Probability Homework 3

Michael Nelson

Problem 3.14

Problem 3.14.a

Let $0 < p < 1$. For each $n \in \mathbb{Z}_{\geq 1}$, let $f_X(n) = -(1-p)^n/(n \log p)$. Note that $f_X(n) > 0$ since $\log p < 0$. Also, note that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f_X(n) &= \sum_{n=1}^{\infty} f_X(n) \\
&= \sum_{n=1}^{\infty} \frac{-(1-p)^n}{n \log p} \\
&= \frac{1}{\log p} \sum_{n=1}^{\infty} \frac{-(1-p)^n}{n} \\
&= \frac{1}{\log p} \log p \\
&= 1,
\end{aligned}$$

where we used the fact that the Taylor series for $\log x$ centered at $x = 1$ is given by

$$\log x = \sum_{n=1}^{\infty} \frac{-(1-x)^n}{n},$$

which has radius of convergence $|x| < 1$. Thus f_X is a legitimate probability function.

Problem 3.14.b

We now wish to find the mean and variance of f_X. First we find the mean. We have

$$\begin{aligned}
\mathrm{E}X &= \sum_{n=-\infty}^{\infty} n f_X(n) \\
&= \sum_{n=1}^{\infty} n f_X(n) \\
&= \sum_{n=1}^{\infty} \frac{-(1-p)^n}{\log p} \\
&= \frac{-1}{\log p} \sum_{n=1}^{\infty} (1-p)^n \\
&= \frac{-1}{\log p} \left(\frac{1-p}{1-(1-p)} \right) \\
&= \frac{p-1}{p \log p},
\end{aligned}$$

where we used the fact that the Taylor series for $x/(1-x)$ centered at $x = 0$ is given by

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} x^n,$$

which has radius of convergence $|x| < 1$.

Next we find the variance. First we calculate

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{n=-\infty}^{\infty} n^2 f_X(n) \\
&= \sum_{n=1}^{\infty} n^2 f_X(n) \\
&= \sum_{n=1}^{\infty} \frac{-n(1-p)^n}{\log p} \\
&= \frac{-1}{\log p} \sum_{n=1}^{\infty} n(1-p)^n \\
&= \frac{-1}{\log p} \left(\frac{1-p}{(1-(1-p))^2} \right) \\
&= \frac{p-1}{p^2 \log p},
\end{aligned}$$

where we used the fact that the Taylor series for $x/(1-x)^2$ centered at $x = 0$ is given by

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n,$$

which has radius of convergence $|x| < 1$. Now we calculate

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - (\mathrm{E}X)^2 \\
&= \frac{p-1}{p^2 \log p} - \left(\frac{p-1}{p \log p} \right)^2 \\
&= \frac{(p-1)\log p - (p-1)^2}{p^2 \log^2 p} \\
&= -\frac{(1-p)\log p + (1-p)^2}{p^2 \log^2 p}
\end{aligned}$$

Problem 3.24

Exponential Distribution

We recall that the pdf of an exponential distribution is given by

$$f(x;\beta) = \begin{cases} \beta e^{-\beta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\beta > 0$ is the paramater of the distribution. The cdf of an exponential distribution is given by

$$F(x;\beta) = \begin{cases} 1 - e^{-\beta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Gamma Distribution

We recall that the pdf of a gamma distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $\alpha > 0$ is the shape paramater of the distribution and $\beta > 0$ is the scale parameter of the distribution, and where the gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x.$$

Problem 3.24.a

Let $\gamma, \beta > 0$, let $X \sim \text{exponential}(\beta)$, and let $Y = g(X) = X^{1/\gamma}$. Clearly $\mathcal{Y} = \mathbb{R}_{\geq 0}$ and $g(x)$ is monotone increasing on $\mathcal{X} = \mathbb{R}_{\geq 0}$. Indeed, we have $g(0) = 0$ and

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x^{1/\gamma}) \\ &= \frac{1}{\gamma}x^{(1-\gamma)/\gamma} \\ &> 0 \end{aligned}$$

for all $x \geq 0$. The inverse function is given by $g^{-1}(y) = y^\gamma$ for all $y \in \mathcal{Y} = \mathbb{R}_{\geq 0}$. Thus if $y \in \mathcal{Y}$, then

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \\ &= \beta e^{-\beta y^\gamma} \cdot \gamma y^{\gamma-1} \\ &= \beta\gamma y^{\gamma-1} e^{-\beta y^\gamma}. \end{aligned}$$

Otherwise, if $y \notin \mathcal{Y}$, then $f_Y(y) = 0$. Let us verify that this is in fact a pdf. First note that $f_Y(y) \geq 0$ for all y. Next, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\mathrm{d}y &= \int_0^\infty f_Y(y)\mathrm{d}y \\ &= \int_0^\infty \beta\gamma y^{\gamma-1} e^{-\beta y^\gamma}\mathrm{d}y \\ &= -e^{-\beta y^\gamma}\Big|_0^\infty \\ &= 0 - (-1) \\ &= 1. \end{aligned}$$

Thus f_Y is in fact a pdf.

Now we calculate the mean and variance. First we calculate the mean. We have

$$\begin{aligned} \mathrm{E}Y &= \int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y \\ &= \int_0^\infty y f_Y(y)\mathrm{d}y \\ &= \int_0^\infty \beta\gamma y^\gamma e^{-\beta y^\gamma}\mathrm{d}y \\ &= \beta^{-1/\gamma}\int_0^\infty u^{1/\gamma}e^{-u}\mathrm{d}u \\ &= \beta^{-1/\gamma}\Gamma(1/\gamma + 1). \end{aligned}$$

where we did a u-substitution with $u = \beta y^\gamma$.

Next we calculate the variance. First we calculate

$$\begin{aligned} \mathrm{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\ &= \int_0^\infty y^2 f_Y(y)\mathrm{d}y \\ &= \int_0^\infty \beta\gamma y^{\gamma+1} e^{-\beta y^\gamma}\mathrm{d}y \\ &= \beta^{-2/\gamma}\int_0^\infty u^{2/\gamma}e^{-u}\mathrm{d}u \\ &= \beta^{-2/\gamma}\Gamma(2/\gamma + 1). \end{aligned}$$

where we did a u-substitution with $u = \beta y^\gamma$. Thus the variance is given by

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}Y)^2 \\ &= \beta^{-2/\gamma}\left(\Gamma(2/\gamma+1) - \Gamma(1/\gamma+1)^2\right). \end{aligned}$$

Problem 3.24.b

Let $\beta > 0$, let $X \sim$ exponential(β), and let $Y = g(X) = (2X/\beta)^{1/2}$. Clearly $\mathcal{Y} = \mathbb{R}_{\geq 0}$ and $g(x)$ is monotone increasing on $\mathcal{X} = \mathbb{R}_{\geq 0}$. The inverse function is given by $g^{-1}(y) = (\beta/2)y^2$ for all $y \in \mathcal{Y}$. Thus if $y \in \mathcal{Y}$, then

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \\ &= \beta e^{-\beta(\beta/2)y^2} \cdot \beta y \\ &= \beta^2 y e^{-\beta^2 y^2/2}. \end{aligned}$$

Otherwise, if $y \notin \mathcal{Y}$, then $f_Y(y) = 0$. Let us verify that this is in fact a pdf. First note that $f_Y(y) \geq 0$ for all y. Next, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\mathrm{d}y &= \int_0^{\infty} f_Y(y)\mathrm{d}y \\ &= \int_0^{\infty} \beta^2 y e^{-\beta^2 y^2/2}\mathrm{d}y \\ &= -e^{-\beta^2 y^2/2}\Big|_0^{\infty} \\ &= 0 - (-1) \\ &= 1. \end{aligned}$$

Thus f_Y is in fact a pdf.

Now we calculate the mean and variance. First we calculate the mean. We have

$$\begin{aligned} \mathrm{E}Y &= \int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y \\ &= \int_0^{\infty} y f_Y(y)\mathrm{d}y \\ &= \int_0^{\infty} \beta^2 y^2 e^{-\beta^2 y^2/2} y \\ &= \frac{2}{\beta\sqrt{2}} \int_0^{\infty} u^{1/2} e^{-u}\mathrm{d}u \\ &= \frac{2}{\beta\sqrt{2}} \Gamma(3/2) \\ &= \frac{1}{\beta}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

where we did a u-substitution with $u = \beta^2 y^2/2$.

Next we calculate the variance. First we calculate

$$\begin{aligned} \mathrm{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\ &= \int_0^{\infty} y^2 f_Y(y)\mathrm{d}y \\ &= \int_0^{\infty} \beta^2 y^3 e^{-\beta^2 y^2/2} y \\ &= \frac{2}{\beta^2} \int_0^{\infty} u e^{-u}\mathrm{d}u \\ &= \frac{2}{\beta^2} \end{aligned}$$

where we did a u-substitution with $u = \beta^2 y^2/2$. Thus the variance is given by

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}Y)^2 \\
&= \frac{2}{\beta^2} - \left(\frac{1}{\beta}\sqrt{\frac{\pi}{2}}\right)^2. \\
&= \frac{2}{\beta^2} - \frac{\pi}{2\beta^2} \\
&= \frac{4-\pi}{2\beta^2}.
\end{aligned}$$

Problem 3.24.c

Let $a, b > 0$, let $X \sim \mathrm{gamma}(a, b)$, and let $Y = g(X) = 1/X$. Clearly $\mathcal{Y} = \mathbb{R}_{>0}$ and $g(x)$ is monotone decreasing on $\mathcal{X} = \mathbb{R}_{>0}$. The inverse function is given by $g^{-1}(y) = 1/y$ for all $y \in \mathcal{Y}$. Thus if $y \in \mathcal{Y}$, then

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| \\
&= \frac{1}{\Gamma(a)b^a} y^{1-a} e^{-1/by} \cdot 1/y^2 \\
&= \frac{1}{\Gamma(a)b^a} y^{-1-a} e^{-1/by}
\end{aligned}$$

Otherwise, if $y \notin \mathcal{Y}$, then $f_Y(y) = 0$. Let us verify that this is in fact a pdf. First note that $f_Y(y) \geq 0$ for all y. Next, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Y(y)\mathrm{d}y &= \int_0^{\infty} f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{1}{\Gamma(a)b^a} y^{-1-a} e^{-1/by} \mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} y^{-1-a} e^{-1/by} \mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} b^a \int_0^{\infty} u^{a-1} e^{-u} \mathrm{d}u \\
&= \frac{1}{\Gamma(a)b^a} b^a \Gamma(a) \\
&= 1.
\end{aligned}$$

where we did a u-substitution with $u = 1/by$. Thus f_Y is in fact a pdf.

Now we calculate the mean and variance. First we calculate the mean. We have

$$\begin{aligned}
\mathrm{E}Y &= \int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} y f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{1}{\Gamma(a)b^a} y^{-a} e^{-1/by} \mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} y^{-a} e^{-1/by} \mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} b^{a-1} u^{a-2} e^{-u} \mathrm{d}u \\
&= \frac{b^{a-1}}{\Gamma(a)b^a} \int_0^{\infty} u^{a-2} e^{-u} \mathrm{d}u \\
&= \frac{\Gamma(a-1)}{b\Gamma(a)} \\
&= \frac{1}{(a-1)b}
\end{aligned}$$

where we did a u-substitution with $u = 1/by$.

Next we calculate the variance. First we calculate

$$\begin{aligned}
\mathrm{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{1}{\Gamma(a)b^a} y^{1-a} e^{-1/by}\mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} y^{1-a} e^{-1/by}\mathrm{d}y \\
&= \frac{1}{\Gamma(a)b^a} \int_0^{\infty} b^{a-2} u^{a-3} e^{-u}\mathrm{d}u \\
&= \frac{b^{a-2}}{\Gamma(a)b^a} \int_0^{\infty} u^{a-3} e^{-u}\mathrm{d}u \\
&= \frac{\Gamma(a-2)}{\Gamma(a)b^2} \\
&= \frac{1}{(a-1)(a-2)b^2}
\end{aligned}$$

where we did a u-substitution with $u = 1/by$. Thus the variance is given by

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}Y)^2 \\
&= \frac{1}{(a-1)(a-2)b^2} - \left(\frac{1}{(a-1)b}\right)^2 \\
&= \frac{1}{(a-1)(a-2)b^2} - \frac{1}{(a-1)^2 b^2} \\
&= \frac{1}{(a-1)(a-2)b^2} - \frac{1}{(a-1)^2 b^2} \\
&= \frac{a-1}{(a-1)^2(a-2)b^2} - \frac{a-2}{(a-1)^2(a-2)b^2} \\
&= \frac{1}{(a-1)^2(a-2)b^2}.
\end{aligned}$$

Problem 3.24.d

Let $\beta > 0$, let $X \sim \text{gamma}(3/2, \beta)$, and let $Y = g(X) = (X/\beta)^{1/2}$. Clearly $\mathcal{Y} = \mathbb{R}_{>0}$ and $g(x)$ is monotone increasing on $\mathcal{X} = \mathbb{R}_{>0}$. The inverse function is given by $g^{-1}(y) = \beta y^2$ for all $y \in \mathcal{Y}$. Thus if $y \in \mathcal{Y}$, then

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left|\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y)\right| \\
&= \frac{1}{\Gamma(2/3)\beta^{3/2}} \beta^{3/2-1} y^{2(3/2)-2} e^{-\beta y^2/\beta} \cdot 2\beta y \\
&= \frac{2}{\Gamma(3/2)} y^2 e^{-y^2}
\end{aligned}$$

Otherwise, if $y \notin \mathcal{Y}$, then $f_Y(y) = 0$. Let us verify that this is in fact a pdf. First note that $f_Y(y) \geq 0$ for all y. Next, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Y(y)\mathrm{d}y &= \int_0^{\infty} f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{2}{\Gamma(3/2)} y^2 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} 2y^2 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} u^{1/2} e^{-u}\mathrm{d}u \\
&= \frac{1}{\Gamma(3/2)} \Gamma(3/2) \\
&= 1
\end{aligned}$$

where we did a u-substitution with $u = y^2$. Thus f_Y is in fact a pdf.

Now we calculate the mean and variance. First we calculate the mean. We have

$$\begin{aligned}
\mathrm{E}Y &= \int_{-\infty}^{\infty} y f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} y f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{2}{\Gamma(3/2)} y^3 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} 2y^3 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} u e^{-u}\mathrm{d}u \\
&= \frac{1}{\Gamma(3/2)} \Gamma(2) \\
&= \frac{2}{\sqrt{\pi}}
\end{aligned}$$

where we did a u-substitution with $u = y^2$.

Next we calculate the variance. First we calculate

$$\begin{aligned}
\mathrm{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_0^{\infty} \frac{2}{\Gamma(3/2)} y^4 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} 2y^4 e^{-y^2}\mathrm{d}y \\
&= \frac{1}{\Gamma(3/2)} \int_0^{\infty} u^{3/2} e^{-u}\mathrm{d}u \\
&= \frac{1}{\Gamma(3/2)} \Gamma(5/2) \\
&= \frac{3}{2}
\end{aligned}$$

where we did a u-substitution with $u = y^2$. Thus the variance is given by

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}Y)^2 \\
&= \frac{3}{2} - \frac{4}{\pi} \\
&= \frac{3\pi - 8}{2\pi}.
\end{aligned}$$

Problem 3.24.e

Let $\alpha \in \mathbb{R}$, let $\beta \in \mathbb{R}_{>0}$ (we are reserving γ to denote the Euler-Mascheroni constant), let $X \sim$ exponential(1), and let $Y = g(X) = \alpha - \beta \log X$. Clearly $\mathcal{Y} = \mathbb{R}$ and $g(x)$ is monotone decreasing on $\mathcal{X} = \mathbb{R}_{\geq 0}$. The inverse function is given by $g^{-1}(y) = e^{\frac{\alpha-y}{\beta}}$ for all $y \in \mathcal{Y}$. Thus if $y \in \mathcal{Y}$, then

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| \\
&= e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)} \frac{1}{\beta} e^{\frac{\alpha-y}{\beta}} \\
&= \frac{1}{\beta} e^{\frac{\alpha-y}{\beta}} e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)}
\end{aligned}$$

Let us verify that this is in fact a pdf. First note that $f_Y(y) \geq 0$ for all y. Next, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Y(y) \mathrm{d}y &= \int_{-\infty}^{\infty} \frac{1}{\beta} e^{\frac{\alpha-y}{\beta}} e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)} \mathrm{d}y \\
&= \left(e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)} \Big|_{-\infty}^{\infty} \right) \\
&= e^{-0} - 0 \\
&= 1.
\end{aligned}$$

Thus f_Y is in fact a pdf.

Now we calculate the mean and variance. First we calculate the mean. We have

$$\begin{aligned}
\mathrm{E}Y &= \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{\beta} y e^{\frac{\alpha-y}{\beta}} e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)} \mathrm{d}y \\
&= \int_0^{\infty} (\alpha - \beta \log u) e^{-u} \mathrm{d}u \\
&= \alpha \int_0^{\infty} e^{-u} \mathrm{d}u - \beta \int_0^{\infty} \log(u) e^{-u} \mathrm{d}u \\
&= \alpha - \beta \Gamma'(1) \\
&= \alpha + \beta \gamma,
\end{aligned}$$

where we did a u-substitution with $u = e^{\frac{\alpha-y}{\beta}}$ and where γ denotes the Euler-Mascheroni constant. Here we used the fact that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} \Gamma(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \int_0^{\infty} x^{z-1} e^{-x} \mathrm{d}x \\
&= \int_0^{\infty} \partial_z x^{z-1} e^{-x} \mathrm{d}x \\
&= \int_0^{\infty} \log(x) x^{z-1} e^{-x} \mathrm{d}x.
\end{aligned}$$

Next we calculate the variance. First we calculate

$$\begin{aligned}
\mathrm{E}(Y^2) &= \int_{-\infty}^{\infty} y^2 f_Y(y)\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{\beta} y^2 e^{\frac{\alpha-y}{\beta}} e^{-\left(e^{\frac{\alpha-y}{\beta}}\right)} \mathrm{d}y \\
&= \int_0^{\infty} (\alpha - \beta \log u)^2 e^{-u} \mathrm{d}u \\
&= \int_0^{\infty} (\alpha^2 - 2\alpha\beta \log u + \beta^2 \log^2 u) e^{-u} \mathrm{d}u \\
&= \alpha^2 \int_0^{\infty} e^{-u}\mathrm{d}u - 2\alpha\beta \int_0^{\infty} \log(u) e^{-u}\mathrm{d}u + \beta^2 \int_0^{\infty} \log^2(u) e^{-u}\mathrm{d}u \\
&= \alpha^2 - 2\alpha\beta\Gamma'(1) + \beta^2\Gamma''(1) \\
&= \alpha^2 + 2\alpha\beta\gamma + \beta^2 \left(\gamma^2 + \frac{\pi^2}{6}\right).
\end{aligned}$$

where we did a u-substitution with $u = e^{\frac{\alpha-y}{\beta}}$. Here we used the fact that

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \int_0^{\infty} \log(x) x^{z-1} e^{-x} \mathrm{d}x \\
&= \int_0^{\infty} \partial_z \log(x) x^{z-1} e^{-x} \mathrm{d}x \\
&= \int_0^{\infty} \log^2(x) x^{z-1} e^{-x} \mathrm{d}x.
\end{aligned}$$

Thus the variance is given by

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{E}(Y^2) - (\mathrm{E}Y)^2 \\
&= \alpha^2 + 2\alpha\beta\gamma + \beta^2 \left(\gamma^2 + \frac{\pi^2}{6}\right) - (\alpha + \beta\gamma)^2 \\
&= \alpha^2 + 2\alpha\beta\gamma + \beta^2\gamma^2 + \beta^2\frac{\pi^2}{6} - \alpha^2 - 2\alpha\beta\gamma - \beta^2\gamma^2 \\
&= \frac{\beta^2\pi^2}{6}.
\end{aligned}$$

Problem 3.38

We have

$$\begin{aligned}
\mathrm{P}(X > x_\alpha) &= \int_{x_\alpha}^{\infty} f_X(x)\mathrm{d}x \\
&= \int_{\sigma z_\alpha + \mu}^{\infty} \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) \mathrm{d}x \\
&= \int_{z_\alpha}^{\infty} f_Z(z)\mathrm{d}z \\
&= \alpha,
\end{aligned}$$

where we did a u-substitution with $u = (x - \mu)/\sigma$.

Problem 3.41

Problem 3.41.a

Let $\mu_1, \mu_2 \in \mathbb{R}$ with $\mu_1 > \mu_2$, let $\sigma^2 \in \mathbb{R}_{\geq 0}$, let $X_1 \sim \mathrm{n}(\mu_1, \sigma^2)$, and let $X_2 \sim \mathrm{n}(\mu_2, \sigma^2)$. Then for all $t \in \mathbb{R}$, we have

$$\begin{aligned} F_{X_2}(t) &= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu_2)^2/(2\sigma^2)} \mathrm{d}x \\ &= \int_{-\infty}^{t+\mu_1-\mu_2} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu_1)^2/(2\sigma^2)} \mathrm{d}u \\ &= F_{X_1}(t + \mu_1 - \mu_2) \\ &> F_{X_1}(t), \end{aligned}$$

where we did a u-substitution with $u = x + \mu_1 - \mu_2$, and where the last inequality follows from the fact that F_{X_1} is strictly increasing and $\mu_1 - \mu_2 > 0$. It follows that the $\mathrm{n}(\mu, \sigma^2)$ family is stochastically increasing in μ for fixed σ^2.

Problem 3.41.b

Let $\alpha, \beta_1, \beta_2 > 0$ with $\beta_1 > \beta_2$, let $X_1 \sim \mathrm{gamma}(\alpha, \beta_1)$, and let $X_2 \sim \mathrm{gamma}(\alpha, \beta_2)$. Then for all $t \in \mathbb{R}_{>0}$, we have

$$\begin{aligned} F_{X_2}(t) &= \int_{0}^{t} \frac{1}{\Gamma(\alpha)\beta_2^{\alpha}} x^{\alpha-1} e^{-x/\beta_2} \mathrm{d}x \\ &= \int_{0}^{(\beta_1/\beta_2)t} \frac{1}{\Gamma(\alpha)\beta_2^{\alpha}} (\beta_2 u/\beta_1)^{\alpha-1} e^{-(\beta_2 u/\beta_1)/\beta_2} \frac{\beta_2}{\beta_1} \mathrm{d}u \\ &= \int_{0}^{(\beta_1/\beta_2)t} \frac{1}{\Gamma(\alpha)\beta_1^{\alpha}} u^{\alpha-1} e^{-u/\beta_1} \mathrm{d}u \\ &= F_{X_1}((\beta_1/\beta_2)t) \\ &> F_{X_1}(t) \end{aligned}$$

where we did a u-substitution with $u = (\beta_1/\beta_2)x$, and where the last inequality follows from the fact that F_{X_1} is strictly increasing (on $t > 0$) and $\beta_1/\beta_2 > 1$. It follows that the $\mathrm{gamma}(\alpha, \beta)$ family is stochastically increasing in β for fixed α.

Problem 3.42

Problem 3.42.a

Let $f(x)$ be a pdf and let $\mu_1, \mu_2 \in \mathbb{R}$ with $\mu_1 > \mu_2$. Then for all $t \in \mathbb{R}$ we have

$$\begin{aligned} F_{\mu_2}(t) &= \int_{-\infty}^{t} f(x - \mu_2) \mathrm{d}x \\ &= \int_{-\infty}^{t} f(x - \mu_2) \mathrm{d}x \\ &= \int_{-\infty}^{t+\mu_1-\mu_2} f(u - \mu_1) \mathrm{d}u \\ &= F_{\mu_1}(t + \mu_1 - \mu_2) \\ &\geq F_{\mu_1}(t) \end{aligned}$$

where we did a u-substitution with $u = x + \mu_1 - \mu_2$, and where the last inequality follows from the fact that F_{μ_1} is an increasing function and $\mu_1 - \mu_2 > 0$. Note that since $\lim_{t\to\infty} F_{\mu_1}(t) = 1$ and $\lim_{t\to-\infty} F_{\mu_1}(t) = 0$, there must exist some $t_0 \in \mathbb{R}$ such that $F_{\mu_1}(t_0) < F_{\mu_1}(t_0 + \mu_1 - \mu_2)$. This shows that a location family is stochastically increasing in its location parameter.

Problem 3.42.b

Let $f(x)$ be a pdf and let $\sigma_1, \sigma_2 \in \mathbb{R}_{>0}$ with $\sigma_1 > \sigma_2$. Then for all $t \in \mathbb{R}_{>0}$ we have

$$\begin{aligned}
F_{\sigma_2}(t) &= \int_{-\infty}^{t} \frac{1}{\sigma_2} f\left(\frac{x}{\sigma_2}\right) \mathrm{d}x \\
&= \int_{-\infty}^{t(\sigma_1/\sigma_2)} \frac{1}{\sigma_1} f\left(\frac{u}{\sigma_1}\right) \mathrm{d}u \\
&= F_{\sigma_1}(t(\sigma_1/\sigma_2)) \\
&\geq F_{\sigma_1}(t)
\end{aligned}$$

where we did a u-substitution with $u = (\sigma_1/\sigma_2)t$, and where the last inequality follows from the fact that F_{σ_1} is an increasing function and $\sigma_1/\sigma_2 > 1$. Note that since $\lim_{t\to\infty} F_{\sigma_1}(t) = 1$ and $\lim_{t\to 0} F_{\sigma_2}(t) = 0$, there must exist some $t_0 \in \mathbb{R}$ such that $F_{\sigma_1}(t_0) < F_{\sigma_1}((\sigma_1/\sigma_2)t_0)$. This shows that a location family is stochastically increasing in its scale parameter.

Problem 3.47

Let $t > 0$. Then we have

$$\begin{aligned}
\mathrm{P}(|Z| \geq t) &= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z + \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\
&= 2\int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \mathrm{d}z \\
&= \sqrt{\frac{2}{\pi}} \int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z
\end{aligned}$$

To finish the proof, set

$$p(t) = \int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z - \frac{t}{t^2+1} e^{-t^2/2}.$$

We shall show $p(t) > 0$ for all $t \geq 0$. First note that $p(0) > 0$. Next, note that

$$\begin{aligned}
p'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z - \frac{\mathrm{d}}{\mathrm{d}t} \frac{t}{t^2+1} e^{-t^2/2} \\
&= -e^{-t^2/2} + \frac{e^{-t^2/2}(t^4 + 2t^2 - 1)}{(t^2+1)^2} \\
&= \frac{-e^{-t^2/2}(t^4 + 2t^2 + 1)}{(t^2+1)^2} + \frac{e^{-t^2/2}(t^4 + 2t^2 - 1)}{(t^2+1)^2} \\
&= \frac{-2e^{-t^2/2}}{(t^2+1)^2}.
\end{aligned}$$

Thus p is strictly decreasing. Finally, since

$$\begin{aligned}
\lim_{t\to\infty} p(t) &= \lim_{t\to\infty} \left(\int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z - \frac{t}{t^2+1} e^{-t^2/2} \right) \\
&= \lim_{t\to\infty} \left(\int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z \right) - \lim_{t\to\infty} \left(\frac{t}{t^2+1} e^{-t^2/2} \right) \\
&= 0 - 0 \\
&= 0,
\end{aligned}$$

we see that $p(t)$ must always be positive. Thus we continue with our proof:

$$\begin{aligned}
\mathrm{P}(|Z| \geq t) &= \sqrt{\frac{2}{\pi}} \int_{t}^{\infty} e^{-z^2/2} \mathrm{d}z \\
&> \sqrt{\frac{2}{\pi}} \frac{t}{t^2+1} e^{-t^2/2}.
\end{aligned}$$