An Alternating Series

Proposition 0.1. *For each $n \in \mathbb{N}$, let*

$$a_n = \sum_{k=n^2}^{(n+1)^2-1} \frac{1}{k}.$$

Then the series

$$\sum_{n=1}^{\infty} (-1)^n a_n \tag{1}$$

converges. Moreover, for each odd positive integer N, we can estimate the Nth tail of the series (1) by

$$\ln\left(\frac{N-1}{N}\right) \leq \sum_{n=N}^{\infty} (-1)^n a_n \leq \ln\left(\frac{N}{N+1}\right).$$

Proof. We show that the series (1) converges by applying the alternating series test. First note that each a_n is clearly positive. Next we check that the sequence (a_n) is eventually decreasing. First note that whenever $n > 1$, we have

$$\begin{aligned} a_n &= \sum_{k=n^2}^{(n+1)^2-1} \frac{1}{k} \\ &\geq \int_{n^2}^{(n+1)^2} \frac{\mathrm{d}x}{x} \\ &= \ln\left(\frac{(n+1)^2}{n^2}\right) \end{aligned}$$

and

$$\begin{aligned} a_n &= \sum_{k=n^2}^{(n+1)^2-1} \frac{1}{k} \\ &\leq \int_{n^2-1}^{(n+1)^2-1} \frac{\mathrm{d}x}{x} \\ &= \ln\left(\frac{(n+1)^2-1}{n^2-1}\right) \\ &= \ln\left(\frac{n(n+2)}{(n-1)(n+1)}\right). \end{aligned}$$

Thus for all $n > 1$, we have

$$\ln\left(\frac{(n+1)^2}{n^2}\right) \leq a_n \leq \ln\left(\frac{n(n+2)}{(n-1)(n+1)}\right). \tag{2}$$

In particular, for all $n > 1$, we have

$$\begin{aligned} a_{n+1} &\leq \ln\left(\frac{(n+1)(n+3)}{n(n+2)}\right) \\ &< \ln\left(\frac{(n+1)^2}{n^2}\right). \\ &\leq a_n \end{aligned}$$

since

$$\frac{(n+1)}{n} > \frac{(n+3)}{(n+2)}.$$

Thus the sequence (a_n) is eventually decreasing. In fact, we can drop the qualifer "eventually" here since

$$\begin{aligned} a_1 &= 1 \\ &\geq \frac{1}{2} + \frac{1}{3} \\ &= a_2, \end{aligned}$$

and so the the sequence (a_n) is a decreasing sequence. The final step is to check that $a_n \to 0$ as $n \to \infty$, but this follows from (2). Thus (1) satisfies all the conditions in the alternating series test, and hence must be convergent.

Now we prove the last part of the proposition. Choose an odd integer $N > 1$ and and an even integer $M > 1$. Observe that

$$\begin{aligned}
\sum_{n=N}^{N+M}(-1)^n a_n &= -a_N + a_{N+1} - \cdots + a_{N+M-1} - a_{N+M} \\
&\leq -\ln\left(\frac{(N+1)^2}{N^2}\right) + \ln\left(\frac{(N+1)(N+3)}{N(N+2)}\right) - \cdots + \ln\left(\frac{(N+M-1)(N+M+1)}{(N+M-2)(N+M)}\right) - \ln\left(\frac{(N+M+1)^2}{(N+M)^2}\right) \\
&= \ln\left(\frac{N^2(N+1)(N+3)\cdots(N+M-1)(N+M+1)(N+M)^2}{(N+1)^2N(N+2)\cdots(N+M-2)(N+M)(N+M+1)^2}\right) \\
&= \ln\left(\frac{N(N+M)}{(N+1)(N+M+1)}\right).
\end{aligned}$$

Letting $M \to \infty$, we see that

$$\sum_{n=N}^{\infty}(-1)^n a_n \leq \ln\left(\frac{N}{N+1}\right)$$

Similarly, observe that

$$\begin{aligned}
\sum_{n=N}^{N+M}(-1)^n a_n &= -a_N + a_{N+1} - \cdots + a_{N+M-1} - a_{N+M} \\
&\geq -\ln\left(\frac{N(N+2)}{(N-1)(N+1)}\right) + \ln\left(\frac{(N+2)^2}{(N+1)^2}\right) - \cdots + \ln\left(\frac{(N+M)^2}{(N+M-1)^2}\right) - \ln\left(\frac{(N+M)(N+M+2)}{(N+M-1)(N+M+1)}\right) \\
&= \ln\left(\frac{(N-1)(N+1)(N+2)^2\cdots(N+M)^2(N+M-1)(N+M+1)}{N(N+2)(N+1)^2\cdots(N+M-1)^2(N+M)(N+M+2)}\right) \\
&= \ln\left(\frac{(N-1)(N+M+1)}{N(N+M+2)}\right).
\end{aligned}$$

Letting $M \to \infty$, we see that

$$\ln\left(\frac{N-1}{N}\right) \leq \sum_{n=N}^{\infty}(-1)^n a_n.$$

Therefore we have the inequality

$$\ln\left(\frac{N-1}{N}\right) \leq \sum_{n=N}^{\infty}(-1)^n a_n \leq \ln\left(\frac{N}{N+1}\right).$$

□