Measure Theory Homework 2

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Throughout this homework, let $(X, \mathcal{M}, \mu)$ be a measure space. We say that $(X, \mathcal{M}, \mu)$ is a **finite** measure space if $\mu(X) < \infty$. Observe that in this case, we have $\mu(A) < \infty$ for all $A \in \mathcal{M}$, by monotonicity of μ.

Problem 1

Proposition 0.1. *Let (E_n) be a sequence of sets in $\mathcal{M}$. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Proof. Disjointify[1] (E_n) into the sequence (D_n); set $D_1 := E_1$ and

$$D_n := E_n \backslash \left(\bigcup_{i=1}^{n-1} E_n\right)$$

for all $n > 1$. Then we have

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} D_n\right) \\ &= \sum_{n=1}^{\infty} \mu(D_n) \\ &\leq \sum_{n=1}^{\infty} \mu(E_n),\end{aligned}$$

where we use countable additivity of μ to get from the first line to the second line and where we used monotonicity of μ to get from the second line to the third line. □

Problem 2

Proposition 0.2. *Let $(Y, \mathcal{N}, \nu)$ be a measure space and suppose $f \colon X \to Y$ is a function. Then $(X, f^{-1}(\mathcal{N}), f^*\mu)$ is a measure space, where*

$$f^{-1}(\mathcal{N}) = \{f^{-1}(B) \subseteq X \mid B \in \mathcal{N}\}$$

and where $f^*\mu \colon f^{-1}(\mathcal{N}) \to [0, \infty]$ is defined by

$$(f^*\mu)(f^{-1}(B)) = \mu(B)$$

for all $f^{-1}(B) \in f^{-1}(\mathcal{N})$. In particular, if $\iota \colon A \to X$ denotes the inclusion map, then $(X, \mathcal{M}_A, \mu_A)$ is a measure space.

Proof. We first show that $f^{-1}(\mathcal{N})$ is a σ-algebra. This follows from the fact that f^{-1} commutes with unions and compliments:

$$f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}\left(B_j\right) \quad \text{and} \quad f^{-1}\left(Y \backslash B\right) = f^{-1}(Y) \backslash f^{-1}(B)$$

[1] See Appendix for details on disjointification.

for all subsets B and B_j of Y for all $j \in J$. Indeed, we have

$$\begin{aligned} x \in \bigcup_{j \in J} f^{-1}\left(B_j\right) &\iff x \in f^{-1}(B_j) \text{ for some } j \in J \\ &\iff f(x) \in B_j \text{ for some } j \in J \\ &\iff f(x) \in \bigcup_{j \in J} B_j \\ &\iff x \in f^{-1}\left(\bigcup_{j \in J} B_j\right) \end{aligned}$$

and we have

$$\begin{aligned} x \in f^{-1}\left(Y \backslash B\right) &\iff f(x) \in Y \backslash B \\ &\iff f(x) \in Y \text{ and } f(a) \notin B \\ &\iff x \in f^{-1}(Y) \text{ and } a \notin f^{-1}(B) \\ &\iff x \in f^{-1}(Y) \backslash f^{-1}(B). \end{aligned}$$

Now we show that the function $f^*\mu$ is a measure. First observe that $f^{-1}(\emptyset) = \emptyset$, and so

$$\begin{aligned} (f^*\mu)(\emptyset) &= \mu(\emptyset) \\ &= 0. \end{aligned}$$

Next, let $(f^{-1}(B_n))$ be a sequence of pairwise disjoint members of $f^{-1}(\mathcal{N})$. Then (B_n) is a sequence of pairwise disjoint members of $\mathcal{N}$, and so we have

$$\begin{aligned} (f^*\mu)\left(\bigcup_{n=1}^{\infty} f^{-1}(B_n)\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \sum_{n=1}^{\infty} (f^*\mu)(f^{-1}(B_n)). \end{aligned}$$

This implies $f^*\mu$ is a measure. □

Problem 3

Definition 0.1. A set $E \subseteq X$ is called **locally measurable** if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with finite measure.

Proposition 0.3. *Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable subsets of X. Then $\mathcal{M} = \widetilde{\mathcal{M}}$.*

Proof. Let first show that $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$. Let $E \in \mathcal{M}$. Then $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ since $\mathcal{M}$ is closed under finite intersections. In particular, this implies $E \in \widetilde{\mathcal{M}}$. Thus $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$.

Now we show the reverse inclusion $\mathcal{M} \supseteq \widetilde{\mathcal{M}}$. Let $E \in \widetilde{\mathcal{M}}$. Since $\mu(X) < \infty$ and E is locally measurable, we have

$$\begin{aligned} E &= E \cap X \\ &\in \mathcal{M}. \end{aligned}$$

Thus $\mathcal{M} \supseteq \widetilde{\mathcal{M}}$. □

Problem 4

Lemma 0.1. *Let $A, B \in \mathcal{M}$ such that $A \subseteq B$. If $\mu(A) < \infty$, then*

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

Proof. By finite additivity of μ, we have

$$\begin{aligned}\mu(B) &= \mu((B \backslash A) \cup A) \\ &= \mu(B \backslash A) + \mu(A).\end{aligned}$$

If moreover $\mu(A) < \infty$, then we may subtract $\mu(A)$ from both sides to obtain

$$\mu(B \backslash A) = \mu(B) - \mu(A).$$

□

Problem 4.a

Proposition 0.4. *Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space. Then*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

for all $A, B \in \mathcal{M}$.

Proof. Let $A, B \in \mathcal{M}$. Then by finite additivity of μ, we have

$$\begin{aligned}\mu(A \cup B) &= \mu(A \cup (B \backslash A)) \\ &= \mu(A) + \mu(B \backslash A) \\ &= \mu(A) + \mu(B \backslash (A \cap B)) \\ &= \mu(A) + \mu(B) - \mu(A \cap B),\end{aligned}$$

where the last equality follows Lemma (0.1) since $(X, \mathcal{M}, \mu)$ is a finite measure space. □

Problem 4.b

Proposition 0.5. *Let (A_n) be a sequence of "almost pairwise disjoint" members of $\mathcal{M}$, in the sense that $\mu(A_i \cap A_j) = 0$ whenever $i \neq j$. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Proof. First note that countable subadditivity of μ implies

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

so it suffices to show the reverse inequality. Before doing so, we first prove by induction on $N \geq 1$, that

$$\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu(A_n). \tag{1}$$

The base case $N = 1$ holds trivially. Assume that we have shown (1) holds for some $N > 1$. Then

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{N+1} A_n\right) &= \mu\left(\left(\bigcup_{n=1}^{N} A_n\right) \cup A_{N+1}\right) \\
&= \mu\left(\bigcup_{n=1}^{N} A_n\right) + \mu(A_{N+1}) - \mu\left(\left(\bigcup_{n=1}^{N} A_n\right) \cap A_{N+1}\right) \\
&= \sum_{n=1}^{N} \mu\left(A_N\right) + \mu(A_{N+1}) - \mu\left(\bigcup_{n=1}^{N} (A_n \cap A_{N+1})\right) \\
&\geq \sum_{n=1}^{N+1} \mu(A_n) - \sum_{n=1}^{N} \mu(A_n \cap A_{N+1}) \\
&= \sum_{n=1}^{N+1} \mu(A_n) - \sum_{n=1}^{N} 0 \\
&= \sum_{n=1}^{N+1} \mu(A_n),
\end{aligned}$$

where we used the induction hypothesis to get from the second line to the third line, and where we used finite subadditivity of μ to get from the third line to the fourth line. We already have

$$\mu\left(\bigcup_{n=1}^{N+1} A_n\right) \leq \sum_{n=1}^{N+1} \mu(A_n)$$

by finite subadditivity of μ, and so it follows that

$$\mu\left(\bigcup_{n=1}^{N+1} A_n\right) = \sum_{n=1}^{N+1} \mu(A_n).$$

Therefore (1) holds for all $N \in \mathbb{N}$ induction.

Now we prove the reverse inequality: for each $N \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=1}^{N} \mu(A_n) &= \mu\left(\bigcup_{n=1}^{N} A_n\right) \\
&\subseteq \mu\left(\bigcup_{n=1}^{\infty} A_n\right)
\end{aligned}$$

by monotonicity of μ. By taking $N \to \infty$, we see that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

□

Problem 5

Proposition 0.6. *Let $\mathcal{A}$ be the collection of all finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \leq a \leq b \leq \infty$. Then*

1. *$\mathcal{A}$ is an algebra of subsets of $\mathbb{Q}$;*
2. *$\sigma(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$ where $\mathcal{P}(\mathbb{Q})$ is the collection of all subsets of $\mathbb{Q}$;*
3. *the function $\mu \colon \mathcal{A} \to [0, \infty]$ defined by $\mu(\varnothing) = 0$ and $\mu(A) = \infty$ for all nonempty $A \in \mathcal{A}$ is a measure on $\mathcal{A}$;*
4. *there is more than one measure on $\sigma(\mathcal{A})$ whose restriction to $\mathcal{A}$ is μ;*

Proof.
1. In Homework 1, it was shown that $(\mathbb{R} \cup \{\infty\}, \mathcal{T})$ was a semialgebra, where $\mathcal{T}$ consisted of all subintervals of $\mathbb{R} \cup \{\infty\}$ of the form $(a, b]$ where $-\infty \leq a \leq b \leq \infty$. If we let $\iota \colon \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ denote the inclusion map, then we see that $\iota^{-1}(\mathcal{T}) = \mathcal{S}$, where $\mathcal{S}$ denotes the collection of all subintervals of $\mathbb{Q}$ of the form $(a, b] \cap \mathbb{Q}$. It follows easily from Proposition (0.2) that $\mathcal{S}$ is a semialgebra of subsets of $\mathbb{Q}$.[2]
Therefore the set of all finite disjoint unions of members of $\mathcal{S}$ forms an algebra, and as any finite union of members of $\mathcal{S}$ can be expressed as a finite disjoint union of members of $\mathcal{S}$ (since $\mathcal{S}$ is a semialgebra), we see that $\mathcal{A}$ is an algebra.

2. Clearly $\mathcal{P}(\mathbb{Q}) \supseteq \sigma(\mathcal{A})$. Let us prove the reverse inclusion. We first observe that $\{r\} \in \sigma(\mathcal{A})$ for all $r \in \mathbb{Q}$. Indeed, if $r \in \mathbb{Q}$, then we have

$$\{r\} = \bigcap_{n \in \mathbb{N}} (r - 1/n, r] \cap \mathbb{Q} \in \sigma(\mathcal{A})$$

Now let $S \in \mathcal{P}(\mathbb{Q})$. Then since S is countable, we have

$$S = \bigcup_{s \in S} \{s\} \in \sigma(\mathcal{A}).$$

3. We have $\mu(\emptyset) = 0$ by definition. Let (A_n) be a sequence of pairwise disjoint members of $\mathcal{A}$ whose union also belongs to $\mathcal{A}$. If $\bigcup_{n=1}^{\infty} A_n \neq \emptyset$, then $A_n \neq \emptyset$ for some $n \in \mathbb{N}$, and thus

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \infty \\ &= \mu(A_n) \\ &= \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

Similarly, if $\bigcup_{n=1}^{\infty} A_n = \emptyset$, then $A_n = \emptyset$ for all $n \in \mathbb{N}$, and thus

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= 0 \\ &= \sum_{n=1}^{\infty} 0 \\ &= \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

In both cases, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

4. We define $\mu_1 \colon \mathcal{P}(\mathbb{Q}) \to [0, \infty]$ and $\mu_2 \colon \mathcal{P}(\mathbb{Q}) \to [0, \infty]$ by

$$\mu_1(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{else} \end{cases} \quad \text{and} \quad \mu_2(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \neq \emptyset \end{cases}$$

for all $A \in \mathcal{P}(\mathbb{Q})$. Both μ_1 and μ_2 restrict to μ as functions since every member of $\mathcal{A}$ is infinite. They are also both distinct as functions since, for example, $\mu_1(\{x\}) = 1$ and $\mu_2(\{x\}) = \infty$ for any $x \in \mathbb{Q}$. Thus it suffices to show that they are measures. That μ_2 is a measure follows from a similar argument as in the case of μ, so we

[2]Technically we showed that the inverse image of a σ-algebra is a σ-algebra. However the same reasoning used in that proof shows that the inverse image of a semialgebra is a semialgebra: namely f^{-1} commutes with complements and unions.

just show that μ_1 is a measure. We have $\mu_1(\emptyset) = 0$ since $|\emptyset| = 0$. Next we show it is finitely additive. Let A and B be members of $\mathcal{P}(\mathbb{Q})$ such that $A \cap B = \emptyset$. If $A = \emptyset$, then

$$\begin{aligned}
\mu_1(A \cup B) &= \mu_1(\emptyset \cup B) \\
&= \mu_1(B) \\
&= 0 + \mu_1(B) \\
&= \mu_1(\emptyset) + \mu_1(B) \\
&= \mu_1(A) + \mu_1(B).
\end{aligned}$$

Similarly, if $B = \emptyset$, then $\mu_1(A \cup B) = \mu_1(A) \cup \mu_1(B)$. So assume neither A nor B is the emptyset. Write them as

$$A = \{x_1, \ldots x_m\} \quad \text{and} \quad B = \{y_1, \ldots, y_n\}.$$

Then

$$A \cup B = \{x_1, \ldots, x_m, y_1, \ldots, y_n\},$$

and so

$$\begin{aligned}
\mu_1(A \cup B) &= m + n \\
&= \mu_1(A) + \mu_1(B).
\end{aligned}$$

It follows that μ_1 is finitely additive.

Now, let (A_n) be a sequence of pairwise disjoint members of $\mathcal{P}(\mathbb{Q})$. Suppose that $A_n \neq \emptyset$ for only finitely many n, say $n_1, \ldots, n_k$. Then it follows from finite additivity of μ_1 and the fact that $\mu(\emptyset) = 0$ that

$$\begin{aligned}
\mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu_1\left(\bigcup_{i=1}^{k} A_{n_i}\right) \\
&= \sum_{i=1}^{k} \mu(A_{n_i}) \\
&= \sum_{n=1}^{\infty} \mu(A_n).
\end{aligned}$$

Now suppose that $A_n \neq \emptyset$ for infinitely many n. By taking a subsequence of (A_n) if necessary, we may assume that $A_n \neq \emptyset$ for all n. Then $\bigcup_{n=1}^{\infty} A_n$ is infinite, and so

$$\begin{aligned}
\mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) &= \infty \\
&\geq \sum_{n=1}^{\infty} \mu_1(A_n) \\
&\geq \sum_{n=1}^{\infty} 1 \\
&= \infty.
\end{aligned}$$

It follows that

$$\mu_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty = \sum_{n=1}^{\infty} \mu_1(A_n).$$

Therefore μ_1 and μ_2 are distinct measures which restrict to μ. □

Remark. Note that the extension theorem does not apply here as μ is not a finite measure.

Problem 6

Proposition 0.7. *Let $A, B \in \mathcal{P}(X)$. Then the following properties hold*

1. $A\Delta A = \emptyset$;
2. $(A\Delta B)\Delta C = A\Delta(B\Delta C)$;
3. $(A\Delta B)\Delta(B\Delta C) = A\Delta C$;
4. $(A\Delta B)\Delta(C\Delta D) = (A\Delta C)\Delta(B\Delta D)$;
5. $|1_A - 1_B| = 1_{A\Delta B}$.

Proof.
1. We have

$$\begin{aligned} A\Delta A &= (A\backslash A)\cup(A\backslash A)\\ &= \emptyset\cup\emptyset\\ &= \emptyset. \end{aligned}$$

2. We have

$$\begin{aligned} (A\Delta B)\Delta C &= ((A\Delta B)\cup C)\cap((A\Delta B)\cap C)^c\\ &= ((A\Delta B)\cup C)\cap((A\Delta B)^c\cup C^c)\\ &= (((A\cup B)\cap(A\cap B)^c)\cup C))\cap((A\cap B^c)\cup(A^c\cap B))^c\cup C^c)\\ &= (((A\cup B)\cap(A^c\cup B^c))\cup C))\cap(((A\cap B^c)^c\cap(A^c\cap B)^c)\cup C^c)\\ &= (A\cup B\cup C)\cap(A^c\cup B^c\cup C)\cap((A^c\cup B)\cap(A\cup B^c))\cup C^c)\\ &= (A\cup B\cup C)\cap(A^c\cup B^c\cup C)\cap(A^c\cup B\cup C^c)\cap(A\cup B^c\cup C^c)\\ &= (B\cup C\cup A)\cap(B^c\cup C^c\cup A)\cap(B^c\cup C\cup A^c)\cap(B\cup C^c\cup A^c)\\ &= ((B\cup C\cup A)\cap(B^c\cup C^c\cup A))\cap((B^c\cup C)\cap(B\cup C^c))\cup A^c)\\ &= ((B\cup C\cup A)\cap(B^c\cup C^c\cup A))\cap(((B\cap C^c)^c\cap(B^c\cap C)^c)\cup A^c)\\ &= (((B\cup C)\cap(B\cap C)^c)\cup A))\cap((B\cap C^c)\cup(B^c\cap C))^c\cup A^c)\\ &= ((B\Delta C)\cup A)\cap((B\Delta C)^c\cup A^c)\\ &= ((B\Delta C)\cup A)\cap((B\Delta C)\cap A)^c\\ &= (B\Delta C)\Delta A\\ &= A\Delta(B\Delta C) \end{aligned}$$

3. We have

$$\begin{aligned} (A\Delta B)\Delta(B\Delta C) &= A\Delta B\Delta B\Delta C\\ &= A\Delta\emptyset\Delta C\\ &= A\Delta C. \end{aligned}$$

4. We have

$$\begin{aligned} (A\Delta B)\Delta(C\Delta D) &= A\Delta B\Delta C\Delta D\\ &= A\Delta C\Delta B\Delta D\\ &= (A\Delta C)\Delta(B\Delta D) \end{aligned}$$

5. Let $x \in X$. If $x \notin A \cup B$, then

$$\begin{aligned} |1_A(x) - 1_B(x)| &= |0-0| \\ &= 0 \\ &= 0-0 \\ &= 1_{A\cup B}(x) - 1_{A\cap B}(x) \\ &= 1_{A\Delta B}(x). \end{aligned}$$

If $x \in A \backslash B$, then

$$\begin{aligned} |1_A(x) - 1_B(x)| &= |1-0| \\ &= 1 \\ &= 1-0 \\ &= 1_{A\cup B}(x) - 1_{A\cap B}(x) \\ &= 1_{A\Delta B}(x). \end{aligned}$$

If $x \in B \backslash A$, then

$$\begin{aligned} |1_A(x) - 1_B(x)| &= |0-1| \\ &= 1 \\ &= 1-0 \\ &= 1_{A\cup B}(x) - 1_{A\cap B}(x) \\ &= 1_{A\Delta B}(x). \end{aligned}$$

If $x \in A \cap B$, then

$$\begin{aligned} |1_A(x) - 1_B(x)| &= |1-1| \\ &= 0 \\ &= 1-1 \\ &= 1_{A\cup B}(x) - 1_{A\cap B}(x) \\ &= 1_{A\Delta B}(x). \end{aligned}$$

Thus $|1_A(x) - 1_B(x)| = 1_{A\Delta B}(x)$ for all $x \in X$ and hence $|1_A - 1_B| = 1_{A\Delta B}$. □

Problem 7

Proposition 0.8. *Let (A_n) and (B_n) be two sequences of sets. Then*

$$\left(\bigcup_{m=1}^{\infty} A_m\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \Delta B_n) \quad \text{and} \quad \left(\bigcap_{m=1}^{\infty} A_m\right) \Delta \left(\bigcap_{n=1}^{\infty} B_n\right) \subseteq \bigcup_{n=1}^{\infty} (A_n \Delta B_n).$$

Proof. We have

$$\begin{aligned} \left(\bigcup_{m=1}^{\infty} A_m\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n\right) &= \left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right) \Big\backslash \left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) \\ &= \left(\bigcup_{n=1}^{\infty} (A_n \cup B_n)\right) \Big\backslash \left(\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty} (A_m \cap B_n)\right) \\ &\subseteq \left(\bigcup_{n=1}^{\infty} (A_n \cup B_n)\right) \Big\backslash \left(\bigcup_{n=1}^{\infty} (A_n \cap B_n)\right) \\ &\subseteq \bigcup_{n=1}^{\infty} (A_n \cup B_n) \backslash (A_n \cap B_n) \\ &= \bigcup_{n=1}^{\infty} (A_n \Delta B_n). \end{aligned}$$

Similarly, we have

$$\begin{aligned}
\left(\bigcap_{m=1}^{\infty} A_m\right) \Delta \left(\bigcap_{n=1}^{\infty} B_n\right) &= \left(\bigcap_{m=1}^{\infty} (A_m^c)^c\right) \Delta \left(\bigcap_{n=1}^{\infty} (B_n^c)^c\right) \\
&= \left(\bigcup_{m=1}^{\infty} A_m^c\right)^c \Delta \left(\bigcup_{n=1}^{\infty} B_n^c\right)^c \\
&= \left(\bigcup_{m=1}^{\infty} A_m^c\right) \Delta \left(\bigcup_{n=1}^{\infty} B_n^c\right) \\
&\subseteq \bigcup_{n=1}^{\infty} (A_n^c \Delta B_n^c) \\
&= \bigcup_{n=1}^{\infty} (A_n \Delta B_n).
\end{aligned}$$

□

Appendix

Disjointification

Proposition 0.9. *Let $\mathcal{A}$ be an algebra of subsets of X and let (A_n) be a sequence of sets in $\mathcal{A}$. Then there exists a sequence (D_n) of sets in $\mathcal{A}$ such that*

1. *$D_n \subseteq A_n$ for all $n \in \mathbb{N}$.*
2. *$D_m \cap D_n = \emptyset$ for all $m, n \in \mathbb{N}$ such that $m \neq n$.*
3. *$\bigcup_{m=1}^{n} D_m = \bigcup_{m=1}^{n} A_m$ for all $n \in \mathbb{N}$.*

*We say the sequence (D_n) is the **disjointification** of the sequence (A_n) or that we **disjointify** the sequence (A_n) to the sequence (D_n).*

Proof. Set $D_1 := A_1$ and

$$D_n := A_n \setminus \left(\bigcup_{m=1}^{n-1} A_m\right)$$

for all $n > 1$. It is clear that $D_n \in \mathcal{A}$ and that $D_n \subseteq A_n$ for all $n \in \mathbb{N}$. Let us show that $D_m \cap D_n = \emptyset$ whenever $m \neq n$. Without loss of generality, we may assume that $m < n$. Then since $D_m \subseteq A_m$ and $D_n \cap A_m = \emptyset$, we have $D_m \cap D_n = \emptyset$. It remains to show

$$\bigcup_{m=1}^{n} D_m = \bigcup_{m=1}^{n} A_m$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since $D_m \subseteq A_m$ for all $m \leq n$, we have

$$\bigcup_{m=1}^{n} D_m \subseteq \bigcup_{m=1}^{n} A_m.$$

To show the reverse inclusion, let $x \in \bigcup_{m=1}^{n} A_m$. Then $x \in A_m$ for some $m = 1, \ldots, n$. Choose m to be the smallest natural number such that $x \in A_m$. Then x belongs to A_m but does not belong to $A_1, \ldots, A_{m-1}$. In other words,

$$x \in D_m \subseteq \bigcup_{k=1}^{n} D_k.$$

This implies the reverse inclusion

$$\bigcup_{m=1}^{n} D_m \supseteq \bigcup_{m=1}^{n} A_m.$$

□