Section 2.2: Measures of Change at a Point – Graphical

The **(instantaneous) rate of change** of a continuous function f at a point x is given by the **slope** of the **tangent line** to the graph at that point (unless the tangent line is vertical at x or there is a sharp corner at x). It is denoted by $f'(x)$ and read as "*f prime of x*". The unit of measure for rate of change is **output units per input unit**.

Instantaneous rate of change expresses how quickly a quantity is changing at a single point.

The **slope of a graph** at a point refers to the slope of the tangent line to the graph at that point.

The **percentage rate of change** at a point expresses the rate of change as a percent. If the rate of change $f'(a)$ exists for input a and $f(a) \neq 0$, then

$$\textbf{percentage rate of change} = \frac{\textit{rate of change at a point}}{\textit{value of the function at that point}} \cdot 100\,\% = \frac{f'(a)}{f(a)} \cdot 100\,\%$$

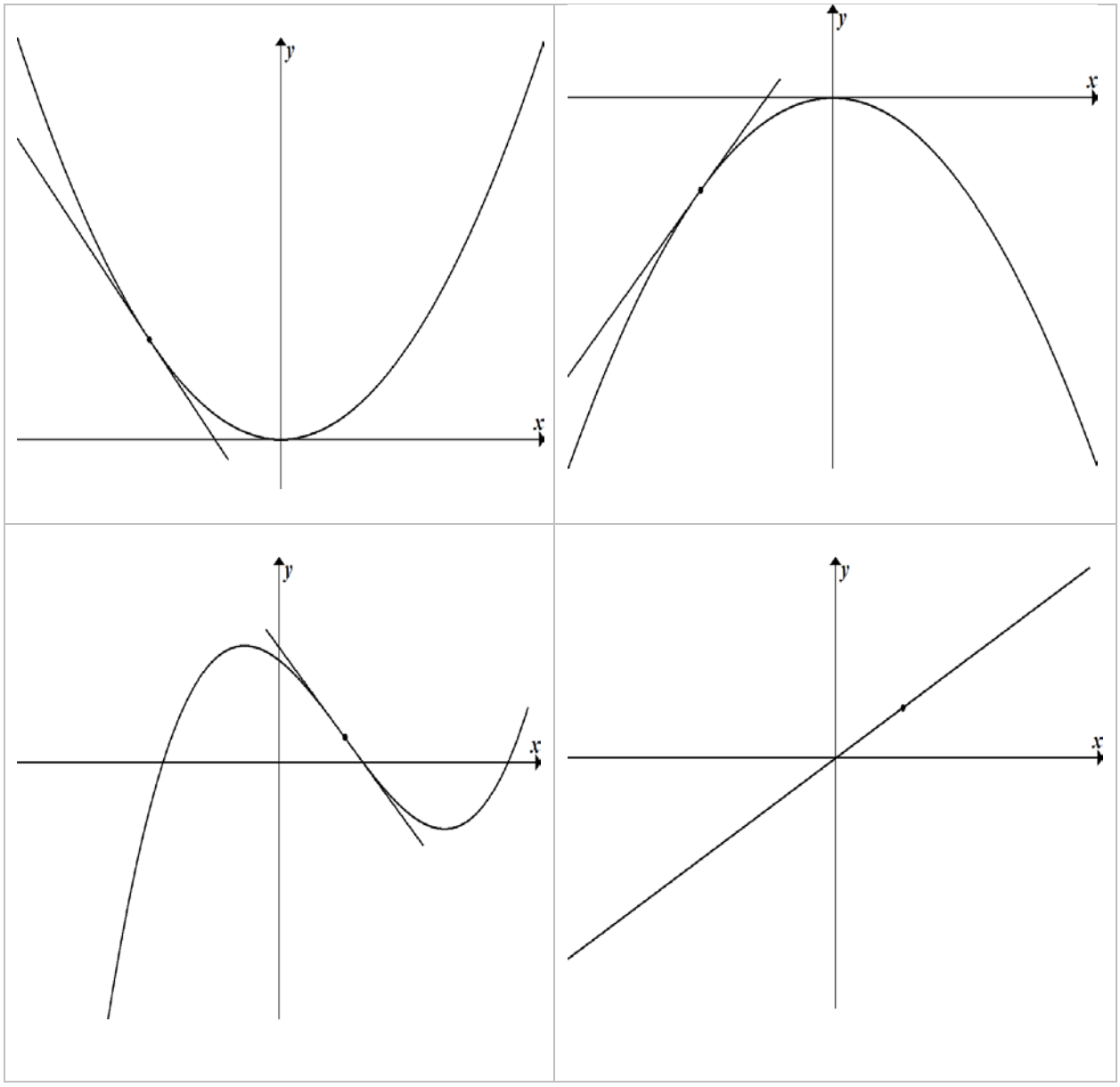
The unit of measure for percentage rate of change is **% per input unit**.

A sentence of **interpretation** for **rate of change** or **percentage rate of change** at a point uses ordinary conversational language to answer the questions:

- *When?* refers to the single point.
- *What?* refers to the output description for the function.
- *Increas**ing** or Decreas**ing**?*
- *By how much?* refers to the rate of change or percentage rate of change calculation, and includes its corresponding units.

Example 1:

Examples of graphs with a tangent line are shown below. Describe the concavity of the graph at the point of tangency and state whether the tangent line lies above the graph, below the graph, cuts through the graph, or coincides with the graph.



To draw a **tangent line** to a graph at a point, draw a line that goes through that point (called the **point of tangency)** and follows the slope of the graph.

- If the graph is **concave up** at the point of tangency, the tangent line lies below the graph.
- If the graph is **concave down** at the point of tangency, the tangent line lies above the graph.
- If there is an **inflection point** at the point of tangency, the tangent line cuts through the graph. It lies above the concave down portion and below the concave up portion of the graph.
- If the graph is of a **linear function**, the tangent line at the point of tangency will coincide with the graph of the line.

Use the idea of **local linearity** to help draw a tangent line. Notice that over a small input interval, the curve appears to be linear. Extend this apparent straight line while using the above information to draw a tangent line.

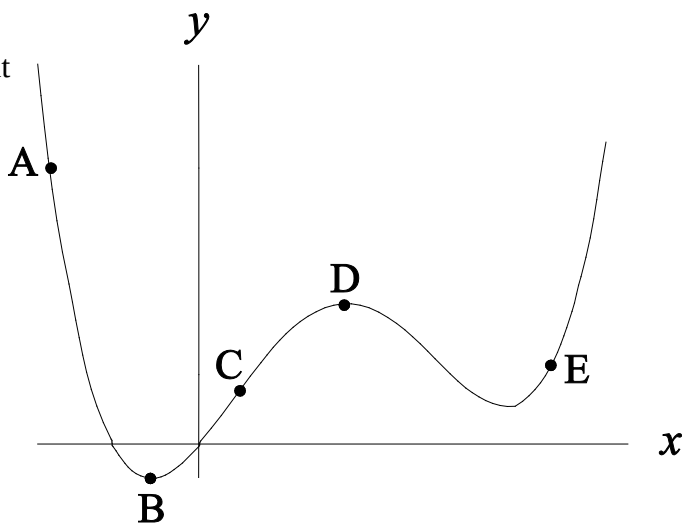
Example 2: (CC5e p. 147, Activity 1)

a. Draw a tangent line at each labeled point on the graph. State whether the graph is concave up, concave down, or has an inflection point at each labeled point.

A: D:

B: E:

C:



b. State whether the instantaneous rate of change is positive, negative or zero at each of the labeled points.

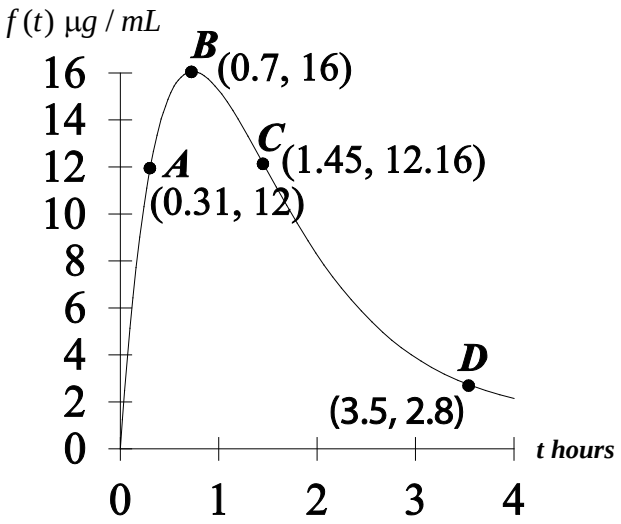
A: B: C: D: E:

c. Is the graph **steeper** at point *A* or at point *C*?

d. Is the slope **greater** at point *A* or at point *C*?

Example 3: (CC5e p. 143)

The graph shows the mean plasma concentration of acetaminophen in adults, in micrograms per milliliter $(\mu g / mL)$, t hours after being dosed with two 500 mg caplets of Tylenol.



- The slope of the line tangent to the graph at point A is approximately $22.4\ \mu g / mL$ per hour. The input at point A is 0.31 hours.
- The slope of the graph is zero at point B.
- Mean plasma concentration is decreasing most rapidly at point C, where the rate is $-7.61\ \mu g / mL$ per hour.
- The instantaneous rate of change of mean plasma concentration at point D is approximately $-1.67\ \mu g / mL$ per hour.

a. Draw tangent lines to the curve at each of the points A, B, C, and D.

b. List points A, B, C, and D in order from **least to greatest slope**.

c. List points A, B, C, and D in order from **least to greatest steepness**.

d. Write a sentence of interpretation for the **rate of change** in mean plasma concentration at each of the points A and C.

A:

C:

e. Find the **percentage rate of change** in mean plasma concentration of acetaminophen 18.6 minutes (0.31hours) after being dosed with two 500 mg caplets of Tylenol. Write a sentence of interpretation.

The **tangent line at point** T can be defined as follows:
Given a point T and close points P_n on the graph of a smooth continuous function, the **tangent line at point** T is the limiting position of the secant lines through point T and increasingly close points P_n.

Example 4:
Draw secant lines through P_1 and T, P_2 and T, etc. on the left side of T. Repeat on the right side of T by drawing secant lines through P_5 and T, P_6 and T, etc. on the right side of T. Finally, draw the tangent line at T, noting how it is the limiting position of the secant lines.