Section 2.4: Rates of Change – Numerical Limits and Nonexistence

The rate of change, or derivative, of a function $f(x)$ at input *a*, is the limit of secant slopes:
$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists.

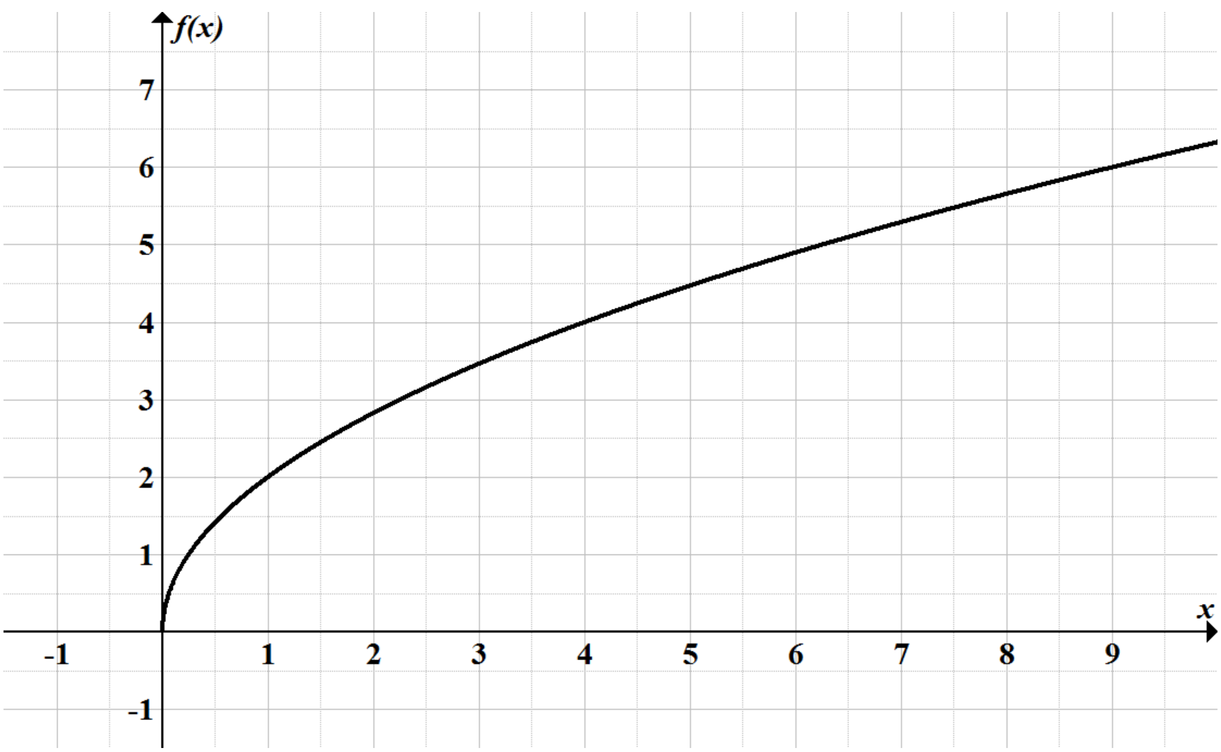
Graphically, $f'(a)$ is the slope of the line tangent to the graph of $f(x)$ at input *a*.

$f'(a)$ can be estimated **numerically** by calculating slopes of secant lines between a point of tangency $(a, f(a))$ and nearby points $(x, f(x))$ and then finding the limit of the secant slopes as nearby those points get closer to the point of tangency. Points on both sides of *a* must be used to verify that the limit of the secant slopes exists at *a*.

Example 1: (CC5e p.160)

Estimate $f'(4)$ for the function $f(x) = 2\sqrt{x}$, both graphically and numerically.

a. Graphically estimate $f'(4)$ by finding the slope of the tangent line to $f(x)$ at $x =$ ______.



Graphically, $f'(4) \approx$ _____.

b. Numerically estimate $f'(4)$ by finding the limit of the secant slopes.

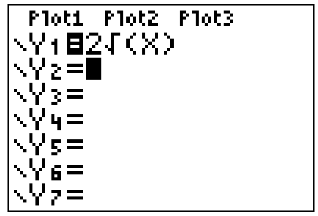
➢ Let $(x, f(x))$ be a point very close to the point $(4, f(4))$.
Write the slope formula for the secant line connecting these two points.

➢ If a nearby point has input $x = 3.9$, what is the slope of the secant line between the points $(3.9, f(3.9))$ and $(4, f(4))$? Enter the answer in the table below, rounding to four decimal places. Repeat for other input values as points get closer and closer to $x = 4$, from both the left and the right.

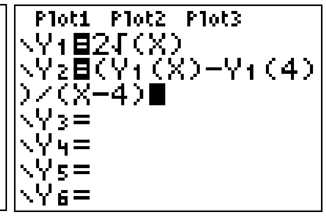
$x \to 4^-$	$\dfrac{f(x)-f(4)}{x-4}$	$x \to 4^+$	$\dfrac{f(x)-f(4)}{x-4}$
3.9		4.1	
3.99		4.01	
3.999			
3.9999			

Numerically estimating a rate of change:

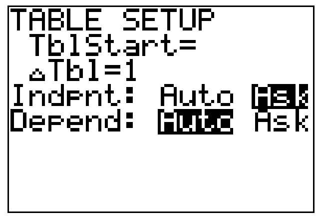
- Enter $f(x)$ into Y1.



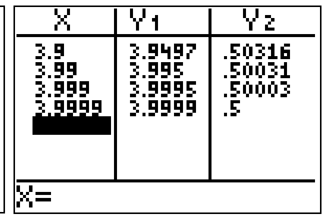
- Enter the formula for the slope of the secant line into Y2:
  (Y1 (X) - Y1 (4)) / (X – 4)



- **2ND** **WINDOW** [TBLSET] returns the table setup.

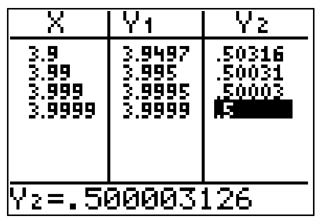


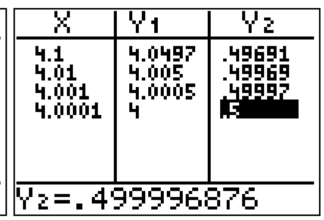
- **2ND** **GRAPH** [TABLE] returns the table.



- Use **DEL** to clear any values in the X column.

- In the X column, type **3.9** **ENTER** **3.99** **ENTER** **3.999** **ENTER** **3.9999** **ENTER**. The function values are in the Y1 column and the slope of the secant lines between each entered x value and $x = 4$ are in the Y2 column.





- Move the cursor over the values in Y2 to see the unrounded slopes.

- Repeat for x values to the right of 4.

➢ The **limit** of the slopes of the secant lines from the left, $\lim\limits_{x \to 4^-} \dfrac{f(x) - f(4)}{x - 4} =$ ________.

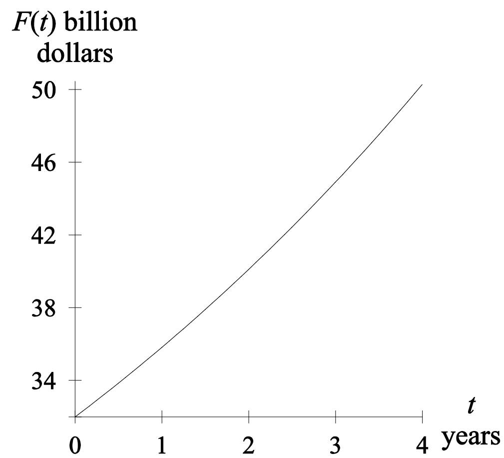
The **limit** of the slopes of the secant lines from the right, $\lim\limits_{x \to 4^+} \dfrac{f(x) - f(4)}{x - 4} =$ _________.

Since the left-hand and right-hand limits are equal, the limit of the slopes of the secant lines ***exists*** and $\lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x - 4} =$ ________.

Numerically, $f'(4) \approx$ _________ .

Example 2: (CC5e p.161)

A multinational corporation invests 432 billion of its assets in the global market, resulting in an investment with a future value of $F(t) = 32(1.12^t)$ billion dollars after t years. The graph of $F(t)$ is shown to the right.



a. Draw the line on the graph of $F(t)$ whose slope represents the rate of change of the future value of the investment at $t = 3.5$, in the middle of the fourth year.

b. Find the secant slopes between $(t, F(t))$ and $(3.5, F(3.5))$ and enter them in the table. Round answers to three decimal places.

$t \to 3.5^-$	$\frac{F(t)-F(3.5)}{t-3.5}$	$t \to 3.5^+$	$\frac{F(t)-F(3.5)}{t-3.5}$
3.49	5.389	3.51	5.395
3.499		3.501	
3.4999		3.5001	

c. Find the left-hand and right-hand limits of the secant slopes.

$$\lim_{t \to 3.5^-} \frac{F(t)-F(3.5)}{t-3.5} = \_\_\_\_\_\_\_\_ \quad \text{and} \quad \lim_{t \to 3.5^+} \frac{F(t)-F(3.5)}{t-3.5} = \_\_\_\_\_\_\_\_$$

d. $F'(3.5) = \lim_{t \to 3.5} \frac{F(t)-F(3.5)}{t-3.5} = \_\_\_\_\_\_\_\_$

e. Complete the following sentence of interpretation for the answer to part d.

In the middle of the fourth year, the value of the multinational corporation's investment is

____.

f. Find and interpret the percentage rate of change when $t = 3.5$.

If the derivative of a function exists at a point, the function is said to be **differentiable** at that point. If the derivative of a function exists for every point whose input is in an open interval, the function is **differentiable over that open interval.**

The **derivative does not exist**:

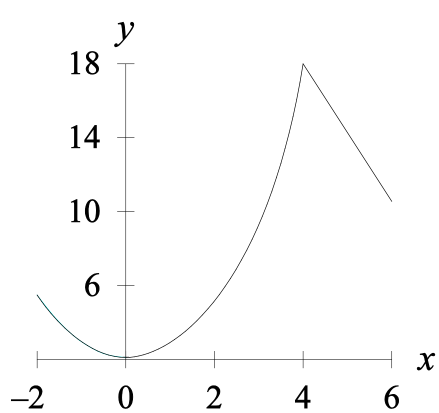
- at a point P where the function is ***not continuous.***
  At such a point, a tangent line cannot be drawn. The derivative does not exist because the limit of the secant slopes does not exist.

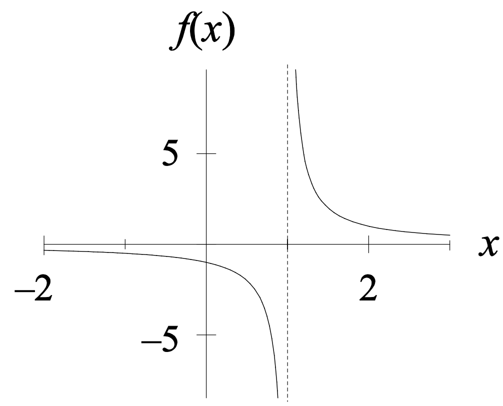
- at a point P where the function ***has a sharp point*** (the function is not smooth).
  At such a point, a tangent line cannot be drawn. The derivative does not exist because the limit of the secant slopes does not exist.

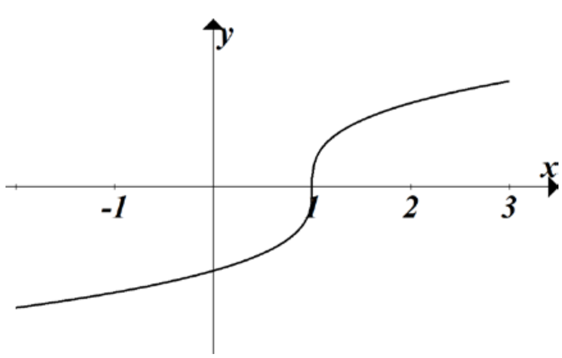
- at a point P where the function is continuous but ***the tangent line is vertical***.
  At such a point, the derivative does not exist because the slope of the vertical line is undefined.

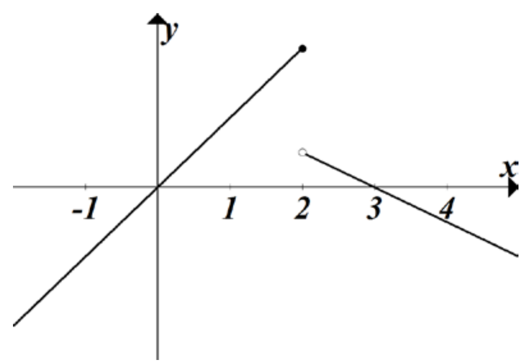
Example 3: (CC5e p. 165-166, Activities 15 and 17)

a. Identify input values (other than endpoints) at which the function is **not differentiable**. State the reason(s) the function is not differentiable at each input value identified.









b. Circle the graphs that are **continuous** at the point where the function is not differentiable. Note that the other graphs are **not continuous** at the point where the function is not differentiable.