Section 3.5

So far we know how to compute the derivative of simple functions like x^5 or e^x. We also know how to compute the derivative of composite functions, like e^{x^5}: indeed we learned the chain rule last section:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^5}\right) &= e^{x^5} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^5) \\ &= e^{x^5} \cdot 5x^4.\end{aligned}$$

Now we want to know how to calculate the derivative of a product of two functions like $x^5 e^x$.

The way that we will do it is via the product rule: The product rule says that if you have two functions $f(x)$ and $g(x)$, then the derivative of their product is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \cdot g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \cdot g(x) + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(g(x)).$$

So for example

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^5 e^x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^5)e^x + x^5\frac{\mathrm{d}}{\mathrm{d}x}(e^x) \\ &= 5x^4 e^x + x^5 e^x.\end{aligned}$$

If we use the prime notation for the derative, then the product rule looks like this:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Let's looks at some examples.

Example 0.1. Suppose $g(x) = 5x^6$ and $h(x) = \ln(x)$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 5x^6 \ln(x).$$

The derivative of their product is given by

$$\begin{aligned}(g \cdot h)'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}((g \cdot h)(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(5x^6 \ln(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(5x^6)\ln(x) + 5x^6\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) \\ &= 30x^5 \ln(x) + 5x^6 \cdot \frac{1}{x} \\ &= 30x^5 \ln(x) + 5x^5 \\ &= 5x^5(6\ln(x) + 1).\end{aligned}$$

Example 0.2. Suppose $g(x) = 2(3^x)$ and $h(x) = 3x^2 - 2x + 1$. Then their product is given by

$$(g \cdot h)(x) = g(x)h(x) = 2(3^x)(3x^2 - 2x + 1).$$

The derivative of their product is given by

$$\begin{aligned}
(g\cdot h)'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}((g\cdot h)(x)) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(2(3^x)(3x^2-2x+1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(2(3^x))(3x^2-2x+1) + 2(3^x)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-2x+1) \\
&= 2\frac{\mathrm{d}}{\mathrm{d}x}(3^x)(3x^2-2x+1) + 2(3^x)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-2x+1) \\
&= 2\ln(3)3^x(3x^2-2x+1) + 2(3^x)(6x-2).
\end{aligned}$$

Example 0.3. $s(x)$ million students gives the number of full time students enrolled in American public colleges and universities, where x is the number of years since the fall semester, 1999.

$t(x)$ thousand dollars per student gives the average tuition paid by a full-time student in an American public colleges and universities, where x is the number of years since the fall semester, 1999.

We denote their product by

$$\begin{aligned}
E(x) &= s(x) \text{ million students} \cdot t(x)\ \frac{\text{thousand dollars}}{\text{student}} \\
&= 1000000s(x) \text{ students} \cdot 1000t(x)\ \frac{\text{dollars}}{\text{student}} \\
&= 1000000000s(x)\cdot t(x) \text{ dollars} \\
&= s(x)\cdot t(x) \text{ billion dollars}
\end{aligned}$$

So billion dollars is the output unit of $E(x)$ and years (since the fall semester, 1999) is the input unit of $E(x)$.

The total amount spent on tuition by students enrolled full time in American public colleges and universities in the fall semester of 2009 is given by

$$\begin{aligned}
E(10) &= s(10)\cdot t(10) \text{ billion dollars} \\
&= 8.3\cdot 4 \text{ billion dollars} \\
&= 33.2 \text{ billion dollars.}
\end{aligned}$$

The amount spent on tuition by sutdents enrolled full time in American public colleges and universities was changing by

$$\begin{aligned}
E'(10) &= s'(10)\ \frac{\text{million students}}{\text{years}}\cdot t(10)\ \frac{\text{thousdand dollars}}{\text{student}} + s(10) \text{ million students}\cdot t'(10)\ \frac{\frac{\text{thousand dollars}}{\text{student}}}{\text{year}} \\
&= s'(10)\cdot t(10)\ \frac{\text{billion dollars}}{\text{years}} + s(10) \text{ million students}\cdot t'(10)\ \frac{\text{thousand dollars}}{\text{student}\cdot\text{year}} \\
&= s'(10)\cdot t(10)\ \frac{\text{billion dollars}}{\text{year}} + s(10)\cdot t'(10)\ \frac{\text{billion dollars}}{\text{year}} \\
&= (s'(10)\cdot t(10) + s(10)\cdot t'(10))\ \frac{\text{billion dollars}}{\text{year}} \\
&= (0.1\cdot 4 + 8.3\cdot 0.2)\ \frac{\text{billion dollars}}{\text{year}} \\
&= (0.1\cdot 4 + 8.3\cdot 0.2)\ \frac{\text{billion dollars}}{\text{year}} \\
&= 2.06\ \frac{\text{billion dollars}}{\text{year}}.
\end{aligned}$$

Example 0.4. Let's answer part a. The input units for $(f\cdot g)(m)$ are months and the output units for $(f\cdot g)(m)$ is given by eggs since

$$\begin{aligned}
(f\cdot g)(m) &= f(m) \text{ layers}\cdot g(m)\ \frac{\text{eggs}}{\text{layer}} \\
&= f(m)\cdot g(m) \text{ eggs.}
\end{aligned}$$

Since $f(3) = 30000$ and $g(3) = 21$, egg production in March 2010 was $30000 \cdot 21 = 630000$ eggs.

Egg product was changing in March by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}m}(f(m)g(m))\mid_{m=3} &= f'(3)\ \frac{\text{layers}}{\text{month}} \cdot g(3)\ \frac{\text{eggs}}{\text{layer}} + f(3)\ \text{layers} \cdot g'(3)\ \frac{\frac{\text{eggs}}{\text{layer}}}{\text{month}} \\
&= f'(3) \cdot g(3)\ \frac{\text{eggs}}{\text{month}} + f(3)\ \text{layers} \cdot g'(3)\ \frac{\text{eggs}}{\text{layer}\cdot\text{month}} \\
&= f'(3) \cdot g(3)\ \frac{\text{eggs}}{\text{month}} + f(3) \cdot g'(3)\ \frac{\text{eggs}}{\text{month}} \\
&= (f'(3) \cdot g(3) + f(3) \cdot g'(3))\frac{\text{eggs}}{\text{month}} \\
&= (500 \cdot 21 + 30000 \cdot 0.2)\frac{\text{eggs}}{\text{month}} \\
&= 16500\frac{\text{eggs}}{\text{month}}.
\end{aligned}$$

Sentence of Interpretation: The egg production was increasing by 16500 eggs per month in March.

Example 0.5. To write a model, we first calculate

$$\begin{aligned}
(f \cdot s)(x) &= f(x)\ \text{percent} \cdot s(x)\ \text{million students} \\
&= \frac{f(x)}{100} \cdot s(x)\ \text{million students} \\
&= \left(\frac{3.09x + 54.18}{100}\right)(0.76\ln(x) + 6.5)\ \ \text{million students} \\
&= (0.0309x + 0.5418)\,(0.76\ln(x) + 6.5)\ \ \text{million students} \\
&= 0.0309x \cdot 0.076\ln(x) + 0.0309x \cdot 6.5 + 0.5418 \cdot 0.76\ln(x) + 0.5418 \cdot 6.5\ \text{million students} \\
&= 0.0023484x\ln(x) + 0.20085x + 0.411768\ln(x) + 3.5217\ \text{million students} \\
&= (0.0023484\ln(x) + 0.20085)x + 0.411768\ln(x) + 3.5217\ \text{million students}
\end{aligned}$$

Model: $(f \cdot s)(x) = (0.0023484\ln(x) + 0.20085)x + 0.411768\ln(x) + 3.5217$ **million students**

Completely Defined Model: $(f \cdot s)(x) = (0.0023484\ln(x) + 0.20085)x + 0.411768\ln(x) + 3.5217$ **million students** gives the number of full-time students with some form of financial aid in the fall of $1999 + x$, where x is the number of years after 1999, $1 \leq x$.

Sentence of Interpretation for $(f \cdot s)(11) \approx 7.338$: The number of students that had fincancial aid in fall 2010 is given by $(f \cdot s)(11) \approx 7.338$ million students

Sentence of Interpretation for $(f \cdot s)'(11) \approx 0.318$**:** The number of students that had fincancial aid was increasing by $(f \cdot s)'(11) \approx 0.318$ million students per year in fall 2010.

Example 0.6. The value of the inverstor's stock 10 weeks after the stock was first offered is 1904.545 dollars. The value of the investor's stock was increasing by 73.496 dollars per week, 10 weeks after the stock was first offered.

Example 0.7. Suppose $f(x) = g(x) \cdot h(x)$ and that

$$\begin{aligned}
g(-1) &= 5 \\
g'(-1) &= 0.02 \\
h(-1) &= -8 \\
h'(-1) &= -0.025.
\end{aligned}$$

Then we have

$$\begin{aligned}
f(-1) &= g(-1) \cdot h(-1) \\
&= 5 \cdot -8 \\
&= -40,
\end{aligned}$$

and

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(g(x) \cdot h(x)) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(g(x)) \cdot h(x) + g(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(h(x)) \end{aligned}$$

and

$$\begin{aligned} f'(-1) &= g'(-1) \cdot h(-1) + g(-1) \cdot h'(-1) \\ &= 0.02 \cdot -8 + 5 \cdot -0.025 \\ &= -0.16 - 0.125 \\ &= -0.141. \end{aligned}$$