Abstract Algebra Homework 5

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Problem 1

Problem 1.a

Definition 0.1. Let R be a commutative ring. A subset $S \subset R$ is called **multiplicatively closed** if $1 \in S$ and $s, t \in S$ implies $st \in S$.

Definition 0.2. Let S be a multiplicatively closed subset of R. We define the **localization of R with respect to S**, denoted R_S or $S^{-1}R$, as follows: as a set R_S is given by

$$R_S := \left\{ \frac{a}{s} \mid a \in R,\, s \in S \right\}$$

where a/s denotes the equivalence class of $(a, s) \in R \times S$ with respect to the following equivalence relation:

$$(a, s) \sim (a', s') \text{ if and only if there exists } s'' \in S \text{ such that } s''s'a = s''sa'. \tag{1}$$

We give R_S a ring structure by defining addition and multiplication on R_S by

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1 + s_1a_2}{s_1s_2} \quad \text{and} \quad \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}, \tag{2}$$

for a_1/s_1 and a_2/s_2 in R_S, where $1/1$ is the multiplicative identity element in R_S and $0/0$ is the additive identity in R_S. The ring R_S comes equipped with a natural ring homomoprhism $\rho_S \colon R \to R_S$, given by

$$\rho_S(a) = \frac{a}{1}$$

for all $a \in R$.

Proposition 0.1. *With the notation as above, R_S is a ring. Furthermore, $\rho_S \colon R \to R_S$ is a ring homomorphism.*

Proof. There are several things we need to check. We will break them into steps

Step 1: We show that the relation (1) is in fact a equivalence relation. First we show reflexivity of $\sim$. Let $(a, s) \in R \times S$. Then since $1 \in S$ and $1 \cdot sa = 1 \cdot sa$, we have $(a, s) \sim (a, s)$. Next we show symmetry of $\sim$. Suppose $(a, s) \sim (a', s')$. Choose $s'' \in S$ such that $s''s'a = s''sa'$. Then by symmetry of equality, we have $s''sa' = s''s'a$. Therefore $(a', s') \sim (a, s)$. Finally, we show transitivity of $\sim$. Suppose $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$. Choose $s_{12}, s_{23} \in S$ such that

$$s_{12}s_2a_1 = s_{12}s_1a_2 \quad \text{and} \quad s_{23}s_3a_2 = s_{23}s_2a_3$$

Then $s_{23}s_{12}s_2 \in S$ and

$$\begin{aligned}
(s_{23}s_{12}s_2)(s_3a_1) &= s_{23}(s_{12}s_2a_1)s_3 \\
&= s_{23}(s_{12}s_1a_2)s_3 \\
&= s_{12}s_1(s_{23}s_3a_2) \\
&= s_{12}s_1(s_{23}s_2a_3) \\
&= (s_{12}s_{23}s_2)(s_1a_3).
\end{aligned}$$

Thus $\sim$ is in fact an equivalence relation.

Step 2: Addition and multiplication defined in (2) are well-defined. Suppose $a_1/s_1 = a_1'/s_1'$ and $a_2/s_2 = a_2'/s_2'$. Choose $s_1'', s_2'' \in S$ such that

$$s_1''s_1'a_1 = s_1''s_1a_1' \quad \text{and} \quad s_2''s_2'a_2 = s_2''s_2a_2'.$$

Then $s_1''s_2'' \in S$ and

$$\begin{aligned} s_1''s_2''(s_2a_1 + s_1a_2)s_1's_2' &= s_2''s_2(s_1''s_1'a_1)s_2' + s_1''s_1(s_2''s_2'a_2)s_1' \\ &= s_2''s_2(s_1''s_1a_1')s_2' + s_1''s_1(s_2''s_2a_2')s_1' \\ &= s_2''s_2(s_1''s_1a_1')s_2' + s_1''s_1(s_2''s_2a_2')s_1' \\ &= s_1''s_2''(s_2'a_1' + s_1'a_2')s_1s_2 \end{aligned}$$

implies

$$\frac{s_2a_1 + s_1a_2}{s_1s_2} = \frac{s_2'a_1' + s_1'a_2'}{s_1's_2'}.$$

Similarly, $s_1''s_2'' \in S$ and

$$\begin{aligned} s_1''s_2''a_1a_2s_1's_2' &= (s_1''s_1'a_1)(s_2''s_2'a_2) \\ &= (s_1''s_1a_1')(s_2''s_2a_2') \\ &= s_1''s_2''a_1'a_2's_1s_2 \end{aligned}$$

implies

$$\frac{a_1a_2}{s_1s_2} = \frac{a_1'a_2'}{s_1's_2'}.$$

Thus we have shown that addition and multiplication in (2) are well-defined.

Step 3: Now we show that addition and multiplication in (2) gives us a ring structure. First let us show that addition in (2) gives us an abelian group with $0/1$ being the additive identity. We begin by checking associativity. Let $a_1/s_1, a_2/s_2, a_3/s_3 \in R_S$. Then

$$\begin{aligned} \left(\frac{a_1}{s_1} + \frac{a_2}{s_2}\right) + \frac{a_3}{s_3} &= \frac{s_2a_1 + s_1a_2}{s_1s_2} + \frac{a_3}{s_3} \\ &= \frac{s_3(s_2a_1 + s_1a_2) + (s_1s_2)a_3}{(s_1s_2)s_3} \\ &= \frac{s_3(s_2a_1) + s_3(s_1a_2) + (s_1s_2)a_3}{s_1(s_2s_3)} \\ &= \frac{(s_2s_3)a_1 + s_1(s_3a_2) + s_1(s_2a_3)}{s_1(s_2s_3)} \\ &= \frac{(s_2s_3)a_1 + s_1(s_3a_2 + s_2a_3)}{s_1(s_2s_3)} \\ &= \frac{a_1}{s_1} + \frac{s_3a_2 + s_2a_3}{s_2s_3} \\ &= \frac{a_1}{s_1} + \left(\frac{a_2}{s_2} + \frac{a_3}{s_3}\right). \end{aligned}$$

Thus addition in (2) is associative. Now we check commutativity. Let $a_1/s_1, a_2/s_2 \in R_S$. Then

$$\begin{aligned} \frac{a_1}{s_1} + \frac{a_2}{s_2} &= \frac{s_2a_1 + s_1a_2}{s_1s_2} \\ &= \frac{s_1a_2 + s_2a_1}{s_2s_1} \\ &= \frac{a_2}{s_2} + \frac{a_1}{s_1}. \end{aligned}$$

Thus addition in (2) is commutative. Now we check that $0/1$ is the identity. Let $a/s \in R_S$. Then

$$\begin{aligned}
\frac{0}{1}+\frac{a}{s} &= \frac{s\cdot 0+1\cdot a}{1\cdot s}\\
&= \frac{0+a}{s}\\
&= \frac{a}{s}.
\end{aligned}$$

Thus addition in (2) is commutative. Thus $0/1$ is the identity. Finally we check that every element has an inverse. Let $a/s \in R_S$. Then

$$\begin{aligned}
\frac{a}{s}+\frac{-a}{s} &= \frac{a-a}{s}\\
&= \frac{0}{s}\\
&= \frac{0}{1}.
\end{aligned}$$

implies $-a/s$ is the inverse to a/s. Therefore $(R_S,+)$ forms an abelian group with $0/1$ being identity element.

Now let us show that $(R_S,+,\cdot)$ is a ring. We first check that multiplication in (2) is associative. Let $a_1/s_1, a_2/s_2, a_3/s_3 \in R_S$. Then

$$\begin{aligned}
\left(\frac{a_1}{s_1}\frac{a_2}{s_2}\right)\frac{a_3}{s_3} &= \frac{a_1a_2}{s_1s_2}\frac{a_3}{s_3}\\
&= \frac{(a_1a_2)a_3}{(s_1s_2)s_3}\\
&= \frac{a_1(a_2a_3)}{s_1(s_2s_3)}\\
&= \frac{a_1}{s_1}\frac{a_2a_3}{s_2s_3}\\
&= \frac{a_1}{s_1}\left(\frac{a_2}{s_2}\frac{a_3}{s_3}\right).
\end{aligned}$$

Therefore multiplication in (2) is associative. Next we check that multiplication in (2) distributes over addition. Let $a_1/s_1, a_2/s_2, a_3/s_3 \in R_S$. Then

$$\begin{aligned}
\frac{a_1}{s_1}\left(\frac{a_2}{s_2}+\frac{a_3}{s_3}\right) &= \frac{a_1}{s_1}\left(\frac{s_3a_2+s_2a_3}{s_2s_3}\right)\\
&= \frac{a_1(s_3a_2+s_2a_3)}{s_1s_2s_3}\\
&= \frac{a_1s_3a_2+a_1s_2a_3}{s_1s_2s_3}\\
&= \frac{s_3a_1a_2+s_2a_1a_3}{s_1s_2s_3}\\
&= \frac{s_3a_1a_2}{s_1s_2s_3}+\frac{s_2a_1a_3}{s_1s_2s_3}\\
&= \frac{a_1a_2}{s_1s_2}+\frac{a_1a_3}{s_1s_3}\\
&= \frac{a_1}{s_1}\frac{a_2}{s_2}+\frac{a_1}{s_1}\frac{a_3}{s_3}
\end{aligned}$$

Thus multiplication in (2) distributes over addition. Finally, let us check that $1/1$ is the identity element in R_S under multplication. Let $a/s \in R_S$. Then

$$\begin{aligned}
\frac{1}{1}\cdot\frac{a}{s} &= \frac{1\cdot a}{1\cdot s}\\
&= \frac{a}{s}.
\end{aligned}$$

Thus $1/1$ is the identity element in R_S under multiplication.

Step 4: For the final step, we prove that $\rho_S \colon R \to R_S$ is a ring homomorphism. First note that it sends the identity to the identity. Next, let $a, b \in R$. Then

$$\begin{aligned}\rho_S(a+b) &= \frac{a+b}{1} \\ &= \frac{1 \cdot a + 1 \cdot b}{1 \cdot 1} \\ &= \frac{a}{1} + \frac{b}{1} \\ &= \rho_S(a) + \rho_S(b)\end{aligned}$$

and

$$\begin{aligned}\rho_S(ab) &= \frac{ab}{1} \\ &= \frac{ab}{1 \cdot 1} \\ &= \frac{a}{1} \cdot \frac{b}{1} \\ &= \rho_S(a)\rho_S(b).\end{aligned}$$

Thus ρ_S is a ring homomorphism. □

Definition 0.3. Let S be a multiplicatively closed subset of R and let M be an R-module. We define the **localization of** M **with respect to** S, denoted M_S or $S^{-1}M$, as follows: as a set M_S is given by

$$M_S := \left\{ \frac{u}{s} \mid u \in M, s \in S \right\}$$

where u/s denotes the equivalence class of $(u, s) \in M \times S$ with respect to the following equivalence relation:

$$(u,s) \sim (u',s') \text{ if and only if there exists } s'' \in S \text{ such that } s''s'u = s''su'. \tag{3}$$

We give M_S an R_S-module structure by ring defining addition and scalar multiplication on M_S by

$$\frac{u_1}{s_1} + \frac{u_2}{s_2} = \frac{s_2u_1 + s_1u_2}{s_1s_2} \quad \text{and} \quad \frac{a}{s}\frac{u}{t} = \frac{au}{st}, \tag{4}$$

for $u_1/s_1, u_2/s_2, u/t \in M_S$ and $a/s \in R_S$, with $0/0$ being the additive identity in M_S.

Proposition 0.2. *With the notation above, M_S is an R_S-module. By restricting scalars via the ring the homomoprhism $\rho_S \colon R \to R_S$, it is also an R-module. More specifically, the R-module scalar multiplication is given by*

$$a \cdot \frac{u}{s} = \frac{au}{s}$$

for all $a \in R$ and $u/s \in M_S$.

Proof. The proof of this is similar to the proof of (0.1), but we include it here for completeness. Again, there are several things we need to check, so we break it up into steps.

Step 1: We show that the relation (1) is in fact a equivalence relation. First we show reflexivity of $\sim$. Let $(u,s) \in M \times S$. Then since $1 \in S$ and $1 \cdot su = 1 \cdot su$, we have $(u,s) \sim (u,s)$. Next we show symmetry of $\sim$. Suppose $(u,s) \sim (u',s')$. Choose $s'' \in S$ such that $s''s'u = s''su'$. Then by symmetry of equality, we have $s''su' = s''s'u$. Therefore $(u',s') \sim (u,s)$. Finally, we show transitivity of $\sim$. Suppose $(u_1,s_1) \sim (u_2,s_2)$ and $(u_2,s_2) \sim (u_3,s_3)$. Choose $s_{12}, s_{23} \in S$ such that

$$s_{12}s_2u_1 = s_{12}s_1u_2 \quad \text{and} \quad s_{23}s_3u_2 = s_{23}s_2u_3$$

Then $s_{23}s_{12}s_2 \in S$ and

$$\begin{aligned}(s_{23}s_{12}s_2)(s_3u_1) &= s_{23}(s_{12}s_2u_1)s_3 \\ &= s_{23}(s_{12}s_1u_2)s_3 \\ &= s_{12}s_1(s_{23}s_3u_2) \\ &= s_{12}s_1(s_{23}s_2u_3) \\ &= (s_{12}s_{23}s_2)(s_1u_3).\end{aligned}$$

Thus $\sim$ is in fact an equivalence relation.

Step 2: Addition and multiplication in (4) are well-defined. Suppose $u_1/s_1 = u_1'/s_1'$ and $u_2/s_2 = u_2'/s_2'$. Choose $s_1'', s_2'' \in S$ such that

$$s_1''s_1'u_1 = s_1''s_1u_1' \quad \text{and} \quad s_2''s_2'u_2 = s_2''s_2u_2'.$$

Then $s_1''s_2'' \in S$ and

$$\begin{aligned}s_1''s_2''(s_2u_1 + s_1u_2)s_1's_2' &= s_2''s_2(s_1''s_1'u_1)s_2' + s_1''s_1(s_2''s_2'u_2)s_1' \\ &= s_2''s_2(s_1''s_1u_1')s_2' + s_1''s_1(s_2''s_2u_2')s_1' \\ &= s_2''s_2(s_1''s_1u_1')s_2' + s_1''s_1(s_2''s_2u_2')s_1' \\ &= s_1''s_2''(s_2'u_1' + s_1'u_2')s_1s_2\end{aligned}$$

implies

$$\frac{s_2u_1 + s_1u_2}{s_1s_2} = \frac{s_2'u_1' + s_1'u_2'}{s_1's_2'}.$$

Similarly, $s_1''s_2'' \in S$ and

$$\begin{aligned}s_1''s_2''u_1u_2s_1's_2' &= (s_1''s_1'u_1)(s_2''s_2'u_2) \\ &= (s_1''s_1u_1')(s_2''s_2u_2') \\ &= s_1''s_2''u_1'u_2's_1s_2\end{aligned}$$

implies

$$\frac{a_1a_2}{s_1s_2} = \frac{a_1'a_2'}{s_1's_2'}.$$

Thus we have shown that addition and scalar multiplication in (4) are well-defined.

Step 3: Now we show that addition and multiplication in (4) gives us an R_S-module structure. First let us show that addition in (4) gives us an abelian group with $0/1$ being the additive identity. We begin by checking associativity. Let $u_1/s_1, u_2/s_2, u_3/s_3 \in M_S$. Then

$$\begin{aligned}\left(\frac{u_1}{s_1} + \frac{u_2}{s_2}\right) + \frac{u_3}{s_3} &= \frac{s_2u_1 + s_1u_2}{s_1s_2} + \frac{u_3}{s_3} \\ &= \frac{s_3(s_2u_1 + s_1u_2) + (s_1s_2)u_3}{(s_1s_2)s_3} \\ &= \frac{s_3(s_2u_1) + s_3(s_1u_2) + (s_1s_2)u_3}{s_1(s_2s_3)} \\ &= \frac{(s_2s_3)u_1 + s_1(s_3u_2) + s_1(s_2u_3)}{s_1(s_2s_3)} \\ &= \frac{(s_2s_3)u_1 + s_1(s_3u_2 + s_2u_3)}{s_1(s_2s_3)} \\ &= \frac{u_1}{s_1} + \frac{s_3u_2 + s_2u_3}{s_2s_3} \\ &= \frac{u_1}{s_1} + \left(\frac{u_2}{s_2} + \frac{u_3}{s_3}\right).\end{aligned}$$

Thus addition in (4) is associative. Now we check commutativity. Let $u_1/s_1, u_2/s_2 \in M_S$. Then

$$\begin{aligned}\frac{u_1}{s_1}+\frac{u_2}{s_2} &= \frac{s_2u_1+s_1u_2}{s_1s_2}\\ &= \frac{s_1u_2+s_2u_1}{s_2s_1}\\ &= \frac{u_2}{s_2}+\frac{u_1}{s_1}.\end{aligned}$$

Thus addition in (4) is commutative. Now we check that $0/1$ is the identity. Let $u/s \in M_S$. Then

$$\begin{aligned}\frac{0}{1}+\frac{u}{s} &= \frac{s\cdot 0+1\cdot u}{1\cdot s}\\ &= \frac{0+u}{s}\\ &= \frac{u}{s}.\end{aligned}$$

Thus $0/1$ is the identity. Finally we check that every element has an inverse. Let $u/s \in M_S$. Then

$$\begin{aligned}\frac{u}{s}+\frac{-u}{s} &= \frac{u-u}{s}\\ &= \frac{0}{s}\\ &= \frac{0}{1}.\end{aligned}$$

implies $-u/s$ is the inverse to u/s. Therefore $(M_S, +)$ forms an abelian group with $0/1$ being the identity element.

Now let us show that $(M_S, +, \cdot)$ is an R_S-module. We first check that scalar multiplication in (4) is associative. Let $a_1/s_1, a_2/s_2 \in R_S$ and let $u/s \in M_S$. Then

$$\begin{aligned}\left(\frac{a_1}{s_1}\frac{a_2}{s_2}\right)\frac{u}{s} &= \frac{a_1a_2}{s_1s_2}\frac{u}{s}\\ &= \frac{(a_1a_2)u}{(s_1s_2)s}\\ &= \frac{a_1(a_2u)}{s_1(s_2s)}\\ &= \frac{a_1}{s_1}\frac{a_2u}{s_2s}\\ &= \frac{a_1}{s_1}\left(\frac{a_2}{s_2}\frac{u}{s}\right).\end{aligned}$$

Therefore scalar multiplication in (4) is associative. Next we check that scalar multiplication in (4) distributes over addition. Let $a/s \in R_S$ and $u_1/s_1, u_2/s_2 \in M_S$. Then

$$\begin{aligned}\frac{a}{s}\left(\frac{u_1}{s_1}+\frac{u_2}{s_2}\right) &= \frac{a}{s}\left(\frac{s_2u_1+s_1u_2}{s_1s_2}\right)\\ &= \frac{a(s_2u_1+s_1u_2)}{ss_1s_2}\\ &= \frac{as_2u_1+as_1u_2}{ss_1s_2}\\ &= \frac{s_2au_1+sau_2}{ss_1s_2}\\ &= \frac{s_2au_1}{ss_1s_2}+\frac{sau_2}{ss_1s_2}\\ &= \frac{au_1}{ss_1}+\frac{au_2}{ss_2}\\ &= \frac{a}{s}\frac{u_1}{s_1}+\frac{a}{s}\frac{u_2}{s_2}.\end{aligned}$$

Similarly, let $a_1/s_1, a_2/s_2 \in R_S$ and $u/s \in M_S$. Then

$$\begin{aligned}\left(\frac{a_1}{s_1}+\frac{a_2}{s_2}\right)\frac{u}{s} &= \left(\frac{s_2a_1+s_1a_2}{s_1s_2}\right)\frac{u}{s} \\ &= \frac{(s_2a_1+s_1a_2)u}{s_1s_2s} \\ &= \frac{s_2a_1u+s_1a_2u}{s_1s_2s} \\ &= \frac{s_2a_1u+s_1a_2u}{s_2s_1s} \\ &= \frac{s_2a_1u}{s_2s_1s}+\frac{s_1a_2u}{s_1s_2s} \\ &= \frac{a_1u}{s_1s}+\frac{a_2u}{s_2s} \\ &= \frac{a_1}{s_1}\frac{u}{s}+\frac{a_2}{s_2}\frac{u}{s}.\end{aligned}$$

Thus multiplication in (4) distributes over addition. Finally, let us check that $1/1$ fixes M_S. Let $u/s \in M_S$. Then

$$\begin{aligned}\frac{1}{1}\cdot\frac{u}{s} &= \frac{1\cdot u}{1\cdot s} \\ &= \frac{u}{s}.\end{aligned}$$

Thus $1/1$ fixes M_S. □

Problem 1.b

Lemma 0.1. *Let N be an R-module. Every element in $R_S \otimes_R N$ can be expressed as an elementary tensor of the form $(1/s) \otimes v$ with $s \in S$ and $v \in N$.*

Proof. Let $\sum_{i=1}^n (a_i/s_i) \otimes v_i$ be a general tensor in $R_S \otimes_R N$. Then

$$\begin{aligned}\frac{a_1}{s_1}\otimes v_1+\cdots+\frac{a_n}{s_n}\otimes v_n &= \frac{a_1s_2\cdots s_n}{s_1s_2\cdots s_n}\otimes v_1+\cdots+\frac{s_1s_2\cdots a_n}{s_1s_2\cdots s_n}\otimes v_n \\ &= \frac{1}{s_1s_2\cdots s_n}\otimes a_1s_2\cdots s_nv_1+\cdots+\frac{1}{s_1s_2\cdots s_n}\otimes s_1s_2\cdots a_nv_n \\ &= \frac{1}{s_1s_2\cdots s_n}\otimes(a_1s_2\cdots s_nv_1+\cdots+s_1s_2\cdots a_nv_n) \\ &= \frac{1}{s}\otimes v,\end{aligned}$$

where $s = s_1s_2\cdots s_n$ and $v = a_1s_2\cdots s_nv_1+\cdots+s_1s_2\cdots a_nv_n$. □

Problem 1.c

Proposition 0.3. *Let S be a multiplicatively closed subset of R. Then we have a natural isomorphism between functors*

$$R_S \otimes_R - \colon \mathbf{Mod}_R \to \mathbf{Mod}_{R_S} \quad \textit{and} \quad -_S \colon \mathbf{Mod}_R \to \mathbf{Mod}_{R_S}$$

Proof. For each R-module M, we define $\eta_M \colon R_S \otimes_R M \to M_S$ by

$$\eta_M\left(\frac{1}{s}\otimes u\right) = \frac{u}{s}$$

for all $(1/s)\otimes u \in R_S \otimes_R M$. Every tensor in $R_S \otimes_R M$ can be expressed as an elementary tensor of the form $(1/s)\otimes u$, and so η_M really is defined on all of $R_S \otimes M$. Also η_M is a well-defined R-linear map since the map $R_S \times M \to M_S$ given by

$$\left(\frac{1}{s}, u\right) \mapsto \frac{u}{s}$$

is readily seen to be R-bilinear. The map η_M is surjective since every element in M_S can be expressed in the form u/s. Let us show that η_M is injective. Suppose $(1/s) \otimes u \in \ker \eta_M$. Then $u/s = 0$. Thus there exists a $t \in S$ such that

$$\begin{aligned} tu &= ts \cdot 0 \\ &= 0. \end{aligned}$$

Then this implies

$$\begin{aligned} \frac{1}{s} \otimes u &= \frac{t}{st} \otimes u \\ &= \frac{1}{st} \otimes tu \\ &= \frac{1}{st} \otimes 0 \\ &= 0. \end{aligned}$$

Thus η_M is injective, and hence an isomorphism.

Now we will show that η is a natural transformation. Let $\varphi \colon M \to N$ be an R-linear map. We need to show that the diagram below commutes

$$\begin{tikzcd} R_S \otimes_R M \arrow[r, "\eta_M"] \arrow[d, "1 \otimes \varphi"'] & M_S \arrow[d, "\varphi_S"] \\ R_S \otimes_R N \arrow[r, "\eta_N"'] & N_S \end{tikzcd} \tag{5}$$

Let $(1/s) \otimes u \in R_S \otimes_R M$. Then

$$\begin{aligned} (\varphi_S \eta_M)\left(\frac{1}{s} \otimes u\right) &= \varphi_S\left(\eta_M\left(\frac{1}{s} \otimes u\right)\right) \\ &= \varphi_S\left(\frac{u}{s}\right) \\ &= \frac{\varphi(u)}{s} \\ &= \eta_N\left(\frac{1}{s} \otimes \varphi(u)\right) \\ &= \eta_N\left((1 \otimes \varphi)\left(\frac{1}{s} \otimes u\right)\right) \\ &= (\eta_N(1 \otimes \varphi))\left(\frac{1}{s} \otimes u\right). \end{aligned}$$

Therefore the diagram (5) commutes. □

Problem 1.d

Corollary. *Let $(1/s) \otimes v$ be a tensor in $R_S \otimes_R N$. Then $(1/s) \otimes v = 0$ if and only if there exists a $t \in S$ such that $tv = 0$.*

Proof. We have

$$\begin{aligned} \frac{1}{s} \otimes v = 0 &\iff \eta_N\left(\frac{1}{s} \otimes v\right) = 0 \\ &\iff \frac{v}{s} = 0 \\ &\iff \text{there exists a } t \in S \text{ such that } tv = 0. \end{aligned}$$

□

Problem 1.e

Proposition 0.4. *Let S be a multiplicatively closed subset of R. Then R_S is a flat R-module.*

Proof. Let $\varphi\colon M \to N$ be an injective R-linear map. We must show that $1 \otimes \varphi\colon R_S \otimes_R M \to R_S \otimes_R N$ is injective. Suppose $(1/s) \otimes u \in \ker 1 \otimes \varphi$. Thus $(1/s) \otimes \varphi(u) = 0$. By the corollary above, this implies there exists a $t \in S$ such that $t\varphi(u) = 0$. Thus

$$\begin{aligned} 0 &= t\varphi(u) \\ &= \varphi(tu). \end{aligned}$$

Since φ is injective, this implies $tu = 0$. Applying corollary above again, we see that $(1/s) \otimes u = 0$. Therefore $\ker 1 \otimes \varphi = 0$ and hence $1 \otimes \varphi$ is injective. Thus R_S is a flat R-module. □

Problem 1.f

Proposition 0.5. *$\mathbb{Q}$ is a flat $\mathbb{Z}$-module that is not projective.*

Proof. It follows from Proposition (0.4) that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, so we just need to show that $\mathbb{Q}$ is not projective. Let $\varphi\colon \bigoplus_{i \in \mathbb{N}} \mathbb{Z} \to \mathbb{Q}$ be the unique $\mathbb{Z}$-linear map defined on the standard basis $\{e_n\}$ of $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ by

$$\varphi(e_n) = \frac{1}{n}$$

for all $n \in \mathbb{N}$, and let $\psi\colon \mathbb{Q} \to \mathbb{Q}$ be the identity map. Observe that φ is surjective since if $m/n \in \mathbb{Q}$, then $\varphi(me_n) = m/n$. However there is no $\widetilde{\psi}\colon \mathbb{Q} \to \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ such that $\psi = \varphi\widetilde{\psi}$. Indeed, observe that the injective map

$$\bigoplus_{n \in \mathbb{N}} \mathbb{Z} \to \prod_{n \in \mathbb{N}} \mathbb{Z}$$

induces the injective map

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\right) \to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n \in \mathbb{N}} \mathbb{Z}\right)$$

since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$ is a left-exact covariant functor. Therefore the injection

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\right) &\to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n \in \mathbb{N}} \mathbb{Z}\right) \\ &\cong \prod_{n \in \mathbb{N}} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \\ &\cong 0 \end{aligned}$$

implies

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\right) \cong 0.$$

Thus the only $\mathbb{Z}$-linear map from $\mathbb{Q}$ to $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ is the zero map. □

Problem 2

Proposition 0.6. *Let R be an integral domain with quotient field K, let M be an R-module, and let M_{tor} denote the set of all torsion elements of M. Then*

1. *M/M_{tor} is torsion free.*
2. *$M \otimes_R K \cong M/M_{\text{tor}} \otimes_R K$.*

Proof.
1. Suppose $a \in R\backslash\{0\}$ and $\overline{u} \in M/M_{\text{tor}}$ such that $a\overline{u} = \overline{0}$. Then there exists a $v \in M_{\text{tor}}$ such that $au = v$. Since $v \in M_{\text{tor}}$, there exists a $b \in R\backslash\{0\}$ such that $bv = 0$. Then

$$\begin{aligned}(ba)u &= b(au) \\ &= bv \\ &= 0\end{aligned}$$

implies $u \in M_{\text{tor}}$. Thus $\overline{u} = \overline{0}$.

2. The quotient map $\pi \colon M \to M/M_{\text{tor}}$ induces an R-linear map $\pi \otimes 1 \colon M \otimes_R K \to M/M_{\text{tor}} \otimes_R K$. We claim that $\pi \otimes 1$ is an isomorphism. We will show this by constructing an inverse. Define $\varphi \colon M/M_{\text{tor}} \otimes_R K \to M \otimes_R K$ by

$$\varphi\left(\overline{u} \otimes \frac{a}{s}\right) = u \otimes \frac{a}{s} \tag{6}$$

for all $\overline{u} \otimes (a/s) \in M/M_{\text{tor}} \otimes_R K$. We claim that (6) is well-defined. Indeed, choose another representative of the coset class $\overline{u}$, say $v \in M$. So $u - v \in M_{\text{tor}}$, which means that there exists a nonzero $b \in R$ such that $b(u-v) = 0$. Then

$$\begin{aligned}\varphi\left(\overline{v} \otimes \frac{a}{s}\right) &= v \otimes \frac{a}{s} \\ &= v \otimes \frac{ba}{bs} \\ &= bv \otimes \frac{a}{bs} \\ &= bu \otimes \frac{a}{bs} \\ &= u \otimes \frac{ba}{bs} \\ &= u \otimes \frac{a}{s} \\ &= \varphi\left(\overline{u} \otimes \frac{a}{s}\right).\end{aligned}$$

Also, (6) is R-bilinear in $\overline{u}$ and a/s. Thus φ is well-defined. It is also clearly the inverse to $\pi \otimes 1$. Hence $\pi \otimes 1$ is an isomorphism. □

Problem 3

Lemma 0.2. *Let M_1, M_2, M_3 be R-modules. Then*

$$\operatorname{Hom}_R(M_1, \operatorname{Hom}_R(M_2, M_3)) \cong \operatorname{Hom}_R(M_1 \otimes_R M_2, M_3). \tag{7}$$

Moreover (7) is natural in M_3.

Remark. It is also natural in M_1 and M_2, but we omit the proof of this.

Proof. We define

$$\Psi_{M_3} \colon \operatorname{Hom}_R(M_1, \operatorname{Hom}_R(M_2, M_3)) \to \operatorname{Hom}_R(M_1 \otimes_R M_2, M_3)$$

to be the map which sends a $\psi \in \operatorname{Hom}_R(M_1, \operatorname{Hom}_R(M_2, M_3))$ to the map $\Psi_{M_3}(\psi) \in \operatorname{Hom}_R(M_1 \otimes_R M_2, M_3)$ defined by

$$\Psi_{M_3}(\psi)(u_1 \otimes u_2) = (\psi(u_1))(u_2) \tag{8}$$

for all elementary tensors $u_1 \otimes u_2 \in M_1 \otimes_R M_2$. Note that $\Psi_{M_3}(\psi)$ is a well-defined R-linear map since the map $M_1 \times M_2 \to M_3$ given by

$$(u_1, u_2) \mapsto (\psi(u_1))(u_2)$$

is R-bilinear. Indeed, let $a \in R$. Then we have

$$\begin{aligned}(\psi(au_1))(u_2) &= (a\psi(u_1))(u_2) \\ &= (\psi(u_1))(au_2) \\ &= a((\psi(u_1))(u_2))\end{aligned}$$

since both ψ and $\psi(u_1)$ are R-linear. Similarly, if $v_1 \in M_1$, then

$$\begin{aligned}(\psi(u_1+v_1))(u_2) &= (\psi(u_1)+\psi(v_1))(u_2)\\ &= (\psi(u_1))(u_2)+(\psi(v_1))(u_2),\end{aligned}$$

and if $v_2 \in M_2$, then

$$(\psi(u_1))(u_2+v_2) = (\psi(u_1))(u_2)+(\psi(u_1))(v_2).$$

Thus $\Psi_{M_3}(\psi)$ is a well-defined R-linear map.

Let us check that Ψ_{M_3} is R-linear. Let $a, b \in R$ and $\psi, \varphi \in \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$. Then for all $u_1 \otimes u_2 \in M_1 \otimes_S M_2$, we have

$$\begin{aligned}\Psi_{M_3}(a\varphi+b\psi)(u_1\otimes u_2) &= ((a\varphi+b\psi)(u_1))(u_2)\\ &= (a\varphi(u_1)+b\psi(u_1))(u_2)\\ &= (a\varphi(u_1))(u_2)+(b\psi(u_1))(u_2)\\ &= a(\varphi(u_1))(u_2)+b(\psi(u_1))(u_2)\\ &= a\Psi_{M_3}(\varphi)(u_1\otimes u_2)+b\Psi_{M_3}(\psi)(u_1\otimes u_2)\end{aligned}$$

Thus Ψ_{M_3} is R-linear.

To see that Ψ_{M_3} is an isomorphism, we construct an inverse function. Define

$$\Phi_{M_3}\colon \mathrm{Hom}_R(M_1\otimes_R M_2, M_3) \to \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$$

to be the map which sends $\varphi \in \mathrm{Hom}_R(M_1\otimes_R M_2, M_3)$ to the map $\Phi_{M_3}(\varphi) \in \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$ defined by

$$(\Phi(\varphi)(u_1))(u_2) = \varphi(u_1\otimes u_2)$$

for all $u_1 \in M_1$ and $u_2 \in M_2$. We claim that Ψ_{M_3} and Φ_{M_3} are inverse to each other. Indeed, we have

$$\begin{aligned}(\Psi_{M_3}(\Phi_{M_3}(\varphi)))(u_1\otimes u_2) &= (\Phi_{M_3}(\varphi)(u_1))(u_2)\\ &= \varphi(u_1\otimes u_2).\end{aligned}$$

for all $u_1 \otimes u_2$ and $\varphi \in \mathrm{Hom}_R(M_1\otimes_R M_2, M_3)$. It follows that

$$\Psi_{M_3}\Phi_{M_3} = 1_{\mathrm{Hom}_R(M_1\otimes_R M_2, M_3)}.$$

Similarly, we have

$$\begin{aligned}(\Phi_{M_3}(\Psi_{M_3}(\psi))(u_1))(u_2) &= \Psi_{M_3}(\psi)(u_1\otimes u_2)\\ &= (\psi(u_1))(u_2)\end{aligned}$$

for all $u_1 \in M_1$, $u_2 \in M_2$, and $\psi \in \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$. It follows that

$$\Phi_{M_3}\Psi_{M_3} = 1_{\mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))}.$$

Thus Ψ_{M_3} is an isomorphism.

Naturality in M_3 means that if $\lambda\colon M_3 \to M_3'$ is an R-module homomorphism, then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3)) \arrow[r, "\Psi_{M_3}"] \arrow[d, "(\lambda_*)_*"'] & \mathrm{Hom}_R(M_1\otimes_R M_2, M_3) \arrow[d, "\lambda_*"] \\
\mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3')) \arrow[r, "\Psi_{M_3'}"'] & \mathrm{Hom}_R(M_1\otimes_S M_2, M_3')
\end{tikzcd}$$

Thus we want to show for all $\psi \in \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$, we have

$$\lambda_* \left(\Psi_{M_3}(\psi) \right) = \Psi_{M_3'}((\lambda_*)_*(\psi)) \tag{9}$$

To see that (9) is equal, we apply all elementary tensors to both sides. Let $u_1 \otimes u_2 \in M_1 \otimes_R M_2$. We have

$$\begin{aligned} \left(\lambda_* \left(\Psi_{M_3}(\psi)\right)\right)(u_1 \otimes u_2) &= \lambda\left(\left(\Psi_{M_3}(\psi)\right)(u_1 \otimes u_2)\right) \\ &= \lambda\left((\psi(u_1))(u_2)\right) \\ &= (\lambda_*(\psi(u_1)))(u_2) \\ &= ((\lambda_*)_*(\psi))(u_1))(u_2) \\ &= \left(\Psi_{M_3'}((\lambda_*)_*(\psi))\right)(u_1 \otimes u_2). \end{aligned}$$

□

Problem 4

Proposition 0.7. *Let P and Q be projective R-modules. Then $P \otimes_R Q$ is a projective R-module.*

Proof. It suffices to show that $\mathrm{Hom}_R(P \otimes_R Q, -)$ is exact. Let

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{10}$$

be a short exact sequence. Then since Q is projective, the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(Q, M_1) \longrightarrow \mathrm{Hom}_R(Q, M_2) \longrightarrow \mathrm{Hom}_R(Q, M_3) \longrightarrow 0$$

is exact. Then since P is projective, the induced sequence

$$0 \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_1)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_2)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_3)) \to 0$$

is exact. By tensor-hom adjointness, we have a commutative diagram[1]

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_1)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_2)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, M_3)) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, M_1) \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, M_2) \arrow[r] & \mathrm{Hom}_R(P \otimes_R Q, M_3) \arrow[r] & 0 \end{tikzcd}$$

where the columns are isomorphisms and where the top row is exact. It follows from the 3×3 lemma that the bottom row is exact too.

□

[1]Note how we need naturality in the third argument to get a commutative diagram.