Section 1.11: Cubic Functions and Models

A **cubic** function has an equation of the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, *b*, *c*, and *d* are constants.

The graph of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ has an **inflection point** because its graph shows a change in concavity. An inflection point is a point at which a function is increasing or decreasing the most or least rapidly on an interval around that inflection point.

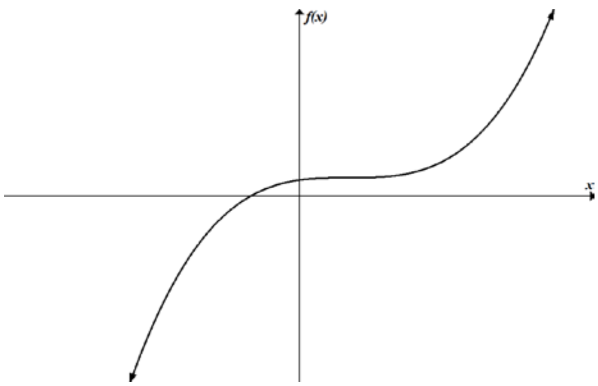
The sign of *a* determines the behavior of a cubic function*:*

- For $a < 0$, f is concave up, followed by concave down, with end behavior $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$
- For $a > 0$, f is concave down, followed by concave up, with end behavior $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$

Some cubic functions appear to be strictly increasing or strictly decreasing. Other cubic functions change direction and have both increasing and decreasing intervals, showing both a relative maximum value and a relative minimum value.

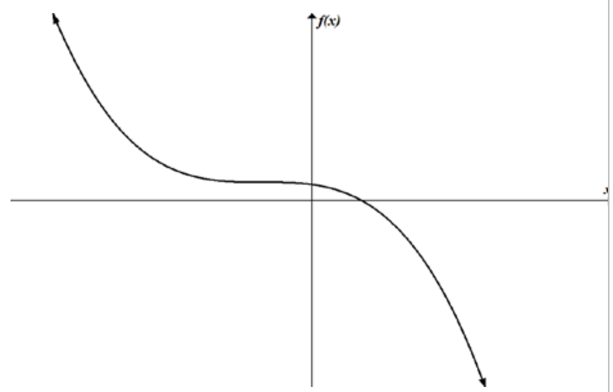
Example 1:

a. Label each of the following graphs of $f(x) = ax^3 + bx^2 + cx + d$ as either $a < 0$ or $a > 0$.
b. Identify the *inflection point* on each graph by marking an “X” on the graph. Note that the concavity changes at this point.
c. Complete the limit statements that describe the end behavior.
d. If a *relative maximum* and/or *relative minimum* occur, identify and label these points on the graph.



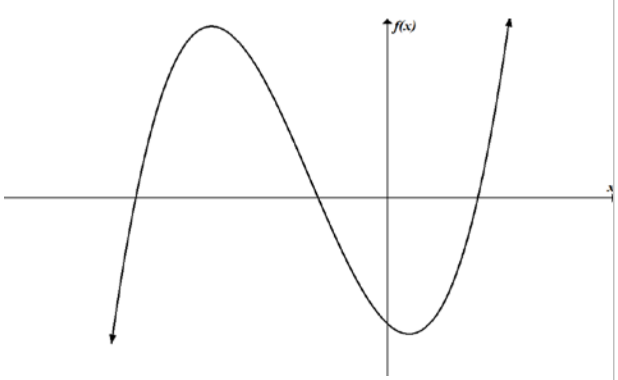
Is $a < 0$ or $a > 0$?

$\lim_{x \to -\infty} f(x) =$ ________; $\lim_{x \to \infty} f(x) =$ ________



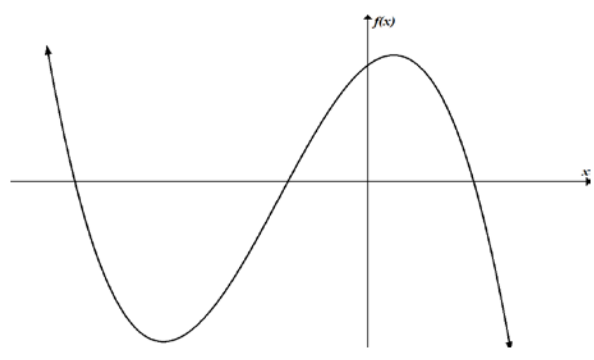
Is $a < 0$ or $a > 0$?

$\lim_{x \to -\infty} f(x) =$ ________; $\lim_{x \to \infty} f(x) =$ ________



Is $a < 0$ or $a > 0$?

$\lim_{x \to -\infty} f(x) =$ ________; $\lim_{x \to \infty} f(x) =$ ________



Is $a < 0$ or $a > 0$?

$\lim_{x \to -\infty} f(x) =$ ________; $\lim_{x \to \infty} f(x) =$ ________

Example 2: (CC5e p. 108, Activity 15)

The table shows the yearly monetary value of loss resulting from identity fraud between 2004 and 2008.

Year	Loss, in billion dollars
2004	60
2005	57
2006	51
2007	45
2008	48

a. Align the data to the number of years since 2004. Write a cubic model for the amount of loss due to identity fraud.

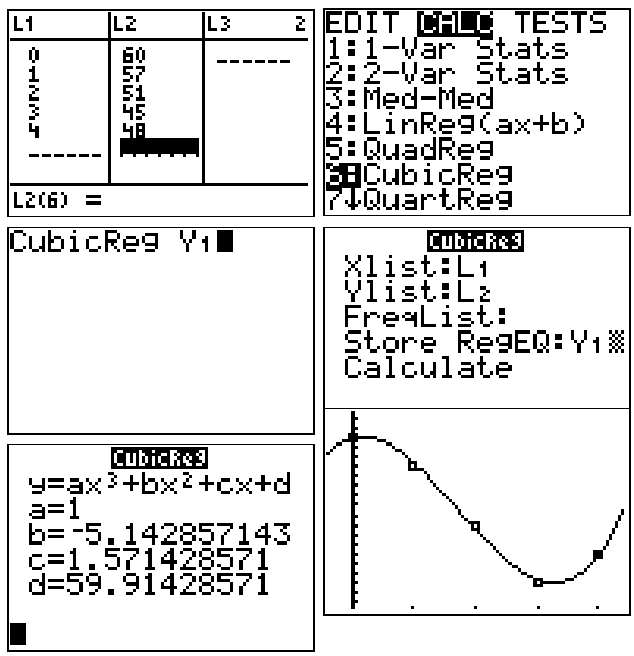
Finding, storing, and viewing a cubic function:

- With the aligned data in L1 and L2, **STAT** ▸ [CALC] ▾ to 6 [CubicReg] **ENTER** returns CubicReg on the Home Screen. **VARS** ▸ [Y-VARS] **1** [Function] **1** [Y1] returns Y1 **ENTER**

  OR

  STAT ▸ [CALC] ▾ to 6 [CubicReg] **ENTER** returns the CubicReg Screen
  Xlist: **2nd** **1** [L1]
  Ylist: **2nd** **2** [L2]
  Store RegEQ: **VARS** ▸ [Y-VARS] **1** [Function] **1** [Y1]
  Move cursor to Calculate and hit **ENTER**

- Hit **ZOOM** **9** [ZoomStat] to view the function and the scatter plot



b. Use the model to estimate the amount of loss in 2009. Comment on the usefulness of this estimate.

c. Does a model output value corresponding to an input of 2.5 make sense in context? Explain.

d. Estimate the input and output values of the inflection point.

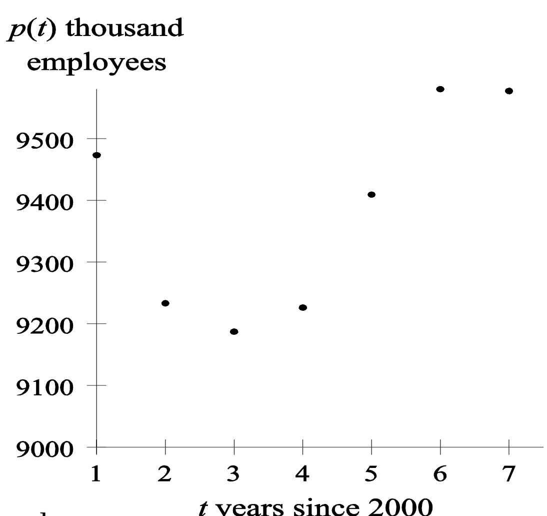
e. Interpret the inflection point value and location in a sentence.

Example 3: (CC5e p. 105)

The number of 20 to 24-year-olds who were employed full time during a given year is shown in the table below.

Year	2001	2002	2003	2004	2005	2006	2007
Employees, in thousands	9473	9233	9187	9226	9409	9580	9577

a. Align the data to the number of years since 2000 and view a scatter plot of the data. Why is a cubic model more appropriate for the data than a logistic model?



b. Write a completely defined cubic model for the data.

c. What is the concern in using the model to extrapolate beyond 2007?

d. Use the model in part b to estimate the employment of 20 to 24-year-olds in 2008.

e. According to the model, find the approximate time periods in which the number of 20 to 24-year-olds employees exceeded 9400 thousand.

Strategies for Choosing the Best-Fit Function to Model a Data Set

1. Look at the curvature of the scatter plot.

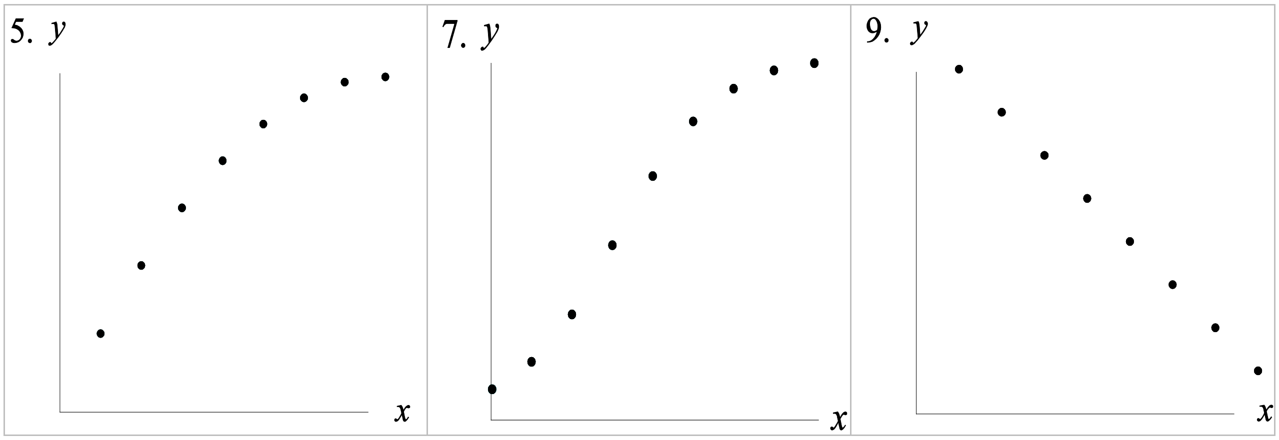
   - **No curvature** suggests a *linear* function.
   - **One concavity, with no inflection point**, suggests a *quadratic, exponential,* or *logarithmic* function.
   - **Two concavities, with an inflection point** suggests a *cubic* or *logistic* function.

2. Find the best fit among the functions (two functions, at most) with the same amount of concavity.

3. Consider the **end behavior** and the given **context** in choosing the best function to model the data, especially if two functions fit the data equally well.

Example 4: (CC5e pp. 99, Activities 5, 7, 9)

a. For each of the following graphs, determine the amount of curvature displayed (no curvature, one concavity, two concavities).
b. For each of the following graphs, what type(s) of function(s) might be appropriate to model the data represented by each scatter-plot: linear, exponential, logarithmic, quadratic, logistic, or cubic?



Example 5:

In 2008, a community began a local campaign to increase voter turnout in local elections. The percentages of all eligible voters voting in the yearly local elections are shown in the table below.

Year	2008	2009	2010	2011	2012	2013
Voter turnout, in percent	2.0	4.1	5.32	6.2	6.8	7.2

a. Examine the scatterplot of the data. What *two* functions would you consider for modeling this data?

b. If you were planning to use one of these models to estimate voter turnout in subsequent years, which of the two models do you think you would use and why?

c. Align the data to years after 2007. Write a completely defined model based on your answer to part b.

Example 6: (CC5e p. 99, based on Activity 16)

The age-specific likelihood for a woman to develop breast cancer in the next ten years is given in the table below.

Age	20	30	40	50	60	70
Likelihood of breast cancer, in percent	0.05	0.43	1.43	2.51	3.51	3.88

a. Examine the scatterplot of the data. What *two* functions would you consider for modeling this data?

b. If you were planning to use one of these models to predict the likelihood of breast cancer occurring in the next ten years for an 80-year old woman, which of the two functions do you think you would use to model the data and why?

c. Write a completely defined model using the function based on your answer to part b:

Example 7:

The table below is organized according to the amount of concavity displayed in a scatter plot. Use it to summarize the six functions studied in this chapter.

For each of the functions, sketch two possible graphs. Describe the end behavior using limit notation. List the special features (horizontal or vertical asymptote(s), maximum, minimum, inflection point, etc.) of each function. Write the equation(s) of any asymptotes.

SIX FUNCTION SUMMARY			
Function	**Graphs**	**End Behavior**	**Special Features**
No Curvature			
Linear $f(x) = ax + b$		When $a > 0$, $\lim_{x\to\infty} f(x) =$ $\lim_{x\to-\infty} f(x) =$	
One Concavity (Up or Down)			
Exponential $f(x) = a\left(b^x\right)$		When $a > 0$, $b > 1$, $\lim_{x\to\infty} f(x) =$ $\lim_{x\to-\infty} f(x) =$	
Logarithmic $f(x) = a + b\ln x$		When $b > 0$, $\lim_{x\to0^+} f(x) =$ $\lim_{x\to\infty} f(x) =$	
Quadratic $f(x) = ax^2 + bx + c$		When $a > 0$, $\lim_{x\to\infty} f(x) =$ $\lim_{x\to-\infty} f(x) =$	
Two Concavities			
Logistic $f(x) = \dfrac{L}{1 + ae^{-bx}}$		When $b < 0$, $\lim_{x\to\infty} f(x) =$ $\lim_{x\to-\infty} f(x) =$	
Cubic $f(x) = ax^3 + bx^2 + cx + d$		When $a > 0$, $\lim_{x\to\infty} f(x) =$ $\lim_{x\to-\infty} f(x) =$	