Linear Analysis Homework 6

Michael Nelson

Throughout this homework, let $\mathcal{H}$ be a Hilbert space.

Problem 1

Proposition 0.1. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. There exists unique self-adjoint operators $A\colon \mathcal{H} \to \mathcal{H}$ and $B\colon \mathcal{H} \to \mathcal{H}$ such that $T = A + iB$.*

Proof. Define

$$A = \frac{1}{2}(T + T^*) \quad \text{and} \quad B = \frac{-i}{2}(T - T^*).$$

Then

$$\begin{aligned} A + iB &= \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) T + \left(\frac{1}{2} - \frac{1}{2}\right) T^* \\ &= T \end{aligned}$$

Furthermore, A and B are self-adjoint. Indeed,

$$\begin{aligned} A^* &= \left(\frac{1}{2}(T + T^*)\right)^* \\ &= \frac{1}{2}(T^* + T^{**}) \\ &= \frac{1}{2}(T^* + T) \\ &= \frac{1}{2}(T + T^*) \\ &= A, \end{aligned}$$

and similarly

$$\begin{aligned} B^* &= \left(\frac{-i}{2}(T - T^*)\right)^* \\ &= \frac{i}{2}(T^* - T^{**}) \\ &= \frac{i}{2}(T^* - T) \\ &= \frac{-i}{2}(T - T^*) \\ &= B. \end{aligned}$$

This establishes existence.

For uniqueness, suppose that $A'\colon \mathcal{H} \to \mathcal{H}$ and $B'\colon \mathcal{H} \to \mathcal{H}$ are two other self-adjoint operators such that $T = A' + iB'$. Then since

$$\begin{aligned} T^* &= (A + iB)^* \\ &= A^* - iB^* \\ &= A - iB, \end{aligned}$$

and since

$$\begin{aligned}T^* &= (A' + iB')^* \\ &= A'^* - iB'^* \\ &= A' - iB',\end{aligned}$$

we have

$$\begin{aligned}A + iB &= A' + iB' \\ A - iB &= A' - iB'.\end{aligned}$$

Adding these together gives us $2A = 2A'$, and hence $A = A'$. Similarly, subtracting these gives us $2iB = 2iB'$, and hence $B = B'$. □

Problem 2

Definition 0.1. A self-adjoint operator $T\colon \mathcal{H} \to \mathcal{H}$ is said to be **positive** if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We say T is **strictly positive** if $\langle Tx, x\rangle > 0$ for all $x \in \mathcal{H}\backslash\{0\}$.

Remark. Equivalently, $T\colon \mathcal{H} \to \mathcal{H}$ is positive if and only if $\langle x, Tx\rangle \geq 0$ for all $x \in \mathcal{H}$. Indeed, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$, then $\langle Tx, x\rangle$ is real for all $x \in \mathcal{H}$, and so

$$\begin{aligned}\langle x, Tx\rangle &= \overline{\langle Tx, x\rangle} \\ &= \langle Tx, x\rangle \\ &\geq 0.\end{aligned}$$

Similarly, $\langle x, Tx\rangle \geq 0$ for all $x \in \mathcal{H}$ implies $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$.

Problem 2.a

Proposition 0.2. *Let $S\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then S^*S is positive.*

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}\langle S^*Sx, x\rangle &= \langle Sx, Sx\rangle \\ &\geq 0\end{aligned}$$

by positive-definiteness of the inner-product. It follows that S^*S is positive. □

Remark. I think we do not need S to be bounded here, but we only defined the adjoint of a bounded operator in class.

Problem 2.b

Proposition 0.3. *Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. Then the orthogonal projection $\mathrm{P}_{\mathcal{K}}\colon \mathcal{H} \to \mathcal{H}$ is positive.*

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}0 &= \langle x - \mathrm{P}_{\mathcal{K}}x, \mathrm{P}_{\mathcal{K}}x\rangle \\ &= \langle x, \mathrm{P}_{\mathcal{K}}x\rangle - \langle \mathrm{P}_{\mathcal{K}}x, \mathrm{P}_{\mathcal{K}}x\rangle \\ &= \langle x, \mathrm{P}_{\mathcal{K}}x\rangle - \|\mathrm{P}_{\mathcal{K}}x\|^2.\end{aligned}$$

It follows that $\langle x, \mathrm{P}_{\mathcal{K}}x\rangle = \|\mathrm{P}_{\mathcal{K}}x\|^2 \geq 0$ which implies $\mathrm{P}_{\mathcal{K}}$ is positive by Remark (). □

Problem 3

Problem 3.a

Proposition 0.4. *(Another Version of Polarization Identity) Let $T\colon \mathcal{H} \to \mathcal{H}$ be any operator. Then*

$$4\langle Tx, y\rangle = \sum_{i=0}^{3}\langle T(x + i^k y), x + i^k y\rangle \tag{1}$$

Proof. We have

$$\begin{aligned}\langle T(x+y), x+y\rangle &= \langle Tx+Ty, x+y\rangle \\ &= \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle\end{aligned}$$

and

$$\begin{aligned}i\langle T(x+iy), x+iy\rangle &= i\langle Tx+iTy, x+iy\rangle \\ &= i\langle Tx, x\rangle + i\langle Tx, iy\rangle + i\langle iTy, x\rangle + i\langle iTy, iy\rangle \\ &= i\langle Tx, x\rangle + \langle Tx, y\rangle - \langle Ty, x\rangle + i\langle Ty, y\rangle\end{aligned}$$

and

$$\begin{aligned}-\langle T(x-y), x-y\rangle &= -\langle Tx-Ty, x-y\rangle \\ &= -\langle Tx, x\rangle - \langle Tx, -y\rangle - \langle -Ty, x\rangle - \langle -Ty, -y\rangle \\ &= -\langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle - \langle Ty, y\rangle\end{aligned}$$

and

$$\begin{aligned}-i\langle T(x-iy), x-iy\rangle &= -i\langle Tx-iTy, x-iy\rangle \\ &= -i\langle Tx, x\rangle - i\langle Tx, -iy\rangle - i\langle -iTy, x\rangle - i\langle -iTy, -iy\rangle \\ &= -i\langle Tx, x\rangle + \langle Tx, y\rangle - \langle Ty, x\rangle - i\langle Ty, y\rangle.\end{aligned}$$

Adding these together gives us our desired result. □

Problem 3.b

Proposition 0.5. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be any operator such that $\langle Tx, x\rangle = 0$ for all $x \in \mathcal{H}$. Then $T = 0$.*

Proof. Let $x \in \mathcal{H}$. Then it follows from the polarization identity proved above that

$$\begin{aligned}4\langle Tx, y\rangle &= \sum_{i=0}^{3}\langle T(x+i^k y), x+i^k y\rangle \\ &= \sum_{i=0}^{3} 0 \\ &= 0\end{aligned}$$

for all $y \in \mathcal{H}$. It follows that $\langle Tx, y\rangle = 0$ for all $y \in \mathcal{H}$. This implies $Tx = 0$ by positive-definiteness of the inner-product. Since x was arbitrary, this implies $T = 0$. □

Problem 4

Problem 4.a

Proposition 0.6. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a strictly positive self-adjoint operator. Define a map $\langle\cdot,\cdot\rangle_T\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ by*

$$\langle x, y\rangle_T = \langle Tx, y\rangle$$

for all $x, y \in \mathcal{H}$. Then $\langle\cdot,\cdot\rangle_T$ is an inner-product.

Proof. We first check that $\langle\cdot,\cdot\rangle_T$ is linear in the first argument. Let $x, y, z \in \mathcal{H}$. Then

$$\begin{aligned}\langle x+z, y\rangle_T &= \langle T(x+z), y\rangle \\ &= \langle Tx+Tz, y\rangle \\ &= \langle Tx, y\rangle + \langle Tz, y\rangle \\ &= \langle x, y\rangle_T + \langle z, y\rangle_T.\end{aligned}$$

Next we check that $\langle\cdot,\cdot\rangle_T$ is conjugate-symmetric. Let $x, y \in \mathcal{H}$. Then since T is self-adjoint, we have

$$\begin{aligned}\langle x, y\rangle_T &= \langle Tx, y\rangle \\ &= \overline{\langle y, Tx\rangle} \\ &= \overline{\langle Ty, x\rangle} \\ &= \overline{\langle y, x\rangle}_T.\end{aligned}$$

Next we check that $\langle \cdot, \cdot \rangle_T$ is positive-definite. Let $x \in \mathcal{H}$. Then since T is strictly positive, we have

$$\begin{aligned} \langle x, x \rangle_T &= \langle Tx, x \rangle \\ &> 0, \end{aligned}$$

where $\langle x, x \rangle_T = 0$ if and only if $x = 0$. □

Problem 4.b

Proposition 0.7. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a strictly positive self-adjoint operator. Then*

$$|\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle \tag{2}$$

for all $x, y \in \mathcal{H}$.

Proof. We have

$$\begin{aligned} |\langle Tx, y \rangle|^2 &= |\langle x, y \rangle_T|^2 \\ &= \leq \|x\|_T^2 \|y\|_T^2 \\ &= \langle x, x \rangle_T \langle y, y \rangle_T \\ &= \langle Tx, x \rangle \langle Ty, y \rangle, \end{aligned}$$

where we applied Cauchy-Schwarz for the $\langle \cdot, \cdot \rangle_T$ inner-product. □

Problem 4.c

Proposition 0.8. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a strictly positive self-adjoint operator. Then*

$$\|Tx\|^2 \leq \|T\| \langle Tx, x \rangle \tag{3}$$

for all $x \in \mathcal{H}$.

Proof. Let $x \in \mathcal{H}$. Then

$$\begin{aligned} \|Tx\|^4 &= \langle Tx, Tx \rangle^2 \\ &\leq \langle Tx, x \rangle \langle T^2 x, Tx \rangle \\ &\leq \langle Tx, x \rangle \|T^2 x\| \|Tx\| \\ &\leq \langle Tx, x \rangle \|T\| \|Tx\| \|Tx\| \\ &= \langle Tx, x \rangle \|T\| \|Tx\|^2, \end{aligned}$$

where we used (2) to get from the first line to the second line. Now dividing both sides by $\|Tx\|^2$[1], we obtain $\|Tx\|^2 \leq \langle Tx, x \rangle \|T\|$. □

Problem 5

Proposition 0.9. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then T is self-adjoint if and only if $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.*

Proof. Suppose that T is self-adjoint. Let $x \in \mathcal{H}$. Then

$$\begin{aligned} \langle Tx, x \rangle &= \langle x, Tx \rangle \\ &= \overline{\langle Tx, x \rangle} \end{aligned}$$

implies $\langle Tx, x \rangle \in \mathbb{R}$.

Conversely, suppose that $\langle Tx, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. Then

$$\begin{aligned} \langle (T - T^*)x, x \rangle &= \langle Tx - T^*x, x \rangle \\ &= \langle Tx, x \rangle - \langle T^*x, x \rangle \\ &= \overline{\langle x, Tx \rangle} - \langle x, Tx \rangle \\ &= \langle x, Tx \rangle - \langle x, Tx \rangle \\ &= 0 \end{aligned}$$

for all $x \in \mathcal{H}$. Therefore by Proposition (0.5), we see that $T - T^* = 0$, i.e. $T = T^*$. □

[1] If $Tx = 0$, then we clearly have (3), thus we assume $Tx \neq 0$.

Problem 6

Problem 6.a

Proposition 0.10. *Let $T: \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Then*

$$\|T^n\|^2 \leq \|T^{n+1}\|\|T^{n-1}\| \tag{4}$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and let $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Then

$$\begin{aligned}
\|T^n x\|^2 &= \langle T^n x, T^n x \rangle \\
&= \langle T^{n+1} x, T^{n-1} x \rangle \\
&\leq \|T^{n+1} x\|\|T^{n-1} x\| \\
&\leq \|T^{n+1}\|\|x\|\|T^{n-1}\|\|x\| \\
&\leq \|T^{n+1}\|\|T^{n-1}\|,
\end{aligned}$$

which implies (4). □

Problem 6.b

Proposition 0.11. *Let $T: \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Then*

$$\|T^n\| = \|T\|^n \tag{5}$$

for all $n \in \mathbb{N}$.

Proof. We prove (5) by induction on $n \geq 0$. The base case $n = 0$ and the case $n = 1$ are trivial. Assume that (5) holds for some $n \geq 1$. Then by (4), we have

$$\begin{aligned}
\|T^{n+1}\| &\geq \|T^{n-1}\|^{-1}\|T^n\|^2 \\
&= \|T\|^{1-n}\|T\|^{2n} \\
&= \|T\|^{n+1},
\end{aligned}$$

where we used the induction step to get from the first line to the second line.

For the reverse inequality, let $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Then

$$\begin{aligned}
\|T^{n+1} x\| &\leq \|T^n x\|\|Tx\| \\
&\leq \|T^n\|\|x\|\|Tx\| \\
&\leq \|T^n\|\|Tx\| \\
&\leq \|T^n\|\|T\| \\
&= \|T\|^n\|T\| \\
&= \|T\|^{n+1},
\end{aligned}$$

where we used the induction step to get from the fourth line to the fifth line. It follows that $\|T^{n+1}\| \leq \|T\|^{n+1}$. □