MATH 9500
FALL 2020
HOMEWORK 1

Due Monday, August 31, 2020

1. (5 pts) Give an example of a commutative ring (necessarily without identity) that does not have a maximal proper ideal (that is, given any ideal proper ideal $I \subsetneq R$ there is an ideal $J \subsetneq R$ such that $I \subsetneq J$).

2. Let R be a commutative ring with identity, and let $I \subsetneq R$ be a proper ideal. We define the radical of I to be $\text{rad(I)} = \sqrt{I} = \{x \in R | x^n \in I \text{ for some } n \in \mathbb{N}\}$. We also define (as per class) that the set of nilpotents of R is $N(R) = \{x \in R | x^n = 0 \text{ for some } n \in \mathbb{N}\}$.

   a) (5 pts) Show that $\text{rad}(I)$ is contained in the intersection of all prime ideals that contain I.
   b) (5 pts) Show the other containment (that is, show that $\text{rad}(I) = \bigcap_{P \supseteq I} P$ where the intersection is taken over all prime ideals containing I).
   c) (5 pts) Show that $N(R)$ is the intersection of all prime ideals in R.

3. (5 pts) Let R be a commutative ring with identity. We define the *Jacobson radical*, $J(R)$, to be the intersection of all the maximal ideals of R. Show that $x \in J(R)$ if and only if $1 + rx$ is a unit for all $r \in R$.

4. (5 pts) Let R be an integral domain and $\mathfrak{M} \subsetneq R$ be a maximal ideal. We recall that $R_{\mathfrak{M}} = \{\frac{r}{s} | r \in R, s \notin \mathfrak{M}\}$. Show that

$$R = \bigcap_{\mathfrak{M}:\text{maximal}} R_{\mathfrak{M}}$$

where the intersection ranges over all of the maximal ideals of R.