Measure Theory Homework 5

Michael Nelson

Problem 1

Proposition 0.1. *Let $f\colon X \to \mathbb{R}$ be a function. Then f is measurable if and only if for every $q \in \mathbb{Q}$ the set $f^{-1}(-\infty, q)$ is measurable.*

Proof. If f is measurable, then certainly $f^{-1}(-\infty, c) \in \mathcal{M}$ for any $c \in \mathbb{R}$ (and hence for any $c \in \mathbb{Q}$). Conversely, suppose $f^{-1}(-\infty, q) \in \mathcal{M}$ for any $q \in \mathbb{Q}$. Let $c \in \mathbb{R}$. For each $n \in \mathbb{N}$ choose $q_n \in \mathbb{Q}$ such that

$$c < q_n < c + \frac{1}{n}.$$

Such a choice for each n can be made since $\mathbb{Q}$ is dense in $\mathbb{R}$. We claim that

$$f^{-1}(-\infty, c] = \bigcap_{n=1}^{\infty} f^{-1}(-\infty, q_n)$$

To see this, first note that the inclusion

$$f^{-1}(-\infty, c] \subseteq \bigcap_{n=1}^{\infty} f^{-1}(-\infty, q_n)$$

is clear since each $f^{-1}(-\infty, q_n)$ contains $f^{-1}(-\infty, c)$ (as $c < q_n$). For the reverse inclusion, suppose $x \in \bigcap_{n=1}^{\infty} f^{-1}(-\infty, q_n)$, so $f(x) < q_n$ for all n. Since $q_n \to c$, this implies $f(x) \leq c$. Thus $x \in f^{-1}(-\infty, c]$. It follows that f is measurable. $\square$

Remark 1. Note that we needed to use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ in order to prove this.

Problem 2

Before we answer this problem, we give a more general definition of what it means for a function to be measurable with respect to σ-algebras $\mathcal{M}$ and $\mathcal{N}$. Then we show that this more general definition is equivalent to the definition we've been using when $\mathcal{N}$ is the Borel σ-algebra on $\mathbb{R}$.

Definition 0.1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and let $f\colon X \to Y$ be a function. We say f is **measurable with respect to** $\mathcal{M}$ **and** $\mathcal{N}$ if $f^{-1}(\mathcal{N}) \subseteq \mathcal{M}$ where

$$f^{-1}(\mathcal{N}) = \{f^{-1}(B) \mid B \in \mathcal{N}\}.$$

In other words, f is measurable with respect to $\mathcal{M}$ and $\mathcal{N}$ if for all $B \in \mathcal{N}$ we have

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \in \mathcal{M}.$$

If $\mathcal{M} = \mathcal{N}$, then we will just say f is measurable with respect to $\mathcal{M}$. If the σ-algebras $\mathcal{M}$ and $\mathcal{N}$ are clear from context, then we will just say f is measurable.

Let us now show that when $Y = \mathbb{R}$ and $\mathcal{N} = \mathcal{B}(\mathbb{R})$, that this definition is equivalent to the definition we gave in class. We first prove the following two propositions:

Proposition 0.2. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and let $f\colon X \to Y$ be a function. Suppose that $\mathcal{N}$ is generated as a σ-algebra by the collection $\mathcal{C}$ of subsets of Y. Then $f^{-1}(\mathcal{N}) \subseteq \mathcal{M}$ if and only if $f^{-1}(\mathcal{C}) \subseteq \mathcal{M}$.*

Proof. One direction is clear, so we just prove the other direction. Suppose $f^{-1}(\mathcal{C}) \subseteq \mathcal{M}$. Observe that

$$\{B \in \mathcal{P}(Y) \mid f^{-1}(B) \in \mathcal{M}\}$$

is a σ-algebra which contains $\mathcal{C}$. Indeed, it is a σ-algebra since f^{-1} maps the emptyset set to the emptyset and maps the whole space Y to the whole space X, and since f^{-1} commutes with unions and complements. Furthemore, this σ-algebra contains $\mathcal{C}$ since $f^{-1}(\mathcal{C}) \subseteq \mathcal{M}$. Since $\mathcal{N}$ is the *smallest* σ-algebra which contains $\mathcal{C}$, it follows that

$$\mathcal{N} \subseteq \{B \in \mathcal{P}(Y) \mid f^{-1}(B) \in \mathcal{M}\}.$$

In particular, if $B \in \mathcal{N}$, then $f^{-1}(B) \in \mathcal{M}$. Thus f is measurable. $\square$

Proposition 0.3. *Let $\mathcal{C} = \{(-\infty, c) \mid c \in \mathbb{R}\}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$.*

Proof. Let $\mathcal{I}_n$ be the collection of all subintervals of $[n, n+1)$ and let $\mathcal{B}_n = \sigma(\mathcal{I}_n)$. So

$$\mathcal{B}(\mathbb{R}) = \{E \subseteq \mathbb{R} \mid E \cap [n, n+1) \in \mathcal{B}_n \text{ for all } n \in \mathbb{Z}\}.$$

Let $c \in \mathbb{R}$. Then since $(-\infty, c) \cap [n, n+1]$ is a subinterval of $[n, n+1]$ for all $n \in \mathbb{Z}$, it follows that $(-\infty, c) \in \mathcal{B}$ for all $n \in \mathbb{Z}$. Thus $\mathcal{C} \subseteq \mathcal{B}$ which implies $\sigma(\mathcal{C}) \subseteq \mathcal{B}$ (as $\sigma(\mathcal{C})$ is the *smallest* σ-algebra which contains $\mathcal{C}$). Conversely, note that $\sigma(\mathcal{C})$ contains all subintervals of $[n, n+1)$ for all $n \in \mathbb{Z}$. Thus $\sigma(\mathcal{C}) \supseteq \mathcal{B}_n$ for all $n \in \mathbb{Z}$ (as $\mathcal{B}_n$ is the *smallest* σ-algebra which contains all subintervals of $[n, n+1)$. Since $\mathcal{B}(\mathbb{R}) = \sigma\left(\bigcup_{n\in\mathbb{Z}} \mathcal{B}_n\right)$, it follows that $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{C})$. $\square$

Corollary 1. *Let $(X, \mathcal{M})$ be a measurable space and let $f \colon X \to \mathbb{R}$ be a function. Then f is measurable with respect to $\mathcal{M}$ and $\mathcal{B}(\mathbb{R})$ if and only if $f^{-1}(-\infty, c) \in \mathcal{M}$ for all $c \in \mathbb{R}$.*

Proof. Follows from Proposition (0.3) and Proposition (0.2). $\square$

Corollary 2. *Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra on $\mathbb{R}$. Suppose that $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$ is a collection of sets in $\mathcal{B}(\mathbb{R})$ such that $\mathcal{B}(\mathbb{R}) \subseteq \sigma(\mathcal{C})$. Then $f \colon X \to \mathbb{R}$ is measurable with respect to $\mathcal{M}$ and $\mathcal{B}(\mathbb{R})$ if and only if $f^{-1}(C) \in \mathcal{M}$ for all $C \in \mathcal{C}$.*

Proof. Follows from Proposition (0.2) and from Corollary (1). $\square$

Problem 3

Problem 3.i

Proposition 0.4. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is measurable with respect to $\mathcal{B}(\mathbb{R})$.*

Proof. For each $q, r \in \mathbb{Q}$ and $n \in \mathbb{N}$, let

$$\mathrm{B}_{1/n}(q) = \{x \in \mathbb{R} \mid |x - q| < 1/n\}$$

Then the collection

$$\mathscr{B} = \{\mathrm{B}_{1/n}(q) \mid n \in \mathbb{N} \text{ and } q \in \mathbb{Q}\}$$

forms a countable basis for the usual topology on $\mathbb{R}$. In particular, if U be an open subset of $\mathbb{R}$, then we can express U as a union of the form

$$U = \bigcup_{\lambda \in \Lambda} B_\lambda$$

where $B_\lambda \in \mathscr{B}$ and where the index set Λ is *countable*. In particular, it follows that $\tau(\mathscr{B}) \subseteq \mathcal{B}(\mathbb{R})$, where $\tau(\mathscr{B})$ is the usual Euclidean topology on $\mathbb{R}$. Thus since $(-\infty, c)$ is an open subset of $\mathbb{R}$ for any $c \in \mathbb{R}$, it follows that $f^{-1}(-\infty, c)$ can be expressed a countable union of open subsets of $\mathbb{R}$ (by definition of what it means to be continuous, the inverse image of an open set under f is open). Since every open subset of $\mathbb{R}$ is Borel measurable, it follows that $f^{-1}(-\infty, c) \in \mathcal{B}(\mathbb{R})$. Thus f is measurable with respect to $\mathcal{B}(\mathbb{R})$. $\square$

Problem 3.ii

Proposition 0.5. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function. Then f is $\mathcal{B}(\mathbb{R})$-measurable.*

Proof. Let $c \in \mathbb{R}$. We want to show that $f^{-1}(-\infty, c) \in \mathcal{B}(\mathbb{R})$. If $f^{-1}(-\infty, c) = \emptyset$ or $f^{-1}(-\infty, c) = \mathbb{R}$, then we are done, so assume $f^{-1}(-\infty, c) \neq \emptyset$ and $f^{-1}(-\infty, c) \neq \mathbb{R}$. Choose $y \in \mathbb{R}$ such that $c \leq f(y)$. Observe that if $x \in f^{-1}(-\infty, c)$, then

$$f(x) < c \leq f(y),$$

which implies $x \leq y$ since f is monotone increasing. Thus y is an upper bound of the set $f^{-1}(-\infty, c)$. Since $f^{-1}(-\infty, c)$ is nonempty and bounded above, it follows that its supremum exists. Denote its supremum by y_0. So

$$y_0 = \sup\{x \in \mathbb{R} \mid f(x) < c\}.$$

We claim that

$$f^{-1}(-\infty,c)=\begin{cases}(-\infty,y_0) & \text{if } f(y_0)\geq c\\ (-\infty,y_0] & \text{if } f(y_0)<c.\end{cases}$$

Indeed, since y_0 is an upper bound $f^{-1}(-\infty,c)$, it must be greater than or equal to all elements in $f^{-1}(-\infty,c)$. In other words, if $x\in f^{-1}(-\infty,c)$, then $x\leq y_0$. Thus

$$f^{-1}(-\infty,c)\subseteq\begin{cases}(-\infty,y_0) & \text{if } f(y_0)\geq c\\ (-\infty,y_0] & \text{if } f(y_0)<c.\end{cases}$$

Conversely, suppose $x\in(-\infty,y_0)$, so $x<y_0$. Then x is not an upper bound of the set $f^{-1}(-\infty,c)$ (since y_0 is the *least* upper bound), which means that there exists an $x'\in\mathbb{R}$ such that $x\leq x'$ and $f(x')<c$. But since $f(x)\leq f(x')$, this implies $f(x)<c$, and hence $x\in f^{-1}(-\infty,c)$. Thus

$$f^{-1}(-\infty,c)\supseteq\begin{cases}(-\infty,y_0) & \text{if } f(y_0)\geq c\\ (-\infty,y_0] & \text{if } f(y_0)<c.\end{cases}$$

In any case, we see that $f^{-1}(-\infty,c)\in\mathcal{B}(\mathbb{R})$. □

Problem 4

Proposition 0.6. *Let $(X,\mathcal{M},\mu)$ be a measure space. Suppose $(f_n\colon X\to[0,\infty])$ is a sequence of non-negative measurable functions. Define $f\colon X\to[0,\infty]$ by*

$$f(x)=\sum_{n=1}^{\infty}f_n(x)$$

for all $x\in X$. Then f is a non-negative measurable function and

$$\int_X f\mathrm{d}\mu=\sum_{n=1}^{\infty}\int_X f_n\mathrm{d}\mu.$$

Proof. For each $N\in\mathbb{N}$, let $s_N=\sum_{n=1}^{N}f_n$. Then s_N converges pointwise to f since

$$\begin{aligned}f(x)&=\sum_{n=1}^{\infty}f_n(x)\\&=\lim_{N\to\infty}\sum_{n=1}^{N}f_n(x)\\&=\lim_{N\to\infty}s_N(x)\end{aligned}$$

for all $x\in X$. Each s_N is a nonnegative measurable function since it is a finite sum of nonnegative measurable functions, and so (s_N) is a sequence of nonnegative functions which converges pointwise to f. This implies f is a nonnegative measurable function. Furthermore, s_N is an increasing sequence since if $M\leq N$, then

$$\begin{aligned}s_M(x)&=\sum_{n=1}^{M}f_n(x)\\&\leq\sum_{n=1}^{N}f_n(x)\\&=s_N(x)\end{aligned}$$

for all $x\in X$, where the inequality follows from the fact that each f_n is nonnegative. Therefore we may apply the Monotone Convergence Theorem to obtain

$$\begin{aligned}\int_X f\mathrm{d}\mu&=\lim_{N\to\infty}\int_X s_N\mathrm{d}\mu\\&=\lim_{N\to\infty}\int_X\sum_{n=1}^{N}f_n\mathrm{d}\mu\\&=\lim_{N\to\infty}\sum_{n=1}^{N}\int_X f_n\mathrm{d}\mu\\&=\sum_{n=1}^{\infty}\int_X f_n\mathrm{d}\mu,\end{aligned}$$

where we obtained the third line from the fourth line from the fact that this is a finite sum. □

Problem 5

Proposition 0.7. *Let $(X, \mathcal{M}, \mu)$ be measure space and let $g\colon X \to [0, \infty)$ be a nonnegative measurable function. Define $\nu_g\colon \mathcal{M} \to [0, \infty]$ by*

$$\nu_g(E) = \int_X g 1_E \mathrm{d}\mu$$

for all $E \in \mathcal{M}$. Then ν is a measure on $(X, \mathcal{M})$.

Proof. First note that

$$\begin{aligned}
\nu_g(\emptyset) &= \int_X g 1_\emptyset \mathrm{d}\mu \\
&= \int_X g \cdot 0 \cdot \mathrm{d}\mu \\
&= \int_X 0 \cdot \mathrm{d}\mu \\
&= 0.
\end{aligned}$$

Next we show that ν_g is finitely additive. Let $(E_n)_{n=1}^N$ be a finite sequence of pairwise disjoint sets in $\mathcal{M}$. Then

$$\begin{aligned}
\nu_g\left(\bigcup_{n=1}^N E_n\right) &= \int_X g 1_{\bigcup_{n=1}^N E_n} \mathrm{d}\mu \\
&= \int_X g \sum_{n=1}^N 1_{E_n} \mathrm{d}\mu \\
&= \int_X \sum_{n=1}^N g 1_{E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^N \int_X g 1_{E_n} \mathrm{d}\mu \\
&= \sum_{n=1}^N \nu_g(E_n),
\end{aligned}$$

where we used the fact that each $g1_{E_n}$ is a nonnegative measurable function in order to commute the finite sum with the integral. Thus it follows that ν_g is finitely additive.

It remains to show that ν_g is countably subadditive. In problem 3 in HW4, we showed that for any nonnegative simple function $\varphi\colon X \to [0, \infty)$, the function $\nu_\varphi\colon \mathcal{M} \to [0, \infty]$ defined by

$$\nu_\varphi(E) = \int_X \varphi 1_E \mathrm{d}\mu$$

is a measure. We will use this fact in what follows:

Choose an increasing sequence $(\varphi_n\colon X \to [0, \infty])$ of nonnegative simple functions which converges pointwise to g (we can do this since g is a nonnegative measurable function). Then $(\varphi_n 1_E)$ is an increasing sequence of nonnegative simple functions which converges pointwise to $g1_E$ for all $E \in \mathcal{M}$. It follows from the Monotone Convergence Theorem that

$$\begin{aligned}
\nu_{\varphi_n}(E) &= \int_X \varphi_n 1_E \mathrm{d}\mu \\
&\to \int_X g 1_E \mathrm{d}\mu \\
&= \nu_g(E)
\end{aligned}$$

for any $E \in \mathcal{M}$. In particular, for any $E \in \mathcal{M}$ and $\varepsilon > 0$, we can find a $N_{E,\varepsilon} \in \mathbb{N}$ (which depends on E and ε) such that

$$\nu_g(E) < \nu_{\varphi_n}(E) + \varepsilon \tag{1}$$

for all $n \geq N_{E,\varepsilon}$. However we will don't need estimate $\nu_g(E)$ to this level of precision. We just need to know thta for any $\varepsilon > 0$, we can find an $n \in \mathbb{N}$ such that (1) holds.

Now let (E_k) be a sequence of measurable sets and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\nu_g\left(\bigcup_{k=1}^\infty E_k\right) < \nu_{\varphi_n}\left(\bigcup_{k=1}^\infty E_k\right) + \varepsilon$$

Then we have

$$\begin{aligned}
\nu_g\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \nu_{\varphi_n}\left(\bigcup_{k=1}^{\infty} E_k\right) + \varepsilon \\
&\leq \sum_{k=1}^{\infty} \nu_{\varphi_n}(E_k) + \varepsilon \\
&\leq \sum_{k=1}^{\infty} \nu_g(E_k) + \varepsilon
\end{aligned}$$

where we obtained the second line from the first line using countable subadditivity of ν_{φ_n}, and where we obtained the third line from the second line from the fact that $\varphi_n 1_{E_k} \leq g 1_{E_k}$ for all k and from monotonicity of integration. Taking $\varepsilon \to 0$ gives us countable subadditivity of ν_g. □

Problem 6

Proposition 0.8. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(f_n \colon X \to [0, \infty])$ is a decreasing sequence of nonnegative measurable functions which converges pointwise to $f \colon X \to [0, \infty]$. If $\int_X f_1 \mathrm{d}\mu < \infty$, then*

$$\lim_{n\to\infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu. \tag{2}$$

Proof. For each $n \in \mathbb{N}$, set $g_n = f_{n+1} - f_n$. Since (f_n) is a decreasing sequence, each g_n is nonnegative. Furthermore each g_n is measurable since it is a difference of two measurable functions. Set $g = \sum_{n=1}^{\infty} g_n$ and observe that

$$\begin{aligned}
g &= \sum_{n=1}^{\infty} g_n \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} g_n \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} (f_{n+1} - f_n) \\
&= \lim_{N\to\infty} (f_N - f_1) \\
&= f - f_1.
\end{aligned}$$

It follows from problem 4 that

$$\begin{aligned}
\int_X f \mathrm{d}\mu - \int_X f_1 \mathrm{d}\mu &= \int_X (f - f_1) \mathrm{d}\mu \\
&= \int_X g \mathrm{d}\mu \\
&= \sum_{n=1}^{\infty} \int_X g_n \mathrm{d}\mu \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X g_n \mathrm{d}\mu \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X (f_{n+1} - f_n) \mathrm{d}\mu \\
&= \lim_{N\to\infty} \int_X \sum_{n=1}^{N} (f_{n+1} - f_n) \mathrm{d}\mu \\
&= \lim_{N\to\infty} \int_X (f_N - f_1) \mathrm{d}\mu \\
&= \lim_{N\to\infty} \int_X f_N \mathrm{d}\mu - \int_X f_1 \mathrm{d}\mu.
\end{aligned}$$

Since $\int_X f_1 \mathrm{d}\mu < 0$, we can cancel it from both sides to get (2). □

Problem 7

Proposition 0.9. *Fatou's Lemma remains valid if the hypothesis that all $f_n\colon X \to [0,\infty]$ are nonnegative measurable functions is replaced by the hypothesis that $f_n\colon X \to \mathbb{R}$ are measurable and there exists a nonnegative integrable function $g\colon X \to [0,\infty]$ such that $-g \le f_n$ pointwise for all $n \in \mathbb{N}$.*

Proof. Observe that $(g+f_n)$ is a sequence of nonnegative measurable functions which converges pointwise to the nonnegative measurable function $g+f$. Then it follows from Fatou's Lemma that

$$\begin{aligned}
\int_X g\mathrm{d}\mu + \int_X f\mathrm{d}\mu &= \int_X g\mathrm{d}\mu + \int_X ((g+f)-g)\mathrm{d}\mu \\
&= \int_X g\mathrm{d}\mu + \int_X (g+f)\mathrm{d}\mu - \int_X g\mathrm{d}\mu \\
&= \int_X (g+f)\mathrm{d}\mu \\
&\le \liminf_{n\to\infty} \int_X (g+f_n)\mathrm{d}\mu \\
&= \liminf_{n\to\infty} (\int_X g\mathrm{d}\mu + \int_X (g+f_n)\mathrm{d}\mu - \int_X g\mathrm{d}\mu) \\
&= \liminf_{n\to\infty} (\int_X g\mathrm{d}\mu + \int_X ((g+f_n)-g)\mathrm{d}\mu) \\
&= \liminf_{n\to\infty} (\int_X g\mathrm{d}\mu + \int_X f_n\mathrm{d}\mu) \\
&= \int_X g\mathrm{d}\mu + \liminf_{n\to\infty} \int_X f_n\mathrm{d}\mu.
\end{aligned}$$

Since $\int_X g\mathrm{d}\mu < \infty$, we can cancel it from both sides to obtain

$$\int_X f\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n\mathrm{d}\mu.$$

□

Problem 8

Exercise 1. Compute the following integrals

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{(1+x/n)^n}\mathrm{d}x \tag{3}$$

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\mathrm{d}x \tag{4}$$

Solution 1. We first compute (3). For each $n \in \mathbb{N}$, let $f_n = \sin(x/n)(1+x/n)^{-n}$. Observe that each f_n is measurable since each f_n is continuous (the denominator is only zero when $x=-n$). Let us check that each f_n is integrable: we have

$$\begin{aligned}
\int_0^\infty |f_n|\mathrm{d}x &= \int_0^\infty \left| \frac{\sin(x/n)}{(1+x/n)^n} \right| \\
&\le \int_0^\infty \left| (1+x/n)^{-n} \right| \mathrm{d}x \\
&= \int_0^\infty (1+x/n)^{-n}\mathrm{d}x \\
&\le \int_0^\infty e^{-x}\mathrm{d}x \\
&= 1.
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus each f_n is integrable.

Next we observe that f_n converges pointwise to 0 since $(1+x/n)^n \to e^x$ and $\sin(x/n) \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$. Finally, note that $e^{-x} \ge |f_n|$ pointwise and e^{-x} is integrable ($\int_0^\infty |e^{-x}|\mathrm{d}x = 1$). It follows from the Lebesgue Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(x/n)}{(1+x/n)^n}\mathrm{d}x = \int_0^\infty 0\mathrm{d}x = 0.$$

Next we compute (4). For each $n \in \mathbb{N}$, let $f_n = (1+nx^2)(1+x^2)^{-n}$. Observe that each f_n is measurable since each f_n is continuous (the denominator is never zero for any $x \in \mathbb{R}$). Furthermore, each f_n is nonnegative (since taking squares makes everything nonnegative). We claim that f_n is a decreasing sequence. Indeed

$$\begin{aligned}
\frac{f_n}{f_{n+1}} &= \left(\frac{1+nx^2}{(1+x^2)^n}\right)\left(\frac{(1+x^2)^{n+1}}{1+(n+1)x^2}\right) \\
&= \frac{(1+nx^2)(1+x^2)}{1+(n+1)x^2} \\
&= \frac{nx^4+(n+1)x^2+1}{(n+1)x^2+1} \\
&\geq \frac{(n+1)x^2+1}{(n+1)x^2+1} \\
&= 1.
\end{aligned}$$

Thus (f_n) is a decreasing sequence which is bounded below, so it must converge pointwise to some function. For $x = 0$, it's easy to see that $f_n(0) \to 0$. For $x \neq 0$, we use L'Hopital's rule to get

$$\begin{aligned}
\lim_{n\to\infty} f_n &= \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n} \\
&= \lim_{n\to\infty} \frac{x^2}{\ln(1+x^2)(1+x^2)^n} \\
&= 0.
\end{aligned}$$

Thus (f_n) converges pointwise to 0. Since

$$\begin{aligned}
\int_0^1 f_1 \mathrm{d}x &= \int_0^1 \frac{1+x^2}{1+x^2} \mathrm{d}x \\
&= \int_0^1 \mathrm{d}x \\
&= 1 \\
&< \infty,
\end{aligned}$$

it follows from problem 6 that

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} \mathrm{d}x &= \lim_{n\to\infty} \int_0^1 f_n \mathrm{d}x \\
&= \int_0^1 0 \mathrm{d}x \\
&= 0.
\end{aligned}$$