MATH 8000

Final Exam

Problem 1: Consider a guidance system which is designed to deliver a bomb to a target site, which we denote by (x_0, y_0). As with all things, the guidance system that we consider is imperfect and will deliver the bomb to a random point $(X, Y)'$, where the joint PDF of this random vector, given the point of target, is

$$f_{X,Y}(x, y|x_0, y_0) = \frac{1}{2\pi}\exp\left\{-\frac{(x-x_0)^2}{2} - \frac{(y-y_0)^2}{2}\right\}$$

where both X and Y are measured in feet. If a particular target, located at the point $(3, 2)$, will be destroyed if a bomb is delivered within $r = 2$ feet of it, what is the probability that the target will be destroyed if 10 bombs are fired at it? You should assume that the targeting system is set to target the point $(3, 2)$, and that the point of impact for each bomb is independent. To gain full credit you should be explicit in your calculations.

Problem 2: Let $F_X : \mathbb{R} \to [0, 1]$ and $F_Y : \mathbb{R} \to [0, 1]$ be univariate cumulative distribution functions (CDFs) and suppose $-1 \leq \alpha \leq 1$. Define $F_{X,Y}^{(\alpha)} : \mathbb{R}^2 \to [0, 1]$ by

$$F_{X,Y}^{(\alpha)}(x, y) = F_X(x)F_Y(y)\{1 + \alpha[1 - F_X(x)][1 - F_Y(y)]\}.$$

The collection $\{F_{X,Y}^{(\alpha)} : -1 \leq \alpha \leq 1\}$ is called the *Farlie-Morgenstern family* of bivariate CDFs corresponding to F_X and F_Y.

(a) Starting with the expression for $F_{X,Y}^{(\alpha)}(x, y)$, show that the marginal cdfs of X and Y are given by $F_X(x)$ and $F_Y(y)$, respectively.

(b) What value of α corresponds to X and Y being independent? Explain.

(c) As a special case, suppose that X and Y are each distributed as exponential with mean 1. In this case, for any $\alpha \in [-1, 1]$, show that the joint probability density function of $(X, Y)'$ is

$$f_{X,Y}^{(\alpha)}(x, y) = \{1 + \alpha[(1 - 2e^{-x})(1 - 2e^{-y})]\}e^{-(x+y)}I(x > 0)I(y > 0).$$

(d) For the special case described in part (c) derive the correlation between X and Y as a function of α.

Problem 3: Let X_i, for $i = 1, 2, ...,$ be independent uniform(0,1) random variables. Define the integer valued random variable N which has the following PMF

$$f_N(n) = \frac{c}{n!} I(n \in \{1, 2, ...\}),$$

where $c = 1/(e-1)$. Derive the probability density function of $T = \min_{1 \leq i \leq N}(X_i)$ and use it to find the $E(T)$. Then using the law of iterated expectations confirm your calculation of $E(T)$. Note $e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$.

Problem 4: Let $X_1, X_2 \overset{iid}{\sim}$ Exponential(1), and define the random variables $\overline{X} = (X_1 + X_2)/2$ and $Y = X_2 - \overline{X}$. Are $\overline{X}$ and Y independent? Justify your answer.

Problem 5: Find the marginal distribution of $Y = \frac{X_1}{X_1 + X_2}$, where $X_i \overset{iid}{\sim}$ Gamma(α, β), for $i = 1, 2$.

Problem 6: A nonnegative random variable X is defined as $Z = \log(X)$, where $E(Z) = 0$ and $V(Z) = \sigma^2 > 0$. Is $E(X)$ greater than, equal to, or less than one? Justify your answer.

Problem 7: Assume that $\lambda \sim$ Uniform$(0, \theta)$, and that given λ, $X_1, X_2, ...$ are independent Poisson(λ). Let $Y = \min\{i | X_i = 0\}$ be the index of the first zero. Compute the marginal PMF of Y, as well as the $E(Y)$ and $V(Y)$.