DG Algebra Associativity

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1 Setup

Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $\mathfrak{a}$ be an ideal in R. Fix a minimal free resolution F of $R/\mathfrak{a}$ over R. The differential on F is denoted by $\partial\colon F \to F$ and the augmentation map is denoted by $\tau\colon F \to R/\mathfrak{a}$. We also fix $\{e_1, \dots, e_n\}$ to be a basis of F as a free graded R-module. Thus

$$F = \bigoplus_{i=1}^{n} Re_i.$$

The differential ∂ can be characterized interms of this basis. Indeed, for each $1 \leq i \leq n$, we have

$$\partial(e_i) = \sum_{j=1}^{n} a_i^j e_j$$

where $a_i^j \in R$ for each $1 \leq j \leq n$. The condition $\partial^2 = 0$ translates to the condition $\sum_{j=1}^{n} a_i^j a_j^k = 0$ for all $1 \leq i, k \leq n$. The condition that ∂ is graded of degree -1 translated to the condition $a_i^j = 0$ if $|e_j| \neq |e_i| - 1$ for all $1 \leq i, j \leq n$ where $|\cdot|$ denotes the homological degree of a homogeneous element in F.

Definition 1.1. With the notation above, a **multiplication** on F is a chain map $\mu\colon F \otimes F \to F$ which lifts the usual multiplication $\mathrm{m}\colon R/\mathfrak{a} \otimes R/\mathfrak{a} \to R/\mathfrak{a}$ where $\mathrm{m}(\overline{x} \otimes \overline{y}) = \overline{xy}$ for all $x, y \in R$. In other words, μ is a chain map which satisfies $\tau\mu = \mathrm{m}(\tau \otimes \tau)$.

Let us fix μ to be a multiplication on F. For convenience, we often denote $\mu(\alpha, \beta) = \alpha \star_\mu \beta$ for all $\alpha, \beta \in F$. If the multiplication μ is understood from context, then we will drop the μ from the subscript altogether and just write $\mu(\alpha, \beta) = \alpha \star \beta$ for all $\alpha, \beta \in F$. Just like the differential, the multiplication μ can be characterized in terms of the basis. Indeed, for each $1 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=1}^{n} c_{i,j}^k e_k$$

where $c_{i,j}^k \in R$ for all $1 \leq k \leq n$. In this case, μ being graded translates to $c_{i,j}^k = 0$ if $|e_i| + |e_j| \neq |e_k|$ for all $1 \leq i, j, k \leq n$. Also μ satisfying Leibniz law translates to the identity

$$\sum_{1 \leq k.l \leq n} c_{i,j}^k a_k^l = \sum_{1 \leq k,l \leq n} c_{k,j}^l a_i^k + (-1)^{|e_i|} c_{i,k}^l a_j^k$$

for all $1 \leq i, j \leq n$.

1.1 Associatior

With the notation above, the **associator** with respect to μ is the map $[\cdot,\cdot,\cdot]_\mu\colon F \otimes F \otimes F \to F$ defined by

$$[\alpha, \beta, \gamma]_\mu = (\alpha \star_\mu \beta) \star_\mu \gamma - \alpha \star_\mu (\beta \star_\mu \gamma)$$

for all $\alpha, \beta, \gamma \in F$. If μ is understood from context, then we suppress μ from the subscript in $[\cdot,\cdot,\cdot]_\mu$. It is easy to see that $[\cdot,\cdot,\cdot]$ is a graded trilinear map precisely because μ is a graded bilinear map. Note that since μ satisfies Leibniz law, the associatior satisfies

$$\partial[\alpha, \beta, \gamma] = [\partial\alpha, \beta, \gamma] + (-1)^{|\alpha|}[\alpha, \partial\beta, \gamma] + (-1)^{|\alpha|+|\beta|}[\alpha, \beta, \partial\gamma]. \tag{1}$$

for all homogeneous $\alpha, \beta, \gamma \in F$. Indeed, we have

$$\begin{aligned}
\partial[\alpha,\beta,\gamma] &= \partial((\alpha \star \beta) \star \gamma) - \alpha \star (\beta \star \gamma)) \\
&= \partial((\alpha \star \beta) \star \gamma) - \partial(\alpha \star (\beta \star \gamma)) \\
&= \partial(\alpha \star \beta) \star \gamma + (-1)^{|\alpha|+|\beta|}(\alpha \star \beta) \star \partial\gamma - \partial\alpha \star (\beta \star \gamma) - (-1)^{|\alpha|}\alpha \star \partial(\beta \star \gamma) \\
&= (\partial\alpha \star \beta + (-1)^{|\alpha|}\alpha \star \partial\beta) \star \gamma + (-1)^{|\alpha|+|\beta|}(\alpha \star \beta) \star \partial\gamma - \partial\alpha \star (\beta \star \gamma) - (-1)^{|\alpha|}\alpha \star (\partial\beta \star \gamma + (-1)^{|\beta|}\beta \star \partial\gamma) \\
&= (\partial\alpha \star \beta) \star \gamma + (-1)^{|\alpha|}(\alpha \star \partial\beta) \star \gamma + (-1)^{|\alpha|+|\beta|}(\alpha \star \beta) \star \partial\gamma - \partial\alpha \star (\beta \star \gamma) - (-1)^{|\alpha|}\alpha \star (\partial\beta \star \gamma) - (-1)^{|\alpha|+|\beta|}\alpha \star (\beta \star \partial\gamma) \\
&= [\partial\alpha,\beta,\gamma] + (-1)^{|\alpha|}[\alpha,\partial\beta,\gamma] + (-1)^{|\alpha|+|\beta|}[\alpha,\beta,\partial\gamma].
\end{aligned}$$

The identity (1) is what makes $[\cdot,\cdot,\cdot]$ a chain map and not just a graded trilinear map.

1.2 Homology of $[F, F, F]$

Let us denote by $[F, F, F]$ to be the image of the trilinear map $[\cdot,\cdot,\cdot]$. Thus $[F, F, F]$ is an R-subcomplex of F. We wish to understand the homology of $[F, F, F]$.

Proposition 1.1. *The following conditions are equivalent*

1. *μ is not associative.*
2. *$[F, F, F] \neq 0$.*
3. *$\mathrm{H}[F, F, F] \neq 0$.*

Proof. That 1 and 2 are equivalent is trivial. That 3 implies 2 is also trivial. Let us show that 2 implies 3. Suppose $[F, F, F] \neq 0$. Choose $m \in \mathbb{N}$ minimal so that $[F, F, F]_m \neq 0$ and $[F, F, F]_{m-1} = 0$. Note that necessarily we have $m \geq 3$. By Nakayama's Lemma, we can find a triple (e_i, e_j, e_k) such that $|e_i| + |e_j| + |e_k| = m$ and such that $[e_i, e_j, e_k] \notin \mathfrak{m}[F, F, F]_m$. By minimality of m, we have $\partial[e_i, e_j, e_k] = 0$. Also, since F is minimal, we have $\partial[F, F, F] \subseteq \mathfrak{m}[F, F, F]$. Thus $[e_i, e_j, e_k]$ represents a nontrivial element in homology. □

Proposition (1.1) tells us that we can characterize the failure of μ being associative in terms of the homology of $[F, F, F]$. With this in mind, we make the following definition:

Definition 1.2. Let $(R, \mathfrak{m})$ be a local Noetherian ring, let $\mathfrak{a}$ be an ideal in R, and fix a minimal free resolution F of $R/\mathfrak{a}$ over R. We define

$$\mathrm{Assoc}(R/\mathfrak{a}) = \sup_{\mu}\{\inf_{i}\{\mathrm{H}_i[F, F, F]_\mu \neq 0\}\},$$

where the supremum is taken over all multiplications μ on F and the infimum is taken over all $i \in \mathbb{Z}$.

1.3 DG-Algebra Associated to F and μ

We now wish to associate to F and μ a DG-algebra. Let $S_\mu = R[e_1, \dots, e_n]$ be the free differential graded R-algebra generated by variables $e_1, \dots, e_n$. In particular, in S_μ we have

$$e_i e_j = (-1)^{|i||j|} e_j e_i$$

for all $1 \leq i, j \leq n$. We view F as a subcomplex of S_μ. The differential on S_μ is denoted d_μ and is defined by

$$\mathrm{d}_\mu = \sum_{i=1}^{n} \partial(e_i)\partial_{e_i}.$$

The differential d_μ extends the differential ∂ in the sense that $\mathrm{d}_\mu|_F = \partial$. For instance, we have

$$\begin{aligned}
\mathrm{d}_\mu(e_i e_j) &= \mathrm{d}_\mu(e_i)e_j + (-1)^{|e_i|}e_i \mathrm{d}_\mu(e_j) \\
&= \partial(e_i)e_j + (-1)^{|e_i|}e_i\partial(e_j) \\
&= \sum_{k=1}^{n} a_i^k e_k e_j + (-1)^{|e_i|} a_j^k e_i e_k.
\end{aligned}$$

Finally let I_μ be the ideal in S_μ given by

$$I_\mu = \langle \{e_i e_j - e_i \star e_j \mid 1 \leq i, j \leq n\} \rangle.$$

As usual, we supress μ from the subscript in S_μ, d_μ, and I_μ whenever μ is understood from context. Note that I is d-stable. Thus $(S/I, \overline{\mathrm{d}})$ is differential graded R-algebra where $\overline{\mathrm{d}}$ is the differential on S/I induced by d.

1.4 Homology of S/I

We now wish to study the homology of S/I. First we need a lemma.

Lemma 1.1. *We have* $[F, F, F] = I \cap F$.

Proof. For each $1 \le i, j \le n$, write $f_{i,j} = e_i e_j - e_i \star e_j$. Thus $I = \langle \{f_{i,j}\} \rangle$. We wish to cancel the lead terms in each $f_{i,j}$. To this end, let $S_{i,j,k} = e_i f_{j,k} - f_{i,j} e_k$ for each $1 \le i, j, k \le n$. Observe that

$$\begin{aligned}
S_{i,j,k} &= e_i f_{j,k} - f_{i,j} e_k \\
&= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
&= e_i(e_j e_k) - (e_i e_j) e_k + (e_i \star e_j) e_k - e_i(e_j \star e_k) \\
&= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
&= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l
\end{aligned}$$

In particular, we see that

$$\begin{aligned}
S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k} e_i e_l - \sum_l c_{i,j} f_{l,k} + \sum_l c_{j,k} f_{i,l} \\
&= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) - \sum_l c_{j,k} (f_{i,l} - e_i e_l) \\
&= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k].
\end{aligned}$$

Therefore $[e_i, e_j, e_k] \in I$ for all $1 \le i, j, k \le n$ and hence $[F, F, F] \subseteq I \cap F$.

Conversely, suppose $\sum_{r=1}^s g_r f_{i_r, j_r} \in I \cap F$ where $g_r \in S$ for all $1 \le r \le s$. We may assume that $i_r < j_r$ for each $1 \le r \le s$ and that $(i_r, j_r) \ne (i_{r'}, j_{r'})$ whenever $r \ne r'$. Then since $\sum_{r=1}^s g_r f_{i_r, j_r} \in F$, we have

$$\sum_{r=1}^s g_r f_{i_r, j_r} = \sum_{k=1}^n b_k e_k$$

where $b_k \in R$ for all $1 \le k \le n$. Thus

$$\begin{aligned}
\sum_{k=1}^n b_k e_k &= \sum_{r=1}^s g_r f_{i_r, j_r} \\
&= \sum_{r=1}^s g_r (e_{i_r} e_{j_r} - e_{i_r} \star e_{j_r}) \\
&= \sum_{r=1}^s g_r e_{i_r} e_{j_r} - \sum_{r=1}^s g_r (e_{i_r} \star e_{j_r}) \\
&= \sum_{r=1}^s g_r e_{i_r} e_{j_r} - \sum_{\substack{1 \le r \le s \\ 1 \le k \le n}} c_{i_r, j_r}^k g_r e_k.
\end{aligned}$$

Now for each $1 \le r \le s$, we express g_r as $g_r = g_{r,0} + g_{r,1} + \cdots + g_{r,d_r}$ where d_r is the degree of g_r and $g_{r,k} = \sum a_{i_1, \dots, i_k} e_{i_1} \cdots e_{i_k}$ is its degree k part for each $1 \le k \le d_r$. It follows that

$$\begin{aligned}
b_k &= \sum_{r=1}^s c_{i_r, j_r}^k g_{r,0} \\
0 &= \sum_{r=1}^s g_{r,0} e_{i_r} e_{j_r} - \sum_{r=1}^s c_{i_r, j_r}^k g_{r,1} e_k
\end{aligned}$$

for each k. Thus

$$\begin{aligned}
b_k &= \sum_{r=1}^s c_{i_r, j_r}^k g_{r,0} \\
&=
\end{aligned}$$

$$
\begin{aligned}
0 &= \sum_{r=1}^{s} a_r[e_{i_r}, e_{j_r}, e_{k_r}] \\
&= \sum_{r=1}^{s} a_r((e_{i_r} \star e_{j_r}) \star e_{k_r} - e_{i_r} \star (e_{j_r} \star e_{k_r})) \\
&= \sum_{\substack{1\leq r\leq s \\ 1\leq l\leq n}} a_r(c^l_{i_r,j_r} e_l \star e_{k_r} - c^l_{j_r,k_r} e_{i_r} \star e_l) \\
&= \sum_{\substack{1\leq r\leq s \\ 1\leq l,m\leq n}} (a_r c^l_{i_r,j_r} c^m_{l,k_r} - a_r c^l_{j_r,k_r} c^m_{i_r,l}) e_m.
\end{aligned}
$$

□

Theorem 1.2. *We have*

$$\mathrm{H}_i(S/I) \cong \mathrm{H}_{i-1}([F,F,F])$$

for all $i \in \mathbb{Z}$.

Proof. First note that every element in S/I can be represented by an element in F, that is, every element in S/I has the form $\overline{\alpha}$ where $\alpha \in F$. In particular, the chain map $\pi\colon F \to S/I$ given by $\pi(\alpha) = \overline{\alpha}$ for all $\alpha \in F$ is surjective. The kernel of π is $I \cap F = [F,F,F]$. Thus π induces an isomorphism of R-complexes

$$\overline{\pi}\colon F/[F,F,F] \to S/I.$$

Thus to understand $\mathrm{H}(S/I)$, we just need to study $\mathrm{H}(F/[F,F,F])$. Using the fact that $\mathrm{H}(F) \cong 0$, observe that short exact sequence of R-complexes

$$0 \longrightarrow [F,F,F] \longrightarrow F \longrightarrow F/[F,F,F] \longrightarrow 0$$

induces isomorphisms

$$\mathrm{H}_{i-1}([F,F,F]) \cong \mathrm{H}_i(F/[F,F,F]) \cong \mathrm{H}_i(S/I)$$

for all $i \in \mathbb{Z}$. □

Lemma 1.3. *Let (A, d_A) be a differential graded R-algebra such that* $\mathrm{im}\,(\mathrm{d}_A)_1 = \mathfrak{a}$ *and let $x \in \mathfrak{a}$. Then the multiplication map $\mathrm{m}_x\colon A \to A$ is null-homotopic. In particular, $\mathfrak{a}\mathrm{H}(A) \cong 0$.*

Proof. Choose $\alpha \in A_1$ such that $\mathrm{d}_A(\alpha) = x$ and let $\mathrm{h}\colon A \to A$ be the unique graded homomorphism of degree 1 given by $\mathrm{h}(\beta) = \alpha\beta$ for all $\beta \in A$. Then observe that for all $\beta \in A$, we have

$$
\begin{aligned}
(\mathrm{d}_A\mathrm{h} + \mathrm{h}\mathrm{d}_A)(\beta) &= (\mathrm{d}_A\mathrm{h} + \mathrm{h}\mathrm{d}_A)(\beta) \\
&= \mathrm{d}_A\mathrm{h}(\beta) + \mathrm{h}\mathrm{d}_A(\beta) \\
&= \mathrm{d}_A(\alpha\beta) + \mathrm{h}\mathrm{d}_A(\beta) \\
&= \mathrm{d}_A(\alpha)\beta - \alpha\mathrm{d}_A(\beta) + \mathrm{h}\mathrm{d}_A(\beta) \\
&= x\beta - \alpha\mathrm{d}_A(\beta) + \alpha\mathrm{d}_A(\beta) \\
&= x\beta \\
&= \mathrm{m}_x(\beta).
\end{aligned}
$$

It follows that $\mathrm{m}_x = \mathrm{d}_A\mathrm{h} + \mathrm{h}\mathrm{d}_A$. Thus m_x is null-homotopic. In particular, this means $x\mathrm{H}(A) \cong 0$, and since $x \in \mathfrak{a}$ was arbitrary, we have $\mathfrak{a}\mathrm{H}(A) \cong 0$. □

Corollary. *We have* $\mathfrak{a}\mathrm{H}([F,F,F]) \cong 0$.

Proof. Note that □

1.5 Example

In this subsection, we apply our theory to a specific example. Let K be a field. For simplicity, we assume $\mathrm{char}\, K = 2$. Suppose $R = K[x,y,z,w]$ and $\mathfrak{a} = \langle x^2, w^2, zw, xy, y^2z^2 \rangle$. Then the minimal free resolution F of $R/\mathfrak{a}$ over R is supported on a simplicial complex.

2 Extra

Suppose $\sum_{r=1}^{s} a_r[e_{i_r}, e_{j_r}, e_{k_r}] = 0$. Then

$$\begin{aligned}
0 &= \sum_{r=1}^{s} a_r[e_{i_r}, e_{j_r}, e_{k_r}] \\
&= \sum_{r=1}^{s} a_r((e_{i_r} \star e_{j_r}) \star e_{k_r} - e_{i_r} \star (e_{j_r} \star e_{k_r})) \\
&= \sum_{\substack{1\leq r\leq s \\ 1\leq l\leq n}} a_r(c_{i_r,j_r}^{l} e_l \star e_{k_r} - c_{j_r,k_r}^{l} e_{i_r} \star e_l) \\
&= \sum_{\substack{1\leq r\leq s \\ 1\leq l,m\leq n}} (a_r c_{i_r,j_r}^{l} c_{l,k_r}^{m} - a_r c_{j_r,k_r}^{l} c_{i_r,l}^{m}) e_m.
\end{aligned}$$

It follows that

$$\sum_{\substack{1\leq r\leq s \\ 1\leq l,m\leq n}} a_r(c_{i_r,j_r}^{l} c_{l,k_r}^{m} - c_{j_r,k_r}^{l} c_{i_r,l}^{m}) = 0.$$

Conversely, suppose $\sum_{1\leq i<j\leq n} g_{i,j} f_{i,j} \in I \cap F$ where $g_{i,j} \in S$ for all $1 \leq i < j \leq n$. Then since $\sum_{1\leq i<j\leq n} g_{i,j} f_{i,j} \in F$, we have

$$\sum_{1\leq i<j\leq n} g_{i,j} f_{i,j} = \sum_{k=1}^{n} a_k e_k \tag{2}$$

where $a_k \in R$ for all $1 \leq k \leq n$. Now for each $1 \leq i < j \leq n$, we express $g_{i,j}$ as $g_{i,j} = g_{i,j,0} + g_{i,j,1} + \cdots + g_{i,j,s_{i,j}}$ where $s_{i,j}$ is the degree of $g_{i,j}$ and $g_{i,j,r}$ and is the degree r part of $g_{i,j}$ for each $1 \leq r \leq s_{i,j}$. The degree 1 part of (2) gives us

$$\begin{aligned}
-\sum_{k} a_k e_k &= \sum_{i,j} g_{i,j,0} e_i \star e_j \\
&= \sum_{i,j} g_{i,j,0} c_{i,j}^{k} e_k.
\end{aligned}$$

Thus for each k we have $-\sum_{i,j} g_{i,j,0} c_{i,j}^{k} = a_k$. Thus

$$\begin{aligned}
\sum_{k} a_k e_k &= -\sum_{k}\sum_{i,j} g_{i,j,0} c_{i,j}^{k} e_k \\
&= -\sum_{i,j} g_{i,j,0} \sum_{k} c_{i,j}^{k} e_k \\
&= -\sum_{i,j} g_{i,j,0} (e_i \star e_j)
\end{aligned}$$

The degree 2 part of (2) gives us

$$\begin{aligned}
0 &= \sum_{i,j} g_{i,j,0} e_i e_j - g_{i,j,1}(e_i \star e_j) \\
&= \sum_{i,j} g_{i,j,0} e_i e_j - g_{i,j,1} \sum_{k} c_{i,j}^{k} e_k
\end{aligned}$$

$$\sum_{i,j}$$

$$-\sum_{i,j} g_{i,j,0} e_i \star e_j = \sum_{k} a_k e_k$$

$$
\begin{aligned}
\sum_{k=1}^{n} b_k e_k &= \sum_{r=1}^{s} g_r f_{i_r,j_r} \\
&= \sum_{r=1}^{s} g_r (e_{i_r} e_{j_r} - e_{i_r} \star e_{j_r}) \\
&= \sum_{r=1}^{s} g_r e_{i_r} e_{j_r} - \sum_{r=1}^{s} g_r (e_{i_r} \star e_{j_r}) \\
&= \sum_{r=1}^{s} g_r e_{i_r} e_{j_r} - \sum_{\substack{1 \le r \le s \\ 1 \le k \le n}} c^k_{i_r,j_r} g_r e_k.
\end{aligned}
$$

Now for each $1 \le r \le s$, we express g_r as $g_r = g_{r,0} + g_{r,1} + \cdots + g_{r,d_r}$ where d_r is the degree of g_r and $g_{r,k} = \sum a_{i_1,\dots,i_k} e_{i_1} \cdots e_{i_k}$ is its degree k part for each $1 \le k \le d_r$. It follows that

$$
\begin{aligned}
b_k &= \sum_{r=1}^{s} c^k_{i_r,j_r} g_{r,0} \\
0 &= \sum_{r=1}^{s} g_{r,0} e_{i_r} e_{j_r} - \sum_{r=1}^{s} c^k_{i_r,j_r} g_{r,1} e_k
\end{aligned}
$$

for each k. Thus

$$
\begin{aligned}
b_k &= \sum_{r=1}^{s} c^k_{i_r,j_r} g_{r,0} \\
&=
\end{aligned}
$$

$$
\begin{aligned}
0 &= \sum_{r=1}^{s} a_r [e_{i_r}, e_{j_r}, e_{k_r}] \\
&= \sum_{r=1}^{s} a_r ((e_{i_r} \star e_{j_r}) \star e_{k_r} - e_{i_r} \star (e_{j_r} \star e_{k_r})) \\
&= \sum_{\substack{1 \le r \le s \\ 1 \le l \le n}} a_r (c^l_{i_r,j_r} e_l \star e_{k_r} - c^l_{j_r,k_r} e_{i_r} \star e_l) \\
&= \sum_{\substack{1 \le r \le s \\ 1 \le l,m \le n}} (a_r c^l_{i_r,j_r} c^m_{l,k_r} - a_r c^l_{j_r,k_r} c^m_{i_r,l}) e_m.
\end{aligned}
$$

Suppose

Attempt

Suppose

$$
\begin{aligned}
b_1 e_1 + b_2 e_2 + b_3 e_3 &= g_{1,2} f_{1,2} + g_{1,3} f_{1,3} \\
&= g_{1,2}(e_1 e_2 + \sum_k c^k_{1,2} e_k) + g_{1,3}(e_1 e_3 + \sum_k c^k_{1,3} e_k) \\
&= \left(a^{(1,0,0)}_{1,2} e_1 + a^{(0,1,0)}_{1,2} e_2 + a^{(0,0,1)}_{1,2} e_3 + a^{(0,0,0)}_{1,2} \right) (e_1 e_2 + c^1_{1,2} e_1 + c^2_{1,2} e_2 + c^3_{1,2} e_3) + \left(a^{(1,0,0)}_{1,3} e_1 + a^{(0,1,0)}_{1,3} e_2 + a^{(0,0,1)}_{1,3} e_3 + a^{(0,0,0)}_{1,3} \right) (e \\
&= \\
&= a^{(1,0,0)}_{1,2} e_1^2 e_2 + a^{(0,1,0)}_{1,2} e_1 e_2^2 + a^{(1,0,0)}_{1,3} e_1^2 e_3 + a^{(0,0,1)}_{1,3} e_1 e_3^2 + (a^{(0,0,1)}_{1,2} + a^{(0,1,0)}_{1,3}) e_1 e_2 e_3 \\
&\quad + (a^{(1,0,0)}_{1,2} c^1_{1,2} + a^{(1,0,0)}_{1,3} c^1_{1,3}) e_1^2 + (a^{(0,1,0)}_{1,2} c^2_{1,2} + a^{(0,1,0)}_{1,3} c^2_{1,3}) e_2^2 + (a^{(0,0,1)}_{1,2} c^3_{1,2} + a^{(0,0,1)}_{1,3} c^3_{1,3}) e_3^2 + (a^{(1,0,0)}_{1,2} c^2_{1,2} + a^{(0,1,0)}_{1,2} c^1_{1,2} + a^{(1,0,0)}_{1,3} c^2_{1,} \\
&\quad + (a^{(0,0,0)}_{1,2} c^1_{1,2} + a^{(0,0,0)}_{1,3} c^1_{1,3}) e_1 + (a^{(0,0,0)}_{1,2} c^2_{1,2} + a^{(0,0,0)}_{1,3} c^2_{1,3}) e_2 + (a^{(0,0,0)}_{1,2} c^3_{1,2} + a^{(0,0,0)}_{1,3} c^3_{1,3}) e_3
\end{aligned}
$$

This implies

$$
g_{1,2} = a^{(0,0,1)}_{1,2} e_3 + a^{(0,0,0)}_{1,2} \qquad \text{and} \qquad g_{1,3} = a^{(1,0,0)}_{1,3} e_1 + a^{(0,0,0)}_{1,3}
$$

and $a_{1,2}^{(0,0,1)} = a_{1,3}^{(1,0,0)}$ also

$$a_{1,3}^{(1,0,0)} c_{1,3}^{2} + a_{1,2}^{(0,0,0)} = 0$$

In other words

$$a_{1,2}^{(0,0,0)} = a_{1,2}^{(1,0,0)} c_{1,3}^{2}$$

Thus

$$g_{1,2} = a_{1,2}^{(0,0,1)} e_3 + a_{1,2}^{(1,0,0)} c_{1,3}^{2} \qquad \text{and} \qquad g_{1,3} = a_{1,3}^{(1,0,0)} e_1 + a_{1,3}^{(0,0,0)} c_{1,2}^{1}$$

,

Table

We write $g_{i,j} = \sum_{\alpha} a_{i,j}^{\alpha} e_{\alpha}$ for each $1 \leq i < j \leq n$. For instance, if $n = 3$, then $g_{1,2}$ is expressed as

$$g_{1,2} = a_{1,2}^{(0,0,0)} + a_{1,2}^{(1,0,0)} e_1 + a_{1,2}^{(0,1,0)} e_2 + a_{1,2}^{(0,0,1)} e_3 + a_{1,2}^{(2,0,0)} e_1^2 + a_{1,2}^{(1,1,0)} e_1 e_2 + a_{1,2}^{(1,0,1)} e_1 e_3 + a_{1,2}^{(0,1,1)} e_2 e_3 + \cdots$$

Now suppose $n = 3$ and $\deg g_{i,j} \leq 2$ for each $1 \leq i < j \leq 3$. Then since $\sum g_{i,j} f_{i,j}$ lands in $[F, F, F]$, the coefficients for the monomials of degree ≥ 2 in $\sum g_{i,j} f_{i,j}$ must all be equal to 0. The table below summarizes these relations obtained from the monomials in degree 4:

Monomial in degree 4	Relation given by equating coefficients
$e_1^2 e_2 e_3$	$a_{1,2}^{(1,0,1)} + a_{1,3}^{(1,1,0)} + a_{2,3}^{(2,0,0)} = 0$
$e_1 e_2^2 e_3$	$a_{1,2}^{(0,1,1)} + a_{1,3}^{(0,2,0)} + a_{2,3}^{(1,1,0)} = 0$
$e_1 e_2 e_3^2$	$a_{1,2}^{(0,0,2)} + a_{1,3}^{(0,1,1)} + a_{2,3}^{(1,0,1)} = 0$
$e_1^2 e_2^2$	$a_{1,2}^{(1,1,0)} = 0$
$e_1^2 e_3^2$	$a_{1,3}^{(1,0,1)} = 0$
$e_2^2 e_3^2$	$a_{2,3}^{(0,1,1)} = 0$
$e_1^3 e_2$	$a_{1,2}^{(2,0,0)} = 0$
$e_1^3 e_3$	$a_{1,3}^{(2,0,0)} = 0$
$e_1 e_2^3$	$a_{1,2}^{(0,2,0)} = 0$
$e_2^3 e_3$	$a_{2,3}^{(0,2,0)} = 0$
$e_1 e_3^3$	$a_{1,3}^{(0,0,2)} = 0$
$e_2 e_3^3$	$a_{2,3}^{(0,0,2)} = 0$

The table below summarizes these relations obtained from the monomials in degree 3:

Monomial in degree 4	Relation given by equating coefficients
$e_1 e_2 e_3$	$a_{1,2}^{(0,0,1)} + a_{1,3}^{(0,1,0)} + a_{2,3}^{(1,0,0)} + a_{1,2}^{(1,0,1)} c_{1,2}^{(0,1,0)} + a_{1,2}^{(0,1,1)} c_{1,2}^{(1,0,0)} + a_{1,3}^{(1,1,0)} c_{1,3}^{(0,0,1)} + \cdots + a_{2,3}^{(1,1,0)} c_{2,3}^{(0,0,1)} = 0$
$e_1^2 e_2$	$a_{1,2}^{(1,0,0)} + a_{1,3}^{(1,1,0)} c_{1,3}^{(1,0,0)} + a_{2,3}^{(1,1,0)} c_{2,3}^{(1,0,0)} + a_{2,3}^{(2,0,0)} c_{2,3}^{(0,1,0)} = 0$
$e_1^2 e_3$	$a_{1,3}^{(1,0,0)} + a_{1,2}^{(1,0,1)} c_{1,2}^{(1,0,0)} + a_{2,3}^{(1,0,1)} c_{2,3}^{(1,0,0)} + a_{2,3}^{(2,0,0)} c_{2,3}^{(0,0,1)} = 0$
$e_1 e_2^2$	$\vdots$
$e_1 e_3^2$	$\vdots$
$e_2^2 e_3$	$\vdots$
$e_2 e_3^2$	$\vdots$

The table below summarizes these relations obtained from the monomials in degree 2:

Monomial in degree 4	Relation given by equating coefficients
$e_1 e_2$	$a_{1,2}^{(0,0,0)} + a_{1,2}^{(1,0,0)} c_{1,2}^{(0,1,0)} + a_{1,2}^{(0,1,0)} c_{1,2}^{(1,0,0)} + a_{1,3}^{(1,0,0)} c_{1,3}^{(0,1,0)} + a_{1,3}^{(0,1,0)} c_{1,3}^{(1,0,0)} + a_{2,3}^{(1,0,0)} c_{2,3}^{(0,1,0)} + a_{2,3}^{(0,1,0)} c_{2,3}^{(1,0,0)} = 0$
$\vdots$	$\vdots$

The table below summarizes these relations obtained from the monomials in degree 1:

Monomial in degree 4	Relation given by equating coefficients
e_1	$a_{1,2}^{(0,0,0)} c_{1,2}^{(1,0,0)} + a_{1,3}^{(0,0,0)} c_{1,3}^{(1,0,0)} + a_{2,3}^{(0,0,0)} c_{2,3}^{(1,0,0)} = b_1$
e_2	$a_{1,2}^{(0,0,0)} c_{1,2}^{(0,1,0)} + a_{1,3}^{(0,0,0)} c_{1,3}^{(0,1,0)} + a_{2,3}^{(0,0,0)} c_{2,3}^{(0,1,0)} = b_2$
e_3	$a_{1,2}^{(0,0,0)} c_{1,2}^{(0,0,1)} + a_{1,3}^{(0,0,0)} c_{1,3}^{(0,0,1)} + a_{2,3}^{(0,0,0)} c_{2,3}^{(0,0,1)} = b_3$

Another Calculation

We calculate

$$\begin{aligned}
[e_i, e_j, e_k] &= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
&= \sum_m \left(\sum_l (c_{i,j}^l c_{l,k}^m - c_{i,l}^m c_{j,k}^l) \right) e_m
\end{aligned}$$