Manifolds

March 26, 2020

We first recall a few definitions from point-set topology. A topological space is **second countable** if it has a countable basis. A **neighborhood** of a point p in a topological space M is any open set containing p. A topological space M is **Hausdorff** if for every pair of points $x, y \in M$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. An **open cover** of M is a collection $\{U_i\}_{i \in I}$ of open sets in M whose union $\bigcup_{i \in I} U_i$ is M.

The Hausdorff condition and second countability are "hereditary properties"; they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff.

Proposition 0.1. *Let M' be a subspace of a topological space M.*

1. *If M is Hausdorff, then M' is Hausdorff.*
2. *If M is second countable, then M' is second countable.*

Proof. (1) : Suppose $x, y \in M'$. Since $x, y \in M$ and M is Hausdorff, choose a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. Then $U' = U \cap M'$ is a neighboorhood of x in the subspace topology and $V' = V \cap M'$ is a neighborhood of y in the subspace topology and $U' \cap V' = \emptyset$. (2) : If $\{B_i\}_{i \in \mathbb{N}}$ is a countable basis for M, then $\{B_i'\}_{i \in \mathbb{N}}$ is a countable basis for M', where $B_i' = B_i \cap M'$. $\square$

Definition 0.1. A topological space M is **locally Euclidean of dimension** n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system on** U. We say that a chart (U, ϕ) is **centered** at $p \in U$ if $\phi(p) = 0$.

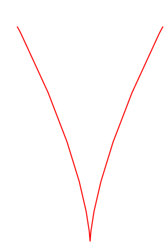
Proposition 0.2. *Let (U, ϕ) be a chart on the topological space M. If V is an open subset U, then $(V, \phi|_V)$ is a chart on M.*

Proof. This follows from the fact that if $\phi : U \to \phi(U)$ is a homeomorphism, then $\phi|_V : V \to \phi(V)$ is a homeomorphism. $\square$

Definition 0.2. A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Example 0.1. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, 1_U)$.

Example 0.2. (A cusp). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold. By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean because it is homeomorphic to $\mathbb{R}$ via the projection $(x, x^{2/3}) \mapsto x$.



Example 0.3. (A cross). The cross can be described as $\{(r, 0) \mid r \in \mathbb{R}\} \cup \{(0, r) \mid r \in \mathbb{R}\}$. We show that the cross in $\mathbb{R}^2$ with the subspace topology is not locally Euclidean at the intersection $p = (0, 0)$, and so cannot be a manifold. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B_\varepsilon(0) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism $U \to B$ restricts to a homemorphism $U \setminus \{p\} \to B \setminus \{0\}$. Now $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components of $n = 1$. Since $U \setminus \{p\}$ has four connected components, there can be no homeomorphism from $U \setminus \{p\}$ to $B \setminus \{p\}$. This contradiction proves that the cross is not locally Euclidean at p.

0.1 Compatible Charts

Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^k **compatible** if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^k. These two maps are called the **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically C^k compatible. To simplify this notation, we will sometimes write U_{ij} for $U_i \cap U_j$ and U_{ijk} for $U_i \cap U_j \cap U_k$. We will also sometimes write ϕ_{ij} for $\phi_i \circ \phi_j^{-1}$. Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of compatible charts.

C^k compatibility is clearly reflexive and symmetric, but not necessarily transitive. Indeed, suppose (U_1, ϕ_1) is C^k compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^k compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2)^{-1} \circ (\phi_2 \circ \phi_1^{-1})$$

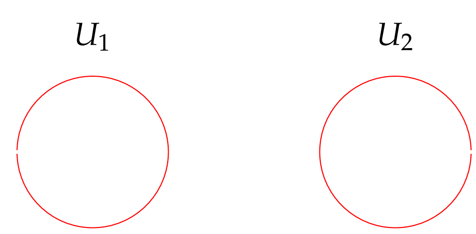
is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$. A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} \setminus U_{123})$.

0.1.1 Atlas

A C^k **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ of pairwise C^k compatible charts such that $\{U_i\}_{i \in I}$ covers M.

Example 0.4. (A C^∞ atlas on a circle). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{2\pi i t} \in \mathbb{C} \mid 0 \le t \le 1\}$. Let U_1 and U_2 be the two open subsets of S^1

$$U_1 = \{e^{2\pi i t} \in \mathbb{C} \mid -\frac{1}{2} < t < \frac{1}{2}\} \qquad U_2 = \{e^{2\pi i t} \in \mathbb{C} \mid 0 < t < 1\}$$



and define $\phi_i : U_i \to \mathbb{R}$ for $i = 1, 2$ by

$$\phi_1(e^{2\pi i t}) = t \qquad \phi_2(e^{2\pi i t}) = t$$

Both ϕ_1 and ϕ_2 are branches of the complex log funciton $(1/i)\log z$ and are homeomorphisms onto their respective images. Thus (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection U_{12} consists of two connected components, the lower half A and the upper half B:

$$A = \{e^{2\pi i t} \mid -\frac{1}{2} < t < 0\} \qquad B = \{e^{2\pi i t} \mid 0 < t < \frac{1}{2}\}$$

with

$$\phi_1(U_{12}) = \phi_1(A \cup B) = \phi_1(A) \cup \phi_1(B) = \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$
$$\phi_2(U_{12}) = \phi_2(A \cup B) = \phi_2(A) \cup \phi_2(B) = \left(\frac{1}{2}, 1\right) \cup \left(0, \frac{1}{2}\right)$$

The transisition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_{12}) \to \phi_2(U_{12})$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 1 & \text{for } t \in \left(-\frac{1}{2}, 0\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Similarly,

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t-1 & \text{for } t \in \left(\frac{1}{2}, 1\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1.

0.1.2 Compatibility With Atlas

Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ be an atlas a locally Euclidean space. We say that a chart (V, ψ) is **compatible with** $\mathcal{A}$ if it is compatible with all the charts (U_i, ϕ_i) in $\mathcal{A}$.

Lemma 0.1. *Let $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\mathcal{A}$, then they are compatible with each other.*

Proof. We will show that $\sigma \circ \psi^{-1}$ is C^k on $\psi(V \cap W)$. That $\psi \circ \sigma^{-1}$ is C^k on $\sigma(V \cap W)$ will follow by a similar argument. For all $i \in I$,

$$\sigma \circ \psi^{-1}|_{\psi(V \cap W \cap U_i)} = (\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})|_{\psi(V \cap W \cap U_i)}$$

is C^k on $\psi(V \cap W \cap U_i)$. In particular, $\sigma \circ \psi^{-1}$ is C^k on

$$\bigcup_{i \in I} \psi(V \cap W \cap U_i) = \psi(V \cap W).$$

□

0.1.3 Standardized and Maximal Atlas

Definition 0.3. We say that $\mathcal{A}$ is **standardized** if it has no repetitions in the sense that whenever $i \neq j$ we have that either $U_i \neq U_j$ or, when $U_i = U_j$, the maps $\phi_i, \phi_j : U_i \to \mathbb{R}^{n_i}$ do not coincide. If $\mathcal{A}$ and $\mathcal{A}'$ are standarized C^k atlases on a locally Euclidean space, then it makes sense to ask if $\mathcal{A} \subseteq \mathcal{A}'$. This means that each $(\phi, U) \in \mathcal{A}$ is equal to some $(\phi', U') \in \mathcal{A}'$, where "equality" means $U = U'$ and the maps $\phi, \phi' : U \rightrightarrows \mathbb{R}^n$ coincide. We say that a standardized atlas $\mathcal{A}'$ **dominates** a standardized atlas $\mathcal{A}$ if $\mathcal{A} \subseteq \mathcal{A}'$ in the senese just defined.

Remark. Unless otherwise specified, we shall always assume that a C^k atlas is standardized.

Definition 0.4. An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be **maximal** if it is not strictly contained in a larger atlas.

0.1.4 C^k Manifolds

Definition 0.5. A C^k manifold is a topological manifold M together with a maximal atlas. If $k = \infty$, then we say M is **smooth** and the maximal atlas is called a **differentiable structure** on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a **curve**. A 2-dimensional manifold is a **surface**, and an n-dimensional manifold an n-manifold.

In practice, to check that a topological manifold M is a C^k manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, because of the following proposition.

Proposition 0.3. *Any atlas $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathcal{A}$ all charts (V_i, ψ_i) that are compatible with $\mathcal{A}$. By Lemma (0.1), the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathcal{A}$ and so by construction belongs to the new atlas. This proves existence. If $\mathfrak{M}'$ is another maximal atlas containing $\mathcal{A}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathcal{A}$ and so by construction must belong to $\mathfrak{M}$. This proves $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. This proves uniqueness. □

In summary, to show that a topological space M is a C^k manifold, it suffices to check that

1. M is Hausdorff and second countable
2. M has a C^k atlas.

Remark. A topological manifold can be endowed with different (non-compatible) differentiable structures. For instance, consider $X = \mathbb{R}$. We can give the space the structure of a C^k manifold using the single chart $(\mathbb{R}, \varphi_1)$, where $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is given by $x \to x$. We can also give the space the structure of a C^k manifold using the single chart $(\mathbb{R}, \varphi_2)$, where $\varphi_2 : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto x^3$. These two charts are not C^∞ compatible since $\varphi_1 \circ \varphi_2^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto x^{\frac{1}{3}}$, and this is *not* C^∞ on $\mathbb{R}$. Indeed, $\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ is not continuous at $x = 0$.

0.1.5 An Atlas For a Product

Proposition 0.4. *If $\mathfrak{U} = \{(U_i, \phi_i) \mid i \in I\}$ and $\mathfrak{V} = \{(V_j, \psi_j) \mid j \in J\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection*

$$\mathfrak{U} \times \mathfrak{V} = \{(U_i \times V_j, \phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n) \mid (i,j) \in I \times J\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. Clearly the set $\{U_i \times V_j \mid (i,j) \in I \times J\}$ covers $M \times N$, so we just need to show that any two charts in $\mathfrak{U} \times \mathfrak{V}$ are pairwise compatible. Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two charts in $\mathfrak{U} \times \mathfrak{V}$. Then $(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1}$ is C^∞, since

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} = \left(\phi_1 \circ \phi_2^{-1}\right) \times \left(\psi_2 \times \psi_2^{-1}\right),$$

and both $\phi_1 \circ \phi_2^{-1}$ and $\psi_2 \times \psi_2^{-1}$ are C^∞ on their respective domains. The same proof shows that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1}$ is C^∞. Thus $\mathfrak{U} \times \mathfrak{V}$ is a collection of pairwise C^∞ compatible charts that cover $M \times N$. □

Example 0.5. It follows from Proposition (0.4) that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds.

0.2 Examples of Smooth Manifolds

0.2.1 Euclidean Space

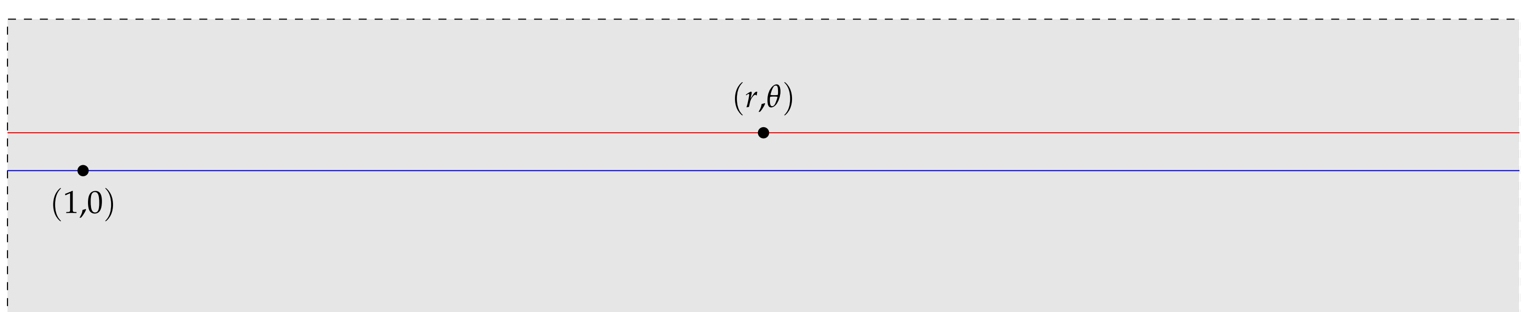
Example 0.6. (Euclidean space). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, \mathrm{id})$. We use $x_1, \dots, x_n$ to denote coordinates functions and $a_1, \dots, a_n$ to denote real numbers. Thus, if $p = (a_1, \dots, a_n)$ is a point in $\mathbb{R}^n$, we have $x_1(p) = a_1$, $x_2(p) = a_2$, and etc...

Example 0.7. The real half line $\mathbb{R}_{>0} : \{a \in \mathbb{R} \mid a > 0\}$ is also a smooth manifold, with a single chart $(\mathbb{R}_{>0}, \mathrm{id})$. In fact, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$. A homeomorphism from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is given by $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

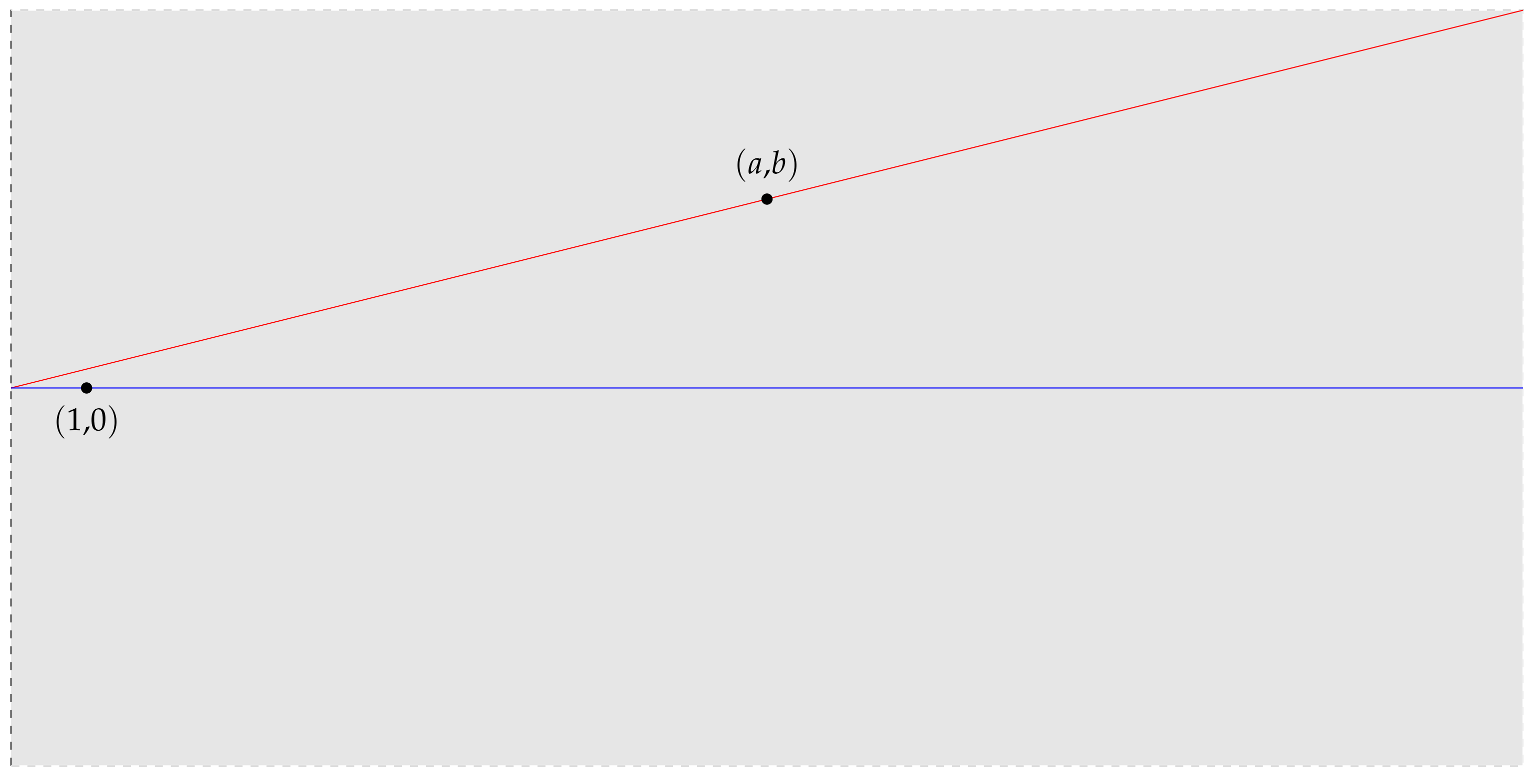
Now consider the half-open interval $(0, 2\pi)$. Open sets of the form (a, b) where and $0 \leq a < b < 2\pi$ form a basis for this topological space.

0.2.2 Right-Half Infinite Strip and the Right-Half Plane

Let $M = \mathbb{R}_{>0} \times \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. We illustrate this space below:



Now let $N = \mathbb{R}_{>0} \times \mathbb{R}$ be the right-half plane. We illustrate this space below:



We can give both M and N the structure of a smooth manifold by simply using the identity charts.

Let $\varphi : M \to N$ be given by $\varphi(r,\theta) = (\varphi_1(r,\theta), \varphi_2(r,\theta))$, where

$$\varphi_1(r,\theta) = r\sin\theta$$
$$\varphi_2(r,\theta) = r\cos\theta$$

Then φ is a diffeomorphism from M to N. The Jacobian of φ at a point $(r,\theta) \in M$:

$$J_{(r,\theta)}(\varphi) = \begin{pmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}$$

The inverse to $\varphi : M \to N$ is $\psi : N \to M$, given by $\psi(a,b) = (\psi_1(a,b), \psi_2(a,b))$ where

$$\psi_1(a,b) = \sqrt{a^2+b^2}$$
$$\psi_2(a,b) = \arctan\left(\frac{a}{b}\right)$$

The Jacobian of ψ at a point $(a,b) \in N$:

$$J_{(a,b)}(\psi) = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

0.2.3 Manifolds of Dimension Zero

Example 0.8. (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

0.2.4 Graph of a Smooth Function

Example 0.9. (Graph of a smooth function). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$, the **graph** of f is defined to be the subset

$$\Gamma(f) = \{(p, f(p)) \in A \times \mathbb{R}^n\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^∞, then the two maps

$$\phi : \Gamma(f) \to U \qquad (p, f(p)) \mapsto p$$

and

$$(1, f) : U \to \Gamma(f) \qquad p \mapsto (p, f(p))$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^n$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

0.2.5 Circle S^1

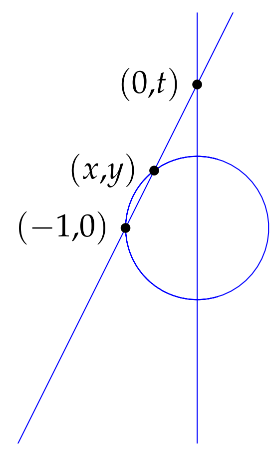
Example 0.10. (Circle) Let S^1 be the unit circle centered at the origin in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We shall describe an atlas on S^1 using stereographic projection. Let $U_1 = S^1 \backslash \{(-1, 0)\}$. Consider the line L which passes through the points $(-1, 0)$ and $(0, t)$ where $t \in \mathbb{R}$. The equation of this line is given by

$$Y = t(X + 1).$$

Since L passes through $(-1, 0)$ and is not tangent to $(-1, 0)$, it must pass through a unique point (x, y) in S^1. This is illustrated in the image below:



Since (x, y) lies on the line L and the unit circle, we get the relations

$$\begin{aligned} x^2 + y^2 - 1 &= 0, \\ y - t(x + 1) &= 0. \end{aligned}$$

Using the second relation, we have $y = t(x + 1)$. Plugging in $t(x + 1)$ for y in the first relation, we get

$$t^2 = \frac{(1 - x)^2}{(1 + x)^2} = \frac{1 - x}{1 + x}.$$

Now we solve for x in terms of t, to get:

$$\begin{aligned} x &= \frac{1 - t^2}{1 + t^2}, \\ y &= \frac{2t}{1 + t^2}. \end{aligned}$$

Now, let $\phi_1 : U_1 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1 + x}.$$

This map is clearly C^∞ in its domain U_1, since $x \neq -1$, and the inverse $\phi_1^{-1} : \mathbb{R} \to U_1$ is given by

$$t \mapsto \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

Next, let $U_2 = S^1 \backslash \{(1, 0)\}$. Following the same line of reasoning as the paragraph above, let $\phi_2 : U_2 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1 - x}.$$

Again, this map is clearly C^∞ in its domain U_2, since $x \neq 1$, and the inverse $\phi_2^{-1} : \mathbb{R} \to U_2$ is given by

$$t \mapsto \left(\frac{t^2 - 1}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

Let us calculate the transition map $\phi_{12} := \phi_1 \circ \phi_2^{-1}$:

$$\begin{aligned} \phi_{12}(t) &= (\phi_1 \circ \phi_2^{-1})(t) \\ &= \phi_1 \left(\frac{t^2 - 1}{1 + t^2}, \frac{2t}{1 + t^2} \right) \\ &= \frac{1}{t}. \end{aligned}$$

Remark. We think of t as a local coordinate of S^1 and x, y as global coordinates of S^1.

0.2.6 Projective Line

Example 0.11. Let $\mathbb{P}^1(\mathbb{R})$ be the projective line over $\mathbb{R}$. Define in $\mathbb{P}^1(\mathbb{R})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0} \in \mathbb{R},$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1} \in \mathbb{R}.$$

These maps are clearly C^∞ in their domains. The inverse maps are given by

$$\phi_0^{-1}(t) = (1 : t) \in U_0 \qquad \phi_1^{-1}(t) = (t : 1) \in U_1.$$

Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1}$:

$$\begin{aligned} \phi_0 \circ \phi_1^{-1}(t) &= \phi_0 \circ \phi_1^{-1}(t) \\ &= \phi_0(t : 1) \\ &= \frac{1}{t}. \end{aligned}$$

Recall that this is the same transition map we calculated in Example (0.10).

0.2.7 Sphere S^2

Example 0.12. (Sphere) Let S^2 be the unit sphere

$$S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres - the front, rear, right , left, upper, and lower hemispheres

$$\begin{aligned} U_1 &= \{(a,b,c) \in S^2 \mid a > 0\} & \phi_1(a,b,c) &= (b,c) \\ U_2 &= \{(a,b,c) \in S^2 \mid a < 0\} & \phi_2(a,b,c) &= (b,c) \\ U_3 &= \{(a,b,c) \in S^2 \mid b > 0\} & \phi_3(a,b,c) &= (a,c) \\ U_4 &= \{(a,b,c) \in S^2 \mid b < 0\} & \phi_4(a,b,c) &= (a,c) \\ U_5 &= \{(a,b,c) \in S^2 \mid c > 0\} & \phi_5(a,b,c) &= (a,b) \\ U_6 &= \{(a,b,c) \in S^2 \mid c < 0\} & \phi_6(a,b,c) &= (a,b) \end{aligned}$$

The open set U_{14} is $\{(a,b,c) \in S^2 \mid b < 0 < a\}$ and $\phi_4(U_{14}) = \{(a,c) \in \mathbb{R}^2 \mid a^2 + c^2 < 1 \text{ and } a > 0\}$.

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned} \phi_{14}(a,c) &= \phi_1 \circ \phi_4^{-1}(a,c) \\ &= \phi_1\left(a, \sqrt{1 - c^2 - a^2}, c\right) \\ &= \left(\sqrt{1 - c^2 - a^2}, c\right). \end{aligned}$$

It is easy to see that this is indeed a smooth map in its domain (since $1 - c^2 - a^2 \neq 0$). The Jacobian of ϕ_{14} at the point (a,c) is

$$J_{(a,c)}(\phi_{14}) = \begin{pmatrix} \frac{a}{\sqrt{1-c^2-a^2}} & \frac{c}{\sqrt{1-c^2-a^2}} \\ 0 & 1 \end{pmatrix}$$

Now let's compute the transition map ϕ_{45}:

$$\begin{aligned} \phi_{45}(a,b) &= \phi_4 \circ \phi_5^{-1}(a,b) \\ &= \phi_4\left(a, b, \sqrt{1 - a^2 - b^2}\right) \\ &= \left(a, \sqrt{1 - a^2 - b^2}\right). \end{aligned}$$

0.2.8 The Sphere S^n

Example 0.13. Using stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $p_N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the north pole and $p_S = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the south pole. Define the maps $\phi_N : S^n \backslash \{p_N\} \to \mathbb{R}^n$ and $\phi_S : S^n \backslash \{p_S\}$, called **stereographic projection** from the north pole (resp. south pole), by

$$\phi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \phi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\phi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2} \left(2x_1, \ldots, 2x_n, -1 + \sum_{i=1}^n x_i^2 \right)$$

and

$$\phi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2} \left(2x_1, \ldots, 2x_n, 1 - \sum_{i=1}^n x_i^2 \right).$$

Thus, if we let $U_N = S^n \backslash \{p_N\}$ and $U_S = S^n \backslash \{p_S\}$, we see that U_N and U_S are two open subsets convering S^n, both homeomorphic to $\mathbb{R}^n$. Furthermore, it is easily checked that on the overlap, $U_N \cap U_S$, the transition maps

$$\phi_S \circ \phi_N^{-1} = \phi_N \circ \phi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $p_O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n \backslash \{O\}$, so we conclude that (U_N, ϕ_N) and (U_S, ϕ_S) form a smooth atlas for S^n.

0.2.9 Real Projective Plane

Example 0.14. (Projective Plane) Let $\mathbb{P}^2(\mathbb{R})$ be the projective plane over $\mathbb{R}$. Define in $\mathbb{P}^2(\mathbb{R})$ the three charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1 : x_2) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \right) =: (a, b)$$

$$U_1 = D(X_1) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_1}, \frac{x_2}{x_1} \right) =: (c, d)$$

$$U_2 = D(X_2) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_2 \neq 0\} \qquad \phi_2(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_2}, \frac{x_1}{x_2} \right) =: (e, f)$$

The reason the map ϕ_1 is a homeomorphism is because given that $x_1 \neq 0$, we use the equivalence relation to write the point $p = (x_0 : x_1 : x_2)$ as $p = \left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1} \right)$. Now $\frac{x_0}{x_1}$ and $\frac{x_2}{x_1}$ are two real rumbers which uniquely determine the point (a, b). We think of a and b as the local coordinates in the (U_0, ϕ_0) chart.

Let U_{01} be the intersection of U_0 and U_1, that is, $U_{01} := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$. Then $\phi_0(U_{01}) = \left\{ \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0 \right\}$ and $\phi_1(U_{01}) = \left\{ \left(\frac{x_0}{x_1}, \frac{x_2}{x_1} \right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0 \right\}$. We can also write this in terms of local coordinates as $\phi_0(U_{01}) = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ and $\phi_1(U_{01}) = \{(c, d) \in \mathbb{R}^2 \mid c \neq 0\}$. Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1} : \phi_1(U_{01}) \to \phi_0(U_{01})$ using the local coordinates. We have

$$\begin{aligned}
\phi_{01}(c, d) &= \phi_0 \circ \phi_1^{-1}(c, d) \\
&= \phi_0 \circ \phi_1^{-1} \left(\frac{x_0}{x_1}, \frac{x_2}{x_1} \right) \\
&= \phi_0 \left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1} \right) \\
&= \phi_0 \left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0} \right) \\
&= \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \right) \\
&= \left(\frac{1}{c}, \frac{d}{c} \right).
\end{aligned}$$

It's easy to see that ϕ_{01} is C^∞. Indeed, writing ϕ_{01}^1 and ϕ_{01}^2 for the components of ϕ_{01} (so $\phi_{01}^1(c,d) = \frac{1}{c}$ and $\phi_{01}^2(c,d) = \frac{d}{c}$), the partial derivatives $\partial_c^m \partial_d^n \phi_{01}^i$ exist and are continuous everywhere in $\phi_1(U_{01})$ for all $m,n \in \mathbb{N}$ and $i = 1,2$. This is because ϕ_{01}^1 and ϕ_{01}^2 are rational functions (i.e. ratio of two polynomials) and are they are defined everywhere since $c \neq 0$ in$\phi_1(U_{01})$.

Similarly, one can easily show that

$$\begin{aligned}
\phi_{10}(a,b) &= \left(\frac{1}{a}, \frac{b}{a}\right) \\
\phi_{20}(a,b) &= \left(\frac{1}{b}, \frac{a}{b}\right) \\
\phi_{02}(e,f) &= \left(\frac{f}{e}, \frac{1}{e}\right) \\
\phi_{12}(e,f) &= \left(\frac{e}{f}, \frac{1}{f}\right) \\
\phi_{21}(c,d) &= \left(\frac{c}{d}, \frac{1}{d}\right)
\end{aligned}$$

It is instructive to check that $\phi_{ij} \circ \phi_{ji} = 1$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$.

0.2.10 Riemann Sphere

Example 0.15. (Riemann sphere) In this example we describe a **complex manifold**. A complex manifold is the complex analogue of a manifold, however in the complex manifold case, we require the transition maps to be holomorphic, and not just C^∞. Let $\mathbb{P}^1(\mathbb{C})$ be the projective line over $\mathbb{C}$ (also known as the Riemann sphere). Define in $\mathbb{P}^1(\mathbb{C})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0}$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1}$$

This time, let $z = \frac{x_0}{x_1}$. The open set U_{01} is $\{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$ and $\phi_1(U_{01}) = \mathbb{C}^\times$. Now

$$\begin{aligned}
\phi_0 \circ \phi_1^{-1}(z) &= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0}\right) \\
&= \frac{x_1}{x_0} \\
&= \frac{1}{z}.
\end{aligned}$$

One can show that the map $z \mapsto \frac{1}{z}$ is holomorphic in the domain $\mathbb{C}^\times$.

0.2.11 Mobius Strip

Example 0.16. Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold. First we consider the set of all nonvertical lines in $\mathbb{R}^2$, which we denote by U_v. A nonvertial is of the form $\ell_{a,b}^v = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell_{a,b}^v \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism. Next we consider the set of all nonhorizontal lines in $\mathbb{R}^2$, which we denote by U_h. A nonhorizontal is of the form $\ell_{c,d}^h = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell_{c,d}^h \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism. Now we have $U_v \cup U_h = \mathcal{L}$. To get a topology on $\mathcal{L}$, we glue the topologies from U_v

and U_h: a set $U \subset \mathcal{L}$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d)\\ &= \varphi_v\left(\ell^h_{c,d}\right)\\ &= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right)\\ &= \left(\frac{1}{c},-\frac{d}{c}\right),\end{aligned}$$

and similarly,

$$\begin{aligned}\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b)\\ &= \varphi_h\left(\ell^v_{a,b}\right)\\ &= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right)\\ &= \left(\frac{1}{a},-\frac{b}{a}\right).\end{aligned}$$

These maps are clearly C^∞. In fact, they look very similar to the transition maps for the projective plane, except they are twisted by a negative sign.

Remark. We can also describe $\mathcal{L}$ as $\mathbb{RP}^2\backslash\{[0:0:1]\}$: Any line in the euclidean plane is of the form $ax+by+c=0$, for some $a,b,c \in \mathbb{R}$. First note that these coefficients uniquely determine the line and they are homogeneous. Hence there is a well defined map $\phi : \mathcal{L} \to \mathbb{RP}^2$, given by mapping the line $\mathbf{V}(ax+by+c)$ to the point $[a:b:c]$. Now ϕ is injective, but not surjective. However if we remove the point $[0:0:1]$, then the induced map $\phi : \mathcal{L} \to \mathbb{RP}^2\backslash\{[0:0:1]\}$ is a bijection.

0.2.12 Grassmannians

The **Grassmannian** $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $v_1,\ldots,v_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the **rank** of a matrix A, denoted by rk$A$, is defined to be the number of linearly independent columns of A. This matrix is called a **matrix representative** of the k-plane.

Two bases $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k,\mathbb{R})$ such that

$$b_j = \sum_{i=1}^k a_i g_{ij}$$

for all $1 \le k \le n$. In matrix notation, this says $B = Ag$. Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \text{ if and only if there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

A **real Grassmann manifold** $G(n,k)$ is defined as the space of all k-dimensional subspaces of the space $\mathbb{R}^n$. The topology in $G(n,k)$ may be described as induced by the embedding $G(n,k) \to \mathrm{End}(\mathbb{R}^n)$ which assigns to a $P \in G(n,k)$, the orthogonal projection $\mathbb{R}^n \to P$ combined with the inclusion map $P \to \mathbb{R}^n$.

In $G(4,2)$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} \sim \begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} = \begin{pmatrix} c_{11}a_{11}+c_{12}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24}\\ c_{21}a_{11}+c_{22}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24}\end{pmatrix}$$

where $c_{11}c_{22} - c_{21}c_{12} \neq 0$.

1 Smooth Maps on a Manifold

Now that we've defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined.

1.1 Smooth Functions

Definition 1.1. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or **smooth at a point** p in M if there is a chart (U_i, ϕ_i) about p in M such that $f \circ \phi_i^{-1}$, a function defined on the open subset $\phi(U_i)$ of $\mathbb{R}^n$, is C^∞ at $\phi_i(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M. For each open subset U of M, we denote by $\mathcal{C}_M^\infty(U)$ to be the $\mathbb{R}$-algebra of smooth functions on U:

$$\mathcal{C}_M^\infty(U) := \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty \text{ on } U\}.$$

Lemma 1.1. *Let M be a smooth manifold of dimension n, U and open subset of M, and let $\{(\phi_i, U_i)\}_{i\in I}$ be an atlas on U. Then*

$$\mathcal{C}_M^\infty(U) \simeq \left\{ (g_i)_i \in \prod_{i\in I} \mathcal{C}_{\mathbb{R}^n}^\infty(U_i) \mid \phi_{ji}^*(g_i|_{U_{ij}}) = g_j|_{U_{ij}} \textit{ for all } i, j \in I \right\}.$$

Proof. Let $f \in \mathcal{C}_M^\infty(U)$. Then for each i, the function $f \circ \phi_i^{-1} \in \mathcal{C}_{\mathbb{R}^n}^\infty(U_i)$. Thus, we get a map

$$\varphi : f \mapsto (f \circ \phi_i^{-1})_i \in \prod_{i\in I} \mathcal{C}_{\mathbb{R}^n}^\infty(U_i),$$

and it is easy to check that $(f \circ \phi_i^{-1})_i$ satisfies the compatibility condition, i.e.

$$\phi_{ji}^*(f \circ \phi_i^{-1}|_{U_{ij}}) = f \circ \phi_j^{-1}|_{U_{ij}}.$$

Conversely, if $(g_i)_i \in \prod_{i\in I} \mathcal{C}_{\mathbb{R}^n}^\infty(U_i)$ such that $\phi_{ji}^*(g_i|_{U_{ij}}) = g_j|_{U_{ij}}$ for all $i, j \in I$, then writing $f_i := g_i \circ \phi_i$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$. These glue to a unique f. □

Remark.

1. The definition of the smoothness of a function f at a point is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) is any other chart about p in M, then

   $$f \circ \psi^{-1} |_{\psi(U\cap V)} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

   is a composition of C^∞ functions and hence must be C^∞ at $\psi(p)$.

2. In the definition above, $f : M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f|_U = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are only interested in functions that are smooth on an open set, there is no loss of generality in assuming at the onset that f is continuous.

Proposition 1.1. *Let M be a smooth manifold of dimension n and let (U, ϕ) be a chart of M.*

Proposition 1.2. *Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

1. *The function $f : M \to \mathbb{R}$ is C^∞.*
2. *The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
3. *For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

Proof. We will prove the proposition as a cyclic chain of implications.

($2 \implies 1$): This follows directly from the definition of a C^∞ function, since by (2) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

($1 \implies 3$): Let (V, ψ) be an arbitrary chart on M and let $p \in V$. By the remark above, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

($3 \implies 2$): Obvious.

□

The smoothness conditions of Proposition (1.2) will be a recurrent motif: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on *every* chart.

Definition 1.2. Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

Remark. In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

Example 1.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by an angle θ and let x, y denote the standard coordinate functions on $\mathbb{R}^2$. Then

$$\begin{aligned} \phi^* x &= (\cos\theta)x - (\sin\theta)y \\ \phi^* y &= (\sin\theta)x + (\cos\theta)y. \end{aligned}$$

Indeed, let e_1, e_2 denote the standard coordinates on $\mathbb{R}^2$; so $x(e_1) =$

$$\begin{aligned} (\phi^* x)(a,b) &= x\left(\phi(a,b)\right) \\ &= x\left(\cos\theta a - \sin\theta b, \sin\theta a + \cos\theta b\right) \\ &= \cos\theta a - \sin\theta b \\ &= ((\cos\theta)x - (\sin\theta)y)(a,b). \end{aligned}$$

1.2 Smooth Maps Between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms "C^∞" and "smooth" interchangeably.

Definition 1.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ **at a point** p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

Remark.

1. In the definition, we needed $F^{-1}(V)$ to be open so that $\phi(F^{-1}(V) \cap U)$ is open. Thus, C^∞ maps between manifolds are by definition continuous.

2. In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ as a chart about $F(p)$ in $\mathbb{R}^m$. According to the definition above, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p in N such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

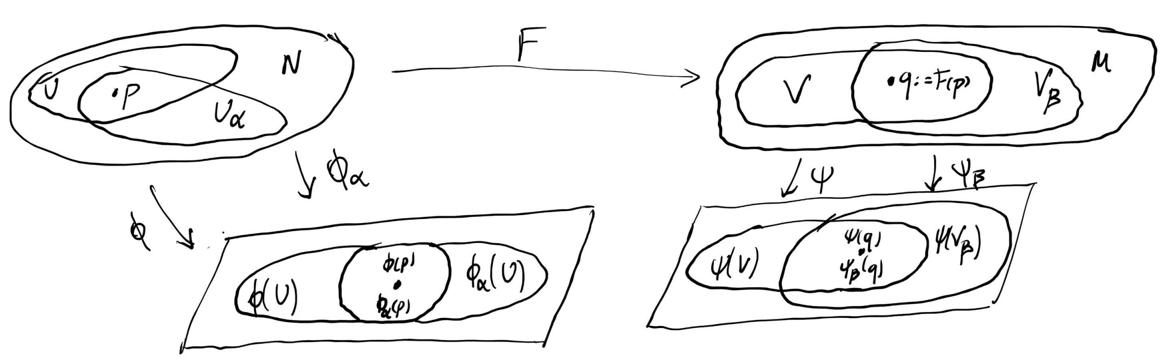
We show now that the definition of the smoothness of a map $F : N \to M$ at a point is independent of the choice of charts.

Proposition 1.3. *Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.*

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1}|_{\phi\left(F^{-1}(V \cap V_\beta) \cap U \cap U_\alpha\right)} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1}),$$

is C^∞ at $\phi(p)$. Therefore $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.



□

Proposition 1.4. *(Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:*

1. *The map $F : N \to M$ is C^∞.*
2. *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map*
$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$
is C^∞.
3. *For every chart (U, ϕ) on N and (V, ψ) on M, the map*
$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$
is C^∞.

Proof.
($2 \implies 1$): Let $p \in N$. Suppose (U, ϕ) is a chart about p in $\mathfrak{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathfrak{V}$. By (2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

($1 \implies 3$): Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \emptyset$. Let $p \in U \cap F^{-1}(V)$. Then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition (1.3), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitary point of $\phi(U \cap F^{-1}(V))$, the map $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞.

($3 \implies 2$): Obvious.

□

Proposition 1.5. *(Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.*

Proof. Let (U, ϕ), (V, ψ), and (W, σ) be charts on N, M, and P respectively. Then
$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}).$$
Since F and G are C^∞, the maps $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are also C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞, and thus $G \circ F$ is C^∞. □

1.2.1 Diffeomorphisms

A **diffeomorpism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 1.6. *If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\left\{(\phi(U), \mathrm{id}_{\phi(U)})\right\}$ with a single chart on $\phi(U)$. Since
$$\mathrm{id}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$$
is the identity map, it is C^∞. By Proposition (1.4), ϕ is C^∞.

To test smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since
$$\phi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} : \phi(U) \to \phi(U)$$
is the identity map, the map ϕ^{-1} is also C^∞. □

Proposition 1.7. *Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition (1.6). As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U, F) is in the atlas. □

1.2.2 Smoothness in Terms of Components

In this subsection, we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 1.8. *(Smoothness of a vector-valued function) Let N be a manifold and let $F : N \to \mathbb{R}^m$ be a continuous map. The following are equivalent:*

1. *The map $F : N \to \mathbb{R}^m$ is C^∞.*
2. *The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*
3. *For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*

Proposition 1.9. *(Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F_1, \dots, F_m : N \to \mathbb{R}$ are all C^∞.*

Proof. The $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the functions $F_i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ if and only if the functions $F_i : N \to \mathbb{R}$ are all C^∞. □

1.3 Germs of C^∞ functions

Let M be an n-dimensional manifold and let p be a point in M. Consider the set of all pairs (f, U), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. Just as in the $\mathbb{R}^n$ case, we introduce an equivalence relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(M)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Let (f, U) be represent a germ in $C_p^\infty(M)$ and suppose (U_0, ϕ) is a chart centered at p. Then $(U_0 \cap U, \phi_{|U})$ is a chart centered at p and clearly we have $(f, U) \sim (f\,|_{U_0 \cap U}, U_0 \cap U)$. Thus we may always assume that a germ can be represented by (f, U) where (U, ϕ) is a chart centered at p. In particular, we obtain an isomorphism

$$\widehat{\phi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \phi^{-1}, \phi(U))$. Of course this map depends on our choice of chart. If (V, φ) was another chart, then we'd obtain another isomorphism

$$\widehat{\varphi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \varphi^{-1}, \varphi(U))$. We can relate these two isomorphisms via the transition function $\phi \circ \varphi^{-1}$.

Let M be a manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on M. We describe the structure of a premanifold as follows: if U is an open subset of M, then we set

$$\mathcal{O}_M(U) := \{f : U \to \mathbb{R} \mid f|_{U \cap U_i} \circ \phi_i^{-1} : \phi_i(U \cap U_i) \to \mathbb{R} \text{ is } C^\infty\} = \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty\}.$$

To see that this is a premanifold, fix $i_0 \in I$. For $U \subseteq U_{i_0}$ open let $f : U \to \mathbb{R}$ be a map such that $f \circ \phi_{i_0}^{-1} : \phi_{i_0}(U_{i_0} \cap U) \to \mathbb{R}$ is a C^∞ function. Then $f \in \mathcal{O}_M(U)$ because the change of charts between i and i_0 are C^∞-diffeomorphisms. Indeed, we have

$$f|_{U \cap U_i} \circ \phi_i^{-1} = (f \circ \phi_{i_0}^{-1}) \circ (\phi_{i_0} \circ \phi_i^{-1}).$$

Therefore ϕ_{i_0} yields an isomorphism $(U_{i_0}, \mathcal{O}_{M|U_{i_0}}) \cong (Y_{i_0}, \mathcal{O}_{i_0})$, where $\mathcal{O}_{Y_0}$ is the sheaf of C^∞ functions on Y_{i_0}. Hence, $(M, \mathcal{O}_M)$ is a ringed space that is locally isomorphic to a manifold. Hence it is a premanifold.

1.4 Examples of Smooth Maps

Example 1.2. We show that the map $F : \mathbb{R} \to S^1$ given by $F(t) = (\cos t, \sin t)$ is C^∞. For $\mathbb{R}$, we use the atlas which consists of a single chart $(\mathbb{R}, \mathrm{id})$. For S^1 we use the atlas which consists of the charts (U_1, ϕ_1), (U_2, ϕ_2), (U_3, ϕ_3) and (U_4, ϕ_4) where

$$\begin{aligned} U_1 &= \{(a, b) \in S^1 \mid a > 0\} & \phi_1(a, b) &= b \\ U_2 &= \{(a, b) \in S^2 \mid a < 0\} & \phi_2(a, b) &= b \\ U_3 &= \{(a, b) \in S^2 \mid b > 0\} & \phi_3(a, b) &= a \end{aligned}$$

$$U_4 = \{(a,b) \in S^2 \mid b < 0\} \qquad \phi_4(a,b) = a$$

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned}\phi_{14}(a) &= \phi_1 \circ \phi_4^{-1}(a)\\ &= \phi_1\left(a, \sqrt{1-a^2}\right)\\ &= \sqrt{1-a^2}.\end{aligned}$$

Similar computations shows that

$$\begin{aligned}\phi_{13}(a) &= \sqrt{1-a^2}\\ \phi_{24}(a) &= \sqrt{1-a^2}\\ \phi_{23}(a) &= \sqrt{1-a^2}.\end{aligned}$$

Now, we need to show that $\phi_i \circ F \circ \mathrm{id}$ is C^∞ for $i = 1,2,3,4$. Let's compute $\phi_1 \circ F \circ \mathrm{id}$:

$$\begin{aligned}(\phi_1 \circ F \circ \mathrm{id})(t) &= \phi_1(F(t))\\ &= (\phi_1((\cos t, \sin t))\\ &= \sin t.\end{aligned}$$

Similar computations shows that

$$\begin{aligned}(\phi_2 \circ F \circ \mathrm{id})(t) &= \sin t\\ (\phi_3 \circ F \circ \mathrm{id})(t) &= \cos t\\ (\phi_4 \circ F \circ \mathrm{id})(t) &= \cos t.\end{aligned}$$

These maps are all C^∞.

Example 1.3. Consider $N = \mathbb{R}$ and $M = \mathbb{R}^2$ and let $f : N \to M$ be given by $f(t) = (t^2, t^3)$.

Example 1.4. Let S^2 be the unit sphere with its smooth structure given in Example (0.12). Let $f : S^2 \to \mathbb{R}$ be given by

$$f(a,b,c) = c^2.$$

We claim that f is C^∞. To see this, we need to show that f is C^∞ at every point $p = (a,b,c)$ in S^2. First assume that $p \in U_6$. Using the chart (U_6, ϕ_6), we find that

$$\begin{aligned}(f \circ \phi_6^{-1})(a,b) &= f\left(\phi_6^{-1}(a,b)\right)\\ &= f\left(a, b, \sqrt{1-a^2-b^2}\right)\\ &= 1 - a^2 - b^2,\end{aligned}$$

which is clearly C^∞.

Example 1.5. Let us show that a C^∞ function $f(x,y)$ on $\mathbb{R}^2$ restricts to a C^∞-function on S^1. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a,b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a,b) = a$ and $y(a,b) = b$. Suppose that we can show that x and y restrict to C^∞-functions on S^1. Then the inclusion map $i : S^1 \to \mathbb{R}^2$, given by $i(p) = (x(p), y(p))$ is C^∞ on S^1, and so the composition $f\,|_{S^1} = f \circ i$ with be C^∞ on S^1 too.

Consider first the function x. We use the following atlas (U_i, ϕ_i) for S^1, where

$$\begin{aligned}U_1 &= \{(a,b) \in S^1 \mid b > 0\} \qquad \phi_1(a,b) = a\\ U_2 &= \{(a,b) \in S^1 \mid b < 0\} \qquad \phi_2(a,b) = a\\ U_3 &= \{(a,b) \in S^1 \mid a > 0\} \qquad \phi_3(a,b) = b\\ U_4 &= \{(a,b) \in S^2 \mid a < 0\} \qquad \phi_4(a,b) = b\end{aligned}$$

Since x is a coordinate function on U_1 and U_2, it is a coordinate function on $U_1 \cup U_2$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1} : \phi_3(U_3) \to \mathbb{R}$.

$$(x \circ \phi_3^{-1})(b) = x\left(\sqrt{1-b^2}, b\right) = \sqrt{1-b^2}.$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1-b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3. On U_4, we have

$$(x \circ \phi_4^{-1})(b) = x\left(-\sqrt{1-b^2}, b\right) = -\sqrt{1-b^2}.$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1. The proof that y is C^∞ on S^1 is similar.

Example 1.6. Let S^2 be the unit sphere with its smooth structure given in Example (0.12). Let's construct a smooth function on S^2. First note that

$$\phi_1(U_{16}) = \{(b,c) \in \mathbb{R}^2 \mid b^2 + c^2 < 1 \text{ and } c < 0\} \qquad \text{and} \qquad \phi_6(U_{16}) = \{(a,b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \text{ and } 0 < a\}.$$

Let $f : \phi_1(U_{16}) \to \mathbb{R}^2$ be given by

$$f(b,c) = b^2 + c^2.$$

Let's pullback $f : \phi_1(U_{16}) \to \mathbb{R}^2$ to $\phi_{16}^*(f) : \phi_6(U_{16}) \to \mathbb{R}^2$ using the transition function ϕ_{16}, where

$$\begin{aligned}
\phi_{16}(a,b) &= \phi_1 \circ \phi_6^{-1}(a,b) \\
&= \phi_1\left(a, b, \sqrt{1-b^2-a^2}\right) \\
&= \left(b, \sqrt{1-b^2-a^2}\right).
\end{aligned}$$

We have,

$$\begin{aligned}
\phi_{16}^*(f)(a,b) &= (f \circ \phi_{16})(a,b) \\
&= f\left(b, \sqrt{1-b^2-a^2}\right) \\
&= 1 - a^2.
\end{aligned}$$

1.4.1 Diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B_1(0)$

Let $\beta : \mathbb{R}^n \to B_1(0)$ be given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1+\sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1+\sum_{i=1}^n x_i^2}} \right) := \beta(x)$$

for all $x \in \mathbb{R}^n$. Then β is a diffeomorphism from $\mathbb{R}^n$ to $B_1(0)$ with inverse given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1-\sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1-\sum_{i=1}^n x_i^2}} \right) := \beta^{-1}(x)$$

for all $x \in B_1(0)$. Indeed, let us first check that $\beta(x) \in B_1(0)$:

$$\begin{aligned}
\|\beta(x)\| &= \sqrt{\left(\frac{x_1}{\sqrt{1+\sum_{i=1}^n x_i^2}}\right)^2 + \cdots + \left(\frac{x_n}{\sqrt{1+\sum_{i=1}^n x_i^2}}\right)^2} \\
&= \sqrt{\frac{\sum_{i=1}^n x_i^2}{1+\sum_{i=1}^n x_i^2}} \\
&< \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2}} \\
&= 1.
\end{aligned}$$

Thus $\beta(x) \in B_1(0)$. Next we check that β is smooth. This comes down to checking the component functions β_i are smooth:

$$x := (x_1, \dots, x_n) \mapsto \frac{x_i}{\sqrt{1+\sum_{i=1}^n x_i^2}} := \beta_i(x).$$

This follows from the fact that $1 + \sum_{i=1}^n x_i^2 > 0$. That β^{-1} is smooth follows by the same reasoning. Finally, checking that $\beta(\beta^{-1}(x)) = x$ is tedious but trivial:

$$\begin{aligned}
\beta(\beta^{-1}(x)) &= \frac{1}{\sqrt{1+\sum_{i=1}^n \left(\frac{x_i}{\sqrt{1-\sum_{i=1}^n x_i^2}}\right)^2}} \left(\frac{x_1}{\sqrt{1-\sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1-\sum_{i=1}^n x_i^2}} \right) \\
&= \frac{1}{\sqrt{1-\sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i^2}} (x_1, \dots, x_n) \\
&= (x_1, \dots, x_n).
\end{aligned}$$

1.4.2 A map smooth map from D^1 to S^2

We want to describe a smooth map from the unit ball D^1 to the unit sphere S^2. The idea will be to work in local charts and find a smooth map there. Then pull this back to a partial map from D^1 to S^2. Finally, we will need to extend it by a continuity argument.

Let (U, φ) be the chart in D^1 given by

$$U := \{x \in D^1 \mid \|x\| \neq 1 \text{ and } x \notin [0,1] \times \{0\}\} \qquad \text{and} \qquad \varphi(x) := \varphi(x_1, x_2) = (\arctan(x_2/x_1), \|x\|) := (\theta, r)$$

Here, U is the unit disk minus its boundary and the line segment from $(0,0)$ to $(1,0)$ and φ corresonds to polar coordinates. Let (V, ψ) be the chart in S^2 given by

$$V := \{x \in S^2 \mid x \notin [0,1] \times \{0\} \times \mathbb{R}^2\} \qquad \text{and} \qquad \psi(x) := \psi(x_1, x_2, x_3) = (\arctan(x_2/x_1), \arcsin(x_3)) := (\theta, \psi)$$

Here, V is the unit sphere minus the arc segment from $(0,1,0)$ to $(0,-1,0)$ and ψ corresponds to spherical coordaintes.

Now with respect to these coordinates, let $\Phi\colon \varphi(U) \to \varphi(V)$ be given by

$$\Phi(\theta, r) = (\theta, \pi(r - 1/2)).$$

Clearly this is a smooth map.