrm{d}_{M_1})(u_1 \otimes u_2) \\
&= (\Psi(\mathrm{d}^\star_{(M_2, M_3)} \psi) + (-1)^n \Psi(\psi \mathrm{d}_{M_1}))(u_1 \otimes u_2).
\end{aligned}$$

It follows that Ψ commutes with the differentials.

Step 4: We will show that Ψ is a bijection. It will then follows that Ψ gives an isomorphism of S-complexes. We construct its inverse as follows: we define

$$\Phi_{M_1,M_2,M_3} \colon \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3) \to \mathrm{Hom}^\star_S(M_1, \mathrm{Hom}^\star_R(M_2, M_3))$$

to be the map given by

$$(\Phi(\varphi)(u_1))(u_2) = \varphi(u_1 \otimes u_2)$$

for all $\varphi \in \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3)$, $u_1 \in M_1$, and $u_2 \in M_2$. We claim that Ψ and Φ are inverse to each other. Indeed, we have

$$\begin{aligned}\Psi(\Phi(\varphi))(u_1 \otimes u_2) &= (\Phi(\varphi)(u_1))(u_2) \\ &= \varphi(u_1 \otimes u_2)\end{aligned}$$

for all $\varphi \in \mathrm{Hom}^\star_R(M_1 \otimes_S M_2, M_3)$ and $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Thus $\Psi\Phi = 1$. Similarly, we have

$$\begin{aligned}(\Phi(\Psi(\psi))(u_1))(u_2) &= \Psi(\psi)(u_1 \otimes u_2) \\ &= (\psi(u_1))(u_2)\end{aligned}$$

for all $\psi \in \mathrm{Hom}^\star_S(M_1, \mathrm{Hom}^\star_R(M_2, M_3))$ and $u_1 \in M_1$ and $u_2 \in M_2$. Thus $\Phi\Psi = 1$.

Step 5: We show naturality in M_1, M_2, and M_3. Naturality in M_1 means that if $\lambda \colon M_1 \to M_1'$ is an R-module homomorphism, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M_1', \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1',M_3}"] \arrow[d, "\lambda^*"'] & \mathrm{Hom}_R(M_1' \otimes_S M_2, M_3) \arrow[d, "(\lambda\otimes 1)^*"] \\
\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1,M_3}"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3)
\end{tikzcd}$$

Thus we want to show for all $\psi \in \mathrm{Hom}^\star_S(M_1', \mathrm{Hom}^\star_R(M_2, M_3))$, we have

$$(\lambda \otimes 1)^* \left(\Psi_{M_1',M_3}(\psi)\right) = \Psi_{M_1,M_3}(\lambda^*(\psi)) \tag{35}$$

To see that (35) is equal, we apply all elementary tensors to both sides. Let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then we have

$$\begin{aligned}\left((\lambda \otimes 1)^* \left(\Psi_{M_1',M_3}(\psi)\right)\right)(u_1 \otimes u_2) &= (\Psi_{M_1,M_3}(\psi))\,((\lambda \otimes 1)(u_1 \otimes u_2)) \\ &= (\Psi_{M_1,M_3}(\psi))\,(\lambda(u_1) \otimes u_2) \\ &= (\psi(\lambda(u_1))(u_2) \\ &= ((\lambda^*(\psi))(u_1))(u_2) \\ &= (\Psi_{M_1,M_3}(\lambda^*(\psi)))\,(u_1 \otimes u_2) \\ &= (\Psi_{M_1,M_3}(\lambda^*(\psi)))\,(u_1 \otimes u_2).\end{aligned}$$

Similarly, naturality in M_3 means that if $\lambda \colon M_3 \to M_3'$ is an R-module homomorphism, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_1,M_3}"] \arrow[d, "(\lambda_*)_*"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3) \arrow[d, "\lambda_*"] \\
\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3')) \arrow[r, "\Psi_{M_1,M_3'}"'] & \mathrm{Hom}_R(M_1 \otimes_S M_2, M_3')
\end{tikzcd}$$

Thus we want to show for all $\psi \in \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(M_2, M_3))$, we have

$$\lambda_* \left(\Psi_{M_1,M_3}(\psi)\right) = \Psi_{M_1,M_3'}((\lambda_*)_*(\psi)) \tag{36}$$

To see that ([36]) is equal, we apply all elementary tensors to both sides. Let $u_1 \otimes u_2 \in M_1 \otimes_S M_2$. Then we have

$$\begin{aligned}(\lambda_* (\Psi_{M_1,M_3}(\psi)))\,(u_1 \otimes u_2) &= \lambda\left((\Psi_{M_1,M_3}(\psi))\,(u_1 \otimes u_2)\right) \\ &= \lambda\left((\psi(u_1))(u_2))\right) \\ &= (\lambda_*(\psi(u_1)))(u_2) \\ &= ((\lambda_*)_*(\psi))(u_1))(u_2) \\ &= \left(\Psi_{M_1,M_3'}((\lambda_*)_*(\psi)))\right)(u_1 \otimes u_2).\end{aligned}$$

□

There is another version of Tensor-Hom adjointness which we will state now but not prove.

Proposition 3.33. *Let S be an R-algebra, let M_2, M_3 be S-complexes, and let M_1 be an R-complex. Then we have an isomorphism of S-complexes*

$$\operatorname{Hom}^\star_R(M_1, \operatorname{Hom}^\star_S(M_2, M_3)) \cong \operatorname{Hom}^\star_S(M_1 \otimes_R M_2, M_3). \tag{37}$$

Moreover ([30]) is natural in M_1, M_2, and M_3.

3.9.6 Hom Commutes with Shifts

Proposition 3.34. *Let $n \in \mathbb{Z}$ and let A and A' be R-complexes. Then*

$$\operatorname{Hom}^\star_R(\Sigma^n A, A') \cong \Sigma^{-n}\operatorname{Hom}^\star_R(A, A') \quad \textit{and} \quad \operatorname{Hom}^\star_R(A, \Sigma^n A') \cong \Sigma^n \operatorname{Hom}^\star_R(A, A')$$

are isomorphisms of R-complexes.

Remark. Thus the covariant functor $\operatorname{Hom}^\star_R(A, -)$ commutes with shifts and the contravariant functor $\operatorname{Hom}^\star_R(-, A')$ anticommutes with shifts.

Proof. We will first show $\operatorname{Hom}^\star_R(\Sigma^n A, A') \cong \Sigma^{-n}\operatorname{Hom}^\star_R(A, A')$. As graded R-modules, we have

$$\begin{aligned}\operatorname{Hom}^\star_R(\Sigma^n A, A') &= \operatorname{Hom}^\star_R(A(-n), A') \\ &= \operatorname{Hom}^\star_R(A, A')(n) \\ &= \Sigma^{-n}\operatorname{Hom}^\star_R(A, A').\end{aligned}$$

We define $\Phi\colon \operatorname{Hom}^\star_R(\Sigma^n A, A') \to \Sigma^{-n}\operatorname{Hom}^\star_R(A, A')$ by

$$\Phi(\alpha) = (-1)^{x_i}\alpha$$

for all $\alpha \in \operatorname{Hom}^\star_R(\Sigma^n A, A')$ where $x_i \in \mathbb{Z}$ satisfies

$$x_i = n + x_{i-1}$$

for all $i \in \mathbb{Z}$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \operatorname{Hom}^\star_R(\Sigma^n A, A')_i$; so $\alpha\colon A \to A'$ is a graded homomorphism of degree $n + i$. Then we have

$$\begin{aligned}(\Sigma^{-n}\mathrm{d}^\star_{(A,A')}\Phi)(\alpha) &= (-1)^{-n}\mathrm{d}^\star_{(A,A')}(\Phi(\alpha)) \\ &= (-1)^{-n+x_i}\mathrm{d}^\star_{(A,A')}(\alpha) \\ &= (-1)^{-n+x_i}(\mathrm{d}_{A'}\alpha - (-1)^{n+i}\alpha\mathrm{d}_A) \\ &= (-1)^{-n+x_i}\mathrm{d}_{A'}\alpha - (-1)^{x_i+i}\alpha\mathrm{d}_A) \\ &= (-1)^{x_{i-1}}\mathrm{d}_{A'}\alpha - (-1)^{i+x_{i-1}+n}\alpha\mathrm{d}_A \\ &= (-1)^{x_{i-1}}\mathrm{d}_{A'}\alpha - (-1)^{i+x_{i-1}}\alpha\mathrm{d}_{\Sigma^n A} \\ &= \Phi(\mathrm{d}_{A'}\alpha - (-1)^i\alpha\mathrm{d}_{\Sigma^n A}) \\ &= \Phi(\mathrm{d}^\star_{(\Sigma^n A,A')}(\alpha)) \\ &= (\Phi\mathrm{d}^\star_{(\Sigma^n A,A')})(\alpha)\end{aligned}$$

Now we will show $\operatorname{Hom}^\star_R(A, \Sigma^n A') \cong \Sigma^n\operatorname{Hom}^\star_R(A, A')$. As graded R-modules, we have

$$\begin{aligned}\operatorname{Hom}^\star_R(A, \Sigma^n A') &= \operatorname{Hom}^\star_R(A, A'(-n)) \\ &= \operatorname{Hom}^\star_R(A, A')(-n) \\ &= \Sigma^n\operatorname{Hom}^\star_R(A, A').\end{aligned}$$

We define $\Phi\colon \mathrm{Hom}^\star_R(A,\Sigma^n A') \to \Sigma^n \mathrm{Hom}^\star_R(A,A')$ by

$$\Phi(\alpha) = (-1)^{x_i}\alpha$$

for all $\alpha \in \mathrm{Hom}^\star_R(A,\Sigma^n A')$ where $x_i \in \mathbb{Z}$ satisfies

$$x_i = x_{i-1}$$

for all $i \in \mathbb{Z}$. Then Φ is a graded isomorphism of the underlying graded R-module. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \mathrm{Hom}^\star_R(A,\Sigma^n A')_i$; so $\alpha\colon A \to A'$ is a graded homomorphism of degree $i-n$. Then we have

$$\begin{aligned}
(\Sigma^n \mathrm{d}^\star_{(A,A')}\Phi)(\alpha) &= (-1)^n \mathrm{d}^\star_{(A,A')}(\Phi(\alpha)) \\
&= (-1)^{n+x_i}\mathrm{d}^\star_{(A,A')}(\alpha) \\
&= (-1)^{n+x_i}(\mathrm{d}_{A'}\alpha - (-1)^{i-n}\alpha\mathrm{d}_A) \\
&= (-1)^{n+x_i}\mathrm{d}_{A'}\alpha - (-1)^{x_i+i}\alpha\mathrm{d}_A) \\
&= (-1)^{x_{i-1}}\mathrm{d}_{\Sigma^n A'}\alpha - (-1)^{x_{i-1}+i}\alpha\mathrm{d}_A \\
&= \Phi(\mathrm{d}_{\Sigma^n A'}\alpha - (-1)^i\alpha\mathrm{d}_A) \\
&= \Phi(\mathrm{d}^\star_{(A,\Sigma^n A')}(\alpha)) \\
&= (\Phi\mathrm{d}^\star_{(A,\Sigma^n A')})(\alpha)
\end{aligned}$$

□

3.9.7 Hom Commutes with Mapping Cone

Proposition 3.35. *Let X and Y be R-complexes and let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then*

$$\mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi)) \cong \mathrm{C}(\mathrm{Hom}^\star_R(X,\varphi)) \quad \text{and} \quad \Sigma\mathrm{Hom}^\star_R(\mathrm{C}(\varphi),Y) \cong \mathrm{C}(\mathrm{Hom}^\star_R(\varphi,Y))$$

are isomorphisms of R-complexes.

Proof. We first show $\mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi)) \cong \mathrm{C}(\varphi_*)$. As graded R-modules, we have

$$\begin{aligned}
\mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi)) &= \mathrm{Hom}^\star_R(X, A' \oplus A(-1)) \\
&\cong \mathrm{Hom}^\star_R(X,A') \oplus \mathrm{Hom}^\star_R(X,A(-1)) \\
&= \mathrm{Hom}^\star_R(X,A') \oplus \mathrm{Hom}^\star_R(X,A)(-1) \\
&= \mathrm{C}(\varphi_*),
\end{aligned}$$

where the graded isomorphism in the second line is given by

$$\alpha \mapsto (\pi_1\alpha, \pi_2\alpha)$$

for all $\alpha \in \mathrm{Hom}^\star_R(X, A' \oplus A(-1))$, where

$$\pi_1\colon A' \oplus A(-1) \to A' \quad \text{and} \quad \pi_2\colon A' \oplus A(-1) \to A(-1)$$

are the natural projection maps.

We define $\Phi\colon \mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi)) \to \mathrm{C}(\varphi_*)$ by

$$\Phi(\alpha) = (\pi_1\alpha, \pi_2\alpha)$$

for all $\alpha \in \mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi))$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \mathrm{Hom}^\star_R(X,\mathrm{C}(\varphi))_i$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi_*)}\Phi)(\alpha) &= \mathrm{d}_{\mathrm{C}(\varphi_*)}(\Phi(\alpha)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi_*)}(\pi_1\alpha, \pi_2\alpha) \\
&= (\mathrm{d}^\star_{(X,A')}(\pi_1\alpha) + \varphi_*(\pi_2\alpha), -\mathrm{d}^\star_{(X,A)}(\pi_2\alpha)) \\
&= (\mathrm{d}_{A'}\pi_1\alpha - (-1)^i\pi_1\alpha\mathrm{d}_X + \varphi\pi_2\alpha, -\mathrm{d}_A\pi_2\alpha - (-1)^i\pi_2\alpha\mathrm{d}_X) \\
&= (\pi_1\mathrm{d}_{\mathrm{C}(\varphi)}\alpha - (-1)^i\pi_1\alpha\mathrm{d}_X, \pi_2\mathrm{d}_\varphi\alpha - (-1)^i\pi_2\alpha\mathrm{d}_X) \\
&= \Phi(\mathrm{d}_{\mathrm{C}(\varphi)}\alpha - (-1)^i\alpha\mathrm{d}_X) \\
&= \Phi(\mathrm{d}^\star_{(X,\mathrm{C}(\varphi))}(\alpha)) \\
&= (\Phi\mathrm{d}^\star_{(X,\mathrm{C}(\varphi))})(\alpha)
\end{aligned}$$

where we used the fact that $-\mathrm{d}_A\pi_2 = \pi_2\mathrm{d}_\varphi$ and $\pi_1\mathrm{d}_\varphi = \mathrm{d}_{A'}\pi_1 + \varphi\pi_2$.

Now we show $\Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) \cong \mathrm{C}(\varphi^*)$. As graded R-modules, we have

$$\begin{aligned}
\Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) &= \mathrm{Hom}_R^\star(A' \oplus A(-1), Y)(-1) \\
&\cong \mathrm{Hom}_R^\star(A', Y)(-1) \oplus \mathrm{Hom}_R^\star(A(-1), Y))(-1) \\
&= \mathrm{Hom}_R^\star(A', Y)(-1) \oplus \mathrm{Hom}_R^\star(A, Y)) \\
&\cong \mathrm{Hom}_R^\star(A, Y) \oplus \mathrm{Hom}_R^\star(A', Y)(-1) \\
&= \mathrm{C}(\varphi_*),
\end{aligned}$$

where the graded isomorphism in the second line is given by

$$\alpha \mapsto (\alpha\iota_1, \alpha\iota_2)$$

for all $\alpha \in \mathrm{Hom}_R^\star(X, A' \oplus A(-1))$, where

$$\iota_1 \colon A' \to A' \oplus A(-1) \quad \text{and} \quad \iota_2 \colon A(-1) \to A' \oplus A(-1)$$

are the natural inclusion maps.

We define $\Phi \colon \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y) \to \mathrm{C}(\varphi_*)$ by

$$\Phi(\alpha) = (\alpha\iota_2, \alpha\iota_1)$$

for all $\alpha \in \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y)$. Then Φ is a graded isomorphism of the underlying graded R-modules. We claim that it also commutes with the differentials, making it into an isomorphism of R-complexes. Indeed, let $\alpha \in \Sigma\mathrm{Hom}_R^\star(\mathrm{C}(\varphi), Y)_i$. Then we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{C}(\varphi^*)}\Phi)(\alpha) &= \mathrm{d}_{\mathrm{C}(\varphi^*)}(\Phi(\alpha)) \\
&= \mathrm{d}_{\mathrm{C}(\varphi^*)}(\alpha\iota_2, \alpha\iota_1) \\
&= (\mathrm{d}^\star_{(A,Y)}(\alpha\iota_2) + \varphi^*(\alpha\iota_1), -\mathrm{d}^\star_{(A',Y)}(\alpha\iota_1)) \\
&= (\mathrm{d}_Y\alpha\iota_2 + (-1)^i\alpha\iota_2\mathrm{d}_A + \alpha\iota_1\varphi, -\mathrm{d}_Y\alpha\iota_1 + (-1)^i\alpha\iota_1\mathrm{d}_{A'}) \\
&= (-\mathrm{d}_Y\alpha\iota_2 + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}\iota_2, -\mathrm{d}_Y\alpha\iota_1 + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}\iota_1) \\
&= \Phi(-\mathrm{d}_Y\alpha + (-1)^i\alpha\mathrm{d}_{\mathrm{C}(\varphi)}) \\
&= \Phi(-\mathrm{d}^\star_{(\mathrm{C}(\varphi),Y)}(\alpha)) \\
&= (\Phi\Sigma\mathrm{d}^\star_{(\mathrm{C}(\varphi),Y)})(\alpha)
\end{aligned}$$

where we used the fact that $\iota_2\mathrm{d}_A = \iota_1\varphi - \mathrm{d}_{\mathrm{C}(\varphi)}\iota_2$ and $\mathrm{d}_{\mathrm{C}(\varphi)}\iota_1 = \iota_1\mathrm{d}_{A'}$. □

3.9.8 Hom Preserves Homotopy Equivalences

Proposition 3.36. *Let B be an R-complex, let $\varphi \colon A \to A'$ and $\psi \colon A \to A'$ be two chain maps of R-complexes, and suppose $\varphi \sim \psi$. Then $\mathrm{Hom}_R^\star(\varphi, B) \sim \mathrm{Hom}_R^\star(\psi, B)$.*

Proof. Choose a homotopy $h \colon A \to A'$ from φ to ψ (so $\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A$). To ease the notation in the following calculation, we write $\varphi^\star = \mathrm{Hom}_R^\star(\varphi, B)$, $\psi^\star = \mathrm{Hom}_R^\star(\psi, B)$, and $h^\star = \mathrm{Hom}_R^\star(h, B)$. We claim that $h^\star \colon \mathrm{Hom}_R^\star(A', B) \to \mathrm{Hom}_R^\star(A, B)$ is a homotopy from $\varphi^\star$ to $\psi^\star$. Indeed, let $\alpha \colon A' \to B$ be a graded R-linear map of degree i. Then observe that

$$\begin{aligned}
(\mathrm{d}^\star_{(A,B)}h^\star + h^\star\mathrm{d}^\star_{(A',B)})(\alpha) &= (-1)^i\mathrm{d}^\star_{(A,B)}(\alpha h) + h^\star(\mathrm{d}_B\alpha - (-1)^i\alpha\mathrm{d}_{A'}) \\
&= (-1)^i\mathrm{d}_B\alpha h + (-1)^i(-1)^i\alpha h\mathrm{d}_A - (-1)^i\mathrm{d}_B\alpha h - (-1)^i(-1)^{i+1}\alpha\mathrm{d}_{A'}h \\
&= \alpha h\mathrm{d}_A + \alpha\mathrm{d}_{A'}h \\
&= \alpha(h\mathrm{d}_A + \mathrm{d}_{A'}h) \\
&= \alpha(\varphi - \psi) \\
&= (\varphi^\star - \psi^\star)(\alpha)
\end{aligned}$$

Thus $h^\star$ is indeed a homotopy from $\varphi^\star$ to $\psi^\star$. □

Corollary. *Suppose $\varphi \colon A \to A'$ is a homotopy of equivalence of R-complexes. Then $\mathrm{Hom}_R^\star(\varphi, B) \colon \mathrm{Hom}_R^\star(A', B) \to \mathrm{Hom}_R^\star(A, B)$ is a homotopy equivalence of R-complexes.*

Proof. Let $\varphi' \colon A' \to A$ be the homotopy inverse to φ. Thus $\varphi\varphi' \sim 1_{A'}$ and $\varphi'\varphi \sim 1_A$. It follows that

$$\begin{aligned} 1_{\mathrm{Hom}_R^\star(A',B)} &= \mathrm{Hom}_R^\star(1_{A'}, B) \\ &\sim \mathrm{Hom}_R^\star(\varphi\varphi', B) \\ &= \mathrm{Hom}_R^\star(\varphi', B)\mathrm{Hom}_R^\star(\varphi, B). \end{aligned}$$

Similarly, we have $1_{\mathrm{Hom}_R^\star(A,B)} \sim \mathrm{Hom}_R^\star(\varphi, B)\mathrm{Hom}_R^\star(\varphi', B)$. Therefore $\mathrm{Hom}_R^\star(\varphi, B)$ is a homotopy equivalence of R-complexes. □

3.9.9 Twisting the hom complex with a chain map

Definition 3.20. Let (A, d) be an R-complex and let $\alpha \colon A \to A$ be a chain map. We define an R-complex $\mathrm{Hom}_R^{\star_\alpha}(A, A)$ as follows: as a graded R-module, $\mathrm{Hom}_R^{\star_\alpha}(A, A)$ is just $\mathrm{Hom}_R^\star(A, A)$. We define the differential $\mathrm{d}_\alpha^\star \colon \mathrm{Hom}_R^{\star_\alpha}(A, A) \to \mathrm{Hom}_R^{\star_\alpha}(A, A)$ on graded R-linear map $\varphi \colon A \to A$ of degree i by

$$\mathrm{d}_\alpha^\star(\varphi) = \mathrm{d}\varphi + (-1)^i \alpha\varphi\mathrm{d} \tag{38}$$

and then we extend $\mathrm{d}_\alpha^\star$ linearly everywhere else. Note that $\mathrm{d}_\alpha^\star$ is graded of degree -1 since α is a chain map. Let us show that we have $\mathrm{d}_\alpha^\star\mathrm{d}_\alpha^\star = 0$. Let $\varphi \colon A \to A$ be a graded R-linear map of degree i . Then we have

$$\begin{aligned} \mathrm{d}_\alpha^\star\mathrm{d}_\alpha^\star(\varphi) &= \mathrm{d}_\alpha^\star(\mathrm{d}\varphi + (-1)^i\alpha\varphi\mathrm{d}) \\ &= \mathrm{d}\mathrm{d}\varphi + (-1)^{i-1}\alpha\mathrm{d}\varphi\mathrm{d} + (-1)^i\mathrm{d}\alpha\varphi\mathrm{d} + (-1)^{i-1}\alpha\alpha\varphi\mathrm{d}\mathrm{d} \\ &= (-1)^{i-1}\alpha\mathrm{d}\varphi\mathrm{d} + (-1)^i\alpha\mathrm{d}\varphi\mathrm{d} \\ &= 0. \end{aligned}$$

It follows that $\mathrm{d}_\alpha^\star$ is a differential.

4 Ext and Tor

4.1 Projective Resolutions

Definition 4.1. Let M be an R-module. An **augmented projective resolution of** M **over** R is an R-complex (P, d) such that

1. P is a projective R-module. Equivalently, P_i is a projective R-module for all $i \in \mathbb{Z}$;
2. $P_i = 0$ for all $i < 0$;
3. $\mathrm{H}_0(P) \cong M$ and $\mathrm{H}_i(P) = 0$ for all $i > 0$.

Theorem 4.1. *Let (P, d) and (P', d') be two projective resolutions of M over R. Then (P, d) and (P', d') are homotopically equivalent.*

Proof. For each $i \geq 0$, let $M_i' := \operatorname{im} \mathrm{d}_i'$ and let $M_i := \operatorname{im} \mathrm{d}_i$. We build a chain map $\varphi \colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ by constructing R-module homomorphism $\varphi_i \colon P_i \to P_i'$ which commute with the differentials using induction on $i \geq 0$. First consider the base case $i = 0$. Since $P_0/M_1 \cong P_0'/M_1'$, there exists a homomorphism $\psi_0 \colon P_0 \to P_0'/M_0'$. Then since P_0 is projective and since $\mathrm{d}_0' \colon P_0' \to P_0'/M_1$ is a surjective homomorphism, we can lift $\psi_0 : P_0 \to P_0'/M_0'$ along $\mathrm{d}_0' : P_0' \to P_0'/M_1$ to a homomorphism $\varphi_0 \colon P_0 \to P_0'$ such that $\mathrm{d}_0'\varphi_0 = \psi_0$.

Now suppose for some $i > 0$ we have constructed R-module homomorphisms $\varphi_0, \varphi_1, \ldots, \varphi_i$ which commute with the differentials. We need to construct an R-module homomorphism $\varphi_{i+1} \colon P_{i+1} \to P_{i+1}'$ which commutes with the differentials. First, we claim that $\operatorname{im}(\varphi_i\mathrm{d}_{i+1}) \subseteq M_{i+1}'$. To see this, note that

$$\begin{aligned} \mathrm{d}_i'\varphi_i\mathrm{d}_{i+1} &= \varphi_{i-1}\mathrm{d}_i\mathrm{d}_{i+1} \\ &= 0. \end{aligned}$$

Thus, since $i > 0$, we have

$$\begin{aligned} \operatorname{im}(\varphi_i\mathrm{d}_{i+1}) &\subseteq \ker \mathrm{d}_i \\ &= \operatorname{im} \mathrm{d}_{i+1}' \\ &= M_{i+1}'. \end{aligned}$$

Now since P_{i+1} is projective and $\mathrm{d}_{i+1}' \colon P_{i+1} \to M_{i+1}$ is surjective, we can lift $\varphi_i\mathrm{d}_{i+1} : P_{i+1} \to M_{i+1}'$ along $\mathrm{d}_{i+1}' : P_{i+1}' \to M_{i+1}'$ to a homomorphism $\varphi_{i+1} \colon P_{i+1} \to P_{i+1}'$ such that $\mathrm{d}_{i+1}'\varphi_{i+1} = \varphi_i\mathrm{d}_{i+1}$.

By a similar construction as above, we get a chain map $\varphi' : (P', \mathrm{d}') \to (P, \mathrm{d})$. Now we claim that $\varphi'\varphi$ is homotopic to id_P and similarly $\varphi\varphi'$ is homotopic to $\mathrm{id}_{P'}$. It suffices to show that $\varphi'\varphi \sim \mathrm{id}_P$ (a similar argument will give $\varphi\varphi' \sim \mathrm{id}_{P'}$). The idea is to build the homotopy $h : (P, \mathrm{d}) \to (P, \mathrm{d})$ using induction on $i \geq 0$. The homotopy equation that we need is

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d}, \tag{39}$$

where we write 1 instead of id_P is clean notation. Since P_0 is projective and $\mathrm{d}_1 : P_1 \to P_0$ is a surjective morphism, there exists a homomorphism $h_0 : P_0 \to P_1$ such that

$$\varphi_0'\varphi_0 - 1 = \mathrm{d}_1 h_0. \tag{40}$$

In homological degree $i = 0$, the equation (39) becomes (40). Thus, we are on the right track.

Now we use induction. Suppose for $i > 0$ we have constructed an R-module homomorphism $h_i : P_i \to P_{i+1}$ such that

$$\varphi_i'\varphi_i - 1 = \mathrm{d}_{i+1} h_i + h_{i-1}\mathrm{d}_i. \tag{41}$$

Observe that $\mathrm{Im}(\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i) \subseteq M_{i+1}$. Indeed, note that

$$\begin{aligned}
\mathrm{d}_i(\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i) &= \mathrm{d}_i\varphi_i'\varphi_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \varphi_{i-1}'\mathrm{d}_i'\varphi_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \varphi_{i-1}'\varphi_{i-1}\mathrm{d}_i - \mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= (\varphi_{i-1}'\varphi_{i-1} - 1)\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= (\mathrm{d}_i h_{i-1} + h_{i-2}\mathrm{d}_{i-1})\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \mathrm{d}_i h_{i-1}\mathrm{d}_i + h_{i-2}\mathrm{d}_{i-1}\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= \mathrm{d}_i h_{i-1}\mathrm{d}_i - \mathrm{d}_i h_{i-1}\mathrm{d}_i \\
&= 0.
\end{aligned}$$

Therefore since P_{i+1} is projective and since $\mathrm{d}_{i+2} : P_{i+2} \to M_{i+2}$ is a surjective homomorphism, there exists $h_{i+1} : P_{i+1} \to P_{i+2}$ such that

$$\varphi_i'\varphi_i - 1 - h_{i-1}\mathrm{d}_i = \mathrm{d}_{i+2}h_{i+1},$$

which is the homotopy equation in degree $i + 1$. □

4.2 Projective Dimension

Definition 4.2. Let M be an R-module. The **projective dimenion of** M **over** R, denoted $\mathrm{pd}_R(M)$, is defined to be

$$\mathrm{pd}_R(M) = \inf \{\sup P \mid P \text{ is a projective resolution of } M \text{ over } R\}.$$

The **global dimension** of R, denoted $\mathrm{gldim}\, R$, is defined to be

$$\mathrm{gldim}\, R = \sup \{\mathrm{pd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

In fact, it is a theorem from Auslander that it is enough to take the supremum for finitely generated R-modules. That is,

$$\mathrm{gldim}\, R = \sup \{\mathrm{pd}_R(M) \mid M \text{ is a finitely generated } R\text{-module}\}.$$

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated nonzero R-module. Then*

$$\mathrm{pd}_R(M) = \inf_{i \in \mathbb{Z}} \left\{ \mathrm{Tor}_{i+1}^R(R/\mathfrak{m}, M) = 0 \right\}.$$

Thus the global dimension of R is equal to $\mathrm{pd}_R(R/\mathfrak{m})$.

Proof. Denote $n = \mathrm{pd}_R(M)$ and $m = \inf_{i \in \mathbb{N}} \left\{ \mathrm{Tor}_{i+1}^R(R/\mathfrak{m}, M) = 0 \right\}$. Choose a minimal projective resolution of M over R, say (P, d). Then

$$\mathrm{Tor}_{i+1}^R(R/\mathfrak{m}, M) \cong \mathrm{H}_{i+1}(R/\mathfrak{m} \otimes_R P) \cong 0$$

for all $i \geq n$. In particular, this implies $m \leq n$. On the other hand, since P is minimal, the differential on $R/\mathfrak{m} \otimes_R P$ is the zero map: $\overline{1} \otimes \mathrm{d} = 0$. In particular, this implies

$$\mathrm{Tor}_i^R(R/\mathfrak{m}, M) \cong P_i \not\cong 0.$$

for all $0 \leq i \leq n$. Thus $m \geq n$. The last part of the proposition follows from symmetry of Tor. □

Proposition 4.2. *Suppose $(R, \mathfrak{m})$ is a regular local ring of dimension n. Then the global dimension of R is n.*

Proof. Let $x_1, \dots, x_n$ generate the maximal ideal $\mathfrak{m}$ of R. Then the Koszul complex $\mathcal{K}(x_1, \dots, x_n)$ is a minimal free resolution of $R/\mathfrak{m}$ over R. It follows that $n = \mathrm{pd}_R(R/\mathfrak{m})$ is equal to the global dimension of R. □

4.2.1 Minimal Projective Resolutions over a Noetherian Local Ring

Definition 4.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R-module, and let (P, d) be a projective resolution of M over R. We say P is **minimal** if $\mathrm{d}(P) \subset \mathfrak{m}P$.

Proposition 4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated R-module, and let (P, d) and (P', d') be two minimal projective resolutions of M over R. Then for each $i \in \mathbb{Z}$, the ranks of P_i and P'_i are finite and equal to each other. We denote this common rank by $\beta_i(M)$, and we call it the **ith Betti number of** M.*

Proof. Choose chain map $\alpha\colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ and $\alpha'\colon (P', \mathrm{d}') \to (P, \mathrm{d})$ together with a homotopy $h\colon (P, \mathrm{d}) \to (P', \mathrm{d}')$ such that

$$\alpha'\alpha - 1 = \mathrm{d}'h + h\mathrm{d}. \tag{42}$$

Since $\mathrm{d}(P) \subset \mathfrak{m}P$ and $\mathrm{d}'(P') \subset \mathfrak{m}P'$, the homotopy equation (42) reduces to

$$\alpha'\alpha - 1 \equiv 0 \bmod \mathfrak{m}P'.$$

In other words, $\alpha\colon P \to P'$ induces an isomorphism $\overline{\alpha}\colon P/\mathfrak{m}P \to P'/\mathfrak{m}P'$ of graded $(R/\mathfrak{m})$-vector spaces. In particular, for each $i \in \mathbb{Z}$, we have isomorphisms

$$\overline{\alpha}_i\colon P_i/\mathfrak{m}P_i \to P'_i/\mathfrak{m}P'_i$$

of $(R/\mathfrak{m})$-vector spaces. Therefore by Nakayama's Lemma, for all $i \in \mathbb{Z}$, we have

$$\begin{aligned} \mathrm{rank}(P_i) &= \dim_{R/\mathfrak{m}}(P_i/\mathfrak{m}P_i) \\ &= \dim_{R/\mathfrak{m}}(P'_i/\mathfrak{m}P'_i) \\ &= \mathrm{rank}(P'_i). \end{aligned}$$

□

4.3 Definition of Tor

Definition 4.4. Let M and N be R-modules. We define the **Tor** with respect to M and N as follows: Choose a projective resolution of M, say (P, d), then set

$$\mathrm{Tor}^R(M, N) := \mathrm{H}(P \otimes_R N).$$

We need to check that this definition does not depend on the choice of a projective resolution of M, so suppose (P', d') is another projective resolution of M. By Theorem (4.1), there exists a homotopy equivalence from (P, d) to (P', d'), say $\varphi\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ and $\varphi'\colon (P', \mathrm{d}') \to (P, \mathrm{d})$ with homotopies $h\colon (P, \mathrm{d}) \to (P, \mathrm{d})$ and $h'\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ such that

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

We claim that $P \otimes_R N$ is homotopically equivalent to $P' \otimes_R N$ via the pair of maps $\varphi \otimes 1\colon P \otimes_R N \to P' \otimes_R N$ and $\varphi' \otimes 1\colon P' \otimes_R N \to P \otimes_R N$ with homotopies given by $h \otimes 1\colon P \otimes_R N \to P' \otimes_R N$ and $h' \otimes_R 1\colon P' \otimes_R N \to P \otimes_R N$ respectively. Indeed, we have

$$\begin{aligned} (\varphi' \otimes 1)(\varphi \otimes 1) - 1 \otimes 1 &= \varphi'\varphi \otimes 1 - 1 \otimes 1 \\ &= (\varphi'\varphi - 1) \otimes 1 \\ &= (\mathrm{d}h + h\mathrm{d}) \otimes 1 \\ &= \mathrm{d}h \otimes 1 + h\mathrm{d} \otimes 1 \\ &= \mathrm{d}^{P \otimes_R N}(h \otimes 1) + (h \otimes 1)\mathrm{d}^{P \otimes_R N}. \end{aligned}$$

A similar calculation shows

$$(\varphi \otimes 1)(\varphi' \otimes 1) = \mathrm{d}^{P' \otimes_R N}(h' \otimes 1) + (h' \otimes 1)\mathrm{d}^{P' \otimes_R N}.$$

Thus $P \otimes_R N$ is homotopically equivalent to $P' \otimes_R N$ and hence

$$\mathrm{H}(P \otimes_R N) = \mathrm{H}(P' \otimes_R N).$$

Therefore the definition of Tor is well-defined.

4.4 Examples of Tor

Example 4.1. Let I and J be ideals in R. We compute $\mathrm{Tor}_1^R(R/I, R/J)$. First we tensor the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

with R/J to get the exact sequence

$$\begin{tikzcd} & 0 \cong \mathrm{Tor}_1^R(R, R/J) \arrow[r] & \mathrm{Tor}_1^R(R/I, R/J) \arrow[dll, rounded corners] & \\ I/IJ \arrow[r] & R/J \arrow[r] & R/(I+J) \arrow[r] & 0 \end{tikzcd}$$

where $\mathrm{Tor}_1^R(R, R/J) \cong 0$ for trivial reasons. From here, it follows that $\mathrm{Tor}_1^R(R/I, R/J)$ is isomorphic to the kernel of the map $I/IJ \to R/J$, which is just $I \cap J/IJ$.

Example 4.2. Let $R = K[x,y,z]$, $I = \langle xy^2z^3, x^2yz^3, x^3yz^2, x^3y^2z, x^2y^3z, xy^3z^2 \rangle$, and $J = \langle x, y \rangle$. We compute $\mathrm{Tor}_i^R(R/I, R/J)$ for all i. An augmented free resolution for R/I comes from the permutohedron of order 3. It is given by

$$0 \longrightarrow R \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^6 \xrightarrow{\varphi_1} R \longrightarrow R/I$$

where

$$\varphi_3 = \begin{pmatrix} xy \\ y^2 \\ yz \\ z^2 \\ xz \\ x^2 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} -x & 0 & 0 & 0 & 0 & y \\ y & -x & 0 & 0 & 0 & 0 \\ 0 & z & -y & 0 & 0 & 0 \\ 0 & 0 & z & -y & 0 & 0 \\ 0 & 0 & 0 & x & -z & 0 \\ 0 & 0 & 0 & 0 & x & -z \end{pmatrix}, \qquad \varphi_1 = \begin{pmatrix} xy^2z^3 & x^2yz^3 & x^3yz^2 & x^3y^2z & x^2y^3z & xy^3z^2 \end{pmatrix}.$$

We now truncate this resolution by replacing the R/I term with 0 and then tensor the truncated resolution with R/J to get:

$$0 \longrightarrow R/J \xrightarrow{\widetilde{\varphi}_3} (R/J)^6 \xrightarrow{\widetilde{\varphi}_2} (R/J)^6 \xrightarrow{\widetilde{\varphi}_1} R/J \longrightarrow 0$$

where $\overline{\varphi}_i$ is given by

$$\overline{\varphi}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{z}^2 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{\varphi}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\overline{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\overline{z} \end{pmatrix}, \qquad \overline{\varphi}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this, we see that

$$\begin{aligned} &\mathrm{Tor}_0^R(R/I, R/J) \cong R/\langle x,y \rangle \\ &\mathrm{Tor}_1^R(R/I, R/J) \cong (R/\langle x,y \rangle)^2 \oplus (R/\langle x,y,z \rangle)^4 \\ &\mathrm{Tor}_2^R(R/I, R/J) \cong (R/\langle x,y \rangle) \oplus \left(R/\langle x,y,z^2 \rangle\right), \end{aligned}$$

and $\mathrm{Tor}_i^R(R/I, R/J) \cong 0$ for all $i \geq 3$.

4.5 Definition of Ext

Definition 4.5. Let M and N be R-modules. We define the **Ext** with respect to M and N as follows: Choose a projective resolution of M, say (P, d), then set

$$\mathrm{Ext}_R(M, N) := \mathrm{H}(\mathrm{Hom}_R^\star(P, N)).$$

We need to check that this definition does not depend on the choice of a projective resolution of M, so suppose (P', d') is another projective resolution of M. By Theorem (4.1), there exists a homotopy equivalence from (P, d) to (P', d'), say $\varphi\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ and $\varphi'\colon (P', \mathrm{d}') \to (P, \mathrm{d})$ with homotopies $h\colon (P, \mathrm{d}) \to (P, \mathrm{d})$ and $h'\colon (P, \mathrm{d}) \to (P, \mathrm{d}')$ such that

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

We claim that $\mathrm{Hom}_R^\star(P, N)$ is homotopically equivalent to $\mathrm{Hom}_R^\star(P', N)$ via the pair of maps $\varphi^\star\colon \mathrm{Hom}_R^\star(P, N) \to \mathrm{Hom}_R^\star(P', N)$ and $\varphi'^\star\colon P' \otimes_R N \to P \otimes_R N$ with homotopies given by $h^\star\colon \mathrm{Hom}_R^\star(P, N) \to \mathrm{Hom}_R^\star(P, N)$ and $h'^\star\colon \mathrm{Hom}_R^\star(P', N) \to \mathrm{Hom}_R^\star(P', N)$ respectively. Indeed, if $\psi \in \mathrm{Hom}_R(P_i, N)$, then we have

$$\begin{aligned}(\varphi'^\star\varphi^\star - 1^\star)(\psi) &= \psi(\varphi'\varphi - 1) \\ &= \psi(\mathrm{d}h + h\mathrm{d}) \\ &= (\mathrm{d}^\star h^\star + h^\star\mathrm{d}^\star)(\psi).\end{aligned}$$

It follows that $\varphi'^\star\varphi^\star - 1^\star = \mathrm{d}^\star h^\star + h^\star\mathrm{d}^\star$. A similar calculation shows $\varphi^\star\varphi'^\star - 1^\star = \mathrm{d}^\star h'^\star + h'^\star\mathrm{d}^\star$. Thus $\mathrm{Hom}_R^\star(P, N)$ is homotopically equivalent to $\mathrm{Hom}_R^\star(P', N)$ and hence

$$\mathrm{H}(\mathrm{Hom}_R^\star(P, N)) = \mathrm{H}(\mathrm{Hom}_R^\star(P', N)).$$

Therefore the definition of Ext is well-defined.

4.6 Balance of Ext

We are striving for balance of Ext: the sketch of that proof goes like this: We have

$$\mathrm{Hom}_R(P, N) \xrightarrow[\varepsilon_*]{\simeq} \mathrm{Hom}_R(P, E) \xleftarrow[\tau^*]{\simeq} \mathrm{Hom}_R(M, E).$$

The quasiisomorphisms are: augment $P \xrightarrow[\simeq]{\tau} M$ and $N \xrightarrow[\simeq]{\varepsilon} E$. Then $\mathrm{Hom}_R(P, \mathrm{C}(\varepsilon)) \cong \mathrm{C}(\varepsilon_*)$ where $\mathrm{C}(\varepsilon)$ is exact because ε is quasiisomorphism and $\mathrm{Hom}_R(P, \mathrm{C}(\varepsilon))$ is exact because P is bounded below complex of projectives. Therefore $\mathrm{C}(\varepsilon_*)$ is exact, which implies ε_* is a quasiisomorphism.

Lemma 4.2. *Let I be a bounded above complex of injective R-modules. Then $\mathrm{Hom}_R(-, I)$ respects exact complexes. That is, if U is exact, then the complex $\mathrm{Hom}_R(U, I)$ is exact.*

Proposition 4.4. *Let P be a bounded below complex of projective R-modules and let I be a bounded above complex of injective R-modules. Then $\mathrm{Hom}_R(P, -)$ and $\mathrm{Hom}_R(-, I)$ respect quasiisomorphisms. That is, given a quasiisomorphism $\phi\colon U \to V$, the chain maps $\phi_*\colon \mathrm{Hom}_R(P, U) \to \mathrm{Hom}_R(P, V)$ and $\phi^*\colon \mathrm{Hom}_R(V, I) \to \mathrm{Hom}_R(U, I)$ are quasiisomorphisms.*

Proof. We have

$$\begin{aligned} V \xrightarrow[\simeq]{\phi} U &\implies \mathrm{C}(\phi) \text{ is exact} \\ &\implies \mathrm{Hom}_R(\mathrm{C}(\phi), I) \text{ is exact} \\ &\implies \mathrm{C}(\mathrm{Hom}_R(\phi, I)) \text{ is exact} \\ &\implies \mathrm{Hom}(\phi, I) = \phi_* \text{ is quasiisomorphism}\end{aligned}$$

□

Theorem 4.3. *(Balance for Ext) Let P be a projective resolution of an R-module M and let I be an injective resolution of an R-module N. Then*

$$\mathrm{Ext}_R^i(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P, N)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(P, I)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M, I)).$$

Proof. Resolution gives us quasiisomorphisms $P \xrightarrow[\simeq]{\tau} M$ and $N \xrightarrow[\simeq]{\varepsilon} I$. Thus

$$\mathrm{Hom}_R(P, N) \xrightarrow[\simeq]{\varepsilon_*} \mathrm{Hom}_R(P, I) \xleftarrow[\simeq]{\tau^*} \mathrm{Hom}_R(M, I).$$

□

4.7 Shift Property of Tor and Ext

Proposition 4.5. *Let A be a ring. Let M and N finitely generated A-modules, and for $i \geq 0$, let M_i and N_i denote there respective nonnegative syzygies. For $j \geq 1$, we have*

$$\begin{aligned} \mathrm{Ext}_A^{j+1}(M_i, N) &\cong \mathrm{Ext}_A^{j}(M_{i+1}, N) \\ \mathrm{Tor}^A_{j+1}(M_i, N) &\cong \mathrm{Tor}^A_{j}(M_{i+1}, N) \\ \mathrm{Tor}^A_{j+1}(M, N_i) &\cong \mathrm{Tor}^A_{j}(M, N_{i+1}) \end{aligned}$$

Moreover, assume A is Gorenstein, M and N are maximal Cohen-Macaulay, and for $i \leq -1$, let M_i and N_i denote their respective nonnegative syzygies. Then for $j \geq 1$, we have

$$\begin{aligned} \mathrm{Ext}_A^{j+1}(M_i, N) &\cong \mathrm{Ext}_A^{j}(M_{i+1}, N) \\ \mathrm{Ext}_A^{j}(M, N_i) &\cong \mathrm{Ext}_A^{j+1}(M, N_{i+1}) \\ \mathrm{Tor}^A_{j+1}(M_i, N) &\cong \mathrm{Tor}^A_{j}(M_{i+1}, N) \\ \mathrm{Tor}^A_{j+1}(M, N_i) &\cong \mathrm{Tor}^A_{j}(M, N_{i+1}) \end{aligned}$$

5 Differential Graded Algebras

5.1 DG Algebras

Let (A, d) be an R-complex. A **graded-multiplication** on A is a graded R-linear map $\mathrm{m}\colon A \otimes_R A \to A$ of the underlying graded R-modules. The universal mapping property on graded tensor products tells us that there exists a unique graded R-bilinear map $B_{\mathrm{m}}\colon A \times A \to A$ such that

$$B_{\mathrm{m}}(a, b) = \mathrm{m}(a \otimes b)$$

for all $(a, b) \in A \times A$. However since B_{m} is *uniquely* determined by m, we often identify B_{m} with m and simply think of m as a graded R-bilinear map. In fact, we often drop m altogether and simply denote this multiplication map by

$$\sum a_i \otimes b_i \mapsto \sum a_i b_i$$

for all $\sum a_i \otimes b_i \in A \otimes_R A$. At the end of the day, context will make everything clear.

Suppose m is a graded multiplication As the name of the definition suggests, a graded-multiplication on A must respect the grading. In particular, this means that if $a \in A_i$ and $b \in A_j$, then $ab \in A_{i+j}$. We can also impose other conditions on a graded-multiplication on A.

Definition 5.1. Let (A, d) be an R-complex and let m be a graded-multiplication on A.

1. We say m is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.

2. We say m is **graded-commutative** if
$$ab = (-1)^i ba$$
for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$.

3. We say m is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all $a \in A_i$ for all i odd.

4. We say m is **unital** if there exists an $e \in A$ such that
$$ae = e = ea$$
for all $a \in A$.

5. We say a graded-multiplication satisfies **Leibniz law** if
$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)$$
for all $a \in A_i$ and $b \in A_j$ for all $i, j \in \mathbb{Z}$. This is equivalent to m being a chain map!

6. We say $(A, \mathrm{m}, \mathrm{d})$ is a **differential graded R-algebra** (or **DG R-algebra**) if m is a graded-multiplication on A which satisfies conditions 1-5.

Remark. If the differential d and the multiplication map m are understood from context, then we will denote a differential graded R-algebra simply as "A" rather than as a triple "$(A, \mathrm{m}, \mathrm{d})$". We will also often introduce a differential grade R-algebra as "A" without specifying how the differential and multiplication map are to be denoted. In this case, the differential is denoted "d_A" and the multiplication map is denoted "m_A".

Definition 5.2. Let (A, d) and (A', d') be two DG R-algebras. A chain map $\varphi\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ is said to be a **DG-algebra morphism** if it respects multiplication and identity. In other words, we need
$$\varphi(ab) = \varphi(a)\varphi(b)$$
for all $a, b \in A$, and we need
$$\varphi(1) = 1.$$

We obtain a category of DG R-algebras.

5.1.1 Tensor Product of DG Algebras is DG Algebra

Proposition 5.1. *Let A and B be two DG R-algebras. Then $A \otimes_R B$ is is a DG R-algebra.*

Proof. Let $\mathrm{m}_A\colon A \otimes_R A \to A$ be the multiplication map for A and let $\mathrm{m}_B\colon B \otimes_R B \to B$ the multiplication map for B. Then
$$\begin{aligned}(A \otimes_R B) \otimes_R (A \otimes_R B) &\cong A \otimes_R (B \otimes_R (A \otimes_R B)) \\ &\cong A \otimes_R ((B \otimes_R A) \otimes_R B) \\ &\cong A \otimes_R ((A \otimes_R B) \otimes_R B) \\ &\cong \end{aligned}$$
$$A \otimes_R B)$$
$\square$

Proposition 5.2. *Let (A, d) and (A', d') be two DG R-algebras. Then $(A \otimes_R A', \mathrm{d}^{A \otimes_R A'})$ is a DG R-algebra.*

Proof. Throughout this proof, denote $\mathrm{d}^{\otimes} := \mathrm{d}^{A \otimes_R A'}$. We define multiplication on $A \otimes_R A'$ by the formula
$$(a \otimes a')(b \otimes b') = (-1)^{i'j} ab \otimes a'b'. \tag{43}$$

for all $a \otimes a' \in A_i \otimes_R A_{i'}$ and $b \otimes b' \in A_j \otimes_R A_{j'}$. It is easy to check that (43) is associative and unital with with unit being $e_A \otimes e_{A'}$ where e_A is the unit of A and $e_{A'}$ is the unit of A'. Let us check that Leibniz law is satisfied. Let $a \otimes a', b \otimes b' \in A \otimes_R A'$. Then we have

$$\begin{aligned}
\mathrm{d}^{\otimes}((a \otimes a')(b \otimes b')) &= (-1)^{i'j}\mathrm{d}^{\otimes}(ab \otimes a'b') \\
&= (-1)^{i'j}(\mathrm{d}(ab) \otimes a'b' + (-1)^{i+j}ab \otimes \mathrm{d}'(a'b')) \\
&= (-1)^{i'j}((\mathrm{d}(a)b + (-1)^{i}a\mathrm{d}(b)) \otimes a'b' + (-1)^{i+j}ab \otimes (\mathrm{d}'(a')b' + (-1)^{i'}a'\mathrm{d}'(b'))) \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i'j+i}a\mathrm{d}(b) \otimes a'b' + (-1)^{i'j+i+j}ab \otimes \mathrm{d}'(a')b' + (-1)^{i'j+i+j+i'}ab \otimes a'\mathrm{d}'(b') \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i+j(i'+1)}ab \otimes \mathrm{d}'(a')b' + (-1)^{i+i'+i'(j+1)}a\mathrm{d}(b) \otimes a'b' + (-1)^{i+i'+j+i'j}(ab \otimes a'\mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a')(b \otimes b') + (-1)^{i}(a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b') + (-1)^{i+i'+j}(a \otimes a')(b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a' + (-1)^{i}a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b' + (-1)^{j}b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}^{\otimes}(a \otimes a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}^{\otimes}(b \otimes b')).
\end{aligned}$$

Thus $\mathrm{d}^{\otimes}$ satifies Leibniz law with respect to (43).

□

Proposition 5.3. *Let F be an R-complex of free modules and let B be a DG R-algebras. Then $\mathrm{Hom}^{\star}_R(F, B)$ is a DG R-algebra.*

Proof. Let $\{e_\lambda\}$ be a homogeneous basis for F indexed over a set Λ. We define a graded-multiplication on $\mathrm{Hom}^{\star}_R(F, B)$ as follows: let $\varphi \in \mathrm{Hom}^{\star}_R(F, B)_i$ and $\psi \in \mathrm{Hom}^{\star}_R(F, B)_j$, then we define $\varphi \smile \psi \in \mathrm{Hom}^{\star}_R(F, B)_{i+j}$ to be the unique graded R-linear map defined on basis elements $\{e_\lambda\}$ by

$$(\varphi \smile \psi)(e_\lambda) = \varphi(s_-^{n-i}e_\lambda)\psi(s_+^{n-j}e_\lambda)$$

for all $\lambda \in \Lambda$. Note that we are defining $\varphi \smile \psi$ on $\{e_\lambda\}$ and *then* extending R-linearly. Thus $(\varphi \smile \psi)(re_\lambda) = r\varphi(e_\lambda)\psi(e_\lambda)$ (not $r^2\varphi(e_\lambda)\psi(e_\lambda)$)! Similarly, $(\varphi \smile \psi)(e_\lambda + e_\mu) = \varphi(e_\lambda)\psi(e_\lambda) + \varphi(e_\mu)\psi(e_\mu)$ (not $\varphi(e_\lambda)\psi(e_\lambda) + \varphi(e_\mu)\psi(e_\mu) + \varphi(e_\lambda)\psi(e_\mu) + \varphi(e_\mu)\psi(e_\lambda)$)!
for all $a \in A$. Observe that

$$\mathrm{d}(\varphi \cdot \psi) = \mathrm{d}\varphi \cdot \psi + (-1)^i \varphi \cdot \mathrm{d}\psi$$

Indeed, we have

$$\begin{aligned}
\mathrm{d}(\varphi \cdot \psi)(a) &= \mathrm{d}(\varphi(a)\psi(a)) \\
&= (\mathrm{d}\varphi(a))\psi(a) + (-1)^{i+n}\varphi(a)(\mathrm{d}\psi(a))
\end{aligned}$$

Now we want to show $\cdot$ induces an R-bilinear map in homology. First let us show that $\mathrm{H}(\varphi \cdot \psi)$ is a graded R-linear map. Let □

5.1.2 Hom of DG Algebras is a Noncommutative DG Algebra

Proposition 5.4. *Let (A, d) be a DG R-algebras. Then $\mathrm{Hom}^{\star}_R(A, A')$ is a noncommutative DG R-algebra.*

Proof. We define multiplication on $\mathrm{Hom}^{\star}_R(A, A)$ via composition of functions. Thus if $\varphi\colon A \to A$ and $\psi\colon A \to A$ are graded homomorphisms of degrees i and j respectively. Then $\varphi\psi\colon A \to A'$ is given by

$$(\varphi\psi)(a) = \varphi(\psi(a))$$

for all $a \in A$. Note that $\varphi\psi$ is a graded R-homomorphism of degree $i + j$. Multiplication is easy seen to satisfy associativity and the identity map $1_A\colon A \to A$ serves as the identity element with respect to this multiplication. Moreover, Leibniz law is satisfied: we have

$$\begin{aligned}
\mathrm{d}^{\star}(\varphi)\psi + (-1)^i\varphi\mathrm{d}^{\star}(\psi) &= (\mathrm{d}\varphi - (-1)^i\varphi\mathrm{d})\psi + (-1)^i\varphi(\mathrm{d}\psi - (-1)^j\psi\mathrm{d}) \\
&= \mathrm{d}\varphi\psi - (-1)^i\varphi\mathrm{d}\psi + (-1)^i\varphi\mathrm{d}\psi - (-1)^{i+j}\varphi\psi\mathrm{d} \\
&= \mathrm{d}\varphi\psi - (-1)^{i+j}\varphi\psi\mathrm{d} \\
&= \mathrm{d}^{\star}(\varphi\psi).
\end{aligned}$$

for all $\varphi \in \mathrm{Hom}^{\star}_R(A, A)_i$ and $\psi \in \mathrm{Hom}^{\star}_R(A, A)_j$. □

5.1.3 DG Algebra Embedding

Proposition 5.5. *Let A be a DG algebra. Define $\varphi\colon A \to \mathrm{Hom}_R^\star(A, A)$ by*

$$\varphi(a) = m_a$$

for all $a \in A$ where $m_a\colon A \to A$ is the homothety map, given by

$$m_a(x) = ax$$

for all $x \in A$. Then φ is an injective DG algebra homomorphism.

Proof. Note that $\varphi\colon A \to \mathrm{Hom}_R^\star(A, A)$ is easily seen to be a graded R-homomorphism. Let us check that it commutes with the differentials so that it is a chain map. Let $a \in A_i$. Observe that

$$\begin{aligned} \mathrm{d}m_a(x) &= \mathrm{d}(ax) \\ &= \mathrm{d}(a)x + (-1)^i a\mathrm{d}(x) \\ &= m_{\mathrm{d}(a)}(x) + (-1)^i m_a(\mathrm{d}(x)) \\ &= (m_{\mathrm{d}(a)} + (-1)^i m_a\mathrm{d})(x) \end{aligned}$$

for all $x \in A$. It follows that

$$\mathrm{d}m_a = m_{\mathrm{d}(a)} + (-1)^i m_a\mathrm{d}.$$

Thus

$$\begin{aligned} (\mathrm{d}^\star\varphi)(a) &= \mathrm{d}^\star(\varphi(a)) \\ &= \mathrm{d}^\star m_a \\ &= \mathrm{d}m_a - (-1)^i m_a\mathrm{d} \\ &= m_{\mathrm{d}(a)} \\ &= \varphi(\mathrm{d}(a)) \\ &= (\varphi\mathrm{d})(a), \end{aligned}$$

and so φ commutes with the differentials. Thus φ is a chain map.

Let us now check that φ is a DG algebra homomorphism. Let $a, b \in A$. Observe that we have

$$\begin{aligned} (m_a m_b)(x) &= m_a(m_b(x)) \\ &= m_a(bx) \\ &= a(bx) \\ &= (ab)x \\ &= m_{ab}(x) \end{aligned}$$

for all $x \in A$. It follows that $m_a m_b = m_{ab}$. Thus

$$\begin{aligned} \varphi(ab) &= m_{ab} \\ &= m_a m_b \\ &= \varphi(a)\varphi(b), \end{aligned}$$

and hence φ respects addition, and also $\varphi(1) = 1_A$, where e is the identity in A and 1_A is the identity in $\mathrm{Hom}_R^\star(A, A)$.

Finally, note that φ is injective. Indeed, suppose $m_a = 0$ for some $a \in A$, then

$$\begin{aligned} 0 &= m_a(1) \\ &= a \cdot 1 \\ &= a \end{aligned}$$

implies $\ker \varphi = 0$. □

Proposition 5.6. *Let R be a ring, let I be an ideal in R, and let (A, d) be a DG algebra resolution of R/I over R. Then I kills $\mathrm{H}(A)$.*

Proof. The embedding of DG Algebras $A \to \mathrm{Hom}_R(A, A)$, given by $a \mapsto m_a$, induces a map in the 0th homology

$$R/I \to \{\text{homotopy classes of chain maps } A \to A\}.$$

In particular, if x is in I, then m_x must be null-homotopic. Hence I kills $\mathrm{H}(A)$. □

Proposition 5.7. *Let R be a ring, let I be an ideal in R, and let (A, d) and (A', d') be two DG algebra resolutions of R/I over R. Then $\mathrm{Hom}_R^\star(A, A)$ is homotopically equivalent to $\mathrm{Hom}_R^\star(A', A')$.*

Proof. Since A and A' are homotopically equivalent, we may choose chain maps $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A$ together with homotopies $h\colon A \to A'$ and $h'\colon A \to A'$ where

$$\varphi'\varphi - 1 = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1 = \mathrm{d}'h' + h'\mathrm{d}'.$$

Define $\gamma\colon \mathrm{Hom}_R^\star(A, A) \to \mathrm{Hom}_R^\star(A', A')$ by

$$\gamma(\alpha) = \varphi\alpha\varphi'$$

for all $\alpha \in \mathrm{Hom}_R^\star(A, A)$. We claim that γ is a chain map. Indeed, it is graded since φ and φ' have degree 0. It is an R-module homomorphism since if $r, s \in R$ and $\alpha, \beta \in \mathrm{Hom}_R^\star(A, A)$, then we have

$$\begin{aligned}\gamma(r\alpha + s\beta) &= \varphi(r\alpha + s\beta)\varphi' \\ &= \varphi r\alpha\varphi' + \varphi s\phi\varphi' \\ &= r\varphi\alpha\varphi' + s\varphi\beta\varphi' \\ &= r\gamma(\alpha) + s\gamma(\beta).\end{aligned}$$

It commutes with the differentials since if $\alpha \in \mathrm{Hom}_R^\star(A, A)_i$, then we have

$$\begin{aligned}(\mathrm{d}^\star_{A'}\gamma)(\alpha) &= \mathrm{d}^\star_{A'}(\gamma(\alpha)) \\ &= \mathrm{d}^\star_{A'}(\varphi\alpha\varphi') \\ &= \mathrm{d}'\varphi\alpha\varphi' + (-1)^i\varphi\alpha\varphi'\mathrm{d}' \\ &= \varphi\mathrm{d}\alpha\varphi' + (-1)^i\varphi\alpha\mathrm{d}\varphi' \\ &= \varphi(\mathrm{d}\alpha + (-1)^i\alpha\mathrm{d})\varphi' \\ &= \gamma(\mathrm{d}\alpha + (-1)^i\alpha\mathrm{d}) \\ &= \gamma(\mathrm{d}^\star_A(\alpha)) \\ &= (\gamma\mathrm{d}^\star_A)(\alpha).\end{aligned}$$

Similarly, we define $\gamma'\colon \mathrm{Hom}_R^\star(A', A') \to \mathrm{Hom}_R^\star(A, A)$ by

$$\gamma'(\alpha') = \varphi'\alpha'\varphi$$

for all $\alpha' \in \mathrm{Hom}_R^\star(A', A')$. We claim that $\gamma'\gamma \sim 1_{\mathrm{Hom}_R^\star(A,A)}$ and $\gamma'\gamma \sim 1_{\mathrm{Hom}_R^\star(A',A')}$. It suffices to show that $\gamma'\gamma \sim 1_{\mathrm{Hom}_R^\star(A,A)}$ as the other homotopy equivalence will follows by a similar argument. Let $H\colon \mathrm{Hom}_R^\star(A, A) \to \mathrm{Hom}_R^\star(A, A)$ be defined by

$$H(\alpha) = h\alpha\mathrm{d}h + h\alpha h\mathrm{d} + h\alpha + \alpha h$$

for all $\alpha \in \mathrm{Hom}_R^\star(A, A)$. Now let $\alpha \in \mathrm{Hom}_R^\star(A, A)_i$. Then we have

$$\begin{aligned}(\gamma'\gamma - 1)(\alpha) &= (\gamma'\gamma)(\alpha) - \alpha \\ &= \gamma'(\gamma(\alpha)) - \alpha \\ &= \gamma'(\varphi\alpha\varphi') - \alpha \\ &= \varphi'\varphi\alpha\varphi'\varphi - \alpha \\ &= (\mathrm{d}h + h\mathrm{d} + 1)\alpha(\mathrm{d}h + h\mathrm{d} + 1) - \alpha \\ &= \mathrm{d}h\alpha\mathrm{d}h + \mathrm{d}h\alpha h\mathrm{d} + \mathrm{d}h\alpha + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h + \alpha h\mathrm{d} + \alpha - \alpha \\ &= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + (\mathrm{d}h + h\mathrm{d})\alpha + \alpha(\mathrm{d}h + h\mathrm{d})\end{aligned}$$

$$\begin{aligned}&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\ &= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i h\alpha h\mathrm{d}\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\ &= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d} - h\alpha\mathrm{d}h)\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} \\ &= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d})\mathrm{d} + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} - (-1)^i h\alpha\mathrm{d}h\mathrm{d} \\ &= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + (-1)^i(h\alpha\mathrm{d}h + h\alpha h\mathrm{d})\mathrm{d} + (h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d}) - (-1)^i(h\alpha\mathrm{d}\mathrm{d}h + h\alpha\mathrm{d}h\mathrm{d}) \\ &= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\ &= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\ &= \mathrm{d}H(\alpha) - (-1)^{i+1} H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha - (-1)^i\alpha\mathrm{d}) \\ &= \mathrm{d}^\star(H(\alpha)) + H(\mathrm{d}^\star(\alpha)) \\ &= (\mathrm{d}^\star H + H\mathrm{d}^\star)(\alpha)\end{aligned}$$

□

$$
\begin{aligned}
(\gamma'\gamma - 1)(\alpha) &= (\gamma'\gamma)(\alpha) - \alpha \\
&= \gamma'(\gamma(\alpha)) - \alpha \\
&= \gamma'(\varphi\alpha\varphi') - \alpha \\
&= \varphi'\varphi\alpha\varphi'\varphi - \alpha \\
&= (\mathrm{d}h + h\mathrm{d} + 1)\alpha(\mathrm{d}h + h\mathrm{d} + 1) - \alpha \\
&= \mathrm{d}h\alpha\mathrm{d}h + \mathrm{d}h\alpha h\mathrm{d} + \mathrm{d}h\alpha + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h + \alpha h\mathrm{d} + \alpha - \alpha \\
&= \mathrm{d}(h\alpha\mathrm{d}h + h\alpha h\mathrm{d}) + h\mathrm{d}\alpha\mathrm{d}h + h\mathrm{d}\alpha h\mathrm{d} + (\mathrm{d}h + h\mathrm{d})\alpha + \alpha(\mathrm{d}h + h\mathrm{d})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{d}h\alpha + \alpha h\mathrm{d} + h\mathrm{d}\alpha + \alpha\mathrm{d}h \\
&= \mathrm{d}h\alpha - (-1)^i\mathrm{d}\alpha h + (-1)^i h\alpha\mathrm{d} + \alpha h\mathrm{d} + h\mathrm{d}\alpha + (-1)^i\mathrm{d}\alpha h - (-1)^i h\alpha\mathrm{d} + \alpha\mathrm{d}h \\
&= \mathrm{d}(h\alpha - (-1)^i\alpha h) + (-1)^i(h\alpha - (-1)^i\alpha h)\mathrm{d} + h\mathrm{d}\alpha + (-1)^i\mathrm{d}\alpha h - (-1)^i h\alpha\mathrm{d} + \alpha\mathrm{d}h \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) + (-1)^i H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha) - (-1)^i H(\alpha\mathrm{d}) \\
&= \mathrm{d}H(\alpha) - (-1)^{i+1} H(\alpha)\mathrm{d} + H(\mathrm{d}\alpha - (-1)^i\alpha\mathrm{d}) \\
&= \mathrm{d}^\star(H(\alpha)) + H(\mathrm{d}^\star(\alpha)) \\
&= (\mathrm{d}^\star H + H\mathrm{d}^\star)(\alpha)
\end{aligned}
$$

5.1.4 Direct Sum of DG Algebras is DG Algebra

Proposition 5.8. *Let* (A, d) *and* (A', d') *be two DG R-algebras. Then* $(A \oplus_R A', \mathrm{d}^{A\oplus_R A'})$ *is a DG R-algebra.*

Proof. Throughout this proof, denote $\mathrm{d}^\oplus := \mathrm{d}^{A\oplus_R A'}$. We define multiplication on $A \oplus_R A'$ by the formula

$$(a, a')(b, b') = (-1)^{i'j}(ab, a'b') \tag{44}$$

for all $a \otimes a' \in A_i \otimes_R A_{i'}$ and $b \otimes b' \in A_j \otimes_R A_{j'}$. It is easy to check that (43) is associative and unital with with unit being $e_A \otimes e_{A'}$ where e_A is the unit of A and $e_{A'}$ is the unit of A'. Let us check that Leibniz law is satisfied. Let $a \otimes a', b \otimes b' \in A \otimes_R A'$. Then we have

$$
\begin{aligned}
\mathrm{d}^\oplus((a,a')(b,b')) &= (-1)^{i'j}\mathrm{d}^\oplus(ab, a'b') \\
&= (-1)^{i'j}\mathrm{d}^\oplus(ab, a'b') \\
&= (-1)^{i'j}((\mathrm{d}(a)b + (-1)^i a\mathrm{d}(b)) \otimes a'b' + (-1)^{i+j} ab \otimes (\mathrm{d}'(a')b' + (-1)^{i'} a'\mathrm{d}'(b'))) \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i'j+i} a\mathrm{d}(b) \otimes a'b' + (-1)^{i'j+i+j} ab \otimes \mathrm{d}'(a')b' + (-1)^{i'j+i+j+i'} ab \otimes a'\mathrm{d}'(b') \\
&= (-1)^{i'j}\mathrm{d}(a)b \otimes a'b' + (-1)^{i+j(i'+1)} ab \otimes \mathrm{d}'(a')b' + (-1)^{i+i'+i'(j+1)} a\mathrm{d}(b) \otimes a'b' + (-1)^{i+i'+j+i'j}(ab \otimes a'\mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a')(b \otimes b') + (-1)^i(a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b') + (-1)^{i+i'+j}(a \otimes a')(b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}(a) \otimes a' + (-1)^i a \otimes \mathrm{d}'(a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}(b) \otimes b' + (-1)^j b \otimes \mathrm{d}'(b')) \\
&= (\mathrm{d}^\otimes(a \otimes a'))(b \otimes b') + (-1)^{i+i'}(a \otimes a')(\mathrm{d}^\otimes(b \otimes b')).
\end{aligned}
$$

Thus $\mathrm{d}^\otimes$ satifies Leibniz law with respect to (43). □

5.1.5 Localization of DG-Algebra

Let (A, d) be a DG R-algebra and let S be a multiplicatively-closed subset of A consisting of homogeneous elements of even degree. The **localization of** (A, d) **with respect to** S is the R-complex (A_S, d_S) where A_S is the graded R-module whose component in degree i is

$$(A_S)_i = \{a/s \mid j \in \mathbb{N}, a \in A_{i+j}, \text{ and } s \in A_j\}.$$

The differential d_S is defined as follows: if $a \in A_{i+j}$ and $s \in A_j$, then $a/s \in (A_S)_i$ and

$$\mathrm{d}_S\left(\frac{a}{s}\right) = \frac{\mathrm{d}(a)s - (-1)^{i+j}a\mathrm{d}(s)}{s^2}.$$

To see that this is well-defined, suppose $a/s = a'/s'$ with both $|s|$ and $|s'|$ even, so $as' = a's$ and $|a| = |a'|$. Applying the differential gives us

$$\mathrm{d}(a)s' + (-1)^{|a|}a\mathrm{d}(s') = \mathrm{d}(a')s + (-1)^{|a'|}a'\mathrm{d}(s).$$

We need to show that

$$\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2} = \frac{\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')}{s'^2}.$$

Or in other words, we need to show

$$\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s'^2 = \left(\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')\right)s^2.$$

We have

$$\begin{aligned}
\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s'^2 &= \mathrm{d}(a)ss'^2 - (-1)^{|a|}a\mathrm{d}(s)s'^2 \\
&= \mathrm{d}(a)s'^2s - (-1)^{|a|}as'^2\mathrm{d}(s) \\
&= (\mathrm{d}(a')s + (-1)^{|a'|}a'\mathrm{d}(s) - (-1)^{|a|}a\mathrm{d}(s'))s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a\mathrm{d}(s')s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a'\mathrm{d}(s')s^2 - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 - (-1)^{|a|}a'\mathrm{d}(s')s^2 + (-1)^{|a'|}a'\mathrm{d}(s)s's - (-1)^{|a|}a'ss'\mathrm{d}(s) \\
&= \mathrm{d}(a')s's^2 - (-1)^{|a'|}a'\mathrm{d}(s')s^2 \\
&= \left(\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')\right)s^2
\end{aligned}$$

Next, we need to check that $\mathrm{d}_S^2 = 0$. We have

$$\begin{aligned}
\mathrm{d}_S^2\left(\frac{a}{s}\right) &= \mathrm{d}_S\left(\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2}\right) \\
&= \frac{\mathrm{d}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)s^2 - (-1)^{|a|-1}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)\mathrm{d}(s^2)}{s^4} \\
&= \frac{((-1)^{|a|-1}\mathrm{d}(a)\mathrm{d}(s) - (-1)^{|a|}\mathrm{d}(a)\mathrm{d}(s))s^2 + (-1)^{|a|}\left(\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)\right)2s\mathrm{d}(s)}{s^4} \\
&= \frac{(-1)^{|a|-1}2\mathrm{d}(a)\mathrm{d}(s)s^2 + (-1)^{|a|}2\mathrm{d}(a)\mathrm{d}(s)s^2 - 2a\mathrm{d}(s)^2s}{s^4} \\
&= \frac{0}{s^4} \\
&= 0.
\end{aligned}$$

Next, we need to check that Leibniz law is satisfies. We have

$$
\begin{aligned}
\mathrm{d}_S\left(\frac{aa'}{ss'}\right) &= \frac{\mathrm{d}(aa')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(ss')}{s^2s'^2} \\
&= \frac{\mathrm{d}(aa')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(ss')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)a'ss' + (-1)^{|a|}a\mathrm{d}(a')ss' - (-1)^{|a|+|a'|}aa'\mathrm{d}(s)s' - (-1)^{|a|+|a'|}aa's\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's' + (-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's' + (-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \frac{\mathrm{d}(a)sa's' - (-1)^{|a|}a\mathrm{d}(s)a's'}{s^2s'^2} + \frac{(-1)^{|a|}as\mathrm{d}(a')s' - (-1)^{|a'|+|a|}asa'\mathrm{d}(s')}{s^2s'^2} \\
&= \left(\frac{\mathrm{d}(a)s - (-1)^{|a|}a\mathrm{d}(s)}{s^2}\right)\frac{a'}{s'} + (-1)^{|a|}\frac{a}{s}\left(\frac{\mathrm{d}(a')s' - (-1)^{|a'|}a'\mathrm{d}(s')}{s'^2}\right) \\
&= \mathrm{d}_S\left(\frac{a}{s}\right)\frac{a'}{s'} + (-1)^{|a|}\frac{a}{s}\mathrm{d}_S\left(\frac{a'}{s'}\right).
\end{aligned}
$$

5.2 DG Modules

Definition 5.3. Let (A, d_A) be a DG R-algebra. A (right) **differential graded A-module** (or DG A-module for short) is an R-complex (M, d_M) equipped with a chain map

$$\star\colon (M \otimes_R A, \mathrm{d}^{M\otimes_R A}) \to (M, \mathrm{d}_M)$$

denoted $u \otimes a \mapsto \star(u \otimes a)$ (or just ua if context is clear). In other words, M has an A-module structure which behaves well with respect to the Leibniz law:

$$\mathrm{d}_M(ua) = \mathrm{d}_M(u)a + (-1)^i u\mathrm{d}_A(a)$$

for all $u \in M_i$ and $a \in A$. If (I, d_I) is an R-complex with $I \subset A$ and $\star$ being the usual multiplication map, then say (I, d_I) is a **DG ideal** in (A, d_A).

Definition 5.4. Let (A, d) be a DG R-algebra and let (M, d_M) and (N, d_N) be DG A-modules. A chain map $\varphi\colon (M, \mathrm{d}_M) \to (N, \mathrm{d}_N)$ is said to be a **DG-module morphism** if it respects A-scaling. In other words, we need

$$\varphi(ua) = \varphi(u)a$$

for all $u \in M$ and $a \in A$ (so the underlying map $\varphi\colon M \to N$ of A-modules is an A-module homomorphism). The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Obtaining a Differential Graded A-Module from an R-Complex

Example 5.1. Let (A, d_A) be a differential graded R-algebra and let (M, d_M) be an R-complex. Then the R-complex $(M \otimes_R A, \mathrm{d}^{M\otimes_R A})$ is a DG A-module.

5.2.1 Completion of DG Algebra with respect to an Ideal

Let (A, d) be a DG R-algebra and let (I, d) be a DG ideal in (A, d). We define the I-adic DG algebra, denoted $(\widehat{A}_I, \widehat{\mathrm{d}}_I)$, where

$$\widehat{A}_I := \varprojlim A/I^n = \{(\overline{a_n}) \in A/I^n \mid a_n \equiv a_m \bmod I^m \text{ whenever } n \geq m\}$$

and where $\widehat{\mathrm{d}}_I$ is defined pointwise:

$$\widehat{\mathrm{d}}_I((\overline{a_n})) = (\overline{\mathrm{d}(a_n)})$$

for all $(\overline{a_n}) \in \widehat{A}_I$. Note that the ith homogeneous component of $\widehat{A}_I$ is

$$(\widehat{A}_I)_i = \varprojlim_n (A_i/I_i^n) = \{(\overline{a_n}) \in A_i/I_i^n \mid a_n \equiv a_m \bmod I_i^m \text{ whenever } n \geq m\}.$$

In particular, if $(\overline{a_n}) \in (\widehat{A}_I)_i$, then $a_n \in A_i$ for all $i \geq 0$. Suppose $(\overline{a_n}) \in \ker \widehat{\mathrm{d}}_I$. Then $\mathrm{d}(a_n) \in I^n$ for all $n \in \mathbb{N}$.

5.2.2 Blowing up DG Algebra with respect to an Ideal

Let (A, d) be a DG R-algebra and let I be a DG ideal in A. Let

$$\mathrm{N}_I(A) := A \oplus A/I \oplus A/I^2 \oplus \cdots = A + (A/I)t + (A/I^2)t^2 + \cdots$$

and let $\mathrm{d}^{\mathrm{N}_I(A)} \colon \mathrm{N}_I(A) \to \mathrm{N}_I(A)$ be the unique graded linear map such that

$$\mathrm{d}^{\mathrm{N}_I(A)}(\overline{a}t^n) = \overline{\mathrm{d}(a)}t^{n-1},$$

for all $\overline{a}t^n \in (A/I^n)t^n$[4].

Proposition 5.9. *Let (A, d) be a DG R-algebra and let I be a DG ideal in A such that $I \subset A_+$. Then*

$$\mathrm{H}_n(\mathrm{N}_I(A)) = 0 \textit{ for } n \gg 0 \textit{ if and only if } \mathrm{H}(A) = 0.$$

Proof. Suppose first that $\mathrm{H}(A) = 0$ and assume for a contradiction that $\mathrm{H}_n(\mathrm{N}_I(A)) \neq 0$ for $n \gg 0$. Choose a $(\overline{a}$
Suppose $k \in \mathbb{Z}$ such that $\mathrm{H}_i(A) = 0$ for all $i \geq k$. We wish to show that ☐

Note that

$$\mathrm{H}_n(\mathrm{N}_I(A)) \cong \frac{\mathrm{d}^{-1}(I^{n-1})}{\mathrm{im}\ \mathrm{d} + I^n}.$$

Thus, we want to show that

$$\mathrm{d}^{-1}(I^{n-1}) = \mathrm{im}\ \mathrm{d} + I^n$$

for $n \gg 0$. The theorem would follow at once if we can show that

$$\mathrm{d}^{-1}(I^{n-1}) \subset I^{n-1}$$

for $n \gg 0$. Assume for a contradiction that we can find $a_n \in A \backslash I^n$ such that $\mathrm{d}(a_n) \in I^n$.

We claim that $\mathrm{H}_i(A) \cong \mathrm{H}_i(\mathrm{N}_I(A))$ for all i

5.3 The Koszul Complex

Throughout this subsection, let $\underline{x} = x_1, \ldots, x_n$ be a sequence in R. We will construct a DG R-algebra called the **Koszul complex of** $\underline{x}$. Before doing so, we need to discuss ordered sets.

5.3.1 Ordered Sets

An **ordered set** is a set with a total linear ordering on it. The **ordered set** $[n]$ is the set $\{1, \ldots, n\}$ equipped with the natural ordering $1 < \cdots < n$. Let σ be a subset of $\{1, \ldots, n\}$. Then the natural ordering on $\{1, \ldots, n\}$ induces a natural ordering on σ. If we want to think of σ as a set equipped with this natural ordering, then we will write $[\sigma]$. If $\sigma = \{\lambda_1, \ldots, \lambda_k\}$, where $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$, then we will also write $[\sigma] = [\lambda_1, \ldots, \lambda_k]$. If we write "suppose $[\sigma] = [\lambda_1, \ldots, \lambda_k]$", then it is understood that $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$. For each $i \in \mathbb{Z}$ such that $0 \leq i \leq n$, we denote

$$S_i[n] := \{\sigma \subseteq \{1, \ldots, n\} \mid |\sigma| = i\}.$$

Compliments

Let $\sigma \subseteq [n]$. We denote by $\sigma^\star$ to be the compliment of σ in $[n]$:

$$\sigma^\star := [n] \backslash \sigma.$$

If $[\sigma] = [\lambda_1, \ldots, \lambda_k]$, then we write $\sigma^\star = [\lambda_1^\star, \ldots, \lambda_{n-k}^\star]$.

[4] Here, the $\overline{a}$ is understood to be a coset in A/I^n with represenative $a \in A$.

Signature

Let σ and τ be two disjoint subsets of $\{1, \ldots, n\}$. Suppose that

$$[\sigma] = [\lambda_1, \ldots, \lambda_k] \quad \text{and} \quad [\sigma'] = [\lambda_{k+1}, \ldots, \lambda_{k+m}].$$

Then

$$[\sigma \cup \sigma'] = [\lambda_{\pi(1)}, \ldots, \lambda_{\pi(k+m)}],$$

where $\pi \colon S_{k+m} \to S_{k+m}$ is the permutation which puts everything in the correct order. We define

$$\langle \sigma, \tau \rangle := \operatorname{sign}(\pi).$$

Remark. Let $\lambda \in \{1, \ldots, n\}$ and let $\sigma \subseteq \{1, \ldots, n\}$. To clean notation, we often drop the curly brackets around singleton elements $\{\lambda\}$ in what follows. For instance, we will write $\sigma \backslash \lambda$ instead of $\sigma \backslash \{\lambda\}$ and $\sigma \cup \lambda$ instead of $\sigma \cup \{\lambda\}$. We will also write $\langle \lambda, \sigma \rangle$ (or $\langle \sigma, \lambda \rangle$) instead of $\langle \{\lambda\}, \sigma \rangle$ (respectively $\langle \sigma, \{\lambda\} \rangle$).

Example 5.2. Consider $n = 4$. We perform some computations:

$$\begin{aligned}
\langle 2, \{1,4\} \rangle &= -1 \\
\langle 2, 3 \rangle &= 1 \\
\langle 3, 2 \rangle &= -1 \\
\langle \{1,4\}, 2 \rangle &= -1 \\
\langle 2, \{1,3,4\} \rangle &= -1 \\
\langle \{1,3,4\}, 2 \rangle &= 1 \\
\langle \{1,3\}, \{2,4\} \rangle &= -1 \\
\langle \{2,4\}, \{1,3\} \rangle &= -1
\end{aligned}$$

Signature Identities

Proposition 5.10. *Let σ, τ, and $\{\lambda\}$ be mutually disjoint subsets of $\{1, \ldots, n\}$. Then*

$$\langle \lambda, \sigma \cup \tau \rangle = \langle \lambda, \sigma \rangle \langle \lambda, \tau \rangle.$$

Proof. The permutation which places $[\lambda] \cup [\sigma \cup \tau]$ into proper order is a composition of the permutation which places $[\lambda] \cup [\sigma]$ into proper order with the permutation which places $[\lambda] \cup [\tau]$ into proper order. □

Proof. The permutation with puts λ in the proper order in $[\lambda] \cup [\sigma \cup \tau]$ is just a composition of the permutation which puts λ in the proper order in $[\lambda] \cup [\sigma]$ with the permutation which puts λ in the proper order in $[\lambda] \cup [\tau]$. □

Proposition 5.11. *Let σ and τ be two disjoint subsets of $\{1, \ldots, n\}$. If $\lambda \in \sigma$, then*

$$\langle \sigma, \tau \rangle = \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle.$$

Similarly, if $\mu \in \tau$, then

$$\langle \sigma, \tau \rangle = \langle \sigma, \mu \rangle \langle \sigma, \tau \backslash \mu \rangle. \tag{45}$$

Proof. Suppose $\lambda \in \sigma$. We can place $[\sigma] \cup [\tau]$ into proper order by moving λ all the way to the left of $[\sigma]$, then place $[\sigma \backslash \lambda] \cup [\tau]$ into proper order, then place $[\lambda] \cup [\sigma \backslash \lambda \cup \tau]$ into proper order. This gives us

$$\begin{aligned}
\langle \sigma, \tau \rangle &= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, (\sigma \backslash \lambda) \cup \tau) \rangle \\
&= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \sigma \backslash \lambda \rangle \langle \lambda, \tau \rangle \\
&= \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle
\end{aligned}$$

An analagous argument gives (45). □

5.3.2 Definition of the Koszul Complex

We are now ready to define the Koszul complex of $\underline{x}$.

Definition 5.5. The **Koszul complex of** $\underline{x}$, denoted $(\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})})$ is the R-complex whose graded R-module $\mathcal{K}(x)$ has

$$\mathcal{K}_i(\underline{x}) := \begin{cases} \bigoplus_{\sigma \in S_i[n]} Re_\sigma & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $\mathrm{d}^{\mathcal{K}(\underline{x})}$ is uniquely determined by

$$\mathrm{d}^{\mathcal{K}(\underline{x})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle x_\lambda e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq \{1, \dots, n\}$.

Exercise 1. Check that $(\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})})$ is an R-complex. In particular, show $\mathrm{d}^{\mathcal{K}(\underline{x})}\mathrm{d}^{\mathcal{K}(\underline{x})} = 0$.

Example 5.3. Here's what the Koszul complex $\mathcal{K}(x_1, x_2, x_3)$ looks like:

$$\begin{tikzcd}
R \arrow[r, "\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \end{pmatrix}"] & R^3 \arrow[r, "\begin{pmatrix} 0 & -x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}"] & R^3 \arrow[r, "\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}"] & R
\end{tikzcd}$$

$$\begin{array}{lll}
e_{\{1,2,3\}} \longmapsto x_1 e_{\{2,3\}} - x_2 e_{\{1,3\}} + x_3 e_{\{1,2\}} & & \\
& e_{\{2,3\}} \longmapsto x_2 e_{\{3\}} - x_3 e_{\{2\}} & \\
& e_{\{1,3\}} \longmapsto x_1 e_{\{3\}} - x_3 e_{\{1\}} & \\
& e_{\{1,2\}} \longmapsto x_1 e_{\{2\}} - x_2 e_{\{1\}} & \\
& & e_{\{1\}} \longmapsto x_1 \\
& & e_{\{2\}} \longmapsto x_2 \\
& & e_{\{3\}} \longmapsto x_3
\end{array}$$

5.3.3 Koszul Complex as Tensor Product

Proposition 5.12. *We have an isomorphism of R-complexes:*

$$(\mathcal{K}(x_1), \mathrm{d}^{\mathcal{K}(x_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(x_n), \mathrm{d}^{\mathcal{K}(x_n)}) \cong (\mathcal{K}(\underline{x}), \mathrm{d}^{\mathcal{K}(\underline{x})}).$$

Remark. Note that Proposition (3.21) gives an unambiguous interpretation for $(\mathcal{K}(x_1), \mathrm{d}^{\mathcal{K}(x_1)}) \otimes_R \cdots \otimes_R (\mathcal{K}(x_n), \mathrm{d}^{\mathcal{K}(x_n)})$.

Proof. For each $1 \leq \lambda \leq n$, write $\mathcal{K}(x_\lambda) = R \oplus Re_\lambda$ (so $\{1\}$ is a basis for $\mathcal{K}(x_\lambda)_0$ and $\{e_\lambda\}$ is a basis for $\mathcal{K}(x_\lambda)_1$). Let

$$\varphi\colon \mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n) \to \mathcal{K}(\underline{x})$$

be the unique graded linear map [5] such that

$$\varphi(1 \otimes \cdots \otimes 1) = 1 \quad \text{and} \quad \varphi(1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1) = e_{\{\lambda_1, \dots, \lambda_i\}}$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism since it restricts to a bijection on basis sets.

For the rest of the proof, denote $\mathrm{d}^{\mathcal{K}} := \mathrm{d}^{\mathcal{K}(\underline{x})}$ and $\mathrm{d}^{\otimes} := \mathrm{d}^{\mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n)}$. To see that φ is an isomorphism of R-complexes, we need to show that

$$\varphi \mathrm{d}^{\otimes} = \mathrm{d}^{\mathcal{K}} \varphi. \tag{46}$$

It suffices to check (**??**) on the basis elements. We have

$$\begin{aligned} \mathrm{d}^{\mathcal{K}} \varphi(1 \otimes \cdots \otimes 1) &= \mathrm{d}^{\mathcal{K}}(1) \\ &= 0 \\ &= \varphi(0) \\ &= \varphi \mathrm{d}^{\otimes}(1 \otimes \cdots \otimes 1), \end{aligned}$$

[5] We say unique graded linear map here because $\mathcal{K}(x_1) \otimes_R \cdots \otimes_R \mathcal{K}(x_n)$ is free with basis elements of the form $1 \otimes \cdots \otimes 1$ and $1 \otimes \cdots \otimes e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_i} \cdots \otimes 1$ for $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$ and φ respects the grading.

and

$$\begin{aligned}
\mathrm{d}^{\mathcal{K}}\varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1) &= \mathrm{d}^{\mathcal{K}}(e_{\{\lambda_1,\dots,\lambda_i\}}) \\
&= \sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}e_{\{\lambda_1,\dots,\lambda_i\}} \\
&= \sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}\varphi(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes\widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1) \\
&= \varphi\sum_{\mu=1}^{i}(-1)^{\mu-1}x_{\lambda_\mu}1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes\widehat{e}_{\lambda_\mu}\otimes\cdots\otimes e_{\lambda_i}\otimes\cdots\otimes 1) \\
&= \varphi\mathrm{d}^{\otimes}(1\otimes\cdots\otimes e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_i}\cdots\otimes 1).
\end{aligned}$$

for all $1\le\lambda_1<\cdots<\lambda_i\le n$. □

5.3.4 Koszul Complex is a DG Algebra

Proposition 5.13. *Let $\underline{x}=x_1,\dots,x_n$ be a sequence of elements in R. The Koszul complex $(\mathcal{K}(\underline{x}),\mathrm{d}^{\mathcal{K}(\underline{x})})$ is a DG algebra, with multiplication being uniquely determined on elementary tensors: for $\sigma,\tau\subseteq[n]$, we map $e_\sigma\otimes e_\tau\mapsto e_\sigma e_\tau$, where*

$$e_\sigma e_\tau=\begin{cases}\langle\sigma,\tau\rangle e_{\sigma\cup\tau} & \text{if } \sigma\cap\tau=\emptyset\\ 0 & \text{if } \sigma\cap\tau\neq\emptyset\end{cases} \tag{47}$$

Proof. Throughout this proof, denote $\mathrm{d}:=\mathrm{d}^{\mathcal{K}(\underline{x})}$. We first want to show that $\mathcal{K}(\underline{x})$ is an associative, unital, and strictly graded-commutative R-algebra. Since $\mathcal{K}(\underline{x})$ is a free R-module with $\{e_\sigma\mid\sigma\subseteq[n]\}$ as a basis, it suffices to check associativity and graded-commutativitythese properties on the basis elements. We first note that $e_\emptyset$ serves as the identity for the multiplication rule (**??**). Indeed, let $\sigma\subseteq[n]$. Then since $\sigma\cap\emptyset=\emptyset$, we have

$$e_\sigma e_\emptyset=e_\sigma=e_\emptyset e_\sigma.$$

Thus the underlying R-algebra $\mathcal{K}(\underline{x})$ is unital.

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is associative. Let $\sigma,\tau,\kappa\subseteq[n]$. If $\sigma\cap\tau\cap\kappa\neq\emptyset$, then it is clear that

$$\begin{aligned}
e_\sigma(e_\tau e_\kappa) &= 0\\
&= (e_\sigma e_\tau)e_\kappa,
\end{aligned}$$

so assume $\sigma\cap\tau\cap\kappa=\emptyset$. Then

$$\begin{aligned}
e_\sigma(e_\tau e_\kappa) &= \langle\tau,\kappa\rangle e_\sigma e_{\tau\cup\kappa}\\
&= \langle\sigma,\tau\cup\kappa\rangle\langle\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&= \langle\sigma,\tau\rangle\langle\sigma,\kappa\rangle\langle\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&= \langle\sigma,\tau\rangle\langle\sigma\cup\tau,\kappa\rangle e_{\sigma\cup\tau\cup\kappa}\\
&= \langle\sigma,\tau\rangle e_{\sigma\cup\tau}e_\kappa\\
&= (e_\sigma e_\tau)e_\kappa.
\end{aligned}$$

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is graded-commutative. Let $\sigma,\tau\subseteq[n]$. If $\sigma\cap\tau\neq\emptyset$, then

$$\begin{aligned}
e_\sigma e_\tau &= 0\\
&= (-1)^{|\sigma||\tau|}e_\tau e_\sigma.
\end{aligned}$$

Suppose $\sigma\cap\tau=\emptyset$. Then

$$\begin{aligned}
e_\sigma e_\tau &= \langle\sigma,\tau\rangle e_{\sigma\cup\tau}\\
&= (-1)^{|\sigma||\tau|}\langle\tau,\sigma\rangle e_{\sigma\cup\tau}\\
&= (-1)^{|\sigma||\tau|}e_\tau e_\sigma.
\end{aligned}$$

Next we check the underlying R-algebra $\mathcal{K}(\underline{x})$ is strictly graded-commutative. Let $\sigma\subseteq[n]$ such that $|\sigma|$ is odd. Then

$$\begin{aligned}
e_\sigma^2 &= e_\sigma e_\sigma\\
&= 0
\end{aligned}$$

since $\sigma \cap \sigma \neq \emptyset$.

Finally, we need to check Leibniz law. First note that multiplication by $e_\emptyset$ and e_σ satisfies Leibniz law:

$$\begin{aligned} \mathrm{d}(e_\sigma)e_\emptyset - e_\sigma d(e_\emptyset) &= \mathrm{d}(e_\sigma)e_\emptyset \\ &= \mathrm{d}(e_\sigma) \\ &= \mathrm{d}(e_\sigma e_\emptyset), \end{aligned}$$

and similarly

$$\begin{aligned} \mathrm{d}(e_\emptyset)e_\sigma + e_\emptyset \mathrm{d}(e_\sigma) &= e_\emptyset \mathrm{d}(e_\sigma) \\ &= \mathrm{d}(e_\sigma) \\ &= \mathrm{d}(e_\emptyset e_\sigma), \end{aligned}$$

Next, let $\lambda \in [n]$ and let $\tau \subseteq [n]$. If $\lambda \in \tau$, then the pair (e_λ, e_τ) satisfies Leibniz law trivially, so suppose that $\lambda \notin \tau$. Then

$$\begin{aligned} \mathrm{d}(e_\lambda)e_\tau - e_\lambda \mathrm{d}(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu} \\ &= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\ &= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle \langle \lambda, \tau \rangle \langle \lambda, \mu \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\ &= x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \rangle \langle \mu, \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\ &= x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \cup \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\ &= \langle \lambda, \tau \rangle \langle \lambda, \tau \rangle x_\lambda e_\tau + \sum_{\mu \in \tau} \langle \lambda, \tau \rangle \langle \mu, \tau \backslash \mu \cup \lambda \rangle x_\mu e_{\tau \backslash \mu \cup \lambda} \\ &= \langle \lambda, \tau \rangle \sum_{\mu \in \tau \cup \lambda} \langle \mu, (\tau \cup \lambda) \backslash \mu \rangle x_\mu e_{(\tau \cup \lambda) \backslash \mu} \\ &= \langle \lambda, \tau \rangle \mathrm{d}(e_{\tau \cup \lambda}) \\ &= \mathrm{d}(e_\lambda e_\tau), \end{aligned}$$

where we used Proposition (5.11) to get from the second line to the third line. Next suppose $\tau \subseteq [n]$ and $\lambda \in \tau$. Then

$$\begin{aligned} \mathrm{d}(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_{\tau \backslash \mu} \\ &= x_\lambda e_\tau - \sum_{\mu \in \tau} \langle \mu, \tau \backslash \mu \rangle x_\mu e_\lambda e_{\tau \backslash \mu} \\ &= x_\lambda e_\tau - \langle \lambda, \tau \backslash \lambda \rangle \langle \lambda, \tau \backslash \lambda \rangle x_\lambda e_\tau \\ &= x_\lambda e_\tau - x_\lambda e_\tau \\ &= 0 \\ &= \mathrm{d}(0) \\ &= \mathrm{d}(e_\lambda e_\tau). \end{aligned}$$

Thus we have shown (**??**) satisfies the Leibniz law for all pairs (λ, τ) where $\lambda \in [n]$ and $\tau \subseteq [n]$. We prove by induction on $|\sigma| = i \geq 1$ that (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$. The base case $i = 1$ was just shown. Now suppose we have shown (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i < n$. Let $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i + 1$. Choose $\lambda \in \sigma$. Then

$$\begin{aligned} \mathrm{d}(e_\sigma e_\tau) &= \mathrm{d}(e_\lambda e_{\sigma \backslash \lambda} e_\tau) \\ &= x_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda \mathrm{d}(e_{\sigma \backslash \lambda} e_\tau) \\ &= x_\lambda e_{\sigma \backslash \lambda} e_\tau - e_\lambda (\mathrm{d}(e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma| - 1} e_{\sigma \backslash \lambda} \mathrm{d}(e_\tau)) \\ &= (x_\lambda e_{\sigma \backslash \lambda} - e_\lambda \mathrm{d}(e_{\sigma \backslash \lambda})) e_\tau + (-1)^{|\sigma|} e_\sigma \mathrm{d}(e_\tau) \\ &= \mathrm{d}(e_\lambda e_{\sigma \backslash \lambda}) e_\tau + (-1)^{|\sigma|} e_\sigma \mathrm{d}(e_\tau) \\ &= \mathrm{d}(e_\sigma) e_\tau + (-1)^{|\sigma| + 1} e_\sigma \mathrm{d}(e_\tau), \end{aligned}$$

where we used the base case on the pairs $(e_\lambda, e_{\sigma \backslash \lambda} e_\tau)$[6] and $(e_\lambda, e_{\sigma \backslash \lambda})$ and where we used the induction hypothesis on the pair $(e_{\sigma \backslash \lambda}, e_\tau)$. and where we used the base case on the pair $(e_\lambda, e_{\sigma \backslash \lambda})$. □

[6] If $e_{\sigma \backslash \lambda} e_\tau = 0$, then obviously Leibniz law holds for the pair $(e_\lambda, e_{\sigma \backslash \lambda} e_\tau)$.

5.3.5 The Dual Koszul Complex

We now want to discuss the dual Koszul complex of $\underline{x}$.

Definition 5.6. The **dual Koszul complex of** $\underline{x}$ is the R-complex

$$\mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R),$$

where R is viewed as a trivial R-complex (trivially grading with $\mathrm{d} = 0$). We denote by $\mathcal{K}^\star(\underline{x})$ to be the graded R-module hom $\mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R)$. We also denote by $\mathrm{d}^{\mathcal{K}^\star(\underline{x})}$ to be the corresponding differential. We can describe the dual Koszul complex more explicitly as follows: the graded R-module $\mathcal{K}^\star(\underline{x})$ has

$$\mathcal{K}_i^\star(\underline{x}) := \begin{cases} \bigoplus_{\sigma \in S_{-i}[n]} Re_\sigma^\star & \text{if } -n \leq i \leq 0 \\ 0 & \text{if } i < n \text{ or if } i > 0. \end{cases}$$

as its ith homogeneous component, where $e_\sigma^\star \colon \mathcal{K}(\underline{x}) \to R$ is uniquely determined by

$$e_\sigma^\star(e_{\sigma'}) = \begin{cases} 1 & \sigma = \sigma' \\ 0 & \text{else.} \end{cases}$$

for all $\sigma, \sigma' \subseteq [n]$. The differential $\mathrm{d}^{\mathcal{K}^\star(\underline{x})}$ is uniquely determined by

$$\mathrm{d}^{\mathcal{K}^\star(\underline{x})}(e_\sigma^\star) = (-1)^{|\sigma|+1} \sum_{\lambda^\star \in \sigma^\star} \langle \lambda^\star, \sigma \rangle r_\lambda e_{\sigma \cup \lambda^\star}^\star$$

for all $\sigma \subseteq [n]$.

Duality

Theorem 5.1. *There exists an isomorphism of R-complexes*

$$S^n \mathrm{Hom}_R^\star(\mathcal{K}(\underline{x}), R) \cong \mathcal{K}(\underline{x}).$$

In particular, we have an isomorphism of R-modules

$$\mathrm{H}_i(\mathcal{K}(\underline{x})) \cong \mathrm{H}_{i-n}(\mathcal{K}^\star(\underline{x}))$$

for all $i \in \mathbb{Z}$.

Proof. Let $i \in \mathbb{Z}$. If $i > n$ or $i < 0$, then theorem is obvious, so we may assume that $0 \leq i \leq n$. Let $\varphi \colon S^n(\mathcal{K}^\star(\underline{r}), d^{\mathcal{K}^\star(\underline{r})}) \to (\mathcal{K}(\underline{r}), d^{\mathcal{K}(\underline{r})})$ be the unique R-module graded homomorphism such that

$$\varphi(e_\sigma^\star) = \langle \sigma^\star, \sigma \rangle e_{\sigma^\star}.$$

for all $1 \leq \lambda_1 < \cdots < \lambda_i \leq n$. Then φ is an isomorphism of graded R-modules since it restricts to a bijection of basis sets. To see that φ is an isomorphism of R-complexes, we need to show that it commutes with the differentials. To do this, we first simplify notation by denoting $d^\star := (d^{\mathcal{K}^\star(\underline{r})})^{\Sigma^n}$ and $d := d^{\mathcal{K}(\underline{r})}$. Now we have

$$\begin{aligned}
d\varphi(e^\star_\sigma) &= d(\langle\sigma^\star,\sigma\rangle e_{\sigma^\star}) \\
&= \langle\sigma^\star,\sigma\rangle d(e_{\sigma^\star}) \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\sigma^\star,\sigma\rangle\langle\lambda^\star,\sigma^\star\backslash\lambda^\star\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma^\star\backslash\lambda^\star\rangle\langle\sigma^\star,\sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma^\star\backslash\lambda^\star\rangle\langle\sigma^\star\backslash\lambda^\star,\sigma\rangle\langle\lambda^\star,\sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\sigma^\star\backslash\lambda^\star,\sigma\cup\lambda^\star\rangle\langle\lambda^\star,\sigma\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma\rangle\langle\sigma^\star\backslash\lambda^\star,\sigma\cup\lambda^\star\rangle r_{\lambda^\star} e_{\sigma^\star\backslash\lambda^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma\rangle\langle(\sigma\cup\lambda^\star)^\star,\sigma\cup\lambda^\star\rangle r_{\lambda^\star} e_{(\sigma\cup\lambda^\star)^\star} \\
&= \sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma\rangle r_{\lambda^\star}\varphi(e^\star_{\sigma\cup\lambda^\star}) \\
&= \varphi\sum_{\lambda^\star\in\sigma^\star} \langle\lambda^\star,\sigma\rangle r_{\lambda^\star} e^\star_{\sigma\cup\lambda^\star} \\
&= \varphi d^\star(e^\star_\sigma)
\end{aligned}$$

where we used the fact that $\sigma^\star\backslash\lambda^\star = (\sigma\cup\lambda^\star)^\star$ and $\langle\sigma^\star,\sigma\rangle = \langle\lambda^\star,\sigma^\star\backslash\lambda^\star\rangle\langle\lambda^\star,\sigma\rangle\langle\sigma^\star\backslash\lambda^\star,\sigma\cup\lambda^\star\rangle$. □

5.3.6 Mapping Cone of Homothety Map as Tensor Product

Proposition 5.14. *Let (A,d) be an R-complex, let $x\in R$, and let $\mu_x\colon(A,\mathrm{d})\to(A,\mathrm{d})$ be the multiplication by x homothety map. Then*

$$(\mathrm{C}(\mu_x),\mathrm{d}^{\mathrm{C}(\mu_x)}) \cong (\mathcal{K}(x),\mathrm{d}^{\mathcal{K}(x)})\otimes_R(A,\mathrm{d}).$$

Proof. Let $\mathcal{K}(x) = R\oplus Re$ (so $\{1\}$ is a basis for $\mathcal{K}(x)_0$ and $\{e\}$ is a basis for $\mathcal{K}(x)_1$). Let $\varphi\colon\mathcal{K}(x)\otimes_R A\to\mathrm{C}(\mu_x)$ be defined by

$$\varphi(1\otimes a + e\otimes b) = (a,b)$$

for all $i\in\mathbb{Z}$, $a\in A_i$, and $b\in A_{i-1}$. Clearly φ is an isomorphism of graded R-modules. To see that φ is an isomorphism of R-complexes, we need to check that

$$\mathrm{d}^{\mathrm{C}(\mu_x)}\varphi = \varphi\mathrm{d}^{\mathcal{K}(x)\otimes_R A} \tag{48}$$

Let $i\in\mathbb{Z}$, $a\in A_i$, and $b\in A_{i-1}$. Then

$$\begin{aligned}
\mathrm{d}^{\mathrm{C}(\mu_x)}\varphi(1\otimes a+e\otimes b) &= \mathrm{d}^{\mathrm{C}(\mu_x)}(a,b) \\
&= (\mathrm{d}(a)+xb,-\mathrm{d}(b)) \\
&= \varphi(1\otimes(\mathrm{d}(a)+xb)+e\otimes(-\mathrm{d}(b))) \\
&= \varphi(1\otimes\mathrm{d}(a)+x\otimes b-e\otimes\mathrm{d}(b)) \\
&= \varphi(\mathrm{d}^{\mathcal{K}(x)\otimes_R A}(1\otimes a)+\mathrm{d}^{\mathcal{K}(x)\otimes_R A}(e\otimes b)) \\
&= \varphi\mathrm{d}^{\mathcal{K}(x)\otimes_R A}(1\otimes a+e\otimes b).
\end{aligned}$$

□

5.3.7 Properties of the Koszul Complex

Proposition 5.15. *Let $\lambda\in[n]$. Then the homothety map*

$$\mu_{x_\lambda}\colon(\mathcal{K}(\underline{x}),\mathrm{d}^{\mathcal{K}(\underline{x})})\to(\mathcal{K}(\underline{x}),\mathrm{d}^{\mathcal{K}(\underline{x})})$$

is null-homotopic. In particular, $x_\lambda\mathrm{H}(\mathcal{K}(\underline{x}))\cong 0$.

Proof. Denote $\mathrm{d} := \mathrm{d}^{\mathcal{K}(\underline{x})}$ and let $h\colon\mathcal{K}(\underline{x})\to\mathcal{K}(\underline{x})$ be the unique graded homomorphism of degree 1 such that

$$h(e_\sigma) = e_\lambda e_\sigma$$

for all $\sigma \subseteq [n]$. Then

$$\begin{aligned}(h\mathrm{d} + h\mathrm{d})(e_\sigma) &= \mathrm{d}(e_\lambda e_\sigma) + e_\lambda \mathrm{d}(e_\sigma) \\ &= x_\lambda e_\sigma - e_\lambda \mathrm{d}(e_\sigma) + e_\lambda \mathrm{d}(e_\sigma) \\ &= x_\lambda e_\sigma\end{aligned}$$

for all $\sigma \subseteq [n]$. It follows that

$$\mathrm{d}h + h\mathrm{d} = \mu_{x_\lambda}$$

on all of $\mathcal{K}(\underline{x})$. Thus the homothety map μ_{x_λ} is null-homotopic. □

Proposition 5.16. *The following conditions are equivalent.*

1. $\langle \underline{x} \rangle = R$,
2. $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$,
3. $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong 0$.

This follows immediately from Proposition (5.15) and the fact that $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong R/\langle \underline{x} \rangle$, but we will give an alternative proof:

Proof. Throughout this proof, we denote $\mathrm{d} := \mathrm{d}^{\mathcal{K}(\underline{x})}$.

($1 \implies 2$) Since $\langle \underline{x} \rangle = R$, there exists $y_1, \dots, y_n \in R$ such that

$$\sum_{\lambda=1}^{n} x_\lambda y_\lambda = 1.$$

Choose such $y_1, \dots, y_n \in R$ and let $\overline{f} \in \mathrm{H}(\mathcal{K}(\underline{x}))$ (so $f \in \ker \mathrm{d}$ is a representative of the coset $\overline{f}$). Then

$$\begin{aligned}\mathrm{d}\left(\sum_{\lambda=1}^{n} y_\lambda e_\lambda f\right) &= \sum_{\lambda=1}^{n} y_\lambda \mathrm{d}(e_\lambda f) \\ &= \sum_{\lambda=1}^{n} y_\lambda (\mathrm{d}(e_\lambda) f - e_\lambda \mathrm{d}(f)) \\ &= \sum_{\lambda=1}^{n} y_\lambda x_\lambda f \\ &= \left(\sum_{\lambda=1}^{n} y_\lambda x_\lambda\right) f \\ &= f.\end{aligned}$$

Thus, $f \in \operatorname{im} \mathrm{d}$, which implies $\mathrm{H}(\mathcal{K}(\underline{x})) = 0$.

($2 \implies 3$) $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$ if and only if $\mathrm{H}_i(\mathcal{K}(\underline{x})) \cong 0$ for all $i \in \mathbb{Z}$. In particular, $\mathrm{H}(\mathcal{K}(\underline{x})) \cong 0$ implies $\mathrm{H}_0(\mathcal{K}(\underline{x})) \cong 0$.

($3 \implies 1$) We have

$$\begin{aligned}0 &\cong \mathrm{H}(\mathcal{K}(\underline{x})) \\ &= R/\langle \underline{x} \rangle,\end{aligned}$$

which implies $\langle \underline{x} \rangle = R$. □

Denote $\mathcal{K}(\underline{x}; M) := \mathcal{K}(\underline{x}) \otimes_R M$.

6 Advanced Homological Algebra

Definition 6.1. Let

$$0 \longrightarrow A \xrightarrow{\varphi} A' \xrightarrow{\varphi'} A'' \longrightarrow 0 \tag{49}$$

be an exact sequence of R-complexes and chain maps. We say (49) is **degree-wise exact** if it is exact when viewed as a sequence of graded R-modules, that is, if for each $i \in \mathbb{Z}$ the sequence

$$0 \longrightarrow A_i \xrightarrow{\varphi_i} A'_i \xrightarrow{\varphi'_i} A''_i \longrightarrow 0 \tag{50}$$

is exact. Similarly, we say (49) is **degree-wise split exact** if (49) is split exact for each $i \in \mathbb{Z}$.

Proposition 6.1. *Let*

be an exact sequence of R-complexes and chain maps. Assume that for all $p \in \mathbb{Z}$ the sequence $\xi_p = (0 \to A_p \xrightarrow{\alpha_p} B_p \xrightarrow{\beta_p} C_p \to 0)$ is split exact. Then for all R-complexes X, Y the sequences $\xi_* = \operatorname{Hom}_R(X, \xi)$ and $\xi^* = \operatorname{Hom}_R(\xi, Y)$ are short exact.

Proof. Focus on ξ^*. First note that $0 \to C^* \xrightarrow{\beta^*} B \xrightarrow{\alpha^*} A^*$ is exact by left exactness. Need to show α^* is surjective. Note that ξ_p split implies $\gamma_p \colon B_p \to A_p$ such that $\gamma_p \alpha_p = 1_{A_p}$. We have

$$\begin{aligned}
\operatorname{Hom}_R(\alpha_p, Y_{p+n}) &= \operatorname{Hom}_R(\gamma_p, Y_{p+n}) \\
&= \operatorname{Hom}_R(\gamma_p \alpha_p, Y_{p+n}) \\
&= \operatorname{Hom}_R(1_{A_p}, Y_{p+n}) \\
&= 1_{\operatorname{Hom}_R(A_p, Y_{p+n})}.
\end{aligned}$$

□

Remark. There is a notion of split exactness for sequences of R-complexes and chain maps. Essentially the splitting map has to commute with the differentials.

Definition 6.2. Exact sequence ξ as above is called **degree-wise split exact**

6.1 Resolutions

Definition 6.3. Let M be an R-complex.

1. A **projective resolution of** M is a bounded below R-complex of projective R-modules P equipped with a quasiisomorphism $\tau \colon P \xrightarrow{\simeq} M$. In this case, we say (P, τ) (or just P if context is clear) is a projective resolution of M.

2. An **injective resolution of** M is a bounded above R-complex of injective R-modules E equipped with a quasiisomorphism $\varepsilon \colon M \xrightarrow{\simeq} E$. In this case, we say (E, ε) (or just E if context is clear) is an injective resolution of M.

6.1.1 Existence of projective resolutions

Proposition 6.2. *Let M, N, and P be R-modules, let $\psi\colon N \to M$ be an R-linear map, and let $\varphi\colon P \twoheadrightarrow M$ be a surjective R-linear map. Define the* ***pullback of*** $\psi\colon N \to M$ ***and*** $\varphi\colon P \twoheadrightarrow M$ *to be the R-module*

$$N \times_M P = \{(u,v) \in N \times P \mid \psi(u) = \varphi(v)\}$$

equipped with the R-linear maps $\pi_1\colon N \times_M P \to N$ and $\pi_2\colon N \times_M P \to P$ given by

$$\pi_1(u,v) = u \quad \text{and} \quad \pi_2(u,v) = v$$

for all $(u,v) \in N \times_M P$. Then there exists an isomorphism $\overline{\varphi}\colon P/\pi_1(N \times_M P) \to M/N$ given by

$$\overline{\varphi}(\overline{v}) = \overline{\varphi(v)}$$

for all $\overline{v} \in P/\pi_1(N \times_M P)$. Moreover, the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & \ker \pi_2 \arrow[r] \arrow[d, "\pi_1|_{\ker \pi_2}"] & N \times_M P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & P \arrow[r] \arrow[d, "\varphi"] & P/\pi_1(N \times_M P) \arrow[r] \arrow[d, "\overline{\varphi}"] & 0 \\
0 \arrow[r] & \ker \psi \arrow[r] & N \arrow[r, "\psi"] & M \arrow[r] & M/\psi(N) \arrow[r] & 0
\end{tikzcd}$$

where π_1 induces an isomorphism $\pi_1\colon \ker \pi_2 \to \ker \psi$.

Proof. We first need to check that $\overline{\varphi}$ is well-defined. Suppose $v + v'$ is another representative of $\overline{v}$ where $v' \in \operatorname{im} \pi_2$. Choose $[u', v'] \in N \times_M P$ such that $\pi_2[u', v'] = v'$ (so $\varphi(v') = \psi(u')$). Then

$$\begin{aligned}
\overline{\varphi}(\overline{v + v'}) &= \overline{\varphi(v + v')} \\
&= \overline{\varphi(v) + \varphi(v')} \\
&= \overline{\varphi(v) + \psi(u')} \\
&= \overline{\varphi(v)}.
\end{aligned}$$

Thus $\overline{\varphi}$ is well-defined. Clearly, $\overline{\varphi}$ is a surjective R-linear map since φ is a surjective R-linear map. It remains to show that $\overline{\varphi}$ is injective. Suppose $\overline{v} \in \ker \overline{\varphi}$. Then $\varphi(v) \in \operatorname{im} \psi$. Choose $u \in N$ such that $\psi(u) = \varphi(v)$. Then $[u, v] \in N \times_M P$ and $v = \pi_2[u, v]$. It follows that $\overline{v} = 0$ in $P/\pi_2(N \times_M P)$.

Let us now check that $\pi_1|_{\ker \pi_2}$ lands in $\ker \psi$. Let $u \in \ker \pi_2$. Then

$$\begin{aligned}
\psi \pi_1(u) &= \varphi \pi_2(u) \\
&= \varphi(0) \\
&= 0
\end{aligned}$$

implies $\pi_1(u) \in \ker \psi$. Thus $\pi_1|_{\ker \pi_2}$ lands in $\ker \psi$. Now we check that $\pi_1|_{\ker \pi_2}$ is an R-linear isomorphism. It is clearly an R-linear isomomorphism since it is the restriction of the homomorphism π_1. To see that $\pi_1|_{\ker \pi_2}$ is surjective, let $u \in \ker \psi$. Since

$$\begin{aligned}
\psi(u) &= 0 \\
&= \varphi(0),
\end{aligned}$$

we see that $[u, 0] \in N \times_M P$. Moreover we have $\pi_2[u, 0] = 0$ and so $[u, 0] \in \ker \pi_2$, and since $\pi_1[u, 0] = u$, we see that $\pi_1|_{\ker \pi_2}$ is surjetive. To see that $\pi_1|_{\ker \pi_2}$ is injective, suppose $\pi_1[u, v] = 0$ for some $[u, v] \in \ker \pi_2$. Then

$$\begin{aligned}
0 &= \pi_1[u, v] \\
&= u
\end{aligned}$$

implies $u = 0$ and

$$\begin{aligned}
0 &= \pi_2[u, v] \\
&= v
\end{aligned}$$

implies $v = 0$. Thus $[u, v] = [0, 0]$, hence $\pi_1|_{\ker \pi_2}$ is injective. □

Theorem 6.1. *Let (M, d) be an R-complex such that $M_i = 0$ for all $i < 0$. Then there exists a projective resolution of (M, d).*

Proof. We construct an R-complex (P, ∂) together with a chain map $\tau\colon (P, \partial) \to (M, \mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker \partial}\colon \ker \partial \to \ker \mathrm{d}$$

by induction on homological degree as follows: for the base case $i = 0$, we choose a projective R-module P_0 together with a surjective R-linear map $\tau_0\colon P_0 \to M_0$ and we set $\partial_0\colon P_0 \to 0$ to be the zero map. Suppose for some $k > 0$, we have constructed R-linear maps $\tau_i\colon P_i \to M_i$ and $\partial_i\colon P_i \to P_{i-1}$ such that

$$\partial_{i-1} \circ \partial_i = 0 \quad \text{and} \quad \tau_{i-1} \circ \partial_i = \mathrm{d}_i \circ \tau_i$$

and such that τ_i restricts to a surjection

$$\tau_i|_{\ker \partial_i}\colon \ker \partial_i \to \ker \mathrm{d}_i$$

for all $0 < i < k$. We first construct the pullback:

$$\begin{tikzcd}
P_k \arrow[dr, dashed, "\rho_k"] \arrow[drr, dashed, bend left, "\partial_k"] \arrow[ddr, dashed, bend right, "\tau_k"'] & & \\
& M_k \times_{\ker \mathrm{d}_{k-1}} \ker \partial_{k-1} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \ker \partial_{k-1} \arrow[d, "\tau_{k-1}|_{\ker \partial_{k-1}}"] \\
& M_k \arrow[r, "\mathrm{d}_k"] & \ker \mathrm{d}_{k-1}
\end{tikzcd}$$

where the map $\tau_{k-1}|_{\ker \partial_{k-1}}$ lands in $\ker \mathrm{d}_{k-1}$ since the τ_i commute with the differentials. Now we choose a projective R-module P_k together with a surjective R-linear map

$$\rho_k\colon P_k \to M_k \times_{\ker \mathrm{d}_{k-1}} \ker \partial_{k-1}$$

and we set $\partial_k = \pi_2 \circ \rho_k$ and $\tau_k = \pi_1 \circ \rho_k$. Observe that $\operatorname{im} \partial_k \subset \ker \mathrm{d}_k$ implies $\partial_{k-1} \circ \partial_k = 0$ and observe that

$$\begin{aligned}
\tau_{k-1} \circ \partial_k &= \tau_{k-1} \circ \pi_2 \circ \rho_k \\
&= \mathrm{d}_k \circ \pi_1 \circ \rho_k \\
&= \mathrm{d}_k \circ \tau_k
\end{aligned}$$

implies $\tau_{k-1} \circ \partial_k = \mathrm{d}_k \circ \tau_k$. Finally, observe that $\tau_k\colon \ker \partial_k \to \ker \mathrm{d}_k$ is surjective since it is a composition of surjective maps

$$\ker \partial_k = \ker(\pi_2 \circ \rho_k) \xrightarrow{\rho_k} \ker \pi_2 \xrightarrow[\cong]{\pi_1} \ker \mathrm{d}_k$$

where the isomorphism $\ker \pi_2 \cong \ker \mathrm{d}_k$ follows from Proposition (6.2). This completes the induction step.

Therefore we have an R-complex (P, ∂) together with a chain map $\tau\colon (P, \partial) \to (M, \mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker \partial}\colon \ker \partial \to \ker \mathrm{d}.$$

Moreover, Proposition (6.2) implies

$$\begin{aligned}
\mathrm{H}_{k-1}(M) &= \ker \mathrm{d}_{k-1} / \operatorname{im} \mathrm{d}_k \\
&= \ker \mathrm{d}_{k-1} / \mathrm{d}_k(M_k) \\
&\cong \ker \partial_{k-1} / \operatorname{im} \pi_2 \\
&= \ker \partial_{k-1} / \operatorname{im} \partial_k \\
&= \mathrm{H}_{k-1}(P),
\end{aligned}$$

It follows that τ is a quasi-isomorphism. $\square$

6.1.2 Existence of injective resolutions

Lemma 6.2. *Let M, N, and E be R-modules, let $\psi\colon M \to N$ be an R-linear map, and let $\varphi\colon M \to E$ be an injective R-linear map. Define the pushout of $\psi\colon M \to N$ and $\varphi\colon M \to E$ to be the R-module $E +_M N$ given by*

$$E +_M N = E \times N / \{(\varphi(v), 0) - (0, \psi(v)) \mid v \in M\}$$

equipped with the R-linear maps $\iota_1\colon E \to E +_M N$ and $\iota_2\colon N \to E +_M N$ given by

$$\iota_1(u) = [u, 0] \quad \text{and} \quad \iota_2(w) = [0, w]$$

for all $u \in E$ and $w \in N$, where $[u, w]$ denotes the coset class in $E +_M N$ with (u, w) as a representative. Then the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \ker \iota_1 \arrow[r] & E \arrow[r, "\iota_1"] & E +_M N \arrow[r] & E +_M N/E \arrow[r] & 0 \\
0 \arrow[r] & \ker \psi \arrow[r] \arrow[u, "\varphi|_{\ker \varphi}"] & M \arrow[r, "\psi"] \arrow[u, "\varphi"] & N \arrow[r] \arrow[u, "\iota_2"] & N/M \arrow[r] \arrow[u, "\overline{\iota_2}"] & 0
\end{tikzcd}$$

where $\overline{\iota_2}\colon N/M \to E +_M N/E$ is defined by

$$\overline{\iota_2}(\overline{w}) = \overline{[0, w]}$$

for all $\overline{w} \in N/M$ and where $\varphi|_{\ker \psi}\colon \ker \psi \to \ker \iota_1$ is defined by

$$\varphi|_{\ker \varphi}(v) = \varphi(v)$$

for all $v \in \ker \psi$.

Proof. We need to check that $\overline{\iota_2}$ is well-defined. Suppose $w + \psi(v)$ is another representative of $\overline{w}$ where $v \in M$. Then

$$\begin{aligned}
\overline{\iota_2}(\overline{v + \psi(w)}) &= \overline{[0, w + \psi(v)]} \\
&= \overline{[0, w] + [0, \psi(v)]} \\
&= \overline{[0, w] + [\varphi(v), 0]} \\
&= \overline{[0, w]}.
\end{aligned}$$

Thus λ is well-defined. Clearly, λ is a surjective R-linear map since φ is a surjective R-linear map. It remains to show that λ is injective. Suppose $\overline{v} \in P/\pi_2(N \times_M P)$ such that

$$\lambda(\overline{v}) = \overline{\varphi(v)} = \overline{0}.$$

Then $\varphi(v) \in \mathrm{im}(\psi)$. In other words, there exists $u \in N$ such that $\psi(u) = \varphi(v)$. In other words, $(u, v) \in N \times_M P$ and hence

$$\begin{aligned}
v &= \pi_2(u, v) \\
&\in \pi_2(N \times_M P).
\end{aligned}$$

Thus $\overline{v} = \overline{0}$ in $P/\pi_2(N \times_M P)$. $\square$

Theorem 6.3. *Let (M, d) be an R-complex such that $M_i = 0$ for all $i > 0$. Then there exists an injective resolution of (M, d).*

Proof. We construct an R-complex (E, ∂) together with an injective chain map $\varepsilon\colon (M, \mathrm{d}) \to (E, \partial)$ which induces an injective map

$$\overline{\varepsilon}\colon M/\mathrm{im}\, \mathrm{d} \to E/\mathrm{im}\, \partial$$

by induction on homological degree as follows: for $i > 0$, we set $E_i = 0$, $\partial_{i+1} = 0$, and $\varepsilon_i = 0$. For $i = 0$, we choose an injective R-module E_0 together with an injective R-linear map $\varepsilon_0\colon M_0 \to E_0$ and we set $\partial_1\colon E_1 \to E_0$ to be the zero map. Suppose for some $k < 0$, we have constructed R-linear maps $\varepsilon_i\colon M_i \to E_i$ and $\partial_{i+1}\colon E_{i+1} \to E_i$ such that

$$\partial_{i-1}\partial_i = 0 \quad \text{and} \quad \partial_{i+1}\tau_{i+1} = \varepsilon_i \mathrm{d}_{i+1}$$

and such that ε_i induces an injective map

$$\overline{\varepsilon_i}\colon M_i/\mathrm{im}\, \mathrm{d}_{i+1} \to E_i/\mathrm{im}\, \partial_{i+1}$$

for all $i > k$. We first construct the pushout

$$\begin{tikzcd}
E_k/\operatorname{im}\partial_{k+1} \arrow[r, "\iota_1"] & \frac{E_k}{\operatorname{im}\partial_{k+1}} +_{\frac{M_k}{\operatorname{im} \mathrm{d}_{k+1}}} M_{k-1} \\
M_k/\operatorname{im}\mathrm{d}_{k+1} \arrow[u, "\overline{\varepsilon_k}"] \arrow[r, "\mathrm{d}_k"] & M_{k-1} \arrow[u, "\iota_2"']
\end{tikzcd}$$

here the map $\overline{\varepsilon_k}$ is well-defined since ε_k commutes with the differentials. Now we choose an injective R-module E_{k-1} together with an injective R-linear map

$$\rho_k \colon \frac{E_k}{\operatorname{im}\partial_{k+1}} +_{\frac{M_k}{\operatorname{im}\mathrm{d}_{k+1}}} M_{k-1} \to E_{k-1}.$$

and we set $\partial_k = \rho_k \circ \iota_1 \circ \pi$ and $\varepsilon_{k-1} = \rho_k \circ \iota_2$. Observe that $\operatorname{im}\partial_k \subset \ker \mathrm{d}_k$ implies $\partial_{k-1} \circ \partial_k = 0$ and observe that

$$\begin{aligned}
\tau_{k-1} \circ \partial_k &= \tau_{k-1} \circ \pi_2 \circ \rho_k \\
&= \mathrm{d}_k \circ \pi_1 \circ \rho_k \\
&= \mathrm{d}_k \circ \tau_k
\end{aligned}$$

implies $\tau_{k-1} \circ \partial_k = \mathrm{d}_k \circ \tau_k$. Finally, observe that $\tau_k \colon \ker \partial_k \to \ker \mathrm{d}_k$ is surjective since it is a composition of surjective maps

$$\ker \partial_k = \ker(\pi_2 \circ \rho_k) \xrightarrow{\rho_k} \ker \pi_2 \xrightarrow[\cong]{\pi_1} \ker \mathrm{d}_k$$

where the isomorphism $\ker \pi_2 \cong \ker \mathrm{d}_k$ follows from Proposition (6.2). This completes the induction step.

Therefore we have an R-complex (P, ∂) together with a chain map $\tau \colon (P, \partial) \to (M, \mathrm{d})$ which restricts to a surjection

$$\tau|_{\ker \partial} \colon \ker \partial \to \ker \mathrm{d}.$$

Moreover, Proposition (6.2) implies

$$\begin{aligned}
\mathrm{H}_{k-1}(M) &= \ker \mathrm{d}_{k-1}/\operatorname{im}\mathrm{d}_k \\
&= \ker \mathrm{d}_{k-1}/\mathrm{d}_k(M_k) \\
&\cong \ker \partial_{k-1}/\operatorname{im}\pi_2 \\
&= \ker \partial_{k-1}/\operatorname{im}\partial_k \\
&= \mathrm{H}_{k-1}(P),
\end{aligned}$$

It follows that τ is a quasi-isomorphism. □

6.1.3 Extra

Let (M, d) be an R-complex. We know wish to show how to construct a projective resolution of (M, d). That is, we will build an R-complex $(P^{-\infty}, \partial^{-\infty})$ together with a quasiisomorphism $\tau^{-\infty} \colon (P^{-\infty}, \partial^{-\infty}) \to (M, \mathrm{d})$. We proceed as follows: for each $n \in \mathbb{Z}$, let (M^n, d^n) be the truncated R-complex where

$$M_i^n = \begin{cases} M_i & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

and where

$$\mathrm{d}_i^0 = \begin{cases} \mathrm{d}_i & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

Next, choose a projective resolution of (M^0, d^0) as in Theorem (6.1), say (P^0, ∂^0). We construct an R-complex (P^{-1}, ∂^{-1}) together with a chain map $\tau^{-1} \colon (P^{-1}, \partial^{-1}) \to (M^{-1}, \mathrm{d}^{-1})$ which restricts to a surjection

$$\tau|_{\ker \partial} \colon \ker \partial \to \ker \mathrm{d}$$

by induction on homological degree as follows: for the base case $i = 0$, we choose a projective R-module P_{-1}^{-1} together with a surjective R-linear map $\tau_{-1}^{-1} \colon P_{-1}^{-1} \to M_{-1}^{-1}$ and we set $\partial_{-1}^{-1} \colon P_{-1} \to 0$ to be the zero map. Suppose for some $k > 0$, we have constructed R-linear maps $\tau_i \colon P_i \to M_i$ and $\partial_i \colon P_i \to P_{i-1}$

6.2 Semiprojective and semiinjective complexes

Definition 6.4. Let P be an R-complex of projective R-modules and let E be an R-complex of injective R-modules.

1. We say P is **semiprojective** if $\mathrm{Hom}_R^\star(P,-)$ respects quasiisomorphisms. If $\tau\colon P \to X$ is a quasiisiomorphism, then we say P is a **semiprojective resolution** of X.

2. We say E is **seminjective** if $\mathrm{Hom}_R^\star(-,E)$ respects quasiisomorphisms. If $\varepsilon\colon X \to E$ is a quasiisiomorphism, then we say E is a **semiinjective resolution** of X.

Proposition 6.3. *Let P be an R-complex of projective modules and let E be an R-complex of injective modules. Then P is semiprojective if and only if $\mathrm{Hom}_R^\star(P,-)$ takes exact complexes to exact complexes. Similarly, E is seminjective if and only if $\mathrm{Hom}_R^\star(-,E)$ takes exact complexes to exact complexes.*

Proof. First suppose that $\mathrm{Hom}_R^\star(P,-)$ is exact. Let $\varphi\colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned}\varphi\colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact}\\ &\implies \mathrm{Hom}_R^\star(P,\mathrm{C}(\varphi)) \text{ is exact}\\ &\implies \mathrm{C}(\mathrm{Hom}_R^\star(P,\varphi)) \text{ is exact}\\ &\implies \mathrm{Hom}_R^\star(P,\varphi) \text{ is a quasiisomorphism.}\end{aligned}$$

Convsersely, suppose P is semiprojective. Let M be an exact R-complex. Then the zero map $M \to 0$ is a quasiisomorphism. Since P is semiprojective, the induced map $\mathrm{Hom}_R^\star(P,M) \to 0$ is a quasiisomorphism. This implies $\mathrm{Hom}_R^\star(P,M)$ is exact. Thus $\mathrm{Hom}_R^\star(P,-)$ is exact. The proof is similar for the injective case. □

6.2.1 Operations on semiprojective R-complexes

Proposition 6.4. *Let P and P' be semiprojective R-complexes.*

1. *ΣP is semiprojective;*

2. *if $\varphi\colon P \to P'$ is a chain map, then $\mathrm{C}(\varphi)$ is semiprojective;*

3. *$P \oplus P'$ is semiprojective;*

4. *if Q is a complex of projective R-modules, then $\mathrm{C}(1_Q)$ is semiprojective.*

5. *$P \otimes_R P'$ is semiprojective.*

Proof. 1. Let M be an exact R-complex. Then

$$\mathrm{Hom}_R^\star(\Sigma P, M) \cong \Sigma^{-1}\mathrm{Hom}_R^\star(P,M)$$

is exact. It follows that ΣP is semiprojective.

2. Let M be an exact R-complex. Observe that the exact sequence

$$0 \longrightarrow P' \xrightarrow{\iota} \mathrm{C}(\varphi) \xrightarrow{\pi} \Sigma P \longrightarrow 0$$

is degreewise split exact. Therefore the sequence

$$0 \longrightarrow \mathrm{Hom}_R^\star(\Sigma P, M) \xrightarrow{\pi^*} \mathrm{Hom}_R^\star(\mathrm{C}(\varphi), M) \xrightarrow{\iota^*} \mathrm{Hom}_R^\star(P,M) \longrightarrow 0$$

is exact. It follows from the fact that both $\mathrm{Hom}_R^\star(\Sigma P, M)$ and $\mathrm{Hom}_R^\star(P',M)$ are exact and from the long exact sequence in homology that $\mathrm{Hom}_R^\star(\mathrm{C}(\varphi),M)$ is exact.

3. This follows from 2 and the fact that

$$P \oplus P' \cong \mathrm{C}(\Sigma^{-1}P \xrightarrow{0} P').$$

4. Let M be an exact R-complex. Then

$$\begin{aligned}\mathrm{Hom}_R^\star(\mathrm{C}(1_Q),M) &\cong \Sigma^{-1}\mathrm{C}(\mathrm{Hom}_R^\star(1_Q,M))\\ &= \Sigma^{-1}\mathrm{C}(1_{\mathrm{Hom}_R^\star(Q,M)})\end{aligned}$$

is exact.

5. By hom tensor adjointness, $\mathrm{Hom}_R(P \otimes_R P', -) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(P', -))$ is a composition of two exact functors.

$\square$

Theorem 6.4. *Every R-complex has a semiprojective resolution and a semiinjective resolution.*

6.2.2 A bounded below complex of projective R-modules is semiprojective

Lemma 6.5. *Let (P, ∂) be a bounded below complex of projective R-modules and let (M, d) be an exact R-complex. Then*

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(P, M)) \cong 0. \tag{51}$$

Proof. By reindexing if necessary, we may assume that $P_i = 0$ for all $i < 0$. Recall that

$$\mathrm{Hom}_R^\star(P, M) = \{\text{homotopy classes of chain maps } \varphi\colon P \to M\}.$$

Thus in order to obtain (51), we need to show that any two chain maps from P to M are homotopic to each other. Let $\varphi\colon P \to M$ and $\psi\colon P \to M$ be any two chain maps. The idea is to build the homotopy $h\colon P \to M$ using induction on $i \geq 0$. The homotopy equation that needs to be satisfied is

$$\varphi - \psi = \mathrm{d}h + h\partial, \tag{52}$$

First, for each $i < 0$, we set $h_i\colon P_i \to M_{i+1}$ to be the zero map. Next we observe that $\mathrm{im}(\varphi_0 - \psi_0) \subseteq \mathrm{im}\,\mathrm{d}_1$. Indeed,

$$\begin{aligned}
\mathrm{d}_0(\varphi_0 - \psi_0) &= \mathrm{d}_0\varphi_0 - \mathrm{d}_0\psi_0 \\
&= \varphi_{-1}\partial_0 - \psi_{-1}\partial_0 \\
&= (\varphi_{-1} - \psi_{-1})\partial_0 \\
&= (\varphi_{-1} - \psi_{-1}) \circ 0 \\
&= 0
\end{aligned}$$

implies

$$\begin{aligned}
\mathrm{im}(\varphi_0 - \psi_0) &\subseteq \ker \mathrm{d}_0 \\
&= \mathrm{im}\,\mathrm{d}_1.
\end{aligned}$$

Thus since P_0 is projective, $\mathrm{d}_1\colon M_1 \to \mathrm{im}\,\mathrm{d}_1$ is surjective, and $\varphi_0 - \psi_0\colon P_0 \to M_0$ lands in $\mathrm{im}\,\mathrm{d}_1$, there exists an R-linear map $h_0\colon P_0 \to P_1$ such that

$$\varphi_0 - \psi_0 = \mathrm{d}_1 h_0. \tag{53}$$

In homological degree $i = 0$, the equation (52) becomes (53). Thus, we are on the right track.

Now we use induction. Suppose for some $i > 0$ we have constructed an R-module homomorphism $h_i\colon P_i \to P_{i+1}$ such that

$$\varphi_i - \psi_i = \mathrm{d}_{i+1}h_i + h_{i-1}\partial_i. \tag{54}$$

Observe that $\mathrm{im}\,(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) \subseteq \mathrm{im}\,\mathrm{d}_{i+2}$. Indeed,

$$\begin{aligned}
\mathrm{d}_{i+1}(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) &= \mathrm{d}_{i+1}\varphi_{i+1} - \mathrm{d}_{i+1}\psi_{i+1} - \mathrm{d}_{i+1}h_i\partial_{i+1} \\
&= \varphi_i\partial_{i+1} - \psi_i\partial_{i+1} - \mathrm{d}_{i+1}h_i\partial_{i+1} \\
&= (\varphi_i - \psi_i - \mathrm{d}_{i+1}h_i)\partial_{i+1} \\
&= h_{i-1}\partial_i\partial_{i+1} \\
&= h_{i-1} \circ 0 \\
&= 0
\end{aligned}$$

implies

$$\begin{aligned}
\mathrm{im}(\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}) &\subseteq \ker \mathrm{d}_{i+1} \\
&= \mathrm{im}\,\mathrm{d}_{i+2}.
\end{aligned}$$

Therefore since P_{i+1} is projective, $\mathrm{d}_{i+2}\colon M_{i+2} \to \mathrm{im}\,\mathrm{d}_{i+2}$ is surjective, and $\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1}\colon P_{i+1} \to M_{i+1}$ lands in $\mathrm{im}\,\mathrm{d}_{i+2}$, there exists an R-linear map $h_{i+1}\colon P_{i+1} \to P_{i+2}$ such that

$$\varphi_{i+1} - \psi_{i+1} - h_i\partial_{i+1} = \mathrm{d}_{i+2}h_{i+1},$$

which is the homotopy equation in degree $i + 1$. $\square$

Corollary. *Let P be a bounded below complex of projective R-modules. Then $\mathrm{Hom}_R^\star(P,-)$ respects exact complexes. In particular, this implies P is semiprojective.*

Proof. Let M be an exact R-complex. Observe that $\Sigma^i P$ is a bounded below complex of projective R-modules for each $i \in \mathbb{Z}$. It follows from Lemma (6.5) that for each $i \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{H}_i(\mathrm{Hom}_R^\star(P,M)) &= \mathrm{H}_{0-(-i)}(\mathrm{Hom}_R^\star(P,M)) \\ &= \mathrm{H}_0(\Sigma^{-i}\mathrm{Hom}_R^\star(P,M)) \\ &= \mathrm{H}_0(\mathrm{Hom}_R^\star(\Sigma^i P,M)) \\ &= 0. \end{aligned}$$

Therefore $\mathrm{Hom}_R^\star(P,-)$ takes exact complexes to exact complexes.

Now we show that this implies $\mathrm{Hom}_R^\star(P,-)$ takes quasiisomorphisms to quasiisomorphisms. Let $\varphi\colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned} \varphi\colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact} \\ &\implies \mathrm{Hom}_R^\star(P,\mathrm{C}(\varphi)) \text{ is exact} \\ &\implies \mathrm{C}(\mathrm{Hom}_R^\star(P,\varphi)) \text{ is exact} \\ &\implies \mathrm{Hom}_R^\star(P,\varphi) \text{ is a quasiisomorphism.} \end{aligned}$$

Therefore P is semiprojective. □

6.2.3 Lifting Lemma

Lemma 6.6. *Let P be a semiprojective R-complex, let $\varphi\colon A \to A'$ be a quasiisomorphism, and let $\psi\colon P \to A'$ be a chain map. Then*

1. *Then there exists a chain map $\widetilde{\psi}\colon P \to A$ such that $\varphi\widetilde{\psi} \sim \psi$. Furthermore, if $\widetilde{\psi}'\colon P \to A$ is another such chain map which satisfies $\varphi\widetilde{\psi}' \sim \psi$, then $\widetilde{\psi} \sim \widetilde{\psi}'$. We call $\widetilde{\psi}$ a **homotopic lift of ψ with respect to φ**.*
2. *If in addition φ is surjective, then there exists a chain map $\widetilde{\psi}\colon P \to A$ such that $\varphi\widetilde{\psi} = \psi$.*

Proof. 1. Since $\mathrm{Hom}_R^\star(P,-)$ preserves quasiisomorphisms, we see that

$$\varphi_*\colon \mathrm{Hom}_R^\star(P,A) \to \mathrm{Hom}_R^\star(P,A')$$

is a quasiisomorphism. In particular, φ_* induces an isomorphism in the degree 0 part of homology:

$$\mathrm{H}_0(\varphi_*)\colon \mathrm{H}_0(\mathrm{Hom}_R^\star(P,A)) \to \mathrm{H}_0(\mathrm{Hom}_R^\star(P,A')).$$

Now ψ represents the the homology class $[\psi]$ in $\mathrm{H}_0(\mathrm{Hom}_R^\star(P,A'))$, and since $\mathrm{H}_0(\varphi_*)$ is an isomorphism, there exists a homology class $[\widetilde{\psi}]$ in $\mathrm{H}_0(\mathrm{Hom}_R^\star(P,A))$ such that

$$\mathrm{H}_0(\varphi_*)[\widetilde{\psi}] = [\psi].$$

In other words, such that $[\varphi\widetilde{\psi}] = [\psi]$. Since

$$\mathrm{H}_0(\mathrm{Hom}_R^\star(P,A')) = \mathcal{C}(A,A')/\sim,$$

we see that $\varphi\widetilde{\psi} \sim \psi$. For the second statement, suppose $\widetilde{\psi}'\colon P \to A$ is another such chain map which satisfies $\varphi\widetilde{\psi}' \sim \psi$. Then $[\widetilde{\psi}'] = [\widetilde{\psi}]$ since $\mathrm{H}_0(\varphi_*)$ is an isomorphism, hence $\widetilde{\psi} \sim \widetilde{\psi}'$.

2. Now suppose that φ is surjective. Choose a homotopic lift of ψ with respect to φ, say $\widetilde{\psi}$. Choose a homotopy from $\varphi\widetilde{\psi}$ to ψ, say $h\colon P \to A'$. So

$$\varphi\widetilde{\psi} - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_P.$$

Using the fact that P is a projective R-module and φ is surjective, we choose a graded lift of h with respect to φ, say $\widetilde{h}\colon P \to A$. So $\widetilde{h}$ is a graded homomorphism of degree 1 such that $\varphi\widetilde{h} = h$. Then note that $\widetilde{\psi} \sim \widetilde{\psi} - \mathrm{d}_A\widetilde{h} - \widetilde{h}\mathrm{d}_P$ and

$$\begin{aligned} \varphi(\widetilde{\psi} - \mathrm{d}_A\widetilde{h} - \widetilde{h}\mathrm{d}_P) &= \varphi\widetilde{\psi} - \varphi\mathrm{d}_A\widetilde{h} - \varphi\widetilde{h}\mathrm{d}_P \\ &= \varphi\widetilde{\psi} - \mathrm{d}_{A'}\varphi\widetilde{h} - \varphi\widetilde{h}\mathrm{d}_P \\ &= \varphi\widetilde{\psi} - \mathrm{d}_{A'}h - h\mathrm{d}_P \\ &= \mathrm{d}_{A'}h + h\mathrm{d}_P + \psi - \mathrm{d}_{A'}h - h\mathrm{d}_P \\ &= \psi. \end{aligned}$$

□

6.3 Ext Functor

Definition 6.5. Let A and B be R-complexes. We define the graded R-module $\mathrm{Ext}_R(A,B)$ as follows: choose a semiprojective resolution $\tau\colon P \to A$. Then

$$\mathrm{Ext}_R(A,B) := \mathrm{H}(\mathrm{Hom}_R^\star(P,B)).$$

The ith homogeneous component of $\mathrm{Ext}_R(A,B)$ is denoted

$$\mathrm{Ext}_R^i(A,B) := \mathrm{H}_{-i}(\mathrm{Hom}_R^\star(P,B))$$

In our definition of $\mathrm{Ext}_R(A,B)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get an isomorphic object. Thus $\mathrm{Ext}_R(A,B)$ is well-defined *up to isomorphism*.

Theorem 6.7. $\mathrm{Ext}_R(A,B)$ *is well-defined up to isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_1\colon P_1 \to P_2$ of τ_1 with respect to τ_2. Similarly, choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1. We claim that $\widetilde{\tau}_1\colon P_1 \to P_2$ is a homotopy equivalence with $\widetilde{\tau}_2\colon P_2 \to P_1$ being its homotopy inverse. Indeed, observe that

$$\begin{aligned} \tau_1\widetilde{\tau}_2\widetilde{\tau}_1 &\sim \tau_2\widetilde{\tau}_1 \\ &\sim \tau_1 \end{aligned}$$

implies $\widetilde{\tau}_2\widetilde{\tau}_1$ is a homotopic lift of τ_1 with respect to τ_1, but 1_{P_1} is also a homotopic lift of τ_1 with respect to τ_1. Therefore $\widetilde{\tau}_2\widetilde{\tau}_1 \sim 1_{P_1}$. A similar computation gives $\widetilde{\tau}_1\widetilde{\tau}_2 \sim 1_{P_2}$. Now $\mathrm{Hom}_R^\star(-,B)$ preserves homotopy equivalences, and thus $\mathrm{Hom}_R^\star(\widetilde{\tau}_1,B)\colon \mathrm{Hom}_R^\star(P_1,B) \to \mathrm{Hom}_R^\star(P_2,B)$ is a homotopy equivalence. Then since the homology functor takes homotopy equivalences to isomorphisms, we see that

$$\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_1,B))\colon \mathrm{H}(\mathrm{Hom}_R^\star(P_1,B)) \to \mathrm{H}(\mathrm{Hom}_R^\star(P_2,B))$$

is an isomorphism. □

6.3.1 The functor $\mathrm{Ext}_R(A,-)$

Now that we've defined the module $\mathrm{Ext}_R(A,B)$, we want to define the covariant functor

$$\mathrm{Ext}_R(A,-)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Clearly, we want this functor to map an R-complex B to the graded R-module $\mathrm{Ext}_R(A,B)$. Let us show how it should act on chain maps:

Definition 6.6. Let $\psi\colon B \to B'$ be a chain map and let $\tau\colon P \to A$ be a semiprojective resolution of A. We define

$$\mathrm{Ext}_R(A,\psi)\colon \mathrm{Ext}_R(A,B) \to \mathrm{Ext}_R(A,B')$$

by $\mathrm{Ext}_R(A,\psi) := \mathrm{H}(\mathrm{Hom}_R^\star(A,\psi))$.

Again, in our definition of $\mathrm{Ext}_R(A,\psi)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get a *naturally isomorphic* functor. Thus the functor $\mathrm{Ext}_R(A,-)$ is well-defined *up to natural isomorphism*.

Theorem 6.8. $\mathrm{Ext}_R(A,-)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1. Then $\widetilde{\tau}_2$ is a homotopy equivalence, by the same argument as in the proof of Theorem (6.10). Now observe that the diagram

$$\begin{tikzcd} \mathrm{Hom}_R^\star(P_1,B) \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B)"] \arrow[d, "\mathrm{Hom}_R^\star(P_1,\psi)"'] & \mathrm{Hom}_R^\star(P_2,B) \arrow[d, "\mathrm{Hom}_R^\star(P_2,\psi)"] \\ \mathrm{Hom}_R^\star(P_1,B') \arrow[r, "\mathrm{Hom}_R^\star(\widetilde{\tau}_2,B')"'] & \mathrm{Hom}_R^\star(P_2,B') \end{tikzcd}$$

is commutative. Therefore we obtain a commutative diagram after apply homology:

$$\begin{tikzcd}
\mathrm{H}(\mathrm{Hom}^\star_R(P_1,B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\tau}_2,B))"] \arrow[d, "\mathrm{H}(\mathrm{Hom}^\star_R(P_1,\psi))"'] & \mathrm{H}(\mathrm{Hom}^\star_R(P_2,B)) \arrow[d, "\mathrm{H}(\mathrm{Hom}^\star_R(P_2,\psi))"] \\
\mathrm{H}(\mathrm{Hom}^\star_R(P_1,B')) \arrow[r, "\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\tau}_2,B'))"'] & \mathrm{H}(\mathrm{Hom}^\star_R(P_2,B'))
\end{tikzcd}$$

Since the rows are isomorphisms, we see that $\mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\tau}_2,-))$ is a natural isomorphism. □

6.3.2 The functor $\mathrm{Ext}_R(-,B)$

Next we want to define the contravariant functor

$$\mathrm{Ext}_R(-,B)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Again, we want this functor to send and an R-complex A to the graded R-module $\mathrm{Ext}_R(A,B)$. This time, the way it acts on chain maps will be a little more involved than in the covariant case.

Definition 6.7. Let $\varphi\colon A \to A'$ be a chain map, let $\tau\colon P \to A$ be a semiprojective resolution of A, let $\tau'\colon P' \to A'$ be a semiprojective resolution of A', and let $\widetilde{\varphi}\colon P \to P'$ be a homotopic lift of $\varphi\tau$ with respect to τ'. We define

$$\mathrm{Ext}_R(\varphi,B)\colon \mathrm{Ext}_R(A',B) \to \mathrm{Ext}_R(A,B).$$

by $\mathrm{Ext}_R(\varphi,B) := \mathrm{H}(\mathrm{Hom}^\star_R(\widetilde{\varphi},B))$.

This time our definition of the functor $\mathrm{Ext}_R(-,B)$ involves *three choices*; namely, the semiprojective resolutions $\tau\colon P \to A$ and $\tau'\colon P' \to A'$ as well as the homotopic lift $\widetilde{\varphi}\colon P \to P'$. Even though we made three choices, we shall still see that $\mathrm{Ext}_R(-,B)$ is well-defined up to natural isomorphism.

Theorem 6.9. *$\mathrm{Ext}_R(-,B)$ is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A, suppose $\tau_1'\colon P_1' \to A'$ and $\tau_2'\colon P_2' \to A'$ are two semiprojective resolutions of A', and suppose $\widetilde{\varphi_1}\colon P_1 \to P_1'$ is a homotopic lift of $\varphi\tau_1$ with respect to τ_1' and $\widetilde{\varphi_2}\colon P_2 \to P_2'$ is a homotopic lift of $\varphi\tau_2$ with respect to τ_2'. So altogether we have the diagrams

$$\begin{tikzcd}
P_1 \arrow[r, "\widetilde{\varphi_1}"] \arrow[d, "\tau_1"'] & P_1' \arrow[d, "\tau_1'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
P_2 \arrow[r, "\widetilde{\varphi_2}"] \arrow[d, "\tau_2"'] & P_2' \arrow[d, "\tau_2'"] \\
A \arrow[r, "\varphi"'] & A'
\end{tikzcd}$$

which commute up to homotopy.

Choose a homotopic lift $\widetilde{\tau}_2\colon P_2 \to P_1$ of τ_2 with respect to τ_1 and choose a homotopic lift $\widetilde{\tau}_2{}'\colon P_2' \to P_1'$ of τ_2' with respect to τ_1'. Then $\widetilde{\tau}_2$ and $\widetilde{\tau}_2{}'$ are both homotopy equivalences by the same argument as in the proof of Theorem (6.10). Now observe that

$$\begin{aligned}
\tau_1'\widetilde{\tau}_2{}'\widetilde{\varphi_2} &\sim \tau_2'\widetilde{\varphi_2} \\
&\sim \varphi\tau_2 \\
&\sim \varphi\tau_1\widetilde{\tau}_2 \\
&\sim \tau_1'\widetilde{\varphi_1}\widetilde{\tau}_2.
\end{aligned}$$

In particular, both $\widetilde{\tau}_2{}'\widetilde{\varphi_2}\colon P_2 \to P_1'$ and $\widetilde{\varphi_1}\widetilde{\tau}_2\colon P_2 \to P_1'$ are homotopic lifts of $\varphi\tau_2$ with respect to τ_1'. Therefore $\widetilde{\tau}_2{}'\widetilde{\varphi_2} \sim \widetilde{\varphi_1}\widetilde{\tau}_2$, which further implies

$$\begin{aligned}
\mathrm{Hom}^\star_R(\widetilde{\varphi_2},B)\mathrm{Hom}^\star_R(\widetilde{\tau}_2{}',B) &= \mathrm{Hom}^\star_R(\widetilde{\tau}_2{}'\widetilde{\varphi_2},B) \\
&\sim \mathrm{Hom}^\star_R(\widetilde{\varphi_1}\widetilde{\tau}_2,B) \\
&= \mathrm{Hom}^\star_R(\widetilde{\tau}_2,B)\mathrm{Hom}^\star_R(\widetilde{\varphi_1},B)
\end{aligned}$$

since $\mathrm{Hom}^\star_R(-,B)$ respects homotopies. Therefore we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}^\star_R(P_1',B) \arrow[r, "\mathrm{Hom}^\star_R(\widetilde{\tau}_2{}',B)"] \arrow[d, "\mathrm{Hom}^\star_R(\widetilde{\varphi_1},B)"'] & \mathrm{Hom}^\star_R(P_2',B) \arrow[d, "\mathrm{Hom}^\star_R(\widetilde{\varphi_2},B)"] \\
\mathrm{Hom}^\star_R(P_1,B') \arrow[r, "\mathrm{Hom}^\star_R(\widetilde{\tau}_2,B)"'] & \mathrm{Hom}^\star_R(P_2,B)
\end{tikzcd}$$

which commutes up to homotopy. Then since homology takes homotopic maps to equal maps, we see that the diagram

$$\begin{tikzcd} \mathrm{H}(\mathrm{Hom}_R^\star(P_1', B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_2', B))"] \arrow[d, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\varphi_1}, B))"'] & \mathrm{H}(\mathrm{Hom}_R^\star(P_2', B)) \arrow[d, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\varphi_2}, B))"] \\ \mathrm{H}(\mathrm{Hom}_R^\star(P_1, B)) \arrow[r, "\mathrm{H}(\mathrm{Hom}_R^\star(\widetilde{\tau}_2, B))"'] & \mathrm{H}(\mathrm{Hom}_R^\star(P_2, B)) \end{tikzcd}$$

is commutative. Since the rows are isomorphisms, we see that $\mathrm{H}(\mathrm{Hom}_R^\star(-, B))$ is a natural isomorphism. □

6.3.3 Properties of Ext

Proposition 6.5. *Let A, B be R-complexes, let $\{A_\lambda\}$ and $\{B_\lambda\}$ be a collection of R-complexes indexed over a set Λ, and let $S \subseteq R$ be a multiplicatively closed set. Then*

1. $\mathrm{Ext}_R\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) \cong \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R\left(A_\lambda, B\right)$;
2. $\mathrm{Ext}_R\left(A, \prod^\star_{\lambda\in\Lambda} B_\lambda\right) \cong \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A, B_\lambda)$
3. *If A is finitely presented, then* $\mathrm{Ext}_R(A, B)_S \cong \mathrm{Ext}_{R_S}(A_S, B_S)$.

Proof. Choose a semiprojective resolutions $\tau_\lambda \colon P_\lambda \to A_\lambda$ of A_λ for each $\lambda \in \Lambda$. Then $\oplus\tau_\lambda \colon \bigoplus_\lambda P_\lambda \to \bigoplus_\lambda A_\lambda$ is a semiprojective resolution of $\bigoplus_\lambda A_\lambda$. Indeed, the homogeneous piece in degree i of $\bigoplus_\lambda P_\lambda$ is given by $\bigoplus_\lambda P_{\lambda,i}$, where $P_{\lambda,i}$ is the homogeneous piece in degree i of P_λ for all $\lambda \in \Lambda$, and $\bigoplus_\lambda P_{\lambda,i}$ is a projective R-module since each $P_{\lambda,i}$ is a projective R-module. Also, $\oplus\tau_\lambda$ is a quasiisomorphism since each τ_λ is a quasiisomorphism and since homology commutes with direct sums.

Therefore

$$\begin{aligned} \mathrm{Ext}_R\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right) &= \mathrm{H}\left(\mathrm{Hom}_R^\star\left(\bigoplus_{\lambda\in\Lambda} A_\lambda, B\right)\right) \\ &= \mathrm{H}\left(\prod^\star_{\lambda\in\Lambda} \mathrm{Hom}_R^\star(A_\lambda, B)\right) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{H}(\mathrm{Hom}_R^\star(A_\lambda, B)) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A_\lambda, B) \end{aligned}$$

Similarly, choose a semiprojective resolution $\tau \colon P \to A$ of A. Then we have

$$\begin{aligned} \mathrm{Ext}_R\left(A, \prod^\star_{\lambda\in\Lambda} B_\lambda\right) &= \mathrm{H}\left(\mathrm{Hom}_R^\star\left(P, \prod^\star_{\lambda\in\Lambda} B_\lambda\right)\right) \\ &= \mathrm{H}\left(\prod^\star_{\lambda\in\Lambda} \mathrm{Hom}_R^\star\left(P, B_\lambda\right)\right) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{H}(\mathrm{Hom}_R^\star(P, B_\lambda)) \\ &= \prod^\star_{\lambda\in\Lambda} \mathrm{Ext}_R(A, B_\lambda). \end{aligned}$$

For the final equality, observe that $\tau_S \colon P_S \to A_S$ is a semiprojective resolution of A_S. Thus

$$\begin{aligned} \mathrm{Ext}_{R_S}(A_S, B_S) &= \mathrm{H}\left(\mathrm{Hom}_{R_S}^\star(P_S, B_S)\right) \\ &= \mathrm{H}\left(\mathrm{Hom}_R^\star(P, B)_S\right) \\ &= \mathrm{H}(\mathrm{Hom}_R^\star(P, B))_S \\ &= \mathrm{Ext}_R(A, B)_S. \end{aligned}$$

□

6.4 Semiflat complexes

Definition 6.8. Let M be an R-complex of flat R-modules. We say M is **semiflat** if $- \otimes_R M$ respects quasiisomorphisms. If $\tau \colon M \to X$ is a quasiisomorphism, then we say M is a **semiflat resolution** of X.

Remark. Since $- \otimes_R M$ is naturally isomorphic to $M \otimes_R -$, we see that M is semiflat if and only if $M \otimes_R -$ respects quasiisomorphisms.

Proposition 6.6. *Let M be an R-complex of flat R-modules. Then M is semiflat if and only if $M \otimes_R -$ is exact.*

Proof. First suppose that $- \otimes_R M$ is exact. Let $\varphi \colon A \to A'$ be a quasiisomorphism. Then

$$\begin{aligned} \varphi \colon A \to A' \text{ is a quasiisomorphism} &\implies \mathrm{C}(\varphi) \text{ is exact} \\ &\implies \mathrm{C}(\varphi) \otimes_R M \text{ is exact} \\ &\implies \mathrm{C}(\varphi \otimes_R M) \text{ is exact} \\ &\implies \varphi \otimes_R M \text{ is a quasiisomorphism.} \end{aligned}$$

Therefore $- \otimes_R M$ respects quasiisomorphisms.

Conversely, suppose M is semiflat. Let A be an exact R-complex. Then the zero map $M \to 0$ is a quasiisomorphism. Since M is semiflat, the induced map $A \otimes_R M \to 0$ is a quasiisomorphism. This implies $A \otimes_R M$ is exact. Therefore $- \otimes_R M$ is exact. □

6.4.1 Semiprojective complexes are semiflat

Proposition 6.7. *Let P be a semiprojective R-complex. Then P is semiflat.*

Proof. Since projective R-modules are flat, we see that P_i is flat for all $i \in \mathbb{Z}$. Now let A be an exact R-complex and let $\varepsilon \colon P \otimes_R A \to E$ be a semiinjective resolution. Then

$$\begin{aligned} P \otimes_R A \text{ is exact} &\iff \mathrm{Hom}_R^\star(P \otimes_R A, E) \text{ is exact} \\ &\iff \mathrm{Hom}_R^\star(P, \mathrm{Hom}_R^\star(A, E)) \text{ is exact.} \end{aligned}$$

the last line follows from the fact that P is semiprojective and E is semiinjective. □

6.5 Tor Functor

Definition 6.9. Let A and B be R-complexes. We define the graded R-module $\mathrm{Tor}^R(A, B)$ as follows: choose a semiprojective resolution $\tau \colon P \to A$. Then

$$\mathrm{Tor}^R(A, B) := \mathrm{H}(P \otimes_R B).$$

The ith homogeneous component of $\mathrm{Tor}^R(A, B)$ is denoted

$$\mathrm{Tor}_i^R(A, B) := \mathrm{H}_i(P \otimes_R B)$$

In our definition of $\mathrm{Tor}^R(A, B)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get an isomorphic object. Thus $\mathrm{Tor}^R(A, B)$ is well-defined *up to isomorphism*.

Theorem 6.10. $\mathrm{Tor}^R(A, B)$ *is well-defined up to isomorphism.*

Proof. Suppose $\tau_1 \colon P_1 \to A$ and $\tau_2 \colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau}_1 \colon P_1 \to P_2$ of τ_1 with respect to τ_2. Similarly, choose a homotopic lift $\widetilde{\tau}_2 \colon P_2 \to P_1$ of τ_2 with respect to τ_1. As in the proof of Theorem (6.10), $\widetilde{\tau}_1 \colon P_1 \to P_2$ is a homotopy equivalence with $\widetilde{\tau}_2 \colon P_2 \to P_1$ being its homotopy inverse. Now $- \otimes_R B$ preserves homotopy equivalences, and thus $\widetilde{\tau}_1 \otimes_R B \colon P_1 \otimes_R B \to P_2 \otimes_R B$ is a homotopy equivalence. Then since the homology functor takes homotopy equivalences to isomorphisms, we see that

$$\mathrm{H}(\widetilde{\tau}_1 \otimes_R B)) \colon \mathrm{H}(P_1 \otimes_R B) \to \mathrm{H}(P_2 \otimes_R B)$$

is an isomorphism. This isomorphism is unique in a sense. Indeed, if we had chosen another homotopic lift of τ_1 with respect to τ_2, say $\widetilde{\tau}_1' \colon P_1 \to P_2$, then $\widetilde{\tau}_1 \sim \widetilde{\tau}_1{}'$, which implies $\widetilde{\tau}_1 \otimes_R B \sim \widetilde{\tau}_1{}' \otimes_R B$, which implies $\mathrm{H}(\widetilde{\tau}_1 \otimes_R B)) = \mathrm{H}(\widetilde{\tau}_1{}' \otimes_R B))$. □

6.5.1 The functor $\mathrm{Tor}^R(A,-)$

Now that we've defined the module $\mathrm{Tor}^R(A,B)$, we want to define the covariant functor

$$\mathrm{Tor}^R(A,-)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Clearly, we want this functor to map an R-complex B to the graded R-module $\mathrm{Tor}^R(A,B)$. Let us show how it should act on chain maps:

Definition 6.10. Let $\psi\colon B \to B'$ be a chain map and let $\tau\colon P \to A$ be a semiprojective resolution of A. We define

$$\mathrm{Tor}^R(A,\psi)\colon \mathrm{Tor}^R(A,B) \to \mathrm{Tor}^R(A,B')$$

by $\mathrm{Tor}^R(A,\psi) := \mathrm{H}(A \otimes_R \psi)$.

Again, in our definition of $\mathrm{Tor}^R(A,\psi)$, we *chose* a semiprojective resolution of A. Let us now show that had we chosen a different semiprojective resolution of A, we would get a *naturally isomorphic* functor. Thus the functor $\mathrm{Tor}^R(A,-)$ is well-defined *up to natural isomorphism*.

Theorem 6.11. $\mathrm{Tor}^R(A,-)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A. Choose a homotopic lift $\widetilde{\tau_1}\colon P_1 \to P_2$ of τ_1 with respect to τ_2. Then $\widetilde{\tau_1}$ is a homotopy equivalence, by the same argument as in the proof of Theorem (6.10). Now observe that the diagram

$$\begin{tikzcd}
P_1 \otimes_R B \arrow[r, "\widetilde{\tau_1}\otimes_R B"] \arrow[d, "P_1\otimes_R\psi"'] & P_2 \otimes_R B \arrow[d, "P_2\otimes_R\psi"] \\
P_1 \otimes_R B' \arrow[r, "\widetilde{\tau_2}\otimes_R B'"'] & P_2 \otimes_R B'
\end{tikzcd}$$

is commutative where the rows are homotopy equivalences since $-\otimes_R B$ preserves homotopy equivalences. Therefore we obtain a commutative diagram after apply homology

$$\begin{tikzcd}
\mathrm{H}(P_1 \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau_1}\otimes_R B)"] \arrow[d, "\mathrm{H}(P_1\otimes_R\psi)"'] & \mathrm{H}(P_2 \otimes_R B) \arrow[d, "\mathrm{H}(P_2\otimes_R\psi)"] \\
\mathrm{H}(P_1 \otimes_R B') \arrow[r, "\mathrm{H}(\widetilde{\tau_2}\otimes_R B')"'] & \mathrm{H}(P_2 \otimes_R B')
\end{tikzcd}$$

where the rows are isomorphisms since the $\mathrm{H}(-)$ takes homotopy equivalences to isomorphisms. Since the rows are isomorphisms and the diagram commutes, we see that $\mathrm{H}(\mathrm{Tor}^R(\widetilde{\tau_1},-))$ is a natural isomorphism. $\square$

6.5.2 The functor $\mathrm{Tor}^R(-,B)$

Next we want to define the covariant functor

$$\mathrm{Tor}^R(-,B)\colon \mathbf{Comp}_R \to \mathbf{Grad}_R.$$

Again, we want this functor to send and an R-complex A to the graded R-module $\mathrm{Tor}^R(A,B)$.

Definition 6.11. Let $\varphi\colon A \to A'$ be a chain map, let $\tau\colon P \to A$ be a semiprojective resolution of A, let $\tau'\colon P' \to A'$ be a semiprojective resolution of A', and let $\widetilde{\varphi}\colon P \to P'$ be a homotopic lift of $\varphi\tau$ with respect to τ'. We define

$$\mathrm{Tor}^R(\varphi,B)\colon \mathrm{Tor}^R(A,B) \to \mathrm{Tor}^R(A',B).$$

by $\mathrm{Tor}^R(\varphi,B) := \mathrm{H}(\widetilde{\varphi}\otimes_R B)$.

This time our definition of the functor $\mathrm{Tor}^R(-,B)$ involves *three choices*; namely, the semiprojective resolutions $\tau\colon P \to A$ and $\tau'\colon P' \to A'$ as well as the homotopic lift $\widetilde{\varphi}\colon P \to P'$. Even though we made three choices, we shall still see that $\mathrm{Tor}^R(-,B)$ is well-defined up to natural isomorphism.

Theorem 6.12. $\mathrm{Tor}^R(-,B)$ *is well-defined up to natural isomorphism.*

Proof. Suppose $\tau_1\colon P_1 \to A$ and $\tau_2\colon P_2 \to A$ are two semiprojective resolutions of A, suppose $\tau_1'\colon P_1' \to A'$ and $\tau_2'\colon P_2' \to A'$ are two semiprojective resolutions of A', and suppose $\widetilde{\varphi_1}\colon P_1 \to P_1'$ is a homotopic lift of $\varphi\tau_1$ with respect to τ_1' and $\widetilde{\varphi_2}\colon P_2 \to P_2'$ is a homotopic lift of $\varphi\tau_2$ with respect to τ_2'. So altogether we have the diagrams

$$\begin{tikzcd} P_1 \arrow[r, "\widetilde{\varphi_1}"] \arrow[d, "\tau_1"'] & P_1' \arrow[d, "\tau_1'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd} \qquad \begin{tikzcd} P_2 \arrow[r, "\widetilde{\varphi_2}"] \arrow[d, "\tau_2"'] & P_2' \arrow[d, "\tau_2'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd}$$

which commute up to homotopy.

Choose a homotopic lift $\widetilde{\tau}_1 \colon P_1 \to P_2$ of τ_1 with respect to τ_2 and choose a homotopic lift $\widetilde{\tau}_1{}' \colon P_1' \to P_2'$ of τ_1' with respect to τ_2'. Then $\widetilde{\tau}_1$ and $\widetilde{\tau}_1{}'$ are both homotopy equivalences by the same argument as in the proof of Theorem (6.10). Now observe that

$$\begin{aligned} \tau_2'\widetilde{\varphi_2}\widetilde{\tau}_1 &\sim \varphi\tau_2\widetilde{\tau}_1 \\ &\sim \varphi\tau_1 \\ &\sim \tau_1'\widetilde{\varphi_1} \\ &\sim \tau_2'\widetilde{\tau}_1{}'\widetilde{\varphi_1}. \end{aligned}$$

In particular, both $\widetilde{\varphi_2}\widetilde{\tau}_1 \colon P_1 \to P_2'$ and $\widetilde{\tau}_1{}'\widetilde{\varphi_1} \colon P_1 \to P_2'$ are homotopic lifts of $\varphi\tau_1$ with respect to τ_2'. Therefore

$$\widetilde{\varphi_2}\widetilde{\tau}_1 \sim \widetilde{\tau}_1{}'\widetilde{\varphi_1},$$

and since $- \otimes_R B$ respects homotopies, we have a diagram

$$\begin{tikzcd} P_1 \otimes_R B \arrow[r, "\widetilde{\tau}_1 \otimes_R B"] \arrow[d, "\widetilde{\varphi_1} \otimes_R B"'] & P_2 \otimes_R B \arrow[d, "\widetilde{\varphi_2} \otimes_R B"] \\ P_1' \otimes_R B \arrow[r, "\widetilde{\tau}_1{}' \otimes_R B"] & P_2' \otimes_R B \end{tikzcd}$$

which commutes up to homotopy. Finally, since $\mathrm{H}(-)$ takes homotopic maps to equal maps, we see that the diagram

$$\begin{tikzcd} \mathrm{H}(P_1 \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau}_1 \otimes_R B)"] \arrow[d, "\mathrm{H}(\widetilde{\varphi_1} \otimes_R B)"'] & \mathrm{H}(P_2 \otimes_R B) \arrow[d, "\mathrm{H}(\widetilde{\varphi_2} \otimes_R B)"] \\ \mathrm{H}(P_1' \otimes_R B) \arrow[r, "\mathrm{H}(\widetilde{\tau}_1{}' \otimes_R B)"] & \mathrm{H}(P_2' \otimes_R B) \end{tikzcd}$$

which is commutative. Since $\mathrm{H}(-)$ takes homotopy equivalences to isomorphisms, we see that the rows are isomorphisms, and thus $\mathrm{H}(\mathrm{Hom}_R^\star(-, B))$ is a natural isomorphism.

□

6.5.3 Balance of Tor

Proposition 6.8. *Let A and B be R-complexes and let $\sigma \colon P \to A$ and $\tau \colon Q \to B$ be semiprojective resolutions. Then*

$$\mathrm{Tor}^R(A, B) \cong \mathrm{H}(P \otimes_R Q) \cong \mathrm{H}(A \otimes_R Q).$$

Proof. Observe that $P \otimes_R -$ respects quasiisomorphisms since P is semiprojective (and hence semiflat). Therefore $P \otimes_R \tau \colon P \otimes_R Q \to P \otimes_R B$ is a quasiisomoprhism. Thus

$$\mathrm{H}(P \otimes_R \tau) \colon \mathrm{H}(P \otimes_R Q) \to \mathrm{H}(P \otimes_R B)$$

is an isomorphism. Similarly, $- \otimes_R Q$ respects quasiisomorphisms since Q is semiprojective (and hence semiflat). Therefore $\sigma \otimes_R Q \colon P \otimes_R Q \to A \otimes_R Q$ is a quasiisomorphism. Thus

$$\mathrm{H}(\sigma \otimes_R Q) \colon \mathrm{H}(P \otimes_R Q) \to \mathrm{H}(A \otimes_R Q)$$

is an isomorphism. Therefore we have balance of Tor:

$$\begin{aligned} \mathrm{Tor}^R(A, B) &= \mathrm{H}(P \otimes_R B) \\ &\cong \mathrm{H}(P \otimes_R Q) \\ &\cong \mathrm{H}(A \otimes_R Q). \end{aligned}$$

□

6.5.4 Commutativity of Tor

Proposition 6.9. *Let A and B be R-complexes. Then we have an isomorphism of graded R-modules*

$$\mathrm{Tor}^R(A,B) \cong \mathrm{Tor}^R(B,A),$$

which is natural in A and B.

Proof. Let $\sigma\colon P \to A$ be a semiprojective resolution of A and let $\tau\colon Q \to B$ be a semiprojective resolutions of B. We have

$$\begin{aligned}\mathrm{Tor}^R(A,B) &= \mathrm{H}(P \otimes_R B)\\ &\cong \mathrm{H}(P \otimes_R Q)\\ &\cong \mathrm{H}(Q \otimes_R P)\\ &\cong \mathrm{H}(Q \otimes_R A)\\ &= \mathrm{Tor}^R(B,A).\end{aligned}$$

□

6.6 Functors from $\mathbf{Comp}_R$ to $\mathbf{HComp}_R$ and $\mathbf{HComp}_R$ to $\mathbf{HComp}_R$

6.6.1 Semiprojective Version

For every R-complex A we fix a semiprojective resolution $\mathrm{P}_R(A) \xrightarrow{\tau_A} A$ and for every chain map $\varphi\colon A \to B$ we fix a homotopic lift $\mathrm{P}_R(\varphi)\colon \mathrm{P}_R(A) \to \mathrm{P}_R(B)$ of $\varphi\tau_A$ with respect to τ_B. If the ring R is clear from context, then we write $\mathrm{P}(A)$ and $\mathrm{P}(\varphi)$ rather than $\mathrm{P}_R(A)$ and $\mathrm{P}_R(\varphi)$ in order to simplify notation.

Proposition 6.10. *We obtain a well-defined R-linear covariant functor $\mathbb{P}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{P}(A)$ and which takes a chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{P}(\varphi)]$.*

Proof. The well-definedness comes from the fact that we used fixed resolutions and lifts. The functor $\mathbb{P}$ respects identity maps. Indeed, given the identity morphism $1_A\colon A \to A$, we have $\tau_A 1_{\mathrm{P}(A)} = 1_A\tau_A$. In particular, $1_{\mathrm{P}(A)}$ is a homotopic lift of $1_A\tau_A$ with respect to τ_A. Thus $\mathrm{P}(1_A) \sim 1_{\mathrm{P}(A)}$, and thus $[\mathrm{P}(1_A)] = [1_{\mathrm{P}(A)}]$. The functor $\mathbb{P}$ also respects compositions. Indeed, let $\varphi\colon A \to B$ and $\psi\colon B \to C$ be two chain maps. Then

$$\begin{aligned}\tau_C \mathrm{P}(\psi)\mathrm{P}(\varphi) &\sim \psi\tau_B\mathrm{P}(\varphi)\\ &\sim \psi\varphi\tau_A.\end{aligned}$$

Thus $\mathrm{P}(\psi)\mathrm{P}(\varphi)$ is a homotopic lift of $\psi\varphi\tau_A$ with respect to τ_C. Since $\mathrm{P}(\psi\varphi)$ is also a homotopic lift of $\psi\varphi\tau_A$ with respect to τ_C, it follows that $\mathrm{P}(\psi\varphi) \sim \mathrm{P}(\psi)\mathrm{P}(\varphi)$, and thus $[\mathrm{P}(\psi\varphi)] = [\mathrm{P}(\psi)][\mathrm{P}(\varphi)]$.

Now we show that $\mathbb{P}$ is an R-linear functor. Let A and B be R-complexes. We want to show that if $\varphi, \psi \in \mathcal{C}(A,B)$ and $r, s \in R$ then

$$[\mathrm{P}(r\varphi + s\psi)] = [r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)]. \tag{55}$$

To see this, note that $\mathrm{P}(\varphi)$ is a homotopic lift of $\varphi\tau_A$ with respect to τ_B and $\mathrm{P}(\psi)$ is a homotopic lift of $\psi\tau_A$ with respect to τ_B. Now observe that

$$\begin{aligned}\tau_B(r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)) &= r\tau_B\mathrm{P}(\varphi) + s\tau_B\mathrm{P}(\psi)\\ &\sim r\varphi\tau_A + s\psi\tau_A\\ &= (r\varphi + s\psi)\tau_A.\end{aligned}$$

Thus $r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)$ is a homotopic lift of $(r\varphi + s\psi)\tau_A$ with respect to τ_B. Since $\mathrm{P}(r\varphi + s\psi)$ is another homotopic lift of $(r\varphi + s\psi)\tau_A$ with respect to τ_B, it follows that $\mathrm{P}(r\varphi + s\psi) \sim r\mathrm{P}(\varphi) + s\mathrm{P}(\psi)$. In other words, we have (55). □

Definition 6.12. Define $\Omega_R\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ to be functor which sends the R-complex A to the R-complex A and which takes a chain map $\varphi\colon A \to B$ to the homotopy class $[\varphi]$.

Remark. If the ring R is clear from context, then we write Ω rather than Ω_R in order to simplify notation.

Proposition 6.11. *The functor Ω is a well-defined R-linear covariant functor. Moreover it transforms homotopy equivalences to isomorphisms. Furthermore, Ω satisfies the following universal mapping property: for every R-linear covariant functor $F\colon \mathbf{Comp}_R \to \mathcal{C}$ which takes homotopic maps to equal maps, there exists a unique R-linear functor $\widetilde{F}\colon \mathbf{HComp}_R \to \mathcal{C}$ such that $\widetilde{F}\Omega = F$.*

Proof. The first part of the propositions is straightforward. Let us address the universal mapping property. Given such an $F\colon \mathbf{Comp}_R \to \mathcal{C}$, we define $\widetilde{F}\colon \mathbf{HComp}_R \to \mathcal{C}$ to be the functor which takes an R-complex A to the object $F(A)$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$ to the morphism $F(\varphi)\colon F(A) \to F(B)$. Observe that this is well-defined by assumption of F (it takes homotopic chain maps to equal maps). Let us show that $\widetilde{F}$ is a functor. First we check that it respects identity maps. Let $[1_A]$ be the homotopy class of the identity map $1_A\colon A \to A$. Then

$$\begin{aligned}\widetilde{F}[1_A] &= F(1_A) \\ &= 1_{F(A)}.\end{aligned}$$

Thus $\widetilde{F}$ respects identity maps. Next let's check that it respects compositions. Let $[\varphi]$ and $[\psi]$ be the homotopy classes of the chain maps $\varphi\colon A \to B$ and $\psi\colon B \to C$ respectively. Then

$$\begin{aligned}\widetilde{F}[\psi\varphi] &= F(\psi\varphi) \\ &= F(\psi)F(\varphi) \\ &= \widetilde{F}[\psi]\widetilde{F}[\varphi].\end{aligned}$$

Thus $\widetilde{F}$ respects compositions. Now let us check that $\widetilde{F}\Omega = F$. For any R-complex A, we have

$$\begin{aligned}\widetilde{F}\Omega(A) &= \widetilde{F}(A) \\ &= F(A)\end{aligned}$$

and for any chain map $\varphi\colon A \to B$, we have

$$\begin{aligned}\widetilde{F}\Omega(\varphi) &= \widetilde{F}[\mathrm{P}(\varphi)] \\ &= F(\varphi).\end{aligned}$$

Therefore $\widetilde{F}\Omega = F$. Finally, note that uniqueness of $\widetilde{F}$ follows from the fact that we were forced to define $\widetilde{F}$ in this way. Indeed, if $\widetilde{F}'$ was another such functor, then for any R-complex A, we have

$$\begin{aligned}\widetilde{F}'(A) &= \widetilde{F}'\Omega(A) \\ &= F(A) \\ &= \widetilde{F}\Omega(A) \\ &= \widetilde{F}(A),\end{aligned}$$

and for any chain map $\varphi\colon A \to B$, we have

$$\begin{aligned}\widetilde{F}'[\varphi] &= \widetilde{F}'\Omega(\varphi) \\ &= F(\varphi) \\ &= \widetilde{F}\Omega(\varphi) \\ &= \widetilde{F}[\varphi].\end{aligned}$$

□

Remark. One should view Ω as some sort of "localization" functor. Indeed, recall that if S is a multilpicatively closed subset of a commutative ring A and $\rho_S\colon A \to A_S$ is the canonical localization map, then the pair (A_S, ρ_S) satisfies the following universal mapping property: for every ring homomorphism $\varphi\colon A \to B$ such that $\varphi(S) \subseteq B^\times$, there exists a unique ring homomorphism $\widetilde{\varphi}\colon A_S \to B$ such that $\widetilde{\varphi}\rho_S = \varphi$.

Theorem 6.13. *Let $\widetilde{\mathbb{P}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ be the functor which takes an R-complex A to the R-complex $\mathrm{P}(A)$ and which takes a homotopy class $[\varphi]$ of the chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{P}(\varphi)]$ of the chain map $\mathrm{P}(\varphi)\colon \mathrm{P}(A) \to \mathrm{P}(B)$. Then $\widetilde{\mathbb{P}}$ is a well-defined R-linear functor.*

Proof. Note that $\mathbb{P}$ takes homotopic chain maps to equal maps. Thus we may apply Proposition (6.11) to $\mathbb{P}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ (where $\mathcal{C} = \mathbf{HComp}_R$) to get $\widetilde{\mathbb{P}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$. □

6.6.2 Semiinjective Version

For every R-complex A we fix a semiinjective resolution $A \xrightarrow{\varepsilon_A} \mathrm{E}_R(A)$ and for every chain map $\varphi\colon A \to B$ we fix a homotopic lift $\mathrm{E}_R(\varphi)\colon \mathrm{E}_R(A) \to \mathrm{E}_R(B)$ of $\varepsilon_B\varphi$ with respect to ε_A. If the ring R is clear from context, then we write $\mathrm{E}(A)$ and $\mathrm{E}(\varphi)$ rather than $\mathrm{E}_R(A)$ and $\mathrm{E}_R(\varphi)$ in order to simplify notation.

Just like in the semiprojective case, we will denote we obtain a well-defined R-linear covariant functor $\mathbb{E}\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{E}(A)$ and which takes a chain map $\varphi\colon A \to B$ to to the homotopy class $[\mathrm{E}(\varphi)]$ of the chain map $\mathrm{E}(\varphi)\colon \mathrm{E}(A) \to \mathrm{E}(B)$. Similarly, we obtain a well-defined R-linear covariant functor $\widetilde{\mathbb{E}}\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ which takes an R-complex A to the R-complex $\mathrm{E}(A)$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$ to the homotopy class $[\mathrm{E}(\varphi)]$ of the chain map $\mathrm{E}(\varphi)\colon \mathrm{E}(A) \to \mathrm{E}(B)$.

6.6.3 Covariant Hom

Theorem 6.14. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}_R^\star(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}_R^\star(A,\varphi)]$ *of the chain map* $\mathrm{Hom}_R^\star(A,\varphi)\colon \mathrm{Hom}_R^\star(A,B) \to \mathrm{Hom}_R^\star(A,B')$.

2. $\widetilde{\mathbb{H}\mathrm{om}}_R^\star(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}_R^\star(A,\varphi)]$ *of the chain map* $\mathrm{Hom}_R^\star(A,\varphi)\colon \mathrm{Hom}_R^\star(A,B) \to \mathrm{Hom}_R^\star(A,B')$.

Proof. 1. Observe that $\mathbb{H}\mathrm{om}_R^\star(A,-) = \Omega\mathrm{Hom}_R^\star(A,-)$. The composition of two R-linear covariant functors is a well-defined R-linear covariant functor.

2. Observe that $\mathbb{H}\mathrm{om}_R^\star(A,-)$ takes homotopic maps to equal maps. Indeed, if $\varphi\colon B \to B'$ and $\psi\colon B \to B'$ are two chain maps such that $\varphi \sim \psi$, then $\mathrm{Hom}_R^\star(A,\varphi) \sim \mathrm{Hom}_R^\star(A,\psi)$. Therefore $[\mathrm{Hom}_R^\star(A,\varphi)] = [\mathrm{Hom}_R^\star(A,\psi)]$. Thus we may apply the universal mapping property in Proposition (6.11) to $\mathbb{H}\mathrm{om}_R^\star(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ (where $\mathcal{C} = \mathbf{HComp}_R$) to get $\widetilde{\mathbb{H}\mathrm{om}}_R^\star(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$. □

6.6.4 Contravariant Hom

Theorem 6.15. *Let B be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}_R^\star(-,B)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a chain map $\varphi\colon A \to A'$ to the homotopy class* $[\mathrm{Hom}_R^\star(\varphi,B)]$ *of the chain map* $\mathrm{Hom}_R^\star(\varphi,B)\colon \mathrm{Hom}_R^\star(A',B) \to \mathrm{Hom}_R^\star(A,B)$.

2. $\widetilde{\mathbb{H}\mathrm{om}}_R^\star(-,B)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to A'$ to the homotopy class* $[\mathrm{Hom}_R^\star(\varphi,B)]$ *of the chain map* $\mathrm{Hom}_R^\star(\varphi,B)\colon \mathrm{Hom}_R^\star(A,B) \to \mathrm{Hom}_R^\star(A,B')$.

Proof. Proof is similar to the proof of Theorem (6.18). □

6.6.5 Tensor Product

Theorem 6.16. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $A\underline{\otimes}_R-\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex $A \otimes_R B$ and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class $[A \otimes_R \varphi]$ of the chain map $A \otimes_R \varphi\colon A \otimes_R B \to A \otimes_R B'$.*

2. $A\widetilde{\underline{\otimes}}_R-\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex $A \otimes_R B$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class $[A \otimes_R \varphi]$ of the chain map $A \otimes_R \varphi\colon A \otimes_R B \to A \otimes_R B'$.*

Theorem 6.17. *Let B be an R-complex. Then the following are well-defined R-linear functors*

1. $-\underline{\otimes}_R B\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex $A \otimes_R B$ and which takes a chain map $\varphi\colon A \to A'$ to the homotopy class $[\varphi \otimes_R A]$ of the chain map $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$.*

2. $-\widetilde{\underline{\otimes}}_R B\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex A to the R-complex $A \otimes_R B$ and which takes the homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to A'$ to the homotopy class $[\varphi \otimes_R B]$ of the chain map $\varphi \otimes_R B\colon A \otimes_R B \to A' \otimes_R B$.*

Remark. (commutativity) Let A be an R-complex. Then $A\underline{\otimes}_R-$ is naturally isomorphic to $-\underline{\otimes}_R A$. Indeed, we have

$$\begin{aligned} A\underline{\otimes}_R- &= \Omega(A\otimes_R -) \\ &\cong \Omega(-\otimes_R A) \\ &= -\underline{\otimes}_R A, \end{aligned}$$

where the isomorphism at the second line is natural (as shown earlier). Note that this also implies $A\widetilde{\underline{\otimes}}_R-$ is naturally isomorphic to $-\widetilde{\underline{\otimes}}_R A$.

6.6.6 Natural Transformation of Functors

Proposition 6.12. *Let A be an R-complex. The natural chain maps*

$$\mathrm{P}(A) \xrightarrow[\simeq]{\tau_A} A \xrightarrow[\simeq]{\varepsilon_A} \mathrm{E}(A)$$

induce the following natural transformations

1. $\mathbb{P} \xrightarrow{[\tau]} \Omega \xrightarrow{[\varepsilon]} \mathbb{E}$ *of functors from* $\mathbf{Comp}_R$ *to* $\mathbf{HComp}_R$.
2. $\widetilde{\mathbb{P}} \xrightarrow{[\tau]} \mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$ *of functors from* $\mathbf{HComp}_R$ *to* $\mathbf{HComp}_R$.

Proof. We focus $\Omega \xrightarrow{[\varepsilon]} \mathbb{E}$ and $\mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$ since the proof that the other maps are natural transformations is a similar argument. We first consider $\Omega \xrightarrow{[\varepsilon]} \mathbb{E}$. We need to check that for every chain map $\varphi\colon A \to B$, the following diagram commutes in $\mathbf{HComp}_R$:

$$\begin{tikzcd} A \arrow[r, "{[\varepsilon_A]}"] \arrow[d, "{[\varphi]}"'] & \mathrm{E}(A) \arrow[d, "{[\mathrm{E}(\varphi)]}"] \\ B \arrow[r, "{[\varepsilon_B]}"'] & \mathrm{E}(B) \end{tikzcd}$$

This is clear however since $\mathrm{E}(\varphi)$ is a homotopic lift of $\varepsilon_B\varphi$ with respect to ε_A. Thus $\varepsilon_B\varphi \sim \mathrm{E}(\varphi)\varepsilon_A$, which implies

$$\begin{aligned} [\varepsilon_B][\varphi] &= [\varepsilon_B\varphi] \\ &= [\mathrm{E}(\varphi)\varepsilon_A] \\ &= [\mathrm{E}(\varphi)][\varepsilon_A]. \end{aligned}$$

Now we consider $\mathrm{id} \xrightarrow{[\varepsilon]} \widetilde{\mathbb{E}}$. We need to check that for every homotopy class $[\varphi]$ of a chain map $\varphi\colon A \to B$, the following diagram commutes in $\mathbf{HComp}_R$:

$$\begin{tikzcd} A \arrow[r, "{[\varepsilon_A]}"] \arrow[d, "{[\varphi]}"'] & \mathrm{E}(A) \arrow[d, "{[\mathrm{E}(\varphi)]}"] \\ B \arrow[r, "{[\varepsilon_B]}"'] & \mathrm{E}(B) \end{tikzcd}$$

This was done above. □

Theorem 6.18. *Let A be an R-complex. Then the following are well-defined R-linear functors*

1. $\mathbb{H}\mathrm{om}_R^\star(A,-)\colon \mathbf{Comp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}_R^\star(A,\varphi)]$ *of the chain map* $\mathrm{Hom}_R^\star(A,\varphi)\colon \mathrm{Hom}_R^\star(A,B) \to \mathrm{Hom}_R^\star(A,B')$.
2. $\widetilde{\mathbb{H}}\mathrm{om}_R^\star(A,-)\colon \mathbf{HComp}_R \to \mathbf{HComp}_R$ *which takes an R-complex B to the R-complex* $\mathrm{Hom}_R^\star(A,B)$ *and which takes a homotopy class $[\varphi]$ of a chain map $\varphi\colon B \to B'$ to the homotopy class* $[\mathrm{Hom}_R^\star(A,\varphi)]$ *of the chain map* $\mathrm{Hom}_R^\star(A,\varphi)\colon \mathrm{Hom}_R^\star(A,B) \to \mathrm{Hom}_R^\star(A,B')$.

6.7 Triangulated Categories

Exact sequences are useful for studying modules and complexes, but these are poorly behaved in $\mathbf{HComp}_R$. For instance, the natural chain $0 \xrightarrow{\simeq} \mathcal{K}(1)$ is a quasiisomorphism between semiprojective complexes and so thus must be a homotopy equivalence. Thus $\mathcal{K}(1)$ is isomorphic to 0 in the $\mathbf{HComp}_R$. Now the 0 complex fits into a really silly exact sequence, namely $0 \to 0 \to 0$, but it is not clear whether the sequence $0 \to \mathcal{K}(1) \to 0$ should be exact. To solve this, Grothendieck and Verdier introduced the notion of a **triangulated category**, where instead of considering exact sequences, one considers **distinguished triangles**.

6.7.1 Shift Functors, Triangles, and Morphisms of Triangles

Definition 6.13. Let $\mathcal{C}$ be an R-linear category.

1. A **shift functor** (or **translation functor**) on $\mathcal{C}$ is an R-linear functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ with a 2-sided inverse $\Sigma^{-1}\colon \mathcal{C} \to \mathcal{C}$. Sometimes ΣA will be denoted $A[1]$. More generally, $\Sigma^n A = A[n]$. Note that $\Sigma^0 = 1_{\mathcal{C}}$.

2. A **triangle** in $\mathcal{C}$ is a diagram in $\mathcal{C}$ of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} \Sigma A \tag{56}$$

   of morphisms in $\mathcal{C}$. Sometimes we call these **pretriangles** or **candidate triangles**. We shall use the shorthand notation $(A, B, C)_{(\alpha,\beta,\gamma)}$ to denote the triangle in (56).

3. A **morphism** of triangles in $\mathcal{C}$ is a commutative diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r, "\gamma"] \arrow[d, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & \Sigma A' \end{tikzcd} \tag{57}$$

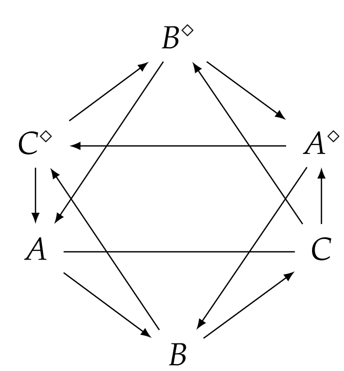
   Such a morphism is called an **isomorphism** if f, g, h are all isomorphisms, that is, the morphism has a 2-sided inverse. We shall use shorthand notation $(f, g, h)\colon (A, B, C)_{(\alpha,\beta,\gamma)} \to (A', B', C')_{(\alpha',\beta',\gamma')}$ to denote the morphism of triangles in (58).

6.7.2 Triangulated Categories

Definition 6.14. A **triangulated R-linear category** is an R-linear category $\mathcal{C}$ equipped with a shift functor Σ and a class of triangles called **distinguished triangles** (or **exact triangles**) such that the following axioms are satisfied.

1. For all objects A in $\mathcal{C}$, the triangle $A \xrightarrow{1_A} A \to 0 \to \Sigma A$ is distinguished.
2. For every morphism $\alpha\colon A \to B$, there exists a distinguished triangle $(A, B, C)_{(\alpha,-,-)}$ (where the $-$ means we aren't specifying that morphism). In this case we call C a **cone of** α (or a **cofiber** of α).
3. Given an isomorphism of triangles $(f, g, h)\colon (A, B, C)_{(\alpha,\beta,\gamma)} \to (A', B', C')_{(\alpha',\beta',\gamma')}$, then $(A, B, C)_{(\alpha,\beta,\gamma)}$ is distinguished if and only if $(A', B', C')_{(\alpha',\beta',\gamma')}$ is distinguished.
4. Given a distinguished triangle $(A, B, C)_{(\alpha,\beta,\gamma)}$, the following **rotated triangles**, $(B, C, \Sigma A)_{(\beta,\gamma,-\Sigma\alpha)}$ and $(\Sigma^{-1}C, A, B)_{(-\Sigma^{-1}\gamma,\alpha,\beta)}$, are both distinguished.
5. Given a diagram in $\mathcal{C}$,
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & B \arrow[r, "\beta"] \arrow[d, "g"] & C \arrow[r, "\gamma"] \arrow[d, dashed, "h"] & \Sigma A \arrow[d, "\Sigma f"] \\ A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r, "\gamma'"] & \Sigma A' \end{tikzcd} \tag{58}$$
where the top and bottom rows are distinguished triangles, then there exists a morphism $h\colon C \to C'$ making diagram commutative.

6. (Octahedral axiom) Start with morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathcal{C}$ and fix distinguished triangles $(A, B, C^\diamond)_{(\alpha,\beta,\gamma_\diamond)}$, $(B, C, A^\diamond)_{(\beta,\gamma^\diamond,\alpha_\diamond)}$, and $(A, C, B^\diamond)_{(\beta\alpha,\widetilde{\beta}_\diamond,\widetilde{\alpha}^\diamond)}$. Then there exists a distinguished triangle $(C^\diamond, B^\diamond, A^\diamond)_{(\widetilde{\beta},\widetilde{\alpha},\widetilde{\gamma})}$ which is compatible with the input data in the following sense
$$\begin{aligned} \gamma^\diamond &= \widetilde{\alpha}\widetilde{\beta}_\diamond \\ \gamma_\diamond &= \widetilde{\alpha}^\diamond\widetilde{\beta} \\ \widetilde{\gamma} &= (\Sigma\beta^\diamond)\alpha_\diamond \\ \alpha_\diamond\widetilde{\alpha} &= (\Sigma\alpha)\widetilde{\alpha}^\diamond \\ \widetilde{\beta}\beta^\diamond &= \widetilde{\beta}_\diamond\beta \end{aligned}$$
We can visualize this axiom via the following diagram



Note that the octahedral axiom is very technical, but it can be interpreted in terms of the third isomorphiss theorem, pullbacks, pushouts, fiber products, and fiber coproducts.

6.7.3 Homotopy Category is a Triangulated Category

Theorem 6.19. $\mathbf{HComp}_R$ *is a triangulated R-linear category, where a triangle is distinguished if and only if it is isomorphic to one of the form $(A, B, \mathrm{C}(\varphi))_{([\varphi],[\iota],[\pi])}$, where $\iota\colon B \to \mathrm{C}(\varphi)$ and $\pi\colon \mathrm{C}(\varphi) \to \Sigma A$ are the natural inclusion and projection maps respectively.*

Proof. Partial proof of TR1: The identity triangle $(A, A, 0)_{([1_A],[0],[0])}$ is distinguished since

$$\begin{array}{ccccccc}
A & \xrightarrow{[1_A]} & A & \xrightarrow{[0]} & 0 & \xrightarrow{[0]} & \Sigma A \\
\downarrow{\scriptstyle [1_A]} & & \downarrow{\scriptstyle [1_A]} & & \downarrow{\scriptstyle [0]} & & \downarrow{\scriptstyle [0]} \\
A & \xrightarrow{[1_A]} & A & \xrightarrow{[\iota]} & \mathrm{C}(A) & \xrightarrow{[\pi]} & \Sigma A
\end{array}$$

is an isomorphism. The only thing to check is that the middle part of the diagram is commutative, that is $[\iota][1_A] = [0][0]$. This is equivalent to ι being null-homotopic, which is clear. □

7 Special Complexes

7.1 Taylor Resolution

Throughout this subsection, let $\underline{m} = m_1, \dots, m_r$ be monomials in $R = K[x_1, \dots, x_n]$. For each subset σ of $\{1, \dots, r\}$ we set $m_\sigma := \operatorname{lcm}(m_\lambda \mid \lambda \in \sigma)$. Let $a_\sigma \in \mathbb{N}^n$ be the exponent vector of m_σ and let $R(-a_\sigma)$ be the free R-module with one generator in multidegree a_σ. The **Taylor resolution** of $R/\langle \underline{m} \rangle$ is the R-complex $(\mathcal{T}(\underline{m}), d^{\mathcal{T}(\underline{m})})$ whose graded R-module $\mathcal{T}(\underline{m})$ has

$$\mathcal{T}_i(\underline{m}) := \begin{cases} \bigoplus_{\sigma \in S_i[n]} Re_\sigma & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{T}(\underline{m})}$ is uniquely determined by

$$d^{\mathcal{T}(\underline{m})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma \backslash \lambda}} e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq [n]$.

Remark. We need to check that the differential defined above really is a differential. Denote $d := d^{\mathcal{T}(\underline{m})}$ and let $\sigma \subseteq [n]$. Then

$$\begin{aligned}
d^2(e_\sigma) &= d(d(e_\sigma)) \\
&= d\left(\sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma \backslash \lambda}} e_{\sigma \backslash \lambda}\right) \\
&= \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma \backslash \lambda}} d(e_{\sigma \backslash \lambda}) \\
&= \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma \backslash \lambda}} \sum_{\mu \in \sigma \backslash \lambda} \langle \mu, \sigma \backslash \{\lambda, \mu\} \rangle \frac{m_{\sigma \backslash \lambda}}{m_{\sigma \backslash \{\lambda, \mu\}}} d(e_{\sigma \backslash \{\lambda, \mu\}}) \\
&= \sum_{\substack{\lambda, \mu \in \sigma \\ \lambda \neq \mu}} \langle \lambda, \sigma \backslash \lambda \rangle \langle \mu, \sigma \backslash \{\lambda, \mu\} \rangle \frac{m_\sigma}{m_{\sigma \backslash \{\lambda, \mu\}}} d(e_{\sigma \backslash \{\lambda, \mu\}}) \\
&= 0,
\end{aligned}$$

where the last part follows from symmetry in μ and λ and

$$\begin{aligned}
\langle \lambda, \sigma \backslash \lambda \rangle \langle \mu, \sigma \backslash \{\lambda, \mu\} \rangle &= \langle \lambda, \sigma \backslash \lambda \rangle \langle \mu, \sigma \backslash \{\lambda, \mu\} \rangle \\
&= \langle \lambda, \sigma \backslash \lambda \rangle \langle \mu, \sigma \backslash \lambda \rangle \langle \mu, \lambda \rangle \\
&= -\langle \lambda, \sigma \backslash \lambda \rangle \langle \lambda, \mu \rangle \langle \mu, \sigma \backslash \lambda \rangle \\
&= -\langle \lambda, \sigma \backslash \{\mu, \lambda\} \rangle \langle \mu, \sigma \backslash \lambda \rangle \\
&= -\langle \mu, \sigma \backslash \mu \rangle \langle \lambda, \sigma \backslash \{\mu, \lambda\} \rangle.
\end{aligned}$$

7.1.1 Taylor Resolution as $\mathbb{N}^n$-Graded k-Algebra

The Taylor resolution has an extra graded structure present which is not necessarily shared by the Koszul complex. The underlying graded R-module $\mathcal{T}(\underline{m})$ has an $\mathbb{N}^n$-graded K-module structure. Indeed, for $\mathbf{b} \in \mathbb{N}^n$, the $\mathbf{b}$th homogeneous component of is given by

$$\mathcal{T}_{\mathbf{b}}(\underline{m}) = \bigoplus_{m_\sigma | \mathbf{x}^{\mathbf{b}}} K \cdot \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma.$$

Moreover, the differential is an $\mathbb{N}^n$-graded K-endormorphism (of degree 0): For any $\sigma \subseteq [n]$ such that $m_\sigma | \mathbf{x}^{\mathbf{b}}$, we have

$$\begin{aligned} d^{\mathcal{T}(\underline{m})}\left(\frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma\right) &= \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} d^{\mathcal{T}(\underline{m})}(e_\sigma) \\ &= \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{m_\sigma}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda} \\ &= \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{\mathbf{x}^{\mathbf{b}}}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda} \\ &\in \mathcal{T}_{\mathbf{b}}(\underline{m}). \end{aligned}$$

In particular, $\ker(d^{\mathcal{T}(\underline{m})})$ and $\mathrm{im}(d^{\mathcal{T}(\underline{m})})$ have induced $\mathbb{N}^n$-graded K-module structures and hence $\mathrm{H}(\mathcal{T}(\underline{m}))$ has an induced $\mathbb{N}^n$-graded K-module structure: For $\mathbf{b} \in \mathbb{N}^n$, the $\mathbf{b}$th homogeneous component of $\mathrm{H}(\mathcal{T}(\underline{m}))$ is

Proposition 7.1. *The Taylor complex is a free resolution of R/I.*

Proof. It suffices to show $\mathrm{H}_{\mathbf{b}}(\mathcal{T}(\underline{m})) \cong 0$ for all $\mathbf{b} \in \mathbb{N}^n \backslash \{0\}$. Observe that the simplicial complex

$$\Delta[\mathbf{x}^{\mathbf{b}}] := \{\sigma \subseteq [n] \mid m_\sigma \text{ divides } \mathbf{x}^{\mathbf{b}}\}$$

□

$$\mathrm{H}_{\mathbf{b}}(\mathcal{T}(\underline{m})) = \frac{\ker_{\mathbf{b}}(d^{\mathcal{T}(\underline{m})})}{\mathrm{im}_{\mathbf{b}}(d^{\mathcal{T}(\underline{m})})}.$$

7.1.2 The K-Complex in Degree b

Let $\mathbf{b} \in \mathbb{N}^n$. The complex $(\mathcal{T}_{\mathbf{b}}(\underline{m}), d^{\mathcal{T}_{\mathbf{b}}(\underline{m})})$ is the K-complex whose underlying graded K-module has

$$\mathcal{T}_{i,\mathbf{b}}(\underline{m}) = \bigoplus_{\substack{m_\sigma | \mathbf{x}^{\mathbf{b}} \\ \sigma \in S_i(n)}} K \cdot \frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma$$

as its ith homogeneous and whose differential is the unique differential such that

$$d^{\mathcal{T}_{\mathbf{b}}(\underline{m})}\left(\frac{\mathbf{x}^{\mathbf{b}}}{m_\sigma} e_\sigma\right) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle \frac{\mathbf{x}^{\mathbf{b}}}{m_{\sigma\backslash\lambda}} e_{\sigma\backslash\lambda}.$$

7.1.3 Taylor Complex is a Free Resolution

In this section, we want to show that the Taylor complex defined above is a free resolution of R/I. We do this by induction on n. The case $n = 1$ is trivial. A

7.1.4 Taylor Complex as a DG Algebra

Proposition 7.2. *Let $I = \langle m_1, \ldots, m_r \rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$. The Taylor resolution $(\mathcal{T}(\underline{m}), d^{\mathcal{T}(\underline{m})})$ is a DG algebra, with multiplication being uniquely determined on elementary tensors: for $\sigma, \tau \subseteq [n]$, we map $e_\sigma \otimes e_\tau \mapsto e_\sigma e_\tau$, where*

$$e_\sigma e_\tau = \begin{cases} \langle \sigma, \tau \rangle \frac{m_\sigma m_\tau}{m_{\sigma\cup\tau}} e_{\sigma\cup\tau} & \text{if } \sigma \cap \tau = \varnothing \\ 0 & \text{if } \sigma \cap \tau \neq \varnothing \end{cases} \tag{59}$$

Proof. Throughout this proof, denote $d := d^{\mathcal{T}(\underline{m})}$. We first note that $e_\varnothing$ serves as the identity for the multiplication rule (**??**). Indeed, let $\sigma \subseteq [n]$. Then since $\sigma \cap \varnothing = \varnothing$, we have

$$e_\sigma e_\varnothing = e_\sigma = e_\varnothing e_\sigma.$$

Moreover, multiplication by $e_\varnothing$ and e_σ given in (**??**) satisfies Leibniz law:

$$\begin{aligned} d(e_\sigma)e_\varnothing - e_\sigma d(e_\varnothing) &= d(e_\sigma)e_\varnothing \\ &= d(e_\sigma) \\ &= d(e_\sigma e_\varnothing), \end{aligned}$$

and similarly

$$\begin{aligned} d(e_\emptyset)e_\sigma + e_\emptyset d(e_\sigma) &= e_\emptyset d(e_\sigma) \\ &= d(e_\sigma) \\ &= d(e_\emptyset e_\sigma), \end{aligned}$$

Next, let $\lambda \in [n]$. Suppose $\tau \subseteq [n]$ and $\lambda \notin \tau$. Then

$$\begin{aligned} d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= m_\lambda e_\tau - e_\lambda \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \frac{m_\tau}{m_{\tau\backslash\mu}} e_{\tau\backslash\mu} \\ &= m_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \frac{m_\tau}{m_{\tau\backslash\mu}} e_\lambda e_{\tau\backslash\mu} \\ &= m_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \langle \lambda, \tau\backslash\mu\rangle \frac{m_\tau}{m_{\tau\backslash\mu}} \frac{m_\lambda m_{\tau\backslash\mu}}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= m_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \langle \lambda, \tau\rangle \langle \lambda, \mu\rangle \frac{m_\lambda m_\tau}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= m_\lambda e_\tau + \sum_{\mu\in\tau} \langle \lambda, \tau\rangle \langle \mu, \tau\backslash\mu\rangle \langle \mu, \lambda\rangle \frac{m_\lambda m_\tau}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= m_\lambda e_\tau + \sum_{\mu\in\tau} \langle \lambda, \tau\rangle \langle \mu, \tau\backslash\mu\cup\lambda\rangle \frac{m_\lambda m_\tau}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= \langle \lambda, \tau\rangle \left(\langle \lambda, \tau\rangle m_\lambda e_\tau + \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\cup\lambda\rangle \frac{m_\lambda m_\tau}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \right) \\ &= \langle \lambda, \tau\rangle \sum_{\mu\in\tau\cup\lambda} \langle \mu, \tau\backslash\mu\cup\lambda\rangle \frac{m_\lambda m_\tau}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= \langle \lambda, \tau\rangle \frac{m_\lambda m_\tau}{m_{\tau\cup\lambda}} \sum_{\mu\in\tau\cup\lambda} \langle \mu, \tau\backslash\mu\cup\lambda\rangle \frac{m_{\tau\cup\lambda}}{m_{\tau\backslash\mu\cup\lambda}} e_{\tau\backslash\mu\cup\lambda} \\ &= \langle \lambda, \tau\rangle \frac{m_\lambda m_\tau}{m_{\tau\cup\lambda}} d(e_{\tau\cup\lambda}) \\ &= d(e_\lambda e_\tau), \end{aligned}$$

Next suppose $\tau \subseteq [n]$ and $\lambda \in \tau$. Then

$$\begin{aligned} d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= m_\lambda e_\tau - e_\lambda \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \frac{m_\tau}{m_{\tau\backslash\mu}} e_{\tau\backslash\mu} \\ &= m_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \frac{m_\tau}{m_{\tau\backslash\mu}} e_\lambda e_{\tau\backslash\mu} \\ &= m_\lambda e_\tau - \langle \lambda, \tau\backslash\lambda\rangle \langle \lambda, \tau\backslash\lambda\rangle \frac{m_\tau}{m_{\tau\backslash\lambda}} \frac{m_\lambda m_{\tau\backslash\lambda}}{m_\tau} e_\tau \\ &= m_\lambda e_\tau - m_\lambda e_\tau \\ &= 0 \\ &= d(0) \\ &= d(e_\lambda e_\tau). \end{aligned}$$

Thus we have shown (**??**) satisfies the Leibniz law for all pairs (λ, τ) where $\lambda \in [n]$ and $\tau \subseteq [n]$. We prove by induction on $|\sigma| = i \geq 1$ that (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$. The base case $i = 1$ was just shown. Now suppose we have shown (**??**) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$

such that $|\sigma| = i < n$. Let $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i+1$. Choose $\lambda \in \sigma$. Then

$$\begin{aligned}
\frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} d(e_\sigma e_\tau) &= d\left(\frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} e_\sigma e_\tau\right) \\
&= d(e_\lambda e_{\sigma\backslash\lambda} e_\tau) \\
&= m_\lambda e_{\sigma\backslash\lambda} e_\tau - e_\lambda d(e_{\sigma\backslash\lambda} e_\tau) \\
&= m_\lambda e_{\sigma\backslash\lambda} e_\tau - e_\lambda (d(e_{\sigma\backslash\lambda}) e_\tau + (-1)^{|\sigma|-1} e_{\sigma\backslash\lambda} d(e_\tau)) \\
&= (m_\lambda e_{\sigma\backslash\lambda} - e_\lambda d(e_{\sigma\backslash\lambda})) e_\tau + (-1)^{|\sigma|} \frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} e_\sigma d(e_\tau) \\
&= d(e_\lambda e_{\sigma\backslash\lambda}) e_\tau + (-1)^{|\sigma|} \frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} e_\sigma d(e_\tau) \\
&= d\left(\frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} e_\sigma\right) e_\tau + (-1)^{|\sigma|+1} \frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} e_\sigma d(e_\tau), \\
&= \frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma} \left(d(e_\sigma) e_\tau + (-1)^{|\sigma|+1} e_\sigma d(e_\tau)\right)
\end{aligned}$$

where we used the base case on the pairs $(e_\lambda, e_{\sigma\backslash\lambda} e_\tau)$[7] and $(e_\lambda, e_{\sigma\backslash\lambda})$ and where we used the induction hypothesis on the pair $(e_{\sigma\backslash\lambda}, e_\tau)$. and where we used the base case on the pair $(e_\lambda, e_{\sigma\backslash\lambda})$. Canceling $\frac{m_\lambda m_{\sigma\backslash\lambda}}{m_\sigma}$ on both sides completes the proof. □

Lemma 7.1. *(DG Algebra Criterion) Let (A, d) be an R-complex such that A is an associative and unital graded R-algebra. Let G be a set of generators for the graded R-algebra A. Suppose the Leibniz law is true for all pairs (a, b) where $a, b \in G$ such that $\deg(a) = 1$. Further suppose that each $a \in G$ is divisible by some $a_1 \in G$ such that $\deg(a_1) = 1$. Then (A, d) is a DG algebra.*

Proof. It suffices to check that the Leibniz law holds for all pairs (a, b) where $a, b \in G$. Indeed, if $x \in A_k$ and $y \in A_l$ and

$$x = \sum_i r_i a_i \quad \text{and} \quad y = \sum_j s_j b_j,$$

then

$$\begin{aligned}
d(xy) &= d\left(\sum r_i a_i \sum s_j b_j\right) \\
&= \sum\sum r_i s_j d(a_i b_j) \\
&= \sum\sum r_i s_j (d(a_i) b_j + (-1)^{\deg(a_i)} a_i d(b_j)) \\
&= \sum\sum r_i s_j d(a_i) b_j + \sum\sum r_i s_j (-1)^{\deg(a_i)} a_i d(b_j)) \\
&= d\left(\sum r_i a_i\right) \sum s_j b_j + (-1)^{\deg(x)} \sum r_i a_i d\left(\sum s_j b_j\right) \\
&= d(x) y + (-1)^{\deg(x)} x d(y).
\end{aligned}$$

First observe that the Leibniz law is satisfied for all pairs $(1, a)$ where $1 \in A$ is the identity and $a \in A$. Indeed, we have

$$\begin{aligned}
d(1)a + 1d(a) &= 0 \cdot a + 1 \cdot d(a) \\
&= d(a) \\
&= d(1 \cdot a).
\end{aligned}$$

Similarly, the Leibniz law is satisfied for all pairs $(a, 1)$ where $1 \in A$ is the identity and $a \in A$. Indeed, we have

$$\begin{aligned}
d(a) \cdot 1 + (-1)^{\deg(a)} a d(1) &= d(a) + (-1)^{\deg(a)} a \cdot 0 \\
&= d(a) \\
&= d(a \cdot 1).
\end{aligned}$$

Now we want to show that the Leibniz law holds for all pairs (a, b) where $a, b \in A$ such that $\deg(a) \geq 1$ by using induction on $\deg(a)$. The base case ($\deg(a) = 1$) is the assumption in the lemma. Now assume that the Leibniz law is satisfied for all pairs (a, b) where $\deg(a) = i \geq 1$. Let $a, b \in A$ such that $\deg(a) = i+1$. Choose $a_1 \in A_1$

[7] If $e_{\sigma\backslash\lambda} e_\tau = 0$, then obviously Leibniz law holds for the pair $(e_\lambda, e_{\sigma\backslash\lambda} e_\tau)$.

such that $a_1|a$. Then $a = a_1 a_i$, for some $a_i \in A_i$. Then

$$\begin{aligned} d(ab) &= d(a_1 a_i b) \\ &= d(a_1)a_i b - a_1 d(a_i b) \\ &= d(a_1)a_i b - a_1(d(a_i)b + (-1)^i a_i d(b)) \\ &= d(a_1)a_i b - a_1 d(a_i)b + (-1)^{i+1} a_1 a_i d(b)) \\ &= (d(a_1)a_i - a_1 d(a_i))b + (-1)^{i+1} a_1 a_i d(b)) \\ &= d(a_1 a_i))b + (-1)^{i+1} a_1 a_i d(b)), \\ &= d(a)b + (-1)^{i+1} a d(b). \end{aligned}$$

□

7.1.5 Taylor Complex is a Free Resolution

In this section, we want to show that the Taylor complex defined above is a free resolution of R/I. We do this by induction on r. The case $r = 1$ being trivial. Let $\underline{m}' = m_2, \dots, m_r$. By induction, $\mathcal{T}(\underline{m})$ is a free resolution of $R/\langle \underline{m}' \rangle$.

7.2 Generalizing Taylor Complex

Let R and S be rings such that $R \subset S$. Let (A, d) be an S-complex. Suppose A is an $\mathbb{N}^n$-graded R-module and d is homogeneous with respect to the $\mathbb{N}^n$-grading. Then for each $\boldsymbol{\alpha} \in \mathbb{N}^n$ we obtain an R-complex $(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}})$ whose graded R-module in degree i is $A_{i,\boldsymbol{\alpha}} := A_i \cap A_{\boldsymbol{\alpha}}$ and whose differential $\mathrm{d}_{\boldsymbol{\alpha}} := \mathrm{d}|_{A_{\boldsymbol{\alpha}}}$ is the restriction of d to A_α. Moreover, we have

$$\begin{aligned} \mathrm{H}(A, \mathrm{d}) &:= \ker \mathrm{d} / \operatorname{im} \mathrm{d} \\ &= \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \ker \mathrm{d}_{\boldsymbol{\alpha}} \right) / \left(\bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \operatorname{im} \mathrm{d}_{\boldsymbol{\alpha}} \right) \\ &\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \ker \mathrm{d}_{\boldsymbol{\alpha}} / \operatorname{im} \mathrm{d}_{\boldsymbol{\alpha}} \\ &:= \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \mathrm{H}(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}}) \\ &\cong \bigoplus_{\boldsymbol{\alpha} \in \mathbb{N}^n} \bigoplus_{i \in \mathbb{Z}} \mathrm{H}_{i,\boldsymbol{\alpha}}(A_{\boldsymbol{\alpha}}, \mathrm{d}_{\boldsymbol{\alpha}}.) \end{aligned}$$

8 Some Category Theory

8.1 Preadditive and Additive Categories

8.1.1 Preadditive Categories

Definition 8.1. A category $\mathcal{A}$ is called **preadditive** if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(y, z) \times \mathrm{Mor}(x, y) \to \mathrm{Mor}(x, z)$$

are bilinear. A functor $F\colon \mathcal{A} \to \mathcal{B}$ of preadditive categories is called **additive** if and only if

$$F\colon \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$$

is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

Remark. In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 8.1. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent:*

1. *x is an initial object;*
2. *x is a final object;*
3. $\mathrm{id}_x = 0$ *in* $\mathrm{Mor}(x, x)$.

Definition 8.2. In a preadditive category $\mathcal{A}$, we call **zero object**, and denote it by 0 any final and initial object as in the Lemma above.

Lemma 8.2. *Let $\mathcal{A}$ be a preadditive category and let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \amalg y$. If the coproduct $x \amalg y$ exists, then so does the product $x \times y$. In this case also $x \amalg y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p\colon z \to x$ and $q\colon z \to y$. Denote $i\colon x \to z$ the morphism corresponding to $(1, 0)$. Denote $j\colon y \to z$ the morphism corresponding to $(0, 1)$. Thus we have a commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "1"] \arrow[dr, "i"] & & x \\
& z \arrow[ur, "p"] \arrow[dr, "q"] & \\
y \arrow[ur, "j"] \arrow[rr, "1"] & & y
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q\colon z \to z$ is the identity since it is a morphism which upon composing p gives p and upon composing q gives q. Suppose given morphisms $a\colon x \to w$ and $b\colon y \to w$. Then we can form the map $a \circ p + b \circ q\colon z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \amalg y$. $\square$

Definition 8.3. Given a pair of objects x, y in a preadditive categore $\mathcal{A}$, the **direct sum** $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma (8.2).

Lemma 8.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be preadditive categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. A direct sum z of x and y is characterized by having morphisms $i\colon x \to z$, $j\colon y \to z$, $p\colon z \to x$, and $q\colon z \to y$ such that $p \circ i = 1_x$, $q \circ j = 1_y$, $p \circ j = 0$, $q \circ i = 0$, and $i \circ p + j \circ q = 1_z$. Clearly $F(x)$, $F(y)$, $F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums. $\square$

8.1.2 Additive Category

Definition 8.4. A category $\mathcal{A}$ is called **additive** if it is preadditive and finite products exist. In other words, it has a zero object and direct sums.

Definition 8.5. Let $\mathcal{A}$ be a preadditive category and let $f\colon x \to y$ be a morphism.

1. A **kernel** of f is an equalizer of $f\colon x \to y$ and $0\colon x \to y$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* kernel of f and denote it by $\iota\colon \ker f \to x$. Thus we have $f\iota = 0$ and if $\iota'\colon z \to x$ is an other morphism such that $f\iota' = 0$, then there exists a unique morphism $g\colon z \to \ker f$ such that $\iota' = \iota g$.

2. A **cokernel** of f is a coequalizer of $f\colon x \to y$ and $0\colon x \to y$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* cokernel of f and denote it by $\pi\colon y \to \operatorname{coker} f$. Thus we have $\pi f = 0$ and if $\pi'\colon y \to z$ is an other morphism such that $\pi' f = 0$, then there exists a unique morphism $g\colon \operatorname{coker} f \to z$ such that $\pi' = g\pi$.

3. If a kernel of f exists, then a **coimage** of f is a cokernel of the morphism $\ker f \to x$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* coimage of f and denote it by $x \to \operatorname{coim} f$.

4. If a cokernel of f exists, then a **image** of f is a kernel of the morphism $y \to \operatorname{coker} f$. If it exists, then it is unique up to unique isomorphism. With this in mind, we talk about *the* image of f and denote it by $\operatorname{im} f \to y$.

Lemma 8.4. *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma (8.2) be a direct sum in $\mathcal{C}$. Then $i\colon x \to x \oplus y$ is a kernel of $q\colon x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f\colon z' \to x \oplus y$ be a morphism such that $qf = 0$. We have to show taht there exists a unique morphism $g\colon z' \to x$ such that $f = ig$. SInce $ip + jq$ is the identity on $x \oplus y$ we see that

$$\begin{aligned} f &= (ip + jq)f \\ &= ipf \end{aligned}$$

and hence $g = pf$ works. Uniqueness holds because pi is the idenity on x. The proof of the second statement is dual. □

Lemma 8.5. *Let $\mathcal{C}$ be a preadditive category. Let $f\colon x \to y$ be a morphism in $\mathcal{C}$.*

1. *If a kernel of f exists, then this kernel is a monomorphism.*
2. *If a cokernel of f exists, then this cokernel is an epimorphism.*
3. *If a kernel and coimage of f exist, then the coimage is an epimorphism.*
4. *If a cokernel and image of f exist, then the image is a monomorphism.*

Lemma 8.6. *Let $f\colon x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image, and coimage all exist. Then f can be factored uniquely as*

$$x \to \operatorname{coim} f \to \operatorname{im} f \to y.$$

Proof. There is a canonical morphism $\operatorname{coim} f \to y$ because $\ker f \to x \to y$ is zero. The composition $\operatorname{coim} f \to y \to \operatorname{coker} f$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \operatorname{coker} f$ which is zero. Hence $\operatorname{coim} f \to y$ factors uniquely through $\operatorname{im} f \to y$, which gives us the desired map. □

8.2 Abelian Category

An abelian category is a category satisfying just enough axioms so the snake lemma holds.

Definition 8.6. A category $\mathcal{A}$ is called **abelian** if

1. it is additive;
2. all kernels and cokernels exist;
3. the natural map $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism for all morphisms f in $\mathcal{A}$.

Definition 8.7. Let $f\colon x \to y$ be a morphism in an abelian category.

1. We say f is **injective** if $\ker f = 0$.
2. We say f is **surjective** if $\operatorname{coker} f = 0$.
3. If $x \to y$ is injective, then we say that x is a **subobject** of y and we use the notation $x \subseteq y$ to denote this. If $x \to y$ is surjective, then we say y is a **quotient** of x.

Lemma 8.7. *Let $f\colon x \to y$ be a morphism in an abelian category $\mathcal{A}$. Then*

1. *f is injective if and only if f is a monomorphism.*
2. *f is surjective if and only if f is an epimorphism.*

Lemma 8.8. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

8.3 R-Linear Categories

Definition 8.8. An **R-linear category** $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition law

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \times \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{A}}(x,z)$$

is R-bilinear. Thus composition determines an R-linear map

$$\mathrm{Hom}_{\mathcal{A}}(y,z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{A}}(x,z)$$

of R-modules. A functor $F\colon \mathcal{A} \to \mathcal{B}$ of R-linear categories is called **R-linear** if the map

$$F\colon \mathrm{Hom}_{\mathcal{A}}(x,y) \to \mathrm{Hom}_{\mathcal{A}}(F(x),F(y))$$

is an R-linear map.

Example 8.1. The category Mod_R of all R-modules and R-linear maps is an R-linear category. Indeed, for each R-module M and N, we have an R-module $\mathrm{Hom}_R(M,N)$. Composition

$$\mathrm{Hom}_R(M_2,M_3) \times \mathrm{Hom}_R(M_1,M_2) \to \mathrm{Hom}_R(M_1,M_3),$$

defined by $(\varphi_2, \varphi_1) \mapsto \varphi_2 \circ \varphi_1$, is easily checked to be R-bilinear.

8.3.1 Additive functor from Graded Modules Induces Functor on Complexes

Proposition 8.1. *Let $\mathcal{F}\colon \mathrm{Grad}_R \to \mathrm{Grad}_R$ be an additive functor. Then $\mathcal{F}$ induces a functor*

$$\mathcal{F}\colon \mathrm{Comp}_R \to \mathrm{Comp}_R,$$

where an R-complex (A, d) gets mapped to the R-complex $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$.

Proof. Let (A, d) be an R-complex. We first need to show that $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$ is an R-complex. Indeed, $\mathcal{F}(A)$ is a graded R-module and $\mathcal{F}(\mathrm{d})$ is a graded homomorphism of degree -1. Moreover,

$$\begin{aligned}\mathcal{F}(\mathrm{d})\mathcal{F}(\mathrm{d}) &= \mathcal{F}(\mathrm{dd}) \\ &= \mathcal{F}(0) \\ &= 0.\end{aligned}$$

Thus $(\mathcal{F}(A), \mathcal{F}(\mathrm{d}))$ is an R-complex.

Next, let $\varphi\colon A \to A'$ be a chain map of R-complexes. Then

$$\begin{aligned}\mathcal{F}(\varphi)\mathcal{F}(\mathrm{d}) &= \mathcal{F}(\varphi\mathrm{d}) \\ &= \mathcal{F}(\mathrm{d}\varphi) \\ &= \mathcal{F}(\mathrm{d})\mathcal{F}(\varphi).\end{aligned}$$

Thus $\mathcal{F}(\varphi)$ is also a chain map. □

8.4 Functors Which Preserve Homotopy

8.4.1 Tensor Product

Proposition 8.2. *Let N be an R-module, let $\varphi\colon M \to M'$ and $\psi\colon M \to M'$ be two chain maps of R-complexes and suppose $\varphi \sim \psi$. Then $\varphi \otimes N \sim \psi \otimes N$.*

Proof. Choose a homotopy $h\colon M \to M'$ from φ to ψ. So

$$\varphi - \psi = \mathrm{d}_{M'}h + h\mathrm{d}_M.$$

We claim that $h \otimes N\colon M \otimes_R N \to M' \otimes_R N$ is a homotopy from $\varphi \otimes N$ to $\psi \otimes N$. Indeed, let $u \otimes v \in M \otimes_R N$ with $u \in M_i$ and $v \in N_j$. Then we have

$$\begin{aligned}(\mathrm{d}^{\otimes}_{(M',N)}(h \otimes N) + (h \otimes N)\mathrm{d}^{\otimes}_{(M,N)})(u \otimes v) &= \mathrm{d}^{\otimes}_{(M',N)}(h(u) \otimes v) + (h \otimes N)(\mathrm{d}_M(u) \otimes v + (-1)^i u \otimes \mathrm{d}_N(v)) \\ &= \mathrm{d}_{M'}h(u) \otimes v - (-1)^i h(u) \otimes \mathrm{d}_N(v) + h\mathrm{d}_M(u) \otimes v + (-1)^i h(u) \otimes \mathrm{d}_N(v)) \\ &= \mathrm{d}_{M'}h(u) \otimes v + h\mathrm{d}_M(u) \otimes v \\ &= (\mathrm{d}_{M'}h(u) + h\mathrm{d}_M(u)) \otimes v \\ &= ((\mathrm{d}_{M'}h + h\mathrm{d}_M)(u)) \otimes v \\ &= (\varphi - \psi)(u)) \otimes v \\ &= \varphi(u) \otimes v - \psi(u) \otimes v \\ &= (\varphi \otimes N)(u \otimes v) - (\psi \otimes N)(u \otimes v) \\ &= (\varphi \otimes N - \psi \otimes N)(u \otimes v).\end{aligned}$$

It follows that

$$\varphi \otimes N - \psi \otimes N = \mathrm{d}^{\otimes}_{(M',N)}(h \otimes N) + (h \otimes N)\mathrm{d}^{\otimes}_{(M,N)}.$$

□

8.4.2 R-linear Functor Preserves Homotopy

Proposition 8.3. *Let $\varphi\colon A \to A'$ and $\psi\colon A \to A'$ be two chain maps of R-complexes which are homotopic to each other, and let $F\colon \mathrm{Comp}_R \to \mathrm{Comp}_R$ be an R-linear functor. Then $F(\varphi)$ is homotopic to $F(\psi)$.*

Proof. Choose a homotopy $h\colon A \to A'$ from φ to ψ. So

$$\varphi - \psi = \mathrm{d}_{A'}h + h\mathrm{d}_A.$$

We claim that $F(h)\colon F(A) \to F(A')$ is a homotopy from $F(\varphi)$ to $F(\psi)$. Indeed, let $a \in F(A)$ with $a \in F(A)_i$. Then we have

$$\begin{aligned}(\mathrm{d}_{F(A')}F(h) + F(h)\mathrm{d}_{F(A)})(a) & \\ &= (F(\varphi) - F(\psi))(a).\end{aligned}$$

It follows that ☐

Proposition 8.4. *Let (A, d) and (A', d') be R-complexes and let $F\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$ be an R-linear functor. Suppose A is homotopically equivalent to A'. Then $(F(A), F(\mathrm{d}))$ is homotopically equivalent to $(F(A'), F(\mathrm{d}'))$.*

Proof. Choose chain maps $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A$ together with homotopies $h\colon A \to A'$ and $h'\colon A \to A'$ where

$$\varphi'\varphi - 1_A = \mathrm{d}h + h\mathrm{d} \qquad \text{and} \qquad \varphi\varphi' - 1_{A'} = \mathrm{d}'h' + h'\mathrm{d}'.$$

Then observe that

$$\begin{aligned} F(\varphi')F(\varphi) - 1_{F(A)} &= F(\varphi')F(\varphi) - F(1_A) \\ &= F(\varphi'\varphi - 1_A) \\ &= F(\mathrm{d}h + h\mathrm{d}) \\ &= F(d)F(h) + F(h)F(d). \end{aligned}$$

Thus $\mathcal{F}(\varphi')\mathcal{F}(\varphi) \sim 1_{\mathcal{F}(A)}$. A similar argument shows $\mathcal{F}(\varphi)\mathcal{F}(\varphi) \sim 1_{\mathcal{F}(A')}$. Therefore $\mathcal{F}(A)$ is homotopically equivalent to $\mathcal{F}(A')$. ☐

8.5 Epimorphisms and Monomorphisms

Definition 8.9. Let $\mathcal{C}$ be a category and let $f\colon x \to y$ be a morphism in $\mathcal{C}$.

1. We say f is an **epimorphism** if it is right-cancellative: $g_1 f = g_2 f$ implies $g_1 = g_2$ for all $g_1\colon y \to z$ and $g_2\colon y \to z$.
2. We say f is a **split epimorphism** if it has a right-sided inverse: there exists $g\colon y \to x$ such that $fg = 1_x$.
3. We say f is a **monomorphism** if it is left-cancellative: $fg_1 = fg_2$ implies $g_1 = g_2$ for all $g_1\colon w \to x$ and $g_2\colon w \to x$.
4. We say f is a **split monomorphism** if it has a left-sided inverse: there exists $g\colon y \to x$ such that $gf = 1_x$.
5. We say f is a **bimorphism** if it is both a monomorphism and an epimorphism.
6. We say f is an **isomorphism** if it is both a split monomomorphism and a split epimorphism.

8.5.1 Epimorphisms and Monomorphisms in $\mathbf{Comp}_R$

Proposition 8.5. *Let $\varphi\colon A \to B$ be a chain map. Then φ is an epimorphism if and only if φ is surjective*

8.6 Adjunctions

Definition 8.10. An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that for all objects x in $\mathcal{C}$ and y in $\mathcal{D}$ we have a bijection

$$\tau_{y,x}\colon \mathrm{Hom}_{\mathcal{C}}(Gy, x) \to \mathrm{Hom}_{\mathcal{D}}(y, Fx)$$

which is natural in x and y. We also say G is **left adjoint to** F and F is **right adjoint to** G.

Proposition 8.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ to left-adjoint to $G\colon \mathcal{D} \to \mathcal{C}$. Then F preserves colimits and G preserves limits.*

Proof. Let us show that F preserves colimits. Let ($\square$

Proposition 8.7. *Let M be a graded R-module. The functor $- \otimes_R M\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$ is left adjoint to the functor $\mathrm{Hom}_R(M, -)\colon \mathbf{Grad}_R \to \mathbf{Grad}_R$. In particular, $- \otimes_R M$ preserves direct limits and $\mathrm{Hom}^\star_R(M, -)$ preserves inverse limits.*

Proof. Let us show that $- \otimes_R M$ being left adjoint to $\mathrm{Hom}^\star_R(M, -)$ implies $- \otimes_R M$ preserves direct limits. Let $(M_\lambda, \varphi_{\lambda\mu})$ be a direct system of graded R-modules and graded R-linear maps indexed over a preordered set $(\Lambda, \leq)$. Since $- \otimes_R M$ is a covariant functor, $(M_\lambda \otimes_R M, \varphi_{\lambda\mu} \otimes 1_M)$ is a direct system of graded R-modules and graded R-linear maps indexed over a preordered set $(\Lambda, \leq)$. Furthermore, $\square$