MATH 8500: HOMEWORK 1

DUE WEDNESDAY, JANUARY 20TH

You may work on homework together, but you must write up your solutions individually and write the names of the individuals with whom you worked. If you use any materials outside of the course materials, e.g., the internet, a different book, or discuss the problems with *anyone other than me*, make sure to provide a short citation.

PROBLEMS:

(1) Consider the system of equations
$$\begin{aligned} x^2 + y^2 - 1 &= 0 \\ xy - 1 &= 0. \end{aligned}$$
These equations describe the intersection of a circle and a parabola.
  (a) Symbolically find all four solutions to this system of equations.
  (b) Find a polynomial of degree four whose roots are x-values of the solutions you found in Part (a).
  (c) Show that the polynomial that you got from Part (b) lies in the ideal $\langle x^2 + y^2 - 1, xy - 1\rangle$.

(2) Let I be an ideal of $k[x_1, \ldots, x_n]$. Show that $\mathcal{G} = \{g_1, \ldots, g_s\} \subseteq I$ is a Gröbner basis of I if and only if the leading term of any element of I is divisible by a leading term of some g_i.

(3) An ideal I is a *radical ideal* if whenever $f^k \in I$, then $f \in I$. The *radical* of an ideal I is defined as $\sqrt{I} = \{f : f^k \in I \text{ for some } k \in \mathbb{N}\}$.
  (a) Let $X \subseteq \mathbb{A}^n_k$ and prove that $\mathcal{I}(X)$ is a radical ideal.
  (b) Let I and J be ideals such that $\sqrt{I} = \sqrt{J}$. Prove that $\mathcal{V}(I) = \mathcal{V}(J)$.

(4) Let V be an algebraic variety. V is *reducible* if there exist algebraic varieties V_1 and V_2 that are properly contained in V such that $V = V_1 \cup V_2$. A variety is *irreducible* if it is not reducible. Prove that V is irreducible if and only if $\mathcal{I}(V)$ is a prime ideal.

(5) Complete the introductory quiz