Commutative Algebra Homework 2

Michael Nelson

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Problem 1

Exercise 1. Let R be an integral domain. Then R is a PID if and only if every prime ideal is principal.

Solution 1. If R is a PID, then every ideal in R is principal, so every prime ideal is principal. Conversely, suppose every prime ideal is principal. Let I be an ideal in R and assume for a contradiction that I is not principal. Consider the partially order set $(\Gamma, \subseteq)$ where

$$\Gamma = \{\text{ideals } \mathfrak{a} \mid I \subseteq \mathfrak{a} \subseteq R \text{ and } \mathfrak{a} \text{ not principal}\}$$

and where $\subseteq$ is set inclusion. Note that Γ is nonempty since $I \in \Gamma$. Also note that every totally ordered subset in Γ has an upper bound. Indeed, if $(\mathfrak{a}_\lambda)_{\lambda \in \Lambda}$ is a totally ordered subset, then $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is an upper bound of $(\mathfrak{a}_\lambda)$: the set $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is an ideal which contains I since $(\mathfrak{a}_\lambda)$ is totally ordered and each $\mathfrak{a}_\lambda$ contains I. Also, if $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is principal, then there must exist some $\mathfrak{a}_\lambda$ which is principal (again since $(\mathfrak{a}_\lambda)$ is totally ordered), thus $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is *not* principal. Hence

$$\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda \in \Gamma.$$

Thus using Zorn's Lemma, we see that Γ has a maximal element, say $\mathfrak{p} \in \Gamma$. We claim that $\mathfrak{p}$ is a prime ideal. To see this, assume for a contradiction that $\mathfrak{p}$ is not a prime ideal. Choose $a, b \in R$ such that $ab \in \mathfrak{p}$ and $a, b \notin \mathfrak{p}$. Then observe that $\langle \mathfrak{p}, a \rangle$ and $\langle \mathfrak{p}, b \rangle$ both properly contain $\mathfrak{p}$. By maximality of $\mathfrak{p}$, they must both be principal ideals, say $\langle \mathfrak{p}, a \rangle = \langle x \rangle$ and $\langle \mathfrak{p}, b \rangle = \langle y \rangle$. Then observe that

$$\begin{aligned} \mathfrak{p} &\subseteq \langle \mathfrak{p}, a \rangle \langle \mathfrak{p}, b \rangle \\ &= (\mathfrak{p} + \langle a \rangle)(\mathfrak{p} + \langle b \rangle) \\ &= \mathfrak{p} + \langle a \rangle \mathfrak{p} + \mathfrak{p} \langle b \rangle + \langle ab \rangle \\ &\subseteq \mathfrak{p}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathfrak{p} &= \langle \mathfrak{p}, a \rangle \langle \mathfrak{p}, b \rangle \\ &= \langle x \rangle \langle y \rangle \\ &= \langle xy \rangle. \end{aligned}$$

This is a contradiction since $\mathfrak{p} \in \Gamma$. Thus $\mathfrak{p}$ is a prime ideal. However by assumption *all* prime ideals are principal, so $\mathfrak{p}$ being prime implies $\mathfrak{p}$ is principal. But this again contradicts the fact that $\mathfrak{p} \in \Gamma$. Thus every ideal in R must be principal.

Problem 2

Exercise 2. Let R be a commutative ring with identity. Show that the following conditions are equivalent:

1. Every ascending chain of ideals in R stabilizes: if (I_n) is ascending chain of ideals in R, meaning $I_n \subseteq I_{n+1}$ for all $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ such that $I_N = I_n$ for all $n \geq N$.
2. Every ideal of R is finitely generated.

Solution 2. Suppose every chain of ideal in R stabilizes and let I be an ideal in R. Assume for a contradiction that I is not finitely generated. Choose any $x_1 \in I$. Since I is not finitely generated, we have

$$\langle x_1 \rangle \subset I$$

where the inclusion is proper. Next we choose $x_2 \in I \backslash \langle x_1 \rangle$. Again, since I is not finitely generated, we have

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset I$$

where each inclusion is proper. Proceeding inductively on $n \geq 3$, we choose $x_n \in I \backslash \langle x_1, \ldots, x_{n-1} \rangle$. Then since I is not finitely generated, we have

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \subset \langle x_1, x_2, \ldots, x_n \rangle \subset I$$

where each inclusion is proper. Continuing in this manner, we construct an ascending chain of ideals

$$(\langle x_1, x_2 \ldots, x_n \rangle)_{n \in \mathbb{N}}$$

which never stabilizes since $\langle x_1, x_2, \ldots, x_n \rangle$ is properly contained in $\langle x_1, x_2, \ldots, x_n, x_{n+1} \rangle$ for all $n \in \mathbb{N}$. This contradicts the hypothesis that every chain of ideal in R stabilizes. Thus every ideal in R is finitely generated.

Now let us show the converse. Suppose every ideal in R is finitely generated. Let (I_n) be an ascending chain of ideals. Then $\bigcup_{n=1}^{\infty} I_n$ is an ideal in R since (I_n) is totally ordered, thus it must be finitely generated, say

$$\bigcup_{n=1}^{\infty} I_n = \langle x_1, \ldots, x_m \rangle.$$

Observe that $x_i \in I_{n_i}$ for some $n_i \in \mathbb{N}$ for each $1 \leq i \leq m$. Set $N = \max_{1 \leq i \leq m}\{n_i\}$. Then $x_i \in I_N$ for each $1 \leq i \leq m$ since (I_n) is totally ordered. It follows that for any $n \geq N$, we have

$$\begin{aligned} I_N &\subseteq I_n \\ &\subseteq \bigcup_{n=1}^{\infty} I_n \\ &= \langle x_1, \ldots, x_m \rangle \\ &\subseteq I_N. \end{aligned}$$

In particular we have $I_N = I_n$ for all $n \geq N$. Thus every chain of ideals in R stabilizes.

Problem 3

Exercise 3. Let R be an integral domain and let A be an overring of R: that is, $R \subseteq A \subseteq K$ where K is the field of fractions of R.

1. Show that R is a PID if and only if R is a UFD and $\dim R \leq 1$.
2. Show that if R is a UFD then any localization of R is a UFD.
3. Show that if R is a PID, then A is a localization of R.
4. Is 3 true for UFDs? Prove or give a counterexample.

Solution 3. 1. Suppose R is a UFD and $\dim R \leq 1$. If $\dim R = 0$, then R must a field: the zero ideal is prime since R is a domain and if $\dim R = 0$, then no prime ideal can contain the zero ideal, so the zero ideal must be maximal, hence $R/\langle 0 \rangle \cong R$ shows that R is a field. So we may assume $\dim R = 1$. To show that R is a PID, it suffices to show that every nonzero prime ideal in R is principal, by problem 1. But this is easy! Indeed, let $\mathfrak{p}$ be a nonzero prime ideal in R. Since R is a UFD, $\mathfrak{p}$ contains a nonzero prime element, say $p \in \mathfrak{p}$. Then we have

$$0 \subset \langle p \rangle \subseteq \mathfrak{p},$$

where the first inclusion is proper since p is nonzero. Thus R having dimension 1 forces $\langle p \rangle = \mathfrak{p}$. Thus every prime ideal in R is principal, and we are done.

Conversely, suppose R is a PID. To show that R is a UFD, we just need to show that all prime ideals in R contains a nonzero prime element. However this is clear as every prime ideal is principal and hence generated by a prime element. See the Appendix for an alternative proof of the fact that all PIDs are UFDs. It remains to show that $\dim R \leq 1$. Assume for a contradiction that $\langle p \rangle$ and $\langle q \rangle$ are prime ideals in R with p and q being nonzero prime elements in R such that $\langle q \rangle$ properly contains $\langle p \rangle$, so $p = aq$ for some $a \in R$. Since $q \notin \langle p \rangle$ we see that $a \in \langle p \rangle$ which implies $a = bp$ for some $b \in R$. Thus

$$\begin{aligned} p &= aq \\ &= bpq \\ &= pbq \end{aligned}$$

implies $1 = bq$ since R is an integral domain. However this means q is a unit, which is a contradiction since q is prime. Thus we cannot have a proper inclusion of nonzero prime ideals in R. This implies $\dim R \leq 1$.

2. Let R be a UFD and let S be a multipicatively closed subset of R. We want to show that R_S is a UFD also. To do this, we will show that every prime ideal in R_S contains a nonzero prime element. Let $\mathfrak{p}_S$ be a prime ideal in R_S where $\mathfrak{p}$ is a prime ideal in R such that $\mathfrak{p} \cap S = \emptyset$ (every prime ideal in R_S has this form by Theorem (0.2)). Since R is a UFD, the prime $\mathfrak{p}$ contains a nonzero prime element, say $p \in \mathfrak{p}$. Then the ideal generated by p is a prime ideal, and furthermore, it intersects S trivially since it is contained in $\mathfrak{p}$; that is

$$\langle p \rangle \cap S = \emptyset.$$

It follows that $\langle p \rangle_S$ is a prime ideal in R_S (again by Theorem (0.2)). Note $\langle p \rangle_S = \langle p/1 \rangle$ where $\langle p/1 \rangle$ denotes the ideal in R_S generated by $p/1$. Therefore $p/1$ is a prime element in R_S which is clearly contained in $\mathfrak{p}_S$. Thus R_S is a UFD.

3. Let $S = \{y \in R \mid 1/y \in A\}$. Observe that S is a multiplicatively closed subset of R since if $y_1, y_2 \in S$, then $y_1 y_2 \in S$ since

$$1/(y_1 y_2) = (1/y_1)(1/y_2) \in A.$$

Every element in R_S has the form x/y where $x \in R$, $1/y \in A$ and $\gcd(x, y) = 1$. Since $R \subseteq A$, we see that any $x/y \in R_S$ is an element of A, thus $R_S \subseteq A$. To show the reverse inclusion, let $x/y \in A$, where $x, y \in R$ and $\gcd(x, y) = 1$. We need to show that $1/y \in A$. Since R is a PID and $\gcd(x, y) = 1$, we have $\langle x, y \rangle = 1$. Thus there exists $a, b \in R$ such that $ax + by = 1$. Then observe that

$$\begin{aligned} \frac{1}{y} &= \frac{ax + by}{y} \\ &= a\left(\frac{x}{y}\right) + b \\ &\in A. \end{aligned}$$

It follows that $x/y \in R_S$. Thus $A \subseteq R_S$.

4. No. Let k be a field, let $R = k[X, Y]$, let $A = k[X, Y, X/Y]$, and let $K = k(X, Y)$ be the field of fractions of R. Then A is an overring of R which is contained in K. However A is not the localization of R at any multiplicative set S. Indeed, assume for a contradiction that S is a multiplicative subset of R such that $R_S = A$. Then since $X/Y \in A$, we have

$$X/Y = f/g$$

for some $f \in R$ and $g \in S$, where we may assume (by canceling common factors if necessary) that $\gcd(f, g) = 1$. Then we have

$$gX = Yf.$$

Since $K[X, Y]$ is a UFD and $\gcd(X, Y) = \gcd(f, g) = 1$, we see that $g = \alpha Y$ where $\alpha \in K^\times$. However $1/\alpha Y \notin A$, so this is a contradiction.

Appendix

PIDs are UFDs

Theorem 0.1. *Let R be a principal ideal domain. Then R is a unique factorization domain.*

Proof. Let a be nonzero nonunit in R. Since R is a Noetherian, an irreducible factorization of a exists, so it suffices to check that such an irreducible factorization is unique. Let

$$p_1 \cdots p_m = a = q_1 \cdots q_n \tag{1}$$

be two irreducible factorizations of a. By relabeling if necessary, we may assume that $m \leq n$. We will prove by induction on $m \geq 1$ that $m = n$ and (perhaps after relabeling) we have $p_i \sim q_i$ for all $1 \leq i \leq m$. For base case $m = 1$, we have

$$p_1 = a = q_1 \cdots q_n.$$

The first step will be to show that $n = 1$. To prove this, we assume for a contradiction that $n > 1$. Since R is a principal ideal domain, every irreducible is a prime. In particular, p_1 is prime. Thus $p_1 \mid q_i$ for some $1 \leq i \leq n$.

By relabeling necessary, we may assume that $p_1 \mid q_1$. In terms of ideals, this means $\langle q_1 \rangle \subseteq \langle p_1 \rangle$. Since both $\langle q_1 \rangle$ and $\langle p_1 \rangle$ are maximal ideals, this implies $\langle q_1 \rangle = \langle p_1 \rangle$. Thus $q_1 = xp_1$ for some $x \in R^\times$. This implies

$$\begin{aligned} 0 &= p_1 - q_1 q_2 \cdots q_n \\ &= p_1 - xp_1 q_2 \cdots q_n \\ &= p_1(1 - xq_2 \cdots q_n). \end{aligned}$$

Again $p_1 \neq 0$ and R an integral domain implies $xq_2 \cdots q_n = 1$, thus $q_2 \cdots q_n \in R^\times$. This is a contradiction as each $q_2, \dots, q_n$ are irreducible! Thus $n = 1$, and clearly in this case, we have $p_1 \sim q_1$ (as $p_1 = q_1$).

Now suppose $m > 1$ and we have shown that if a has an irreducible factorization of length k where $1 \leq k < m$, then it has a unique irreducible factorization. Again, let (1) be two irreducible factorizations of a where we may assume that $m \leq n$. Arguing as above, p_1 is prime, and since $q_1 \cdots q_n \in \langle p_1 \rangle$, we must have $q_i \in \langle p \rangle$ for some $1 \leq i \leq n$. By rebaling if necessary, we may assume that $q_1 \in \langle p_1 \rangle$. Thus $\langle q_1 \rangle \subseteq \langle p_1 \rangle$, and since both $\langle q_1 \rangle$ and $\langle p_1 \rangle$ are maximal ideals, we must in fact have $\langle q_1 \rangle = \langle p_1 \rangle$. In particular, $q_1 = p_1 x$ for some $x \in R^\times$. This implies

$$\begin{aligned} 0 &= p_1 p_2 \cdots p_m - q_1 q_2 \cdots q_n \\ &= p_1 p_2 \cdots p_m - p_1 x q_2 \cdots q_n \\ &= p_1(p_2 \cdots p_m - xq_2 \cdots q_n). \end{aligned}$$

Since $p_1 \neq 0$ and R is an integral domain, this implies

$$p_2 \cdots p_m = xq_2 \cdots q_n.$$

Note that xq_2 is an irreducible element, and thus we may apply induction step to get $m = n$ and (perhaps after relabeling) $p_i \sim q_i$ for all $2 \leq i \leq m$. Since already we have $p_1 \sim q_1$, we are done. □

Prime Ideals in R_S

Theorem 0.2. *Let S be a multiplicatively closed subset of R. Then we have a bijection*

$$\Psi \colon \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\} \to \operatorname{Spec} R_S$$

given by $\Psi(\mathfrak{p}) = \mathfrak{p}_S$ for all prime ideals $\mathfrak{p}$ in R such that $\mathfrak{p} \cap S = \varnothing$. Then inverse to Ψ, which we denote by

$$\Phi \colon \operatorname{Spec} R_S \to \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\}$$

is given by $\Phi(\mathfrak{q}) = \rho^{-1}(\mathfrak{q})$ for all prime ideals $\mathfrak{q}$ in R_S where $\rho \colon R \to R_S$ is the canonical localization map.

Proof. First note that both Ψ and Φ land in their designated target spaces. Indeed, for any prime ideal $\mathfrak{q}$ in $\operatorname{Spec} R_S$, the ideal $\rho^{-1}(\mathfrak{q})$ is easily seen to be prime in R. Also if $\mathfrak{p}$ is a prime ideal in R such that $\mathfrak{p} \cap S = \varnothing$, then $\mathfrak{p}_S$ is a prime ideal in R_S. Indeed, let $x/s, y/t \in \mathfrak{p}_S$, where $x, y \in \mathfrak{p}$ and $s, t \in S$, and suppose $(x/s)(y/t) \in \mathfrak{p}_S$. Then $xy/st \in \mathfrak{p}_S$, which implies $xy \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we have either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Without loss of generality, say $x \in \mathfrak{p}$. Then clearly $x/s \in \mathfrak{p}_S$. This implies $\mathfrak{p}_S$ is prime.

We now want to show that these two maps are inverse to each other. First let us show that Ψ is injective. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be two distinct primes in R such that $\mathfrak{p} \cap S = \mathfrak{p}' \cap S = \varnothing$. Without loss of generality, say $\mathfrak{p} \not\subseteq \mathfrak{p}'$. Choose $x \in \mathfrak{p} \backslash \mathfrak{p}'$. Then observe that $x/1 \in \mathfrak{p}_S$. Furthermore, we also have $x/1 \notin \mathfrak{p}'_S$. Indeed, assume for a contradiction $x/1 \in \mathfrak{p}'_S$. Then $x/1 = y/s$ with $y \in \mathfrak{p}'_S$ and $s \in S$. Then there exists $t \in S$ such that $tsx = ty \in \mathfrak{p}'$. As $\mathfrak{p}'$ is prime and $s, t \notin \mathfrak{p}'$, we must have $x \in \mathfrak{p}'$, which is a contradiction. This shows that $\mathfrak{p}_S$ and $\mathfrak{p}'_S$ are distinct, and hence Ψ is injective.

Now we will show Ψ is surjective. Let $\mathfrak{q} \in \operatorname{Spec} R_S$. We claim that $\mathfrak{q} = \rho^{-1}(\mathfrak{q})_S$. Indeed, we have

$$\begin{aligned} \rho^{-1}(\mathfrak{q})_S &= \{x/s \mid x \in \rho^{-1}(\mathfrak{q}) \text{ and } s \in S\} \\ &= \{x/s \mid x/1 \in \mathfrak{q} \text{ and } s \in S\} \\ &= \mathfrak{q}, \end{aligned}$$

where equality in the last line follows from the fact that $\mathfrak{q}$ is prime: if $x/s \in \mathfrak{q}$, then $x/1 \in \mathfrak{q}$ since $1/s \notin \mathfrak{q}$ and $x/s = (x/1)(1/s)$. Thus Ψ is surjective and hence a bijection. In proving that Ψ is surjective, we also see that the inverse of Ψ is Φ. □