Matrix Analysis Exam 1

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Problem 1

Problem 1.1

We compute the reduced row Echelon form (which we denote by A') of the matrix A. Then we use this informatoin to give the rank of A. Write

$$
\begin{aligned}
\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 & -1 \end{pmatrix} = e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A \\
&\to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -2 \end{pmatrix} = e_{43}^{-1}e_{12}e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A \\
&\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -2 \end{pmatrix} = e_{14}e_{43}^{-1}e_{12}e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A \\
&\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} = s_{34}s_{24}e_{14}e_{43}^{-1}e_{12}e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A \\
&\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} = e_{34}^{-1}s_{34}s_{24}e_{14}e_{43}^{-1}e_{12}e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A \\
&\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} = d_4(-1)d_3(-1)d_2(-1)e_{34}^{-1}s_{34}s_{24}e_{14}e_{43}^{-1}e_{12}e_{41}^{-1}e_{31}^{-1}e_{21}^{-1}A := A'
\end{aligned}
$$

Now we have

$$
\begin{aligned}
\mathrm{Rank}(A) &= \mathrm{Rank}(A') \\
&= \dim(\mathrm{Im}(A')) \\
&= 4.
\end{aligned}
$$

Problem 1.2

Since $\dim(\mathrm{Im}(A)) = 4$ and the domain space of A has dimension 6, it follows that $\dim(\mathrm{Ker}(A)) = 2$. To find a basis for $\mathrm{Ker}(A')$, let $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^\top$. Solving $A'\mathbf{x} = 0$ gives us the system of equations

$$
\begin{aligned}
x_1 &= x_6 \\
x_2 &= -2x_6 \\
x_3 &= -x_4 - 2x_6 \\
x_4 &= x_4 \\
x_5 &= x_6 \\
x_6 &= x_6
\end{aligned}
$$

Therefore a basis for $\mathrm{Ker}(A) = \mathrm{Ker}(A')$ is given by

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Problem 1.3

The matrix A (viewed as a linear map) is not injective since $\dim(\mathrm{Ker}(A)) > 0$. On the other hand, A is onto since the dimension of the target space is 4 and since $\dim(\mathrm{Im}(A)) = 4$.

Problem 2

Write

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{u} = (u_1, \ldots, u_n)^\top, \quad \text{and} \quad \mathbf{v} = (v_1, \ldots, v_n)^\top.$$

Thus $\mathbf{v}^\top = \mathbf{u}^\top M$[1] implies

$$v_i = \sum_{j=1}^{n} a_{ji} u_j \tag{1}$$

for all $1 \le i \le n$.

Problem 2.1

Since $\det(M) \neq 0$, there exists an $\mathbf{a} = (a_1, \ldots, a_n)^\top \in K^n \backslash \{\mathbf{0}\}$ such that $M\mathbf{a} = \mathbf{0}$. Choose such an $\mathbf{a} \in K^n \backslash \{\mathbf{0}\}$. Then

$$\begin{aligned} a_1 v_1 + \cdots + a_n v_n &= \mathbf{v}^\top \mathbf{a} \\ &= (\mathbf{u}^\top M)\mathbf{a} \\ &= \mathbf{u}^\top (M\mathbf{a}) \\ &= \mathbf{u}^\top \mathbf{0} \\ &= 0 \end{aligned}$$

implies $\mathbf{v}^\top$ is linearly dependent since $\mathbf{a} \neq \mathbf{0}$.

Problem 2.2

Suppose $\mathbf{u}^\top$ is linearly independent. Let $W = \mathrm{Span}_K(\mathbf{u}^\top)$ (so $\mathbf{u}^\top$ is an ordered basis for W). Let $T \colon W \to W$ be the unique linear map such that

$$T(u_i) = \sum_{j=1}^{n} a_{ji} u_j.$$

for all $1 \le i \le n$. Thus the matrix representation of T with respect to the ordered basis $\mathbf{u}^\top$ is given by

$$[T]_{\mathbf{u}^\top}^{\mathbf{u}^\top} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = M$$

Since $\det(M) \neq 0$, we see that T is injective. In particular, T maps a linearly independent set to a linearly independent set. Thus since $\mathbf{u}^\top$ is linearly independent and since $T(u_i) = v_i$ for all $1 \le i \le n$ (by (1)), we see that $\mathbf{v}^\top$ is linearly independent.

Now since $\det(M) \neq 0$, the inverse of M exists, and moreover we have $\mathbf{u}^\top = \mathbf{v}^\top M^{-1}$. Thus if we assume that $\mathbf{v}^\top$ is linearly independent, then we can show that $\mathbf{u}^\top$ is linearly independent by swapping M with M^{-1} in the argument above.

[1] We use bold letters to denote column vectors.

Problem 3

$(\implies)$ Suppose f is surjective. Choose a basis $\{w_\lambda\}_{\lambda\in\Lambda}$ of W. Since f is surjective, there exists $v_\lambda \in V$ such that

$$f(w_\lambda) = v_\lambda$$

for all $\lambda \in \Lambda$. Choose such v_λ for all $\lambda \in \Lambda$. Let $g\colon W \to V$ be the unique linear map such that

$$g(w_\lambda) = v_\lambda$$

for all $\lambda \in \Lambda$. Then

$$\begin{aligned}(f \circ g)(w_\lambda) &= f(g(w_\lambda)) \\ &= f(v_\lambda) \\ &= w_\lambda \\ &= 1_W(w_\lambda)\end{aligned}$$

for all $\lambda \in \Lambda$. Since every linear map is *uniquely* determined by where it maps the basis elements, it follows that $f \circ g = 1_W$.

$(\impliedby)$ Let $g\colon W \to V$ be such a linear map and let $w \in W$. Then we have

$$\begin{aligned}f(g(w)) &= (f \circ g)(w) \\ &= 1_W(w) \\ &= w.\end{aligned}$$

This imlpies f is surjective[2] (as w is arbitrary).

Problem 4

Problem 4.1

Let $A \in K^{n\times n}$. Then

$$\begin{aligned}(\sigma^2 - 2\sigma)(A) &= \sigma^2(A) - 2\sigma(A) \\ &= \sigma(\sigma(A)) - 2\sigma(A) \\ &= \sigma(A - A^t) - 2(A - A^t) \\ &= (A - A^t) - (A - A^t)^t - 2(A - A^t) \\ &= A - A^t - A^t - (A^t)^t - 2A - 2A^t \\ &= A - A^t - A^t - A - 2A - 2A^t \\ &= 0.\end{aligned}$$

Problem 4.2

The Kernel is given by

$$\operatorname{Ker}(\sigma) = \{A \in K^{n\times n} \mid A = A^t\}.$$

A basis for $\operatorname{Ker}(\sigma)$ is given by

$$\beta = \{E_{ij} + E_{ji} \mid 1 \le i < j \le n\} \cup \{E_{ii} \mid 1 \le i \le n\}.$$

Indeed, first note that each matrix in β belongs to $\operatorname{Ker}(\sigma)$ since each matrix in β is symmetric. Also, if $A = (a_{ij}) \in \operatorname{Ker}(\sigma)$, then $a_{ij} = a_{ji}$ for all $i \ne j$, and this implies

$$\begin{aligned}A &= \sum_{1\le i,j\le n} a_{ij}E_{ij} \\ &= \sum_{1\le i\le n} a_{ii}E_{ij} + \sum_{1\le i<j\le n} a_{ij}(E_{ij} + E_{ji}) \in \operatorname{Span}(\beta).\end{aligned}$$

[2] When we write "let $w \in W$", then we are introducing w as an arbitrary element in W. Thus our proof that $f(g(w)) = w$ it is understood to apply *for all* $w \in W$.

In particular, this implies $\operatorname{Span}(\beta) = \operatorname{Ker}(\sigma)$. Finally note that β is linearly independent since

$$\begin{aligned} 0 &= \sum_{1 \le i \le n} a_{ii} E_{ij} + \sum_{1 \le i < j \le n} a_{ij}(E_{ij} + E_{ji}) \\ &= \sum_{1 \le i \le n} a_{ii} E_{ij} + \sum_{1 \le i < j \le n} a_{ij} E_{ij} + \sum_{1 \le i < j \le n} a_{ij} E_{ji} \end{aligned}$$

implies $a_{ii} = 0$ and $a_{ij} = 0$ for all $1 \le i, j \le n$ with $i \ne j$ (by linear independence of $\{E_{ij} \mid 1 \le i, j \le n\}$).

$$0 = \sum_{1 \le i \le n} a_{ii} E_{ij} + \sum_{1 \le i < j \le n} a_{ij}(E_{ij} + E_{ji})$$

From this, we conclude that

$$\begin{aligned} \dim(\operatorname{Ker}(\sigma)) &= \#\beta \\ &= n + \frac{n(n-1)}{2}. \end{aligned}$$

Problem 4.3

Let $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. We calculate

$$\begin{aligned} \sigma(E_{11}) &= 0 \\ \sigma(E_{12}) &= E_{12} - E_{21} \\ \sigma(E_{21}) &= -E_{12} + E_{21} \\ \sigma(E_{22}) &= 0. \end{aligned}$$

Thus the matrix represenation of σ with respect to the order basis γ is given by

$$[\sigma]_\gamma^\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the characteristic polynomial of σ is given by

$$\begin{aligned} \chi_\sigma(t) &= \det([\sigma]_\gamma^\gamma - tI) \\ &= \det \begin{pmatrix} -t & 0 & 0 & 0 \\ 0 & 1-t & -1 & 0 \\ 0 & -1 & 1-t & 0 \\ 0 & 0 & 0 & -t \end{pmatrix} \\ &= -t \det \begin{pmatrix} 1-t & -1 & 0 \\ -1 & 1-t & 0 \\ 0 & 0 & -t \end{pmatrix} \\ &= t^2 \det \begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix} \\ &= t^2((1-t)^2 - 1) \\ &= t^3(t-2). \end{aligned}$$