Graded Modules and Hilbert Functions

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1 Graded Rings and Graded Modules

1.1 Graded Rings and Graded K-Algebras

Definition 1.1. Let H be an additive semigroup with identity 0. An H-**graded ring** A is a ring together with a direct sum decomposition

$$A = \bigoplus_{h \in H} A_h,$$

where the A_h are abelian groups which satisfy the property that if $a_{h_1} \in A_{h_1}$ and $a_{h_2} \in A_{h_2}$, then $a_{h_1} a_{h_2} \in A_{h_1+h_2}$ (an equivalent way of saying this is $A_{h_1} A_{h_2} \subseteq A_{h_1+h_2}$). The A_h are called **homogeneous components** and the elements of A_h are called **homogeneous elements** of **degree** h. A **graded** K**-algebra** is a K-algebra which is an N_0-graded ring such that A_i is a K-vector space for all $i \geq 0$ and $A_0 = K$.

Remark.

1. We are mostly interested in the case where $H = \mathbb{N}_0$ or $H = \mathbb{Z}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Unless otherwise specified, when we omit H and simply say "let A be a graded ring", we mean A is an N_0-graded ring.
2. Let $A = \bigoplus_{i \geq 0} A_i$ be an graded ring, then A_0 is a subring of A. This follows since $1 \cdot 1 = 1$, hence $1 \in A_0$. This makes A into a graded A_0-algebra. For a K-algebra A, this implies already $K \subset A_0$, but to be a graded K-algebra, we require even $K = A_0$.
3. Let A be any ring, then $A_0 := A$ and $A_i := 0$ for all $i > 0$ defines a trivial structure of a graded ring for A.

One of the most basic examples of a graded K-algebra is the polynomial ring $A := K[x, y, z]$: Let A_i be the K-vector space generated by the monomials $x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \in A$ such that $\alpha_1 + \alpha_2 + \alpha_3 = i$. We clearly have $A_i A_j \subseteq A_{i+j}$. We also have a direct sum decomposition

$$A = \bigoplus_{i \geq 0} A_i,$$

The first few homogeneous components of A start out as

$$\begin{aligned}
A_0 &= K \\
A_1 &= Kx + Ky + Kz \\
A_2 &= Kx^2 + Kxy + Kxz + Ky^2 + Kyz + Kz^2 \\
&\vdots
\end{aligned}$$

The next proposition gives us a generalization of this construction.

Proposition 1.1. *Let $A = K[x_1, \ldots, x_n]$, $w = (w_1, \ldots, w_n)$ be a vector of positive integers, and let A_d be the K-vector space generated by all monomials of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $w_1\alpha_1 + \cdots + w_n\alpha_n = d$. Then $A = \oplus_{i \geq 0} A_i$ is a graded K-algebra.*

Proof. We clearly have $A_0 = K$. If $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in A_d$ and $x_1^{\beta_1} \cdots x_n^{\beta_n} \in A_{d'}$, then

$$w_1(\alpha_1 + \beta_1) + \cdots + w_n(\alpha_n + \beta_n) = w_1\alpha_1 + w_1\beta_1 \cdots + w_n\alpha_n + w_n\beta_n = d + d'$$

implies $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot x_1^{\beta_1} \cdots x_n^{\beta_n} \in A_{d+d'}$. □

Remark. Note that for each i we have $x_i \in A_{w_i}$. The elements of A_d are called **quasihomogeneous** or **weighted homogeneous** polynomials of **weighted degree** d with respect to the weights $w_1, \ldots, w_n$. If $w_1 = \cdots = w_n = 1$, we obtain the usual notion of homogeneous polynomials.

For example, let A be the polynomial ring $K[x, y, z]$. There is a direct sum decomposition

$$A = \bigoplus_{0 \leq i} A_i,$$

where A_i is K-vector space generated by the monomials $x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \in A$ where $\alpha_1 + 2\alpha_2 + 3\alpha_3 = i$. This gives A the structure of a graded K-algebra with respect to the weights $w = (1, 2, 3)$. The homogeneous components of A start out as

$$\begin{aligned}
A_0 &= K \\
A_1 &= Kx \\
A_2 &= Kx^2 + Ky \\
A_3 &= Kx^3 + Kxy + Kz \\
&\vdots
\end{aligned}$$

1.2 Graded Modules

Definition 1.2. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring. An A-module M, together with a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ into abelian groups is called a **graded A-module** if $A_i M_j \subset M_{i+j}$ for all $i \geq 0$ and $j \in \mathbb{Z}$. The elements of M_i are called **homogeneous** of **degree** i. If $m = \sum_i m_i$, with $m_i \in M_i$, then m_i is called the **homogeneous part** of **degree** i of m.

Remark. Again, we can easily generalize this construction to H-graded modules, but for our purposes, we are mainly interested in $H = \mathbb{Z}$ or $H = \mathbb{N}_0$.

Example 1.1. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded K-algebra and consider the free module $A^m = \bigoplus_{i=1}^m Ae_i$ where e_i denotes the standard basis element in A^m. Let $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$, define $\deg(e_i) := k_i$, and let M_k be the A_0-module generated by all fe_i with $f \in A_{k-k_i}$. Then $A^m = \bigoplus_{k \in \mathbb{Z}} M_k$ is a graded A-module.

Example 1.2. Continuing Example (4.2), let M be the graded A-module A^2 with weights $k = (1, 2)$. The homogeneous components of M start out as

$$
\begin{aligned}
&\vdots \\
M_0 &= 0 \\
M_1 &= Ke_1 \\
M_2 &= Kxe_1 + Ke_2 \\
M_3 &= Kx^2e_1 + Kye_1 + Kxe_2 \\
&\vdots
\end{aligned}
$$

Definition 1.3. Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A-module and define $M(d) := \bigoplus_{i \in \mathbb{Z}} M(d)_i$ with $M(d)_i := M_{i+d}$. Then $M(d)$ is a graded A-module, especially $A(d)$ is a graded A-module. $M(d)$ is called the d'**th twist** or the d'**th shift** of M.

Example 1.3. The module M in Example (1.2) is isomorphic to $A(-1) \oplus A(-2)$.

Lemma 1.1. *Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A-module and $N \subset M$ a submodule. The following conditions are equivalent:*

1. *N is graded with the induced grading, that is $N = \bigoplus_{i \in \mathbb{Z}} (M_i \cap N)$.*
2. *N is generated by homogeneous elements.*
3. *Let $m = \sum m_i$ with $m_i \in M_i$. Then $m \in N$ if and only if $m_i \in N$ for all i.*

Definition 1.4. A submodule $N \subset M$ satisfying the equivalent conditions of Lemma (1.1) is called a **graded** (or **homogeneous**) submodule. A graded submodule of a graded ring is called a **graded** (or **homogeneous**) ideal.

Remark. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring, and let $I \subset A$ be a homogeneous ideal. Then the quotient A/I has an induced structure as a graded ring: $A/I = \bigoplus_{i \geq 0} (A_i + I)/I \cong \bigoplus_{i \geq 0} A_i/(I \cap A_i)$.

Example 1.4. Let $A = K[x, y]$ and $I = \langle xy + y^2, x^3 \rangle$. Then I is a homogeneous ideal, and therefore is graded with the induced grading $I = \bigoplus_{i \in \mathbb{Z}} (A_i \cap I)$. Before we write down the first few homogeneous components of I, we first use Singular to compute a Gröbner basis of G of I with respect to graded reverse lex order. We obtain $G = \{f_1, f_2, f_3\}$. where $f_1 = xy + y^2$, $f_2 = x^3$, and $f_3 = y^4$. Now we write the first few homogeneous components of I:

$$
\begin{aligned}
I_0 &= 0 \\
I_1 &= 0 \\
I_2 &= Kf_1 \\
I_3 &= Kxf_1 + Kyf_1 + Kf_2 \\
I_4 &= Kx^2 f_1 + Kxyf_1 + Ky^2 f_1 + Kxf_2 + Kyf_2 \\
I_5 &= A_5 \\
&\vdots
\end{aligned}
$$

The quotient A/I is also graded. Using the Gröbner basis we just calculated, we see that the homogeneous components of the quotient start out as

$$
\begin{aligned}
(A/I)_0 &= K \cdot \overline{1} \\
(A/I)_1 &= K\overline{x} + K\overline{y} \\
(A/I)_2 &= K\overline{x}^2 + K\overline{y}^2 \\
(A/I)_3 &= K\overline{y}^3 \\
(A/I)_4 &= 0 \\
&\vdots
\end{aligned}
$$

Example 1.5. Let $S = K[x_1, \dots, x_n]$ and I be a homogeneous ideal in S, so S/I is a graded K-algebra. Define $S_I := \operatorname{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle)$. There is an obvious decompostion of S_I into homogeneous pieces $(S_I)_i$, where $(S_I)_i = \operatorname{Span}_K(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle$ and $|\alpha| = i)$. In fact, S/I and S_I are isomorphic as graded K-algebras.

To see this, let $G = \{g_1, g_2 \dots, g_r\}$ be the reduced Gröbner basis for I with respect to a fixed monomial ordering. Recall that $f \in K[x_1, \dots, x_n]$ can be written in the form $f = g + r$, where $g \in I$ and no term of r is divisible by any element of $\mathrm{LT}(I)$, and, moreover, g and r are uniquely determined. We use the notation $f^G := r$ and call this the **normal form of** f **with respect to** I (or simply the **normal form of** f if the there is no confusion of the ideal I). It follows from uniqueness of f^G and $f - f^G$ that taking the normal form of a polynomial is a K-linear map:

$$c_1 f_1^G + c_2 f_2^G = (c_1 f_1 + c_2 f_2)^G \qquad \text{for all } c_1, c_2 \in K \text{ and } f_1, f_2 \in S. \tag{1}$$

The isomorphism from S/I to S_I is given by mapping $\overline{f} \in (S/I)$ to $f^G \in S_I$, where K-linearity follows from (1). The inverse to this isomorphism is given by mapping $f \in S_I$ to $\overline{f} \in S/I$.

Using this isomorphism, we can carry multiplication from S/I over to S_I to turn S_I into a K-algebra: For $f_1, f_2 \in S_I$, we define multiplication as

$$f_1 \cdot f_2 = (f_1 f_2)^G.$$

Bilinearity of $\cdot$ follows from bilinearity of multiplication and linearity of $-^G$. Also, $-^G$ preserves homogeneity, and so S_I is isomorphic to S/I as a graded K-algebra.

Example 1.6. Let $A = K[x, y, z]$ and $I = \langle y^3 - z^2, x^3 - z \rangle$. Then I is homogeneous if we consider A as a graded ring with respect to the weights $w = (1, 2, 3)$. Next let $M = \langle (y^3 - z^2)e_1 + (x^3 - z)e_2, x^3 e_1 + e_2 \rangle$. Then M is a homogeneous submodule of A^2 if we consider A^2 as a graded A-module with respect to weights $k = (0, 3)$.

Example 1.7. Let $A = K[x, y, z]$ and $I = \langle y^5 - z^2, x^3 - z, x^6 - y^5 \rangle$. Can we give A a grading so that I is a homogeneous ideal? Yes. To find a grading for A such that I is a homogeneous ideal, we need to solve the following system of equations

$$
\begin{aligned}
5w_2 - 2w_3 &= 0 \\
3w_1 - w_3 &= 0 \\
6w_1 - 5w_2 &= 0
\end{aligned}
$$

where $w_1, w_2, w_3 \in \mathbb{Z}$. A solution to this is given by $w_1 = 5$, $w_2 = 6$, and $w_3 = 15$. On the other hand, $J = \langle y^5 - z^2, x^3 - z, x^7 - y^5 \rangle$ cannot be made into a homogeneous ideal with respect to some grading since

$$\left| \begin{pmatrix} 0 & 5 & -2 \\ 3 & 0 & -1 \\ 7 & -5 & 0 \end{pmatrix} \right| = -5 \neq 0.$$

Definition 1.5. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring and $M = \bigoplus_{i \in \mathbb{Z}} M_i$, $N = \bigoplus_{i \in \mathbb{Z}} N_i$ be graded A-modules. A homomorphism $\varphi : M \to N$ is called **homogeneous** (or **graded**) of degree d if $\varphi(M_i) \subset N_{i+d}$ for all i. If φ is homogeneous of degree zero we call φ just **homogeneous**.

Example 1.8. Let be the graded K-algebra $K[x, y, z, t]$ with respect to the weights $w = (1, 1, 1, 1)$. Then the matrix

$$\varphi = \begin{pmatrix} x + y + z & w^2 - x^2 & x^3 \\ 1 & x & xy + z^2 \end{pmatrix}$$

defines a homomorphism $\varphi : A(-1) \oplus A(-2) \oplus A(-3) \to A \oplus A(-1)$ which is graded of degree zero.

Definition 1.6. Let A be a ring and let Q be an ideal in A. The **associated graded ring of A with respect to Q** is

$$\mathrm{Gr}_Q(A) = \bigoplus_{i=0}^{\infty} Q^i/Q^{i+1}.$$

The multiplication in $\mathrm{Gr}_Q(A)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$, and $\mathrm{Gr}_Q(A)$ is a graded ring with $\mathrm{Gr}_Q(A)_0 = A/Q$. If M is an A-module, one similarly constructs the **associated graded module**

$$\mathrm{Gr}_Q(M) = \bigoplus_{i=0}^{\infty} Q^i M/Q^{i+1}M.$$

It is straightforward to verify that $\mathrm{Gr}_Q(M)$ is a graded $\mathrm{Gr}_Q(A)$-module.

Example 1.9. Let $A = K[x,y,z]$ and let $Q = \langle x^2, xy\rangle$. We want to compute $\mathrm{Gr}_Q(A)$. An easy computation shows that $Q^2 = \langle x^4, x^3y, x^2y^2\rangle$. Let us write down the first few homogeneous components of $\mathrm{Gr}_Q(A)$ using a K-basis:

$$\begin{aligned}
\mathrm{Gr}_Q(A)_0 &= A/Q = K + K\overline{x} + K\overline{y} + K\overline{y}^2 + K\overline{y}^3 + K\overline{y}^4 + K\overline{y}^5 + K\overline{y}^6 + K\overline{y}^7 + K\overline{y}^8 \cdots \\
\mathrm{Gr}_Q(A)_1 &= Q/Q^2 = K\overline{x}^2 + K\overline{xy} + K\overline{x}^3 + K\overline{x}^2\overline{y} + K\overline{xy}^2 + K\overline{xy}^3 + K\overline{xy}^4 + K\overline{xy}^5 + \cdots \\
&\vdots
\end{aligned}$$

This way of writing things down isn't very illuminating. However, there is another way to think of $\mathrm{Gr}_Q(A)$. First we note that we have a surjective morphism of graded (A/Q)-algebras

$$\varphi : (A/Q)[s,t] \to \mathrm{Gr}_Q(A)$$

where φ is the map induced by mapping $s \mapsto \overline{x}^2 \in Q/Q^2$ and $t \mapsto \overline{xy} \in Q/Q^2$. However, this map is not injective, because

$$\begin{aligned}
\varphi(\overline{y}s - \overline{x}t) &= \overline{y}\varphi(s) - \overline{x}\varphi(t) \\
&= \overline{yx^2} - \overline{xxy} \\
&= 0.
\end{aligned}$$

and $\overline{y}s - \overline{x}t \neq 0$ in $(A/Q)[s,t]$. This isn't the only nontrivial relation though.

Lemma 1.2. *Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian graded ring and let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded A-module. Then*

1. *There exists $m \in \mathbb{Z}$ such that $M_i = \langle 0\rangle$ for $i < m$;*
2. *M_i is a finitely generated A_0-module for all $i \in \mathbb{Z}$. In particular, if A is a Noetherian graded K-algebra, then $\dim_K(M_i)$ is finite for all $i \in \mathbb{Z}$.*

Proof.

1. Is obvious because M is finitely generated and a graded A-module.
2. First we show A_i is a finitely generated A_0-module for each $i \geq 0$. Since A is Noetherian, $\langle A_i\rangle$ is finitely generated, say by $x_1, \ldots, x_{\lambda_i} \in A$. Since $1 \in A_0$, $1 \cdot x_j = x_j$ implies $x_j \in A_i$ for all $j = 1, \ldots, \lambda_i$. Then

$$\begin{aligned}
\langle A_i\rangle &= \langle x_1, \ldots, x_{\lambda_i}\rangle \\
&= Ax_1 + \cdots + Ax_{\lambda_i} \\
&= (A_0x_1 + \cdots + A_0x_{\lambda_i}) \oplus (A_1x_1 + \cdots + A_1x_{\lambda_i}) \oplus (A_2x_1 + \cdots + A_2x_{\lambda_i}) \oplus \cdots .
\end{aligned}$$

Clearly $A_i = A_0x_1 + \cdots + A_0x_{\lambda_i}$, since $A_jx_1 + \cdots + A_jx_{\lambda_i} \in A_{i+j}$ for all $j > 0$, which shows A_i is a finitely generated A_0-module. Next, since M is finitely generated, there exists finitely many homogeneous elements $m_1, \ldots, m_k$ in M such that

$$M = Am_1 + \cdots + Am_k$$

where $m_i \in M_{e_i}$ for all $i = 1, \ldots, k$. Then

$$M_n = A_{n-e_1}m_1 + \cdots + A_{n-e_k}m_k.$$

This implies that M_n is a finitely generated A_0-module because the A_i are finitely generated A_0-modules.

□

Definition 1.7. Let H be an additive semigroup with identity 0. A **semigroup ordering** on H is a partial ordering $>$ on H such that

1. $>$ is a total ordering, i.e. either $h_\alpha > h_\beta$ or $h_\beta > h_\alpha$ for all $h_\alpha, h_\beta \in H$.
2. $>$ is translate invariant, i.e. $h_\alpha > h_\beta$ implies $h_\alpha + h_\gamma > h_\beta + h_\gamma$ for all $h_\alpha, h_\beta, h_\gamma \in H$.
3. $>$ is a well-ordering, i.e. every non-empty subset of H has a least element in this ordering.

Example 1.10. The integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$ can be equipped with the usual semigroup ordering $>$.

Theorem 1.3. *Let M be a Noetherian graded module over a Noetherian graded ring A, where the grading is by a semigroup H equipped with a semigroup ordering $>$. Then every associated prime $\mathfrak{p}$ of M is a homogeneous ideal.*

Proof. If $\mathfrak{p}$ is an associated prime of M, it is the annihilator of a nonzero element

$$u = u_{j_1} + \cdots + u_{j_t} \in M,$$

where the u_{j_ν} are nonzero homogeneous elements of degrees $j_1 < \cdots < j_t$. Choose u such that t is as small as possible. Suppose that

$$a = a_{i_1} + \cdots + a_{i_s}$$

kills u, where for every ν, a_{i_ν} has degree i_ν, and $i_1 < \cdots < i_s$. We shall show that every a_{i_ν} kills u, which proves that $\mathfrak{p}$ is homogeneous. It suffices to show that a_{i_1} kills u (since $a - a_{i_1}$ kills u and we can proceed by induction). Since $au = 0$, the unique least degree term $a_{i_1}u_{j_1} = 0$. Therefore

$$u' = a_{i_1}u = a_{i_1}u_{j_2} + \cdots + a_{i_1}u_{j_t}.$$

If this element is nonzero, its annihilator is still $\mathfrak{p}$, since $Au \cong A/\mathfrak{p}$ and every nonzero element has annihilator $\mathfrak{p}$. Since $a_{i_1}u_{j_\nu}$ is homogeneous of degree $i_1 + j_\nu$, or else is 0, u' has fewer nonzero homogeneous components than u does, contradicting our choice of u. □

Corollary. *If I is a homogeneous ideal of a Noetherian ring A graded by a semigroup H equipped with a semigroup ordering $>$, then every minimal prime of I is homogeneous.*

Proof. This is immediate, since the minimal primes of I are among the associated primes of A/I. □

Proposition 1.2. *Let A be a graded ring, where the grading is by a semigroup H equipped with a semigroup ordering $>$ and let I be a homogeneous ideal. Then $\sqrt{I}$ is homogeneous.*

Proof. Let

$$f_{i_1} + \cdots + f_{i_k} \in \sqrt{I}$$

with $i_1 < \cdots < i_k$ and each f_{i_j} nonzero of degree i_j. We need to show that every $f_{i_j} \in \sqrt{I}$. If any of the components are in $\sqrt{I}$, we may subtract them off, giving a similar sum whose terms are the homogeneous components not in $\sqrt{I}$. Therefore it suffices to show that $f_{i_1} \in \sqrt{I}$. But

$$\left(f_{i_1} + \cdots + f_{i_k}\right)^N \in I$$

for some $N > 0$. When we expand, there is a unique term formally of least degree, namely $f_{i_1}^N$, and therefore this term is in I, since I is homogeneous. But this means that $f_{i_1} \in \sqrt{I}$, as required. □

Corollary. *Let A be a finitely generated graded K-algebra and let $\mathfrak{m} = \bigoplus_{d=1}^\infty A_d$ be the homogeneous maximal ideal of A. Then $dim(A) = height(\mathfrak{m}) = dim(A_\mathfrak{m})$.*

Proof. The dimension of A will be equal to the dimension of $A/\mathfrak{p}$ for one of the minimal primes $\mathfrak{p}$ of A. Since $\mathfrak{p}$ is minimal, it is an associated prime and therefore is homogeneous. Hence, $\mathfrak{p} \subseteq \mathfrak{m}$. The domain $A/\mathfrak{p}$ is finitely generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathfrak{m}/\mathfrak{p}$. Thus,

$$\dim(A) = \dim(A/\mathfrak{p}) = \dim((A/\mathfrak{p})_\mathfrak{m}) \leq \dim(A_\mathfrak{m}) \leq \dim(A),$$

and so equality holds throughout, as required. □

2 Hilbert Function and Dimension

The Hilbert function of a graded module associates to an integer n the dimension of the nth graded part of the given module. For sufficiently large n, the values of this function are given by a polynomial, the Hilbert polynomial.

Definition 2.1. Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian graded K-algebra and let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded A-module. The **Hilbert function** $H_M : \mathbb{Z} \to \mathbb{Z}$ of M is defined by

$$H_M(n) := \dim_K(M_n),$$

and the **Hilbert-Poincare series HP**$_M$ of M is defined by

$$\mathrm{HP}_M(t) := \sum_{n\in\mathbb{Z}} H_M(n)t^n \in \mathbb{Z}[[t]][t^{-1}].$$

By definition, H_M (and, hence, HP_M) depend only on the graded structure of M, i.e. the M_i are K-vector spaces, hence, if $\varphi : B \to A$ is a graded K-algebra map, then it does not matter whether we consider M as an A-module or B-module. In particular, since $A/\mathrm{Ann}_A(M)$ is a graded A-algebra, we may always consider M as an $A/\mathrm{Ann}_A(M)$-module when computing the hilbert function.

2.1 Properties of the Hilbert Function and Hilbert-Poincare Series

Lemma 2.1. *Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian graded K-algebra, and let M be a finitely generated graded A-module.*

1. *Let $N \subset M$ be a graded submodule, then*

$$H_M(n) = H_N(n) + H_{M/N}(n)$$

for all n, in particular, $HP_M(t) = HP_N(t) + HP_{M/N}(t)$.

2. *Let d be an integer, then*

$$H_{M(d)}(n) = H_M(n+d)$$

for all n, in particular, $HP_{M(d)}(t) = t^{-d}HP_M(t)$.

3. *Let d be a non-negative integer, let $f \in A_d$, and let $\varphi : M(-d) \to M$ be defined by $\varphi(m) := f \cdot m$. Then $Ker\varphi$ and $Coker\varphi$ are graded (A/f)-modules with the induced gradings and*

$$H_M(n) - H_M(n-d) = H_{Coker(\varphi)}(n) - H_{Ker(\varphi)}(n-d),$$

in particular, $HP_M(t) - t^d HP_M(t) = HP_{Coker(\varphi)}(t) - t^d HP_{Ker(\varphi)}(t)$.

Proof.

1. Holds, because $N_i = N \cap M_i$ and $(M/N)_i = M_i/N_i$.
2. An immediate consequence of the definition of $M(d)$.
3. Consequence of (1) and (2).

□

2.1.1 Reading off the Hilbert Function from a Free Resolution

Proposition 2.1. *Let $A = \bigoplus_{i\geq 0} A_i$ be a Noetherian graded K-algebra on n generators and let $M = \bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded A-module. Suppose*

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} A(-m_{r,j}) \xrightarrow{\varphi_r} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbb{Z}} A(-m_{1,j}) \xrightarrow{\varphi_1} \bigoplus_{j\in\mathbb{Z}} A(-m_{0,j}) \longrightarrow M \longrightarrow 0$$

is an exact sequence of graded A-modules. Then

$$HP_M(t) = \frac{\sum_{i=0}^{r}(-1)^i \left(\sum_j t^{m_{i,j}}\right)}{(1-t)^n}.$$

Proof. The exact sequence of graded A-modules

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} A(-m_{r,j}) \xrightarrow{\varphi_r} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbb{Z}} A(-m_{1,j}) \xrightarrow{\varphi_1} \bigoplus_{j\in\mathbb{Z}} A(-m_{0,j}) \longrightarrow M \longrightarrow 0$$

gives rise to an exact sequence of K-vector spaces

$$0 \longrightarrow \bigoplus_{j\in\mathbb{Z}} (A(-m_{r,j}))_i \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbb{Z}} (A(-m_{1,j}))_i \longrightarrow \bigoplus_{j\in\mathbb{Z}} (A(-m_{0,j}))_i \longrightarrow M \longrightarrow 0$$

for each $i \in \mathbb{Z}$. Now apply Lemma (2.1). □

2.1.2 The Structure of the Hilbert-Poincare Series

Theorem 2.2. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded K-algebra, and assume that A is generated, as a K-algebra, by $x_1, \ldots, x_r \in A_1$. Then, for any finitely generated (positively) graded A-module $M = \bigoplus_{i\geq 0} M_i$,*

$$HP_M(t) = \frac{Q(t)}{(1-t)^r}$$

for some $Q(t) \in \mathbb{Z}[t]$.

Proof. We prove the theorem using induction on r. In the case $r = 0$, M is a finite dimensional K-vector space, and therefore, there exists an integer n such that $M_i = \langle 0 \rangle$ for $i \geq n$. This implies $\mathrm{HP}_M(t) \in \mathbb{Z}[t]$.

Assume that $r \geq 0$ and consider the map $\varphi : M(-1) \to M$ defined by $\varphi(m) := x_1 \cdot m$. Using Lemma (2.1), we obtain

$$(1-t) \cdot \mathrm{HP}_M(t) = \mathrm{HP}_{\mathrm{Coker}(\varphi)}(t) - t \cdot \mathrm{HP}_{\mathrm{Ker}(\varphi)}(t).$$

Now both $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are graded (A/x_1)-modules. Using the induction hypothesis we obtain $\mathrm{HP}_{\mathrm{Coker}(\varphi)}(t) = Q_1(t)/(1-t)^{r-1}$ and $\mathrm{HP}_{\mathrm{Ker}(\varphi)}(t) = Q_2(t)/(1-t)^{r-1}$ for some $Q_1, Q_2 \in \mathbb{Z}[t]$. This implies

$$\mathrm{HP}_M(t) = \frac{Q_1(t) - tQ_2(t)}{(1-t)^r}$$

□

Theorem 2.3. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded K-algebra, and assume that A is generated, as a K-algebra, by $x_1, \ldots, x_r$ where $x_i \in A_{w_i}$. Then, for any finitely generated (positively) graded A-module $M = \bigoplus_{i\geq 0} M_i$,*

$$HP_M(t) = \frac{Q(t)}{(1-t^{w_1})(1-t^{w_2})\cdots(1-t^{w_n})}$$

for some $Q(t) \in \mathbb{Z}[t]$.

Proof. The proof is nearly identical to the proof of Theorem (2.2). □

3 Hilbert polynomial and the second Hilbert series

Let $A = \bigoplus_{\nu\geq 0} A_\nu$ be a Noetherian graded K-algebra, and let $M = \bigoplus_{\nu\geq 0} M_\nu$ be a finitely generated (positively) graded A-module. From Theorem (2.2), we know that $\mathrm{HP}_M(t) = Q(t)/(1-t)^r$, where $Q(t) \in \mathbb{Z}[t]$. After canceling all common factors in the numerator and denominator of $\mathrm{HP}_M(t)$, and we obtain

$$\mathrm{HP}_M(t) = \frac{G(t)}{(1-t)^s}, \qquad 0 \leq s \leq r, \qquad G(t) = \sum_{i=0}^{d} g_i t^i \in \mathbb{Z}[t],$$

such that $g_d \neq 0$ and $G(1) \neq 0$, that is, s is the pole order of $\mathrm{HP}_M(t)$ at $t = 1$.

1. The polynomial $Q(t)$, respectively $G(t)$, defined above, is called the **first Hilbert series**, respectively the **second Hilbert series**, of M.

2. Let d be the degree of the second Hilbert series $G(t)$, and let s be the pole order of the Hilbert-Poincare series $\mathrm{HP}_M(t)$ at $t = 1$, then

$$P_M := \sum_{i=0}^{d} g_i \cdot \binom{s-1+n-i}{s-1} \in \mathbb{Q}[n]$$

is called the **Hilbert polynomial** of M (with $\binom{n}{k} = 0$ for $k < 0$).

Lemma 3.1. *Let $P(x) \in \mathbb{Q}(x)$ be a polynomial of degree $s-1$. Then the following conditions are equivalent:*

1. *$P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.*
2. *There exists $a_0, \ldots, a_{s-1} \in \mathbb{Z}$ such that*
$$P(x) = \sum_{i=0}^{s-1} a_i \binom{x}{i}.$$

Proof. (2) implies (1) is trivial. For the converse, observe that the polynomials $\binom{x}{i}$, where $i \in \mathbb{N}$, form a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$. Therefore $P(x) = \sum_{i=0}^{s-1} a_i \binom{x}{i}$ with $a_i \in \mathbb{Q}$. Let $\Delta : \mathbb{Q}[x] \to \mathbb{Q}[x]$ denote the forward difference operator, given by $(\Delta f)(x) = f(x+1) - f(x)$. Then

$$a_k = (\Delta^k P)(0) = P(k) - P(0) \in \mathbb{Z}.$$

$\square$

Corollary. *With the above assumptions, P_M is a polynomial in n with rational coefficients, of degree $s-1$, and satisfies $P_M(n) = H_M(n)$ for $n \geq d$. Moreover, there exist $a_i \in \mathbb{Z}$ such that*

$$P_M(n) = \sum_{i=0}^{s-1} a_i \cdot \binom{n}{i} = \frac{a_{s-1}}{(s-1)!} \cdot n^{s-1} + \text{ lower terms in } n,$$

where $a_{s-1} = G(1) > 0$.

Proof. The equality $1/(1-t)^s = \sum_{i=0}^{\infty} \binom{s-1+i}{s-1} \cdot t^i$ implies

$$\sum_{i=0}^{\infty} H_M(i) t^i = \mathrm{HP}_M(t) = \left(\sum_{i=0}^{d} g_i t^i \right) \cdot \sum_{j=0}^{\infty} \binom{s-1+j}{s-1} \cdot t^j.$$

After expressing $P_M(n)$ in long form notation, we see that the leading term of P_M is $G(1) \cdot n^{s-1}/(s-1)!$. In particular, we obtain $\deg(P_M) = s-1$. Next, we have to prove that $P_M = \sum_{i=0}^{s-1} a_i \cdot \binom{n}{i}$ for suitable $a_i \in \mathbb{Z}$ and $a_{s-1} > 0$. Suppose that we can find such $a_i \in \mathbb{Z}$. Then

$$P_M(n) = \frac{a_{s-1}}{(s-1)!} \cdot n^{s-1} + \text{ lower terms in } n.$$

Now, $P_M(n) = H_M(n) > 0$ for n sufficiently large implies $a_{s-1} > 0$. Finally, the existence of suitable integer coefficients a_i is a consequence Lemma (3.1), since $P_M(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. $\square$

Example 3.1. Let's compute some explicit examples of hilbert polynomials. First assume $d = 2$. Then

$$H_M(t) = \left(g_0 + g_1 t + g_2 t^2\right) \left(\binom{s-1}{s-1} + \binom{s-1+1}{s-1} t + \binom{s-1+2}{s-1} t^2 + \binom{s-1+3}{s-1} t^3 + \cdots \right).$$

After expanding, we see that

$$\begin{aligned}
H_M(0) &= g_0 \binom{s-1}{s-1} \\
H_M(1) &= g_0 \binom{s-1+1}{s-1} + g_1 \binom{s-1}{s-1} \\
H_M(2) &= g_0 \binom{s-1+2}{s-1} + g_1 \binom{s-1+1}{s-1} + g_2 \binom{s-1}{s-1} = P_M(2) \\
H_M(3) &= g_0 \binom{s-1+3}{s-1} + g_1 \binom{s-1+2}{s-1} + g_2 \binom{s-1+1}{s-1} = P_M(3) \\
&\vdots
\end{aligned}$$

So we've defined a rational polynomial P_M in a way so that $P_M(n) = H_M(n)$ for $n \geq 2$. Now assume $s = 1$. Then

$$P_M(n) = g_0 \binom{n}{0} + g_1 \binom{n-1}{0} + g_2 \binom{n-2}{0} = g_0 + g_1 + g_2$$

Now assume $s = 2$. Then

$$P_M(n) = g_0 \binom{n+1}{1} + g_1 \binom{n}{1} + g_2 \binom{n-1}{1} = (g_0 + g_1 + g_2)n + (g_0 - g_2)$$

Now assume $s = 3$. Then

$$P_M(n) = g_0\binom{n+2}{2} + g_1\binom{n+1}{2} + g_2\binom{n}{2} == \frac{(g_0+g_1+g_2)n^2 + (3g_0+g_1-g_2)n + 2}{2}$$

Now assume $s = 4$. Then

$$P_M(n) = g_0\binom{n+3}{3} + g_1\binom{n+2}{3} + g_2\binom{n+1}{3} = \frac{(g_0+g_1+g_2)n^3 + (6g_0+3g_1)n^2 + (11g_0+2g_1-g_2)n + 6}{6}$$

3.1 Properties of the Hilbert Polynomial

In this section we prove that, for a graded K-algebra $A = K[x_1, \dots, x_r]/I$, we have $\dim(A) - 1$ is equal to the degree of the Hilbert polynomial P_A.

Definition 3.1. Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a Noetherian graded K-algebra, and let $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ be a finitely generated, (not necessarily positively) graded A-module. Then we introduce

$$M^{(0)} := \bigoplus_{\nu \geq 0} M_\nu,$$

and define the **Hilbert polynomial** of M to be the Hilbert polynomial of $M^{(0)}$, that is, $P_M := P_{M^{(0)}}$.

Example 3.2. Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a Noetherian graded K-algebra. Then

$$P_{A(d)}(n) = P_A(n+d) = P_A(n) + \text{terms of lower degree in } n.$$

Definition 3.2. Let A be a Noetherian graded K-algebra and M a finitely generated graded A-module, and let $P_M = \sum_{i=0}^{s-1} a_i \cdot \binom{n}{i}$ be the Hilbert polynomial of M. Then we set

$$d(M) := \deg(P_M) = s - 1,$$

and we define the **degree** of M as

$$\deg(M) := a_{s-1}.$$

Remark. If M is positively graded and $\mathrm{HP}_M(t) = G(t)/(1-t)^s$ with $G(1) \neq 0$, then $d(M) = s - 1$ and $\deg(M) = G(1)$.

Proposition 3.1. *Let A be a Noetherian graded K-algebra, and let M, N be finitely generated graded A-modules. Then*

1. *If there is a surjective graded morphism $\varphi : M \to N$, then $d(M) \geq d(N)$.*
2. *$d(M) \leq d(A)$.*
3. *If there is a homogeneous element $m \in M$ such that $Ann_A(m) = 0$, then $d(M) = d(A)$.*
4. *Let $x \in A_d$ be a homogeneous nonzerodivisor for M. Then*

$$d(M/xM) = d(M) - 1, \qquad deg(M/xM) = d \cdot deg(M).$$

Proof.

1. Let $\varphi : M \to N$ be a graded and surjective homomorphism of A-modules. Then, for all n, the restriction to M_n, denoted $\varphi_{|M_n} : M_n \to N_n$, is surjective too. This implies

$$H_M(n) = \dim_K(M_n) \geq \dim_K(N_n) = H_N(n).$$

   Hence $P_M(n) \geq P_N(n)$ for all n sufficiently large, which is only possible if $\deg(P_M) \geq \deg(P_N)$, since the leading coefficients are positive.

2. Since M is finitely generated, we may choose homogeneous generators $m_1, \dots, m_k$ of degree $d_1, \dots, d_k$. Now consider the map

$$\varphi : \bigoplus_{i=1}^{k} A(-d_i) \to M$$

   defined by $\varphi(a_1, \dots, a_k) = \sum_{i=1}^{k} a_i m_i$. Obviously, φ is graded and surjective. Using (1), we obtain

$$d\left(\bigoplus_{i=1}^{k} A(-d_i)\right) \geq d(M).$$

On the other hand, for n sufficiently large, we have

$$\begin{aligned} P_{\bigoplus_{i=1}^k A(-d_i)}(n) &= \sum_{i=1}^k P_{A(-d_i)}(n) \\ &= \sum_{i=1}^k P_A(n-d_i) \\ &= k \cdot P_A(n) + \text{terms of lower degree in } n, \end{aligned}$$

which implies

$$d\left(\bigoplus_{i=1}^k A(-d_i)\right) = d(A).$$

3. Let $m \in M_d$ such that $\mathrm{Ann}_A(m) = 0$. Then $\varphi : A(-d) \to M$ defined by $\varphi(a) := am$ is graded and injective. This implies that, for n sufficiently large, $P_A(n-d) = P_{A(-d)}(n) \leq P_M(n)$, which is only possible if $\deg(P_M) \geq \deg(P_A)$. Together with (2), this implies $d(M) = d(A)$.

4. Using the exact sequence

$$0 \longrightarrow M(-d) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

of graded A-modules, we obtain, by Lemma (2.1),

$$(1-t^d)\mathrm{HP}_M(t) = \mathrm{HP}_{M/xM}(t).$$

If $\mathrm{HP}_M(t) = G(t)/(1-t)^{d(M)+1}$ with $G(1) \neq 0$, then

$$\mathrm{HP}_{M/xM}(t) = \frac{G(t)(1-t^d)}{(1-t)^{d(M)}(1-t)} = \frac{G(t) \cdot \sum_{\nu=0}^{d-1} t^\nu}{(1-t)^{d(M)}}.$$

Then $\mathrm{HP}_{M/xM}$ has pole order $d(M)$ at $t=1$, hence,

$$d(M/xM) = d(M) - 1 \quad \text{and} \quad \deg(M/xM) = \left(G(t) \cdot \sum_{\nu=0}^{d-1} t^\nu\right)_{|t=1} = \deg(M) \cdot d.$$

□

Theorem 3.2. *Let $I \subset K[x_1, \ldots, x_r]$ be a homogeneous ideal. Then*

$$dim(K[x_1, \ldots, x_r]/I) = d(K[x_1, \ldots, x_r]/I) + 1.$$

Proof. Using Noether normalization, $K[x_1, \ldots, x_r]/I$ can be considered as a finitely generated graded $K[y_1, \ldots, y_s]$-module. The assumptions of Proposition (3.1) (3) are satisfied and, therefore,

$$\begin{aligned} \deg(P_{K[x_1,\ldots,x_r]/I}) &= \deg(P_{K[y_1,\ldots,y_s]}) \\ &= s - 1 \\ &= \dim(K[x_1, \ldots, x_r]/I) - 1. \end{aligned}$$

□

4 Examples

We now wish to give several examples which demonstrate concepts introduced above.

Example 4.1. Let A be the graded ring $K[x,y,z]$ with respect to the weights $w = (1,1,1)$. Then

$$\mathrm{HP}_A(t) = \frac{1}{(1-t)^3} \quad \text{and} \quad P_A(n) = \binom{2+n}{2} = \frac{n^2+3n+2}{2}.$$

Example 4.2. Let A be the graded ring $K[x,y,z]$ with respect to the weights $w = (1,2,3)$. Then

$$H_A(n) = \{(a,b,c) \in \mathbb{Z}_{\geq 0} \mid a + 2b + 3c = n\} \quad \text{and} \quad \mathrm{HP}_A(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)}.$$

Example 4.3. Let A be the graded ring $K[x,y,z]$ with respect to the weights $w=(1,2,3)$, B be the graded ring $K[x,y,z]$ with respect to the weights $w=(1,1,3)$, M be the graded A-module A^2 with respect to weights $k=(1,2)$, and N be the graded B-module B with weight $k_1=1$. Then

$$\begin{aligned}\mathrm{HP}_M(t) &= (t+t^2)\mathrm{HP}_A(t)\\ &= \frac{t+t^2}{(1-t)(1-t^2)(1-t^3)}\\ &= \frac{t}{(1-t)^2(1-t^3)}\\ &= \mathrm{HP}_N(t).\end{aligned}$$

Therefore M and N have the same graded structure. Moreover, one can check that the map from $M_n \to N_n$ given by $e_2 \mapsto ye_1$, $y \mapsto y^2$, and fixing everything else the same, is an isomorphism of K-vector spaces. We can interpret $H_M(n)$ as the number of elements $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ such that $a+2b+3c=n-1$ plus the number of elements $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ such that $a+2b+3c=n-2$. Similarly, we can interpret $H_N(n)$ as the number of elements $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$ such that $a+b+3c=n-1$.

Example 4.4. Let A be the graded ring $K[x,y,z]$ with respect to weights $w=(1,1,1)$ and let $I=\langle x^2,y^2,z^2\rangle$. Since I is homogeneous, A/I is a graded A-module. The homogenous components of A/I are

$$\begin{aligned}(A/I)_0 &= K\\ (A/I)_1 &= K\bar{x}+K\bar{y}+K\bar{z}\\ (A/I)_2 &= K\bar{x}\bar{y}+K\bar{x}\bar{z}+K\bar{y}\bar{z}\\ (A/I)_3 &= K\bar{x}\bar{y}\bar{z}.\end{aligned}$$

In particular, A/I is an 8-dimensional K-vector space.

Example 4.5. Let A be the graded ring $K[x,y,z]$ with respect to weights $w=(1,1,1)$, let $I=\langle x^3+y^3+z^3\rangle$, and let $J=\langle x^3+y^3+z^3,x\rangle$. Since I and J are homogeneous, A/I and A/J are graded A-modules. We have an exact sequence of graded A-modules:

$$0 \longrightarrow A(-3) \xrightarrow{\cdot(x^3+y^3+z^3)} A \longrightarrow A/I \longrightarrow 0$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}\mathrm{HP}_{A/I}(t) &= \frac{1-t^3}{(1-t)^3}\\ &= \frac{1+t+t^2}{(1-t)^2}\end{aligned}$$

From the reduced expression of $\mathrm{HP}_{A/I}(t)$, we see that

$$P_{A/I}(n) = \binom{n+1}{1}+\binom{n}{1}+\binom{n-1}{1} = 3n.$$

By tensoring the exact sequence above with $A(-1) \xrightarrow{\cdot x} A$, we obtain another exact sequence of graded A-modules

$$0 \longrightarrow A(-4) \xrightarrow{\begin{pmatrix}-x^3-y^3-z^3\\ x\end{pmatrix}} A(-1)\oplus A(-3) \xrightarrow{\begin{pmatrix}x & x^3+y^3+z^3\end{pmatrix}} A \longrightarrow A/J \longrightarrow 0$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}\mathrm{HP}_{A/J}(t) &= \frac{1-t-t^3+t^4}{(1-t)^3}\\ &= \frac{1+t+t^2}{1-t}.\end{aligned}$$

Notice that $(1-t)\mathrm{HP}_{A/J}(t) = \mathrm{HP}_{A/I}(t)$ because of the way tensoring complexes works. From the reduced expression of $\mathrm{HP}_{A/J}(t)$, we see that

$$P_{A/J}(n) = \binom{n}{0} + \binom{n-1}{0} + \binom{n-2}{0} = 3.$$

The graded parts of A/J starts out as

$$\begin{aligned}
&\vdots \\
(A/J)_0 &= K \\
(A/J)_1 &= K\bar{y} + K\bar{z} \\
(A/J)_2 &= K\bar{y}^2 + K\bar{y}\bar{z} + K\bar{z}^2 \\
(A/J)_3 &= K\bar{y}^2\bar{z} + K\bar{y}\bar{z}^2 + K\bar{z}^3 \\
(A/J)_4 &= K\bar{y}^2\bar{z}^2 + K\bar{y}\bar{z}^3 + K\bar{z}^4 \\
&\vdots
\end{aligned}$$

As we can see, the dimension of the graded parts eventually agrees with the hilbert polynomial, which is just 3. Also, note that we could have also calculated $P_{A/J}(n)$ by

$$P_{A/J}(n) = P_A(n) - P_A(n-3) - P_A(n-1) + P_A(n-4).$$

Now let $\mathrm{LT}(J)$ be the ideal generated by lead terms of elements in J with respect to graded lex ordering. From the theory of Gröbner bases, this is just $\mathrm{LT}(J) = \langle x, y^3 + z^3\rangle$. The graded parts of $A/\mathrm{LT}(J)$ starts out as

$$\begin{aligned}
&\vdots \\
(A/\mathrm{LT}(J))_0 &= K \\
(A/\mathrm{LT}(J))_1 &= K\bar{y} + K\bar{z} \\
(A/\mathrm{LT}(J))_2 &= K\bar{y}^2 + K\bar{y}\bar{z} + K\bar{z}^2 \\
(A/\mathrm{LT}(J))_3 &= K\bar{y}^2\bar{z} + K\bar{y}\bar{z}^2 + K\bar{z}^3 \\
(A/\mathrm{LT}(J))_4 &= K\bar{y}^2\bar{z}^2 + K\bar{y}\bar{z}^3 + K\bar{z}^4 \\
&\vdots
\end{aligned}$$

Notice that A/J and $A/\mathrm{LT}(J)$ have the same graded structure.

$$G = \{g_0, g_1, f_1, f_2, f_3\}$$

$$\begin{aligned}
g_0 &= xz + z^2 \\
g_1 &= xy + y^2 \\
g_2 &= y^2z - yz^2
\end{aligned}$$

$$g_0, g_1$$

$$xg_0, yg_0, zg_0, xg_1, yg_1, g_2$$
$$x^2g_0, xyg_0, y^2g_0, yzg_0, z^2g_0, x^2g_1, xyg_1, y^2g_1$$

Formally set

$$d(g_0) = d(g_1) = d(g_2) = 0$$

Now what about S/I? First observe that we have an isomorphism

$$S/S \cap I \to \mathrm{Span}_K\{\text{monomials } m \in S \mid m \notin \mathrm{LT}(I)\},$$

given by mapping $\overline{f}$ to $\overline{f}^G$, where $\overline{f}^G$ is division of $f \in I$ by G.

$$\begin{aligned}
yg_0 - zg_1 &= z^2y - y^2z \\
S(g_1, g_2) &= y^3z + xyz^2
\end{aligned}$$

Example 4.6. Let A be the graded ring $K[x,y]$ with respect to weights $w = (1,1)$ and let I be the ideal in A given by $I = \langle x^2, xy \rangle$. Since I is a homogeneous ideal of A, A/I is a graded A-module. A free resolution of A/I is given by

$$A(-3) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} A(-2)^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} A \longrightarrow A/I$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned} \mathrm{HP}_{A/I}(t) &= \frac{1-2t^2+t^3}{(1-t)^2} \\ &= \frac{1+t-t^2}{1-t}. \end{aligned}$$

From the reduced expression of $\mathrm{HP}_{A/I}(t)$, we see that

$$P_{A/I}(n) = \binom{n}{0} + \binom{n-1}{0} - \binom{n-2}{0} = 1.$$

Example 4.7. Let A be the graded ring $K[x,y,z]$ with respect to weights $w = (1,1,1)$ and let I be the ideal in A given by $I = \langle xz, yz \rangle$. Since I is a homogeneous ideal of A, A/I is a graded A-module. A free resolution of A/I is given by

$$A(-3) \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} A(-2)^2 \xrightarrow{\begin{pmatrix} xz & yz \end{pmatrix}} A \longrightarrow A/I$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned} \mathrm{HP}_{A/I}(t) &= \frac{1-2t^2+t^3}{(1-t)^3} \\ &= \frac{1+t-t^2}{(1-t)^2}. \end{aligned}$$

From the reduced expression of $\mathrm{HP}_{A/I}(t)$, we see that

$$P_{A/I}(n) = \binom{n+1}{1} + \binom{n}{1} - \binom{n-1}{1} = n+2.$$

The graded parts of A/I starts out as

$$\begin{aligned} &\vdots \\ (A/I)_0 &= K \\ (A/I)_1 &= K\bar{x} + K\bar{y} + K\bar{z} \\ (A/I)_2 &= K\bar{x}^2 + K\bar{x}\bar{y} + K\bar{y}^2 + K\bar{z}^2 \\ (A/I)_3 &= K\bar{x}^3 + K\bar{x}^2\bar{y} + K\bar{x}\bar{y}^2 + K\bar{y}^3 + K\bar{z}^3 \\ &\vdots \end{aligned}$$

Notice that

$$\begin{aligned} (A/I)_n &= (A/\langle xz, yz \rangle)_n \\ &= (A/(\langle z \rangle \cap \langle x,y \rangle))_n \\ &\cong ((A/\langle z \rangle) \oplus (A/\langle x,y \rangle))_n \\ &\cong (A/\langle z \rangle)_n \oplus (A/\langle x,y \rangle)_n \end{aligned}$$

Example 4.8. Let A be the graded ring $K[x,y,z]$ with weights $w = (1,2,3)$, I be the ideal in A given by $I = \langle x^3 + z \rangle$, and B be the graded ring $K[s,t]$ with respect to weights $w = (1,2)$. Since I is a homogeneous ideal of A, A/I is a graded A-module. We have an exact sequence of graded A-modules

$$0 \longrightarrow A(-3) \xrightarrow{\cdot(x^3+z)} A \longrightarrow A/I \longrightarrow 0$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}
\mathrm{HP}_{A/I}(t) &= \frac{1-t^3}{(1-t)(1-t^2)(1-t^3)} \\
&= \frac{1}{(1-t)(1-t^2)} \\
&= \mathrm{HP}_B(t).
\end{aligned}$$

Example 4.9. Let A be the graded ring $K[x,y,z]$ with weights $w=(1,2,3)$ and let $I=\langle x^3+z, y^3+z^2\rangle$. Since I is a homogeneous ideal of A, A/I is a graded A-module. We have an exact sequence of graded A-modules

$$0 \longrightarrow A(-9) \xrightarrow{\begin{pmatrix} -y^3-z^2 \\ x^3+z \end{pmatrix}} A(-3)\oplus A(-6) \xrightarrow{\begin{pmatrix} x^3+z & y^3+z^2 \end{pmatrix}} A \longrightarrow A/I \longrightarrow 0$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}
\mathrm{HP}_{A/I}(t) &= \frac{1-t^3-t^6+t^9}{(1-t)(1-t^2)(1-t^3)} \\
&= \frac{(1-t+t^2)(1+t+t^2)}{1-t}.
\end{aligned}$$

Example 4.10. (Twisted Cubic) Let A be the graded ring $K[x,y,z,w]$ with respect to weights $w=(1,1,1,1)$, B be the graded ring $K[s,t]$ with respect to weights $w=(1,1)$, I be the ideal in A given by $I=\langle xz-y^2, yw-z^2, xw-yz\rangle$, and M be the B-module B^3 with respect to weights $k=(0,1,1)$. Since I is a homogeneous ideal of A, A/I is a graded A-module. A free resolution of A/I is given by

$$A(-3)^2 \xrightarrow{\begin{pmatrix} w & z \\ y & x \\ -z & -y \end{pmatrix}} A(-2)^3 \xrightarrow{\begin{pmatrix} xz-y^2 & yw-z^2 & xw-yz \end{pmatrix}} A \longrightarrow A/I$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}
\mathrm{HP}_{A/I}(t) &= \frac{1-3t^2+2t^3}{(1-t)^4} \\
&= \frac{1+2t}{(1-t)^2} \\
&= \mathrm{HP}_M(t).
\end{aligned}$$

Let's write out the graded components of A/I and M side by side:

$$A/I = \langle 1\rangle \oplus \langle x,w,y,z\rangle \oplus \langle x^2,w^2,xw,xy,xz,yw,zw\rangle \oplus\cdots \qquad M=\langle e_0\rangle \oplus \langle se_0, te_0 e_1, e_2\rangle \oplus \langle s^2e_0, ste_0, t^2e_0, se_1, se_2, te_1, te_2\rangle \oplus\cdots$$

It's easy to see that we get an isomorphism of K-vector spaces $(A/I)_n \to M_n$ by mapping $x\mapsto se_0$, $w\mapsto te_0$, $y\mapsto e_1$, $z\mapsto e_2$, and treating e_0 as the identity. The idea is that the twisted cubic is really a one dimensional object, which is why the $K[x,y,z,w]$-module A/I and the $K[s,t]$-module M have the same graded structure. From the reduced expression of $\mathrm{HP}_{A/I}(t)$ and $\mathrm{HP}_M(t)$, we see that

$$P_{A/I}(n) = P_M(n) = \binom{n+1}{1} + 2\binom{n}{1} = 3n+1.$$

Example 4.11. Let A be the graded ring $K[x_1,\dots,x_r]$ with respect to weights $w=(1,1\dots,1)$ and let $f\in A$ be a homogeneous polynomial of degree d. Since $\langle f\rangle$ is a homogeneous ideal of A, A/f is a graded A-module. We have an exact sequence of graded A-modules

$$0 \longrightarrow A(-d) \xrightarrow{\cdot f} A \longrightarrow A/f \longrightarrow 0$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}
\mathrm{HP}_{A/f}(t) &= \frac{1-t^d}{(1-t)^r} \\
&= \frac{1+t+t^2+\cdots+t^{d-1}}{(1-t)^{r-1}}
\end{aligned}$$

From the reduced expression of $\mathrm{HP}_{A/I}(t)$, we see that

$$P_{A/f}(n) = \binom{r-2+n}{r-2} + \binom{r-2+n-1}{r-2} + \cdots + \binom{r-2+n-d-1}{r-2} = \frac{d}{(r-2)!}n^{r-2} + \text{ terms of lower degree.}$$

For example, if $r = 3$ and $d = 4$, we have

$$P_{A/f} = \binom{n+1}{1} + \binom{n}{1} + \binom{n-1}{1} + \binom{n-2}{1} = 4n - 2.$$

5 Filtrations and the Lemma of Artin-Rees

Throughout this section, let A be a Noetherian ring and $Q \subset A$ be an ideal.

Definition 5.1. A set $\{M_n\}_{n\geq 0}$ of submodules of an A-module M is called a Q-**filtration** of M if

1. $M = M_0 \supset M_1 \supset M_2 \supset \cdots$
2. $QM_n \subset M_{n+1}$ for all $n \geq 0$.

A Q-filtration $\{M_n\}_{n\geq 0}$ of M is called **stable** if $QM_n = M_{n+1}$ for all sufficiently large n.

Example 5.1. Let M be an A-module and $M_n := Q^n M$ for $n \geq 0$. Then $\{M_n\}_{n\geq 0}$ is a stable Q-filtration of M.

Lemma 5.1. *Let $\{M_n\}_{n\geq 0}$ and $\{N_n\}_{n\geq 0}$ be two stable Q-filtrations of M. Then there exists some non-negative integer n_0 such that $M_{n_0+n} \subset N_n$ and $N_{n+n_0} \subset M_n$ for all $n \geq 0$.*

Proof. Without loss of generality, assume $N_n := Q^n M$. Now $\{M_n\}_{n\geq 0}$ being stable implies that there exists some non-negative integer n_0 such that $M_{n_0+n} = Q^n M_{n_0}$ for all $n \geq 0$. Therefore

$$\begin{aligned} M_{n+n_0} &= Q^n M_{n_0} \\ &\subset Q^n M \\ &= N_n. \end{aligned}$$

Conversely, as $\{M_n\}_{n\geq 0}$ is a Q-filtration, we have $QM_n \subset M_{n+1}$ for all $n \geq 0$, which implies, in particular

$$\begin{aligned} N_{n+n_0} &\subset N_n \\ &= Q^n M \\ &= Q^n M_0 \\ &\subset M_n. \end{aligned}$$

On the other hand, $Q^n M_{n_0} \subset Q^n M = N_n$ implies $M_{n+n_0} \subset N$. ☐

Lemma 5.2. *Let $\varphi : N \to M$ be an A-linear map of A-modules, and let $\{M_n\}_{n\geq 0}$ be a Q-filtration of M. Then $\{\varphi^{-1}(M_n)\}_{n\geq 0}$ is a Q-filtration of N*

Proof. We have $Q\varphi^{-1}(M_n) \subset \varphi^{-1}(QM_n) \subset \varphi^{-1}(M_{n+1})$ for all $n \geq 0$. ☐

Definition 5.2. Let A be a ring, $Q \subset A$ an ideal, and M and A-module. The **blowup algebra of** Q **in** A is the A-algebra

$$B_Q(A) := A + tQ + t^2Q^2 + t^3Q^3 + \cdots.$$

The multiplication in $B_Q(A)$ is induced by the multiplication $Q^i \times Q^j \to Q^{i+j}$. We also define the **blowup module**

$$B_Q(M) := M + tQM + t^2Q^2M + t^3Q^3M + \cdots.$$

Remark. Note that $B_Q(A)/QB_Q(A) \cong \mathrm{Gr}_Q(A)$ and $B_Q(M)/QB_Q(M) \cong \mathrm{Gr}_Q(M)$.

Proposition 5.1. *Let A be a Noetherian ring and $Q \subset A$ an ideal. Then $B_Q(A)$ is a Noetherian ring.*

Proof. Since A is Noetherian, Q is finitely generated, say $Q = \langle f_1, \ldots, f_r \rangle$. Then the map $\varphi : A[x_1, \ldots, x_r] \to B_Q(A)$, induced by $\varphi(x_i) = tf_i$, is a surjective ring homomorphism from a Noetherian ring. Therefore $B_Q(A)$ is a Noetherian ring. ☐

Example 5.2. Let $A = K[x,y]/\langle y^2 - x^3 - x^2\rangle$ and $Q = \langle x, y\rangle$. Then the map $\varphi : K[x,y,u,v] \to B_Q(A)$, induced by $u \mapsto xt$, and $v \mapsto yt$, is a surjective ring homomorphism. The kernel of φ is an ideal which is homogeneous in the variables u, v:

$$\operatorname{Ker}(\varphi) = \langle v^2 - (x+1)u^2, xv - yu, y^2 - x^3 - x^2\rangle.$$

In particular, $B_Q(A)$ corresponds to an algebraic subset $Z \subset \mathbb{A}^2 \times \mathbb{P}^1$.

Example 5.3. Let $R = \mathbb{F}_2[x,y]/\langle y^2 + x^3 + x^2\rangle$ and $\mathfrak{m} = \langle \overline{x}, \overline{y}\rangle \subset R$. Then the map $\varphi : \mathbb{F}_2[x,y,u,v] \to B_{\mathfrak{m}}(R)$, induced by $u \mapsto \overline{x}t$, and $v \mapsto \overline{y}t$, is a surjective ring homomorphism. The kernel of φ is an ideal which is homogeneous in the variables u, v and is given by $\operatorname{Ker}(\varphi) = \langle f_1, f_2, f_3, f_4, f_5\rangle$, where

$$\begin{aligned}
f_1 &= yu^3 + u^2v + v^3 \\
f_2 &= xu^2 + u^2 + v^2 \\
f_3 &= x^2u + xu + yv \\
f_4 &= xv + yu \\
f_5 &= x^3 + x^2 + y^2
\end{aligned}$$

Therefore, $K[x,y,u,v]/\langle f_1, f_2, f_3, f_4, f_5\rangle \cong B_{\mathfrak{m}}(R)$.

$$\operatorname{Ker}(\varphi) = \langle yu^3 + u^2v + v^3, xu^2 + u^2 + v^2, x^2u + xu + yv, xv + yu, x^3 + x^2 + y^2\rangle.$$

In particular, $B_Q(A)$ corresponds to an algebraic subset $Z \subset \mathbb{A}^2 \times \mathbb{P}^1$.

Remark. Let $\varphi : K[y_1, \dots, y_m] \to K[x_1, \dots, x_n]/I$ be a K-algebra homomorphism, induced by mapping $y_i \mapsto \overline{f}_i$, and let J be the ideal in $K[y_1, \dots, y_m, x_1, \dots, x_n]$ given by

$$J = IK[y_1, \dots, y_m, x_1, \dots, x_n] + \langle f_1 - y_1, \dots, f_m - y_m\rangle.$$

Then $\operatorname{Ker}(\varphi) = J \cap K[y_1, \dots, y_m]$. In the example above, we can view φ as ring homomorphism from $K[x,y,u,v]$ to $K[x,y,t]/\langle y^2 - x^3 - x^2\rangle$.

Example 5.4. Let $A = K[x,y]/\langle x^2 + y^2 - z^2\rangle$ and $Q = \langle x, y, z\rangle$. Then the map $\varphi : K[x,y,z,u,v,w] \to B_Q(A)$, induced by $u \mapsto xt$, $v \mapsto yt$, and $w \mapsto zt$ is a surjective ring homomorphism. Since $B_Q(A) \subset A[t] = K[t,x,y,z]/\langle x^2 + y^2 - z^2\rangle$, we can view φ as a map from $K[x,y,z,u,v,w]$ to $K[t,x,y,z]/\langle x^2 + y^2 - z^2\rangle$. Then the kernel of φ can be computed by eliminating t from the ideal generated by $J = \langle u - xt, v - yt, w - zt, x^2 + y^2 - z^2\rangle \subset K[t,x,y,z,u,v,w]$. We obtain

$$\operatorname{Ker}(\varphi) = \langle u^2 + v^2 - w^2, yw - zv, xw - zu, xv - yu, xu + yv - zw, x^2 + y^2 - z^2\rangle.$$

Lemma 5.3. *(Artin-Rees) Let $\{M_n\}_{n\geq 0}$ be a stable Q-filtration of the finitely generated A-module M and $N \subset M$ a submodule, then $\{M_n \cap N\}_{n\geq 0}$ is a stable Q-filtration of N.*

To prove the lemma, we need a criterion for stability. Let M be a finitely generated A-module and $\{M_n\}_{n\geq 0}$ be a Q-filtration. Let

$$\begin{aligned}
B_Q(A) &:= A + tQ + t^2Q^2 + t^3Q^3 + \cdots \\
B_Q(M) &:= M + tQM + t^2Q^2M + t^3Q^3M + \cdots \\
\overline{M} &:= M + tM_1 + t^2M_2 + t^3M_3 + \cdots \\
\overline{M}_1 &:= M + tM_1 + t^2QM_1 + t^3Q^2M_1 + \cdots \\
\overline{M}_2 &:= M + tM_1 + t^2M_2 + t^3QM_2 + \cdots \\
\overline{M}_n &:= M + M_1t + \cdots + M_{n-1}t^{n-1} + B_Q(A)M_nt^n
\end{aligned}$$

Lemma 5.4. *(Criterion for stability). $\overline{M}$ is a finitely generated $B_Q(A)$-module if and only if $\{M_n\}_{n\geq 0}$ is Q-stable.*

Proof. Since A is Noetherian and M is finitely generated, it follows that the submodules M_n, $n \geq 0$, are finitely generated. Let

$$\begin{aligned}
\overline{M}_n &:= M + M_1t + \cdots + M_{n-1}t^{n-1} + B_Q(A)M_nt^n \\
&= M + M_1t + \cdots + M_{n-1}t^{n-1} + M_nt^n + QM_nt^{n+1} + Q^2M_nt^{n+2} + \cdots
\end{aligned}$$

then $\overline{M}_n$ is a finitely generated $B_Q(A)$-module, because $\bigoplus_{i=0}^n M_i$ is a finitely generated A-module. Moreover, $\overline{M}_n \subset \overline{M}_{n+1}$ for all $n \geq 0$ and $\bigcup_{n=0}^\infty \overline{M}_n = \overline{M}$.

By Proposition (5.1), $B_Q(A)$ is a Noetherian ring. This implies that $\overline{M}$ is a finitely generated $B_Q(A)$-module if and only if there exists a non-negative integer n_0 such that $\overline{M}_{n_0} = \overline{M}$. This is the case if and only if $M_{n_0+r} = Q^rM_{n_0}$ for all $r \geq 0$. $\square$

Corollary. *Let A be a Noetherian ring, $\mathfrak{p}$ be a prime ideal of A, and I be an ideal of A. For any map $\varphi : I \to A/\mathfrak{p}$, there exists a number d such that φ factors through $I/(\mathfrak{p}^d \cap I) \cong (\mathfrak{p}^d + I)/\mathfrak{p}^d$.*

Proof. By Artin-Rees, $\{I \cap \mathfrak{p}^n\}_{n\geq 0}$ is a stable $\mathfrak{p}$-filtration. Therefore $I \cap \mathfrak{p}^d = \mathfrak{p}\left(I \cap \mathfrak{p}^{d-1}\right)$ for some $d \geq 1$. This implies $I \cap \mathfrak{p}^d \subset \mathrm{Ker}(\varphi)$. □

Proposition 5.2. *Let A be a ring, Q an ideal in A, and let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence of A-modules. Then

$$0 \longrightarrow B_Q(M_1) \longrightarrow B_Q(M_2) \longrightarrow B_Q(M_3)$$

is exact.

Proof. □

6 The Hilbert-Samuel Function

In the previous section, we defined Hilbert functions and Hilbert polynomials for graded modules over a Noetherian graded K-algebra. It turns out that we can define something analagous for modules over local Noetherian rings. Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$. We assume (for simplicity) that $K = A/\mathfrak{m} \subset A$. Moreover, let Q be an $\mathfrak{m}$-primary ideal and M a finitely generated A-module.

Lemma 6.1. *Let $\{M_n\}_{n\geq 0}$ be a stable Q-filtration of M and let*

$$HS_{\{M_n\}_{n\geq 0}}(k) := dim_K(M/M_k).$$

Moreover, suppose that Q is generated by r elements. Then

1. *$HS_{\{M_n\}_{n\geq 0}}(k) < \infty$ for all $k \geq 0$;*
2. *There exists a polynomial $HSP_{\{M_n\}_{n\geq 0}}(t) \in \mathbb{Q}[t]$ of degree at most r such that $HS_{\{M_n\}_{n\geq 0}}(k) = HSP_{\{M_n\}_{n\geq 0}}(k)$ for all sufficiently large k;*
3. *The degree of $HSP_{\{M_n\}_{n\geq 0}}$ and its leading coefficient do not depend on the choice of a stable Q-filtration $\{M_n\}_{n\geq 0}$.*

Proof.

1. Recall that $\mathrm{Gr}_Q(A) = \bigoplus_{i\geq 0} Q^i/Q^{i+1}$ is a graded (A/Q)-algebra which is generated by r elements of degree 1. Now, let
$$\mathrm{Gr}_{\{M_n\}}(M) := \overline{M}/Q\overline{M} = \bigoplus_{i\geq 0} M_i/M_{i+1}.$$
Since $\{M_n\}_{n\geq 0}$ is a stable Q-filtration, $\overline{M}$ is a finitely generated $B_Q(A)$-module. Thus, $\mathrm{Gr}_{\{M_n\}}(M)$ is a finitely generated $\mathrm{Gr}_Q(A)$-module. Now as $QM_i \subset M_{i+1}$, the quotients M_i/M_{i+1}, $i \geq 0$, are annihilated by Q and, therefore, are finitely generated (A/Q)-modules, but A/Q is a finite dimensional K-vector space since Q is $\mathfrak{m}$-primary. Hence $\dim_K(M_i/M_{i+1}) < \infty$, and therefore
$$\dim_K(M/M_n) = \sum_{i=1}^{n} \dim_K(M_{i-1}/M_i) < \infty.$$
2. Note that $H_{\mathrm{Gr}_{\{M_n\}}(M)}(k) = \dim_K(M_k/M_{k+1})$. For sufficiently large k, $H_{\mathrm{Gr}_{\{M_n\}}(M)}(k) = P_{\mathrm{Gr}_{\{M_n\}}(M)}(k)$, and $P_{\mathrm{Gr}_{\{M_n\}}(M)}$ is a polynomial of degree at most $r-1$. Let
$$P_{\mathrm{Gr}_{\{M_n\}}(M)}(k) = \sum_{i=0}^{r-1} a_i \binom{k}{i},$$
then we have
$$\begin{aligned} \mathrm{HS}_{\{M_n\}_{n\geq 0}}(k+1) - \mathrm{HS}_{\{M_n\}_{n\geq 0}}(k) &= \dim_K(M/M_{k+1}) - \dim_K(M/M_k) \\ &= \dim_K(M_k/M_{k+1}) \\ &= H_{\mathrm{Gr}_{\{M_n\}}(M)}(k) \\ &= P_{\mathrm{Gr}_{\{M_n\}}(M)}(k), \end{aligned}$$

for sufficiently large k. On the other hand

$$\sum_{i=1}^{r} a_{i-1}\binom{k+1}{i} - \sum_{i=1}^{r} a_{i-1}\binom{k}{i} = \sum_{i=0}^{r-1} a_i \binom{k}{i} = \mathrm{HS}_{\{M_n\}_{n\geq 0}}(k+1) - \mathrm{HS}_{\{M_n\}_{n\geq 0}}(k).$$

Hence $\mathrm{HS}_{\{M_n\}_{n\geq 0}}(k) - \sum_{i=1}^{r} a_{i-1}\binom{k}{i}$ is constant if k is sufficiently large. Let C be this constant and set $\mathrm{HSP}_{\{M_n\}_{n\geq 0}}(k) := \sum_{i=1}^{r} a_{i-1}\binom{k}{i} + C$. Then $\mathrm{HS}_{\{M_n\}_{n\geq 0}}(k) = \mathrm{HSP}_{\{M_n\}_{n\geq 0}}(k)$, a polynomial of degree at most r, for sufficiently large k.

3. Let $\{M'_n\}_{n\geq 0}$ be another stable Q-filtration of M, and choose k_0 such that $M_{k+k_0} \subset M'_k$ and $M'_{k+k_0} \subset M_k$ for all $k \geq 0$. This implies the inequalities $\mathrm{HS}_{\{M_n\}_{n\geq 0}}(k) \leq \mathrm{HS}_{\{M'_n\}_{n\geq 0}}(k+k_0)$ and $\mathrm{HS}_{\{M'_n\}_{n\geq 0}}(k) \leq \mathrm{HS}_{\{M_n\}_{n\geq 0}}(k+k_0)$ and, therefore

$$1 = \lim_{k\to\infty} \frac{\mathrm{HS}_{\{M_n\}}(k)}{\mathrm{HS}_{\{M'_n\}}(k)} = \lim_{k\to\infty} \frac{\mathrm{HSP}_{\{M_n\}_{n\geq 0}}(k)}{\mathrm{HSP}_{\{M'_n\}_{n\geq 0}}(k)}.$$

$\square$

Definition 6.1. With the notation of Lemma (6.1) we define:

1. $\mathrm{HS}_{M,Q} := \mathrm{HS}_{\{Q^n M\}_{n\geq 0}} = \dim_K(M/Q^n M)$ is called the **Hilbert-Samuel function** of M with respect to Q.
2. $\mathrm{HSP}_{M,Q} := \mathrm{HSP}_{\{Q^n M\}_{n\geq 0}}$ is called the **Hilbert-Samuel polynomial** of M with respect to Q;
3. Let $\mathrm{HSP}_{M,Q}(k) = \sum_{\nu=0}^{d} a_\nu k^\nu$ with $a_d \neq 0$. Then $\mathrm{mult}(M,Q) := d! a_d$ is called the **Hilbert-Samuel multiplicity** of M with respect to Q.
4. $\mathrm{mult}(M) := \mathrm{mult}(M,\mathfrak{m})$ is called the **Hilbert-Samuel multiplicity** of M.

Proposition 6.1. *Let*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

be an exact sequence of finitely generated A-modules, and Q an $\mathfrak{m}$-primary ideal. Then

$$HSP_{M,Q} = HSP_{M/N,Q} + HSP_{N,Q} - R,$$

where R is a polynomial of degree strictly smaller than that of $HSP_{N,Q}$.

Proof. The exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow N/(Q^n M \cap N) \longrightarrow M/Q^n M \longrightarrow (M/N)/Q^n(M/N) \longrightarrow 0.$$

Therefore,

$$\mathrm{HSP}_{M,Q} = \mathrm{HSP}_{M/N,Q} + \mathrm{HSP}_{\{Q^n M \cap N\}}.$$

The proof of Lemma (6.1) shows that, indeed,

$$\mathrm{HSP}_{\{Q^n M\cap N\}} = \mathrm{HSP}_{N,Q} - R,$$

where R is a polynomial of degree strictly smaller than that of $\mathrm{HSP}_{N,Q}$. $\square$

In the proof of (Lemma (6.1), we actually proved more than just the claim. We can summarize the additional results as a comparison between the Hilbert-Samuel polynomial of M with respect to Q and the Hilbert polynomial of the graded $\mathrm{Gr}_Q(A)$-module $\mathrm{Gr}_Q(M)$.

Corollary. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, $Q \subset A$ be an $\mathfrak{m}$-primary ideal, and M a finitely generated A-module. Then*

1. $HSP_{M,Q}(k+1) - HSP_{M,Q}(k) = P_{Gr_Q(M)}(k)$.

2. *If* $P_{Gr_Q(M)}(k) = \sum_{\nu=0}^{s-1} a_\nu \binom{k}{\nu}$, *then*
$$HSP_{M,Q}(k) = \sum_{\nu=1}^{s} a_{\nu-1} \binom{k}{\nu} + c$$
with $c = dim_K(M/Q^\ell M) - \sum_{\nu=1}^{s} a_{\nu-1}\binom{\ell}{\nu}$ *for any sufficiently large* ℓ. *In particular, we obtain* $mult(M,Q) = deg(Gr_Q(M))$ *and*
$$deg(HSP_{M,Q}) = deg(P_{Gr_Q(M)}) + 1.$$

Example 6.1. Let $A = K[x,y,z]_{\langle x,y,z\rangle}/\langle x^2+y^3+z^4, xy+xz+z^3\rangle$. A standard basis for $\langle x^2+y^3+z^4, xy+xz+z^3\rangle$ with respect to ds order is given by

$$\begin{aligned} f_1 &= x^2+y^3+z^4 \\ f_2 &= xy+xz+z^3 \\ f_3 &= y^4+y^3z-xz^3+yz^4+z^5 \end{aligned}$$

Therefore $\mathrm{Gr}_{\mathfrak{m}}(A) \cong K[x,y,z]/\langle x^2, xy+xz, y^4+y^3z-xz^3\rangle$. A free resolution $K[x,y,z]$ of $\mathrm{Gr}_{\mathfrak{m}}(A)$ is given by

$$A(-3)\oplus A(-5) \xrightarrow{\begin{pmatrix} x & y^3 \\ -y-z & -z^3 \\ 0 & -x \end{pmatrix}} A(-2)\oplus A(-2)\oplus A(-4) \xrightarrow{\begin{pmatrix} xy+xz & x^2 & y^4+y^3z-xz^3 \end{pmatrix}} A \longrightarrow A/I$$

$$\mathrm{Gr}_{\mathfrak{m}}(A) \cong K[x,y,z]/\langle x^2, xy+xz, y^4+y^3z-xz^3\rangle$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned} \mathrm{HP}_{\mathrm{Gr}_{\mathfrak{m}}(A)}(t) &= \frac{1-(t^2+t^2+t^4)+(t^3+t^5)}{(1-t)^3} \\ &= \frac{1+2t+t^2+t^3}{1-t}. \end{aligned}$$

In particular, $\deg(\mathrm{Gr}_{\mathfrak{m}}(A)) = 5$ and $\deg(P_{\mathrm{Gr}_{\mathfrak{m}}(A)}) = 0$. Therefore $\mathrm{mult}(A,\mathfrak{m}) = 5$ and $\deg(\mathrm{HSP}_{M,Q}) = 1$. Finally, we list the first few graded pieces of $\mathrm{Gr}_{\mathfrak{m}}(A)$:

$$\begin{aligned} A/\mathfrak{m} &= K \\ \mathfrak{m}/\mathfrak{m}^2 &= Kx+Ky+Kz \\ \mathfrak{m}^2/\mathfrak{m}^3 &= Kxz+Ky^2+Kyz+Kz^2 \\ \mathfrak{m}^3/\mathfrak{m}^4 &= Kxz^2+Ky^3+Ky^2z+Kyz^2+Kz^3 \\ \mathfrak{m}^4/\mathfrak{m}^5 &= Kxz^3+Ky^3z+Ky^2z^2+Kyz^3+Kz^4 \\ \mathfrak{m}^5/\mathfrak{m}^6 &= Kxz^4+Ky^3z^2+Ky^2z^3+Kyz^4+Kz^5 \\ &\vdots \end{aligned}$$

7 Characterization of the Dimension of Local Rings

Proposition 7.1. *Let A be a Noetherian local ring and M a finitely generated A-module such that $Ann_A(M) = 0$. Then*
$$deg(HSP_{M,\mathfrak{m}}) = deg(HSP_{A,\mathfrak{m}}).$$

Proof. □

Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and assume $k = A/\mathfrak{m} \subset A$. We shall prove that the dimension of a local ring is equal to the degree of the Hilbert-Samuel polynomial and equal to the least number of generators of an $\mathfrak{m}$-primary ideal.

Definition 7.1. We introduce the following non-negative integers:

- $\delta(A) :=$ the minimal number of generators of an $\mathfrak{m}$-primary ideal of A,
- $d(A) := \deg(\mathrm{HSP}_{A,\mathfrak{m}})$,
- $\mathrm{edim}(A) :=$ the **embedding dimension** of A, defined as the minimal number of generators for $\mathfrak{m}$. Hence, $\mathrm{edim}(A) = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$, by Nakayama's Lemma.

Theorem 7.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then, with the above notation, $\delta(A) = d(A) = dim(A)$.*

We first prove the following proposition:

Proposition 7.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated A-module, and let Q be an $\mathfrak{m}$-primary ideal. Then*

1. *$deg(HSP_{M,Q}) = deg(HSP_{M,\mathfrak{m}})$*
2. *Moreover, if $x \in A$ is a nonzerodivisor for M, then $deg(HSP_{M/xM,Q}) \leq deg(HSP_{M,Q}) - 1$.*

Proof.

1. Suppose $\mathfrak{m} = \langle x_1, \ldots, x_r \rangle$. Choose s such that $\mathfrak{m} \supset Q \supset \mathfrak{m}^s$. Then $\mathfrak{m}^k \supset Q^k \supset \mathfrak{m}^{sk}$ for all k implies $\mathrm{HSP}_{M,\mathfrak{m}}(k) \leq \mathrm{HSP}_{M,Q}(k) \leq \mathrm{HSP}_{M,\mathfrak{m}}(sk)$ for sufficiently large k. But this is only possible if $\deg(\mathrm{HSP}_{M,Q}) = \deg(\mathrm{HSP}_{M,\mathfrak{m}})$.
2. Apply Proposition (6.1) to the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0$$

and conclude that $\deg(\mathrm{HSP}_{M/xM,Q}) \leq \deg(\mathrm{HSP}_{M,Q}) - 1$.

□

Definition 7.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $d = \dim(A)$, $\{x_1, \ldots, x_d\}$ is called a **system of parameters** of A, if $\langle x_1, \ldots, x_d \rangle$ is $\mathfrak{m}$-primary. If moreover, $\langle x_1, \ldots, x_d \rangle = \mathfrak{m}$, then it is called a **regular system of parameters**.

Example 7.1. Let $>$ be a local degree ordering, $A = K[x_1, \ldots, x_r]$, and let $I \subset \langle x \rangle = \langle x_1, \ldots, x_r \rangle \subset K[x]$. Then since

$$\begin{aligned}(A/I)/\mathfrak{m}^k(A/I) &\cong (A/I)/(\mathfrak{m}^k/I \cap \mathfrak{m}^k)\\ &\cong (A/I)/((I + \mathfrak{m}^k)/I)\\ &\cong A/(I + \mathfrak{m}^k)\\ &= A/(I + \langle x \rangle^k),\end{aligned}$$

we see that $\mathrm{HS}_{A/I,\mathfrak{m}}(k) = \dim_K(A/(I + \langle x \rangle^k))$.

Proposition 7.3. *Let $>$ be a local degree ordering on $K[x] = K[x_1, \ldots, x_r]$, and let $I \subset \langle x \rangle = \langle x_1, \ldots, x_r \rangle \subset K[x]$ be an ideal. Then*

$$HS_{K[x]_{\langle x \rangle}/I,\langle x \rangle} = HS_{K[x]_{\langle x \rangle}/L(I),\langle x \rangle}.$$

Proof. We have to prove that

$$\dim_K K[x]_{\langle x \rangle}/(I + \langle x \rangle^k)_{\langle x \rangle} = \dim_K K[x]_{\langle x \rangle}/(L(I) + \langle x \rangle^k)_{\langle x \rangle}.$$

Clearly, for each $k \geq 0$, the set $S := \{x^\alpha \notin L(I) \mid \deg(x^\alpha) < k\}$ represents a K-basis of $K[x]_{\langle x \rangle}/(L(I) + \langle x \rangle^k)_{\langle x \rangle} \cong \big(K[x]/(L(I) + \langle x \rangle^k)\big)_{\langle x \rangle} \cong K[x]/(L(I) + \langle x \rangle^k)$. On the other hand, using reduction by a standard basis of I, we can write each $f \in K[x]$ as

$$f = g + \sum_{x^\alpha \in S} c_\alpha x^\alpha \bmod \langle x \rangle^k$$

for some $g \in I$ and uniquely determined $c_\alpha \in K$. This is possible without multiplying f by a unit, because we are working modulo $\langle x \rangle^k$. Therefore, S also represents a K-basis of $K[x]/(I + \langle x \rangle^k) \cong \big(K[x]/(L(I) + \langle x \rangle^k)\big)_{\langle x \rangle} \cong K[x]_{\langle x \rangle}/(L(I) + \langle x \rangle^k)_{\langle x \rangle}$. □

Example 7.2. Let $A = K[x,y,z]_{\langle x,y,z\rangle}$, $\mathfrak{m} = \langle x,y,z\rangle$, $I = \langle y^2\rangle$, and let $J = \langle y^2, xyz + y^4\rangle$. The table below gives the first few values various Hilbert-Samuel functions:

k	$\mathrm{HS}_{A,\mathfrak{m}}(k)$	$\mathrm{HS}_{A/I,\mathfrak{m}}(k)$	$\mathrm{HS}_{A/J,\mathfrak{m}}(k)$
1	1	1	1
2	4	4	4
3	10	$10-1=9$	$10-1=9$
4	20	$20-3=17$	$20-3-1=16$
5	35	$35-6=29$	$35-6-3+1=27$
6	56	$56-10=46$	$56-10-6+3=43$

Example 7.3. Let $A = K[x,y,z]_{\langle x,y,z\rangle}/\langle z^2 - z, yz - y\rangle$. The table below gives the first few values various Hilbert-Samuel functions:

k	$\mathrm{HS}_{A,\mathfrak{m}}(k)$	$\mathrm{HS}_{A/I,\mathfrak{m}}(k)$	$\mathrm{HS}_{A/J,\mathfrak{m}}(k)$
1	1	1	1
2	4	4	4
3	10	$10-1=9$	$10-1=9$
4	20	$20-3=17$	$20-3-1=16$
5	35	$35-6=29$	$35-6-3+1=27$
6	56	$56-10=46$	$56-10-6+3=43$

Example 7.4. Let $A = K[x,y]_{\langle x,y\rangle}/\langle y^2 - x^3 - x^2\rangle$ and $\mathfrak{m} = \langle x,y\rangle$. Then

$$\begin{aligned}
\mathfrak{m} &= \langle x,y\rangle \\
\mathfrak{m}^2 &= \langle x^2, xy, y^2\rangle \\
\mathfrak{m}^3 &= \langle x^2 - y^2, xy^2, y^3\rangle \\
\mathfrak{m}^4 &= \langle x^2 - y^2 + x^3, xy^3, y^4\rangle \\
&\vdots
\end{aligned}$$

The ring homomorphism $\varphi : K[s,t] \to \mathrm{Gr}_{\mathfrak{m}}(A)$ given by $s \mapsto x$ and $t \mapsto y$ has kernel $\langle s^2 - t^2\rangle$. Therefore we have an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(A) \cong K[s,t]/\langle s^2 - t^2\rangle$.

Example 7.5. Let A be the graded ring $K[x,y,z]$ with respect to weights $w = (1,1,1)$ and let I be the ideal in A given by $I = \langle x^2, xy + xz, y^4 + y^3z - xz^3\rangle$. Since I is a homogeneous ideal of A, A/I is a graded A-module. A free resolution of A/I is given by

$$A(-3)\oplus A(-5) \xrightarrow{\begin{pmatrix} x & y^3 \\ -y-z & -z^3 \\ 0 & -x \end{pmatrix}} A(-2)\oplus A(-2)\oplus A(-4) \xrightarrow{\begin{pmatrix} xy+xz & x^2 & y^4+y^3z-xz^3 \end{pmatrix}} A \longrightarrow A/I$$

Therefore by Proposition (2.1), we conclude that

$$\begin{aligned}
\mathrm{HP}_{A/I}(t) &= \frac{1 - \left(t^2 + t^2 + t^4\right) + \left(t^3 + t^5\right)}{(1-t)^3} \\
&= \frac{1 + 2t + t^2 + t^3}{1-t}.
\end{aligned}$$

The graded parts of A/I starts out as

$$\begin{aligned}
&\vdots \\
(A/I)_0 &= K \\
(A/I)_1 &= K\bar{x} + K\bar{y} + K\bar{z} \\
(A/I)_2 &= K\bar{x}^2 + K\bar{x}\bar{y} + K\bar{y}^2 + K\bar{z}^2 \\
(A/I)_3 &= K\bar{x}^3 + K\bar{x}^2\bar{y} + K\bar{x}\bar{y}^2 + K\bar{y}^3 + K\bar{z}^3 \\
&\vdots
\end{aligned}$$

Notice that

$$\begin{aligned}
(A/I)_n &= (A/\langle xz, yz\rangle)_n \\
&= (A/(\langle z\rangle \cap \langle x,y\rangle))_n \\
&\cong ((A/\langle z\rangle) \oplus (A/\langle x,y\rangle))_n \\
&\cong (A/\langle z\rangle)_n \oplus (A/\langle x,y\rangle)_n
\end{aligned}$$