Section 4.4: Inflection Points and Second Derivatives

The **second derivative** of a function $f(x)$ is the derivative of the derivative function $f'(x)$. It is denoted by $f''(x)$ and read as "f *double prime of* x". The unit of measure for the second derivative is *output units of* f' *per input unit of* f', or in terms of the function f, *output units of* f *per input unit of* f *per input unit of* f.

An **inflection point** is a point at which a continuous function f changes concavity. On a graph of a differentiable function f, it is either the point of greatest slope (*most rapid change*) or the point of least slope (*least rapid change*) and it corresponds to a relative extreme point of $f'(x)$.

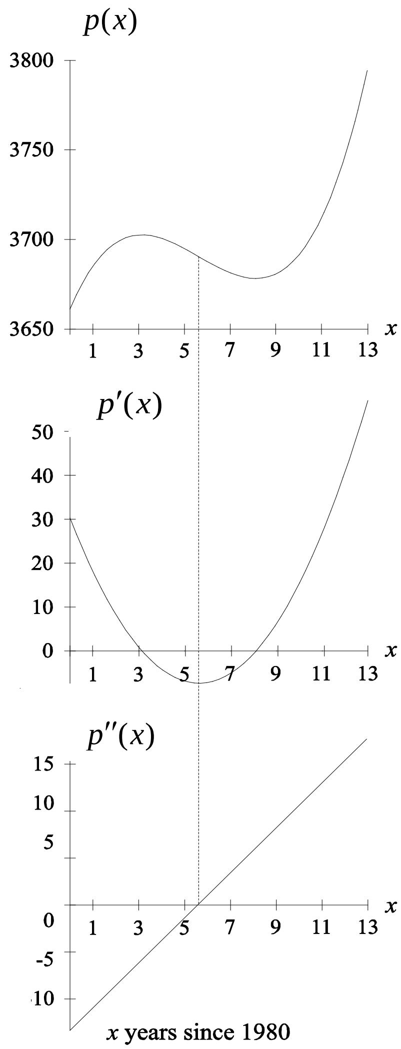
If there is an **inflection point** at $x = c$, then $f''(c) = 0$ or $f''(c)$ does not exist. Solutions to the equation $f''(c) = 0$ or points at which $f''(c)$ does not exist are *possible* location(s) of inflection points of the function $f(x)$.

Second Derivatives and Concavity:

- On an interval on which $f''(c) > 0$, a function f is concave up.
- On an interval on which $f''(c) < 0$, a function f is concave down.

For a continuous function f, if f'' changes from positive to negative as x increases through c, or from negative to positive as x increases through c, then f has an inflection point at c. Graphically, if the second derivative graph crosses the x-axis or lies on opposite sides of the x-axis at c, f has an inflection point at c.

Example 1: (CC5e p. 273)



The population of Kentucky can be modeled as $p(x) = 0.395x^3 - 6.67x^2 + 30.3x + 3,661$ thousand people where x is the number of years since 1980, $0 \le x \le 13$.

a. Find the equation and write the output units for $p'(x)$.

b. Find the equation and write the output units for $p''(x)$.

c. Place an "X" on the graph of $p(x)$ that indicates the inflection point, the point at which the population of Kentucky is *decreasing most rapidly* on the given interval.

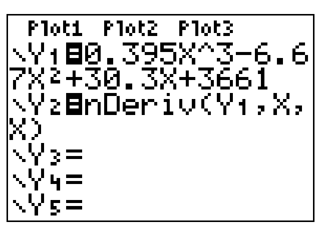
d. The graph of $p'(x)$ has a *relative maximum /relative minimum* at the inflection point on $p(x)$.

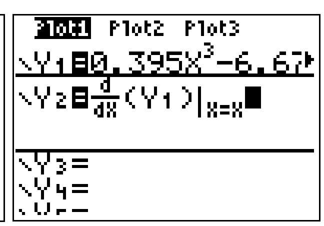
e. Use the calculator to find the point at which the population of Kentucky is *decreasing most rapidly* on the given interval.

Between 1980 and 1993, Kentucky's population was decreasing most rapidly _____________ years after 1980.

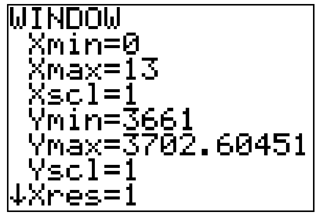
Finding both coordinates of an inflection point:

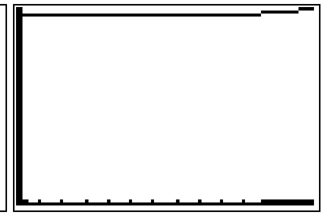
- Enter $p(x)$ in Y1
- Enter **nDeriv (Y1, X, X)** or $\frac{d}{dx}(Y_1)\big|_{X=X}$ in Y2 using **MATH 8**



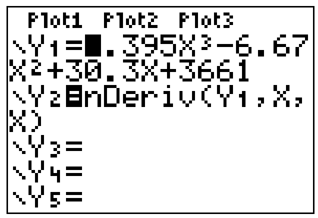


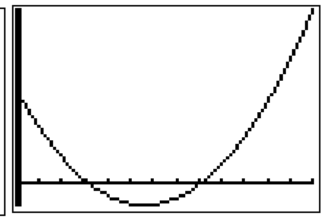
- In **WINDOW**, set Xmin = **0** and Xmax = **13**
- **ZOOM** **0** [ZoomFit] returns the graphs of p and p' which are difficult to see in this example. If both functions are visible, refer to example 8 at the end of this section for an alternate method.



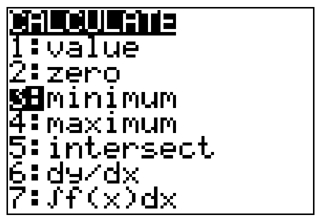


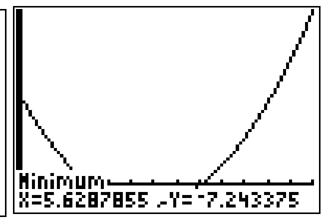
- Turn Y1 off and ZoomFit again to view the graph of p' by itself.





- Find the relative minimum of $p'(x)$ using the process outlined in the previous section. The x-coordinate of the relative minimum of p' is the same as the x-coordinate of the inflection point of p.

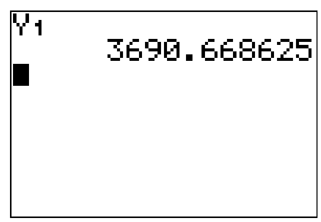




f. How quickly is the population changing at that time?

g. What is Kentucky's population at that time?

- To eliminate intermediate rounding of the input value, use **MODE** to double-check that the calculator is set to FLOAT the number of decimals. (see Chapter 1.1)
- Return to the Home Screen and evaluate **Y1** **ENTER** which finds the y-coordinate of the inflection point.



h. The graph of $p''(x) =$ _______ at the inflection point on $p(x)$.
Solve the equation $p''(x) = 0$. Is the solution the same as the solution in part e?

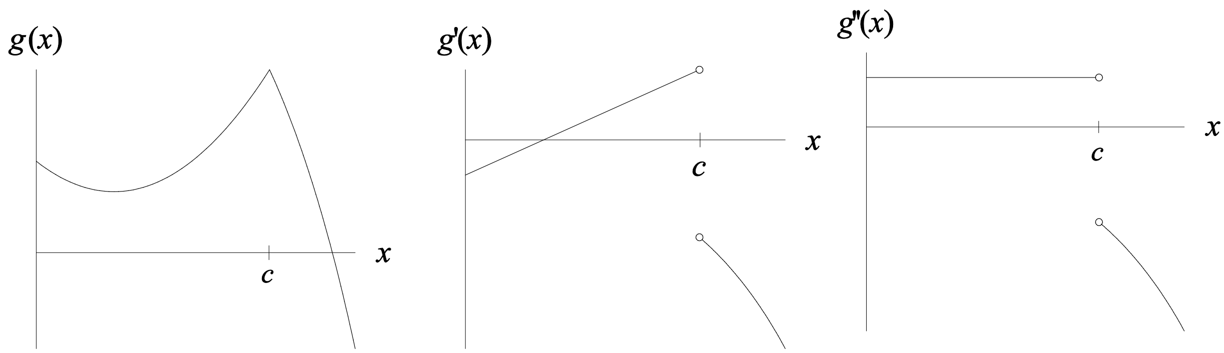
i. The change in concavity at the inflection point is indicated by a change in sign on the second derivative graph from *positive to negative / negative to positive*.

On the interval to the left of the inflection point, the graph of $p(x)$ is *concave up/concave down* and the graph of $p''(x)$ lies *above the x-axis/below the x-axis.*

On the interval to the right of the inflection point, the graph of $p(x)$ is *concave up/concave down* and the graph of $p''(x)$ lies *above the x-axis/below the x-axis.*

Example 2: (CC5e p. 279)

The three graphs, g, g', and g'', are shown below.



a. At an input value of $x = c$, is there a change in concavity in $g(x)$?

   Does the function $g(x)$ have an inflection point at $x = c$?

b. Which is correct: $g''(c) = 0$ or $g''(c)$ does not exist?

c. For input values $x < c$, $\underline{g''(c) < 0 \,/\, g''(c) > 0}$ and $g(x)$ is concave *up/down.*

   For input values $x > c$, $\underline{g''(c) < 0 \,/\, g''(c) > 0}$ and $g(x)$ is concave *up/down.*

Second Derivative Test for Relative Extrema (to determine whether a critical point is a relative extreme point):

Suppose a function f is continuous over an interval containing c,

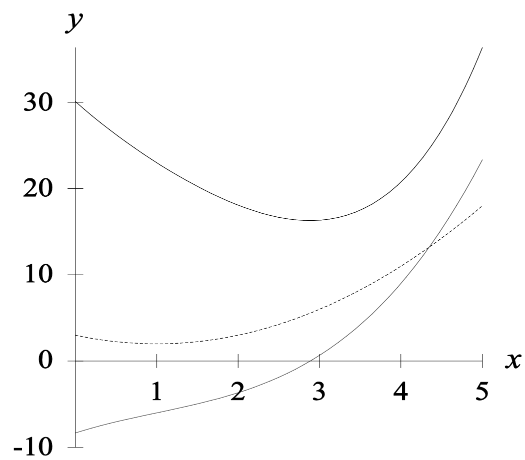
- If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c.

Example 3:

In section 4.2, $p(x) = 0.395x^3 - 6.67x^2 + 30.3x + 3661$ was found to have a critical point at $x = 3.156$. The *first derivative test* confirmed that there is in fact a relative maximum at this point (see graphs in Example 1 above). The *second derivative test* also confirms this: $\underline{p''(3.156) > 0 \,/\, p''(3.156) < 0}$, which means that $p(x)$ is concave *up/down* at $x = 3.156$, and thus a relative maximum occurs at $x = 3.156$.

Example 4: (CC5e p. 277)

The figure to the right shows graphs of a function f, its first derivative f', and its second derivative f'' on the interval $0 \le x \le 5$.



a. Identify and label the graphs of f, f', and f''.

Complete the following statements:

b. The graph of the function f is concave *up/down* on the given interval and has a relative *maximum/minimum* near $x = 3$.

c. The graph of the first derivative f' lies *above/below* the x-axis for $x < 3$, lies *above/below* the x-axis for $x > 3$, and is *increasing/decreasing* on the interval $0 \le x \le 5$.

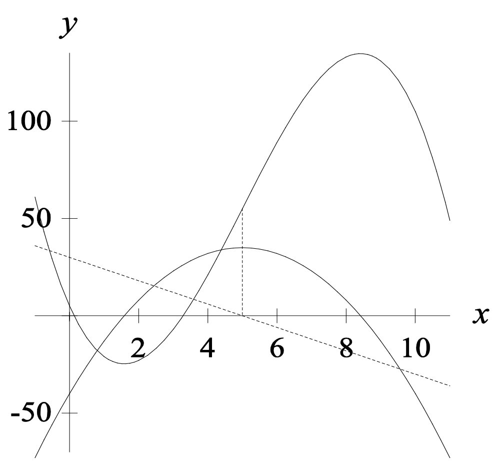
f' has a ___________________________ near $x = 3$.

d. The graph of the second derivative f'' lies *above/below* the x-axis and is *positive/negative* on the interval $0 \le x \le 5$.

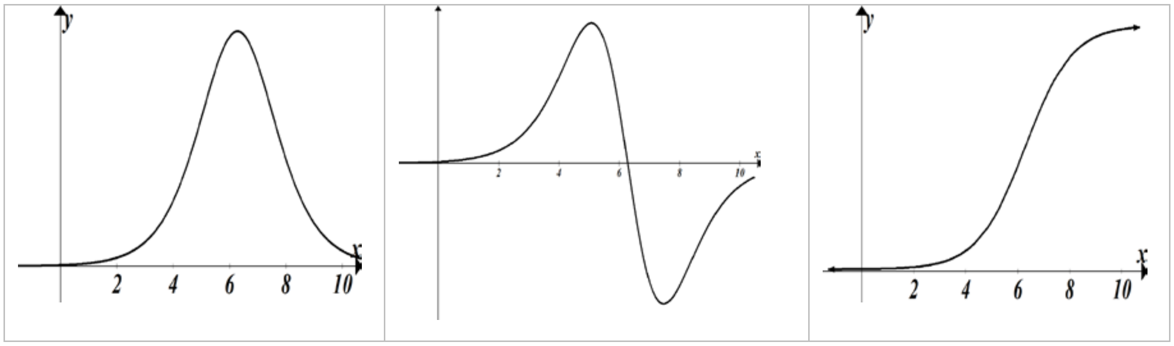
Example 5: (part a, CC5e p. 274)

Use relative extreme points, inflection points, direction, and concavity to determine which of the three graphs shown is the graph of f, f', and f''.

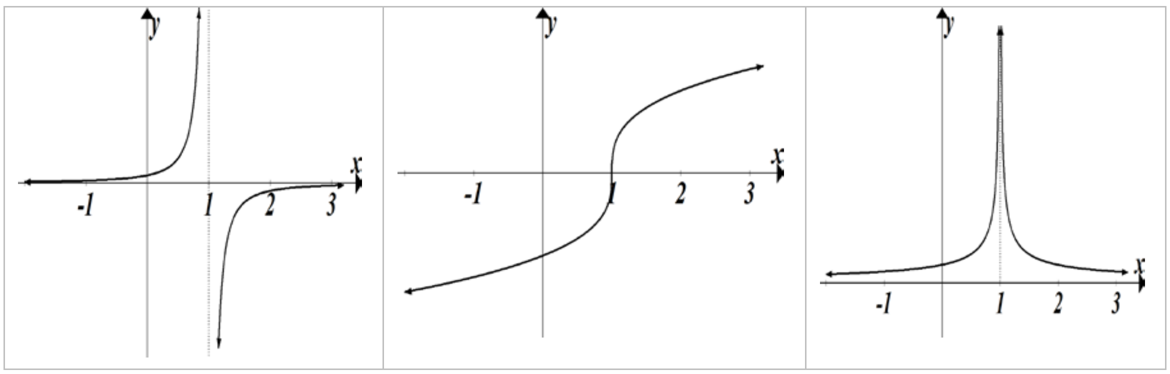
a.



b.

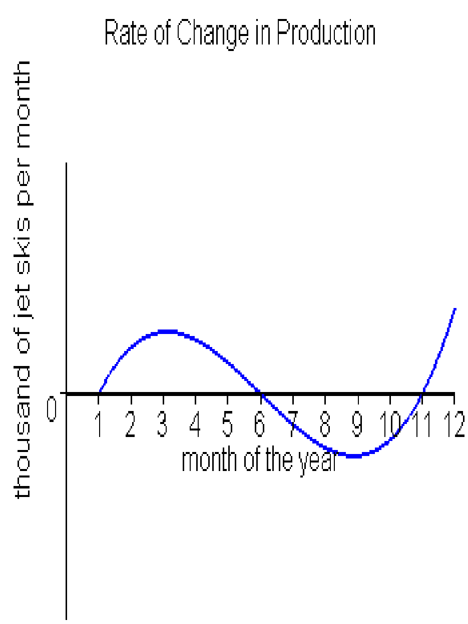


c.



Example 6:

The derivative graph to the right shows the rate of change in last year's production level of jet skis at a manufacturing company. 1= end of January, etc.



Complete the following statements by naming the appropriate months.

a. Last year, the production of jet skis was increasing from the end of _____________through the end of _____________________ and again from the end of _____________through the end of ______________.

b. Last year, the production of jet skis was decreasing most rapidly at the end of ____________________.

c. Last year the production of jet skis reached a relative maximum at the end of _______________________.

d. Last year the production of jet skis reached a relative minimum at the end of ________________________.

Example 7:

Suppose that $f(x)$ is a continuous and differentiable function.

a. Using the graph of $f'(x)$ shown to the left below, complete the following statements.

At $x \approx -1.3$, $f(x)$ has a(n) *relative maximum/ relative minimum/inflection point*.

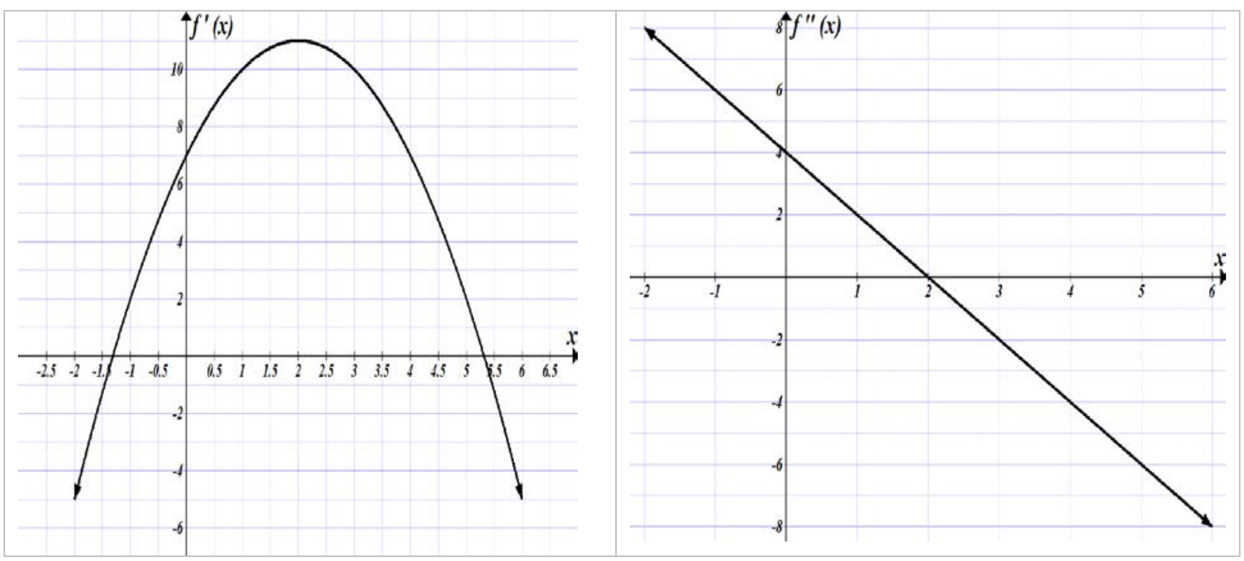
At $x \approx 5.3$, $f(x)$ has a(n) *relative maximum/ relative minimum/inflection point*.

b. Using the graph of $f''(x)$ shown to the right below, complete the following statements.

At $x = 2$, $f(x)$ has a(n) *relative maximum/ relative minimum/inflection point*.

On the interval $x < 2$, $f(x)$ is *concave up/concave down.*

On the interval $x > 2$, $f(x)$ is *concave up/concave down.*

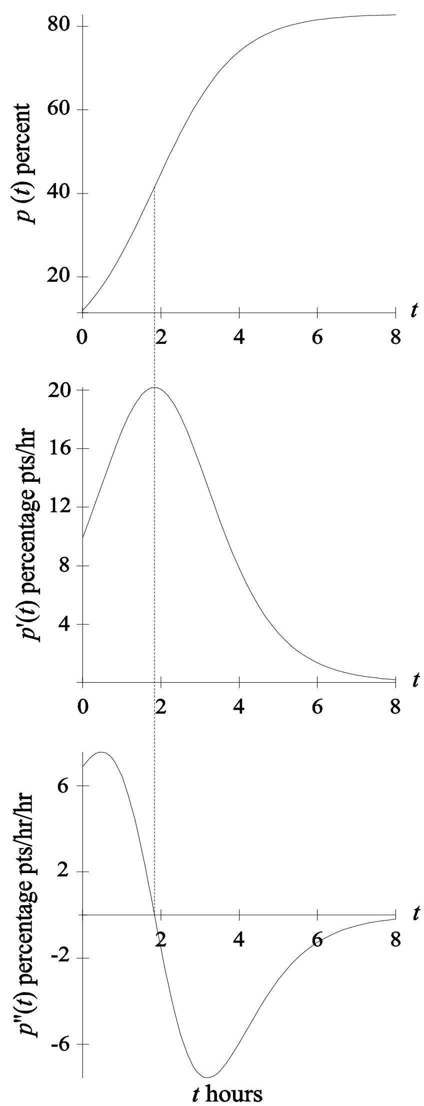


If an inflection point appears on an *increasing* function that is *concave down to the right* of the inflection point, that point is regarded as the **point of diminishing returns** since each additional unit of input results in a smaller gain in output.

Example 8: (CC5e p. 276)

$p(t) = \dfrac{83}{1+5.94e^{-0.969t}}$ percent gives the percentage of new material that an average college student retains after studying for t hours without a break, $0 \le t \le 8$.

The graphs of p, p', and p'' are shown to the right.



a. How much of the material is retained by a student who has not started studying $(t = 0)$?

b. What is the point of diminishing returns? After how many hours of study is a student's retention rate increasing most rapidly?

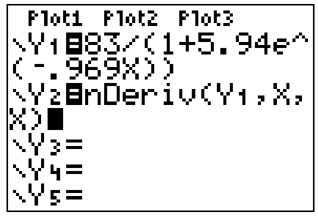
c. At the point of diminishing returns, how quickly is the retention rate changing and what is the retention rate? Include units with your answers.

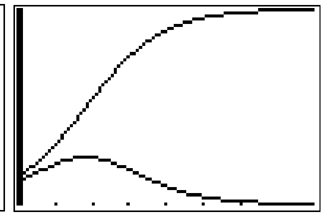
d. Why is the second derivative graph below the t-axis to the right of the point of diminishing returns?

Another example of finding both coordinates of an inflection point:

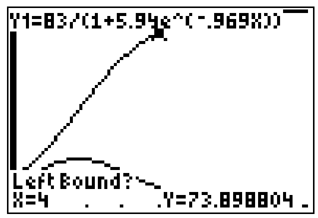
- Enter $p(t)$ in Y1 and $p'(t)$ in Y2

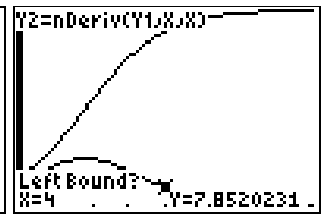
- Set the WINDOW and ZOOMFIT to get the graph of Y1 and at Y2.



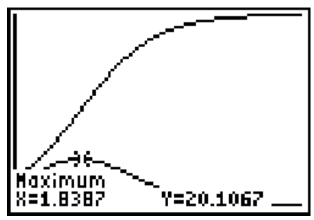


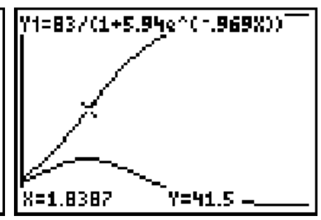
- Use the process described earlier in this section to find the maximum of $p'(t)$. *Initially the header indicates Y1 is selected. Use ▼ to select the derivative function Y2.*





- The first screen shows the maximum of $p'(t)$.
- Use ▲ to move the cursor to the function and obtain the output value at the inflection point.





Example 9:

Suppose that $f(x)$ is a continuous and differentiable function. For each characteristic of the graph of $f(x)$ in the table below, describe the corresponding feature on the graph of $f'(x)$. Then, in the last three rows, also describe the corresponding feature on the graph of $f''(x)$.

Graph of $f(x)$ is/has ….	**Graph of $f'(x)$ is/has…**	**Graph of $f''(x)$ is/has…**
Relative Maximum/Minimum (not at a sharp point)		
Increasing		
Decreasing		
Concave up		
Concave down		
Inflection point (without a vertical tangent)		