MATH 8500: HOMEWORK 2

DUE WEDNESDAY, JANUARY 27TH

You may work on homework together, but you must write up your solutions individually and write the names of the individuals with whom you worked. If you use any materials outside of the course materials, e.g., the internet, a different book, or discuss the problems with *anyone other than me*, make sure to provide a short citation.

PROBLEMS:

(1) Let $\mathbb{F}_2$ be the field with two elements. In other words, $\mathbb{F}_2 = \{0, 1\}$ where addition is performed modulo 2 and multiplication is the standard multiplication.
   (a) Show that $x^2y + y^2x$ vanishes on $\mathbb{F}_2^2$.
   (b) Prove that $\langle x^2 - x, y^2 - y\rangle \subseteq \mathcal{I}(\mathbb{F}_2^2)$.
   (c) Show that every $f \in \mathbb{F}_2[x, y]$ can be written as $f = A(x^2 - x) + B(y^2 - y) + axy + bx + cy + d$ where $A, B \in \mathbb{F}_2[x, y]$ and $a, b, c, d \in \mathbb{F}_2$.
   (d) Show that $axy + bx + cy + d \in \mathcal{I}(\mathbb{F}_2^2)$ if and only if $a = b = c = d = 0$.
   (e) From here, conclude that $\langle x^2 - x, y^2 - y\rangle = \mathcal{I}(\mathbb{F}_2^2)$

(2) Let $f = x^3 - x^2y - x^2z$, $f_1 = x^2y - z$ and $f_2 = xy - 1$.
   (a) Use the lexicographic order to compute the remainder r_1 of f when divided by (f_1, f_2) and the remainder r_2 of f when divided (f_2, f_1).
   (b) Find an expression for $r = r_1 - r_2$ in $\langle f_1, f_2\rangle$, i.e., find $A, B \in k[x, y, z]$ such that $r = Af_1 + Bf_2$ for r.

(3) A basis (generating set) $\{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ for a monomial ideal I is *minimal* if no x^{α_i} divides any x^{α_j} for $i \neq j$.
   (a) Prove that every monomial ideal has a minimal basis.
   (b) Prove that every monomial ideal has a *unique* minimal basis.

(4) Consider $\mathbb{Z}^n \subseteq \mathbb{C}^n$. Prove that if f vanishes on $\mathbb{Z}^n$, then f is the zero polynomial. From this, conclude that $\mathcal{I}(\mathbb{Z}^n) = \langle 0\rangle$.

(5) Consider the system of equations

$$2x^2 + y^2 = 3$$
$$x^2 + xy + y^2 = 3$$

   (a) Compute a Gröbner basis for the corresponding ideal using the lexicographic order.
   (b) Symbolically find the four common solutions to these equations.
   (c) Let f be the smallest degree polynomial in I in y (i.e., x does not appear in the polynomial). Symbolically, find the roots of f and compare them to what you found in Part (b).