MATH 9500
FALL 2020
HOMEWORK 2

Due Monday, September 14, 2020

1. (5 pts) Show that the domain R is a PID if and only if every *prime* ideal is principal.

2. (5 pts) Let R be a commutative ring with identity. Show that the following conditions are equivalent.

(1) Every ascending chain of ideals of R

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

stabilizes (that is, there exists $n \in \mathbb{N}$ such that $I_n = I_{n+k}$ for all $k \in \mathbb{N}$).

(2) Every ideal of R is finitely generated.

If R is a commutative ring with identity we say that the chain of (proper) prime ideals

$$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_n$$

has *length* n (note that we begin counting at 0). We define the *Krull dimension* of R ($\dim(R)$)to be the supremum over all lengths of chains of prime ideals in R.

3. Let R be an integral domain and T an overring of R (that is, $R \subseteq T \subseteq K$ where K is the quotient field of R.

a) (5 pts) Show that R is a PID if and only if R is a UFD and $\dim(R) \leq 1$.

b) (5 pts) Show that if R is a UFD then any localization of R is a UFD.

d) (5 pts) Show that if R is a PID, then T is a localization of R.

e) (5 pts) Is the statement in d) true for UFDs? Prove or give a counterexample.