Category Theory Extra Notes

August 2, 2016

Contents

Analogy/Reference Tables

monic	**characteristic arrow**	**subobject classifier**
$\tau : F \hookrightarrow G$	$(\chi_\tau)_V(x) = \{U \subseteq V \mid G_U^V(x) \in F(U)\}$	$\Omega(V)$ is the set of all V-sieves
$i : (X, \lambda) \to (Y, \mu)$	$\chi_i(y) = \{m \mid \mu_m(y) \in X\}$	L_m set of all left ideals
$k : (A, q) \to (B, p)$	$\chi_k(b) = (p(b), [p(A \cap U_b)]_{p(b)})$	$B_\Omega = \bigcup_{x \in X} \Omega_x$

$\top : 1 \to \Omega$	$\bot : 1 \to \Omega$
$\top_U(0) = \mathbf{O}(V)$ (varies with U)	$\mathbf{O}(\emptyset)$
$\top(0) = M$ (fixed)	$\bot(x) = \emptyset$
$\top(x) = (x, [X]_x)$ (varies with x)	$\bot(x) = (x, [\emptyset]_x)$

Part I
Presheaf Topoi

Let's discuss a few examples of presheaf categories and describe them as elementary topoi. Recall that an elementary topos is a cartesian closed category **C** which has a subobject classifier Ω. So we just need to show that the category has:

1. pullbacks and a terminal object (finitely complete).
2. an exponential object (finitely complete + exponential object = cartesian closed).
3. a subobject classifier.

1 Set $\times$ Set

Example 1.1. Let **C** be a *discrete* category with two objects. So it looks like this:

$$\begin{tikzcd} \bullet \arrow[loop left, "1_\bullet"] & & \bullet \arrow[loop right, "1_\bullet"] \end{tikzcd}$$

Thus, the presheaf category $\mathbf{Set}^{\mathbf{C}}$ is just $\mathbf{Set} \times \mathbf{Set}$. Here is what the objects and morphisms look like:

<u>**Objects**</u>	<u>**Morphisms**</u>
$A \qquad B$	$A \xrightarrow{f} C$ $B \xrightarrow[g]{} D$
object (A, B)	arrow (f, g)

1.0.1 terminal object

A terminal object in this category is a pair $(\{0\}, \{0\})$ of terminal objects in **Set**.

$$0 \qquad 0$$

pullback
Given two morphisms $(f, g) : (A, B) \to (E, F)$ and $(h, k) : (C, D) \to (E, F)$, the pullback is just a pair of pullbacks in **Set**.

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "i"'] & C \arrow[d, "h"] \\ A \arrow[r, "f"'] & E \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r, "v"] \arrow[d, "u"'] & D \arrow[d, "k"] \\ B \arrow[r, "g"'] & F \end{tikzcd}$$

1.0.2 exponential object

$(A, B)^{(C,D)} = (A^C, B^D)$

1.0.3 subobject classifier

The subobject classifier is a pair (Ω, Ω) of subobject classifiers in **Set**.

2 $\mathbf{Set^n}$

2.1 Arrow Category

Example 2.1. Let **2** be a category with only two objects 0 and 1, and only one non-identity morphism, $0 \to 1$. This category is typically called the **arrow category**, and it looks like this:

$$\begin{tikzcd} 0 \arrow[loop above, "id_0"] \arrow[r] & 1 \arrow[loop above, "id_1"] \end{tikzcd}$$

$\mathbf{Set^2}$ is called the **arrow category** of **Set**. In general, for any category **C**, the presheaf category $\mathbf{C^2}$ is called the **arrow category** of **C**. How do we describe objects and morphisms in $\mathbf{Set^2}$? Here's how:

Objects	**Morphisms**
$S_0 \xrightarrow{S_{01}} S_1$	$\begin{tikzcd} S_0 \arrow[r, "S_{01}"] \arrow[d, "f_0"'] & S_1 \arrow[d, "f_1"] \\ T_0 \arrow[r, "T_{01}"'] & T_1 \end{tikzcd}$
object S_{01}	arrow (f_0, f_1)

Here, S_0, S_1, T_0, T_1 are sets, and f_0, f_1, S_{01}, T_{01} are functions. We now describe the topos structure of $\mathbf{Set^2}$.

2.1.1 terminal object

The terminal object 1 is the identity function from the terminal set to itself $id_{\{0\}} : \{0\} \to \{0\}$.

$$\{0\} \xrightarrow{id_{\{0\}}} \{0\}$$

2.1.2 pullback

Given two morphisms $(f_0, f_1) : S_{01} \to T_{01}$ and $(g_0, g_1) : R_{01} \to T_{01}$, we obtain part of a cube

$$\begin{tikzcd}
 & & & R_0 \arrow[dl, "R_{01}"'] \arrow[dd, "g_0"] \\
 & & R_1 \arrow[dd, "g_1"] & \\
 & S_0 \arrow[rr, "f_0"] \arrow[dl, "S_{01}"'] & & T_0 \arrow[dl, "T_{01}"] \\
S_1 \arrow[rr, "f_1"'] & & T_1 &
\end{tikzcd}$$

to complete the cube, form the pullbacks

$$\begin{tikzcd}
Z_0 \arrow[r, "p_0"] \arrow[d, "q_0"'] & R_0 \arrow[d, "g_0"] \\
S_0 \arrow[r, "f_0"'] & T_0
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_1 \arrow[r, "p_1"] \arrow[d, "q_1"'] & R_1 \arrow[d, "g_1"] \\
S_1 \arrow[r, "f_1"'] & T_1
\end{tikzcd}$$

in **Set**, and now we get something that looks like this

$$\begin{tikzcd}
 & Z_0 \arrow[rr, "p_0"] \arrow[dl, "Z_{01}"'] \arrow[dd, "q_0"' near end] & & R_0 \arrow[dl, "R_{01}"'] \arrow[dd, "g_0"] \\
Z_1 \arrow[rr, "p_1" near end] \arrow[dd, "q_1"'] & & R_1 \arrow[dd, "g_1" near start] & \\
 & S_0 \arrow[rr, "f_0" near start] \arrow[dl, "S_{01}"'] & & T_0 \arrow[dl, "T_{01}"] \\
S_1 \arrow[rr, "f_1"'] & & T_1 &
\end{tikzcd}$$

Here, Z_{01} is uniquely determined by the universal property of Z_1. Let's understand why this is. All we need to show is that Z_0 is a contender, i.e.:

$$f_1 \circ S_{01} \circ q_0 = T_{01} \circ g_0 \circ p_0$$

We know that $f_1 \circ S_{01} = T_{01} \circ f_0$ and $f_0 \circ q_0 = g_0 \circ p_0$ by the commutative diagrams, so using this we get:

$$f_1 \circ S_{01} \circ q_0 = T_{01} \circ f_0 \circ q_0 = T_{01} \circ g_0 \circ p_0$$

And we are done.

2.1.3 exponential object

2.1.4 subobject classifier

The object Ω is the function $\Omega_{01} : \Omega_0 \to \Omega_1$ where:

$$\Omega_0 = \{0, \frac{1}{2}, 1\} \text{ and } \Omega_1 = \{0, 1\}$$

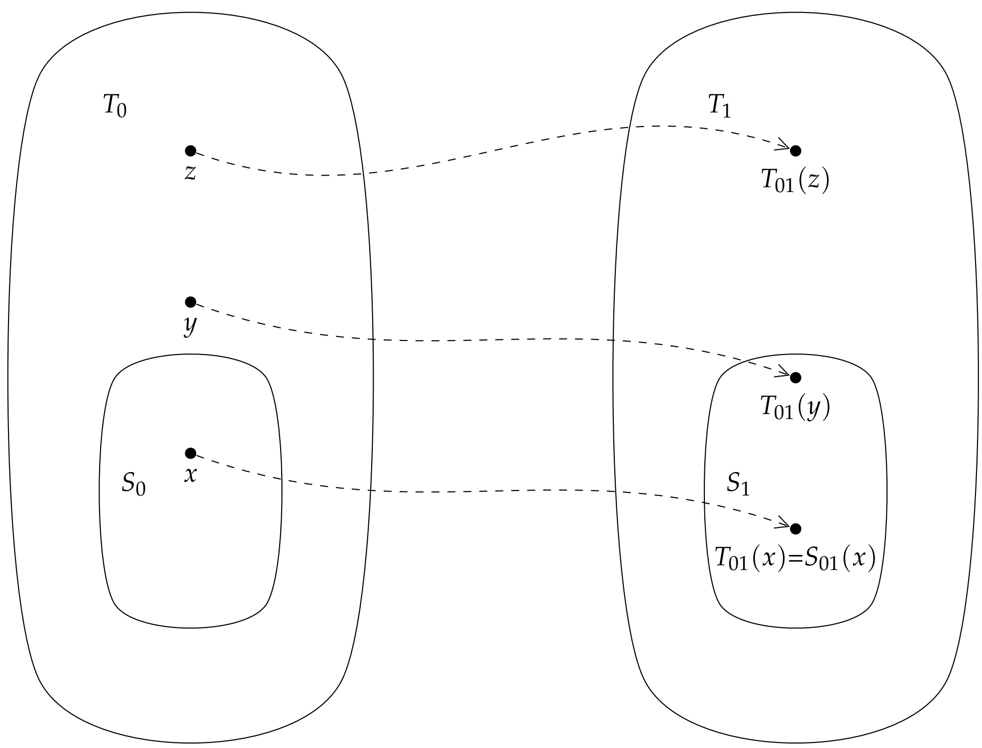
$$\Omega_{01}(0) = 0, \Omega_{01}(\frac{1}{2}) = 1, \text{ and } \Omega(1) = 1$$

The classifier arrow $\top : 1 \to \Omega$, is given by a pair

$$\top_0 : \{0\} \to \Omega_0 \mid \top_0(0) = 1$$

$$\top_1 : \{0\} \to \Omega_1 \mid \top_1(0) = 1$$

The reason we need an extra element $\frac{1}{2}$ in the domain is because we might have a subobject $(i_0, i_1) : S_{01} \to T_{01}$, with i_0 and i_1 being inclusions, that looks like this:



Let's see what goes wrong if Ω is just a function $id_{\{0,1\}} : \{0,1\} \to \{0,1\}$: Given the subobject $(i_0, i_1) : S_{01} \to T_{01}$ which we drew above, we could try defining a characteristic function $\chi_{S_{01}} := (\chi_0, \chi_1) : T_{01} \to id_{\{0,1\}}$ by letting $\chi_0 = \chi_{S_0}$ and $\chi_1 = \chi_{S_1}$ (i.e. the characteristic functions in set). But then this wouldn't be a commutative diagram: χ_{S_0}

$$\begin{tikzcd}
S_{01} \arrow[r, "{(i_0,i_1)}"] \arrow[d, "!"'] & T_{01} \arrow[d, "\chi_{S_{01}}"] \\
1 \arrow[r, "\top"'] & \Omega
\end{tikzcd}$$

$T_{01}(y)$ is sent to 1, since it is contained in S_1, and y to 0, since it is not contained in S_0. This is why we need to introduce a new element $\frac{1}{2}$ in the domain of Ω_{01}. The idea is that the element y will be contained in S_1, but not yet! In general, an element $t \in T_0$ is classified in three ways

1. $t \in S_0$, or
2. $t \notin S_0$, but $T_{01}(t) \in S_1$, or
3. $t \notin S_0$, and $T_{01}(t) \notin S_1$

define $\chi_0 : T_0 \to \Omega_0$ by

$$\chi_0(x) = \begin{cases} 1 & \text{(1) holds} \\ \frac{1}{2} & \text{(2) holds} \\ 0 & \text{(3) holds} \end{cases}$$

and define $\chi_1 : T_1 \to \Omega_1$ to be the characteristic function of S_1 in **Set**, χ_{S_1}.
Now form the cube demonstrates (χ_0, χ_1) is the unique arrow that corresponds to the monic (i_0, i_1) .

$$\begin{tikzcd}
& S_0 \arrow[rr, "i_0"] \arrow[dl, "S_{01}"'] \arrow[dd, "!"' near end] & & T_0 \arrow[dl, "T_{01}"'] \arrow[dd, "\chi_0"] \\
S_1 \arrow[rr, "i_1" near start] \arrow[dd, "!"'] & & T_q \arrow[dd, "\chi_1" near start] & \\
& \{0\} \arrow[rr, "\top_0" near start] \arrow[dl, "id_{\{0\}}"'] & & \Omega_0 \arrow[dl, "\Omega_{01}"] \\
\{0\} \arrow[rr, "\top_1"'] & & \Omega_1 &
\end{tikzcd}$$

2.2 "Sets through time"

Example 2.2. We can easily generalize everything we did in the arrow category of set by replacing the poset category **2** with the poset ω, i.e. the poset of all finite ordinals under their natural ordering. It looks like this:

$$\begin{tikzcd}
0 \arrow[r] \arrow[loop above, "id_0"] & 1 \arrow[r] \arrow[loop above, "id_1"] & 2 \arrow[r] \arrow[loop above, "id_2"] & \cdots \arrow[r] & n \arrow[r] \arrow[loop above, "id_n"] & n+1 \arrow[r] \arrow[loop above, "id_{n+1}"] & \cdots
\end{tikzcd}$$

How do we describe objects and morphisms in the presheaf category $\mathbf{Set}^{\omega}$, here's how:

Objects

$$\begin{tikzcd}
S_0 \arrow[r, "S_{01}"] & S_1 \arrow[r, "S_{12}"] & S_2 \arrow[r, "S_{23}"] & \cdots
\end{tikzcd}$$

object $(S_{01}, S_{12}, \dots)$

Morphisms

$$\begin{tikzcd}
S_0 \arrow[r, "S_{01}"] \arrow[d, "f_0"] & S_1 \arrow[r, "S_{12}"] \arrow[d, "f_1"] & S_2 \arrow[r, "S_{23}"] \arrow[d, "f_2"] & \cdots \\
T_0 \arrow[r, "T_{01}"'] & T_1 \arrow[r, "T_{12}"'] & T_2 \arrow[r, "T_{23}"'] & \cdots
\end{tikzcd}$$

arrow $(f_0, f_1, \dots)$

Here, $S_0, S_1, \dots, T_0, T_1, \dots$ are sets, and $f_0, f_1, \dots, S_{01}, S_{12}, \dots, T_{01}, T_{12}, \dots$ are functions. It has been suggested by Lawvere that such an object $(S_{01}, S_{12}, \dots)$ be considered as a "set through time", where each S_n is regarded as the state of the variable set S at the (discrete) time n. If $m < n-1$, we define:

$$S_{mn} := S_{(n-1)n} \circ \cdots \circ S_{m(m+1)}$$

For example, $S_{02} = S_{12} \circ S_{01}$. We now describe the topos structure of $\mathbf{Set}^{\omega}$.

2.2.1 terminal object

The terminal object 1 is simply $(id_{\{0\}}, id_{\{0\}}, \dots)$.

$$\begin{tikzcd}
\{0\} \arrow[r, "id_{\{0\}}"] & \{0\} \arrow[r, "id_{\{0\}}"] & \{0\} \arrow[r, "id_{\{0\}}"] & \cdots
\end{tikzcd}$$

2.2.2 pullback

Pullbacks in $\mathbf{Set}^{\omega}$ are constructed very similarly to the arrow category of $\mathbf{Set}$. Given two morphisms $(f_0, f_1, f_2, \dots) : (S_{01}, S_{12}, \dots) \to (T_{01}, T_{12}, \dots)$ and $(g_0, g_1, g_2, \dots) : (R_{01}, R_{12}, \dots) \to (T_{01}, T_{12}, \dots)$ we first find the pullbacks at each n:

$$\begin{tikzcd}
Z_0 \arrow[r, "p_0"] \arrow[d, "q_0"'] & R_0 \arrow[d, "g_0"] & Z_1 \arrow[r, "p_1"] \arrow[d, "q_1"'] & R_1 \arrow[d, "g_1"] & \cdots \\
S_0 \arrow[r, "f_0"'] & T_0 & S_1 \arrow[r, "f_1"'] & T_1 &
\end{tikzcd}$$

in $\mathbf{Set}$, and by the universal properties of $Z_1, Z_2, \dots,$ we get unique maps $Z_{01} : Z_0 \to Z_1$, $Z_{12} : Z_1 \to Z_2$, and so on. The pullback then is $(Z_{01}, Z_{12}, \dots)$.

2.2.3 exponential object

2.2.4 subobject classifier

In $\mathbf{Set}^{\omega}$, Ω is the object $(\Omega_{01}, \Omega_{12}, \dots)$ where:

$$\{\emptyset, 0, 1, 2, \dots\} = \Omega_0 = \Omega_1 = \Omega_2 = \dots$$

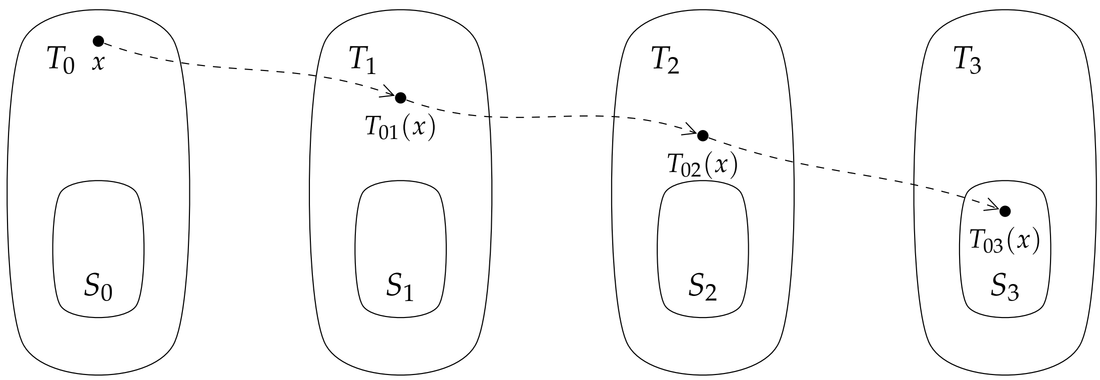
$$\Omega_{01}(\emptyset) = \emptyset, \Omega_{01}(0) = 0,\ \Omega_{01}(1) = 0,\ \Omega_{01}(2) = 1,\ \Omega_{01}(3) = 2,\ \dots$$
$$\Omega_{12}(\emptyset) = \emptyset, \Omega_{12}(0) = 0,\ \Omega_{12}(1) = 1,\ \Omega_{12}(2) = 1,\ \Omega_{12}(3) = 2,\ \dots$$
$$\dots$$

The classifier arrow $\top : 1 \to \Omega$, is given by:

$$\top_0 : \{0\} \to \Omega_0 \mid \top_0(0) = 0$$

$$\top_1 : \{0\} \to \Omega_1 \mid \top_1(0) = 0$$

$$\dots$$



The characteristic function $\chi_S = (\chi_0, \chi_1, \dots)$ has a nice interpretation. For example, in the image above, $x \notin S_0$, but $T_{03}(x) \in S_3$. We think of this as saying, "x is in S three seconds later". So we set $\chi_0(x) = 3$

3 SSets

3.1 Simplex Category

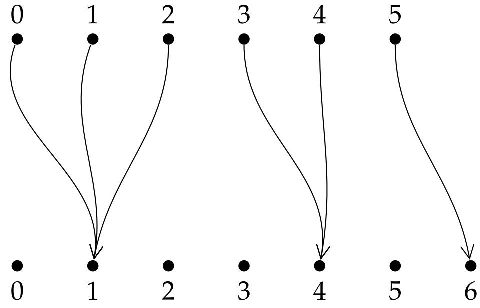
Definition 3.1. There's a category Δ_{alg}- the algebraist's category of simplices- where the objects are finite totally ordered sets and morphisms are order pereserving functions, i.e. functions with

$$x \leq y \ \Rightarrow\ f(x) \leq f(y).$$

Objects

object $6 = \{0, 1, 2, 3, 4, 5\}$

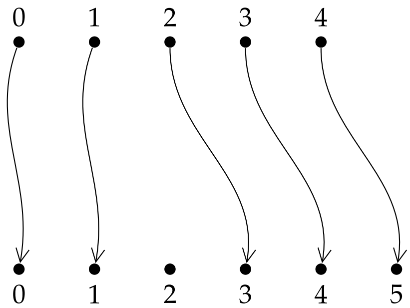
Morphisms



arrow $f : 6 \to 7$

Δ_{alg} is generated by the objects $0, 1, 2, \ldots$ and certain special morphisms like:

δ_2^6

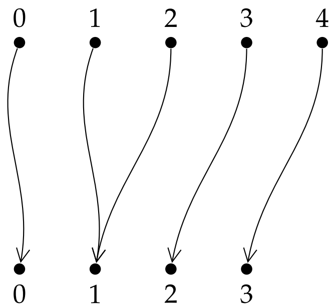


More generally, we need **face maps**

$$\delta_i^n : n - 1 \hookrightarrow n \qquad (n > 0 \text{ and } 0 \leq i < n)$$

the order preserving injective map whose image only fails to contain i. We also need ones like:

σ_1^4



more generally, we need **degeneracy maps**

$$\sigma_i^n : n + 1 \to n \qquad (n > 0 \text{ and } 0 \leq i < n)$$

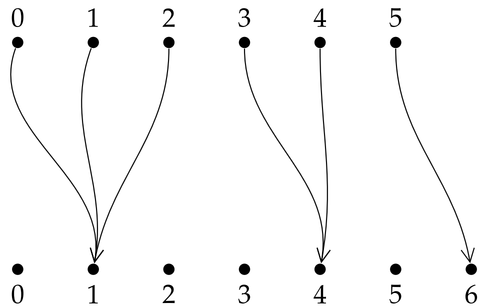
the order preserving surjective map such that $\sigma_i(i) = \sigma_i(i+1) = i$. By composing various face and degeneracy maps we can build any morphism in Δ_{alg}, subject to the following relations, called the **simplicial identities**

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n \qquad (0 \leq i < j \leq n)$$

$$\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1} \qquad (0 \leq i \leq j < n)$$

$$\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1} & 0 \leq i < j < n \\ id_n & 0 \leq j < n \quad \text{and} \quad i = j \quad \text{or} \quad i = j+1 \\ \delta_{i-1}^n \circ \sigma_j^{n-1} & 0 \leq j \quad \text{and} \quad j+1 < i \leq n \end{cases}$$

For example, one can easily verify that this order preserving map:

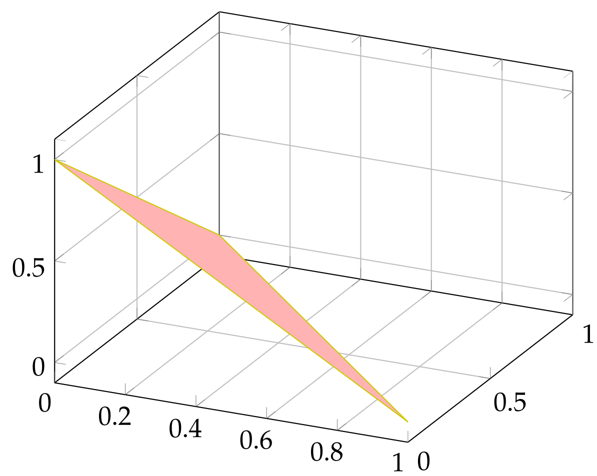


is just

$$\delta_5^7 \circ \sigma_4^6 \circ \delta_3^7 \circ \delta_2^6 \circ \sigma_1^5 \circ \sigma_1^6 \circ \delta_0^7$$

We can turn n into a space, the **standard** $(n-1)$-simplex:

$$\Delta(n-1) = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_i \geq 0 \text{ and } \sum x_i = 0\}$$



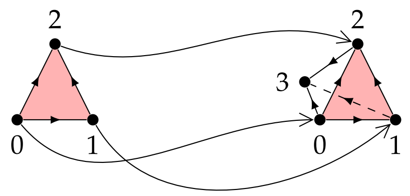
And we can turn any morphism $f : n \to m$ into an affine map

$$\Delta(f) : \Delta(n-1) \to \Delta(m-1)$$

which is the unique affine map sending the i-th vertex of $\Delta(n-1)$ to the $f(i)$-th vertex of $\Delta(m-1)$. Indeed, we have a functor

$$\Delta : \Delta_{alg} \to \mathbf{Top}$$

For example, $\Delta(\delta_3^4) : 3 \to 4$ looks like:



When $n = 0$, this functor give "the standard (-1)-simplex which is $\emptyset$. If we want to avoid this, we can restrict Δ_{alg} to Δ_{top}, the category of *nonempty* totally ordered sets and order preserving functions. This is the topologist's category of simplices. Topologists call this Δ and call Δ_{alg} the **augmented** category of simplices.

Definition 3.2. A **simplicial set** is a functor

$$F : \Delta_{top}^{op} \to \mathbf{Set}$$

So a simplicial set looks like this:

$$\begin{tikzcd}
F(1) \arrow[r, "F(\sigma_0)"] & F(2) \arrow[l, bend right, "F(\delta_0)"'] \arrow[l, bend left, "F(\delta_0)"] \arrow[r, bend left=20, "F(\sigma_0)"] \arrow[r, bend right=20, "F(\sigma_1)"'] & F(3) \cdots \arrow[l, "F(\delta_1)"'] \arrow[l, bend right=60, "F(\delta_0)"'] \arrow[l, bend left=60, "F(\delta_2)"]
\end{tikzcd}$$

Here, we think of

$$F(1) = \{vertices\}$$
$$F(2) = \{edges\}$$
$$F(3) = \{triangles\}$$

And we also make write

$$\partial_i = F(\delta_i)$$
$$\epsilon_i = F(\sigma_i)$$

3.1.1 Subobject Classifier

Definition 3.3. An **abstract simplicial complex** S on a vertex set V is a set of subsets of V such that

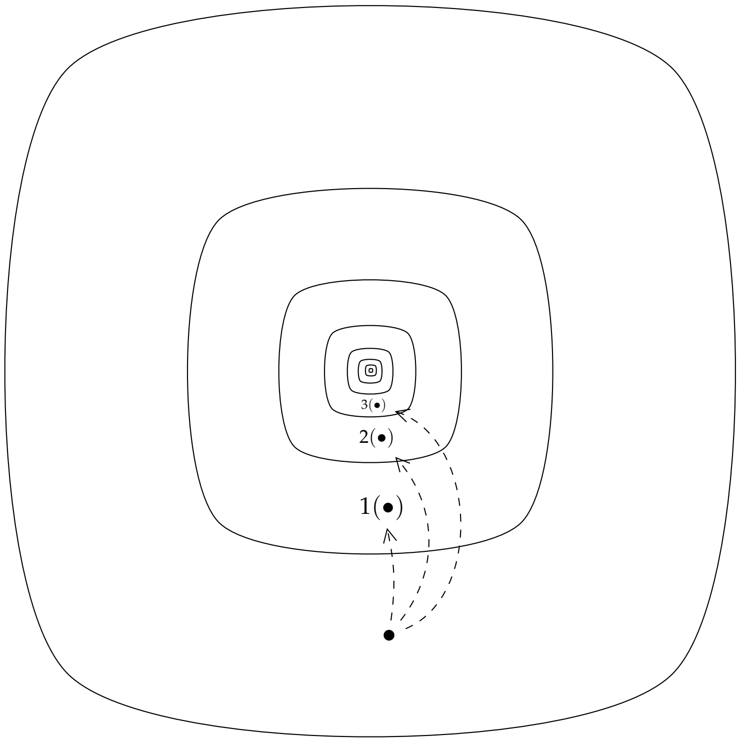
$$\{v_i\} \in \Delta \text{ if } v_i \in V$$

if $\sigma \subseteq \tau \in S$, then $\sigma \in S$.

3.2 Cohomology of spaces:

$$\mathbf{Top} \xrightarrow{S} hom(\Delta^{op}, \mathbf{Set}) \xrightarrow{\mathbb{Z}} hom(\Delta^{op}, \mathbf{AbGp}) \xrightarrow{\cong} \text{[Chain complexes]}$$

4 M-set



Example 4.1. Let M be a monoid (which may be thought of as a category with one object) an object in $\mathbf{Set}^M$ is called an M-set. We can think of an M-set as a pair (X, λ), where $\lambda : M \times X \to X$ is a function such that gives us for each m a function λ_m such that:

- $\lambda_e(x) = x$ for e the identity in M.
- $\lambda_{m \cdot n} = \lambda_m \circ \lambda_n$

A morphism $f : (X, \lambda) \to (Y, \mu)$ is an equivariant, or action-preserving function $f : X \to Y$, i.e. one such that for all $m \in M$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_m"'] & Y \arrow[d, "\mu_m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

4.0.1 terminal object

A terminal object in this category is $(\{0\}, \lambda)$, where $\{0\}$ is the terminal object in **Set** and $\lambda_m(0) = 0$ for all $m \in M$.

4.0.2 pullback

Given arrows $g : (Y, \mu) \to (Z, \gamma)$ and $f : (X, \lambda) \to (Z, \gamma)$, the pullback of f and g is $(X \times_Z Y, \delta)$, where $X \times_Z Y$ is the pullback of f and g in **Set**, and $\delta_m = \lambda_m \times \mu_m$.

4.0.3 subobject classifier

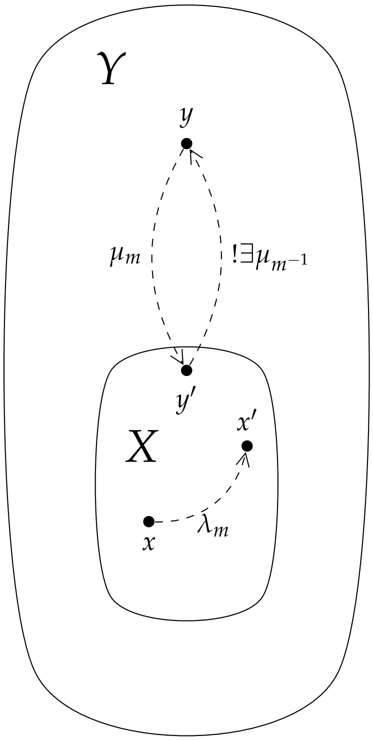
Definition 4.1. A **left ideal** of a monoid M is a subset I of M which is closed under left-multiplication, i.e. $m \cdot i \in I$ for all $i \in I$ and $m \in M$. If the monoid is commutative, we will drop the adjective and just say "ideal".

Remark. Don't get this confused with left ideals in ring theory. Every ideal in a ring is a left ideal of the underlying monoid $(R, \cdot, 1)$, but there are usually many more ideals of the underlying monoid that are not ideals of the ring. This is because an ideal J in a ring is required to be compatible with the underlying group $(R, +, 0)$ of the ring, i.e. non-empty and for all $x, y \in J$, $x - y \in J$.

Define $\Omega = (L_M, \omega)$ where L_M is the set of left ideals of M, and $\omega(I) = I : m$, where $I : m$ is defined as $\{n \in M \mid n \cdot m \in I\}$. The classifier arrow $\top : 1 \to \Omega$ is the function that picks out the largest left ideal of M, namely $\top(0) = M$. Here are some important facts about $I : m$:

- $m \in I$ if and only if $I : m = M$.
- If $m \in I'$ and $I' \cap I = \emptyset$, then $I : m = \emptyset$.
- $\emptyset : m = \emptyset$ and $M : m = M$.
- If M is a *commutative* monoid, then $I \subseteq I : m$ for all $m \in M$.

Let's try to understand how the subobject classifier works. If $i : (X, \lambda) \to (Y, \mu)$ is monic, then we may assume that X is a subset of Y and the action μ of Y restricts to the action λ of X. Since i is equivariant, for each $m \in M$, the action λ_m must map an element in X to another element in X. On the other hand, μ_m might map an element in Y to an element in X. The character $\chi_i : (Y, \mu) \to \Omega$ of k is defined by $\chi_i(y) = \{m \mid \mu_m(y) \in X\}$.



Since the classifier arrow $\top : 1 \to \Omega$ picks out the largest ideal M, then if $y \in X$, $\chi_i(y)$ must be M. This is true since μ_m restricts to λ_m on X, and λ_m maps every element in X to another element in X by the equivariance of i. Notice that if $y \notin X$ and μ_m maps y into X, then $\mu_{m^{-1}}$ cannot exist, which just means that m is not invertible. Also notice that if $y \notin X$ and μ_m maps y into X, then $\mu_{n \cdot m} = \mu_n \circ \mu_m$ maps y into X too, so $\chi_i(y)$ really is a left ideal.

4.0.4 exponential object

Given (X, λ) and (Y, μ), we define the exponential $(Y, \mu)^{(X,\lambda)} = (E, \sigma)$, where E is the set of equivariant maps f of the form $f : (M, \cdot) \times (X, \lambda) \to (Y, \mu)$ and $\sigma_m : E \to E$ takes such an f to the function $\sigma_m(f) : M \times X \to Y$ given by $g(n, x) = f(m \cdot n, x)$. The evaluation arrow $ev : (E, \sigma) \times (X, \lambda) \to (Y, \mu)$ have $ev(f, x) = f(e, x)$. Then given an arrow $f : (X, \lambda) \times (Y, \mu) \to (Z, \delta)$, the exponential adjoint $\hat{f} : (X, \lambda) \to (Z, \delta)^{(Y,\mu)}$ takes x to the equivariant map $\hat{f}_x : M \times Y \to Z$ having $\hat{f}_x(m, y) = f(\lambda_m(x), y)$. Categories of the form **M**-set frovide a rich source of examples, particularly of topoi that have "non-classical" properties.

Example 4.2. $(\mathbb{N}, +, 0)$ is the monoid of natural numbers under addtion. The set of left ideals are of the form $[n] = \{n, n+1, \dots\}$, hence the set of left ideals corresponds with the set of natural numbers. The ideal $[n] : m$ is also easy to describe.

$$\text{If } m \geq n \text{, then } [n] : m = \mathbb{N}.$$

$$\text{If } m < n \text{, then } [n] : m = [n - m].$$

Example 4.3. Recall the monoid from the other set of notes:

$$1 \circlearrowright \bullet \circlearrowleft f$$

with this multiplication table:

$\circ$	e	f
e	e	f
f	f	f

Let's denote this monoid $\mathbf{G_2}$. Now, what is the subobject classifier in the category of $\mathbf{G_2}$-sets? We need to compute the left ideals of $\mathbf{G_2}$, and the corresponding action. This is presented in the table below:

ω	$\{e, f\}$	$\{f\}$	$\varnothing$
e	$\{e, f\}$	$\{f\}$	$\varnothing$
f	$\{e, f\}$	$\{e, f\}$	$\varnothing$

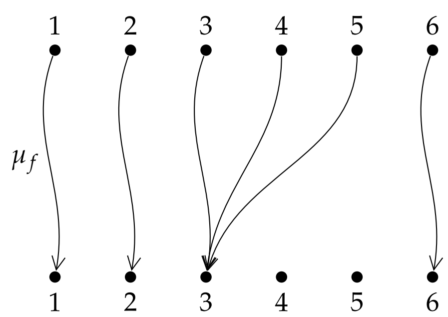
Let's go over an example of a $\mathbf{G_2}$-set Y with six elements. So suppose $Y = \{1, 2, 3, 4, 5, 6\}$, i.e. just a set with five elements. We need two functions μ_f,μ_e from Y to Y which satisfy the following identities:

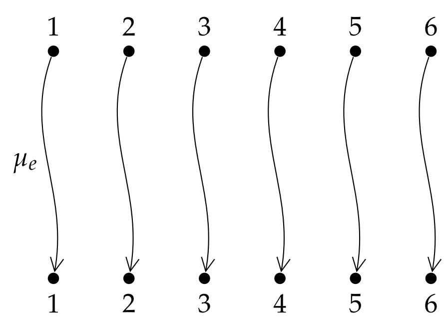
$$\mu_e \circ \mu_f = \mu_f \circ \mu_e = \mu_f$$

$$\mu_e \circ \mu_e = \mu_e$$

$$\mu_f \circ \mu_f = \mu_f$$

These work:





Now, let's find a subobject $i : (X, \lambda) \to (Y, \mu)$ of (Y, μ). Since i is monic, can assume that X is a subset of Y, say $X = \{1, 2, 3, 4\}$. We mentioned earlier that we can then assume that λ_m is just the restriction of μ_m onto X. Let's explicitly show why this is the case. So we need two functions λ_f,λ_e from X to X which satisfies the following identities:

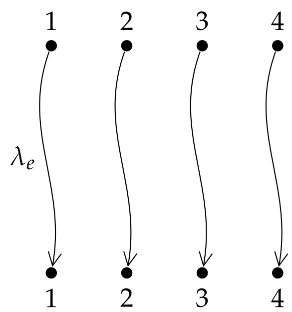
$$\lambda_e \circ \lambda_f = \lambda_f \circ \lambda_e = \lambda_f$$
$$\lambda_e \circ \lambda_e = \lambda_e$$
$$\lambda_f \circ \lambda_f = \lambda_f$$

We also need λ_f,λ_e to satisfy the following commutative diagrams:

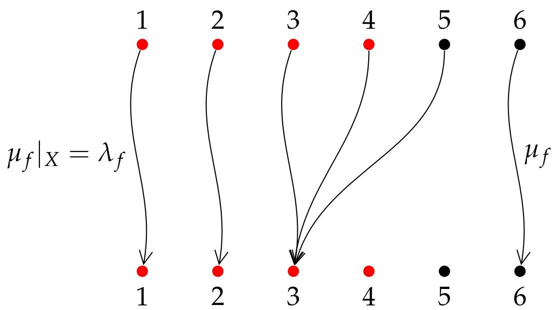
$$\begin{tikzcd} \bullet \arrow[d, "f"'] \\ \bullet \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\lambda_f"'] & Y \arrow[d, "\mu_f"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

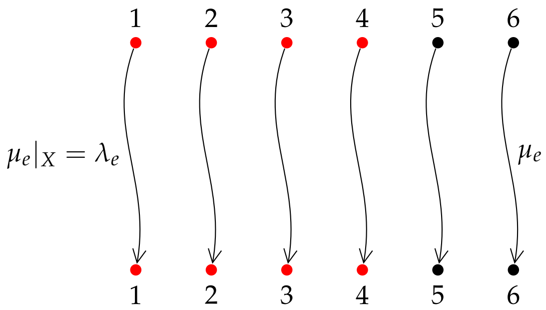
$$\begin{tikzcd} \bullet \arrow[d, "id_\bullet"'] \\ \bullet \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\lambda_e"'] & Y \arrow[d, "\mu_e"] \\ X \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

The arrows $f : \bullet \to \bullet$ and $id_\bullet : \bullet \to \bullet$ are there to remind the reader that i is a natural transformation. Now since we already specified what i is the commutative diagrams above force us to take:



Now, let's highlight the subset X of Y in red:





Now, how does $\Omega = (L_m, \omega)$ classify this subobject? Recall, that $\chi_i(y) = \{m \mid \mu_m(y) \in X\}$. So we have:

$\chi_i(n)$	1	2	3	4	5	6
χ_i	$\{e, f\}$	$\{e, f\}$	$\{e, f\}$	$\{e, f\}$	$\{f\}$	$\varnothing$

So far, this looks similar to how we classified subobjects in $\mathbf{Set}^2$, but there are some important fundamental differences. We will discuss this more at a later section.

5 Slice Categories

5.1 Bundles

Definition 5.1. The **slice category** $\mathbf{C}/c$ of a category $\mathbf{C}$ over an object $c \in \mathbf{C}$ has

- objects are all arrows f with codomain c.
- morphisms from f to f' are all arrows g such that $f' \circ g = f$

Objects

$$d \xrightarrow{f} c$$

object (d, f)

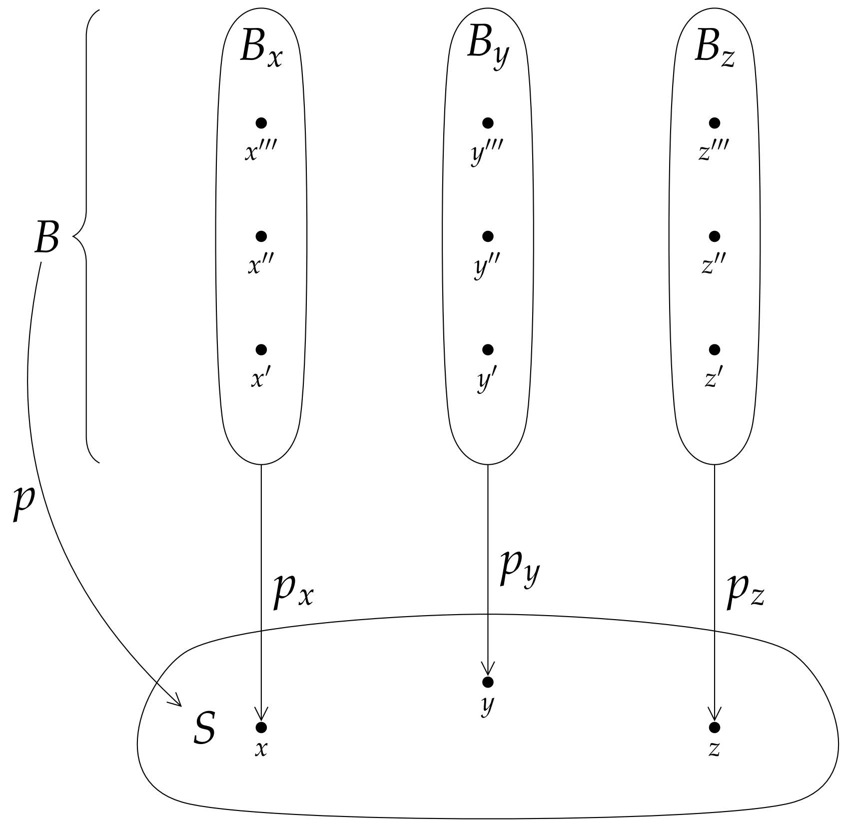
Morphisms

$$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$$

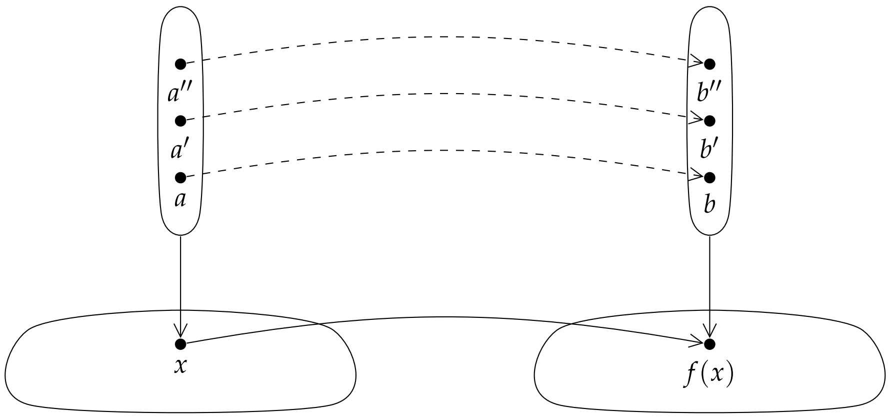
morphism g

Note. $\mathbf{C}/c$ is sometimes called the **over category**.

Example 5.1. An object $(B, p) \in \mathbf{Set}/S$ can be pictured like this:



Let's take a moment to reflect on this image, because it will serve as a nice visualization tool for other categories. For each element $s \in S$, there is an associated set B_s, which is just the inverse image of s under p, i.e. $p^{-1}(s) = B_s$. B_s is typically called the **stalk**, or **fiber** of p over s, and elements of B_s are called *germs* at s. Notice that for any distinct $s, s' \in S$, $B_s \cap B_{s'} = \emptyset$, and that $\bigcup_s B_s = B$. We also have functions p_s, which is just the restriction of p to B_s. The commutativity condition for morphisms in the slice category tells us that a morphism from (B, p) to (B', p') maps at s in (B, p) to germs at s in (B', p'). The whole structure is called a **bundle** of sets over the **base space** X, with B being called the **total space** and p being called the **projection**. What do morphisms $(A, q) \to (B, q)$ look like in $\mathbf{Set}/S$? The commutative diagram tells us that they should be functions $f : A \to B$ that preserve fibers likeso:



Now, let's describe the topos structure of **Set**/S.

5.1.1 terminal object

The terminal object for **Set**/S is (S, id_S). The stalk of id_S over s is $\{s\}$, which is just the terminal object in **Set**. So the terminal object in **Set**/S is just a "bundle" of terminal objects in **Set** over S. A morphism from the terminal object (S, id_S) to any other object (B, p) in **Set**/S is just a section of p.

5.1.2 pullback

Given arrows $f : (B, p) \to (A, q)$ and $g : (C, r) \to (A, q)$, the pullback object of f and g in **Set**/S will have the form (Z, j). To figure out what this object is, we forget about the arrows p, q, r, and j, and compute the object Z by forming the pullback of f and g in **Set**

$$\begin{tikzcd}
Z \arrow[r, "m"] \arrow[d, "n"'] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & A
\end{tikzcd}$$

Now that we have Z, the j is the unique arrow which makes the following diagram commutative

$$\begin{tikzcd}
Z \arrow[rr, "m"] \arrow[dd, "n"'] \arrow[dr, "j"] & & B \arrow[dd, "f"] \arrow[dl, "p"] \\
& S & \\
C \arrow[rr, "g"'] \arrow[ur, "r"] & & A \arrow[ul, "q"']
\end{tikzcd}$$

Now if B_s, A_s, C_s are the stalks over s for the bundles (B, p), (A, q), and (C, r) respectively, then the pullback of

$$\begin{tikzcd} & B_s \arrow[d, "f_s"] \\ C_s \arrow[r, "g_s"'] & A_s \end{tikzcd}$$

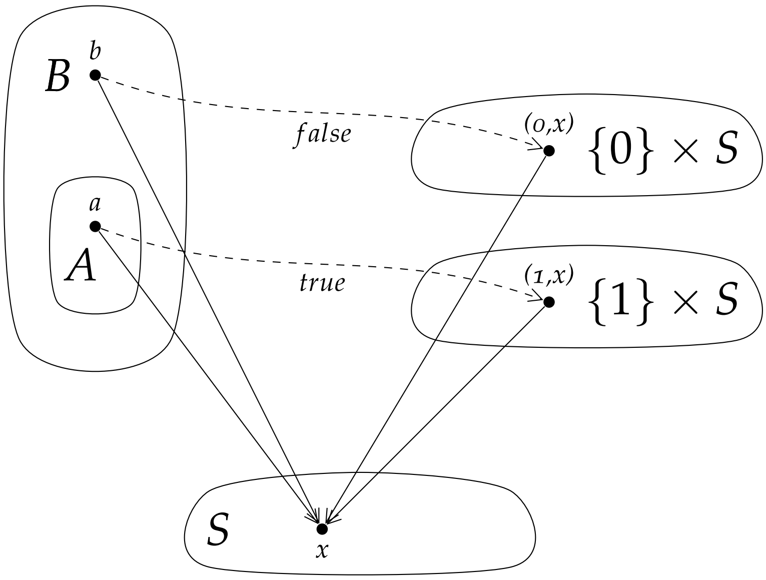
is the same as the stalk over s for (Z, j). Thus, the pullback (Z, j) is a "bundle" of pullbacks in **Set**.

5.1.3 subobject classifier

Ω is $(2 \times S, p_S)$ where $2 = \{0, 1\}$ and $p_S(x, y) = y$. So the stalk over a typical element s consists of two elements: $\{0, s\}$ and $\{1, s\}$. The arrow $\top : 1 \to \Omega$ is a section of Ω, which is defined by $\top(s) = (1, s)$. $\top$ can be thought of as a bundle of copies of the set function *true*. Now, given a monic $i : (A, q) \to (B, p)$, we may assume $A \subseteq B$ and $p|_A = q$. We wish to define the character $\chi_i : (B, p) \to \Omega$ so that the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "q"] \arrow[dd, "q"'] & & B \arrow[dl, "p"'] \arrow[dd, "\chi_i"] \\ & S & \\ S \arrow[ur, "id_S"] \arrow[rr, "\top"'] & & 2 \times S \arrow[ul, "p_S"'] \end{tikzcd}$$

It's easy to see that we get a "bundle" of characteristic functions.



5.2 Etale Spaces

Now, let's consider an object $(B, p) \in \mathbf{Top}/X$. This time, X and B are topological spaces and p is continuous.

Definition 5.2. A **local homeomorphism** is a continuous map $p : B \to X$ with the additional property that for each point $b \in B$ there exists an open neighborhood U_b in B such that $p(U_b)$ is open in X, and p restricts to a homeomorphism $p|_{U_b} : U_b \to p(U_b)$.

Remark. Intuitively, a local homeomorphism preserves "local structure". For example, B is locally compact if and only if $p(B)$ is.

Proposition 5.1. *Let $p : B \to X$ be a local homeomorphism, then p is an open map, i.e. it maps open sets to open sets.*

Proof. Let U be open in B, we need to show that $p(U)$ is open in X. For each $b \in U$, $U \cap U_b$ is open in U_b, and since homeomorphisms are open maps, $p|_{U_b}(U \cap U_b)$ is open in $p(U_b)$ in the subspace topology. Since $p(U_b)$ is open in X, $p(U \cap U_b)$ is open in X too. Finally, since $\bigcup\limits_{b \in U} p(U \cap U_b) = p(U)$, $p(U)$ is open in X.

□

Example 5.2. Every covering map is a local homeomorphism.

Example 5.3. If X is a topological space and Y is a discrete space, then the projection $Y \times X \to X$ is a local homeomorphism.

Example 5.4. On the other hand, the projection map $\mathbb{R} \times X \to X$ is never a local homeomorphism, because no product neighborhood is projected homeomorphically into X. For much the same reason, a nontrivial vector bundle is never a locally homeomorphism either.

Definition 5.3. An **etale space** over X is an object $(B, p) \in \mathbf{Top}/X$ such that p is a local homeomorphism. $\mathbf{Etale}(X)$ is the category whose objects are pairs **etale spaces** over X and morphisms being the same as in $\mathbf{Top}/X$.

Remark. $\mathbf{Etale}(X)$ is a full subcategory of $\mathbf{Top}/X$.

Here's how $\mathbf{Etale}(X)$ looks as a topos:

5.2.1 terminal object

The terminal object for $\mathbf{Etale}(X)$ is (X, id_X).

Definition 5.4. A morphism s from the terminal object (X, id_X) to any other object (B, p) in $\mathbf{Etale}(X)$ is called a **global section** of p.

$$\begin{tikzcd} X \arrow[rr, "s"] \arrow[dr, "id_X"'] & & B \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

So a global section is just a continuous map $s : X \to B$ such that $p \circ s$ is the identity map $id_X : X \to X$. We will also be interested in **local sections** of p, which are continuous maps defined on an open subset $U \subset X$ to B, such that $p \circ s$ is the inclusion $U \hookrightarrow X$. A **section** of a local homeomorphism p will always mean a local section s of p defined on some open subset $U \subset X$. Sometimes the words section and local section are used interchangeably, however we will always include the adjective "global" when referring to global sections.

Proposition 5.2. *If $p : B \to X$ is a local homeomorphism and $U \subset X$ is open, the pullback*

$$\begin{tikzcd} B_U \arrow[r] \arrow[d] & B \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

$B_U \to U$ is also a local homeomorphism.

Remark. A local section of p is just a global section of B_U, for some open subset U of X.

Proposition 5.3. *If $p : B \to X$ is a local homeomorphism and b is any element of B, then there exists a local section of p defined on some open subset V, such that $b \in s(V)$*

Proof. $p|_{U_b}^{-1}$ is a local section of p defined on $p(U_b)$, it maps $p(b)$ to b. □

Definition 5.5. We say that s and t have the same **germ** at a point $x \in X$ when they agree in some open neighborhood of x. We write $[s] \sim_x [t]$ if s and t have the same germ at x.

Proposition 5.4. *Given $b \in B$, $x \in X$, with $p(b) = x$, any local section s of p with b in its image has the same germ as $p|_{U_b}^{-1}$ at x.*

Proof. Suppose s maps x to b. Set$W = s^{-1}(U_b)$, which is an open neighborhood of x. Since s and $p|_{U_b}^{-1}$ agree on W, $[s] \sim_x [p|_{U_b}^{-1}]$. □

Proposition 5.5. *A local section s of p defined on an open set $U \subset X$ is an open map. The images sU of all local sections form a base for the topology of B. If s and t are two local sections, then the set $W = \{x \mid sx = tx\}$ of points where they are both defined and agree is open in X.*

Exercise 1. Prove this.

For ease of notation, we simply write p instead of $p|_{U_b}$. Whenever there is a potential source of confusion, we will use $p|_{U_b}$.

5.2.2 subobject classifier

Let $\mathbf{O}(X)$ be the collection of open subset of X. Consider the following equivalence relation on this collection: $U \sim_x V$ if and only if there is some open set W such that $x \in W$ and $U \cap W = V \cap W$. The idea is that $U \sim_x V$ when the points in U that are close to x are the same as those that are in V and close to x, i.e. "locally" around x, U and V look the same, i.e. the statement "$U = V$" is "locally true" at x. The equivalence class $[U]_x = \{V : U \sim_x V\}$ is called the germ of U at x. It "represents" the collection of all points in U that are "close" to x. The stalk then is $\Omega_x = \{(x, [U]_x) \mid U \subseteq X\}$. Now let

$B_\Omega = \bigcup_{x \in X} \Omega_x$ be the total space $\qquad$ $p : B_\Omega \to X$ the obvious projection map

Then we have $\Omega = (B_\Omega, p)$. The topology on the total space $\hat{\Omega}$ has as base all sets of the form

$$[U, V] = \{(x, [U]_x) \mid x \in V \text{ and } U \subseteq V\}$$

Here are some important facts about germs of open sets:

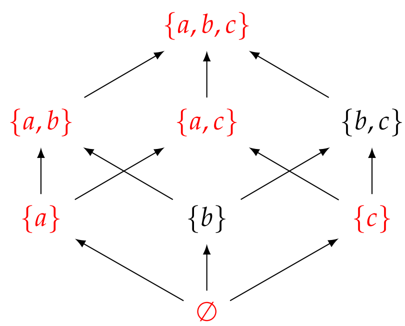
- $[U]_x = [X]_x$ if and only if $x \in U$, i.e. $[X]_x$ is the collection of all open neighborhoods of x.
- $[U]_x = [\varnothing]_x$ if and only if x is **separated** from U, i.e. there exists an open neighborhood V of x and $U \cap V = \varnothing$.

Proposition 5.6. *Every open set in X gives rise to a continuous global section $s_U(x) = (x, [U]_x)$, conversely, if $s : 1 \to \Omega$ is any continuous global section of Ω then $s^{-1}([X, X]) = \{x \mid s(x) = (x, [X]_x)\}$ is open and $s = s_{s^{-1}([X,X])}$. Thus, there is a one to one correspondence between the truth values in* **Top**(X) *and the open subsets of X.*

Remark. Compare this with **Bn**(X), where we found that there was a one to one correspondence between truth values in **Bn**(S) and subsets of S.

Exercise 2. Prove this.

Example 5.5. Consider the topology on the finite set $X = \{a, b, c\}$ where the open sets are highlighted in red:



What is the stalk at point a? We have two elements:

$$X \sim_a \{a, b\} \sim_a \{a, c\} \sim_a \{a\}$$

$$\varnothing \sim_a \{c\}$$

What is the stalk at point b? We have three elements:

$$X \sim_b \{a, b\}$$

$$\{a\} \sim_b \{a, c\}$$

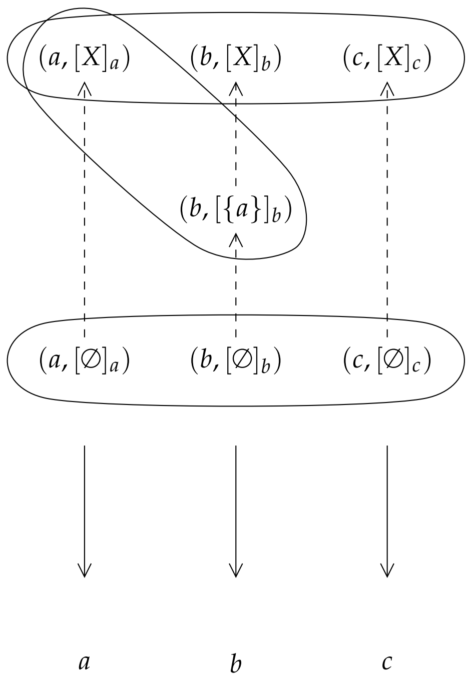
$$\varnothing \sim_b \{c\}$$

What is the stalk at point c? We have two elements:

$$X \sim_c \{a, c\} \sim_c \{c\}$$

$$\varnothing \sim_c \{a\} \sim_c \{a, b\}$$

So our entire space Ω consists of 7 elements:



Notice that for each point in $x \in \Omega$ there exists an open neighborhood U_x (not necessarily unique!) of b such that $p : U_x \to p(U_x)$ is a homeomorphism. In the example above, the three neighborhoods we picked out are:

$$[X, X] = \{(a, [X]_a), (b, [X]_b), (c, [X]_c)\}$$

$$[\varnothing, X] = \{(a, [\varnothing]_a), (b, [\varnothing]_b), (c, [\varnothing]_c)\}$$

$$[\{a\}, \{a, b\}] = \{(a, [X]_a), (b, [\{a\}]_b)\}$$

Now, you may be wondering what the dotted arrows mean. Well, it turns out that there is a partial ordering $\leq$ on each stalk Ω_x which comes from the partial ordering $\subseteq$ on $\mathbf{O}(X)$:

$[U]_x \leq [V]_x$ if and only if there is some open set W such that $x \in W$ and $U \cap W \subseteq V \cap W$.

i.e. if and only if the statement "$U \subseteq V$" is locally true at x. We will use this partial ordering to measure "closeness".

Now let's see how the subobject classifier works. The arrow $\top : 1 \to \Omega$ is the continuous section $\top : X \to B_\Omega$ that has $\top(x) = (x, [X]_x)$, for all $x \in X$. Now suppose we have a monic arrow $k : (A, q) \to (B, p)$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, hook, "k"] \arrow[dr, "q"'] & & B \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

We may think of A as an open subset of B. The character $\chi_k : (B, p) \to \Omega$ associated to the monic k is defined as follows: If $b \in B$, choose a neighborhood of U_b of b on which p restricts to a homeomorphism $p : U_b \to p(U_b)$. Then $\chi_k(b) = (x, [p(A \cap U_b)]_x)$, where $p(b) = x$. Intuitively, the germ of $p(A \cap U_b)$ at x represents the set of points in A close to b. For example, if $b \in A$, we would say that b is as close to A as possible, so this should mean

$$(x, [p(A \cap U_b)]_x) = (x, [X]_x) = \top(x)$$

which is indeed true since $p(A \cap U_b)$ is an open neighborhood of x. On the other hand, if b is separated from A, then we would say that b is as far away from A as possible, and this should mean

$$(x, [p(A \cap U_b)]_x) = (x, [\emptyset]_x)$$

which again is true. Now in set theory, these are our only two choices; either $b \in A$ or $b \notin A$. However now, using the partial ordering $\leq$ on the germs over x, we can obtain more subtle distinctions. Under this partial ordering over a given stalk Ω_x, $(x, [X]_x)$ is always the largest germ, $(x, [\emptyset]_x)$ is always the smallest germ, and everything else is in between. Thus, the larger the germ$[p(A \cap U_b)]_x$ is in terms of this ordering, the closer b will be to A.

Example 5.6. Continuing on the previous example, how close is b to the open set $\{a\}$? Well it's certainly not in $\{a\}$, at the same time, it's not separated from $\{a\}$ either; it's somewhere in between. Now the open set $\{a\}$ is a subobject of 1, hence there exists a unique map $\chi_{\{a\}} : 1 \to \Omega$, i.e. a continuous global section of B_Ω, and indeed b is mapped to the germ $(b, [\{a\}]_b)$.

5.2.3 exponential object

...

6 $\mathbf{Set}^{\mathbf{O}(\mathbf{X})^{\mathbf{op}}}$

6.1 Presheaves and Sheaves

Definition 6.1. $\mathbf{O}(X)$ is the poset category with open subsets of X as objects and inclusions as morphisms.

Remark. $\mathbf{O}(X)$ is an example of a complete heyting algebra, or a poset that is cartesian closed and with finite sums (finite co-complete).

Recall that a presheaf is just a functor F from a category $\mathbf{C}^{op}$ to the category **Set**. Historically however, presheaves were first defined on the category $\mathbf{O}(X)^{op}$.

Definition 6.2. Let X be any topological space. A **presheaf** F **on** X assigns to each open set U in X a set, $F(X)$, and to every pair of nested open sets $U \subset V \subset X$, a restriction map

$$F_U^V : F(V) \to F(U)$$

satisfying the basic properties that

$$F_U^U = identity$$

and for all $U \subset V \subset W \subset X$, we have

$$F_U^V \circ F_V^W = F_U^W$$

The elements $F(U)$ are called sections of F over U; elements of $F(X)$ are called global sections.

Exercise 3. Check that this agrees with the functor definition.

Presheaves on a topological space X are top-down constructions; we can restrict information from larger to smaller sets. However, many objects in mathematics are bottom-up constructions; they are defined by locally, which we then piece together to obtain something global. Presheaves do not provide the means to deduce global properties from the properties we find locally in the open sets of X. This is where the idea of sheaves come in.

Definition 6.3. A **sheaf** on X is a presheaf F on X which satisfies the following **sheaf axiom:**:

- If (U_i) is an open covering of an open set U, with $i \in I$ for some indexing set I, and if for each i a section $s_i \in F(U_i)$ is given such that for each pair U_j,$U_k \in$ (U_i) of the covering sets the restrictions of $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$, then there exists a unique section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.

Definition 6.4. $\mathbf{Sh}(X)$ is the full subcategory of $\mathbf{Set}^{\mathbf{O}(X)^{op}}$, whose objects are sheaves on X and morphisms being natural transformations.

Proposition 6.1. *The sheaf axioms imply that any sheaf has exactly one section of the empty set.*

Proof. The empty set $\emptyset$ can be written as the union of an empty family (that is, the indexing set I is $\emptyset$). The condition given for the sheaf property is vacuously true. So there must exist a unique section in $F(\emptyset)$. $\square$

Exercise 4. Show that the sheaf property can be stated in a more categorical way like this: the following diagram is an equalizer:

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

6.1.1 Examples of Sheaves

Example 6.1. Given a topological space X, a point $x \in X$, and a set S, the skyscraper sheaf $skyscr_x(S)$ is defined as follows:

$$skyscr_x(S)(U) = \begin{cases} S & \text{if } x \in U \\ * & \text{otherwise} \end{cases}$$

Remark. The skyscraper sheaf $skyscr_x(S)$ is the direct image of S under the geometric morphism $x : \mathbf{Set} \to \mathbf{Sh}(X)$ which defines the point of a topos given by $x \in X$.

Example 6.2. Let $X = \{a, b, c\}$ be given the discrete topology so that every subset of X is open. Sheaves on X are completely determined by their values on the singleton sets. Namely, if F is a sheaf which takes values $\{a\} \to A$, $\{b\} \to B$, $\{c\} \to C$ for some sets A, B, C, then F will take this diagram:

$$\begin{tikzcd}
& \{a,b,c\} & \\
\{a,b\} \arrow[ur] & \{a,c\} \arrow[u] & \{b,c\} \arrow[ul] \\
\{a\} \arrow[u] \arrow[ur] & \{b\} \arrow[ul] \arrow[ur] & \{c\} \arrow[ul] \arrow[u] \\
& \varnothing \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and turn it into this diagram:

$$\begin{tikzcd}
& A \times B \times C \arrow[dl] \arrow[d] \arrow[dr] & \\
A \times B \arrow[d] \arrow[dr] & A \times C \arrow[dl] \arrow[dr] & B \times C \arrow[dl] \arrow[d] \\
A \arrow[dr] & B \arrow[d] & C \arrow[dl] \\
& \{0\} &
\end{tikzcd}$$

Example 6.3. Let $X = \{a, b, c\}$ with the topology given as in example 1.10, where the open sets are highlighted in red:

$$\begin{tikzcd}
& \{a,b,c\} & \\
\{a,b\} \arrow[ur] & \{a,c\} \arrow[u] & \{b,c\} \arrow[ul] \\
\{a\} \arrow[u] \arrow[ur] & \{b\} \arrow[ul] \arrow[ur] & \{c\} \arrow[ul] \arrow[u] \\
& \varnothing \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

This time a sheaf F on X is completely determined by its values on$\{a\}$, $\{c\}$, and $\{a,b\}$. So let's say F takes values $\{a\} \to A$, $\{c\} \to C$, and $\{a,b\} \to B$. The sheaf F will take this diagram:

$$\begin{tikzcd}
& \{a,b,c\} & \\
\{a,b\} \arrow[ur] & \{a,c\} \arrow[u] & \\
\{a\} \arrow[u] \arrow[ur] & & \{c\} \arrow[ul] \\
& \varnothing \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and turn it into this diagram:

$$\begin{tikzcd}
& B \times C \arrow[dl] \arrow[d] & \\
B \arrow[d] & A \times C \arrow[dl] \arrow[dr] & \\
A \arrow[dr] & & C \arrow[dl] \\
& \{0\} &
\end{tikzcd}$$

Note: It may seem as if sheaves take colimits to limits. This isn't true for all diagrams! However, if our diagram is a collection of open subsets that is closed under finite intersections, then will take a colimit of this diagram, namely the union of this collection, to a limit of the corresponding diagram in **Set**.

6.1.2 Presheaves that are not Sheaves

Example 6.4. Let F be a presheaf on the real line $\mathbb{R}$, defined as follows:

$$F(U) = \begin{cases} \{0,1\} & \textit{if } U{=}\mathbb{R} \\ \{0\} & \textit{otherwise} \end{cases}$$

locally, this presheaf is just $\{0\}$, so where did this extra element 1 come from? It can't be a sheaf. It would, however, be a sheaf if we got rid of the element 1.

Example 6.5. Let F be the presheaf of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Why is this not a sheaf? Hint: Try gluing a bunch of continuous bounded functions to form a continuous unbounded function on $\mathbb{R}$. We can turn this thing into a sheaf if we adjoin continuous unbounded functions.

Note: Notice in the previous two examples we either needed to remove elements to turn a presheaf into a sheaf, or we needed to adjoin elements to turn a presheaf into a sheaf. This process of removing or adjoining elements of a presheaf to obtain a sheaf is known as **sheafification**.

6.1.3 Stalks

In general, if for each $x \in X$, we can find the smallest neighborhood containing x, say U_x, we can determine the sheaf completely by computing the values of the sheaf on these open sets. The problem is that the limit of the diagram which consists of all open neighborhoods of x may not exist, i.e. we may not have a smallest open neighborhood. However, colimits do exist in **Set**, and there's nothing stopping from looking at colimits in the diagram of F-images of neighborhoods of x. It turns out this is precisely what we need.

Definition 6.5. Let F be a presheaf on a topological space X. For each $x \in X$, the collection $\{F(V) \mid x \in V\}$ of F-images of neighborhoods of x, together with their associated restricting maps, forms a diagram in **Set**. The **stalk** of F at x, denoted F_x, is defined to be the colimit $\varinjlim_{x \in V} F(V)$ of this diagram. Explicitly, an equivalence relation $\sim_x$ is defined on $\cup\{F(V) \mid x \in V\}$: if $s_i \in F(V_i)$ and $s_j \in F(V_j)$ (where V_i and V_j are neighborhoods of x), we put

$$s_i \sim_x s_j \quad \text{if and only if} \quad F^i_k(s_i) = F^j_k(s_j) \quad \text{for some } x\text{-neighborhood } V_k \subseteq V_i \cap V_j$$

Remark. Thus, $s_i \sim_x s_j$ when they are "locally equal".

The elements in F_x are called **germs**. This terminology was also used when discussed various objects in slice categories. Is there a connection? Yes, there is. And it's not just any old connection, but in fact an adjunction! Stay tuned for that.

6.2 $Set^{O(X)^{op}}$ as a topos

Now we discuss the topos aspect to $\mathbf{Set}^{\mathbf{O}(X)^{op}}$

6.2.1 terminal object

Easy exercise.

6.2.2 pullback

Easy exercise.

6.2.3 subobject classifier

Definition 6.6. Let X be a topological space. Given an object $V \in \mathbf{O}(X)$, a V-**sieve**, S_V, is a collection $\mathcal{C}$ of open subsets of V such that if $V \in C$ and W is an open subset of U, then $W \in \mathcal{C}$.

The classifying presheaf in the presheaf category $\mathbf{Set}^{\mathbf{O}(X)^{op}}$ is defined as follows: $\Omega(V)$ is the set of all V-sieves. The classifier arrow is a natural transformation $\top : 1 \to \Omega$, which has components $\top_V : \{0\} \to \Omega(V)$ given by:

$$\top_V(0) = \mathbf{O}(V) \qquad \text{the largest } V\text{-sieve, i.e. the set of all open subsets of } V.$$

Given a monic arrow $\tau : F \to G$ of presheaves, with each τ_V being the inclusion $F(V) \hookrightarrow G(V)$, the character $\chi_\tau : G \to \Omega$ has components given by:

$$\chi_V(x) = \{U \subseteq V \mid G^V_U(x) \in F(U)$$

Example 6.6. Let X be finite topological space we considered earlier. Let's compute some sieves on it. How many sieves are there on $\{a, b, c\}$? There are eight:

$$S^X_X = \{X, \{a, b\}, \{a, c\}, \{a\}, \{c\}, \emptyset\} \qquad \text{generated by } X$$

$$S^{\{a,b\},\{a,c\}}_X = \{\{a, b\}, \{a, c\}, \{a\}, \{c\}, \emptyset\} \qquad \text{generated by } \{a, b\} \text{ and } \{a, c\}$$

$$S^{\{a,c\}}_X = \{\{a, c\}, \{a\}, \{c\}, \emptyset\} \qquad \text{generated by } \{a, c\}$$

$$S^{\{a,b\}}_X = \{\{a, b\}, \{a\}, \emptyset\} \qquad \text{generated by } \{a, b\}$$

$$S^{\{a\},\{c\}}_X = \{\{a\}, \{c\}, \emptyset\} \qquad \text{generated by } \{a\} \text{ and } \{c\}$$

$$S^{\{a\}}_X = \{\{a\}, \emptyset\} \qquad \text{generated by } \{a\}$$

$$S^{\{c\}}_X = \{\{c, \emptyset\} \qquad \text{generated by } \{c\}$$

$$S^{\emptyset}_X = \{\emptyset\} \qquad \text{generated by } \emptyset$$

The superscript of S denotes the generators of the sieve, and the subscript denotes the object the sieve is on. Is this a sheaf? The answer no: The open sets $\{a, c\}$ and $\{a, b\}$ cover X, (i.e. $X \subseteq \{a, c\} \cup \{a, b\}$). We also have $\{a, c\} \cap \{a, b\} = \{a\}$. Let's compute $\Omega(\{a\})$, $\Omega(\{a, c\})$, and $\Omega(\{a, b\})$:

$$\Omega(\{a\}) = \{S^{\{a\}}_{\{a\}}, S^{\emptyset}_{\{a\}}\}$$

$$\Omega(\{a,c\}) = \{S^{\{a,c\}}_{\{a,c\}}, S^{\{a\},\{c\}}_{\{a,c\}}, S^{\{c\}}_{\{a,c\}}, S^{\{a\}}_{\{a,c\}}, S^{\varnothing}_{\{a,c\}}\}$$
$$\Omega(\{a,b\}) = \{S^{\{a,b\}}_{\{a,b\}}, S^{\{a\}}_{\{a,b\}}, S^{\varnothing}_{\{a,b\}}\}$$

Since $S^{\{a,b\}}_{\{a,b\}}$ is defined on $\{a,b\}$ and restricts to $S^{\{a\}}_{\{a\}}$ on $\{a\}$, and $S^{\{a,c\}}_{\{a,c\}}, S^{\{a\},\{c\}}_{\{a,c\}}, S^{\{a\}}_{\{a,c\}}$ are all defined on $\{a,c\}$ and restrict to $S^{\{a\}}_{\{a\}}$ on $\{a\}$, then if Ω were a sheaf, we should be able to glue these together to form three unique elements in $\Omega(X)$. But there are several problems we run into. For one, $S^{\{a,b\}}_{\{a,b\}}$ and $S^{\{a,c\}}_{\{a,c\}}$ can glue together to form $S^{\{a,b\},\{a,c\}}_{X}$ or S^{X}_{X}, which is not unique. Also, we can't even glue $S^{\{a,b\}}_{\{a,b\}}$ and $S^{\{a\},\{c\}}_{\{a,c\}}$ together, it doesn't exist. These sieves generated by more than one open set are causing us problems. If we were to remove them, would we then get a sheaf? Indeed we would.

6.3 $Sh(X)$ as a topos

Now we discuss the topos structure of $\mathbf{Sh}(X)$

6.3.1 terminal object

Easy exercise.

6.3.2 pullback

Easy exercise.

6.3.3 subobject classifier

The classifying sheaf is defined as follows: $\Omega_j(U)$ is the set of all open subsets of U. This is also the same thing as the set of all U-sieves generated by a single element! The classifier arrow is a natural transformation $\top : 1 \to \Omega$, which has components $\top_U : \{0\} \to \Omega_j(U)$ given by:

$\top_U(0) = \mathbf{O}(V)$ the largest U-sieve, i.e. the set of all open subsets of V.

Now, given a monic arrow $\tau : F \to G$ of presheaves, with each τ_U being the inclusion $F(U) \hookrightarrow G(U)$, the character $\chi_\tau : G \to \Omega$ has components given by:

$$(\chi_\tau)_U(x) = \{V \subseteq U \mid G^U_V(x) \in F(U)$$

Example 6.7. Let X be the finite topological space we considered earlier. $\Omega_j(U)$ is the set of open subsets of U. In our case, we have:

$$\Omega_j(\varnothing) = \{\varnothing\}$$
$$\Omega_j(\{a\}) = \{\{a\},\varnothing\}$$

$$\Omega_j(\{c\}) = \{\{c\}, \varnothing\}$$

$$\Omega_j(\{a,b\}) = \{\{a,b\}, \{a\}, \varnothing\}$$

$$\Omega_j(\{a,c\}) = \{\{a,c\}, \{a\}, \{c\}, \varnothing\}$$

$$\Omega_j(X) = \{X, \{a,b\}, \{a,c\}, \{a\}, \{c\}, \varnothing\}$$

Now we ask, what is the relationship between Ω_j and Ω? To answer this, we first need to define two natural transformations $j : \Omega \to \Omega$, $e : \Omega_j \to \Omega$.

$$\begin{tikzcd} V & \Omega(V) \arrow[r, "j_V"] \arrow[d, "\Omega^V_U"'] & \Omega(V) \arrow[d, "\Omega^V_U"] \\ U \arrow[u, hook] & \Omega(U) \arrow[r, "j_U"'] & \Omega(U) \end{tikzcd}$$

let $e : \Omega_j \to \Omega$ be the natural transformation which has components $e_V(\mathbf{O}(V)) \to \Omega(V)$ given by:

$$e_V(U) = \mathbf{O}(U) = S^U_V$$

So e acts like an inclusion. Using our example above, we have $e_X(X) = S^X_X$, $e_X(\{a,b\}) = S^{\{a,b\}}_X$, and so on. Every sieve that is generated by a single open set is mapped to, and conversely every open set is mapped to a sieve generated by one open set. This gives us a one-one correspondence between sieves generated by a single open set (also known as **principal sieves**, and open sets. We are now ready to state the relationship between Ω_j and Ω.

Proposition 6.2. *The following diagram is an equalizer* $\mathbf{Set}^{\mathbf{O}(X)^{op}}$*:*

$$\begin{tikzcd} \Omega_j \arrow[r, "e"] & \Omega \arrow[r, shift left, "1_\Omega"] \arrow[r, shift right, "j"'] & \Omega \end{tikzcd}$$

6.4 Sheaves with Algebraic Structure

Definition 6.7. A **presheaf** P **of abelian groups** on a space X is defined to be an abelian group object in $\mathbf{Set}^{\mathbf{O}(X)^{op}}$.

Thus, a presheaf of abelian groups consists of natural transformations:
"group multiplication" $m : P \times P \to P$

$$\begin{tikzcd} V & P(V) \times P(V) \arrow[r, "m_V"] \arrow[d, "P^V_U \times P^V_U"'] & P(V) \arrow[d, "P^V_U"] \\ U \arrow[u, hook] & P(U) \times P(U) \arrow[r, "m_U"'] & P(U) \end{tikzcd}$$

An "inclusion of identity element" $e : 1 \to P$:

$$\begin{tikzcd} V & \{0\} \arrow[r, "e_V"] \arrow[d, "!"'] & P(V) \arrow[d, "P^V_U"] \\ U \arrow[u, hook] & \{0\} \arrow[r, "e_U"'] & P(U) \end{tikzcd}$$

An "inverse operation" $inv : P \to P$:

$$\begin{tikzcd} V & & P(V) \arrow[r, "inv_V"] \arrow[d, "P^V_U"'] & P(V) \arrow[d, "P^V_U"] \\ U \arrow[u, hook] & & P(U) \arrow[r, "inv_U"'] & P(U) \end{tikzcd}$$

7 Functor Categories

With these examples in mind, we now describe the topos structure of $\mathbf{Set}^{\mathbf{C}}$, where $\mathbf{C}$ is any small category.

7.0.1 terminal object

A terminal object in this category is the constant functor $1 : \mathbf{C} \to \mathbf{Set}$, which takes every object $c \in \mathbf{C}$ to the terminal object in $\mathbf{Set}$, $\{0\}$, and it takes every morphism $f : c \to c'$ in $\mathbf{C}$ to the identity morphism $id_{\{0\}}\{0\} \to \{0\}$.

7.0.2 pullback

All limits and colimits in $\mathbf{Set}^{\mathbf{C}}$ are defined componentwise (hence, pullbacks are defined componentwise in particular). So given $\tau : F \to H$ and $\sigma : G \to H$, for each object $c \in \mathbf{C}$, form the pullback of:

$$\begin{tikzcd} Z(c) \arrow[r, "\mu_c"] \arrow[d, "\lambda_c"'] & G(c) \arrow[d, "\sigma_c"] \\ F(c) \arrow[r, "\tau_c"'] & H(c) \end{tikzcd}$$

in $\mathbf{Set}$ of the components τ_c and σ_c. Now, the assignment c to $Z(c)$ for each object c establishes a functor $\mathbf{C} \to \mathbf{Set}$. Given an arrow $f : c \to c'$ in $\mathbf{C}$, we get a unique function $Z(f) : Z(c) \to Z(c')$ which comes from the cube:

$$\begin{tikzcd} & Z(c) \arrow[rr, "\mu_c"] \arrow[dl, "Z(f)"'] \arrow[dd, "\lambda_c" near start] & & G(c) \arrow[dl, "G(f)"'] \arrow[dd, "\sigma_c"] \\ Z(c') \arrow[rr, "\mu_{c'}" near start] \arrow[dd, "\lambda_{c'}"'] & & G(c') \arrow[dd, "\sigma_{c'}" near start] & \\ & F(c) \arrow[rr, "\tau_c" near start] \arrow[dl, "F(f)"'] & & H(c) \arrow[dl, "H(f)"] \\ F(c') \arrow[rr, "\tau_{c'}"'] & & H(c') & \end{tikzcd}$$

The λ_c's and μ_c's are components for the natural transformations $\lambda : Z \to F$ and $\mu : Z \to G$ that make the following diagram a pullback in $\mathbf{Set}^{\mathbf{C}}$:

$$
\begin{tikzcd}
Z \arrow[r, "\mu"] \arrow[d, "\lambda"'] & G \arrow[d, "\sigma"] \\
F \arrow[r, "\tau"'] & H
\end{tikzcd}
$$

Compare this with the arrow category of **Set**.

7.0.3 subobject classifier

Definition 7.1. A c-**cosieve** in $\mathbf{C}$ is a collection S_c of arrows in $\mathbf{C}$ with common domain c, which is closed under left composition, i.e. if $f : c \to c'$ is in S_c and $g : c' \to c''$ is any other arrow, then $g \circ f : c \to c''$ is in S_c. Dually, a c-**sieve** in $\mathbf{C}$ is a collection S_c of arrows in $\mathbf{C}$ with common codomain c, which is closed under right composition, i.e. if $f : c' \to c$ is in S_c and $g : c'' \to c'$ is any other arrow, then $f \circ g : c \to c''$ is in S_c.

Remark. More precisely, a c-cosieve is a subfunctor of $Hom(c, -)$, and a c-sieve is a subfunctor of $Hom(-, c)$. So a c-cosieve in $\mathbf{C}$ is a c-sieve in the opposite category $\mathbf{C}^{op}$.

Example 7.1. Let M be a monoid, i.e. a category with one object $\bullet$. A $\bullet$-cosieve is a left ideal in the set of arrows in M.

Example 7.2. n-sieves in the simplex category Δ correspond to abstract simplicial complexes S. The n-sieve corresponding to a given simplicial complex S consists of all order preserving maps $f : m \to n$ such that $\{f(0), \dots, f(m)\} \in S$.

Example 7.3. Let $(P, \leq)$ be a poset. A P-sieve is an upward closed subset of $(P, \leq)$.

Definition 7.2. Given an arrow $f : c' \to c$ and a c-sieve S_c, we can construct a c'-sieve denoted $f^*(S_c)$

$$
f^*(S_c) = \{g : c' \to c'' \mid g \circ f \in S_c\}
$$

Define $\Omega(c)$ to be the set of all c-cosieves, and for an arrow $f : c \to c'$ in $\mathbf{C}$, let $\Omega(f) : \Omega(c) \to \Omega(c')$ be the function that takes the c-cosieve S_c to the c'-cosieve $f^*(S_c)$. Define $\top : 1 \to \Omega$ to be the natural transformation that has components $\top_c : \{0\} \to \Omega(c)$ given by $\top_c(0) = Hom(c, -)$ (the largest c-cosieve). Notice that if $f \in S_c$, then $\Omega(f)$ sends S_c to $\mathrm{Hom}(c, -)$. Suppose $\tau : F \to G$ is a monic arrow in $\mathbf{Set}^{\mathbf{C}}$.

$$
\begin{tikzcd}
c \arrow[d, "f"'] & F(c) \arrow[r, "\tau_c"] \arrow[d, "F(f)"'] & G(c) \arrow[r, "(\chi_\tau)_c"] \arrow[d, "G(f)"'] & \Omega(c) \arrow[d, "\Omega(f)"'] \\
c' & F(c') \arrow[r, hook, "\tau_{c'}"'] & G(c') \arrow[r, "(\chi_\tau)_{c'}"'] & \Omega(c')
\end{tikzcd}
$$

The character χ_τ of τ is to be a natural transformation with the component $(\chi_\tau)_c$ a set function from $G(c)$ to $\Omega(c)$. It is defined as

$$
(\chi_\tau)_c(x) = \{f : c \to c' \mid G(f)(x) \in \tau_{c'}(F(c'))\}.
$$

7.1 Representable Functors

Recall, the functor $Hom(c,-) : \mathbf{C} \to \mathbf{Set}$, which takes an object $d \in \mathbf{C}$ to the set of all morphisms from c to d, and takes a morphism $f : d \to d'$ to a function $Hom(c,f) : Hom(c,d) \to Hom(c,d')$, sending $g : c \to d$ to $f \circ g : c \to d \to d'$:

$$\begin{tikzcd} d \arrow[d, "f"'] & Hom(c,d) \arrow[d, "Hom(c,f)"'] & g \arrow[d, mapsto] \\ d' & Hom(c,d') & f \circ g \end{tikzcd}$$

Example 7.4. Let $\mathbf{O}(X)$ be the poset category with open sets as objects and inclusions are morphisms. $Hom(U,-)$ is easy to describe:

$$Hom(U,V) = \begin{cases} 0 & \text{if } U \nsubseteq V \\ 1 & \text{if } U \subseteq V \end{cases}$$

and given an inclusion $i : W \hookrightarrow V$

$$Hom(U,i) : \{U \hookrightarrow W\} \to \{U \hookrightarrow W \hookrightarrow V\} = \{U \hookrightarrow V\}$$

This is actually a sheaf, can you see why?

Lemma 7.1. *Let F be a functor from a category $\mathbf{C}$ to $\mathbf{Set}$. The set of natural transformations from $Hom(c,-)$ to F are in one-to-one correspondence with the elements of $F(c)$, i.e.*

$$Nat(Hom(c,-),F) \cong F(c)$$

Proof. The idea is that any natural transformation $\tau : Hom(c,-) \to F$ is completely determined by $\tau_c(id_c)$.

$$\begin{tikzcd} c \arrow[d, "f"'] & Hom(c,c) \arrow[r, "\tau_c"] \arrow[d, "Hom(c,f)"'] & F(c) \arrow[d, "F(f)"] \\ d & Hom(c,d) \arrow[r, "\tau_d"'] & F(d) \end{tikzcd}$$

For each $d \in \mathbf{C}$, we must show how τ_d is completely determined by$\tau_c(id_c)$. The commutative diagram above tells us that $\tau_d(f) = F(f)(\tau_c(id_c))$. So as soon as we assign a value to $\tau_c(id_c)$, then $\tau_d(f)$ is determined for all objects d and morphisms $f : c \to d$.

$\square$

Part II
Sheaves as Etale Spaces

8 An equivalence of categories

We start with the main theorem:

Theorem 8.1. *For any space X there is a pair of adjoint functors*

$$\mathbf{Top}/X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set}^{O(X)^{op}}$$

where Γ assigns to each object in $(B, p) \in \mathbf{Top}/X$, the sheaf of all sections F_p of p, while its left adjoint Λ assigns to each preasheaf F, the etale space (B_F, p_F). There are natural transformations

$$\iota_F : F \to \Gamma\Lambda F \qquad \epsilon_B : \Lambda\Gamma B \to B$$

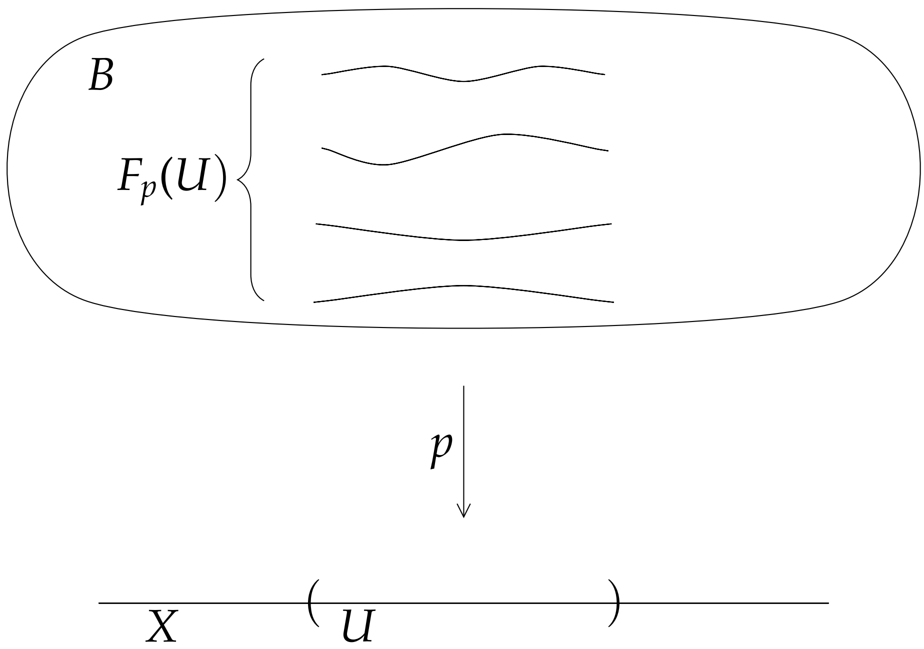
for a presheaf F and an object $(B, p) \in \mathbf{Top}/X$, which are unit and counit making Λ a left adjoint for Γ. If F is a sheaf, η_F is an isomorphism, while if B is etale, ϵ_B is an isomorphism.

We will

8.0.1 From $\mathbf{Top}/X$ to $\mathbf{Sh}(X)$

We are now ready to discuss the relationship between sheaves and etale spaces. Given an object $(B, p) \in \mathbf{Top}/X$, we can associate a sheaf F_p as follows:

$$F_p(U) = \{s \mid s \text{ is a section of } p \text{ defined on } U\}$$



For an inclusion $U \hookrightarrow V$, $(F_p)^U_V$ is the obvious restriction map that assigns to each section $s : V \to B$, its restriction $s|_U : U \to B$ to U. F_p is called the **sheaf of sections of** p. We need to check that this is indeed a sheaf, and not just a presheaf, which certainly isn't obvious.

Proposition 8.1. *F_p is a sheaf.*

Proof. Suppose (U_i) is an open covering of an open set U, with $i \in I$ for some indexing set I, and that for each i a local section $s_i \in F_p(U_i)$ is given such that for each pair U_j,$U_k \in$ (U_i) of the covering sets the restrictions of $s_j|_{U_j \cap U_k} = s_k|_{U_j \cap U_k}$. We need to show that there exists a unique section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$. We can construct a unique s like this: if $x \in U$, choose some U_i that has $x \in U_i$, and set $s(x) = s_i(x)$. This isn't as obvious as it looks. We need to check a few things:

- **Why is it well defined?**

Whenever you make a construction where you have to choose something, you always need to check and see if your construction is well-defined. In the construction we just made, we chose an open neighborhood of x. Now it's easy to see that this choice is irrelevant since we could have chosen another open neighborhood U_j of x, and set $s(x) = s_j(x)$, and because $s_i = s_j$ on $U_i \cap U_j$, we'd obtain the same value. This is a basic example of what is called a **gluing construction**. In more general contexts, gluing constructions will look very similar to what we just did. Namely, there will be a pair (s_i, U_i), where (U_i) covers U, however the equality between s_i and s_j may be replaced with an equivalence, a group action, or maybe something even more general.

- **Why is s continuous?**

Because given an open subset V of B, $s^{-1}(V) = \bigcup_i s_i^{-1}(V)$, hence $s^{-}(V)$ is open.

- **Why is s unique?**

The uniqueness of s comes from how we constructed it in the first place, for if $t : U \to B$ has $t|U_i = s_i$ for all $i \in I$, then $t = s$.

$\square$

Let $k : (A, q) \to (B, p)$ be an arrow in $\mathbf{Top}/X$. For each open set U, define $k_U : F_q(U) \to F_p(U)$ to be the function that maps a section $s \in F_q(U)$ to $k \circ s$. The maps k_U are the components of a natural transformation from $F_q \to F_p$. Thus, we have constructed a functor $\Gamma : \mathbf{Top}/X \to \mathbf{Sh}(X)$.

8.0.2 From $\mathbf{Set}^{\mathbf{O(X)^{op}}}$ to $\mathbf{Etale}(X)$

The functor from $\Lambda : \mathbf{Set}^{\mathbf{O}(X)^{op}} \to \mathbf{Etale}(X)$ is constructed using the idea of the stalk F_x of a presheaf F at a given point x. We remind the reader that elements in the stalk space F_x are equivalence classes $[s]_x = \{t \mid s \sim_x t\}$, where the equivalence relation $\sim_x$ is defined on $\cup\{F(V) \mid x \in V\}$: if $s_i \in F(V_i)$ and $s_j \in F(V_j)$ (where V_i and V_j are neighborhoods of x), we put

$$s_i \sim_x s_j \quad \text{if and only if} \quad F^i_k(s_i) = F^j_k(s_j) \quad \text{for some } x\text{-neighborhood } V_k \subseteq V_i \cap V_j.$$

For convenience, we write elements in F_x like this: $(x, [s]_x)$. Now, suppose F is a presheaf on X. We can construct a local homeomorphism $p_F : B_F \to X$ as follows: First set $B_F = \bigcup_x F_x$, and let p_F be the obvious projection map. For each open subset V of X and $s \in F(V)$, let $[s, V] = \{(x, [s]_x) \mid x \in V\}$. The collection of all $[s, V]$ generates a topology on B_F, making p_F a local homeomorphism.

Example 8.1. Let X be the finite topological space we considered earlier. We constructed the classifying sheaf Ω_j in this example. Let's compute the stalk of this sheaf at point b. We know that they need to consist of equivalence classes like this: $[s]_b$, where s is a section defined on some open set containing b. Since every element in $\Omega_j(\{a, b\})$ is a restriction of an element in $\Omega_j(X)$, we can choose representatives of this equivalence class in $\Omega_j(X)$, so given that

$$\Omega_j(X) = \{X, \{a, b\}, \{a, c\}, \{a\}, \{c\}, \varnothing\}$$

It's easy to check that we have three elements:

$$X \sim_b \{a, b\}$$

$$\{a\} \sim_b \{a, c\}$$

$$\varnothing \sim_b \{c\}$$

We already computed this earlier though. We also said from earlier that the collection of sets $[U, V]$ where U, V are open and $U \subseteq V$ generate a topology on B_{Ω_j}, which is the same thing as saying $[s, V]$ generates a topology on B_{Ω_j}, where s is a section on V. Notice that the global sections of our classifying sheaf Ω_j are all open subsets of X.

Example 8.2. Now let's sheafify the classifying presheaf Ω in $\mathbf{Set}^{\mathbf{O}(X)^{op}}$ on X. Let's compute the stalk of this presheaf at point b. We know that they need to consist of equivalence classes like this: $[s]_b$, where s is a section defined on some open set containing b. Since every element in $\Omega(\{a, b\})$ is a restriction of an element in $\Omega(X)$, we can choose representatives of this equivalence class in $\Omega(X)$, so given that

$$\Omega(X) = \{S_X^{\{a,b,c\}}, S_X^{\{a,b\}\{a,c\}}, S_X^{\{a,c\}}, S_X^{\{a,b\}}, S_X^{\{a\}\{c\}}, S_X^{\{a\}}, S_X^{\{c\}}, S_X^{\varnothing}\}$$

It's easy to check that we have three elements:

$$S_X^{\{a,b,c\}} \sim_b S_X^{\{a,b\}} \sim_b S_X^{\{a,b\}\{a,c\}}$$

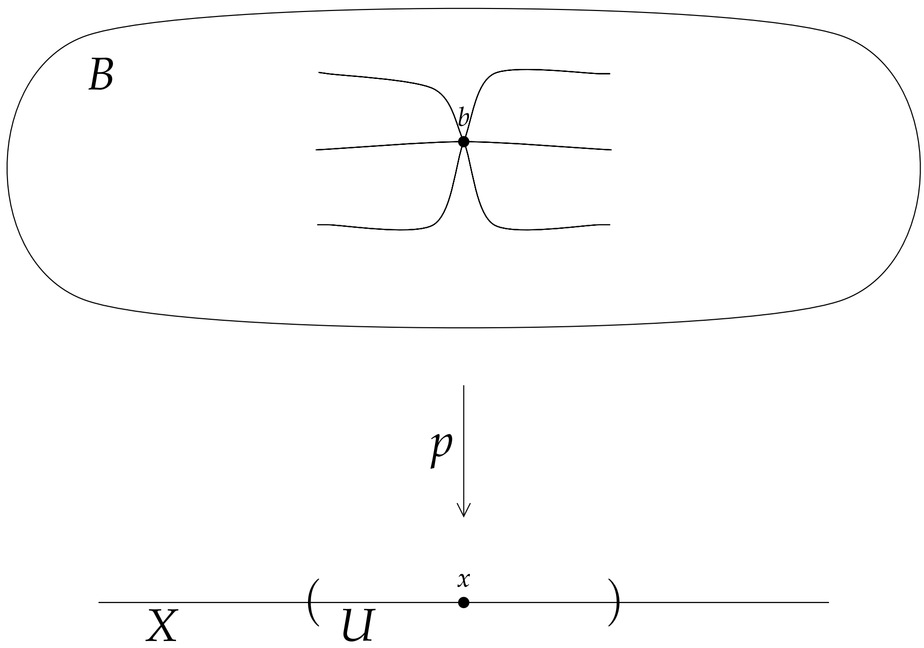
$$S_X^{\{a\}} \sim_b S_X^{\{a,c\}} \sim_b S_X^{\{a\}\{c\}}$$

$$S_X \sim_b S_X^{\{c\}}$$

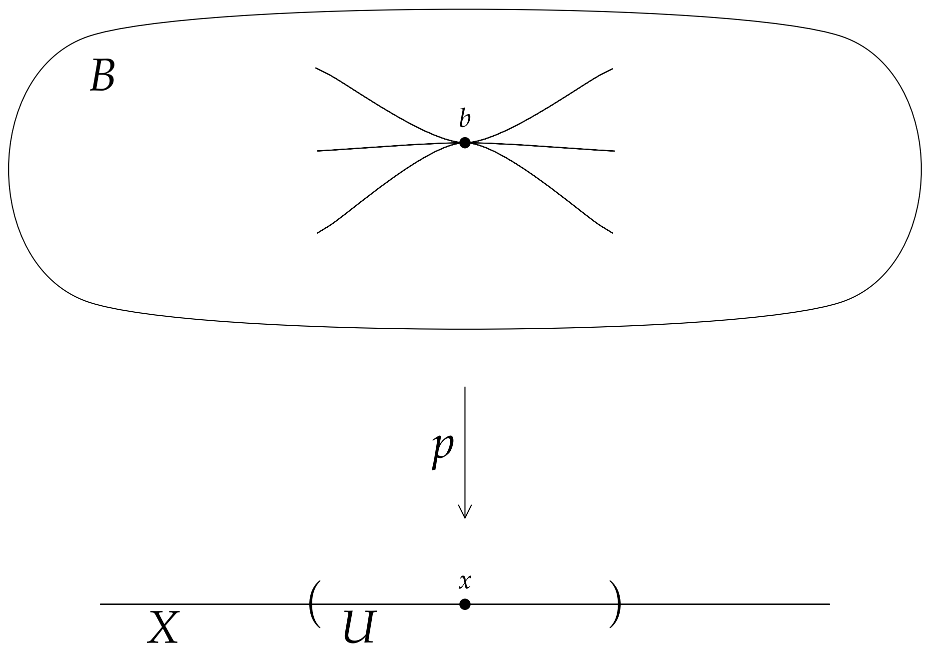
For example, the reason we have $S_X^{\{a,b\}} \sim_b S_X^{\{a,b\},\{a,c\}}$ is because they both restrict to $S_{\{a,b\}}^{\{a,b\}}$ on $\{a, b\}$. So the elements with two generators restrict to an element with one generator on a small enough open set. Indeed, it can easily be shown that $S_X^{\{U\},\{V\}} \sim_x S_X^{\{U \cup V\}}$ for two open sets U, V. So the sheafification of the classifying presheaf is "the same thing" as the sheafification of the classifying sheaf.

8.0.3 Etalification

Let (B, p) be an object in **Top**$/X$. There will be several sections through any point $b \in B$:



As the picture above suggests, they may give different germs at x. The counit, $\epsilon_B : (B_{F_p}, p_{F_p}) \to (B, p)$, maps all of the germs of sections through b to b. Now, suppose that $p : B \to X$ is a local homeomorphism. Then ϵ_B is actually an isomorphism; there exists a local section s of p through b, defined on an open set V, i.e. $b \in s(V)$ (we proved this earlier) and every section through b gives rise to the same germ (we proved this earlier too). If $p : B \to X$ is a local homeomorphism, all sections through a given point $b \in B$ look the same (locally):



8.0.4 Sheafification

Now we want to describe the unit $\iota_F : F \to F_{p_F}$. Define $(\iota_F)_U : F(U) \to F_{p_F}(U)$ by setting, for each $s \in F(U)$, $(\iota_F)_U(s) = s_U$, where $s_U : U \to B_F$ is defined by $s_U(x) = (x, [s]_x)$ for all $x \in U$. The sets of the form $[s, V]$ generate the topology of B_F which makes p_F a local homeomorphism.

Example 8.3. Let's return to the previous example. The map $(\iota_\Omega)_X$ sends $S_X^{\{a,b,c\}}$ and $S_X^{\{a,b\},\{a,c\}}$ to the same section since $\{\{a,b\},\{a,c\}\}$ covers our space $\{a,b,c\}$.

8.0.5 Equivalence of Categories

Definition 8.1. A full subcategory $i : \mathbf{C} \to \mathbf{D}$ is **reflective** if the inclusion functor i has a left adjoint L:

$$(L \dashv i) : \mathbf{C} \underset{i}{\overset{L}{\rightleftarrows}} \mathbf{D}$$

dually, a full subcategory is **coreflective** if the inclusion functor has a right adjoint R:

$$(i \dashv R) : \mathbf{C} \underset{R}{\overset{i}{\rightleftarrows}} \mathbf{D}$$

Corollary. *The functors* Γ *and* Λ *restrict to an equivalence of categories*

$$\mathbf{Sh}(X) \rightleftarrows \mathbf{Etale}(X)$$

Moreover, $\mathbf{Sh}(X)$ *is a reflective subcategory of* $\mathbf{Set}^{\mathbf{O}(X)^{op}}$, *and* $\mathbf{Etale}(X)$ *is a coreflective subcategory of* $\mathbf{Top}/X$.

Thus, sheafification preserves arbitrary colimits and limits of sheaves may be computed on the underlying presheaves. Dually, etalification preserves arbitrary limits and colimits of etale spaces may be computed on the underlying object in $\mathbf{Top}/X$.

8.1 Geometric Morphisms

Let $\mathbf{C}$ be a category with pullbacks. Then an arrow $f : c \to c'$ in $\mathbf{C}$ induces a "pulling-back" functor

$$f^* : \mathbf{C}/c' \to \mathbf{C}/c.$$

which sends an arrow $k : (d, g) \to (e, h)$ to the unique arrow which makes this diagram commute:

$$\begin{tikzcd}
f^*(d) \arrow[rr] \arrow[dd, "f^*(g)"'] \arrow[dr, dashed] & & d \arrow[dd, "g"'] \arrow[dr, "k"] & \\
& f^*(e) \arrow[rr] \arrow[dl, "f^*(h)"] & & e \arrow[dl, "h"] \\
c \arrow[rr, "f"'] & & c' &
\end{tikzcd}$$

The "pulling-back" has a left adjoint called the "composing with f" functor

$$\Sigma_f : \mathbf{C}/c \to \mathbf{C}/c'$$

which takes an object $g : d \to c$ to $f \circ g : d \to c'$, and an arrow

$$\begin{tikzcd}
d \arrow[rr, "k"] \arrow[dr, "g"'] & & e \arrow[dl, "h"] \\
& c &
\end{tikzcd}$$

to the arrow

$$\begin{tikzcd}
d \arrow[rr, "k"] \arrow[dr, "f \circ g"'] & & e \arrow[dl, "f \circ h"] \\
& c' &
\end{tikzcd}$$

Now an arrow k from $\Sigma_f(g)$ to $p : e \to c'$ in $\mathbf{C}/c'$ looks like this

$$\begin{tikzcd}
d \arrow[rr, "k"] \arrow[dr, "f \circ g"'] & & e \arrow[dl, "p"] \\
& c' &
\end{tikzcd}$$

The unique arrow k' in $\mathbf{C}/c$ that it corresponds comes from the universal property of the pullback

$$\begin{tikzcd}
d \arrow[rrd, bend left, "k"] \arrow[rd, dotted, "k'"] \arrow[rdd, bend right, "g"'] & & \\
& f^*(e) \arrow[r] \arrow[d, "f^*(p)"] & e \arrow[d, "p"] \\
& c \arrow[r, "f"'] & c'
\end{tikzcd}$$

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous precisely when each member of $\mathbf{O}(Y)$ pulls back under f to a member of $\mathbf{O}(X)$, i.e.

$$V \in \mathbf{O}(Y) \quad \text{only if} \quad f^{-1}(V) \in \mathbf{O}(X).$$

A continuous map of spaces, $f : X \to Y$, will induce functors in both directions, forward and backward, on the associated categories of sheaves.

Definition 8.2. Given a presheaf F on X and a continuous map $f : X \to Y$, the pushforward of F under f is a sheaf f_*F on Y is defined on open sets U of Y by:

$$(f_*F)(U) = F(f^{-1}U)$$

Moreover, f_* is a functor from $\mathbf{Sh}(X)$ to $\mathbf{Sh}(Y)$. Given the natural transformation τ:

$$\begin{tikzcd}
V & F(V) \arrow[r, "\tau_V"] \arrow[d] & G(V) \arrow[d] \\
U \arrow[u, hook] & F(U) \arrow[r, "\tau_U"'] & G(U)
\end{tikzcd}$$

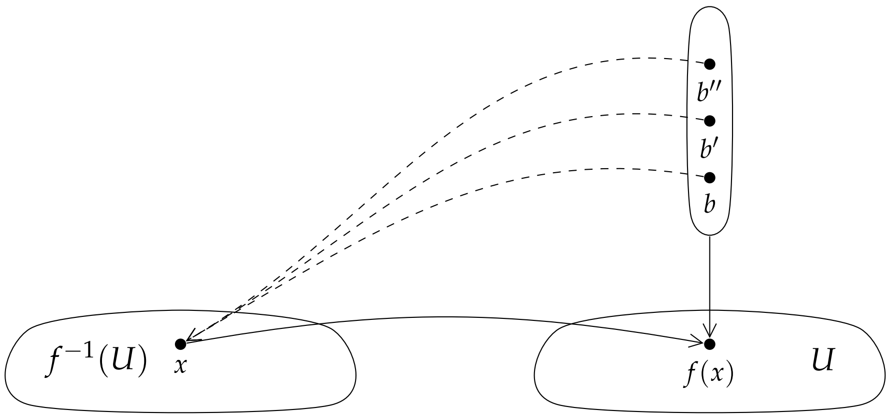
f_* turns this into:

$$\begin{tikzcd}
f^{-1}(V) & F(f^{-1}(V)) \arrow[r, "\tau_{f^{-1}(V)}"] \arrow[d] & G(f^{-1}(V)) \arrow[d] \\
f^{-1}(U) \arrow[u, hook] & F(f^{-1}(U)) \arrow[r, "\tau_{f^{-1}(V)}"'] & G(f^{-1}(U))
\end{tikzcd}$$

Now that we have defined the direct image of a presheaf F under f, let's try to define an pullback presheaf f^*. We could try something like this:

$$(f^*F)(U) = F(fU)$$

we run into a problem though: $f(U)$ is not necessarily open. On the other hand, given an object in $(B, p) \in \mathbf{Top}/Y$, and a continuous map $f : X \to Y$, the pullback functor $f^* : \mathbf{Top}/Y \to \mathbf{Top}/X$ is easy to define. For example, the stalk over x is given by $B_{f(x)}$.



Definition 8.3. A **geometric morphism** $f : \mathbf{C} \to \mathbf{D}$ of elementary topoi $\mathbf{C}$ and $\mathbf{D}$, is a pair (f^*, f_*) of functors of the form

$$\mathbf{C} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{D}$$

such that f^* is left exact and left adjoint to f_*. f^* is called the inverse image part, and f_* the direct image part, of the geometric morphism.

So f^* preserves finite limits and arbitrary colimits.

Part III
Grothendieck Topologies

9 Motivation

The fundamental group $\pi_1(X)$ of a space X can be thought of as a Galois (automorphism) group $\mathrm{Gal}(\hat{X}/X)$, where $\hat{X}$ is the universal covering space of X.

Definition 9.1. Assume X is connected, a continuous map $p : Y \to X$ is call a **covering** if for any $x \in X$, there is an open neighborhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically onto U by p. More formally, the pullback $Y \times_X U$ is a coproduct:

$$Y \times_X U = U \amalg U \amalg U \cdots$$

The set of homeomorphisms $\alpha : Y \to Y$ over X such that $p \circ \alpha = p$ forms a group, $\mathrm{Aut}(Y/X)$, called the group of covering transformations of $p : Y \to X$.

Example 9.1. Coverings of S^1.

- Define $p_n : S^1 \to S^1$ to be the function $p_n(z) = z^n$ where n is a positive integer and we view z as a complex number with norm 1.
- Define $p_\infty : \mathbb{R}^1 \to S^1$ to be the function $p_\infty(x) = (\cos 2\pi x, \sin 2\pi x)$.

We now restrict our attention to connected covering spaces, since a general covering space is a disjoint union of connected ones.

Definition 9.2. A covering $p : Y \to X$ is called **Galois** if for each $x \in X$ and each pair of lifts $\tilde{x}$, $\tilde{x}'$ of x , there is a covering transformation taking $\tilde{x}$ to $\tilde{x}'$. For a Galois covering $p : Y \to X$, we call $\mathrm{Aut}(Y/X)$ the Galois group of Y over X and denote it by $\mathrm{Gal}(Y/X)$

Intuitively, a Galois covering is one with maximal symmetry, in analogy with Galois extensions, which as splitting fields of polynomials, can be considered "maximally symmetric." Recall that the main theorem of Galois theory gives a bijective correspondence between intermediate field extensions and subgroups of the Galois group. There is a similar version of the main theorem for coverings, which relates connected coverings of a given space X and subgroups of $\pi_1(X)$.

Theorem 9.1. *The induced map $p_* : \pi_1(Y, y) \to \pi_1(X, x)$ is injective, and there is a bijection*

$$\{\text{connected coverings } p : Y \to X\}/\text{isom.} \quad \cong \quad \{\text{subgroups of } \pi_1(X, x)\}/\text{conj.}$$

with the property that $p : Y \to X$ is a Galois covering if and only if $p_*(\pi_1(Y, y))$ is a normal subgroup of $\pi_1(X, x)$. In this case,

$$Gal(Y/X) \cong \pi_1(X, x)/p_*(\pi_1(Y, y)).$$

So the analogy between Galois coverings and Galois extensions clearly involves some sort of dualization. Spaces over X are epimorphisms to X, while extensions of a field K are monomorphisms from K. In the category of spaces $Y \to X$, the product is the pullback $Y \times_X Y' \to X$, the coproduct is the disjoint union $Y \amalg Y' \to X$. The category of fields however, does not have duals of these operations, but if we embed fields in the larger category of commutative rings, we do get products and coproducts. The product of any indexed family of rings R_i is the cartesian product $\prod R_i$ with termwise ring operations. The coproduct of two rings R, S is the tensor product $R \otimes S$.

Definition 9.3. The covering $p : \hat{X} \to X$ which corresponds to the identity subgroup of $\pi_1(X, x)$ is called the **universal covering** of X.

Galois Coverings	**Galois Extensions**
Spaces over X	extensions of a field K
Product given by $Y \times_X Y' \to X$	Product given by $\prod R_i$
Coproduct given by $Y \amalg Y' \to X$	Coproduct given by $R \otimes_K R'$
$\pi_1(Y, y)$	$Gal(N/K)$
factorizations $Y \to Y' \to X$	extensions $K \to L \to N$
subgroups of $\pi_1(Y, y)$	subgroups of $Gal(N/K)$
$Y \times_X U = U \amalg U \amalg U \cdots$	$L \otimes_K N \cong N[x]/(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$
monomorphism $U \to X$ "splits" covering space	splitting field N splits extension $K \to L$.

$L \otimes_K N \cong N[x]/(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$ example: $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt[3]{2})$, $N = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ where ζ_3 is a nontrivial cube root of unity (for example $\zeta_3 = \exp^{\frac{2\pi i}{3}}$).

9.1 Coverage

Definition 9.4. A **coverage** on a category $\mathbf{C}$ consists of a function assigning to each object $c \in \mathbf{C}$ a collection of families of morphisms $\{f_i : c_i \to c\}$, called **covering families**, such that:

- If $\{f_i : c_i \to c\}$ is a covering family and $h : d \to c$ is a morphism, then there exists a covering family $\{g_j : d_j \to d\}$ such that each composite $h \circ g_j$ factors through some f_i.

$$\begin{tikzcd} d_j \arrow[r, "k"] \arrow[d, "g_j"'] & c_i \arrow[d, "f_i"] \\ d \arrow[r, "h"'] & c \end{tikzcd}$$

A **site** is a category equipped with a coverage.

If $\mathbf{C}$ has pullbacks, then it is natural to impose the following stronger condition:

- If $\{f_i : c_i \to c\}$ is a covering family and $g : d \to c$ is a morphism, then the family of pullbacks $\{g_i : d \times_c c_i \to d\}$ is a covering family of d.

Sheaves on a pretopology have a simple description. For each covering family $\{f_i : c_i \to c\} \in Cov_{\mathbf{C}}(c)$, the diagram

$$F(c) \longrightarrow \prod_i F(c_i) \rightrightarrows \prod_{i,j} F(c_i \textstyle\prod_c c_j)$$

must be an equalizer. Here's another way we can describe this.

$$\begin{tikzcd} c_i \times_c c_j \arrow[r] \arrow[d] & c_j \arrow[d, "f_j"] \\ c_i \arrow[r, "f_i"'] & c \end{tikzcd} \qquad \begin{tikzcd} F(c_i \times_c c_j) & F(c_j) \arrow[l] \\ F(c_i) \arrow[u] & F(c) \arrow[l, "F(f_i)"] \arrow[u, "F(f_x)"'] \end{tikzcd}$$

Given any cover $\{f_i : c_i \to c\} \in Cov_{\mathbf{C}}(c)$ of c, and any selection of elements $x_i \in F(c_i)$ that are pairwise compatible, i.e. $F^i_j(c_i) = F^j_i(c_j)$, then there is exactly one $x \in F(c)$ such that $F_i(c) = s_i$.

Example 9.2. $(\mathbf{O}(X), Cov_{\mathbf{O}(X)})$ is a site, where for open $U \in \mathbf{O}(X)$, $Cov_{\mathbf{O}(X)}(U) = \{U_i \mid U_i \text{ is open and } \bigcup_i U_i = U\}$.

Example 9.3. Consider the poset category $\mathbf{3}$, which looks like this:

$$\begin{tikzcd} 0 \arrow[loop above, "id_0"] \arrow[r] & 1 \arrow[loop above, "id_1"] \arrow[r] & 2 \arrow[loop above, "id_2"] \end{tikzcd}$$

Consider these collections of sets:

$Cov_3(2)$	$\{\{2 \to 2\},\{1 \to 2, 2 \to 2\}\}$
$Cov_3(1)$	$\{\{1 \to 1\},\{1 \to 1, 0 \to 1\},\{0 \to 1\}\}$
$Cov_3(0)$	$\{\{0 \to 0\}\}$

Is this a pretopology? No, because these collection of sets fail to satisfy the second condition of our definition. We can turn this into a pretopology by adding in some sets into $Cov_3(2)$ like this:

$Cov_3(2)$	$\{\{2 \to 2\},\{1 \to 2, 2 \to 2\},\{0 \to 2, 2 \to 2\},\{0 \to 2, 1 \to 2, 2 \to 2\}\}$
$Cov_3(1)$	$\{\{1 \to 1\},\{1 \to 1, 0 \to 1\},\{0 \to 1\}\}$
$Cov_3(0)$	$\{\{0 \to 0\}\}$

This is related to multicategories. For example, axiom 2 says add in compositions in a multicategory.

10 Sheaves on pretopologies

Sheaves on a pretopology have a simple description. For each covering family $\{f_i : c_i \to c\} \in Cov_{\mathbf{C}}(c)$, the diagram

$$F(c) \longrightarrow \prod_i F(c_i) \rightrightarrows \prod_{i,j} F(c_i \times_c c_j)$$

must be an equalizer. Here's another way we can describe this.

$$\begin{tikzcd} c_i \times_c c_j \arrow[r] \arrow[d] & c_j \arrow[d, "f_j"] \\ c_i \arrow[r, "f_i"'] & c \end{tikzcd} \qquad \begin{tikzcd} F(c_i \times_c c_j) & F(c_j) \arrow[l] \\ F(c_i) \arrow[u] & F(c) \arrow[l, "F(f_i)"] \arrow[u, "F(f_x)"'] \end{tikzcd}$$

Given any cover $\{f_i : c_i \to c\} \in Cov_{\mathbf{C}}(c)$ of c, and any selection of elements $x_i \in F(c_i)$ that are pairwise compatible, i.e. $F^i_j(c_i) = F^j_i(c_j)$, then there is exactly one $x \in F(c)$ such that $F_i(c) = s_i$.

Example 10.1. $(\mathbf{O}(X), Cov_{\mathbf{O}(X)})$ is a site, where for open $U \in \mathbf{O}(X)$, $Cov_{\mathbf{O}(X)}(U) = \{U_i \mid U_i \text{ is open and } \bigcup_i U_i = U\}$.

Part IV
Applications

11 Origins in Algebraic Geometry

The basic problem of algebraic geometry is to understand the set of solutions $(x_1, \ldots, x_n) \in \mathbb{K}^n$ of a system of polynomial equations

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ f_2(x_1, \ldots, x_n) &= 0 \\ &\ldots \\ f_k(x_1, \ldots, x_n) &= 0 \end{aligned}$$

where $f_i \in \mathbb{K}[x]$, and $\mathbb{K}$ a field. Classical algebraic geometry assumes $\mathbb{K}$ to be closed. The study of varieties over $\mathbb{Q}$ belongs to arithmetic geometry.

Definition 11.1. Given a set S of polynomials, define the **variety of zeros** $\mathcal{V}(S)$ as the set of common zeros of polynomials in S:

$$\mathcal{V}(S) = \{x \in \mathbb{K}^n \mid f(x) = 0 \text{ for all } f \in S\}$$

Given a subset X of $\mathbb{K}[x_1, x_2, \ldots, x_n]$, define the **ideal of polynomials** $\mathcal{I}(X)$ as the set of all polynomials that vanish on X:

$$\mathcal{I}(X) = \{f \in \mathbb{K}[x_1, x_2, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}$$

Proposition 11.1. *There is an adjunction between lattice of ideals* $(\mathbf{I}(R), \subseteq)$ *and the lattice of varieties* $(\mathbf{V}(X), \subseteq)^{op}$*:*

$$(I, \subseteq) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set}^{O(X)^{op}}$$

$$X = \mathcal{V}(\mathcal{I}(X)) \text{ and } I \subseteq \mathcal{I}(\mathcal{V}(I))$$

Given ideals $I, J \in \mathbf{I}(R)$, we have:

$$\begin{gathered} \mathcal{V}(I + J) = \mathcal{V}(I) \cap \mathcal{V}(J) \\ \mathcal{V}(I \otimes J) = \mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J) \end{gathered}$$

In a noetherian ring we have primary decomposition:

$$I = P_1 \cap P_2 \cap \cdots \cap P_k$$

The corresponding variety is:

$$V(I) = V(P_1) \cup V(P_2) \cup \cdots \cup V(P_k)$$

Using this adjunction, we can translate geometric concepts to algebraic ones, and vice-versa. For example, a variety X is **irreducible**, i.e. it can not be written as a union $X = X_1 \cup X_2$ of two non-empty proper subvarieties X_1, X_2 of X, if and only if $\mathcal{I}(X)$ is a prime ideal.

Definition 11.2. The **radical** of and ideal I, denoted $\sqrt{I}$ is:

$$\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$$

Remark. So it "contains all nth roots"

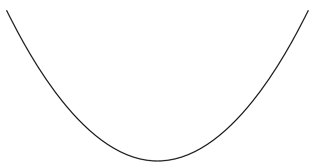
Definition 11.3. The **colon ideal** $I : J$ is defined as:

$$I : J = \{f \in R \mid fJ \subseteq I\}$$

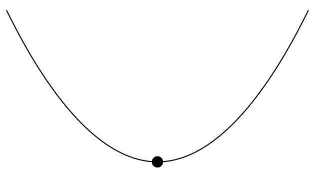
Remark. Given $f \in R$, we use the shorthand notation $Q : f$ for $Q : (f)$.

$\mathcal{V}(I : J)$ is the smallest variety containing $\mathcal{V}(I) \setminus V(J)$.

Example 11.1. A useful way to think of morphisms is as an object equipped with additional structure. For example, the ring $k[x,y]/\langle y - x^2\rangle$ $\frac{k[x,y]}{\langle y-x^2\rangle}$ can be thought of as a parabola, however the ring morphism $\phi : \frac{k[x,y]}{\langle y-x^2\rangle} \to k$ can be thought of as a parabola with a point on it.



$$\frac{k[x,y]}{\langle y-x^2\rangle}$$



$$\frac{k[x,y]}{\langle y-x^2\rangle} \xrightarrow{\phi} k$$

11.1 Primary Ideals

Definition 11.4. A proper ideal Q of a commutative ring R is said to be **primary** if whenever $fg \in Q$, then either $f \in Q$ or $g^n \in Q$, for some $n \in \mathbb{N}$.

Remark. This definition is symmetric. If neither f nor g belongs to a primary ideal Q, and fg belongs to Q, then f^n and g^m must belong to Q for some $m, n \in \mathbb{N}$.

Proposition 11.2. *If Q is primary, then $\sqrt{Q} = P$, where P is a prime ideal. Moreover, P is the smallest prime ideal containing Q.*

For this reason, we say that Q is P-primary.

Proposition 11.3. *If Q_1 and Q_2 are P-primary, then $Q_1 \cap Q_2$ is P-primary.*

Proof. Suppose $fg \in Q_1 \cap Q_2$ and $f \notin Q_1 \cap Q_2$. We need to show that $g^n \in Q_1 \cap Q_2$ for some $n \in \mathbb{N}$. Since $f \notin Q_1 \cap Q_2$, either $f \notin Q_1$ or $f \notin Q_2$. Without loss of generality, say $f \notin Q_2$, then $g^n \in Q_2$, for some $n \in \mathbb{N}$. But then this also means that $g \in P$, since $\sqrt{Q_2} = P$. And since $P = \sqrt{Q_1}$, $g^m \in Q_1$, for some $m \in \mathbb{N}$. So $g^{\gcd(m,n)} \in Q_1 \cap Q_2$.

Remark. Notice that we used the fact that these are P-primary ideals. The intersection of two primary ideals doesn't have to be primary! Take $(2) \cap (3) = (6)$ in $\mathbb{Z}$ for example.

$\square$

Proposition 11.4. *If Q is P-primary, and $f \notin Q$, then $Q : f$ is P-primary.*

Proof. If $fg \in Q$, then since $f \notin Q$, we must have $g^n \in Q$ for some $n \in \mathbb{N}$, so $g \in P$.

$$Q \subseteq Q : f \subseteq P \quad \text{so } \sqrt{Q : f} = P,$$

Showing that this is primary is left as an exercise.

$\square$

Here is the picture locally at a prime ideal P:

$$\begin{tikzcd}
& P = \sqrt{Q_i} & & \\
& & & Q_i : f \\
Q_1 \arrow[uur, bend left] & & Q_2 \arrow[uul, bend right] & Q_i \arrow[u] \\
& Q_1 \cap Q_2 \arrow[ul, bend left] \arrow[ur, bend right] & &
\end{tikzcd}$$

In a Noetherian ring, any ideal can be written as a finite intersection of primary ideals (called the **primary decomposition**).

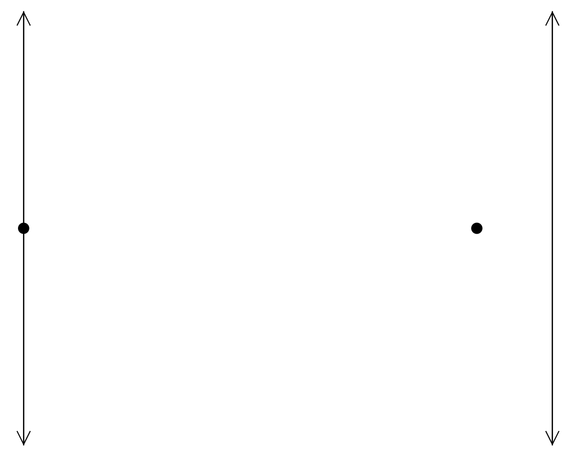
Definition 11.5. A primary decomposition $I = \bigcap_{i=1}^{n} Q_i$ is **irredundant** if for each $j \in \{1, \ldots, n\}$

$$\bigcap_{i \neq j} Q_i \neq I.$$

Remark. So there are no "extraneous" factors".

Definition 11.6. An associated prime P_i which does not properly contain any other associated prime P_j is called a **minimal** associated prime. The non-minimal associated primes are called **embedded** associated primes.

Example 11.2. Let $I = \langle x^2, xy \rangle$. Clearly $I = \langle x^2, y \rangle \cap \langle x \rangle$.



$\langle x^2, xy \rangle$ $\qquad\qquad$ $\langle x^2 - x, xy \rangle$

11.2 Affine Schemes

Definition 11.7. Let R be a commutative unitary ring, $Spec(R)$ be the set of all prime ideals in R. For any $f \in R$, define $D(f)$ to be

$$D(f) = \{p \in Spec(R) \mid f \notin p\}.$$

Proposition 11.5. $D(gf) = D(f) \cap D(g)$.

Proof. Suppose $p \in D(gf)$, i.e. $gf \notin p$. We need to show that $p \in D(f) \cap D(g)$, i.e. $f \notin p$ and $g \notin p$. If $f \in p$, then since p is an ideal, $gf \in p$, which is a contradiction. Thus,

$$D(gf) \subseteq D(f) \cap D(g).$$

Now suppose $p \in D(f) \cap D(g)$, i.e. $f \notin p$ and $g \notin p$. We need to show that $p \in D(gf)$, i.e. $gf \notin p$. If $gf \in p$, then since p is a prime ideal, either $f \in p$ or $g \in p$, which is a contradiction. Thus,

$$D(f) \cap D(g) \subseteq D(gf).$$

□

Define the topology on $Spec(R)$ for which $D(f)$, $f \in R$, form a base of open sets.
$D(g) \subseteq D(f)$ if and only if $g^n = rf$ for some $n > 0$, $u \in R$.

11.3 Graded Rings and Modules

Definition 11.8. A $\mathbb{Z}$-**graded ring** R is a ring with a direct sum decomposition into homogeneous pieces

$$R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

such that if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$. Also, given a $\mathbb{Z}$-graded ring R, a $\mathbb{Z}$-**graded** module M is an R-module with a direct sum decomposition into homogeneous pieces

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

such that if $r_i \in R_i$ and $m_j \in M_j$, then $r_i m_j \in M_{i+j}$.

Remark. You can replace $\mathbb{Z}$ with an arbitrary group G to get a more general definition, however for now on, when we say graded, we'll assume the group is $\mathbb{Z}$.

One of the most basic (and important!) examples of a graded module is just the ring itself, but with the grading shifted.

Example 11.3. Let R be a graded ring. Define $R(i)$ to be the graded module with R as the underlying R-module, and the grading given by

$$R(i)_j = R_{i+j}$$

So for example, let $R = k[x, y, z]$, $I = \langle x^3 + y^3 + z^3 \rangle$, and $J = \langle x \rangle$. We can visualize the degrees like this basis table

degree i	-3	-2	-1	0	1	2	3	$\dots$
basis of R_i	0	0	0	1	$\langle x, y, z \rangle$	$\langle x^2, yx, \dots \rangle$	$\dots$	$\dots$
basis of $R(3)_i$	1	$\langle x, y, z \rangle$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
basis of $R(-2)_i$	0	0	0	0	0	1	$\langle x, y, z \rangle$	$\dots$
basis of I_i	0	0	0	0	0	0	$\langle x^3 + y^3 + z^3 \rangle$	$\dots$
basis of J_i	0	0	0	0	$\langle x \rangle$	$\langle x^2, yx, zx \rangle$	$\dots$	$\dots$
basis of $(R/I)_i$	0	0	0	1	$\langle x, y, z \rangle$	$\langle x^2, yx, \dots \rangle$	$\dots$	$\dots$

or a dimension table

degree i	-5	-4	-3	-2	-1	0	1	2	3	4	5	$\dots$
dimension of R_i	0	0	0	0	0	1	3	6	10	15	21	$\dots$
dimension of $R(3)_i$	0	0	1	3	6	10	15	21	28	36	45	$\dots$
dimension of $R(-2)_i$	0	0	0	0	0	0	0	1	3	6	10	$\dots$
dimension of I_i	0	0	0	0	0	0	0	0	1	3	6	$\dots$
dimension of J_i	0	0	0	0	0	0	1	3	6	10	15	$\dots$
dimension of $(R/I)_i$	0	0	0	0	0	1	3	6	9	12	15	$\dots$

Notice that from the dimension perspective, all of these numbers look like shifted versions of the triangular numbers, except for the last row, which instead looks like the second row (R_i) minus the fifth row (I_i). This motivates us to study the following function.

Definition 11.9. The **Hilbert function** of a finitely-generated graded module M is

$$HF(M, i) = \dim_k M_i.$$

11.4 Stanley-Reisner Ring

Definition 11.10. Given a simplicial complex S, the **Stanley-Reisner ring** R associated to S is the ring generated by the vertices $v_1, v_2, \ldots, v_n$ of the simplicial complex over a field k, modulo the ideal I generated by monomials corresponding to nonfaces of S.

Example 11.4. For the boundary of the octahedron, the ideal I is generated by the three edges $v_i v_j$ which do not lie on the boudnary. The picture below has vertices labeled A,B,C,D,E, F. In that case, the nonfaces are AC, BD, EF.
The Stanley-Reisner ring of S encodes lots of useful topological and combinatorial information about S. One really important combinatorial invariant of S is the f-vector. The letter f stands for face: f(i) is the number of i-dimensional faces of S; by convention there is one empty face, so the f-vector starts with a 1 at position -1. So for example, the f-vector of a line segment connecting two vertices is (1,2,1). For the boundary of the octahedron, the f-vector is (1,6,12,8). And the f-vector of the n-simplex is given by the nth row of pascal's triangle! Now suppose we fix a number of vertices, what f-vectors can arise as the f-vector of the boundary of some simplicial polytope? What's the biggest f-vector that can occur? Should the biggest f-vector exist? The answers to these questions use the Stanley-Reisner ring!
There is a really nice dictionary between S and R! For starters, R has a really nice primary decomposition which is described in terms of S. The ideal I is simply the intersection over all minimal cofaces (i.e. the compliments of the maximal faces). So for example, if S is a tetrahedron with three missing faces:
$A, B, C, A, B, A, C, B, C, A, D, B, D, C, D, A, B, C, D,$
Then the maximal faces are
{{A,B,C}, {A,D}, {B,D}, {C,D}}
And thus the minimal cofaces are
{{D}, {B,C}, {A,C}, {A,B}}
And this means that I is the intersection of the ideals: <D>, <BC>, <AC>, <AB>
Also, given the f-vector for S, you can use it to construct something called the h-vector, which turns out to be the coefficients of the hilbert polynomial for R. And apparently, the h-vector also turns out to be the even betti numbers for the toric variety associated with S, but I'm not too familiar with this yet. All of this is relatively new stuff too!
In the Stanley-Reisner case, we are turning a poset into a ring, with filters going to ideals.

12 Posets

12.1 Lattices

Definition 12.1. A **distributive lattice** is a lattice $(L, \leq)$ that satisfies the distributive law, i.e. for all a, b, c in the lattice:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \forall a, b, c \in L$$

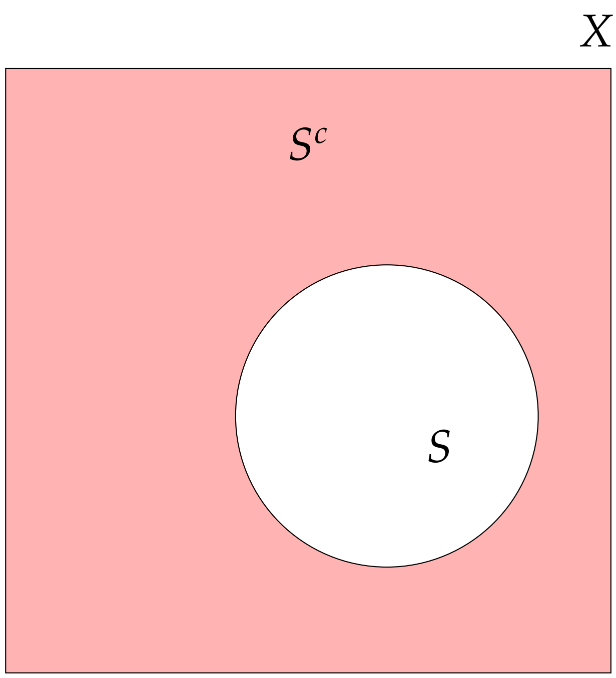
Remark. Notice that every distributive lattice is modular.

Definition 12.2. An element y in a lattice $(L, \leq)$ is said to be a **complement** of x if:

$$x \vee y = 1 \text{ and } x \wedge y = 0$$

Definition 12.3. A **boolean algebra** is a distributive lattice with the property that every element has a complement.

Example 12.1. $(\mathbf{P}(X), \subseteq)$ is a complete boolean algebra, with set inclusion $\subseteq$ being the partial order, set intersection $\cap$ being meet, and set union $\cup$ being join. Given a subset $S \subset X$, the lattice complement of S is the set complement $S^c = X \setminus S = \{x \in X \mid x \notin S\}$. The initial object is $\emptyset$. The terminal object is X.



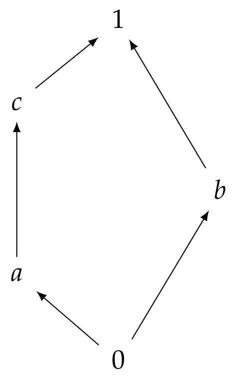
Definition 12.4. A **modular lattice** is a lattice $(L, \leq)$ that satisfies the self-dual condition:

$$a \leq c \text{ implies } a \vee (b \wedge c) = (a \vee b) \wedge c \quad \forall a, b, c \in L$$

Remark. Notice that every distributive lattice is modular.

Example 12.2. Smallest example of a non-modular lattice:

$$a \vee (b \wedge c) = a \vee 0 = x \neq c = 1 \wedge c = (a \vee b) \wedge c$$



Definition 12.5. Given a commutative ring with identity R, let $\mathbf{I}(R)$ be the set of all ideals in R.

Example 12.3. $(\mathbf{I}(R), \subseteq)$ is a complete modular lattice, with ideal inclusion $\subseteq$ being the partial order, the sum $+$ of ideals being join, and the $\cap$ intersection of ideals being meet. The initial object is $\emptyset$. The terminal object is R.

Definition 12.6. If A is a subset of a lattice $(L, \leq)$, then $z \in L$ is an **upper bound** of A, denoted $A \leq z$, if $x \leq z$ whenever $x \in A$. If moreover, $z \leq y$ whenever $A \leq y$, then z is a **least upper bound** of A.

Definition 12.7. We say that z is the **greatest element** of A if z is a least upper bound of A and also a member of A. Thus, A has a greatest element precisely when one of its members is a least upper bound of A.

Definition 12.8. If $(L, \leq)$ is a lattice and $x \in L$, then $y \in L$ is the **pseudo complement** of x if and only if y is the greatest element of L disjoint from x, i.e. greatest element of the set:

$$\{z \in L \mid x \wedge z = 0\}$$

If every member of L has a pseudo complement, L is called a **pseudo complemented lattice**.

Example 12.4. If M is a monoid, then the lattice of left ideals $(L_M, \subseteq)$ is a pseudo complemented lattice. Given a left ideal $I \in L_M$, define $\neg I = \{m \mid I : m = \emptyset\}$. Then $\neg I$ is the pseudo complement of I, since $J \subseteq \neg I$ if and only if $J \cap I = \emptyset$ for all left ideals $J \in L_M$. Let's show this: First, we show that $\neg I$ is an ideal. If $m \in \neg I$, then $n \cdot m$ must be in $\neg I$ too. since if $k \in I : n \cdot m$, then $k \cdot (n \cdot m) \in I$, but this means that $(k \cdot n) \cdot m \in I$, which is a contradiction. Now we show $\neg I \cap I = \emptyset$. If $m \in \neg I \cap I$, then $I : m = M$, which is a contradiction. Finally, we show that $\neg I$ is the greatest element of the set of all left ideals disjoint from I. If $j \in J$ and $J \cap I = \emptyset$, then $I : j = \emptyset$ (J swallows everything up!). So $\neg I$ contains all left ideals J disjoint from I.

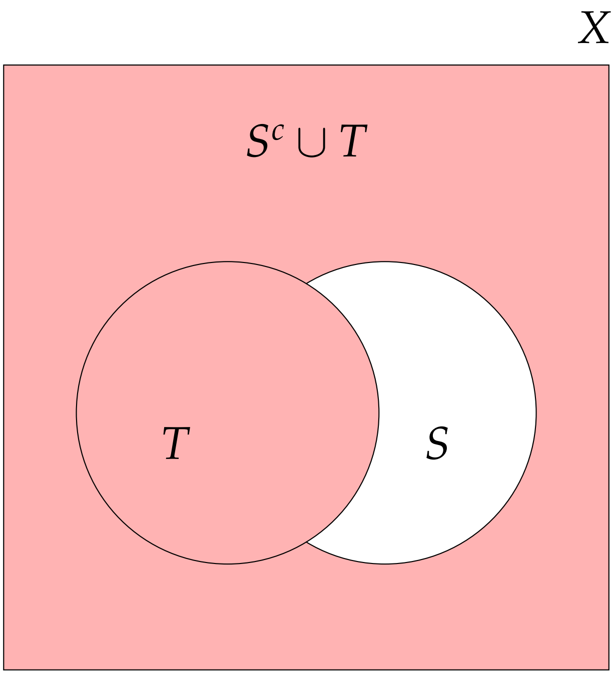
Example 12.5. If X is a topological space, then the lattice of open sets $(\mathbf{O}(X), \subseteq)$ is a pseudo complemented lattice. Even though the set complement $X \setminus U$ is not necessarily open, the interior $(X \setminus U)^{\circ}$, i.e. the largest open set contained in $X \setminus U$, is. So, $(X \setminus U)^{\circ}$ is the pseudo complement of U since have $V \subseteq (X \setminus U)^{\circ}$ if and only if $U \cap V = \emptyset$, for all open $V \in \mathbf{O}(X)$. In this case, $(X \setminus U)^{\circ}$ is simply the union of all open sets disjoint from U. Since arbitrary unions of open sets are open, we are done. Notice that this is not the argument we used for $(L_M, \subseteq)$.

Definition 12.9. The notion of pseudo complement can be generalized by replacing the initial object 0 by some other object y, to obtain the pseudo complement of x **relative** to y. This, if it exists, is the greatest element of the set:

$$\{w \mid x \wedge w \leq y\}$$

So, the pseudo complement of x relative to y is the greatest element z such that $x \wedge z \leq y$.

Example 12.6. If $S, T \in (\mathbf{P}(X), \subseteq)$, then $S^c \cup T$ is the pseudo complement of S relative to T.



Example 12.7. If $I, J \in (L_M, \subseteq)$, then the pseudo complement of I relative to J is $\{m \mid I : m \subseteq J : m\}$. If $I \subseteq J$, then the pseudo complement of I relative to J is M. If $I \cap J = \emptyset$, then the pseudo complement of I relative to J contains J, but is disjoint from I.

Example 12.8. If $U, V \in (\mathbf{O}(X), \subseteq)$, then the pseudo complement of U relative to V is $((X \setminus U) \cup V)^{\circ}$.

Definition 12.10. Given a lattice $(L, \leq)$, the pseudo complement of x relative to y, when it exists, is denoted $x \Rightarrow y$. And if $x \Rightarrow y$ exists for every $x, y \in L$, then L is called a **Heyting Algebra**. We use the notation $(H, \leq)$ to denote a Heyting Algebra. We also define $\neg : H \to H$ by setting $\neg x$ equal to $x \Rightarrow 0$.

Exercise 5. Check that this is the same definition as we gave in the other set of notes, i.e. a Heyting Algebra is a poset that has finite limits, finite colimits, and is cartesian closed.

Proposition 12.1. *A Heyting Algebra is distributive.*

Definition 12.11. A **locale** is a complete lattice in which the infinite distributivity law

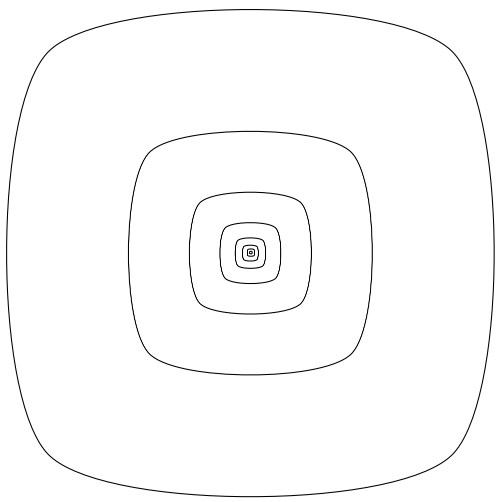
$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i)$$

holds for all elements a, b_i and every set I of indices.

12.2 Filters

Definition 12.12. Let $(P, \leq)$ be a poset. A subset $\mathcal{F}$ of $(P, \leq)$ is a **filter** if the following conditions hold:

1. $\mathcal{F}$ is nonempty
2. $\mathcal{F}$ is **downward directed**, i.e. for every $x, y \in \mathcal{F}$, there is some element $z \in \mathcal{F}$ such that $z \leq x$ and $z \leq y$.
3. $\mathcal{F}$ is **upward closed**, i.e. for every $x \in \mathcal{F}$ and $y \in P$, $x \leq y$ implies that $y \in \mathcal{F}$.



Example 12.9. A filter in $(\mathbf{P}(S), \subseteq)$ is a nonempty family $\mathcal{F}$ of nonempty subsets of X that is closed under finite intersections and satisfies the condition that, whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, the set B must also belong to $\mathcal{F}$.

Definition 12.13. A **filter base** is any downward directed non-empty set.

Remark. An so becomes a filter when closed under axiom (3).

Definition 12.14. A filter $\mathcal{F}_1$ is said to be a **refinement** of a filter $\mathcal{F}_2$ if every element of $\mathcal{F}_2$ belongs to $\mathcal{F}_1$. The set of all filters in a poset $(P, \leq)$ is itself a poset under the refinement, denoted $(\text{Filter}(P), \leq)$

Definition 12.15. A **principal filter** for a given element $p \in P$ is the smallest filter that contains p. The principal filter for p is just given by the set

$$\mathcal{F}^p = \{x \in P \mid p \leq x\}.$$

We say that $\mathcal{F}^p$ is the filter **filter** by p.

Proposition 12.2.

Definition 12.16. An **ultrafilter** is a maximal filter, i.e. there doesn't exist another filter that is a refinement of it.

Remark. Every filter is included in an ultrafilter. This is an easy consequence of Zorn's lemma.

Proposition 12.3. *Suppose that $\mathcal{F}$ is a filter in $(\mathbf{P}(S), \subseteq)$. The following conditions are equivalent to one another.*

1. *The filter $\mathcal{F}$ is an ultrafilter.*
2. *Given any subset E of X, either E or $X \setminus E$ must belong to $\mathcal{F}$.*

12.3 Filters in Topology

In topology and analysis, filters are used to define convergence in a manner similar to the role of sequences in a metric space.

Definition 12.17. Let X be a topological space and x a point of X.
Take N_x to be the neighborhood filter at point x. Define $\mathbf{O}(X)_x$ to be the set of all topological neighborhoods of the point x. It is easily verified that $\mathbf{O}(X)_x$ is a filter in $(\mathbf{O}(X), \subseteq)$. It is called the **neighborhood filter** at a point x. A **neighborhood base** $\mathcal{N}$ at x is just a filter base that generates $\mathbf{O}(X)_x$, i.e. an open set V is a neighborhood of x if and only if there exists $W \in \mathcal{N}$ such that $W \subseteq V$.

Definition 12.18. Let Ω/k be a Galois extension (possibly infinite). The filter

$$\{Gal(\Omega/L) \mid \ L/k \text{ is a finite Galois extension, } L \subset \Omega\}$$

is a neighborhood base of the identity in the Krull Topology on $Gal(\Omega/k)$.

Definition 12.19. Let X be a topological space. A filter $\mathcal{F}$ in $(\mathbf{O}(X), \leq)$ **converges** to a point $x \in X$ if every neighborhood of x belongs to $\mathcal{F}$. A filter $\mathcal{F}$ **clusters** at a point x if every neighborhood of x intersects every element of $\mathcal{F}$.

13 Propositional Calculus

14 Cohomology

14.1 Group Cohomology

Definition 14.1. The set of maps $f : G^n \to A$ is denoted by $C^n(G; A)$ and is called the **n-cochains** on G with values in A. Define

$$d : C^n(G; A) \to C^{n+1}(G; A)$$

by $(g_1, \ldots, g_{n+1} \in G)$

$$d(f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) + \sum_{i=1} (-1)^i f(g_1, \ldots, g_i g_{i+1} \cdots g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)$$

One readily verifies that $dd = 0$ and the **nth cohomology group** $(n \geq 0)$ of G with **coefficients** in M, $H^n(G; A)$, is defined by

$$H^n(G; A) = \frac{(\ker d : C^n(G; A) \to C^{n+1}(G; A))}{(\mathrm{im} d : C^{n-1}(G; A) \to C^n(G; A))}$$

where we set $C^n(G; A) = 0$ if $n < 0$.

For example, we have:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow C^0(G, A) \xrightarrow{d} C^1(G, A) \xrightarrow{d} C^2(G, A) \xrightarrow{d} \cdots$$

$$(df_0)(\sigma) = \sigma \cdot f_0(\bullet) - f_0(\bullet)$$
$$(df_1)(\sigma, \tau) = \sigma \cdot f_1(\tau) - f_1(\sigma\tau) + f_1(\sigma)$$
$$(df_2)(\sigma, \tau, \rho) = \sigma f_2(\tau, \rho) - f_2(\sigma\tau, \rho) + f_2(\sigma, \tau\rho) - f_2(\sigma, \tau)$$

14.2 Hochschild Cohomology

Let R be a commutative unitary ring, A an R-algebra, and M an A-bimodule. The Hochschild cohomology $H^n(A, M)$ of the algebra A with coefficients in M is defined as the cohomology of the following cochain complex:

$$C^n(A, M) = Hom_k(A^{\otimes n}, M)$$

$$d(f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1} (-1)^i f(a_1, \ldots, a_i a_{i+1} \ldots a_{n+1}) + (-1)^{n+1} f(a_1, \ldots, a_n)$$

14.3 Koszul Complex