Linear Analysis Homework 8

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Throughout this homework, let $\mathcal{H}$ be a separable Hilbert space. If $x \in \mathcal{H}$ and $r > 0$, then we write

$$B_r(x) := \{y \in \mathcal{H} \mid \|y - x\| < r\}$$

for the open ball centered at x and of radius r. We also write

$$B_r[x] := \{y \in \mathcal{H} \mid \|y - x\| \leq r\}$$

for the closed ball centered at x and of radius r.

Problem 1

Proposition 0.1. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then T is compact if and only if $\overline{T(B_1[0])}$ is a compact space.*

Proof. Suppose T is compact. To show that $\overline{T(B_1[0])}$ is compact, it suffices to show that $T(B_1[0])$ is precompact, by Proposition (0.9) (stated and proved in the Appendix). Let (Tx_n) be a sequence in $T(B_1[0])$. Then (x_n) is a bounded sequence in $B_1[0]$. Since T is compact, it follows that (Tx_n) has a convergent subsequence (by homework 7 probem 5). It follows that $T(B_1[0])$ is precompact.

Conversely, suppose $\overline{T(B_1[0])}$ is compact. Then $T(B_1[0])$ is precompact by Proposition (0.9). Let (x_n) be a bounded sequence in $\mathcal{H}$. Choose $M > 0$ such that $\|x_n\| < M$ for all $n \in \mathbb{N}$. Then $(T(x_n/M))$ is a sequence in the precompact space $T(B_1[0])$, and hence must have a convergent subsequence, say $(T(x_{\pi(n)}/M))$. This impies $(T(x_{\pi(n)}))$ is a convergent subsequence $(T(x_n))$. Thus, T is compact (again by homework 7 probem 5). □

Problem 2

Proposition 0.2. *Let $(T_n\colon \mathcal{H} \to \mathcal{H})$ be a sequence of compact operators that converges in the operator norm to an operator $T\colon \mathcal{H} \to \mathcal{H}$. Then T is compact.*

Proof. Let (x_k) be a weakly convegent sequence. We claim that (Tx_k) is Cauchy. Indeed, let $\varepsilon > 0$. Since (x_k) is weakly convergent, it must be bounded. Choose $M > 0$ such that $\|x_k\| \leq M$ for all $k \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $\|T - T_N\| < \varepsilon/3M$. Since the sequence $(T_N x_k)_{k\in\mathbb{N}}$ is Cauchy, there exists $K \in \mathbb{N}$ such that $j, k \geq K$ implies $\|T_N x_k - T_N x_j\| < \varepsilon/3$. Choose such a $K \in \mathbb{N}$. Then $j, k \geq K$ implies

$$\begin{aligned}
\|Tx_k - Tx_j\| &= \|Tx_k - T_N x_k + T_N x_k - T_N x_j + T_N x_j - Tx_j\| \\
&\leq \|Tx_k - T_N x_k\| + \|T_N x_k - T_N x_j\| + \|T_N x_j - Tx_j\| \\
&\leq \|T - T_N\|\|x_k\| + \|T_N x_k - T_N x_j\| + \|T_N - T\|\|x_j\| \\
&< \frac{\varepsilon}{3M} \cdot M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} \cdot M \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Thus (Tx_k) is a Cauchy sequence. It follows that T is compact. □

Problem 3

Proposition 0.3. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator and let (e_n) and (f_m) be any two orthonormal bases for $\mathcal{H}$. Then*

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{m=1}^{\infty} \|T^* f_m\|^2.$$

Proof. Since $\mathcal{H}$ is a separable Hilbert space, we have

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \quad \text{and} \quad \|x\|^2 = \sum_{m=1}^{\infty} |\langle x, f_m \rangle|^2$$

for every $x \in \mathcal{H}$. Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} \|Te_n\|^2 &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle Te_n, f_m \rangle|^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle Te_n, f_m \rangle|^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle e_n, T^* f_m \rangle|^2 \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle T^* f_m, e_n \rangle|^2 \\
&= \sum_{m=1}^{\infty} \|T^* f_m\|^2,
\end{aligned}$$

where we are justified in changing the order of the infinite sums by Lemma (0.1) (stated and proved in the Appendix). By swapping the roles of T with T^* in the proof above, we see that the quantity $\sum_{n=1}^{\infty} \|Te_n\|^2$ doesn't depend on the choice of the orthonormal basis (e_n). □

Problem 4

Definition 0.1. An operator $T \colon \mathcal{H} \to \mathcal{H}$ is said to be a **Hilbert-Schmidt** operator if if

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$$

for some or equivalently any orthonormal basis (e_n) of $\mathcal{H}$. In this case, the Hilbert-Schmidt norm of T is defined by

$$\|T\|_{\mathrm{HS}} := \sqrt{\sum_{n=1}^{\infty} \|Te_n\|^2}.$$

Problem 4.a

Proposition 0.4. *Let (e_n) be an orthonormal basis of $\mathcal{H}$. For each $k \in \mathbb{N}$ define a projection operator $P_k \colon \mathcal{H} \to \mathcal{H}$ onto span$\{e_1, e_2, \dots, e_k\}$ by*

$$P_k(x) = \sum_{n=1}^{k} \langle x, e_n \rangle e_n$$

for all $x \in \mathcal{H}$. If $T \colon \mathcal{H} \to \mathcal{H}$ is a Hilbert-Schmidt operator, then $\|T - P_k T\| \to 0$ as $k \to \infty$.

Proof. Let $\varepsilon > 0$ and let $x \in B_1[0]$. Since the sum $\sum_{n=1}^{\infty} \|T^* e_n\|^2$ converges, there exists $K \in \mathbb{N}$ such that

$$\sum_{n=K}^{\infty} \|T^* e_n\|^2 < \varepsilon.$$

Choose such $K \in \mathbb{N}$. Then $k \geq K$ implies

$$\begin{aligned}
\|Tx - \mathrm{P}_k Tx\|^2 &= \left\| \sum_{n=1}^{\infty} \langle Tx, e_n \rangle e_n - \sum_{n=1}^{k} \langle Tx, e_n \rangle e_n \right\|^2 \\
&= \left\| \sum_{n=k+1}^{\infty} \langle Tx, e_n \rangle e_n \right\|^2 \\
&= \left\| \sum_{n=k+1}^{\infty} \langle x, T^* e_n \rangle e_n \right\|^2 \\
&= \sum_{n=k+1}^{\infty} |\langle x, T^* e_n \rangle|^2 \\
&\leq \sum_{n=k+1}^{\infty} \|T^* e_n\|^2 \\
&\leq \sum_{n=K}^{\infty} \|T^* e_n\|^2 \\
&< \varepsilon.
\end{aligned}$$

This implies $\|T - \mathrm{P}_k T\| \to 0$ as $k \to \infty$ by Remark () (stated in the Appendix). □

Problem 4.b

Proposition 0.5. *Every Hilbert-Schmidt operator is compact.*

Proof. Let $T \colon \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator. To show that T is compact, it suffices to show that $\mathrm{P}_k T$ is compact for all $k \in \mathbb{N}$ since Proposition (0.4) implies $\|\mathrm{P}_k T - T\| \to 0$ as $k \to \infty$ and Proposition (0.2) would then imply T is compact.

Let $k \in \mathbb{N}$ and let (x_n) be a weakly convergent sequence in $\mathcal{H}$, say $x_n \xrightarrow{w} x$. We claim that $\mathrm{P}_k x_n \to \mathrm{P}_k x$ as $n \to \infty$. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\langle x_n, e_m \rangle - \langle x, e_m \rangle| < \frac{\varepsilon}{k}$$

for all $m = 1, \ldots, k$. Then $n \geq N$ implies

$$\begin{aligned}
\|\mathrm{P}_k x_n - \mathrm{P}_k x\| &= \left\| \sum_{m=1}^{k} \langle x_n, e_m \rangle e_m - \sum_{m=1}^{k} \langle x, e_m \rangle e_m \right\| \\
&= \left\| \sum_{m=1}^{k} (\langle x_n, e_m \rangle - \langle x, e_m \rangle) e_m \right\| \\
&\leq \sum_{m=1}^{k} |\langle x_n, e_m \rangle - \langle x, e_m \rangle| \\
&< \sum_{m=1}^{k} \frac{\varepsilon}{k} \\
&= \varepsilon.
\end{aligned}$$

□

Problem 4.c

Proposition 0.6. *Let $T \colon \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator. Then $\|T\| \leq \|T\|_{HS}$.*

Proof. Let $x \in B_1[0]$. Then

$$\begin{aligned}\|Tx\|^2 &= \sum_{n=1}^{\infty} |\langle Tx, e_n\rangle|^2 \\ &= \sum_{n=1}^{\infty} |\langle x, T^*e_n\rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \|T^*e_n\|^2 \\ &= \|T\|_{\text{HS}}^2.\end{aligned}$$

In particular this implies

$$\begin{aligned}\|T\|^2 &= \sup\{\|Tx\|^2 \mid x \in B_1[0]\} \\ &\leq \|T\|_{\text{HS}}^2,\end{aligned}$$

where the first line is justified in the Appendix. Thus $\|T\| = \|T\|_{\text{HS}}$. □

Problem 5

Proposition 0.7. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Suppose $T^m = 0$ for some $m \in \mathbb{N}$. Then we must have $T = 0$.*

Proof. If $T^m = 0$ for some $m \in \mathbb{N}$, then 0 is the only eigenvalue for T. Indeed, suppose λ is an eigenvalue of T. Choose an eigenvector of λ, say x. Then

$$\begin{aligned}0 &= T^m x \\ &= \lambda^m x,\end{aligned}$$

which implies $\lambda^m = 0$, and hence $\lambda = 0$. Now choose an orthonormal basis (e_n) consisting of eigenvectors of T (the existence of such basis is guaranteed by the spectral theorem for compact self-adjoint operators). Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned}Tx &= \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n \\ &= \sum_{n=1}^{\infty} 0 \cdot \langle x, e_n\rangle e_n \\ &= 0.\end{aligned}$$

□

Problem 6

Proposition 0.8. *Let $\mathcal{H}$ be a separable Hilbert space and let $T\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. Then there exists a sequence T_m of operators with finite dimensional range such that $\|T - T_m\| \to 0$ and $m \to \infty$.*

Proof. Choose an orthonormal basis (e_n) consisting of eigenvectors of T and let (λ_n) be the corresponding sequence of eigenvalues. By reindexing if necessary, we may assume that $|\lambda_n| \geq |\lambda_{n+1}|$ for all $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, we define $T_m\colon \mathcal{H} \to \mathcal{H}$ by

$$T_m x = \sum_{n=1}^{m} \lambda_n \langle x, e_n\rangle e_n$$

for all $x \in \mathcal{H}$. Observe that $\text{im}(T_m) = \text{span}(\{e_1, \dots, e_m\})$ is finite dimensional. We claim that $\|T - T_m\| \to 0$ and $m \to \infty$. Indeed, let $\varepsilon > 0$ and let Λ denote the set of all eigenvalues of T. If Λ is finite, then the claim is clear by the spectral theorem for compact self-adjoint operators, so assume Λ is infinite. Then 0 must be an accumulation point of Λ. In particular, $|\lambda_n| \to 0$ as $n \to \infty$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|\lambda_n| < \varepsilon$. Then for all

$x \in B_1[0]$, we have

$$
\begin{aligned}
\|Tx - T_m x\|^2 &= \left\| \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n - \sum_{n=1}^{m} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \\
&= \left\| \sum_{n=m+1}^{\infty} \lambda_n \langle x, e_n \rangle e_n \right\|^2 \\
&= \sum_{n=m+1}^{\infty} |\lambda_n \langle x, e_n \rangle|^2 \\
&\leq |\lambda_N|^2 \sum_{n=m+1}^{\infty} |\langle x, e_n \rangle|^2 \\
&\leq |\lambda_N|^2 \|x\|^2 \\
&< \varepsilon^2.
\end{aligned}
$$

This implies $\|T - T_m\| \to 0$ and $m \to \infty$. $\square$

Appendix

Problem 1

Definition 0.2. A subspace $A \subseteq \mathcal{H}$ is said to be **precompact** if every sequence in A has a convergent subsequence.

Proposition 0.9. *Let A be a subspace of $\mathcal{H}$. Then A is precompact if and only if $\overline{A}$ is compact.*

Proof. Suppose A is precompact. Let (a_n) be a sequence in $\overline{A}$. For each $n \in \mathbb{N}$ choose $b_n \in A$ such that

$$
\|a_n - b_n\| < \frac{1}{n}.
$$

Choose a convergent subsequence of (b_n), say $(b_{\pi(n)})$ (we can do this since A is precompact). We claim that the sequence $(a_{\pi(n)})$ is Cauchy, and hence convergent subsequence of (a_n). Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\pi(n) \geq \pi(m) \geq N$ implies

$$
\|b_{\pi(n)} - b_{\pi(m)}\| < \frac{\varepsilon}{3} \quad \text{and} \quad \frac{1}{\pi(m)} < \frac{\varepsilon}{3}.
$$

Then $\pi(n) \geq \pi(m) \geq N$ implies

$$
\begin{aligned}
\|a_{\pi(n)} - a_{\pi(m)}\| &= \|a_{\pi(n)} - b_{\pi(n)} + b_{\pi(n)} - b_{\pi(m)} + b_{\pi(m)} - a_{\pi(m)}\| \\
&\leq \|a_{\pi(n)} - b_{\pi(n)}\| + \|b_{\pi(n)} - b_{\pi(m)}\| + \|b_{\pi(m)} - a_{\pi(m)}\| \\
&< \frac{1}{\pi(n)} + \frac{\varepsilon}{3} + \frac{1}{\pi(m)} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

Finally, since $(a_{\pi(n)})$ is Cauchy and since $\mathcal{H}$ is a Hilbert space, we must have $a_{\pi(n)} \to a$ for some $a \in \overline{A}$. Therefore $\overline{A}$ is compact.

Conversely, suppose $\overline{A}$ is compact. Let (a_n) be a sequence in A. Then (a_n) is a sequence in $\overline{A}$. Since $\overline{A}$ is compact, the sequence (a_n) has a convergent subsequence. Therefore A is precompact. $\square$

Convergence in Operator Norm

Remark. Let $\mathcal{V}$ be an inner-product space and let $(T_n : \mathcal{V} \to \mathcal{V})$ be a sequence of bounded linear operators. If we want to show $\|T_n - T\| \to 0$ as $n \to \infty$, then it suffices to show that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|T_n x - Tx\| < \varepsilon$$

for all $x \in B_1[0]$. Indeed, assuming this is true, choose $M \in \mathbb{N}$ such that $n \geq M$ implies

$$\|T_n x - Tx\| < \varepsilon/2$$

for all $x \in B_1[0]$. Then $n \geq M$ implies

$$\begin{aligned}\|T_n - T\| &= \sup\{\|T_n x - Tx\| \mid x \in B_1[0]\} \\ &\leq \varepsilon/2 \\ &< \varepsilon.\end{aligned}$$

Problem 3

Lemma 0.1. *Let f be a nonnegative function defined on $\mathbb{N} \times \mathbb{N}$. Then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n).$$

Proof. Let $M \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{m=1}^{M}\sum_{n=1}^{\infty} f(m,n) &= \sum_{m=1}^{M}\lim_{N\to\infty}\sum_{n=1}^{N} f(m,n) \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}\sum_{m=1}^{M} f(m,n) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{M} f(m,n) \\ &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n).\end{aligned}$$

Taking the limit as $M \to \infty$ gives us

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n).$$

A similar argument gives us

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) \geq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n).$$

□

Problem 4.c

Proposition 0.10. *Let $T : \mathcal{U} \to \mathcal{V}$ be a bounded linear operator. Then*

$$\|T\|^2 = \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}$$

Proof. For any $x \in \mathcal{U}$ such that $\|x\| \leq 1$, we have $\|Tx\|^2 \leq \|T\|^2$. Thus

$$\|T\|^2 \geq \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}. \tag{1}$$

To show the reverse inequality, we assume (for a contradiction) that (1) is a strict inequality. Choose $\delta > 0$ such that

$$\|T\|^2 - \delta > \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}.$$

Now let $\varepsilon = \delta/2\|T\|$, and choose $x \in \mathcal{U}$ such that $\|x\| \leq 1$ and such that

$$\|T\| - \varepsilon < \|Tx\|.$$

Then

$$\begin{aligned}\|Tx\|^2 &> (\|T\| - \varepsilon)^2 \\ &= \|T\|^2 - 2\varepsilon\|T\| + \varepsilon^2 \\ &\geq \|T\|^2 - 2\varepsilon\|T\| \\ &= \|T\|^2 - \delta\end{aligned}$$

gives us a contradiction. □