Linear Analysis Homework 7

Michael Nelson

Throughout this homework, let $\mathcal{H}$ be a Hilbert space.

Problem 1

Problem 1.a

Proposition 0.1. *Let (x_n) and (y_n) be two sequences in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ and let $\alpha, \beta \in \mathbb{C}$. Then*

$$\alpha x_n + \beta y_n \xrightarrow{w} \alpha x + \beta y.$$

Proof. Let $z \in \mathcal{H}$. Then

$$\begin{aligned}
\lim_{n\to\infty}(\langle \alpha x_n + \beta y_n, z\rangle) &= \lim_{n\to\infty}(\alpha\langle x_n, z\rangle + \beta\langle y_n, z\rangle) \\
&= \alpha \lim_{n\to\infty}(\langle x_n, z\rangle) + \beta \lim_{n\to\infty}(\langle y_n, z\rangle) \\
&= \alpha\langle x, z\rangle + \beta\langle y, z\rangle \\
&= \langle \alpha x + \beta y, z\rangle.
\end{aligned}$$

Therefore $\alpha x_n + \beta y_n \xrightarrow{w} \alpha x + \beta y$. □

Problem 1.b

Proposition 0.2. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator and let (x_n) be a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$. Then*

$$Tx_n \xrightarrow{w} Tx.$$

Proof. Let $z \in \mathcal{H}$. Then

$$\begin{aligned}
\lim_{n\to\infty}(\langle Tx_n, z\rangle) &= \lim_{n\to\infty}(\langle x_n, T^*z\rangle) \\
&= \langle x, T^*z\rangle \\
&= \langle Tx, z\rangle.
\end{aligned}$$

Therefore $Tx_n \xrightarrow{w} Tx$. □

Remark. Note that we may not have $Tx_n \to Tx$. Indeed, suppose $\mathcal{H}$ is separable. Let (e_n) be an orthonormal sequence in $\mathcal{H}$ and let $T\colon \mathcal{H} \to \mathcal{H}$ be the identity map. Then $e_n \xrightarrow{w} 0$ but $Te_n = e_n \not\to 0$. In fact (Te_n) doesn't even converge.

Problem 2

Problem 2.a

Proposition 0.3. *Let $\mathcal{Y}$ be a dense subset of $\mathcal{H}$. Let (x_n) be a bounded sequence of elements in $\mathcal{H}$ and suppose $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{Y}$. Then $x_n \xrightarrow{w} x$.*

Proof. Let $z \in \mathcal{H}$. Choose $M \geq 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Choose $y \in \mathcal{Y}$ such that

$$\|z - y\| < \frac{\varepsilon}{3\max\{\|x\|, M\}}$$

(we can do this since $\mathcal{Y}$ is dense in $\mathcal{H}$). Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\langle x_n, y\rangle - \langle x, y\rangle| < \frac{\varepsilon}{3}.$$

Then $n \geq N$ implies

$$\begin{aligned}
|\langle x_n, z\rangle - \langle x, z\rangle| &= |\langle x_n, z\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle + \langle x, y\rangle - \langle x, z\rangle| \\
&\leq |\langle x_n, z-y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| + |\langle x, y-z\rangle| \\
&\leq M\|z-y\| + |\langle x_n, y\rangle - \langle x, y\rangle| + \|x\|\|y-z\| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Therefore $x_n \xrightarrow{w} x$. □

Problem 2.b

Lemma 0.1. *Let $\mathcal{H}$ be a separable Hilbert space, let (e_m) be an othonormal basis in $\mathcal{H}$, let (x_n) be a bounded sequence in $\mathcal{H}$, and let $x \in \mathcal{H}$. Then $x_n \xrightarrow{w} x$ if and only if $\langle x_n, e_m\rangle \to \langle x, e_m\rangle$ for all $m \in \mathbb{N}$.*

Proof. Suppose $x_n \xrightarrow{w} x$. Then $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{H}$, so certainly $\langle x_n, e_m\rangle \to \langle x, e_m\rangle$ for all $m \in \mathbb{N}$. Conversely, suppose $\langle x_n, e_m\rangle \to \langle x, e_m\rangle$ for all $m \in \mathbb{N}$. We first show that $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{Y}$, where $\mathcal{Y} = \operatorname{span}\{e_m \mid m \in \mathbb{N}\}$, so let $y \in \mathcal{Y}$. Then

$$y = \lambda_1 e_{m_1} + \cdots + \lambda_r e_{m_r}$$

for some (unique) $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ and $m_1, \ldots, m_r \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \langle x_n, y\rangle &= \lim_{n\to\infty} \langle x_n, \lambda_1 e_{m_1} + \cdots + \lambda_r e_{m_r}\rangle \\
&= \overline{\lambda}_1 \lim_{n\to\infty} \langle x_n, e_{m_1}\rangle + \cdots + \overline{\lambda}_r \lim_{n\to\infty} \langle x_n, e_{m_r}\rangle \\
&= \overline{\lambda}_1 \langle x, e_{m_1}\rangle + \cdots + \overline{\lambda}_r \langle x, e_{m_r}\rangle \\
&= \langle x, \lambda_1 e_{m_1} + \cdots + \lambda_r e_{m_r}\rangle \\
&= \langle x, y\rangle.
\end{aligned}$$

Therefore $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{Y}$. Now since $\overline{\mathcal{Y}} = \mathcal{H}$, we may use Proposition (0.3) to conclude that $\langle x_n, z\rangle \to \langle x, z\rangle$ for all $z \in \mathcal{H}$. In other words, we have $x_n \xrightarrow{w} x$. □

Corollary. *Let (x^n) be a sequence in $\ell^2(\mathbb{N})$ that converges coordinate-wise to $x = (x_m) \in \ell^2(\mathbb{N})$. Then $x^n \xrightarrow{w} x$.*

Proof. Saying (x^n) converges coordinate-wise to $x = (x_m)$ is equivalent to saying

$$\begin{aligned}
\lim_{n\to\infty} (\langle x^n, e^m\rangle) &= x_m \\
&= \langle x, e^m\rangle
\end{aligned}$$

for all $m \in \mathbb{N}$. Thus Lemma (0.1) implies $x^n \xrightarrow{w} x$. □

Problem 3

Proposition 0.4. *Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. If (x_n) is a sequence of elements in $\mathcal{K}$ and $x_n \xrightarrow{w} x$, then $x \in \mathcal{K}$.*

Proof. Let $y \in \mathcal{K}^\perp$. Then

$$\begin{aligned}
0 &= \lim_{n\to\infty} (0) \\
&= \lim_{n\to\infty} (\langle x_n, y\rangle) \\
&= \langle x, y\rangle.
\end{aligned}$$

This implies $x \in (\mathcal{K}^\perp)^\perp = \mathcal{K}$. □

Remark. The same proof shows that if A is a subset of $\mathcal{H}$ and (x_n) is a sequence of elements in A and $x_n \xrightarrow{w} x$, then $x \in \overline{\operatorname{span}}(A)$.

Problem 4

Proposition 0.5. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator and let $S\colon \mathcal{H} \to \mathcal{H}$ be a compact operator. Then $TS\colon \mathcal{H} \to \mathcal{H}$ and $ST\colon \mathcal{H} \to \mathcal{H}$ are both compact operators.*

Proof. We first show ST is compact. Let (x_n) be a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$. Since T is a bounded operator, we have $Tx_n \xrightarrow{w} Tx$ by Proposition (0.2). Since S is compact, we have $S(Tx_n) \to S(Tx)$. Thus, ST is compact. Now we show TS is compact. Let (x_n) be a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$. Since S is compact, we have $Sx_n \to Sx$. Since T is bounded (and in particular continuous) we have $T(Sx_n) \to T(Sx)$. Thus, TS is compact. □

Problem 5

Lemma 0.2. *Let (x_n) be a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$ and let $(x_{\pi(n)})$ be a subsequence of (x_n). Then $x_{\pi(n)} \xrightarrow{w} x$.*

Remark. Here we view π as a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$ whose range consists of the indices in the subsequence.

Proof. Let $y \in \mathcal{H}$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\langle x_n, y\rangle - \langle x, y\rangle| < \varepsilon.$$

Then $\pi(n) \geq N$ implies

$$|\langle x_{\pi(n)}, y\rangle - \langle x, y\rangle| < \varepsilon$$

□

Proposition 0.6. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator with the property that for any bounded sequence (x_n) in $\mathcal{H}$, the sequence (Tx_n) has a convergent subsequence. Then T is compact.*

Proof. Let (x_n) be a sequence in $\mathcal{H}$ such that $x_n \xrightarrow{w} x$. Assume (for a contradiction) that $Tx_n \not\to Tx$. Choose $\varepsilon > 0$ and choose a subsequence $(Tx_{\pi(n)})$ of (Tx_n) such that $\|Tx_{\pi(n)} - Tx\| > \varepsilon$ for all $n \in \mathbb{N}$. By the (baby version) of the uniform boundedness principle, the sequence (x_n) is bounded, and hence the subsequence $(x_{\pi(n)})$ must be bounded too. Thus the sequence $(Tx_{\pi(n)})$ has a convergent subsequence (by the hypothesis on T). Choose a convergent subsequence of $(Tx_{\pi(n)})$, say $(Tx_{\rho(n)})$. Since $(x_{\rho(n)})$ is a subsequence of (x_n), we must have $x_{\rho(n)} \xrightarrow{w} x$ by Lemma (0.2), and since T is a bounded operator, we must have $Tx_{\rho(n)} \xrightarrow{w} Tx$ by Proposition (0.2). Since $(Tx_{\rho(n)})$ is a convergent sequence and since $Tx_{\rho(n)} \xrightarrow{w} Tx$, we must in fact have $Tx_{\rho(n)} \to Tx$. But since $(Tx_{\rho(n)})$ is a subsequence of $(Tx_{\pi(n)})$, we have $\|Tx_{\rho(n)} - Tx\| > \varepsilon$ for all $n \in \mathbb{N}$, and so $Tx_{\rho(n)} \not\to Tx$. This is a contradiction. □

Problem 6

Let $\mathcal{K}$ be a Hilbert space.

Problem 6.a

Proposition 0.7. *If every bounded sequence in $\mathcal{K}$ has a convergent subsequence, then $\mathcal{K}$ must be finite-dimensional.*

Proof. We prove the contrapositive statement: if $\mathcal{K}$ is infinite-dimensional, then there exists a bounded sequence which has no convergent subsequence. Assume $\mathcal{K}$ is infinite-dimensional. Let (e_n) be an orthonormal sequence in $\mathcal{K}$. Then the sequence (e_n) cannot have a convergent subsequence. Indeed, the distance squared between any two elements e_n, e_m (with $m \neq n$) in the sequence is

$$\begin{aligned}\|e_n - e_m\|^2 &= \|e_n\|^2 + \|e_m\|^2 \\ &= 1 + 1 \\ &= 2,\end{aligned}$$

by the Pythagorean Theorem. Thus any subsequence of (e_n) will fail to be Cauchy (and in particular will fail to converge). □

Problem 6.b

Proposition 0.8. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be compact. Then $\ker(1_{\mathcal{H}} - T)$ is finite-dimensional.*

Proof. Let (x_n) be a bounded sequence in $\ker(1_{\mathcal{H}} - T)$. Choose a subsequence $(x_{\pi(n)})$ of (x_n) such that $x_{\pi(n)} \xrightarrow{w} x$ for some $x \in \mathcal{H}$ (such a subsequence exists by a theorem proved in class). Since T is compact, we have

$$\begin{aligned} x_{\pi(n)} &= Tx_{\pi(n)} \\ &\to Tx \\ &= x. \end{aligned}$$

Thus $(x_{\pi(n)})$ is a convergent subsequence of (x_n). It follows from Proposition (0.7) that $\ker(1_{\mathcal{H}} - T)$ is finite-dimensional. □

Problem 6.c

Proposition 0.9. *Let $S\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator whose image $\mathrm{im}(S)$ is closed and infinite-dimensional. Then S cannot be compact.*

Proof. Let (y_n) be a bounded sequence in $\mathrm{im}(S)$ and let $\mathcal{K} := \ker(S)$. Choose $N > 0$ such that $\|y_n\| \leq N$ for all $n \in \mathbb{N}$. Choose a sequence (x_n) in $\mathcal{H}$ such that $Sx_n = y_n$ for all $n \in \mathbb{N}$. By replacing x_n with $x_n - \mathrm{P}_{\mathcal{K}} x_n$ if necessary, we may assume that $\langle x_n, z\rangle = 0$ for all $n \in \mathbb{N}$ and for all $z \in \mathcal{K}$. We claim that for each $z \in \mathcal{H}$, the sequence $(|\langle x_n, z\rangle|)$ is bounded. Indeed, since $\mathrm{im}(S)$ is closed, we have $\mathcal{H} = \ker(S) \oplus \mathrm{im}(S^*)$. We've already showing $|\langle x_n, z\rangle|$ is bounded for all $z \in \ker(S)$. Thus it suffices to check that $\langle x_n, S^*x\rangle$ is bounded for all $S^*x \in \mathrm{im}(S^*)$. Let $S^*x \in \mathrm{im}(S^*)$. Then

$$\begin{aligned} |\langle x_n, S^*x\rangle| &= |\langle Sx_n, x\rangle| \\ &= |\langle y_n, x\rangle| \\ &\leq \|y_n\|\|x\| \\ &\leq N\|x\|. \end{aligned}$$

Thus for each $z \in \mathcal{H}$, the sequence $(|\langle x_n, z\rangle|)$ is bounded. It follows from the Uniform Boundedness Principle (stated in Theorem (0.5) in the Appendix) that the sequence (x_n) is bounded. Since S is compact and (x_n) is bounded, the sequence $(Sx_n) = (y_n)$ must have a convergent subsequence. This contradicts the fact that $\mathcal{K}$ is infinite-dimensional. □

Proof using Open Mapping Theorem:

Proof. Assume (for a contradiction) that S is compact. Denote $\mathcal{K} := \mathrm{im}(S)$ and let (y_n) be a bounded sequence in $\mathcal{K}$. Let $M > 0$ and choose $N > 0$ such that if $y \in \mathcal{K}$ and $\|y\| < N$, then there exists an $x \in \mathcal{H}$ such that $Sx = y$ and $\|x\| < M$ (this follows from the Open Mapping Theorem). By scaling the sequence (y_n) if necessary, we may assume that $\|y_n\| < N$ for all $n \in \mathbb{N}$. Thus there exists $x_n \in \mathcal{H}$ such that $Sx_n = y$ and $\|x_n\| < M$ for all $n \in \mathbb{N}$. Thus (x_n) is a bounded sequence in $\mathcal{H}$. Since S is compact and (x_n) is bounded, the sequence $(Sx_n) = (y_n)$ must have a convergent subsequence. This contradicts the fact that $\mathcal{K}$ is infinite-dimensional. □

Appendix

Open Mapping Theorem

Theorem 0.3. *(Open Mapping Theorem) Let $T\colon \mathcal{H}_1 \twoheadrightarrow \mathcal{H}_2$ be a surjective bounded linear operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Then T is an open map.*

Let's describe in more detail what it means for T to be an open map. Let $x \in \mathcal{H}_1$ and denote $y := Tx$. Then for all ε-neighborhoods $B_\varepsilon(x)$ of x there exists a δ-neighborhood $B_\delta(y)$ of y such that

$$B_\delta(y) \subseteq T(B_\varepsilon(x)).$$

In other words, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $y' \in \mathcal{H}_2$ and $\|y' - y\| < \delta$, then there exists $x' \in \mathcal{H}_1$ such that $Tx' = y'$ and $\|x' - x\| < \varepsilon$.

Uniform Boundedness Principle

Lemma 0.4. *Let (x_n) be a sequence in $\mathcal{H}$. Assume there exists an $M > 0$ and a closed ball $\overline{B}_r(a) \subseteq \mathcal{H}$ such that*

$$|\langle x_n, y\rangle| \leq M$$

for all $n \in \mathbb{N}$ and for all $y \in \overline{B}_r(a)$. Then the sequence (x_n) is bounded.

Proof. The key is to translate everything from the closed ball $\overline{B}_r(a)$ to the closed ball $\overline{B}_1(0)$. Let $z \in \overline{B}_1(0)$. Then

$$\begin{aligned}
|\langle x_n, z\rangle| &= \frac{1}{r}|\langle x_n, rz\rangle| \\
&= \frac{1}{r}|\langle x_n, rz + a - a\rangle| \\
&= \frac{1}{r}|\langle x_n, rz + a\rangle - \langle x_n, a\rangle| \\
&\leq \frac{1}{r}|\langle x_n, rz + a\rangle| + \frac{1}{r}|\langle x_n, a\rangle \\
&\leq \frac{1}{r}M + \frac{1}{r}M \\
&= \frac{2M}{r}
\end{aligned}$$

for all $n \in \mathbb{N}$. In particular, fixing x_n and setting $z = x_n/\|x_n\|$, we have $\|x_n\| \leq 2M/r$. Since x_n was arbitrary, we have $\|x_n\| \leq 2M/r$ for all $n \in \mathbb{N}$. □

Theorem 0.5. *(Uniform Boundedness Principle) Let $\mathcal{H}$ be a Hilbert space and let (x_n) be a sequence in $\mathcal{H}$. Assume that for every $y \in \mathcal{H}$ there exists an $M_y > 0$ such that*

$$|\langle x_n, y\rangle| \leq M_y \tag{1}$$

for all $n \in \mathbb{N}$. Then the sequence (x_n) is bounded.

Proof. We claim that there exists $M > 0$ and a closed ball $\overline{B}_r(a) \subseteq \mathcal{H}$ such that

$$|\langle x_n, y\rangle| \leq M$$

for all $n \in \mathbb{N}$ and for all $y \in \overline{B}_r(a)$. This will imply (x_n) is bounded by Lemma (0.4).

For each $m \in \mathbb{N}$, define

$$E_m := \{y \in \mathcal{H} \mid |\langle x_n, y\rangle| \leq m \text{ for all } n \in \mathbb{N}\}.$$

It's easy to see that (E_m) is an increasing sequence of closed sets (view it as an infinite intersection of closed sets). Moreover, for any $y \in \mathcal{H}$, (1) implies $y \in \bigcup_{m=1}^{\infty} E_m$. Thus

$$\mathcal{H} = \bigcup_{m=1}^{\infty} E_m.$$

Now to prove the claim, it suffices to show that one of the E_m's contains an open ball of the form $B_r(a)$ (the closure $\overline{B}_r(a)$ will then also belong to E_m). Let $B_{r_1}(a_1)$ be any open ball. If $B_{r_1}(a_1) \subseteq E_1$, then we are done. So assume $B_{r_1}(a_1) \cap E_1^c \neq \emptyset$. Choose $a_2 \in \mathcal{H}$ and $r_2 > 0$ such that $r_2 < r_1/2$ and $\overline{B}_{r_2}(a_2) \subseteq B_{r_1}(a_1) \cap E_1^c$ (we can find an open ball $B_{r_2}(a_2)$ which is contained in $B_{r_1}(a_1) \cap E_1^c$ since the intersection is nonempty, and by replacing r_2 by a smaller positive number if necessary, we may assume that $\overline{B}_{r_2}(a_2)$ is contained in $B_{r_1}(a_1) \cap E_1^c$). Continuing this process, we will either stop after finitely many iterations (and we'd be done!) or we can construct a sequence (a_n) in $\mathcal{H}$ such that $r_{n+1} < r_n/2 < \cdots < r_1/2^n$ and $\overline{B}_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(a_n) \cap E_n^c$ for all $n \in \mathbb{N}$.

Assume (for a contradiction) that such a sequence has been constructed. We claim that (a_n) is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Since $r_n \to 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $2r_n < \varepsilon$. Then $n, m \geq N$ implies

$$\begin{aligned}
\|a_n - a_m\| &\leq \|a_n - a_N\| + \|a_N - a_m\| \\
&< r_N + r_N \\
&= 2r_N \\
&< \varepsilon.
\end{aligned}$$

since $a_n, a_m \in \overline{B}_{r_N}(a_N)$ for all $n, m \geq N$. Thus (a_n) is a Cauchy sequence and hence converges (since we are in a Hilbert space) say to $a \in \mathcal{H}$. Now observe that for any $m \in \mathbb{N}$, we have $a \in \overline{B}_{r_m}(a_m) \subseteq E^c_{m-1}$. In particular

$$
\begin{aligned}
a &= \bigcap_{m=1}^{\infty} E_m^c \\
&= \bigcap_{m=1}^{\infty} (\mathcal{H} \backslash E_m) \\
&= \mathcal{H} \backslash \left(\bigcup_{m=1}^{\infty} E_m \right) \\
&= \mathcal{H} \backslash \mathcal{H} \\
&= \emptyset,
\end{aligned}
$$

which is a contradiction. $\square$