Complex Analysis

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Contents

Notation

Open and Closed Balls

Throughout these notes, I denotes the closed interval $[0,1]$ in $\mathbb{R}$. For all $r > 0$ and $z \in \mathbb{C}$, we write $B_r(z)$ (respectively $\overline{B}_r(z)$) to denote the open (respectively closed) ball centered at z and of radius r:

$$B_r(z) := \{w \in \mathbb{C} \mid |z-w| < r\} \quad \text{and} \quad \overline{B}_r(z) := \{w \in \mathbb{C} \mid |z-w| \leq r\}.$$

Similarly, for all $r > 0$ and $z \in \mathbb{C}$, we write $C_r(z)$ to denote the circle centered at z and of radius r:

$$C_r(z) := \{w \in \mathbb{C} \mid |z-w| = r\}.$$

Complex Numbers

Let z be a nonzero complex number. A **polar representation** of z is given by

$$z := |z|e^{2\pi i(\mathrm{Arg}(z)+n)}$$

for a unique $|z| \in \mathbb{R}_{\geq 0}$, a unique $\mathrm{Arg}(z) \in [-\pi, \pi)$, and a choice of $n \in \mathbb{Z}$. We call $|z|$ the **modulus** of z. We define the **argument** of z to be the set

$$\arg(z) := \{\mathrm{Arg}(z) + n \mid n \in \mathbb{Z}\}$$

and we call $\mathrm{Arg}(z)$ the **main branch** of the argument of z.

Part I
Convergence of Sequences of Functions and Power Series

1 Convergence

Definition 1.1. Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})$ be a sequence of functions. Then

1. The sequence (f_n) converges **pointwise** on D to a function f if for all $z \in D$ and for all $\varepsilon > 0$ there exists $N_{z,\varepsilon} \in \mathbb{N}$ (which depends on $z \in D$ and $\varepsilon > 0$) such that

   $$n \geq N_{z,\varepsilon} \text{ implies } |f_n(z) - f(z)| < \varepsilon$$

2. The sequence (f_n) converges **uniformly** on D to a function f if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ (which depends on $\varepsilon > 0$) such that

   $$n \geq N_\varepsilon \text{ implies } |f_n(z) - f(z)| < \varepsilon$$

   for all $z \in D$.

3. The sequence (f_n) is **uniformly Cauchy** on D if for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

   $$m, n \geq N_\varepsilon \text{ implies } |f_m(z) - f_n(z)| < \varepsilon$$

   for all $z \in D$.

4. The series $\sum f_n$ converges **pointwise** on D to a function f if the sequence of partial sums $(\sum_{m=1}^n f_m)_{n\in\mathbb{N}}$ converges uniformly on D to f.

5. The series $\sum f_n$ converges **uniformly** on D to a function f if the sequence of partial sums $(\sum_{m=1}^n f_m)_{n\in\mathbb{N}}$ converges uniformly on D to f.

The main advantage in determining whether or not a sequence of functions (f_n) is uniformly Cauchy is that we do not need to know what (f_n) converges to. In contrast, the definition of (f_n) converging uniformly assumes that we already know what it converges to from the outset. Fortunately, since $\mathbb{C}$ is complete, we only need to know that (f_n) is uniformly Cauchy to determine whether it converges uniformly or not.

Theorem 1.1. *Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})$ be a sequence of functions.*

1. *The sequence (f_n) converges uniformly on D to a function $f\colon D \to \mathbb{C}$ if and only if (f_n) is uniformly cauchy on D.*

2. *(Weierstrass M-test) Suppose that for each $n \in \mathbb{N}$ there exists $M_n \in [0, \infty)$ such that $\sum_{n=1}^{\infty} M_n < \infty$ and $|f_n(z)| \leq M_n$ for all $z \in D$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on D.*

Proof.

1. First we assume that (f_n) is uniformly cauchy on D. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$m, n \geq N \text{ implies } |f_m(z) - f_n(z)| < \varepsilon \tag{1}$$

for all $z \in D$. Then for each $z \in D$, the sequence $(f_n(z))$ is a Cauchy sequence in $\mathbb{C}$, and by completeness of $\mathbb{C}$, it must converge to a limit. Let $f(z)$ denote this limit. As we vary $z \in D$, we obtain a function $f\colon D \to \mathbb{C}$, given by

$$f(z) := \lim_{n \to \infty} f_n(z).$$

Clearly (f_n) converges pointwise to $f\colon D \to \mathbb{C}$. To see that it converges *uniformly* to f, we fix $m \in \mathbb{N}$ and let $n \to \infty$ in (1) and we see that

$$m \geq N \text{ implies } |f_m(z) - f(z)| \leq \varepsilon$$

for all $z \in D$.

Now we assume that (f_n) converges uniformly on D to a function $f\colon D \to \mathbb{C}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$n \geq N \text{ implies } |f_n(z) - f(z)| < \frac{\varepsilon}{2}$$

for all $z \in D$. Then for all $m, n \geq N$, we have

$$\begin{aligned} |f_m(z) - f_n(z)| &\leq |f_m(z) - f(z)| + |f_n(z) - f(z)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $z \in D$. Thus, (f_n) is uniformly cauchy.

2. By 1, it suffices to show that the sequence $(\sum_{m=1}^{n} f_m)_{n \in \mathbb{N}}$ of partial sums is uniformly Cauchy on D. Let $\varepsilon > 0$. Since the series $\sum_{n=1}^{\infty} M_n$ converges, the sequence $(\sum_{k=1}^{n} M_k)_{k \in \mathbb{N}}$ of partial sums is necessarily a Cauchy sequence. Therefore, there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \text{ implies } \left| \sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k \right| < \varepsilon.$$

In particular, $m, n \geq N$ implies

$$\begin{aligned} \left| \sum_{k=1}^{m} f_k(z) - \sum_{k=1}^{n} f_k(z) \right| &= \left| \sum_{k=m+1}^{n} f_k(z) \right| \\ &\leq \sum_{k=m+1}^{n} |f_k(z)| \\ &\leq \sum_{k=m+1}^{n} M_k \\ &= \left| \sum_{k=m+1}^{n} M_k \right| \\ &= \left| \sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k \right| \\ &< \varepsilon. \end{aligned}$$

for all $z \in D$. □

1.1 Uniform Norm

Proposition 1.1. *Let D be a nonempty subset of $\mathbb{C}$ and let $(f_n \colon D \to \mathbb{C})$ be a sequence of continuous functions. If (f_n) converges to f uniformly on D, then f is continuous on D.*

Proof. Choose any $a \in D$. We will show that f is continuous at a. Let $\varepsilon > 0$. Since (f_n) converges to f uniformly, there exists an $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon/3$ for all $n \geq N$ and $z \in D$. Since f_N is continuous at a, there exists $\delta > 0$ such that $|z - a| < \delta$ implies $|f_N(z) - f_N(a)| < \varepsilon/3$. Combining these together, we see that $|z - a| < \delta$ implies

$$
\begin{aligned}
|f(z) - f(a)| &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(a)| + |f_N(a) - f(a)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

It follows that f is continuous at a. □

Examining the proof in Proposition (1.1) reveals that we can weaken the hypothesis under certain conditions. Let K be a compact subset of $\mathbb{C}$ and let $\mathcal{B}(K, \mathbb{C})$ be the $\mathbb{C}$-vector space set of all bounded functions from D to $\mathbb{C}$. We define the **uniform norm** on $\mathcal{B}(K, \mathbb{C})$ by

$$\|f\|_K = \sup\{|f(x)| \mid x \in K\}$$

for all $f \in \mathcal{B}(K, \mathbb{C})$. The pair $(\mathcal{B}(K, \mathbb{C}), \|\cdot\|_K)$ is easily checked to be a normed vector space. This normed vector space gives rise to a metric space in the usual way. Namely, we define the metric $\mathrm{d}_K \colon \mathcal{B}(K, \mathbb{C}) \times \mathcal{B}(K, \mathbb{C}) \to \mathbb{R}$ by

$$\mathrm{d}_K(f, g) = \|f - g\|_K$$

for all $f, g \in \mathcal{B}(K, \mathbb{C})$. A sequence $(f_n \colon K \to \mathbb{C})$ of bounded functions converges *uniformly* to a function f (which must necessarily be bounded) if and only if it converges to f with respect to metric d_K (check!). This is where the name *uniform* norm comes from.

Proposition 1.2. *Let K be a nonempty compact subset of $\mathbb{C}$ and let (f_n) be a sequence of continuous functions in $(\mathcal{B}(K, \mathbb{C}), \mathrm{d}_K)$. If f is a limit point of (f_n), then f is continuous on K.*

2 Power Series

A **power series centered at** a is a series of the form $\sum a_n (z - a)^n$, where z is a complex variable, a is a given complex number, and (a_n) is a sequence of complex numbers.

2.1 Limit Supremum

To study the convergence of a power series, we study the notion of the limit supremum of a positive real-valued sequence. Let (a_n) be a sequence of positive real numbers. We define the **limit supremum** of (a_n), denoted $\limsup(a_n)$, to be

$$\limsup(a_n) := \lim_{m \to \infty} (\sup\{a_n \mid n \geq m\}).$$

Since $\sup\{a_n \mid n \geq m\}$ is a non-increasing function of m, the limit always exists or equals $+\infty$.

Properties of Limit Supremum

Proposition 2.1. *Let (a_n) be a sequence of positive real-valued numbers.*

1. *If $\limsup(a_n) = A$, then for each $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists some $n \geq N$ such that $a_n > A - \varepsilon$.*
2. *If $\limsup(a_n) = A$, then for each $\varepsilon > 0$, there exists $N \in N$ such that $a_n < A + \varepsilon$ for all $n \geq N$.*
3. *Conversely, if $A \in \mathbb{R}_{\geq 0}$ satisfies 1 and 2, then $\limsup(a_n) = A$.*

Proof.

1. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$. To obtain a contradiction, assume that there does not exist an $n \geq N$ such that $a_n > A - \varepsilon$. Then $A - \varepsilon > a_n$ for all $n > N$. This implies $\sup\{a_n \mid n \geq N\} < A$. This is a contradiction since $\sup\{a_n \mid n \geq m\}$ is a non-increasing function of m.

2. Let $\varepsilon > 0$. To obtain a contradiction, assume that there does not exist an $N \in \mathbb{N}$ such that $a_n < A + \varepsilon$ for all $n \geq N$. Then $\sup\{a_n \mid n \geq N\} \geq A + \varepsilon$ for all $N \in \mathbb{N}$. This implies $\text{limsup}(a_n) \geq A + \varepsilon$, which is a contradiction.

3. Let $A' = \text{limsup}(a_n)$. Assume that $A < A'$. Let $\varepsilon = A' - A$. Then by 2, there exists $N \in \mathbb{N}$ such that $a_n < A + \varepsilon/2$ for all $n \geq N$. So we choose such an $N \in \mathbb{N}$. On the other hand, by 1, there must exist an $n \geq N$ such that $a_n > A' - \varepsilon/2 = A + \varepsilon/2$. Contradiction. An analagous argument gives a contradiction when we assume $A > A'$. Therefore $A = A'$.

□

Lemma 2.1. *Let (a_n) and (b_n) be two sequences of positive real numbers such that limsup$(a_n) = A$ and* $\lim(b_n) = B$. *Then*

1. *limsup$(a_n b_n) = AB$*
2. *limsup$(a_n + b_n) = A + B$*

Proof.

1. Let $\nu > 0$ and let $\delta > 0$ such that $\delta A + \delta B + \delta^2 < \nu$. Choose $N \in \mathbb{N}$ such that $a_n < A + \delta$ and $b_n < B + \delta$ for all $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned} a_n b_n &< (A+\delta)(B+\delta) \\ &= AB + \delta A + \varepsilon B + \delta^2 \\ &< AB + \nu. \end{aligned}$$

Next, let $\varepsilon > 0$, let $N \in \mathbb{N}$, and set $\varepsilon' = \varepsilon/(A+B)$. Choose $n \geq N$ such that $a_n > A - \varepsilon'$ and $b_n > B - \varepsilon'$. Then

$$\begin{aligned} a_n b_n &> (A-\varepsilon')(B-\varepsilon') \\ &= AB - \varepsilon' A - \varepsilon' B + \varepsilon'^2 \\ &> AB - \varepsilon' A - \varepsilon' B \\ &= AB - \varepsilon'(A+B) \\ &= AB - \varepsilon. \end{aligned}$$

Therefore, we must have

$$\text{limsup}(a_n b_n) = AB.$$

2. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $a_n < A + \varepsilon/2$ and $b_n < B + \varepsilon/2$ for all $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned} a_n + b_n &< A + \varepsilon/2 + B + \varepsilon/2 \\ &= A + B + \varepsilon. \end{aligned}$$

Next, let $\varepsilon > 0$ and let $N \in \mathbb{N}$. Choose $n \geq N$ such that $a_n > A - \varepsilon/2$ and $b_n > B - \varepsilon/2$. Then

$$\begin{aligned} a_n + b_n &> A - \varepsilon/2 + B - \varepsilon/2 \\ &= A + B - \varepsilon \end{aligned}$$

Therefore, we must have

$$\text{limsup}(a_n + b_n) = A + B.$$

□

Example 2.1. Let (a_n) be a sequence of complex numbers and let $a \in \mathbb{C}$. Suppose that $\text{limsup}(|a_n|^{1/n}) = A$. Then since $\lim(n^{1/n}) = 1$, we have $\text{limsup}(|na_n|^{1/n}) = A$.

Limit Supremum Test of Convergence of Power Series

Theorem 2.2. *Let (a_n) be a sequence of complex numbers and let $a \in \mathbb{C}$. Suppose that limsup$(|a_n|^{1/n}) = L$.*

1. *If $L = 0$, then the power series $\sum a_n(z-a)^n$ centered at a converges for all $z \in \mathbb{C}$.*
2. *If $L = \infty$, then the the power series $\sum a_n(z-a)^n$ centered at a converges for $z = 0$ only.*
3. *If $0 < L < \infty$, set $R = 1/L$. For any r with $0 < r < R$ the series $\sum a_n(z-a)^n$ converges absolutely and uniformly on the closed disk $\overline{B}_r(a)$ and diverges for $z \notin \overline{B}_R(a)$. In this case, R is called the **radius of convergence** of the power series.*

Proof. We only prove 3, leaving 1 and 2 as easy exercises. Choose r such that $0 < r < R$. Let $\varepsilon = (R-r)/2rR$ (so $r = 1/(L+2\varepsilon)$). Choose $N \in \mathbb{N}$ such that $|a_n|^{1/n} < L + \varepsilon$ for all $n \geq N$. Then

$$|a_n|^{1/n}|z-a| < \frac{L+\varepsilon}{L+2\varepsilon}$$

for all $z \in \overline{B}_r(a)$. Therefore, letting $M = (L+\varepsilon)/(L+2\varepsilon)$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n(z-a)^n| &= \sum_{n=1}^{N} |a_n(z-a)^n| + \sum_{n=N+1}^{\infty} |a_n(z-a)^n| \\ &\leq \sum_{n=1}^{N} |a_n(z-a)^n| + \sum_{n=N+1}^{\infty} M^n \\ &\leq \sum_{n=1}^{N} |a_n(z-a)^n| + \frac{1}{1-M}.\end{aligned}$$

for all $z \in \overline{B}_r(a)$. Thus, the series converges absolutely in $\overline{B}_r(a)$. The series also converges uniformly in $\overline{B}_r(a)$, by Weierstrass M-test, with $M_n = M^n$.

On the other hand, if $z \notin \overline{B}_R(a)$, then

$$\text{limsup}\left(|a_n|^{1/n}|z-a|\right) > 1,$$

so that for infinitely many values of n, $a_n(z-a)^n$ has absolute value greater than 1 and thus $\sum a_n(z-a)^n$ diverges. □

Power Series Examples

Example 2.2.

1. The power series $\sum_{n=1}^{\infty} nz^n$ centered at 0 has radius of convergence 1 since limsup$(n^{1/n}) = 1$.
2. The power series $\sum_{n=0}^{\infty} z^{n^2}$ centered at 0 has radius of convergence 1 since limsup$(a_n^{1/n}) = 1$, where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

3. The generating function for the Catalan numbers C_n is given by

$$f(z) = (z^2+z)^2+z)^2+z)^2+\cdots) = z + 2z^2 + 5z^3 + 14z^4 + \cdots.$$

Since limsup$(C_n^{1/n}) = 4$, we see that $\sum C_n z^n$ has radius of convergence $1/4$.

Properties of Sums

Lemma 2.3. *Let $\sum a_n(z-a)^n$ be a power series centered at a and suppose R is its radius of convergence. Then for all r such that $0 < r < R$, we have the estimate*

$$|a_k| \leq r^{-k}\|f(z)\|_{C_r(a)}.$$

for all $k \geq 0$.

Proof. The partial sum f_n of the power series is a polynomial of degree at most n, and hence the coefficient formula tells us that

$$r^k|a_k| \leq \frac{1}{n+1}\sum_{m=0}^{n} |f_n(r\omega^m)| \leq \sup_{|z-a|=r} |f_n(z)|.$$

Now the assertion follows since f_n converges uniformly to f on the disk $|z| \leq r$. □

2.2 Functions Representable by a Power Series

A function f defined on an open set Ω is said to be **representable by a power series in** Ω if, whenever $a \in \Omega$ and $r > 0$ and the disk $B_r(a)$ is included in Ω, there exists a sequence (a_n) of complex numbers such that the equation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

holds for every $z \in B_r(a)$.

Proposition 2.2. *Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a function, and let $a \in \Omega$ and $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $z \in B_r(a)$. Then $f'(z)$ exists for all $z \in B_r(a)$ and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

Proof. Let $z \in B_r(a)$. Choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\
&= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.
\end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m\colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned}
|g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \right| \\
&< \left| m a_m r^{m-1} \right|.
\end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. $\square$

Corollary. *Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a function, let $a \in \Omega$, and let $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $z \in B_r(a)$. Then $f^{(m)}(z)$ exists for all $m \geq 1$ and $z \in B_r(a)$, and moreover

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} (z-a)^n. \tag{2}$$

In particular, we have $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$.

Proof. The first part follows from an easy induction on m, with Proposition (3.1 giving the base case and the induction step. To get $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$, we set $z = a$ in 2). $\square$

Part II
Analytic Functions

3 Definition of an Analytic Function

Definition 3.1. A function $f\colon D \to \widehat{\mathbb{C}}$ is said to be **analytic at the point** z_0 in D if there exists a nonempty disk $B_r(z_0)$ centered at z_0 such that the restriction of f to $B_r(z_0)$ is the sum of a convergent power series with center z_0, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{3}$$

for all $z \in B_r(z_0)$.

3.1 Uniqueness of Representation

In principle, an analytic function could have different reprsentations (3) as power series at z_0. In order to prove that this cannot happen, we investigate to which extent the coefficients of a power series are determined by the values of its sums.

Theorem 3.1. *(Uniqueness Principle, Local Identity Theorem) Let f and g be the sums of two power series with center z_0, and assume that both converge in an open disk $B_r(z_0)$, say*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n. \tag{4}$$

If there exists a sequence $(z_m) \subset B_r(z_0)\backslash\{z_0\}$ such that $z_m \to z_0$ as $m \to \infty$ and $f(z_m) = g(z_m)$ for all $m \in \mathbb{N}$, then $a_n = b_n$ for all $n \in \mathbb{N}$ and $f(z) = g(z)$ for all $z \in B_r(z_0)$.

Proof. The functions f and g are continuous at z_0, and hence

$$\begin{aligned} a_0 &= f(z_0) \\ &= \lim_{m\to\infty} f(z_m) \\ &= \lim_{m\to\infty} g(z_m) \\ &= g(z_0) \\ &= b_0. \end{aligned}$$

Using the arithmetic rules for convergent sequences, we obtain the representations

$$f_1(z) := \frac{f(z) - a_0}{z - z_0} = \sum_{n=0}^{\infty} a_{n+1}(z - z_0)^n \quad \text{and} \quad g_1(z) := \frac{g(z) - b_0}{z - z_0} = \sum_{n=0}^{\infty} b_{n+1}(z - z_0)^n$$

for all $z \in B_r(z_0)\backslash\{z_0\}$. Because of $a_0 = b_0$ we have $f_1(z_m) = g_1(z_m)$ for all $m \in \mathbb{N}$, which implies $a_1 = b_1$, as just been shown. Proceeding inductively, we get $a_n = b_n$ for all n, and finally $f(z) = g(z)$ for all $z \in B_r(z_0)$. $\square$

3.2 Taylor Coefficients

The coefficients a_n of the power series (3) representing a function f analytic at z_0 are referred to as the **Taylor coefficients of** f **at** z_0. The series (3) itself is said to be the **Taylor series of** f **at** z_0. So we can say that a function f is analytic at z_0 if it admits a convergent Taylor series at z_0.

Proposition 3.1. *Let Ω be an open set and let $f\colon \Omega \to \mathbb{C}$ be analytic at a point a in Ω. Then f is holomorphic at a. Moreover, if*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

for all $z \in B_r(a)$, then f is holomorphic on $B_r(a)$, and we have $f'(z)$

$$f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(z-a)^n$$

for all $z \in B_r(a)$. In particular, f' is analytic at a.

Proof. Let $z \in B_r(a)$. Choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\
&= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.
\end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m \colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned}
|g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \right| \\
&< \left| m a_m r^{m-1} \right|.
\end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. □

Corollary. *Let Ω be an open set, let $f \colon \Omega \to \mathbb{C}$ be a function, let $a \in \Omega$, and let $r > 0$ such that $B_r(a) \subset \Omega$. Suppose that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

for all $z \in B_r(a)$. Then $f^{(m)}(z)$ exists for all $m \geq 1$ and $z \in B_r(a)$, and moreover

$$f^{(m)}(z) = \sum_{n=0}^{\infty} \frac{(n+m)!}{n!} a_{n+m} (z-a)^n. \tag{5}$$

In particular, we have $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$.

Proof. The first part follows from an easy induction on m, with Proposition (3.1 giving the base case and the induction step. To get $a_m = f^{(m)}(a)/m!$ for all $m \geq 0$, we set $z = a$ in 2). □

3.3 Operations Involving Analytic Functions

In this subsection, our goal is to prove the following theorem.

Theorem 3.2. *If f and g are analytic at z_0, then $f+g$, $f-g$, and fg are analytic at z_0. If, moreover, $g(z_0) \neq 0$, then f/g is analytic at z_0. If f is analytic at z_0 and g is analytic at $w_0 := f(z_0)$, then $g \circ f$ is analytic at z_0.*

Note that the composition $g \circ f$ need not exist on the domain of f, but just in a sufficiently small neighborhood of z_0. The analyticity of $f+g$ and $f-g$ are trivial. The proofs of the remaining assertions are more demanding.

3.3.1 Cauchy Product

The next result is a more sophisticated statement about the analyticity of a product fg, which includes an algorithm for computing the Taylor coefficients of fg from the coefficients of the factors f and g.

Theorem 3.3. *(Cauchy Product) Assume that the power series (4) for f and g converge in an open disk $B_R(z_0)$ centered at z_0 and of radius R, and let*

$$c_n := \sum_{m=0}^{n} a_m b_{n-m}.$$

Then the power series $\sum c_n(z-z_0)^n$ converges in $B_R(z_0)$ to the product $f(z)g(z)$, that is

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \tag{6}$$

for all $z \in B_R(z_0)$.

Proof. Let f_n, g_n, and p_n be the partial sums of the series in (4) and (6) respetively. Then a rearrangement of the finite sums yields

$$\begin{aligned} f_n(z)g_n(z) &= \sum_{m=0}^{n} a_m(z-z_0)^m \sum_{m=0}^{n} b_m(z-z_0)^m \\ &= \sum_{m=0}^{n}\left(\sum_{k=0}^{m} a_k b_{m-k}\right)(z-z_0)^m + \sum_{m=n+1}^{2n}\left(\sum_{k=m-n}^{n} a_k b_{m-k}\right)(z-z_0)^m \\ &= p_n(z) + \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} a_k b_{m-k}(z-z_0)^m. \end{aligned}$$

Fix $z \in B_R(z_0)$. Choose r such that $|z-z_0| < r < R$ and choose a constant c such that $|a_k| \le cr^{-k}$ and $|b_k| \le cr^{-k}$ for all k. Setting $q := |z-z_0|/r < 1$, we use the triangle inequality to estimate

$$\begin{aligned} |f_n(z)g_n(z) - p_n(z)| &\le \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} |a_k||b_{m-k}||z-z_0|^m \\ &\le \sum_{m=n+1}^{2n}\sum_{k=m-n}^{n} c^2 r^{-k}||z-z_0|^k \\ &\le \sum_{m=n+1}^{2n} (2n-m+1)c^2 q^k \\ &\le n^2 c^2 q^{n+1}. \end{aligned}$$

Since the right-hand side tends to zero as $n \to \infty$, the assertion follows. □

Example 3.1. Let $x, y \in \mathbb{C}$. Computing the Cauchy product of the Taylor series of $\exp x$ and $\exp y$, we obtain the **addition theorem** of the exponential function

$$\begin{aligned} e^x e^y &= \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{j=0}^{\infty}\frac{y^j}{j!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{x^j y^{k-j}}{j!(k-j)!}\right) \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}x^j y^{k-j} \\ &= \sum_{k=0}^{\infty}\frac{(x+y)^k}{k!} \\ &= e^{x+y}. \end{aligned}$$

When this identity is applied to $z = x + iy$ with $x, y \in \mathbb{R}$, it yields a representation of the complex exponential function by familiar real functions,

$$e^{x+iy} = e^x(\cos y + i \sin y),$$

which implies that for all $z \in \mathbb{C}$,

$$|e^z| = e^{\mathrm{Re}(z)}, \quad \arg(e^z) = \mathrm{Im}(z), \quad e^{z+2\pi i} = e^z.$$

In particular, the exponential function has no zeros and is **periodic** with purely imaginary period $2\pi i$.

3.3.2 Reciprocal Functions

Theorem 3.4. *If f is analytic at z_0 and $f(z_0) \neq 0$, then $1/f$ is analytic at z_0. The Taylor coefficients b_k of $1/f$ at z_0 can be computed recursively from the Taylor coefficients a_k of f by $b_0 := 1/a_0$ and*

$$b_k := -\frac{1}{a_0}(a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_k b_0) \tag{7}$$

for all $k \in \mathbb{N}$.

Proof. In the first step we *assume* that the function $1/f$ is analytic at z_0. Then the Taylor series

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} b_0 (z - z_0)^n \tag{8}$$

converges in a neighborhood of z_0 and its Cauchy product with the Taylor series of f is the constant function 1. The latter is equivalent to the infinite system of equations

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ &\vdots \end{aligned}$$

Since $a_0 \neq 0$, this triangular system can be solved with respect to the coefficients b_k, which yields the recursion (7).

It remains to prove that the series (8), with coefficients b_k given by the recursion (7), indeed has a positive radius of convergence. Choose positive numbers c and r such that $|a_n| \leq c r^{-n}$ for all $n \in \mathbb{N}$. We set $q := 1 + c/|a_0|$ and show that

$$|b_n| \leq \frac{c}{|a_0|^2} \frac{q^{n-1}}{r^n} \tag{9}$$

for all $n \in \mathbb{N}$. For $n = 1$, we have $b_1 = -a_1/a_0^2$ and $|a_1| \leq c/r$, so that indeed

$$\begin{aligned} |b_1| &= \frac{a_1}{a_0^2} \\ &\leq \frac{c}{|a_0|^2} \frac{1}{r}. \end{aligned}$$

Now assume that (9), holds for all $n = 1, 2, \ldots, k-1$ and consider the case where $n = k$. Using $|b_0| = 1/|a_0|$, the recursive definition of b_k, and the triangle inequality, we estimate

$$\begin{aligned} |b_k| &\leq \frac{1}{|a_0|} \left(|a_k b_0| + \sum_{j=1}^{k-1} |a_{k-j}||b_j| \right) \\ &\leq \frac{1}{|a_0|} \left(|a_k b_0| + \sum_{j=1}^{k-1} \frac{c}{r^{k-j}} \frac{c}{r^j |a_0|^2} q^{j-1} \right) \\ &\leq \frac{c}{r^k |a_0|^2} \left(1 + \frac{c}{|a_0|} \sum_{j=0}^{k-2} q^j \right) \\ &\leq \frac{c}{r^k |a_0|^2} \left(1 + \frac{c}{|a_0|} \frac{q^{k-1} - 1}{q - 1} \right) \\ &= \frac{c}{r^k |a_0|^2} q^{k-1}, \end{aligned}$$

which gives (9) for $n = k$ and thus for all n. Consequently, the power series (8) has radius of convergence not less than r/q. $\square$

Example 3.2. Let the function f be defined on the complex plane by $f(z) := (e^z - 1)/z$ if $z \neq 0$ and $f(0) = 1$. Representing e^z by its Taylor series, we obtain the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}$$

which converges in the entire complex plane and attans the correct value $f(0) = 1$ at $z = 0$. SInce $f(0) \neq 0$, the reciprocal function $1/f$ is also analytic at $z_0 = 0$. Writing the Taylor series of $g := 1/f$ in the form

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \tag{10}$$

the numbers B_k are determined by the equations $B_0 = b_0 = 1/a_0 = 1$ and

$$\begin{aligned} 0 &= \sum_{j=0}^{k} a_{k-j} b_j \\ &= \sum_{j=0}^{k} \frac{B_j}{(k-j+1)!j!} \\ &= \frac{1}{(k+1)!} \sum_{j=0}^{k} \binom{k+1}{j} B_j, \end{aligned}$$

for $j \in \mathbb{N}$. Solving this system recursively, we get

$$B_k = -\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j$$

for $j \in \mathbb{N}$. The numbers B_k are called **Bernoulli numbers**. For n odd, all B_n are zero, except B_1 which equals $-1/2$. The first Bernoulli numbers for n ever are

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

Note that the series (10) converges for $|z| < 2\pi$.

3.3.3 Composition of Power Series

The final step in proving Theorem (3.2) is concerned with the composition $g \circ f$ of functions given by power series. In order to ensure that the composition makes sense at least locally, we assume that f is analytic at z_0, while g is supposed to be analytic at the image point $w_0 := f(z_0)$. Then, by continuity, f maps a neighborhood of z_0 into the disk of convergence of g. Our goal is to find a convergent power series for $g \circ f$ from the given power series of f and g. The approach is straightforward: we assume that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{and} \quad g(w) = \sum_{k=0}^{\infty} b_k (w - w_0)^k. \tag{11}$$

substitute $w - w_0 = \sum_{n=1}^{\infty} a_n (z - z_0)^n$ in the series for g, rearrange the double sum according to the powers of $z - z_0$, and show that the resulting series converges to $g \circ f$ in a neighborhood of z_0. The details will be worked out next.

For $n \in \mathbb{N}$, the nth power $(f - a_0)^n$ is analytic at z_0 and the n leading terms of its Taylor series at z_0 vanish. Denoting by a_{nk} the Taylor coefficients of this function, we have

$$(f(z) - a_0)^n = \sum_{k=1}^{\infty} a_{nk} (z - z_0)^k = \sum_{k=n}^{\infty} a_{nk} (z - z_0)^k \tag{12}$$

in some neighborhood of z_0. Substituting the term $w - w_0$ in the power series of $g - b_0$ by the power series of $f - a_0$ (recall that $a_0 = f(z_0) = w_0$), we obtain formally

$$\begin{aligned} \sum_{n=0}^{\infty} b_n (w - w_0)^n &= \sum_{n=1}^{\infty} b_n \left(\sum_{k=n}^{\infty} a_{nk} (z - z_0)^k \right) \\ &= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} b_n a_{nk} \right) (z - z_0)^k. \end{aligned}$$

Before we justify that changing the order of summation is possible, we state the result

Theorem 3.5. *If f is analytic at z_0 and g is analytic at $w_0 := f(z_0)$, then $g \circ f$ is analytic at z_0. Let f, g, and $(f - a_0)^n$ be represented by the series (11) and (12) respectively. Then the Taylor coefficients c_k of $g \circ f$ at z_0 are given by*

$$c_0 = b_0, \qquad c_k = \sum_{n=1}^{k} b_n a_{nk}$$

for all $k \in \mathbb{N}$.

3.4 Weierstrass Rearrangement Theorem

Theorem 3.6. *(Weierstrass Rearrangement Theorem) The sum of a power series is analytic at any point in its disk of convergence. If f is given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{13}$$

for all $z \in B_r(z_0)$, and if $z_1 \in B_r(z_0)$, then

$$f(z) = \sum_{m=0}^{\infty} b_m (z - z_1)^m$$

for all $z \in B_{r_1}(z_1)$, where $r_1 := r - |z_1 - z_0|$ and the coefficients b_k are given by the convergent series

$$b_m = \sum_{n=m}^{\infty} \binom{n}{m} a_n (z_1 - z_0)^{n-m}$$

for all $k \in \mathbb{N}_0$.

Proof. Let $z \in B_{r_1}(z_1)$. Substituting $z - z_0 = (z - z_1) + (z_1 - z_0)$ into (13), we obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n ((z - z_1) + (z_1 - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} (z - z_1)^m (z_1 - z_0)^{n-m} \end{aligned}$$

In order to prove the assertion, it only remains to change the order of summation in the double series. It suffices to show that this series converges absolutely. To this end we remark that

$$\sum_{n=0}^{\infty} |a_n| \sum_{m=0}^{n} \binom{n}{m} |z - z_1|^m |z_1 - z_0|^{n-m} = \sum_{n=0}^{\infty} |a_n| (|z - z_1| + |z_1 - z_0|)^n.$$

The last sum converges because $|z - z_1| + |z_1 - z_0| < r$, so that the power series (13) converges absolutely at the point $z = z_0 + |z - z_1| + |z_1 - z_0|$. □

3.5 Definition of Analytic Function

Definition 3.2. A complex function $f \colon D \subseteq \mathbb{C} \to \mathbb{C}$ is said to be **analytic on** A if A is a subset of D and f is analytic at every point of A. We say that f is **analytic** if it is analytic on its domain set. A function which is analytic on the entire complex plane is called **entire**.

Lemma 3.7. *For any complex function $f \colon D \subseteq \mathbb{C} \to \mathbb{C}$ the set A_f of all points in D at which f is analytic is open.*

Proof. If A_f is empty, there is nothing to prove. If $z_0 \in A_f$, then f has a Taylor expansion at z_0 which converges in a open disk D_0 centered at z_0. By Theorem (3.6), $D_0 \subset A_f$. □

3.5.1 Jacobi Theta Function

An interesting family of entire functions are the **Jacobi Theta functions**, given by the series

$$\vartheta(z; q) := \sum_{n \in \mathbb{Z}} q^{n^2} e^{2\pi i n z}$$

for all $z \in \mathbb{C}$, where q is a complex parameter with modulus less than one. In order to show that ϑ is entire, we consider the power series

$$f(z) := \sum_{n=1}^{\infty} q^{n^2} z^n = qz + q^4 z^2 + q^9 z^3 + \cdots .$$

This series converges for all $z \in \mathbb{C}$ because

$$\limsup \left(\left| q^{n^2} \right|^{1/n} \right) = \limsup \left(|q^n| \right) = 0,$$

and thus the function f is entire. The function g defined by $g(z) := e^{2\pi i z}$ is also entire and has no zeros in $\mathbb{C}$, so that its reciprocal $1/g$ is also entire. Finally, $\vartheta(z) = 1 + 2f(g(z))$

$$\vartheta(z; q) = 1 + f(g(z)) + f(1/g(z))$$

for all $z \in \mathbb{C}$.

The function g, and consequently ϑ, is periodic with period 1. The parameter q is said to be the **nome** of the Theta function. It is often represented as $q = e^{\pi i \tau}$, where τ is a complex number with $\mathrm{Im}(\tau) > 0$.

3.5.2 Local Normal Forms

Theorem 3.8. *(Local Normal Form) Let $f\colon D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic on D. If f is not constant in a neighborhood of $z_0 \in D$, then there exist a positive integer m and an analytic function $g\colon D \subset \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $g(z_0) \neq 0$ such that*

$$f(z) = f(z_0) + (z - z_0)^m g(z) \tag{14}$$

for all $z \in D$. The integer m and the function g are uniquely determined.

Proof. Assume that the Taylor series $f(z) = \sum a_k(z - z_0)^k$ of f at z_0 converges in a disk D_0. Denoting by a_m the first non-zero coefficient among $a_1, a_2, a_3, \ldots$, we have

$$f(z) = f(z_0) + (z - z_0)^m \sum_{k=m}^{\infty} a_k(z - z_0)^{k-m}$$

for all $z \in D_0$. The sum $g_0(z)$ of the series $\sum_{k=m}^{\infty} a_k(z - z_0)^{k-m}$ is an analytic function in D_0 with $g_0(z_0) = a_m \neq 0$. The function g defined in D by

$$g(z) := \begin{cases} \frac{f(z)-f(z_0)}{(z-z_0)^m} & \text{if } z \in D\backslash\{z_0\} \\ a_m & \text{if } z = z_0 \end{cases}$$

is analytic on $D\backslash\{z_0\}$. Since it coincides with g_0 in D_0 it is also analytic at z_0.

For proving uniqueness we assume that $(z - z_0)^n g_1(z) = (z - z_0)^m g_2(z)$ with $n > m$ for all $z \in D$. Then $(z - z_0)^{n-m} g_1(z) = g_2(z)$, and the left-hand side vanishes at z_0 while $g_2(z_0) \neq 0$. So $m = n$ and then $g_1 = g_2$ is obvious. □

Definition 3.3. The integer m in the representation (14) is called the **order** of the function f at z_0 and is denoted by $\operatorname{ord}(f, z_0)$. If f is constant in a neighborhood of z_0 we set $\operatorname{ord}(f, z_0) = \infty$. If in particular $f(z_0) = 0$, then m is said to be the **order of the zero** z_0.

As an immediate corollary of Theorem (3.8) we get the following result which shows, in particular, that all zeros of non-constant analytic functions are isolated.

Lemma 3.9. *If f is analytic at z_0 and $a := f(z_0)$, then there exists a disk D_0 with center z_0 such that either $f(z) = a$ for all $z \in D_0$ or $f(z) \neq a$ for all $z \in D_0\backslash\{z_0\}$.*

3.6 Analytic Functions in Planar Domain

As we have already seen, it is natural to require that the domain set D of an analytic function is open. From now on, we shall also assume that D is a nonempty connected open subset of $\mathbb{C}$, i.e. that D is a **domain**. This assumption is not too strong, since any open set in $\mathbb{C}$ is the disjoint union of domains, but it simplifies life a lot. In particular it is important when local statements about power series will be "lifted" to global results for analytic functions. This will be demonstrated in the proof of the following theorem.

Theorem 3.10. *(Identity Theorem, Uniqueness Principle) Let f and g be analytic functions in a domain D. If there exists a sequence $(z_n) \subset D\backslash\{z_0\}$ such that $z_n \to z_0 \in D$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = g(z)$ for all $z \in D$.*

Proof. The function $h := f - g$ has a sequence of zeros which converge to $z_0 \in D$. Continuity of h implies that $h(z_0) = 0$, so that z_0 is a zero of h which is not isolated. Since h is analytic in D, we infer from Lemma (3.9) that $h(z) = 0$ in some disk D_0 with center z_0.

We pick any point z_1 in D and show that $h(z_1) = 0$. Since D is open and connected, it must be path-connected. So choose a path $\gamma\colon I \to D$ from z_0 to z_1. Then the set

$$S := \{s \in I \mid h(\gamma(t)) = 0 \text{ for all } t \in [0, s]\}$$

is not empty and we denote by s_0 its supremum. Continuity of h implies that $h(\gamma(s_0)) = 0$. Since $h(\gamma(t)) = 0$ for all $t \in [0, s_0]$, Lemma (3.9) tells us that $h(z) = 0$ in a neighborhood of $\gamma(s_0)$. This is only possible if $s_0 = 1$, because otherwise $h(\gamma(t)) = 0$ for all t in an interval $[0, s_1]$ with $s_1 > s_0$. □

3.6.1 Zeros of Analytic Function

The last theorem establishes the surprising fact that a function which is analytic in a domain is completely determined by its values in an arbitrarily small disk. We state another result concerning the zeros of such a function.

Corollary. *If $f \neq 0$ is analytic in a domain D and K is a compact subset of D, then the number of zeros of f in K is finite.*

Proof. If f had infinitely many zeros in K, there would exist a sequence (z_n) of such zeros which converge to a point $z_0 \in K \subset D$. But then $f = 0$ on D by Theorem (3.10). □

Nevertheless an analytic function $f \neq 0$ can have infinitely many zeros in D. If this happens, the zeros must have an accumulation point z_0 on $\widehat{\mathbb{C}}$. Since z_0 cannot lie in D, it must be on the boundary of D (considered as a subset of $\widehat{\mathbb{C}}$). For an entire function, the only possible accumulation point of zeros is the point at infinity.

Example 3.3. The function $\sin(1/z)$ is analytic in $\mathbb{C}\backslash\{0\}$ and has the zeros $z_k = 1/(k\pi)$ with $k = \pm 1, \pm 2, \ldots$, which accumulate at the origin.

3.6.2 Extremal Values

Theorem 3.11. *(Maximum and Minimum Principle) Let $f\colon D \subset \mathbb{C} \to \mathbb{C}$ be a non-constant analytic function. Then $|f|$ has no local maximum in D, and very local minimum of $|f|$ is a zero of f.*

Proof. Assume that $|f|$ attains a maximum or minimum at $z_0 \in D$. By Theorem (3.10) f is not locally constant, so that we can apply Theorem (3.8) and write

$$f(z) = f(z_0) + (z - z_0)^m g(z),$$

where g is analytic in D and $g(z_0) \neq 0$. □

3.7 Analytic Continuation

3.7.1 Direct Analytic Continuation

Theorem 3.12. *(Direct Analytic Continuation) Let the functions $f_1\colon D_1 \to \mathbb{C}$ and $f_2\colon D_2\colon \to \mathbb{C}$ be analytic in the domains D_1 and D_2, respectively. Assume that the intersection $D_0 := D_1 \cap D_2$ is nonempty and that $f_1 = f_2$ on D_0. Then there is a unique analytic function f on $D := D_1 \cup D_2$ which coincides with f_1 on D_1, namely*

$$f(z) := \begin{cases} f_1(z) & \textit{if } z \in D_1 \\ f_2(z) & \textit{if } z \in D_2. \end{cases}$$

Proof. The function f is analytic on D because any point $z \in D$ belongs to D_1 or D_2, so that f coincides with f_1 or f_2 in a neighborhood of z. Since $D_1 \cup D_2$ is a domain, and $D_1 \cap D_2 \neq \emptyset$ is open, uniqueness of f follows from the identity theorem. □

Under the assumptions of Theorem (3.12), the function f is said to be an **analytic continuation of f_1 onto** D. Interchanging the roles of f_1 and f_2, we see that f is also the (unique) analytic extension of f_2 onto D. So direct analytic continuation may extend a function to a larger domain, but this says nothing about how to *find* such an extension. The key to a constructive approach is Weierstrass rearrangement theorem for power series.

3.7.2 Analytic Function Elements

Assume that an analytic function f is given as the sum of a power series which has center z_0 and disk of convergence D_0. It can happen that the rearrangement of that power series to a series centered at a point z_1 in D_0 has a disk of convergence D_1 which protrudes out of D_0. Then by Theorem (3.12), f admits an analytic extension to $D_0 \cup D_1$. In order to explore this further we introduce some notation.

Definition 3.4.

1. An **(analytic) function element** if a pair (f, D) consisting of a disk D and an analytic function $f\colon D \to \mathbb{C}$. The center of the disk D is also referred to as the **center of the function element**.

2. If (f_1, D_1) and (f_2, D_2) are two function elements which satisfy the assumption of Theorem (3.12), we say that (f_2, D_2) is the **direct analytic continuation** of (f_1, D_1) (or vice versa) and write $(f_1, D_1) \bowtie (f_2, D_2)$.

3. A finite sequence $(f_0, D_0), (f_1, D_1), \ldots, (f_n, D_n)$ of funciton elements is said to be a **chain** if any function element (except the first) is the direct analytic continuation of its predecessor,

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_n, D_n). \tag{15}$$

   We then call (f_n, D_n) an **analytic continuation of** (f_0, D_0) **along the chain**.

4. A function element (f_n, D_n) is an **analytic continuation** of (f_0, D_0) if a chain of function elements satisfying (15) exists. We then write $(f_0, D_0) \sim (f_n, D_n)$.

To understand the procedures that follow better it is essential to recognize some subtleties of these definitions. While it is easy to see that $\sim$ is an **equivalence relation**, the relation $\bowtie$ is reflexive and symmetric, but *not transitive*.

Example 3.4. The binomial series

$$f_0(z) = \sum_{n=0}^{\infty} \binom{1/2}{n} (z-1)^n \tag{16}$$

has radius of convergence one and thus defines a function element (f_0, D_0) with $D_0 := B_1(1) = \{z \in \mathbb{C} \mid |z-1| < 1\}$. If z is real and $0 < z < 1$, we have $f_0(z) = \sqrt{z}$. For $k = 0, 1, \ldots, 8$ we denote by $\omega_k = e^{2\pi i k/9}$ the 9th roots of unity and let $D_k := \{z \in \mathbb{C} \mid |z - \omega_k| < 1\}$. All power series

$$f_k(z) := e^{ik\pi/18} \sum_{n=0}^{\infty} e^{-ik\pi/9} \binom{1/2}{n} (z - \omega_k)^n$$

have radius of convergence one and the nine function elements form a chain

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_8, D_8)$$

where neighbors are direct analytic continuations of each other. Consequently any two elements (f_j, D_j) and (f_k, D_k) are **analytic continuations** of each other. Moreover, for $k = 1, 2, 3, 4$, the element (f_k, D_k) is a *direct* analytic continuation of (f_0, D_0), but not for $k = 5, 6, 7, 8$. Since (f_6, D_6) is also a direct analytic continuation of (f_3, D_3), we have

$$(f_0, D_0) \bowtie (f_3, D_3) \bowtie (f_6, D_6) \not\bowtie (f_0, D_0),$$

which again shows that the relation $\bowtie$ is not transitive.

Lemma 3.13. *If $D_1 \cap D_2 \cap D_3 \neq \emptyset$, $(f_1, D_1) \bowtie (f_2, D_2)$ and $(f_2, D_2) \bowtie (f_3, D_3)$, then $(f_1, D_1) \bowtie (f_3, D_3)$.*

Proof. The functions f_1 and f_3 are analytic in the domain $D_1 \cap D_3$ and coincide (with f_2) on its open subset $D_1 \cap D_2 \cap D_3$. Thus $f_1 = f_3$ on $D_1 \cap D_3$. □

3.7.3 Analytic Continuation Along a Path

Definition 3.5. Let $\gamma\colon I \to \mathbb{C}$ be a path. A chain of function elements

$$(f_0, D_0) \bowtie (f_1, D_1) \bowtie \cdots \bowtie (f_n, D_n), \tag{17}$$

is said to be a **chain along** γ, if the chain of disks $(D_0, D_1, \ldots, D_n)$ covers γ in the sense of the Path Covering Lemma.

Let (f_0, D_0) and (f, D) be function elements with centers at $\gamma(0)$ and $\gamma(1)$, respectively. We say that (f, D) is an **analytic continuation of** (f_0, D_0) **along** γ, if there exists a chain of function elements (17) along γ such that $(f, D) = (f_n, D_n)$.

It is essential that analytic continuation along a path does not depend on the special choice of the chain of function elements. This statement is made precise in the next lemma.

Lemma 3.14. *Let $(f_0, D_0) \bowtie \cdots \bowtie (f_n, D_n)$ and $(g_0, \widetilde{D}_0) \bowtie \cdots \bowtie (g_m, \widetilde{D}_m)$ be two chains of function elements along a path γ. If $(f_0, D_0) \bowtie (g_0, \widetilde{D}_0)$, then it is also true that $(f_n, D_n) \bowtie (g_m, \widetilde{D}_m)$.*

Proof. Let $\gamma\colon I \to \mathbb{C}$ be a path and let

$$0 = t_0 < t_1 < \cdots < t_n = 1, \qquad 0 = s_0 < s_1 < \cdots < s_m = 1,$$

be partitions of I such that for all $k = 1, \ldots, n$ and $j = 1, \ldots, m$ we have

$$\gamma([t_{k-1}, t_k]) \subset D_k, \qquad \gamma([s_{j-1}, s_j] \subset \widetilde{D}_j.$$

Intuitively, the following procedure can be described as a walk along the path γ, where the left foot is only allowed to step on disks D_k, the right foot is restricted to the disks $\widetilde{D}_j$, and the function elements (f_k, D_k) and $(f_j, \widetilde{D}_j)$ underneath both feet must be direct analytic continuations of each other. We shall show that one can walk step-by-step all the way along γ, following just a simple rule: don't move the foot which is ahead. □

3.7.4 Function Elements and Germs

Though analytic continuation along a path γ is *essentially* independent of the choice of the function elements which cover γ, these elements are by no means uniquely defined. In fact not even the elements at the endpoints of γ are unique, Lemma (3.14) only tells us that the terminal elements of the chain *coincide on some disk* if the initial elements have this property. The redundancy in this process of analytic continuation is sometimes disturbing and makes formulations cumbersome. To eliminate this drawback we utilize the standard technique of forming classes.

Definition 3.6. Two function elements (f_1, D_1) and (f_2, D_2) centered at z_0 are said to be **equivalent** if $f_1(z) = f_2(z)$ in some neighborhood of z_0. A **germ** at z_0 is a class of equivalent function elements centered at z_0. The germ which contains a function element (f, D) is denoted by $f^\star$. We denote by $\mathcal{O}_{z_0}^{\mathrm{an}}$ to be the set of all germs at z_0. One easily checks that $\mathcal{O}_{z_0}^{\mathrm{an}}$ is a $\mathbb{C}$-algebra.

Depending on the situation, one can choose an appropriate **representative** of a germ $f^\star$. The **canonical representative** of a germ $f^\star$ is that function element (f, D) in $f^\star$ which has the disk D of maximal radius (here we allow $D = \mathbb{C}$).

The **value** $f^\star(z_0)$ **of a germ** $f^\star$ at z_0 is the value $f(z_0)$ of any function element (f, D) which represents $f^\star$. Note that the value of a germ is only defined at its center. On the other hand, the germ of a function element (f, D) is *not* determined by the value of f at its center z_0 alone, but by the complete list of its Taylor coefficients. To explain this idea more precisely, let $\mathbb{C}^\infty$ be the set of all sequences $(z_n)_{n\geq 0}$ in $\mathbb{C}$. Then $\mathbb{C}^\infty$ forms a $\mathbb{C}$-algebra, where addition is defined pointwise and where multiplication is defined by the Cauchy product; namely if (a_n) and (b_n) are two sequences in $\mathbb{C}$, then

$$(a_n) + (b_n) = (a_n + b_n) \quad \text{and} \quad (a_n)(b_n) = (c_n),$$

where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$. Finally, let $\varphi\colon \mathcal{O}_{z_0}^{\mathrm{an}} \to \mathbb{C}^\infty$ be the morphism of $\mathbb{C}$-algebras given by sending a function element (f, D) to the Taylor sequence $(f^{(n)}(z_0))$. The identity theorem implies that this morphism is well-defined and injective.

The concept of germs is not restricted to function elements. If the function f is analytic at a point z, it is analytic in a neighborhood of z, and thus it induces a germ at z which we denote by $f_z^\star$.

3.7.5 Analytic Continuation of Germs

Definition 3.7. We say that a germ $f^\star$ at b is an analytic continuation of a germ $f_a^\star$ at a along a path γ from a to b if there exists a chain of function elements

$$(f_0, D_0) \bowtie \cdots \bowtie (f_n, D_n)$$

along γ such that (f_0, D_0) represents $f_a^\star$ and (f_n, D_n) represents $f^\star$, respectively.

Whenever an analytic continuation of a germ along a path γ exists, Lemma (3.14) tells us that the terminal germ is uniquely determined and does not depend on the specific choice of the function element along γ. We thus can speak of *the* analytic continuation $f^\star(\gamma)$ of a germ $f^\star$ along a path γ.

3.7.6 The Monodromy Principle

In the next step we study analytic continuation of a germ along *different paths* with the same endpoints.

Theorem 3.15. *(Monodromy Principle I) Let γ_s, with $s \in I$, be a family of homotopic paths with fixed endpoints. If the germ $f^\star$ admits an analytic continuation $f^\star(\gamma_s)$ along any parth γ_s, then $f^\star(\gamma_0) = f^\star(\gamma_1)$.*

Example 3.5. (The Complex Logarithm) Our starting point is the function element (f_0, D_0) in the disk $D_0 := B_1(1)$ with

$$f_0(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k = \log|z| + i\mathrm{Arg}z. \tag{18}$$

In order to prove that this function element admits an unrestricted analytic continuation in $\mathbb{C}\backslash\{0\}$, we consider any path $\gamma\colon I \to \mathbb{C}\backslash\{0\}$ with initial point $z_0 = 1$ and arbitrary terminal point z_1.

In order to construct function elements of an analytic continuation of (f_0, D_0) along γ, we first pick a point $z_t := \gamma(t)$ on γ and denote by $D_t := B_{|z_t|}(z_t)$ for all $t \in I$ (the largest disk around z_t contained in $\mathbb{C}\backslash\{0\}$). To find an appropriate argument of z_t, we denote by $t \mapsto a(t)$ the continuous branch of the argument along γ which is equal to the principle value $\mathrm{Arg}1 = 0$ at its initial point and set $\arg_\gamma z_t := a(t)$. Finally, we define the function element (f_t, D_t) by

$$f_t(z) := \log|z_t| + i\arg_\gamma z_t + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{kz_t^k}(z - z_t)^k$$

for all $z \in D_t$. The series on the right-hand side results from substituting z by z/z_t in (18), so that D_t is indeed its disk of convergence.

Part III
Holomorphic Functions

4 Definition of Holomorphic Function

Let Ω be an open subset of $\mathbb{C}$ and let f be a complex-valued function defined on Ω. The function f is said to be **holomorphic at the point** $z \in \Omega$ if the quotient

$$\frac{f(z+h) - f(z)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z + h \in \Omega$ so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by $f'(z)$, and is called the **derivative of** f **at** z:

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

The function f is said to be **holomorphic on** Ω if it is holomorphic at every point of Ω. If C is a closed subset of $\mathbb{C}$, we say that f is **holomorphic on** C if f is holomorphic in some open set containing C. Note that if f is holomorphic at a point $z \in \mathbb{C}$, then it is not necessarily holomorphic on $\{z\}$. If f is holomorphic on all of $\mathbb{C}$ we say that f is **entire**.

4.1 Examples of Holomorphic Functions

Example 4.1.

1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z$ for all $z \in \mathbb{C}$. Then f is entire. Indeed, let $z \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{z + h - z}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \bar{z}$ for all $z \in \mathbb{C}$. Then f is continuous everywhere in $\mathbb{C}$ but is not holomorphic at any point in $\mathbb{C}$. To see this, let $z \in \mathbb{C}$. Then

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} \\ &= \lim_{h \to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} \\ &= \lim_{h \to 0} \frac{\bar{h}}{h}. \end{aligned}$$

   But this limit doesn't exist. Indeed, assume it did exist (to obtain a contradiction). Then setting $h = \varepsilon$, where $\varepsilon \in \mathbb{R}$, and taking $\varepsilon \to 0$, we see that

$$\begin{aligned} \lim_{h \to 0} \frac{\bar{h}}{h} &= \lim_{\varepsilon \to 0} \frac{\bar{\varepsilon}}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} \\ &= 1. \end{aligned}$$

On the other hand, setting $h = i\varepsilon$, where $\varepsilon \in \mathbb{R}$, and taking $\varepsilon \to 0$, we see that

$$\begin{aligned}\lim_{h\to 0}\frac{\overline{h}}{h} &= \lim_{i\varepsilon\to 0}\frac{\overline{i\varepsilon}}{i\varepsilon} \\ &= \lim_{i\varepsilon\to 0}\frac{-i\varepsilon}{i\varepsilon} \\ &= -1.\end{aligned}$$

This is a contradiction. We conclude that the limit does not exist.

3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = |z|^2 = z\overline{z}$ for all $z \in \mathbb{C}$. Then f is holomorphic at the point 0 but nowhere else. Indeed,

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0}\frac{|h|^2}{h} \\ &= \lim_{h\to 0}\frac{h\overline{h}}{h} \\ &= \lim_{h\to 0}\overline{h} \\ &= 0,\end{aligned}$$

implies that f is holomorphic at 0. Now assume (to obtain a contradiction) that f is holomorphic at some $w \neq 0$. Let $g\colon \mathbb{C} \to \mathbb{C}$ be the identity function, given by $g(z) = z$ for all $z \in \mathbb{C}$. Then since g is holomorphic at w and $g(w) \neq 0$, the quotient f/g must be holomorphic at w as well. But this is a contradiction since the quotient is the complex-conjugation function, given by $f(z)/g(z) = \overline{z}$ for all $z \in \mathbb{C}$, which we know is not holomorphic anywhere in $\mathbb{C}$. This example demonstrates that a function being holomorphic at a point does *not* imply it being holomorphic in some neighborhood of that point.

4. Let $f\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be given by $f(z) = 1/z$ for all $z \in \mathbb{C}$. The f is holomorphic in its domain $\mathbb{C}\backslash\{0\}$. Indeed, let $z \in \mathbb{C}\backslash\{0\}$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}\frac{\frac{1}{z+h}-\frac{1}{z}}{h} \\ &= \lim_{h\to 0}\frac{\frac{-h}{(z+h)z}}{h} \\ &= \lim_{h\to 0}\frac{-1}{(z+h)z} \\ &= \frac{-1}{z^2}.\end{aligned}$$

z_{0}

4.2 Holomorphic Functions form a $\mathbb{C}$-Vector Space

Proposition 4.1. *Let f and g be complex-valued functions defined in a neighborhood of a point z in the complex plane. Then for all $a, b \in \mathbb{C}$, the function $af + bg$ is holomorphic at z. Moreover, we have*

$$(af+bg)'(z) = af'(z) + bg'(z).$$

Proof. This follows from linearity of the limit operator:

$$\begin{aligned}(af+bg)'(z) &= \lim_{h\to 0}\frac{(af+bg)(z+h)-(af+bg)(z)}{h} \\ &= \lim_{h\to 0}\frac{af(z+h)+bg(z+h)-af(z)-bg(z)}{h} \\ &= a\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} + b\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} \\ &= af'(z)+bg'(z).\end{aligned}$$

□

4.3 Chain Rule and Product Rule

Let Ω be an open subset of $\mathbb{C}$ and f a complex-valued function on Ω. Recall that f is continuous at $z_0 \in \Omega$ if and only if there exists a small neighborhood $U \subseteq \Omega$ of z_0 and a function $\psi : U \to \mathbb{C}$ where $\psi(z) \to 0$ as $z \to z_0$ such that

$$f(z) = f(z_0) + \psi(z), \tag{19}$$

for all $z \in U$. Indeed, we can pick $U = \Omega$ and define $\psi : U \to \mathbb{C}$ by $\psi(z) = f(z) - f(z_0)$ for all $z \in U$. Then continuity of f at z_0 implies $\psi(z) \to 0$ as $z \to z_0$, and conversely the expression (19) together with the fact that $\psi(z) \to 0$ as $z \to z_0$ implies f is continuous at z_0.

When f is holomorphic at $z_0 \in \Omega$, there is an even better approximation of f at z_0:

Lemma 4.1. *Let Ω be an open subset of $\mathbb{C}$ and f a complex-valued function on Ω. Then f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a (which is necessarily equal to $f'(z_0)$ as limits are unique) such that*

$$f(z_0 + h) = f(z_0) + ah + \psi(h)h, \tag{20}$$

where ψ is a function defined for all small h and $\lim_{h \to 0} \psi(h) = 0$.

Remark. By setting $h = z - z_0$, we can rewrite (20) as

$$f(z) = f(z_0) + a(z - z_0) + \psi_0(z)(z - z_0),$$

where $\psi_0(z) = \psi(z - z_0)$ (and so $\psi_0(z) \to 0$ as $z \to z_0$).

Proof. Assume (20) holds. Then

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{ah + \psi(h)h}{h} = a$$

implies f is holomorphic at z_0. Conversely, assume that f is holomorphic at z_0. Define ψ as

$$\psi(h) = \frac{f(z_0 + h) - f(z_0) - f'(z_0)h}{h}.$$

Then ψ satisfies the desired properties and (20) holds. □

Let us now show how we can use Lemma (4.1) to prove the Chain Rule and Product Rule. First we prove the Chain Rule:

Proposition 4.2. *Let f be holomorphic at z_0 and g be holomorphic at $f(z_0)$. Then $g \circ f$ is holomorphic at z_0 and, moreover, the Chain Rule holds*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. Since f is holomorphic at z_0, we can express f locally at z_0 as

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \psi_1(h)h$$

where ψ_1 is a function defined for all small h and $\psi_1(h) \to 0$ as $h \to 0$. Since g is holomorphic at $f(z_0)$, we can express g locally at $f(z_0)$ as

$$g(f(z_0) + h) = g(f(z_0)) + g'(f(z_0))h + \psi_2(h)h$$

where ψ_2 is a function defined for all small h and $\psi_2(h) \to 0$ has $h \to 0$. Using these local expressions, we can now express $g \circ f$ locally at z_0:

$$\begin{aligned}
(g \circ f)(z_0 + h) &= g(f(z_0 + h)) \\
&= g(f(z_0) + f'(z_0)h + \psi_1(h)h) \\
&= g(f(z_0)) + g'(f(z_0))(f'(z_0)h + \psi_1(h)h) + \psi_2(h)(f'(z_0)h + \psi_1(h)h) \\
&= g(f(z_0)) + g'(f(z_0))f'(z_0)h + \psi_3(h)h
\end{aligned}$$

where $\psi_3(h) = g'(f(z_0))\psi_1(h)h + \psi_2(h)f'(z_0)h + \psi_1(h)\psi_2(h)h$. Since ψ_3 is a function defined for all small h and $\psi_3(h) \to 0$ has $h \to 0$, it follows from uniqueness of limits that

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

□

Corollary. *Let f be holomorphic at z_0. If $f(z_0) \neq 0$, then $1/f$ is holomorphic at z_0*

Proof. The function $1/f$ can be viewed as the composition of $g \circ f$, where g is given by $g(z) = 1/z$. Then $1/f$ is holomorphic at z_0 since f is holomorphic at z_0 and g is holomorphic at $f(z_0)$ (because $f(z_0) \neq 0$). □

Now we will prove the Product Rule:

Proposition 4.3. *Let f and g be holomorphic at z_0. Then fg is holomorphic at z_0 and, moreover, the Product Rule holds*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Proof. Since f is holomorphic at z_0, we can express f locally at z_0 as

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \psi_1(h)h$$

where ψ_1 is a function defined for all small h and $\psi_1(h) \to 0$ as $h \to 0$. Since g is holomorphic at z_0, we can express f locally at z_0 as

$$g(z_0 + h) = g(z_0) + g'(z_0)h + \psi_2(h)h$$

where ψ_2 is a function defined for all small h and $\psi_2(h) \to 0$ as $h \to 0$. Then

$$\begin{aligned}
(fg)(z_0 + h) &= f(z_0 + h)g(z_0 + h) \\
&= (f(z_0) + f'(z_0)h + \psi_1(h)h)(g(z_0) + g'(z_0)h + \psi_2(h)h) \\
&= f(z_0)g(z_0) + (f(z_0)g'(z_0) + f'(z_0)g(z_0))h + \psi_3(h)h,
\end{aligned}$$

where $\psi_3(h) = f(z_0)\psi_2(h) + f'(z_0)g'(z_0)h + g(z_0)\psi_1(h)$. Since ψ_3 is a function defined for all small h and $\psi_3(h) \to 0$ as $h \to 0$, it follows from uniqueness of limits that

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$

□

Since the function $f : \mathbb{C} \to \mathbb{C}$, given by $f(z) = z$, is holomorphic, it follows from Proposition (4.1), Proposition (**??**) and the fact that the function $f : \mathbb{C} \to \mathbb{C}$, given by $f(z) = z$, is entire, that polynomials are entire.

4.4 Analytic Functions are Holomorphic

Let Ω be an open set and let $f \colon \Omega \to \mathbb{C}$. We say f is **analytic** if at each $a \in \Omega$, there exists an open neighborhood U of a and a power series $\sum a_n(z-a)^n$ centered at a such that $U \subseteq \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in U$.

Proposition 4.4. *Let Ω be an open set and let $f \colon \Omega \to \mathbb{C}$ be analytic. Then f is holomorphic.*

Proof. Let $a \in \Omega$. Choose $r > 0$ and a power series $\sum a_n(z-a)^n$ centered at a such that $B_r(a) \subset \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in B_r(a)$. We claim that f is holomorphic in $B_r(a)$. Let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$ and let $z \in B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m\right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\
&= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.
\end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m \colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z - a + h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned}|g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z - a + h)^{m-k}(z-a)^{k-1} \right| \\ &< \left| m a_m r^{m-1} \right|.\end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. □

One of the great triumphs of complex analysis is that the converse to Proposition (4.4) is also true, namely holomorphic functions are analytic.

4.5 Cauchy-Riemann Equations

Throughout this subsection, let f be a complex-valued function defined on some open subset Ω of $\mathbb{C}$ and fix a point $z_0 = x_0 + iy_0$ in Ω. Since $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-vector space, there is a unique decomposition of a complex number z as

$$z = x + iy,$$

where x and y are real numbers. Similarly, there is a unique decomposition of f as

$$f = u + iv,$$

where u and v are real-valued functions defined on Ω.

We define a map $\widetilde{\cdot} : \mathbb{C} \to \mathbb{R}^2$, given by mapping the complex number $z = x + iy$ to the vector $\widetilde{z} = (x, y)$. We also define $\widetilde{u} : \mathbb{R}^2 \to \mathbb{R}$ and $\widetilde{v} : \mathbb{R}^2 \to \mathbb{R}$ by the formulas $\widetilde{u}(x, y) = u(x + iy)$ and $\widetilde{v}(x, y) = v(x + iy)$ respectively. Similarly, we define $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula $\widetilde{f}(x, y) = (\widetilde{u}(x, y), \widetilde{v}(x, y))$.

We say $\widetilde{f}$ is **differentiable** at $\widetilde{z}_0$ if there exists a linear transformation $J\widetilde{f}(\widetilde{z}_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|\widetilde{f}(\widetilde{z}_0 + \widetilde{h}) - \widetilde{f}(\widetilde{z}_0) - J\widetilde{f}(\widetilde{z}_0)(\widetilde{h})\|}{\|\widetilde{h}\|} \to 0$$

as $\widetilde{h} \to 0$ where $\widetilde{h} = (h_1, h_2) \in \mathbb{R}^2$. Equivalently, we can write

$$\widetilde{f}(\widetilde{z}_0 + \widetilde{h}) = \widetilde{f}(\widetilde{z}_0) + J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) + \|\widetilde{h}\|\psi(\widetilde{h}),$$

where $\widetilde{\psi}(\widetilde{h}) \to 0$ as $\widetilde{h} \to 0$. The linear transformation $J\widetilde{f}(\widetilde{z}_0)$ is unique and is called the **derivative** of $\widetilde{f}$ at $\widetilde{z}_0$. If $\widetilde{f}$ is differentiable, then the partial derivatives of its component functions exist, and the linear transformation $J\widetilde{f}(\widetilde{z}_0)$ is described in the standard basis of $\mathbb{R}^2$ by the Jacobian matrix of $\widetilde{f}$:

$$J\widetilde{f}(\widetilde{z}_0) = \begin{pmatrix} \partial_x \widetilde{u}(\widetilde{z}_0) & \partial_y \widetilde{u}(\widetilde{z}_0) \\ \partial_x \widetilde{v}(\widetilde{z}_0) & \partial_y \widetilde{v}(\widetilde{z}_0) \end{pmatrix}.$$

In the case of complex-differentiation the complex-derivative is a complex number $f'(z_0)$, while in the case of real-derivatives, it is a matrix. There is, however, a connection between these two notions, which is given in terms of special relations that are satisfied by the entries of the Jacobian matrix, that is, the partials of u and v.

Theorem 4.2. *If f is holomorphic at z_0, then the partial derivatives $\partial_x \widetilde{u}$, $\partial_y \widetilde{u}$, $\partial_x \widetilde{v}$, and $\partial_y \widetilde{v}$ exist and satisfy the **Cauchy-Riemann Equations** at z_0:*

$$\begin{aligned}\partial_x \widetilde{u}(\widetilde{z}_0) &= \partial_y \widetilde{v}(\widetilde{z}_0) \\ -\partial_y \widetilde{v}(\widetilde{z}_0) &= \partial_x \widetilde{v}(\widetilde{z}_0).\end{aligned}$$

Moreover, $\widetilde{f}$ is real-differentiable and its Jacobian at the point $\widetilde{z}_0$ satisfies

$$det\ J\widetilde{f}(\widetilde{z}_0) = |f'(z_0)|^2.$$

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned}
f'(z_0) &= \lim_{\varepsilon\to 0} \frac{f(z_0+\varepsilon)-f(z_0)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{u((x_0+\varepsilon)+iy_0)+iv((x_0+\varepsilon)+iy_0)-u(x_0+iy_0)-iv(x_0+iy_0)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{u((x_0+\varepsilon)+iy_0)-u(x_0+iy_0)}{\varepsilon} + i\lim_{\varepsilon\to 0} \frac{v((x_0+\varepsilon)+iy_0)-v(x_0+iy_0)}{\varepsilon} \\
&= \partial_x \widetilde{u}(\widetilde{z}_0) + i\partial_x \widetilde{v}(\widetilde{z}_0).
\end{aligned}$$

Similarly,

$$\begin{aligned}
f'(z_0) &= \lim_{i\varepsilon\to 0} \frac{f(z_0+i\varepsilon)-f(z_0)}{i\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{u(x_0+i(y_0+\varepsilon))+iv(x_0+i(y_0+\varepsilon))-u(x_0+iy_0)-v(x_0+iy_0)}{i\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{v(x_0+i(y_0+\varepsilon))-v(x_0+iy_0)}{\varepsilon} - i\lim_{\varepsilon\to 0} \frac{u(x_0+i(y_0+\varepsilon))-u(x_0+iy_0)}{\varepsilon} \\
&= -i\partial_y \widetilde{u}(\widetilde{z}_0) + \partial_y \widetilde{v}(\widetilde{z}_0).
\end{aligned}$$

Equating the two formulas for $f'(z_0)$ above yields the Cauchy-Riemann equations.

To see that $\widetilde{f}$ is differentiable, it suffices to observe that if $\widetilde{h} = (h_1, h_2)$ and $h = h_1 + ih_2$, then the Cauchy-Riemann equations imply

$$J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) = \left(\partial_x \widetilde{u} - i\partial_y \widetilde{u}\right)(h_1 + ih_2) =$$

complex-differentiability of f at z_0 implies

$$f(z_0+h) - f(z_0) = f'(z_0)(h) + h\psi(h),$$

note that

$$\begin{aligned}
\widetilde{f}(\widetilde{z}_0+\widetilde{h}) - \widetilde{f}(\widetilde{z}_0) &= J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) + \|\widetilde{h}\|\widetilde{\psi}(\widetilde{h}), \\
\widetilde{f}(\widetilde{z}_0+\widetilde{h}) - \widetilde{f}(\widetilde{z}_0) &= J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) + \|\widetilde{h}\|\widetilde{\psi}(\widetilde{h}),
\end{aligned}$$

$$\det J\widetilde{f}(\widetilde{z}_0) = |f'(z_0)|^2.$$

□

Define the two differential operators

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) \text{ and } \partial_{\overline{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Proposition 4.5. *If f is holomorphic at $z_0 = x_0 + iy_0$, then*

$$\partial_{\overline{z}} f(z_0) = 0 \textit{ and } f'(z_0) = \partial_z f(z_0) = 2\partial_z u(z_0).$$

Also, if we consider f as a function in two real variables x, y, then f is real-differentiable $p = (x_0, y_0)$ and

$$\textit{det } J\widetilde{f}(p) = |f'(z_0)|^2.$$

Proof. First note that the Cauchy-Riemann equations at z_0 are equivalent to $\partial_{\overline{z}} f(z_0) = 0$. Indeed,

$$\begin{aligned}
0 &= \partial_{\overline{z}} f(z_0) \\
&= \frac{1}{2}\left(\partial_x + i\partial_y\right) f(z_0) \\
&= \frac{1}{2}\left(\partial_x f(z_0) + i\partial_y f(z_0)\right) \\
&= \frac{1}{2}\left(\partial_x u(z_0) + i\partial_x v(z_0) + i\partial_y u(z_0) - \partial_y v(z_0)\right) \\
&= \frac{1}{2}\partial_x u(z_0) - \partial_y v(z_0) + \frac{i}{2}\left(\partial_y u(z_0) + \partial_x v(z_0)\right),
\end{aligned}$$

and equating the real and imaginary parts gives us the Cauchy-Riemann equations.

Moreover, we have

$$
\begin{aligned}
f'(z_0) &= \frac{1}{2}\left(\partial_x f(z_0) - i\partial_y f(z_0)\right) \\
&= \partial_x f(z_0),
\end{aligned}
$$

and the Cauchy-Riemann equations give $\partial_z f = 2\partial_z u$.

To prove that f is real-differentiable, it suffices to observe that if $H = (h_1, h_2)$ and $h = h_1 + ih_2$, then the Cauchy-Riemann equations imply

$$
Jf(p)(h) = (\partial_x u - i\partial_y u)(h_1 + ih_2) = f'(z_0)h
$$

□

We can clarify the situation further by defining two differential operators

$$
\partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right) \quad \text{and} \quad \partial_{\overline{z}} := \frac{1}{2}\left(\partial_x + i\partial_y\right)
$$

Example 4.2. Let $f(z) = \overline{z} = x - iy$. Then

$$
\partial_x u = 1 \neq -1 = \partial_y v,
$$

so f is not differentiable.

Corollary.

1. *If f is holomorphic on $D_r(z_0)$ with $f'(z) = 0$ for all $z \in D_r(z_0)$, then f is constant on $D_r(z_0)$.*

2. *If f is holomorphic on $D_r(z_0)$ and real-valued on $D_r(z_0)$, then f is constant.*

Proof.

1. Since $f'(z) = 0$, we have $\partial_x u + i\partial_x v = 0 = \partial_y v - i\partial_y u$. This implies $\partial_x u = \partial_y u = \partial_x v = \partial_y v = 0$. Therefore u and v are constant functions, and hence, f is constant.

2. Let $f = u + iv$. Then $v = 0$ on $D_r(z_0)$. So $\partial_x v = \partial_y v = 0$ implies $\partial_x u = \partial_y u = 0$. Therefore u and v are constant, and so f is constant.

□

Example 4.3. The function $f(z) = |z|^2$ is not analytic anywhere because it is real-valued and non-constant.

Theorem 4.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ have radius of convergence $R > 0$. Then f is analytic on $D_R(z_0)$ with*

$$
f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}
$$

for $|z - z_0| < R$.

Part IV
Complex Integration

5 Paths

Throughout this section, let D be a nonempty subset of $\mathbb{C}$.

5.1 Definition of a Path

Definition 5.1. A **path** is a continuous function of the form $\gamma\colon [a,b] \to \mathbb{C}$ where $[a,b]$ is a closed interval in $\mathbb{R}$. The image set $[\gamma] := \gamma\left([a,b]\right)$ is said to be the **trace** (or **trajectory**) of γ. If $[\gamma] \subseteq D$, then we say γ is a **path in** D. The space of all paths in D is denoted $\mathcal{P}(D)$. The points $\gamma(a)$ and $\gamma(b)$ are called the **source** and **target** of γ, respectively. We say γ is **simple** if $\gamma(s) = \gamma(t)$ with $s < t$ implies $s = a$ and $t = b$. We say γ is a **loop in** D **based at** z if $\gamma(a) = z = \gamma(b)$. The space of all loops in D is denoted $\mathcal{L}(D)$ and the space of all loops in D based at z is denoted $\mathcal{L}(D, z)$.

5.2 Reparametrization

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. Suppose $\varphi\colon [c,d] \to [a,b]$ is a continuous function from the closed interval $[c,d]$ to the closed interval $[a,b]$. Then $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a path and is called a **reparametrization** of γ. More specifically,

1. if $\varphi(c) = a$ and $\varphi(d) = b$, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **positive reparametrization** of γ.
2. if $\varphi(c) = b$ and $\varphi(d) = a$, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **negative reparametrization** of γ.
3. if φ is a linear map, then we call $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ a **linear reparametrization** of γ.
4. if $\gamma \circ \varphi$ is a positive linear reparametrization whose domain is I, then the map $\varphi\colon I \to [a,b]$ is uniquely determined: it is given by the formula $\varphi(t) = a(1-t) + tb$ for all $t \in I$. In this case, we call $\gamma \circ \varphi\colon I \to \mathbb{C}$ *the* **normalized form** of γ. Any path whose domain is I is called a **normal path**.

As it turns out, the choice of the parameter interval $[a,b]$ is not that essential for the definition of a path. Thus we will usually only work with normal paths. Any contstruction we describe which uses normal paths can easily be extended to all paths by taking their normalized forms.

5.3 Standard Examples

Example 5.1. Let $w, z \in D$, $r > 0$, and let $\gamma\colon I \to D$ be a path in D.

1. The **constant path at** z is the path $c_z\colon I \to \mathbb{C}$ defined by $c_z(t) = z$ for all $t \in I$.
2. The **oriented line segment from** w **to** z is the path $[w,z]\colon I \to D$ defined by $[w,z](t) = w(1-t) + tz$ for all $t \in I$.
3. The **standard parametrization of** $C_r(z)$ is the path $\gamma_r(z)\colon I \to \mathbb{C}$ defined by $\gamma_r(z)(t) = z + re^{2\pi i t}$ for all $t \in I$.
4. The **reversed path** (or **negative path**) γ^- of γ is the path $\gamma^-\colon I \to \mathbb{C}$ defined by the formula $\gamma^-(t) = \gamma(1-t)$ for all $t \in I$.

5.4 Concatenation of Paths

let $\gamma_1\colon I \to \mathbb{C}$ and $\gamma_2\colon I \to \mathbb{C}$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then their **concatenation** $\gamma_2 \oplus \gamma_1$ is a path $\gamma_2 \oplus \gamma_1\colon I \to \mathbb{C}$ defined by the formula

$$(\gamma_2 \oplus \gamma_1)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2\left(2\left(t - \frac{1}{2}\right)\right) & \frac{1}{2} < t \leq 1. \end{cases}$$

for all $t \in [0,1]$. The idea is that we traverse γ_1 and then γ_2 all in one day. The $2t$ in $\gamma_1(2t)$ comes from the fact that we need to traverse γ_1 twice as fast, the $t - \frac{1}{2}$ in $\gamma_2\left(2\left(t - \frac{1}{2}\right)\right)$ comes from the fact that we need to wait half a day before we start traversing γ_2, and the $2\left(t - \frac{1}{2}\right)$ in $\gamma_2\left(2\left(t - \frac{1}{2}\right)\right)$ comes from the fact that we need to traverse γ_2 twice as fast.

5.5 Polygonal Paths and Paraxial Paths

A **polygonal path** is the sum $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_n$ of segments $\gamma_k = [z_{k-1}, z_k]$ A polygonal path is called **paraxial** if each of its segments is parallel to the real or imaginary axis.

6 Homotopy

Throughout this subsection, let D be a nonempty subset of $\mathbb{C}$.

6.1 Homotopy of Paths

Let $\alpha\colon I \to D$ and $\beta\colon I \to D$ be two paths in D. We say that α is **homotopic** to β as paths, denoted $\alpha \sim \beta$, if there exists a continuous function $H\colon I \times I \to D$ such that $H(0,t) = \alpha(t)$ and $H(1,t) = \beta(t)$ for all $t \in I$. The map H is called a **homotopy** joining α to β. It is easy to check that $\sim$ is an equivalence relation in the set $\mathcal{P}(D)$.

6.2 Homotopy of Paths with Fixed Endpoints

Let $\alpha\colon I \to D$ and $\beta\colon I \to D$ be two paths in D with the same source and target, say $\alpha(0) = w = \beta(0)$ and $\alpha(1) = z = \beta(1)$. Then a homotopy H from α to β is called a **homotopy with fixed endpoints** if we additionally have $H(s,0) = w$ and $H(s,1) = z$ for all $s \in I$.

6.3 Homotopy of Loops

Let α and β be two loops in D based at z. We say α is homotopic to β as loops based at z, denoted $\alpha \sim_z \beta$, if there exists a homotopy $H\colon I \times I \to D$ with fixed endpoints from α to β. The space of all loops in D based at z is denoted $\mathcal{L}(D,z)$. It is easy to check that $\sim_z$ is an equivalence relation in the set $\mathcal{L}(D,z)$.

6.4 Free Homotopy of Loops

There is a slightly more general notion of homotopy in the context of loops which we will also consider. Let α and β be two loops in D based at z. We say α is **freely homotopic** to β, denoted $\alpha \sim_{\text{free}} \beta$ if there exists a homotopy $H\colon I \times I \to D$ from α to β such that $H(s,0) = H(s,1)$ for all $s \in I$. Such a homotopy is called a **free homotopy** from α to β. The added condition "$H(s,0) = H(s,1)$ for all $s \in I$" ensures that for each $s \in I$, the function $H(s,-)\colon I \to D$ is a loop. So the base point is allowed to move freely as s goes from 0 to 1.

6.5 The Fundamental Group $\pi(D,z)$

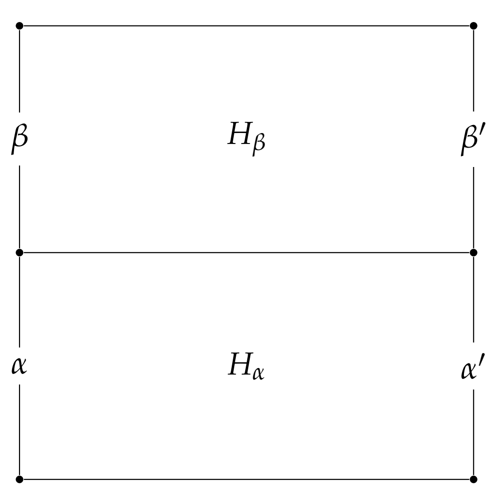
Let $z \in D$. Concatenation serves as a natural binary operation on $\mathcal{L}(D,z)$. However, this binary operation is rather complicated. For example, it isn't associative, it has no identity, and it has no inverses. It turns out however, if we consider loops up to homotopy with fixed endpoints, then we do get these properties. In fact, we will show that $(\mathcal{L}(D,z)/\sim_z, \oplus)$ forms a group, called the **fundamental group of** D **based at** z. Let us show this in the following sequence of steps:

6.5.1 Concatenation Passes to Quotient

First we need to show that concatenation passes to the quotient $\mathcal{L}(D,z)/\sim_z$, thus giving a well-defined binary operation on $\mathcal{L}(D,z)/\sim_z$. Let $\alpha\colon I \to D$ and $\beta\colon I \to D$ be two loops based at z. Suppose α' and β' are two loops based at z such that $\alpha' \sim_z \alpha$ and $\beta' \sim_z \beta$. Let H_α and H_β be their respective homotopies with endpoints. Define $H\colon I \times I \to X$ by the formula

$$H(s,t) = \begin{cases} H_\alpha(s,2t) & 0 \le t \le \frac{1}{2} \\ H_\beta(s,2t-1) & \frac{1}{2} < t \le 1. \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy with fixed endpoints from $\beta \oplus \alpha$ to $\beta' \oplus \alpha'$. One may visualize this homotopy as below:

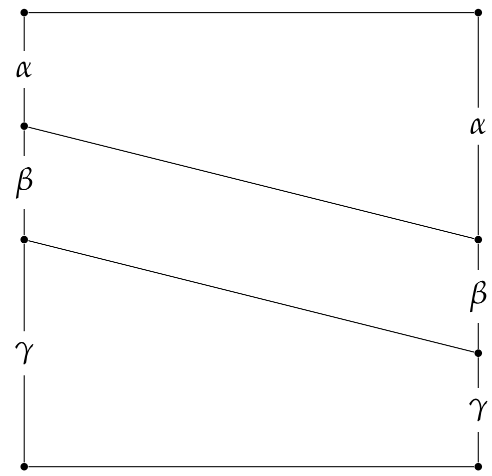


6.5.2 Associativity

Next we show that $(\mathcal{L}(D,z)/\sim_z, \oplus)$ is associative. Suppose α, β, and γ are loops based at z. Define $H\colon I \times I \to X$ by the formula

$$H(s,t) = \begin{cases} \gamma\left(\left(\frac{4}{2-s}\right) \cdot t\right) & 0 \le t \le \frac{2-s}{4} \\ \beta\left(4 \cdot \left(t - \left(\frac{2-s}{4}\right)\right)\right) & \frac{2-s}{4} < t \le \frac{3-s}{4} \\ \alpha\left(\left(\frac{4}{1+s}\right) \cdot \left(t - \left(\frac{3-s}{4}\right)\right)\right) & \frac{3-s}{4} < t \le 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy with fixed endpoints from $(\alpha \oplus \beta) \oplus \gamma$ to $\alpha \oplus (\beta \oplus \gamma)$. One may visualize this homotopy as below:

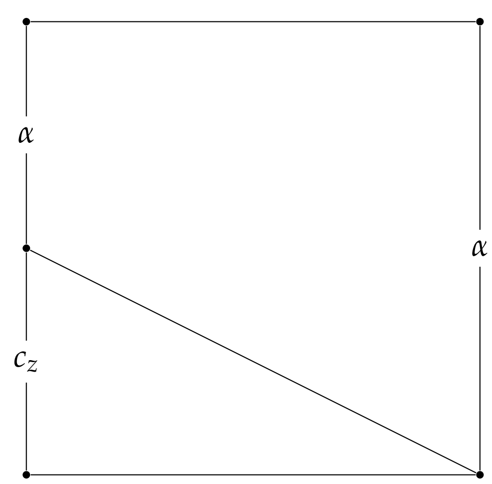


6.5.3 Identity

Next, we want to show that c_z represents the identity element in $(\mathcal{L}(D,z)/\sim_z, \oplus)$. Let α be a loop in D based at z. Define $H\colon I \times I \to X$ by the formula

$$H(s,t) = \begin{cases} c_z\left(\left(\frac{2}{1-s}\right)t\right) & 0 \leq t < \frac{1-s}{2} \\ \alpha\left(\left(\frac{2}{1+s}\right)\left(t - \left(\frac{1-s}{2}\right)\right)\right) & \frac{1-s}{2} \leq t \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy fixed at z from $\alpha \oplus c_z$ to c_z. One may visualize this homotopy as below:



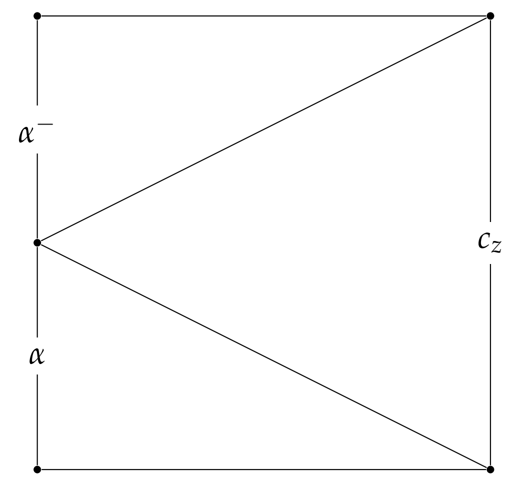
A similar argument gives a homotopy fixed at z from $c_z \oplus \alpha$ to α.

6.5.4 Inverses

Finally, we want to show that $(\mathcal{L}(D,z)/\sim_z, \oplus)$ has inverses. Suppose α is a loop based at z. Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} \alpha\left(\left(\frac{2}{1-s}\right)t\right) & 0 \leq t < \frac{1-s}{2} \\ c_z\left(\left(\frac{1}{s}\right)\left(t - \left(\frac{1-s}{2}\right)\right)\right) & \frac{1-s}{2} \leq t < \frac{1+s}{2} \\ \alpha^-\left(\left(\frac{2}{1-s}\right)\left(t - \frac{2-s}{2}\right)\right) & \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

for all $(s,t) \in I \times I$. Then H is easily seen to be a homotopy fixed at z from $\alpha^- \oplus \alpha$ to c_z. One may visualize this homotopy as below:



A similar argument gives a homotopy fixed at z from $\alpha \oplus \alpha^-$ to c_z.

6.5.5 Changing the Base Point

We have shown that $(\mathcal{L}(D,z)/\sim_z, \oplus)$ forms a group. This group is called the **fundamental group of D based at z** and is denoted as $\pi_1(D,z)$. We now what to consider what happens if we change the base point $z \in D$ to another basepoint, say $w \in D$.

Proposition 6.1. *If D is path connected, then $\pi_1(D,z) \cong \pi_1(D,w)$.*

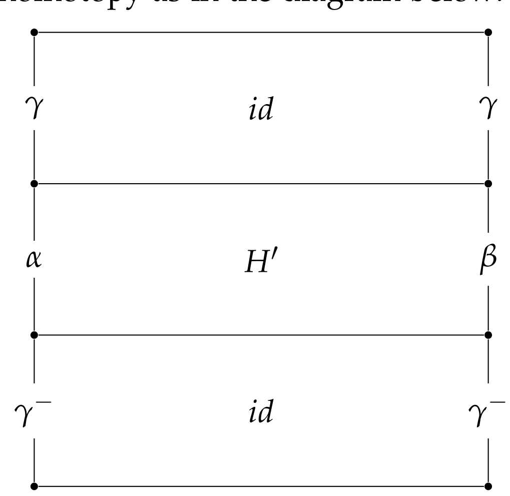
Proof. Choose a path γ from z to w. Define $-^{\gamma}\colon \pi_1(D,z) \to \pi_1(D,w)$ by the formula

$$\overline{\alpha}^{\gamma} = \overline{\gamma \oplus \alpha \oplus \gamma^-}$$

for all $\overline{\alpha} \in \pi_1(D,z)$. We need to show that this is a well-defined map, so choose another representative of the equivalence class $\overline{\alpha}$, say β, and let H' be a homotopy with fixed endpoints from α to β. Define $H\colon I \times I \to D$ by the formula

$$H(s,t) = \begin{cases} \gamma^-(3t) & 0 \leq t < \frac{1}{3} \\ H'(s,t) & \frac{1}{3} \leq t < \frac{2}{3} \\ \gamma(3t-2) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Then H is easily checked to be a homotopy with fixed endpoints from $\gamma \circ \alpha \circ \gamma^-$ to $\gamma \circ \beta \circ \gamma^-$. Thus, $-^{\gamma}$ is well-defined. One may visualize this homotopy as in the diagram below:



The map $-^{\gamma}$ is easily checked to be a group isomorphism with inverse being given by $-^{\gamma^-}$. □

6.5.6 Simply Connected Domains

A domain D is called **simply connected** if for any two paths $\alpha\colon I \to D$ and $\beta\colon I \to D$ which share the same source and target are homotopic with fixed endpoints in D.

6.5.7 Null-Homotopic

A loop which is freely homotopic in D to a constant path is said to be **null-homotopic** in D.

Lemma 6.1. *For any path γ in D, the loop $\gamma \oplus \gamma^-$ is null-homotopic in D to its base point $c_{\gamma(0)}$.*

Proof. Define the function $H\colon I \times I \to D$ by the formula

$$H(s,t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1-s}{2} \\ c_{\gamma(1-s)}(t) & \text{if } \frac{1-s}{2} < t \le \frac{1+s}{2} \\ \gamma^-(2t-1-s) & \text{if } \frac{1+s}{2} < t \le 1 \end{cases}$$

Then H is easily checked to be a homotopy with fixed endpoints from $\gamma^- \oplus \gamma$ to $c_{\gamma(0)}$. □

Lemma 6.2. *If a loop with base point z_0 is null-homotopic in D, then it is also homotopic with fixed endpoints to the constant path c_{z_0}.*

Proof. Let H be a homotopy from the given path γ_0 to a point z_1. We define γ_s and γ_s^+ by the formulas

$$\gamma_s(t) = H(s,t) \quad \text{and} \quad \gamma_s^+(t) = H(st, 0)$$

for all $s, t \in I$, and we set $\gamma_s^- = (\gamma_s^+)^-$. Then the path γ_s^+ lies in D and connects z_0 with the moving basepoint $z_s = H(s,0) = H(s,1)$ of the loop γ_s. The family of loops

$$\gamma_s^* = \gamma_s^+ \oplus \gamma_s \oplus \gamma_s^-$$

has fixed base point z_0 and all paths in this family are homotopic in D. Now γ_0 is homotopic to γ_0^*, γ_0^* is homotopic to γ_1^*, and γ_1^* is homotopic to $\gamma_1^+ \oplus \gamma_1^-$, and the latter is homotopic to the base point z_0. □

7 Smooth Paths

7.1 Definition of Smooth Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path. We say that γ is **smooth** if it is continuously differentiable on $[a,b]$ (i.e. $\gamma'(t)$ exists for $t \in [a,b]$ and the function $t \mapsto \gamma'(t)$ is continuous). At the points $t = a$ and $t = b$, the quantities $\gamma'(a)$ and $\gamma'(b)$ are interpreted as one-sided limits

$$\gamma'(a) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{\gamma(a+h) - \gamma(a)}{h} \quad \text{and} \quad \gamma'(b) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{\gamma(b+h) - \gamma(b)}{h}.$$

In general, these quantities are called the right-hand derivative of $\gamma(t)$ at a, and the left-hand derivative of $\gamma(t)$ at b, respectively. More generally, we say that γ is **piecewise smooth** if there exists a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k, t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \dots, n-1$. In particular, the right-hand derivative of γ at t_k may differ from the left-hand derivative of γ at t_k, for $k = 1, \dots, n-1$.

Example 7.1. The paths described in Example (5.1) are all smooth.

7.2 Integrating along a Smooth Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path and suppose f is a continuous function defined on $[\gamma]$. Then the **integral of** f **along** γ is defined by the formula

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If γ is piecewise smooth, then we choose a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k, t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \dots, n-1$, and we define

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)dt.$$

7.3 Reparametrizing a Smooth Path

Definition 7.1. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path. Suppose $\varphi\colon [c,d] \to [a,b]$ is a continuously differentiable function from the closed interval $[c,d]$ to the closed interval $[a,b]$. Then $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a path and is called a **smooth reparametrization** of γ.

Remark. Note that a linear reparametrization is a smooth reparametrization. Thus the normalized form of a smooth path is also a smooth path.

As in the continuous case, the parameter interval $[a,b]$ is not that essential for the definition of a smooth path. For example, if $\gamma \circ \varphi\colon [c,d] \to \mathbb{C}$ is a smooth positive reparametrization of $\gamma\colon [a,b] \to \mathbb{C}$, then the change of variables formula implies

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_c^d f(\gamma(\varphi(s)))\gamma'(\varphi(s))\varphi'(s)ds \\ &= \int_c^d f((\gamma\circ\varphi)(s))(\gamma\circ\varphi)'(s)ds \\ &= \int_{\gamma\circ\varphi} f(z)dz.\end{aligned}$$

Thus we will usually only work with smooth normal paths. Any construction we describe which uses smooth normal paths can easily be extended to all paths by taking their normalized forms.

7.4 Defining the Length of a Path

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path. Then the **length** of γ, denoted $\mathrm{length}(\gamma)$, is defined by the formula

$$\mathrm{length}(\gamma) := \int_a^b |\gamma'(t)|\, dt.$$

If γ is piecewise-smooth, then we choose a partition

$$a = t_0 < t_1 < \cdots < t_n = b,$$

of the interval $[a,b]$ such that the restriction $\gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \dots, n-1$, and we define

$$\mathrm{length}(\gamma) = \sum_{k=0}^{n-1} \mathrm{length}\left(\gamma|_{[t_k,t_{k+1}]}\right).$$

7.5 Length of a Smooth Path equals the Length of its Normalized Form

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a smooth path and let $\varphi\colon I \to [a,b]$ be given by $\varphi(t) = a(1-t) + bt$ for all $t \in I$ (so $\gamma\circ\varphi\colon I \to \mathbb{C}$ is the normalized form of $\gamma\colon [a,b] \to \mathbb{C}$). Then by the change of variables formula, we have

$$\begin{aligned}\mathrm{length}(\gamma) &= \int_a^b |\gamma'(t)|\, dt. \\ &= \int_0^1 |\gamma'(\varphi(s))|\, \varphi'(s)ds \\ &= \int_0^1 |\gamma'(\varphi(s))|\, |\varphi'(s)|\, ds \\ &= \int_0^1 |\gamma'(\varphi(s))\varphi'(s)|\, ds \\ &= \int_0^1 |(\gamma\circ\varphi)'(s)|\, ds \\ &= \mathrm{length}(\gamma\circ\varphi),\end{aligned}$$

where we used the fact that for all $s \in I$, we have $\varphi'(s) = b - a > 0$.

7.6 Properties of Integration

Throughout this subsection, let Ω be an open subset of $\mathbb{C}$.

7.6.1 Linearity of Integration

Proposition 7.1. *Let $\gamma\colon I \to \Omega$ be a smooth path in Ω, let f, g be complex-valued functions defined on Ω, and let $\alpha, \beta \in \mathbb{C}$. Then*

$$\int_\gamma (\alpha f + \beta g)(z)dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

Proof. This follows from linearity of the Riemann integral. Indeed, we have

$$\begin{aligned}
\int_\gamma (\alpha f + \beta g)(z)dz &= \int_0^1 (\alpha f + \beta g)(\gamma(t))\gamma'(t)dt \\
&= \int_0^1 (\alpha f(\gamma(t))\gamma'(t) + \beta g(\gamma(t))\gamma'(t))dt \\
&= \alpha \int_0^1 f(\gamma(t))\gamma'(t)dt + \beta \int_0^1 g(\gamma(t))\gamma'(t)dt \\
&= \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.
\end{aligned}$$

□

7.6.2 Additivity of Concatenation of Smooth Paths

Proposition 7.2. *Let $\gamma_1\colon I \to \Omega$ and $\gamma_2\colon I \to \Omega$ be smooth paths in Ω such that $\gamma_1(1) = \gamma_2(0)$, and let $f\colon \Omega \to \mathbb{C}$ be a function. Then*

$$\int_{\gamma_2 \oplus \gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz + \int_{\gamma_1} f(z)dz.$$

Proof. This follows from additivity of the Riemann Integral. Indeed, we have

$$\begin{aligned}
\int_{\gamma_2 \oplus \gamma_1} f(z)dz &= \int_0^1 f((\gamma_2 \oplus \gamma_1)(t))(\gamma_2 \oplus \gamma_1)'(t)dt \\
&= 2\int_0^{1/2} f(\gamma_1(2t))\gamma_1'(2t)dt + 2\int_{1/2}^1 f(\gamma_2(2t-1))\gamma_2'(2t-1)dt \\
&= \int_0^1 f(\gamma_1(u))\gamma_1'(u)du + \int_0^1 f(\gamma_2(v))\gamma_2'(v)dv \\
&= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,
\end{aligned}$$

where we used the change of variables $u = 2t$ and $v = 2t - 1$.

□

7.6.3 Negativity of Reverse Orientation

Proposition 7.3. *Let $\gamma\colon I \to \Omega$ be a smooth paths in Ω, and let $f\colon \Omega \to \mathbb{C}$ be a function. Then*

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$$

Proof. This follows from a straightforward calculation:

$$\begin{aligned}
\int_{\gamma^-} f(z)dz &= -\int_0^1 f(\gamma(1-t))\gamma'(1-t)dt \\
&= \int_1^0 f(\gamma(s))\gamma'(s)ds \\
&= -\int_0^1 f(\gamma(s))\gamma'(s)ds \\
&= -\int_\gamma f(z)dz,
\end{aligned}$$

where we used the change of variable $s = 1 - t$. □

7.6.4 Useful Inequality

Proposition 7.4. *Let $\gamma\colon I \to \Omega$ be a smooth path in Ω, and let $f\colon \Omega \to \mathbb{C}$ be a function. Then one has the inquality*

$$\left|\int_\gamma f(z)dz\right| \leq \|f(z)\|_{[\gamma]} \cdot \mathit{length}(\gamma).$$

Proof. This follows from a straightforward calculation:

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &= \left|\int_0^1 f(\gamma(t))\gamma'(t)dt\right| \\ &\leq \int_0^1 \left|f(\gamma(t))\gamma'(t)\right| dt \\ &\leq \sup_{t\in I} |f(\gamma(t))| \cdot \int_0^1 \left|\gamma'(t)\right| dt \\ &= \sup_{z\in[\gamma]} |f(z)| \cdot \mathrm{length}(\gamma)\end{aligned}$$

□

7.6.5 Primitives

Definition 7.2. Let $f\colon \Omega \to \mathbb{C}$ be a function. A **primitive** for f on Ω is a function $F\colon \Omega \to \mathbb{C}$ that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 7.1. *Let $\gamma\colon I \to \Omega$ be a piecewise smooth path in Ω from z_1 to z_2, and let $f\colon \Omega \to \mathbb{C}$ be a function. Suppose that $F\colon \Omega \to \mathbb{C}$ is a primitive for f in Ω. Then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

In particular the integral vanishes if γ is a loop.

Proof. We first assume that γ is smooth. Then by the chain rule and the fundamental theorem of calculus, we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\ &= \int_0^1 F'(\gamma(t))\gamma'(t)dt \\ &= \int_0^1 (F\circ\gamma)'(t)dt \\ &= (F\circ\gamma)(b) - (F\circ\gamma)(a) \\ &= F(\gamma(b)) - F(\gamma(a)). \\ &= F(z_2) - F(z_1).\end{aligned}$$

Now we assume γ is piecewise smooth. Choose a partition

$$0 = t_0 < t_1 < \cdots < t_n = 1,$$

of the interval I such that the restriction $\gamma_k := \gamma|_{[t_k,t_{k+1}]}\colon [t_k, t_{k+1}] \to \mathbb{C}$ is smooth for each $k = 0, \ldots, n-1$. Then

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{k=0}^{n-1} \int_{\gamma_k} f(z)dz \\ &= \sum_{k=0}^{n-1} F(\gamma(t_{k+1})) - F(\gamma(t_k)) \\ &= F(\gamma(t_n)) - F(\gamma(t_0)) \\ &= z_2 - z_1.\end{aligned}$$

□

Corollary. *Assume that Ω is path-connected. Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function and suppose that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant on Ω.*

Proof. Fix a point $z_0 \in \Omega$. It suffices to show that $f(z) = f(z_0)$ for all $z \in \Omega$. Let $z \in \Omega$. Since Ω is connected, there exists a path $\gamma\colon I \to \Omega$ from z_0 to z. Since f is clearly a primitive for f', we have

$$\int_\gamma f'(w)dw = f(z) - f(z_0).$$

By assumption, $f' = 0$, so the integral on the left is 0, and we conclude that $f(z) = f(z_0)$ as desired. $\square$

Integral Representation of the Taylor Coefficients

Theorem 7.2. *Let f be the sum of the power series $\sum a_n(z-a)^n$ with radius of convergence R. Then for any $n \geq 0$ and r such that $0 < r < R$, we have*

$$a_m = \frac{1}{r^m}\int_0^1 f(a + re^{2\pi it})e^{-2\pi imt}dt$$

Proof. By uniform convergence of the power series $\sum a_n(z-a)^n$ on $C_r(a)$, we have

$$\begin{aligned}\int_0^1 f(a + re^{2\pi it})e^{-2\pi imt}dt &= \int_0^1 \sum_{n=0}^\infty a_n r^n e^{2\pi i(n-m)t}dt \\ &= \sum_{n=0}^\infty a_n r^n \int_0^1 e^{2\pi i(n-m)t}dt \\ &= a_m r^m.\end{aligned}$$

$\square$

8 More on Paths

8.1 Path Covering Lemma

Let $\gamma\colon I \to \mathbb{C}$ be a path. A **chain of disks covering** γ is a finite sequence $(D_0, D_1, \ldots, D_n)$ of open disks D_k with the following properties

1. There exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval I such that $\gamma(t_k)$ is the center of D_k for $k = 0, 1, \ldots, n$.
2. The section of γ between $\gamma(t_{k-1})$ and $\gamma(t_{k+1})$ is contained in D_k, more precisely,

$$\begin{array}{lll}\gamma(t) \subset D_0, & t_0 \leq t \leq t_1, & \\ \gamma(t) \subset D_k, & t_{k-1} \leq t \leq t_{k+1}, & (k = 1, \ldots, n-1) \\ \gamma(t) \subset D_n, & t_{n-1} \leq t \leq t_n. & \end{array}$$

Lemma 8.1. *(Path Covering Lemma) Let Ω be a nonempty open connected subset of $\mathbb{C}$ and let $\gamma\colon I \to \Omega$ be a path in Ω. Then there exists a chain of disks which is contained in Ω and covers γ. Moreover, the radii of all disks can be chosen to be of the same size and arbitrarily small.*

Proof. Since γ is continuous on the compact interval I, its trace $[\gamma]$ is a compact subset of D. The complement of Ω in $\mathbb{C}$ is closed, and hence the distance d between $[\gamma]$ and $\mathbb{C}\backslash\Omega$ is positive. If $0 < r < d$, then all disks with radius r and centers on $[\gamma]$ are contained in Ω. Because γ is uniformly continuous, there exists a positive number δ such that $s, t \in I$ and $|s - t| < \delta$ imply that $|\gamma(s) - \gamma(t)| < r$. So all requirements are satisfied if the partition $0 = t_0 < t_1 < \cdots < t_n = 1$ is chosen such that $t_k - t_{k-1} < \delta$. $\square$

8.2 Homotopic Paths with Specific Properties

Technically it is of great importance that any path in D can be approximated by homotopic paths with specific properties.

Lemma 8.2. *Let $\gamma\colon I \to D$ be a path in an open set $D \subseteq \mathbb{C}$. Then there exists a smooth path $\widetilde{\gamma}\colon I \to D$ and a paraxial path $\widehat{\gamma}\colon I \to D$ which are homotopic to γ in D. For each $\varepsilon > 0$ both paths can be chosen such that*

$$\|\gamma - \widetilde{\gamma}\|_I < \varepsilon \quad \textit{and} \quad \|\gamma - \widehat{\gamma}\|_I < \varepsilon.$$

Proof. By the path covering lemma, γ can be covered by a sequence of disks D_k with radii less than $\varepsilon/2$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a subdivision of the parameter interval I, and denote by $z_k = \gamma(t_k)$ the centers of the covering disks D_k. Then the restriction γ_k of γ to $[t_{k-1}, t_k]$ is homotopic in D_k (and hence in D) to the line segment $[z_{k-1}, z_k]$ for all $k = 1, \ldots, n$. Indeed, D_k is convex, so the map $H\colon I \times I \to D_k$ given by

$$H(s,t) = \gamma_k(t)(1-s) + [z_{k-1}, z_k](t)s$$

for all $s, t \in I$ serves as a homotopy from γ_k to $[z_{k-1}, z_k]$.

This induces a homotopy between γ and the polygonal path $\widehat{\gamma} := [z_0, z_1] \oplus \cdots \oplus [z_{n-1}, z_n]$. Smoothing the function $\widehat{\gamma}$ at the points t_k appropriately, we also obtain a smooth path $\widetilde{\gamma}$ which is homotopic to $\widehat{\gamma}$ and hence to γ. Finally, the segments $[z_{k-1}, z_k]$ are homotopic in D_k to the sum $[z_{k-1}, \mathrm{Re}(z_k) + i\mathrm{Im}(z_{k-1})] \oplus [\mathrm{Re}(z_k) + i\mathrm{Im}(z_{k-1}), z_k]$ of two segments which are parallel to the real and imaginary axis, respectively. □

8.3 Winding Numbers

We now introduce a geometric characteristic of loops which describes how many times they "wind around" some point in the plane.

Lemma 8.3. *Let $\gamma\colon I \to \mathbb{C}\backslash\{0\}$ be a path. Then there exist continuous functions $a\colon I \to \mathbb{R}$ and $r\colon I \to \mathbb{R}_+$ such that*

$$\gamma(t) = r(t)e^{ia(t)} \tag{21}$$

for all $t \in I$.

Proof. The function $r(t) := |\gamma(t)|$ is continuous and positive. So the proof amounts to finding an appropriate argument $a(t)$ of $\gamma(t)$ such that $t \mapsto a(t)$ is continuous. For this purpose, we use the path covering lemma with $D := \mathbb{C}\backslash\{0\}$.

At the initial point of γ we choose the principal branch of the argument, $a(0) := \mathrm{Arg}(\gamma(0))$. If $t \in [t_0, t_1]$, all points $\gamma(t)$ lie in the disk D_0. Since D_0 does not contain the origin, it is contained in a sector with vertex at 0 and opening angle less than π. Consequently the argument $a(t) = \arg(\gamma(t))$ can be chosen such that $|a(t) - a(0)| < \pi/2$, which yields a continuous function a on $[0, t_1]$.

Suppose that such a function has already been constructed on some interval $[0, t_k]$. Then it can be prolongated to $[0, t_{k+1}]$ by choosing $a(t) = \arg(\gamma(t))$ on $[t_k, t_{k+1}]$ such that $|a(t) - a(t_k)| < \pi/2$, which is possible since $\gamma(t) \in D_k$ and $0 \notin D_k$. By induction, a can be extended to all of I. □

Any continuous function a satisfying (21) is called a **continuous branch** of the argument along the path γ. The difference of two such functions a_1 and a_2 on I is a constant integral multiple of 2π. If a is continuous branch of the argument along a loop, then $a(1) - a(0)$ is an integral multiple of 2π which does not depend on the special choice of the branch a.

Definition of Winding Numbers

Let γ be a loop in $\mathbb{C}\backslash\{0\}$ and denote by a a continuous branch of the argument along γ. Then the integer

$$\mathrm{wind}(\gamma) := \frac{1}{2\pi}(a(1) - a(0))$$

is called the **winding number** (or **index**) **of** γ. If $z_0 \in \mathbb{C}$ and γ is a loop in $\mathbb{C}\backslash\{z_0\}$, the **winding number of** γ **about** z_0 is defined by

$$\mathrm{wind}(\gamma, z_0) := \mathrm{wind}(\gamma - z_0).$$

Stability of Winding Numbers

Next we prove the intuitive fact that small perturbations of a loop do not change its winding number. Recall that the distance between two paths γ and γ_0 is defined in terms of the uniform norm:

$$\|\gamma - \gamma_0\|_I := \max_{t \in I} |\gamma(t) - \gamma_0(t)|.$$

Lemma 8.4. *Let $\gamma_0\colon I \to \mathbb{C}\backslash\{0\}$ be a loop, and denote by d the distance of its trace $[\gamma_0]$ from the origin. Then for all loops $\gamma\colon I \to \mathbb{C}$ with $\|\gamma - \gamma_0\|_I < d$,*

$$\mathit{wind}(\gamma) = \mathit{wind}(\gamma_0).$$

Proof. Since $[\gamma_0]$ is a compact subset of $\mathbb{C}\backslash\{0\}$, its distance from the origin is positive. Then $|\gamma(t) - \gamma_0(t)| < d$ implies that $\gamma(t)/\gamma_0(t)$ lies in the right half-plane.

Let a_0 be a continuous branch of the argument along γ_0. If we choose a continuous branch of the argument along γ such that $|a(0) - a_0(0)| < \pi/2$, then $|a(t) - a_0(t)| < \pi/2$ for all $t \in I$. Invoking the triangle inequality we see that

$$|(a(1) - a(0)) - (a_0(1) - a_0(0))| < \pi,$$

and since this number is an integral multiple of 2π, it must be zero. □

Lemma 8.5. *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $z_0 \in D$. Then two loops γ_0 and γ_1 are homotopic in the punctured domain $D\backslash\{z_0\}$ if and only if they have the same winding number about z_0.*

9 Cauchy's Theorem and its Applications

Roughly speaking, Cauchy's Theorem states that if f is holomorphic in an open set Ω and γ is a whose interior is also contained in Ω, then

$$\int_\Gamma f(z)dz = 0.$$

Many results that follow, and in particular the calculus of residues, are related in one way or another to this fact. We first prove this in the special case that our curve Γ is a triangle:

9.1 Goursat's Theorem

Theorem 9.1. *(Goursat's Theorem) If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)dz = 0,$$

whenever f is holomorphic in Ω.

Proof. We call $T^{(0)}$ our original triangle (with a fixed orientation which we choose to be positive), and let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$, respectively. The first step in our construction consists of bisecting each side of the triangle and connected the midpoints. This creates four new smaller triangles, denote $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, and $T_4^{(1)}$ that are similar to the original triangle. The orientation is chosen to be consistent with that of the original triangle, and so after cancellations arising from integration over the same side in two opposite directions, we have

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz.$$

By triangle inequality, we must have

$$\left|\int_{T^{(0)}} f(z)dz\right| \le 4\left|\int_{T_j^{(1)}} f(z)dz\right|$$

for some $j \in \{1,2,3,4\}$. Without loss of generality, assume $j = 1$. Observe that if $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of $T^{(1)}$, respectively, then $d^{(1)} = (1/2)d^{(0)}$ and $p^{(1)} = (1/2)p^{(0)}$. We now repeat this process for the triangle $T^{(1)}$, bisecting into four smaller triangles. Continuing this process, we obtain a sequence of triangles

$$T^{(0)}, T^{(1)}, \dots, T^{(n)}, \dots$$

with the properties that

$$\left|\int_{T^{(0)}} f(z)dz\right| \le 4^n\left|\int_{T^{(n)}} f(z)dz\right|$$

and

$$d^{(n)} = 2^{-n}d^{(0)}, \quad p^{(n)} = 2^{-n}p^{(0)},$$

where $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$, respectively. We also denote $\mathcal{T}^{(n)}$ the *solid* closed triangle with boundary $T^{(n)}$, and observe that our construction yields a sequence of nested compact sets

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots \supset \mathcal{T}^{(n)} \supset \cdots$$

whose diameter goes to 0. Thus, by Proposition (**??**), there exists a unique point z_0 that belongs to all the solid triangles $\mathcal{T}^{(n)}$. Since f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$. Since the constant $f(z_0)$ and the linear function $f'(z_0)(z - z_0)$ have primitives, we can integrate the above equality and, using Corollary (**??**), we obtain

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz.$$

Now z_0 belongs to the closure of the triangle $\mathcal{T}^{(n)}$ and z to its boundary, so we must have $|z - z_0| \leq d^{(n)}$, and so we estimate

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n d^{(n)} p^{(n)},$$

where $\varepsilon_n = \|\psi\|_{z\in T^{(n)}} \to 0$ as $n \to \infty$. Therefore

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n 4^{-n} d^{(0)} p^{(0)},$$

which yields our final estimate

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n \left|\int_{T^{(n)}} f(z)dz\right| \leq \varepsilon_n d^{(0)} p^{(0)}.$$

Letting $n \to \infty$ concludes the proof since $\varepsilon_n \to 0$. □

Corollary. *If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then*

$$\int_R f(z)dz = 0.$$

Proof. We simply decompose the rectangle into two triangles, and integrate over the two triangles. □

9.2 Local existence of primitives and Cauchy's theorem in a disc

Theorem 9.2. *A holomorphic function in an open disc has a primitive in that disc.*

Proof. After a translation, we may assume without loss of generality that the disc, say D, is centered at the origin. Given a point $z \in D$, consider the piecewise-smooth curve that joins 0 to z first by moving in the horizontal direction form 0 to $\mathrm{Re}(z)$, and then in the vertical direction from $\mathrm{Re}(z)$ to z. We choose the orientation from 0 to z, and denote this polygonal line by Γ_z. Define

$$F(z) = \int_{\Gamma_z} f(w)dw.$$

The choice of Γ_z gives an unambigious definition of the function F. We contend that F is holomorphic in D and that $F'(z) = f(z)$ for all $z \in D$. To prove this, fix $z \in D$ and let $h \in \mathbb{C}$ be sufficiently small so that $z + h$ also belongs to the disc. Now consider the difference

$$F(z+h) - F(z) = \int_{\Gamma_{z+h}} f(w)dw - \int_{\Gamma_z} f(w)dw.$$

The function f is first integrated along Γ_{z+h} with the original orientation, and then along Γ_z with the reverse orientation. Since we integrate f over the line segment starting at the origin in two opposite directions, it cancels. Then we complete the square and triangle, so that after an application of Goursat's theorem for triangles and rectangles, we are left with the line segment from z to $z + h$. Hence, we have

$$F(z+h) - F(z) = \int_\eta f(w)dw,$$

where η is the straight line segment from z to $z + h$. Since f is continuous at z we can write

$$f(w) = f(z) + \psi(w)$$

where $\psi(w) := f(w) - f(z) \to 0$ as $w \to z$. Therefore

$$F(z+h) - F(z) = \int_\eta f(z)dw + \int_\eta \psi(w)dw = f(z)\int_\eta dw + \int_\eta \psi(w)dw.$$

On the one hand, the constant 1 has w as a primitive, so the first integral is simply h. On the other hand, we have

$$\left|\int_\eta \psi(w)dw\right| \leq \sup_{w\in\eta} |\psi(w)| \cdot |h|.$$

Since the supremum above goes to 0 as h tends to 0, we conclude that

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

thereby proving that F is a primitive for f on the disc. □

Theorem 9.3. *If f is holomorphic in a disc, then*

$$\int_\Gamma f(z)dz = 0$$

for any closed curve Γ in that disc.

Proof. Since f has a primitive, we can apply Corollary (**??**). □

Example 9.1. We show that if $y \in \mathbb{R}$, then

$$e^{-\pi y^2} = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x y} dx.$$

This gives a new proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $y = 0$, then the equality becomes

$$1 = \int_{\mathbb{R}} e^{-\pi x^2} dx.$$

Now suppose that $y > 0$ and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the toy contour Γ_R, where Γ_r consists of a rectangle with vertices R, $R + iy$, $-R + iy$, $-R$ and the positive counterclockwise orientation. By Cauchy's theorem,

$$\int_{\Gamma_R} f(z)dz = 0.$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2} dx,$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$I(R) = \int_0^y f(R+it)idt = \int_0^y e^{-\pi(R^2+2iRt-t^2)} idt.$$

This integral goes to 0 as $R \to \infty$ since y is fixed and we may estimate it by

$$|I(R)| \leq Ce^{-\pi R^2}.$$

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons. Finally, the integral over the horizontal segment on top is

$$\int_R^{-R} e^{-\pi(x+iy)^2} dx = e^{-\pi x^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x y} dx.$$

Therefore we find in the limit as $R \to \infty$ that

$$0 = 1 - e^{-\pi x^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x y} dx,$$

and our desired formula is established. In the case $y < 0$, we then consider the symmetric rectangle in the lower half-plane.

9.3 Differentiable and Analytic Functions

Lemma 9.4. *Let $f\colon B_R(z_0) \to \mathbb{C}$ be a holomorphic function and let $r > 0$ such that $0 < r < R$. If F is a primitive of f, then for all points $z \in B_r(z_0)$, we have*

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_r(z_0)} \frac{F(w)}{w-z}dw.$$

Proof. 1. We begin with an auxiliary result. Let $z_0 \in B_R(a)$ be fixed. Define a function $\varphi\colon B_R(a - z_0) \to \mathbb{C}$ by the formula

$$\varphi(h) := F(z_0 + h) - F(z_0) - f(z_0)h - \frac{1}{2}f'(z_0)h^2$$

for all $h \in B_R(a - z_0)$. The function φ is differentiable in $B_R(a - z_0)$ and its derivative with respect to h can be computed by the chain rule,

$$\begin{aligned}\varphi'(h) &= F'(z_0 + h) - f(z_0) - f'(z_0)h \\ &= f(z_0 + h) - f(z_0) - f'(z_0)h.\end{aligned}$$

Since f is differentiable at z_0, the right-hand side is of order $o(h)$ as $h \to 0$, that is, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varphi'(h)| \le \varepsilon|h|$ whenever $|h| < \delta$.

The function φ' is continuous, whence the mapping $I \to \mathbb{C}$ given by $t \mapsto \varphi(th)$ is continuously differentiable (with respect to the real variable t), so that $\varphi(h)$ can be represented by the fundamental theorem of calculus,

$$\varphi(h) = \int_0^1 \frac{d}{dt}(\varphi(th))dt = \int_0^1 \varphi'(th)hdt.$$

Using the standard estimate for integrals in combination with $|\varphi'(h)| \le \varepsilon|h|$, we conclude that $|\varphi(h)| \le h^2\varepsilon$ for all h with $|h| < \delta$. Since ε can be chosen arbitrarily small, we have

$$\lim_{h\to 0} \varphi(h)/h^2 = 0. \tag{22}$$

2. The function G defined by

$$G(z) := \begin{cases} \frac{F(z)-F(z_0)}{z-z_0} & \text{if } z \in B_R(a)\backslash\{z_0\} \\ f(z_0) & \text{if } z = z_0 \end{cases}$$

is differentiable in $z \in B_R(a)\backslash\{z_0\}$. In order to prove that G is also differentiable at z_0, we consider its difference quotient at z_0. By definition of G and φ, we have

$$\begin{aligned}\frac{G(z) - G(z_0)}{z - z_0} &= \frac{(F(z) - F(z_0) - (z - z_0)f(z_0))}{(z - z_0)^2} \\ &= \frac{1}{2}f'(z_0) + \frac{\varphi(z - z_0)}{(z - z_0)^2},\end{aligned}$$

and using (22), we find $G'(z_0) = (1/2)f'(z_0)$.
3. Because G is differentiable in the disk $B_R(a)$, we can apply Goursat's lemma, which tells us that the integral of G along the closed path $\gamma_r(a)$ vanishes. Hence

$$\begin{aligned}0 &= \int_{\gamma_r(a)} G(z)dz \\ &= \int_{\gamma_r(a)} \frac{F(z) - F(z_0)}{z - z_0}dz \\ &= \int_{\gamma_r(a)} \frac{F(z)}{z - z_0}dz - F(z_0)\int_\gamma \frac{dz}{z - z_0} \\ &= \int_{\gamma_r(a)} \frac{F(z)}{z - z_0}dz - F(z_0) \cdot 2\pi i \cdot \mathrm{wind}(\gamma, z_0).\end{aligned}$$

□

9.3.1 Cauchy Integrals

Definition 9.1. Let γ be a piecewise smooth path in $\mathbb{C}$ and assume that the function $\varphi\colon [\gamma] \to \mathbb{C}$ is continuous. The function $f\colon \mathbb{C}\backslash[\gamma] \to \mathbb{C}$ defined by

$$f(z) := \frac{1}{2\pi i}\int_\gamma \frac{\varphi(w)}{w - z}dw \tag{23}$$

for all $z \in \mathbb{C}\backslash[\gamma]$ is said to be the **Cauchy integral with density φ along γ.**

Theorem 9.5. *Let γ be a piecewise smooth path in $\mathbb{C}$ and assume that $\varphi\colon [\gamma] \to \mathbb{C}$ is continuous. Then the function f defined by the Cauchy integral (23) is analytic on $D := \mathbb{C}\backslash[\gamma]$ and tends to zero as $z \to \infty$. For any disk $D_0 \subset D$ with center z_0, the taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

of f at z_0 converges in D_0 and its coefficients satisfy

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(z)}{(z - z_0)^{n+1}} dz.$$

Proof. Fix $z \in D_0$. Invoking a compactness argument, we know of the existence of a constant $q < 1$ such that

$$\left| \frac{z - z_0}{w - z_0} \right| \leq q < 1$$

for all $w \in [\gamma]$. Consequently,

$$\begin{aligned} \frac{\varphi(w)}{w - z} &= \frac{\varphi(w)}{(w - z_0) - (z - z_0)} \\ &= \sum_{n=0}^{\infty} \frac{\varphi(w)}{w - z_0} \left(\frac{z - z_0}{w - z_0} \right)^n. \end{aligned}$$

The function $w \mapsto \varphi(w)/(w - z_0)$ is bounded on the compact set $[\gamma]$, the series converges uniformly with respect to $w \in [\gamma]$ (apply Weierstrass M-test with $M_n = Mq^n$, where M is a bound for the function $w \mapsto \varphi(w)/(w - z_0)$). Interchanging the order of summation and integration, we obtain

$$\begin{aligned} 2\pi i f(z) &= \int_{\gamma} \frac{\varphi(w)}{w - z} dw \\ &= \sum_{n=0}^{\infty} \left(\int_{\gamma} \frac{\varphi(w)}{(w - z_0)^{n+1}} dw \right) (z - z_0)^n \end{aligned}$$

for all $z \in D_0$, which proves the claim. Finally, the standard integral estimate yields that $f(z) \to 0$ as $z \to \infty$. □

9.4 Cauchy's Integral Formula

Theorem 9.6. *(Cauchy's Integral Formula) Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, let $a \in \Omega$, and let $r > 0$ such that $\overline{B}_r(a) \subset \Omega$. For every $z \in B_r(a)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r(a)} \frac{f(w)}{w - z} dw.$$

To get a feel for how this theorem works, let us assume that f is analytic at a. Then we can choose $\varepsilon > 0$ such that $\overline{B}_\varepsilon(a) \subset B_r(z_0)$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)$$

for all $z \in \overline{B}_\varepsilon(a)$. Then we'd have

$$\begin{aligned} \int_{\gamma_r(z_0)} \frac{f(z)}{z - a} dz &= \int_{\gamma_\varepsilon(a)} \frac{f(z)}{z - a} dz \\ &= \int_{\gamma_\varepsilon(a)} \sum_{n=0}^{\infty} a_n (z - a)^{n-1} dz \\ &= \sum_{n=0}^{\infty} a_n \int_{\gamma_\varepsilon(a)} (z - a)^{n-1} dz \\ &= 2\pi i f(a). \end{aligned}$$

Theorem 9.7. *(Cauchy's Integral Formula) Let Ω be an open subset of $\mathbb{C}$, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, let $a \in \Omega$, and let $r > 0$ such that $C_r(a) \subset \Omega$. For every $z \in B_r(a)$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r(a)} \frac{f(w)}{w - z} dw.$$

Proof. Let $z \in B_r(a)$. Define $\gamma\colon I \to \Omega$ to be the path

$$\gamma := \oplus e^{2\pi i \operatorname{Arg}(z-a)} \gamma_r(a)$$

$$\zeta$$

If $a \neq z$, let $\zeta = \min\{w - z \mid w \in C_r(a)\}$, be the and let $\gamma_{\delta,\varepsilon}$ be the loop in Ω be given by

$$\gamma_{\delta,\varepsilon} = -$$

and consider the "keyhole" $\gamma_{\delta,\varepsilon}$ which omits the point z. Here δ is the width of the corridor, and ε is the radius of the small circle centered at z. Since the function $f(w)/(w-z)$ is holomorphic away from the point $w = z$, we have

$$\int_{\Gamma_{\delta,\varepsilon}} \frac{f(w)}{w-z} dw = 0$$

by Cauchy's theorem for the chosen toy contour. Now we make the corridor narrower by letting δ tend to 0, and use the continuity of $f(w)/(w-z)$ to see that in the limit, the integrals over the two sides of the corridor cancel out. The remaining part consists of two curves, the large boundary circle C with the positive orientation, and a small circle C_ε centered at z of radius ε and oriented negatively, that is, clockwise. To see what happens to the integral over the small circle, we write

$$\frac{f(w)}{w-z} = \frac{f(w) - f(z)}{w-z} + \frac{f(z)}{w-z}$$

and note that since f is holomorphic, the first term on the right-hand is bounded so that its integral over C_ε tends to 0 as $\varepsilon \to 0$. To conclude the proof, it suffices to observe that

$$\begin{aligned}
\int_{C_\varepsilon} \frac{f(z)}{w-z} dw &= f(z) \int_{C_\varepsilon} \frac{dw}{w-z} \\
&= -f(z) \int_0^{2\pi} \frac{\varepsilon i e^{-it}}{\varepsilon e^{-it}} dt \\
&= -f(z) 2\pi i
\end{aligned}$$

so that in the limit we find

$$0 = \int_C \frac{f(w)}{w-z} dw - 2\pi i f(z),$$

as was to be shown. □

Integral Representation of the Taylor Coefficients

Theorem 9.8. *Let f be the sum of the power series $\sum a_n (z-a)^n$ with radius of convergence R. Then for any $n \geq 0$ and r such that $0 < r < R$, we have*

$$a_m = \frac{1}{r^m} \int_0^1 f(a + re^{2\pi i t}) e^{-2\pi i m t} dt$$

Proof. By uniform convergence of the power series $\sum a_n (z-a)^n$ on $C_r(a)$, we have

$$\begin{aligned}
\int_0^1 f(a + re^{2\pi i t}) e^{-2\pi i m t} dt &= \int_0^1 \sum_{n=0}^{\infty} a_n r^n e^{2\pi i (n-m) t} dt \\
&= \sum_{n=0}^{\infty} a_n r^n \int_0^1 e^{2\pi i (n-m) t} dt \\
&= a_m r^m.
\end{aligned}$$

□

Corollary. *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw$$

for all z in the interior of C.

Proof. Let f be the sum of the power series $\sum a_n(z-a)^n$ with radius of convergence R. Then

$$\begin{aligned}
f^{(m)}(a) &= m!a_m \\
&= \frac{m!}{r^m}\int_0^1 f(a+re^{2\pi it})e^{-2\pi imt}dt \\
\\
&= \frac{m!}{2\pi i}\int_0^1 \frac{f(z)}{(re^{2\pi it})^{n+1}}2\pi ire^{2\pi it}dz \\
&= \frac{m!}{2\pi i}\int_{\gamma_r(a)} \frac{f(z)}{(z-a)^{n+1}}dz
\end{aligned}$$

□

Corollary. *If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw$$

for all z in the interior of C.

Proof. The proof is by induction on n, the case $n=0$ being simply the Cauchy integral formula. Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^n}dw.$$

Now for small h, the difference quotient for $f^{(n-1)}$ takes the form

$$\begin{aligned}
\frac{f^{(n-1)}(z+h)-f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{h}\left(\left(\frac{1}{w-(z+h)}\right)^n - \left(\frac{1}{w-z}\right)^n\right)dw \\
&= \frac{(n-1)!}{2\pi i}\int_C \frac{f(w)}{h}\left(\frac{1}{(w-(z+h))} - \frac{1}{(w-z)}\right)\left(\sum_{m=0}^{n-1}\left(\frac{1}{w-(z+h)}\right)^{n-m-1}\left(\frac{1}{w-z}\right)^m\right)dw \\
&= \frac{(n-1)!}{2\pi i}\int_C f(w)\left(\frac{1}{(w-(z+h))(w-z)}\right)\left(\sum_{m=0}^{n-1}\left(\frac{1}{w-(z+h)}\right)^{n-m-1}\left(\frac{1}{w-z}\right)^m\right)dw
\end{aligned}$$

Now observe that if h is small, then $z+h$ and z stay at a finite distance from the boundary circle C, so in the limit as h tends to 0, we find that the quotient converges to

$$\frac{(n-1)!}{2\pi i}\int_C f(w)\left(\frac{1}{(w-z)^2}\right)\left(\frac{n}{(w-z)^{n-1}}\right)dw = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw,$$

which completes the induction argument and proves the theorem. □

9.4.1 Taylor's Theorem

Theorem 9.9. *Let Ω be an open set, let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function, and let $a \in \Omega$. Then there exists $r > 0$ and a power series $\sum a_n(z-a)^n$ centered at a such that $\overline{B}_r(a) \subset \Omega$*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in \overline{B}_r(a)$. Furthermore, the coefficients are given by

$$a_n = \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{(w-a)^{n+1}}dw$$

for all $n \geq 0$.

Proof. Let $z \in B_r(a)$. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-a}\left(\frac{1}{1-\left(\frac{z-a}{w-a}\right)}\right)dw \\ &= \frac{1}{2\pi i}\int_{\gamma_r(a)} \frac{f(w)}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r(a)}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n. \\ &= \sum_{n=0}^{\infty} a_n(z-a)^n. \end{aligned}$$

where we are allowed to interchange the integral with the sum since the series $\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$ converges uniformly in $w \in C_r(a)$. □

9.4.2 Limit of Holomorphic Functions Converging Uniformly on Compact Subsets is Holomorphic

Theorem 9.10. *Let Ω be a nonempty open subset of $\mathbb{C}$ and let (f_n) be a sequence of analytic functions on Ω that converges uniformly to f on each compact subset of Ω. Then f is holomorphic and (f_n') converges uniformly to f' on each compact subset.*

Proof. Let $a \in \Omega$. Choose $z_0 \in \Omega$ and $r > 0$ such that $a \in \overline{B}_r(z_0) \subset \Omega$. Then

$$f_n(a) = \frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f_n(z)}{z-a}dz$$

for all $n \in \mathbb{N}$. Since (f_n) converges to f uniformly on $\overline{B}_r(z_0)$, the function f is continuous on $\overline{B}_r(z_0)$, and so $f(z)/(z-a)$ is integrable along $\gamma_r(z_0)$. Thus

$$\begin{aligned} \left|\frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f_n(z)}{z-a}dz - \frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f(z)}{z-a}dz\right| &= \left|\frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f_n(z)-f(z)}{z-a}dz\right| \\ &\le \frac{1}{2\pi}\|f_n - f\|_{\overline{B}_r(z_0)}\left|\int_{\gamma_r(z_0)}\frac{dz}{z-a}\right| \\ &= \|f_n - f\|_{\overline{B}_r(z_0)}, \end{aligned}$$

which tends to 0 as n tends to ∞. Therefore

$$\begin{aligned} \frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f(z)}{z-a}dz &= \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\gamma_r(z_0)}\frac{f_n(z)}{z-a}dz \\ &= \lim_{n\to\infty} f_n(a) \\ &= f(a). \end{aligned}$$

This implies f is holomorphic in Ω.

To show $f_n' \to f'$ uniformly on compact subsets of Ω, it suffices to work with closed discs. Let $\overline{D}$ be a closed disc in Ω with radius $r > 0$. Choose a in the interior of $\overline{D}$. Then

$$f_n'(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f_n(z)}{(z-a)^2}dz, \quad \text{and} \quad f'(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^2}dz.$$

Therefore

$$\begin{aligned} |f_n'(a) - f'(a)| &\le \frac{\|f_n - f\|_{\overline{D}}}{r} \\ &\to 0. \end{aligned}$$

□

9.4.3 Cauchy's Inequalities

Corollary. *(Cauchy's inequality) Let f is holomorphic in a given set that contains the closure of a disc D centered at z_0 and of radius R, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} \sup_{z\in C} |f(z)|.$$

Proof. Applying Cauchy's Integral Formula for $f^{(n)}(z_0)$, we have

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta\right| \\ &= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+Re^{it})}{R^n e^{int}}d\zeta\right| \\ &\leq \frac{n!}{R^n}\sup_{z\in C}|f(z)|.\end{aligned}$$

□

9.4.4 Louiville's Theorem

Theorem 9.11. *(Louiville's Theorem) Every bounded entire function must be constant.*

Proof. Let f be a bounded entire function. Suppose $\sum a_n z^n$ is the power series representation of f at 0 and choose a constant M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then for every $r > 0$ and $n \geq 1$, we have

$$\begin{aligned}|a_n| &= \left|\frac{1}{2\pi i}\int_{\gamma_r(0)} \frac{f(z)}{z^{n+1}}dz\right| \\ &= \left|\int_0^1 \frac{f(re^{2\pi it})}{r^n e^{2\pi int}}dt\right| \\ &\leq \int_0^1 \left|\frac{f(re^{2\pi it})}{r^n e^{2\pi int}}\right| dt \\ &\leq \frac{M}{r^n}.\end{aligned}$$

This implies $a_n = 0$ for every $n \geq 1$. Thus $f(z) = a_0$, which proves the theorem. □

9.4.5 Fundamental Theorem of Algebra

Corollary. *Every non-constant polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with complex coefficients has a root in $\mathbb{C}$.*

Proof. If $P(z)$ has no roots, then $Q(z) := 1/P(z)$ is a bounded holomorphic function. To see this, we can of course assume that $a_n \neq 0$ and write

$$Q(z) = \frac{1}{a_n z^n + \cdots + a_1 z + a_0} = \left(\frac{1}{z^n}\right)\left(\frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n}\right).$$

As $z \to \infty$, the denominator of the second term in the round brackets converges to $a_n \neq 0$, hence the second term itself goes to $1/a_n$. But the first term tends to zero, hence

$$\lim_{z\to\infty} Q(z) = 0.$$

In particular, $|Q(z)|$ is bounded by 1 outside of some circle $|z| = r$. Inside this circle, $|Q(z)|$ is continuous, hence bounded. Thus $|Q(z)|$, and therefore $Q(z)$ itself is bounded on the whole complex plane. By Liousville's theorem, we then conclude that $Q(z)$ is constant. This contradicts our assumption that $P(z)$ is nonconstant and proves the corollary. □

Theorem 9.12. *(Identity Theorem) Let f, g be holomorphic functions on a connected open set D of $\mathbb{C}$. If $f = g$ on a nonempty open subset of D, then $f = g$ on D.*

Remark. This says that a holomorphic function is completely determined by its values on a (possibly quite small) neighborhood in D. This is not true for real-differentiable functions. In comparison, holomorphy is a much more rigid notion.

Proof. Let S be the set of all $z \in D$ such that $f(z) = g(z)$. We show that S is open and closed, and hence must be D. Since $f - g$ is continuous, and $S = (f-g)^{-1}\{0\}$, we see that S is closed. To show, that S is open, suppose w lies in S. Then, because the Taylor series of f and g at w have non-zero radius of convergence, the open disk $B_r(w)$ also lies in S for some r. □

9.5 Further Applications

9.5.1 Morera's Theorem

Theorem 9.13. *Suppose f is a continuous function in the open disc D such that for any triangle T cotained in D,*

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. By the proof of Theorem (9.2), the function f has a primitive F in D that satisfies $F' = f$. By the regularity theorem, we know that F is indefinitely (and hence twice) complex differentiable, and therefore f is holomorphic. □

Theorem 9.14. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.*

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\int_T f_n(z)dz = 0$$

for all n. By assumption, $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\int_T f_n(z)dz \to \int_T f(z)dz.$$

As a result, we find $\int_T f(z)dz = 0$ and by Morera's theorem, we conclude that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω. □

9.5.2 Sequence of Holomorphic Functions

Theorem 9.15. *Let $F(z,s)$ be defined for $(z,s) \in \Omega \times [0,1]$ where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties:*

1. *$F(z,s)$ is holomorphic in z for each s.*
2. *F is continuous on $\Omega \times [0,1]$.*

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z,s)ds$$

is holomorphic.

Remark. The second condition says that F is jointly continuous in both arguments. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω, and by Morera's theorem this could be achieved by showing that for any triangle T contained in D we have

$$\int_T \int_0^1 F(z,s)dsdz = 0.$$

Interchanging the order of integration, and using property (1) would then yield the desired result. We can, however, get around the issue of justifying the change in the order of integration by arguing differently. The idea is to interpret the integral as a "uniform" limit of Riemann sums, and then apply the results of the previous section.

Proof. For each $n \geq 1$, we consider the Riemann sum

$$f_n(z) = (1/n) \sum_{k=1}^{n} F(z, k/n).$$

Then f_n is holomorphic in all of Ω by property (1), and we claim that on any disc D whose closure is contained in Ω, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. To see this, we recall that a continuous function on a compact set is uniformly continuous, so if $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{z \in D} |F(z,s_1) - F(z,s_2)| < \varepsilon \quad \text{whenever} \quad |s_1 - s_2| < \delta.$$

Then if $n > 1/\delta$, and $z \in D$ we have

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} F(z, k/n) - F(z, s) ds \right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)| \, ds \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{n} \\ &< \varepsilon. \end{aligned}$$

This proves the claim, and by Theorem (9.14), we conclude that f is holomorphic in D. As a consequence, f is holomorphic in Ω, as was to be shown. □

9.5.3 Scwarz reflection principle

Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, that is $z \in \Omega$ if and only if $\bar{z} \in \Omega$. Let Ω^+ denote the part of Ω that lies in the upper half-plane and Ω^- that part that lies in the lower half-plane. Also, let $I = \Omega \cap \mathbb{R}$ so that I denotes the interior of that part of the boundary of Ω^+ and Ω^- that lies on the real axis. Then we have

$$\Omega^+ \cup I \cup \Omega^- = \Omega$$

and the only interesting case of the next theorem occurs, of course, when I is nonempty.

Theorem 9.16. *If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I and $f^+(x) = f^-(x)$ for all $x \in I$, then the function f defined on Ω by*

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+, \\ f^+(z) = f^-(z) & \text{if } z \in I, \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω.

Proof. One first notes that f is continuous on Ω. The only difficulty is to prove that f is holomorphic at points of I. Suppose D is a disc centered at a point on I and entirely contained in Ω. We prove that f is holomorphic in D by Morera's theorem. Suppose T is a triangle in D. If T does not intersect I, then

$$\int_T f(z) dz = 0$$

since f is holomorphic in the upper and lower half-discs. Suppose now that one side or vertex of T is contained in I, and the rest of$T$ is in, say the upper half-disc. If T_ε is the triangle obained from T by slightly raising the edge or vertex which lies on I, we have

$$\int_{T_\varepsilon} f(z) dz = 0$$

since T_ε is entirely contained in the upper half-disc. We then let $\varepsilon \to 0$, and by continuity we conclude that

$$\int_T f(z) dz = 0.$$

If the interior of T intersects I, we can reduce the siutation to the previous one by writing T as the union of triangles each of which has an edge or vertex on I. By Morera's theorem, we conclude that f is holomorphic in D, as was to be shown. □

Theorem 9.17. *(Schwarz reflection principle) Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in all of Ω such that $F = f$ on Ω^+.*

Proof. The idea is simply to define $F(z)$ for $z \in \Omega^-$ by $F(z) = \overline{f(\bar{z})}$. To prove that F is holomorphic in Ω^- we note that if $z, z_0 \in \Omega^-$, then $\bar{z}, \bar{z}_0 \in \Omega^+$ and hence, the power series expansion of f near $\bar{z}_0$ gives

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n \left(\bar{z} - \bar{z}_0\right)^n.$$

As a consequence we see that

$$F(z) = \sum_{n=0}^{\infty} \bar{a}_n \left(z - z_0\right)^n$$

and F is holomorphic in Ω^-. Since f is real valued on I, we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence F extends continuously up to I. The proof is complete once we invoke the symmetry principle. □

Theorem 9.18. *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.*

Proof. Suppose that $z_0 \in \Omega$ is a limit point for the sequence $\{w_k\}_{k=1}^{\infty}$ and that $f(\omega_k) = 0$. First, we show that f is idenitcally zero in a small disc containing z_0. For that, we choose a disc D centered at z_0 and contained in Ω, and consider the power series expansion of f in that disc

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If f is not identically zero, there exists a smallest integer m such that $a_m \neq 0$. But then we can write

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0)),$$

where $g(z - z_0)$ converges to 0 as $z \to z_0$. Taking $z = w_k \neq z_0$ for a sequence of points converging to z_0, we get a contradiction since $a_m (w_k - z_0)^m \neq 0$ and $1 + g(w_k - z_0) \neq 0$, but $f(w_k) = 0$.

We conclude the proof using the fact that Ω is connected. Let U denote the interior of the set of points where $f(z) = 0$. Then U is open by definition and nonempty by the argument just given. The set U is also closed since if $z_n \in U$ and $z_n \to z$, then $f(z) = 0$ by continuity, and f vanishes in a neighborhood of z by the argument above. Hence, $z \in U$. Now if we let V denote the complement of U in Ω, we conclude that U and V are both open, disjoint, and

$$\Omega = U \cup V.$$

Since Ω is connected we conclude that either U or V is empty. Since $z_0 \in U$, we find that $U = \Omega$ and the proof is complete.

□

Corollary. *Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some nonempty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ througout Ω.*

Suppose we are given a pair of functions f and F analytic in regions Ω and Ω', respectively, with $\Omega \subset \Omega'$. If the two functions agree on the smaller set Ω, we say that F is an **analytic continuiation** of f into the region Ω' The corollary then guarantees that there can be only one such analytic continuation, since F is uniquely determined by f.