Complex Analysis Homework 5

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(4) : Recall that if $p, q \in \mathbb{R}[x]$ such that p and q share no common factor, $\deg(q) \geq \deg(p) + 1$, and $q(x) \neq 0$ for all $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{ix} dx = 2\pi i \sum_{r=1}^{k} \text{res}\left(\frac{p(z)}{q(z)} e^{iz}, z_r\right),$$

where z_r denotes the zeros of q in the upper half-plane. These conditions are satisfied with $p(z) = z$ and $q(z) = z^2 + a^2$. The only zero of q in the upper half plane is $z = ai$ of order 1. We first calculate $\text{res}\left(\frac{z}{z^2+a^2} e^{iz}, ai\right)$:

$$\begin{aligned}
\text{res}\left(\frac{p(z)}{q(z)} e^{iz}, ai\right) &= \lim_{z \to ai} \left(\frac{(z - ai)p(z)e^{iz}}{q(z)}\right) \\
&= \lim_{z \to ai} \left(\frac{ze^{iz}}{z + ai}\right) \\
&= \frac{1}{2e^a}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2 + a^2} dx &= \text{Im}\left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx\right) \\
&= \text{Im}\left(2\pi i \left(\frac{1}{2e^a}\right)\right) \\
&= \frac{\pi}{e^a}.
\end{aligned}$$

(6) : First we do a change of variable with $x = \tan(y)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{n+1}} dx &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(1 + \tan^2(y))^{n+1}} \sec^2(y) dy \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sec^{2n+2}(y)} \sec^2(y) dy \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}(y) dy.
\end{aligned}$$

Denote $I_{2n} := \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}(y) dy$ and do integration by parts, with

$$\begin{aligned}
u &= \cos^{2n-1}(y) & v &= \sin(y) \\
du &= (1 - 2n) \cos^{2n-2}(y) \sin(y) dy & dv &= \cos(y) dy
\end{aligned}$$

we obtain

$$\begin{aligned}
I_{2n} &= \cos^{2n-1}(y)\sin(y)_{|_{-\pi/2}^{\pi/2}} - \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1-2n)\cos^{2n-2}(y)\sin^2(y)dy \\
&= -\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1-2n)\cos^{2n-2}(y)\sin^2(y)dy \\
&= (2n-1)\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n-2}(y)(1-\cos^2(y))dy \\
&= (2n-1)\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n-2}(y)dy - (2n-1)\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n}(y)dy \\
&= (2n-1)I_{2n-2} - (2n-1)I_{2n},
\end{aligned}$$

where $\cos^{2n-1}(y)\sin(y)_{|_{-\pi/2}^{\pi/2}} = 0$ since $\cos^{2n-1}(y)\sin(y)$ is odd. Solving for I_{2n}, we obtain

$$\begin{aligned}
I_{2n} &= \frac{2n-1}{2n} I_{2n-2} \\
&= \frac{(2n-1)(2n-3)}{(2n)(2n-2)} I_{2n-4} \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)(2n-2)\cdots 4\cdot 2} I_0 \\
&= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)(2n-2)\cdots 4\cdot 2} \cdot \pi,
\end{aligned}$$

since $I_0 = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} dy = \pi$.

$(6')$: Recall that if $p, q \in \mathbb{R}[x]$ such that p and q share no common factor, $\deg(q) \geq \deg(p) + 2$, and $q(x) \neq 0$ for all $x \in \mathbb{R}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{r=1}^{k} \operatorname{res}\left(\frac{p}{q}, z_r\right)$$

where z_r denotes the zeros of q in the upper half-plane. These conditions are satisfied with $p(z) = 1$ and $q(z) = (1+z^2)^{n+1}$. The only zero of q in the upper half plane is $z = i$ of order $n+1$. We first calculate $\operatorname{res}\left(\frac{1}{(1+z^2)^{n+1}}, i\right)$:

$$\begin{aligned}
\operatorname{res}\left(\frac{1}{(1+z^2)^{n+1}}, i\right) &= \frac{1}{n!} \lim_{z\to i} \frac{d^n}{dz^n}\left(\frac{(z-i)^{n+1}}{(1+z^2)^{n+1}}\right) \\
&= \frac{1}{n!} \lim_{z\to i} \frac{d^n}{dz^n}\left(\frac{1}{(z+i)^{n+1}}\right) \\
&= \frac{1}{n!} \lim_{z\to i} \left(-(n+1)\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{(z+i)^{n+2}}\right)\right) \\
&= \frac{1}{n!} \lim_{z\to i} \left((n+1)(n+2)\frac{d^{n-2}}{dz^{n-2}}\left(\frac{1}{(z+i)^{n+3}}\right)\right) \\
&= \frac{1}{n!} \lim_{z\to i} \left(\frac{(-1)^n(n+1)(n+2)\cdots(2n)}{(z+i)^{2n+1}}\right) \\
&= \frac{-i(n+1)(n+2)\cdots(2n)}{2^{2n+1}n!} \\
&= \frac{-i\cdot 2n!}{2^{2n+1}\cdot n!\cdot n!} \\
&= \frac{-i}{2}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}
\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx &= 2\pi i \left(\frac{-i}{2} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right) \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \pi.\end{aligned}$$

(12) : Recall that if $f = p/q$ where $p, q \in \mathbb{R}[x]$ such that p and q share no common factor, $\deg(q) \geq \deg(p) + 2$, and $q(n) \neq 0$ for all $n \in \mathbb{Z}$. Then

$$\sum_{n \in \mathbb{Z}} f(n) = -\sum_{k=1}^{\ell} \operatorname{res}(f(z)\pi \cot(\pi z), z_k),$$

where $z_1, \ldots, z_\ell$ are the zeros of q in $\mathbb{C}$. These conditions are satisfied with $p(z) = 1$ and $q(z) = (u+z)^2$. The zero of q is $z = -u$ of order 2. Thus

$$\begin{aligned}\operatorname{res}\left(\frac{\pi \cot \pi z}{(u+z)^2}, -u \right) &= \lim_{z \to -u} \left(\frac{d}{dz} (\pi \cot \pi z) \right) \\ &= \lim_{z \to -u} \left(-\pi^2 \csc^2(\pi x) \right) \\ &= \pi^2 \csc^2(\pi u).\end{aligned}$$

Therefore

$$\sum_{n \in \mathbb{Z}} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

(13) : We may assume $z_0 = 0$. Let g be given by $g(z) = zf(z)$. Then g is holomorphic in $D_r(0) \backslash \{0\}$ and $|f(z)| \leq A|z|^{-1+\varepsilon}$ implies $|g(z)| \leq A|z|^{\varepsilon}$. In particular, g is bounded in $D_r(0) \backslash \{0\}$, and thus has a removable singularity at 0. Writing f and g in terms of their Laurent series at $z = 0$,

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \qquad g(z) = \sum_{n \in \mathbb{Z}} a_{n-1} z^n,$$

this implies $a_n = 0$ for $n \leq -2$. In fact, taking $z \to 0$, we have $|g(z)| \leq A|z|^{\varepsilon}$ implies $0 = g(0) = a_{-1}$. Thus, $a_n = 0$ for all $n \leq -1$, which implies f has a removable singularity at 0.