Introduction to Manifolds

September 10, 2020

Contents

1 Euclidean Spaces

The Euclidean space $\mathbb{R}^n$ is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. A good understanding of $\mathbb{R}^n$ is essential in generalizing differential and integral calculus to a manifold.

Definition 1.1. Let k be a nonnegative integer and U be an open subset in $\mathbb{R}^n$. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_j}} f$$

of all orders $j \leq k$ exist and are continuous at p. The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k at p for all $k \geq 0$. A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its component functions $f_1, \ldots, f_n$ are C^k at p. We say that $f : U \to \mathbb{R}^m$ is C^k on U if it is C^k at every point in U. A similar defintion holds for a C^∞ function on an open set U. We treat the terms "C^∞" and "smooth" as synonymous.

Example 1.1.

1. A C^0 function on U is a continuous function on U.

2. The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

   Thus the function f is C^0 but not C^1 at $x = 0$. On the other hand, f is C^1 on the open subset $\{x \in \mathbb{R} \mid x \neq 0\} \subseteq \mathbb{R}$. Now let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$\int_0^x f(t)dt = \int_0^x t^{1/3} dt = \frac{3}{4} x^{4/3}.$$

   Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is smooth and even bijective with inverse f^{-1} given by $f^{-1}(x) = x^{1/3}$, but f^{-1} is not smooth, as shown above.

5. Continuity of a function can often be seen by inspection, but the smoothness of a function always requires a formula. The graph of $y = x^{5/3}$ looks perfectly smooth, but it is in fact not smooth at $x = 0$, since its second derivative $y'' = (10/9)x^{-1/3}$ is not defined there.

6. Consider the norm function from $\mathbb{R}^n$ to $\mathbb{R}$, given by sending $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \in \mathbb{R}$. We will do this in detail. First we claim that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial and $k \geq 1$. We prove this by induction on k. The base case is trivial:

$$\partial_{x_n} (\|x\|) = \frac{x_n}{\|x\|}$$

   Now suppose that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$ where $f(x)$ is a polynomial. Then

$$\begin{aligned} \partial_{x_n}^{k+1} (\|x\|) &= \partial_{x_n} \left(\frac{f(x)}{\|x\|^{2k-1}} \right) \\ &= \frac{(\partial_{x_n} f)(x)}{\|x\|^{2k-1}} + \frac{(1-2n)x_n f(x)}{\|x\|^{2k+1}} \\ &= \frac{(\partial_{x_1} f)(x) \left(x_1^2 + \cdots + x_n^2\right) + (1-2n)x_n f(x)}{\|x\|^{2n+1}}. \end{aligned}$$

   This establishes our claim. Now given that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial, it is clear that $\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} (\|x\|)$ has the form $g(x)/\|x\|^{2(k_1+\cdots+k_n)-1}$, where $g(x)$ is a polynomial (just use the same induction proof). Now since $\|x\| = 0$ if and only if $x = \mathbf{0}$, we see that the norm function is smooth in $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

Proposition 1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be smooth. Then $g \circ f = \mathbb{R}^n \to \mathbb{R}$ is smooth.*

Proof. We will only sketch the proof here. By the chain rule, we have

$$\partial_{x_n}(g \circ f) = (g' \circ f) \cdot \partial_{x_n} f$$

By the product rule we have

$$\partial^2_{x_n}(g \circ f) = (g'' \circ f) \cdot (\partial_{x_n} f)^2 + (g' \circ f) \cdot \partial^2_{x_n} f$$

Similarly, we have

$$\partial^3_{x_n}(g \circ f) = (g''' \circ f)(\partial_{x_n} f)^3 + 3(g'' \circ f)(\partial_{x_n} f)(\partial^2_{x_n} f) + (g' \circ f)\partial^3_{x_n} f.$$

More generally, we will have a pattern which involves stirling numbers. □

Definition 1.2. Let $p = (p_1, \ldots, p_n)$ be a point in $\mathbb{R}^n$. A **neighborhood** of p in $\mathbb{R}^n$ is an open set containing p. The function f is **real-analytic** at p if in some neighborhood of p it is equal to its Taylor series at p:

$$f(x) = f(p) + \sum_i \partial_{x_i} f(p)(x_i - p_i) + \frac{1}{2!} \sum_{i,j} \partial_{x_i} \partial_{x_j} f(p)(x_i - p_i)(x_j - p_j) + \cdots + \frac{1}{k!} \sum_{i_1, \ldots, i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f(p)(x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) + \cdots$$

A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

then term-by-term differentiation gives

$$f'(x) = \cos x = 1 - \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - \cdots$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0.

Example 1.2. (A C^∞ function very flat at 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Clearly $\frac{d^n}{dx^n}(0) = 0$. Also,

$$\frac{d^n}{dx^n}\left(e^{-1/x}\right) = e^{-1/x}\left(\sum_{i=1}^{n}(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}\right)$$

Where $L(n,i)$ are the Lah numbers. Both $e^{-1/x}$ and $\sum_{i=1}^n(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ are well defined for $x > 0$ and $\frac{d^n}{dx^n}\left(e^{-1/x}\right)$ as $x \to 0$ (since $e^{-1/x}$ approaches 0 much faster than $\sum_{i=1}^n(-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ approaches ∞). So this function is clearly C^∞ on $\mathbb{R}$. On the other hand, the Taylor series of this function at the origin is identically zero in any neighborhood of the origin since $\frac{d^n f}{dx^n}(0) = 0$ for all $n \geq 1$. Therefore $f(x)$ cannot be equal to its Taylor series and thus $f(x)$ is not real-analytic at 0.

1.1 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. We say that a subset S of $\mathbb{R}^n$ is **star-shaped** with respect to a point p in S if for every x in S, the line segment from p to x lies in S. The line segment from p to x is parametrized by $\gamma : [0,1] \to \mathbb{R}^n$ where $\gamma(t) = (1-t)p + tx$. S is star-shaped with respect to p if for every x in S, $(1-t)p + tx$ is in S for all $t \in (0,1)$.

Lemma 1.1. *(Taylor's theorem with remainder). Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p_1, \ldots, p_n)$ in U. Then there are functions $g_1(x), \ldots, g_n(x) \in C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n}(x_i - p_i)g_i(x) \qquad g_i(p) = \partial_{x_i} f(p)$$

Remark. The idea behind this proof is to differentiate $f(p+t(x-p))$ and then integerate it.

Proof. Since U is star-shaped with respect to p, for any $x \in U$ the line segment $p+t(x-p)$, $0 \le t \le 1$ lies in U. So $f(p+t(x-p))$ is defined for $0 \le t \le 1$. By the chain rule

$$\begin{aligned}\frac{df}{dt}(p+t(x-p)) &= \frac{df}{dt}(p_1+t(x_1-p_1),\dots,p_n+t(x_n-p_n)) \\ &= (\partial_{x_1} f)(p+t(x-p))\partial_t(p_1+t(x_1-p_1))+\cdots+(\partial_{x_n} f)(p+t(x-p))\partial_t(p_n+t(x_n-p_n)) \\ &= \sum_{i=1}^{n}(x_i-p_i)\partial_{x_i} f(p+t(x-p)).\end{aligned}$$

If we integrate both sides with respect to t from 0 to 1, we get

$$f(p+t(x-p))|_0^1 = \sum(x_i-p_i)\int_0^1 \partial_{x_i} f(p+t(x-p))dt \tag{1}$$

Now let $g_i(x) = \int_0^1 \partial_{x_i} f(p+t(x-p))dt$. ☐

Example 1.3. We want to apply this proof to the function $f(x)$ on $\mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Let $p = 0$ and let $g(x) = \int_0^1 \frac{df}{dx}(p+t(x-p))dt$. Then

$$\begin{aligned} g(x) &= \int_0^1 \frac{df}{dx}(tx)dt \\ &= \int_0^1 \frac{-e^{-1/tx}}{tx^2}dt \\ &= \frac{e^{-1/tx}}{x}|_0^1 \\ &= \frac{e^{-1/x}}{x}. \end{aligned}$$

Thus,

$$f(x) = f(0) + x\left(\frac{e^{-1/x}}{x}\right).$$

1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

In elementary calculus we normally represent a vector at a point p in $\mathbb{R}^3$ algebraically as a column of numbers

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

or geometrically as an arrow emanating from p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p. Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a Euclidean space, and so would not apply to the projective plane, for example, which does not sit inside an $\mathbb{R}^n$ in any natural way.

1.2.1 The Directional Derivative

Let $p = (p_1,\dots,p_n)$ be a point with direction $v = (v_1,\dots,v_n)$ in $\mathbb{R}^n$. The line through the point p in the direction v can be parametized by $\ell := (\ell_1,\dots,\ell_n) : \mathbb{R} \to \mathbb{R}^n$, where

$$\ell(t) := p+tv = (p_1+tv_1,\dots,p_i+tv_i,\dots,p_n+tv_n) =: (\ell_1(t),\dots,\ell_i(t),\dots,\ell_n(t)).$$

Now let f be a C^∞ in a neighborhood of p in $\mathbb{R}^n$. The **directional derivative** of f in the direction of v at p is defined to be

$$\begin{aligned} D_v f &:= \lim_{t\to 0}\left(\frac{f(\ell(t)) - f(p)}{t}\right) \\ &= \partial_t f(\ell(t))|_{t=0} \\ &= \sum_{i=1}^{n} \partial_{x_i} f(\ell(0)) \cdot \partial_t \ell_i(0) \\ &= \sum_{i=1}^{n} v_i \partial_{x_i} f(p) \end{aligned}$$

In the notation $D_v f$, it is understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v_i \partial_{x_i}|_p$$

for the map that sends a function f to the number $D_v f$. To simplify the notation we often omit the subscript p if it is clear from the context.

1.2.2 Germs of Functions

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We introduce a relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. It is easy to check that this is an equivalence relation by showing it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(\mathbb{R}^n)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Remark. What happens if we weaken the relation a bit? Say $(f_1, U_1) \sim (f_2, U_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$. In this case, we no longer have an equivalence relation. The reason is because this relation is not transitive: Suppose $(f_1, U_1) \sim (f_2, U_2)$ and $(f_2, U_2) \sim (f_3, U_3)$. Then $f_1 = f_2$ on $U_1 \cap U_2$ and $f_2 = f_3$ on $U_2 \cap U_3$, but this merely implies that $f_1 = f_3$ on $U_1 \cap U_2 \cap U_3$.

Example 1.4. The functions

$$f(x) = \frac{1}{1-x}$$

with domain $\mathbb{R} \setminus \{1\}$ and

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

with domain the open interval $(-1, 1)$ have the same germ at any point p in the open interval $(-1, 1)$.

The addition and multiplication of functions induce corresponding operations on C_p^∞ making it into an $\mathbb{R}$-algebra. Indeed, let (f_1, U_1) and (f_2, U_2) be two representatives. Then multiplication is given by

$$(f_1, U_1) \cdot (f_2, U_2) = (f_1 f_2, U_1 \cap U_2).$$

We need to check that this is well-defined, so let (f_1', U_1') and (f_2', U_2') be two different represenatives respectively. Then

$$f_1 = f_1' \text{ on } W_1 \subset U_1 \cap U_1' \text{ and } f_2 = f_2' \text{ on } W_2 \subset U_2 \cap U_2'$$

This implies

$$f_1 f_2 = f_1' f_2' \text{ on } W_1 \cap W_2 \subset U_1 \cap U_2,$$

and thus

$$(f_1 f_2, U_1 \cap U_2) \sim (f_1' f_2', U_1 \cap U_2)$$

and hence this is well-defined. Similarly, addition is given by

$$(f_1, U_1) + (f_2, U_2) = (f_1 + f_2, U_1 \cap U_2).$$

Example 1.5. This example requires some knowledge of Algebraic Geometry. Let X be an affine algebraic set over an algebraically closed field K, let $R = A(X)$ be its coordinate ring, let p be a point in X, and let $\mathfrak{m}$ be the maximal ideal in R given by the set of all $f \in R$ which vanish at p. There are two equivalent ways to define the local ring $O_{X,p}$ at p.

One way is to define $O_{X,p}$ to be the local ring $R_\mathfrak{m}$. Elements in $R_\mathfrak{m}$ are equivalence classes of elements of the form f/g, where $f, g \in R$ and $g \notin \mathfrak{m}$. We say f_1/g_1 is equivalent to f_2/g_2 if there is an $h \in R$ such that $h \notin \mathfrak{m}$ and $h(g_2 f_1 - g_1 f_2) = 0$.

The other way is to define $O_{X,p}$ to be the ring of all germs of polynomial functions defined on a neighborhood of p. A "polynomial function defined on a neighborhood of p" is of the form f/g where $f, g \in R$ and $g(p) \neq 0$. We can think of f/g here as being the germ $(f/g, D(g))$, where $D(g)$ is the set of all points such that $g \neq 0$. Two such polynomial functions f_1/g_1 (or germ $(f_1/g_1, D(g_1))$) and f_2/g_2 (or germ $(f_2/g_2, D(g_2))$ represent the same germ if they agree on some small neighborhood of p. A small open neighborhood of p in the Zariski topology is simply something of the form $D(h)$ where h does not vanish at p. Thus, we need $f_1/g_1 = f_2/g_2$ on $D(h) \cap D(g_1) \cap D(g_2)$. Another way of saying this is $g_1 g_2 h(f_1/g_1 - f_2/g_2) = 0$ as a function on X; this matches precisely the criterion for f_1/g_1 and f_2/g_2 to be equal in the local ring $R_{\mathfrak{m}}$.

1.2.3 Derivations at a Point

We claim that D_v gives a map from C_p^∞ to $\mathbb{R}$. Indeed we just need to check that it is well-defined: suppose $(f, U) \sim (g, V)$. Then $f|_W = g|_W$ for some open set $W \subseteq U \cap V$. In particular,

$$\partial_{x_i} f(p) = \lim_{h \to 0} \frac{f(p_1, \ldots, p_i + h, \ldots, p_n)}{h} = \lim_{h \to 0} \frac{g(p_1, \ldots, p_i + h, \ldots, p_n)}{h} = \partial_{x_i} g(p).$$

for all $i = 1, \ldots, n$, which implies

$$\begin{aligned} D_v f &= \sum_{i=1}^n v_i \partial_{x_i} f(p) \\ &= \sum_{i=1}^n v_i \partial_{x_i} g(p) \\ &= D_v g. \end{aligned}$$

Thus $D_v : C_p^\infty \to \mathbb{R}$ is a well-defined map. In fact, D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = (D_v f) g(p) + f(p) D_v g, \tag{2}$$

precisely because the partial derivatives $\partial_{x_i}|_p$ have these properties.

In general, any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (2) is called a **derivation at** p or a **point-derivation** of C_p^∞. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiplication of a derivation at p are again derivations at p.

Thus far, we know that directional derivatives at p are all derivations at p, so there is a map

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n),$$

where a vector $v = (v_1, \ldots, v_n)$ in $T_p(\mathbb{R}^n)$ is mapped to the point-derivation $D_v = \sum_{i=1}^n v_i \partial_{x_i}|_p$. Since D_v is clearly linear in v, the map ϕ is a linear map of vector spaces.

Lemma 1.2. *If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.*

Proof. By $\mathbb{R}$-linearity, $D(c) = cD(1)$, so it suffices to prove that $D(1) = 0$. By the the Leibniz rule (2), we have

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Substracting $D(1)$ from both sides gives $D(1) = 0$. □

The **Kronecker delta** δ is a useful notation that we frequently call upon:

$$\delta_j^i = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{if } i \neq j. \end{cases}$$

Theorem 1.3. *The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined above is an isomorphism of vector spaces.*

Proof. To prove injectivity, suppose $D_v = 0$ for $v = (v_1, \ldots, v_n) \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x_j gives

$$\begin{aligned} 0 &= D_v x_j \\ &= \sum_i v_i \partial_{x_i} x_j \mid_p \\ &= v_j. \end{aligned}$$

Hence $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, V) be a representative of a germ in C_p^∞. Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped. By Taylor's theorem with remainder, there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum_{i=1}^{n} (x_i - p_i) g_i(x),$$

where $g_i(p) = \partial_{x_i} f(p)$. Applying D to both sides and noting that $Df(p) = 0$ and $D(p_i) = 0$ by Lemma (1.2), we get by the Leibniz rule (2)

$$\begin{aligned}
Df &= D(f(p) + \sum_{i=1}^{n} (x_i - p_i) g_i(x)) \\
&= D(f(p)) + \sum_{i=1}^{n} D\left((x_i - p_i) g_i(x)\right) \\
&= \sum_{i=1}^{n} D\left((x_i - p_i) g_i(x)\right) \\
&= \sum_{i=1}^{n} (D(x_i - p_i) g_i(p) + (p_i - p_i) D g_i) \\
&= \sum_{i=1}^{n} (D(x_i) - D(p_i)) g_i(p) \\
&= \sum_{i=1}^{n} D(x_i) g_i(p) \\
&= \sum_{i=1}^{n} D(x_i) \partial_{x_i} f(p)
\end{aligned}$$

This proves that $D = D_v$ for $v = (Dx_1, \ldots, Dx_n)$. □

1.2.4 Vector Fields

A **vector field** $\vec{v}$ on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector $\vec{v}(p)$ in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\{\partial_{x_i}|_p\}$, the vector $\vec{v}(p)$ is a linear combination

$$\vec{v}(p) = \sum_{i=1}^{n} \vec{v}_i(p) \partial_{x_i}(p),$$

where $\vec{v}_i(p) \in \mathbb{R}$. Thus we may write $\vec{v} = \sum_{i=1}^{n} \vec{v}_i \partial_{x_i}$, where the $\vec{v}_i$ are now functions on U. We say that a vector field $\vec{v}$ is C^∞ on U if the coefficient functions $\vec{v}_i$ are all C^∞ on U.

Example 1.6.

1. On $\mathbb{R}^2 \backslash \{0\}$, we have the vector field

$$\vec{v} = \frac{-y}{\sqrt{x^2 + y^2}} \partial_x + \frac{x}{\sqrt{x^2 + y^2}} \partial_y.$$

2. On $\mathbb{R}^2$, we have the vector field

$$\vec{v} = x\partial_x - y\partial_y.$$

The ring of C^∞ on an open set U is commonly denoted by $C^\infty(U)$. Multiplication of vector fields by functions on U is defined pointwise:

$$(f\vec{v})(p) = f(p)\vec{v}(p).$$

Clearly if $\vec{v} = \sum_{i=1}^{n} \vec{v}_i \partial_{x_i}$ is a C^∞ vector field and f is a C^∞ function on U, then

$$f\vec{v} = \sum_{i=1}^{n} f\vec{v}_i \partial_{x_i}$$

is a C^∞ vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $\mathrm{Vec}(U)$, is a $C^\infty(U)$-module.

1.3 Vector Fields as Derivations

If $\vec{v}$ is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function on U by

$$(\vec{v}f)(p) = \vec{v}(p)f$$

for all $p \in U$. Writing $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$, we get

$$(\vec{v}f)(p) = \sum_{i=1}^n \vec{v}_i(p)\partial_{x_i} f(p)$$

or $\vec{v}f = \sum_{i=1}^n \vec{v}_i \partial_{x_i} f$, which shows that $\vec{v}f$ is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$C^\infty(U) \to C^\infty(U), \qquad f \mapsto \vec{v}f.$$

Proposition 1.2. *(Leibniz rule for a vector field) If $\vec{v}$ is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $\vec{v}(fg)$ satisfies the Leibniz rule:*

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

Proof. At each point $p \in U$, the vector $\vec{v}(p)$ satisfies the Leibniz rule:

$$\vec{v}(p)(fg) = \vec{v}(p)(f) \cdot g(p) + f(p) \cdot \vec{v}(p)(g),$$

as p varies over U, this becomes an inequality of functions:

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

$\square$

If A is an algebra over a field K, a **derivation** of A is a K-linear map $D : A \to A$ such that

$$D(ab) = (Da)b + a(Db),$$

for all $a, b \in A$. The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted by $\mathrm{Der}(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\varphi : \mathrm{Vec}(U) \to \mathrm{Der}(C^\infty(U)), \qquad \vec{v} \mapsto (f \mapsto \vec{v}f).$$

Just as the tangent vectors at a point p can be identified with the point-derivations of C_p^∞, so the vector fields on an open set U can be identified with the derivations of the algebra $C^\infty(U)$, i.e. the map φ is an isomorphism of vector spaces.

1.4 The Exterior Algebra of Multicovectors

The basic principle of manifold theory is the linearization principle, according to which every manifold can be locally approximated by its tangent space at a point, a linear object. In this way linear algebra enters into manifold theory.

Instead of working with tangent vectors, it turns out to be more fruitful to adopt the dual point of view and work with linear functions on a tangent space. After all, there is only so much that one can do with tangent vectors, which are essentially arrows, but functions, far more flexible, can be added, multiplied, and composed with other maps.

1.5 Dual Spaces

Definition 1.3. Let V and W be two $\mathbb{R}$-vector spaces. We denote by $\mathrm{Hom}_{\mathbb{R}}(V, W)$ the vector space of all linear maps $\varphi : V \to W$. Define the **dual space** $V^\vee$ of V to be the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The elements of $V^\vee$ are called **covectors** or 1-**covectors** on V.

Let V be a finite dimensional $\mathbb{R}$-vector space with basis $\{e_1, \ldots, e_n\}$. Then every $v \in V$ can be uniquely expressed as $\sum_{i=1}^n v_i e_i$ with $v_i \in \mathbb{R}$. Let $\underset{\sim}{e}_i \in V^\vee$ be the linear function that picks out the ith coordinate, $\underset{\sim}{e}_i(v) = v_i$. Note that $\underset{\sim}{e}_i$ is characterized by

$$\underset{\sim}{e}_i(e_j) = \delta^i_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Proposition 1.3. *The functions $\underline{e}_1, \ldots, \underline{e}_n$ form a basis for $V^\vee$.*

Proof. We first show that $\underline{e}_1, \ldots, \underline{e}_n$ span $V^\vee$. Suppose $\ell \in V^\vee$. For all $v \in V$, we have

$$\begin{aligned}
\ell(v) &= \ell\left(\sum_{i=1}^{n} v_i e_i\right) \\
&= \sum_{i=1}^{n} v_i \ell(e_i) \\
&= \sum_{i=1}^{n} \underline{e}_i(v)\ell(e_i) \\
&= \sum_{i=1}^{n} \ell(e_i)\underline{e}_i(v) \\
&= \left(\sum_{i=1}^{n} \ell(e_i)\underline{e}_i\right)(v)
\end{aligned}$$

Therefore $\ell = \sum_{i=1}^{n} \ell(e_i)\underline{e}_i \in \mathrm{Span}(\{\underline{e}_1, \ldots, \underline{e}_n\})$. Next we show the set $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is linearly independent over $\mathbb{R}$. Suppose

$$\sum_{i=1}^{n} c_i \underline{e}_i = 0, \tag{3}$$

where $e_i \in \mathbb{R}$. By applying e_i to both sides of equation (3), we obtain $c_i = 0$, for all $i = 1, \ldots, n$. $\square$

Remark. We say $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is the **dual basis** of $\{e_1, \ldots, e_n\}$.

Proposition 1.4. *Let V be a finite-dimensional vector space and let $\ell \in V^\vee$. Then $Ker(\ell)$ is a hyperplane in V.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V and let $\{\underline{e}_1, \ldots, \underline{e}_n\}$ be its dual basis. Write ℓ in terms of the dual basis:

$$\ell = \sum_{i=1}^{n} a_i \underline{e}_i,$$

where $a_i \in \mathbb{R}$. A vector $\sum_{i=1}^{n} x_i e_i$ belongs to the kernel of ℓ if and only if $\sum_{i=1}^{n} x_i a_i = 0$. Thus

$$\mathrm{Ker}(\ell) = V\left(\sum_{i=1}^{n} a_i X_i\right) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = 0\right\}.$$

$\square$

Proposition 1.5. *Let V be an n-dimensional vector space and let $\ell_1, \ldots, \ell_k \in V^\vee$. Then $\{\ell_1, \ldots, \ell_k\}$ is linearly independent if and only if*

$$dim\left(\bigcap_{1 \le i \le k} Ker(\ell_i)\right) = n - k.$$

Proof. Suppose $\{\ell_1, \ldots, \ell_k\}$ is linearly independent. We may assume that we are working in $(\mathbb{R}^n)^\vee$ and that $\ell_i = \underline{e}_i$. Then

$$\begin{aligned}
\dim\left(\bigcap_{1 \le i \le k} \mathrm{Ker}(\ell_i)\right) &= \dim\left(\bigcap_{1 \le i \le k} V(X_i)\right) \\
&= \dim\left(V(X_1, \ldots, X_k)\right) \\
&= n - k.
\end{aligned}$$

The converse is trivial. $\square$

1.6 Differential Forms on $\mathbb{R}^n$

The **cotangent space** to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$. In parallel with the definition of a vector field, a **covector field** or **differential 1-form** on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p \in U} T_p^*(\mathbb{R}^n), \qquad p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

We call a differential 1-form a **1-form** for short.

1.7 Jacobian

Let $f = (f_1, \ldots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open subset U of $\mathbb{R}^n$. The **Jacobian** of f at a point $p \in U$ is the $m \times n$ matrix

$$J(f)(p) := \begin{pmatrix} (\partial_{x_1} f_1)(p) & \cdots & (\partial_{x_n} f_1)(p) \\ \vdots & \ddots & \vdots \\ (\partial_{x_1} f_m)(p) & \cdots & (\partial_{x_n} f_m)(p) \end{pmatrix}.$$

The Jacobian satisfies the following property: for all $p \in U$, we have

$$\frac{\|f(p+\varepsilon) - f(p) - J(f)_p(\varepsilon)\|}{\|\varepsilon\|} \to 0$$

as $\varepsilon \to 0$ in $\mathbb{R}^n$. One can view the Jacobian as a smooth linear map

$$J(f)(p) = (J(f)(p)_1, \ldots, J(f)(p)_m) : \mathbb{R}^n \to \mathbb{R}^m,$$

where the ith component $J(f)(p)_i$ is given by

$$J(f)(p)_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} (\partial_{x_j} f_i)(p) x_j.$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

If $m = n$, then f is a function from $\mathbb{R}^n$ to itself and the Jacobian matrix is a square matrix. In particular, we can compute its determinant, known as the **Jacobian determinant**. The Jacobian determinant at a given point gives important information about the behavior of f near that point. For instance, the inverse function theorem tells us that f is invertible near a point $p \in \mathbb{R}^n$ if and only if the Jacobian determinant is non-zero. Furthermore, if the Jacobian determinant at p is positive, then f preserves orientation near p.

Example 1.7.

1. Consider $f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and where $U = \{(x, y) \in \mathbb{R}^2 \mid xy \in (-\pi/2, \pi/2) \text{ and } x + y \in (0, \infty)\}$ and where $f_1(x, y) = \tan(xy)$ and $f_2(x, y) = \ln(x + y)$ for all $(x, y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in U$ is given by
$$J(f)(x_0, y_0) = \begin{pmatrix} y_0 \sec^2(x_0 y_0) & x_0 \sec^2(x_0 y_0) \\ \frac{1}{x_0 + y_0} & \frac{1}{x_0 + y_0} \end{pmatrix}.$$
The Jacobian determinant is then
$$\det\left(J(f)(x_0, y_0)\right) = \frac{(y_0 - x_0)\sec^2(x_0 y_0)}{x_0 + y_0}.$$

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = x^2 + xy + y$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by
$$J(f)(x_0, y_0) = \begin{pmatrix} 2x_0 + y_0 \\ x_0 + 1 \end{pmatrix}.$$
Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that
$$\begin{aligned} f(x_0 + \varepsilon_1, y_0 + \varepsilon_2) &= (x_0 + \varepsilon_1)^2 + (x_0 + \varepsilon_1)(y_0 + \varepsilon_2) + (y_0 + \varepsilon_2) \\ &= x_0^2 + x_0 y_0 + y_0 + (2x_0 + y_0)\varepsilon_1 + (x_0 + 1)\varepsilon_2 + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \\ &= f(x_0, y_0) + J(f)(x_0, y_0)(\varepsilon_1, \varepsilon_2) + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \end{aligned}$$

3. Consider $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ where $f_1(x, y) = x^2 y$ and $f_2(x, y) = y^2 + x$ for all $(x, y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by
$$J(f)(x_0, y_0) = \begin{pmatrix} 2x_0 y_0 & x_0^2 \\ 1 & 2y_0 \end{pmatrix}.$$
Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that
$$\begin{aligned} f(x_0 + \varepsilon_1, y_0 + \varepsilon_2) &= ((x_0 + \varepsilon_1)^2 (y_0 + \varepsilon_2), (y_0 + \varepsilon_2^2) + (x_0 + \varepsilon_1)) \\ &= (x_0^2 y_0, y_0^2 + x_0) + (2x_0 y_0 \varepsilon_1 + x_0^2 \varepsilon_2, \varepsilon_1 + 2y_0 \varepsilon_2) + (y_0 \varepsilon_1^2 + \varepsilon_1^2 \varepsilon_2, \varepsilon_2^2) \\ &= f(x_0, y_0) + J(f)(x_0, y_0)(\varepsilon_1, \varepsilon_2) + (y_0 \varepsilon_1^2 + \varepsilon_1^2 \varepsilon_2, \varepsilon_2^2) \end{aligned}$$

Proposition 1.6. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map from an open subset U of $\mathbb{R}^m$ and let p be a point in $\mathbb{R}^m$. Then*

$$f(p+\varepsilon) = f(p) + J(f)_p(\varepsilon) + \psi(\varepsilon),$$

where ψ is a smooth map such that $\|\psi(\varepsilon)\|/\|\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Proof. Define $\psi : U \to \mathbb{R}^n$ by

$$\psi(\varepsilon) := f(p+\varepsilon) - f(p) - J(f)_p(\varepsilon).$$

□

2 Manifolds

We first recall a few definitions from point-set topology. A topological space is **second countable** if it has a countable basis. A **neighborhood** of a point p in a topological space M is any open set containing p. A topological space M is **Hausdorff** if for every pair of points $x, y \in M$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. An **open cover** of M is a collection $\{U_i\}_{i \in I}$ of open sets in M whose union $\bigcup_{i \in I} U_i$ is M.

The Hausdorff condition and second countability are "hereditary properties"; they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff.

Proposition 2.1. *Let M' be a subspace of a topological space M.*

1. *If M is Hausdorff, then M' is Hausdorff.*
2. *If M is second countable, then M' is second countable.*

Proof. (1) : Suppose $x, y \in M'$. Since $x, y \in M$ and M is Hausdorff, choose a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. Then $U' = U \cap M'$ is a neighboorhood of x in the subspace topology and $V' = V \cap M'$ is a neighborhood of y in the subspace topology and $U' \cap V' = \emptyset$. (2) : If $\{B_i\}_{i \in \mathbb{N}}$ is a countable basis for M, then $\{B'_i\}_{i \in \mathbb{N}}$ is a countable basis for M', where $B'_i = B_i \cap M'$. □

Definition 2.1. A topological space M is **locally Euclidean of dimension** n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system on** U. We say that a chart (U, ϕ) is **centered** at $p \in U$ if $\phi(p) = 0$.

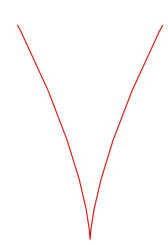
Proposition 2.2. *Let (U, ϕ) be a chart on the topological space M. If V is an open subset U, then $(V, \phi|_V)$ is a chart on M.*

Proof. This follows from the fact that if $\phi : U \to \phi(U)$ is a homeomorphism, then $\phi|_V : V \to \phi(V)$ is a homeomorphism. □

Definition 2.2. A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Example 2.1. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, 1_U)$.

Example 2.2. (A cusp). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold. By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean because it is homeomorphic to $\mathbb{R}$ via the projection $(x, x^{2/3}) \mapsto x$.



Example 2.3. (A cross). The cross can be described as $\{(r,0) \mid r \in \mathbb{R}\} \cup \{(0,r) \mid r \in \mathbb{R}\}$. We show that the cross in $\mathbb{R}^2$ with the subspace topology is not locally Euclidean at the intersection $p = (0,0)$, and so cannot be a manifold. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B_\varepsilon(0) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism $U \to B$ restricts to a homemorphism $U \setminus \{p\} \to B \setminus \{0\}$. Now $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components of $n = 1$. Since $U \setminus \{p\}$ has four connected components, there can be no homeomorphism from $U \setminus \{p\}$ to $B \setminus \{p\}$. This contradiction proves that the cross is not locally Euclidean at p.

2.1 Compatible Charts

Suppose $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 2.3. Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞-**compatible** if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called the **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞ compatible. To simplify this notation, we will sometimes write U_{ij} for $U_i \cap U_j$ and U_{ijk} for $U_i \cap U_j \cap U_k$. We will also sometimes write $\phi_{ij} = \phi_i \circ \phi_j^{-1}$. Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of compatible charts.

C^∞ compatibility is clearly reflexive and symmetric, but not necessarily transitive. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

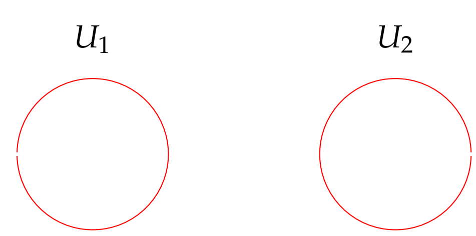
$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2)^{-1} \circ (\phi_2 \circ \phi_1^{-1})$$

is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$. A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} \setminus U_{123})$.

Definition 2.4. A C^∞ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_i, \phi_i)\}_{i \in I}$ of pairwise C^∞-compatible charts that cover M, i.e. such that $M = \bigcup_{i \in I} U_i$.

Example 2.4. (A C^∞ atlas on a circle). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{2\pi it} \in \mathbb{C} \mid 0 \leq t \leq 1\}$. Let U_1 and U_2 be the two open subsets of S^1

$$U_1 = \{e^{2\pi it} \in \mathbb{C} \mid -\frac{1}{2} < t < \frac{1}{2}\} \qquad U_2 = \{e^{2\pi it} \in \mathbb{C} \mid 0 < t < 1\}$$



and define $\phi_i : U_i \to \mathbb{R}$ for $i = 1, 2$ by

$$\phi_1(e^{2\pi it}) = t \qquad \phi_2(e^{2\pi it}) = t$$

Both ϕ_1 and ϕ_2 are branches of the complex log funciton $(1/i)\log z$ and are homeomorphisms onto their respective images. Thus (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection U_{12} consists of two connected components, the lower half A and the upper half B:

$$A = \{e^{2\pi it} \mid -\frac{1}{2} < t < 0\} \qquad B = \{e^{2\pi it} \mid 0 < t < \frac{1}{2}\}$$

with

$$\phi_1(U_{12}) = \phi_1(A \cup B) = \phi_1(A) \cup \phi_1(B) = \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$

$$\phi_2(U_{12}) = \phi_2(A \cup B) = \phi_2(A) \cup \phi_2(B) = \left(\frac{1}{2}, 1\right) \cup \left(0, \frac{1}{2}\right)$$

The transisition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_{12}) \to \phi_2(U_{12})$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t+1 & \text{for } t \in \left(-\frac{1}{2}, 0\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Similarly,

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t-1 & \text{for } t \in \left(\frac{1}{2}, 1\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1.

We say that a chart (V, ψ) is **compatible with an atlas** $\{(U_i, \phi_i)\}_{i \in I}$ if it is compatible with all the charts (U_i, ϕ_i) of the atlas.

Lemma 2.1. *Let $\{(U_i, \phi_i)\}_{i\in I}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_i, \phi_i)\}_{i\in I}$, then they are compatible with each other.*

Proof. We want to show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. For all $i \in I$, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is C^∞ on $\psi(V \cap W \cap U_i)$. Therefore $\sigma \circ \psi^{-1}$ is C^∞ on $\bigcup_{i\in I} \psi(V \cap W \cap U_i) = \psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. □

Remark. The domain of $\sigma \circ \psi^{-1}$ is $\psi(V \cap W)$ and the domain of $(\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is $\psi(U \cap V \cap W)$. What the equality means in the proof above is that the two maps are equal on their common domain.

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas; in other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 2.5. A **smooth** or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a **differentiable structure** on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a **curve**. A 2-dimensional manifold is a **surface**, and an n-dimensional manifold an n-manifold.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, because of the following proposition.

Proposition 2.3. *Any atlas $\mathfrak{U} = \{(U_i, \phi_i)\}_{i\in I}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Lemma (2.1), the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves existence. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{U}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{U}$ and so by construction must belong to $\mathfrak{M}$. This proves $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. This proves uniqueness. □

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that

1. M is Hausdorff and second countable
2. M has a C^∞ atlas.

From now on, a "manifold" will mean a C^∞ manifold. We use the terms "smooth" and "C^∞" interchangeably. In the context of manifolds, we denote the standard coordinates of $\mathbb{R}^n$ by $r^1, \ldots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. Thus for $p \in U$, $(x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \ldots, x^n$ are called **coordinates** or **local coordinates** on U. By abuse of notation, we sometimes omit the p. So the notations $(x^1, \ldots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$.

Remark. A topological manifold can be endowed with different (non-compatible) differentiable structures. For instance, consider $X = \mathbb{R}$. We can give the space the structure of a C^∞-manifold using the chart $(\mathbb{R}, \varphi_1)$, where φ_1 maps $x \to x$. We can also give the space the structure of a C^∞ manifold using the chart $(\mathbb{R}, \varphi_2)$, where φ_2 maps $x \mapsto x^3$. These two charts are not C^∞-compatible since $\varphi_1 \circ \varphi_2^{-1}$ maps $x \mapsto x^{\frac{1}{3}}$, and this is *not* C^∞ on $\mathbb{R}$: $\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ is not continuous at $x = 0$.

2.1.1 An Atlas For a Product

Proposition 2.4. *If $\mathfrak{U} = \{(U_i, \phi_i) \mid i \in I\}$ and $\mathfrak{V} = \{(V_j, \psi_j) \mid j \in J\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection*

$$\mathfrak{U} \times \mathfrak{V} = \{(U_i \times V_j, \phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n) \mid (i,j) \in I \times J\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. Clearly the set $\{U_i \times V_j \mid (i,j) \in I \times J\}$ covers $M \times N$, so we just need to show that any two charts in $\mathfrak{U} \times \mathfrak{V}$ are pairwise compatible. Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two charts in $\mathfrak{U} \times \mathfrak{V}$. Then $(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1}$ is C^∞, since

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} = \left(\phi_1 \circ \phi_2^{-1}\right) \times \left(\psi_2 \times \psi_2^{-1}\right),$$

and both $\phi_1 \circ \phi_2^{-1}$ and $\psi_2 \times \psi_2^{-1}$ are C^∞ on their respective domains. The same proof shows that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1}$ is C^∞. Thus $\mathfrak{U} \times \mathfrak{V}$ is a collection of pairwise C^∞ compatible charts that cover $M \times N$. □

Example 2.5. It follows from Proposition (2.4) that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds.

2.2 Examples of Smooth Manifolds

2.2.1 Euclidean Space

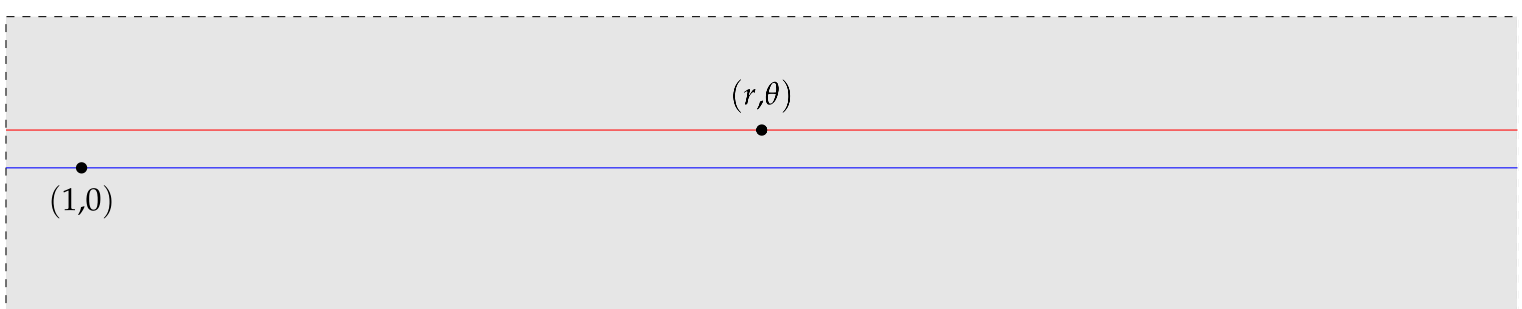
Example 2.6. (Euclidean space). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, \mathrm{id})$. We use $x_1, \dots, x_n$ to denote coordinates functions and $a_1, \dots, a_n$ to denote real numbers. Thus, if $p = (a_1, \dots, a_n)$ is a point in $\mathbb{R}^n$, we have $x_1(p) = a_1$, $x_2(p) = a_2$, and etc...

Example 2.7. The real half line $\mathbb{R}_{>0} : \{a \in \mathbb{R} \mid a > 0\}$ is also a smooth manifold, with a single chart $(\mathbb{R}_{>0}, \mathrm{id})$. In fact, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$. A homeomorphism from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is given by $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

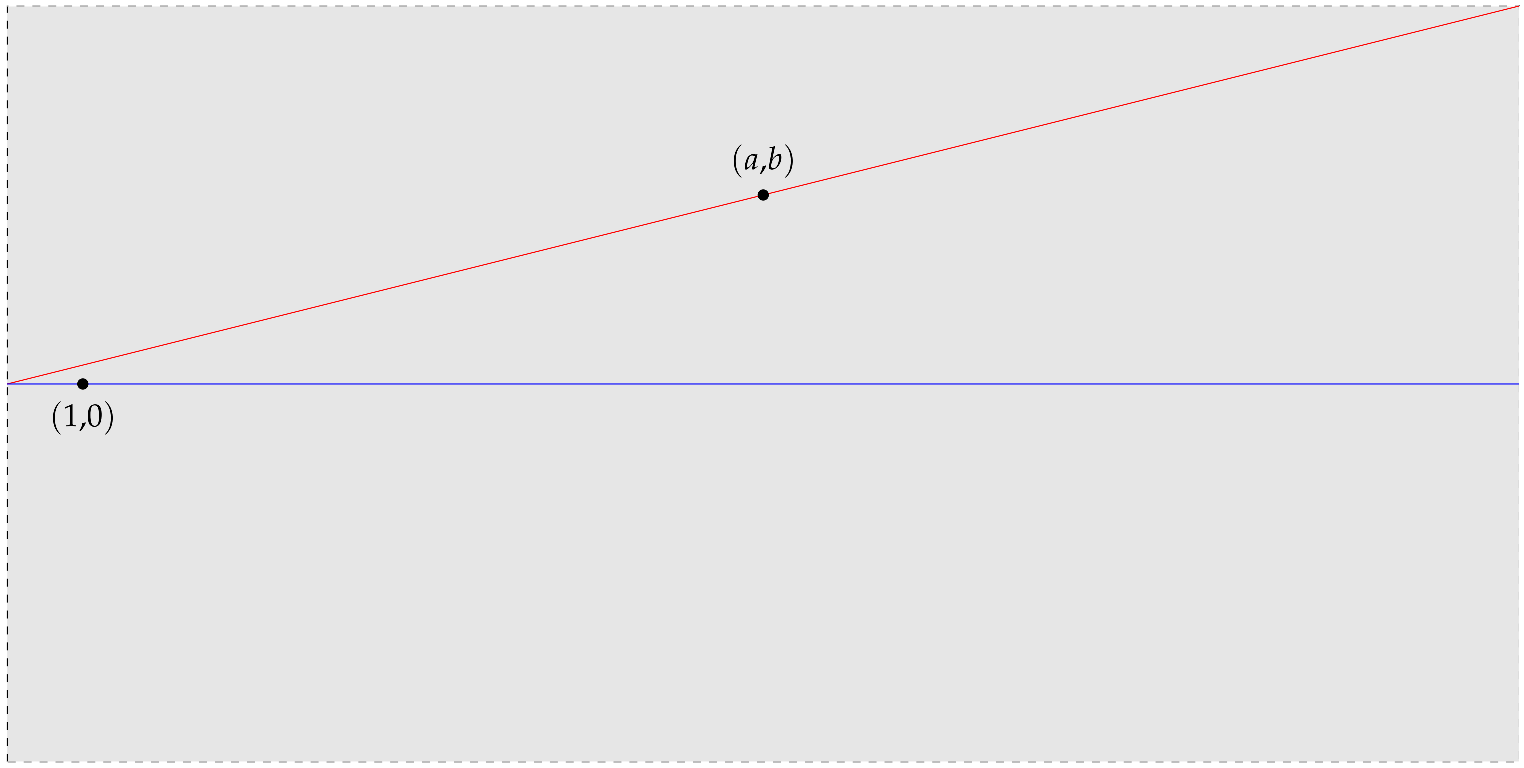
Now consider the half-open interval $(0, 2\pi)$. Open sets of the form (a, b) where and $0 \leq a < b < 2\pi$ form a basis for this topological space.

2.2.2 Right-Half Infinite Strip and the Right-Half Plane

Let $M = \mathbb{R}_{>0} \times \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. We illustrate this space below:



Now let $N = \mathbb{R}_{>0} \times \mathbb{R}$ be the right-half plane. We illustrate this space below:



We can give both M and N the structure of a smooth manifold by simply using the identity charts.

Let $\varphi : M \to N$ be given by $\varphi(r,\theta) = (\varphi_1(r,\theta), \varphi_2(r,\theta))$, where

$$\varphi_1(r,\theta) = r\sin\theta$$
$$\varphi_2(r,\theta) = r\cos\theta$$

Then φ is a diffeomorphism from M to N. The Jacobian of φ at a point $(r,\theta) \in M$:

$$J_{(r,\theta)}(\varphi) = \begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$$

The inverse to $\varphi : M \to N$ is $\psi : N \to M$, given by $\psi(a,b) = (\psi_1(a,b), \psi_2(a,b))$ where

$$\psi_1(a,b) = \sqrt{a^2+b^2}$$
$$\psi_2(a,b) = \arctan\left(\frac{a}{b}\right)$$

The Jacobian of ψ at a point $(a,b) \in N$:

$$J_{(a,b)}(\psi) = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

2.2.3 Manifolds of Dimension Zero

Example 2.8. (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

2.2.4 Graph of a Smooth Function

Example 2.9. (Graph of a smooth function). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$, the **graph** of f is defined to be the subset

$$\Gamma(f) = \{(p, f(p)) \in A \times \mathbb{R}^n\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^∞, then the two maps

$$\phi : \Gamma(f) \to U \qquad (p, f(p)) \mapsto p$$

and

$$(1, f) : U \to \Gamma(f) \qquad p \mapsto (p, f(p))$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^n$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

2.2.5 Circle S^1

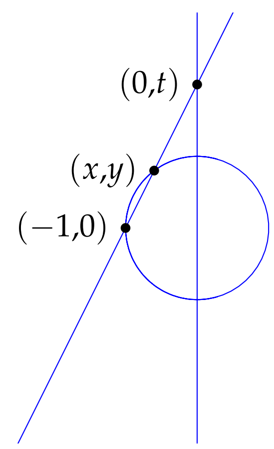
Example 2.10. (Circle) Let S^1 be the unit circle centered at the origin in $\mathbb{R}^2$:

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We shall describe an atlas on S^1 using stereographic projection. Let $U_1 = S^1 \backslash \{(-1,0)\}$. Consider the line L which passes through the points $(-1,0)$ and $(0,t)$ where $t \in \mathbb{R}$. The equation of this line is given by

$$Y = t(X+1).$$

Since L passes through $(-1,0)$ and is not tangent to $(-1,0)$, it must pass through a unique point (x,y) in S^1. This is illustrated in the image below:



Since (x,y) lies on the line L and the unit circle, we get the relations

$$\begin{aligned} x^2 + y^2 - 1 &= 0, \\ y - t(x+1) &= 0. \end{aligned}$$

Using the second relation, we have $y = t(x+1)$. Plugging in $t(x+1)$ for y in the first relation, we get

$$t^2 = \frac{(1-x)^2}{(1+x)^2} = \frac{1-x}{1+x}.$$

Now we solve for x in terms of t, to get:

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2}, \\ y &= \frac{2t}{1+t^2}. \end{aligned}$$

Now, let $\phi_1 : U_1 \to \mathbb{R}$ be given by

$$(x,y) \mapsto \frac{y}{1+x}.$$

This map is clearly C^∞ in its domain U_1, since $x \neq -1$, and the inverse $\phi_1^{-1} : \mathbb{R} \to U_1$ is given by

$$t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

Next, let $U_2 = S^1 \backslash \{(1,0)\}$. Following the same line of reasoning as the paragraph above, let $\phi_2 : U_2 \to \mathbb{R}$ be given by

$$(x,y) \mapsto \frac{y}{1-x}.$$

Again, this map is clearly C^∞ in its domain U_2, since $x \neq 1$, and the inverse $\phi_2^{-1} : \mathbb{R} \to U_2$ is given by

$$t \mapsto \left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2} \right).$$

Let us calculate the transition map $\phi_{12} := \phi_1 \circ \phi_2^{-1}$:

$$\begin{aligned} \phi_{12}(t) &= (\phi_1 \circ \phi_2^{-1})(t) \\ &= \phi_1 \left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2} \right) \\ &= \frac{1}{t}. \end{aligned}$$

Remark. We think of t as a local coordinate of S^1 and x, y as global coordinates of S^1.

2.2.6 Projective Line

Example 2.11. Let $\mathbb{P}^1(\mathbb{R})$ be the projective line over $\mathbb{R}$. Define in $\mathbb{P}^1(\mathbb{R})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0} \in \mathbb{R},$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1} \in \mathbb{R}.$$

These maps are clearly C^∞ in their domains. The inverse maps are given by

$$\phi_0^{-1}(t) = (1 : t) \in U_0 \qquad \phi_1^{-1}(t) = (t : 1) \in U_1.$$

Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1}$:

$$\begin{aligned}
\phi_0 \circ \phi_1^{-1}(t) &= \phi_0 \circ \phi_1^{-1}(t) \\
&= \phi_0 (t : 1) \\
&= \frac{1}{t}.
\end{aligned}$$

Recall that this is the same transition map we calculated in Example (2.10).

2.2.7 Sphere S^2

Example 2.12. (Sphere) Let S^2 be the unit sphere

$$S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres - the front, rear, right , left, upper, and lower hemispheres

$$\begin{aligned}
U_1 &= \{(a, b, c) \in S^2 \mid a > 0\} & \phi_1(a, b, c) &= (b, c) \\
U_2 &= \{(a, b, c) \in S^2 \mid a < 0\} & \phi_2(a, b, c) &= (b, c) \\
U_3 &= \{(a, b, c) \in S^2 \mid b > 0\} & \phi_3(a, b, c) &= (a, c) \\
U_4 &= \{(a, b, c) \in S^2 \mid b < 0\} & \phi_4(a, b, c) &= (a, c) \\
U_5 &= \{(a, b, c) \in S^2 \mid c > 0\} & \phi_5(a, b, c) &= (a, b) \\
U_6 &= \{(a, b, c) \in S^2 \mid c < 0\} & \phi_6(a, b, c) &= (a, b)
\end{aligned}$$

The open set U_{14} is $\{(a, b, c) \in S^2 \mid b < 0 < a\}$ and $\phi_4(U_{14}) = \{(a, c) \in \mathbb{R}^2 \mid a^2 + c^2 < 1 \text{ and } a > 0\}$.

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned}
\phi_{14}(a, c) &= \phi_1 \circ \phi_4^{-1}(a, c) \\
&= \phi_1 \left(a, \sqrt{1 - c^2 - a^2}, c\right) \\
&= \left(\sqrt{1 - c^2 - a^2}, c\right).
\end{aligned}$$

It is easy to see that this is indeed a smooth map in its domain (since $1 - c^2 - a^2 \neq 0$). The Jacobian of ϕ_{14} at the point (a, c) is

$$J_{(a,c)}(\phi_{14}) = \begin{pmatrix} \frac{a}{\sqrt{1-c^2-a^2}} & \frac{c}{\sqrt{1-c^2-a^2}} \\ 0 & 1 \end{pmatrix}$$

Now let's compute the transition map ϕ_{45}:

$$\begin{aligned}
\phi_{45}(a, b) &= \phi_4 \circ \phi_5^{-1}(a, b) \\
&= \phi_4 \left(a, b, \sqrt{1 - a^2 - b^2}\right) \\
&= \left(a, \sqrt{1 - a^2 - b^2}\right).
\end{aligned}$$

2.2.8 The Sphere S^n

Example 2.13. Using stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $p_N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the north pole and $p_S = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the south pole. Define the maps $\phi_N : S^n \backslash \{p_N\} \to \mathbb{R}^n$ and $\phi_S : S^n \backslash \{p_S\}$, called **stereographic projection** from the north pole (resp. south pole), by

$$\phi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \phi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\phi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, -1 + \sum_{i=1}^n x_i^2\right)$$

and

$$\phi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, 1 - \sum_{i=1}^n x_i^2\right).$$

Thus, if we let $U_N = S^n \backslash \{p_N\}$ and $U_S = S^n \backslash \{p_S\}$, we see that U_N and U_S are two open subsets convering S^n, both homeomorphic to $\mathbb{R}^n$. Furthermore, it is easily checked that on the overlap, $U_N \cap U_S$, the transition maps

$$\phi_S \circ \phi_N^{-1} = \phi_N \circ \phi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $p_O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n \backslash \{O\}$, so we conclude that (U_N, ϕ_N) and (U_S, ϕ_S) form a smooth atlas for S^n.

2.2.9 Real Projective Plane

Example 2.14. (Projective Plane) Let $\mathbb{P}^2(\mathbb{R})$ be the projective plane over $\mathbb{R}$. Define in $\mathbb{P}^2(\mathbb{R})$ the three charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1 : x_2) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) =: (a, b)$$

$$U_1 = D(X_1) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) =: (c, d)$$

$$U_2 = D(X_2) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_2 \neq 0\} \qquad \phi_2(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) =: (e, f)$$

The reason the map ϕ_1 is a homeomorphism is because given that $x_1 \neq 0$, we use the equivalence relation to write the point $p = (x_0 : x_1 : x_2)$ as $p = \left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right)$. Now $\frac{x_0}{x_1}$ and $\frac{x_2}{x_1}$ are two real rumbers which uniquely determine the point (a, b). We think of a and b as the local coordinates in the (U_0, ϕ_0) chart.

Let U_{01} be the intersection of U_0 and U_1, that is, $U_{01} := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$. Then $\phi_0(U_{01}) = \left\{\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$ and $\phi_1(U_{01}) = \left\{\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$. We can also write this in terms of local coordinates as $\phi_0(U_{01}) = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ and $\phi_1(U_{01}) = \{(c, d) \in \mathbb{R}^2 \mid c \neq 0\}$. Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1} : \phi_1(U_{01}) \to \phi_0(U_{01})$ using the local coordinates. We have

$$\begin{aligned}
\phi_{01}\,(c, d) &= \phi_0 \circ \phi_1^{-1}\,(c, d) \\
&= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right) \\
&= \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \\
&= \left(\frac{1}{c}, \frac{d}{c}\right).
\end{aligned}$$

It's easy to see that ϕ_{01} is C^∞. Indeed, writing ϕ_{01}^1 and ϕ_{01}^2 for the components of ϕ_{01} (so $\phi_{01}^1(c,d) = \frac{1}{c}$ and $\phi_{01}^2(c,d) = \frac{d}{c}$), the partial derivatives $\partial_c^m \partial_d^n \phi_{01}^i$ exist and are continuous everywhere in $\phi_1(U_{01})$ for all $m,n \in \mathbb{N}$ and $i = 1,2$. This is because ϕ_{01}^1 and ϕ_{01}^2 are rational functions (i.e. ratio of two polynomials) and are they are defined everywhere since $c \neq 0$ in$\phi_1(U_{01})$.

Similarly, one can easily show that

$$\begin{aligned}
\phi_{10}(a,b) &= \left(\frac{1}{a}, \frac{b}{a}\right) \\
\phi_{20}(a,b) &= \left(\frac{1}{b}, \frac{a}{b}\right) \\
\phi_{02}(e,f) &= \left(\frac{f}{e}, \frac{1}{e}\right) \\
\phi_{12}(e,f) &= \left(\frac{e}{f}, \frac{1}{f}\right) \\
\phi_{21}(c,d) &= \left(\frac{c}{d}, \frac{1}{d}\right)
\end{aligned}$$

It is instructive to check that $\phi_{ij} \circ \phi_{ji} = 1$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$.

2.2.10 Riemann Sphere

Example 2.15. (Riemann sphere) In this example we describe a **complex manifold**. A complex manifold is the complex analogue of a manifold, however in the complex manifold case, we require the transition maps to be holomorphic, and not just C^∞. Let $\mathbb{P}^1(\mathbb{C})$ be the projective line over $\mathbb{C}$ (also known as the Riemann sphere). Define in $\mathbb{P}^1(\mathbb{C})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0}$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1}$$

This time, let $z = \frac{x_0}{x_1}$. The open set U_{01} is $\{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$ and $\phi_1(U_{01}) = \mathbb{C}^\times$. Now

$$\begin{aligned}
\phi_0 \circ \phi_1^{-1}(z) &= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0}\right) \\
&= \frac{x_1}{x_0} \\
&= \frac{1}{z}.
\end{aligned}$$

One can show that the map $z \mapsto \frac{1}{z}$ is holomorphic in the domain $\mathbb{C}^\times$.

2.2.11 Mobius Strip

Example 2.16. Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold. First we consider the set of all nonvertical lines in $\mathbb{R}^2$, which we denote by U_v. A nonvertial is of the form $\ell_{a,b}^v = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell_{a,b}^v \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism. Next we consider the set of all nonhorizontal lines in $\mathbb{R}^2$, which we denote by U_h. A nonhorizontal is of the form $\ell_{c,d}^h = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell_{c,d}^h \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism. Now we have $U_v \cup U_h = \mathcal{L}$. To get a topology on $\mathcal{L}$, we glue the topologies from U_v

and U_h: a set $U \subset \mathcal{L}$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d) \\ &= \varphi_v\left(\ell^h_{c,d}\right) \\ &= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right) \\ &= \left(\frac{1}{c},-\frac{d}{c}\right),\end{aligned}$$

and similarly,

$$\begin{aligned}\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b) \\ &= \varphi_h\left(\ell^v_{a,b}\right) \\ &= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right) \\ &= \left(\frac{1}{a},-\frac{b}{a}\right).\end{aligned}$$

These maps are clearly C^∞. In fact, they look very similar to the transition maps for the projective plane, except they are twisted by a negative sign.

Remark. We can also describe $\mathcal{L}$ as $\mathbb{RP}^2\backslash\{[0:0:1]\}$: Any line in the euclidean plane is of the form $ax+by+c=0$, for some $a,b,c \in \mathbb{R}$. First note that these coefficients uniquely determine the line and they are homogeneous. Hence there is a well defined map $\phi : \mathcal{L} \to \mathbb{RP}^2$, given by mapping the line $\mathbf{V}(ax+by+c)$ to the point $[a:b:c]$. Now ϕ is injective, but not surjective. However if we remove the point $[0:0:1]$, then the induced map $\phi : \mathcal{L} \to \mathbb{RP}^2\backslash\{[0:0:1]\}$ is a bijection.

2.2.12 Grassmannians

The **Grassmannian** $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $v_1,\dots,v_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the **rank** of a matrix A, denoted by rk$A$, is defined to be the number of linearly independent columns of A. This matrix is called a **matrix representative** of the k-plane.

Two bases $a_1,\dots,a_k$ and $b_1,\dots,b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k,\mathbb{R})$ such that

$$b_j = \sum_{i=1}^{k} a_i g_{ij}$$

for all $1 \le k \le n$. In matrix notation, this says $B = Ag$. Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \text{ if and only if there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

A **real Grassmann manifold** $G(n,k)$ is defined as the space of all k-dimensional subspaces of the space $\mathbb{R}^n$. The topology in $G(n,k)$ may be described as induced by the embedding $G(n,k) \to \mathrm{End}(\mathbb{R}^n)$ which assigns to a $P \in G(n,k)$, the orthogonal projection $\mathbb{R}^n \to P$ combined with the inclusion map $P \to \mathbb{R}^n$.

In $G(4,2)$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} \sim \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} = \begin{pmatrix} c_{11}a_{11}+c_{12}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24} \\ c_{21}a_{11}+c_{22}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24}\end{pmatrix}$$

where $c_{11}c_{22} - c_{21}c_{12} \neq 0$.

3 Smooth Maps on a Manifold

Now that we've defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined.

3.1 Smooth Functions

Definition 3.1. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or **smooth at a point** p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

Remark.

1. The definition of the smoothness of a function f at a point is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) is any other chart about p in M, then on $\psi(U \cap V)$,
$$f \circ \psi^{-1} |_{\psi(U \cap V)} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$
which is C^∞ at $\psi(p)$. Thus $f \circ \psi^{-1}$ must be C^∞ at $\psi(p)$.

2. In the definition above, $f : M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are only interested in functions that are smooth on an open set, there is no loss of generality in assuming at the onset that f is continuous.

Proposition 3.1. *Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

1. *The function $f : M \to \mathbb{R}$ is C^∞.*
2. *The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
3. *For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

Proof. We will prove the proposition as a cyclic chain of implications.

$(2 \implies 1)$: This follows directly from the definition of a C^∞ function, since by (2) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

$(1 \implies 3)$: Let (V, ψ) be an arbitrary chart on M and let $p \in V$. By the remark above, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

$(3 \implies 2)$: Obvious.

□

The smoothness conditions of Proposition (3.1) will be a recurrent motif: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on *every* chart.

Definition 3.2. Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

Remark. In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

Example 3.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by an angle θ and let x, y denote the standard coordinate functions on $\mathbb{R}^2$. Then
$$\phi^* x = (\cos\theta)x - (\sin\theta)y$$
$$\phi^* y = (\sin\theta)x + (\cos\theta)y.$$

Indeed, let e_1, e_2 denote the standard coordinates on $\mathbb{R}^2$; so $x(e_1) =$
$$\begin{aligned}(\phi^* x)(a, b) &= x\left(\phi(a, b)\right) \\ &= x\left(\cos\theta a - \sin\theta b, \sin\theta a + \cos\theta b\right) \\ &= \cos\theta a - \sin\theta b \\ &= ((\cos\theta)x - (\sin\theta)y)(a, b).\end{aligned}$$

3.2 Smooth Maps Between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms "C^∞" and "smooth" interchangeably.

Definition 3.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ **at a point** p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

Remark.

1. In the definition, we needed $F^{-1}(V)$ to be open so that $\phi(F^{-1}(V) \cap U)$ is open. Thus, C^∞ maps between manifolds are by definition continuous.
2. In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ as a chart about $F(p)$ in $\mathbb{R}^m$. According to the definition above, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p in N such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

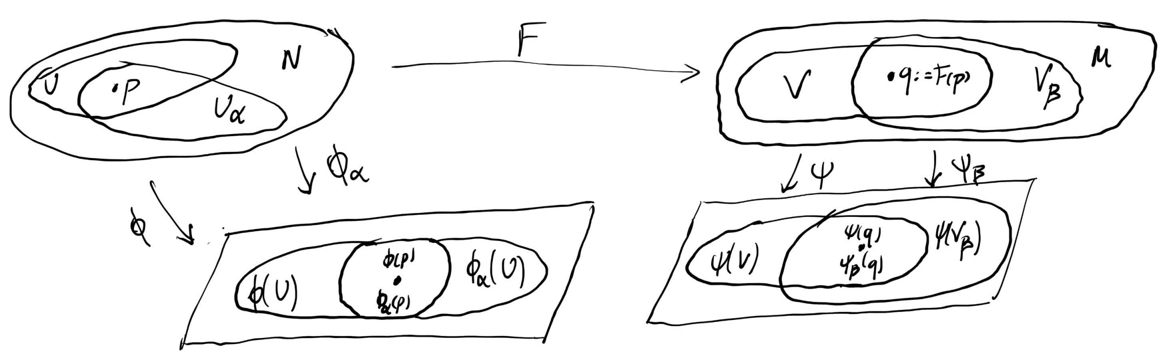
We show now that the definition of the smoothness of a map $F : N \to M$ at a point is independent of the choice of charts.

Proposition 3.2. *Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.*

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1} |_{\phi\left(F^{-1}(V \cap V_\beta) \cap U \cap U_\alpha\right)} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1}),$$

is C^∞ at $\phi(p)$. Therefore $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.



□

Proposition 3.3. *(Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:*

1. *The map $F : N \to M$ is C^∞.*
2. *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map*
$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$
is C^∞.
3. *For every chart (U, ϕ) on N and (V, ψ) on M, the map*
$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$
is C^∞.

Proof.
(2 $\implies$ 1): Let $p \in N$. Suppose (U, ϕ) is a chart about p in $\mathfrak{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathfrak{V}$. By (2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

(1 $\implies$ 3): Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \emptyset$. Let $p \in U \cap F^{-1}(V)$. Then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition (3.2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitary point of $\phi(U \cap F^{-1}(V))$, the map $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞.

(3 $\implies$ 2): Obvious.

□

Proposition 3.4. *(Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.*

Proof. Let (U, ϕ), (V, ψ), and (W, σ) be charts on N, M, and P respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}).$$

Since F and G are C^∞, the maps $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are also C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞, and thus $G \circ F$ is C^∞. □

3.2.1 Diffeomorphisms

A **diffeomorpism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 3.5. *If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\left\{(\phi(U), \mathrm{id}_{\phi(U)})\right\}$ with a single chart on $\phi(U)$. Since

$$\mathrm{id}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$$

is the identity map, it is C^∞. By Proposition (3.3), ϕ is C^∞.
To test smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since

$$\phi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} : \phi(U) \to \phi(U)$$

is the identity map, the map ϕ^{-1} is also C^∞. □

Proposition 3.6. *Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition (3.5). As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U, F) is in the atlas. □

3.2.2 Smoothness in Terms of Components

In this subsection, we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 3.7. *(Smoothness of a vector-valued function) Let N be a manifold and let $F : N \to \mathbb{R}^m$ be a continuous map. The following are equivalent:*

1. *The map $F : N \to \mathbb{R}^m$ is C^∞.*
2. *The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*
3. *For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*

Proposition 3.8. *(Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F_1, \ldots, F_m : N \to \mathbb{R}$ are all C^∞.*

Proof. The $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the functions $F_i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ if and only if the functions $F_i : N \to \mathbb{R}$ are all C^∞. □

3.3 Germs of C^∞ functions

Let M be an n-dimensional manifold and let p be a point in M. Consider the set of all pairs (f, U), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. Just as in the $\mathbb{R}^n$ case, we introduce an equivalence relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(M)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Let (f, U) be represent a germ in $C_p^\infty(M)$ and suppose (U_0, ϕ) is a chart centered at p. Then $(U_0 \cap U, \phi_{|U})$ is a chart centered at p and clearly we have $(f, U) \sim (f\,|_{U_0 \cap U}, U_0 \cap U)$. Thus we may always assume that a germ can be represented by (f, U) where (U, ϕ) is a chart centered at p. In particular, we obtain an isomorphism

$$\widehat{\phi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \phi^{-1}, \phi(U))$. Of course this map depends on our choice of chart. If (V, φ) was another chart, then we'd obtain another isomorphism

$$\widehat{\varphi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \varphi^{-1}, \varphi(U))$. We can relate these two isomorphisms via the transition function $\phi \circ \varphi^{-1}$.

Let M be a manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on M. We describe the structure of a premanifold as follows: if U is an open subset of M, then we set

$$\mathcal{O}_M(U) := \{f : U \to \mathbb{R} \mid f|_{U \cap U_i} \circ \phi_i^{-1} : \phi_i(U \cap U_i) \to \mathbb{R} \text{ is } C^\infty\} = \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty\}.$$

To see that this is a premanifold, fix $i_0 \in I$. For $U \subseteq U_{i_0}$ open let $f : U \to \mathbb{R}$ be a map such that $f \circ \phi_{i_0}^{-1} : \phi_{i_0}(U_{i_0} \cap U) \to \mathbb{R}$ is a C^∞ function. Then $f \in \mathcal{O}_M(U)$ because the change of charts between i and i_0 are C^∞-diffeomorphisms. Indeed, we have

$$f|_{U \cap U_i} \circ \phi_i^{-1} = (f \circ \phi_{i_0}^{-1}) \circ (\phi_{i_0} \circ \phi_i^{-1}).$$

Therefore ϕ_{i_0} yields an isomorphism $(U_{i_0}, \mathcal{O}_{M|U_{i_0}}) \cong (Y_{i_0}, \mathcal{O}_{i_0})$, where $\mathcal{O}_{Y_0}$ is the sheaf of C^∞ functions on Y_{i_0}. Hence, $(M, \mathcal{O}_M)$ is a ringed space that is locally isomorphic to a manifold. Hence it is a premanifold.

3.4 Examples of Smooth Maps

Example 3.2. We show that the map $F : \mathbb{R} \to S^1$ given by $F(t) = (\cos t, \sin t)$ is C^∞. For $\mathbb{R}$, we use the atlas which consists of a single chart $(\mathbb{R}, \mathrm{id})$. For S^1 we use the atlas which consists of the charts (U_1, ϕ_1), (U_2, ϕ_2), (U_3, ϕ_3) and (U_4, ϕ_4) where

$$\begin{aligned}
U_1 &= \{(a, b) \in S^1 \mid a > 0\} \qquad & \phi_1(a, b) &= b \\
U_2 &= \{(a, b) \in S^2 \mid a < 0\} \qquad & \phi_2(a, b) &= b \\
U_3 &= \{(a, b) \in S^2 \mid b > 0\} \qquad & \phi_3(a, b) &= a \\
U_4 &= \{(a, b) \in S^2 \mid b < 0\} \qquad & \phi_4(a, b) &= a
\end{aligned}$$

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned}
\phi_{14}(a) &= \phi_1 \circ \phi_4^{-1}(a) \\
&= \phi_1\left(a, \sqrt{1 - a^2}\right) \\
&= \sqrt{1 - a^2}.
\end{aligned}$$

Similar computations shows that

$$\begin{aligned}
\phi_{13}(a) &= \sqrt{1 - a^2} \\
\phi_{24}(a) &= \sqrt{1 - a^2} \\
\phi_{23}(a) &= \sqrt{1 - a^2}.
\end{aligned}$$

Now, we need to show that $\phi_i \circ F \circ \mathrm{id}$ is C^∞ for $i = 1, 2, 3, 4$. Let's compute $\phi_1 \circ F \circ \mathrm{id}$:

$$\begin{aligned}
(\phi_1 \circ F \circ \mathrm{id})(t) &= \phi_1(F(t)) \\
&= (\phi_1((\cos t, \sin t)) \\
&= \sin t.
\end{aligned}$$

Similar computations shows that

$$\begin{aligned}(\phi_2 \circ F \circ \mathrm{id})(t) &= \sin t \\ (\phi_3 \circ F \circ \mathrm{id})(t) &= \cos t \\ (\phi_4 \circ F \circ \mathrm{id})(t) &= \cos t.\end{aligned}$$

These maps are all C^∞.

Example 3.3. Consider $N = \mathbb{R}$ and $M = \mathbb{R}^2$ and let $f : N \to M$ be given by $f(t) = (t^2, t^3)$.

Example 3.4. Let S^2 be the unit sphere with its smooth structure given in Example (2.12). Let $f : S^2 \to \mathbb{R}$ be given by

$$f(a,b,c) = c^2.$$

We claim that f is C^∞. To see this, we need to show that f is C^∞ at every point $p = (a,b,c)$ in S^2. First assume that $p \in U_6$. Using the chart (U_6, ϕ_6), we find that

$$\begin{aligned}(f \circ \phi_6^{-1})(a,b) &= f\left(\phi_6^{-1}(a,b)\right) \\ &= f\left(a, b, \sqrt{1-a^2-b^2}\right) \\ &= 1 - a^2 - b^2,\end{aligned}$$

which is clearly C^∞.

Example 3.5. Let us show that a C^∞ function $f(x,y)$ on $\mathbb{R}^2$ restricts to a C^∞-function on S^1. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a,b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a,b) = a$ and $y(a,b) = b$. Suppose that we can show that x and y restrict to C^∞-functions on S^1. Then the inclusion map $i : S^1 \to \mathbb{R}^2$, given by $i(p) = (x(p), y(p))$ is C^∞ on S^1, and so the composition $f|_{S^1} = f \circ i$ with be C^∞ on S^1 too.

Consider first the function x. We use the following atlas (U_i, ϕ_i) for S^1, where

$$\begin{aligned}U_1 &= \{(a,b) \in S^1 \mid b > 0\} \qquad & \phi_1(a,b) &= a \\ U_2 &= \{(a,b) \in S^1 \mid b < 0\} \qquad & \phi_2(a,b) &= a \\ U_3 &= \{(a,b) \in S^1 \mid a > 0\} \qquad & \phi_3(a,b) &= b \\ U_4 &= \{(a,b) \in S^2 \mid a < 0\} \qquad & \phi_4(a,b) &= b\end{aligned}$$

Since x is a coordinate function on U_1 and U_2, it is a coordinate function on $U_1 \cup U_2$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1} : \phi_3(U_3) \to \mathbb{R}$.

$$(x \circ \phi_3^{-1})(b) = x\left(\sqrt{1-b^2}, b\right) = \sqrt{1-b^2}.$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1-b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3. On U_4, we have

$$(x \circ \phi_4^{-1})(b) = x\left(-\sqrt{1-b^2}, b\right) = -\sqrt{1-b^2}.$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1. The proof that y is C^∞ on S^1 is similar.

Example 3.6. Let S^2 be the unit sphere with its smooth structure given in Example (2.12). Let's construct a smooth function on S^2. First note that

$$\phi_1(U_{16}) = \{(b,c) \in \mathbb{R}^2 \mid b^2 + c^2 < 1 \text{ and } c < 0\} \qquad \text{and} \qquad \phi_6(U_{16}) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \text{ and } 0 < a\}.$$

Let $f : \phi_1(U_{16}) \to \mathbb{R}^2$ be given by

$$f(b,c) = b^2 + c^2.$$

Let's pullback $f : \phi_1(U_{16}) \to \mathbb{R}^2$ to $\phi_{16}^*(f) : \phi_6(U_{16}) \to \mathbb{R}^2$ using the transition function ϕ_{16}, where

$$\begin{aligned}\phi_{16}(a,b) &= \phi_1 \circ \phi_6^{-1}(a,b) \\ &= \phi_1\left(a, b, \sqrt{1-b^2-a^2}\right) \\ &= \left(b, \sqrt{1-b^2-a^2}\right).\end{aligned}$$

We have,

$$\begin{aligned}\phi_{16}^*(f)(a,b) &= (f \circ \phi_{16})(a,b) \\ &= f\left(b, \sqrt{1-b^2-a^2}\right) \\ &= 1 - a^2.\end{aligned}$$

3.4.1 Diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B_1(0)$

Let $\beta : \mathbb{R}^n \to B_1(0)$ be given by

$$x := (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right) := \beta(x)$$

for all $x \in \mathbb{R}^n$. Then β is a diffeomorphism from $\mathbb{R}^n$ to $B_1(0)$ with inverse given by

$$x := (x_1, \ldots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \ldots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) := \beta^{-1}(x)$$

for all $x \in B_1(0)$. Indeed, let us first check that $\beta(x) \in B_1(0)$:

$$\begin{aligned} \|\beta(x)\| &= \sqrt{\left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2 + \cdots + \left(\frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2} \\ &= \sqrt{\frac{\sum_{i=1}^n x_i^2}{1 + \sum_{i=1}^n x_i^2}} \\ &< \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2}} \\ &= 1. \end{aligned}$$

Thus $\beta(x) \in B_1(0)$. Next we check that β is smooth. This comes down to checking the component functions β_i are smooth:

$$x := (x_1, \ldots, x_n) \mapsto \frac{x_i}{\sqrt{1 + \sum_{i=1}^n x_i^2}} := \beta_i(x).$$

This follows from the fact that $1 + \sum_{i=1}^n x_i^2 > 0$. That β^{-1} is smooth follows by the same reasoning. Finally, checking that $\beta(\beta^{-1}(x)) = x$ is tedious but trivial:

$$\begin{aligned} \beta(\beta^{-1}(x)) &= \frac{1}{\sqrt{1 + \sum_{i=1}^n \left(\frac{x_i}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right)^2}} \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \cdots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) \\ &= \frac{1}{\sqrt{1 - \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i^2}} (x_1, \ldots, x_n) \\ &= (x_1, \ldots, x_n). \end{aligned}$$

3.5 Inverse Function Theorem

We say that a C^∞ map $F : N \to M$ is **locally invertible** or a **local diffeomorphism** at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism. Given n smooth functions $F_1, \ldots, F_n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F_1, \ldots, F_n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 3.1. *(Inverse function theorem for $\mathbb{R}^n$). Ler $F : W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant $det(J(F)_p)$ is not zero.*

Theorem 3.2. *(Inverse function theorem for manifolds). Let $F : N \to M$ be a C^∞*

4 Tangent Spaces

By definition, the **tangent space** to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map, called its **differential**, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map. In this sense, the differential of a map between manifolds is a generalization of the derivative of a map between Euclidean spaces.

4.1 The Tangent Space at a Point

Just as for $\mathbb{R}^n$, we define a **germ** of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; which scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a **derivation at a point** in a manifold M, or a **point-derivation** of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition 4.1. A **tangent vector** at a point p in a manifold M is a derivation at p.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by mapping (x, y) to $x^3 + y^3 + x + 1 := t$. Let $p = (x_0, y_0)$ be a point in $\mathbb{R}^2$. Then f induces a map $T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ by taking a derivation D in $T_p\mathbb{R}^2$ to the derivation f_*D in $T_{f(p)}\mathbb{R}$, where $(f_*D)(g) = D(g \circ f)$.

Example 4.1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^3 - x := t$. Then $\partial_x \mapsto (3x_0^2 - 1)\partial_t$.

Example 4.2. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \mapsto (f_1(x, y), f_2(x, y))$, where

$$\begin{aligned} f_1(x, y) &= x \\ f_2(x, y) &= \frac{xy^2}{y^2 + 1} \end{aligned}$$

Then

$$J_{(x_0,y_0)}(f_1, f_2) = \begin{pmatrix} 1 & 0 \\ \frac{y_0^2}{y_0^2+1} & \frac{2x_0y_0}{(y_0^2+1)^2} \end{pmatrix}$$

$$\frac{2x_0y_0}{(y_0^2 + 1)^2} = 0$$

Let M be an n-dimensional manifold and let p be a point in M. We describe T_pM in another way. Let $\mathcal{P}_p$ be the set of paths through p:

$$\mathcal{P}_p := \{\gamma : (-a, a) \to M \mid \gamma \text{ is } C^\infty \text{ and } \gamma(0) = p\}.$$

We define an equivalence relation on $\mathcal{P}_p$ as follows: we say $\gamma_1 \sim \gamma_2$ if there exist a chart (U, ϕ) centered at p such that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Here, $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are paths in $\mathbb{R}^n$.

Remark. This is independent of the choice of chart. If (V, ψ) is another chart centered at p, then

$$\begin{aligned} (\psi \circ \gamma_1)' &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_1)' \\ &= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_1)' \\ &= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_2)' \\ &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\ &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\ &= (\psi \circ \gamma_2)' \end{aligned}$$

Here, $(\psi \circ \varphi^{-1})'$ is the Jacobian.

Definition 4.2. The tangent space at p in M is

$$T_pM := \mathcal{P}_p / \sim .$$

Example 4.3. Let $M = \{(\cos\alpha, \sin\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$ be the cylinder. Define the two charts (U, φ) and (M, ψ) where $U = M \cap \left\{ (\frac{-\pi}{,2}, \frac{\pi}{2}) \times (\frac{-\pi}{,2}, \frac{\pi}{2}) \times \mathbb{R} \right\}$ and

$$\varphi\,(\cos\alpha, \sin\alpha, \beta) = (\alpha, \beta) \qquad \text{and} \qquad \psi\,(\cos\alpha, \sin\alpha, \beta) = (\cos\alpha, \beta).$$

Now let γ_1 and γ_2 be two paths in M given by

$$\gamma_1(t) = (\cos(t^2), \sin(t^2), t) \qquad \text{and} \qquad \gamma_2(t) = (0, 1, t).$$

Using the chart (U, φ), we have

$$(\varphi \circ \gamma_1)(t) = (t^2, t) \qquad \text{and} \qquad (\varphi \circ \gamma_2)(t) = \left(\frac{\pi}{2}, t\right).$$

Therefore

$$(\varphi \circ \gamma_1)'(0) = (0, 1) = (\varphi \circ \gamma_2)'(0)$$

and so $\gamma_1 \sim \gamma_2$.

4.2 Partial Derivatives

On a manifold M of dimension n, let (U, ϕ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, ϕ has n components $\phi_1, \ldots, \phi_n$. This means that if $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$, then $\phi_i = x_i \circ \phi$. For $p \in U$, we define the **partial derivative of f with respect to ϕ_i**, denoted $\partial_{\phi_i} f$, to be

$$\partial_{\phi_i} |_p \, f := (\partial_{\phi_i} f)(p) := \partial_{x_i}(f \circ \phi^{-1})(\phi(p)) := \partial_{x_i} |_{\phi(p)} \, (f \circ \phi^{-1}).$$

Example 4.4. Consider the projective plane $\mathbb{P}^2(\mathbb{R})$. Use the chart (U, ϕ) where

$$U = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2(\mathbb{R}) \mid a_0 \neq 0\} \qquad \phi(a_0 : a_1 : a_2) = \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right).$$

Then

$$\begin{aligned}
\phi_1(a_0 : a_1 : a_2) &= (x_1 \circ \phi)(a_0 : a_1 : a_2) \\
&= x_1(\phi(a_0 : a_1 : a_2)) \\
&= x_1\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right) \\
&= \frac{a_1}{a_0}.
\end{aligned}$$

Similarly, $\phi_2(a_0 : a_1 : a_2) = a_2/a_0$.

4.2.1 Polar Coordinates

Consider the following smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\theta &= \arctan\left(\frac{y}{x}\right)
\end{aligned}$$

Then

$$\begin{aligned}
dr &= \frac{x}{\sqrt{x^2 + y^2}} dx + \frac{y}{\sqrt{x^2 + y^2}} dy \\
d\theta &= \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy
\end{aligned}$$

Then

$$\begin{aligned}
rdrd\theta &= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy\right)\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) \\
&= (xdx + ydy)\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) \\
&= \frac{1}{x^2+y^2}(xdx+ydy)(-ydx+xdy) \\
&= \frac{1}{x^2+y^2}\left(-xydxdx + x^2dxdy - y^2dydx + xydydy\right) \\
&= \frac{1}{x^2+y^2}\left(x^2dxdy - y^2dydx\right) \\
&= \frac{1}{x^2+y^2}\left(x^2dxdy + y^2dxdy\right) \\
&= \frac{x^2+y^2}{x^2+y^2}dxdy \\
&= dxdy.
\end{aligned}$$

Therefore, we can integrate the Gaussian as follows:

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}rdrd\theta.$$

The inverse map is given by

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

4.3 Immersion, Embedding, Submersion

Let $F : N \to M$ be a C^∞ map and let p be a point in N. Then

1. F is called an **immersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is injective.
2. F is called an **immersion** if it is an immersion at every point in N.
3. F is called a **submersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective.
4. F is called an **submersion** if it is an submersion at every point in N.

Example 4.5. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$i(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi(a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) = (a_1, \ldots, a_m).$$

Example 4.6. If U is an open subset of a manifold M, then the inclusion $i : U \to M$ is both an immersion and submersion. This example shows in particular that a submersion need not be onto.

4.3.1 Critical Point

Definition 4.3. Let $F : N \to M$ be a C^∞ map, p a point in N, and q a point in M. Then

1. We say p is a **critical point** of F if $F_{*,p}$ is not surjective.
2. We say q is a **critical value** of F if the set $F^{-1}(q) := \{p \in N \mid F(p) = q\}$ contains a critical point.

Theorem 4.1. *Let M be a manifold and let f be a C^∞ function. Then the set of critical values of f has measure zero.*

Measure Theory on $\mathbb{R}$

Let μ be the Lebesgue measure on $\mathbb{R}$. Recall that

- If $I = (a, b)$, then $\mu(I) = b - a$.
- If I and J are disjoint intervals, then $\mu(I \cup J) = \mu(I) + \mu(J)$.
- A set $E \subset \mathbb{R}$ has measure 0 if for all $\varepsilon > 0$, you can cover E by a union of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\mu\left(\bigcup_n I_n\right) < \varepsilon$.

Example 4.7. Let $E = \{0, 1\}$. For each $n \in \mathbb{N}$, define the set

$$A_n := \left(\frac{-1}{4n}, \frac{1}{4n}\right) \cup \left(1 - \frac{1}{4n}, 1 + \frac{1}{4n}\right)$$

Then for each $n \in \mathbb{N}$, A_n is a disjoint union of intervals which covers E and

$$\begin{aligned}\mu(A_n) &= \frac{1}{4n} - \frac{-1}{4n} + 1 + \frac{1}{4n} - \left(1 - \frac{1}{4n}\right) \\ &= \frac{1}{n}.\end{aligned}$$

As $n \to \infty$, $\mu(A_n) \to 0$. Thus, E has measure 0.

Example 4.8. Let $M = \{(x, x + \sin(x)) \mid x \in \mathbb{R}\}$ and let $\pi : M \to \mathbb{R}$ be the projection onto the y-axis map, given by $(x, x + \sin(x)) \mapsto x + \sin(x)$.

Definition 4.4. A critical point is **degenerate** if the associated Hessian matrix is **singular** (i.e. has determinant equal to 0).

Example 4.9. Let $M = \{(\cos\theta, \theta, \sin\theta + 2) \mid \theta \in \mathbb{R}\}$ and $N = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Note that M is homeomorphic to $\mathbb{R}$ and N is homeomorphic to $\mathbb{R}^2$. Let $\varphi : M \to N$ be the projection map, given by $(\cos\theta, \theta, \sin\theta + 2) \mapsto (\cos\theta, \theta, 0)$. Let $\{(M, \psi_M)\}$ be an atlas on M and $\{(N, \psi_N)\}$ be an atlas on N where

$$\psi_M(\cos\theta, \theta, \sin\theta + 2) = \theta \qquad \text{and} \qquad \psi_N(x, y, 0) = (x, y)$$

What are the coordinates of $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, 2 + \frac{\sqrt{2}}{2}\right) \in M$? Then answer is $\frac{\pi}{4}$.

Theorem 4.2. *Let f be a continuous function*

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$$

A critical point is **degenerate** if the associated Hessian matrix is **singular**

4.4 Tangent Bundle

Let M be an n-dimensional manifold. The **Tangent Bundle** of M is

$$TM := \bigcup_{p\in M} T_pM.$$

Let $\mathcal{A} := \{(U_i, \phi_i)\}$ be an atlas for M. Then an atlas for TM is given by

$$\mathcal{A}_T := \{(U_i \times \mathbb{R}^n, \phi_i \times \mathrm{id})\}$$

Thus, if we denote $\Phi_i := \phi_i \times \mathrm{id}$. Then $\Phi_i : U_i \times \mathbb{R}^n \to \mathbb{R}^{2n}$ and we think of $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ as the local coordinates of TM, where $(x_1, \ldots, x_n)$ is a point in M and $(y_1, \ldots, y_n)$ is a vector.

Example 4.10. The tangent bundle of the circle S^1 is diffeomorphic to the cylinder.

Remark. There exist a canonical map $\pi : TM \to M$ given by $(p, v) \mapsto v$.

Definition 4.5. A **vector field** is a smooth function ω from M to TM such that $\pi \circ \omega = \mathrm{id}$.

Remark. Intuitively, a vector field is the data of a vector at every point in M.

A vector field ω comes with two gadgets. The first gadget is called a 1-**parameter** flow and is denoted ω^t. The second gadget is called a **differential operator** and is denoted L_ω.

4.5 Vector Bundles

On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi : TM \to M$, given by $\pi(p, v) = p$, makes TM into a C^∞ **vector bundle** over M, which we now define.

Given any map $\pi : E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the **fiber** at p. The fiber at p is often written as E_p. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be **fiber-preserving** if $\phi(E_p) \subset E'_p$ for all $p \in M$. Equivalently, this says that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank** r if

1. Each fiber E_p has the structure of a vector space of dimension r.

2. For each $p \in M$, there are an open neighborhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for every $q \in U$ the restriction

$$\phi\,|_{E_q} : E_q \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. Such an open set U is called a **trivializing open set** for E, and ϕ is called a **trivialization** of E over U.

The collection $\{(U, \phi)\}$, with $\{U\}$ and open cover of M, is called a **local trivialization** for E, and $\{U\}$ is called a **trivializing open cover** of M for E. A C^∞**vector bundle of rank** r is a triple (E, M, π) consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the **total space** of the vector bundle and M the **base space**. By abuse of language, we say that E is a **vector bundle over** M. Properly speaking, the tangent bundle of a manifold M is a triple (TM, M, π), and TM is the total space of the tangent bundle. In common usage, TM is often referred to as the tangent bundle.

4.5.1 Gluing

Given two local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ and $\phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\phi_j \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R}).$$

These are called **transition functions** of the bundle, and since they come from $\phi_j \circ \phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$, as well as the cocycle condition

$$g_{ij} g_{jk} g_{ki} = \mathrm{id}\,|_{U_i \cap U_j \cap U_k}$$

Example 4.11. To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$. Then glue these together on the double overlaps $\{U_i \cap U_j\}$ via functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.

Example 4.12. Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection to the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r, called the **product bundle** of rank r over M. The vector space structure on the fiber $\pi^{-1}(p) = \{(p, v) \mid v \in \mathbb{R}^r\}$ is the obvious one:

$$(p, u) + (p, v) = (p, u + v) \text{ and } b \cdot (p, v) = (p, bv) \text{ for } b \in \mathbb{R}.$$

A local trivialization on $M \times \mathbb{R}$ is given by the identity map $1_{M \times \mathbb{R}} : M \times \mathbb{R} \to M \times \mathbb{R}$. For example, the infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle.

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles, possibly of different ranks. A **bundle map** from E to F is a pair of maps $(f, \tilde{f})$, where $f : M \to N$ and $\tilde{f} : E \to F$ such that

1. The diagram

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative.

2. $\widetilde{f}$ is linear one each fiber; i.e. $\widetilde{f} : E_p \to F_{f(p)}$ is a linear map of vector spaces for each $p \in M$.

The collection of all vector bundles together with bundle maps between them forms a category.

Example 4.13. A smooth map $f : N \to M$ of manifolds induces a bundle map $(f, \widetilde{f})$, where $\widetilde{f} : TN \to TM$ is given by

$$\widetilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subset TM$$

for all $v \in T_pN$. This gives rise to a covariant functor T from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps: to each manifold M, we associate its tangent bundle TM, and to each C^∞ map $f : N \to M$ of manifolds, we associate the bundle map $T(f) = (f, \widetilde{f})$.

If E and F are two vector bundles over the same manifold M, then a bundle map from E to F over M is a bundle map in which the base map is the identity 1_M. For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundles maps over M. In this category it makes sense to speak of an isomorphism of vector bundles over M. Any vector bundle over M is isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a **trivial bundle**.

Example 4.14. Let

$$M = \left\{ (x, y) \in \mathbb{R}^2 \mid \det \begin{pmatrix} x & 1-y \\ y & x \end{pmatrix} = 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}.$$

This can be realized as the circle of radius $\frac{1}{2}$ centered at the point $(0, 1/2)$ in the plane. There is natural vector bundle associated to M. Indeed, to each point $(x, y) \in M$, let $E_p := \operatorname{Ker} \begin{pmatrix} x & 1-y \\ y & x \end{pmatrix}$. Note that E_p is nonzero since $\det \begin{pmatrix} x & 1-y \\ y & x \end{pmatrix} = 0$.

4.5.2 Smooth Sections

A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = 1_M$. This condition means precisely that for each p in M, s maps p into the fiber E_p. We say that a section is **smooth** if it is smooth as a map from M to E. A **vector field** X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point p in M. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is **smooth** if it is smooth as a map from M to TM.

Example 4.15. The formula

$$X_{(x,y)} = -y\partial_x + x\partial_y$$

defines a smooth vector field on $\mathbb{R}^2$.

4.5.3 Whitney Sum

Let (E, M, π) and (E', M, π') be two vector bundles. We can construct a new vector bundle called the **Whitney sum**, given by $(\pi, \pi') : E \oplus E' \to M$.

Example 4.16. Suppose $E = L \oplus L'$ where L and L' are line bundles. Then we can make a new bundle called $\det(E)$.

Throughout this section, let R be a commutative ring.

Definition 4.6. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally ringed** R-**space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ to be the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Remark. As every ring has a unique structure as a $\mathbb{Z}$-algebra, we simply say **(locally) ringed space** instead of **(locally)** $\mathbb{Z}$-**ringed space**. Usually we will denote a (locally) R-ringed space by $(X, \mathcal{O}_X)$ simply by X.

Example 4.17. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by C_X^∞ the sheaf of C^∞-functions, i.e.

$$C_X^\infty(U) := \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty \text{ function}\}.$$

Then C_X^∞ is a sheaf of $\mathbb{R}$-algebras.

5 Differential Forms

5.1 Differential 1-Forms

Let M be a smooth manifold and p a point in M. The **cotangent space** of M at p, denoted by T_p^*M, is defined to be the dual space of the tangent space T_pM :

$$T_p^*M = (T_pM)^\vee = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}).$$

An element of the cotangent space T_p^*M is called a **covector** at p. Thus, a covector ω_p at p is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A **covector field**, also called a **differential 1-form** or more simply a **1-form**, on M is a function ω that assigns to each point p in M a covector ω_p at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p. There are many reasons for the great utility of differential forms in manifold theory, among which is the fact that they can be pulled back under a map. This is in contrast to vector fields, which in general cannot be pushed forward under a map.

5.1.1 The Differential of a Function

If f is a C^∞ real-valued function on a manifold M, its **differential** is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$, we have

$$(df)_p(X_p) = X_pf.$$

Instead of $(df)_p$ we also write $df\,|_p$ for the value of the 1-form df at p.

6 Bump Functions and Partitions of Unity

A partition of unity on a manifold is a collection of nonnegative functions that sum to 1. Usually one demands in addition that the partition of unity be **subordinate** to an open cover $\{U_i\}_{i\in I}$. What this means is that the partition of unity $\{\rho_i\}_{i\in I}$ is indexed over the same set as the open cover $\{U_i\}_{i\in I}$, and for each i in the index I, the support of ρ_i is contained in U_i.

The existence of a C^∞ partition of unity is one of the most technical tools in the theory of C^∞ manifolds. It is the single feature that makes the behavior of C^∞ manifolds so different from that of real-analytic or complex manifolds. In this section we construct C^∞ bump functions on any manifold and prove the existence of a C^∞ partition of unity on a compact manifold. The proof of the existence of a C^∞ partition of unity of a general manifold is more technical and is postponed.

A partition of unity is used in two ways:

1. to decompose a global object on a manifold into a locally finite sum of local objects on the open sets U_i of an open cover.
2. to patch together local objects on the open sets U_i into a global object on the manifold.

Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local coordinates on a manifold, there may be no global coordinates.

6.1 C^∞ Bump Functions

The **support** of a real-valued function f on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$:

$$\mathrm{supp} f := \overline{f^{-1}(\mathbb{R}\backslash\{0\})}.$$

Let q be a point in M, and U a neighborhood of q. By a **bump function at q supported in U** we mean any continuous nonnegative function ρ on M that is 1 in a neighborhood of q with $\mathrm{supp}\rho \subset U$.

Example 6.1. The support of the function $f : (-1, 1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is the open interval $(-1, 1)$, and not the closed interval $[-1, 1]$, because the closure of $f^{-1}(\mathbb{R}\backslash\{0\})$ is taken in the domain $(-1, 1)$ and not in $\mathbb{R}$.

Recall from Example (1.2) the smooth function f defined on $\mathbb{R}$ by the formula

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

We wish to build a smooth bump function function from f. The main challenge in building a smooth bump function from f is to construct a smooth version of a step function. We seek $g(t)$ by dividing $f(t)$ by a positive function $\ell(t)$, for the quotient $f(t)/\ell(t)$ will be zero for $t \leq 0$. The denominator $\ell(t)$ should be a positive function that agrees with $f(t)$ for $t \geq 1$, for then $f(t)/\ell(t)$ will be identically 1 for $t \geq 1$. The simplest way to construct such an $\ell(t)$ is to add to $f(t)$ a nonnegative function that vanishes for $t \geq 1$. One such nonnegative function is $f(1-t)$. This suggests that we take $\ell(t) = f(t) + f(1-t)$ and consider

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}.$$

Given two positive real numbers $a < b$, we make a linear change of variables to map $[a^2, b^2]$ to $[0, 1]$:

$$x \mapsto \left(\frac{x - a^2}{b^2 - a^2} \right):$$

Let $h : \mathbb{R} \to [0, 1]$ be given by

$$h(x) = g\left(\frac{x - a^2}{b^2 - a^2} \right).$$

Then h is a C^∞ step function such that

$$h(x) = \begin{cases} 0 & \text{if } x \leq a^2 \\ 1 & \text{if } x \geq b^2. \end{cases}$$

Now replace x by x^2 to make the function symmetric in x: $k(x) = h(x^2)$. Finally, set

$$\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2} \right).$$

This $\rho(x)$ is a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. For any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

It is easy to extend the construction of a bump function from $\mathbb{R}$ to $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0}$ in $\mathbb{R}^n$ that is 1 on the closed balled $\overline{B_a(\mathbf{0})}$ and has support in the closed ball $\overline{B_b(\mathbf{0})}$, set

$$\sigma(x) = \rho\left(\|x\|\right) = 1 - g\left(\frac{x_1^2 + \cdots + x_n^2 - a^2}{b^2 - a^2} \right).$$

As a composition of C^∞ functions, σ is C^∞. To get a C^∞ bump function at q in $\mathbb{R}^n$, take $\sigma(x - q)$.

6.1.1 Extending C^∞ Bump Functions to M

Now suppose we have manifold M, and open subset U of M, and a point q in U. Choose a chart (ϕ_i, U_i) such that $U_i \subseteq U$ and $\phi_i(U_i) \cong B_b(\phi(q))$, for some $b > 0$, and choose an open neighborhood V_i of p such that $V_i \subseteq U_i$ and $\phi_i(V_i) \cong B_a(\phi(q))$ for some $a < b$. We've shown now to construct a bump function ρ at $\phi_i(q)$ such that $\rho(x) = 1$ for all $x \in B_a(\phi_i(q))$ and such that $\rho(x) = 0$ outside $B_b(\phi_i(q))$. Now we pull back ρ by ϕ to get a bump function on U_i:

$$(\phi_i^* \rho)(q') = \rho(\phi_i(q')) \text{ for all } q' \in U_i.$$

Finally we extend this function a bump function $\widetilde{\rho}$ on M by setting

$$\widetilde{\rho}(q') = \begin{cases} (\phi_i^* \rho)(q') & \text{if } q' \in U_i \\ 0 & \text{if } q' \notin U_i \end{cases}$$

Let us show that this function is C^∞. For $q' \in U_i$, we simply choose the chart (ϕ_i, U_i). Then

$$(\widetilde{\rho} \circ \phi_i^{-1})(x) = (\phi^* \rho)(\phi_i^{-1}(x)) = \rho(x),$$

shows that $\widetilde{\rho}$ is C^∞ in (ϕ_i, U_i). For $q' \notin U_i$, we choose a chart (ϕ_j, U_j) such that $U_i \cap U_j = \emptyset$. Then

$$(\widetilde{\rho} \circ \phi_j^{-1})(x) = \widetilde{\rho}(\phi_j^{-1}(x)) = 0.$$

Thus, the function $\widetilde{\rho}$ we constructed is C^∞ everywhere.

6.1.2 C^∞ Extension of a Function

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on M; an example is the function $\sec x$ on the open interval $(-\pi/2, \pi/2)$ in $\mathbb{R}$. However, if we require that the global function on M agree with the given function only on some neighborhood of a point in U, then a C^∞ extension is possible.

Proposition 6.1. *(C^∞ extension of a function) Suppose f is a C^∞ function defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ function $\widetilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p.*

Proof. Choose a C^∞ bump map $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p. Define

$$\widetilde{f}(q) = \begin{cases} \rho(q)f(q) & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

As the product of two C^∞ functions on U, $\widetilde{f}$ is C^∞ on U. If $q \notin U$, then $q \notin \operatorname{supp}\rho$, and so there is an open set containing q on which $\widetilde{f}$ is 0, since $\operatorname{supp}\rho$ is closed. Therefore $\widetilde{f}$ is also C^∞ at every point $q \notin U$. Finally, since $\rho = 1$ on V, the function $\widetilde{f}$ agrees with f on V. □

Remark. This proposition says that the natural map $C^\infty(M) \to C^\infty_p(M)$ is surjective. Thus, every germ in $C^\infty_p(M)$ can be represented by (f, M), where f is a C^∞ function on M.

6.2 Partitions of Unity

If $\{U_i\}_{i \in I}$ is a finite open cover of M, a C^∞ **partition of unity subordinate to** $\{U_i\}_{i \in I}$ is a collection of nonnegative functions $\{\rho_i : M \to \mathbb{R}\}$ such that $\operatorname{supp}\rho_i \subset U_i$ and

$$\sum_{i \in I} \rho_i = 1. \tag{4}$$

When I is an infinite set, for the sum in (4) to make sense, we will impose a **local finiteness** condition. A collection $\{A_\alpha\}$ of subsets of a topological space X is said to be **locally finite** if every point x in X has a neighborhood that meets only finitely many of the sets A_α. In particular, every point $x \in X$ is contained in only finitely many of the A_α's.

Example 6.2. Let $U_{r,n}$ be the open interval $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ on the real line $\mathbb{R}$. Then the open cover $\{U_{r,n} \mid r \in \mathbb{Q},\ n \in \mathbb{N}\}$ of $\mathbb{R}$ is not locally finite.

Definition 6.1. A C^∞ **partition of unity** on a manifold is a collection of nonnegative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i \in I}$ such that

1. The collection of supports, $\{\operatorname{supp}\rho\}_{i \in I}$, is locally finite,
2. $\sum_{i \in I} \rho_i = 1$.

Given an open cover $\{U_i\}_{i \in I}$ of M, we say that a partition of unity $\{\rho_i\}_{i \in I}$ is **subordinate to the open cover** $\{U_i\}_{i \in I}$ if $\operatorname{supp}\rho_i \subset U_i$ for every $i \in I$.

Remark. Since the collection of supports, $\{\operatorname{supp}\rho_i\}_{i \in I}$, is locally finite, every point q lies in only finitely many of the sets $\operatorname{supp}\rho_i$. Hence $\rho_i(q) \neq 0$ for only finitely many i. It follows that the sum $\sum_{i \in I} \rho_i(q)$ is finite.

Example 6.3. Let U and V be the open intervals $(-\infty, 2)$ and $(-1, \infty)$ in $\mathbb{R}$ respectively, and let ρ_V be a smooth step function which is equal to 0 on $(-\infty, 0)$ and equal to 1 on $(1, \infty)$. Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp}\rho_V \subset V$ and $\operatorname{supp}\rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

6.3 Existence of a Partition of Unity

In this subsection we begin a proof of the existence of a C^∞ partition of unity on a manifold. Because the case of a compact manifold is somewhat easier and already has some of the features of the general case, for pedagogical reasons we give a separate proof for the compact case.

Lemma 6.1. *If $\rho_1, \ldots, \rho_m$ are real-valued funcitons on a manifold M, then*

$$supp\left(\sum_{i=1}^m \rho_i\right) \subset \bigcup_{i=1}^m supp\rho_i.$$

Proof. Suppose $q \in \operatorname{supp}\left(\sum_{i=1}^m \rho_i\right)$. Thus $\sum_{i=1}^m \rho_i(q) \neq 0$. In particular, we must have $\rho_i(q) \neq 0$ for some $i = 1, \dots, m$. Thus, $q \in \operatorname{supp}\rho_i \subset \bigcup_{i=1}^m \operatorname{supp}\rho_i$. □

Proposition 6.2. *Let M be a compact manifold and $\{U_i\}_{i \in I}$ an open cover of M. There exists a C^∞ partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$.*

Proof. For each $q \in M$, find an open set U_i containing q from the given cover and let ψ_q be a C^∞ bump function at q supported in U_i. Because $\psi_q(q) > 0$, there is a neighborhood W_q of q on which $\psi_q > 0$. By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \dots, W_{q_m}\}$. Let $\psi_{q_1}, \dots, \psi_{q_m}$ be the corresponding bump functions. Then $\psi := \sum_{j=1}^m \psi_{q_j}$ is positive at every point q in M because $q \in W_{q_j}$ for some i. Define

$$\varphi_j = \frac{\psi_{q_j}}{\psi}, \quad j = 1, \dots, m.$$

Clearly $\sum_{j=1}^m \varphi_j = 1$. Moreover, since $\psi > 0$, $\varphi_j(q) \neq 0$ if and only if $\psi_{q_j}(q) \neq 0$, so

$$\operatorname{supp}\varphi_j = \operatorname{supp}\psi_{q_j} \subset U_i$$

for some $i \in I$. This shows that $\{\varphi_i\}$ is a partition of unity such that for every j, $\operatorname{supp}\varphi_j \subset U_i$ for some $i \in I$.

The next step is to make the index set of the partition of unity the same as that of the open cover. For each $j = 1, \dots, m$, choose $\tau(j) \in I$ to be an index such that

$$\operatorname{supp}\varphi_j \subset U_{\tau(j)}.$$

We group the collection of functions $\{\varphi_j\}$ into subcollections according to $\tau(j)$ and define for each $i \in I$,

$$\rho_i = \sum_{\tau(j)=i} \varphi_j;$$

if there is no j for which $\tau(j) = i$, the sum is empty and we define $\rho_i = 0$. Then

$$\sum_{i \in I} \rho_i = \sum_{i \in I} \sum_{\tau(j)=i} \varphi_j = \sum_{j=1}^m \varphi_j = 1.$$

Moreover by Lemma (6.1),

$$\operatorname{supp}\rho_i \subset \bigcup_{\tau(j)=i} \operatorname{supp}\varphi_j \subset U_i.$$

So $\{\rho_i\}$ is a partition of unity subordinate to $\{U_i\}$. □

7 Integration on Manifolds

7.1 Riemann Integral of a Function on $\mathbb{R}^n$

A **closed rectangle** in $\mathbb{R}^n$ is a Cartesian product $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ of closed intervals in $\mathbb{R}$, where $a_i, b_i \in \mathbb{R}$. Let $f : R \to \mathbb{R}$ be a bounded function defined on a closed rectangle R. The **volume vol**(R) of the closed rectangle R is defined to be

$$\operatorname{vol}(R) := \prod_{i=1}^n (b_i - a_i).$$

A **partition** of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \dots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A **partition** of the rectangle R is a collection $P = \{P_1, \dots, P_n\}$, where each P_i is a partition of $[a_i, b_i]$. The partition P divides the rectangle R into closed subrectangles, which we denote by R_j.

We define the **lower sum** and the **upper sum** of f with respect to the partition P to be

$$L(f, P) := \sum_{R_j} \left(\inf_{R_j} f\right) \operatorname{vol}(R_j), \qquad U(f, P) := \sum_{R_j} \left(\sup_{R_j} f\right) \operatorname{vol}(R_j),$$

where each sum runs over all subrectangles of the partition P. For any partition P, clearly $L(f, P) \le U(f, P)$. In fact, more is true: for any two partitions P and P' of the rectangle R,

$$L(f, P) \le U(f, P'),$$

which we show next.

A partition $P' = \{P'_1, \ldots, P'_n\}$ is a **refinement** of the partition $P = \{P_1, \ldots, P_n\}$ if $P_i \subset P'_i$ for all $i = 1, \ldots, n$. If P' is a refinement of P, then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of P', and it is easily seen that

$$L(f, P) \leq L(f, P'),$$

because if $R'_{jk} \subset R_j$, then $\inf_{R_j} f \leq \inf_{R'_{jk}} f$. Similarly, if P' is a refinement of P, then

$$U(f, P') \leq U(f, P).$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i = P_i \cup P'_i$, and thus

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions P of R is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions of R. We define these two numbers to be the **lower integral** $\underline{\int}_R f$ and the **upper integral** $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f := \sup_P L(f, P), \qquad \overline{\int}_R f := \inf_P L(f, P).$$

Definition 7.1. Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be **Riemann integrable** if $\underline{\int}_R f = \overline{\int}_R f$; in this case, the Riemann integral of f is this common value, denoted by $\int_R f(x) dx_1 \cdots dx_n$, where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$.

Example 7.1. Let f be a bounded monotone increasing function on $[-1, 1]$. Then f is Riemann integrable. Indeed, consider the partition $P_n = \{p_0 < p_1 < \cdots < p_{2n-1} < p_{2n}\}$ where $p_i = -1 + i/3$. Then

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \frac{1}{n} \sum_{i=1}^{2n} f(-1 + i/3) - \frac{1}{n} \sum_{i=0}^{2n-1} f(-1 + i/3) \\
&= \frac{1}{n} \left(\sum_{i=1}^{2n} f(-1 + i/3) - \sum_{i=0}^{2n-1} f(-1 + i/3) \right) \\
&= \frac{1}{n} \left(f(1) - f(-1) \right),
\end{aligned}$$

which tends to 0 as $n \to \infty$.

If $f : A \subset \mathbb{R}^n \to \mathbb{R}$, then the **extension of** f **by zero** is the function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Now suppose $f : A \to \mathbb{R}$ is a bounded function on a bounded set A in $\mathbb{R}^n$. Enclose A in a closed rectangle R and define the Riemann integral of f over A to be

$$\int_A f(x) dx_1 \cdots dx_n = \int_R \widetilde{f}(x) dx_1 \cdots dx_n$$

if the right-hand side exists. In this way we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$. The **volume** $\mathrm{vol}(A)$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1 dx_1 \cdots dx_n$ if the integral exists.

7.2 Integrability Conditions

In this section we describe some conditions under which a function defined on an open subset of $\mathbb{R}^n$ is Riemann integrable.

Definition 7.2. A set $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a countable cover $\{R_i\}_{i=1}^{\infty}$ of A by closed rectangles R_i such that $\sum_{i=1}^{\infty} \mathrm{vol}(R_i) < \varepsilon$.

Theorem 7.1. *(Lebesgue's theorem) A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set $Disc(\widetilde{f})$ of discontinuities of the extended function $\widetilde{f}$ has measure zero.*

Proposition 7.1. *If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.*

Proof. Being continuous on a compact set, the function f is bounded. Being compact, the set $\mathrm{supp}(f)$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\widetilde{f}$ is continuous.

Since $\widetilde{f}$ agrees with f on U, the extended function $\widetilde{f}$ is continuous on U. It remains to show that $\widetilde{f}$ is continuous on the complement of U in $\mathbb{R}^n$ as well. If $p \notin U$, then $p \notin \mathrm{supp}(f)$. Since $\mathrm{supp}(f)$ is a closed subset of $\mathbb{R}^n$, there is an open ball B containing p and disjoint from $\mathrm{supp}(f)$. On this open ball, $\widetilde{f} = 0$, which implies that $\widetilde{f}$ is continuous at $p \notin U$. Thus, $\widetilde{f}$ is continuous on $\mathbb{R}^n$. By Lebesgue's theorem, f is Riemann integrable on U. □

Example 7.2. The continuous function $f : (-1, 1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is defined on an open subset of finite length in $\mathbb{R}$, but it not bounded. The support of f is the open interval $(-1, 1)$, which is not compact. Thus, the function f does not satisfy the hypotheses of either Lebesgue's theorem or Proposition (7.1). Note that it is not Riemann integrable.

Definition 7.3. A subset $A \subset \mathbb{R}^n$ is called a **domain of integration** if it is bounded and its topological boundary $\mathrm{bd}(A)$ is a set of measure zero.

Proposition 7.2. *Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.*

Proof. Let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero. Since f is continuous on A the extension $\widetilde{f}$ is necessarily continuous at all interior points of A. Clearly, $\widetilde{f}$ is continuous at all exterior points of A also, because every exterior point has a neighborhood contained entirely in $\mathbb{R}^n \backslash A$, on which $\widetilde{f}$ is identically zero. Therefore, the set $\mathrm{Disc}(\widetilde{f})$ of discontinuities of $\widetilde{f}$ is a subset of $\partial(A)$, a set of measure zero. By Lebesgue's theorem, f is Riemann integrable. □

7.3 The Integral of an n-Form on $\mathbb{R}^n$

Once a set of coordinates $x_1, \ldots, x_n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 7.4. Let $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x_1, \ldots, x_n$. Its **integral** over a subset $A \subset U$ is defined to be the Riemann integral of $f(x)$:

$$\int_A \omega = \int_A f(x) dx_1 \wedge \cdots \wedge dx_n := \int_A f(x) dx_1 \cdots dx_n,$$

if the Riemann integral exists.

Example 7.3. If f is a bounded continuous function defined on a domain of integration A in $\mathbb{R}^n$, then the integral $\int_A f(x) dx_1 \wedge \cdots \wedge dx_n$ exists.

Let us see how the integral of an n-form $\omega = f dx_1 \wedge \cdots \wedge dx_n$ on an open subset $U \subset \mathbb{R}^n$ transforms under a change of variables. A change of variables on U is given by a diffeomorphism $T : \mathbb{R}^n \supset V \to U \subset \mathbb{R}^n$. Let $x_1, \ldots, x_n$ be the standard coordinates on U and $y_1, \ldots, y_n$ be the standard coordinates on V. Then $T_i := x_i \circ T$ is the ith component of T. We will assume that U and V are connected, and write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$dT_1 \wedge \cdots \wedge dT_n = \det(J(T))\, dy_1 \wedge \cdots \wedge dy_n.$$

Hence,

$$\begin{aligned}
\int_V T^*\omega &= \int_V (T^* f) T^* dx_1 \wedge \cdots \wedge T^* dx_n \\
&= \int_V (f \circ T) dT_1 \wedge \cdots \wedge dT_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \wedge \cdots \wedge dy_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \cdots dy_n.
\end{aligned}$$

On the other hand, the change-of-variables formula from advanced calculus gives

$$\int_U \omega = \int_U f dx_1 \cdots dx_n = \int_V (f \circ T) |\det(J(T))| dy_1 \cdots dy_n,$$

with an absolute-value sign around the Jacobian determinant. Hence,

$$\int_V T^*\omega = \pm \int_U \omega,$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative. In particular, the integral of a differential form is not invariant under all diffeomorphisms of V with U, but only under orientation-preserving diffeomorphisms.

7.4 Integral of a Differential Form over a Manifold

Integration of an n-form on $\mathbb{R}^n$ is not so different from integration of a function. Our approach to integration over a general manifold has several distinguishing features:

1. The manifold must be oriented.
2. On a manifold of dimension n, one can integrate only n-forms, not functions.
3. The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of M. Denote by $\Omega_c^k(M)$ the vector space of C^∞ k-forms with compact support on M. Suppose (U, ϕ) is a chart in this atlas. If $\omega \in \Omega_c^n(U)$ is an n-form with compact support on U, then because $\phi : U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega = \int_{\phi(U)} (\phi^{-1})^*\omega.$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\phi \circ \psi^{-1}\psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^*(\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega.$$

Thus, the integral $\int_U \omega$ on a chart U of the atlas is well defined, independent of the choice of coordinates on U. By linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a *finite* sum. Also since $\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}(\rho_\alpha) \cap \operatorname{supp}(\omega)$, $\operatorname{supp}(\rho_\alpha\omega)$ is a closed subset of the compact set $\operatorname{supp}(\omega)$. Hence, $\operatorname{supp}(\rho_\alpha\omega)$ is compact. Since $\rho_\alpha\omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{5}$$

For this integral to be well defined, we must show that it is independent of the choices of oriented atlas and partition of unity. Let $\{V_\beta, \psi_\beta\}$ be another oriented atlas of M specifying the orientation of M, and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\{(U_\alpha \cap V_\beta, \phi_\alpha|_{U_\alpha \cap V_\beta})\}$ and $\{(U_\alpha \cap V_\beta, \psi_\beta|_{U_\alpha \cap V_\beta})\}$ are two new atlases of M specifying the orientation of M, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega && \text{(because } \textstyle\sum_\beta \chi_\beta = 1\text{)} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega && \text{(these are finite sums)} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega,
\end{aligned}$$

where the last line follows from the fact that the support of $\rho_\alpha\chi_\beta$ is contained in $U_\alpha \cap V_\beta$. By symmetry, $\sum_\beta \int_{V_\beta} \chi_\beta\omega$ is equal to the same sum. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega,$$

proving that the integral (5) is well defined.

8 Quotients and Gluing

There are many important topological spaces (and manifolds) that are constructed by “identifying” pieces of spaces. This typically takes the form of gluing along open sets or passing to quotients by (reasonable) equivalence relations.

8.1 The Quotient Topology

Recall that an equivalence relation on a set X is a reflexive, symmetric, and transitive relation. The **equivalence class** $[x]$ of $x \in X$ is the set of all elements in X equivalent to x. An equivalence relation on X partitions X into disjoint equivalence classes. We denote the set of equivalence classes by $X/\sim$ and call this set the **quotient** of X by the equivalence relation $\sim$. There is a natural **projection map** $\pi : X \to X/\sim$ that sends $x \in X$ to its equivalence class $[x]$.

Assume now that X is a topological space. We define a topology on $X/\sim$ by declaring a set U in $X/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in X. Clearly, both the empty set $\emptyset$ and the entire quotient $X/\sim$ are open. Further, since

$$\pi^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i) \text{ and } \pi^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} \pi^{-1}(U_i),$$

the collection of open sets in $X/\sim$ is closed under arbitrary unions and finite intersections, and is therefore a topology. It is called the **quotient topology** on $X/\sim$. With this topology, $X/\sim$ is called the **quotient space** of X by the equivalence relation $\sim$. The way we defined the topology on $X/\sim$ makes the projection map π continuous.

8.1.1 Continuity of a Map on a Quotient

Suppose f is a map from X to Y and is constant on each equivalence class. Then it induces a map $\overline{f} : X/\sim \to Y$, given by $\overline{f}([x]) = f(x)$ where $x \in X$.

Proposition 8.1. *The induced map $\overline{f} : X/\sim \to Y$ is continuous if and only if the map $f : X \to Y$ is continuous.*

Proof. If $\overline{f}$ is continuous, then f is continuous since $f = \overline{f} \circ \pi$ is a composition of two continuous functions. Conversely, suppose f is continuous. Let V be an open set in Y. Then $f^{-1}(V) = \pi^{-1}\left(\overline{f}^{-1}(V)\right)$ is open in X. By the definition of quotient topology, $\overline{f}^{-1}(V)$ is open in $X/\sim$. Thus $\overline{f}$ is continuous since V was arbitary. $\square$

8.1.2 Identification of a Subset to a Point

If A is a subspace of a topological space X, we can define a relation $\sim$ on X by declaring

$$x \sim x \text{ for all } x \in X \text{ and } x \sim y \text{ for all } x, y \in A.$$

This is an equivalence relation on X. We say that the quotient space $X/\sim$ is obtained from X by **identifying** A **to a point**.

Example 8.1. Let I be the unit interval $[0,1]$ and $I/\sim$ be the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1$, given by $f(x) = e^{2\pi i x}$, assumes the same value at 0 and 1, and so induces a function $\overline{f} : I/\sim \to S^1$. Since f is continuous, $\overline{f}$ is continuous. As the continuous image of a compact set I, the quotient $I/\sim$ is compact. Thus $\overline{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. Hence it is a homeomorphism.

8.2 Open Equivalence Relations

An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection map $\pi : X \to X/\sim$ is open. In other words, the equivalence relation $\sim$ on X is open if and only if for every open set U in X, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U} [x]$$

of all points equivalent to some point of U is open.

Example 8.2. Let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 and let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map. Then π is not an open map. Indeed, let V be the open interval $(-2, 0)$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\},$$

which is not open in $\mathbb{R}$.

Given an equivalence relation $\sim$ on X, let R be the subset of $X \times X$ that defines the relation

$$R = \{(x, y) \in X \times X \mid x \sim y\}.$$

We call R the **graph** of the equivalence relation $\sim$.

Theorem 8.1. *Suppose $\sim$ is an open equivalence relation on a topological space X. Then the quotient space $X/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $X \times X$.*

Proof. There is a sequence of equivalent statements: R is closed in $X \times X$ iff $(X \times X)\backslash R$ is open in $X \times X$ iff for every $(x, y) \in (X \times X)\backslash R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \emptyset$ iff for every pair $x \not\sim y$ in X, there exist neighborhoods U of x and V of y in X such that no element of U is equivalent to an element of V iff for any two points $[x] \neq [y]$ in $X/\sim$, there exist neighborhoods U of x and V of y in X such that $\pi(U) \cap \pi(V) = \emptyset$ in $X/\sim$.

We now show that this last statement is equivalent to $X/\sim$ being Hausdorff. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $X/\sim$ containing $[x]$ and $[y]$ respectively, so $X/\sim$ is Hausdorff. Conversely, suppose $X/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $X/\sim$. Then there exist disjoint open sets A and B in $X/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$. Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $X/\sim$. □

Theorem 8.2. *Let $\sim$ be an open equivalence relation on a topological space X. If $\mathcal{B} = \{B_\alpha\}$ is a basis for X, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $X/\sim$.*

Proof. Since π is an open map, $\{\pi(B(\alpha)\}$ is a collection of open sets in $X/\sim$. Let W be an open set in $X/\sim$ and $[x] \in W$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that $x \in B \subset \pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B) \subset W$, which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary. *If $\sim$ is an open equivalence relation on a second-countable space X, then the quotient space is second-countable.*

8.3 Quotients by Group Actions

Many important manifolds are constructed as quotients by actions of groups on other manifolds, and this often provides a useful way to understand spaces that may have been constructed by other means. As a basic example, the Klein bottle will be defined as a quotient of $S^1 \times S^1$ by the action of a group of order 2. The circle as defined concretely in $\mathbb{R}^2$ is isomorphic to the quotient of $\mathbb{R}$ by additive translation by $\mathbb{Z}$.

Definition 8.1. Let X be a topological space and G a discrete group. A right action of G on X is **continuous** if for each $g \in G$ the action map $X \to X$ defined by $x \mapsto x.g$ is continuous (and hence a homeomorphism, as the action of g^{-1} gives an inverse). The action is **free** if for each $x \in X$ the stabilizer subgroup $\{g \in G \mid x.g = x\}$ is the trivial subgroup (in other words, $x.g = x$ implies $g = 1$). The action is **properly discontinuous** when it is continuous for the discrete topology on G and each $x \in X$ admits an open neighborhood U_x so that the G-translate $U_x.g$ meets U_x for only finitely many $g \in G$.

Proposition 8.2. *A right action of G on X is continuous if $\pi : X \times G \to X$ is continuous.*

Remark. Here, G has the discrete topology.

Proof. Suppose we have a right action of G on X which is continuous. Let U be an open set in X. For each $g \in G$, let $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) = \bigcup_{g \in G} U_g \times \{g\},$$

which is open. Conversely, suppose π is continuous and let $g \in G$. Let U be open in X and set $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) \cap X \times \{g\} = U_g \times \{g\},$$

which shows that g is continuous since $\pi^{-1}(U)$ and $X \times \{g\}$ are open in $X \times G$. □

Example 8.3. Suppose that X is a locally Hausdorff space, and that G acts on X on the right via a properly discontinuous action. For each $x \in X$, we get an open subset U_x such that U_x meets $U_x.g$ for only finitely many $g \in G$. This property is unaffected by replacing U_x with a smaller open subset around x, so by the locally Hausdorff property we can assume that U_x is Hausdorff. The key is that we can do better: there exists an open set $U_x' \subseteq U_x$ such that U_x' meets $U_x'.g$ if and only if $x = x.g$. Thus, if the action is also free then U_x' is disjoint from $U_x'.g$ for all $g \in G$ with $g \neq 1$.

To find U_x', let $g_1, \ldots, g_n \in G$ be an enumeration of the finite set of elements $g \in G$ such that U_x meets $U_x.g$. For any open subset $U \subseteq U_x$ we can only have $U \cap U.g \neq \emptyset$ for g equal to one of the g_i's, so it suffices to show that for each i with $x.g_i \in U_x \backslash \{x\}$ there is an open subset $U_i \subseteq U_x$ such that $U_i \cap (U_i).g_i = \emptyset$ (and then we may take U_x' to be the intersection of the U_i's over the finitely many i such that $x.g_i \neq x$). By the Hausdorff property of U_x, when $x.g_i \in U_x \backslash \{x\}$ there exist disjoint opens $V_i, V_i' \subseteq U_x$ around x and $x.g_i$ respectively. By continuity of the action on X by $g_i \in G$ there is an open $W_i \subseteq X$ around x such that $(W_i).g_i \subseteq V_i'$. Thus $U_i = W_i \cap V_i$ is disjoint from V_i' yet satisfies $(U_i).g_i \subseteq V_i'$, so $U_i \cap (U_i).g_i = \emptyset$. This completes the construction of U_x'.

The interest in free and properly discontinuous actions is that for such actions in the locally Hausdorff case we may find an open U_x around each $x \in X$ such that U_x is disjoint from $U_x.g$ whenever $g \neq 1$. Thus, for such actions we may say that in X/G we are identifying points in the same G-orbit with this identification process not "crushing" the space X by identifying points in X that are arbitrarily close to each other. An example where things go horribly wrong is the action of $G = \mathbb{Q}$ on $\mathbb{R}$ via additive translations. This is a continuous action, but the quotient $\mathbb{R}/\mathbb{Q}$ is very bad: any two $\mathbb{Q}$-orbits in $\mathbb{R}$ contain arbitrarily close points!

Here are some examples of free and properly discontinuous actions.

Example 8.4. The antipodal map on S^n, given by $(a_1, \ldots, a_{n+1}) \mapsto (-a_1, \ldots, -a_{n+1})$, viewed as an action of the integers mod 2 is free and properly discontinuous: freeness is clear, as is continuity, and for any $x \in S^n$ the points near x all have their antipodes far away!

Example 8.5. Consider the curve $X := \mathbf{V}(x^3 + y^3 + z^3 - 1) \subseteq \mathbb{C}^3$. Then the action $(a_1, a_2, a_3) \mapsto (\zeta_3 a_1, \zeta_3 a_2, \zeta_3)$, viewed as an action of the integers mod 3 is free and properly discontinuous.

Example 8.6. Let $X = S^1 \times S^1$ be a product of two circles, where the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is viewed as a topological group (using multiplication in $\mathbb{C}$, so both the group law and inversion $z \mapsto 1/z = \overline{z}$ on S^1 are continuous). The visibly continuous map $(z, w) \mapsto (1/z, -w) = (\overline{z}, -w)$ reflects through the x-axis in the first circle and rotates 180-degree in the second circle, and it is its own inverse. Thus, this give an action by the order-2 group G of integers mod 2. The associated quotient X/G will be called the (set-theoretic) **Klein bottle**.

Theorem 8.3. *Let X be a locally Hausdorff topological space with a free and properly discontinuous action by a group G. There is a unique topology on X/G such that the quotient map $\pi : X \to X/G$ is a continuous map that is a local homeomorphism (i.e. each $x \in X$ admits a neighborhood mapping homeomorphically onto an open subset of X/G). Moreover, the quotient map is open.*

A subset $S \subseteq X/G$ is open if and only if its preimage in X is open, and if $U \subseteq X$ is an open set that is disjoint from $U.g$ for all nontrivial $g \in G$ then the map $U \to X/G$ is a homeomorphism onto its open image $\overline{U}$ and the natural map $U \times G \to \pi^{-1}(\overline{U})$ over $\overline{U}$ given by $(u, g) \mapsto u.g$ is a homeomorphism when G is given the discrete topology.

Remark. The topology in this theorem is called the **quotient topology**, and it is locally Hausdorff since $X \to X/G$ is a local homeomorphism.

Proof. Sketch: we show that π is an open map. Let $x \in X$ and pick U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$. We first show that $\pi(U_x)$ is open. The inverse image of $\pi(U_x)$ under π is a disjoint union of open sets $\bigcup_{g \in G} U_x.g$. Therefore $\pi(U_x)$ is open. Now let U be any open subset of X. For each $x \in X$, choose U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$ and $U_x \subset U$. Then

$$\pi(U) = \pi\left(\bigcup_{x \in U} U_x\right) = \bigcup_{x \in U} \pi(U_x)$$

implies $\pi(U)$ is open. $\square$

Example 8.7. (Möbius Strip) Choose $a > 0$. Let $X = (-a, a) \times S^1$, and let the group of order 2 act on it with the non-trivial element acting by $(t, w) \mapsto (-t, -w)$. This is easily checked to be a continuous action for the discrete topology of the group of order 2, and it is free and properly discontinuous. The quotient M_a is the **Möbius strip** of height $2a$.

To check that the Möbius strip M_a is Hausdorff, we use the quotient criterion: the set of points in $X \times X$ with the form $((t,w),(t',w'))$ with $(t',w') = (t,w)$ or $(t',w') = (-t,-w)$ is checked to be closed by using the sequential criterion in $X \times X$: suppose $(t_n, w_n) \sim (t'_n, w'_n)$ are sequences in $X \times X$ which converge (t,w) and (t',w') respectively. Then we need to show that $(t,w) \sim (t',w')$. Assume that $(t,w) \neq (t',w')$. Choose open neighborhoods U of (t,w) and U' of (t',w') respectively such that $U \cap U' = \emptyset$ and such that eventually $(t_n, w_n) \neq (t'_n, w'_n)$ (We can do this because they converge to different limits and our space $X \times X$ is Hausdorff). Thus, eventually we have $(t'_n, w'_n) = (-t_n, -w_n) \to (-t, -w)$.

8.4 Möbius Strip in $\mathbb{R}^3$

Recall that the Möbius strip M_a (with height $2a$) was defined as an abstract smooth manifold made as a quotient of $(-a,a) \times S^1$ by a free and properly discontinuous action by the group of order 2. Using the C^∞ isomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and the circle $S^1 \subseteq \mathbb{R}^2$ via $\theta \mapsto (\cos\theta, \sin\theta)$, we consider the standard parameter $\theta \in \mathbb{R}$ as a local coordinate on S^1. For finite $a > 0$, consider the C^∞ map $f : (-a,a) \times S^1 \to \mathbb{R}^3$ defined by

$$(t,\theta) \mapsto (2a\cos 2\theta + t\cos\theta\cos 2\theta, 2a\sin 2\theta + t\cos\theta\sin 2\theta, t\sin\theta).$$

Since $f(-t, \pi+\theta) = f(t,\theta)$ by inspection, it follows from the universal property of the quotient map $(-a,a) \times S^1 \to M_a$ that f uniquely factors through this via a C^∞ map $\overline{f} : M_a \to \mathbb{R}^3$. Our goal is to prove that $\overline{f}$ is an embedding and to use this viewpoint to understand some basic properties of the Möbius strip.

8.4.1 Embedding

Theorem 8.4. *The map $\overline{f}$ is an immersion.*

Proof. We first reduce the problem to working with f, as f is given by a simple explicit formula across its entire domain (M_a does not have global coordinates. Of course, working locally for $\overline{f}$ is "the same" as working locally for f, so the reduction step to working with f isn't really necessary if one says things a little differently. However, it seems a bit cleaner to just make the reduction step right away and so to thereby work with the map f that feel a bit more concrete than the map $\overline{f}$ at the global level.)

Let $p : (-a,a) \times S^1 \to M_a$ be the natural quotient map. Each point in M_a has the form $p(\xi_0)$ for some ξ_0 and the Chain Rule gives that the injection $df(\xi_0)$ factors as $d\overline{f}(p(\xi_0)) \circ dp(\xi_0)$ with $dp(\xi_0)$ an isomorphism (as p is a local C^∞ isomorphism, via the theory of quotients by free and properly discontinuous group actions). Hence, the tangent map for $\overline{f}$ is injective at $p(\xi_0)$ if and only if the tangent map for f is injective at ξ_0. It is therefore enough (even equivalent!) to prove that f is an immersion.

□

8.5 Construction of Manifolds From Gluing Data

The definition of a manifold assumes that the underlying set, M, is already known. However, there are situations where we only have some indirect information about the overlap of the domains U_i, of the local charts defining our manifold, M, in terms of the transition functions

$$\phi_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

but where M itself is not known. Our goal in this subsection is to try and reconstruct a manifold M by gluing open subsets of $\mathbb{R}^n$ using the transition functions ϕ_{ij}.

Definition 8.2. Let n be an integer with $n \geq 1$ and let k be either an integer with $k \geq 1$ or $k = \infty$. A set of **gluing data** is a triple

$$\mathcal{G} = ((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\phi_{ji})_{(i,j) \in K}),$$

satisfying the following properties, where I is a (nonempty) countable set and $K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\}$:

1. For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{R}^n$ called a **parametrization domain**, for short, p**-domain**, and the Ω_i are pairwise disjoint (i.e. $\Omega_i \cap \Omega_j) = \emptyset$ for all $i \neq j$).

2. For every pair $(i,j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ij} \neq \emptyset$ if and only if $\Omega_{ji} \neq \emptyset$. Each nonempty Ω_{ij} (with $i \neq j$) is called a **gluing domain**.

3. The maps $\phi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i,j) \in K$ called a **transition function** (or **gluing function**) and the following condition holds:

(a) The **cocycle condition** holds: for all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ and

$$\phi_{ki}(x) = (\phi_{kj} \circ \phi_{ji})(x)$$

for all $x \in \phi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

4. For every pair $(i, j) \in K$ with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and every $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}.

Remark.

1. In practical applications, the index set, I, is of course finite and the open subsets, Ω_i, may have special properties (for example, connected; open simplicies, etc.).

2. Observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, as Ω_i and Ω_j are disjoint, so are Ω_{ij} and Ω_{ji}.

3. The cocycle condition may seem overly complicated but it is actually needed to guarantee the transitivity of the relation, $\sim$, which we will define shortly. Since the ϕ_{ji} are bijections, the cocycle condition implies the following conditions

   (a) $\phi_{ii} = \mathrm{id}_{\Omega_i}$ for all $i \in I$. This follows by setting $i = j = k$.

   (b) $\phi_{ij} = \phi_{ji}^{-1}$ for all $(i, j) \in K$. This follows from (a) and by setting $k = i$.

4. Let M be a C^k manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on it. Then set $\Omega_i = \phi_i(U_i)$, $\Omega_{ij} = \phi_i(U_i \cap U_j)$, and let $\phi_{ij} : \Omega_{ji} \to \Omega_{ij}$ be the corresponding transition maps. Then it's easy to check that the open sets Ω_i, Ω_{ij}, and the gluing functions ϕ_{ij}, satisfy the conditions of Definition (8.2). Indeed,

$$\begin{aligned}
\phi_{ji}^{-1}(\Omega_{jk}) &= (\phi_i \circ \phi_j^{-1})(\phi_j(U_j \cap U_k)) \\
&= \phi_i(U_j \cap U_k) \\
&= \phi_i(U_i \cap U_j \cap U_k) \\
&\subseteq \phi_i(U_i \cap U_k) \\
&= \Omega_{ik}.
\end{aligned}$$

Let us show that a set of gluing data defines a C^k manifold in a natural way.

Proposition 8.3. *For every set of gluing data $\mathcal{G} = (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\phi_{ji})_{(i,j) \in K})$, there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the ϕ_{ji}'s.*

Proof. Define the binary relation, $\sim$, on the disjoint union, $\Omega := \coprod_{i \in I} \Omega_i$, of the open sets, Ω_i, as follows: For all $x, y \in \Omega$,

$$x \sim y \text{ if and only if there exists } (i, j) \in K \text{ such that } x \in \Omega_{ij},\ y \in \Omega_{ji}, \text{ and } y = \phi_{ji}(x).$$

The cocycle condition ensures that this is an equivalence relation. Indeed, (a) implies reflexivity and (b) implies symmetry. The crucial step is to check transitivity. Assume that $x \sim y$ and $y \sim z$. Then there are some i, j, k such that $\phi_{ji}(x) = y$ and $\phi_{kj}(y) = z$. But then $(\phi_{kj} \circ \phi_{ji})(x) = \phi_{ki}(x) = z$. That is, $x \sim z$, as desired.

Since $\sim$ is an equivalence relation, let

$$M_{\mathcal{G}} := \Omega / \sim$$

be the quotient space by the equivalence relation $\sim$. We claim that $\sim$ is an open equivalence relation. Indeed, let $U := \coprod_{i \in I} U_i$ be an open subset of Ω, where U_i is an open subset of Ω_i for each i. Then

$$\pi^{-1}(\pi(U)) = \coprod_{i \in I} \left(\bigcup_{j \in I} \phi_{ij}(U_j \cap \Omega_{ji}) \cup U_i \right),$$

which is open in Ω since $\phi_{ij}(U_j \cap \Omega_{ji})$ is open in Ω_i for all $i \in I$. Therefore, $M_{\mathcal{G}}$ is second-countable since Ω is second-countable.

Since $\sim$ is an open equivalence relation, we can use Theorem (8.1) to show that $M_{\mathcal{G}}$ is Hausdorff by showing that the graph

$$R = \{(x, y) \in \Omega \times \Omega \mid x \sim y\}$$

is closed in $\Omega \times \Omega$. We do this by showing that if (x_n, y_n) is a sequence in R that converges to $(x, y) \in \Omega \times \Omega$, then $(x, y) \in R$. That is to say, if $x_n \sim y_n$, then $x \sim y$. Since $(x, y) \in \Omega_i \times \Omega_j$, we may assume that $(x_n, y_n) \in \Omega_i \times \Omega_j$ (since it will eventually be in there anyways). If $i = j$, then $x_n = y_n$, and hence $x = y$, so assume $i \neq j$.

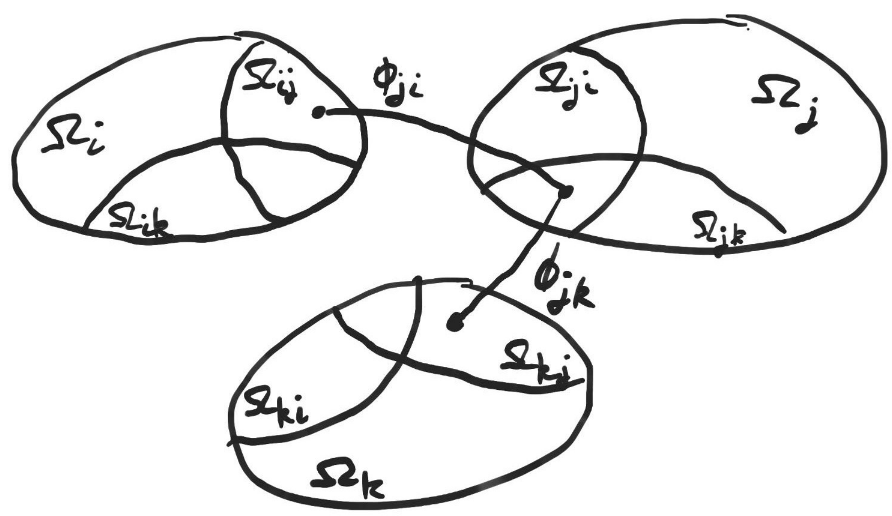
In order for us to have $x_n \sim y_n$, we must have $x_n \in \Omega_{ij}$ and $y_n \in \Omega_{ji}$. If $x \in \Omega_{ij}$, then it is easy to see that $y \in \Omega_{ji}$ and that $x \sim y$, since $x_n \sim y_n$ in arbitrarily small neighborhoods of x and y. Thus we need to show that either $x \in \Omega_{ij}$ or $y \in \Omega_{ji}$. Assume for a contradiction, that $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$. Choose open balls V_x and V_y centered at x and y so that no point in $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}. But this implies that no point in V_x is equivalent to some point in V_y. This contradicts the fact that $x_n \to x$ and $y_n \to y$, as the sequence (x_n, y_n) must eventually be in the neighborhoods V_x and V_y. Therefore $M_{\mathcal{G}}$ is Hausdorff.

Finally, for every $i \in I$, let $\mathrm{in}_i : \Omega_i \to \coprod_{i \in I} \Omega_i$ be the natural injection and let

$$\tau_i := \pi \circ \mathrm{in}_i : \Omega_i \to M_{\mathcal{G}}.$$

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we conclude that every τ_i is injective. If we let $U_i = \tau_i(\Omega_i)$ and $\phi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, ϕ_i) are charts and this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. $\square$

Remark. Note that the condition $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ is needed in order for $\sim$ to be transitive. The picture below illustrates how things could go wrong:



8.5.1 Mobius Strip

Example 8.8. Let X be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold.

Let U_v be the set of all nonvertical lines in $\mathbb{R}^2$. A nonvertical is of the form $\ell^v_{a,b} = \{(x, y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a, b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell^v_{a,b} \mapsto (a, b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism.

Next let U_h be the set of all nonhorizontal lines in $\mathbb{R}^2$. A nonhorizontal is of the form $\ell^h_{c,d} = \{(x, y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c, d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell^h_{c,d} \mapsto (c, d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism.

Now we have $U_v \cup U_h = X$. To get a topology on X, we glue the topologies from U_v and U_h: a set $U \subset X$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}
\varphi_{vh}(c, d) &= \varphi_v \circ \varphi_h^{-1}(c, d) \\
&= \varphi_v\left(\ell^h_{c,d}\right) \\
&= \varphi_v\left(\ell^v_{\frac{1}{c}, -\frac{d}{c}}\right) \\
&= \left(\frac{1}{c}, -\frac{d}{c}\right),
\end{aligned}$$

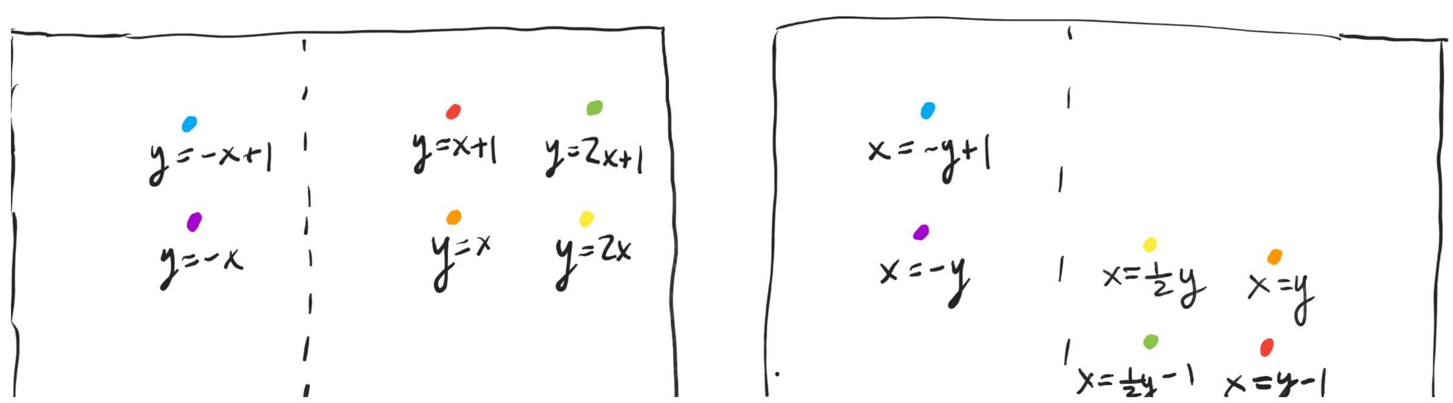
which is C^∞ whenever $c \neq 0$. Similarly,

$$\begin{aligned}\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b) \\ &= \varphi_h\left(\ell^v_{a,b}\right) \\ &= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right) \\ &= \left(\frac{1}{a}, -\frac{b}{a}\right),\end{aligned}$$

which is C^∞ whenever $a \neq 0$. Altogether, our gluing data consists of

$$\Omega_1 = \Omega_2 = \mathbb{R}^2, \qquad \Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}, \qquad \phi_{12} : (a,b) \mapsto \left(\frac{1}{a}, -\frac{b}{a}\right).$$

This manifold is called the **Mobius strip**. We can visualize it as below:



Remark. We can describe this manifold in another way as follows: let G be the group given by

$$G := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a, b \in \mathbb{R} \right\}.$$

The group G has a natural open subgroup

$$\mathrm{Aff}(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a, b \in \mathbb{R} \text{ and } a \neq 0 \right\}.$$

Clearly G can be identified with $\Omega_1 = \Omega_2 = \mathbb{R}^2$ and $\mathrm{Aff}(\mathbb{R})$ can be identified with $\Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}$. Using these identifications, the transition map ϕ_{12} is identified with the inverse map! Indeed, the inverse of $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1/a & -b/a \\ 0 & 1 \end{smallmatrix}\right)$.

Given a set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, it is natural to consider the collection of manifolds, M, parametrized by maps, $\theta_i : \Omega_i \to M$, whose domains are the Ω_i's and whose transition functions are given by the ϕ_{ji}'s, that is, such that

$$\phi_{ji} = \theta_j^{-1} \circ \theta_i.$$

We will say that such manifolds are **induced** by the set of gluing data $\mathcal{G}$.

The parametrization maps τ_i satisfy the property: $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ if and only if $(i,j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Furthermore, they also satisfy the consistency condition:

$$\tau_i = \tau_j \circ \phi_{ji},$$

for all $(i,j) \in K$. If M is a manifold induced by the set of gluing data $\mathcal{G}$, then because the θ_i's are injective and $\phi_{ji} = \theta_j^{-1} \circ \theta_i$, the two properties stated above for the τ_i's also hold for the θ_i's. We will see that the manifold $M_{\mathcal{G}}$ is a "universal" manifold induced by $\mathcal{G}$ in the sense that every manifold induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by some C^k map.

Interestingly, it is possible to characterize when two manifolds induced by the same set of gluing data are isomorphic in terms of a condition on their transition functions.

Proposition 8.4. *Given any set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, for any two manifolds M and M' induced by $\mathcal{G}$ given by families of parametrizations $(\Omega_i, \theta_i)_{i\in I}$ and $(\Omega_i, \theta'_i)_{i\in I}$, respectively, if $f : M \to M'$ is a C^k isomorphism, then there are C^k bijections, $\rho_i : W_{ij} \to W'_{ij}$, for some open subsets $W_{ij}, W'_{ij} \subseteq \Omega_i$, such that*

$$\phi'_{ji}(x) = \rho_j \circ \phi_{ji} \circ \rho_i^{-1}(x),$$

for all $x \in W_{ij}$ with $\phi_{ji} = \theta_j^{-1} \circ \theta_i$ and $\phi'_{ji} = {\theta'}_j^{-1} \circ \theta'_i$. Furthermore, $\rho_i = ({\theta'}_i^{-1} \circ f \circ \theta_i) |_{W_{ij}}$ and if ${\theta'}_i^{-1} \circ f \circ \theta_i$ is a bijection from Ω_i to itself and ${\theta'}_i^{-1} \circ f \circ \theta_i(\Omega_{ij}) = \Omega_{ij}$ for all i, j, then $W_{ij} = W'_{ij} = \Omega_i$.

9 Ringed Spaces

Definition 9.1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally R-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

9.1 From C^p-Structures to Maximal C^p-Atlases

Let $\mathcal{O}$ be a C^p-structure on X. Let $\mathcal{A}$ be the set of all pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : (U, \mathcal{O}\mid_U) \to \mathbb{R}^n$ is a C^p-isomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$ (with $\mathbb{R}^n$ given its usual C^p-structure). The collection $\mathcal{A}$ is a C^p-atlas because of two facts: a composite of C^p maps is C^p, and for maps between opens in finite-dimensional $\mathbb{R}$-vector spaces the "old" notion of C^p is the same as the "new" notion (in terms of structured $\mathbb{R}$-spaces). It is obvious that $\mathcal{A}$ is standardized. We want to prove that the standardized C^p-atlas $\mathcal{A}$ is maximal.

9.2 From Maximal C^p-Atlases to C^p-Structures

Let $\mathcal{A}$ be a maximal standarized C^p-atlas on X. For any non-empty open set $U_0 \subseteq X$, we define $\mathcal{O}(U_0)$ to be the set of functions $f : U_0 \to \mathbb{R}$ such that for all $(U, \phi) \in \mathcal{A}$, the composite map

$$f \circ \phi^{-1} : \phi(U \cap U_0) \to \mathbb{R}$$

is a C^p function on the open subset $\phi(U \cap U_0)$ in the Euclidean space $\mathbb{R}^n$ that is the target of ϕ. Also define $\mathcal{O}(\emptyset) = \{0\}$.

Lemma 9.1. *The correspondence $U_0 \mapsto \mathcal{O}(U_0)$ is an $\mathbb{R}$-space structure on X. For any $(U, \phi) \in \mathcal{A}$ and open $U_0 \subseteq U$, $\mathcal{O}(U_0)$ is the set of $f : U_0 \to \mathbb{R}$ such that $f \circ \phi^{-1} : \phi(U_0) \to \mathbb{R}$ is a C^p function on the open domain $\phi(U_0)$ in a Euclidean space.*

Proof. The usual notion of C^p function on an open set in a Euclidean space is preserved under restirction to smaller opens and can be checked by working on an open covering. Thus, the first claim in the lemma follows easily from the definition of $\mathcal{O}$. $\square$

10 Homological Algebra

10.1 Chain Complexes over R

Let R be a ring. A **chain complex** $A := (A_\bullet, d_\bullet)$ **over** R, or simply a **chain complex** if the context is clear, is a sequence of R-modules A_i and morphisms $d_i : A_i \to A_{i-1}$

$$(A, d) := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots \tag{6}$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. The condition $d_i \circ d_{i+1} = 0$ is equivalent to the condition $\mathrm{Ker}(d_i) \supset \mathrm{Im}(d_{i+1})$. With this in mind, we define the **ith homology of the chain complex** A to be

$$H_i(A) := \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

A **chain map** between two complexes $A = (A_\bullet, d_\bullet)$ and $A' = (A'_\bullet, d'_\bullet)$ is a sequence $\varphi_\bullet$ of R-module homomorphisms $\varphi_i : A_i \to A'_i$ such that $d_i \varphi_{i-1} = \varphi_i d'_{i-1}$ for all $i \in \mathbb{Z}$. It follows from the definition that a chain map $\varphi : A \to A'$ induces map of homologies $\varphi_i : H_i(A) \to H_i(A')$ for all $i \in \mathbb{Z}$.

To simplify notation in what follows, we think of R as a trivially graded ring. If

$$A := \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots \tag{7}$$

is a chain complex over R, then we can think of A as a graded R-module together with an graded endomorphism $d : A \to A$ of degree -1 such that $d^2 = 0$. We think of d_i as being the restriction of d to A_i. We define the **homology** of A to be $H(A) := \mathrm{Ker}(d)/\mathrm{Im}(d)$. Note that $H(A) = \bigoplus_{i \in \mathbb{Z}} H_i(A)$. An element in $\mathrm{Ker}(d)$ is called a **cycle** of A and an element in $\mathrm{Im}(d)$ is called a **boundary** of A.

Let A and A' be chain complexes over R with differentials d and d' respectively. A chain map $\varphi : A \to A'$ can be thought of as a homogeneous homomorphism of graded R-modules such that $\varphi d = d' \varphi$. It follows from the definition that φ takes $\mathrm{Ker}(d)$ to $\mathrm{Ker}(d')$ and $\mathrm{Im}(d)$ to $\mathrm{Im}(d')$. Thus φ gies rise to an **induced map on homology**, which we also call $\varphi : H(A) \to H(A')$. Note that the restriction of φ to $H_i(A)$ is $\varphi_i : H_i(A) \to H_i(A')$.

10.1.1 Homotopy Equivalence

Let φ and ψ be chain maps of chain complexes A and A' with differentials d and d' respectively. We say φ is **homotopic** to ψ if there is a graded homomorphism $h : A \to A'$ of degree 1 such that $\varphi - \psi = d'h + hd$.

Proposition 10.1. *Let φ and ψ be chain maps of chain complexes A and A' with differentials d and d' respectively such that φ is homotopic to ψ. Then φ and ψ induce the same map on homology.*

Proof. It suffices to show that $\varphi - \psi$ induces the 0 map on homology. Thus we may simplify the notation by replacing φ by $\varphi - \psi$, and assume from the outset that $\psi = 0$. Let h be a homotopy, so $\varphi = d'h + dh$. Let $a \in \mathrm{Ker}(d)$ be a cycle of A. From the formula for the homotopy h we get

$$\begin{aligned} \varphi(a) &= d'(h(a)) + h(d(a)) \\ &= d'(h(a)) + h(0) \\ &= d'(h(a)). \end{aligned}$$

Therefore $\varphi(a)$ is a boundary of A'. This implies that φ induces the 0 map on homology. $\square$

10.2 Exact Sequences of Chain Complexes over R

Let A, A', and A'' be chain complexes over R with differentials d,d', and d'' respectively, and let $\varphi : A' \to A$ and $\psi : A \to A''$ be chain maps. Then we say that

$$0 \longrightarrow A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

is a **short exact sequence** of chain complexes if

$$0 \longrightarrow A'_i \xrightarrow{\varphi_i} A_i \xrightarrow{\psi_i} A''_i \longrightarrow 0$$

is exact for all $i \in \mathbb{Z}$. In other words, the rows in the diagram below are exact.

$$\begin{tikzcd}
 & \vdots \arrow[d, "d'_{i+2}"] & \vdots \arrow[d, "d_{i+2}"] & \vdots \arrow[d, "d''_{i+2}"] & \\
0 \arrow[r] & A'_{i+1} \arrow[r, "\varphi_{i+1}"] \arrow[d, "d'_{i+1}"] & A_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "d_{i+1}"] & A''_{i+1} \arrow[r] \arrow[d, "d''_{i+1}"] & 0 \\
0 \arrow[r] & A'_i \arrow[r, "\varphi_i"] \arrow[d, "d'_i"] & A_i \arrow[r, "\psi_i"] \arrow[d, "d_i"] & A''_i \arrow[r] \arrow[d, "d''_i"] & 0 \\
0 \arrow[r] & A'_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "d'_{i-1}"] & A_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "d_{i-1}"] & A''_{i-1} \arrow[r] \arrow[d, "d''_{i-1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given such a short exact sequence, we get induced maps $\varphi_i : H_i(A') \to H_i(A)$ and $\psi_i : H_i(A) \to H_i(A'')$, and **connecting homomorphisms** $\gamma_i : H_i(A'') \to H_{i-1}(A')$ which gives rise a long exact sequence in homology:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{i+1}(A'') \arrow[dll, rounded corners, "\gamma_{i+1}"] \\
H_i(A') \arrow[r, "\varphi_i"] & H_i(A) \arrow[r, "\psi_i"] & H_i(A'') \arrow[dll, rounded corners, "\gamma_i"] \\
H_{i-1}(A') \arrow[r, "\varphi_{i-1}"] & H_{i-1}(A) \arrow[r, "\psi_{i-1}"] & \cdots
\end{tikzcd}$$

11 deRham Cohomology

Suppose $F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a smooth vector field representing a force on an open subset U of $\mathbb{R}^2$, and C is a parametrized curve $c(t) = (x(t), y(t))$ in U from a point p to a point q with $a \leq t \leq b$. Then the work done by the force in moving a particle from p to q along C is given by the line integral $\int_C Pdx + Qdy$.

Such a line integral is easy to compute if the vector field F is the gradient of a scalar function $f(x,y)$:

$$F = \text{grad} f = \langle \partial_x f, \partial_y f\rangle.$$

By Stoke's theorem, the line integral is simply

$$\int_C \partial_x f dx + \partial_y f dy = \int_C df = f(q) - f(p).$$

A necessary condition for the vector field $F = \langle P, Q\rangle$ to be a gradient is that

$$P_y = \partial_y \partial_x f = \partial_x \partial_y f = Q_x.$$

The question is now the following: if $P_y - Q_x = 0$, is the vector field $F = \langle P, Q\rangle$ on U the gradient of some scalar function $f(x,y)$ on U? In terms of differential forms, the question becomes the following: if the 1-form $\omega = Pdx + Qdy$ is closed on U, is it exact? The answer to this question is sometimes yes and sometimes no, depending on the topology of U.

11.1 de Rham Complex

Let M be a manifold and let R denote the ring $\Omega^0(M) := C^\infty(M)$. Then we have the following cochain complex over R

$$(\Omega(M), d) := \qquad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \tag{8}$$

where d denotes the exterior derivative. We denote $H_{\text{dR}}(M)$ to be the cohomology of $(\Omega(M), d)$ and call it the **deRham cohomology** of M. We denote by $Z(M)$ to be the cycles of $(\Omega(M), d)$ and $B(M)$ to be the boundaries of $(\Omega(M), d)$.

Proposition 11.1. *If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered- r-tuple of real numbers, each real number representing a constant function on a connected component of M.*

Proof. Since there are no nonzero exact 0-forms,

$$H^0(M) = Z^0(M).$$

Suppose f is a closed 0-form on M, i.e. f is a C^∞ function on M such that $df = 0$. On any chart $(U, x_1, \dots, x_n)$, we have

$$df = \sum_{\lambda=1}^{n} (\partial_{x_\lambda} f) dx_\lambda.$$

Thus $df = 0$ on U if and only if all the partial derivatives $\partial_{x_\lambda} f$ vanish identically on U. This in turn is equivalent to f being locally constant on U. Hence, the closed 0-forms on M are precisely the locally constant functions on M. Such a function must be constant on each connected component on M. If M has r connected components, then a locally constant function on M can be specified by an ordered set of r real numbers. Thus, $Z^0(M) = \mathbb{R}^r$. $\square$

Proposition 11.2. *On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.*

Proof. At any point $p \in M$, then tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, the space of alternating k-linear functions on T_pM. If $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form on M is the zero form. $\square$

11.1.1 Examples of de Rham Cohomology

Example 11.1. (De Rham cohomology of the real line) Since the real line $\mathbb{R}^1$ is connected, we have

$$H^0(\mathbb{R}^1) = \mathbb{R}.$$

For dimensional reasons, there are no nonzero 2-forms on $\mathbb{R}^1$. This implies that every 1-form on $\mathbb{R}^1$ is closed. A 1-form $f(x)dx$ on $\mathbb{R}^1$ is exact if and only if three is a C^∞ function $g(x)$ on $\mathbb{R}^1$ such that

$$f(x)dx = dg = g'(x)dx,$$

where $g'(x)$ is the calculus derivative of g with respect to x. Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example

$$g(x) = \int_0^x f(t)dt.$$

This proves that every 1-form on $\mathbb{R}^1$ is exact. Therefore, $H^1(\mathbb{R}^1) = 0$.

Example 11.2. (De Rham cohomology of the circle) Let S^1 be the unit circle in the xy-plane. Since S^1 is connected, we have $H^0(S^1) = \mathbb{R}$, and since S^1 is one-dimensional, we have $H^k(S^1) = 0$ for all $k \geq 2$. It remains to compute $H^1(S^1)$.

Let $h : \mathbb{R} \to S^1$ be given by $h(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$ and let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restricting the domain of h to $[0, 2\pi]$ gives a parametrization $F := h \circ i : [0, 2\pi] \to S^1$ of the circle. A nowhere-vanishing 1-form on S^1 is given by $\omega = -ydx + xdy$. Note that

$$\begin{aligned} h^*\omega &= -\sin t d(\cos t) + \cos t d(\sin t) \\ &= (\sin^2 t + \cos^2 t)dt \\ &= dt. \end{aligned}$$

Thus

$$\begin{aligned} F^*\omega &= i^*h^*\omega \\ &= i^*dt \\ &= dt, \end{aligned}$$

and so

$$\begin{aligned} \int_{S^1} \omega &= \int_{F([0,2\pi])} \omega \\ &= \int_{[0,2\pi]} F^*\omega \\ &= \int_0^{2\pi} dt \\ &= 2\pi. \end{aligned}$$

Since the cirle has dimension 1, all 1-forms on S^1 are closed, so $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1-forms on S^1 defines a linear map

$$\varphi : Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}, \quad \varphi(\alpha) = \int_{S^1} \alpha.$$

Because $\varphi(\omega) = 2\pi \neq 0$, the linear map $\varphi : \Omega^1(S^1) \to \mathbb{R}$ is onto.

By Stokes's theorem, the exact 1-forms on S^1 are in $\mathrm{Ker}(\varphi)$. Conversely, we will show that all 1-forms in $\mathrm{Ker}(\varphi)$ are exact. Suppose $\alpha = f\omega$ is a smooth 1-form on S^1 such that $\varphi(\alpha) = 0$. Let $\overline{f} = h^*f = f \circ h \in \Omega^0(\mathbb{R})$. Then $\overline{f}$ is periodic of period 2π and

$$\begin{aligned} 0 &= \int_{S^1} \alpha \\ &= \int_{F([0,2\pi])} F^*\alpha \\ &= \int_{[0,2\pi]} (i^*h^*f)(t) \cdot F^*\omega \\ &= \int_0^{2\pi} \overline{f}(t)dt. \end{aligned}$$

11.2 The C^∞ Hairy Ball Theorem

Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ with $n > 0$. If n is odd then there exists a nowhere-vanishing smooth vector field on S^n. Indeed, if $n = 2k+1$ the consider the vector field $\vec{v}$ on $\mathbb{R}^{n+1} = \mathbb{R}^{2k+2}$ given by

$$\vec{v} = (-x_2\partial_{x_1} + x_1\partial_{x_2}) + \cdots + (-x_{2k+2}\partial_{x_{2k+1}} + x_{2k+1}\partial_{x_{2k+2}}) = \sum_{j=0}^{k}(-x_{2j+2}\partial_{x_{2j+1}} + x_{2j+1}\partial_{x_{2j+2}}).$$

For any point $p \in S^n$ it is easy to see that $\vec{v}(p) \in T_p(\mathbb{R}^{2k+2})$ is perpendicular to the line spanned by $\sum_i x_i(p)\partial_{x_i}|_p$, so it lies in the hyperplane $T_p(S^n)$ orthogonal to this line. In other words, the smooth section $\vec{v}|_{S^n}$ of the pullback bundle $(T(\mathbb{R}^{n+1}))|_{S^n}$ over S^n takes values in the subbundle $T(S^n)$, which is to say that $\vec{v}|_{S^n}$ is a smooth vector field on the manifold S^n. This is a visibly nowhere-vanishing vector field.

The above construction does not work if n is even, so there arises the question of whether there exists a nowhere-vanishing smooth vector field on S^n for even n. The answer is negative, and is called the **hairy ball** theorem.

Theorem 11.1. *A smooth vector field on S^n must vanish somewhere if n is even.*

Proof. Let $\vec{v}$ be a smooth vector field on S^n, and assume that it is nowhere-vanishing. For each $p \in S^n$, let $\gamma_p : [0, \pi/\|\vec{v}(p)\|] \to S^n$ be the smooth parametric great circle (with constant speed) going from p to $-p$ with velocity vector $\gamma'_p(0) = \vec{v}(p) \neq 0$ at $t = 0$ (This would not make sense if $\vec{v}(p) = 0$). Working in the plane spanned by $p \in \mathbb{R}^{n+1}$ and $\vec{v}(p) \in T_p(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$, we get the formula

$$\gamma_p(t) = \cos(t\|\vec{v}(p)\|)\, p + \sin(t\|\vec{v}(p))\, \frac{\vec{v}(p)}{\|\vec{v}(p)\|} \in S^n \subseteq \mathbb{R}^{n+1}.$$

(These algebraic formulas would not make sense if $\vec{v}$ vanishes somewhere on S^n). Consider the "flow" mapping

$$F : S^n \times [0,1] \to S^n,$$

defined by $(p,t) \mapsto \gamma_p(\pi t/\|\vec{v}(p)\|)$. The formula for $\gamma_p(t)$ makes it clear that F is a smooth map (and is continuous if $\vec{v}$ is merely continuous and nowhere-vanishing). Now obviously $F(p,0) = p$ for all $p \in S^n$ and $F(p,1) = -p$ for all $p \in S^n$. Hence, F defines a smooth homotopy from the identity map on S^n to the antipodal map $p \mapsto -p$ on S^n (and is a continuous homotopy if $\vec{v}$ is merely continuous and nowhere-vanishing). Thus, to prove the hairy ball theorem we just have to prove that if n is even then the identity and antipodal maps $S^n \to S^n$ are not smoothly homotopic to each other; likewise to get the continuous version we just need to prove that there is no continuous homotopy deforming one of these maps into the other.

To prove the *non-existence* of such a homotopy, we shall use the (smooth) homotopy invariance of deRham cohomology. Indeed, by this homotopy-invariance we get that under the existence of such a $\vec{v}$ the antipodal map $A : S^n \to S^n$ induces the identity map $A^* : \mathrm{H}^k_{\mathrm{dR}}(S^n) \to \mathrm{H}^k_{\mathrm{dR}}(S^n)$ on the kth deRham cohomology of S^n for all $k \geq 0$. Let us focus on the case $k = n$. To get a contradiction, we just have to prove that if n is even then A^* as a self-map of $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is *not* the identity map.

Consider the n-form on $\mathbb{R}^{n+1}$ defined by

$$\omega = \sum_{i=1}^{n+1}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Clearly $d\omega = (n+1)dx_1 \wedge \cdots \wedge dx_{n+1}$, so for the unit ball $B^{n+1} \subseteq \mathbb{R}^{n+1}$ with its standard orientation we have

$$\int_{B^{n+1}} d\omega = (n+1)\mathrm{vol}(B^{n+1}) \neq 0.$$

By Stokes' theorem for B^{n+1}, if we let $\eta = \omega|_{S^n}$ and we give $S^n = \partial B^{n+1}$ the induced boundary orientation, then

$$\int_{S^n} \eta = \int_{B^{n+1}} d\omega \neq 0.$$

Hence, by Stokes' theorem for the boundaryless smooth compact oriented manifold S^n we conclude that the top-degree differential form η on S^n is not exact. That is, its deRham cohomology class $[\eta] \in H^n_{\mathrm{dR}}(S^n)$ is non-zero. (Note that ω is not closed as an n-form on $\mathbb{R}^{n+1}$, but its pullback η on S^n is necessarily closed on S^n purely for elementary reasons, as S^n is n-dimensional.)

By the existence of the smooth homotopy between A and the identity map, it follows that A^* on $H^n_{\mathrm{dR}}(S^n)$ is the identity map, so $[A^*(\eta)] = A^*([\eta])$ is equal to $[\eta]$. That is, the top-degree differential forms $A^*(\eta)$ and η on S^n differ by an exact form. But the antipodal map $A : S^n \to S^n$ is induced by the negation map $N : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, and by inspection of the definition of $\omega \in \Omega^n_{\mathbb{R}^{n+1}}(\mathbb{R}^{n+1})$ we have $N^*(\omega) = (-1)^{n+1}\omega$. Hence, pulling back this equality to the sphere gives $A^*(\eta) = (-1)^{n+1}\eta$ in $\Omega^n_{S^n}(S^n)$. Thus, in $H^n_{\mathrm{dR}}(S^n)$ we have

$$[\eta] = A^*([\eta]) = [A^*(\eta)] = [(-1)^{n+1}\eta] = (-1)^{n+1}[\eta].$$

If n is even we therefore have $[\eta] = -[\eta]$, so $[\eta] = 0$. But we have already seen via Stokes' theorem for the boundaryless manifold S^n and for the manifold with boundary B^{n+1} that $[\eta]$ is nonzero. This completes the proof. $\square$

12 Exercises

12.1 $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \right\} \subset \mathbb{R}^4$. Let $\gamma : [0,1] \to \mathrm{SL}_2(\mathbb{R})$ be a path in $\mathrm{SL}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

such that $\gamma(0) = I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then by differentiating the identity $a(t)d(t) - b(t)c(t) = 1$ and evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= \dot{a}(0)d(0) + a(0)\dot{d}(0) - \dot{b}(0)c(0) - b(0)\dot{c}(0) \\ &= \dot{a}(0) + \dot{d}(0). \end{aligned}$$

Or $\dot{a}(0) = -\dot{d}(0)$. In particular, this means that $\mathrm{Tr}\,(\dot{\gamma}(0)) = 0$.

Conversely, suppose we have a matrix A such that $\mathrm{Tr}(A) = 0$. Can we find a path γ in $\mathrm{SL}_2(\mathbb{R})$ such that $\dot{\gamma}(0) = A$? Indeed, we can. The matrix exponential works:

$$e^{tA} := I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots.$$

This is because

$$\begin{aligned} \frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left(I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \cdots\right) \\ &= \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \frac{1}{2}\frac{d}{dt}(t^2A^2) + \frac{1}{6}\frac{d}{dt}(t^3A^3) + \cdots \\ &= A + tA^2 + \frac{1}{2}t^2A^3 + \cdots \end{aligned}$$

Thus, $\frac{d}{dt}\left(e^{tA}\right)_{|t=0} = A$. Also we have $e^{tA} \in \mathrm{SL}_2(\mathbb{R})$ since

$$\begin{aligned} \det(e^{tA}) &= e^{\mathrm{Tr}(tA)} \\ &= e^0 \\ &= 1. \end{aligned}$$

12.2 $\mathrm{SO}_2(\mathbb{R})$

Let $\mathrm{SO}_2(\mathbb{R}) := \left\{ A \in \mathrm{SL}_2(\mathbb{R}) \mid AA^t = I \right\}$. Let $\gamma : [0,1] \to \mathrm{SO}_2(\mathbb{R})$ be a path in $\mathrm{SO}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

such that $\gamma(0) = I := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then by differentiating the identity $I = \gamma(t)\gamma(t)^t$ and evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= \dot{\gamma}(0)\gamma(0)^t + \gamma(0)\dot{\gamma}(0)^t \\ &= \dot{\gamma}(0) + \dot{\gamma}(0)^t. \end{aligned}$$

In particular, this means that $\dot{\gamma}(0)$ is a skew-symmetric matrix.

12.3 Vector Field in $\mathbb{R}^3$

Let ω be a vector field in $\mathbb{R}^3$ given by $\omega := (0, y, 0) := y\partial_y$. Let's find a path γ in $\mathbb{R}^3$ such that $\dot{\gamma} = \omega(\gamma)$. A general path γ in $\mathbb{R}^3$ has the form

$$\gamma(t) := (a(t), b(t), c(t)) \qquad \text{and} \qquad \dot{\gamma}(t) = (\dot{a}(t), \dot{b}(t), \dot{c}(t)).$$

Therefore $\omega($So we need

$$\begin{aligned} \dot{a}(t) &= 0 \\ \dot{b}(t) &= b(t) \\ \dot{c}(t) &= 0. \end{aligned}$$

In particular, $\gamma(t) = (a(0), b(0)e^t, c(0))$ works.

12.4 Lie Groups

Definition 12.1. A **Lie group** is a C^∞ manifold G that is also a group such that the two group operations, multiplication

$$G \times G \to G, \qquad (a, b) \mapsto ab,$$

and inverse

$$G \to G, \qquad a \mapsto a^{-1},$$

are C^∞.

For $a \in G$, denote by $\ell_a : G \to G$, where $\ell_a(x) = ax$, the operation of **left multiplication by** a, and by $r_a : G \to G$, where $r_a(x) = xa$, the operation of **right multiplication** by a. We also call left and right multiplications **left** and **right translations**.

Actually smoothness of invsersion can be dropped from the definition of a Lie Group.

Theorem 12.1. *Let G be a C^∞ manifold and suppose it is equipped with a group structure such that the composition law $m : G \times G \to G$ is C^∞. Then the inversion $G \to G$ is C^∞.*

Proof. Consider the "shearing transformation"

$$\Sigma : G \times G \to G \times G,$$

defined by $\Sigma(g, h) = (g, gh)$. This is bijective since we are using a group law, and it is C^∞ since the composition law m is assumed to be C^∞. (Recall that if M, M', M'' are C^∞ manifolds, a map $M \to M' \times M''$ is C^∞ if and only if its component maps $M \to M'$ and $M \to M''$ are C^∞, due to the nature of product manifold structures.)

We claim that Σ is a diffeomorphism. Granting this,

$$G = \{e\} \times G \longrightarrow G \times G \xrightarrow{\Sigma^{-1}} G \times G$$

is C^∞, but explicitely this composite map is $g \mapsto (g, g^{-1})$, so its second component $g \mapsto g^{-1}$ is C^∞ as desired. Since Σ is a C^∞ bijection, the C^∞ property for its inverse is equivalent to Σ being a **local isomorphism** (i.e. each point in its source has an open neighborhood carried diffeomorphically onto an open neighborhood in the target). By the Inverse Function Theorem, this is equivalent to the isomorphism property for the tangent map

$$d\Sigma(g, h) : T_g(G) \oplus T_h(G) = T_{(g,h)}(G \times G) \to T_{(g,gh)}(G \times G) = T_g(G) \oplus T_{gh}(G)$$

for all $g, h \in G$.

We shall now use left and right translations to reduce this latter "linear" problem to the special case $g = h = e$, and in that special case we will be able to compute the tangent map explicitely and see the isomorphism property by inspection.

□