Linear Analysis Homework 5

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Throughout this homework, let $\mathcal{H}$ be a Hilbert space.

Problem 1

Proposition 0.1. *Let $T\colon \mathcal{H} \to \mathcal{H}$ and $S\colon \mathcal{H} \to \mathcal{H}$ be two bounded operators. Then*

$$(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^* \tag{1}$$

for all $\alpha, \beta, \in \mathbb{C}$.

Proof. Let $\alpha, \beta, \in \mathbb{C}$ and let $y \in \mathcal{H}$. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle x, (\alpha T + \beta S)^* y \rangle &= \langle (\alpha T + \beta S)x, y \rangle \\ &= \alpha \langle Tx, y \rangle + \beta \langle Sx, y \rangle \\ &= \alpha \langle x, T^* y \rangle + \beta \langle x, S^* y \rangle \\ &= \langle x, (\overline{\alpha} T^* + \overline{\beta} S^*) y \rangle \end{aligned}$$

In particular, this implies $(\alpha T + \beta S)^* y = (\overline{\alpha} T^* + \overline{\beta} S^*) y$ for all $y \in \mathcal{H}$ (by positive-definiteness of the inner-product) which implies (1). □

Problem 2

Proposition 0.2. *Let $T\colon \mathcal{H} \to \mathcal{H}$ and $S\colon \mathcal{H} \to \mathcal{H}$ be two bounded operators. Then*

1. *TS is bounded and $\|TS\| \leq \|T\|\|S\|$;*
2. *$(TS)^* = S^* T^*$.*

Proof.

1. Let $x \in \mathcal{H}$ such that $\|x\| = 1$. Then

$$\begin{aligned} \|TSx\| &\leq \|T\|\|Sx\| \\ &\leq \|T\|\|S\|\|x\| \\ &= \|T\|\|S\|. \end{aligned}$$

Thus TS is bounded and $\|TS\| \leq \|T\|\|S\|$.

2. Let $y \in \mathcal{H}$. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle x, (TS)^* y \rangle &= \langle TSx, y \rangle \\ &= \langle Sx, T^* y \rangle \\ &= \langle x, S^* T^* y \rangle. \end{aligned}$$

In particular, this implies $(TS)^* y = S^* T^* y$ for all $y \in \mathcal{H}$, which implies $(TS)^* = S^* T^*$. □

Problem 3

Proposition 0.3. *Let $u, v \in \mathcal{H}$ be fixed vectors.*

1. *The operator $T\colon \mathcal{H} \to \mathcal{H}$ defined by*

   $$Tx = \langle x, u \rangle v$$

   for all $x \in \mathcal{H}$ is bounded. Moreover, we have $\|T\| = \|u\|\|v\|$.

2. *The adjoint of T is given by*

   $$T^* y = \langle y, v \rangle u$$

   for all $y \in \mathcal{H}$.

Proof.

1. Let $x \in \mathcal{H}$. Then

$$\begin{aligned}\|Tx\| &= \|\langle x, u\rangle v\| \\ &= |\langle x, u\rangle| \|v\| \\ &\leq \|x\|\|u\|\|v\|,\end{aligned}$$

where we used Cauchy-Schwarz to get from the second to the third line. This implies $\|T\| \leq \|u\|\|v\|$. We have equality at the Cauchy-Schwarz step if and only if $x = \lambda u$ for some $\lambda \in \mathbb{C}$. In particular, setting $x = u/\|u\|$ gives us $\|T\| = \|u\|\|v\|$.

2. Let $y \in \mathcal{H}$. Then

$$\begin{aligned}\langle x, T^* y\rangle &= \langle Tx, y\rangle \\ &= \langle \langle x, u\rangle v, y\rangle \\ &= \langle x, u\rangle \langle v, y\rangle \\ &= \langle x, \overline{\langle v, y\rangle} u\rangle \\ &= \langle x, \langle y, v\rangle u\rangle\end{aligned}$$

for all $x \in \mathcal{H}$. This implies $T^* y = \langle y, v\rangle u$ for all $y \in \mathcal{H}$. $\square$

Problem 4

Corollary. *Let $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be operator defined by*

$$T(x)_n = \sum_{m=1}^{\infty} \frac{x_m}{2^n 3^m},$$

for all $x = (x_m) \in \ell^2(\mathbb{N})$, where $T(x)_n$ denotes the n-th coordinate of $T(x) \in \ell^2(\mathbb{N})$. Then T is bounded with

$$\|T\| = \sqrt{\frac{1}{24}}.$$

The adjoint of T is given by

$$T^*(y)_n = \sum_{m=1}^{\infty} \frac{y_m}{2^m 3^n},$$

for all $y \in \ell^2(\mathbb{N})$.

Proof. Set $u = (1/3^m)$ and $v = (1/2^n)$. Then

$$\begin{aligned}T(x)_n &= \sum_{m=1}^{\infty} \frac{x_m}{2^n 3^m} \\ &= \langle x, u\rangle \frac{1}{2^n} \\ &= \langle x, u\rangle v_n\end{aligned}$$

for all $x \in \mathcal{H}$. Thus $Tx = \langle x, u\rangle v$ for all $x \in \mathcal{H}$. Therefore we can apply Proposition (0.3) and obtain

$$\begin{aligned}\|T\| &= \|u\|\|v\| \\ &= \sqrt{\sum_{n=1}^{\infty} 9^{-n}} \sqrt{\sum_{n=1}^{\infty} 4^{-n}} \\ &= \sqrt{\left(\frac{1}{1-\frac{1}{9}} - 1\right)\left(\frac{1}{1-\frac{1}{4}} - 1\right)} \\ &= \sqrt{\frac{1}{24}}.\end{aligned}$$

The adjoint of T is given by

$$\begin{aligned} T^*(y)_n &= \langle y, v\rangle u_n \\ &= \sum_{m=1}^{\infty} \frac{y_m}{2^m 3^n} \end{aligned}$$

for all $y \in \mathcal{H}$. □

Problem 5

Proposition 0.4. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then*

1. $\|T^*T\| = \|T\|^2$;
2. $Ker(T^*T) = Ker(T)$.

Proof.

1. First note that Proposition (0.2) implies $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. For the reverse inequality, let $x \in \mathcal{H}$ such that $\|x\| = 1$. Then

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle x, T^*Tx\rangle \\ &\leq \|x\|\|T^*Tx\| \\ &= \|T^*Tx\|, \end{aligned}$$

where we used Cauchy-Schwarz to get from the second line to the third line. In particular, this implies

$$\begin{aligned} \|T\|^2 &= \sup\{\|Tx\|^2 \mid \|x\| \leq 1\} \\ &\leq \sup\{\|T^*Tx\| \mid \|x\| \leq 1\} \\ &= \|T^*T\|, \end{aligned}$$

where the first line is justifed in the Appendix.

2. Let $x \in \mathrm{Ker}(T)$. Then

$$\begin{aligned} T^*Tx &= T^*(Tx) \\ &= T^*(0) \\ &= 0 \end{aligned}$$

implies $x \in \mathrm{Ker}(T^*T)$. Thus $\mathrm{Ker}(T) \subseteq \mathrm{Ker}(T^*T)$.
For the reverse inclusion, let $x \in \mathrm{Ker}(T^*T)$. Then

$$\begin{aligned} \langle Tx, Tx\rangle &= \langle x, T^*Tx\rangle \\ &= \langle x, 0\rangle \\ &= 0 \end{aligned}$$

implies $Tx = 0$ (by positive-definiteness of inner-product) which implies $x \in \mathrm{Ker}(T)$. Therefore $\mathrm{Ker}(T) \supseteq \mathrm{Ker}(T^*T)$. □

Problem 6

Proposition 0.5. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then*

1. $Ker(T^*) = Im(T)^{\perp}$;
2. $Ker(T)^{\perp} = \overline{Im(T^*)}$.

Proof.

1. Let $x \in \mathrm{Ker}(T^*)$. Then

$$\begin{aligned} \langle Ty, x\rangle &= \langle y, T^*x\rangle \\ &= \langle y, 0\rangle \\ &= 0 \end{aligned}$$

for all $Ty \in \mathrm{Im}(T)$. This implies $x \in \mathrm{Im}(T)^{\perp}$ and so $\mathrm{Ker}(T^*) \subseteq \mathrm{Im}(T)^{\perp}$.
For the reverse inclusion, let $x \in \mathrm{Im}(T)^{\perp}$. Then

$$\begin{aligned} 0 &= \langle x, TT^*x \rangle \\ &= \langle T^*x, T^*x \rangle \end{aligned}$$

implies $T^*x = 0$ (by positive-definiteness of inner-product) which implies $x \in \mathrm{Ker}(T^*)$.

2. Let us first show that $\mathrm{Ker}(T)^{\perp}$ contains $\mathrm{Im}(T^*)$. Let $T^*y \in \mathrm{Im}(T^*)$. Then for all $x \in \mathrm{Ker}(T)$, we have

$$\begin{aligned} \langle x, T^*y \rangle &= \langle Tx, y \rangle \\ &= \langle 0, y \rangle \\ &= 0. \end{aligned}$$

In particular, this implies $\overline{\mathrm{Im}(T^*)} \subseteq \mathrm{Ker}(T)^{\perp}$ (as $\mathrm{Ker}(T)^{\perp}$ is a closed subspace which contains $\mathrm{Im}(T^*)$).
For the reverse inclusion, we have

$$\begin{aligned} \mathrm{Ker}(T)^{\perp} &= \mathrm{Ker}((T^*)^*)^{\perp} \\ &= (\mathrm{Im}(T^*)^{\perp})^{\perp} \\ &= (\overline{\mathrm{Im}(T^*)}^{\perp})^{\perp} \\ &= \overline{\mathrm{Im}(T^*)}, \end{aligned}$$

where we used part 1 of this proposition to get from the first line to the second line. □

Problem 7

Definition 0.1. An **isometry** between normed vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ is an operator $T \colon \mathcal{V}_1 \to \mathcal{V}_2$ such that

$$\|Tx - Ty\| = \|x - y\|$$

for all $x, y \in \mathcal{V}$.

Proposition 0.6. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be inner-product spaces and let $T \colon \mathcal{V}_1 \to \mathcal{V}_2$ be an operator. Then T is an isometry (where $\mathcal{V}_1$ and $\mathcal{V}_2$ are viewed as the induced normed vector spaces with respect to their inner-products) if and only if*

$$\langle x, y \rangle = \langle Tx, Ty \rangle \tag{2}$$

for all $x, y \in \mathcal{V}_1$.

Proof. Suppose (2) holds for all $x, y \in \mathcal{V}_1$. Then

$$\begin{aligned} \|Tx - Ty\| &= \sqrt{\langle Tx - Ty, Tx - Ty \rangle} \\ &= \sqrt{\langle Tx, Tx \rangle - \langle Tx, Ty \rangle - \langle Ty, Tx \rangle + \langle Ty, Ty \rangle} \\ &= \sqrt{\langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle} \\ &= \sqrt{\langle x - y, x - y \rangle} \\ &= \|x - y\|. \end{aligned}$$

for all $x, y \in \mathcal{V}_1$. Thus T is an isometry.
Conversely, suppose T is an isometry and let $x, y \in \mathcal{V}_1$. Then

$$\begin{aligned} \|x\|^2 - 2\mathrm{Re}(\langle x, y \rangle) + \|y\|^2 &= \langle x - y, x - y \rangle \\ &= \langle Tx - Ty, Tx - Ty \rangle \\ &= \|Tx\|^2 - 2\mathrm{Re}(\langle Tx, Ty \rangle) + \|Ty\|^2 \\ &= \|x\|^2 - 2\mathrm{Re}(\langle Tx, Ty \rangle) + \|y\|^2 \end{aligned}$$

implies $\mathrm{Re}(\langle x, y \rangle) = \mathrm{Re}(\langle Tx, Ty \rangle)$ for all $x, y \in \mathcal{V}_1$. Note that this also implies

$$\begin{aligned} \mathrm{Im}(\langle x, y \rangle) &= -\mathrm{Re}(i\langle x, y \rangle) \\ &= -\mathrm{Re}(\langle ix, y \rangle) \\ &= -\mathrm{Re}(\langle T(ix), Ty \rangle) \\ &= -\mathrm{Re}(i\langle Tx, Ty \rangle) \\ &= \mathrm{Im}(\langle Tx, Ty \rangle) \end{aligned}$$

for all $x, y \in \mathcal{V}_1$. Thus we have (2) for all $x, y \in \mathcal{V}_1$. □

Proposition 0.7. *Let $T: \mathcal{H} \to \mathcal{H}$ be a bounded operator. Then*

1. *T is an isometry if and only if $T^*T = 1_{\mathcal{H}}$.*
2. *There exists isometries T such that $TT^* \neq 1_{\mathcal{H}}$.*

Proof.

1. Suppose T is an isometry. Then for all $y \in \mathcal{H}$, we have

$$\begin{aligned}\langle x, 1_{\mathcal{H}} y\rangle &= \langle x, y\rangle \\ &= \langle Tx, Ty\rangle \\ &= \langle x, T^*Ty\rangle\end{aligned}$$

for all $x \in \mathcal{H}$. In particular, this implies $T^*Ty = 1_{\mathcal{H}}y$ for all $y \in \mathcal{H}$, which implies $T^*T = 1_{\mathcal{H}}$.
Conversely, suppose $T^*T = 1_{\mathcal{H}}$. Then

$$\begin{aligned}\langle Tx, Ty\rangle &= \langle x, T^*Ty\rangle \\ &= \langle x, 1_{\mathcal{H}} y\rangle \\ &= \langle x, y\rangle\end{aligned}$$

for all $x, y \in \mathcal{H}$. This implies T is an isometry.

2. Consider the shift operator $S: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, given by

$$S(x_n) = (x_{n-1})$$

for all $(x_n) \in \ell^2(\mathbb{N})$, where $x_0 = 0$. In class, it was shown that

$$S^*(x_n) = (x_{n+1})$$

for all $(x_n) \in \ell^2(\mathbb{N})$. Thus, whenever $x_1 \neq 0$, we have

$$\begin{aligned}SS^*(x_n) &= SS^*(x_1, x_2, \dots) \\ &= S(x_2, x_3, \dots) \\ &= (0, x_2, x_3, \dots) \\ &\neq (x_n).\end{aligned}$$

On the other hand, S is an isometry. Indeed, let $(x_n), (y_n) \in \ell^2(\mathbb{N})$. Then

$$\begin{aligned}\langle S(x_n), S(y_n)\rangle &= \langle (x_{n-1}), (y_{n-1})\rangle \\ &= \sum_{n=1}^{\infty} x_{n-1}\overline{y}_{n-1} \\ &= \sum_{m=0}^{\infty} x_m \overline{y}_m \\ &= x_0 y_0 + \sum_{m=1}^{\infty} x_m \overline{y}_m \\ &= \sum_{m=1}^{\infty} x_m \overline{y}_m \\ &= \langle (x_n), (y_n)\rangle.\end{aligned}$$

□

Appendix

Proposition 0.8. *Let $T: \mathcal{U} \to \mathcal{V}$ be a bounded linear operator. Then*

$$\|T\|^2 = \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}$$

Proof. For any $x \in \mathcal{U}$ such that $\|x\| \leq 1$, we have $\|Tx\|^2 \leq \|T\|^2$. Thus

$$\|T\|^2 \geq \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}. \tag{3}$$

To show the reverse inequality, we assume (for a contradiction) that (3) is a strictly inequality. Choose $\delta > 0$ such that

$$\|T\|^2 - \delta > \sup\{\|Tx\|^2 \mid \|x\| \leq 1\}.$$

Now let $\varepsilon = \delta/2\|T\|$, and choose $x \in \mathcal{U}$ such that $\|x\| \leq 1$ and such that

$$\|T\| - \varepsilon < \|Tx\|.$$

Then

$$\begin{aligned}
\|Tx\|^2 &> (\|T\| - \varepsilon)^2 \\
&= \|T\|^2 - 2\varepsilon\|T\| + \varepsilon^2 \\
&\geq \|T\|^2 - 2\varepsilon\|T\| \\
&= \|T\|^2 - \delta
\end{aligned}$$

gives us a contradiction. □