Final Exam

Michael Nelson

Problem 1

We first calculate the probability that the target is destroyed after 1 bomb is fired. Let $D_r(x_0, y_0)$ denote the disc centered at (x_0, y_0) of radius r. Then

$$
\begin{aligned}
\mathrm{P}(\text{target destroyed after 1 bomb fired}) &= \int_{D_2(3,2)} f_{X,Y}(x,y|3,2)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{D_2(3,2)} e^{-\frac{1}{2}\left((x-3)^2+(y-2)^2\right)}\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{D_2(0,0)} e^{-\frac{1}{2}(u^2+v^2)}\mathrm{d}u\mathrm{d}v \qquad u = x-3, \quad v = y-2 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 e^{-\frac{1}{2}r^2} r\mathrm{d}r\mathrm{d}\theta \qquad r = \sqrt{u^2+v^2}, \quad \theta = \tan^{-1}(u/v) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(-e^{-\frac{1}{2}r^2}\Big|_0^2\right)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}(1-e^{-2})d\theta \\
&= 1-e^{-2}.
\end{aligned}
$$

Since the point of impact for each bomb is independent, the probability that the target is destroyed after 10 bombs are fired is

$$
\begin{aligned}
\mathrm{P}(\text{target destroyed after 10 bombs fired}) &= 1 - \mathrm{P}(\text{target not destroyed after 10 bombs fired}) \\
&= 1-(1-(1-e^{-2}))^{10} \\
&= 1-e^{-20}.
\end{aligned}
$$

Problem 2

Problem 2.a

Since $F^{\alpha}_{X,Y}(x,y)$ is symmetric with respect to swapping X with Y and x with y, it suffices to show that the marginal cdf of X is $F_X(x)$. We have

$$
\begin{aligned}
F^{\alpha}_X(x) &= \lim_{y\to\infty} F^{\alpha}_{X,Y}(x,y) \\
&= \lim_{y\to\infty} F_X(x)F_Y(y)\left(1+\alpha(1-F_X(x))(1-F_Y(y))\right) \\
&= F_X(x)\cdot 1\cdot(1+\alpha(1-F_X(x))(1-1)) \\
&= F_X(x).
\end{aligned}
$$

It follows that the marginal cdf of X is $F_X(x)$.

Problem 2.b

The random variables X and Y are independent precisely when $\alpha = 0$. Indeed, they are independent when $\alpha = 0$ since their joint pdf can be expressed as the product of the marginal pdfs when $\alpha = 0$, that is, since

$$
F^{\alpha}_{X,Y}(x,y) = F_X(x)F_Y(y)\left(1+\alpha(1-F_X(x))(1-F_Y(y))\right)
$$

we have in particular

$$F^0_{X,Y}(x,y) = F_X(x)F_Y(y).$$

Also if $\alpha \neq 0$, then X and Y are independent if and only if $1+\alpha(1-F_X(x))(1-F_Y(y)) = c$ where c is a constant. In other words, X and Y are independent if and only if

$$(1-F_X(x))(1-F_Y(y)) = (c-1)/\alpha. \tag{1}$$

This is impossible however since on the one hand, taking $x \to \infty$ and $y \to \infty$ in (1) gives us $0 = (c-1)/\alpha$, and on the other hand taking taking $x \to -\infty$ and $y \to -\infty$ in (1) gives us $1 = (c-1)/\alpha$, which is a contradiction.

Problem 2.c

In general, if X and Y are continuous random variables, then we have

$$\begin{aligned}
f^\alpha_{X,Y}(x,y) &= \partial_x\partial_y F^\alpha_{X,Y}(x,y) \\
&= \partial_x\partial_y \left(F_X(x)F_Y(y)\left(1+\alpha(1-F_X(x))(1-F_Y(y))\right)\right) \\
&= \partial_x\partial_y \left(F_X(x)F_Y(y) + \alpha F_X(x)F_Y(y) - \alpha F_X^2(x)F_Y(y) - \alpha F_X(x)F_Y^2(y) + \alpha F_X^2(x)F_Y^2(y)\right) \\
&= \partial_x \left(F_X(x)f_Y(y) + \alpha F_X(x)f_Y(y) - \alpha F_X^2(x)f_Y(y) - 2\alpha F_X(x)F_Y(y)f_Y(y) + 2\alpha F_X^2(x)F_Y(y)f_Y(y)\right) \\
&= f_X(x)f_Y(y) + \alpha f_X(x)f_Y(y) - 2\alpha F_X(x)f_X(x)f_Y(y) - 2\alpha f_X(x)F_Y(y)f_Y(y) + 4\alpha F_X(x)F_Y(y)f_X(x)f_Y(y) \\
&= f_X(x)f_Y(y)\left(1+\alpha - 2\alpha F_X(x) - 2\alpha F_Y(y) + 4\alpha F_X(x)F_Y(y)\right) \\
&= f_X(x)f_Y(y)\left(1+\alpha(1-2F_X(x))(1-2F_Y(y))\right).
\end{aligned}$$

Thus we have the formula

$$f^\alpha_{X,Y}(x,y) = f_X(x)f_Y(y)\left(1+\alpha(1-2F_X(x))(1-2F_Y(y))\right). \tag{2}$$

Note that (2) shows that the support of $f^\alpha_{X,Y}(x,y)$ is $\operatorname{supp}(X)\times\operatorname{supp}(Y)$. So in this particular problem, we have $\operatorname{supp}(f^\alpha_{X,Y}) = \mathbb{R}^2_{>0}$. Using (2), we calculate

$$\begin{aligned}
f^\alpha_{X,Y}(x,y) &= e^{-x}e^{-y}\left(1+\alpha(1-2(1-e^{-x}))(1-2(1-e^{-y}))\right) \\
&= e^{-(x+y)}\left(1+\alpha(1-2+2e^{-x}))(1-2+2e^{-y})\right) \\
&= e^{-(x+y)}\left(1+\alpha(-1+2e^{-x}))(-1+2e^{-y})\right) \\
&= e^{-(x+y)}\left(1+\alpha(1-2e^{-x}))(1-2e^{-y})\right),
\end{aligned}$$

for all $(x,y) \in \mathbb{R}^2_{>0}$.

Problem 2.d

We have

$$\begin{aligned}
\mathrm{E}(XY) &= \int_0^\infty\int_0^\infty xye^{-(x+y)}\left(1+\alpha(1-2e^{-x})(1-2e^{-y})\right)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty y\int_0^\infty xe^{-(x+y)}\left(1+\alpha(1-2e^{-x})(1-2e^{-y})\right)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty y\left(\int_0^\infty xe^{-(x+y)}\mathrm{d}x + \alpha(1-2e^{-y})\int_0^\infty xe^{-(x+y)}(1-2e^{-x})\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^\infty y\left(\int_0^\infty xe^{-(x+y)}\mathrm{d}x + \alpha(1-2e^{-y})\left(\int_0^\infty xe^{-(x+y)}\mathrm{d}x - \int_0^\infty 2xe^{-(2x+y)}\mathrm{d}x\right)\right)\mathrm{d}y \\
&= \int_0^\infty y\left(e^{-y} + \alpha(1-2e^{-y})\left(e^{-y} - \frac{1}{2}e^{-y}\right)\right)\mathrm{d}y \\
&= \int_0^\infty ye^{-y}\mathrm{d}y + \frac{\alpha}{2}\int_0^\infty y(1-2e^{-y})e^{-y}\mathrm{d}y \\
&= 1 + \frac{\alpha}{4}.
\end{aligned}$$

Therefore

$$\begin{aligned}\operatorname{Cov}(X,Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) \\ &= 1 + \frac{\alpha}{4} - 1 \cdot 1 \\ &= \frac{\alpha}{4}.\end{aligned}$$

It follows that

$$\begin{aligned}\rho_{XY} &= \frac{\operatorname{Cov}(X,Y)}{\sigma_X \sigma_Y} \\ &= \frac{\alpha}{4}.\end{aligned}$$

Problem 3

First note that $\operatorname{supp} T = (0,1)$. Now let $t \in (0,1)$ and let $n \in \mathbb{Z}_{\geq 1}$. Then

$$\begin{aligned}F_{T|n}(t|n) &= \mathrm{P}(T < t | N = n) \\ &= \mathrm{P}(X_i < t \text{ for some } 1 \leq i \leq n) \\ &= 1 - \mathrm{P}(X_i \geq t \text{ for all } 1 \leq i \leq n) \\ &= 1 - (1-t)^n.\end{aligned}$$

It follows that $f_{T|n}(t|n) = n(1-t)^{n-1}$. Therefore

$$\begin{aligned}f_T(t) &= \sum_{n=1}^{\infty} f_{T,N}(t,n) \\ &= \sum_{n=1}^{\infty} f_{T|n}(t|n) f_N(n) \\ &= \sum_{n=1}^{\infty} n(1-t)^{n-1} \frac{c}{n!} \\ &= c \sum_{n=1}^{\infty} \frac{(1-t)^{n-1}}{(n-1)!} \\ &= c \sum_{m=0}^{\infty} \frac{(1-t)^m}{m!} \\ &= ce^{1-t}.\end{aligned}$$

Thus the expected value of T is

$$\begin{aligned}\mathrm{E}(T) &= \int_0^1 cte^{1-t} \mathrm{d}t \\ &= c(e-2) \\ &= \frac{e-2}{e-1}.\end{aligned}$$

Now we verify this calculation using the law of iterated expectation. We have

$$\begin{aligned}
\mathrm{E}(T) &= \mathrm{E}(\mathrm{E}(T|N)) \\
&= \sum_{n=1}^{\infty} \mathrm{E}(T|n) f_N(n) \\
&= \sum_{n=1}^{\infty} \int_0^1 nt(1-t)^{n-1}\mathrm{d}t \frac{c}{n!} \\
&= \sum_{n=1}^{\infty} \frac{1}{n+1}\frac{c}{n!} \\
&= c\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \\
&= c\sum_{m=2}^{\infty} \frac{1}{m!} \\
&= c \cdot (e-2) \\
&= \frac{e-2}{e-1},
\end{aligned}$$

as expected (no pun intended; I'm sure you've heard that before!).

Problem 4

Set $\mathcal{A} = \mathbb{R}^2_{>0}$ and define $g = (g_1, g_2) \colon \mathcal{A} \to \mathbb{R}^2$ by

$$g_1(x_1, x_2) = \frac{x_1 + x_2}{2}, \qquad \text{and} \qquad g_2(x_1, x_2) = \frac{-x_1 + x_2}{2}$$

for all $(x_1, x_2) \in \mathcal{A}$. Denote $\mathcal{B} = \operatorname{im} g = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq |v| < u\}$ and denote $U = g_1(X_1, X_2)$ and $V = g_2(X_1, X_2)$. Note that in our notation we have $\overline{X} = U$ and $Y = V$. Now observe that g is a diffeomorphism (it's just a linear transformation) with inverse $h = (h_1, h_2) \colon \mathcal{B} \to \mathcal{A}$ defined by

$$h_1(u, v) = u - v \qquad \text{and} \qquad h_2(u, v) = u + v$$

for all $(u, v) \in \mathcal{B}$. The absolute value of the Jacobian of h at (u, v) is given by

$$\begin{aligned}
|J_{u,v}(h)| &= \left| \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right| \\
&= 2.
\end{aligned}$$

Therefore the joint distribution of U and V is given by

$$\begin{aligned}
f_{U,V}(u, v) &= f_{X,Y}(h(u, v)) \cdot |J_{u,v}(h)| \\
&= 2 f_{X,Y}(u - v, u + v) \\
&= 2 f_X(u - v) f_Y(u + v) \\
&= 2e^{-(u-v)} e^{-(u+v)} \\
&= 2e^{-2u}.
\end{aligned}$$

for all $(u, v) \in \mathcal{B}$. Now note that $\mathcal{B}$ is *not* a cross product, that is, we do not have $\mathcal{B} = A \times B$ for some $A, B \subseteq \mathbb{R}$. Indeed, to check membership of $(u, v) \in \mathcal{B}$, we must check not only $0 < u < \infty$ but also $0 \leq |v| < u$. It follows that U and V are not independent.

Problem 5

Set $\mathcal{A} = \mathbb{R}^2_{>0}$ and define $g = (g_1, g_2) \colon \mathcal{A} \to \mathbb{R}^2$ by

$$g_1(x_1, x_2) = \frac{x_1}{x_1 + x_2}, \qquad \text{and} \qquad g_2(x_1, x_2) = x_2$$

for all $(x_1, x_2) \in \mathcal{A}$. Denote $\mathcal{B} = \operatorname{im} g = (0,1) \times \mathbb{R}_{>0}$ and denote $U = g_1(X_1, X_2)$ and $V = g_2(X_1, X_2)$. Note that in our notation we have $Y = U$. Now observe that g is a diffeomorphism with inverse $h = (h_1, h_2) \colon \mathcal{B} \to \mathcal{A}$ defined by

$$h_1(u,v) = \frac{uv}{1-u} \qquad \text{and} \qquad h_2(u,v) = v$$

for all $(u,v) \in \mathcal{B}$. The absolute value of the Jacobian of h at (u,v) is given by

$$\begin{aligned} |J_{u,v}(h)| &= \left| \begin{pmatrix} \frac{v}{(1-u)^2} & \frac{u}{1-u} \\ 0 & 1 \end{pmatrix} \right| \\ &= \frac{v}{(1-u)^2}. \end{aligned}$$

Therefore the joint distribution of U and V is given by

$$\begin{aligned} f_{U,V}(u,v) &= f_{X_1,X_2}(h(u,v)) \cdot |J_{u,v}(h)| \\ &= f_{X_1}\left(\frac{uv}{1-u}\right) f_{X_2}(v) \frac{v}{(1-u)^2} \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{uv}{1-u}\right)^{\alpha-1} e^{-\frac{1}{\beta}\frac{uv}{1-u}} \frac{1}{\Gamma(\alpha)\beta^\alpha} v^{\alpha-1} e^{-\frac{1}{\beta}v} \frac{v}{(1-u)^2} \\ &= \frac{1}{\Gamma(\alpha)^2\beta^{2\alpha}} \frac{u^{\alpha-1}v^{2\alpha-1}}{(1-u)^{\alpha+1}} e^{-\frac{1}{\beta}\left(v + \frac{uv}{1-u}\right)} \\ &= \frac{1}{\Gamma(\alpha)^2\beta^{2\alpha}} \frac{u^{\alpha-1}v^{2\alpha-1}}{(1-u)^{\alpha+1}} e^{-\frac{v}{\beta(1-u)}} \end{aligned}$$

for all $(u,v) \in \mathcal{B}$. Therefore the marginal distribution of U is given by

$$\begin{aligned} f_U(u) &= \int_0^\infty \frac{1}{\Gamma(\alpha)^2\beta^{2\alpha}} \frac{u^{\alpha-1}v^{2\alpha-1}}{(1-u)^{\alpha+1}} e^{-\frac{v}{\beta(1-u)}} \, \mathrm{d}v \\ &= \frac{1}{\Gamma(\alpha)^2\beta^{2\alpha}} \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}} \int_0^\infty v^{2\alpha-1} e^{-\frac{v}{\beta(1-u)}} \, \mathrm{d}v \\ &= \frac{1}{\Gamma(\alpha)^2\beta^{2\alpha}} \frac{u^{\alpha-1}}{(1-u)^{\alpha+1}} \Gamma(2\alpha)(\beta(1-u))^{2\alpha} && \text{gamma distribution integral} \\ &= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} u^{\alpha-1}(1-u)^{\alpha-1} \end{aligned}$$

for all $u \in (0,1)$. It follows that $U \sim \mathrm{beta}(\alpha, \alpha)$.

Problem 6

Since the function $z \mapsto e^z$ is convex, it follows from Jensen's inequality that

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}(e^Z) \\ &\geq e^{\mathrm{E}(Z)} \\ &= 1. \end{aligned}$$

where inequality is strict unless $\mathrm{P}(Z=0) = 1$, but since $\mathrm{V}(Z) > 0$, we cannot have $\mathrm{P}(Z=0) = 1$ ($\mathrm{V}(0) = 0 \neq \mathrm{V}(Z)$), so the inequality is strict.

Problem 7

First note that $\operatorname{supp} Y = \mathbb{Z}_{\geq 1}$. Let $n \in \mathbb{Z}_{\geq 1}$. For each $i \in \mathbb{Z}_{\geq 1}$, the probability that $X_i|\lambda = 0$ is $p = e^{-\lambda}$. Since the sequence $(X_i|\lambda)$ of random variables is pairwise independent, we can view $(X_i|\lambda)$ as a sequence of coin flips, where $X_i|\lambda = 0$ translates to "the ith coin lands heads" and $X_i|\lambda \neq 0$ translates to "the ith coins lands tails". In this case, the probability that $Y|\lambda = n$ translates to the probability that "the nth coin is the first to land heads". Thus

$$\begin{aligned}
\mathrm{P}(Y = n|\lambda) &= \mathrm{P}(n\text{th coin is first to land heads}) \\
&= (1-p)^{n-1}p \\
&= (1-e^{-\lambda})^{n-1}e^{-\lambda}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mathrm{P}(Y = n) &= \frac{1}{\theta}\int_0^\theta \mathrm{P}(Y = n|\lambda)\mathrm{d}\lambda \\
&= \frac{1}{\theta}\int_0^\theta (1-e^{-\lambda})^{n-1}e^{-\lambda}\mathrm{d}\lambda \\
&= \frac{1}{\theta}\int_0^\theta (1-e^{-\lambda})^{n-1}e^{-\lambda}\mathrm{d}\lambda \\
&= \frac{1}{\theta}\left(\left.\frac{(1-e^{-\lambda})^n}{n}\right|_0^\theta \right) \\
&= \frac{(1-e^{-\theta})^n}{\theta n}.
\end{aligned}$$

In particular, this implies Y has a logarithmic distribution, namely $Y \sim \text{logarithmic}(1-e^{-\theta})$. The expectation and variance of logarithmic distributions are well known, but let's calculate them again anyway. The mean is given by

$$\begin{aligned}
\mathrm{E}(Y) &= \sum_{n=1}^\infty n\mathrm{P}(Y = n) \\
&= \frac{1}{\theta}\sum_{n=1}^\infty (1-e^{-\theta})^n \\
&= \frac{1}{\theta}\frac{1-e^{-\theta}}{1-(1-e^{-\theta})} \\
&= \frac{1}{\theta}\frac{1-e^{-\theta}}{e^{-\theta}} \\
&= \frac{1}{\theta}(e^\theta - 1).
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
\mathrm{E}(Y^2) &= \sum_{n=1}^\infty n^2\mathrm{P}(Y = n) \\
&= \frac{1}{\theta}\sum_{n=1}^\infty n(1-e^{-\theta})^n \\
&= \frac{1}{\theta}\frac{1-e^{-\theta}}{(1-(1-e^{-\theta}))^2} \\
&= \frac{1}{\theta}\frac{1-e^{-\theta}}{e^{-2\theta}} \\
&= \frac{1}{\theta}e^\theta(e^\theta - 1).
\end{aligned}$$

Therefore the variance is given by

$$\begin{aligned}
\mathrm{V}(Y^2) &= \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 \\
&= \frac{1}{\theta}e^\theta(e^\theta - 1) - \frac{1}{\theta^2}(e^\theta - 1)^2 \\
&= \frac{1}{\theta}(e^\theta - 1)\left(e^\theta - \frac{1}{\theta}(e^\theta - 1)\right) \\
&= \frac{1}{\theta}(e^\theta - 1)\left(\frac{\theta e^\theta - e^\theta + 1}{\theta}\right) \\
&= \frac{1}{\theta^2}(e^\theta - 1)\left(e^\theta(\theta - 1) + 1\right).
\end{aligned}$$