MATH 9500
FALL 2020
HOMEWORK 5

Due Monday, October 26, 2020

If R is a commutative ring with 1, we say that for $x, y \neq 0$, the greatest common divisor of x and y (if it exists) is a common divisor that is "maximal". That is, $gcd(x, y) = d$ if d divides both x and y and if α is another common divisor of x and y then α divides d. We say that R is a GCD domain if every pair of nonzero elements $x, y \in R$, $gcd(x, y)$ exists.

1. Let R be a GCD domain and let $a, b, x \in R$ be nonzero. Show the following.
   a) (5 pts) $\gcd(ax, bx) = x(\gcd(a, b))$.
   b) (5 pts) If $d = \gcd(a, b)$ then $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.
   c) (5 pts) If $\gcd(x, a) = \gcd(x, b) = 1$ then $\gcd(x, ab) = 1$.
   d) (5 pts) If $\gcd(x, a) = 1$ and x divides ab then x divides b.
   e) (5 pts) Show that R is integrally closed.
   f) (5 pts) Show that R is Bezout if and only if $\gcd(a, b)$ is a linear combination of a and b.

2. (5 pts) Show that if R is semiquasilocal and I is an invertible ideal, then I is principal.

3. (5 pts) Build a Noetherian domain of infinite Krull dimension.