Homework #4 Solutions

Let me know if you have any questions!

Sentences of Interpretations

(1) $T(x)$ degrees Fahrenheit gives the temperature of an oven x minutes after it's been turned.

(1.a) Write a sentence of interpretation for $\frac{dT}{dx}|_{x=2} = 26$.

Sentence of Interpretation: The temperature of the oven is increasing by 26 degrees Fahrenheit per minute, 2 minute after it has been turned on.
What: The temperature of the oven
I/D: increasing (with an "ing" at the end because we are considering only one input, namely at 2 minutes).
By How Much: by 26 degrees Fahrenheit per minute (where we use the word *per* here because this is a sentence of interpretation for a *rate* of change).
When: 2 minutes after it has been turned on.

(1.b) Write a sentence of interpretation for $T(2) = 120$.

Sentence of Interpretation: The temperature of the oven is 120 degrees Fahrenheit, 2 minutes after it has been turned on.
What: The temperature of the oven
How Much: is 120 degrees Fahrenheit
When: 2 minutes after it has been turned on.

(1.c) The table below gives the temperature of the oven at various minutes after it's been turned on.

x minutes	$T(x)$ degrees Fahrenheit
0	70
2	120
5	200
9	400

Write a sentence of interpretation of the average rate of change of $T(x)$ between 1 minute and 9 minutes.

Calculation: We first calculate average rate of change of $T(x)$ between 2 minute and 9 minutes. This is given by

$$\begin{aligned} \frac{T(9) - T(2)}{9 - 2} &= \frac{400 - 120}{9 - 2} \\ &= \frac{280}{7} \\ &= 40. \end{aligned}$$

Sentence of Interpretation: The temperature of the oven increased on average by 40 degrees Fahrenheit per minute between 2 minutes and 9 minutes after it has been turned on.
What: The temperature of the oven
I/D: increased on average (with an "ed" at the end because we are considering two input points, namely 2 minutes and 9 minutes).

By How Much: by 40 degrees Fahrenheit per minute (where we use the word *per* here because this is a sentence of interpretation for an average *rate* of change).
When: between 2 minutes and 9 minutes after it has been turned on.

(1.d) Write a sentence of interpretation for the percent change of $T(x)$ between 2 minutes and 9 minutes.

Calculation: We first calculate the percent change of $T(x)$ between 2 minutes and 9 minutes. This is given by

$$\begin{aligned}\frac{T(9) - T(2)}{T(2)} \cdot 100 &= \frac{400 - 120}{120} \cdot 100 \\ &\approx 233.333\end{aligned}$$

Sentence of Interpretation: The temperature of the oven increased by 233.333 percent between 2 minutes and 9 minutes after it has been turned on.
What: The temperature of the oven
I/D: increased (with an "ed" at the end because we are considering two input points, namely 2 minutes and 9 minutes).
By How Much: by 233.333 percent
When: between 2 minutes and 9 minutes after it has been turned on.

(1.e) Write a sentence of interpretation for the percent rate of change of $T(x)$ at $x = 2$.

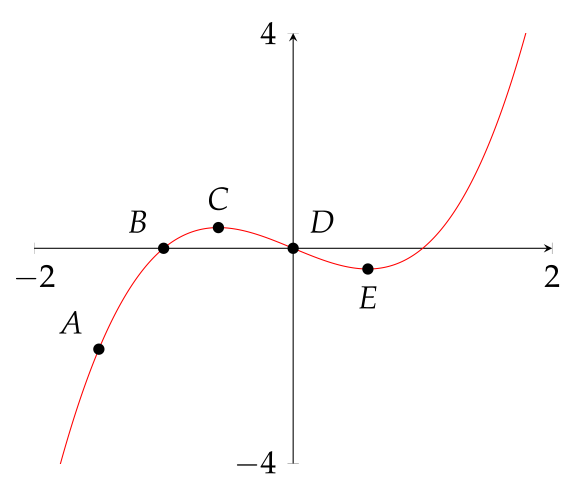
Calculation: We first calculate the percent rate of change of $T(x)$ at $x = 2$. This is given by

$$\begin{aligned}\frac{T'(2)}{T(2)} \cdot 100 &= \frac{26}{120} \cdot 100 \\ &\approx 21.667\end{aligned}$$

Sentence of Interpretation: The temperature of the oven is increasing by 21.667 percent per minute, 2 minutes after it has been turned on.
What: The temperature of the oven
I/D: increasing
By How Much: by 21.667 percent per minute
When: 2 minutes after it has been turned on.

Slope Graphs

(2) Let $f(x)$ be the function whose graph is given below:



(2.a) List the points A, B, C, D, E from least to greatest steepness.

C, E, D, B, A

(2.b) List the points A, B, C, D, E from least to greatest slope.

D, C, E, B, A

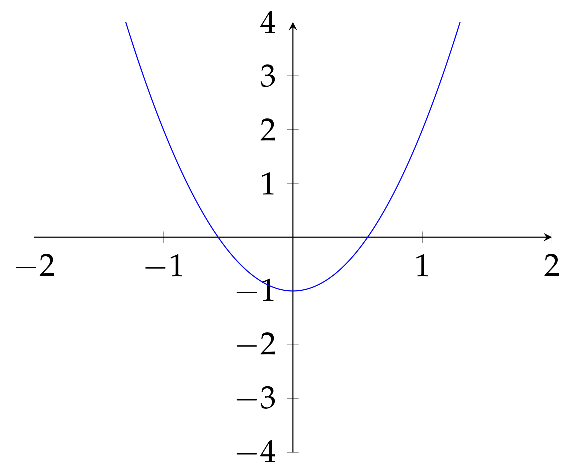
(2.c) How many x-intercepts will the slope graph of $f(x)$ have? Which points out of A, B, C, D, E gives us this information.

The slope graph of $f(x)$ will have two x-intercept points. The points C and E tells us this since the tangent line at those points is a horizontal line (and hence has slope 0).

(2.d) How many relative mins/maxes will the slope graph of $f(x)$ have? Which points out of A, B, C, D, E gives us this information.

The slope graph of $f(x)$ will have one relative min/max. The point D tells us this because D is an inflection point.

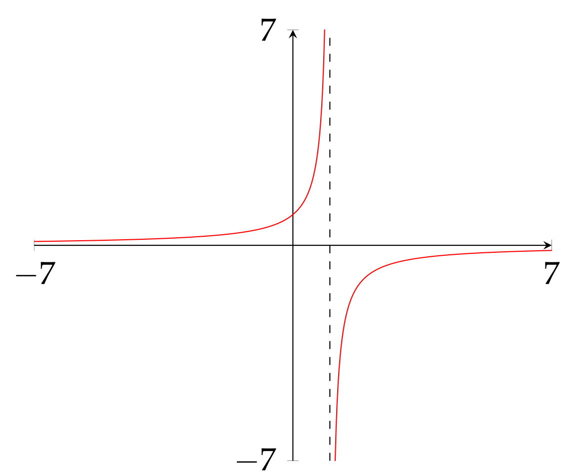
(2.e) Sketch the slope graph below



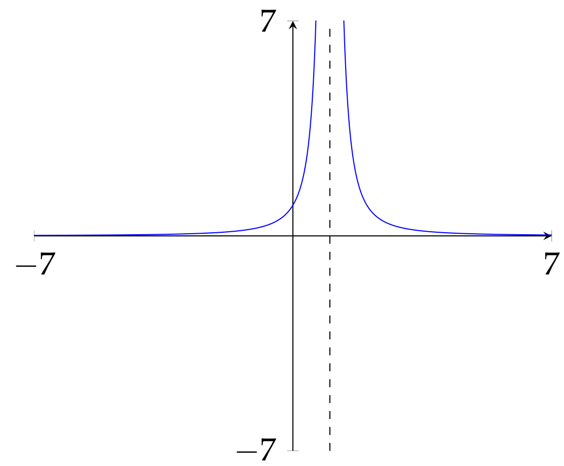
(2.f) Let $g(x)$ be a function and let a be a real number (say $a = 0$ if you want). List three reasons why $g'(a)$ may not exist.

1. $g(x)$ is not continuous at $x = a$.
2. The graph of $g(x)$ has a sharp corner at the point $(a, g(a))$.
3. The line tangent to the graph of $g(x)$ at the point $(a, g(a))$ is a vertical line.

(2.g) Let $h(x)$ be the function whose graph is given below



Give a sketch of the slope graph of $h(x)$. (You don't need to be completely accurate here. Just draw tangent lines of various points on the graph of $h(x)$. Are the slopes of these tangent lines positive/negative? If positive, then plot a positive output at the same input in the slope graph of $h(x)$)



Compounding Interest

(3.a) How much will a \$100 investment, compounded quarterly at 5% APR yield in 10 months? How long will it take to double your investment?

We first insert the function

$$F_4(t) = 100\left(1 + \frac{0.05}{4}\right)^{4t}$$

into our calculator. Then we go to tblset (2nd window) and set ΔTbl=1/4. We also make sure Indpnt and Depend are set to auto. Now we go to table (2nd graph). Since 10 months corresponds to $10/12 \approx 0.833$ years, we will have 103.80 dollars (rounding to two decimal places) in 10 months.

To find the time it takes for our intitial investment to double, we scroll down the table until the output value reaches a number ≥ 200. In this case, it reaches 200.50 dollars in 14 years. Thus the doubling time is 14 years.

(3.b) How much will a \$100 investment, compounded continuously at 5% APR yield in 10 months? How long will it take to double your investment?

We first insert the function

$$F_\infty(t) = 100e^{0.05t}$$

into our calculator. Then we evaluate[1] $F_\infty(10/12)$ in our calculator (don't do table method for continuous compounding).

To find the time it takes for our intitial investment to double, we solve[2] the equation

$$F_\infty(t) = 200.$$

We find that $t \approx 13.863$ years (rounded to three decimal places).

(3.c) What is the APY for an investment that pays 3% APR compounded quarterly? (If you need to, go to section 1.6 in the book and look up the formula for APY. This type of question will be on the test, so make sure you know it.)

Using the formula given in the book, we have

$$\text{APY}_4 = \left(\left(1 + \frac{0.03}{4}\right)^4 - 1\right) \cdot 100\% \approx 3.034\%$$

(3.d) What is the APY for an investment that pays 3% APR compounded continuously? (The formula for APY is different for continuous compounding than the one for quarterly compounding. Again, look up the formula in section 1.6 if you don't know it.)

[1] To do this you enter vars →$y$vars → function → y_1 then enter (10/12) in calculator.

[2] You can either use the intersect method or the math solver method. The math solver method goes like this: math → solver → $0 = y_1 - 200$ → enter → solve (alpha enter). If you aren't sure how to solve an equation, then go back to section 1.1 in the book where they explain how to do so in more detail.

Using the formula given in the book, we have

$$\text{APY}_\infty = \left(e^{0.03} - 1\right) \cdot 100\% \approx 3.045\%$$

Algebra

(4.a) Let a, b be real numbers (i.e. constants). Simplify the following expressions. You need to write each as $x^{\text{something}}$ (I'll do the first for you).

$$\frac{1}{x^a} = x^{-a}$$

$$x^a x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$\frac{1}{x^a} = x^{-a}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$$

(4.b) It's important to keep track of notation. Write three ways of expressing (in terms of notation, i.e. symbols) the derivative of a function $f(x)$. Then write two ways of expressing the derivative of a function $f(x)$ *evaluated* at a number a.

1. $f'(x) = \frac{df}{dx} = \frac{d}{dx}(f(x))$
2. $f'(a) = \frac{df}{dx}|_{x=a}$.

(4.c) Let $f(x)$ and $g(x)$ be functions and let a and b be real numbers (i.e. constants). Complete the expressions below (I'll do the first two for you).

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(af(x)) = a\frac{d}{dx}(f(x))$$

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

$$\frac{d}{dx}(a^x) = \ln(a)a^x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

(4.d) Find the derivative of the following functions (I'll do the first one for you)

$$p(x) = x^3 + 3x^2$$

$$\begin{aligned} p'(x) &= \frac{d}{dx}(p(x)) \\ &= \frac{d}{dx}(x^3 + 3x^2) \\ &= \frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2) \\ &= \frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) \\ &= 3x^{3-1} + 3 \cdot 2x^{2-1} \\ &= 3x^2 + 6x \end{aligned}$$

$$h(t) = t^2 + 5\sqrt{t} + \frac{2}{t^3}$$

$$\begin{aligned} h'(t) &= \frac{d}{dt}(h(t)) \\ &= \frac{d}{dt}\left(t^2 + 5\sqrt{t} + \frac{2}{t^3}\right) \\ &= \frac{d}{dt}(t^2) + \frac{d}{dt}(5\sqrt{t}) + \frac{d}{dt}\left(\frac{2}{t^3}\right) \\ &= \frac{d}{dt}(t^2) + \frac{d}{dt}(5t^{\frac{1}{2}}) + \frac{d}{dt}(2t^{-3}) \\ &= \frac{d}{dt}(t^2) + 5\frac{d}{dt}(t^{\frac{1}{2}}) + 2\frac{d}{dt}(t^{-3}) \\ &= 2t^{2-1} + 5 \cdot \frac{1}{2}t^{\frac{1}{2}-1} + 2 \cdot -3t^{-3-1} \\ &= 2t + \frac{5}{2}t^{-\frac{1}{2}} - 6t^{-4}. \end{aligned}$$

$$h(x) = \sqrt{\frac{1}{\sqrt[3]{x}}} + \sqrt{x^3}$$

$$\begin{aligned}
h'(x) &= \frac{d}{dx}(h(x)) \\
&= \frac{d}{dx}\left(\sqrt{\frac{1}{\sqrt[3]{x}}} + \sqrt{x^3}\right) \\
&= \frac{d}{dx}\left(\sqrt{\frac{1}{\sqrt[3]{x}}}\right) + \frac{d}{dx}\left(\sqrt{x^3}\right) \\
&= \frac{d}{dx}\left(\left(\frac{1}{x^{\frac{1}{3}}}\right)^{\frac{1}{2}}\right) + \frac{d}{dx}((x^3)^{\frac{1}{2}}) \\
&= \frac{d}{dx}((x^{-\frac{1}{3}})^{\frac{1}{2}}) + \frac{d}{dx}((x^3)^{\frac{1}{2}}) \\
&= \frac{d}{dx}(x^{-\frac{1}{6}}) + \frac{d}{dx}(x^{\frac{3}{2}}) \\
&= -\frac{1}{6}x^{-\frac{1}{6}-1} + \frac{3}{2}x^{\frac{3}{2}-1} \\
&= -\frac{1}{6}x^{-\frac{7}{6}} + \frac{3}{2}x^{\frac{1}{2}}.
\end{aligned}$$

$$g(t) = \frac{3t^3 - t^2 + 1}{t}$$

$$\begin{aligned}
g'(t) &= \frac{d}{dt}(g(t)) \\
&= \frac{d}{dt}\left(\frac{3t^3 - t^2 + 1}{t}\right) \\
&= \frac{d}{dt}\left(\frac{3t^3}{t} - \frac{t^2}{t} + \frac{1}{t}\right) \\
&= \frac{d}{dt}(3t^2 - t + t^{-1}) \\
&= \frac{d}{dt}(3t^2) + \frac{d}{dt}(-t) + \frac{d}{dt}(t^{-1}) \\
&= 3\frac{d}{dt}(t^2) - \frac{d}{dt}(t) + \frac{d}{dt}(t^{-1}) \\
&= 3 \cdot 2t^{2-1} - 1 - t^{-1-1} \\
&= 6t - 1 - \frac{1}{t^2}.
\end{aligned}$$

$$g(x) = 2\ln x + 3^x$$

$$\begin{aligned}
g'(x) &= \frac{d}{dx}(g(x)) \\
&= \frac{d}{dx}(2\ln x + 3^x) \\
&= \frac{d}{dx}(2\ln x) + \frac{d}{dx}(3^x) \\
&= 2\frac{d}{dx}(\ln x) + \frac{d}{dx}(3^x) \\
&= 2 \cdot \frac{1}{x} + \ln(3)3^x \\
&= \frac{2}{x} + \ln(3)3^x.
\end{aligned}$$

$$f(x) = 5e^x + e^e$$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(f(x)) \\
&= \frac{d}{dx}(5e^x + e^e) \\
&= \frac{d}{dx}(5e^x) + \frac{d}{dx}(e^e) \\
&= 5\frac{d}{dx}(e^x) + \frac{d}{dx}(e^e) \\
&= 5e^x + 0 \\
&= 5e^x.
\end{aligned}$$

(4.e) In the limit definition of the derivative, what does h represent? What does $\frac{f(x+h)-f(x)}{h}$ represent? What does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ represent?

h represents a small number, $\frac{f(x+h)-f(x)}{h}$ represents the slope of the secant line from $(x, f(x))$ to $(x+h, f(x+h))$[3], and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ represents the limit of the slopes of the secant lines from $(x, f(x))$ to $(x+h, f(x+h))$ as h tends to 0.

(4.f) Find the derivative of $f(x) = 2x^2 + 0.1x - 2$ using the *limit* definition. Show all steps.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h)^2 + 0.1(x+h) - 2 - (2x^2 + 0.1x - 2)}{h} \\
\text{(factor)} &= \lim_{h\to 0} \frac{2(x^2 + 2hx + h^2) + 0.1(x+h) - 2 - (2x^2 + 0.1x - 2)}{h} \\
\text{(distribute)} &= \lim_{h\to 0} \frac{2x^2 + 2hx + h^2 + 0.1x + 0.1h - 2 - 2x^2 - 0.1x + 2}{h} \\
\text{(combine like terms)} &= \lim_{h\to 0} \frac{(2-2)x^2 + (2h + 0.1 - 0.1)x + (h^2 + 0.1h - 2 + 2)}{h} \\
\text{(simplify)} &= \lim_{h\to 0} \frac{2hx + h^2 + 0.1h}{h} \\
\text{(factor the } h \text{ out)} &= \lim_{h\to 0} \frac{h(2x + h + 0.1)}{h} \\
\text{(cancel the } h\text{)} &= \lim_{h\to 0} 2x + h + 0.1 \\
\text{(set } h = 0\text{)} &= 2x + 0.1
\end{aligned}$$

[3]Make sure you remember this one especially!