Analysis Prelim Solutions

Contents

1 Winter 2020

1.1 Problem 1

Exercise 1. Let $\mathcal{V}$ be an inner-product space.

1. Let (x_n) be a convergent sequence in $\mathcal{V}$. Then (x_n) is bounded.
2. Let (x_n) and (y_n) be two convergent sequences in $\mathcal{V}$. Prove that if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Solution 1. 1. Let (x_n) be a convergent sequence in $\mathcal{V}$. In particular, it must be a Cauchy sequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|x_n - x_m\| < \varepsilon$. Set $M = \max\{\|x_1\|, \dots \|x_N\|\}$. Observe that if $n \geq N$, then we have

$$\begin{aligned}
\|x_n\| &= \|x_n - x_N + x_N\| \\
&\leq \|x_n - x_N\| + \|x_N\| \\
&< \varepsilon + \|x_N\| \\
&\leq \varepsilon + M.
\end{aligned}$$

In particular, we see that $M + \varepsilon$ is an upper bound of (x_n).

2. Choose $M \in \mathbb{N}$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\|x_n - x\| < \varepsilon/2M \text{ and } \|y_n - y\| < \varepsilon/2\|x\|.$$

Then $n \geq N$ implies

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
&\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\
&\leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| \\
&\leq \|x_n - x\|M + \|x\|\|y_n - y\| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

This implies $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

1.2 Problem 2

Exercise 2. Let $\mathcal{V}$ be a normed linear space and let $\mathcal{W} \subset \mathcal{V}$ be a proper subspace. Prove that $\mathrm{Int}(\mathcal{W}) = \emptyset$.

Solution 2. Let $y \in \mathcal{V} \backslash \mathcal{W}$, let $x \in \mathcal{W}$, and let $\varepsilon > 0$. Assume for a contradiction $\mathrm{B}_\varepsilon(x) \subseteq \mathcal{W}$, where

$$\mathrm{B}_\varepsilon(x) = \{z \in \mathcal{V} \mid \|z - x\| < \varepsilon\}.$$

Then observe that $x + \frac{\varepsilon}{2\|y\|} y \in \mathrm{B}_\varepsilon(x) \subseteq \mathcal{W}$. However this implies $y \in \mathcal{W}$, which is a contradiction. Therefore $\mathrm{B}_\varepsilon(x) \nsubseteq \mathcal{W}$ for any $x \in \mathcal{W}$ and for any $\varepsilon > 0$. In particular, the only open subset of $\mathcal{V}$ which is contained in $\mathcal{W}$ is the empty set.

1.3 Problem 3

Exercise 3. Let $\ell^2(\mathbb{N})$ be the space of square summable sequences and define $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T((x_n)) = (x_{n+1} - x_n)$$

for all $(x_n) \in \ell^2(\mathbb{N})$. Prove that T is bounded and find $\|T\|$.

Solution 3. Let $(x_n) \in \ell^2(\mathbb{N})$ such that $\|(x_n)\| = \sum_{n=1}^{\infty} |x_n|^2 \leq 1$. Then we have

$$\begin{aligned}
\|T(x_n)\| &= \|(x_{n+1} - x_n)\| \\
&= \sum_{n=1}^{\infty} |x_{n+1} - x_n|^2 \\
&\leq \sum_{n=1}^{\infty} ((|x_{n+1}| + |x_n|)^2) \\
&= \sum_{n=1}^{\infty} |x_{n+1}|^2 + \sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |x_{n+1}||x_n| \\
&\leq \sum_{n=1}^{\infty} |x_{n+1}|^2 + \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} (|x_{n+1}|^2 + |x_n|^2) \\
&\leq 4.
\end{aligned}$$

It follows that T is bounded. In fact, we claim that $\|T\| = 4$. Indeed, to see this, let $n \in 2\mathbb{N}$ and consider the sequence

$$\mathbf{x}_n = (1/\sqrt{n}, -1/\sqrt{n}, 1/\sqrt{n}, \dots, -1/\sqrt{n}, 1/\sqrt{n}, 0, \dots),$$

where the first n terms are nonzero and every term after the nth term is zero. Then note that

$$T\mathbf{x}_n = (-2/\sqrt{n}, 2/\sqrt{n}, \dots, 2/\sqrt{n}, -2/\sqrt{n}, 0, \dots),$$

where the first $n-1$ terms are nonzero and every term after the $(n-1)$th term is zero. Then we have $\|\mathbf{x}_n\| = 1$ and $\|T\mathbf{x}_n\| = 4(n-1)/n$. By taking $n \to \infty$, we obtain a sequence $(\mathbf{x}_n)$ in $\ell^2(\mathbb{N})$ where $\|\mathbf{x}_n\| = 1$ for all $n \in 2\mathbb{N}$ such that the $\|T\mathbf{x}_n\| \to 4$ as $n \to \infty$. It follows that $\|T\| = 4$.

1.4 Problem 4

Exercise 4. Let (X, d) be a compact metric space and let $f \colon X \to X$ be a continuous function. Suppose that for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $\mathrm{d}(x_\varepsilon, f(x_\varepsilon)) < \varepsilon$. Prove that there exists $x \in X$ such that $f(x) = x$.

Solution 4. Observe that the function $g \colon X \to \mathbb{R}_{\geq 0}$ given by $g(x) = \mathrm{d}(x, f(x))$ for all $x \in X$ is continuous. Indeed, it is the composite of continuous functions $X \to X \times X \to \mathbb{R}_{\geq 0}$ given by $x \mapsto (x, f(x)) \mapsto \mathrm{d}(x, f(x))$ for all $x \in X$. Since X is compact, the continuous function must attain a global minimum. Since for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $\mathrm{d}(x_\varepsilon, f(x_\varepsilon)) < \varepsilon$, we see that 0 is the global minimum. Thus there exists an $x \in X$ such that $\mathrm{d}(x, f(x)) = 0$. Since d is positive-definite, this implies $x = f(x)$.

1.5 Problem 5 (need to finish)

Exercise 5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Prove that if

$$\sup\{|\langle x, y\rangle| \mid x \in \mathcal{M},\ y \in \mathcal{N} \text{ and } \|x\| = \|y\| = 1\} < 1,$$

then $\mathcal{M} + \mathcal{N} = \{x + y \mid x \in \mathcal{M},\ y \in \mathcal{N}\}$ is a closed subspace of $\mathcal{H}$.

Solution 5. Let us check that $\mathcal{M} + \mathcal{N}$ is a subspace of $\mathcal{H}$. First note $\mathcal{M} + \mathcal{N}$ is nonempty since $0 \in \mathcal{M} + \mathcal{N}$. Next, let $\lambda, \lambda' \in \mathbb{C}$ and let $x + y, x' + y' \in \mathcal{M} + \mathcal{N}$. Then

$$\lambda(x + y) + \lambda'(x' + y') = (\lambda x + \lambda' x') + (\lambda y + \lambda' y') \in \mathcal{M} + \mathcal{N}.$$

Thus $\mathcal{M} + \mathcal{N}$ is a subspace of $\mathcal{H}$. Now we need to check that it is a *closed* subspace of $\mathcal{H}$.

We first note that for any nonzero $x \in \mathcal{M}$ and nonzero $y \in \mathcal{N}$, we have $|\langle x, y\rangle| < \|x\|\|y\|$. Indeed,

$$\begin{aligned}
1 &> \left| \langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \rangle \right| \\
&= \frac{1}{\|x\|\|y\|} |\langle x, y\rangle|.
\end{aligned}$$

In particular, this implies $\mathcal{M} \cap \mathcal{N} = 0$ (if $z \in \mathcal{M} \cap \mathcal{N}$ is nonzero, then $\|z\|^2 = |\langle z, z\rangle|$, which is a contradiction). Therefore we have a direct sum $\mathcal{M} \oplus \mathcal{N}$.

Let $(x_n + y_n)$ be a sequence in $\mathcal{M} + \mathcal{N}$ such that $x_n + y_n \to z$ where $z \in \mathcal{H}$. Using the fact that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, write

$$z = x + w$$

where $x \in \mathcal{M}$ and $w \in \mathcal{M}^\perp$. We want to show that $x_n \to x$. Assume for a contradiction that the (x_n) does not converge to x. Then there exists $\varepsilon > 0$ and a subsequence $(x_{\pi(n)})$ of (x_n) such that

$$\|x_{\pi(n)} - x\| \geq \varepsilon$$

for all $n \in \mathbb{N}$. Now choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\|z - x_{\pi(n)} - y_{\pi(n)}\| < \varepsilon.$$

Then $n \geq N$ implies

$$\begin{aligned}
\varepsilon^2 &> \|z - x_{\pi(n)} - y_{\pi(n)}\|^2 \\
&= \|x + w - x_{\pi(n)} - y_{\pi(n)}\|^2 \\
&= \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 + \langle x - x_{\pi(n)}, w - y_{\pi(n)}\rangle + \langle w - y_{\pi(n)}, x - x_{\pi(n)}\rangle \\
&> \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 - \|x - x_{\pi(n)}\|\|w - y_{\pi(n)}\| - \|w - y_{\pi(n)}\|\|x - x_{\pi(n)}\| \\
&= \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 - 2\|x - x_{\pi(n)}\|\|w - y_{\pi(n)}\| \\
&= (\|x - x_{\pi(n)}\| - \|w - y_{\pi(n)}\|)^2.
\end{aligned}$$

which implies

$$-\varepsilon < \|x - x_{\pi(n)}\| - \|w - y_{\pi(n)}\| < \varepsilon$$

It suffices to show that (x_n) is a Cauchy sequence. Indeed, if this is case, then
Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$\|(x_n + y_n) - (x_m + y_m)\|^2 < \varepsilon.$$

Then $m, n \geq N$ implies

$$\begin{aligned}
\varepsilon^2 &> \|(x_n + y_n) - (x_m + y_m)\|^2 \\
&= \langle x_n - x_m + y_n - y_m, x_n - x_m + y_n - y_m\rangle \\
&= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 + \langle x_n - x_m, y_n - y_m\rangle + \langle y_n - y_m, x_n - x_m\rangle \\
&> \|x_n - x_m\|^2 + \|y_n - y_m\|^2 - \|x_n - x_m\|\|y_n - y_m\| - \|y_n - y_m\|\|x_n - x_m\| \\
&= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 - 2\|x_n - x_m\|\|y_n - y_m\| \\
&= (\|x_n - x_m\| - \|y_n - y_m\|)^2.
\end{aligned}$$

which implies

$$-\varepsilon < \|x_n - x_m\| - \|y_n - y_m\| < \varepsilon.$$

In particular, $n \geq N$ implies

$$|\|x_N - x_n\| - \|y_N - y_n\|| < \varepsilon$$

Thus if $\|x_n - x_m\|$ is sufficiently small, then so to must $\|y_n - y_m\|$.

$$\|x_n - x_m + y_n - y_m\| < \varepsilon$$

1.6 Problem 6

Exercise 6. Let $(X, \mathcal{S})$ be a measurable space and let (E_n) be a sequence of measurable sets. Prove that the set E consisting of all points $x \in X$ that belong to infinitely many of the sets E_n is measurable.

Solution 6. We claim that

$$E = \bigcap_{N \geq 1} \bigcup_{n \geq N} E_n. \tag{1}$$

Indeed,

$$\begin{aligned}
x \in \bigcap_{k \geq 1} \bigcup_{n \geq k} E_n &\iff x \in \bigcup_{n \geq k} E_n \text{ for all } k \\
&\iff x \in E_{\pi(k)} \text{ for some } \pi(k) \geq k \text{ for all } k \\
&\iff x \in E_{\pi(k)} \text{ for some sequence } (\pi(k)) \text{ of } (k) \\
&\iff x \text{ belongs to infinitely many } E_n \\
&\iff x \in E.
\end{aligned}$$

Now the expression (1) shows that E is measurable.

1.7 Problem 7

Exercise 7. Let $(X, \mathcal{S}, \mu)$ be measure space and let $f \colon X \to \mathbb{R}$ be an integrable function. Suppose (E_n) is a sequence of members of $\mathcal{S}$ such that $\lim_{n \to \infty} \mu(E_n) = 0$. Prove that

$$\lim_{n \to \infty} \int_X f 1_{E_n} \mathrm{d}\mu = 0$$

Solution 7. Since $\int_X f 1_{E_n} \mathrm{d}\mu \leq \int_X |f| 1_{E_n} \mathrm{d}\mu$ for all $n \in \mathbb{N}$, it suffices to show

$$\lim_{n \to \infty} \int_X |f| 1_{E_n} \mathrm{d}\mu = 0.$$

In fact, by replacing f with $|f|$ if necessary, we may as well assume f is a nonnegative integrable function. Then $(f 1_{E_n})$ is a sequence of integerable functions which converges pointwise a.e. to the zero function since $\lim_{n \to \infty} \mu(E_n) = 0$. Furthermore, the sequence $(f 1_{E_n})$ is dominated by the integrable function f. It follows from the dominated convergence theorem that

$$\lim_{n \to \infty} \int_X f 1_{E_n} \mathrm{d}\mu = 0.$$

1.8 Problem 8

Exercise 8. Let $(X, \mathcal{S})$ be a measurable space and let (μ_n) be a sequence of measures on $(X, \mathcal{S})$ such that $\mu_n(X) = 1$ for all $n \in \mathbb{N}$. Prove that $\lambda \colon \mathcal{S} \to [0, \infty]$ defined by

$$\lambda(F) = \sum_{n=1}^{\infty} \frac{\mu_n(F)}{2^n}$$

for all $F \in \mathcal{S}$ is a measure on $(X, \mathcal{S})$ with $\lambda(X) = 1$.

Solution 8. First note that $\lambda(\emptyset) = 0$ since $\mu_n(\emptyset) = 0$ for all $n \in \mathbb{N}$. Next let (F_k) be a sequence of pairwise disjoint sets in $\mathcal{S}$. Then

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} F_k\right) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n \left(\bigcup_{k=1}^{\infty} F_k\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \mu_n(F_k) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_n(F_k)}{2^n} \\
&= \sum_{k=1}^{\infty} \lambda(F_k).
\end{aligned}$$

It follows that λ is a measure on $(X, \mathcal{S})$. For the last part of the problem, we have

$$\begin{aligned}\lambda(X) &= \sum_{n=1}^{\infty} \frac{\mu_n(F)}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \frac{1/2}{1-1/2} \\ &= 1.\end{aligned}$$

1.9 Problem 9

Exercise 9. Let $f \in L^2[0,\infty)$ and let $G\colon (0,\infty) \to \mathbb{R}$ be defined by

$$G(t) = \int_0^\infty \frac{f(x)}{1+tx} \mathrm{d}x.$$

Prove the following:

1. $\lim_{t\to\infty} G(t) = 0$;
2. G is continuous at every point of $(0,\infty)$.

Solution 9. 1. For each $t \in (0,\infty)$, we define $g_t\colon [0,\infty) \to \mathbb{R}$ by

$$g_t(x) = \frac{1}{1+tx}$$

for all $x \in [0,\infty)$. Observe that

$$\begin{aligned}\int_0^\infty |g_t(x)|^2 \mathrm{d}x &= \int_0^\infty \frac{1}{(1+tx)^2} \mathrm{d}x \\ &= \frac{-1}{t(1+tx)}\Big|_0^\infty \\ &= 0 + 1/t \\ &= 1/t.\end{aligned}$$

Therefore $g_t \in L^2[0,\infty)$ with $\|g_t\|_2 = 1/t$. Also, note that $G(t) = \langle f, g_t\rangle$. In particlar, by Cauchy-Schwarz we have

$$\begin{aligned}|G(t)| &= |\langle f, g_t\rangle| \\ &\leq \|f\|_2 \|g_t\| \\ &= \|f\|_2/t.\end{aligned}$$

So taking $t \to \infty$ gives us $|G(t)| \to 0$, which implies $\lim_{t\to\infty} G(t) = 0$.

2. Note that G is the composite of the maps $[0,\infty) \to L^2[0,\infty)$, given by $t \mapsto g_t$, with the map $L^2[0,\infty) \to \mathbb{R}$, given by $g \mapsto \langle f, g\rangle$. The latter map is continuous, so to show G is continuous, it suffices to show the former map is continuous. That is, let $t \in (0,\infty)$ and let (t_n) be a sequence in $(0,\infty)$ such that $t_n \to t$ as $n \to \infty$. Then

we need to show that $g_{t_n} \to g_t$ as $n \to \infty$. Observe that

$$\begin{aligned}
\|g_{t_n} - g_t\|_2^2 &= \int_0^\infty \left| \frac{1}{1+t_n x} - \frac{1}{1+tx} \right|^2 \mathrm{d}x \\
&= \int_0^\infty \left| \frac{(t-t_n)x}{(1+tx)(1+t_n x)} \right|^2 \mathrm{d}x \\
&= |t-t_n|^2 \int_0^\infty \frac{x^2}{(1+tx)^2(1+t_n x)^2} \mathrm{d}x \\
&= |t-t_n|^2 \left(\int_0^1 \frac{x^2}{(1+tx)^2(1+t_n x)^2} \mathrm{d}x + \int_1^\infty \frac{x^2}{(1+tx)^2(1+t_n x)^2} \mathrm{d}x \right) \\
&\leq |t-t_n|^2 \left(\int_0^1 \frac{x^2}{(1+tx)^2(1+t_n x)^2} \mathrm{d}x + \int_1^\infty \frac{x^2}{tt_n x^4} \mathrm{d}x \right) \\
&\leq |t-t_n|^2 \left(\int_0^1 x^2 \mathrm{d}x + \frac{1}{tt_n} \int_1^\infty \frac{1}{x^2} \mathrm{d}x \right) \\
&= |t-t_n|^2 \left(\frac{1}{3} + \frac{1}{tt_n} \right).
\end{aligned}$$

In particular, we see that $g_{t_n} \to g_t$ as $n \to \infty$.

1.10 Problem 10

Exercise 10. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \colon X \to [0, \infty)$ be a nonnegative measurable function. Suppose that for every $s > 0$ we have

$$\int_X e^{sf} \mathrm{d}\mu \leq e^{s^2}.$$

Prove that for every $t > 0$ we have

$$\mu\{f > t\} \leq e^{\frac{-t^2}{4}}.$$

Solution 10. Let $s > 0$ and $t > 0$. First note that

$$\begin{aligned}
f > t &\iff sf > st \\
&\iff e^{sf} > e^{st}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\mu\{f > t\} &= \mu\{e^{sf} > e^{st}\} \\
&\leq \frac{1}{e^{st}} \int_X e^{sf} \mathrm{d}\mu \\
&\leq \frac{1}{e^{st}} e^{s^2} \\
&= e^{s(s-t)},
\end{aligned}$$

where we applied Chebyshev's inequality to get from the first line to the second line. In particular, setting $s = t/2$ gives us the desired result.

2 Winter 2019

2.1 Problem 1

Exercise 11. Let $\mathcal{X}$ be a normed linear space and let (x_n) be a sequence in $\mathcal{X}$. Suppose that every subsequence of (x_n) contains a convergent subsequence with limit $x_0 \in X$. Show that $x_n \to x_0$ as $n \to \infty$.

Solution 11. Assume for a contradiction that $x_n \not\to x_0$. Then there exists $\varepsilon > 0$ and a subsequence $(x_{\pi(n)})$ of (x_n) such that

$$\|x_{\pi(n)} - x_0\| \geq \varepsilon \tag{2}$$

for all $n \in \mathbb{N}$. In particular, (2) implies no subsequence of $(x_{\pi(n)})$ can converge to x_0, which is a contradiction.

2.2 Problem 2

Exercise 12. Let $P[0,1]$ be the collection of all polynomials with indeterminate t on $[0,1]$, namely,

$$P[0,1] = \left\{ \sum_{i=0}^{n} a_i t^i \mid a_i \in \mathbb{R} \text{ and } n \in \mathbb{N}_0 \right\}.$$

Define $\mathrm{d}\colon P[0,1] \times P[0,1] \to \mathbb{R}$ by

$$\mathrm{d}(p,q) = \int_0^1 |p(t) - q(t)| \mathrm{d}t.$$

Prove or disprove: $(P[0,1], \mathrm{d})$ is a complete metric space.

Solution 12. This is false. For each $n \in \mathbb{N}$, define $f_n \in P[0,1]$ by

$$f_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!}.$$

The sequence (f_n) converges uniformly to e^t on $[0,1]$. Therefore it converges in the L^1-norm to e^t (as the measure of $[0,1]$ is finite). In particular, the sequence (f_n) is a Cauchy sequence in $P[0,1]$ which cannot converge to a polynomial. To see why this is the case, note that if it did converge to some polynomial, say $p(t)$, then $p(t)$ and e^t must agree almost everywhere. However since $p(t)$ and e^t are continuous on $(0,1)$, they in fact must agree everywhere. Indeed, if $c \in (0,1)$ such that $p(c) \neq e^c$. Then since $p(t) - e^t$ is continuous, there exists an open neighborhood of c, say

$$\mathrm{B}_\varepsilon(c) = \{x \in (0,1) \mid |x - c| < \varepsilon\},$$

such that $p(x) \neq e^x$ for all $x \in \mathrm{B}_\varepsilon(c)$. However $\mathrm{m}(\mathrm{B}_\varepsilon(c)) = 2\varepsilon \neq 0$, contradicting the fact that $p(t)$ and e^t agree almost everywhere.

2.3 Problem 3

Exercise 13. Let (X, d) be a metric space with the property that there are $A \subseteq X$ and $\varepsilon > 0$ such that A is uncountable for any dictinct elements $a, b \in A$ we have $\mathrm{d}(a,b) \geq \varepsilon$. Show that X is not separable.

Solution 13. Assume for a contradiction that X is separable. Choose a countable dense subset of X, say $Y \subseteq X$. For each $a \in A$, we choose $y_a \in Y$ such that $\mathrm{d}(a, y_a) < \varepsilon/2$. Observe that this gives rise to a function $y_{(-)}\colon A \to Y$, given by

$$y_{(-)}(a) = y_a$$

for all $a \in A$. We claim that $y_{(-)}$ is injective. Indeed, if $y_a = y_b$ for some distinct pair $a, b \in A$, then we have

$$\begin{aligned} \mathrm{d}(a,b) &\leq \mathrm{d}(a, y_a) + \mathrm{d}(y_b, b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

which is a contradiction. Thus $y_{(-)}$ is an injective function, which contradicts the fact that A is uncountable. Thus X is separable.

2.4 Problem 4

Exercise 14. Recall the distance betwen two subsets A and B of a metric space (X, d) is defined as

$$\mathrm{d}(A,B) = \inf_{(a,b) \in A \times B} \mathrm{d}(a,b).$$

Show that if both A and B are compact, then there exists $x \in A$ and $y \in B$ such that

$$\mathrm{d}(x,y) = \mathrm{d}(A,B).$$

Solution 14. The function $\mathrm{d}\colon A \times B \to \mathbb{R}_{\geq 0}$ is continuous, so if A and B are both compact, then $A \times B$ is compact, which implies d attains a minimum, say at $(x,y) \in A \times B$. Thus for any $(a,b) \in A \times B$, we have $\mathrm{d}(x,y) \leq \mathrm{d}(a,b)$. This implies

$$\mathrm{d}(A,B) \leq \mathrm{d}(x,y) \leq \mathrm{d}(A,B).$$

Therefore $\mathrm{d}(x,y) = \mathrm{d}(A,B)$.

2.5 Problem 5

Exercise 15. Let $\mathcal{H}$ be a Hilbert space and let T be a nonzero linear operator on $\mathcal{H}$ such that $T^2 = T$. Show that the following are equivalent:

1. T is an orthogonal projection.
2. $\|T\| = 1$.
3. $\ker T = (\operatorname{im} T)^{\perp}$.

Solution 15. We first show 1 implies 2. Let $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Then we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx \rangle \\ &= \langle x, T^2 x \rangle \\ &= \|x\|^2 \\ &= 1. \end{aligned}$$

Thus T is bounded with $\|T\| \leq 1$. To see that $\|T\| = 1$, we just choose a $Ty \in \operatorname{im} T$ such that $\|Ty\| = 1$ (this can be done since $\operatorname{im} T \neq 0$). Then

$$\begin{aligned} \|T(Ty)\| &= \|T^2 y\| \\ &= \|Ty\| \\ &= 1. \end{aligned}$$

Thus $\|T\| = 1$.

Now we show 2 implies 3. Let $x \in \ker T$. Then for all $Ty \in \operatorname{im} T$, we have

$$\begin{aligned} \langle x, Ty \rangle &= \langle x, T^2 y \rangle \\ &= \end{aligned}$$

2.6 Problem 6

Exercise 16.

1. State the monotone convergence theorem and the dominated convergence theorem.
2. Show that
$$\lim_{n \to \infty} \int_0^{\infty} \frac{e^{-x}}{1 + (x/n)^2} \mathrm{d}x = 1.$$

Solution 16. 1. The monotone convergence theorem is:

Theorem 2.1. *(MCT) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n \colon X \to [0, \infty])$ be an increasing sequence of nonnegative measurable functions which converges pointwise to a nonnegative function $f \colon X \to [0, \infty]$. Then*

$$\lim_{n \to \infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

The dominated convergence theorem is:

Theorem 2.2. *(DCT) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $g \colon X \to [0, \infty]$ be a nonnegative integrable function. Suppose $(f_n \colon X \to \mathbb{R})$ is a sequence of integrable functions such that*

1. *(f_n) converges pointwise to $f \colon X \to \mathbb{R}$.*
2. *$|f_n| \leq g$ pointwise for all $n \in \mathbb{N}$.*

Then

$$\lim_{n \to \infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

2. For each $n \in \mathbb{N}$ set $f_n = e^{-x}/(1+(x/n)^2)$. Note that (f_n) is an increasing sequence. Indeed, if $m < n$, then $(x/m)^2 > (x/n)^2$ for each $x \in \mathbb{R}_{>0}$, which implies

$$\frac{e^{-x}}{1+(x/n)^2} > \frac{e^{-x}}{1+(x/m)^2}$$

for each $x \in \mathbb{R}_{>0}$.

Next observe that (f_n) converges pointwise to e^{-x}. Indeed, for each $x \in \mathbb{R}_{>0}$, we have

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty}\left(\frac{e^{-x}}{1+(x/n)^2}\right)\\ &= e^{-x}\lim_{n\to\infty}\left(\frac{1}{1+(x/n)^2}\right)\\ &= e^{-x}.\end{aligned}$$

In particular, since (f_n) is increasing and converges pointwise to e^{-x}, it follows from MCT that

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty \frac{e^{-x}}{1+(x/n)^2}\mathrm{d}x &= \int_0^\infty \lim_{n\to\infty}\frac{e^{-x}}{1+(x/n)^2}\mathrm{d}x\\ &= \int_0^\infty e^{-x}\mathrm{d}x\\ &= e^{-x}\Big|_0^\infty\\ &= 1.\end{aligned}$$

2.7 Problem 7 (need to finish)

Exercise 17. Let $E \subseteq \mathbb{R}$ have finite Lebesgue measure and let $f\colon E \to \mathbb{R}$ be a measurable function such that that $f(x) > 0$ for a.e. $x \in E$. Show that if (E_n) is a sequence of subsets of E such that

$$\lim_{n\to\infty}\int_{E_n} f\mathrm{d}x = 0,$$

then $\lim_{n\to\infty}\mathrm{m}(E_n) = 0$.

Solution 17. If $\mathrm{m}(E) = 0$, then clearly $\mathrm{m}(E_n) = 0$ for all $n \in \mathbb{N}$, which implies the result, so assume $\mathrm{m}(E) \neq 0$. Assume for a contradiction that $\lim_{n\to\infty}\mathrm{m}(E_n) \neq 0$. Then there exists $\varepsilon > 0$ and a subsequence $(E_{\pi(n)})$ of (E_n) such that $\mathrm{m}(E_{\pi(n)}) \geq \varepsilon$ for all $n \in \mathbb{N}$. By replacing (E_n) with the subsequence $(E_{\pi(n)})$ if necessary, we may as well assume that $\mathrm{m}(E_n) \geq \varepsilon$ for all $n \in \mathbb{N}$.

Let $A = \{f > 0\}$ and for each $k \in \mathbb{N}$ let $A_k = \{f \geq 1/k\}$. Then observe that

$$A = \bigcup_{k=1}^\infty A_k.$$

Since $\mathrm{m}(A) = \mathrm{m}(E) \neq 0$, there must exist some k such that $\mathrm{m}(A_k) \neq 0$. Indeed, if $\mathrm{m}(A_k) = 0$ for all k, then

$$\begin{aligned}0 &\neq \mathrm{m}(A)\\ &= \mathrm{m}\left(\bigcup_{k=1}^\infty A_k\right)\\ &\leq \sum_{k=1}^\infty \mathrm{m}(A_k)\\ &= 0\end{aligned}$$

would give us a contradiction. In particular, we can choose a $c > 0$ such that the set $C = \{f \geq c\}$ has nonzero measure. Now observe that for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\int_{E_n} f \mathrm{d}\mu &= \int_{E_n \cap C} f \mathrm{d}\mu + \int_{E_n \cap C^c} f \mathrm{d}\mu \\
&\geq c\mathrm{m}(E_n \cap C) \\
&= c\mathrm{m}(E_n) + c\mathrm{m}(C) - c\mathrm{m}(E_n \cup C) \\
&\geq c\varepsilon + c\mathrm{m}(C) - c\mathrm{m}(E_n \cup C)
\end{aligned}$$

$$\begin{aligned}
\int_{E_n} f \mathrm{d}\mu &\geq \int_{E_n \cap C} f \mathrm{d}\mu \\
&\geq c\mathrm{m}(E_n \cap C) \\
&= c\mathrm{m}(E_n) + c\mathrm{m}(C) - c\mathrm{m}(E_n \cup C) \\
&\geq c\varepsilon + c\mathrm{m}(C) - c\mathrm{m}(E_n \cup C)
\end{aligned}$$

So choose k such that $\mathrm{m}(A_k) \neq 0$. Then observe that for each n we have

$$\begin{aligned}
\int_{E_n} f \mathrm{d}x &\geq \frac{1}{k} \int_{E_n \cap A_k} \mathrm{d}x \\
&= \frac{1}{k}\mathrm{m}(E_n \cap A_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{m}(E_n) &= \mathrm{m}(E_n \cap A) \\
&= \mathrm{m}\left(E_n \cap \left(\bigcup_{k=1}^{\infty} A_k\right)\right) \\
&= \mathrm{m}\left(\bigcup_{k=1}^{\infty} (E_n \cap A_k)\right) \\
&\leq \sum_{k=1}^{\infty} \mathrm{m}(E_n \cap A_k)
\end{aligned}$$

In particular, taking $n \to \infty$ implies $\frac{1}{k}\mathrm{m}(E_n \cap A_k) \to 0$. However note that

$$\begin{aligned}
\frac{1}{k}\mathrm{m}(E_n \cap A_k) &= \frac{1}{k}\left(\mathrm{m}(E_n) + \mathrm{m}(A_k) - \mathrm{m}(E_n \cup A_k)\right) \\
&= \frac{1}{k}\mathrm{m}(E_n) + \frac{1}{k}\mathrm{m}(A_k) - \frac{1}{k}\mathrm{m}(E_n \cup A_k).
\end{aligned}$$

Thus as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathrm{m}(E_n) =$$

$$0 = \frac{1}{k}\mathrm{m}(A_k) + \lim_{n\to\infty}\left(\frac{1}{k}\mathrm{m}(E_n) - \frac{1}{k}\mathrm{m}(E_n \cup A_k)\right)$$

We have

$$\begin{aligned}
\int_{E_n} f \mathrm{d}\mu &\geq \int_{E_n \cap A} f \mathrm{d}\mu \\
&\geq c\mathrm{m}(E_n \cap A) \\
&= c\mathrm{m}(E_n) + c\mathrm{m}(A) - c\mathrm{m}(E_n \cap A) \\
&\geq c\varepsilon + c\mathrm{m}(A) - c\mathrm{m}(E_n \cap A) \\
&\geq c\varepsilon + c\mathrm{m}(E_n \cap A) - c\mathrm{m}(E_n \cap A) \\
&= c\varepsilon,
\end{aligned}$$

where we needed to use the fact that E has finite measure in order to get the third line from the second line. Thus as $n \to \infty$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathrm{m}(E_n \cap A) \\ &= \lim_{n\to\infty} \mathrm{m}(E_n) + \lim_{n\to\infty} \mathrm{m}(A) - \lim_{n\to\infty} \mathrm{m}(A \cup E_n) \\ &= \lim_{n\to\infty} \mathrm{m}(E_n) + \mathrm{m}(A) - \lim_{n\to\infty} \mathrm{m}(A \cup E_n) \\ &\geq \varepsilon + \mathrm{m}(A) - \lim_{n\to\infty} \mathrm{m}(A \cup E_n) \end{aligned}$$

$$\lim_{n\to\infty} c\mathrm{m}(E_n \cap A) = 0.$$

2.8 Problem 8

Exercise 18. Is there a measurable function $f\colon [0,1] \to \mathbb{R}$ such that both of the identities

$$\int_0^1 |f(x) - \sin(2\pi x)|^2 \mathrm{d}x = \frac{1}{9} \quad \text{and} \quad \int_0^1 |f(x) - \cos(2\pi x)|^2 \mathrm{d}x = \frac{1}{9}$$

hold? Justify your answer.

Solution 18. No. Indeed, assume for a contradiction that such a function did exist. First note that that f must be L^2-integrable since

$$\begin{aligned} \|f\|_2 &= \|f - \sin(2\pi x) + \sin(2\pi x)\|_2 \\ &\leq \|f - \sin(2\pi x)\|_2 + \|\sin(2\pi x)\|_2 \\ &= \frac{1}{9} + \frac{1}{2} \\ &= \frac{11}{18}. \end{aligned}$$

Next, we calculate

$$\begin{aligned} \|\cos(2\pi x) - \sin(2\pi x)\|_2 &= \int_0^1 |\cos(2\pi x) - \sin(2\pi x)|^2 \mathrm{d}x \\ &= \int_0^{1/8} (\cos(2\pi x) - \sin(2\pi x))^2 \mathrm{d}x + \int_{1/8}^{5/8} (\sin(2\pi x) - \cos(2\pi x))^2 \mathrm{d}x + \int_{5/8}^{1} (\cos(2\pi x) - \sin(2\pi x))^2 \mathrm{d}x \\ &= \int_0^1 (\cos(2\pi x) - \sin(2\pi x))^2 \mathrm{d}x \\ &= \int_0^1 \left(\cos^2(2\pi x)\mathrm{d}x + \sin^2(2\pi x)\right) \mathrm{d}x - 2\int_0^1 \cos(2\pi x)\sin(2\pi x)\mathrm{d}x \\ &= 1 - 2\int_0^1 \cos(2\pi x)\sin(2\pi x)\mathrm{d}x \\ &= 1. \end{aligned}$$

However this is a contradiction since

$$\begin{aligned} \|\cos(2\pi x) - \sin(2\pi x)\|_2 &= \|\cos(2\pi x) - f + f - \sin(2\pi x)\|_2 \\ &\leq \|\cos(2\pi x) - f\|_2 + \|f - \sin(2\pi x)\|_2 \\ &= \frac{1}{9} + \frac{1}{9} \\ &= \frac{2}{9}. \end{aligned}$$

2.9 Problem 9

Exercise 19. Suppose $f\colon (0,\infty) \to \mathbb{R}$ is bounded and measurable, so that $\lim_{x\to\infty} |xf(x)| = 0$. Show that

$$\lim_{n\to\infty} \int_0^1 n\sqrt{x} f(nx)\mathrm{d}x = 0.$$

Solution 19. By taking absolute values if necessary, we may assume that f is nonnegative. Choose $M \in \mathbb{N}$ such that $f \leq M$. Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ such that $x \geq N_\varepsilon$ implies $xf(x) < \varepsilon$. Then for $n \geq N$ implies

$$\begin{aligned}
\int_0^1 n\sqrt{x} f(nx)\mathrm{d}x &= \int_0^1 \frac{1}{\sqrt{x}} nxf(nx)\mathrm{d}x \\
&= \int_0^{N/n} \frac{1}{\sqrt{x}} nxf(nx)\mathrm{d}x + \int_{N/n}^1 \frac{1}{\sqrt{x}} nxf(nx)\mathrm{d}x \\
&< Mn \int_0^{N/n} \sqrt{x}\mathrm{d}x + \varepsilon \int_{N/n}^1 \frac{1}{\sqrt{x}}\mathrm{d}x \\
&= Mn\left(\frac{2}{3}x^{3/2}\Big|_0^{N/n}\right) + 2\varepsilon\left(x^{1/2}\Big|_{N/n}^1\right) \\
&= \frac{2}{3}MN^{3/2}n^{-1/2} + 2\varepsilon(1 - \sqrt{N/n}).
\end{aligned}$$

In particular taking $n \to \infty$ gives us

$$\lim_{n\to\infty} \int_0^1 n\sqrt{x} f(nx)\mathrm{d}x < 2\varepsilon.$$

Finally taking $\varepsilon \to 0$ gives us

$$\lim_{n\to\infty} \int_0^1 n\sqrt{x} f(nx)\mathrm{d}x = 0.$$

2.10 Problem 10

Exercise 20. Let $(X, \mathcal{M}, \mu)$ be a measure space and let (A_n) be a sequence of $\mathcal{M}$-measurable sets. Assume that $\sum_n \mu(A_n) < 0$. Show that $\mu(\limsup A_n) = 0$.

Solution 20. Note that the sequence

$$\left(\bigcup_{n\geq N} A_n\right)_{N\in\mathbb{N}}$$

is a descending sequence in N. This together with the fact that

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n=1}^{\infty} \mu(A_n) \\
&< \infty
\end{aligned}$$

implies

$$\begin{aligned}
\mu(\limsup A_n) &= \mu\left(\bigcap_{N=1}^{\infty}\left(\bigcup_{n\geq N} A_n\right)\right) \\
&= \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N} A_n\right) \\
&\leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(E_n) \\
&= 0,
\end{aligned}$$

where the last equality follows since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$.

3 Summer 2019

3.1 Problem 1

Exercise 21. Let (X, d) be a metric space and let $A, B \subseteq X$. Prove or disprove the following statements:

1. If A and B are dense in X, then $A \cap B$ is also dense in X.

2. If A and B are open and dense in X, then $A \cap B$ is also open and dense in X.

Solution 21. 1. This is false. For instance, consider $(X, \mathrm{d}) = (\mathbb{R}, |\cdot|)$, $A = \mathbb{Q}$, and $B = \mathbb{R} \backslash \mathbb{Q}$. Then both A and B are dense in $\mathbb{R}$, but $A \cap B = \emptyset$, which is not dense in $\mathbb{R}$.

2. This is true. First note that $A \cap B$ is open since it is the intersection of two open sets, so we just need to show that it is dense in X. Let U be a nonempty open subset of X. Since A is an open dense subset of X, we see that $A \cap U$ is a nonempty open subset of X. Since B dense in X, we see that $B \cap A \cap U$ is nonempty.

3.2 Problem 2

Exercise 22. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace. Then $\mathcal{K} = \mathcal{K}^{\perp\perp}$.

Solution 22. Let $x \in \mathcal{K}$. Then for any $y \in \mathcal{K}^{\perp}$, we have $\langle x, y \rangle = 0$. In particular, this implies $x \in \mathcal{K}^{\perp\perp}$. Thus $\mathcal{K} \subseteq \mathcal{K}^{\perp\perp}$. For the reverse direction, let $x \in \mathcal{K}^{\perp\perp}$. Then we have, in particular, $\langle x, x - \mathrm{P}_{\mathcal{K}} x \rangle = 0$. This implies $\|x\|^2 = \langle x, \mathrm{P}_{\mathcal{K}} x \rangle = \|\mathrm{P}_{\mathcal{K}} x\|^2$, which implies $x = \mathrm{P}_{\mathcal{K}} x \in \mathcal{K}$.

3.3 Problem 3

Exercise 23. Let $(x_n) \in l^\infty(\mathbb{N})$ be a bounded sequence. Define for each $k \in \mathbb{N}$ the simple function

$$S_k((x_n)) = \sum_{n=1}^{k} x_n \chi_{[2^{-n}, 2^{1-n}]}$$

Prove the following:

1. For each fixed $(x_n) \in l^\infty(\mathbb{N})$, the sequence $(S_k((x_n)))_{k \in \mathbb{N}}$ converges in $L^2[0,1]$.
2. Let $T \colon l^\infty(\mathbb{N}) \to L^2[0,1]$ be defined by

$$T((x_n)) = \lim_{k \to \infty} S_k((x_n)),$$

where the limit is the L^2-limit. Prove that T is bounded and find $\|T\|$.

Solution 23. 1. Let $(x_n) \in \ell^\infty(\mathbb{N})$. Let $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon / M^2$. Then $m \geq k \geq N$ implies

$$\begin{aligned}
\|S_m((x_n)) - S_k((x_n))\|_2 &= \int_0^1 \left| \sum_{n=k+1}^{m} x_n \chi_{[2^{-n}, 2^{1-n}]} \right|^2 \mathrm{d}x \\
&= \int_0^1 \sum_{n=k+1}^{m} |x_n|^2 \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&\leq \int_0^1 \sum_{n=k+1}^{m} M^2 \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&= M^2 \int_0^1 \sum_{n=k+1}^{m} \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&= M^2 \int_0^1 \chi_{[2^{-m}, 2^{-k}]} \mathrm{d}x \\
&\leq M^2 2^{-k}. \\
&\leq M^2 2^{-N} \\
&< M^2 \frac{\varepsilon}{M^2} \\
&= \varepsilon.
\end{aligned}$$

This implies $(S_k((x_n)))_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^2[0,1]$. In particular, it must converge since $L^2[0,1]$ is complete.

2. Let $(x_n) \in \ell^\infty(\mathbb{N})$ such that $|x_n| \le 1$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\|T((x_n))\|_2 &= \left\| \sum_{n=1}^{\infty} x_n \chi_{[2^{-n}, 2^{1-n}]} \right\|_2 \\
&= \int_0^1 \sum_{n=1}^{\infty} |x_n|^2 \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&\le \int_0^1 \sum_{n=1}^{\infty} \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&= \int_0^1 \chi_{[0,1]} \mathrm{d}x \\
&= 1.
\end{aligned}$$

This implies T is bounded with $\|T\| \le 1$. In fact, we claim that $\|T\| = 1$. Indeed, we just take the constant sequence $(x_n) = (1)$. Then clearly in this case we have

$$\begin{aligned}
\|T((1))\|_2 &= \int_0^1 \sum_{n=1}^{\infty} \chi_{[2^{-n}, 2^{1-n}]} \mathrm{d}x \\
&= \int_0^1 \chi_{[0,1]} \mathrm{d}x \\
&= 1.
\end{aligned}$$

3.4 Problem 4

Exercise 24. Let X, Y be normed linear spaces. A linear operator $T\colon X \to Y$ is called **compact** if for each bounded sequence (x_n) in X, the sequence (Tx_n) in Y contains a convergent subsequence.

1. Show that if $T\colon X \to Y$ is compact, then it is bounded.
2. Assume Y is a Banach space and $(T_n\colon X \to Y)$ is a sequence of compact linear operators that converges uniformly to a linear operator $T\colon X \to Y$, namely $\|T_n - T\| \to 0$ as $n \to \infty$. Show that T is also compact.

Solution 24. 1. Assume for a contradiction that T is not bounded. Then for each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\|x_n\| \le 1$ and $\|Tx_n\| \ge n$. The sequence (x_n) is bounded, and since T is compact, the sequence (Tx_n) must contain a convergent subequence, say $(Tx_{\pi(n)})$. However, $(Tx_{\pi(n)})$ cannot be convergent since $\|Tx_{\pi(n)}\| \ge \pi(n)$ for all $n \in \mathbb{N}$ implies

$$\lim_{n \to \infty} \|Tx_{\pi(n)}\| = \infty.$$

Indeed, if $Tx_{\pi(n)} \to y$ for some $y \in Y$, then we must have

$$\lim_{n \to \infty} \|Tx_{\pi(n)}\| = \|y\| \neq \infty.$$

2. Let (x_k) be a bounded sequence in X. Assume for a contradiction that (Tx_k) does not contain a convergent subsequence in Y. Then there exists an $\varepsilon > 0$ such that for all subsequences $(Tx_{\pi(k)})$ of (Tx_k) there exists a subsequence $(Tx_{\rho(k)})$ of $(Tx_{\pi(k)})$ such that

$$\|Tx_{\rho(k)} - Tx_{\rho(m)}\| \ge \varepsilon$$

for all $k, m \in \mathbb{N}$. Another way of phrasing this is that for each $k \in \mathbb{N}$, there exists an $m > k$ such that

$$\|Tx_{\pi(k)} - Tx_{\pi(m)}\| \ge \varepsilon.$$

We fix such an ε and will derive a contradiction.

Now, choose $M > 0$ such that $\|x_k\| \le M$ for all $k \in \mathbb{N}$. Also, choose $n \in \mathbb{N}$ such that

$$\|T_n - T\| < \frac{\varepsilon}{3M}.$$

Since T_n is compact and (x_k) is bounded, the sequence $(T_n x_k)$ contains a convergent subsequence, say $(T_n x_{\pi(k)})$. In particular, the sequence $(T_n x_{\pi(k)})$ is Cauchy, and so we can choose a $K \in \mathbb{N}$ such that $m, k \geq K$ implies

$$\|T_n x_{\pi(k)} - T_n x_{\pi(m)}\| < \frac{\varepsilon}{3}.$$

Then $m, k \geq K$ implies

$$\begin{aligned}
\|Tx_{\pi(k)} - Tx_{\pi(m)}\| &= \|Tx_{\pi(k)} - T_n x_{\pi(k)} + T_n x_{\pi(k)} - T_n x_{\pi(m)} + T_n x_{\pi(m)} - Tx_{\pi(m)}\| \\
&\leq \|Tx_{\pi(k)} - T_n x_{\pi(k)}\| + \|T_n x_{\pi(k)} - T_n x_{\pi(m)}\| + \|T_n x_{\pi(m)} - Tx_{\pi(m)}\| \\
&\leq \|T - T_n\|\|x_{\pi(k)}\| + \|T_n x_{\pi(k)} - T_n x_{\pi(m)}\| + \|T_n - T\|\|x_{\pi(m)}\| \\
&< \frac{\varepsilon}{3M} \cdot M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} \cdot M \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

This is a contradiction as ε was chosen in a such way that so that we can find a subsequence $(Tx_{\rho(k)})$ of $(Tx_{\pi(k)})$ such that

$$\|Tx_{\rho(k)} - Tx_{\rho(m)}\| \geq \varepsilon$$

for all $k, m \in \mathbb{N}$. However, there can be no such subsequence since, as we've just shown, we have

$$\|Tx_{\pi(k)} - Tx_{\pi(m)}\| < \varepsilon$$

for all $k, m \geq K$.

3.5 Problem 5

Exercise 25. Let (X, d) be a compact metric space and let $f \colon X \to X$ be a continuous function.

1. Prove that the set

   $$f(X) = \{f(x) \mid x \in X\}$$

   is compact.

2. Assume, in addition, that $f \colon X \to X$ is an isometry of (X, d) (that is, $\mathrm{d}(f(x), f(y)) = \mathrm{d}(x, y)$ for all $x, y \in X$). Prove that f is surjective.

Solution 25. 1. We shall a prove a more general result. Suppose X and Y are topological space and suppose $f \colon X \to Y$ is a surjective continuous function. We will show that if X is compact, then Y is compact. In other words, the image of a compact set under a continuous function is compact. Let $\{V_j\}_{j \in J}$ be an open covering of Y. Then $\{f^{-1}(V_j)\}_{j \in J}$ is an open covering of X. Since X is compact, there exists a finite subcovering of $\{f^{-1}(V_j)\}_{j \in J}$, say $\{f^{-1}(V_{j_1}), \dots, f^{-1}(V_{j_n})\}$. We claim that $\{V_{j_1}, \dots, V_{j_n}\}$ is a finite subcovering of $\{V_j\}_{j \in J}$. Indeed, it suffices to show that

$$Y = \bigcup_{k=1}^{n} V_{j_k}. \tag{3}$$

Indeed, this follows from the fact that f is surjective: if $y \in Y$, then we choose $x \in X$ such that $f(x) = y$, then since $\{f^{-1}(V_{j_1}), \dots, f^{-1}(V_{j_n})\}$ is an open covering of X, we see that $x \in f^{-1}(V_{j_k})$ for some k, and this implies $y \in V_{j_k}$. So $Y \subseteq \bigcup_{k=1}^{n} V_{j_k}$, and since the reverse inclusion is trivial, we have (3).

2. Let $x \in X$ and let $d = \inf\{\mathrm{d}(f(y), x) \mid y \in X\}$. We will first show that $d = 0$. Assume for a contradiction that $d > 0$. Then observe that for all $n \in \mathbb{N}$, we have

$$\mathrm{d}(f^n(x), x) \geq d.$$

In particular, since f is an isometry, this implies

$$\mathrm{d}(f^n(x), f^m(x)) \geq d$$

for all $n, m \in \mathbb{N}$. In particular, the sequence $(f^n(x))$ has no convergent subsequence since the distance between any two terms in the sequence is always greater than d. This contradicts the fact that $f(X)$ is compact. Therefore $d = 0$.

Now let $g\colon X \to \mathbb{R}_{\geq 0}$ be the function given by

$$g(y) = \mathrm{d}(f(y), x)$$

for all $y \in X$. Note that g is continuous since it is the composite of the continuous function $X \to X \times X$, given by $y \mapsto (f(x), f(y))$, with the continuous function $X \times X \to \mathbb{R}_{\geq 0}$, given by $(x, y) \mapsto \mathrm{d}(x, y)$. Therefore it attains a minimum value, say at $x_0 \in X$. In particular, we have $\mathrm{d}(f(x_0), x) = 0$, which implies $f(x_0) = x$. Thus f is surjective.

3.6 Problem 6

Exercise 26.

1. State (without proof) the monotone convergence theorem and the dominated convergence theorem.
2. Evaluate (with justification) the limit

$$\lim_{n\to\infty} \int_1^\infty \frac{1}{(1 + x/n)^n} \mathrm{d}x$$

Solution 26. 1. The monotone convergence theorem is:

Theorem 3.1. *(MCT) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n\colon X \to [0, \infty])$ be an increasing sequence of nonnegative measurable functions which converges pointwise to a nonnegative function $f\colon X \to [0, \infty]$. Then*

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

The dominated convergence theorem is:

Theorem 3.2. *(DCT) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $g\colon X \to [0, \infty]$ be a nonnegative integrable function. Suppose $(f_n\colon X \to \mathbb{R})$ is a sequence of integrable functions such that*

1. *(f_n) converges pointwise to $f\colon X \to \mathbb{R}$.*
2. *$|f_n| \leq g$ pointwise for all $n \in \mathbb{N}$.*

Then

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

2. For each $n \in \mathbb{N}$ set $f_n = (1 + x/n)^{-n}$. Note that (f_n) is a decreasing sequence of nonnegative integrable functions each of which converges pointwise to e^{-x}. Indeed, if $m \leq n$, then we have

$$\begin{aligned}
(1 + x/n)^{-n} \leq (1 + x/m)^{-m} &\iff \log((1 + x/n)^{-n}) \leq \log((1 + x/m)^{-m}) \\
&\iff -n \log((1 + x/n)) \leq -m \log((1 + x/m)) \\
&\iff n \log((1 + x/n)) \geq m \log((1 + x/m))
\end{aligned}$$

where the last inequality follows from the fact that

$$\begin{aligned}
n \log((1 + x/n))\Big|_{x=0} &= n \\
&\geq m \\
&= m \log((1 + x/m))\Big|_{x=0}
\end{aligned}$$

and from the fact that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} (n \log((1 + x/n))) &= \frac{1}{1 + x/n} \\
&\geq \frac{1}{1 + x/m} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} (m \log((1 + x/m)))
\end{aligned}$$

for all $x \geq 0$. Since

$$\begin{aligned}\int_1^\infty f_2 \mathrm{d}x &= \int_1^\infty \frac{1}{(1+x/2)^2}\mathrm{d}x \\ &= -\frac{2}{1+x/2}\Big|_1^\infty \\ &= \frac{4}{3},\end{aligned}$$

it follows from the decreasing version of MCT that

$$\begin{aligned}\lim_{n\to\infty}\int_1^\infty \frac{1}{(1+x/n)^n}\mathrm{d}x &= \int_1^\infty e^{-x}\mathrm{d}x \\ &= -e^{-x}\Big|_1^\infty \\ &= 1/e.\end{aligned}$$

3.7 Problem 7

Exercise 27. Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that $(f_n\colon X \to \mathbb{R})$ is a sequence of measurable functions which converges pointwise to $f\colon X \to \mathbb{R}$. Prove that f is measurable.

Solution 27. The standard trick here is to first prove that $\sup f_n$ and $\inf f_n$ are measurable. For $\sup f_n$, we have

$$\{\sup f_n > c\} = \bigcup_{n=1}^\infty \{f_n > c\}$$

for all $c \in \mathbb{R}$. It follows that $\sup f_n$ is measurable. For $\inf f_n$, we have

$$\{\inf f_n < c\} = \bigcup_{n=1}^\infty \{f_n < c\}$$

for all $c \in \mathbb{R}$. Next we have

$$\limsup f_n = \inf_{N\geq 1}\sup_{n\geq N} f_n \quad \text{and} \quad \liminf f_n = \sup_{N\geq 1}\inf_{n\geq N} f_n,$$

and so $\limsup f_n$ and $\liminf f_n$ are both measurable. Finally, since $\lim f_n = f$, we have

$$\limsup f_n = f = \liminf f_n.$$

Thus f is measurable.

3.8 Problem 8

Exercise 28. Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that $(f_n\colon X \to \mathbb{R})$ is a sequence of measurable functions, and there is a nonnegative integrable function $f\colon X \to [0, \infty)$ such that $|f_n| \leq f$ for every $n \in \mathbb{N}$. Prove that

$$\limsup \int_X f_n \mathrm{d}\mu \leq \int_X \limsup f_n \mathrm{d}\mu.$$

Solution 28. Observe that $(f - f_n)$ is a sequence of nonnegative measurable functions. Thus by Fatou's Lemma, we have

$$\begin{aligned}\int_X g\mathrm{d}\mu - \int_X \limsup f_n \mathrm{d}\mu &= \int_X (g - \limsup f_n)\mathrm{d}\mu \\ &\leq \liminf \int_X (g - f_n)\mathrm{d}\mu \\ &= \int_X g\mathrm{d}\mu - \limsup \int f_n \mathrm{d}\mu.\end{aligned}$$

Subtracting $\int_X g\mathrm{d}\mu$ from both sides and negating both sides gives us the desired inequality.

4 Summer 2018

4.1 Problem 1

Exercise 29. Let (a_n) be a sequence of real numbers such that $a_n \to 0$. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = 0. \tag{4}$$

Solution 29. Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies $-\varepsilon < a_n < \varepsilon$. Then for all $k \in \mathbb{N}$, we have

$$\frac{1}{N_\varepsilon + k}\sum_{n=1}^{N_\varepsilon} a_n - \frac{k\varepsilon}{N_\varepsilon + k} \leq \frac{1}{N_\varepsilon + k}\sum_{n=1}^{N_\varepsilon + k} a_n \leq \frac{1}{N_\varepsilon + k}\sum_{n=1}^{N_\varepsilon} a_n + \frac{k\varepsilon}{N_\varepsilon + k} \tag{5}$$

Taking $k \to \infty$ in (5) gives us

$$-\varepsilon \leq \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary it follows that (4) holds.

4.2 Problem 2

Exercise 30. Let X be a normed linear subspace and $\emptyset \neq Y \subseteq X$ be a subset with the property that $X \backslash Y$ is a linear subspace. Show that Y is dense in X.

Solution 30. Since $Y \neq \emptyset$ we see that $X\backslash Y$ is a proper subspace of X. It follows that $\mathrm{int}(X\backslash Y) = \emptyset$ (see winter 2020 problem 2), or equivalently, Y is dense in X.

4.3 Problem 4

Exercise 31. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two closed linear subspaces of $\mathcal{H}$. Denote P_1 and P_2 to be the orthogonal projections onto $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Show that $\|\mathrm{P}_1 - \mathrm{P}_2\| \leq 1$.

Solution 31. Let $x \in \mathcal{H}$. We have

$$\begin{aligned}
\|(\mathrm{P}_1 - \mathrm{P}_2)(x)\|^2 &= \|\mathrm{P}_1 x - \mathrm{P}_2 x\|^2 \\
&= \|\mathrm{P}_1(\mathrm{P}_1 x - \mathrm{P}_2 x)\|^2 + \|\mathrm{P}_1 x - \mathrm{P}_2 x - \mathrm{P}_1(\mathrm{P}_1 x - \mathrm{P}_2 x)\|^2 \\
&= \|\mathrm{P}_1 x - \mathrm{P}_1\mathrm{P}_2 x\|^2 + \|\mathrm{P}_1\mathrm{P}_2 x - \mathrm{P}_2 x\|^2 \\
&= \|\mathrm{P}_1(x - \mathrm{P}_2 x)\|^2 + \|\mathrm{P}_2 x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&\leq \|x - \mathrm{P}_2 x\|^2 + \|\mathrm{P}_2 x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&= \|x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&\leq \|x\|^2.
\end{aligned}$$

It follows that $\|\mathrm{P}_1 - \mathrm{P}_2\| \leq 1$.

5 Winter 2016

5.1 Problem 1

Exercise 32. Evaluate the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{6}$$

Solution 32. The series converges by the alternating series test. Recall the Mclaurin expansion for $\log(1-x)$ is given by

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

with radius of convegence $r = 1$. Since the series (6) converges, we find that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

6 Summer 2016

6.1 Problem 1

Exercise 33. Let $f_n \colon [0,1] \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{x^n}{1+x^n}$$

for every $n \in \mathbb{N}$.

1. Prove or disprove: (f_n) converges uniformly on $[0,1]$.
2. Show that
$$\lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x = 0.$$

Solution 33. 1. First note that (f_n) converges pointwise to the function $f \colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/2 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1). \end{cases}$$

Indeed, if $x \in [0,1)$, then

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \left(\frac{x^n}{1+x^n} \right) \\ &\leq \lim_{n\to\infty} x^n \\ &= 0, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \left(\frac{x^n}{1+x^n} \right) = 0.$$

If $x = 1$, then

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{1^n}{1+1^n} \right) &= \lim_{n\to\infty} \left(\frac{1}{2} \right) \\ &= \frac{1}{2}. \end{aligned}$$

So if (f_n) converges uniformly, then it must converge uniformly to f. However each f_n is a continuous function, whereas f is not continuous. This is a contradiction.

2. As noted in part 1, (f_n) converges pointwise to f. Also the sequence (f_n) is dominated by the integrable constant function 1. Indeed, for any $x \in [0,1]$, we have

$$\begin{aligned} f_n(x) &= \frac{x^n}{1+x^n} \\ &\leq x^n \\ &\leq 1. \end{aligned}$$

Thus by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x &= \int_0^1 f(x)\mathrm{d}x \\ &= 0, \end{aligned}$$

where the last equality holds since $f = 0$ almost everywhere.

6.2 Problem 2

Exercise 34. Let $X = \{(x_n) \subseteq \mathbb{R} \mid x_n \neq 0 \text{ for finitely many } n \in \mathbb{N}\}$ and consider the metric $\mathrm{d}\colon X \times X \to \mathbb{R}$ defined by

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

for all $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ in X. Prove or disprove: (X, d) is a complete metric space.

Solution 34. It is not a complete metric space. To see why, consider the sequence $(\mathbf{x}^n)$ in X where

$$\mathbf{x}_n = (1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots).$$

First we claim that $(\mathbf{x}_n)$ is a Cauchy sequence. It is clearly a sequence in X since for each $n \in \mathbb{N}$ only finitely many components in $\mathbf{x}_n$ is nonzero. Let us now show that it is Cauchy. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then $n \geq m \geq N$ implies

$$\begin{aligned} \mathrm{d}(\mathbf{x}_m, \mathbf{x}_n) &= \frac{1}{m} \\ &\leq \frac{1}{N} \\ &< \varepsilon. \end{aligned}$$

It follows that $(\mathbf{x}_n)$ is a Cauchy sequence in X.

However note that $(\mathbf{x}_n)$ cannot converge to an element in X. To see why, assume for a contradiction that $\mathbf{x}_n \to \mathbf{x} = (x_n)$ where $\mathbf{x} \in X$. Then only finitely many x_n's are nonzero. In particular, we can choose $N \in \mathbb{N}$ so that $x_N = 0$. Then $n \geq N$ implies

$$\mathrm{d}(\mathbf{x}, \mathbf{x}_n) \geq \frac{1}{N}.$$

This contradicts our assumption that $\mathbf{x}_n \to \mathbf{x}$.

6.3 Problem 3

Exercise 35. Let (X, d) be a metric space and let (x_n) be a convergent sequence in X which converges to $x_0 \in X$. Show that

$$K = \{x_n \mid n \in \mathbb{N} \cup \{0\}\}$$

is a compact set.

Solution 35. It suffices to show that every sequence in K has a convergent subsequence with a limit in K. Let $(x_{\pi(n)})$ be a sequence in K. Here, π is viewed as a function from $\mathbb{N} \to \mathbb{N}$, which is not necessarily increasing. If for some $k \in \mathbb{N}$ we have $\pi(n) = k$ infinitely many $n \in \mathbb{N}$, then we can view the constant sequence $(x_k)_{n \in \mathbb{N}}$ with k fixed as a subsequence of $(x_{\pi(n)})$. So assume we can't do this. We construct a subsequence of $(x_{\pi(n)})$ as follows. First, we start with any $n_1 \in \mathbb{N}$ and we set $\rho(1) = \pi(1)$. Next, we choose $n_2 \in \mathbb{N}$ such that $\pi(n_2) > \pi(n_1)$ and we set $\rho(2) = \pi(n_2)$. Note that we can do this since, otherwise the function π takes a value less than or equal to $\pi(n_1)$ infinitely many times. We proceed inductively: at the kth step, we choose $n_{k+1} \in \mathbb{N}$ such that $\pi(n_{k+1}) > \pi(n_k)$ and we set $\rho(k+1) = \pi(n_{k+1})$. Thus we have constructed a function $\rho\colon \mathbb{N} \to \mathbb{N}$ which is strictly increasing. In particular, $(x_{\rho(n)})$ is both a subsequence of $(x_{\pi(n)})$ and of (x_n). Since $(x_{\rho(n)})$ is a subsequence of (x_n), it must converge to x_0 also. Thus $(x_{\rho(n)})$ is a convergent subsequence of $(x_{\pi(n)})$.

6.4 Problem 4

Exercise 36. Let X be a normed linear space and let T, S be two different bounded linear operators on X such that $T^2 = T$, $S^2 = S$, and $TS = ST$. Show that $\|T - S\| \geq 1$.

Solution 36. Since $T^2 = T$, $S^2 = S$, and $TS = ST$, we have $(T - S)^3 = T - S$. Therefore for any $x \in X$, we have

$$\begin{aligned} \|(T - S)x\| &= \|(T - S)^3 x\| \\ &\leq \|T - S\| \|(T - S)^2 x\| \\ &\leq \|T - S\|^2 \|(T - S)x\|. \end{aligned}$$

It follows that $\|T-S\|^2 \geq 1$, which implies $\|T-S\| \geq 1$. Note that we also have $||T+S\| \geq 1$. Indeed, for any $x \in X$, we have

$$\begin{aligned}
\|(T-S)x\| &= \|Tx - Sx\| \\
&= \|T^2x - S^2x\| \\
&= \|(T^2 - S^2)x\| \\
&= \|(T+S)(T-S)x\| \\
&\leq \|T+S\|\|(T-S)x\|
\end{aligned}$$

It follows at once that $\|T+S\| \geq 1$.

6.5 Problem 5

Exercise 37. Let $\mathcal{H}$ be a Hilbert space and let (x_n) be a sequence of elements in $\mathcal{H}$ that satisfies the following two conditions:

1. There exists $M > 0$ such that $\|x_n\| \leq M$

Solution 37.

7 Winter 2015

7.1 Problem 1

Exercise 38. For $n \in \mathbb{N}$, let $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that $f_n \to 0$ in $L^1([0,1])$ while $f_n \not\to 0$ in $L^\infty([0,1])$.

Solution 38. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|f_n\|_1 &= \int_0^1 |f_n(x)| \mathrm{d}x \\
&= \int_0^1 \frac{nx}{1+n^2x^2} \mathrm{d}x \\
&= \frac{1}{2n} \int_1^{1+n^2} \frac{1}{u} \mathrm{d}u && u = 1 + n^2x^2 \\
&= \frac{1}{2n} \ln(1+n^2).
\end{aligned}$$

Thus, by L'Hospital's rule, we see that

$$\begin{aligned}
\lim_{n\to\infty} \|f_n\|_1 &= \lim_{n\to\infty} \left(\frac{1}{2n} \ln(1+n^2) \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} \frac{2n}{1+n^2} \right) \\
&= 0.
\end{aligned}$$

Thus $\|f_n\|_1 \to 0$ as $n \to \infty$ which implies $f_n \to 0$ in $L^1([0,1])$ as $n \to \infty$.
On the other hand, for each $n \in \mathbb{N}$, observe that

$$\begin{aligned}
\|f_n\|_\infty &= \sup_{x\in[0,1]} \left(\frac{nx}{1+n^2x^2} \right) \\
&\geq \frac{1}{1+1} \\
&= 1/2,
\end{aligned}$$

where the inequality follows from setting $x = 1/n$.

7.2 Problem 2

Exercise 39. Let $f\colon (-1,1) \to \mathbb{R}$ be convex, that is,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in (-1,1)$ and $t \in [0,1]$. Show that f is continuous but not necessarily differentiable.

7.3 Problem 6

Exercise 40. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be a bounded linear operator such that

$$\langle Tx, x \rangle \ge \|x\|^2$$

for all $x \in \mathcal{H}$. Show that the equation $Tx = y$ has a unique solution for every $y \in \mathcal{H}$ and it satisfies $\|x\| \le \|y\|$.

Solution 39. We first show T is injective. Let $x \in \ker T$. Then observe that

$$\begin{aligned} 0 &= \langle 0, x \rangle \\ &= \langle Tx, x \rangle \\ &\ge \|x\|^2 \end{aligned}$$

implies $x = 0$. Thus T is injective.

Next we show $\operatorname{im} T$ is closed. First observe that for each $x \in \mathcal{H}$,

$$\begin{aligned} \|x\|^2 &\le \langle Tx, x \rangle \\ &\le \|Tx\| \|x\| \end{aligned}$$

implies $\|x\| \le \|Tx\|$. Now let (Tx_n) is a Cauchy sequence in $\operatorname{im} T$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $m, n \ge N$ implies $\|Tx_n - Tx_m\| < \varepsilon$. Then $m, n \ge N$ implies

$$\begin{aligned} \varepsilon &> \|Tx_n - Tx_m\| \\ &= \|T(x_n - x_m)\| \\ &\ge \|x_n - x_m\|. \end{aligned}$$

In particular, we see that (x_n) is a Cauchy sequence. Let $x \in \mathcal{H}$ such that $x_n \to x$. Then it follows that $Tx_n \to Tx$ since T is continuous. Thus $\operatorname{im} T$ is closed.

Finally we show that T is surjective. Observe that

$$\operatorname{im} T = \overline{\operatorname{im} T} = (\ker T^*)^{\perp}.$$

Thus to show that $\operatorname{im} T = \mathcal{H}$, we just need to show that $\ker T^* = 0$, that is, that T^* is injective. However the same proof which showed T is injective also shows T^* is injective. Indeed, let $x \in \ker T^*$, then

$$\begin{aligned} 0 &= \langle x, 0 \rangle \\ &= \langle x, T^* x \rangle \\ &= \langle Tx, x \rangle \\ &\ge \|x\|^2 \end{aligned}$$

implies $x = 0$. Thus T^* is injective, which implies T is surjective.

Thus since T is a bijection, there is a unique $x \in \mathcal{H}$ such that $Tx = y$. Futhermore, we have

$$\begin{aligned} \|y\| &= \|Tx\| \\ &\ge \|x\|, \end{aligned}$$

as shown above.

8 Winter 2010

8.1 Problem 1

Exercise 41. Prove the following two statements that look similar but are different.

1. $E \subseteq \mathbb{R}$ is bounded and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous implies $f(E)$ is bounded.

2. $E \subseteq \mathbb{R}$ is bounded and $f\colon E \to \mathbb{R}$ is uniformly continuous implies $f(E)$ is bounded.

Find a counterexample for the following false statement: $E \subseteq \mathbb{R}$ is bounded and $f\colon E \to \mathbb{R}$ is continuous implies $f(E)$ is bounded.

Solution 40. 1. Choose $M > 0$ such that $E \subseteq [-M, M]$. Since f is continuous, the image of a compact set is a compact set. In particular, $f([-M, M])$ is compact. By the Heine-Borel theorem, $f([-M, M])$ is closed and bounded. In particular, $f(E)$ is bounded.

2. We want to show that f can be extended to a continuous function $\widetilde{f}\colon \overline{E} \to \mathbb{R}$. We define $\widetilde{f}$ as follows: let $x \in \overline{E}$. Choose a sequence (x_n) in E such that $x_n \to x$. Then we define

$$\widetilde{f}(x) = \lim_{n\to\infty} f(x_n). \tag{7}$$

We need to make sure that this definition makes since. First, note that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Choose $\delta > 0$ such that $|y - z| < \delta$ implies

$$|f(y) - f(z)| < \varepsilon$$

for all $y, z \in E$. Next, we use the fact that (x_n) is a Cauchy sequence to choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies

$$|x_n - x_m| < \delta.$$

Then $n, m \geq N$ implies

$$|f(x_n) - f(x_m)| < \varepsilon.$$

Thus $(f(x_n))$ is a Cauchy sequence, so the limit in (7) makes since. Finally we note that $\widetilde{f}$ extends f since f is continuous.

Now $\overline{E}$ is a closed and bounded subset of $\mathbb{R}$, so by the Heine-Borel theorem, it must be compact. Therefore $\widetilde{f}(\overline{E})$ is compact, and again by the Heine-Borel theorem, $\widetilde{f}(\overline{E})$ is closed and bounded. In particular, $f(E)$ is bounded.

Now let us counterexample to the last statement. Consider the function $f(x) = 1/x$ defined on the interval $E = (0, 1)$. Even though E is bounded and f is continuous on E, we see that $f(E)$ is not bounded since

$$\begin{aligned} \lim_{n\to\infty} f(1/n) &= \lim_{n\to\infty} \frac{1}{1/n} \\ &= \lim_{n\to\infty} n \\ &= \infty. \end{aligned}$$

Exercise 42. Let (X, d_X) be a compact metric space and let (Y, d_Y) be a (not necessarily complete) metric space.

1. Prove that for any continuous bijection $f\colon X \to Y$, the inverse function $f^{-1}\colon Y \to X$ is also continuous.

2. Find an example that shows (1) is not true in general if X is not compact.

Solution 41. 1. It suffices to show that f is a closed mapping (takes closed sets to closed sets). Let $E \subseteq X$ be a closed set. Since X is compact, E must also be compact. Since f is continuous, $f(E) \subseteq Y$ is also compact. Now since Y is Hausdorff, this implies $f(E)$ is closed.

2. Let (X, d) be the set of real numbers equipped with the discrete metric: that is $X = \mathbb{R}$ as sets and

$$\mathrm{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$. In particular, X is discrete and not compact. Then the identity function $f\colon X \to \mathbb{R}$, given by $f(x) = x$, is continuous (since any function out of a discrete space is continuous). However the inverse function is not continuous ($\{x\} \subseteq X$ is open in X, but $\{f(x)\}$ is not open in $\mathbb{R}$).