Section 2.5: Rates of Change Defined over Intervals

The rate of change, or derivative, of a function f at an input x, is the limit of secant slopes between $(x, f(x))$ and nearby points $(x+h, f(x+h))$ and is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, provided the limit exists.

This is called the **limit definition of the derivative**.

Example 1: Evaluate $f(x+h)$ for the following functions. Completely simplify.

The above limit definition of the derivative involves evaluating the given function at input value $x+h$. Practice with the functions below.

$f(x) = 3x - 4$	$f(x+h) =$
$f(x) = x^2 + 1$	$f(x+h) =$
$f(x) = 3x^2 - 2x + 1$	$f(x+h) =$
$f(x) = \frac{1}{x}$	$f(x+h) =$
$f(x) = \sqrt{x}$	$f(x+h) =$
$f(x) = x^3$	$f(x+h) =$

Example 2: Use the limit definition to find the derivative $f'(x)$ for $f(x) = 2x^2 - 1$.

a. Let $(x+h, f(x+h))$ be a point very close to the point of tangency $(x, f(x))$.
The formula for the slope of a secant line connecting these two points is given.

Simplify: $\dfrac{f(x+h) - f(x)}{(x+h) - x} =$

b. $f'(x)$ is the limit of the secant slopes, as $h \to 0$, i.e. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Follow each of the steps below to find $f'(x)$ for $f(x) = 2x^2 - 1$, using the limit definition.

1. Write the general limit definition of the derivative	$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
2. Rewrite the limit definition using the given function. This step will require finding $f(x+h)$.	$= \lim\limits_{h \to 0} \dfrac{[\qquad\qquad] - [\qquad\qquad]}{h}$
3. Simplify the algebraic expression, showing one step at a time: *3a) Expand $(x+h)^2$*	$= \lim\limits_{h \to 0}$
3b) Distribute	$= \lim\limits_{h \to 0}$
3c) Combine like terms	$= \lim\limits_{h \to 0}$
3d) Factor out and cancel a common factor of h	$= \lim\limits_{h \to 0}$
4. Show the limit of a completely simplified algebraic expression.	$= \lim\limits_{h \to 0} (4x + 2h)$
*5. **Evaluate** the limit as $h \to 0$.*	$=$
6. Conclusion. State the derivative formula.	Thus, $f'(x) = 4x$

c. Use the derivative formula $f'(x) = 4x$ to find the following.

$f'(-2) =$ $\qquad$ $f'(-1) =$ $\qquad$ $f'(0) =$

$f'(1) =$ $\qquad$ $f'(2) =$ $\qquad$ $f'(3) =$

Example 3: (CC5e pp. 169-70)

a. Use the limit definition to find the derivative $f'(x)$ for $f(x) = -1.6x^2 + 15.6x - 6.4$. Clearly show all steps, as illustrated in the previous example, including equal signs and limit notation.

b. The amount of coal used quarterly for synthetic-fuel plants in the United States between 2001 and 2004 can be modeled as $f(x) = -1.6x^2 + 15.6x - 6.4$ million short tons, where x is the number of years since the beginning of 2000.

Find $f'(3.5)$ and write a sentence of interpretation.

c. Find the percentage rate of change in the amount of coal used quarterly in synthetic-fuel plants when $x = 3.5$. Include units with the answer.

Example 4: In finding the derivative of $f(x) = 2x^2 - x + 3$ below, there is one missing parentheses in step 2, there are four incorrect terms due to distribution errors in step 3b, two limit notations missing, and one crucial missing step.

Write-in the missing parentheses, correct the four terms, write in the limits, and fill in the missing step.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[2(x+h)^2 - (x+h) + 3] - 2x^2 - x + 3}{h}$$

$$= \lim_{h \to 0} \frac{[2(x^2 + 2xh + h^2) - (x+h) + 3] - [2x^2 - x + 3]}{h}$$

$$= \lim_{h \to 0} \frac{2x^2 + 2xh + 2h^2 - x + h + 3 - 2x^2 - x + 3}{h}$$

$$= \frac{4xh + 2h^2 - h}{h}$$

$$= \frac{h(4x + 2h - 1)}{h}$$

$$=$$

$$= 4x - 1$$

$$Thus,\ f'(x) = 4x - 1$$

Example 5: (CC5e pp. 168-9)

The pressure on a scuba diver underwater is $f(x)=\dfrac{1}{33}x+1$ atm at x feet below the surface of the water.

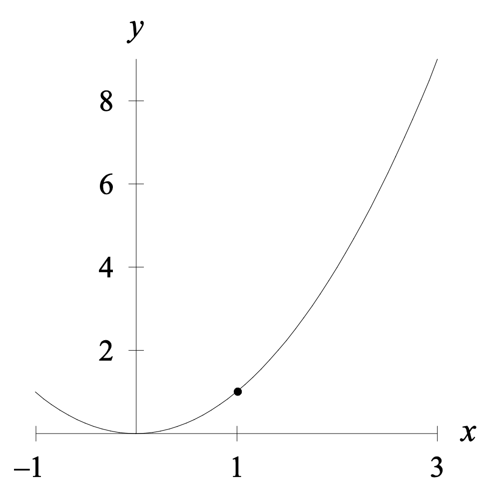
a. Use the limit definition to find the equation for the rate-of-change of f .

b. Find $f(100)$, $f'(100)$, and the percentage rate of change in the pressure on a scuba diver that is 100 feet below the water. Include units with all answers.

Example 6: (CC5e pp. 170-71)

Compare three methods for finding the derivative for the function $f(x)=x^2$ at $x=1$.

a. The figure shows a graph of the function $f(x)=x^2$. Use a tangent line to **graphically** *estimate* $\left.\dfrac{df}{dx}\right|_{x=1} = f'(1) \approx$ ___________.



b. Use the limit of the slopes of secant lines to ***numerically*** *estimate*. Round entries in the table to three decimal places.

$x \to 1^-$	$\frac{f(x)-f(1)}{x-1}$	$x \to 1^+$	$\frac{f(x)-f(1)}{x-1}$
0.9		1.1	

$\lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} =$

$\lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} =$

Conclusion: $f'(1) = \lim_{x \to 1} \frac{f(x)-f(1)}{x-1} =$

c. Find a formula for $f'(x)$ using the **limit definition** of the derivative, clearly showing all steps. Then use the formula to evaluate $f'(1)$.