Math 9853: Homework #7

Due Thursday, October 31, 2019

(a) Let $f\colon V \to V$ be any linear map where V is a vector space of dimension n over a field K. Suppose there is a vector $v \in V$ so that $v, f(v), \ldots, f^{n-1}(v)$ are independent, hence form a basis of V.

(a.1) Show that there exist $a_0, a_1, \ldots, a_{n-1} \in K$ so that

$$f^n(v) = a_{n-1}f^{n-1}(v) + \cdots + a_1 f(v) + a_0 v.$$

(a.2) Use (a.1) to find the matrix of f under the basis $\{v, f(v), \ldots, f^{n-1}(v)\}$.

(a.3) Find the characteristic polynomial of f.

(b) Let $f\colon V \to V$ be any linear map of vector spaces over a field K. Recall that, for any polynomial $p(X) = \sum_{i=0}^{n} c_i X^i \in K[X]$ and any $v \in V$,

$$p(X) \cdot v = p(f)(v) = \sum_{i=0}^{n} c_i f^i(v).$$

The kernel of $p(X)$ is defined to be

$$\mathrm{Ker}(p(X)) = \{v \in V : p(X) \cdot v = 0\}.$$

(b.1) Show that $\mathrm{Ker}(p(X))$ is a linear subspace of V. When $p(X) = X - \lambda$ where $\lambda \in K$, explain that $\mathrm{Ker}(P(X))$ is the eigenspace of f with respect to λ.

(b.2) Let $p(X)$ and $q(X)$ be polynomials in $K[X]$ so that $\gcd(p(X), q(X)) = 1$. Show that

$$\mathrm{Ker}(p(X)q(X)) = \mathrm{Ker}(p(X)) + \mathrm{Ker}(q(X))$$

is a direct sum of subspaces. (**Hint**: Use the fact that if $\gcd(p(X), q(X)) = 1$ then there exist $a(X), b(X) \in K[X]$ so that $a(X)p(X) + b(X)q(X) = 1$.)

(b.3) Let $c(X) \in K[X]$ be any nonzero polynomial so that $c(f) = 0$ (for example $c(X)$ is the characteristic polynomial of f). Suppose

$$c(X) = p_1(X)p_2(X) \cdots p_m(X)$$

where each $p_i(X) \in K[X]$ and $\gcd(p_i(X), p_j(X)) = 1$ for all pairs $1 \le i < j \le m$. Then

$$V = \mathrm{Ker}(p_1(X)) + \mathrm{Ker}(p_2(X)) + \cdots + \mathrm{Ker}(p_m(X))$$

is a direct sum of subspaces.

(b.4) Let λ_i, $1 \le i \le t$, be different eigenvalues of f. Let $B_i = \{u_{ij} : 1 \le j \le m_i\}$ be a basis for the eigenspace of λ_i for $1 \le i \le t$. Use (b.3) to show that $B_1 \cup B_2 \cup \cdots \cup B_t$ is independent.