MATH 8000

Homework Assignment 2

Problem 1. In class we discussed the *probability integral transformation* for a continuous random variable X. Now we generalize this transformation for all random variables. Specifically, you will need to prove that the random variable $U = F(X, V) \sim \text{Uniform}(0, 1)$, where $F(x, \lambda) = P_X(X < x) + \lambda P_X(X = x)$ and the random variables X and V are independent with $V \sim \text{Uniform}(0, 1)$. This result, so I have been told, is integral in proving the existence of copulas, a highly controversial statistical technique that is applied in financial modeling.

Problem 2. Consider the following function

$$g(x, z) = \begin{cases} 0, & \text{if } \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} = 0, \\ z, & \text{if } \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} = 1, \\ 1, & \text{if } \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)} = \infty, \end{cases}$$

where $\theta_0 < \theta_1 < \theta_0 + 1$ and $f_X(x|\theta)$ denotes the PDF of a uniform random variable over the interval $(\theta, \theta + 1)$; i.e., $f(x|\theta) = 1(\theta \leq x \leq \theta + 1)$.

a.) Consider the random variable $Y_0 = g(X_0, Z)$ where $X_0 \sim \text{Uniform}(\theta_0, \theta_0 + 1)$, $Z \sim \text{Bernoulli}(p)$, and X_0 and Z are independent. Find the value of p such that $P_{Y_0}(Y_0 = 1) = \alpha$, for $\alpha \in (0, 1)$.

b.) Define the random variable $Y_1 = g(X_1, Z)$ where $X_1 \sim \text{Uniform}(\theta_1, \theta_1 + 1)$, $Z \sim \text{Bernoulli}(p)$, and X_1 and Z are independent. Using the value of p you found in part a.) find $P_{Y_1}(Y_1 = 1)$.

Problem 3. The PDF of the logistic distribution is given by

$$f_X(x|\theta) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2} I(x \in \mathbb{R}),$$

where $\theta \in \mathbb{R}$. Consider the following function

$$g(x) = \frac{f_X(x|\theta_1)}{f_X(x|\theta_0)},$$

where $\theta_0 < \theta_1$ and f_X is the PDF of the logistic distribution.

a.) Consider the random variable $Y_0 = I\{g(X_0) > k\}$, where $X_0 \sim \text{Logistic}(\theta_0)$. Find the value of k such that $E(Y_0) = \alpha$, for $\alpha \in (0, 1)$.

b.) Define the random variable $Y_1 = I\{g(X_1) > k\}$, where $X_1 \sim \text{Logistic}(\theta_1)$ and k is the value you found in part a.). Find $E(Y_1)$.

Additional problems from text:
From Chapter 2: 2.6, 2.7, 2.14, 2.32, and 2.38.