MATH 8500: HOMEWORK 3

DUE WEDNESDAY, FEBRUARY 3RD

You may work on homework together, but you must write up your solutions individually and write the names of the individuals with whom you worked. If you use any materials outside of the course materials, e.g., the internet, a different book, or discuss the problems with *anyone other than me*, make sure to provide a short citation.

PROBLEMS:

(1) Get an ssh client. Most operating systems include a command-line ssh client, but ccit also provides a free graphical client:
`https://ccit.clemson.edu/services/software-hardware/software/web-downloads/`

Connect to the Palmetto cluster by ssh-ing
`ssh <username>@login.palmetto.clemson.edu`
You will need two-factor authentication.

To create a new interactive terminal, run the following command:
`qsub -I -l select=1:ncpus=24:mem=60gb:interconnect=fdr,walltime=24:00:00`
It is important that the number of cpus is 24, otherwise `Macaulay2` will crash.

To load the software for this class, run the following commands:
```
export MODULEPATH="/zfs/courses/math8500/ModuleFile:$MODULEPATH"
module add MATH8500
```
You can set this command to be automatically run when you log in if you put int into `.bashrc` (if you know how to do this).

At this point, you can run `Macaulay2` via the command `M2`.

The `Macaulay2` website has some good information for new and experienced users:
- A guide for new users to test out commands:
`https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.15/share/doc/Macaulay2/Macaulay2Doc/html/_a_spfirst_sp__Macaulay2_spsession.html`
- A guide to the `Macaulay2` language:
`https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.15/share/doc/Macaulay2/Macaulay2Doc/html/___The_sp__Macaulay2_splanguage.html`
- An index to most built-in commands:
`https://faculty.math.illinois.edu/Macaulay2/doc/Macaulay2-1.15/share/doc/Macaulay2/Macaulay2Doc/html/master.html`

There is no question for this part, just write on your homework that you were successful in accessing `Macaulay2` on the Palmetto cluster.

(2) Suppose that f_1, f_2, and f_3 are three polynomials in $\mathbb{Q}[x,y,z]$ of degree $3d$ whose coefficients are all integers of absolute value at most 20. Experiment with `Macaulay2` to see how large the coefficients in a Gröbner basis for $\mathcal{V}(f_1,f_2,f_3)$ can be.

(3) Find an example of an ideal $I \subseteq \mathbb{R}[x,y]$ such that $\mathcal{V}(I_1) = \mathbb{R}$, but $\overline{\pi_1(\mathcal{V}(I))} = \emptyset$.

(4) Let $f(x,y) \in k[x,y]$ and $V = \mathcal{V}(f)$. Let p be a point on V. We say that p is a *singular point* of $\mathcal{V}(f)$ if both partial derivatives $\partial_x f(p) = 0$ and $\partial_y f(p) = 0$. If at least one of these partial derivatives is nonzero, then p is a nonsingular point of $\mathcal{V}(f)$.

Let p be a point on V and $\ell(t)$ be a line passing through p when the parameter $t = 0$. Without using calculus, we say that ℓ is *tangent* to V at p if $f(\ell(t))$ has a root of multiplicity greater than one as a function of t at $t = 0$.

(a) Show that the algebraic variety $\mathcal{V}(x^3 - xy + y^2 - 1)$ has no singular points.

(b) Check that the only tangent line to $\mathcal{V}(x^3 - xy + y^2 - 1)$ at the point $(1,1)$ is the one given by calculus.

(c) Construct an algebraic variety whose points correspond to tangent lines to $\mathcal{V}(x^3 - xy + y^2 - 1)$, i.e., the set lines which are tangent at some point of $\mathcal{V}(x^3 - xy + y^2 - 1)$.

(5) Prove that an algebraically closed field is infinite.