DUE DATE: Upload it on Canvas by Monday 11:59pm, April 27.

INSTRUCTIONS: You should submit the following problems: 2, 3, 5, 6, 7, 9, 10, and 13(i). The rest of the problems are for practice.

Problem 1. Let $f : X \times Y \to \mathbb{R}$ be a simple function on the product measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$. Prove that for each $x \in X$ and $y \in Y$ the sections $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$ are $\mathcal{N}$ simple and $\mathcal{M}$ simple respectively.

Problem 2. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. Prove that $A \times B \in \mathcal{M} \otimes \mathcal{N}$ if and only if $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Problem 3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two finite measure spaces. Prove that if $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h(x, y) = f(x)g(y) \in L^1(\mu \otimes \nu)$. Prove also that

$$\int_{X \times Y} h(x, y) d(\mu \times \nu) = \int f(x) d\mu(x) \int g(y) d\nu(y).$$

Problem 4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two finite measure spaces. Prove that if $E, F \in \mathcal{M} \otimes \mathcal{N}$ such that $\nu(E_x) = \nu(F_x)$ for μ a.e. $x \in X$, then $\mu \times \nu(E) = \mu \times \nu(F)$.

Problem 5. Let $\mathcal{B}$ be the σ-algebra of all Borel measurable sets in $[0, 1]$, m the Lebesgue measure, and let $([0, 1] \times [0, 1], \mathcal{B} \times \mathcal{B}, m \times m)$ be the corresponding product measure space. Prove that the function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

is not integrable in the product measure space.

Problem 6. Let $\mathcal{B}$ be the σ-algebra of all Borel measurable sets in $[0, 1]$ equipped with the Lebesgue measure m. Let $\mathcal{N} = \mathcal{P}(\mathbb{N})$ be the σ-algebra consisting of all the subsets of the integers $\mathbb{N}$ equipped with the counting measure μ. State the Fubini and Tonelli Theorems explicitly for this case.

Problem 7. Let μ be a (positive) measure and $f \in L^1(\mu)$. Prove that the set function ν defined by $\nu(E) := \int f 1_E d\mu$ is a finite signed measure. Describe the Hahn and Jordan decomposition of the signed measure ν in terms of f and μ.

Problem 8. Let $(X, \mathcal{M})$ be a measurable space and let ν be a signed measure. Prove that E is a null set for ν if and only if $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

Problem 9. Let $(X, \mathcal{M})$ be a measurable space, let ν be a signed measure, and let μ be a (usual) measure.Prove that $\nu \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Problem 10. Let $(X, \mathcal{M})$ be a measure space and ν be a signed measure. Prove that for every $f \in L^1(|\nu|)$ the following inequality holds:

$$\left| \int f d\nu \right| \leq \int |f| \, d \, |\nu| \, .$$

Problem 11. Let $(X, \mathcal{M})$ be a measure space and ν be a signed measure. Prove that if μ and σ are (positive) measures on $(X, \mathcal{M})$ such that $\nu = \mu - \sigma$, then $\nu^+ \leq \mu$ and $\nu^- \leq \sigma$.

Problem 12. Prove that for any two signed measures ν_1 and ν_2 the following inequality holds: $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Problem 13. Let $(X, \mathcal{M})$ be a measure space and ν be a signed measure. Prove that

(i) $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subseteq E\}$, and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subseteq E\}$.

(ii) $|\nu| \, (E) = \sup\{\sum_{i=1}^n |\nu(E_i)|\}$, where the supremum is taken over all finite disjoint partitions of $E = \cup_{i=1}^n E_i$.

Problem 14. Let $(X, \mathcal{M})$ be a measurable space, let ν be a signed measure, and let μ be a measure. Prove that $\nu << \mu$ if and only if $|\nu| << \mu$. Prove also that $\nu << \mu$ if and only if $\nu^+ << \mu$ and $\nu^- << \mu$.

Problem 15. Prove the Lebesgue decomposition theorem: Let $(X, \mathcal{M})$ be a measurable space, let ν be a signed measure, and let μ be a measure. There exist unique measures $\sigma << \mu$ and $\lambda \perp \mu$ such the $\nu = \sigma + \lambda$.

Problem 16. Let ν be a signed measure. Prove that $\nu << |\nu|$ and that the Radon-Nikodym derivative $f = \frac{d\nu}{d|\nu|}$ of ν with respect to $|\nu|$ satisfies $|f| = 1$ a.e. $|\nu|$. This is sometimes called a polar decomposition of ν.