# Linear Analysis Homework 1

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## Problem 1

**Proposition 0.1.** *(Polarization Identity) For $x, y \in \mathscr{V}$ we have*

$$4\langle x, y\rangle = \|x+y\|^2 + i\|x+iy\|^2 - \|x-y\|^2 - i\|x-iy\|^2$$

*Proof.* We calculate

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle, \end{aligned}$$

and

$$\begin{aligned} i\|x+iy\|^2 &= i\langle x+iy, x+iy\rangle \\ &= i\langle x, x\rangle + i\langle x, iy\rangle + i\langle iy, x\rangle + i\langle iy, iy\rangle \\ &= i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle + i\langle y, y\rangle, \end{aligned}$$

and

$$\begin{aligned} -\|x-y\|^2 &= -\langle x-y, x-y\rangle \\ &= -\langle x, x\rangle - \langle x, -y\rangle - \langle -y, x\rangle - \langle -y, -y\rangle \\ &= -\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle, \end{aligned}$$

and

$$\begin{aligned} -i\|x-iy\|^2 &= -i\langle x-iy, x-iy\rangle \\ &= -i\langle x, x\rangle - i\langle x, -iy\rangle - i\langle -iy, x\rangle - i\langle -iy, -iy\rangle \\ &= -i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle - i\langle y, y\rangle. \end{aligned}$$

Adding these together gives us our desired result. □

## Problem 2

**Proposition 0.2.** *(Parallelogram Identity) For $x, y \in \mathscr{V}$ we have*

$$\|x-y\|^2 + \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

*Proof.* We calculate

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + 2\mathrm{Re}(\langle x, y\rangle) + \|y\|^2. \end{aligned}$$

and

$$\begin{aligned} \|x-y\|^2 &= \langle x-y, x-y\rangle \\ &= \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle \\ &= \|x\|^2 - 2\mathrm{Re}(\langle x, y\rangle) + \|y\|^2. \end{aligned}$$

Adding these together gives us our desired result. □

The geometric interpretation of Proposition (0.2) in the case where $\mathscr{V} = \mathbb{R}^3$ can be seen below:

## Problem 3

**Proposition 0.3.** *(Pythagorean Theorem) Let $x$ and $y$ be nonzero vectors in $\mathscr{V}$ such that $\langle x, y \rangle = 0$ (we call such vectors* ***orthogonal*** *to one another). Then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

*Proof.* We have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

□

## Problem 4

**Proposition 0.4.** *Let $(x_n)$ and $(y_n)$ be two sequences in $\mathscr{V}$. Then the following statements hold:*

1. *If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.*
2. *If $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$. In particular, $\|x_n\| \to \|x\|$.*

*Proof.*

1. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $\|x_n - x\| < \varepsilon/2$ and $\|y_n - y\| < \varepsilon/2$. Then $n \geq N$ implies

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &\leq \|x_n - x\| + \|y_n - y\| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

2. Since $y_n \to y$, there exists $M \geq 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Choose such an $M$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $\|x_n - x\| < \varepsilon/2M$ and $\|y_n - y\| < \varepsilon/2\|x\|$. Then $n \geq N$ implies

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\ &\leq |\langle x_n, y_n \rangle - \langle x, y_n \rangle| + |\langle x, y_n \rangle - \langle x, y \rangle| \\ &= |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \\ &\leq \|x_n - x\| M + \|x\| \|y_n - y\| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

To see that $\|x_n\| \to \|x\|$, we just set $y_n = x_n$. Then

$$\begin{aligned}\|x_n\| &= \sqrt{\langle x_n, x_n\rangle} \\ &\to \sqrt{\langle x, x\rangle} \\ &= \|x\|,\end{aligned}$$

where we were allowed to take limits inside the square root function since the square root function is continuous on $\mathbb{R}_{\geq 0}$.

□

## Problem 5

**Proposition 0.5.** *Let $\langle \cdot, \cdot \rangle \colon M_{m\times n}(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \to \mathbb{R}$ be given by*

$$\langle A, B\rangle = Tr(B^\top A),$$

*for all $A, B \in M_n(\mathbb{C})$. Then the pair $(M_n(\mathbb{C}), \langle \cdot, \cdot \rangle)$ forms an inner-product space.*

*Proof.* Linearity in the first argument follows from distributivity of matrix multiplication and from linearity of the trace function: Let $A, B, C \in \mathrm{M}_{m\times n}(\mathbb{R})$. Then

$$\begin{aligned}\langle A + B, C\rangle &= \mathrm{Tr}(C^\top (A + B)) \\ &= \mathrm{Tr}(C^\top A + C^\top B) \\ &= \mathrm{Tr}(C^\top A) + \mathrm{Tr}(C^\top B) \\ &= \langle A, C\rangle + \langle B, C\rangle.\end{aligned}$$

Symmetry of $\langle \cdot, \cdot \rangle$ follows from the fact that $\mathrm{Tr}(A) = \mathrm{Tr}(A^\top)$ for all $A \in \mathrm{M}_{m\times n}(\mathbb{R})$: Let $A, B \in \mathrm{M}_{m\times n}(\mathbb{R})$. Then

$$\begin{aligned}\langle A, B\rangle &= \mathrm{Tr}(B^\top A) \\ &= \mathrm{Tr}((B^\top A)^\top) \\ &= \mathrm{Tr}(A^\top B) \\ &= \langle B, A\rangle.\end{aligned}$$

Finally, to see positive-definiteness of $\langle \cdot, \cdot \rangle$, let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in \mathrm{M}_{m\times n}(\mathbb{R}).$$

Then

$$\begin{aligned}\langle A, A\rangle &= \mathrm{Tr}(A^\top A) \\ &= \mathrm{Tr} \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}^2.\end{aligned}$$

is a sum of its entries squared. This implies positive-definiteness. □

## Problem 6a

**Proposition 0.6.** *Let $\langle \cdot, \cdot \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by*

$$\langle x, y\rangle = \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \sum_{i=1}^{n} x_i \overline{y}_i.$$

*for all $x, y \in \mathbb{C}^n$. Then the pair $(\mathbb{C}^n, \langle \cdot, \cdot \rangle)$ forms an inner-product space.*

*Proof.* For linearity in the first argument follows from linearity, let $x, y, z \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle x+y, z\rangle &= \sum_{i=1}^{n}(x_i + y_i)\overline{z}_i \\ &= \sum_{i=1}^{n} x_i\overline{z}_i + \sum_{i=1}^{n} y_i\overline{z}_i \\ &= \langle x, z\rangle + \langle y, z\rangle.\end{aligned}$$

For conjugate symmetry of $\langle\cdot,\cdot\rangle$, let $x, y \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle x, y\rangle &= \sum_{i=1}^{n} x_i\overline{y}_i \\ &= \sum_{i=1}^{n} \overline{\overline{x_i\overline{y}_i}} \\ &= \sum_{i=1}^{n} \overline{y_i\overline{x}_i} \\ &= \overline{\langle y, x\rangle}.\end{aligned}$$

For positive-definiteness of $\langle\cdot,\cdot\rangle$, let $x \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle x, x\rangle &= \sum_{i=1}^{n} x_i\overline{x}_i \\ &= \sum_{i=1}^{n} |x_i|^2.\end{aligned}$$

is a sum of its components absolute squared. This implies positive-definiteness. □

## Problem 6b

This follows from an easy application of Cauchy-Schwarz, but here's another method (which turns out to be equivalent to Cauchy-Schwarz). We need the following two lemmas:

**Lemma 0.1.** *Let $a$ and $b$ be nonnegative real numbers. Then we have*

$$2ab \leq a^2 + b^2. \tag{1}$$

*Proof.* We have

$$\begin{aligned}0 &\leq (a-b)^2 \\ &= a^2 - 2ab + b^2.\end{aligned}$$

Therefore the inequality (1) follows from adding $2ab$. □

**Lemma 0.2.** *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

*Proof.* We have

$$\begin{aligned}\left(\sum_{i=1}^{n} a_i b_i\right)^2 &= \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{1\leq i<j\leq n} 2a_i b_j a_j b_i \\ &\leq \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{1\leq i<j\leq n} (a_i^2 b_j^2 + a_j^2 b_i^2) \\ &= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)\end{aligned}$$

where the inequality in the second line follows from Lemma (0.1) applied to $a_i b_j$ and $a_j b_i$. □

**Corollary.** *Let $x, y \in \mathbb{C}^n$. Then*
$$\sum_{i=1}^{n} |x_i||y_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2}\sqrt{\sum_{i=1}^{n} |y_i|^2}.$$

*Proof.* This follows from by taking squares on both sides and applying Lemma (0.2) since the $|x_i|$ and $|y_i|$ are nonnegative real numbers. □

## Problem 7a

**Proposition 0.7.** *Let $\ell^2(\mathbb{N})$ be the set of all sequence $(x_n)$ in $\mathbb{C}$ such that*

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty$$

*and let $\langle \cdot, \cdot \rangle \colon \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{C}$ be given by*

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n.$$

*for all $(x_n), (y_n) \in \ell^2(\mathbb{N})$. Then the pair $(\ell^2(\mathbb{N}), \langle \cdot, \cdot \rangle)$ forms an inner-product space.*

*Proof.* We first need to show that $\ell^2(\mathbb{N})$ is indeed a vector space. In fact, we will show that $\ell^2(\mathbb{N})$ is a subspace of $\mathbb{C}^N$, the set of all sequences in $\mathbb{C}$. Let $(x_n), (y_n) \in \ell^2(\mathbb{N})$ and $\lambda \in \mathbb{C}$. Then Lemma (0.1) implies

$$\begin{aligned}
\sum_{n=1}^{\infty} |\lambda x_n + y_n|^2 &\leq \sum_{n=1}^{\infty} |\lambda x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2 + \sum_{n=1}^{\infty} 2|\lambda x_n||y_n| \\
&\leq \lambda^2 \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} |y_n|^2 + \lambda^2 \sum_{n=1}^{\infty} |x_n|^2 + |y_n|^2 \\
&< \infty.
\end{aligned}$$

Therefore $(\lambda x_n + y_n) \in \ell^2(\mathbb{N})$, which implies $\ell^2(\mathbb{N})$ is a subspace of $\mathbb{C}^{\mathbb{N}}$.

Next, let us show that the inner product converges, and hence is defined everywhere. Let $(x_n), (y_n) \in \ell^2(\mathbb{N})$. Then it follows from Lemma (0.1) that

$$\begin{aligned}
\sum_{n=1}^{\infty} |x_n \overline{y}_n| &= \sum_{n=1}^{\infty} |x_n||y_n| \\
&\leq \sum_{n=1}^{\infty} \frac{|x_n|^2 + |y_n|^2}{2} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} |x_n|^2 + \frac{1}{2} \sum_{n=1}^{\infty} |y_n|^2 \\
&< \infty.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} x_n \overline{y}_n$ is absolutely convergent, which implies it is convergent. (We can't use Cauchy-Schwarz here since we haven't yet shown that $\langle \cdot, \cdot \rangle$ is in fact an inner-product).

Finally, let us shows that $\langle \cdot, \cdot \rangle$ is an inner-product. Linearity in the first argument follows from distrubitivity of multiplication and linearity of taking infinite sums. For conjugate symmetry, let $(x_n), (y_n) \in \ell^2(\mathbb{N})$. Then

$$\begin{aligned}
\langle (x_n), (y_n) \rangle &= \sum_{n=1}^{\infty} x_n \overline{y}_n \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} x_n \overline{y}_n \\
&= \overline{\overline{\lim_{N\to\infty} \sum_{n=1}^{N} x_n \overline{y}_n}} \\
&= \overline{\lim_{N\to\infty} \sum_{n=1}^{N} \overline{x_n \overline{y}_n}} \\
&= \overline{\lim_{N\to\infty} \sum_{n=1}^{N} y_n \overline{x}_n} \\
&= \overline{\sum_{n=1}^{\infty} y_n \overline{x}_n} \\
&= \overline{\langle (y_n), (x_n) \rangle},
\end{aligned}$$

where we were allowed to bring the conjugate inside the limit since the conjugate function is continuous on $\mathbb{C}$. For positive-definiteness, let $(x_n) \in \ell^2(\mathbb{N})$. Then

$$\begin{aligned}
\langle (x_n), (x_n) \rangle &= \sum_{n=1}^{\infty} x_n \overline{x}_n \\
&= \sum_{n=1}^{\infty} |x_n|^2 \\
&\geq 0.
\end{aligned}$$

If $\sum_{n=1}^{\infty} |x_n|^2 = 0$, then clearly we must have $x_n = 0$ for all $n$. □

## Problem 7b

**Proposition 0.8.** *Let $(x_n) \in \ell^2(\mathbb{N})$ such that $\sum_{n=1}^{\infty} |x_n|^2 = 1$. Then*

$$\sum_{n=1}^{\infty} \frac{|x_n|}{2^n} \leq \frac{1}{\sqrt{3}}. \tag{2}$$

*where the inequality (2) becomes an equality if and only if $|x_n| = \sqrt{3} \cdot 2^{-n}$ for all $n$.*

*Proof.* By Cauchy-Schwarz, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{|x_n|}{2^n} &= |\langle (|x_n|), (2^{-n}) \rangle| \\
&\leq \|(|x_n|)\| \|(2^{-n})\| \\
&= \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \sqrt{\sum_{n=1}^{\infty} 2^{-2n}} \\
&1 \cdot \sqrt{\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n - 1} \\
&= \sqrt{\frac{1}{1-1/4} - 1} \\
&= \sqrt{\frac{4}{3} - 1} \\
&= \frac{1}{\sqrt{3}}.
\end{aligned}$$

where the inequality becomes an equality if and only if $(|x_n|)$ and $(2^{-n})$ are linearly dependent. This means that there is a $\lambda \in \mathbb{C}$ such that $|x_n| = \lambda 2^{-n}$ for all $n$. To find this $\lambda$, write

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} |x_n|^2 \\ &= \sum_{n=1}^{\infty} |\lambda 2^{-n}|^2 \\ &= |\lambda|^2 \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \frac{|\lambda|^2}{3}. \end{aligned}$$

Thus, any $\lambda \in \mathbb{C}$ such that $|\lambda| = \sqrt{3}$ works. (Actually, we must have $\lambda = \sqrt{3}$ since $\lambda = |x_n|2^n$ is positive). □

## Problem 8

**Proposition 0.9.** *Let $f \in C[0,1]$ such that $\int_0^1 |f(x)|^2 dx = 1$. Then*

$$\int_0^1 |f(x)| \sin(\pi x) dx \leq \frac{1}{\sqrt{2}},$$

*where the inequality becomes an equality if and only if $|f(x)| = \sqrt{2}\sin(\pi x)$.*

*Proof.* First note that

$$\begin{aligned} \int_0^1 \sin^2(\pi x) dx &= \int_0^1 \cos^2(\pi x) dx \\ &= \int_0^1 (1 - \sin^2(\pi x)) dx \end{aligned}$$

implies $\int_0^1 \sin^2(\pi x) dx = 1/2$, where in the first equality above we used integration by parts with $u = \sin(\pi x)$ and $dv = \sin(\pi x) dx$. Therefore, by Cauchy-Schwarz, we have

$$\begin{aligned} \int_0^1 |f(x)| \sin(\pi x) dx &\leq \sqrt{\int_0^1 |f(x)|^2 dx} \cdot \sqrt{\int_0^1 \sin^2(\pi x) dx} \\ &= 1 \cdot \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}, \end{aligned}$$

where the inequality becomes an equality if and only if $|f(x)|$ and $\sin(\pi x)$ and linearly dependent. This means that there is a $\lambda \in \mathbb{C}$ such that $|f(x)| = \lambda \sin(\pi x)$ for all $x$. To find this $\lambda$, write

$$\begin{aligned} 1 &= \int_0^1 |f(x)|^2 dx \\ &= \int_0^1 |\lambda \sin(\pi x)|^2 dx \\ &= |\lambda|^2 \int_0^1 \sin^2(\pi x) dx \\ &= \frac{|\lambda|^2}{2}. \end{aligned}$$

Thus, any $\lambda \in \mathbb{C}$ such that $|\lambda| = \sqrt{2}$ works. (Actually, we must have $\lambda = \sqrt{2}$ since $\lambda = |f(x)|/\sin(\pi x)$ is positive). □

*Remark.* If we tried to apply Lemma (0.1) at each $x \in [0,1]$, we'd only get the weaker result:

$$\begin{aligned} \int_0^1 |f(x)| \sin(\pi x) dx &\leq \frac{1}{2}\left(\int_0^1 |f(x)|^2 dx + \int_0^1 \sin^2(\pi x) dx\right) \\ &= \frac{1}{2} + \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$