Associativity Test Using Gröbner Bases

1 Setup

1.1 Graded Algebra Setup

Let K be a field, let A be an n-dimensional graded K-vector space, and let $\star\colon A \otimes_K A \to A$ be a graded K-linear map. If $\sum_{i=1}^m a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under $\star$ by

$$\star\left(\sum_{i=1}^m a_i \otimes b_i\right) = \sum_{i=1}^m a_i \star b_i.$$

We think of $\star$ as giving A the structure of a graded K-algebra. Now suppose $\{e_1, \dots, e_n\}$ is a basis for A as graded K-vector space. Then for each $1 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{1 \leq k \leq n} c_{i,j}^k e_k$$

where $c_{i,j}^k \in K$ for all $1 \leq k \leq n$. The $c_{i,j}^k$ uniquely determine the graded-multiplication map $\star$; they are called the **structure coefficients**. Since $\star$ is a *graded*-multiplication, we have $c_{i,j}^k = 0$ if $|e_i| + |e_j| \neq |e_k|$, where $|a|$ denotes the degree of a homogeneous element $a \in A$. We assume that $\star$ is commutative, however we will not assume that $\star$ is associative. To measure the failure of associativity, we define the associatior with respect to the triple $(a, b, c) \in A^3$ as

$$[a, b, c] = (a \star b) \star c - a \star (b \star c).$$

It is clear that A is associative if and only if $[a, b, c] = 0$ for all $a, b, c \in A$.

1.2 Monomial Ordering Setup

Let $S = K[x_1, \dots, x_n]$. For each $1 \leq i, j \leq n$, let $f_{i,j} = x_i x_j - x_i \star x_j$ where $x_i \star x_j = \sum_k c_{i,j}^k x_k$. Note that since both $\star$ and $\cdot$ are commutative, we have $f_{i,j} = f_{j,i}$ for all $1 \leq i, j \leq n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i \leq j \leq n\}$ and let I be the homogeneous ideal in S generated by $\mathcal{F}$. We equip S with a weighted lexicographic ordering where x_i is assigned the weight $|x_i| = n + 1 - |e_i|$[1]. Thus given two monomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{and} \quad x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n},$$

we say $x^\alpha >_{\mathrm{Wp}} x^\beta$ if either

1. $|\alpha| > |\beta|$ where $|\alpha| = \sum_{i=1}^n \alpha_i |x_i|$ and $|\beta| = \sum_{i=1}^n \beta_i |e_i|$ or;
2. $|\alpha| = |\beta|$ and there exists $1 \leq i \leq n$ such that $\alpha_i = \beta_i$ andf

$$\begin{aligned} \alpha_1 &= \beta_1 \\ &\vdots \\ \alpha_{i-1} &= \beta_{i-1} \\ \beta_{i-1} &= \beta_i \end{aligned}$$

Observe that for each $1 \leq i, j \leq n$, we have $\mathrm{LT}(f_{i,j}) = x_i x_j$. In particular, if $\mathcal{M} = \{\text{monomials } m \mid m \notin \mathrm{LT}(I)\}$, then we see that $\mathcal{M}$ is a subset of $\{x_1, \dots, x_n\}$.

[1] the reason we assign x_i the weight $n + 1 - |e_i|$ and not $|e_i|$ is so that this becomes a global ordering.

2 Main Theorem

We are now ready to state and prove the main theorem.

Theorem 2.1. *The following statements are equivalent:*

1. $\star$ *is associative.*

2. $\mathcal{F}$ *is a Gröbner basis.*

3. $\mathcal{M} = \{x_1, \ldots, x_n\}$.

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{i,j}$ and $f_{j,k}$ where $1 \le i \le j < k \le n$. We have

$$\begin{aligned} S(f_{i,j}, f_{j,k}) &= x_k f_{i,j} - x_i f_{j,k} \\ &= x_k(x_i x_j - x_i \star x_j) - x_i(x_j x_k - x_j \star x_k) \\ &= x_i(x_j \star x_k) - x_k(x_i \star x_j) \\ &= x_i\left(\sum_l c^l_{j,k} x_l\right) - x_k\left(\sum_l c^l_{i,j} x_l\right) \\ &= \sum_l c^l_{j,k} x_i x_l - \sum_l c^l_{i,j} x_k x_l. \end{aligned}$$

In particular, we see that

$$\begin{aligned} S(f_{i,j}, f_{j,k}) - \sum_l c^l_{j,k} f_{i,l} + \sum_l c^l_{i,j} f_{k,l} &= \sum_l c^l_{j,k} x_i x_l - \sum_l c^l_{i,j} x_k x_l - \sum_l c^l_{j,k} f_{i,l} + \sum_l c^l_{i,j} f_{k,l} \\ &= \sum_l c^l_{j,k}(x_i x_l - f_{i,l}) + \sum_l c^l_{i,j}(f_{k,l} - x_k x_l) \\ &= \sum_l c^l_{j,k}(x_i x_l - x_i x_l + x_i \star x_l) + \sum_l c^l_{i,j}(x_k x_l - x_k \star x_l - x_k x_l) \\ &= \sum_l c^l_{j,k} x_i \star x_l - \sum_l c^l_{i,j} x_k \star x_l \\ &= x_i \star \left(\sum_l c^l_{j,k} x_l\right) - \left(\sum_l c^l_{i,j} x_l\right) \star x_k \\ &= x_i \star (x_j \star x_k) - (x_i \star x_j) \star x_k \\ &= [x_i, x_j, x_k]. \end{aligned}$$

It follows that $S(f_{i,j}, f_{j,k})^{\mathcal{F}} = [x_i, x_j, x_k]$. A straightforward comptutation also shows that $S(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \le i \le n$. Similarly, we have $S(f_{i,j}, f_{k,l})^{\mathcal{F}} = 0$ for all $1 \le i \le j < k \le l \le n$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. $\square$

2.1 Example

Let A be the graded K-vector space with basis $\{e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{24}, e_{34}, e_{123}, e_{124}, e_{134}, e_{234}, e_{1234}\}$ with the obvious grading. We attempt to put a graded-multiplication on A in the Singular code below:

```
intvec w=(4,4,4,4,3,3,3,3,3,3,2,2,2,2,1);
ring A=2,(x1,x2,x3,x4,x12,x13,x14,x23,x24,x34,x123,x124,x134,x234,x1234),Wp(w);
ideal I= x1*x1,
x2*x2,
x3*x3,
x4*x4,
x1*x2+x12,
x1*x3+x14+x34,
x1*x4+x12+x24,
x2*x3+x23,
x2*x4+x12+x13+x34,
x3*x4+x14+x13,
x2*x13+x123,
x2*x14+x134,
```

```
x4*x23+x124+x234,
x2*x24+x123+x234,
x2*x124+x1234;
groebner(I);

_[1]=x1234 // not associative
_[2]=x123+x124+x134+x234 = [x2,x3,x4] // not associative
_[3]=x234^2
_[4]=x134*x234
_[5]=x134^2

....

_[78]=x1*x4+x12+x24
_[79]=x1*x3+x14+x34
_[80]=x1*x2+x12
_[81]=x1^2
```

We see that the multiplication that we are attempting to build is already not associative.