Prevarieties

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Throughout these notes, let k be an algebraically closed field and let K be an arbitrary field (not necessarily algebraically closed).

1 Affine Algebraic Sets

Definition 1.1. Let S be a set of polynomials in $K[T_1, \ldots, T_n]$. The set of common zeros of the polynomials in S is denoted by

$$V(S) = \{x \in K^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

Remark 1.

1. If $f_1, \ldots, f_m \in K[T_1, \ldots, T_n]$, then we shorten $V(\{f_1, \ldots, f_m\})$ to $V(f_1, \ldots, f_m)$.

2. If S and T are sets of polynomials $K[T_1, \ldots, T_n]$ such that $S \subset T$, then $V(S) \supset V(T)$. This is called the **inclusion reversing** property of V.

3. Let $\mathfrak{a}$ be the ideal generated by the set S. Then $V(\mathfrak{a}) = V(S)$. Indeed, if $x \in V(S)$, then $f(x) = 0$ for all $f \in S$. But this also means

$$\begin{aligned}(g_1 f_1 + \cdots + g_n f_n)(x) &= g_1(x) f_1(x) + \cdots + g_n(x) f_n(x) \\ &= g_1(x) \cdot 0 + \cdots + g_n(x) \cdot 0 \\ &= 0\end{aligned}$$

for all $g_1 f_1 + \cdots + g_n f_n \in \mathfrak{a}$. So $x \in V(\mathfrak{a})$, hence $V(S) \subseteq V(\mathfrak{a})$. The reverse inclusion follows from the remark above and the fact that $S \subseteq \mathfrak{a}$.

4. By the Hilbert Basis Theorem, $K[T_1, \ldots, T_n]$ is a **Noetherian ring**. This means every ideal in $K[T_1, \ldots, T_n]$ is finitely generated. In particular, there exists $f_1, \ldots, f_m \in S$ such that $I = \langle f_1, \ldots, f_m \rangle$. So $V(S) = V(I) = V(f_1, \ldots, f_k)$.

1.0.1 A Useful Reformulation

Let $x = (x_1, \ldots, x_n) \in K^n$ and let $\mathrm{ev}_x : K[T_1, \ldots, T_n] \to K$ be the unique K-algebra map given by $\mathrm{ev}_x(T_\lambda) = x_\lambda$ for all $\lambda = 1, \ldots, n$. We denote by $\mathfrak{m}_x$ to be the kernel of ev_x. Thus,

$$\mathfrak{m}_x = \{f \in K[T_1, \ldots, T_n] \mid f(x) = 0\}.$$

Clearly, $\mathfrak{m}_x$ is a maximal ideal since $K[T_1, \ldots, T_n]/\mathfrak{m}_x \cong K$.

Now let $\mathfrak{a}$ be an ideal in $K[T_1, \ldots, T_n]$. We can use $\mathfrak{m}_x$ to reformulate what it means for $x \in V(\mathfrak{a})$. Indeed, $x \in V(\mathfrak{a})$ if and only if $\mathfrak{m}_x \supset \mathfrak{a}$. We will use this reformulation many times throughout this article.

1.1 The Zariski Topology

Lemma 1.1. *The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is an inclusion reversing map from the set of ideals of $K[T_1, \ldots, T_n]$ to the set of subsets of K^n. Moreover, the following relations hold:*

1. *$V(0) = K^n$ and $V(1) = \emptyset$.*

2. *For two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have*
$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

3. *For every family $\{\mathfrak{a}_i\}_{i \in I}$ of ideals, we have*

$$V\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

Proof.

1. Trivial.

2. Since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{b}$, it follows that $V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a} \cap \mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b})$ from the inclusion reversing property of V. It remains to show that $V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. We just need to show that $\mathfrak{m}_x \supset \mathfrak{a}\mathfrak{b}$ implies either $\mathfrak{m}_x \supset \mathfrak{a}$ or $\mathfrak{m}_x \supset \mathfrak{b}$ for all $x \in K^n$. But this follows from the fact that $\mathfrak{m}_x$ is a prime ideal.

3. That $V\left(\bigcup_{i\in I}\mathfrak{a}_i\right) = V\left(\sum_{i\in I}\mathfrak{a}_i\right)$ follows from the fact that $\sum_{i\in I}\mathfrak{a}_i$ is the ideal generated by $\bigcup_{i\in I}\mathfrak{a}_i$. That $V\left(\sum_{i\in I}\mathfrak{a}_i\right) = \bigcap_{i\in I} V(\mathfrak{a}_i)$ follows from the fact that $\mathfrak{m}_x \supset \sum_{i\in I}\mathfrak{a}_i$ if and only if $\mathfrak{m}_x \supset \mathfrak{a}_i$ for all $i \in I$ and for all $x \in K^n$.

□

Remark 2. It is very important to pay close attention to what is actually used in proofs. For example, in the proof of the second statement of this lemma, we only used the fact that $\mathfrak{m}_x$ is a prime ideal (even though it is a maximal ideal). This gives us an idea for how we can generalize things. In particular, we will be replacing maximal ideals of the form $\mathfrak{m}_x$ with arbitrary prime ideals. Keep this in mind!

This lemma implies that there is a unique topology on K^n for which the closed subsets are exactly the affine varieties. We call this topology the **Zariski topology** and write $\mathbb{A}^n(K)$ to mean the set K^n equipped with the Zariski topology. We call $\mathbb{A}^n(K)$ an n**-dimensional affine space**.

1.1.1 Affine Algebraic Sets

Definition 1.2. Closed subspaces of $\mathbb{A}^n(K)$ are called **affine algebraic sets**.

Example 1.1. Sets consisting of one point $x = (x_1, \ldots, x_n) \in \mathbb{A}^n(K)$ are closed because $\{x\} = V(\mathfrak{m}_x)$.

1.2 Hilbert's Nullstellensatz

The connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz.

Theorem 1.2. *(Hilbert's Nullstellensatz) Let A be a finitely generated K-algebra. Then A is **Jacobson**, that is, for every prime ideal $\mathfrak{p} \subset A$, we have*

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m}\supset\mathfrak{p} \\ \mathfrak{m} \text{ is maximal}}} \mathfrak{m}$$

If $\mathfrak{m} \subset A$ is a maximal ideal, the field extension $K \subseteq A/\mathfrak{m}$ is finite.

We will base the proof of the theorem on Noether's normalization.

Theorem 1.3. *(Noether's normalization theorem) Let K be a field and let A be a finitely generated K-algebra. Then there exists an integer $n \geq 0$ and $t_1, \ldots, t_n \in A$ such that the K-algebra homomorphism $K[T_1, \ldots, T_n] \to A$, given by $T_i \mapsto t_i$ for all $i = 1, \ldots, n$, is injective and finite.*

Lemma 1.4. *Let $\varphi : A \to B$ be an ring map.*

1. *If $\mathfrak{p} \subset B$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p}) \subset A$ is a prime ideal.*
2. *If φ is an integral extension and $\varphi(a) \in \varphi(A)$ is a unit in B, then $\varphi(a)$ is a unit in the ring $\varphi(A)$ too.*
3. *If φ is an integral extension and B is an integral domain, then B is a field if and only if $A/Ker(\varphi) \cong \varphi(A)$ is a field.*
4. *If φ is an integral extension and $\mathfrak{m} \subset B$ is a maximal ideal, then $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal in A. In particular, if A and B are local rings, then φ is a local ring homomorphism.*

Proof.

1. The inverse image of any ideal in B is an ideal in A, so it suffices to show that $\varphi^{-1}(\mathfrak{p})$ is prime. Suppose $a, b \in A$ such that $ab \in \varphi^{-1}(\mathfrak{p})$. Then $\varphi(a)\varphi(b) = \varphi(ab) \in \mathfrak{p}$. This implies either $\varphi(a)$ or $\varphi(b)$ belongs to $\mathfrak{p}$, which implies either a or b belongs to $\varphi^{-1}(\mathfrak{p})$.
2. Let $\varphi(a) \in \varphi(A)$ and suppose $\varphi(a)b = 1$ for some $b \in B$. Since B is integral over A, there exists $a_0, \ldots, a_{n-1} \in A$ such that
$$b^n + \varphi(a_{n-1})b^{n-1} + \cdots + \varphi(a_0) = 0.$$
Multiplication with $\varphi(a)^{n-1}$ on both sides gives
$$b + \varphi(a_{n-1}) + \cdots + \varphi(a^{n-1}a_0) = 0.$$
In particular, $b \in \varphi(A)$.

3. Assume B is a field. Then $\varphi(A) \subset B$ must be a field too, since every element $\varphi(a) \in \varphi(A)$ is a unit in B, which implies it is a unit in $\varphi(A)$. Now assume $\varphi(A)$ is a field. Let b be a nonzero element in B. Since b is integral over $\varphi(A)$, there exists $a_0, \ldots, a_{n-1} \in A$ such that
$$b^n + \varphi(a_{n-1})b^{n-1} + \cdots + \varphi(a_0) = 0,$$
where we may assume that n is minimal. Then $\varphi(a_0) \neq 0$ since n is minimal and B is an integral domain. Then a straightforward calculation shows that
$$-\varphi(a_0)^{-1}(b^{n-1} + \varphi(a_{n-1})b^{n-2} + \cdots + \varphi(a_1))$$
is the inverse to b.

4. The inverse image of any ideal in B is an ideal in A, so it suffices to show that $\varphi^{-1}(\mathfrak{m})$ is maximal. This is a consequence of 3 because the monomorphism $A/\varphi^{-1}(\mathfrak{m}) \to B/\mathfrak{m}$ is again an integral extension.

□

Lemma 1.5. *Let K be a field and let L be a field extension of K that is finitely generated as a K-algebra. Then L is a finite extension of K.*

Proof. We apply to L Noether's normalization theorem and obtain a finite injective homomorphism $K[T_1, \ldots, T_n] \to L$ of K-algebras. By Lemma (1.4), we must have $n = 0$ which shows that $K \to L$ is a finite extension. □

Proof. (Hilbert's Nullstellensatz) Lemma (1.5) implies at once the second assertion: If $\mathfrak{m} \subset A$ is a maximal ideal, then $A/\mathfrak{m}$ is a field extension of K which is finitely generated as a K-algebra.

For the proof of the first assertion we start with a remark. If L is a finite field extension of K and $\varphi : A \to L$ is a K-algebra homomorphism, then the image of φ is an integral domain that is finite over K. Thus $\mathrm{Im}\varphi$ is a field by Lemma (1.4) and therefore $\mathrm{Ker}\varphi$ is a maximal ideal of A.

We now show that A is Jacobson. Let $\mathfrak{p} \subset A$ be a prime ideal. Replacing A by $A/\mathfrak{p}$ it suffices to show that in an integral finitely generated K-algebra, the intersection of all maximal ideals is the zero ideal. Assume there existed $x \neq 0$ that is contained in all maximal ideals of A. Since x is a nonzero divisor, $A[x^{-1}]$ is a nonzero finitely generated K-algebra. Let $\mathfrak{n}$ be a maximal ideal of $A[x^{-1}]$. Then $L := A[x^{-1}]/\mathfrak{n}$ is a finite extension of K by the second assertion of the Nullstellensatz. The kernel of the composition $\varphi : A \to A[x^{-1}] \to L$ is a maximal ideal by the above remark, but it does not contain x. Contradiction. □

If $K = k$ is an algebraically closed field, then the Nullstellensatz implies:

Corollary 1.

1. *Let A be a finitely generated k-algebra and let $\mathfrak{m} \subset A$ be a maximal ideal. Then $A/\mathfrak{m} \cong k$.*

2. *Let $\mathfrak{m} \subset k[T_1, \ldots, T_n]$ be a maximal ideal. Then there exists $x = (x_1, \ldots, x_n) \in \mathbb{A}^n(k)$ such that $\mathfrak{m} = \mathfrak{m}_x := \langle T_1 - x_1, \ldots, T_n - x_n \rangle$.*

Proof.

1. From the Nullstellenatz, we know that $k \to A/\mathfrak{m}$ is a finite extension of fields. Since k is algebraically closed, we must have $A/\mathfrak{m} \cong k$.

2. Let x_i be the image of T_i by the homomorphism $k[T_1, \ldots, T_n] \to k[T_1, \ldots, T_n]/\mathfrak{m} \cong k$. Then $\mathfrak{m}$ is a maximal ideal which contains the maximal ideal $\mathfrak{m}_x = \langle T_1 - x_1, \ldots, T_n - x_n \rangle$. Therefore both are equal.

□

Remark 3. Note that we really do need k to be algebraically closed here. For instance, $\langle T^2 + 1 \rangle$ is a maximal ideal in $\mathbb{R}[T]$ and
$$\mathbb{R}[T]/\langle T^2 + 1 \rangle \cong \mathbb{C}.$$
Thus, there are more maximal ideals in $\mathbb{R}[T]$ than just the ones which correspond to points ($\mathfrak{m}_x = \langle T - x \rangle$ where $x \in \mathbb{R}$). On the other hand, the Nullstellensatz guarentees that for all maximal ideals $\mathfrak{m}$ in $\mathbb{R}[T]$, we have either $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{C}$ or $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{R}$. If $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{R}$, then $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in \mathbb{R}$, by the same proof as in the proof of Corollary (1). If on the other hand $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{C}$, then it is an easy exercise to show that $\mathfrak{m} = \mathfrak{m}_{z,\bar{z}} = \langle (T - z)(T - \bar{z}) \rangle$ where z is a complex number in the upper-half plane ($\mathrm{Im}(z) > 0$).

Corollary 2. *Let A and B be finitely generated K-algebras and let $\varphi : A \to B$ be a ring map. If $\mathfrak{m} \subset B$ is a maximal ideal, then $\varphi^{-1}(\mathfrak{m}) \subset A$ is a maximal ideal.*

Proof. Since $B/\mathfrak{m}$ is a finite field extension of K, the map $A \to B/\mathfrak{m}$, is an integral extension, and since $B/\mathfrak{m}$ is a field, Lemma (1.4) implies that $A/\varphi^{-1}(\mathfrak{m})$ is a field. Therefore $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal. □

1.3 The Correspondence Between Radical Ideals and Affine Algebraic Sets

Let A be a ring. If A is a finitely generated K-algebra, we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{a}\subseteq\mathfrak{p}\subset A \\ \text{prime ideal}}} \mathfrak{p} = \bigcap_{\substack{\mathfrak{a}\subseteq\mathfrak{m}\subset A \\ \text{maximal ideal}}} \mathfrak{m}.$$

Indeed, the first equality holds in arbitrary commutative rings and the second equality follows immediately from the Nullstellensatz.

We now study the question when two ideals describe the same closed subset of $\mathbb{A}^n(k)$. Clearly this may happen: As $f^r(x) = 0$ if and only if $f(x) = 0$, we always have the equality $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. If $Z \subseteq \mathbb{A}^n(k)$ is a subset, we denote by

$$I(Z) := \{f \in k[T_1, \dots, T_n] \mid f(x) = 0 \text{ for all } x \in Z\}$$

the ideal of functions that vanish on Z. For $f \in k[T_1, \dots, T_n]$ and $x \in \mathbb{A}^n(k)$ we have $f(x) = 0$ if and only if $f \in \mathfrak{m}_x$. Thus we find

$$I(Z) = \bigcap_{x\in Z} \mathfrak{m}_x.$$

We have the following consequence of Hilbert's Nullstellensatz.

Proposition 1.1.

1. *Let $\mathfrak{a} \subseteq k[T_1, \dots, T_n]$ be an ideal. Then*
$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

2. *Let $Z \subseteq \mathbb{A}^n(k)$ be a subset and let $\overline{Z}$ be its closure. Then*
$$V(I(Z)) = \overline{Z}.$$

Proof.

1. As $x \in V(\mathfrak{a})$ is equivalent to $\mathfrak{a} \subseteq \mathfrak{m}_x$, we have
$$I(V(\mathfrak{a})) = \bigcap_{x\in V(\mathfrak{a})} \mathfrak{m}_x = \bigcap_{\substack{\mathfrak{m}\supseteq\mathfrak{a} \\ \text{maximal ideal}}} \mathfrak{m} = \sqrt{\mathfrak{a}}.$$

2. This is a simple assertion for which we do not need the Nullstellensatz. On one hand we have $Z \subseteq V(I(Z))$ and $V(I(Z))$ is closed. This shows $V(I(Z)) \supseteq \overline{Z}$. On the other hand let $V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ be a closed subset that contains Z. Then we have $f(x) = 0$ for all $x \in Z$ and $f \in \mathfrak{a}$. This shows $\mathfrak{a} \subseteq I(Z)$ and hence $V(I(Z)) \subseteq V(\mathfrak{a})$.

□

The proposition implies:

Corollary 3. *The maps*

$$\{\textit{radical ideals } \mathfrak{a} \textit{ of } k[T_1, \dots, T_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\textit{closed subsets } Z \textit{ of } A\}$$

are mutually inverse bijections, whose restritions define a bijection

$$\{\textit{maximal ideals of } k[T_1, \dots, T_n]\} \longleftrightarrow \{\textit{points of } \mathbb{A}^n(k)\}.$$

1.4 Morphisms of Affine Algebraic Sets

Definition 1.3. Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic sets. A **morphism** $f : X \to Y$ of affine algebraic sets is a map $f : X \to Y$ of the underlying sets such that there exist polynomials $f_1, \dots, f_n \in k[T_1, \dots, T_n]$ with $f(x) = (f_1(x), \dots, f_n(x))$ for all $x \in X$. We say that the n-tuple of polynomials

$$(f_1, \dots, f_n) \in (k[T_1, \dots, T_m])^n$$

represents f. The f_i are the **components** of this representation. We denote the set of morphisms from X to Y by $\mathrm{Hom}(X, Y)$.

Remark 4.

1. To say that f is a morphism from $X \subseteq \mathbb{A}_K^m$ to $Y \subseteq \mathbb{A}_K^n$ represented by $(f_1, \dots, f_n)$ means that $(f_1(x), \dots, f_n(x))$ must satisfy the defining equations of Y for all points $x \in X$.

2. Of particular interest is the case $Y = \mathbb{A}_K^1$, where f simply becomes a scalar **polynomial function** defined on X.

Example 1.2.

1. The map $\mathbb{A}^1(k) \to V(T_2 - T_1^2) \subset \mathbb{A}^2(k)$, given by $x \mapsto (x^2, x)$, is a morphism of affine algebraic sets. It is even an isomorphism with inverse morphism $(x, y) \mapsto y$.

2. The map $\mathbb{A}^1(k) \to V(T_2^2 - T_1^2(T_1 + 1))$, given by $x \mapsto (x^2 - 1, x(x^2 - 1))$, is a morphism of affine algebraic sets. For $\operatorname{char}(k) \neq 2$, it is not bijective: 1 and -1 are both mapped to the origin $(0, 0)$. In $\operatorname{char}(k) = 2$, it is bijective but not an isomorphism.

3. We identify the space $M_n(k)$ of $(n \times n)$-matrices with $\mathbb{A}^{n^2}(k)$, thus giving $M_n(k)$ the structure of an affine algebraic set. Then sending a matrix $A \in M_n(k)$ to its determinant $\det(A)$ is a morphism $M_n(k) \to \mathbb{A}^1(k)$ of affine algebraic sets.

4. For $k = \mathbb{C}$, consider the exponential function $\exp : \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$. This is *not* a morphism of algebraic sets. Indeed, this map is not even continuous with respect to the Zariski topology: The preimage of 1 is $\{2\pi i n \mid n \in \mathbb{Z}\}$, which is not closed in the Zariski topology.

5. Let $X = V(T_2 - T_1^2, T_3 - T_1^3) \subset \mathbb{A}^3(k)$ and $Y = V(T_2' - T_1' - T_1'^2) \subset \mathbb{A}^2(k)$. Then the map $f : X \to Y$ whose components are $(T_1 T_2, T_1^2 T_2^2 + T_3)$ is a morphism from X to Y. Indeed, given a point $x = (x_1, x_2, x_3) \in X$, we have $f(x) = (x_1 x_2, x_1^2 x_2^2 + x_3) = (y_1, y_2) \in Y$. We verify this by showing that the point satisfies the defining equation of Y: Indeed, we have $x_3 - x_1 x_2 = 0$, since $(x_1, x_2, x_3) \in X$. Therefore
$$\begin{aligned}(T_2' - T_1' - T_1'^2)(y_1, y_2) &= y_1 - y_2 - y_2^2 \\ &= (x_3 + x_1^2 x_2^2) - (x_1 x_2) - (x_1 x_2)^2 \\ &= x_3 - x_1 x_2 \\ &= 0.\end{aligned}$$
Note that the map f induces a map
$$f^* : K[T_1', T_2'] / \langle T_2' - T_1' - T_1'^2 \rangle \to K[T_1, T_2, T_3] / \langle T_2 - T_1^2, T_3 - T_1^3 \rangle,$$
where f^* is the K-algebra map induced by mapping
$$\begin{aligned} T_1' &\mapsto T_1 T_2 \\ T_2' &\mapsto T_1^2 T_2^2 + T_3. \end{aligned}$$

6. Let $X = V(1 - T_1 T_2) \subseteq \mathbb{A}^2(k)$ and $f : \mathbb{A}^2(k) \to \mathbb{A}^1(k)$ be given by $(x, y) \mapsto x$. Then $f(X) = \mathbb{A}^1(k) \backslash \{0\}$ is not an algebraic set. This shows that the image of an algebraic set is not necessarily an algebraic set.

The notion of an affine algebraic set is still not satisfactory. We list three problems:

- Open subsets of affine algebraic sets do not carry the structure of an affine algebraic set in a natural way. In particular, we cannot glue affine algebraic sets along open subsets (although this is a "natural operation" for geometric objects.
- Intersections of affine algebraic sets in $\mathbb{A}^n(k)$ are closed and hence again affine algebraic sets. But we cannot distinguish between $V(X) \cap V(Y) \subset \mathbb{A}^2(k)$ and $V(Y) \cap V(X^2 - Y) \subset \mathbb{A}^2(k)$ although the geometric situation seems to be different.
- Affine algebraic sets seem not to help in studying solutions of polynomial equations in more general rings than algebraically closed fields.

1.4.1 Morphisms are Zariski-Continuous Maps

We now want to show that morphisms are continuous with respect to the Zariski topology. We first need a lemma:

Lemma 1.6. *Let $Y \subseteq \mathbb{A}^n(k)$ be an affine algebraic set and let $f : \mathbb{A}^m(k) \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Suppose $Y = V(p_1, \dots, p_r)$ and let Z be a closed subset of Y. Then Z has the form

$$Z = V(p_1, \dots, p_r) \cap V(q_1, \dots, q_s) = V(p_1, \dots, p_r, q_1, \dots, q_s).$$

where $V(q_1, \dots, q_s)$ is a closed subset of $\mathbb{A}^n(k)$. In particular

$$f^{-1}(Z) = V(p_1 \circ f, \dots, p_r \circ f, q_1 \circ f, \dots, q_s \circ f)$$

is a closed subset in $\mathbb{A}^m(k)$. □

Proposition 1.2. *Let $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$ be an affine algebraic sets and let $f : X \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Lift f to a morphism $\widetilde{f} : \mathbb{A}^m(k) \to Y$ such that $\widetilde{f}|_X = f$[1]. By Lemma (1.6), $\widetilde{f}$ is continuous. Therefore its restriction $\widetilde{f}|_X = f$ must be continuous also. □

Example 1.3. Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic sets and let $\pi_i : \mathbb{A}^n(k) \to \mathbb{A}^1(k)$ be the projection to the ith coordinate map (i.e. $\pi_i(y_1, \dots, y_i, \dots, y_n) = y_i$). Then $\pi_i|_X$ is continuous with respect to the Zariski topology. Let us show this directly: let $\{z_1, \dots, z_n\}$ be a closed subset of $\mathbb{A}^1(k)$ where the z_j are distinct points in $\mathbb{A}^1(k)$ (every closed subset in $\mathbb{A}^1(k)$ is just a finite set of points). Then

$$\pi_i|_X^{-1}(\{z_1, \dots, z_n\}) = X \cap \left(\bigcap_{j=1}^{n} V(\pi_i - z_j) \right).$$

Thus the inverse image a closed subset in $\mathbb{A}^1(k)$ is a closed subset in X.

1.4.2 Zariski-Continuous Maps are not Necessarily Morphisms

A continuous map with respect to the Zariski topology does not have to be a morphism. Indeed, consider the complex conjugation map $\bar{\cdot} : \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$. This map is continuous with respect to the Zariski topology. To see why, let $V(p_1, \dots, p_r)$ be a closed subset of $\mathbb{A}^1(\mathbb{C})$. Then the inverse image of $V(p_1, \dots, p_r)$ under $\bar{\cdot}$ is the closed subset $V(\overline{p}_1, \dots, \overline{p}_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $\overline{p}_i = \sum_{j=1}^{n_i} \overline{a}_{n_j} z^{n_j}$. On the other hand, $\bar{\cdot}$ does not have a polynomial representation: the only root of $\bar{\cdot}$ is $z = 0$, but $\bar{\cdot} \neq T^m$ for any $m \in \mathbb{N}$.

More generally, if L/K galois extension with galois group $G = \mathrm{Gal}(L/K)$. Then for all $g \in G$ the map $g\cdot : \mathbb{A}^1(L) \to \mathbb{A}^1(L)$, given by $x \mapsto g \cdot x$, is Zariski continuous because the inverse image of a closed subset $V(p_1, \dots, p_r)$ of $\mathbb{A}^1(L)$ under $g\cdot$ is the closed subset $V(g \cdot p_1, \dots, g \cdot p_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $g \cdot p_i = \sum_{j=1}^{n_i} (g \cdot a_{n_j}) z^{n_j}$. On the other hand, $g\cdot$ does not have a polynomial representation: the only root of $g\cdot$ is $z = 0$, but $g\cdot \neq T^m$ for any $m \in \mathbb{N}$. Note that $g\cdot$ gives rise to a K-algebra homomorphism $\Gamma(g\cdot) : L[T] \to L[T]$ (and not an L-algebra homomorphism).

Remark 5. One may wonder why we restrict our morphisms between affine algebraic sets in the first place. Why do we not consider $\mathrm{Hom}(X, Y)$ to be the set of all Zariski-continuous maps? The point is that the category of affine algebraic sets are naturally thought of as being objects in the category of locally ringed spaces (we will define what these are later on) rather than just in the category of topological spaces.

1.5 Affine Algebraic Sets as Reduced Finitely-Generated k-Algebras

Let $X \subseteq \mathbb{A}^n(k)$ be a closed subspace. Every polynomial $f \in k[T_1, \dots, T_n]$ induces a morphism $X \to \mathbb{A}^1(k)$ of affine algebraic sets. The set $\mathrm{Hom}(X, \mathbb{A}^1(k))$ carries in a natural way the structure of a k-algebra with addition and multiplication

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x).$$

To elements of k we associate the corresponding constant functions. The homomorphism

$$k[T_1, \dots, T_n] \to \mathrm{Hom}(X, \mathbb{A}^1(k))$$

is a surjective homomorphism of k-algebras with kernel $I(X)$.

[1] Choosing a lift of f is equivalent to choosing a polynomial representation $(f_1, \dots, f_n) \in (k[T_1, \dots, T_m])^n$ of f.

Definition 1.4. Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic set. The k-algebra

$$\Gamma(X) := k[T_1, \ldots, T_n]/I(X) \cong \mathrm{Hom}(X, \mathbb{A}^1(k))$$

is called the **affine coordinate ring**.

Remark 6. Let $\pi : k[T_1, \ldots, T_n] \to \Gamma(X)$ be the projection map and let $f \in \Gamma(X)$. We can think of f as a function on X as follows: for all $x \in X$, we set

$$f(x) := \widetilde{f}(x),$$

where $\widetilde{f}$ is any lift[2] To see that this is well-defined, let $\widetilde{f}'$ be another lift of f. Then $\widetilde{f}' = \widetilde{f} + \widetilde{g}$, where $\widetilde{g} \in I(X)$. Thus,

$$\begin{aligned} f(x) &= \widetilde{f}'(x) \\ &= \widetilde{f}(x) + \widetilde{g}(x) \\ &= \widetilde{f}(x). \end{aligned}$$

For $x = (x_1, \ldots, x_n) \in X$ we denote by $\mathfrak{m}_x$ the ideal

$$\mathfrak{m}_x := \{f \in \Gamma(x) \mid f(x) = 0\} \subset \Gamma(X).$$

It is the image of the maximal ideal $(T_1 - x_1, \ldots, T_n - x_n)$ of $\Gamma(\mathbb{A}^n(k)) = k[T_1, \ldots, T_n]$ under the projection $\pi : k[T_1, \ldots, T_n] \to \Gamma(X)$. In other words, $\mathfrak{m}_x$ is the kernel of the evaluation homomorphism $\Gamma(X) \to k$, given by $f \mapsto f(x)$. As the evaluation homorphism is clearly surjective, $\mathfrak{m}_x$ is a maximal ideal and we find that $\Gamma(X)/\mathfrak{m}_x \cong k$.

If $\mathfrak{a} \subseteq \Gamma(X)$ is an ideal, consider

$$V(\mathfrak{a}) := \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Note that $V(\mathfrak{a}) = V(\pi^{-1}(\mathfrak{a})) \cap X$. Indeed, let $x \in V(\pi^{-1}(\mathfrak{a})) \cap X$ and $f \in \mathfrak{a}$. Then $f = \pi(\widetilde{f})$ for some $\widetilde{f} \in \pi^{-1}(\mathfrak{a})$ since π is surjective, so

$$\begin{aligned} f(x) &= \pi(\widetilde{f}(x)) \\ &= \pi(0) \\ &= 0. \end{aligned}$$

Conversely, let $x \in V(\mathfrak{a})$ and $\widetilde{f} \in \pi^{-1}(\mathfrak{a})$. Then $\widetilde{f}$ is a lift of some polynomial $f \in \mathfrak{a}$. Therefore $0 = f(x) = \widetilde{f}(x)$ implies $x \in V(\pi^{-1}(\mathfrak{a}))$, and since $V(\mathfrak{a}) \subset X$, we have $V(\mathfrak{a}) \subset V(\pi^{-1}(\mathfrak{a})) \cap X$. Thus the $V(\mathfrak{a})$ are precisely the closed subsets of X if we consider X as a subspace of $\mathbb{A}^n(k)$. This topology is again called the **Zariski topology**. For $f \in \Gamma(X)$ we set

$$D(f) := \{x \in X \mid f(x) \neq 0\} = X \setminus V(f).$$

These are open subsets of X, called **principal open subsets**.

Lemma 1.7. *The open sets $D(f)$, for $f \in \Gamma(X)$, form a basis of the topology (i.e. finite intersections of principal open subsets are again principal open subsets and for every open subset $U \subseteq X$ there exist $f_i \in \Gamma(X)$ with $U = \bigcup_i D(f_i)$.*

Proof. Clearly we have $D(f) \cap D(g) = D(fg)$ for $f, g \in \Gamma(X)$. It remains to show that every open subset U is a union of principal open subsets. We write $U = X \setminus V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. For generators $f_1, \ldots, f_n$ of this ideal we find $V(\mathfrak{a}) = \bigcap_{i=1}^n V(f_i)$, and hence $U = \bigcup_{i=1}^n D(f_i)$. □

Remark 7. Let $f : X \to Y$ be a morphism of affine algebraic sets. Then f is continuous with respect to the Zariski topology. Indeed, if $D(g)$ is a basic open set in Y, then $f^{-1}(D(g)) = D(f^*g)$.

Proposition 1.3. *Let X be an affine algebraic set. The affine coordinate ring $\Gamma(X)$ is a reduced finitely generated k-alebra. Moreover, X is irreducible if and only if $\Gamma(X)$ is an integral domain.*

Proof. As $\Gamma(X) = k[T_1, \ldots, T_n]/I(X)$, it is a finitely generated k-algebra. As $I(X) = \sqrt{I(X)}$, we find that $\Gamma(X)$ is reduced. Also, X is irreducible if and only if $I(X)$ is prime if and only if $\Gamma(X)$ is an integral domain. □

Proposition 1.4. *Let $f : X \to Y$ be a morphism between affine algebraic sets. Then*

1. *$f(X)$ is dense in Y if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is injective.*

[2] i.e. any $\widetilde{f} \in k[T_1, \ldots, T_n]$ such that $\pi(\widetilde{f}) = f$.

2. *$f(X) \subset Y$ is a closed subvariety and $f : X \to f(X)$ is an isomorphism if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is surjective.*

Proof.

1. First assume that $f(X)$ is dense in Y. Suppose that $f^*h = f^*g$ where $g, h \in \Gamma(Y)$. Then for all $x \in X$, we have $h(f(x)) = g(f(x))$. Or in other words, we have $(h - g)(y) = 0$ for all $y \in f(X)$. Since $f(X)$ is dense in Y and $h - g$ is continuous, we must therefore have $(h - g)(y) = 0$ for all $y \in Y$. Thus, $h = g$, which shows that f^* is injective. Conversely, assume that f^* is injective. Suppose $f(X)$ is not dense in Y. Denote $Z := \overline{f(X)}$ and pick $y \in Y$ such that $y \notin Z$. Then $Z \subset Y$ implies $I(Z) \supset I(Y)$. Thus, we can find an $g \in I(Z)$ such that $g \notin I(Y)$. This means $g(z) = 0$ for all $z \in Z$ and there exists $y \in Y$ such that $g(y) \neq 0$. But f^* is injective, $f^*g = f^*0$ implies $g = 0$, which is a contradiction.

□

Remark 8. We say $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y.

1.5.1 Equivalence of Categories Between Affine Algebraic Sets and Reduced Finitely Generated k-Algebras

Let $f : X \to Y$ be a morphism of affine algebraic sets. The map

$$\Gamma(f) : \mathrm{Hom}(Y, \mathbb{A}^1(k)) \to \mathrm{Hom}(X, \mathbb{A}^1(k)),$$

given by $g \mapsto f^*g := g \circ f$, defines a homomorphism of k-algebras. We obtain a functor

$$\Gamma : (\text{affine algebraic sets})^{\text{opp}} \to (\text{reduced finitely generated } k\text{-algebras}).$$

Proposition 1.5. *The functor Γ induces an equivalence of categories. By restriction one obtains an equivalence of categories*

$$\Gamma : (\textit{irreducible affine algebraic sets})^{opp} \to (\textit{integral finitely generated } k\textit{-algebras}).$$

Proof. A functor induces an equivalence of categories if and only if it is fully faithful and essentially surjective. We first show that Γ is fully faithful, i.e. that for affine algebraic sets $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$, the map $\Gamma : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\Gamma(Y), \Gamma(X))$ is bijective. We define an inverse map. If $\varphi : \Gamma(Y) \to \Gamma(X)$ is given, there exists a k-algebra homomorphism $\widetilde{\varphi}$ that makes the following diagram commutative

$$\begin{tikzcd} k[T_1', \ldots, T_m'] \arrow[r, "\widetilde{\varphi}"] \arrow[d] & k[T_1, \ldots, T_n] \arrow[d] \\ \Gamma(Y) \arrow[r, "\varphi"] & \Gamma(X) \end{tikzcd}$$

We define $f : X \to Y$ by

$$f(x) := (\widetilde{\varphi}(T_1')(x), \ldots, \widetilde{\varphi}(T_n')(x))$$

and obtain the desired inverse homomorphism.

It remains to show that the functor is essentially surjective, i.e. that for every reduced finitely generated k-algebra A there exists an affine algebraic set X such that $A \cong \Gamma(X)$. By hypothesis, A is isomorphic to $k[T_1, \ldots, T_n]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $k[T_1, \ldots, T_n]$ with $\mathfrak{a} = \sqrt{\mathfrak{a}}$. If we set $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$, we have

$$\Gamma(X) = k[T_1, \ldots, T_n]/I(V(\mathfrak{a})) = k[T_1, \ldots, T_n]/\mathfrak{a}.$$

□

Remark 9. Let $X \subseteq \mathbb{A}^m(K)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic sets and let $f : X \to Y$ whose components are f_i for $i = 1, \ldots, m$. Write the affine coordinate rings of X and Y as $\Gamma(X) = K[T_1, \ldots, T_m]/I(X)$ and $\Gamma(Y) = K[T_1', \ldots, T_n']/I(Y)$. Then $\Gamma(f)(T_i) := T_i \circ f = f_i$ for all $i = 1, \ldots, m$. Indeed, for all points $x \in X$, we have

$$\begin{aligned} \Gamma(f)(T_i)(x) &= T_i(f(x)) \\ &= T_i(f_1(x), \ldots, f_i(x), \ldots, f_n(x)) \\ &= f_i(x). \end{aligned}$$

Using the bijective correspondence between points of affine algebraic sets X and maximal ideals of $\Gamma(X)$, we also have the following description of morphisms.

Proposition 1.6. *Let $f : X \to Y$ be a morphism of affine algebraic sets and let $\Gamma(f) : \Gamma(Y) \to \Gamma(X)$ be the corresponding homomorphism of the affine coordinate rings. Then $\Gamma(f)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ for all $x \in X$.*

Proof. This follows from $g(f(x)) = \Gamma(f)(g)(x)$ for all $g \in \Gamma(Y) = \mathrm{Hom}(Y, \mathbb{A}^1(k))$. □

1.6 Affine Algebraic Sets as Spaces with Functions

We will now define the notion of a **space with functions**. For us this will be the prototype of a "geometric object". It is a special case of a so-called ringed space on which the notion of a scheme will be based on.

Definition 1.5.

1. A **space with functions over** K is a topological space X together with a family $\mathcal{O}_X$ of K-subalgebras $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, K)$ for every open subset $U \subseteq X$ that satisfy the following properties:
   (a) If $U' \subseteq U \subseteq X$ are open and $f \in \mathcal{O}_X(U)$, then the restriction $f|_{U'} \in \mathrm{Map}(U', K)$ is an element of $\mathcal{O}_X(U')$.
   (b) Given an open covering $\{U_i\}_{i \in I}$ of an open subset U of X and elements $f_i \in \mathcal{O}_X(U_i)$ such that
   $$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$
   for all $i, j \in I$, then there exists a unique function $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.
2. A **morphism** $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of spaces with functions is a continuous map $g : X \to Y$ such that for all open subsets V of Y and functions $f \in \mathcal{O}_Y(V)$, the function $g^* f := f \circ g|_{g^{-1}(V)} : g^{-1}(V) \to K$ lies in $\mathcal{O}_X(g^{-1}(V))$.

Clearly spaces with functions over K form a category.

Definition 1.6. Let X be a space with functions and let U be an open subset of X. We denote by $(U, \mathcal{O}_{X|U})$ the space U with functions
$$\mathcal{O}_{X|U}(V) = \mathcal{O}_X(V)$$
for $V \subseteq U$ open.

1.6.1 The Space with Functions of an Irreducible Affine Algebraic Set

Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. It is endowed with the Zariski topology and we want to define for every open subset $U \subseteq X$ a k-algebra of functions $\mathcal{O}_X(U)$ such that $(X, \mathcal{O}_X)$ is a space with functions.

As X is irreducible, the k-algebra $\Gamma(X)$ is a domain, and by definition all the sets $\mathcal{O}_X(U)$ will be k-subalgebras of its field of fractions.

Definition 1.7. The field of fractions $K(X) := \mathrm{Frac}(\Gamma(X))$ is called the **function field** of X.

If we consider $\Gamma(X)$ as the set of morphisms $X \to \mathbb{A}^1(k)$, elements of the function field f/g, where $f, g \in \Gamma(X)$ and $g \neq 0$, usually do not define functions on X because the denominator may have zeros on X, but certainly f/g defines a function $D(g) \to \mathbb{A}^1(k)$[3] We will use functions of this kind to make X into a space with functions.

Lemma 1.8. *Let X be an irreducible affine algebraic set and let f_1/g_1 and f_2/g_2 be elements of $K(X)$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$ if and only if there exists a non-empty open subset $U \subseteq D(g_1 g_2)$ with*
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in U$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$.

Proof. First suppose $f_1/g_1 = f_2/g_2$ in $K(X)$. This means $f_1 g_2 = f_2 g_1$ in $\Gamma(X)$. In particular,
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in D(g_1 g_2)$. Conversely, let $U \subseteq D(g_1 g_2)$ be a non-empty open subset such that
$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$
for all $x \in U$. Then the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. $\square$

Proof. We have $(f_1 g_2 - f_2 g_1)(x) = 0$ for all $x \in U$. Therefore the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. $\square$

[3] It might be even defined on a bigger open subset of X as there exist representations of the fraction with different denominators.

Definition 1.8. Let X be an irreducible affine algebraic set and let $U \subseteq X$ be open. We denote by $\mathfrak{m}_x$ the maximal ideal of $\Gamma(X)$ corresponding to $x \in X$ and by $\Gamma(X)_{\mathfrak{m}_x}$ the localization of the affine coordinate ring with respect to $\mathfrak{m}_x$. We define

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \Gamma(X)_{\mathfrak{m}_x} \subset K(X).$$

The localization $\Gamma(X)_{\mathfrak{m}_x}$ can be described in this situation as the union

$$\Gamma(X)_{\mathfrak{m}_x} = \bigcup_{f \in \Gamma(X) \setminus \mathfrak{m}_x} \Gamma(X)_f \subset K(X).$$

Remark 10. Note that

$$\Gamma(X)_{\mathfrak{m}_x} = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \right\}.$$

Indeed, $g(x) \neq 0$ is equivalent to $g \notin \mathfrak{m}_x$. It may be tempting to think that

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \text{ for all } x \in U \right\},$$

but this is not necessarily the case. For instance, let $X \subset \mathbb{A}^4$ be the variety defined by the equation $T_1 T_4 = T_2 T_3$. Then $T_1/T_2 \in \mathcal{O}_X(D(T_2))$ and $T_3/T_4 \in \mathcal{O}_X(D(T_4))$ and by the equation of X, these two functions coincide where they are both defined;

$$\frac{T_1}{T_2}|_{D(T_2T_4)} = \frac{T_3}{T_4}|_{D(T_2T_4)}$$

So this gives rise to a regular function on $D(T_2) \cup D(T_4)$. But there is no representation of this function as a quotient of two polynomials in $K[x_1, x_2, x_3, x_4]$ that works on all of $D(x_2) \cup D(x_4)$; we have to use different representations at different points.

On the other hand, it is true that

$$\mathcal{O}_{\mathbb{A}^n(k)}(U) = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \text{ for all } x \in U \right\}.$$

For instance, let $X = \mathbb{A}^2(k)$ and $U = \mathbb{A}^2(k) \setminus \{0\}$. Suppose $f \in \mathcal{O}_X(U)$ and $x \in U$. Since $f \in O_{X,p}$, we can write

$$f|_{D(g_1)} = \frac{f_1}{g_1},$$

where $g_1(x) \neq 0$. We may assume f_1 and g_1 share no common factors. If g_1 is not a constant, then there exists another point $y \in U$ such that $g_1(y) = 0$. Since $f \in \mathcal{O}_{X,y}$, we must be able to write

$$f|_{D(g_2)} = \frac{f_2}{g_2},$$

where $g_2(y) \neq 0$. This implies

$$\frac{f_1}{f_2}|_{D(g_1g_2)} = f = \frac{f_2}{g_2}|_{D(g_1g_2)}.$$

Thus, $f_1/g_1 = f_2/g_2$ in $K(X)$. But the only way we can have $f_1/g_1 = f_2/g_2$ is if $g_1 = hf_1$ and $g_2 = hf_2$, where $h \in k[T_1, T_2]$.[4] But this implies $g_2(y) = h(y) f_2(y) = 0$, which is a contradiction.

To consider $(X, \mathcal{O}_X)$ as a space with functions, we first have to explain how to identify elements $f \in \mathcal{O}_X(U)$ with functions $U \to k$. Given $x \in U$, the element f is by definition in $\Gamma(X)_{\mathfrak{m}_x}$ and we may write $f = g/h$ where $g, h \in \Gamma(X)$ and $h \notin \mathfrak{m}_x$. But then $h(x) \neq 0$ and we may set $f(x) := g(x)/h(x) \in k$. The value of $f(x)$ is well defined and Lemma (1.8) implies that this construction defines an injective map $\mathcal{O}_X(U) \to \mathrm{Map}(U, k)$.

If $V \subseteq U \subseteq X$ are open subsets we have $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V)$ by definition and this inclusion corresponds via the identification with maps $U \to k$ resp. $V \to k$ to the restriction of functions.

To show that $(X, \mathcal{O}_X)$ is a space with functions, we still have to show that we may glue functions together. But this follows immediately from the definition of $\mathcal{O}_X(U)$ as subsets of the function field $K(X)$. We call $(X, \mathcal{O}_X)$ the **space of functions associated with** X. Functions on principal open subsets $D(f)$ can be explicitely described as follows.

Proposition 1.7. *Let $(X, \mathcal{O}_X)$ be the space with functions associated to the irreducible affine algebraic set X and let $f \in \Gamma(X)$. Then there is an equality*

$$\mathcal{O}_X(D(f)) = \Gamma(X)_f$$

(as subsets of $K(X)$). In particular $\mathcal{O}_X(X) = \Gamma(X)$ (taking $f = 1$).

[4] This is related to the fact that $\langle g_1, g_2 \rangle$ has depth 2.

Proof. Clearly we have $\Gamma(X)_f \subset \mathcal{O}_X(D(f))$. Let $g \in \mathcal{O}_X(D(f))$. If we can show that $f^n g = h$, for some $n \in \mathbb{N}$ and $h \in \Gamma(X)$, then $g = h/f^n$ would show that $g \in \Gamma(X)_f$. To do this, we will work with ideals, because our argument will use Nullstellensatz which is a theorem about ideals. So set

$$\mathfrak{a} = \{q \in \Gamma(X) \mid qg \in \Gamma(X)\}.$$

Obviously $\mathfrak{a}$ is an ideal of $\Gamma(X)$ and we have to show that $f \in \operatorname{rad}\mathfrak{a}$. By Hilbert's Nullstellensatz we have $\operatorname{rad}\mathfrak{a} = I(V(\mathfrak{a}))$. Therefore it suffices to show $f(x) = 0$ for all $x \in V(\mathfrak{a})$. Let $x \in X$ be a point with $f(x) \neq 0$, i.e. $x \in D(f)$. As $g \in \mathcal{O}_X(D(f))$, we find $g_1, g_2 \in \Gamma(X)$ with $g_2 \notin \mathfrak{m}_x$ and $g = g_1/g_2$. Thus $g_2 \in \mathfrak{a}$ and as $g_2(x) \neq 0$ we have $x \notin V(\mathfrak{a})$. $\square$

Remark 11.

1. Note that we needed to use Nullstellensatz here. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1(\mathbb{R})$, but not polynomial.

2. The proposition shows that we could have defined $(X, \mathcal{O}_X)$ also in another way, namely by setting

$$\mathcal{O}_X(D(f)) = \Gamma(X)_f \text{ for } f \in \Gamma(X).$$

   As the $D(f)$ for $f \in \Gamma(X)$ form a basis of the topology, the axiom of gluing implies that at most one such space with functions can exist. It would remain to show the existence of such a space (i.e. that for $f, g \in \Gamma(X)$ with $D(f) = D(g)$ we have $\Gamma(X)_f = \Gamma(X)_g$ and that gluing of functions is possible). This is more or less the same as the proof of Proposition (1.7). The way we chose is more comfortable in our situation. For affine schemes we will use the other approach.

Remark 12. If A is an integral finitely generated k-algebra we may construct the space with functions $(X, \mathcal{O}_X)$ of "the" corresponding irreducible affine algebraic set directly without choosing generators of A. Namely, we obtain X as the set of maximal ideals in A. Closed subsets of X are sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{m} \subset A \text{ maximal} \mid \mathfrak{m} \supseteq \mathfrak{a}\},$$

where $\mathfrak{a}$ is an ideal in A. For an open subset $U \subseteq X$ we finally define

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{m} \in U} A_{\mathfrak{m}} \subset \operatorname{Frac}(A).$$

This defines a space with functions $(X, \mathcal{O}_X)$ which coincides the space with functions of the irreducible affine algebraic set X corresponding in A. This approach is the point of departure for the definition of schemes.

1.6.2 The Functor from the Category of Irreducible Affine Algebraic Sets to the Category of Spaces with Functions

Proposition 1.8. *Let X, Y be irreducible affine algebraic sets and $f : X \to Y$ a map. The following assertions are equivalent.*

1. *The map f is a morphism of affine algebraic sets.*

2. *If $g \in \Gamma(Y)$, then $g \circ f \in \Gamma(X)$.*

3. *The map f is a morphism of spaces with functions, i.e. f is continuous and if $U \subseteq Y$ open and $g \in \mathcal{O}_Y(U)$, then $g \circ f \in \mathcal{O}_X(f^{-1}(U))$.*

Proof. The equivalence of (1) and (2) has already been proved in Proposition (1.5). Moreover, it is clear that (2) is implied by (3) by taking $U = Y$. Let us show that (2) implies (3). Let $f^* : \Gamma(Y) \to \Gamma(X)$ be the homomorphism $h \mapsto h \circ f$. For $g \in \Gamma(Y)$ we have

$$\begin{aligned} f^{-1}(D(g)) &= \{x \in X \mid f(x) \in D(g)\} \\ &= \{x \in X \mid g(f(x)) \neq 0\} \\ &= D(f^*(g)). \end{aligned}$$

As the principal open subsets form a basis of the topology, this shows that f is continuous. The homomorphism f^* induces a homomorphism of the localizations $\Gamma(Y)_g \to \Gamma(X)_{f^*(g)}$. By definition of f^* this is the map $\mathcal{O}_Y(D(g)) \to \mathcal{O}_X(D(f^*(g)))$, given by $h \mapsto h \circ f$. This shows the claim if U is principal open. As we can obtain functions on arbitrary open subsets of Y by gluing functions on principal open subsets, this proves (3). $\square$

Altogether we obtain

Theorem 1.9. *The above construction $X \mapsto (X, \mathcal{O}_X)$ defines a fully faithful functor*

(Irreducible affine algebraic sets) $\mapsto$ (Spaces with functions over k).

2 Prevarieties

We have seen that we can embed the category of irreducible affine algebraic sets into the category of spaces with functions. Of course we do not obtain all spaces with functions in this way. We will now define prevarieties as those connected spaces with functions that can be glued together from finitely many spaces with functions attached to irreducible affine algebraic sets.

2.1 Definition of Prevarieties

We call a space with functions $(X, \mathcal{O}_X)$ **connected**, if the underlying topological space X is connected.

Definition 2.1.

1. An **affine variety** is a space with functions that is isomorphic to a space with functions associated to an irreducible affine algebraic set.
2. A **prevariety** is a connected space with functions $(X, \mathcal{O}_X)$ with the property that there exists a finite covering $X = \bigcup_{i=1}^{n} U_i$ such that the space with functions $(U_i, \mathcal{O}_{X|U_i})$ is an affine variety for all $i = 1, \ldots, n$.
3. A **morphism** of prevarieties is a morphism of spaces with functions.

Corollary 4. *The following categories are equivalent.*

1. *The opposed category of finitely generated k-algebras without zero divisors.*
2. *The category of irreducible affine algebraic sets.*
3. *The category of affine varieties.*

We define an **open affine covering of a prevariety** X to be a family of open subspaces with functions $U_i \subseteq X$ that are affine varieties such that $X = \bigcup_i U_i$.

Proposition 2.1. *Let $(X, \mathcal{O}_X)$ be a prevariety. The topological space X is Noetherian (in particular quasi-compact) and irreducible.*

Proof. The first assertion follows from the fact that X has a finite covering of Noetherian spaces, which implies that X is Noetherian. The second assertion follows from the fact that X is connected and has a finite covering of irreducible spaces, which implies X is irreducible. □

2.1.1 Open Subprevarieties

We are now able to endow open subsets of affine varieties, and more general of prevarieties with the structure of a prevariety. Note that in general open subprevarieties of affine varieties are not affine.

Lemma 2.1. *Let X be an affine variety, $f \in \Gamma(X) = \mathcal{O}_X(X)$, and let $D(f) \subseteq X$ be the corresponding principal open subset. Let $\Gamma(X)_f$ be the localization of $\Gamma(X)$ by f and let $(Y, \mathcal{O}_Y)$ be the affine variety corresponding to this integral finitely generated k-algebra. Then $(D(f), \mathcal{O}_{X|D(f)})$ and $(Y, \mathcal{O}_Y)$ are isomorphic spaces with functions. In particular, $(D(f), \mathcal{O}_{X|D(f)})$ is an affine variety.*

Proof. By Proposition (1.7) we have $\mathcal{O}_X(D(f)) = \Gamma(X)_f$. As two affine varieties are isomorphic if and only if their coordinate rings are isomorphic, it suffices to show that $(D(f), \mathcal{O}_{X|D(f)})$ is an affine variety.

Let $X \subseteq \mathbb{A}^n(k)$ and $\mathfrak{a} = I(X) \subseteq k[T_1, \ldots, T_n]$ be the corresponding radical ideal. We consider $k[T_1, \ldots, T_n]$ as a subring of $k[T_1, \ldots, T_n, T_{n+1}]$ and denote by $\mathfrak{a}' \subseteq k[T_1, \ldots, T_n, T_{n+1}]$ the ideal generated by $\mathfrak{a}$ and the polynomial $fT_{n+1} - 1$. Then the affine coordinate ring of Y is $\Gamma(Y) = \Gamma(X)_f \cong k[T_1, \ldots, T_n, T_{n+1}]/\mathfrak{a}'$, and we can identify Y with $V(\mathfrak{a}') \subseteq \mathbb{A}^{n+1}(k)$.

The projection $\mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ to the first n coordinates induces a bijective map

$$j : Y = \{(x, x_{n+1}) \in X \times \mathbb{A}^1(k) \mid x_{n+1} f(x) = 1\} \to D(f) = \{x \in X \mid f(x) \neq 0\}.$$

We will show that j is an isomorphism of spaces with functions. As a restriction of a continuous map, j is continuous. It is also open, because for $\frac{g}{f^N} \in \Gamma(Y)$, with $g \in \Gamma(X)$, we have

$$j\left(D\left(\frac{g}{f^N}\right)\right) = j(D(gf)) = D(gf).$$

Thus j is a homeomorphism.

It remains to show that for all $g \in \Gamma(X)$ the map $\mathcal{O}_X(D(fg)) \to \Gamma(Y)_g$, given by $s \mapsto s \circ j$, is an isomorphism. But we have

$$\mathcal{O}_X(D(fg)) = \Gamma(X)_{fg} = \Gamma(Y)_g$$

and this identification corresponds to the composition with j. $\square$

Proposition 2.2. *Let $(X, \mathcal{O}_X)$ be a prevariety and let $U \subseteq X$ be a non-empty open subset. Then $(U, \mathcal{O}_{X|U})$ is prevariety and the inclusion $U \hookrightarrow X$ is a morphism of prevarieties.*

Proof. As X is irreducible, U is connected. The previous lemma shows that U can be covered by open affine subsets of X. As X is Noetherian, U is quasi-compact. Thus a finite covering suffices. $\square$

2.1.2 Function Field of a Prevariety

Let X be a prevariety. If $U, V \subseteq X$ are non-empty open affine subvarieties, then $U \cap V$ is open in U and non-empty. We have $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(U \cap V) \subseteq K(U)$ by the definition of functions on U, and therefore $\mathrm{Frac}(\mathcal{O}_X(U \cap V)) = K(U)$. The same argument for V shows $K(U) = K(V)$. Thus the function field of a non-empty open affine subvariety U of X does not depend on U and we denote it by $K(X)$.

Definition 2.2. The field $K(X)$ is called the **function field** of X.

Remark 13. Let $f : X \to Y$ be a morphism of affine varieties. As the corresponding homomorphism $\Gamma(Y) \to \Gamma(X)$ between the affine coordinate rings is not injective in general, it does not induce a homomorphism of function fields $K(Y) \to K(X)$. Thus $K(X)$ is not functorial in X. But if $f : X \to Y$ is a morphism of prevarieties whose image contains a non-empty open (and hence dense) subset, f induces a homomorphism $K(Y) \to K(X)$. Such morphisms will be called **dominant**.

Proposition 2.3. *Let X be a prevariety and $U \subseteq X$ a non-empty open subset. Then $\mathcal{O}_X(U)$ is a k-subalgebra of the function field $K(X)$. If $U' \subseteq U$ is another open subset, the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ is the inclusion of subalgebras of $K(X)$. If $U, V \subseteq X$ are arbitrary open subsets, then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$.*

Proof. Let $f : U \to \mathbb{A}^1(k)$ be an element of $\mathcal{O}_X(U)$. Then its vanishing set $f^{-1}(0) \subseteq U$ is closed because f is continuous and $\{0\} \subseteq \mathbb{A}^1(k)$ is closed. Therefore if the restriction of f to U' is zero, then f is zero because U' is dense in U. This shows that restriction maps are injective. The axiom of gluing implies therefore $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$ for all open subsets $U, V \subseteq X$. $\square$

2.1.3 Closed Subprevarieties

Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. We want to define on Z the structure of a prevariety. For this we have to define functions on open subsets of Z. We define:

$$\mathcal{O}'_Z(U) = \{ f \in \mathrm{Map}(U, k) \mid \text{for all } x \in U \text{, there exists } V \subseteq U \text{ open and } g \in \mathcal{O}_X(V) \text{ such that } f\,|_{U \cap V} = g\,|_{U \cap V} \}.$$

The definition shows that $(Z, \mathcal{O}'_Z)$ is a space with functions and that $\mathcal{O}'_X = \mathcal{O}_X$. Once we have shown the following lemma, we will always write $\mathcal{O}_Z$ (instead of $\mathcal{O}'_Z$).

Remark 14. $\mathcal{O}'_Z$ is the sheafification of the sheaf $\mathcal{O}_{X|Z}$.

Lemma 2.2. *Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set and let $Z \subseteq X$ be an irreducible closed subset. Then the space with functions $(Z, \mathcal{O}_Z)$ associated to the affine algebraic set Z and the above defined space with functions $(Z, \mathcal{O}'_Z)$ coincide.*

Proof. In both case Z is endowed with the topology induced by X. As the inclusion $Z \to X$ is a morphism of affine algebraic sets it induces a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$. The definition of $\mathcal{O}'_Z$ shows that $\mathcal{O}'_Z(U) \subseteq \mathcal{O}_Z(U)$ for all open subsets $U \subseteq Z$.

Conversely, let $f \in \mathcal{O}_Z(U)$. For $x \in U$ there exists $h \in \Gamma(Z)$ with $x \in D(h) \subseteq U$. The restriction $f\,|_{D(h)} \in \mathcal{O}_Z(D(h)) = \Gamma(Z)_h$ has the form $f = g/h^n$ where $n \geq 0$ and $g \in \Gamma(Z)$. We lift g and h to elements in $\widetilde{g}, \widetilde{h} \in \Gamma(X)$, set $V := D(\widetilde{h}) \subseteq X$, and obtain $x \in V$, $\widetilde{g}/\widetilde{h}^n \in \mathcal{O}_X(D(\widetilde{h}))$ and $f\,|_{U \cap V} = \frac{\widetilde{g}}{\widetilde{h}^n}\,|_{U \cap V}$. $\square$

As a corollary of the lemme we obtain:

Proposition 2.4. *Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. Let $\mathcal{O}_Z$ be the system of functions defined above. Then $(Z, \mathcal{O}_Z)$ is a prevariety.*

2.2 Gluing Prevarieties

The most general way to construct prevarieties is to take some affine varieties and patch them together:

Example 2.1. Let X_1 and X_2 be prevarieties, $U_1 \subset X_1$ and $U_2 \subset X_2$ be non-empty open subsets, and let $f : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_2}|_{U_2})$ be an isomorphism. Then we can define a prevariety X, obtained by **gluing** X_1 and X_2 along U_1 and U_2 via the isomorphism f:

- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation $x \sim f(x)$ for all $x \in U_1$.
- As a topological space, we endow X with the so-called **quotient topology** induced by the above equivalence relation, i.e. we say that a subset $U \subset X$ is open if $U \cap X_1 \subset X_1$ is open in X_1 and $U \cap X_2 \subset X_2$ is open in X_2.
- As a ringed space, we define the structure sheaf $\mathcal{O}_X$ by

  $$\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2),\text{ and } s_1 = s_2 \text{ on the overlap (i.e. } f^*(s_2|_{U \cap U_2}) = s_1|_{U \cap U_1})\}$$

Example 2.2. Let $X_1 = X_2 = \mathbb{A}^1(k)$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $t \mapsto \frac{1}{t} := t'$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $k = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$. Let us show that $\mathcal{O}_X(X) \cong k$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2|_{U_2}) = b_m T^{-m} + b_{m-1} T^{1-m} + \cdots + b_0 |_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 |_{U_2}.$$

  The only way this happens is if $a_0 = b_0$ and $a_i = b_j = 0$ for all $i, j > 0$. Thus, $(s_1, s_2) = (a_0, a_0)$.
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X. Let us show that $\mathcal{O}_X(X) \cong k[T]$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2|_{U_2}) = b_m T^m + b_{m-1} T^{m-1} + \cdots + b_0 |_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 |_{U_2}.$$

  The only way this happens is if $m = n$ and $a_i = b_i$ for all $i = 0, \ldots, n$.

Example 2.3. Let X be the complex affine curve

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2 \widetilde{x}^4 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x}, \widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x, y) \mapsto (\widetilde{x}, \widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^4}\right)$$

$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x}, \widetilde{y}) \mapsto (x, y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^4}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

3 Projective Varieties

By far the most important example of prevarieties are projective space $\mathbb{P}^n(k)$ and subvarieties of $\mathbb{P}^n(k)$, called (quasi-)projective varieties.

3.1 Homogeneous Polynomials

To describe the functions on projective space we start with some remarks on homogeneous polynomials.

Definition 3.1. A polynomial $f \in R[X_0, \ldots, X_n]$ is called **homogeneous** of degree $d \in \mathbb{Z}_{\geq 0}$ if f is the sum of monomials of degree d.

Lemma 3.1. *Let R be an integral domain with infinitely many elements. Then a polynomial $f \in R[X_0, \ldots, X_n]$ is homogeneous of degree d if and only if*

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n) \tag{1}$$

for all $x_1, \ldots, x_n \in k$ and $0 \neq \lambda \in R$.

Proof. One direction is obvious. We prove the other direction by induction on the number of terms in f. Suppose f has one term, i.e.

$$f = cX_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

for some $0 \neq c \in R$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}$. Choose $x_1, \ldots, x_n \in k$ such that $f(x_1, \ldots, x_n) \neq 0$. This can be done since R has infinitely many elements. Then for all $0 \neq \lambda \in R$, we have

$$\begin{aligned} c\lambda^d x_1^{\alpha_1} \cdots x_n^{\alpha_n} &= \lambda^d f(x_0, \ldots, x_n) \\ &= f(\lambda x_0, \ldots, \lambda x_n) \\ &= c\lambda^{\alpha_1 + \cdots + \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \end{aligned}$$

This implies that

$$cx_1^{\alpha_1} \cdots x_n^{\alpha_n}(\lambda^d - \lambda^{\alpha_1 + \cdots + \alpha_n}) = 0.$$

Since $cx_1^{\alpha_1} \cdots x_n^{\alpha_n} = f(x_1, \ldots, x_n) \neq 0$, we must have $\lambda^d - \lambda^{\alpha_1 + \cdots + \alpha_n} = 0$ since R is an integral domain. In other words, we have

$$\lambda^{d - \alpha_1 - \cdots - \alpha_n} = 1$$

for all $0 \neq \lambda \in R$. The only way this can happen is if $d = \alpha_1 + \cdots + \alpha_n$.

Now assume we have proven it for all polynomials with at most r terms, and let f be a polynomial with $r + 1$ terms. Write $f = g + cm$, where g has r terms and m is a monomial. Then since g and m satisfy (1), we see that every term in g and m has degree d. In particular, f has degree d.

□

Remark 15. In $\mathbb{F}_p$, we have $x^{p-1} = 1$ for all $x \in \mathbb{F}_p$. Thus, we really need to make an argument for why $\lambda^m = 1$ for all $0 \neq \lambda \in R$ implies $m = 0$. It's going to be related to the fact that R has infinitely many elements. Let us show that $\lambda^m = \lambda$ for all $0 \neq \lambda \in R$ implies $m = 0$. Let us first assume that R has characteristic $n > 0$: $n \cdot 1 = 0$. Let $\{r_i\}_{i \in \mathbb{Z}}$ be an infinite set of distinct nonzero elements in R. Then $r_i^m = 1$ implies $r_i^m - 1 = 0$ implies $(r_i^{m/2} - 1)(r_i^{m/2} + 1) = 0$. Since R is an integral domain, and

The zero polynomial is homogeneous of degree d for all d. We denote by $R[X_0, \ldots, X_n]_d$ the R-submodule of all homogeneous polynomials of degree d. As we can decompose uniquely every polynomial into its homogeneous parts, we have

$$R[X_0, \ldots, X_n] = \bigoplus_{n \geq 0} R[X_0, \ldots, X_n]_d.$$

Lemma 3.2. *Let $i \in \{0, \ldots, n\}$ and $d \geq 0$. There is a bijective R-linear map*

$$\Phi_i = \Phi_i^{(d)} : R[X_0, \ldots, X_n]_d \to \{g \in R[T_0, \ldots, \widehat{T}_i, \ldots, T_n] \mid deg(g) \leq d\}$$

given by $f \mapsto f(T_0, \ldots, 1, \ldots, T_n)$.

Proof. We construct an inverse map. Let g be a polynomial in the right hand side set and let $g = \sum_{j=0}^d g_j$ be its decomposition into homogeneous parts (with respect to T_ℓ for $\ell = 0, \ldots, i-1, i+1, \ldots, n$. Define

$$\Psi_i(g) = \sum_{j=0}^{d} X_i^{d-j} g_j(X_0, \ldots, \widehat{X}_i, \ldots, X_n).$$

It is easy to see that Ψ_i and Φ_i are inverse to each other (as both maps are R-linear, it suffices to check this on monomials). □

Example 3.1. Consider $R[X_0, X_1, X_2]$. Then

$$\Psi_1(\Phi_1(X_0^2X_2 + X_1^3 + X_1X_2^2)) = \Psi_1(T_0^2T_2 + 1 + T_2^2) = X_0^2X_2 + X_1^3 + X_1X_2^2.$$

The map Φ_i is called **dehomogenization**, and the map Ψ_i is called **homogenization** (with respect to X_i). For $f \in R[X_0, \dots, X_n]_d$ and $g \in R[X_0, \dots, X_n]_e$, the product fg is homogeneous of degree $d+e$ and we have

$$\Phi_i^{(d)}(f)\Phi_i^{(e)}(g) = \Phi_i^{(d+e)}(fg). \tag{2}$$

If $R = K$ is a field, we will extend homogenization and dehomogenization to fields of fractions as follows. Let $\mathcal{F}$ be the subset of $K(X_0, \dots, X_n)$ that consists of those elements f/g, where $f, g \in K[X_0, \dots, X_n]$ are homogeneous polynomials of the same degree. It is easy to check that $\mathcal{F}$ is a subfield of $K(X_0, \dots, X_n)$. By (2), we have a well defined isomorphism of K-extensions

$$\Phi_i : \mathcal{F} \to K(T_0, \dots, \widehat{T}_i, \dots, T_n), \tag{3}$$

given by $f/g \mapsto \Phi_i(f)/\Phi_i(g)$. Often, we will identify $K(T_0, \dots, \widehat{T}_i, \dots, T_n)$ with the subring $K\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right)$ of the field $K(X_0, \dots, X_n)$. Via this identification, the isomorphism (3) can also be described as follows. Let $f/g \in \mathcal{F}$ with $f, g \in K[X_0, \dots, X_n]_d$ for some d. Set $\widetilde{f} = f/X_i^d$ and $\widetilde{g} = g/X_i^d$. Then $\widetilde{f}, \widetilde{g} \in K\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$ and $\Phi_i(f/g) = \widetilde{f}/\widetilde{g}$.

Example 3.2. Consider $\frac{1}{T_2^3 + T_0^2} \in K(T_0, T_2)$. Then $\frac{X_1^3}{X_2^3 + X_1X_0^2} \in \mathcal{F}$ is its homogenization.

3.2 Definition of the Projective Space $\mathbb{P}^n(k)$

The projective space $\mathbb{P}^n(k)$ is an extremely important prevariety within algebraic geometry. Many prevarieties of interest are subprevarieties of the projective space. Moreover, the projective space is the correct environment for projective geometry which remedies the "defect" of affine geometry of missing points at infinity.

As a set, we define for every field k (not necessarily algebraically closed)

$$\mathbb{P}^n(k) := \{\text{lines through the origin in } k^{n+1}\} = (k^{n+1} \backslash \{0\})/k^\times.$$

Here a line through the origin is per definition a 1-dimensional k-subspace and we denote by $(k^{n+1} \backslash \{0\})/k^\times$ the set of equivalence classes in $k^{n+1} \backslash \{0\}$ with respect to the equivalence relation

$$(x_0, \dots, x_n) \sim (x_0', \dots, x_n') \text{ if and only if there exists } \lambda \in k^\times \text{ such that } x_i = \lambda x_i' \text{ for all } 0 \le i \le n.$$

The equivalence class of a point $(x_0, \dots, x_n)$ is denoted by $(x_0 : \dots : x_n)$. We call the x_i the **homogeneous coordinates** on $\mathbb{P}^n(k)$.

To endow $\mathbb{P}^n(k)$ with the structure of a prevariety we will assume from now on that k is algebraically closed. The following observation is essential: For $0 \le i \le n$ we set

$$U_i := \{(x_0 : \dots : x_n) \in \mathbb{P}^n(k) \mid x_i \neq 0\} \subset \mathbb{P}^n(k).$$

This subset is well-defined and the union of the U_i is all of $\mathbb{P}^n(k)$. There are bijections

$$U_i \cong \mathbb{A}^n(k), \qquad (x_0 : \dots : x_n) \mapsto \left(\frac{x_0}{x_i}, \dots \frac{\widehat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}\right).$$

Via this bijection we will endown U_i with the structure of a space with functions, isomorphic to $(\mathbb{A}^n(k), \mathcal{O}_{\mathbb{A}^n(k)})$, which we denote by $(U_i, \mathcal{O}_{U_i})$. We want to define on $\mathbb{P}^n(k)$ the structure of a space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ such that U_i becomes an open subset of $\mathbb{P}^n(k)$ and such that $\mathcal{O}_{\mathbb{P}^n(k)} \mid_{U_i} = \mathcal{O}_{U_i}$ for all $i = 0, \dots, n$. As $\bigcup_i U_i = \mathbb{P}^n(k)$, there's at most one way to do this:

We define the topology on $\mathbb{P}^n(k)$ by calling a subset $U \subseteq \mathbb{P}^n(k)$ open if $U \cap U_i$ is open in U_i for all i. This defines a topology on $\mathbb{P}^n(k)$ as for all $i \neq j$ the set $U_i \cap U_j = D(T_j) \subseteq U_i$ is open (we use here on $U_i \cong \mathbb{A}^n(k)$ the coordinates $T_0, \dots, \widehat{T}_i, \dots, T_n$). With this definition, $\{U_i\}_{i \in \{0, \dots, n\}}$ becomes an open covering of $\mathbb{P}^n(k)$.

We still have to define functions on open subsets $U \subseteq \mathbb{P}^n(k)$. We set

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f \in \mathrm{Map}(U, k) \mid f \mid_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i) \text{ for all } i = 0, \dots, n\}.$$

It is clear that this defines the structure of a space with functions on $\mathbb{P}^n(k)$, although we still have to see that $\mathcal{O}_{\mathbb{P}^n(k)} \mid_{U_i} = \mathcal{O}_{U_i}$ for all i. This follows from the following description of the k-algebras $\mathcal{O}_{\mathbb{P}^n(k)}(U)$ using the inverse isomorphism of the function field $k(T_0, \dots, \widehat{T}_i, \dots, T_n)$ of U_i with the subfield $\mathcal{F}$ of $k(X_0, \dots, X_n)$.

Proposition 3.1. *Let $U \subseteq \mathbb{P}^n(k)$ be open. Then*

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f : U \to k \mid \forall x \in U,\ \exists x \in V \subseteq U \text{ open and } g, h \in k[X_0, \dots, X_n]$$
$$\text{homogeneous of same degree such that } h(v) \neq 0 \text{ and } f(v) = g(v)/h(v) \text{ for all } v \in V\}.$$

Proof. Let $f \in \mathcal{O}_{\mathbb{P}^n(k)}(U)$. As $f \mid_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$, the function f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g}, \widetilde{h} \in k[T_0, \dots, \widehat{T}_i, \dots, T_n]$. Applying the inverse of (3) yields the desired form of f.

Conversely, let f be an element of the right hand side. We fix $i \in \{0, \dots, n\}$. Thus locally on $U \cap U_i$ the function f has the form g/h where $g, h \in k[X_0, \dots, X_n]_d$ for some d. Once more applying the isomorphism (3) we obtain that f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g}, \widetilde{h} \in k[T_0, \dots, \widehat{T}_i, \dots, T_n]$. This shows $f \mid_{U \cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i)$. □

Example 3.3. Consider $\mathbb{P}^2(k)$ and

$$f \mid_{U \cap U_1} = \frac{T_2^2 + 1}{T_0 + 1}.$$

Then the inverse of (3) yields

$$\frac{X_2^2 + X_1^2}{X_0^2 + X_1^2}$$

Corollary 5. *Let $i \in \{0, \dots, n\}$. The bijection $U_i \cong \mathbb{A}^n(k)$ induces an isomorphism*

$$(U_i, \mathcal{O}_{\mathbb{P}^n(k)} \mid_{U_i}) \cong \mathbb{A}^n(k).$$

of spaces with functions. The space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a prevariety.

Proof. The first assertion follows from the proof of Proposition (3.1). This shows that $\mathbb{P}^n(k)$ is a space with functions that has a finite open covering by affine varieties. Moreover, $\mathbb{P}^n(k)$ is irreducible since it is connected and is covered by finitely many irreducible open subsets. □

The function field $K(\mathbb{P}^n(k))$ of $\mathbb{P}^n(k)$ is by its very definition the function field $K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right)$ of U_i. Using the isomorphism Φ_i, we usually describe $K(\mathbb{P}^n(k))$ as the field

$$K(\mathbb{P}^n(k)) = \{f/g \mid f, g \in k[X_0, \dots, X_n] \text{ homogeneous of the same degree}\}.$$

For $0 \leq i, j \leq n$ the identification of $K(U_i) \cong K(U_j)$ is then given by $\Phi_j \circ \Phi_i^{-1}$. This can be described explicitely

$$K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \mapsto k\left(\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right) = K(U_j), \qquad \frac{X_\ell}{X_i} \mapsto \frac{X_\ell}{X_i}\frac{X_i}{X_j} = \frac{X_\ell}{X_j}.$$

We use these explicit descriptions to prove the following result.

Proposition 3.2. *The only global functions on $\mathbb{P}^n(k)$ are the constant functions, i.e. $\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = k$. In particular, $\mathbb{P}^n(k)$ is not an affine variety for $n \geq 1$.*

Proof. By Proposition (2.3) we have

$$\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = \bigcap_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}^n(k)}(U_i) = \bigcap_{0 \leq i \leq n} k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] = k,$$

where the intersection is taken in $K(\mathbb{P}^n(k))$. The last assertion follows because if $\mathbb{P}^n(k)$ were affine, its set of points would be in bijection to the set of maximal ideals in the ring $k = \mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k))$. This implies that $\mathbb{P}^n(k)$ consists of only one point, so $n = 0$. □

3.2.1 Gluing $\mathbb{A}^1(k)$ With $\mathbb{A}^1(k)$ to Make $\mathbb{P}^1(k)$

We now want to describe how we can glue $\mathbb{A}^1(k)$ with $\mathbb{A}^1(k)$ to make $\mathbb{P}^1(k)$ in explicit detail. First we start with the rings $k[S]$ and $k[T]$

Let X_0 and X_1 be the homogeneous coordinates of $\mathbb{P}^1(k)$ and denote $T := \frac{X_1}{X_0}$ and $S := \frac{X_0}{X_1}$.

3.3 Projective Varieties

Definition 3.2. A prevariety is called a **projective variety** if it is isomorphic to a closed subprevariety of a projective space $\mathbb{P}^n(k)$.

As in the affine case, we speak of projective varieties rather than prevarieties. Similarly, we will talk about subvarieties of projective space, instead of subprevarieties.

For $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n(k)$ and $f \in k[X_0, \ldots, X_n]$ the value $f(x_0, \ldots, x_n)$ obviously depends on the choice of the representative of x and we cannot consider f as a function on $\mathbb{P}^n(k)$. But if f is homogeneous, at least the question whether the value is zero or nonzero is independent of the choice of a representative. Thus we define for homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$ (not necessarily of the same degree) the vanishing set

$$V_+(f_1, \ldots, f_m) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid f_i(x_0 : \cdots : x_n) = 0 \text{ for all } i = 1, \ldots, m\}.$$

Subsets of the form $V_+(f_1, \ldots, f_m)$ are closed. More precisely we have $i = 0, \ldots, n$:

$$V_+(f_1, \ldots, f_m) \cap U_i = V(\Phi_i(f_1), \ldots, \Phi_i(f_m)),$$

where Φ_i denotes as usual dehomogenization with respect to X_i. We will see that all closed subsets of the projective space are of this form. To do this we consider the map

$$f : \mathbb{A}^{n+1}(k) \backslash \{0\} \to \mathbb{P}^n(k), \qquad (x_0, \cdots, x_n) \mapsto (x_0 : \cdots : x_n).$$

As for all i its restriction $f\mid_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a morphism of prevarieties, this holds for f. If $Z \subseteq \mathbb{P}^n(k)$ is a closed subset, $f^{-1}(Z)$ is a closed subset of $\mathbb{A}^{n+1}(k)\backslash\{0\}$ and we denote by $C(Z)$ its closure in $\mathbb{A}^{n+1}(k)$. Affine algebraic sets $X \subseteq \mathbb{A}^{n+1}(k)$ are called **affine cones** if for all $x \in X$ we have $\lambda x \in X$ for all $\lambda \in k^\times$. Clearly $C(Z)$ is an affine cone in $\mathbb{A}^{n+1}(k)$. It is called the **affine cone of** Z.

Proposition 3.3. *Let $X \subseteq \mathbb{A}^{n+1}(k)$ be an affine algebraic set such that $X \neq \{0\}$. Then the following assertions are equivalent.*

1. *X is an affine cone.*
2. *$I(X)$ is generated by homogeneous polynomials.*
3. *There exists a closed subset $Z \subset \mathbb{P}^n(k)$ such that $X = C(Z)$.*

If in this case $I(X)$ is generated by homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$, then $Z = V_+(f_1, \ldots, f_m)$.

3.3.1 Segre Embedding

Consider $\mathbb{P}^n(k)$ with homogeneous coordinates $X_0, \ldots, X_n$ and $\mathbb{P}^m(k)$ with homogeneous coordinates $Y_0, \ldots, Y_m$. We want to find an easy description of the product $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$.

Let $\mathbb{P}^N(k) = \mathbb{P}^{(n+1)(m+1)-1}$ be projective space with homogeneous coordinates $Z_{i,j}$ where $0 \leq i \leq n$ and $0 \leq j \leq m$. There is an obviously well-defined set-theoretic map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ given by $z_{i,j} = x_i y_j$.

Lemma 3.3. *Let $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ be the set-theoretic map as above. Then:*

1. *The image $X = f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ is a projective variety in $\mathbb{P}^N(k)$, with ideal generated by the homogeneous polynomials $Z_{i,j}Z_{i',j'} - Z_{i,j'}Z_{i',j}$ for all $0 \leq i, i' \leq n$ and $0 \leq j, j' \leq m$.*
2. *The map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to X$ is an isomorphism. In particular, $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is a projective variety.*
3. *The closed subsets of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ are exactly those subsets that can be written as the zero locus of polynomials in $k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ that are bihomogeneous in the X_i and Y_i.*

Proof.

1. It is obvious that the points of $f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ satisfy the given equations. Conversely, let z be a point in $\mathbb{P}^N(k)$ with coordinates $z_{i,j}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to affine coordinates by setting $z_{0,0} = 1$. Then we have $z_{i,j} = z_{i,0}z_{0,j}$; so by setting $x_i = z_{i,0}$ and $y_j = z_{0,j}$ we obtain a point (x, y) in $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ that is mapped to z by f.
2. Continuing the above notation, let $z \in f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ be a point with $z_{0,0} = 1$. If $f(x, y) = z$, it follows that $x_0 \neq 0$ and $y_0 \neq 0$, so we can assume $x_0 = 1$ and $y_0 = 1$ as the x_i and y_j are only determined up to a common scalar. But then it follows that $x_i = z_{i,0}$ and $y_j = z_{0,j}$, i.e. f is bijective. The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps; so f is an isomorphism.

3. It follows by the isomorphism of (2) that any closed subset of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is the zero locus of homogeneous polynomials in the $Z_{i,j}$, i.e. of bihomogeneous polynomials in the X_i and Y_j (of the same degree). Conversely, a zero locus of bihomogeneous polynomials can always be rewritten as a zero locus of bihomogeneous polynomials of the same degree in the X_i and Y_j. But such a polynomial is obviously a polynomial in the $Z_{i,j}$, so it determines an algebraic set in $X \cong \mathbb{P}^n \times \mathbb{P}^m$.

$\square$

Example 3.4. Consider the case where $n = 1$ and $m = 2$. Then Segre embedding $f : \mathbb{P}^1(k) \times \mathbb{P}^2(k) \to \mathbb{P}^5(k)$ is given by

$$([x_0 : x_1], [y_0 : y_1 : y_2]) \mapsto [x_0 y_0 : x_0 y_1 : x_0 y_2 : x_1 y_0 : x_1 y_1 : x_1 y_2] := [z_{00} : z_{01} : z_{02} : z_{10} : z_{11} : z_{12}].$$

By Lemma (3.3), the vanishing ideal of $f(\mathbb{P}^1(k) \times \mathbb{P}^2(k))$ is given by

$$\langle Z_{00} Z_{11} - Z_{01} Z_{10}, Z_{00} Z_{12} - Z_{02} Z_{10}, Z_{01} Z_{12} - Z_{02} Z_{11} \rangle.$$

We can view this as the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \end{pmatrix}.$$

This is an example of a **determinantal variety**.