Section 2.6: Rate-of-Change Graphs

A **slope graph** of a function (or **rate-of-change graph** or **derivative graph)** is the graph that results from plotting the slopes of the function.

If the function graph is continuous, with no sharp corners or points that would have vertical tangent lines, then the **slope graph** of the function will be continuous.

If the function graph is continuous, but has a point with a vertical tangent, the slope graph will have a vertical asymptote at that point. If the function graph is continuous, but has a sharp corner, the **slope graph** will not be continuous at that point.

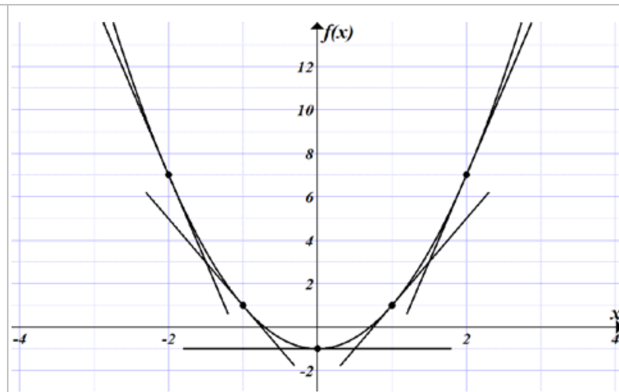
If the function graph is not continuous at a point, the **slope graph** will not be continuous at that point.

Example 1:

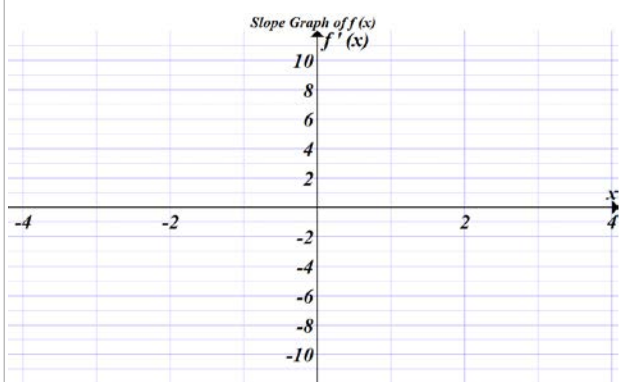
The quadratic function $f(x) = 2x^2 - 1$, has derivative $f'(x) = 4x$. This can be verified using the limit definition of the derivative. Therefore, the graph of $f'(x) = 4x$ is the slope graph of $f(x) = 2x^2 - 1$.

It is possible to sketch the slope graph without knowing the derivative formula.

a. The tangent lines at $x = -2$, $x = -1$, $x = 0$, $x = 1$, and $x = 2$ are drawn on the graph of the quadratic. Write the slopes of each of these tangent lines directly on the graph of the quadratic.



b. Plot the **slopes** of the tangent lines at $x = -2, x = -1, x = 0, x = 1,$ and $x = 2$ on the **slope graph**. Since the slope of the tangent line at $x = -2$ is -8, plot the point $(-2, -8)$. Repeat for the slopes at $x = -1, x = 0, x = 1,$ and $x = 2$.



c. Draw a line through the points to sketch a slope graph for $f(x) = 2x^2 - 1$.

d. Find the equation of the line drawn in part c.

To sketch a slope graph from a graph of a function, it is not necessary to calculate exact slopes of tangent lines. Follow the steps below to sketch a slope graph.

Step 1: Find points on the graph of $f(x)$ with a slope of *zero*. Since relative maxima or relative minima have horizontal tangents, the function has a slope of zero at these points. These are *x-intercepts* on the **slope graph**.

Identify any points on the graph of $f(x)$ in which the slope *does not exist*. The slope does not exist at input values that have a sharp point, a discontinuity, or a vertical tangent. The **slope graph** has *breaks* (either open circles or a vertical asymptote) at such points.

Step 2: Examine each interval between the *x*-intercepts or breaks.

If the function is *increasing*, with *positive* slopes, the **slope graph** lies *above* the *x*-axis on that interval.

If the function is *decreasing*, with *negative* slopes, the **slope graph** lies *below* the *x*-axis on that interval.

Step 3: If the graph of the function is *concave up*, slopes are increasing and the **slope graph** is *increasing*. If the function is *concave down*, slopes are decreasing and the **slope graph** is *decreasing*.

Note that the steeper the tangent lines are, the further away the **slope graph** will be from the *x*-axis at those input values.

Since an inflection point on the graph of $f(x)$ is a point at which the function is increasing or decreasing the most or least rapidly, that is the point at which the **slope graph** has a *relative maximum* or *relative minimum* (except in the case of a vertical tangent).

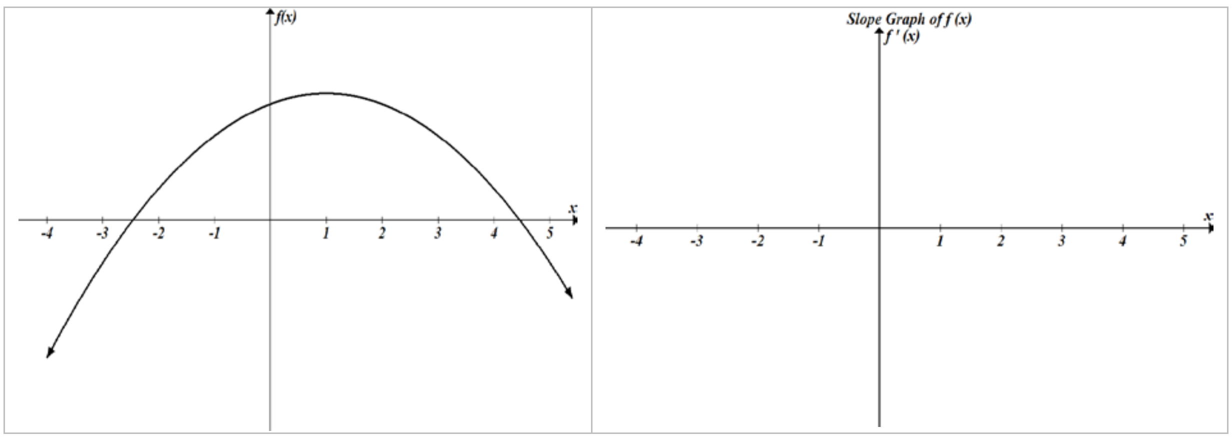
Example 2:

Draw the slope graph for the following quadratic function by following the above three steps.

- What input value has a slope of zero? ________ Graph (1,0) on the slope graph.

- For $x < 1$, slopes are positive. The slope graph is _________ *(above/below)* the x-axis on this interval. Since the graph of the parabola is concave down, and the slopes are becoming less steep (from left to right), the slope graph is ___________________ (*increasing/decreasing)*on this interval.

- For $x > 1$, slopes are negative. The slope graph is ________ *(above/below)* the x-axis on this interval. Since the graph of the parabola is concave down, and the slopes are becoming steeper (from left to right), the slope graph is ___________________ (*increasing/decreasing)* on this interval.



Example 3:

The graph of a cubic function $f(x)$ is shown below on the interval $0 < x < 6.5$.

a. The slope is zero when $x =$____ and $x =$____.

b. For $0 < x < a$, the graph of f is increasing, so the slopes are __________(*positive/negative*).

   For $0 < x < a$, the graph is ________________(*concave up/concave down*). The tangent lines become less steep as *x* approaches *a* from the left, so the slopes are __________________ *(increasing/decreasing).*

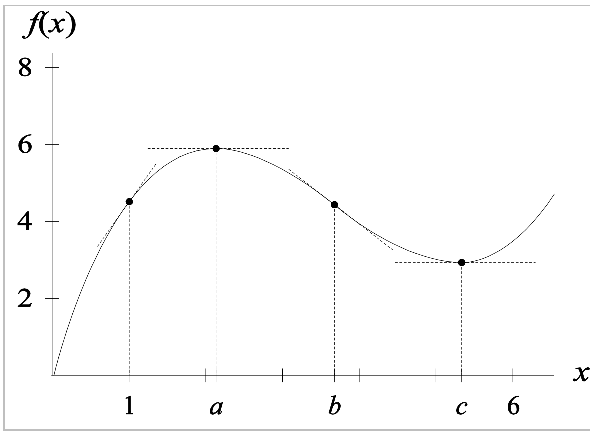
c. For $a < x < c$, the graph of f is decreasing, so the slopes are ________ (*positive/negative*).

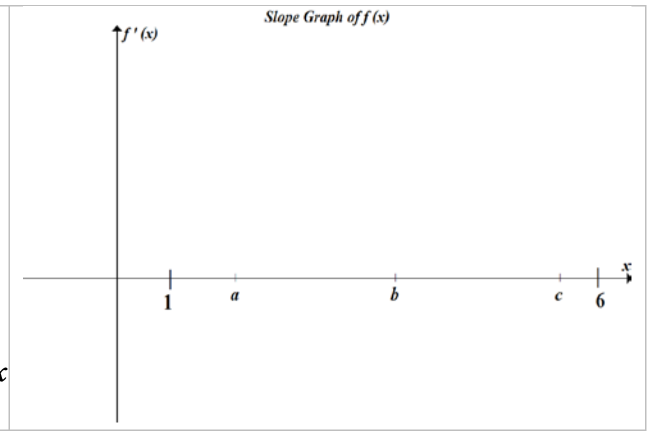
On the interval $a < x < c$, the graph of f has an inflection point at x = ____. This is the point where f is decreasing ______________________ (*most rapidly/least rapidly*).
$f'(b)$ is a ______________________(*relative maximum/relative minimum)* on $0 < x < 6.5$.

d. For $c < x < 6.5$, the graph of f is ______________________(*increasing/decreasing*), so the slopes are __________________ (*positive/negative)*.

For $c < x < 6.5$, the graph is _______________(*concave up/concave down*). The tangent lines become steeper for $x > c$, so the slopes are _____________ *(increasing/decreasing).*

e. Sketch a graph on f' on the axes provided.

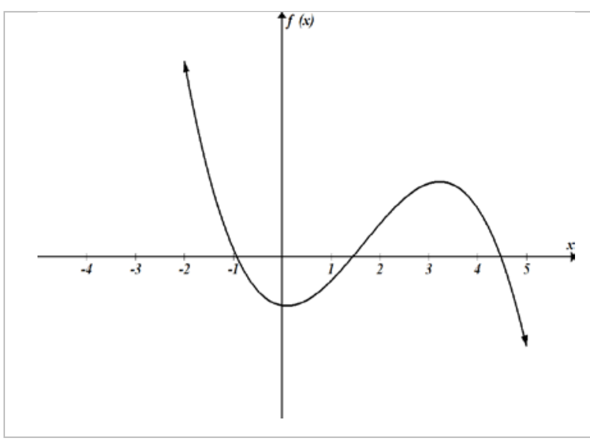


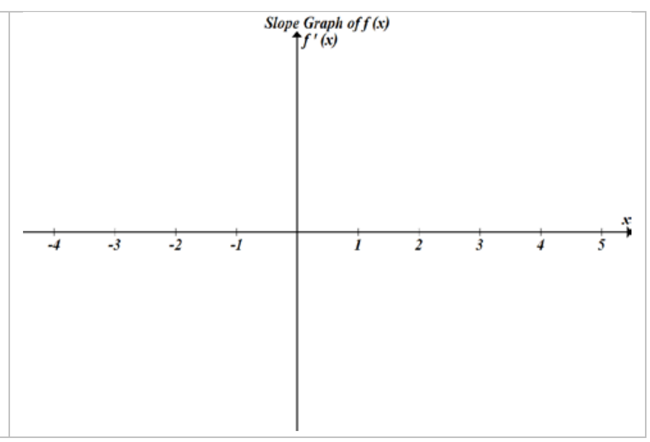


Example 4:

Draw the slope graph for the following cubic function.

$f(x)$ is increasing______________ (*most rapidly/least rapidly*) at approximately $x = 1.5$, where $f'(x)$ has a ________________ (*relative maximum/relative minimum*)..

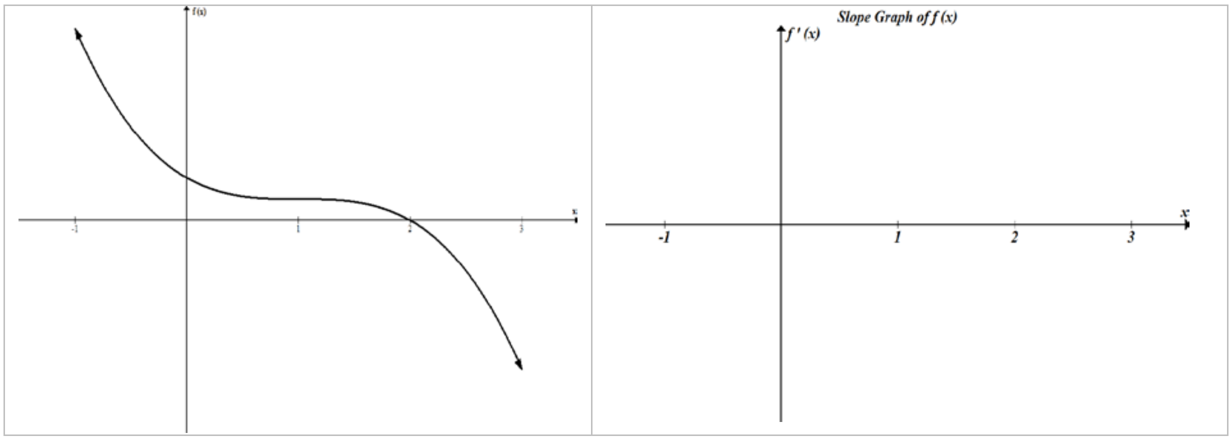




Example 5:

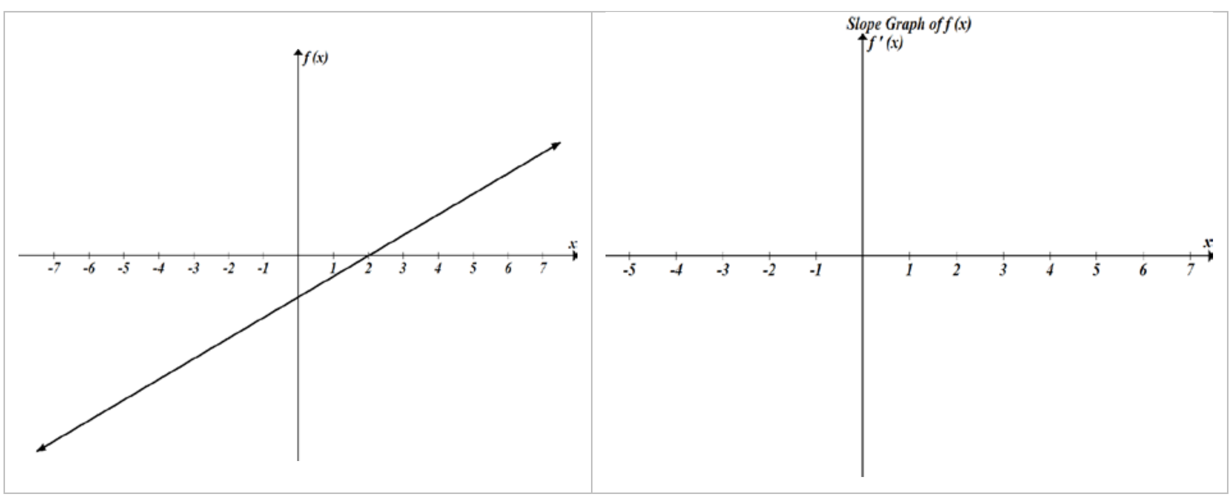
Draw the slope graph for the following cubic function.

$f(x)$ is decreasing______________ (*most rapidly/least rapidly*) at approximately $x =$ ___ , where $f'(x)$ has a ________________ (*relative maximum/relative minimum*).



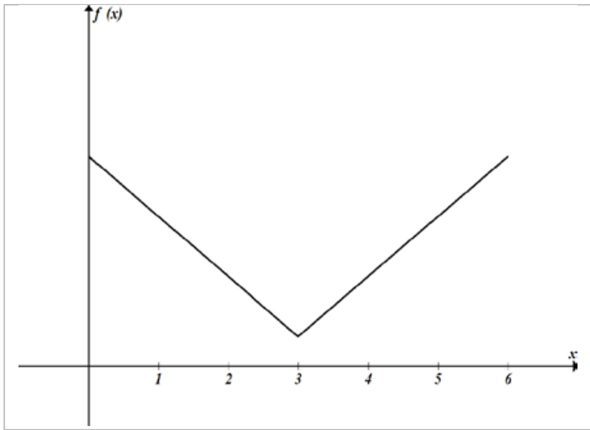
Example 6:

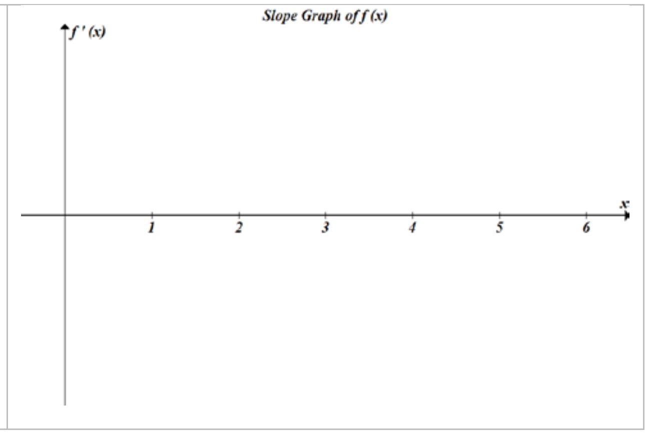
Draw the slope graph for the following linear function. Note that the slope is constant.



Example 7:

Draw the slope graph for the following function. This function is not smooth, since it *has a sharp point at* $x=3$. The derivative does not exist at x=3 because a tangent line cannot be drawn at a sharp point. The slope graph has ________________________________at $x=3$.

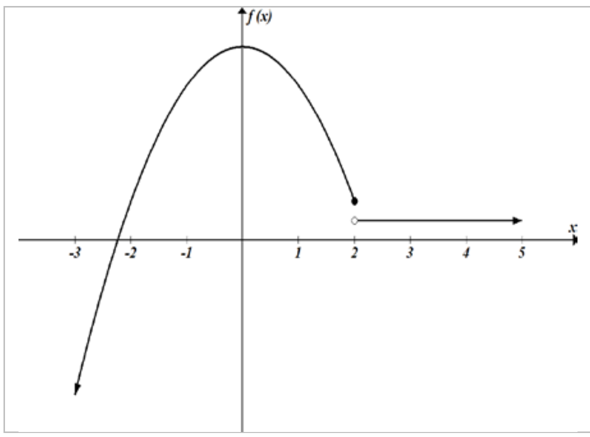


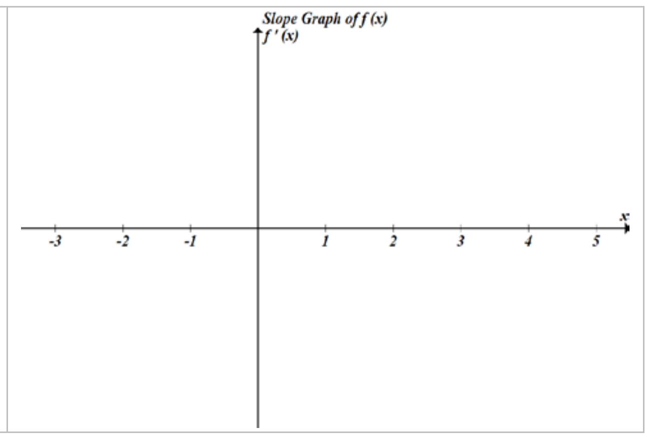


Example 8:

Draw the slope graph for the following function. This function *is not continuous at* $x=2$.
The derivative does not exist at $x=2$ because a tangent line cannot be drawn at a *discontinuity.*

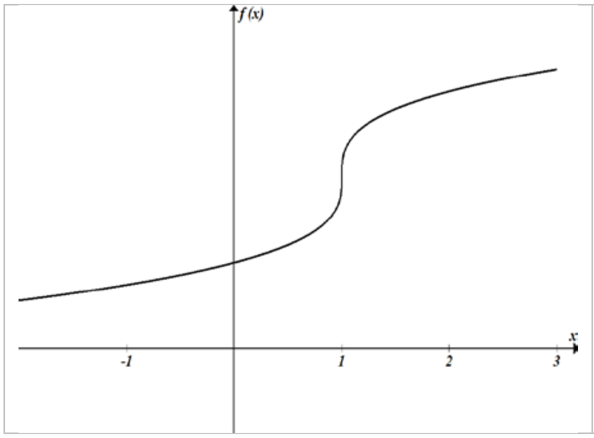
The slope graph has ____________________________at $x=2$.

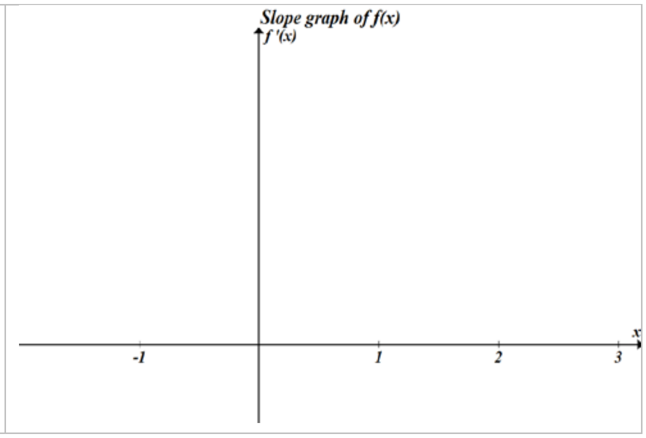




Example 9:

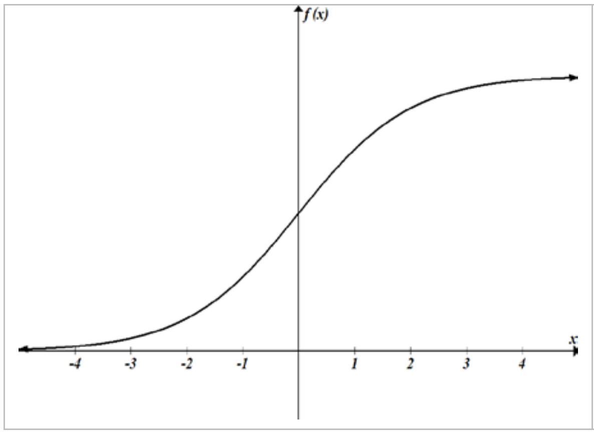
Draw the slope graph for the following function. This function *has a vertical tangent at* $x = 1$. The derivative does not exist at x = 1 because the slope of a vertical tangent line is undefined. The slope graph has ____at $x = 1$.

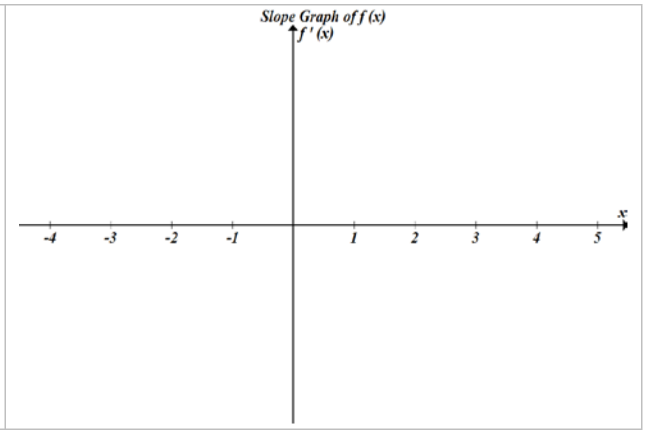




Example 10:

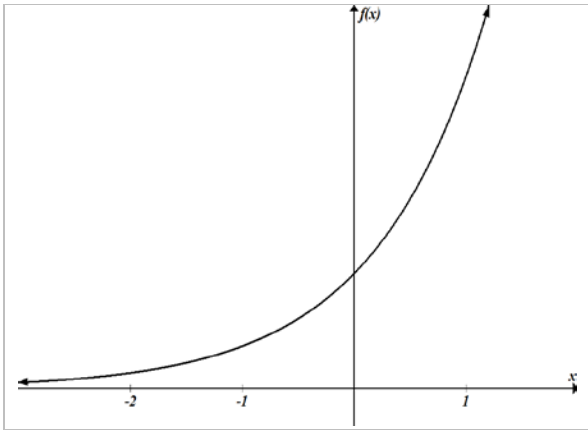
Draw the slope graph for the following logistic function.
$f'(x)$ has a _________________ (*relative maximum/relative minimum)* at approximately $x =$ ___ .

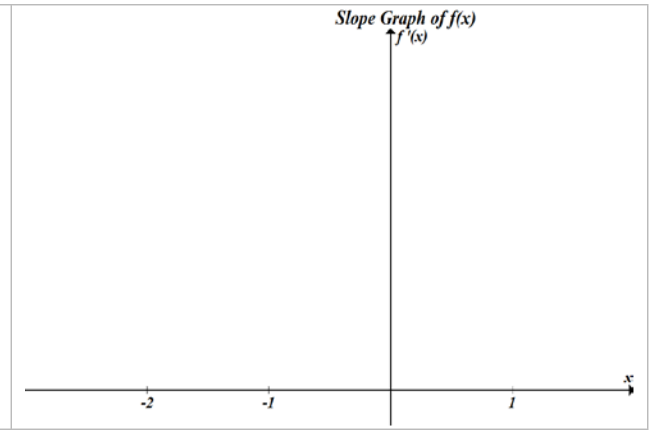




Example 11:

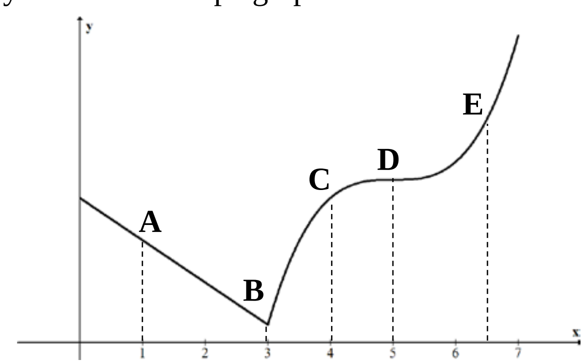
Draw the slope graph for the following exponential function.

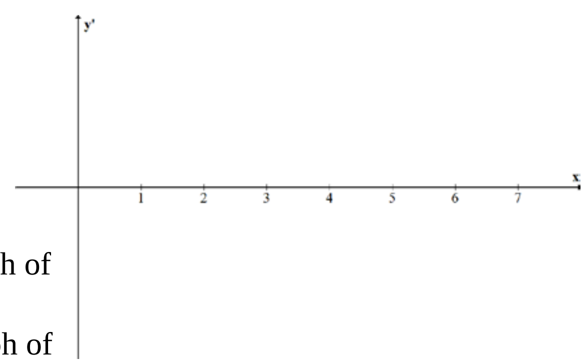




Example 12:

Answer the questions using the graph of $y = f(x)$ shown below and the points labeled points A, B, C, D and E. For parts a - e, list all x-values that apply. Sketch the slope graph on the axes below.





a. $f'(x) > 0$ at $x =$ __________

b. $f'(x) < 0$ at $x =$ ___________

c. $f'(x)$ does not exist at x = _______

d. $f'(x) = 0$ at $x =$ ________

e. A horizontal tangent line would be drawn at $x =$ ______.

f. The average rate of change between point A and B is the same as the instantaneous rate-of-change at $x =$ ________.

g. The tangent line at $x =$ _____ lies above the graph of $y = f(x)$.

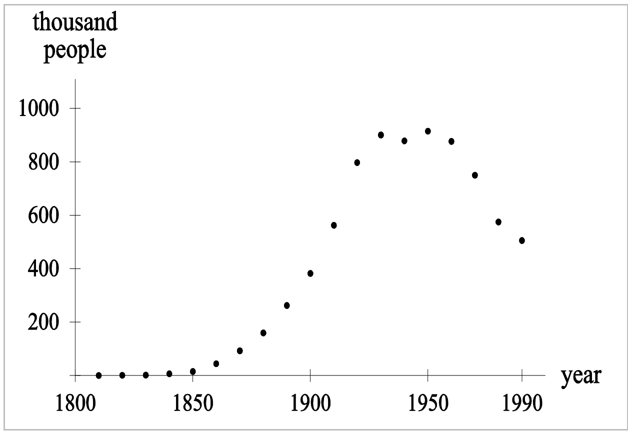
h. The tangent line at $x =$ _____ lies below the graph of $y = f(x)$.

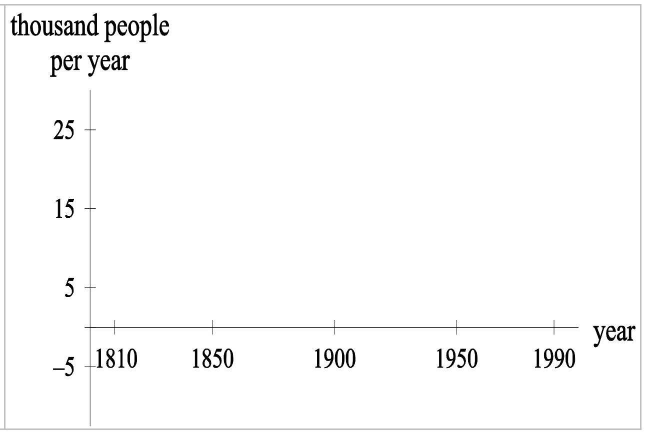
i. The tangent line at $x =$ _____ cuts through the graph of $y = f(x)$.

j. A tangent line cannot be drawn at $x =$ _____.

Example 13: (CC5e pp. 178-79)

A scatter plot of the population data for Cleveland from 1810 through 1990 is shown below.





a. Sketch a smooth curve over the scatter plot. The curve should have one relative maximum and two inflection points.

b. What are the approximate input values at the inflection points?

c. Draw tangent lines and estimate the slope at the inflection points.

d. Sketch the slope graph on the axes shown to the right.

e. Identify the input units and output units on the slope graph.

Example 14:

a. Consider a continuous and differentiable function $f(x)$. For each characteristic of the graph of $f(x)$ described in the table below, identify a corresponding feature on the slope graph $f'(x)$.

Graph of $f(x)$ is/has ….	**Graph of $f'(x)$ is/has…**
Relative Maximum/Minimum (not at a sharp point)	
Increasing	
Decreasing	
Concave up	
Concave down	
Inflection point (without a vertical tangent)	

b. If a function is continuous, but is not differentiable at a point because it has a sharp point, its slope graph will have ________________________________ at that point.

c. If a function is continuous, but is not differentiable at a point because it has a vertical tangent, its slope graph will have ________________________________ at that point.

d. If a function is not continuous at a point, then its slope graph will have ________________________________ at that point.

e. Sketch a graph of a continuous function $f(x)$ on $(-\infty, \infty)$ with the following properties.

$f(0) = 5$

$f'(-3) = 0 = f'(3)$

$f'(x) < 0$ *for* $-3 < x < 3$