# Research Project

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Let us state up front the theorem we wish to prove:

**Theorem 0.1.** *Let $X$ be a compact Hausdorff space, let $C(X)$ be the space of continuous real-valued functions on $X$ equipped with the supremum norm, and let $\ell$ be a linear functional on $C(X)$. Then there exists a unique Baire measure $\mu$ on $X$ such that*

$$\ell(f) = \int_X f \mathrm{d}\mu$$

*for all $f \in C(X)$.*

**Proposition 0.1.** *Let $\mu$ be a signed Baire measure in $\mathcal{M}(X)$. Define $\ell_\mu \colon C(X) \to \mathbb{R}$ by*

$$\ell_\mu(f) = \int_X f \mathrm{d}\mu$$

*for all $f \in C(X)$. The map $\ell_\mu$ is a bounded linear functional*

*Proof.* Linearity of $\ell_\mu$ follows from linearity of integration. To see that $\ell_\mu$ is bounded, note that

$$\begin{aligned} \ell_\mu(f) &= \int_X f \mathrm{d}\mu \\ &\leq \|f\|_\infty \mu(X) \end{aligned}$$

for all $f \in C(X)$. Taking $f$ to be the constant function 1, we see that $\|\ell_\mu\| = \mu(X)$. □

**Definition 0.1.** Let $X$ be a topological space. We say $X$ is **extremally disconnected** if each open subset of $X$ has open closure, that is, if $U$ is an open subset of $X$, then $\overline{U}$ is a clopen subset of $X$. Equivalently, every pair of disjoint open subsets of $X$ have disjoint closures.

**Theorem 0.2.** *Let $X$ be a compact Hausdorff space, let $C(X)$ be the space of continuous real-valued functions on $X$ equipped with the supremum norm, and let $\ell$ be a linear functional on $C(X)$. Then there exists a unique Baire measure $\mu$ on $X$ such that $\ell = \ell_\mu$.*

*Proof.* We first show existence.

**Step 1:** Suppose that $X$ is equipped with the discrete topology.

□

## 0.1 Notation and Conventions

### 0.1.1 Category Theory

In this document, we consider the following categories:

- The category of all compact Hausdorff spaces and continuous maps between them, denoted **Comp**;
- The category of all Banach spaces and bounded linear maps between them, denoted **Ban**;

We will also be interested in the following functors:

- The functor $M \colon \mathbf{Comp} \to \mathbf{Ban}$ defined as follows: given a compact Hausdorff space $X$, we set $M(X)$ to be the Banach space of signed Baire measures on $X$, and given a continuous function $f \colon X \to Y$ between two compact Hausdorff spaces $X$ and $Y$, we set $M(f) \colon M(X) \to M(Y)$ to be the bounded linear map defined by

  $$M(\mu) = \mu \circ f^{-1}$$

  for all $\mu \in M(X)$.

- The functor $C\colon \mathbf{Comp} \to \mathbf{Ban}$ defined as follows: given a compact Hausdorff space $X$, we set $C(X)$ be the Banach space of continuous real-valued functions on $X$ equipped with the supremum norm, and given a continuous function $f\colon X \to Y$ between two compact Hausdorff spaces $X$ and $Y$, we set $C(f) = f^{\#}$ where $f^{\#}\colon C(Y) \to C(X)$ is bounded linear map defined by

$$f^{\#}(g) = g \circ f$$

for all $g \in C(Y)$.

- The functor $C^*\colon \mathbf{Comp} \to \mathbf{Ban}$ defined as follows: given a compact Hausdorff space $X$, we set $C^*(X) = C(X)^*$ to be the dual of $C(X)$, and given a continuous function $f\colon X \to Y$ between two compact Hausdorff spaces $X$ and $Y$, we set $C^*(f) = f^{\#\#}$ where $f^{\#\#}\colon C(X)^* \to C(Y)^*$ is the bounded linear map defined by

$$f^{\#\#}(\ell) = \ell \circ f^{\#}$$

for all $\ell \in C(X)^*$.

# 1 Introduction

Let $X$ be a compact Hausdorff space. We denote by $C(X)$ to be the space of real-valued continuous functions on $X$ equipped with the supremum norm. Recall that if $\mathcal{C}$ is any collection of subsets of $X$, then we denote by $\sigma(\mathcal{C})$ to be the smallest $\sigma$-algebra which contains $\mathcal{C}$. Suppose

$$\mathcal{C} = \{f$$

$\tau$ denotes the collection of all open subset of $\mathcal{C}$, then $\sigma(\tau)$ is the Borel $\sigma$-

## 1.1 Baire $\sigma$-algebra

**Definition 1.1.** Let $X$ be a compact Hausdorff space.

1. The **Borel $\sigma$-algebra** $\mathcal{B}_X$ is the $\sigma$-algebra generated by all open sets subsets of $X$.
2. The **Baire $\sigma$-algebra** $\mathcal{M}_X$ is the $\sigma$-algebra generated by all sets of the form $f^{-1}(U)$ where $U$ is an open subset of $\mathbb{C}$ and where $f \in C(X)$. In particular, $\mathcal{M}_X$ is the smallest $\sigma$-algebra which makes every $f \in C(X)$ measurable.
3. A measure $\mu$ is called a **Baire measure** if it satisfies the following conditions:
   (a) The domain of $\mu$ contains $\mathcal{M}_X$;
   (b) $\mu(K) < \infty$ for all compact Baire measureable sets $K$.
   (c) $\mu$ is inner regular, that is, for each Baire measurable set $E$, we have

$$\mu(E) = \sup\{\mu(K) \mid K \text{ is a compact Baire measurable set such that } K \subseteq E\}$$

   (d) $\mu$ is outer regular, that is, for each Baire measurable set $E$, we have

$$\mu(E) = \inf\{\mu(U) \mid U \text{ is an open Baire measurable set such that } E \subseteq U\}$$

We will prove the following form of the Riesz representation theorem:

**Theorem 1.1.** *Let $X$ be a compact Hausdorff space, let $C(X)$ be the space of continuous real-valued functions on $X$ equipped with the supremum norm, and let $\ell$ be a positive linear functional on $C(X)$. Then there exists a unique Baire measure $\mu$ on $X$ such that*

$$\ell(f) = \int_X f \mathrm{d}\mu$$

*for all $f \in C(X)$.*

*Proof.* ∎

### 1.2 Banach Space of Signed Measures

## 2 Extra

Let $X$ be a compact Hausdorff space. We denote by $C(X)$ be the Banach space of continuous real-valued functions on $X$ equipped with the supremum norm. As usual, we will denote by $C(X)^*$ to be the dual space of $C(X)$. We also denote by $M(X)$ to be the Banach space of signed Baire measures on $X$.