Holomorphic Functions

October 22, 2019

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1 Definition of Holomorphic Function

Let Ω be an open subset of $\mathbb{C}$ and let f be a complex-valued function defined on Ω. The function f is said to be **holomorphic at the point** $z \in \Omega$ if the quotient

$$\frac{f(z+h)-f(z)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$ and $h \neq 0$ with $z+h \in \Omega$ so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by $f'(z)$, and is called the **derivative of** f **at** z:

$$f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

The function f is said to be **holomorphic on** Ω if it is holomorphic at each point in Ω. If C is a closed subset of $\mathbb{C}$, we say that f is **holomorphic on** C if f is holomorphic in some open set containing C. Note that if f is holomorphic at a point $z \in \mathbb{C}$, then it is not necessarily holomorphic on $\{z\}$. If f is holomorphic on all of $\mathbb{C}$ we say that f is **entire**.

1.1 Examples of Holomorphic Functions

Example 1.1.

1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = z$ for all $z \in \mathbb{C}$. Then f is entire. Indeed, let $z \in \mathbb{C}$. Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{z+h-z}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1.\end{aligned}$$

2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \overline{z}$ for all $z \in \mathbb{C}$. Then f is continuous everywhere in $\mathbb{C}$ but is not holomorphic at any point in $\mathbb{C}$. To see this, let $z \in \mathbb{C}$. Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \frac{\overline{z+h}-\overline{z}}{h} \\ &= \lim_{h\to 0} \frac{\overline{z}+\overline{h}-\overline{z}}{h} \\ &= \lim_{h\to 0} \frac{\overline{h}}{h}.\end{aligned}$$

   But this limit doesn't exist. Indeed, assume it did exist (to obtain a contradiction). Then setting $h = \varepsilon$, where $\varepsilon \in \mathbb{R}$, and taking $\varepsilon \to 0$, we see that

$$\begin{aligned}\lim_{h\to 0} \frac{\overline{h}}{h} &= \lim_{\varepsilon\to 0} \frac{\overline{\varepsilon}}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{\varepsilon}{\varepsilon} \\ &= 1.\end{aligned}$$

   On the other hand, setting $h = i\varepsilon$, where $\varepsilon \in \mathbb{R}$, and taking $\varepsilon \to 0$, we see that

$$\begin{aligned}\lim_{h\to 0} \frac{\overline{h}}{h} &= \lim_{i\varepsilon\to 0} \frac{\overline{i\varepsilon}}{i\varepsilon} \\ &= \lim_{i\varepsilon\to 0} \frac{-i\varepsilon}{i\varepsilon} \\ &= -1.\end{aligned}$$

   This is a contradiction. We conclude that the limit does not exist.

3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = |z|^2 = z\overline{z}$ for all $z \in \mathbb{C}$. Then f is holomorphic at the point 0 but nowhere else. Indeed,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{|h|^2}{h} \\
&= \lim_{h\to 0} \frac{h\overline{h}}{h} \\
&= \lim_{h\to 0} \overline{h} \\
&= 0,
\end{aligned}$$

implies that f is holomorphic at 0. Now assume (to obtain a contradiction) that f is holomorphic at some $w \neq 0$. Let $g\colon \mathbb{C} \to \mathbb{C}$ be the identity function, given by $g(z) = z$ for all $z \in \mathbb{C}$. Then since g is holomorphic at w and $g(w) \neq 0$, the quotient f/g must be holomorphic at w as well. But this is a contradiction since the quotient is the complex-conjugation function, given by $f(z)/g(z) = \overline{z}$ for all $z \in \mathbb{C}$, which we know is not holomorphic anywhere in $\mathbb{C}$. This example demonstrates that a function being holomorphic at a point does *not* imply it being holomorphic in some neighborhood of that point.

4. Let $f\colon \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be given by $f(z) = 1/z$ for all $z \in \mathbb{C}$. The f is holomorphic in its domain $\mathbb{C}\backslash\{0\}$. Indeed, let $z \in \mathbb{C}\backslash\{0\}$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{\frac{1}{z+h} - \frac{1}{z}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h}{(z+h)z}}{h} \\
&= \lim_{h\to 0} \frac{-1}{(z+h)z} \\
&= \frac{-1}{z^2}.
\end{aligned}$$

1.2 Holomorphic Functions form a $\mathbb{C}$-Vector Space

Proposition 1.1. *Let f and g be complex-valued functions defined in a neighborhood of a point z in the complex plane. Then for all $a, b \in \mathbb{C}$, the function $af + bg$ is holomorphic at z. Moreover, we have*

$$(af + bg)'(z) = af'(z) + bg'(z).$$

Proof. This follows from linearity of the limit operator:

$$\begin{aligned}
(af + bg)'(z) &= \lim_{h\to 0} \frac{(af+bg)(z+h) - (af+bg)(z)}{h} \\
&= \lim_{h\to 0} \frac{af(z+h) + bg(z+h) - af(z) - bg(z)}{h} \\
&= a \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} + b \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} \\
&= af'(z) + bg'(z).
\end{aligned}$$

□

1.3 Chain Rule and Product Rule

Let Ω be an open subset of $\mathbb{C}$ and f a complex-valued function on Ω. Recall that f is continuous at $z_0 \in \Omega$ if and only if there exists a function $\psi\colon \Omega \to \mathbb{C}$ such that $\psi(z) \to 0$ as $z \to z_0$ and

$$f(z) = f(z_0) + \psi(z), \tag{1}$$

for all $z \in \Omega$. Indeed, we define $\psi\colon \Omega \to \mathbb{C}$ by the formula $\psi(z) = f(z) - f(z_0)$ for all $z \in \Omega$. Then continuity of f at z_0 implies $\psi(z) \to 0$ as $z \to z_0$, and conversely the expression (1) together with the fact that $\psi(z) \to 0$ as $z \to z_0$ implies f is continuous at z_0.

When f is holomorphic at $z_0 \in \Omega$, there is an even better approximation of f at z_0:

Lemma 1.1. *Let Ω be an open subset of $\mathbb{C}$ and f a complex-valued function on Ω. Then f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a (which is necessarily equal to $f'(z_0)$ as limits are unique) and a function $\psi\colon \Omega \to \mathbb{C}$ such that*

$$f(z) = f(z_0) + a(z - z_0) + \psi(z)(z - z_0), \tag{2}$$

where ψ is a function defined for all small h and $\lim_{h\to 0} \psi(h) = 0$.

Proof. Suppose f is holomorphic at z_0. Then we define $\psi\colon \Omega \to \mathbb{C}$ by the formula

$$\psi(z) = \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0}$$

for all $z \in \Omega$. Clearly ψ is well-defined in $\Omega \backslash \{z_0\}$. Moreover, since f is differentiable at z_0, we have $\psi(z) \to 0$ as $z \to z_0$.

Conversely, suppose $a \in \mathbb{C}$ and $\psi\colon \Omega \to \mathbb{C}$ are given and that (2) is satisfied. Then

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} (a + \psi(z)) = a$$

implies f is differentiable at z_0 with $f'(z_0) = a$. □

Let us now show how we can use Lemma (1.1) to prove the Chain Rule and Product Rule. First we prove the Chain Rule:

Proposition 1.2. *Let f be holomorphic at z_0 and g be holomorphic at $f(z_0)$. Then $g \circ f$ is holomorphic at z_0 and, moreover, the Chain Rule holds*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Proof. Since f is holomorphic at z_0, we can express f locally at z_0 as

$$f(z_0 + h) = f(z_0) + f'(z_0)h + \psi_1(h)h$$

where ψ_1 is a function defined for all small h and $\psi_1(h) \to 0$ as $h \to 0$. Since g is holomorphic at $f(z_0)$, we can express g locally at $f(z_0)$ as

$$g(f(z_0) + h) = g(f(z_0)) + g'(f(z_0))h + \psi_2(h)h$$

where ψ_2 is a function defined for all small h and $\psi_2(h) \to 0$ has $h \to 0$. Using these local expressions, we can now express $g \circ f$ locally at z_0:

$$\begin{aligned}
(g \circ f)(z_0 + h) &= g(f(z_0 + h)) \\
&= g(f(z_0) + f'(z_0)h + \psi_1(h)h) \\
&= g(f(z_0)) + g'(f(z_0))(f'(z_0)h + \psi_1(h)h) + \psi_2(h)(f'(z_0)h + \psi_1(h)h) \\
&= g(f(z_0)) + g'(f(z_0))f'(z_0)h + \psi_3(h)h
\end{aligned}$$

where $\psi_3(h) = g'(f(z_0))\psi_1(h)h + \psi_2(h)f'(z_0)h + \psi_1(h)\psi_2(h)h$. Since ψ_3 is a function defined for all small h and $\psi_3(h) \to 0$ has $h \to 0$, it follows from uniqueness of limits that

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

□

Corollary. *Let f be holomorphic at z_0. If $f(z_0) \neq 0$, then $1/f$ is holomorphic at z_0*

Proof. The function $1/f$ can be viewed as the composition of $g \circ f$, where g is given by $g(z) = 1/z$. Then $1/f$ is holomorphic at z_0 since f is holomorphic at z_0 and g is holomorphic at $f(z_0)$ (because $f(z_0) \neq 0$). □

Now we will prove the Product Rule:

Proposition 1.3. *Let f and g be holomorphic at z_0. Then fg is holomorphic at z_0 and, moreover, the Product Rule holds*

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Proof. Since f is holomorphic at z_0, we can express f locally at z_0 as

$$f(z_0+h) = f(z_0) + f'(z_0)h + \psi_1(h)h$$

where ψ_1 is a function defined for all small h and $\psi_1(h) \to 0$ as $h \to 0$. Since g is holomorphic at z_0, we can express f locally at z_0 as

$$g(z_0+h) = g(z_0) + g'(z_0)h + \psi_2(h)h$$

where ψ_2 is a function defined for all small h and $\psi_2(h) \to 0$ as $h \to 0$. Then

$$\begin{aligned}
(fg)(z_0+h) &= f(z_0+h)g(z_0+h) \\
&= (f(z_0) + f'(z_0)h + \psi_1(h)h)(g(z_0) + g'(z_0)h + \psi_2(h)h) \\
&= f(z_0)g(z_0) + (f(z_0)g'(z_0) + f'(z_0)g(z_0))h + \psi_3(h)h,
\end{aligned}$$

where $\psi_3(h) = f(z_0)\psi_2(h) + f'(z_0)g'(z_0)h + g(z_0)\psi_1(h)$. Since ψ_3 is a function defined for all small h and $\psi_3(h) \to 0$ as $h \to 0$, it follows from uniqueness of limits that

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$$

□

Since the function $f : \mathbb{C} \to \mathbb{C}$, given by $f(z) = z$, is holomorphic, it follows from Proposition (1.1), Proposition (**??**) and the fact that the function $f : \mathbb{C} \to \mathbb{C}$, given by $f(z) = z$, is entire, that polynomials are entire.

1.4 Analytic Functions are Holomorphic

Let Ω be an open set and let $f \colon \Omega \to \mathbb{C}$. We say f is **analytic** if at each $a \in \Omega$, there exists an open neighborhood U of a and a power series $\sum a_n(z-a)^n$ centered at a such that $U \subseteq \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in U$.

Proposition 1.4. *Let Ω be an open set and let $f \colon \Omega \to \mathbb{C}$ be analytic. Then f is holomorphic.*

Proof. Let $a \in \Omega$. Choose $r > 0$ and a power series $\sum a_n(z-a)^n$ centered at a such that $B_r(a) \subset \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

for all $z \in B_r(a)$. We claim that f is holomorphic in $B_r(a)$. Let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_r(a)$ and let $z \in B_r(a)$. Then for all $h \in B_\varepsilon(0)$, we have

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \sum_{n=0}^{\infty} a_n \left((z+h-a)^n - (z-a)^n \right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \sum_{m=1}^{n} a_m \left((z+h-a)^m - (z-a)^m \right) \\
&= \lim_{h\to 0} \lim_{n\to\infty} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \lim_{h\to 0} \sum_{m=1}^{n} a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} m a_m (z-a)^{m-1} \\
&= \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.
\end{aligned}$$

We need to justify why we were allowed to swap limits. Let $g_m \colon B_\varepsilon(0) \to \mathbb{C}$ be given by

$$g_m(h) = a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1}.$$

We need to show that the series $\sum g_m$ converges uniformly. In fact, this follows from an easy application of Weierstrass M-test. We first observe that

$$\begin{aligned} |g_m(h)| &= \left| a_m \sum_{k=1}^{m} (z-a+h)^{m-k}(z-a)^{k-1} \right| \\ &< \left| m a_m r^{m-1} \right|. \end{aligned}$$

Now we just set $M_m = |m a_m r^{m-1}|$ and apply Weierstrass M-test. □

One of the great triumphs of complex analysis is that the converse to Proposition (1.4) is also true, namely holomorphic functions are analytic.

1.5 Cauchy-Riemann Equations

Throughout this subsection, let f be a complex-valued function defined on some open subset Ω of $\mathbb{C}$ and fix a point $z_0 = x_0 + iy_0$ in Ω. Since $\mathbb{C}$ is a 2-dimensional $\mathbb{R}$-vector space, there is a unique decomposition of a complex number z as

$$z = x + iy,$$

where x and y are real numbers. Similarly, there is a unique decomposition of f as

$$f = u + iv,$$

where u and v are real-valued functions defined on Ω.

We define a map $\widetilde{\cdot} : \mathbb{C} \to \mathbb{R}^2$, given by mapping the complex number $z = x + iy$ to the vector $\widetilde{z} = (x, y)$. We also define $\widetilde{u} : \mathbb{R}^2 \to \mathbb{R}$ and $\widetilde{v} : \mathbb{R}^2 \to \mathbb{R}$ by the formulas $\widetilde{u}(x, y) = u(x + iy)$ and $\widetilde{v}(x, y) = v(x + iy)$ respectively. Similarly, we define $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula $\widetilde{f}(x, y) = (\widetilde{u}(x, y), \widetilde{v}(x, y))$.

We say $\widetilde{f}$ is **differentiable** at $\widetilde{z}_0$ if there exists a linear transformation $J\widetilde{f}(\widetilde{z}_0) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{\|\widetilde{f}(\widetilde{z}_0 + \widetilde{h}) - \widetilde{f}(\widetilde{z}_0) - J\widetilde{f}(\widetilde{z}_0)(\widetilde{h})\|}{\|\widetilde{h}\|} \to 0$$

as $\widetilde{h} \to 0$ where $\widetilde{h} = (h_1, h_2) \in \mathbb{R}^2$. Equivalently, we can write

$$\widetilde{f}(\widetilde{z}_0 + \widetilde{h}) = \widetilde{f}(\widetilde{z}_0) + J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) + \|\widetilde{h}\|\psi(\widetilde{h}),$$

where $\widetilde{\psi}(\widetilde{h}) \to 0$ as $\widetilde{h} \to 0$. The linear transformation $J\widetilde{f}(\widetilde{z}_0)$ is unique and is called the **derivative** of $\widetilde{f}$ at $\widetilde{z}_0$. If $\widetilde{f}$ is differentiable, then the partial derivatives of its component functions exist, and the linear transformation $J\widetilde{f}(\widetilde{z}_0)$ is described in the standard basis of $\mathbb{R}^2$ by the Jacobian matrix of $\widetilde{f}$:

$$J\widetilde{f}(\widetilde{z}_0) = \begin{pmatrix} \partial_x \widetilde{u}(\widetilde{z}_0) & \partial_y \widetilde{u}(\widetilde{z}_0) \\ \partial_x \widetilde{v}(\widetilde{z}_0) & \partial_y \widetilde{v}(\widetilde{z}_0) \end{pmatrix}.$$

In the case of complex-differentiation the complex-derivative is a complex number $f'(z_0)$, while in the case of real-derivatives, it is a matrix. There is, however, a connection between these two notions, which is given in terms of special relations that are satisfied by the entries of the Jacobian matrix, that is, the partials of u and v.

Theorem 1.2. *If f is holomorphic at z_0, then the partial derivatives $\partial_x \widetilde{u}$, $\partial_y \widetilde{u}$, $\partial_x \widetilde{v}$, and $\partial_y \widetilde{v}$ exist and satisfy the **Cauchy-Riemann Equations** at z_0:*

$$\begin{aligned} \partial_x \widetilde{u}(\widetilde{z}_0) &= \partial_y \widetilde{v}(\widetilde{z}_0) \\ -\partial_y \widetilde{v}(\widetilde{z}_0) &= \partial_x \widetilde{v}(\widetilde{z}_0). \end{aligned}$$

Moreover, $\widetilde{f}$ is real-differentiable and its Jacobian at the point $\widetilde{z}_0$ satisfies

$$\det J\widetilde{f}(\widetilde{z}_0) = |f'(z_0)|^2.$$

Proof. Let $\varepsilon > 0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\varepsilon \to 0} \frac{f(z_0 + \varepsilon) - f(z_0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{u((x_0 + \varepsilon) + iy_0) + iv((x_0 + \varepsilon) + iy_0) - u(x_0 + iy_0) - iv(x_0 + iy_0)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{u((x_0 + \varepsilon) + iy_0) - u(x_0 + iy_0)}{\varepsilon} + i \lim_{\varepsilon \to 0} \frac{v((x_0 + \varepsilon) + iy_0) - v(x_0 + iy_0)}{\varepsilon} \\ &= \partial_x \widetilde{u}(\widetilde{z}_0) + i\partial_x \widetilde{v}(\widetilde{z}_0). \end{aligned}$$

Similarly,

$$\begin{aligned} f'(z_0) &= \lim_{i\varepsilon\to 0} \frac{f(z_0+i\varepsilon)-f(z_0)}{i\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{u(x_0+i(y_0+\varepsilon))+iv(x_0+i(y_0+\varepsilon))-u(x_0+iy_0)-v(x_0+iy_0)}{i\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{v(x_0+i(y_0+\varepsilon))-v(x_0+iy_0)}{\varepsilon} - i\lim_{\varepsilon\to 0} \frac{u(x_0+i(y_0+\varepsilon))-u(x_0+iy_0)}{\varepsilon} \\ &= -i\partial_y\widetilde{u}(\widetilde{z}_0)+\partial_y\widetilde{v}(\widetilde{z}_0). \end{aligned}$$

Equating the two formulas for $f'(z_0)$ above yields the Cauchy-Riemann equations.

To see that $\widetilde{f}$ is differentiable, it suffices to observe that if $\widetilde{h}=(h_1,h_2)$ and $h=h_1+ih_2$, then the Cauchy-Riemann equations imply

$$J\widetilde{f}(\widetilde{z}_0)(\widetilde{h}) = \left(\partial_x\widetilde{u}-i\partial_y\widetilde{u}\right)(h_1+ih_2) =$$

complex-differentiability of f at z_0 implies

$$f(z_0+h)-f(z_0)=f'(z_0)(h)+h\psi(h),$$

note that

$$\widetilde{f}(\widetilde{z}_0+\widetilde{h})-\widetilde{f}(\widetilde{z}_0)=J\widetilde{f}(\widetilde{z}_0)(\widetilde{h})+\|\widetilde{h}\|\widetilde{\psi}(\widetilde{h}),$$
$$\widetilde{f}(\widetilde{z}_0+\widetilde{h})-\widetilde{f}(\widetilde{z}_0)=J\widetilde{f}(\widetilde{z}_0)(\widetilde{h})+\|\widetilde{h}\|\widetilde{\psi}(\widetilde{h}),$$

$$\det J\widetilde{f}(\widetilde{z}_0)=|f'(z_0)|^2.$$

□

Define the two differential operators

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) \text{ and } \partial_{\overline{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right).$$

Proposition 1.5. *If f is holomorphic at $z_0=x_0+iy_0$, then*

$$\partial_{\overline{z}}f(z_0)=0 \text{ and } f'(z_0)=\partial_z f(z_0)=2\partial_z u(z_0).$$

Also, if we consider f as a function in two real variables x,y, then f is real-differentiable $p=(x_0,y_0)$ and

$$\det J\widetilde{f}(p)=|f'(z_0)|^2.$$

Proof. First note that the Cauchy-Riemann equations at z_0 are equivalent to $\partial_{\overline{z}}f(z_0)=0$. Indeed,

$$\begin{aligned} 0 &= \partial_{\overline{z}}f(z_0) \\ &= \frac{1}{2}\left(\partial_x+i\partial_y\right)f(z_0) \\ &= \frac{1}{2}\left(\partial_x f(z_0)+i\partial_y f(z_0)\right) \\ &= \frac{1}{2}\left(\partial_x u(z_0)+i\partial_x v(z_0)+i\partial_y u(z_0)-\partial_y v(z_0)\right) \\ &= \frac{1}{2}\partial_x u(z_0)-\partial_y v(z_0)+\frac{i}{2}\left(\partial_y u(z_0)+\partial_x v(z_0)\right), \end{aligned}$$

and equating the real and imaginary parts gives us the Cauchy-Riemann equations.

Moreover, we have

$$\begin{aligned} f'(z_0) &= \frac{1}{2}\left(\partial_x f(z_0)-i\partial_y f(z_0)\right) \\ &= \partial_x f(z_0), \end{aligned}$$

and the Cauchy-Riemann equations give $\partial_z f = 2\partial_z u$.

To prove that f is real-differentiable, it suffices to observe that if $H=(h_1,h_2)$ and $h=h_1+ih_2$, then the Cauchy-Riemann equations imply

$$Jf(p)(h)=(\partial_x u-i\partial_y u)(h_1+ih_2)=f'(z_0)h$$

□

We can clarify the situation further by defining two differential operators

$$\partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right) \quad \text{and} \quad \partial_{\bar{z}} := \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

Example 1.2. Let $f(z) = \bar{z} = x - iy$. Then

$$\partial_x u = 1 \neq -1 = \partial_y v,$$

so f is not differentiable.

Corollary.

1. *If f is holomorphic on $D_r(z_0)$ with $f'(z) = 0$ for all $z \in D_r(z_0)$, then f is constant on $D_r(z_0)$.*
2. *If f is holomorphic on $D_r(z_0)$ and real-valued on $D_r(z_0)$, then f is constant.*

Proof.

1. Since $f'(z) = 0$, we have $\partial_x u + i\partial_x v = 0 = \partial_y v - i\partial_y u$. This implies $\partial_x u = \partial_y u = \partial_x v = \partial_y v = 0$. Therefore u and v are constant functions, and hence, f is constant.
2. Let $f = u + iv$. Then $v = 0$ on $D_r(z_0)$. So $\partial_x v = \partial_y v = 0$ implies $\partial_x u = \partial_y u = 0$. Therefore u and v are constant, and so f is constant.

□

Example 1.3. The function $f(z) = |z|^2$ is not analytic anywhere because it is real-valued and non-constant.

Theorem 1.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ have radius of convergence $R > 0$. Then f is analytic on $D_R(z_0)$ with*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

for $|z - z_0| < R$.