Free Resolutions Homework 3

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Troughout this homework assignment, let R be a commutative ring with identity and let $\mathbf{x} = x_1, \dots, x_n \in R$.

Exercise 1

Lemma 0.1. *(R-linearity of homology) Let $\varphi, \psi \colon (A, d) \to (A', d')$ be two chain maps and let $r, s \in R$. Then*

$$H(r\varphi + s\psi) = rH(\varphi) + sH(\psi)$$

Proof. Let $\overline{a} \in H(A, d)$. Then

$$\begin{aligned} H(r\varphi + s\psi)(\overline{a}) &= \overline{(r\varphi + s\psi)(a)} \\ &= \overline{r\varphi(a) + s\psi(a)} \\ &= r\overline{\varphi(a)} + s\overline{\psi(a)} \\ &= rH(\varphi)(\overline{a}) + sH(\psi)(\overline{a}). \\ &= (rH(\varphi) + sH(\psi))(\overline{a}). \end{aligned}$$

□

Definition 0.1. Let φ and ψ be two chain maps between R-complexes (A, d) and (A', d'). We say φ is **homotopic** to ψ if there exists a graded homomorphism $h \colon A \to A'$ of degree 1 such that

$$\varphi - \psi = d'h + hd.$$

In the case where $\psi = 0$, then we say φ is **null-homotopic**.

Proposition 0.1. *Let φ and ψ be chain maps of chain complexes (A, d) and (A', d'). If φ is homotopic to ψ, then $H(\varphi) = H(\psi)$.*

Proof. Showing $H(\varphi) = H(\psi)$ is equivalent to showing $H(\varphi - \psi) = 0$. Thus, we may assume that φ is null-homotopic homotopic and that we are trying to show that $H(\varphi) = 0$. Let $\overline{a} \in H(A, d)$. Then $d(a) = 0$, and so

$$\begin{aligned} H(\varphi)(\overline{a}) &= \overline{\varphi(a)} \\ &= \overline{(d'h + hd)(a)} \\ &= \overline{d'(h(a)) + h(d(a))} \\ &= \overline{d'(h(a)) + h(0)} \\ &= \overline{d'(h(a))} \\ &= 0. \end{aligned}$$

□

Exercises 2a,2b

Proposition 0.2. *Let $\lambda \in [n]$. Then the homothety map*

$$(\mathcal{K}(\mathbf{x}), d^{\mathcal{K}(\mathbf{x})}) \xrightarrow{\cdot x_\lambda} (\mathcal{K}(\mathbf{x}), d^{\mathcal{K}(\mathbf{x})})$$

is null-homotopic. In particular, $x_\lambda H(\mathcal{K}(\mathbf{x})) = 0$.

Proof. Denote $d := d^{\mathcal{K}(\mathbf{x})}$ and let $h\colon \mathcal{K}(\mathbf{x}) \to \mathcal{K}(\mathbf{x})$ be the unique graded homomorphism of degree 1 such that

$$h(e_\sigma) = e_\lambda e_\sigma$$

for all $\sigma \subseteq [n]$. Then

$$\begin{aligned}(dh + hd)(e_\sigma) &= d(e_\lambda e_\sigma) + e_\lambda d(e_\sigma)\\ &= x_\lambda e_\sigma - e_\lambda d(e_\sigma) + e_\lambda d(e_\sigma)\\ &= x_\lambda e_\sigma\end{aligned}$$

for all $\sigma \subseteq [n]$. It follows that

$$dh + hd = \mu_{x_\lambda}$$

on all of $\mathcal{K}(\mathbf{x})$. Thus the homothety map μ_{x_λ} is null-homotopic. $\square$

Corollary. *Let $\lambda \in [n]$. Then $x_\lambda H(\mathcal{K}(\mathbf{x})) = 0$.*

Proof. Let $\overline{f} \in H(\mathcal{K}(\mathbf{x}))$. Combining Proposition (0.2) and (Proposition (0.1), we see that

$$\begin{aligned}0 &= H(0)(\overline{f})\\ &= H(\mu_{x_\lambda})(\overline{f})\\ &= \overline{x_\lambda f}\\ &= x_\lambda \overline{f}.\end{aligned}$$

$\square$

Exercise 2c

Proposition 0.3. *The following conditions are equivalent.*

1. $\langle \mathbf{x} \rangle = R$,
2. $H(\mathcal{K}(\mathbf{x})) \cong 0$,
3. $H_0(\mathcal{K}(\mathbf{x})) \cong 0$.

Proof. Throughout this proof, we denote $d := d^{\mathcal{K}(\mathbf{x})}$.

($1 \implies 2$) Since $\langle \mathbf{x} \rangle = R$, there exists $y_1, \dots, y_n \in R$ such that

$$\sum_{\lambda=1}^{n} x_\lambda y_\lambda = 1.$$

Choose such $y_1, \dots, y_n \in R$. Let $\overline{f} \in H(\mathcal{K}(\mathbf{x}))$. So $f \in \mathrm{Ker}(d)$ is a representative of the coset $\overline{f}$ (meaning $d(f) = 0$). Then

$$\begin{aligned}d\left(\sum_{\lambda=1}^{n} y_\lambda e_\lambda f\right) &= \sum_{\lambda=1}^{n} y_\lambda d(e_\lambda f)\\ &= \sum_{\lambda=1}^{n} y_\lambda (d(e_\lambda) f - e_\lambda d(f))\\ &= \sum_{\lambda=1}^{n} y_\lambda x_\lambda f\\ &= \left(\sum_{\lambda=1}^{n} y_\lambda x_\lambda\right) f\\ &= f.\end{aligned}$$

Thus, $f \in \mathrm{Im}(d)$, and this implies $H(\mathcal{K}(\mathbf{x})) = 0$.

($2 \implies 3$) $H(\mathcal{K}(\mathbf{x})) = 0$ if and only if $H_i(\mathcal{K}(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$. In particular, $H(\mathcal{K}(\mathbf{x})) = 0$ implies $H_0(\mathcal{K}(\mathbf{x})) = 0$.

($3 \implies 1$) We have $0 \cong H_0(\mathcal{K}(\mathbf{x})) = R/\langle \mathbf{x} \rangle$, which implies $\langle \mathbf{x} \rangle = R$.

$\square$

Appendix

In this appendix, we introduce notation and show that the Koszul complex is a DG algebra.

Ordered Sets

An **ordered set** is a set with a total linear ordering on it. The **ordered set** $[n]$ is the set $\{1, \dots, n\}$ equipped with the natural ordering $1 < \cdots < n$. Let σ be a subset of $\{1, \dots, n\}$. Then the natural ordering on $\{1, \dots, n\}$ induces a natural ordering on σ. If we want to think of σ as a set equipped with this natural ordering, then we will write $[\sigma]$. If $\sigma = \{\lambda_1, \dots, \lambda_k\}$, where $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$, then we will also write $[\sigma] = [\lambda_1, \dots, \lambda_k]$. For each $i \in \mathbb{Z}$ such that $0 \leq i \leq n$, we denote

$$S_i[n] := \{\sigma \subseteq \{1, \dots, n\} \mid |\sigma| = i\}.$$

Signature

Let $\sigma, \tau \subseteq [n]$ such that $\sigma \cap \tau = \emptyset$. Suppose that

$$[\sigma] = [\lambda_1, \dots, \lambda_k] \quad \text{and} \quad [\sigma'] = [\lambda_{k+1}, \dots, \lambda_{k+m}].$$

where $1 \leq \lambda_1 < \cdots < \lambda_k \leq n$ and $1 \leq \lambda_{k+1} < \cdots < \lambda_{k+m} \leq n$. Then we have

$$[\sigma \cup \sigma'] = [\lambda_{\pi(1)}, \dots, \lambda_{\pi(k)}, \lambda_{\pi(k+1)}, \dots, \lambda_{\pi(k+m)}],$$

where $\pi \colon S_{k+m} \to S_{k+m}$ is the permutation which puts everything in the correct order. We define

$$\langle \sigma, \tau \rangle := \operatorname{sign}(\pi).$$

Remark. Let $\lambda \in [n]$ and let $\sigma \subseteq [n]$. To clean notation, we often drop the curly brackets around singleton elements $\{\lambda\}$. For instance, we will write $\sigma \backslash \lambda$ instead of $\sigma \backslash \{\lambda\}$ and $\sigma \cup \lambda$ instead of $\sigma \cup \{\lambda\}$. We will also write $\langle \lambda, \sigma \rangle$ or $\langle \sigma, \lambda \rangle$ instead of $\langle \{\lambda\}, \sigma \rangle$ or $\langle \sigma, \{\lambda\} \rangle$.

Example 0.1. Consider $n = 4$. We perform some computations:

$$\begin{aligned}
\langle 2, [1,4] \rangle &= -1 \\
\langle 2, 3 \rangle &= 1 \\
\langle [1,4], 2 \rangle &= -1 \\
\langle 2, [1,3,4] \rangle &= -1 \\
\langle [1,3,4], 2 \rangle &= 1 \\
\langle [1,3], [2,4] \rangle &= -1 \\
\langle [2,4], [1,3] \rangle &= -1
\end{aligned}$$

Signature Identities

Proposition 0.4. *Let $\sigma, \tau \subseteq [n]$ such that $\sigma \cap \tau = \emptyset$. If $\lambda \in \sigma$, then*

$$\langle \sigma, \tau \rangle = \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle.$$

Similarly, if $\mu \in \tau$, then

$$\langle \sigma, \tau \rangle = \langle \sigma, \mu \rangle \langle \sigma, \tau \backslash \mu \rangle. \tag{1}$$

Proof. Suppose $\lambda \in \sigma$. We can set $\sigma \cup \tau$ into proper order by moving λ all the way to the left of σ, then set $\sigma \backslash \lambda \cup \tau$ into proper order, then set $\lambda \cup (\sigma \backslash \lambda \cup \tau)$ into proper order. This gives us

$$\begin{aligned}
\langle \sigma, \tau \rangle &= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, (\sigma \backslash \lambda) \cup \tau) \rangle \\
&= \langle \lambda, \sigma \backslash \lambda \rangle \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \sigma \backslash \lambda \rangle \langle \lambda, \tau \rangle \\
&= \langle \sigma \backslash \lambda, \tau \rangle \langle \lambda, \tau \rangle
\end{aligned}$$

An analagous argument gives (1). □

Koszul Complex

Definition 0.2. The **Koszul complex** of $\underline{x}$, denoted $(\mathcal{K}(\underline{x}), d^{\mathcal{K}(\underline{x})})$ is the R-complex whose graded R-module $\mathcal{K}(x)$ has

$$\mathcal{K}_i(\underline{x}) := \begin{cases} \bigoplus_{\sigma \in S_i[n]} Re_\sigma & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \text{ or if } i < 0. \end{cases}$$

as its ith homogeneous component, and whose differential $d^{\mathcal{K}(\underline{r})}$ is uniquely determined by

$$d^{\mathcal{K}(\underline{x})}(e_\sigma) = \sum_{\lambda \in \sigma} \langle \lambda, \sigma \backslash \lambda \rangle x_\lambda e_{\sigma \backslash \lambda}$$

for all nonempty $\sigma \subseteq \{1, \ldots, n\}$.

Differential Graded R-Algebras

Definition 0.3. A **differential graded R-algebra** is an R-complex (A, d) equipped with a chain map

$$m \colon (A \otimes_R A, d^{A \otimes_R A}) \to (A, d),$$

denoted $a \otimes b \mapsto m(a \otimes b)$ (or just $a \otimes b \mapsto ab$ if context is clear) such that the underlying graded R-module A becomes an associative and unital R-algebra with respect to m.

Remark. Let us flesh out what this means. Let $i, j \in \mathbb{Z}$ and let $a \otimes b \in A_i \otimes_R A_j$. Then for m to be a chain map, we need

$$d(ab) = d(a)b + (-1)^i a d(b) \tag{2}$$

We call (**??**) the **Leibniz law**.

Proposition 0.5. *The Koszul complex is a DG algebra, with multiplication being uniquely determined on elementary tensors: for $\sigma, \tau \subseteq [n]$, we map $e_\sigma \otimes e_\tau \mapsto e_\sigma e_\tau$, where*

$$e_\sigma e_\tau = \begin{cases} \langle \sigma, \tau \rangle e_{\sigma \cup \tau} & \text{if } \sigma \cap \tau = \emptyset \\ 0 & \text{if } \sigma \cap \tau \neq \emptyset \end{cases} \tag{3}$$

Proof. Throughout this proof, denote $d := d^{\mathcal{K}(\mathbf{x})}$. We first note that $e_\emptyset$ serves as the identity for the multiplication rule (3). Indeed, let $\sigma \subseteq [n]$. Then since $\sigma \cap \emptyset = \emptyset$, we have

$$e_\sigma e_\emptyset = e_\sigma = e_\emptyset e_\sigma.$$

Moreover, multiplication by $e_\emptyset$ and e_σ given in (3) satisfies Leibniz law:

$$\begin{aligned} d(e_\sigma)e_\emptyset - e_\sigma d(e_\emptyset) &= d(e_\sigma)e_\emptyset \\ &= d(e_\sigma) \\ &= d(e_\sigma e_\emptyset), \end{aligned}$$

and similarly

$$\begin{aligned} d(e_\emptyset)e_\sigma + e_\emptyset d(e_\sigma) &= e_\emptyset d(e_\sigma) \\ &= d(e_\sigma) \\ &= d(e_\emptyset e_\sigma), \end{aligned}$$

Next, let $\lambda \in [n]$. Suppose $\tau \subseteq [n]$ and $\lambda \notin \tau$. Then

$$\begin{aligned}
d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle x_\mu e_{\tau\backslash\mu} \\
&= x_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \langle \lambda, \tau\backslash\mu\rangle x_\mu e_{\tau\backslash\mu\cup\lambda} \\
&= x_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle \langle \lambda, \tau\rangle \langle \lambda, \mu\rangle x_\mu e_{\tau\backslash\mu\cup\lambda} \\
&= x_\lambda e_\tau + \sum_{\mu\in\tau} \langle \lambda, \tau\rangle \langle \mu, \tau\backslash\mu\rangle \langle \mu, \lambda\rangle x_\mu e_{\tau\backslash\mu\cup\lambda} \\
&= x_\lambda e_\tau + \sum_{\mu\in\tau} \langle \lambda, \tau\rangle \langle \mu, \tau\backslash\mu\cup\lambda\rangle x_\mu e_{\tau\backslash\mu\cup\lambda} \\
&= \langle \lambda, \tau\rangle\langle \lambda, \tau\rangle x_\lambda e_\tau + \sum_{\mu\in\tau} \langle \lambda, \tau\rangle \langle \mu, \tau\backslash\mu\cup\lambda\rangle x_\mu e_{\tau\backslash\mu\cup\lambda} \\
&= \langle \lambda, \tau\rangle \sum_{\mu\in\tau\cup\lambda} \langle \mu, (\tau\cup\lambda)\backslash\mu\rangle x_\mu e_{(\tau\cup\lambda)\backslash\mu} \\
&= \langle \lambda, \tau\rangle d(e_{\tau\cup\lambda}) \\
&= d(e_\lambda e_\tau),
\end{aligned}$$

where we used Proposition (0.4) to get from the second line to the third line. Next suppose $\tau \subseteq [n]$ and $\lambda \in \tau$. Then

$$\begin{aligned}
d(e_\lambda)e_\tau - e_\lambda d(e_\tau) &= x_\lambda e_\tau - e_\lambda \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle x_\mu e_{\tau\backslash\mu} \\
&= x_\lambda e_\tau - \sum_{\mu\in\tau} \langle \mu, \tau\backslash\mu\rangle x_\mu e_\lambda e_{\tau\backslash\mu} \\
&= x_\lambda e_\tau - \langle \lambda, \tau\backslash\lambda\rangle\langle \lambda, \tau\backslash\lambda\rangle x_\lambda e_\tau \\
&= x_\lambda e_\tau - x_\lambda e_\tau \\
&= 0 \\
&= d(0) \\
&= d(e_\lambda e_\tau).
\end{aligned}$$

Thus we have shown (3) satisfies the Leibniz law for all pairs (λ, τ) where $\lambda \in [n]$ and $\tau \subseteq [n]$. We prove by induction on $|\sigma| = i \geq 1$ that (3) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$. The base case $i = 1$ was just shown. Now suppose we have shown (3) satisfies the Leibniz law for all pairs (σ, τ) where $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i < n$. Let $\sigma, \tau \subseteq [n]$ such that $|\sigma| = i + 1$. Choose $\lambda \in \sigma$. Then

$$\begin{aligned}
d(e_\sigma e_\tau) &= d(e_\lambda e_{\sigma\backslash\lambda} e_\tau) \\
&= x_\lambda e_{\sigma\backslash\lambda} e_\tau - e_\lambda d(e_{\sigma\backslash\lambda} e_\tau) \\
&= x_\lambda e_{\sigma\backslash\lambda} e_\tau - e_\lambda (d(e_{\sigma\backslash\lambda})e_\tau + (-1)^{|\sigma|-1} e_{\sigma\backslash\lambda} d(e_\tau)) \\
&= (x_\lambda e_{\sigma\backslash\lambda} - e_\lambda d(e_{\sigma\backslash\lambda}))e_\tau + (-1)^{|\sigma|} e_\sigma d(e_\tau) \\
&= d(e_\lambda e_{\sigma\backslash\lambda})e_\tau + (-1)^{|\sigma|} e_\sigma d(e_\tau) \\
&= d(e_\sigma)e_\tau + (-1)^{|\sigma|+1} e_\sigma d(e_\tau),
\end{aligned}$$

where we used the base case on the pairs $(e_\lambda, e_{\sigma\backslash\lambda} e_\tau)$[1] and $(e_\lambda, e_{\sigma\backslash\lambda})$ and where we used the induction hypothesis on the pair $(e_{\sigma\backslash\lambda}, e_\tau)$. and where we used the base case on the pair $(e_\lambda, e_{\sigma\backslash\lambda})$. □

[1]If $e_{\sigma\backslash\lambda} e_\tau = 0$, then obviously Leibniz law holds for the pair $(e_\lambda, e_{\sigma\backslash\lambda} e_\tau)$.