Math 9853: Homework #4

Due Thursday, September 19, 2019

(a) Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear map defined by $f(e_i) = ie_i$ for $1 \leq i \leq 4$ where $(e_1, \ldots, e_4) = I_4$, the basis consisting of unit vectors. Let

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

(a.1) Compute H^tH and H^{-1}. Deduce that the columns of H form a basis for $\mathbb{R}^4$.

(a.2) Find the dual basis of H. (Describe the linear forms explicitly.)

(a.3) Find the matrix of f under the basis H of $\mathbb{R}^4$.

(b) Let V and W be vector spaces of **the same finite dimension** n over a field K and $f\colon V \to W$ any linear map. Prove the following:

(b.1) If f is injective, then for any basis $u_1, \ldots, u_n$ of V, the images $f(u_1), \ldots, f(u_n)$ form a basis for W, hence f is onto;

(b.2) If f is onto, then for any basis $u_1, \ldots, u_n$ of V, the images $f(u_1), \ldots, f(u_n)$ form a basis for W, hence f is injective.

Remark. Note that neither of above is true when the dimension is infinite (see examples in class). In your proof, you may use the fact that any set of independent vectors in a vector space can be extended to basis and any set of dependent vectors contains a subset that is a basis.

(c) Let $f\colon V \to W$ be a linear map of vector spaces over a field K. Let $X \subseteq V$ be a linear subspace such that $X \subseteq \mathrm{Null}(f)$ and let $\tau\colon V \to V/X$ be the natural surjection $\tau(v) = \overline{v}$.

(c.1) Let $\overline{f}\colon V/X \to W$ be given by $\overline{f}(\overline{v}) = f(v)$. Prove that $\overline{f}$ is a linear map such that $\overline{f} \circ \tau = f$ and $\overline{f}(V/X) = f(V)$.

Remark. First show that $\overline{f}$ is well-defined, that is, if $v, v_1 \in V$ are such that $\overline{v}_1 = \overline{v}_2$, then $f(v_1) = f(v_2)$.

(c.2) Prove that if $g\colon V/X \to W$ is a a linear map such that $g \circ \tau = f$, then $g = \overline{f}$.

(c.3) Prove that $\overline{f}$ is 1-1 if and only if $X = \mathrm{Null}(f)$. This proves that $V/\mathrm{Null}(f)$ is isomorphic to $\mathrm{Imag}(f) = \{f(v) : v \in V\} \subseteq W$.