Uniqueness of Measure Extensions

Uniqueness of Extensions when Target Space is Hausdorff

Proposition 0.1. *Let X be a topological space and let $f\colon A \to Y$ be a continuous function from a dense subspace A of X to a Hausdorff space Y. If there exists a continuous extension of f to all of X, then it must be unique. In other words, suppose $\widetilde{f}_1\colon X \to Y$ and $\widetilde{f}_2\colon X \to Y$ are continuous functions such that*

$$\widetilde{f}_1|_A = f = \widetilde{f}_2|_A.$$

Then $\widetilde{f}_1 = \widetilde{f}_2$.

Proof. To prove uniqueness, assume for a contradiction that $\widetilde{f}_1\colon X \to Y$ and $\widetilde{f}_2\colon X \to Y$ are two continuous extensions of f such that $\widetilde{f}_1 \neq \widetilde{f}_2$. Choose $x \in X$ such that $\widetilde{f}_1(x) \neq \widetilde{f}_2(x)$. Since Y is Hausdorff, we may choose open neighborhoods V_1 and V_2 of $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ respectively such that $V_1 \cap V_2 = \emptyset$. Then $\widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$ is an open neighborhood of x, and so it must have a nonempty intersection with A. Choose $a \in A \cap \widetilde{f}_1^{-1}(V_1) \cap \widetilde{f}_2^{-1}(V_2)$. Then

$$\begin{aligned} f(a) &= \widetilde{f}_1(a) \\ &\in V_1. \end{aligned}$$

Similarly,

$$\begin{aligned} f(a) &= \widetilde{f}_2(a) \\ &\in V_2. \end{aligned}$$

Thus $f(a) \in V_1 \cap V_2$, which is a contradiction since V_1 and V_2 were chosen to disjoint from one another. □

Uniqueness of Extension for Measures

Proposition 0.2. *Let μ and ν be two finite measures defined on $\sigma(\mathcal{A})$ which coincide on $\mathcal{A}$. Then $\mu = \nu$.*

Proof. We first note that $\mathrm{d}_\mu = \mathrm{d}_\nu$ since μ and ν agree on $\mathcal{A}$. Indeed, let $A, B \in \sigma(\mathcal{A})$. Then we have

$$\begin{aligned} \mathrm{d}_\mu(A, B) &= \mu^*(A\Delta B) \\ &= \inf\left\{ \sum_{n=1}^\infty \mu(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A\Delta B \subseteq \bigcup_{n=1}^\infty E_n \right\} \\ &= \inf\left\{ \sum_{n=1}^\infty \nu(E_n) \mid (E_n) \text{ is a sequence in } \mathcal{A} \text{ such that } A\Delta B \subseteq \bigcup_{n=1}^\infty E_n \right\} \\ &= \nu^*(A\Delta B) \\ &= \mathrm{d}_\nu(A, B). \end{aligned}$$

Therefore d_μ and d_ν induce a common topology on $\sigma(\mathcal{A})$. Both $\mu\colon \sigma(\mathcal{A}) \to [0, \infty]$ and $\nu\colon \sigma(\mathcal{A}) \to [0, \infty]$ are continuous extensions of $\mu|_\mathcal{A} = \nu|_\mathcal{A}$ with respect to this common topology. Since $[0, \infty]$ is Hausdorff and since $\mathcal{A}$ is dense in $\sigma(\mathcal{A})$ with respect to this common topology, it follows from Proposition (0.1) that $\mu = \nu$. □