Complex Analysis Homework 1

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$(7a)$: Assume $|w| < 1$ and $|z| < 1$. Then $w\overline{w} = |w|^2 < 1$ and $z\overline{z} = |z|^2 < 1$. So

$$\begin{aligned} \left|\frac{w-z}{1-\overline{w}z}\right| &= \sqrt{\left(\frac{w-z}{1-\overline{w}z}\right)\left(\frac{\overline{w}-\overline{z}}{1-w\overline{z}}\right)} \\ &= \sqrt{\frac{w\overline{w}-w\overline{z}-z\overline{w}+z\overline{z}}{1-w\overline{z}-\overline{w}z+w\overline{w}z\overline{z}}} \\ &< \sqrt{\frac{2-w\overline{z}-z\overline{w}}{2-w\overline{z}-\overline{w}z}} \\ &= 1. \end{aligned}$$

Now assume $|z| = 1$. Then $z\overline{z} = |z|^2 = 1$. So

$$\begin{aligned} \left|\frac{w-z}{1-\overline{w}z}\right| &= \sqrt{\left(\frac{w-z}{1-\overline{w}z}\right)\left(\frac{\overline{w}-\overline{z}}{1-w\overline{z}}\right)} \\ &= \sqrt{\frac{w\overline{w}-w\overline{z}-z\overline{w}+z\overline{z}}{1-w\overline{z}-\overline{w}z+w\overline{w}z\overline{z}}} \\ &= \sqrt{\frac{w\overline{w}-w\overline{z}-z\overline{w}+1}{1-w\overline{z}-\overline{w}z+w\overline{w}}} \\ &= 1. \end{aligned}$$

Now assume $|w| = 1$. Then $w\bar{w} = |w|^2 = 1$. So

$$\begin{aligned} \left|\frac{w-z}{1-\overline{w}z}\right| &= \sqrt{\left(\frac{w-z}{1-\overline{w}z}\right)\left(\frac{\overline{w}-\overline{z}}{1-w\overline{z}}\right)} \\ &= \sqrt{\frac{w\overline{w}-w\overline{z}-z\overline{w}+z\overline{z}}{1-w\overline{z}-\overline{w}z+w\overline{w}z\overline{z}}} \\ &= \sqrt{\frac{1-w\overline{z}-z\overline{w}+z\overline{z}}{1-w\overline{z}-\overline{w}z+z\overline{z}}} \\ &= 1. \end{aligned}$$

$(7b)$: Fix $w \in \mathbb{D}$.

1. First observe that $z \in \mathbb{D}$ if and only if $|z| \leq 1$. Then for all $z \in \mathbb{D}$, we have

$$\left|\frac{w-z}{1-\overline{w}z}\right| \leq 1$$

   by part (a). This implies

$$F(z) = \frac{w-z}{1-\overline{w}z} \in \mathbb{D},$$

   So $F : \mathbb{D} \to \mathbb{D}$. To see that F is holomorphic in $\mathbb{D}$, we simply observe that F is the ratio of two holomorphic functions $f(z) = w - z$ and $g(z) = 1 - \overline{w}z$, where $g(z) \neq 0$ for all $z \in \mathbb{D}$.

2. We have
$$F(0) = \frac{w}{1} = w \quad \text{and} \quad F(w) = \frac{w-w}{1-\overline{w}w} = 0.$$

3. If $|z| = 1$, then
$$|F(z)| = \left| \frac{w-z}{1-\overline{w}z} \right| = 1$$
from part (a).

4. First we calculate $F \circ F$:
$$F(F(z)) = \frac{w - \left(\frac{w-z}{1-\overline{w}z}\right)}{1 - \overline{w}\left(\frac{w-z}{1-\overline{w}z}\right)} = \frac{\frac{w-w\overline{w}z-w+z}{1-\overline{w}z}}{\frac{1-\overline{w}z-\overline{w}w+\overline{w}z}{1-\overline{w}z}} = \frac{-w\overline{w}z+z}{1-\overline{w}w} = z.$$
So F is its own inverse. This implies F is bijective.

(10) : By definition, we have
$$\partial_z := \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right) \quad \text{and} \quad \partial_{\overline{z}} := \frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right)$$
And since ∂_x and ∂_y commute with one another and are $\mathbb{C}$-linear, we have
$$\begin{aligned}
\partial_{\overline{z}}\partial_z &= \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right)\frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right) \\
&= \frac{1}{4}\left(\partial_x^2 - \frac{1}{i}\partial_x\partial_y + \frac{1}{i}\partial_y\partial_x + \partial_y^2\right) \\
&= \frac{1}{4}\left(\partial_x^2 + \partial_y^2\right) \\
&= \Delta \\
&= \frac{1}{4}\left(\partial_y^2 + \partial_x^2\right) \\
&= \frac{1}{4}\left(\partial_y^2 + \frac{1}{i}\partial_y\partial_x - \frac{1}{i}\partial_x\partial_y + \partial_x^2\right) \\
&= \frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right)\frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right) \\
&= \partial_z\partial_{\overline{z}}.
\end{aligned}$$

$(16a)$: We have
$$limsup\left(\left(\left|(\log n)^2\right|\right)^{1/n}\right) = limsup\left(|\log n|^{\,2/n}\right) = 1.$$
Therefore the radius of convergence is $R = \frac{1}{1} = 1$.

$(25a)$: Let $\gamma : [0,1) \to \mathbb{C}$ be given by $\gamma(t) = e^{2\pi it}$. Then
$$\int_\gamma z^n dz = 2\pi i \int_0^1 e^{2\pi nit} e^{2\pi it} dt = 2\pi i \int_0^1 e^{2\pi(n+1)it} = \begin{cases} 0 & \text{if } n \in \mathbb{Z}\backslash\{-1\} \\ 2\pi i & \text{if } n = -1. \end{cases}$$