Abstract Algebra Homework 2

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Throughout this homework, let R be a commutative ring.

Problem 1

Proposition 0.1. *Define* $\varphi\colon \mathbb{Z} \to \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ *by*

$$\varphi(a) = (2a, 0)$$

for all $a \in \mathbb{Z}$ *and define* $\psi\colon \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \to (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ *by*

$$\psi(a, \overline{a_1}, \overline{a_2}, \dots) = (\overline{a}, \overline{a_1}, \overline{a_2}, \dots)$$

for all $(a, \overline{a_1}, \overline{a_2}, \dots) \in \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. *Then*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \xrightarrow{\psi} (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \longrightarrow 0 \tag{1}$$

is a short exact sequence which does not split, even though we have $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

Proof. The maps defined above are $\mathbb{Z}$-linear since each component map is $\mathbb{Z}$-linear. The map φ is injective since 2 is a nonzerodivisor in $\mathbb{Z}$, and the map ψ is surjective since the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is surjective. We also have exactness at $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Indeed, let $(a, \overline{a_1}, \overline{a_2}, \dots) \in \ker \psi$. Then

$$\begin{aligned} 0 &= \psi(a, \overline{a_1}, \overline{a_2}, \dots) \\ &= (\overline{a}, \overline{a_1}, \overline{a_2}, \dots) \end{aligned}$$

implies $\overline{a_n} = 0$ for all $n \geq 1$ and $a = 2b$ for some $b \in \mathbb{Z}$. Then

$$\begin{aligned} (a, \overline{a_1}, \overline{a_2}, \dots) &= (2b, 0) \\ &= \varphi(b) \end{aligned}$$

implies $(a, \overline{a_1}, \overline{a_2}, \dots) \in \operatorname{im} \varphi$. Therefore we have exactness at $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, and so (1) is a short exact sequence.

Now we show that (1) does not split. Assume for a contradiction that it did split. Then there exists an R-linear map

$$\widetilde{\psi}\colon (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \to \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

such that $\psi\widetilde{\psi} = 1$. Let

$$\pi_1\colon \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \to \mathbb{Z} \quad \text{and} \quad \pi_2\colon \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \to (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

be the natural projection maps and denote $\widetilde{\psi}_1 = \pi_1 \circ \widetilde{\psi}$ and $\widetilde{\psi}_2 = \pi_2 \circ \widetilde{\psi}$ to be the component maps of $\widetilde{\psi}$. Note that $\widetilde{\psi}_1\colon (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} \to \mathbb{Z}$ must be the zero map since 2 is a nonzerodivisor on $\mathbb{Z}$ and $2 \in \operatorname{Ann}((\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Indeed, we have

$$\begin{aligned} 2\widetilde{\psi}_1((\overline{a_n})) &= \widetilde{\psi}_1((\overline{2a_n})) \\ &= \widetilde{\psi}_1(\overline{0}) \\ &= 0, \end{aligned}$$

which implies $\widetilde{\psi}_1((\overline{a_n})) = 0$ for all $(\overline{a_n}) \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Now let $(\overline{a_n}) \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ with $\overline{a_1} = \overline{1}$ and denote $(b_n) = \widetilde{\psi}_2((\overline{a_n}))$. Then

$$\begin{aligned}(\overline{a_n}) &= \psi\widetilde{\psi}((\overline{a_n})) \\ &= \psi(\widetilde{\psi}_1((\overline{a_n})), \widetilde{\psi}_2((\overline{a_n}))) \\ &= \psi(0, (b_n)) \\ &= (\overline{0}, \overline{b_1}, \overline{b_2}, \dots).\end{aligned}$$

This is a contradiction since $\overline{a_1} = \overline{1}$. ☐

Problem 2

Proposition 0.2. *Suppose for each $i \in \mathbb{Z}$, suppose we are given short exact sequences of the form*

$$0 \longrightarrow K_i \xrightarrow{\phi_i} M_i \xrightarrow{\psi_i} K_{i-1} \longrightarrow 0 \tag{2}$$

Then we can splice these short exact sequences together to get a long exact sequence of the form

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots \tag{3}$$

where $\varphi_i = \phi_{i-1} \circ \psi_i$.

Proof. Let $i \in \mathbb{Z}$. It follows the short exact sequences (2) that

$$\begin{aligned}\ker \varphi_i &= \ker(\phi_{i-1} \circ \psi_i) \\ &= \ker \psi_i \\ &= \operatorname{im} \phi_i \\ &= \operatorname{im}(\phi_i \circ \psi_{i+1}) \\ &= \operatorname{im} \varphi_{i+1}.\end{aligned}$$

As i was arbitrary, it follows that (3) is exact. ☐

Corollary. *Every long exact of R-modules can be formed by splicing together suitable short exact sequences.*

Proof. Let

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots \tag{4}$$

be an exact sequence of R-modules. For each $i \in \mathbb{Z}$, we break (4) into short exact sequences of the form

$$0 \longrightarrow \ker \varphi_i \xrightarrow{\iota_i} M_i \xrightarrow{\widetilde{\varphi}_i} \operatorname{im} \varphi_i \longrightarrow 0 \tag{5}$$

where ι_i is the inclusion map and $\widetilde{\varphi}_i$ is just φ_i but with range $\operatorname{im} \varphi_i$ rather than M_{i-1}. In fact, since $\ker \varphi_{i-1} = \operatorname{im} \varphi_i$, we can rewrite (6) as

$$0 \longrightarrow \ker \varphi_i \xrightarrow{\iota_i} M_i \xrightarrow{\varphi_i} \ker \varphi_{i-1} \longrightarrow 0 \tag{6}$$

Since $\varphi_i = \iota_{i-1} \circ \widetilde{\varphi}_i$, it follows from Proposition (0.2) that splicing these short exact sequences together gives us our original long exact sequence (4) . ☐

Problem 3

Proposition 0.3. *Let K be a field, let V be a vector space of countably infinite dimension over K, and set $A = \operatorname{Hom}_K(V, V)$. Then A is a ring with identity where multiplication is given by function composition. Moreover, A is isomorphic (as an A-module over itself) to $\bigoplus_{i=1}^{n} A$ for every positive integer n.*

Proof. We first show that A is a ring with identity. First note that A has the structure of an abelian group where addition is defined pointwise: let $\varphi, \psi \in A$, then we define $\varphi + \psi \in A$ to be the K-linear map

$$(\varphi + \psi)(v) := \varphi(v) + \psi(v)$$

for all $v \in V$. Addition is associative and commutative since addition in V is associative and commutative. Moreover, the zero map $0\colon V \to V$ defined by

$$0(v) = 0$$

for all $v \in V$ serves as the identity element. We claim that composition gives the abelian group A a ring structure. Indeed, let $\varphi, \psi, \phi \in A$ and let $v \in V$. Then

$$\begin{aligned}(\varphi \circ (\psi + \phi))(v) &= \varphi((\psi + \phi)(v)) \\ &= \varphi((\psi(v) + \phi(v)) \\ &= \varphi((\psi(v)) + \varphi(\phi(v)) \\ &= (\varphi \circ \psi)(v) + (\varphi \circ \phi)(v). \\ &= (\varphi \circ \psi + \varphi \circ \phi)(v)\end{aligned}$$

and

$$\begin{aligned}((\varphi + \psi) \circ \phi)(v) &= (\varphi + \psi)(\phi(v)) \\ &= \varphi(\phi(v)) + \psi(\phi(v)) \\ &= (\varphi \circ \phi)(v) + (\psi \circ \phi)(v) \\ &= (\varphi \circ \phi + \psi \circ \phi)(v).\end{aligned}$$

and

$$\begin{aligned}(\varphi \circ (\psi \circ \phi))(v) &= \varphi((\psi \circ \phi)(v)) \\ &= \varphi(\psi(\phi(v))) \\ &= (\varphi \circ \psi)(\phi(v)) \\ &= ((\varphi \circ \psi) \circ \phi)(v)\end{aligned}$$

It follows that

$$\begin{aligned}\varphi \circ (\psi + \phi) &= \varphi \circ \psi + \varphi \circ \phi; \\ (\varphi + \psi) \circ \phi &= \varphi \circ \phi + \psi \circ \phi; \\ \varphi \circ (\psi \circ \phi) &= (\varphi \circ \psi) \circ \phi.\end{aligned}$$

Thus we have left and right distributivity as well as associativity. The identity map $1_V\colon V \to V$, given by $v \mapsto v$, serves as the identity element in A: all $v \in V$ and $\varphi \in A$, we have

$$\begin{aligned}(1_V \circ \varphi)(v) &= 1_V(\varphi(v)) \\ &= \varphi(v) \\ &= \varphi(1_V(v)) \\ &= (\varphi \circ 1_V)(v).\end{aligned}$$

It follows that

$$1_V \circ \varphi = \varphi = \varphi \circ 1_V$$

for all $\varphi \in A$, and hence 1_V is the identity element in A. This establishes our claim that A is a ring with identity.

Now we want to prove the "moreover" part of the proposition. First note that it suffices to show that $A \cong A \oplus A$. Indeed if this is the case, then an induction argument would gives us

$$\begin{aligned}A^n &= A \oplus A^{n-1} \\ &\cong A \oplus A \\ &\cong A.\end{aligned}$$

Let $\{e_i\}$ be a countable basis for V. Let $\psi_{\mathrm{o}}\colon V \to V$ and $\psi_{\mathrm{e}}\colon V \to V$ be the unique linear maps such that

$$\psi_{\mathrm{o}}(e_i) = \begin{cases} e_{(i+1)/2} & \text{if } i \text{ is odd.} \\ 0 & \text{if } i \text{ is even.} \end{cases} \qquad \text{and} \qquad \psi_{\mathrm{e}}(e_i) = \begin{cases} 0 & \text{if } i \text{ is odd.} \\ e_{i/2} & \text{if } i \text{ is even.} \end{cases}$$

for all $i \in \mathbb{N}$. We claim that $\{\psi_{\mathrm{o}}, \psi_{\mathrm{e}}\}$ is linearly independent and $\mathrm{span}\{\psi_{\mathrm{o}}, \psi_{\mathrm{e}}\} = A$. This will imply $A \cong A \oplus A$.

Let us first show that $\{\psi_{\mathrm{o}}, \psi_{\mathrm{e}}\}$ is linearly independent. Suppose we have the relation

$$\varphi_1\psi_{\mathrm{o}} + \varphi_2\psi_{\mathrm{e}} = 0 \tag{7}$$

for some $\varphi_1, \varphi_2 \in A$. If i is a positive odd integer, then applying e_i to both sides of (7) gives us

$$\varphi_1(e_{(i+1)/2}) = 0.$$

Similarly, if j is a positive even integer, then applying e_j to both sides of (7) gives us

$$\varphi_2(e_{j/2}) = 0.$$

Since every positive integer n can be expressed as $n = (i+1)/2$ and $n = j/2$ where i is a positive odd integer and j is a positive even integer, we see that

$$\varphi_1(e_n) = \varphi_2(e_n) = 0$$

for all $n \in \mathbb{N}$. This implies $\varphi_1 = \varphi_2 = 0$. Thus $\{\psi_{\mathrm{o}}, \psi_{\mathrm{e}}\}$ is linearly independent.

Next we show that $\mathrm{span}\{\psi_{\mathrm{o}}, \psi_{\mathrm{e}}\} = A$. Let $\varphi \in A$ and define $\varphi_{\mathrm{o}}\colon V \to V$ and $\varphi_{\mathrm{e}}\colon V \to V$ be the unique linear maps such that

$$\varphi_{\mathrm{o}}(e_n) = \varphi(e_{2n-1}) \qquad \text{and} \qquad \varphi_{\mathrm{e}}(e_n) = \varphi(e_{2n})$$

for all $n \in \mathbb{N}$. Then if n is a positive odd integer, then we have

$$\begin{aligned} \varphi(e_n) &= \varphi_{\mathrm{o}}(e_{(n+1)/2}) \\ &= \varphi_{\mathrm{o}}(\psi_{\mathrm{o}}(e_n)) \\ &= (\varphi_{\mathrm{o}}\psi_{\mathrm{o}} + \varphi_{\mathrm{e}}\psi_{\mathrm{e}})(e_n), \end{aligned}$$

and if n is a positive even integer, then we have

$$\begin{aligned} \varphi(e_n) &= \varphi_{\mathrm{e}}(e_{n/2}) \\ &= \varphi_{\mathrm{e}}(\psi_e(e_n)) \\ &= (\varphi_{\mathrm{o}}\psi_{\mathrm{o}} + \varphi_{\mathrm{e}}\psi_{\mathrm{e}})(e_n). \end{aligned}$$

Thus $\varphi = \varphi_{\mathrm{o}}\psi_{\mathrm{o}} + \varphi_{\mathrm{e}}\psi_{\mathrm{e}}$ since they agree on the basis $\{e_n\}$. □

Problem 4

Lemma 0.1. *Let E an R-module. The following statements are equivalent;*

1. *E is an injective R-module;*
2. *Every short exact sequence of the form*

$$0 \longrightarrow E \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{8}$$

splits.

3. *If E is a submodule of an R-module M, then E is a direct summand of M.*

Proof. (2 $\implies$ 1) Assume that any short exact sequence of the form (8) splits. This means, equivalently, that any injective R-linear map out of E splits. Let $\varphi\colon M \to N$ be an injective R-linear map and let $\psi\colon M \to E$ be any R-linear map. We need to construct a map $\widetilde{\psi}\colon N \to E$ such that $\widetilde{\psi}\varphi = \psi$. To do this, consider the pushout module

$$E +_M N = (E \times N)/\{(\psi(u), -\varphi(u)) \mid u \in M\}$$

together its natural maps $\iota_1\colon E \to E +_M N$ and $\iota_2\colon N \to E +_M N$, given by

$$\iota_1(v) = [v, 0] \qquad \text{and} \qquad \iota_2(w) = [0, w]$$

for all $v \in E$ and $w \in N$ where $[v, w]$ denotes the equivalence class in $E +_M N$ with (v, w) as one of its representatives. Observe that

$$\begin{aligned} \iota_1(\psi(u)) &= [\psi(u), 0] \\ &= [0, \varphi(u)] \\ &= \iota_2(\varphi(u)) \end{aligned}$$

for all $u \in M$. Therefore, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\psi"'] & N \arrow[d, "\iota_2"] \\ E \arrow[r, "\iota_1"'] & E +_M N \end{tikzcd}$$

We claim that ι_1 is injective. Indeed, suppose $v \in \ker \iota_1$. Then $[v, 0] = [0, 0]$ implies if $(v, 0) = (\psi(u), -\varphi(u))$ for some $u \in M$. Then $\varphi(u) = 0$ implies $u = 0$ since φ is injective, and therefore

$$\begin{aligned} v &= \psi(u) \\ &= \psi(0) \\ &= 0. \end{aligned}$$

Thus ι_1 is injective. Therefore by hypothesis the map $\iota_1\colon E \to E +_M N$ splits, say by $\lambda\colon E +_M N \to E$, where $\lambda\iota_1 = 1_E$. Finally, we obtain a map $\widetilde{\psi}\colon N \to E$ by setting $\widetilde{\psi} := \lambda\iota_2$. Then

$$\begin{aligned} \widetilde{\psi}\varphi &= \lambda\iota_2\varphi \\ &= \lambda\iota_1\psi \\ &= \psi, \end{aligned}$$

shows that $\widetilde{\psi}$ has the desired property.

(1 $\implies$ 2) Assume that E is an injective R-module. Let $\varphi\colon E \to M$ be an injective homomorphism. Since E is an injective R-module and since $1_E\colon E \to E$ is an injective R-module homomorphism, there exists an R-linear map $\widetilde{\varphi}\colon M \to E$ such that $\widetilde{\varphi} \circ \varphi = 1_E$. That is, $\widetilde{\varphi}$ splits $\varphi\colon E \to M$.

(2 $\implies$ 3) Assume that any short exact sequence of the form (8) splits. Let M be an R-module such that $E \subseteq M$. Then the short exact sequence

$$0 \longrightarrow E \xrightarrow{\iota} M \xrightarrow{\pi} M/E \longrightarrow 0$$

splits, where $\iota\colon E \to M$ denotes the inclusion map and $\pi\colon M \to M/E$ denotes the quotient map. Therefore we may choose a $\widetilde{\pi}\colon M/E \to M$ such that $\pi\widetilde{\pi} = 1_{M/E}$. We claim that

$$M = E \oplus \widetilde{\pi}(M/E).$$

Indeed, they are both submodules of M. Furthermore, observe that we have $E \cap \widetilde{\pi}(M/E) = \{0\}$. Indeed, suppose $u \in E \cap \widetilde{\pi}(M/E)$. Then $u \in E$ implies $\pi(u) = 0$. Also $u \in \widetilde{\pi}(M/E)$ implies $u = \widetilde{\pi}(\overline{v})$ for some $\overline{v} \in M/E$. Therefore

$$\begin{aligned} 0 &= \widetilde{\pi}(0) \\ &= \widetilde{\pi}\pi(u) \\ &= \widetilde{\pi}\pi\widetilde{\pi}(\overline{v}) \\ &= \widetilde{\pi}(\overline{v}) \\ &= u. \end{aligned}$$

Finally, note that if $u \in M$, then we can write

$$u = u - \widetilde{\pi}\pi(u) + \widetilde{\pi}\pi(u),$$

where $\widetilde{\pi}\pi(u) \in \widetilde{\pi}(M/E)$ and where $u - \widetilde{\pi}\pi(u) \in E$ since

$$\begin{aligned} \pi(u - \widetilde{\pi}\pi(u)) &= \pi(u) - \pi\widetilde{\pi}\pi(u) \\ &= \pi(u) - \pi(u) \\ &= 0 \end{aligned}$$

implies $u - \widetilde{\pi}\pi(u) \in \ker \pi = E$. This implies $M = E + \widetilde{\pi}(M/E)$.

($3 \implies 2$) Assume that E satisfies the property that if E is a submodule of an R-module M, then it must be a direct summand of M. We show that any short exact sequence of the form (8) splits by showing that any injective R-linear map out of E splits.

Step 1: Before we show that any injective R-linear map out of E splits, we need to show that if $\varphi\colon E \to F$ is an isomorphism of R-modules, then F satisfies the same property as E; namely if N is an R-module such that $F \subseteq N$, then F is a direct summand of N. Let $\varphi\colon E \to F$ be an isomorphism, let $\psi\colon F \to E$ denote its inverse, and let N be an R-module such that $F \subseteq N$. We define an R-module $\psi(N)$, where as a set we have

$$\psi(N) = E \cup \{\psi(v) \mid v \in N \backslash F\},$$

where $\psi(v)$ is understood to be a formal symbol if $v \in N\backslash F$ and is understood to be an element in E if $v \in F$. Here, E is *literally* a subset of $\psi(N)$. We extend the R-linear structure on E to an R-linear structure on $\psi(N)$ by defining addition and scalar multiplication by

$$\psi(v_1) + \psi(v_2) = \psi(v_1 + v_2) \quad \text{and} \quad a\psi(v) = \psi(av).$$

for all $v, v_1, v_2 \in N\backslash F$ and $a \in R$. Defining the R-linear structure on $\psi(N)$ in this way makes it so that $\psi\colon F \to E$ and $\varphi\colon E \to F$ extends to an isomorphism $\psi\colon N \to \psi(N)$ with corresponding inverse $\varphi\colon \psi(N) \to N$.

With this construction in place, we see that E is *literally* a submodule of $\psi(N)$. Therefore $\psi(N)$ is an internal direct sum, say

$$\psi(N) = E \oplus K,$$

where K is another submodule of $\psi(N)$ such that $E \cap K = \{0\}$ and $E + K = \psi(N)$. Then since $\varphi\colon \psi(N) \to N$ is an isomorphism, we see that

$$\begin{aligned} N &= \varphi(E) \oplus \varphi(K) \\ &= F \oplus \varphi(K). \end{aligned}$$

Step 2: Now we will show that any injective R-linear map out of E splits. Let $\varphi\colon E \to M$ be any injective R-linear map. We claim that $\varphi\colon E \to M$ splits if and only if $\iota\colon \varphi(E) \to M$ splits, where ι denotes the inclusion map. Indeed, denote $\varphi^{-1}\colon E \to \varphi(E)$ to be the inverse of $\varphi\colon E \to \varphi(E)$. If $\varphi\colon E \to M$ splits, then there exists an R-linear map $\widetilde{\varphi}\colon M \to E$ such that $\widetilde{\varphi}\varphi = 1_E$. Then $\varphi\widetilde{\varphi}\colon M \to \varphi(E)$ splits $\iota\colon \varphi(E) \to M$ since

$$\begin{aligned} (\varphi\widetilde{\varphi}\iota)(\varphi(u)) &= \varphi\widetilde{\varphi}(\varphi(u)) \\ &= \varphi(\widetilde{\varphi}\varphi(u)) \\ &= \varphi(u) \end{aligned}$$

for all $\varphi(u) \in \varphi(E)$. Similarly, if $\iota\colon \varphi(E) \to M$ splits, then there exists an R-linear map $\widetilde{\iota}\colon M \to \varphi(E)$ such that $\widetilde{\iota}\iota = 1_{\varphi(E)}$. Then $\varphi^{-1}\widetilde{\iota}\colon M \to E$ splits $\varphi\colon E \to M$ since

$$\begin{aligned} (\varphi^{-1}\widetilde{\iota}\varphi)(u) &= (\varphi^{-1}\widetilde{\iota})(\varphi(u)) \\ &= (\varphi^{-1}\widetilde{\iota})(\iota\varphi(u)) \\ &= (\varphi^{-1}\widetilde{\iota}\iota)(\varphi(u)) \\ &= (\varphi^{-1})(\varphi(u)) \\ &= u \end{aligned}$$

for all $u \in E$.

Thus, to show that $\varphi\colon E \to M$ splits, it suffices to show that $\iota\colon \varphi(E) \to M$ splits. In this case, $\varphi(E)$ is a submodule of M, and by step 1, we see that M is an internal direct sum, say

$$M = \varphi(E) \oplus K$$

for some R-module $K \subseteq M$. The projection map $\pi_1\colon M \to \varphi(E)$ is easily seen to split the inclusion map $\iota\colon \varphi(E) \to M$. $\square$