Homework #1 Solutions

You should use this as a study guide for the upcoming exam. Try to understand why/how I wrote the solutions the way I did. If you are confused by a particular solution, then send me an email (mnels07@clemson.edu). Check footnotes for other useful information.

(1) $P(t)$ million people gives the population of the world, t centuries since 1000 AD, $0 \leq t \leq 10$.

(1.a) (3 points) Complete the following table

Output Variable:	$P(t)$
Input Variable:	t
Input Unit:	centuries
Output Unit:	million people
Input Description:	number of centuries since 1000 AD
Output Description:	population of the world
Input Range (or Domain):	$0 \leq t \leq 10$
Slope Unit:	centuries per million people

(1.b) (2 points) Write the following statement in function notation: In 1500 AD, the world population was 500, 000, 000 people.[1]

Solution: $P(5) = 500$

(1.c) (3 points) Write your answer you found in part b as an ordered pair and then give a sentence of interpretation of this ordered pair.

Solution: The corresponding ordered pair is $(5, 500)$. To write a sentence of interpretation for an ordered pair, we need to answer the questions: what, how much, and when.

What: The population of the world
How Much: 500 million people
When: In 1500 AD
Sentence of Interpretation: The population of the world is 500 million people in 1500 AD.

(2) $h(x) = -2.1x + 10.2$ hours is the time it takes to clean a house where x is the number of people working, $0 \leq x \leq 3$.

(2.a) (3 points) Complete the following table

Output Variable:	$h(x)$
Input Variable:	x
Input Unit:	people
Output Unit:	hours
Input Description:	number of people working
Output Description:	the time it takes to clean a house
Input Range (or Domain):	$0 \leq x \leq 3$
Slope Units:	hours per people

[1] Keep in mind that $P(t)$ is just a model. In fact, we don't know the exact populatiion of the world in 1500 AD. All we really know is that it was between 425 million people and 540 million people.

(2.b) (1 point) What type of representation does this function have? (Verbal, Algebraic, Numberical, Graphical)

Solution: Algebraic[2]

(2.c) (2 points) How many hours did it take to clean a house when 4 people were cleaning it? Is this interpolation or extrapolation?

Solution: It took 1.8 hours to clean the house. This is extrapolation since 4 is outside our domain $0 \leq x \leq 3$.

(2.d) (3 points) Write a sentence of interpretation for $h(0) = 10.2$.[3]

What: The time it took to clean a house
How Much: 10.2 hours
When: when 0 people are working
Sentence of Interpretation: The time it took to clean a house was 10.2 hours when 0 people were working.[4]

(2.e) (3 points) Write a sentence of interpretation for the slope of this function between the inputs $x = 0$ and $x = 3$.

Solution: We first need to find the slope! The slope is given by

$$\begin{aligned} \frac{h(3) - h(0)}{3 - 0} &= \frac{3.9 - 8.3}{3 - 0} \\ &= \frac{-4.4}{3} \\ &\approx -1.467. \end{aligned}$$

To write a sentence of interpretation for a slope, we need to answer the questions: what, increased/decreased, how much, and when.

What: The time it took to clean a house
Increased/Decreased: decreased
How Much: by 1.467[5] hours per people[6].
When: between 0 and 3 people.
Sentence of Interpretation: The time it took to clean a house decreased by 1.467 hours per people between 0 and 3 people.

(3) Let $f(x)$ be the function whose graph is given by

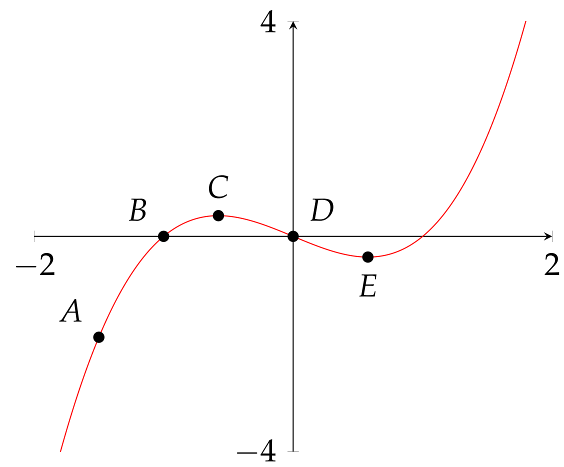
[2]This one is tricky because one could interpret this to be a verbal representation. However that would be wrong. Just remember that if you see an equation somewhere, then it must be an algebraic representation.

[3]In other words, write a sentence of the ordered pair $(0, 8.3)$.

[4]I guess that means the house was cleaning itself, huh? Joking aside, I probably should have set the initial domain to be $1 \leq x \leq 3$ instead of $0 \leq x \leq 3$ so that the context of this problem would make sense.

[5]note that there is not minus sign here as that is already accounted for by the word "decreased".

[6]make sure to include slope units!



(3.a) (1 point) What kind of a function is $f(x)$? (Linear, Quadratic, Cubic, Exponential, Logarithmic, Logistic)

Cubic

(3.b) (1 point) Which point(s) corresponds to a relative max?

C[7]

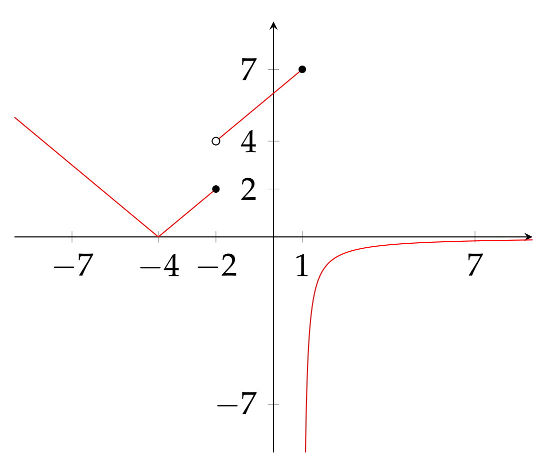
(3.c) (1 point) Which point(s) corresponds to a relative min?

E

(3.d) (1 point) Which point(s) is an inflection point?

D

(4) Let $g(x)$ be the function whose graph is given by



[7] We haven't covered relative mins/maxes yet, but it should be intuitive which points they'll correspond to.

(4.a) (5 points) Complete the following (write DNE for does not exist):

$$\begin{aligned}
\lim_{x \to -4^+} g(x) &= 0 \\
\lim_{x \to -4^-} g(x) &= 0 \\
\lim_{x \to -4} g(x) &= 0 \\
g(-4) &= 0 \\
\lim_{x \to -2^+} g(x) &= 4 \\
\lim_{x \to -2^-} g(x) &= 2 \\
\lim_{x \to -2} g(x) &= \text{DNE} \\
g(-2) &= 2 \\
\lim_{x \to 1^+} g(x) &= -\infty \\
\lim_{x \to 1^-} g(x) &= 7 \\
\lim_{x \to 1} g(x) &= \text{DNE} \\
g(1) &= \text{DNE} \\
\lim_{x \to \infty} g(x) &= 0 \\
\lim_{x \to -\infty} g(x) &= \infty
\end{aligned}$$

(4.b) (3 points) Where is $g(x)$ continuous? Remember to consider *all* points where $g(x)$ is continuous. Use interval notation for your answer.

Solution: The function $g(x)$ is continuous on the intervals $-\infty < x < -2$ and $-2 < x < 1$ and $1 < x < \infty$. It is continuous at $x = -4$ since $g(-4)$ exists and

$$\lim_{x \to -4^+} g(x) = \lim_{x \to -4^-} g(x) = g(-4).$$

It is not continuous at $x = -2$ since

$$\begin{aligned}
\lim_{x \to -2^+} g(x) &= 4 \\
&\neq 2 \\
&= \lim_{x \to -2^-} g(x).
\end{aligned}$$

It is not continuous at $x = 1$ since

$$\begin{aligned}
\lim_{x \to 1^+} g(x) &= \infty \\
&\neq 7 \\
&= \lim_{x \to 1^-} g(x).
\end{aligned}$$

(4.c) (3 points) Does $g(x)$ have any horizontal asymptotes? If so, then state how many and then write down the equation of the corresponding line.

Solution: In general, functions have at most two horizontal asymptotes. Since $\lim_{x \to \infty} g(x) = 0$, the equation for one horizontal asymptote is $y = 0$. Since $\lim_{x \to -\infty} g(x) = \infty$, we do not have another horizontal asymptote. So there is only one horizontal asymptote and it is given by the equation $y = 0$.

(4.d) (3 points) Does $g(x)$ have any vertical asymptotes? If so, then state how many and then write down the equation of the corresponding line.

Solution: Unlike horizontal asymptotes, a function can have more than two vertical asymptotes. In this case however, there is only one vertical asymptote: since $\lim_{x\to 1^+} g(x) = -\infty$, the equation $x = 1$ gives us a vertical asymptote. At all other inputs $c \in \mathbb{R}$, we have $\lim_{x\to c^\pm} g(x) < \infty$, so we only have one vertical asymptote.

(4.e) (3 points) State the interval(s) on which $g(x)$ is increasing.

Solution: It is increasing on the intervals $-4 \le x \le 1$ and $1 < x < \infty$.[8]

(4.f) (3 points) State the interval(s) on which $g(x)$ is decreasing.

Solution: It is decreasing on the interval $-\infty < x \le -4$.

(4.g) (3 points) State the interval(s) on which $g(x)$ is concave down.

Solution: It is concave down on the interval $1 < x < \infty$.

(5) (5 points) Complete the following table[9]

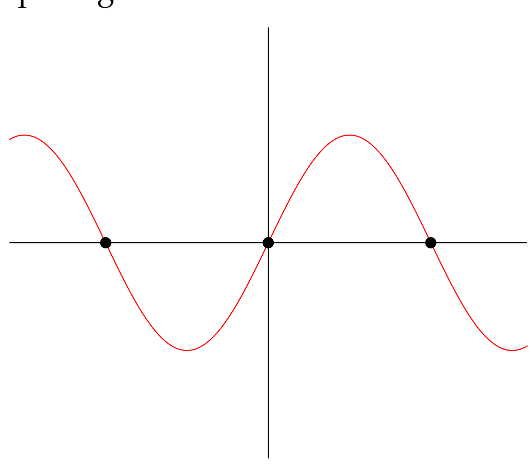
Function	# of Concavities	# of H-Asymptotes	# of V-Asymptotes	# of Inflection Points
Linear	0	0 or 1 (depends)	0	0
Quadratic	1	0	0	0
Cubic	2	0	0	1
Logarithmic	1	0	1	0
Exponential	1	1	0	0
Logistic	2	2 (one of them is always $y = 0$)	0	1

(6) (3 points) During the qth quarter last year, a bicycle company made $P(q)$ dollars in profit, but their total cost was $C(q)$ thousand dollars. Recall that $q = 3$ refers to the 3rd quarter (July 1st to September 30th). Suppose that $P(3) = 1,000,000$ and $C(3) = 100$. How much revenue (in dollars) did the bicycle company make during the third quarter last year?

Solution: Revenue is the total amout of money you bring in. Profit is your revenue minus your total costs. Thus, revenue is your profit plus your total costs. Letting $R(q)$ be the revenue function, we have

$$\begin{aligned}
R(3) \text{ dollars} &= P(3) \text{ dollars } + C(3) \text{ thousand dollars} \\
&= P(3) \text{ dollars } + 1000 \cdot C(3) \text{ dollars} \\
&= 1000000 \text{ dollars } + 1000 \cdot 100 \text{ dollars} \\
&= 1000000 \text{ dollars } + 100000 \text{ dollars} \\
&= 1100000 \text{ dollars.}
\end{aligned}$$

(7) Let $f(x)$ be the function whose graph is given below



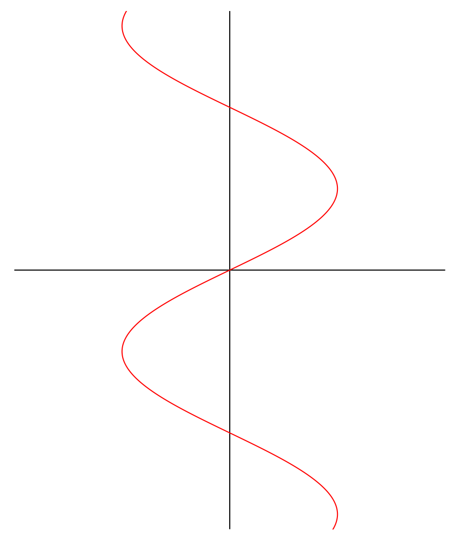
[8]Technically $-4 \le x < \infty$ is incorrect, but if you wrote this then that's okay. Here's the technical definition of what increasing means if you're curious: we say a function $f(x)$ is **increasing** on an interval (a, b) if $a < x_1 \le x_2 < b$ implies $f(x_1) \le f(x_2)$. Like I said, you don't have to worry about this technical stuff on the test; it'll be very easy to state the interval on which the function is increasing.

[9]Make sure you remember this table! It'll be helpful for the test.

(7.a) (3 points) How many inflection points does this function have?

Solution: It has 3 inflection points. They are plotted above.

(7.b) (3 points) Graph the inverse of this function below. Is the inverse a function? Why or why not?



Solution: No because it fails the horizontal line test.

(8) Consider the function $f(x) = \frac{2.53}{1+4.86e^{-0.5x}}$.

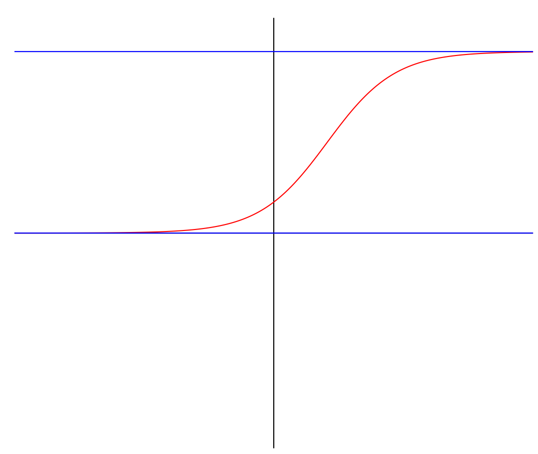
(8.a) (1 point) What kind of function is this?

Solution: Logistic

(8.b) (2 points) Find $f(2)$.

Solution: $f(2) \approx 0.907$

(8.c) (3 points) Sketch a graph of this function below



(8.d) (3 points) How many horizontal asymptotes does this function have? If it has any, then write down their equations.

Solution: There are two horizontal asymptotes. They are given by equations $y = 0$ and $y = 2.53$. Both are drawn in blue above.

(9) (3 points) $R(x) = 332.10(1.02^x)$ million dollars gives revenue that Company A makes in a given year, where x is the number of years after 2000, $0 \leq x \leq 19$. According to the model, how much will the companies revenue increase per year?

Solution: We are given b, so we need to find C. We have

$$\begin{aligned} C\% &= (b-1) \cdot 100\% \\ &= (1.02 - 1) \cdot 100\% \\ &= 0.02 \cdot 100\% \\ &= 2\%. \end{aligned}$$

So the companies revenue will increase by 2% each year.

(10) (3 points) Suppose Company B made 30 million dollars in profits during the year 2000 and that their profits are increasing by 1.3% each year between the years 2000 and 2019. Write a completely defined model for their profit.

Solution: To write a completely defined model, we need to first find the appropriate function. Clearly the appropriate function is an exponential function since there is a constant percent increase. An exponential function has the form

$$P(t) = ab^t.$$

The initial value a is given; namely $a = 30$. We are also given $C = 1.3$, so we use this to find b. We have

$$\begin{aligned} b &= \frac{C + 100}{100} \\ &= \frac{1.3 + 100}{100} \\ &= 1.013. \end{aligned}$$

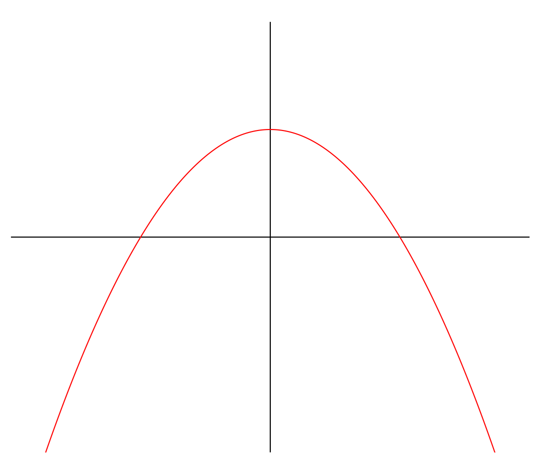
Thus

$$P(t) = 30(1.013^t).$$

To write a completely defined model for the profit of the company, we need to include:

Equation with output units: $P(t) = 30(1.013^t)$ million dollars
Output description: gives the profit Company B made
Input description with input units: t years since 2000
Domain: $0 \leq t \leq 19$
Completely defined model: $P(t) = 30(1.013^t)$ million dollars gives the profit Company B made t years after 2000, $0 \leq t \leq 19$.

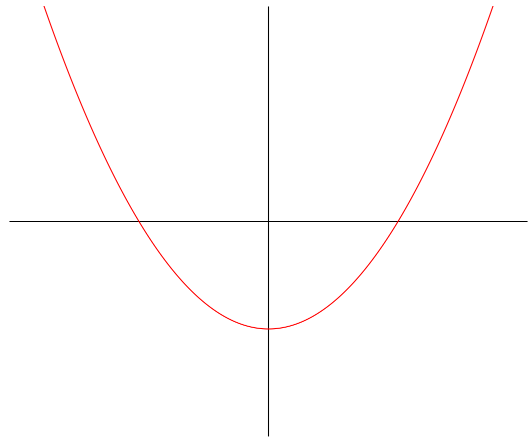
(11) (2 points) Sketch a graph of the function $-x^2 + 1$ below



Does this function have an inflection point? What about a relative min/max?

Solution: It does not have an inflection point, but it does have a relative max at $x = 0$.

(12) (2 points) Sketch a graph of the function $x^2 - 1$ below



Does this function have an inflection point? What about a relative min/max?

Solution: It does not have an inflection point, but it does have a relative min at $x = 0$.

(13) Consider a function $f(x)$.

(13.a) (3 points) What does it mean for $f(x)$ to have a horizontal asymptote?

Solution: The line with equation $y = L$ is a horizontal asymptote for $f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

(13.b) (3 points) What does it mean for $f(x)$ to have a vertical asymptote?

Solution: The line with equation $x = c$ is a vertical asymptote for $f(x)$ if either

$$\lim_{x\to c^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to c^-} f(x) = \infty$$

(13.c) (3 points) What does it mean for $f(x)$ to be continuous at $x = c$?

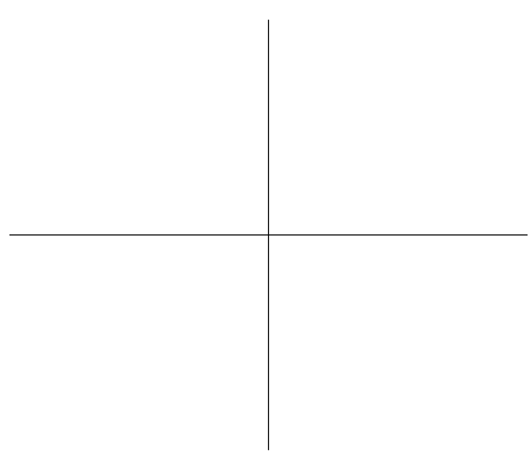
We say $f(x)$ is continous at $x = c$ if $f(c)$ exists and

$$\lim_{x\to c^+} f(x) = f(c) = \lim_{x\to c^-} f(x).$$

(14) The table shows US consumption of energy from biomass fuels in selected years

year	2002	2004	2006	2008	2010	2012
billion BTUs	2745	2921	3278	3861	4288	4310

(14.a) (3 points) Align the data to the number of years after 2000, and then sketch the a scatter plot for this table below



(14.b) (2 points) Based on the scatter plot, which two functions could model the data above?

Solution: Either a logistic or a cubic.

(14.c) (3 points) Write a completely defined cubic model for the data in the table.

Solution: $f(t) = -4.718t^3 + 96.882t^2 - 388.866t + 3185.667$ billion BTUs gives the US consumption of energy from biomass fuels in the year $2000 + t$, where t is the number of years since 2000, $2 \leq t \leq 12$.