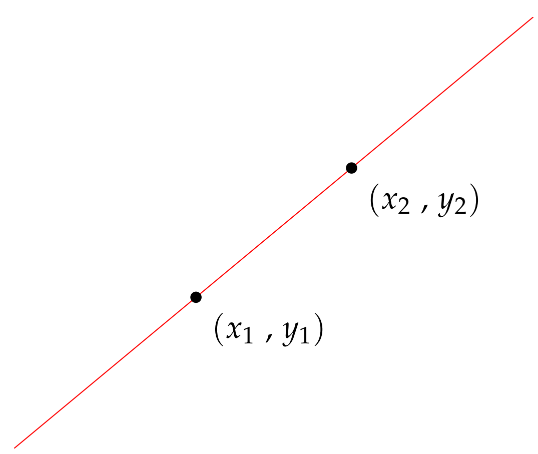
Homework #3 Solutions

Due Wednesday (March 4th)

This homework will be graded out of 30 points based on effort alone. Of course you must put serious effort into each problem for you to be given full credit on that problem. I will post the solutions on Wednesday and we will go over them to review for the test.

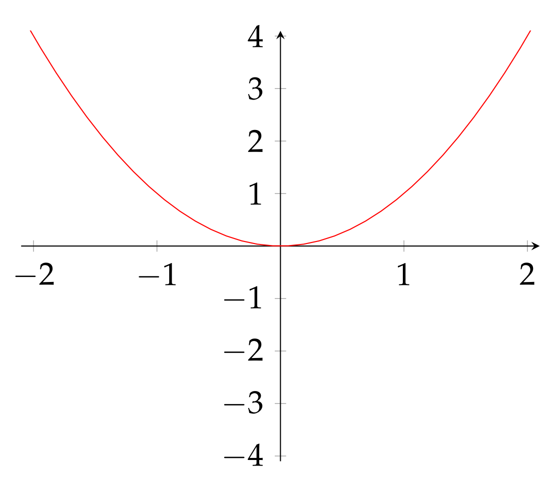
1. Suppose a line passes through the points (x_1, y_1) and (x_2, y_2), as shown below:



What is the slope of this line?

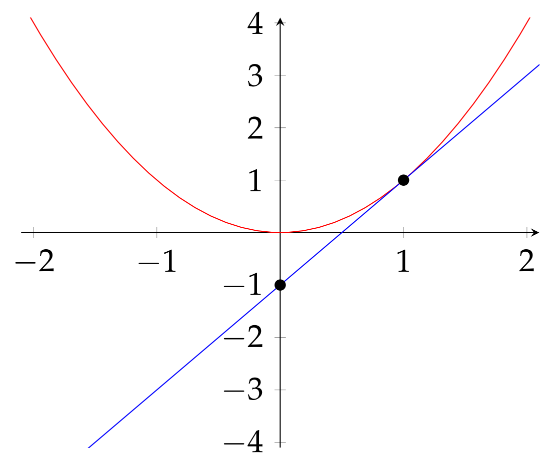
Solution: The slope is given by $\frac{y_2-y_1}{x_2-x_1}$.

2. Let $f(x)$ be the function given by the graph below:



2.a. Use a ruler to draw the tangent line to the graph of the function at the point $(1, 1)$. Find the slope of this line (Hint: You need to find two points this line goes through. Then you need to use the slope formula you wrote down above).

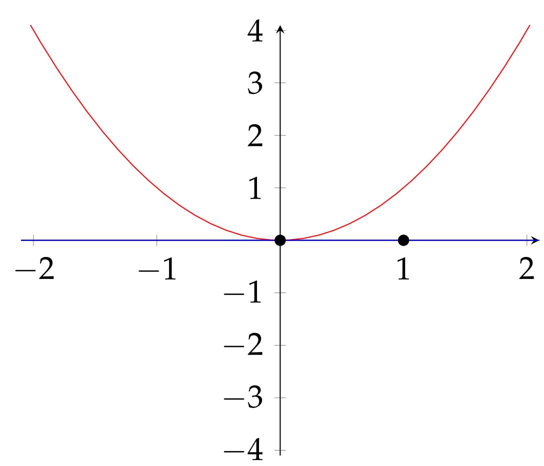
Solution:



The tangent line to the graph of the function at the point $(1,1)$ goes through the points $(1,1)$ and $(0,-1)$. Therefore, the slope of this line is $\frac{-1-1}{0-1} = 2$.

2.b. Use a ruler to draw the tangent line to the graph of the function at the point $(0,0)$. Find the slope of this line.

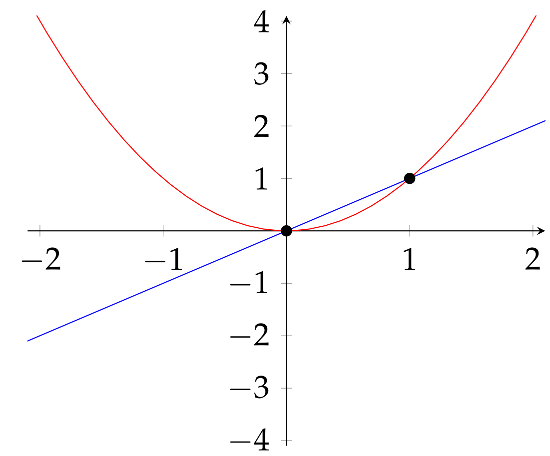
Solution:



The tangent line to the graph of the function at the point $(0,0)$ goes through the points $(0,0)$ and $(1,0)$. Therefore, the slope of this line is $\frac{0-0}{1-0} = 0$.

2.c. Use a ruler to draw the secant line through the points $(0,0)$ and $(1,1)$. Find the slope of this line.

Solution:



The secant line to the graph of the function through the points $(0,0)$ and $(1,1)$ goes through the points $(0,0)$ and $(1,1)$. Therefore, the slope of this line is $\frac{1-0}{1-0} = 1$.

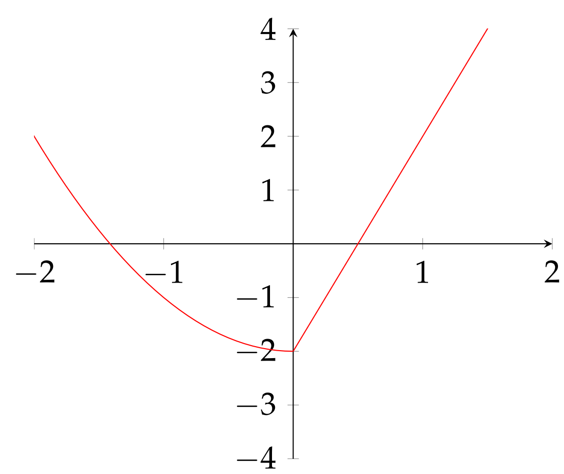
2.d. With the information above, evaluate $f(0)$, $f'(0)$, $f(1)$, and $f'(1)$. What is the average rate of change between the points $(0,0)$ and $(1,1)$?

Solution: We have

$$f(0) = 0$$
$$f'(0) = 0$$
$$f(1) = 1$$
$$f'(1) = 2$$

The average rate of change between the points $(0,0)$ and $(1,1)$ is given by the slope of the secant line which goes through these two points. Thus the average rate of change is 1.[1]

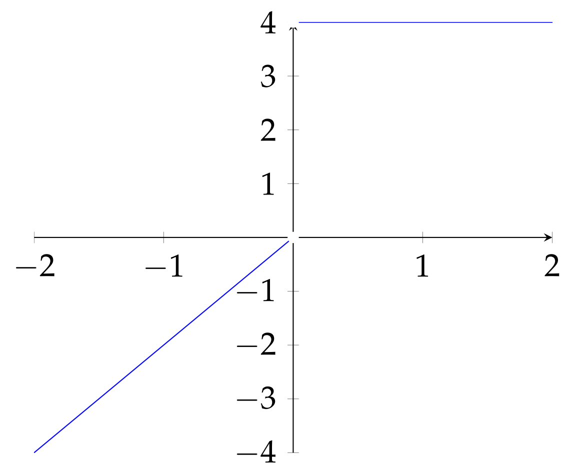
3. Let $f(x)$ be the function given by the graph below:



Sketch the graph of $f'(x)$:

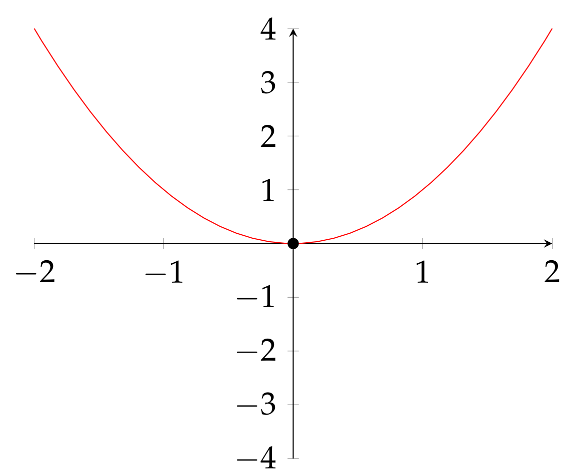
[1]On the test, you'll have to worry about units here. However in this problem, there are no units; we're just talking about functions.

Solution:



(Hint: The function goes through the point $(0.5, 0)$. Find the tangent line to the graph at this point, then find the slope of this line (this is $f'(0.5)$). Now plot the point $(0.5, f'(0.5))$. The function also goes through the point $(1, 2)$. Repeat the same procedure and plot the point $(1, f'(1))$. The function also goes through the point $(-2, 2)$. Repeat the same procedure and plot the point $(-2, f'(-2))$. You should be able to construct $f'(x)$ from this).

4. The function in the graph below is given by $f(x) = x^2$.

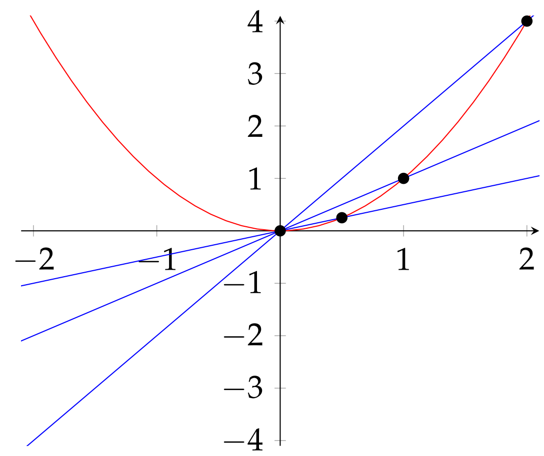


From problem 2, we know what the slope of the tangent line at $(0, 0)$ is (i.e. $f'(0)$). In this exercise, we want to compute $f'(0)$ by taking the limit of slopes of secant lines[2]

(4.a,4.b,4.c) Use your ruler to draw the secant line through the points $(0, 0)$ and $(2, 4)$. Then use your ruler to draw the secant line through the points $(0, 0)$ and $(1, 1)$. Then use your ruler to draw the secant line through the points $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$.

Solution:

[2] In Calculus, almost everything is defined in terms of limits.



4.d Complete the table below.

Solution:

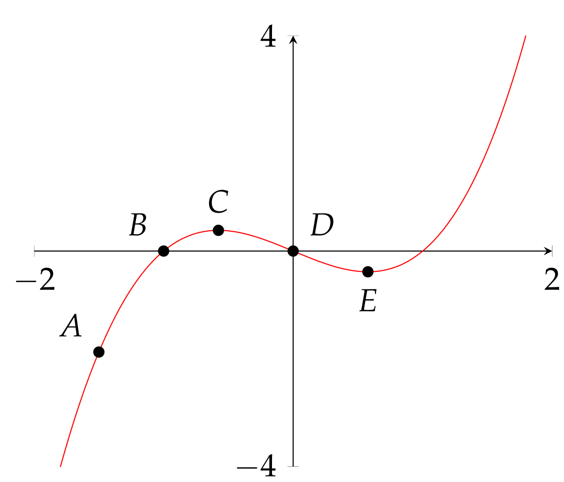
x	Slope of secant line from $(0,0)$ to $(x, f(x))$
2	2
1	1
0.5	0.5
0.1	0.1

What does this table suggest? (Hint use $\lim_{x\to 0^+} f(x)$ notation).

Solution: The table suggests

$$\lim_{x\to 0^+} f(x) = 0.$$

5. Let $f(x)$ be the function whose graph is given below:



(5.a) List the points A, B, C, D, E from least to greatest steepness.

Solution: C, E, D, B, A

(5.b) List the points A, B, C, D, E from least to greatest slope.

Solution: D, C, E, B, A

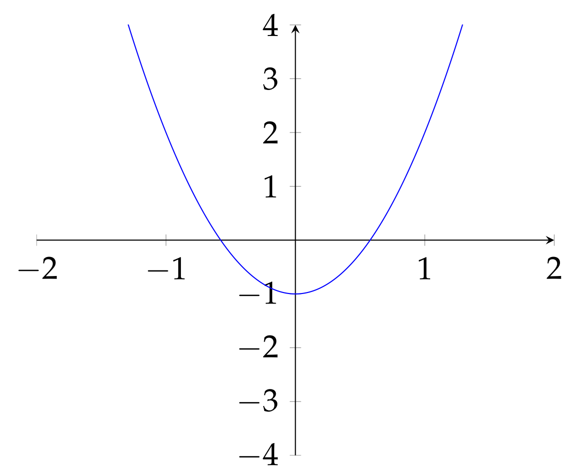
(5.c) How many x-intercepts will the slope graph of $f(x)$ have? Which points out of A, B, C, D, E gives us this information.

Solution: The slope graph of $f(x)$ will have two x-intercept points. The points C and E tells us this since the tangent line at those points is a horizontal line (and hence has slope 0).

(5.d) How many relative mins/maxes will the slope graph of $f(x)$ have? Which points out of A, B, C, D, E gives us this information.

Solution: The slope graph of $f(x)$ will have one relative min/max. The point D tells us this because D is an inflection point.

(5.e) Sketch the slope graph below

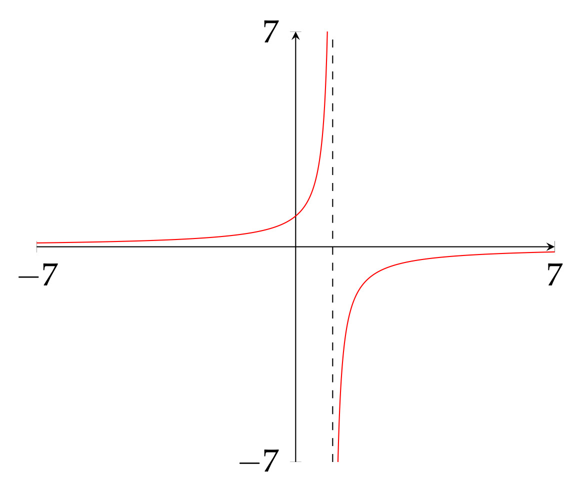


6. Let $g(x)$ be a function and let a be a real number. List three reasons why $g'(a)$ may not exist.

Solution:

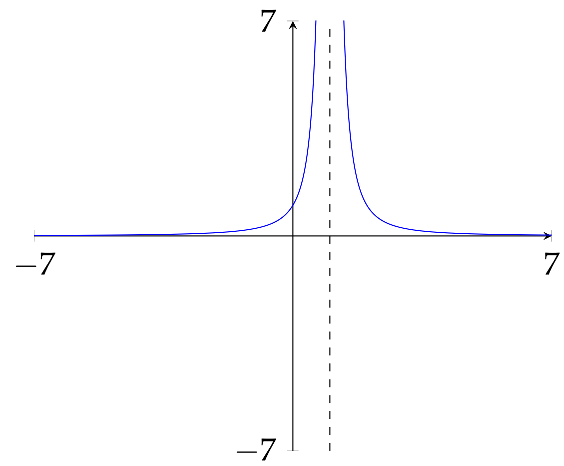
1. $g(x)$ is not continuous at $x = a$.
2. The graph of $g(x)$ has a sharp corner at the point $(a, g(a))$.
3. The line tangent to the graph of $g(x)$ at the point $(a, g(a))$ is a vertical line.

7. Let $h(x)$ be the function whose graph is given below



Give a sketch of the slope graph of $h(x)$. (You don't need to be completely accurate here. Just draw tangent lines of various points on the graph of $h(x)$. Are the slopes of these tangent lines positive/negative? If positive, then plot a positive output at the same input in the slope graph of $h(x)$)

Solution:



8. Find the derivative of $f(x) = 2x^2 + 0.1x - 2$ using the *limit* definition. Show all steps.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)^2 + 0.1(x+h) - 2 - (2x^2 + 0.1x - 2)}{h} \\
\text{(factor)} &= \lim_{h\to 0} \frac{2(x^2 + 2hx + h^2) + 0.1(x+h) - 2 - (2x^2 + 0.1x - 2)}{h} \\
\text{(distribute)} &= \lim_{h\to 0} \frac{2x^2 + 4hx + 2h^2 + 0.1x + 0.1h - 2 - 2x^2 - 0.1x + 2}{h} \\
\text{(combine like terms)} &= \lim_{h\to 0} \frac{(2-2)x^2 + (4h + 0.1 - 0.1)x + (2h^2 + 0.1h - 2 + 2)}{h} \\
\text{(simplify)} &= \lim_{h\to 0} \frac{4hx + 2h^2 + 0.1h}{h} \\
\text{(factor the } h \text{ out)} &= \lim_{h\to 0} \frac{h(4x + 2h + 0.1)}{h} \\
\text{(cancel the } h) &= \lim_{h\to 0} (4x + 2h + 0.1) \\
\text{(set } h = 0) &= 4x + 0.1
\end{aligned}$$

9. Let a, b be real numbers (i.e. constants). Simplify the following expressions. You need to write each as $x^{\text{something}}$ (I'll do the first for you).

Solution:

$$\frac{1}{x^a} = x^{-a}$$

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$\frac{1}{x^a} = x^{-a}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$$

10. It's important to keep track of notation. Write three ways of expressing (in terms of notation, i.e. symbols) the derivative of a function $f(x)$. Then write two ways of expressing the derivative of a function $f(x)$ *evaluated* at a number a.

Solution:

1. $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(f(x))$

2. $f'(a) = \frac{\mathrm{d}f}{\mathrm{d}x}|_{x=a}$.

11. Let $f(x)$ and $g(x)$ be functions and let a and b be real numbers (i.e. constants). Complete the expressions below (I'll do the first two for you).

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}(a) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(af(x)) = a\frac{\mathrm{d}}{\mathrm{d}x}(f(x))$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(g(x))$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = ax^{a-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = \ln(a)a^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

12. Find the derivative of the following functions (I'll do the first one for you)

Solution:

$$p(x) = x^3 + 3x^2$$

$$\begin{aligned}
p'(x) &= \frac{d}{dx}(p(x)) \\
&= \frac{d}{dx}(x^3+3x^2) \\
&= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2) \\
&= \frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) \\
&= 3x^{3-1} + 3\cdot 2x^{2-1} \\
&= 3x^2 + 6x
\end{aligned}$$

$$h(t) = t^2 + 5\sqrt{t} + \frac{2}{t^3}$$

$$\begin{aligned}
h'(t) &= \frac{d}{dt}(h(t)) \\
&= \frac{d}{dt}\left(t^2 + 5\sqrt{t} + \frac{2}{t^3}\right) \\
&= \frac{d}{dt}(t^2) + \frac{d}{dt}(5\sqrt{t}) + \frac{d}{dt}\left(\frac{2}{t^3}\right) \\
&= \frac{d}{dt}(t^2) + \frac{d}{dt}(5t^{\frac{1}{2}}) + \frac{d}{dt}(2t^{-3}) \\
&= \frac{d}{dt}(t^2) + 5\frac{d}{dt}(t^{\frac{1}{2}}) + 2\frac{d}{dt}(t^{-3}) \\
&= 2t^{2-1} + 5 \cdot \frac{1}{2}t^{\frac{1}{2}-1} + 2 \cdot -3t^{-3-1} \\
&= 2t + \frac{5}{2}t^{-\frac{1}{2}} - 6t^{-4}.
\end{aligned}$$

$$h(x) = \sqrt{\frac{1}{\sqrt[3]{x}}} + \sqrt{x^3}$$

$$\begin{aligned}
h'(x) &= \frac{d}{dx}(h(x)) \\
&= \frac{d}{dx}\left(\sqrt{\frac{1}{\sqrt[3]{x}}} + \sqrt{x^3}\right) \\
&= \frac{d}{dx}\left(\sqrt{\frac{1}{\sqrt[3]{x}}}\right) + \frac{d}{dx}\left(\sqrt{x^3}\right) \\
&= \frac{d}{dx}\left(\left(\frac{1}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right) + \frac{d}{dx}((x^3)^{\frac{1}{2}}) \\
&= \frac{d}{dx}((x^{-\frac{1}{3}})^{\frac{1}{2}}) + \frac{d}{dx}((x^3)^{\frac{1}{2}}) \\
&= \frac{d}{dx}(x^{-\frac{1}{6}}) + \frac{d}{dx}(x^{\frac{3}{2}}) \\
&= -\frac{1}{6}x^{-\frac{1}{6}-1} + \frac{3}{2}x^{\frac{3}{2}-1} \\
&= -\frac{1}{6}x^{-\frac{7}{6}} + \frac{3}{2}x^{\frac{1}{2}}.
\end{aligned}$$

$$g(t) = \frac{3t^3 - t^2 + 1}{t}$$

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(g(t)) \\
&= \frac{d}{dt}(\frac{3t^3 - t^2 + 1}{t}) \\
&= \frac{d}{dt}(\frac{3t^3}{t} - \frac{t^2}{t} + \frac{1}{t}) \\
&= \frac{d}{dt}(3t^2 - t + t^{-1}) \\
&= \frac{d}{dt}(3t^2) + \frac{d}{dt}(-t) + \frac{d}{dt}(t^{-1}) \\
&= 3\frac{d}{dt}(t^2) - \frac{d}{dt}(t) + \frac{d}{dt}(t^{-1}) \\
&= 3 \cdot 2t^{2-1} - 1 - t^{-1-1} \\
&= 6t - 1 - \frac{1}{t^2}.
\end{aligned}$$

$$g(x) = 2\ln x + 3^x$$

$$\begin{aligned}
g'(x) &= \frac{d}{dx}(g(x)) \\
&= \frac{d}{dx}(2\ln x + 3^x) \\
&= \frac{d}{dx}(2\ln x) + \frac{d}{dx}(3^x) \\
&= 2\frac{d}{dx}(\ln x) + \frac{d}{dx}(3^x) \\
&= 2 \cdot \frac{1}{x} + \ln(3)3^x \\
&= \frac{2}{x} + \ln(3)3^x.
\end{aligned}$$

$$f(x) = 5e^x + e^e$$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(f(x)) \\
&= \frac{d}{dx}(5e^x + e^e) \\
&= \frac{d}{dx}(5e^x) + \frac{d}{dx}(e^e) \\
&= 5\frac{d}{dx}(e^x) + \frac{d}{dx}(e^e) \\
&= 5e^x + 0 \\
&= 5e^x.
\end{aligned}$$