Abstract Algebra Homework 2

Michael Nelson

Throughout this homework, let R be a commutative ring.

Problem 1

Proposition 0.1. *Suppose the following diagram of R-modules and R-homomorphisms is commutative with exact rows*

$$\begin{tikzcd} M_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & M_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2"] & M_3 \arrow[r, "\varphi_3"] \arrow[d, "\psi_3"] & M_4 \arrow[r, "\varphi_4"] \arrow[d, "\psi_4"] & M_5 \arrow[d, "\psi_5"] \\ M_1' \arrow[r, "\varphi_1'"] & M_2' \arrow[r, "\varphi_2'"] & M_3' \arrow[r, "\varphi_3'"] & M_4' \arrow[r, "\varphi_4'"] & M_5' \end{tikzcd}$$

1. *If ψ_2, ψ_4 are surjective and ψ_5 is injective, then ψ_3 is surjective.*
2. *If ψ_2, ψ_4 are injective and ψ_1 is surjective, then ψ_3 is injective.*

Proof.

1. Suppose ψ_2, ψ_4 are surjective and ψ_5 is injective and let $u_3' \in M_3'$. Since ψ_4 is surjective, we may choose a $u_4 \in M_4$ such that $\psi_4(u_4) = \varphi_3'(u_3')$. Observe that

$$\begin{aligned} \psi_5\varphi_4(u_4) &= \varphi_4'\psi_4(u_4) \\ &= \varphi_4'\varphi_3'(u_3') \\ &= 0. \end{aligned}$$

It follows that $\varphi_4(u_4) = 0$ since ψ_5 is injective. Therefore we may choose a $u_3 \in M_3$ such that $\varphi_3(u_3) = u_4$ (by exactness of the top row). Now observe that

$$\begin{aligned} \varphi_3'(u_3' - \psi_3(u_3)) &= \varphi_3'(u_3') - \varphi_3'\psi_3(u_3) \\ &= \psi_4(u_4) - \psi_4\varphi_3(u_3) \\ &= \psi_4(u_4) - \psi_4(u_4) \\ &= 0. \end{aligned}$$

Therefore we may choose a $u_2' \in M_2'$ such that $\varphi_2'(u_2') = u_3' - \psi_3(u_3)$ (by exactness of the bottom row). Since ψ_2 is surjective, we may choose a $u_2 \in M_2$ such that $\psi_2(u_2) = u_2'$. Finally we see that

$$\begin{aligned} \psi_3(\varphi_2(u_2) + u_3) &= \psi_3\varphi_2(u_2) + \psi_3(u_3) \\ &= \varphi_2'\psi_2(u_2) + \psi_3(u_3) \\ &= \varphi_2'(u_2') + \psi_3(u_3) \\ &= u_3' - \psi_3(u_3) + \psi_3(u_3) \\ &= u_3'. \end{aligned}$$

It follows that ψ_3 is surjective.

2. Suppose ψ_2, ψ_4 are injective and ψ_1 is surjective and let $u_3 \in \ker \psi_3$. Observe that

$$\begin{aligned} \psi_4\varphi_3(u_3) &= \varphi_3'\psi_3(u_3) \\ &= \varphi_3'(0) \\ &= 0. \end{aligned}$$

It follows that $\varphi_3(u_3) = 0$ since ψ_4 is injective. Therefore we may choose a $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$ (by exactness of the top row). Now observe that

$$\begin{aligned} \varphi_2'\psi_2(u_2) &= \psi_3\varphi_2(u_2) \\ &= \psi_3(u_3) \\ &= 0. \end{aligned}$$

Therefore we may choose a $u_1' \in M_1'$ such that $\varphi_1'(u_1') = \psi_2(u_2)$ (by exactness of the bottom row). Since ψ_1 is surjective, we may choose a $u_1 \in M_1$ such that $\psi_1(u_1) = u_1'$. Now observe that

$$\begin{aligned} \psi_2\varphi_1(u_1) &= \varphi_1'\psi_1(u_1) \\ &= \varphi_1'(u_1') \\ &= \psi_2(u_2). \end{aligned}$$

It follows that $\varphi_1(u_1) = u_2$ since ψ_2 is injective. Therefore

$$\begin{aligned} u_3 &= \varphi_2(u_2) \\ &= \varphi_2\varphi_1(u_1) \\ &= 0, \end{aligned}$$

which implies $\ker \psi_3 = 0$. Thus ψ_3 is injective. $\square$

Problem 2

Proposition 0.2. *Consider the following diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & M_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2"] & M_3 \arrow[r] \arrow[d, "\psi_3"] & 0 \\
0 \arrow[r] & M_1' \arrow[r, "\varphi_1'"] \arrow[d, "\psi_1'"] & M_2' \arrow[r, "\varphi_2'"] \arrow[d, "\psi_2'"] & M_3' \arrow[r] \arrow[d, "\psi_3'"] & 0 \\
0 \arrow[r] & M_1'' \arrow[r, "\varphi_1''"] \arrow[d] & M_2'' \arrow[r, "\varphi_2''"] \arrow[d] & M_3'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the columns and top two rows are exact, then the bottom row is exact.

Proof. We first show φ_1'' is injective. Let $u_1'' \in \ker \varphi_1''$. Since ψ_1' is surjective (by exactness of first column) we may choose a $u_1' \in M_1'$ such that $\psi_1'(u_1') = u_1''$. Then

$$\begin{aligned} \psi_2'\varphi_1'(u_1') &= \varphi_1''\psi_1'(u_1') \\ &= \varphi_1''(u_1'') \\ &= 0 \end{aligned}$$

implies $\varphi_1'(u_1') \in \ker \psi_2'$. Therefore there exists a unique $u_2 \in M_2$ such that $\psi_2(u_2) = \varphi_1'(u_1')$ (by exac Then

$$\begin{aligned} \psi_3\varphi_2(u_2) &= \varphi_2'\psi_2(u_2) \\ &= \varphi_2'\varphi_1'(u_1') \\ &= 0 \end{aligned}$$

implies $\varphi_2(u_2) = 0$ since ψ_3 is injective (by exactness of third column). Thus $u_2 \in \ker \varphi_2$ and so there exists a unique $u_1 \in M_1$ such that $\varphi_1(u_1) = u_2$ (by exactness of first row). Therefore

$$\begin{aligned}
\varphi_1'\psi_1(u_1) &= \psi_2\varphi_1(u_1) \\
&= \psi_2(u_2) \\
&= \varphi_1'(u_1')
\end{aligned}$$

implies $\psi_1(u_1) = u_1'$ since φ_1' is injective (by exactness of second row). Thus

$$\begin{aligned}
u_1'' &= \psi_1'(u_1') \\
&= \psi_1'\psi_1(u_1) \\
&= 0.
\end{aligned}$$

Now we show $\ker \varphi_2'' = \operatorname{im} \varphi_1''$. Let $u_2'' \in \ker \varphi_2''$. Since ψ_2' is surjective (by exactness of second colunn), we may choose a $u_2' \in M_2'$ such that $\psi_2'(u_2') = u_2''$. Then

$$\begin{aligned}
\psi_3'\varphi_2'(u_2') &= \varphi_2''\psi_2'(u_2') \\
&= \varphi_2''(u_2'') \\
&= 0
\end{aligned}$$

implies $\varphi_2'(u_2') \in \ker \psi_3'$. Therefore there exists a unique $u_3 \in M_3$ such that $\psi_3(u_3) = \varphi_2'(u_2')$ (by exactness of third column). Since φ_2 is surjective, we may choose a $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$. Then

$$\begin{aligned}
\varphi_2'(\psi_2(u_2) - u_2') &= \varphi_2'\psi_2(u_2) - \varphi_2'(u_2') \\
&= \psi_3\varphi_2(u_2) - \varphi_2'(u_2') \\
&= \psi_3(u_3) - \varphi_2'(u_2') \\
&= \varphi_2'(u_2') - \varphi_2'(u_2') \\
&= 0
\end{aligned}$$

implies $\psi_2(u_2) - u_2' \in \ker \varphi_2'$. Therefore there exists a uniuqe $u_1' \in M_1'$ such that $\varphi_1'(u_1') = \psi_2(u_2) - u_2'$ (by exactness of second row). Therefore

$$\begin{aligned}
\varphi_1''\psi_1'(u_1') &= \psi_2'\varphi_1'(u_1') \\
&= \psi_2'(\psi_2(u_2) - u_2') \\
&= \psi_2'\psi_2(u_2) - \psi_2'(u_2') \\
&= \psi_2'(u_2') \\
&= u_2''.
\end{aligned}$$

It follows that $u_2'' \in \operatorname{im} \varphi_1''$. Thus $\ker \varphi_2'' \subseteq \operatorname{im} \varphi_1''$. For the reverse inclusion, let $u_2'' \in M_2''$. Choose $u_1'' \in M_1''$ such that $\varphi_1''(u_1'') = u_2''$. Since ψ_1' is surjective (by exactness of first column), we may choose a $u_1' \in M_1'$ such that $\psi_1'(u_1') = u_1''$. Then

$$\begin{aligned}
0 &= \psi_3'\varphi_2'\varphi_1'(u_1') \\
&= \varphi_2''\psi_2'\varphi_1'(u_1') \\
&= \varphi_2''\varphi_1''\psi_1'(u_1') \\
&= \varphi_2''\varphi_1''(u_1'') \\
&= \varphi_2''(u_2'')
\end{aligned}$$

implies $u_2'' \in \ker \varphi_2''$. Thus $\ker \varphi_2'' \supseteq \operatorname{im} \varphi_1''$.

The last step is to show φ_2'' is surjective. Let $u_3'' \in M_3''$. Since ψ_3' is surjective (by exactness of third column), we may choose a $u_3' \in M_3'$ such that $\psi_3'(u_3') = u_3''$. Since φ_2' is surjective (by exactness of second row), we may choose a $u_2' \in M_2'$ such that $\varphi_2'(u_2') = u_3'$. Then

$$\begin{aligned}
\varphi_2''\psi_2'(u_2') &= \psi_3'\varphi_2'(u_2') \\
&= \psi_3'(u_3') \\
&= u_3''
\end{aligned}$$

implies φ_2'' is surjective. $\square$

Problem 3

Proposition 0.3. *Consider the following commutative diagram with exact rows*

$$\begin{tikzcd} & M_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & M_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2"] & M_3 \arrow[r] \arrow[d, "\psi_3"] & 0 \\ 0 \arrow[r] & M_1' \arrow[r, "\varphi_1'"] & M_2' \arrow[r, "\varphi_2'"] & M_3' & \end{tikzcd} \tag{1}$$

Then there exists an exact sequence

$$\ker \psi_1 \xrightarrow{\widetilde{\varphi_1}} \ker \psi_2 \xrightarrow{\widetilde{\varphi_2}} \ker \psi_3 \xrightarrow{\partial} \operatorname{coker} \psi_1 \xrightarrow{\overline{\varphi_1'}} \operatorname{coker} \psi_2 \xrightarrow{\overline{\varphi_2'}} \operatorname{coker} \psi_3. \tag{2}$$

Moreover, if φ_1 is injective, then $\widetilde{\varphi_1}$ is injective; and if φ_2' is surjective, then $\overline{\varphi_2'}$ is surjective.

Proof.

Step 1: We first define the maps in question. Define $\widetilde{\varphi_1}\colon \ker \psi_1 \to \ker \psi_2$ by

$$\widetilde{\varphi_1}(u_1) = \varphi_1(u_1)$$

for all $u_1 \in \ker \psi_1$. Note that $\widetilde{\varphi_1}$ lands in $\ker \psi_2$ by the commutativity of the diagram. Indeed,

$$\begin{aligned} \psi_2\widetilde{\varphi_1}(u_1) &= \psi_2\varphi_1(u_1) \\ &= \varphi_1'\psi_1(u_1) \\ &= \varphi_1'(0) \\ &= 0 \end{aligned}$$

implies $\widetilde{\varphi_1}(u_1) \in \ker \psi_2$ for all $u_1 \in \ker \psi_1$. Also note that $\widetilde{\varphi_1}$ is an R-module homomorphism since φ_1 is an R-module homomorphism. Similarly, we define $\widetilde{\varphi_2}\colon \ker \psi_2 \to \ker \psi_3$ by

$$\widetilde{\varphi_2}(u_2) = \varphi_2(u_2)$$

for all $u_2 \in \ker \psi_2$.

Next we define $\partial\colon \ker \psi_3 \to \operatorname{coker} \psi_1$ as follows: let $u_3 \in \ker \psi_3$. Choose $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$ (such an element exists because φ_2 is surjective by exactness of the first row). By the commutativity of the diagram, we have

$$\begin{aligned} \varphi_2'\psi_2(u_2) &= \psi_3\varphi_2(u_2) \\ &= \psi_3(u_3) \\ &= 0. \end{aligned}$$

It follows that $\psi_2(u_2) \in \ker \varphi_2'$. Therefore there exists a unique $u_1' \in M_1'$ such that $\varphi_1'(u_1') = \psi_2(u_2)$ (by exactness of the second row). We set

$$\partial(u_3) = \overline{u_1'}$$

where $\overline{u_1'}$ is the coset in $\operatorname{coker} \psi_1$ with u_1' as a representative. We must check that ∂ defined in this is in fact a well-defined map. There was one choice that we made in our construction, namely the lift of u_3 under φ_2 to u_2. So let v_2 be another element in M_2 such that $\varphi_2(v_2) = u_3$. Denote by v_1' to be the unique element in M_1' such that $\varphi_1'(v_1') = \psi_2(v_2)$. We must show that $\overline{u_1'} = \overline{v_1'}$ in $\operatorname{coker} \psi_1$. In other words, we must show that $v_1' - u_1' \in \operatorname{im} \psi_1$. Observe that

$$\begin{aligned} \varphi_2(v_2 - u_2) &= \varphi_2(v_2) - \varphi_2(u_2) \\ &= u_3 - u_3 \\ &= 0 \end{aligned}$$

implies $v_2 - u_2 \in \ker \varphi_2$. It follows that there exists a unique element $u_1 \in M_1$ such that $\varphi_1(u_1) = v_2 - u_2$ (by exactness of the first row). Then

$$\begin{aligned}\varphi_1'\psi_1(u_1) &= \psi_2\varphi_1(u_1)\\ &= \psi_2(v_2 - u_2)\\ &= \psi_2(v_2) - \psi_2(u_2)\\ &= \varphi_1'(v_1') - \varphi_1'(u_1')\\ &= \varphi_1'(v_1' - u_1')\end{aligned}$$

implies $\psi_1(u_1) = v_1' - u_1'$ since φ_1' is injective (by exactness of the second row). It follows that $v_1' - u_1' \in \operatorname{im}\psi_1$, and hence ∂ is well-defined.

Finally, we define $\overline{\varphi_1'}\colon \operatorname{coker}\psi_1 \to \operatorname{coker}\psi_2$ by

$$\overline{\varphi_1'}(\overline{u_1'}) = \overline{\varphi_1'(u_1')}$$

for all $\overline{u_1'} \in \operatorname{coker}\psi_1$. The map $\overline{\varphi_1'}$ is well-defined by the commutativity of the diagram. Indeed, let v_1' be another representative of the coset $\overline{u_1'}$ in $\operatorname{coker}\psi_1$. Choose $u_1 \in M_1$ such that $v_1' - u_1' = \psi_1(u_1)$. Then

$$\begin{aligned}\psi_2\varphi_1(u_1) &= \varphi_1'\psi_1(u_1)\\ &= \varphi_1'(v_1' - u_1')\\ &= \varphi_1'(v_1') - \varphi_1'(u_1').\end{aligned}$$

It follows that $\varphi_1'(v_1') - \varphi_1'(u_1') \in \operatorname{im}\psi_2$, and hence $\varphi_1'(v_1')$ and $\varphi_1'(u_1')$ represent the same coset in $\operatorname{coker}\psi_2$. Similarly, we define $\overline{\varphi_2'}\colon \operatorname{coker}\psi_2 \to \operatorname{coker}\psi_3$ by

$$\overline{\varphi_2'}(\overline{u_2'}) = \overline{\varphi_2'(u_2')}$$

for all $\overline{u_2'} \in \operatorname{coker}\psi_2$.

Step 2: Now that we've defined the maps in question, we will now show that the sequence (2) is exact as well as prove the "moreover" part of the proposition. First we show exactness at $\ker\psi_2$. Observe that

$$\begin{aligned}\widetilde{\varphi_2}\widetilde{\varphi_1}(u_1) &= \varphi_2\varphi_1(u_1)\\ &= 0\end{aligned}$$

for all $u_1 \in \ker\psi_1$. It follows that $\ker\widetilde{\varphi_2} \supseteq \operatorname{im}\widetilde{\varphi_1}$. Conversely, let $u_2 \in \ker\widetilde{\varphi_2}$. Thus $u_2 \in \ker\varphi_2 \cap \ker\psi_2$. By exactness of the top row in (1), we may choose a $u_1 \in M_1$ such that $\varphi_1(u_1) = u_2$. Moreover,

$$\begin{aligned}\varphi_1'\psi_1(u_1) &= \psi_2\varphi_1(u_1)\\ &= \psi_2(u_2)\\ &= 0\end{aligned}$$

implies $\psi_1(u_1) = 0$ since φ_1' is injective (by exactness of the bottom row in (1)). Therefore $u_1 \in \ker\psi_1$, and so $u_2 \in \operatorname{im}\widetilde{\varphi_1}$. Thus $\ker\widetilde{\varphi_2} \subseteq \operatorname{im}\widetilde{\varphi_1}$.

Next we show exactness at $\ker\psi_3$: let $u_3 \in \ker\partial$. Choose $u_2 \in M_2$ and $u_1' \in M_1'$ such that $\varphi_2(u_2) = u_3$ and $\varphi_1'(u_1') = \psi_2(u_2)$. Then

$$\begin{aligned}0 &= \partial(u_3)\\ &= \overline{u_1'}\end{aligned}$$

implies $u_1' \in \operatorname{im}\psi_1$. Choose $u_1 \in M_1$ such that $\psi_1(u_1) = u_1'$. Then

$$\begin{aligned}\psi_2(u_2 - \varphi_1(u_1)) &= \psi_2(u_2) - \psi_2\varphi_1(u_1)\\ &= \psi_2(u_2) - \varphi_1'\psi_1(u_1)\\ &= \psi_2(u_2) - \varphi_1'(u_1')\\ &= \psi_2(u_2) - \psi_2(u_2)\\ &= 0\end{aligned}$$

implies $u_2 - \varphi_1(u_1) \in \ker \psi_2$. Furthermore, we have

$$\begin{aligned} \varphi_2(u_2 - \varphi_1(u_1)) &= \varphi_2(u_2) - \varphi_2\varphi_1(u_1) \\ &= \varphi_2(u_2) \\ &= u_3. \end{aligned}$$

It follows that $u_3 \in \operatorname{im} \widetilde{\varphi_2}$. Thus $\ker \partial \subseteq \operatorname{im} \widetilde{\varphi_2}$. Convsersely, let $u_3 \in \operatorname{im} \widetilde{\varphi_2}$. Choose $u_2 \in \ker \psi_2$ such that $\varphi_2(u_2) = u_3$. Then $0 \in M_1'$ is the unique element in M_1' which maps to $\psi_2(u_2) = 0$. Thus $\partial(u_3) = \overline{0}$ which implies $\ker \partial \supseteq \operatorname{im} \widetilde{\varphi_2}$.

Next we show exactness at $\operatorname{coker} \psi_1$: let $\overline{u_1'} \in \ker \overline{\varphi_1'}$. Then $\varphi_1'(u_1') = \psi_2(u_2)$ for some $u_2 \in M_2$. Moreover,

$$\begin{aligned} \psi_3\varphi_2(u_2) &= \varphi_2'\psi_2(u_2) \\ &= \varphi_2'\varphi_1'(u_1') \\ &= 0 \end{aligned}$$

implies $\varphi_2(u_2) \in \ker \psi_3$. Also we have $\partial(\varphi_2(u_2)) = \overline{u_1'}$, and so $\overline{u_1'} \in \operatorname{im}\partial$. Thus $\ker \overline{\varphi}_1' \subseteq \operatorname{im}\partial$. Conversely, let $\overline{u_1'} \in \operatorname{im}\partial$. Choose $u_3 \in M_3$ and $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$ and $\psi_2(u_2) = \varphi_1'(u_1')$. It follows that

$$\begin{aligned} \overline{\varphi_1'}(\overline{u_1'}) &= \overline{\varphi_1'(u_1')} \\ &= \overline{\psi_2(u_2)} \\ &= \overline{0} \end{aligned}$$

in $\operatorname{coker} \psi_2$. Thus $\ker \overline{\varphi_1'} \supseteq \operatorname{im}\partial$.

Next we check exactness at $\operatorname{coker} \psi_2$: let $\overline{u_2'} \in \ker \overline{\varphi_2'}$. Choose $u_3 \in M_3$ such that $\psi_3(u_3) = \varphi_2'(u_2')$ and choose $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$. Since

$$\begin{aligned} \varphi_2'(u_2' - \psi_2(u_2)) &= \varphi_2'(u_2') - \varphi_2'\psi_2(u_2) \\ &= \varphi_2'(u_2') - \psi_3\varphi_2(u_2) \\ &= \varphi_2'(u_2') - \psi_3(u_3) \\ &= \varphi_2'(u_2') - \varphi_2'(u_2') \\ &= 0, \end{aligned}$$

it follows that $u_2' - \psi_2(u_2) \in \ker \varphi_2'$. Therefore there exists a unique $u_1' \in M_1'$ such that $\varphi_1'(u_1') = u_2' - \psi_2(u_2)$ (by exactness of the bottom row in (1)). Then

$$\begin{aligned} \overline{\varphi_1'}(\overline{u_1'}) &= \overline{\varphi_1'(u_1')} \\ &= \overline{u_2' - \psi_2(u_2)} \\ &= \overline{u_2'} \end{aligned}$$

in $\operatorname{coker} \psi_2$. It follows that $\overline{u_2'} \in \operatorname{im} \overline{\varphi_2'}$ and hence $\ker \overline{\varphi_2'} \subseteq \operatorname{im} \overline{\varphi_1'}$. Conversely, let $\overline{u_2'} \in \operatorname{im} \overline{\varphi_2'}$. Choose $u_1' \in M_1'$ such that $\varphi_1'(u_1') = u_2'$. Then

$$\begin{aligned} 0 &= \varphi_2'\varphi_1'(u_1') \\ &= \varphi_2'(u_2') \end{aligned}$$

implies $u_2' \in \ker \varphi_2$. Therefore $\overline{\varphi_2'}(\overline{u_2'}) = \overline{0}$ in $\operatorname{coker} \psi_3$, and it follows that $\ker \overline{\varphi_2'} \supseteq \operatorname{im} \overline{\varphi_1'}$.

Finally, we prove the moreover part of this proposition. Suppose that φ_1 is injective. We want to show that $\widetilde{\varphi_1}$ is injective. Let $u_1 \in \ker \widetilde{\varphi_1}$. Then

$$\begin{aligned} 0 &= \widetilde{\varphi_1}(u_1) \\ &= \varphi_1(u_1) \end{aligned}$$

implies $u_1 = 0$ since φ_1 is injective. It follows that $\widetilde{\varphi_1}$ is injective. Now suppose that φ_2' is surjective. We want to show that $\overline{\varphi_2'}$ is surjective. Let $\overline{u_3'} \in \operatorname{coker} \psi_3$. Since φ_2' is surjective, we may choose a $u_2' \in M_2'$ such that $\varphi_2'(u_2') = u_3'$. Then

$$\begin{aligned} \overline{\varphi_2'}(\overline{u_2'}) &= \overline{\varphi_2'(u_2')} \\ &= \overline{u_3'}. \end{aligned}$$

It follows that $\overline{\varphi_2'}$ is surjective. $\square$

Problem 4

Definition 0.1. Let M be an R-module.

1. We say M is **simple** if the only submodules of M are itself and 0.
2. We say M is **cyclic** if there exists a $u \in M$ such that $M = Ru$.

Problem 4.a

Proposition 0.4. *Let M be a simple R-module. Then M is cyclic.*

Proof. If $M = 0$, then the proposition is clear, so assume $M \neq 0$. Choose any nonzero element u in M. Since M is simple, the submodule of M generated by u, given by

$$\langle u \rangle = \{au \mid a \in R\},$$

must either be the zero module or all of M. Since u was chosen to be nonzero, we cannot have $\langle u \rangle = 0$. Thus $\langle u \rangle = M$, which implies M is cyclic. $\square$

Problem 4.b

Proposition 0.5. *Let M be a nonzero simple R-module and let $\varphi\colon M \to M$ be any nonzero R-module homomorphism. Then φ is an isomorphism. Moreover, assuming R is commutative, then we have*

$$\mathrm{Hom}_R(M, M) \cong M. \tag{3}$$

Proof. Since M is simple and φ is nonzero, we must have $\ker \varphi = 0$ and $\operatorname{im} \varphi = M$. Thus φ is an isomorphism.

Now let us show (3). Choose a nonzero element u in M (so $M = \langle u \rangle$). We define $\Psi\colon \mathrm{Hom}_R(M, M) \to M$ by the formula

$$\Psi(\varphi) = \varphi(u)$$

for all $\varphi \in \mathrm{Hom}_R(M, M)$.

Let us show that Ψ is an R-module homomorphism. Let $a, b \in R$ and $\psi, \varphi \in \mathrm{Hom}_R(M, M)$. Then we have

$$\begin{aligned}\Psi(a\varphi + b\psi) &= (a\varphi + b\psi)(u) \\ &= a\varphi(u) + b\psi(u) \\ &= a\Psi(\varphi) + b\Psi(\psi).\end{aligned}$$

It follows that Ψ is an R-module homomorphism.

Next we show that Ψ is injective. Suppose $\varphi \in \ker \Psi$ (so $\varphi(u) = 0$). Since $M = \langle u \rangle$, every element in M has the form au for some $a \in R$, so let au be an arbitary element in M. Then

$$\begin{aligned}\varphi(au) &= a\varphi(u) \\ &= a \cdot 0 \\ &= 0.\end{aligned}$$

This implies $\varphi = 0$ and thus Ψ is injective.

Lastly, we show that Ψ is surjective. Let $bu \in M$ where $b \in R$ and let $m_b\colon M \to M$ be the multiplication by b map, given by

$$m_b(v) = bv$$

for all $v \in M$. Then m_b is an R-module homomorphism (assuming that R is commutative) and moreover we have

$$\begin{aligned}\Psi(m_b) &= m_b(u) \\ &= bu.\end{aligned}$$

This implies Ψ is surjective. $\square$