Abstract Algebra II Take Home Test

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(1) : Let $\varphi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ is a ring automorphism. Then φ is completely determined by where it maps 1 and x, since

$$\begin{aligned}\varphi(a_0 + a_1 x + \cdots + a_n x^n) &= \varphi(a_0) + \varphi(a_1 x) + \cdots + \varphi(a_n x^n) \\ &= a_0 \varphi(1) + a_1 \varphi(x) + \cdots + a_n \varphi(x)^n,\end{aligned}$$

for all $a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$. Since $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)^2$, we must have $\varphi(1)(\varphi(1) - 1) = 0$. Since $\mathbb{Z}[x]$ is an integral domain, we either have $\varphi(1) = 0$ or $\varphi(1) = 1$. If $\varphi(1) = 0$, then $\varphi(a) = 0$ for all $a \in \mathbb{Z}$, which implies φ is not injective, therefore we must have $\varphi(1) = 1$.

Next, suppose $\varphi(x) = c_k x^k + c_{k-1} x^{k-1} + \cdots + c_0$ where $c_k \neq 0$. Since $x \in \mathrm{Im}\varphi$, there is some $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_n \neq 0$, such that $x = \varphi(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0)$. Since $a_n \neq 0$, $c_k^n \neq 0$, the lead term of $\varphi(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0)$ is $a_n c_k^n x^{kn}$. Since the lead term of x and $\varphi(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0)$ must be equal, we must have $kn = 1$ and $a_n c_k^n = 1$. This implies $k = n = 1$ and $c_k = \pm 1$. Therefore $\varphi(x)$ has the form $\varphi(x) = \pm x + c$ for some $c \in \mathbb{Z}$. Conversely, map $\varphi : \mathbb{Z}[x] \to \mathbb{Z}[x]$ given by

$$\varphi(a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1(c \pm x) + \cdots + a_n (c \pm x)^n$$

for all $a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ is a ring automorphism: Let $\sum_{i \in \mathbb{Z}} a_i x^i$ and $\sum_{j \in \mathbb{Z}} b_j x^j$ be in $\mathbb{Z}[x]$, so a_i and b_j are zero for all but finitely many i, j. Then

$$\begin{aligned}\varphi\left(\sum_{i\in\mathbb{Z}} a_i x^i \sum_{j\in\mathbb{Z}} b_j x^j\right) &= \varphi\left(\sum_{n\in\mathbb{Z}}\left(\sum_{m=0}^{n} a_m b_{n-m}\right) x^n\right) \\ &= \sum_{n\in\mathbb{Z}}\left(\sum_{m=0}^{n} a_m b_{n-m}\right)(c \pm x)^n \\ &= \sum_{i\in\mathbb{Z}} a_i (c\pm x)^i \sum_{j\in\mathbb{Z}} b_j (c \pm x)^j \\ &= \varphi\left(\sum_{i\in\mathbb{Z}} a_i x^i\right)\varphi\left(\sum_{j\in\mathbb{Z}} b_j x^j\right)\end{aligned}$$

and

$$\begin{aligned}\varphi\left(\sum_{i\in\mathbb{Z}} a_i x^i + \sum_{j\in\mathbb{Z}} b_j x^j\right) &= \varphi\left(\sum_{i\in\mathbb{Z}} (a_i + b_i) x^i\right) \\ &= \sum_{i\in\mathbb{Z}} (a_i + b_i)(c \pm x)^i \\ &= \sum_{i\in\mathbb{Z}} a_i (c\pm x)^i + \sum_{j\in\mathbb{Z}} b_j (c \pm x)^j \\ &= \varphi\left(\sum_{i\in\mathbb{Z}} a_i x^i\right) + \varphi\left(\sum_{i\in\mathbb{Z}} b_j x^j\right).\end{aligned}$$

(2) : Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be a ring automorphism. Then φ is completely determined by where it maps $(1, 0)$ and $(0, 1)$ since

$$\begin{aligned}\varphi(a, b) &= \varphi(a, 0) + \varphi(0, b) \\ &= a\varphi(1, 0) + b\varphi(0, 1),\end{aligned}$$

for all $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. Since φ is injective, we must have $\varphi(1,0) \neq (0,0)$ and $\varphi(0,1) \neq (0,0)$. Since $(0,0) = \varphi((1,0) \cdot (0,1)) = \varphi(1,0)\varphi(0,1)$, we see that $\varphi(1,0)$ and $\varphi(0,1)$ must be zero divisors. Nonzero zero divisors in $\mathbb{Z} \times \mathbb{Z}$ have the form $(a,0)$ or $(0,b)$ where $a,b \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$: Let $(x,y),(r,s) \in \mathbb{Z} \times \mathbb{Z}$ with $(x,y) \neq (0,0)$ and $(r,s) \neq (0,0)$. Then

$$(x,y)(r,s) = (xr, ys) = (0,0)$$

implies either $x = 0$ or $r = 0$ and either $y = 0$ or $s = 0$, since $\mathbb{Z}$ is an integral domain. We can't have both x and y be zero, so if $x = 0$, then $y \neq 0$, and therefore $s = 0$. Similarly if $y = 0$, then $x \neq 0$, and therefore $r = 0$. So φ can have one of two forms: $\varphi_{(x,y)}(a,b) = (ax, by)$ and $\varphi^{\star}_{(x,y)}(a,b) = (bx, ay)$ for all $a,b \in \mathbb{Z}$ and $x,y \in \mathbb{Z} - \{0\}$. If (a,b) and (c,d) are elements in $\mathbb{Z} \times \mathbb{Z}$, then

$$\begin{aligned}
\varphi_{(x,y)}(a+c, b+d) &= ((a+c)x, (b+d)y) \\
&= (ax + cx, by + dy) \\
&= (ax, by) + (cx, dy) \\
&= \varphi_{(x,y)}(a,b) + \varphi_{(x,y)}(c,d).
\end{aligned}$$

similarly,

$$\begin{aligned}
\varphi^{\star}_{(x,y)}(a+c, b+d) &= ((b+d)x, (a+c)y) \\
&= (bx + dx, ay + cy) \\
&= (bx, ay) + (dx, cy) \\
&= \varphi_{(x,y)}(a,b) + \varphi_{(x,y)}(c,d).
\end{aligned}$$

This shows $\varphi_{(x,y)}$ and $\varphi^{\star}_{(x,y)}$ are additive for all $x,y \in \mathbb{Z} - \{0\}$. However,

$$\begin{aligned}
(x,y) &= \varphi_{(x,y)}(1,1) \\
&= \varphi_{(x,y)}((1,1)(1,1)) \\
&= \varphi_{(x,y)}(1,1)\varphi_{(x,y)}(1,1) \\
&= (x^2, y^2)
\end{aligned}$$

implies $x = 1$ and $y = 1$ in $\varphi_{(x,y)}$ since $\mathbb{Z}$ is an integral domain. Similarly,

$$\begin{aligned}
(x,y) &= \varphi^{\star}_{(x,y)}(1,1) \\
&= \varphi^{\star}_{(x,y)}((1,1)(1,1)) \\
&= \varphi^{\star}_{(x,y)}(1,1)\varphi_{(x,y)}(1,1) \\
&= (x^2, y^2)
\end{aligned}$$

implies $x = 1$ and $y = 1$ in $\varphi^{\star}_{(x,y)}$. So we are limited to two possibilities, $\varphi_{(1,1)}$ and $\varphi^{\star}_{(1,1)}$, and these are ring homomorphisms since they are also multiplicative: $\varphi_{(1,1)}$ is just the identity map, so it's suffices to check $\varphi^{\star}_{(1,1)}$:

$$\begin{aligned}
\varphi^{\star}_{(1,1)}(ac, bd) &= (bd, ac) \\
&= (b,a)(d,c) \\
&= \varphi^{\star}_{(1,1)}(a,b)\varphi^{\star}_{(1,1)}(c,d).
\end{aligned}$$

Moreover, these are ring automorphisms since they can be represented by matrices in $\mathrm{GL}_2(\mathbb{Z})$, namely $\varphi_{(1,1)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\varphi^{\star}_{(1,1)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which are bijections on the set $\mathbb{Z} \times \mathbb{Z}$.

(3) : Let R be a local ring with maximal ideal $\mathfrak{m}$. To show $R - R^{\times}$ is an ideal, we need to show $R - R^{\times}$ is nonempty and that $x + ry \in R - R^{\times}$ for all $x,y \in R - R^{\times}$ and $r \in R$. Since 0 is not a unit, $0 \in R - R^{\times}$, so $R - R^{\times}$ is nonempty. Let $x,y \in R - R^{\times}$ and $r \in R$. Since x and y are not units, $\langle x \rangle \neq \langle 1 \rangle$ and $\langle y \rangle \neq \langle 1 \rangle$. Thus, $\langle x \rangle$ and $\langle y \rangle$ are each contained in their own maximal ideal. Since there is only one maximal ideal, they must both be contained in $\mathfrak{m}$. Therefore $x,y \in \mathfrak{m}$. Since $\mathfrak{m}$ is an ideal, we have $x + ry \in \mathfrak{m}$. Since $\mathfrak{m}$ contains no units (otherwise $\mathfrak{m} = \langle 1 \rangle$), $x + ry$ is not a unit, and therefore $x + ry \in R - R^{\times}$.

Conversely, suppose the set of nonunits $R - R^\times$ forms an ideal. By Zorn's Lemma, it must be contained in some maximal ideal $\mathfrak{m}$. We first show that $R - R^\times$ contains $\mathfrak{m}$ too, so that $R - R^\times = \mathfrak{m}$. Suppose there is an $x \in \mathfrak{m}$ such that $x \notin R - R^\times$. This means x is a unit which belongs to $\mathfrak{m}$. But this is a contradiction since this would imply $\mathfrak{m} = \langle 1 \rangle$. Now we show that this maximal ideal is unique in R. Suppose $\mathfrak{m}'$ is another maximal ideal in R distinct from $\mathfrak{m}$. Then $\mathfrak{m}' \neq \mathfrak{m}$ implies $\mathfrak{m} \not\subset \mathfrak{m}'$ and $\mathfrak{m} \not\supset \mathfrak{m}'$ by maximality of $\mathfrak{m}$ and $\mathfrak{m}'$, so there is some $x \in \mathfrak{m}'$ such that $x \notin \mathfrak{m}$. Since $\mathfrak{m} = R - R^\times$, x must be a unit, but this implies $\mathfrak{m}' = \langle 1 \rangle$, which is a contradiction.