Duality, Canonical Modules, and Gorenstein Rings

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Suppose K is a field and that A is a local zero-dimensional ring that is a finite-dimensional K-algebra. If we wish to imitate the usual duality theory for vector spaces, we might at first try to work with the functor

$$M \mapsto M^{\star} := \mathrm{Hom}_A(M, A).$$

But this is often very badly behaved; for example, it does not usually preserve exact sequences, and if we do it twice we do not get the identity: That is, $\mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A) \ncong M$ in general.

Example 0.1. Let A be a ring and I an ideal in A. Then

$$\mathrm{Hom}_A(A/I, A) \cong \mathrm{Ann}_A(I) := \{x \in A \mid xI = 0\} \qquad \text{and} \qquad \mathrm{Hom}_A(\mathrm{Ann}_A(I), A))$$

Example 0.2. Let $A = K[x, y]/\langle x^2, xy^2, y^3\rangle$ and $I = \langle \overline{x}\rangle$. Then

$$\mathrm{Hom}_A(A/I, A) \cong \mathrm{Ann}(I) = \langle \overline{xy}, \overline{y}^2\rangle.$$

On the other hand, what is $\mathrm{Hom}_A(\langle \overline{xy}, \overline{y}^2\rangle, A)$? Well any $\varphi \in \mathrm{Hom}_A(\langle \overline{xy}, \overline{y}^2\rangle, A)$ is completely determined by where it maps $\overline{xy}$ and $\overline{y}^2$, but we must keep in mind that φ can't send these elements to *any* two elements in A, i.e. we have restrictions. For example, we cannot have $\overline{xy} \mapsto \overline{y}$ since

$$\begin{aligned} \varphi(\overline{x} \cdot \overline{xy}) &= \varphi(0) \\ &= 0 \\ &\neq \overline{xy} \\ &= \overline{x} \cdot \varphi(\overline{xy}). \end{aligned}$$

. Taking the dual again, $\mathrm{Hom}_A\left(\langle \overline{x}, \overline{y}^2\rangle, A\right)$

$$\begin{aligned} \mathrm{Hom}_A(A/I, A) &\cong \mathrm{Ann}_A(I) \\ &\cong \end{aligned}$$

$$\mathrm{Hom}_A(M, A) \cong \mathrm{Ann}\left(\langle x, y^3\rangle\right) = \langle x, y^2\rangle = Kx + Kxy + Ky^2,$$

but one can check that $\mathrm{Hom}_A\left(\langle x, y^2\rangle, A\right)$ is a six-dimensional K-vector space. So $A/I \ncong (A/I)^{\star\star}$.

A good duality theory may be defined in a different way: If M is a finitely generated A-module, we provisionally define the dual of M to be

$$D(M) = \mathrm{Hom}_K(M, K)$$

(we shall give a more intrinsic definition shortly). The vector space $D(M)$ is naturally an A-module by the action

$$(a \cdot \varphi)(m) = \varphi(am)$$

for all $\varphi \in D(M)$, $a \in A$, and $m \in M$.

Example 0.3. Returning to Example (0.8), we see that $D(A/I) = K\varphi_x + K\varphi_{xy}$, where $\varphi_x(x) = 1$, $\varphi_x(xy) = 0$ and $\varphi_{xy}(x) = 0$ and $\varphi_{xy}(xy) = 1$. Then $x \cdot \varphi_x = 0$ since

$$\begin{aligned} x \cdot \varphi_x(xy) &= \varphi_x(x^2y) \\ &= \varphi_x(0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} x \cdot \varphi_x(x) &= \varphi_x(x^2) \\ &= \varphi_x(0) \\ &= 0. \end{aligned}$$

Similarly, one can show that $x \cdot \varphi_{xy} = 0$, $y \cdot \varphi_x = 0$, and $y \cdot \varphi_{xy} = \varphi_x$.

With D defined above, we see that D a contravariant functor from the category of finitely generated A-modules to itself. Since M is finite-dimensional over K, the natural map $M \to D(D(M))$ sending $m \in M$ to the functional $\widehat{m} : \varphi \mapsto \varphi(m)$, for $\varphi \in \operatorname{Hom}_K(M, K)$ is an isomorphism of vector spaces. In fact, it is an isomophism of A-modules. Indeed, we have $\widehat{am} = a \cdot \widehat{m}$ since

$$\begin{aligned} (a \cdot \widehat{m})(\varphi) &= \widehat{m}(a \cdot \varphi) \\ &= (a \cdot \varphi)(m) \\ &= \varphi(am) \\ &= \widehat{am}(\varphi) \end{aligned}$$

for all $\varphi \in D(M)$. Since K is a field, D is **exact** in the sense that it takes exact sequences to exact sequences (with arrows reversed). Thus D is a **dualizing functor** on the category of finitely generated A-modules.

To get an idea of how D acts, note first that if $\mathfrak{p}$ is a maximal ideal of A, then any dualizing functor D takes the simple module $A/\mathfrak{p}$ to itself. Indeed, $D(A/\mathfrak{p})$ must be simple, because else it would have a proper factor module M and $D(M)$ would be a proper submodule of $A/\mathfrak{p}$. As A is local, it has only one simple module, and thus $D(A/\mathfrak{p}) \cong A/\mathfrak{p}$. Since D takes exact sequences to exact sequences, reversing the arrows, D "turns composition series upside down" in the sense that if

$$0 \subset M_1 \subset \cdots \subset M_n \subset M$$

is a chain of modules with simple quotients $M_i/M_{i-1} \cong A/\mathfrak{p}$, then

$$D(M) \supset D(M_n) \supset \cdots \supset D(M_1) \supset D(0) = 0$$

is a chain of surjections whose kernels N_i are simple. In particular, for any module of finite length, then length of $D(M)$ equals the length of M.

A central role in the theory of modules over a local ring $(A, \mathfrak{p})$ is played by what might be thought of as the **top** of a module M, defined to be the quotient $\operatorname{Top}(M) := M/\mathfrak{p}M$; Nakayama's lemma shows that this quotient controls the generators of M. It could be defined categorically as the largest quotient of M that is a direct sum of simple modules. That is,

$$M/\mathfrak{p}M = \bigoplus_i (A/\mathfrak{p}).$$

The dual notion is that of the **socle** of M, denoted $\operatorname{Soc}(M)$: It is defined as the annihilator in M of the maximal ideal $\mathfrak{p}$, or equivalently, as the sum of all the simple submodules of M. Note that since the top of A is $A/\mathfrak{p}$, a simple module, the socle of $D(A)$ must be a simple module as well.

Example 0.4. Let $A = K[x,y]/\langle x^2, y^3 \rangle$. Then $\operatorname{Soc}(A) = Kxy^2$ and $\operatorname{Top}(A) = K$. To calculate $D(A)$, we first write A as a K-vector space:

$$A = K + Kx + Ky + Kxy + Ky^2 + Kxy^2.$$

Then a dual basis for $D(A)$ is given by

$$D(A) = K\varphi_1 + K\varphi_x + K\varphi_y + K\varphi_{xy} + K\varphi_{y^2} + K\varphi_{xy^2}.$$

Then one can check that $\operatorname{Soc}(D(A)) = K\varphi_1$ and $\operatorname{Top}(D(A)) = K\varphi_{xy^2}$.

Remark. This remark is for those who are familiar with the Koszul Complex construction. Let $(A, \mathfrak{p})$ be a local ring and suppose $\mathfrak{p} = \langle x_1, \ldots, x_n \rangle$. Then

$$H_n(K(x_1, \ldots, x_n; M) \cong \mathrm{Soc}(M)$$
$$H_0(K(x_1, \ldots, x_n; M) \cong \mathrm{Top}(M)$$

Any dualizing functor preserves endomorphism rings; more generally, $\mathrm{Hom}_A(D(M), D(N)) \cong \mathrm{Hom}_A(N, M)$. In particular, $D(A)$ is a module with endomorphism ring A. To see this, consider the mappings given by applying D :

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M)) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M)).$$

Since $D^2 \cong 1$, the composite of two successive maps in this sequence is an isomorphism, so each of the maps is an isomorphism too. For instance, suppose $\varphi \in \mathrm{Hom}_A(M, N)$ was in the first map, that is, $D(\varphi) = 0$. Then $D^2(\varphi) = 0$ implies $\varphi = 0$ since D^2 is an isomorphism, which shows the map $D : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M))$ is injective. Next, suppose $\varphi \in \mathrm{Hom}_A(D(N), D(M))$. Since D^2 is an isomorphism, there exists a $\psi \in \mathrm{Hom}_A(D(N), D(M)$ such that $D^2(\psi) = \varphi$. Then $D(\psi) \in \mathrm{Hom}_A(M, N)$ and $D(D(\psi)) = \varphi$, which shows the map $D : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M))$ is surjective.

Proposition 0.1. *Let $(A, \mathfrak{m})$ be a local zero-dimensional ring. If E is any dualizing functor from the category of finitely generated A-modules to itself, then there is an isomorphism of functors $E(-) \cong Hom_A(-, E(A))$. Further, $E(A)$ is isomorphic to the injective hull of $A/\mathfrak{m}$. Thus there is up to isomorphism at most one dualizing functor.*

Proof. Since $E^2 \cong 1$ as functors, the map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(E(N), E(M))$ given by $\varphi \mapsto E(\varphi)$ is an isomorphism. Thus, there is an isomorphism, functorial in M,

$$E(M) \cong \mathrm{Hom}_A(A, E(M)) \cong \mathrm{Hom}_A(E(E(M)), E(A)) \cong \mathrm{Hom}_A(M, E(A)).$$

This proves the first statement.

Since A is projective, $E(A)$ is injective. As we observed above, A has a simple top, so $E(A)$ has a simple socle. Because A is zero-dimensional, every module contains simple submodules. The socle of a module M contains all the simple submodules of M, and thus meets every submodule of M; that is, it is an essential submodule of M. Since $A/\mathfrak{m}$ appears as an essential submodule of $E(A)$, we see that $E(A)$ is an injective hull of $A/\mathfrak{m}$. □

With Proposition (0.1) for justification, we define the **canonical module** ω_A of a local zero-dimensional ring A to be the injective hull of the residue class field of A. By Proposition (0.1), any **dualizing functor** D on the category of finitely generated A-modules must be $D(M) := \mathrm{Hom}_A(M, \omega_A)$, and in fact this functor is always dualizing.

Proposition 0.2. *Let $(A, \mathfrak{m})$ be a local zero-dimensional ring. Then the functor $M \mapsto D(M) := Hom_A(M, \omega_A)$ is a dualizing functor on the category of finitely generated A-modules.*

Proof. The

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M))$$

is given by $\lambda \mapsto \lambda^\star$, where $\lambda^\star(\varphi) = \varphi \circ \lambda$, $\lambda \in \mathrm{Hom}_A(M, N)$ and $\varphi \in D(N)$. Then $(a\lambda + b\mu)^\star = a\lambda^\star + b\mu^\star$ for all $a, b \in A$ and $\lambda, \mu \in \mathrm{Hom}_A(M, N)$ shows D is A-linear. Also, D is exact because ω_A is injective. Thus it suffices to show that D^2 is isomorphic to the identity. Let $\alpha : 1 \to D^2$ be the natural transformation given by maps

$$\alpha_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, \omega_A), \omega_A)$$

given by mapping $m \mapsto \widehat{m}$ where $m \in M$ and $\widehat{m}$ is the homomorphism taking $\varphi \in \mathrm{Hom}_A(M, \omega_A)$ to $\varphi(m)$. We shall show that α is an isomorphism by showing that each α_M is an isomorphism.

We do induction on the length of M. First suppose that the length is 1, so that $M = A/\mathfrak{p}$, where $\mathfrak{p}$ is the maximal ideal of A. Since ω_A is the injective hull of $A/\mathfrak{p}$, the socle of ω_A is $A/\mathfrak{p}$, and we have $\mathrm{Hom}_A(A/\mathfrak{p}, \omega_A) = A/\mathfrak{p}$, generated by any nonzero map $A/\mathfrak{p} \to \omega_A$. Thus $\mathrm{Hom}_A(\mathrm{Hom}_A(A/\mathfrak{p}, \omega_A), \omega_A) = A/\mathfrak{p}$, generated by any nonzero map. But if $1 \in A/\mathfrak{p}$ is the identity, then the map induced by 1 takes the inclusion $A/\mathfrak{p} \hookrightarrow \omega_A$ to the image of 1 under that inclusion, and is thus nonzero, so $\alpha_{A/\mathfrak{p}}$ is an isomorphism.

If the length of M is greater than 1, let M' be any proper submodule and let $M'' = M/M'$. By the naturality of α and the exactness of D^2 it follows that there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\alpha'_M"] & M \arrow[r] \arrow[d, "\alpha_M"] & M'' \arrow[r] \arrow[d, "\alpha'_M"] & 0 \\
0 \arrow[r] & D^2(M') \arrow[r] & D^2(M) \arrow[r] & D^2(M'') \arrow[r] & 0
\end{tikzcd}$$

Both M' and M'' have lengths stricly less than the length of M, so the left-hand and right-hand vertical maps are isomorphisms by induction. It follows by an easy diagram chase that the middle map α_M is an isomorphism too. □

Corollary. *Let A be a local Artinian ring. Then the annihilator of ω_A is 0; the length of ω_A is the same as the length of A; and the endomorphism ring of ω_A is A.*

Proof. The dualizing functor preserves annihilators, lengths, and endomorphism rings, and takes A to ω_A. □

Proposition 0.3. *Let A be a zero-dimensional local ring. Suppose that for some local ring B, A is a B-algebra that is finitely generated as a B-module and the maximal ideal of B maps into that of A. If E is the injective hull of the residue class field of B, then*

$$\omega_A = Hom_B(A, E).$$

In particular, if B is also zero-dimensional, then $\omega_A = Hom_B(A, \omega_B)$.

Proof. $\mathrm{Hom}_B(A, E)$ is an injective A-module (see my notes on injective modules for a proof of this). To show that it is the injective hull of the residue class field k of A, it suffices to show that it is an essential extension of the residue class field k of A. Let $\mathfrak{p}$ be the maximal ideal of A, and $\mathfrak{p}_B$ be the maximal ideal of B. The preimage of $\mathfrak{p}$ is a maximal ideal in B which contains $\mathfrak{p}_B$, so it must be $\mathfrak{p}_B$. Therefore there is an induced homomorphism of the residue class field k_B of B to k. As k is a finite dimensional vector space over k_B, we have $k = \omega_k \cong \mathrm{Hom}_{k_B}(k, k_B)$ as k-modules.

Let $S \subset \mathrm{Hom}_B(A, E)$ be the A-submodule of homomorphisms whose kernel contains $\mathfrak{p}$, or equivalently, such that $\mathfrak{p}\varphi = 0$. The module S is the socle of $\mathrm{Hom}_B(A, E)$ as an A-module. If $\varphi \in S$, then since $\mathfrak{p}_B A \subset \mathfrak{p}$, the image of φ is annihilated by $\mathfrak{p}_B$; that is, the image of φ is in the socle of E as a B-module, and since E is the injective hull of k_B, this is k_B. Since the homomorphisms in S all factor through the projection $A \to A/\mathfrak{p} = k$, we have $S \cong \mathrm{Hom}_B(k, k_B) \cong k$.

If $\varphi : A \to E$ is any B-module homomorphism, then since $\mathfrak{p}$ is nilpotent, φ is annihilated by a power of $\mathfrak{p}$, and thus there is a multiple $a\varphi \neq 0$ that is annihilated by $\mathfrak{p}$. Thus S is an essential A-submodule of $\mathrm{Hom}_B(A, E)$, as required. □

Definition 0.1. A zero-dimensional local ring A is **Gorenstein** if $A \cong \omega_A$.

Example 0.5. Let $A = K[x, y]/\langle x^2, y^3 \rangle$. Then both A and $D(A)$ are graded K-modules, with homogeneous components being

$$\begin{aligned}
A_0 &= K & D(A)_0 &= K\varphi_1 \\
A_1 &= K\overline{x} + K\overline{y} & (S_{I_\Delta^{\mathrm{sq}}})_2 &= K\varphi_x + K\varphi_y \\
A_2 &= K\overline{xy} + K\overline{y}^2 & (S_{I_\Delta^{\mathrm{sq}}})_1 &= K\varphi_{xy} + K\varphi_{y^2} \\
A_3 &= K\overline{xy}^2 & (S_{I_\Delta^{\mathrm{sq}}})_0 &= K\varphi_{xy^2}
\end{aligned}$$

where

$$\varphi_{\mathbf{x}^\alpha}\left(\overline{\mathbf{x}}^\beta\right) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else .} \end{cases}$$

One can check that $A \cong D(A)$ by the map

$$\begin{aligned}
1 &\mapsto \varphi_{xy^2} \\
x &\mapsto \varphi_{y^2} \\
y &\mapsto \varphi_{xy} \\
xy &\mapsto \varphi_y \\
y^2 &\mapsto \varphi_x \\
xy^2 &\mapsto \varphi_1
\end{aligned}$$

So A is a Gorenstein ring.

Proposition 0.4. *Let $(A, \mathfrak{m})$ be a zero-dimensional local ring. The following are equivalent.*

1. *A is Gorenstein.*
2. *A is injective as an A-module.*
3. *The socle of A is simple.*
4. *ω_A can be generated by one element.*

Proof.

$(1 \implies 2)$: ω_A is injective as an A-module, so if $A \cong \omega_A$, then A is injective as an A-module.

$(2 \implies 3)$: As A is a local ring, it is indecomposable as an A-module (Proof: Suppose $A \cong I \oplus J$ for two proper submodules $I, J \subset A$ (i.e. ideals). This implies there exists $x \in I$ and $y \in J$ such that $x + y = 1$. But since $\mathfrak{m}$ is the unique maximal ideal of A, we have $I \subset \mathfrak{m}$ and $J \subset \mathfrak{m}$, and so $1 = x + y \in \mathfrak{m}$ leads to a contradiction.) Since

$$\mathrm{Soc}(A) \subset \bigcup_{n=1}^{\infty} 0 :_A \mathfrak{m}^n = A$$

is an essential extension, if A is injective as an A-module, then it must be the injective hull of its socle. The injective hull of a direct sum is the direct sum of the injective hulls of the summands, so the socle must be simple.

$(3 \implies 4)$: If the socle of A is simple, that is, isomorphic to $A/\mathfrak{m}$, then the "top" of the dual of A, that is the top of ω_A, which is $\omega_A/\mathfrak{m}\omega_A$, is simple. By Nakayama's lemma ω_A can be generated by one element.

$(4 \implies 1)$: If ω_A is generated by one element then it is a homomorphic image of A. But A and ω_A have the same length by Proposition (0.2), so $A \cong \omega_A$. □

Example 0.6. Let $A = K[x, y, z]/\langle x^2, y^2, xz, yz, z^2 - xy\rangle$. Then A is a 0-dimensional Gorenstein ring that is not a complete intersection ring. In more detail: a basis for A as a K-vector space is

$$A = K + Kx + Ky + Kz + Kz^2$$

The ring A is Gorenstein because the socle has dimension 1 as K-vector space, namely $\mathrm{Soc}(A) = Kz^2$. Finally, A is not a complete intersection because it has 3 generators and a minimal set of 5 relations.

Most of the common methods of constructing Gorenstein rings work just as well in the case where A is not zero-dimensional, and we shall postpone them for a moment. However, one technique, Macaulay's method of **inverse systems**, is principally of interest in the zero-dimensional case.

Let $S = K[x_1, \ldots, x_r]$. For each $d \geq 0$, let S_d be the vector space of forms of degree d in the x_i. Let $T = K[x_1^{-1}, \ldots, x_r^{-1}] \subset K(A) = K(x_1, \ldots, x_r)$ be the polynomial ring on the inverses of the x_i. We make T into an S-module as follows: Let x^α be a monomial in A and x^β be a monomial in T, where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r_{\geq 0}$ and $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r_{\leq 0}$. Then

$$x^\alpha \cdot x^\beta = \begin{cases} 0 & \text{if } \alpha_i > \beta_i \text{ for some } i \\ x^{\alpha+\beta} & \text{else.} \end{cases}$$

Theorem 0.1. *With the notation above, there is a one-to-one inclusion reversing correspondence between finitely generated S-modules $M \subset T$ and ideal $I \subset S$ such that $I \subset \langle x_1, \ldots, x_r\rangle$ and A/I is a local zero-dimensional ring, given by*

$$M \mapsto (0 :_S M), \text{ the annihilator of } M \text{ in } S.$$
$$I \mapsto (0 :_T I), \text{ the submodule of } T \text{ annihilated by } I.$$

Proof. The S-module T may be identified with the graded dual $\bigoplus_d \mathrm{Hom}_K(S_d, K)$ of S; indeed the dual basis vector to $x^\alpha \in S_d$ is $x^{-\alpha} \in T$. Moreover, the graded dual is the injective hull of $K = S/\langle x_1, \ldots, x_r\rangle$ as an S-module. □

Canonical Modules and Gorenstein Rings in Higher Dimension

Definition 0.2. Let A be a local Cohen-Macaulay ring. A finitely generated A-module ω_A is a **canonical module for** A if there is a nonzerodivisor $x \in A$ such that $\omega_A / x\omega_A$ is a canonical module for $A/\langle x\rangle$. The ring A is **Gorenstein** if A is itself a canonical module; that is, A is Gorenstein if there is a nonzerodivisor $x \in A$ such that $A/\langle x\rangle$ is Gorenstein.

The induction in this definition terminates because $\dim(A/\langle x\rangle) = \dim(A) - 1$. We may easily unwind the induction, and say that ω_A is a canonical module if some maximal regular sequence $x_1, \ldots, x_d$ on A is also an ω_A-sequence, and $\omega_A/\langle x_1, \ldots, x_d\rangle\omega_A$ is the injective hull of the residue class field of $A/\langle x_1, \ldots, x_d\rangle$. Similarly, A is Gorenstein if and only if $A/\langle x_1, \ldots, x_d\rangle$ is a zero-dimensional Gorenstein ring for some maximal regular sequence $x_1, \ldots, x_d$. By Nakayama's lemma and Proposition (0.4), this is the case if and only if A has a canonical module generated by one element.

For a simple example, consider the case when A is a regular local ring. We claim that A has a canonical module, and in fact $\omega_A = A$. When $\dim(A) = 0$ the result is obvious, since A is a field. For the general case we do inductino on the dimension. If we choose x in the maximal ideal of A, but not its square, then x is a nonzerodivisor and A/x is again a regular local ring, so A/x is a canonical module for A/x . Therefore A is a canonical module for A, by defintion.

There are three problems with these notions. First, it is not at all obvious from the definitions that they are independent of the nonzero divisor x that was chosen. Second, something called a canonical module should at least be unique, and uniqueness is not clear either. Our first goal is to show that this independence and uniqueness do hold.

The third problem is that it is not obvious that a canonical module should even exist. Here we are not quite so lucky: There are local Cohen-Macaulay rings with no canonical module. However, our second goal will be to establish that canonical modules do exist for any Cohen-Macaulay rings that are homomorphic images of regular local rings (and a little more generally). This includes complete local rings and virtually all other rings of interest in algebraic geometry and number theory.

Example 0.7. Let $A = K[x,y,z]_{\langle x,y,z\rangle}/\langle xy, xz, yz\rangle$. Then $x+y+z$ is a nonzerodivisor in A, and

$$A/\langle x+y+z\rangle = K[x,y,z]_{\langle x,y,z\rangle}/\langle x+y+z, xy, xz, yz\rangle \cong K[y,z]_{\langle y,z\rangle}/\langle y^2, yz, z^2\rangle = K + Ky + Kz,$$

which does not have a simple socle, so this is not Gorenstein.

Example 0.8. Let $A = K[x,y,z]_{\langle x,y,z\rangle}/\langle x+y+z, xz, yz\rangle$. Then $x+y+z$ is a nonzerodivisor in A, and

$$A/\langle x+y+z\rangle = K[x,y,z]_{\langle x,y,z\rangle}/\langle x+y+z, xy, xz, yz\rangle \cong K[y,z]_{\langle y,z\rangle}/\langle y^2, yz, z^2\rangle = K + Ky + Kz,$$

which does not have a simple socle, so this is not Gorenstein.

Maximal Cohen-Macaulay Modules

Proposition 0.5. *Let A be a local ring of dimension d, and let M be a finitely generated A-module. The following conditions are equivalent:*

1. *Every system of parameters in A is an M-sequence.*
2. *Some system of parameters in A is an M-sequence.*
3. *depth$(M) = d$.*

If these conditions are satisfied, we say that M is a ***maximal Cohen-Maculay module over*** *A. Every element outside the minimal primes of A is a nonzero divisor on M.*

Proof. The implications $1 \Longrightarrow 2 \Longrightarrow 3$ are immediate from the definitions. Suppose $\operatorname{depth}(M) = d$. If $x_1, \ldots, x_d$ is a system of parameters then $Q = \langle x_1, \ldots, x_d\rangle$ is $\mathfrak{m}$-primary. In particular, $\sqrt{Q} = \mathfrak{m}$. Therefore

$$\operatorname{depth}(Q, M) = \operatorname{depth}(\mathfrak{m}, M) = d,$$

which implies $x_1, \ldots, x_d$ is a regular sequence on M.

To prove the last statement, note that if x_1 is not in any minimal prime of A, then $\dim(A/x_1) = \dim(A) - 1$, so a system of parameters mod x_1 may be lifted to a system of parameters for A beginning with x_1. Thus, x_1 is a nonzero divisor on M. $\square$

Corollary. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, $Q = \langle x_1, \ldots, x_d \rangle$ and $\mathfrak{m}$-primary ideal, and M a maximal Cohen-Macaulay module over A. Then*
$$Gr_{\mathfrak{q}}(M) \cong Gr_{\mathfrak{q}}(A) \otimes_A M.$$

In case A is zero-dimensional, all finitely generated modules are maximal Cohen-Macaulay modules. On the other hand, if A is a regular local ring, then by the Auslander-Buchsbaum formula, the maximal Cohen-Macaulay A-modules are exactly the free A-modules.

More generally, if A is a finitely generated module over some regular local ring S of dimension d, then by the Auslander-Buchsbaum theorem, the maximal Cohen-Macaulay modules over A are those A-modules that are free as S-modules. Thus maximal Cohen-Macaulay modules may be thought of as representations of A as a ring of matrices over a regular local ring–as such they generalize the objects studied in integral representation theory of finite groups under the name **lattices**. We shall exploit the following example. If $B = A/J$ is a homomorphic image of A such that B is again Cohen-Macaulay of dimension d as a ring, then B is a Cohen-Macaulay A-module.

Modules of Finite Injective Dimension

Lemma 0.2. *Let A be a ring, B an A-algebra, E an injective A-module, and P a projective B-module. Then $Hom_A(P, E)$ is an injective B-module, where B acts on $Hom_A(P, E)$ by*

$$(b \cdot \varphi)(x) = \varphi(bx),$$

where $b \in B$, $\varphi \in Hom_A(P, E)$, and $x \in P$.

Proof. We need to show that $\mathrm{Hom}_B(-, \mathrm{Hom}_A(P, E))$ is exact. From the universal property of tensor products, we have
$$\mathrm{Hom}_B(-, \mathrm{Hom}_A(P, E)) \cong \mathrm{Hom}_A(- \otimes_B P, E).$$
Now notice that the functor $- \otimes_B P$ is exact since P is projective. As E is injective, it follows that $\mathrm{Hom}_A(-, E)$ is exact. Combine both to obtain the result. $\square$

Proposition 0.6. *Let A be a ring, M be an A-module, $x \in A$ be an A-regular and M-regular element, and*

$$\mathcal{E} : E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2 \longrightarrow \cdots$$

be a minimal injective resolution of M. Set $E_i' := Hom_A(A/x, E_i) \cong \{m \in E_i \mid xm = 0\}$. The complex

$$\mathcal{E}' : E_1' \xrightarrow{\varphi_2} E_2' \xrightarrow{\varphi_3} E_3' \longrightarrow \cdots$$

is a minimal injective resolution of M/xM over A/x. Thus,

$$id_{A/x}(M/xM) = id_A(M) - 1.$$

and if N is an A-module annihilated by x, then

$$Ext_A^{i+1}(N, M) \cong Ext_{A/x}^i(N, M/xM)$$

for all $i \geq 0$.

Proof. Lemma (0.2) tells us that E_i' are injective (A/x)-modules. The homology of the complex

$$\mathrm{Hom}_A(A/x, \mathcal{E}) : E_0' \xrightarrow{\varphi_1} E_1' \xrightarrow{\varphi_2} E_2' \xrightarrow{\varphi_3} E_3' \longrightarrow \cdots$$

is by definintion $\mathrm{Ext}_A^*(A/x, M)$. On the other hand, M is an essential submodule of E_0, and M contains no submodule annihilated by x, so E_0 contains no submodule annihilated by x. Thus $E_0' = 0$, and we see that $\mathrm{Hom}_A(A/x, \mathcal{E}) = \mathcal{E}'$.

Computing $\mathrm{Ext}_A^*(A/x, M)$ instead from the free resolution

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/x \longrightarrow 0,$$

we see that $\mathrm{Ext}^1_A(A/x, M) = M/xM$ while $\mathrm{Ext}^i_A(A/x, M) = 0$ for $i \neq 1$. Thus, $\mathcal{E}'$ is an injective resolution of M/xM. Note that the numbering of the terms of $\mathcal{E}'$ is such that $\mathrm{Ext}^i_{A/x}(N, M/xM)$ is the homology of $\mathrm{Hom}_{A/x}(N, \mathcal{E}')$ at $\mathrm{Hom}_{A/x}(N, E'_{j+1})$; strictly speaking, we should say that $\mathcal{E}'[1]$ is an injective resolution of M/xM.

To see that $\mathcal{E}'$ is minimal, note that $\mathrm{Ker}\left(\varphi_{n+1} : E'_n \to E'_{n+1}\right)$ is the intersection of the essential submodule $\mathrm{Ker}\left(\varphi_{n+1} : E_n \to E_{n+1}\right)$ with E'_n, and is thus essential in E'_n. It follows at once that $\mathrm{id}_{A/x}(M/xM) = \mathrm{id}_A(M) - 1$.

If x annihilates the A-module N, then every map from N to an E_i has image killed by x, so

$$\mathrm{Hom}_A(N, \mathcal{E}) = \mathrm{Hom}_A(N, \mathcal{E}') = \mathrm{Hom}_{A/x}(N, \mathcal{E}').$$

Taking homology, and taking into account the shift in numbering, we get the last statement of the proposition. □

Remark. Recall that if $(A, \mathfrak{m})$ is a local ring, M is an A-module, and $x \in \mathfrak{m}$ is A-regular and M-regular, then $\mathrm{pd}_{A/x}(M/xM) = \mathrm{pd}_A(M)$. The idea behind that proof was to start with a minimal projective resolution of M,

$$\mathcal{P} : \cdots \longrightarrow P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0$$

and show that the sequence

$$\mathcal{P} \otimes (A/x) : \cdots \longrightarrow P_2/xP_2 \xrightarrow{\overline{\varphi}_2} P_1/xP_1 \xrightarrow{\overline{\varphi}_1} P_0/xP_0$$

was a minimal projective resolution of M/xM.

To exploit this result, we need to know the modules of finite injective dimension over a zero-dimensional ring.

Proposition 0.7. *Let A be a local Cohen-Macaulay ring. If M is a maximal Cohen-Macaulay module of finite injective dimension, then $id_A(M) = dim(A)$. If $dim(A) = 0$, then M is a direct sum of copies of ω_A, and $M \cong \omega_A$ if and only if $End_A(M) = A$.*

Proof. Suppose first that $\dim(A) = 0$. Let $D = \mathrm{Hom}_A(-, \omega_A)$ be the dualizing functor. If M has finite injective dimension, then applying D to an injective resolution of M we see that $D(M)$ is a module of finite projective dimension, and is thus free by the Auslander-Buchsbaum formula. Applying D again we see that $M \simeq D^2(M)$ is a direct sum of copies of $D(A) = \omega_A$. Using D, we see that the endomorphism ring of ω_A^n is the same as the endomorphism ring of A^n. Thus it is equal to A if and only if $n = 1$.

If $\dim(A) = d$ is arbitrary, then we may choose a regular sequence $x_1, \ldots, x_d$ of A that is a regular sequence on M, and use Proposition (0.6) to conclude that

$$\begin{aligned} \mathrm{id}_A(M) &= d + \mathrm{id}_{A/\langle x_1,\ldots,x_d\rangle}\left(M/\langle x_1, \ldots, x_d\rangle M\right) \\ &= d + 0 \\ &= d. \end{aligned}$$

□

Remark. We can generalize some of Proposition (**??**) as follows: Let A be a local Noetherian ring, and M be a finitely generated A-module of finite injective dimension. Then

$$\dim(M) \leq \mathrm{id}_A(M) = \mathrm{depth}(A).$$

For a proof of this, see my notes on Injective Modules.

Proposition 0.8. *Let A be a local Cohen-Macaulay ring of dimension d, and let M be a maximal Cohen-Macaulay module of finite injective dimension.*

1. *If N is a finitely generated A-module of depth e, then $Ext^j_A(N, M) = 0$ for $j > d - e$.*

2. *If x is a nonzerodivisor on M, then x is a nonzerodivisor on $Hom_A(N, M)$. If N is also a maximal Cohen-Macaulay module, then*

   $$Hom_A(N, M)/xHom_A(N, M) \cong Hom_{A/x}(N/xN, M/xM)$$

   by the homomorphism taking the class of a map $\varphi : N \to M$ to the map $N/xN \to M/xM$ induced by φ.

Proof.

1. We do induction on e. By Proposition (0.7), the injective dimension of M is d, so that $\mathrm{Ext}^j_A(N, M) = 0$ for any N if $j > d$. This gives the case $e = 0$. Now suppose $e > 0$, and let x be a nonzerodivisor on N that lies in the maximal ideal of A. From the short exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot x} N \longrightarrow N/xN \longrightarrow 0$$

we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^j_A(N, M) \xrightarrow{\cdot x} \mathrm{Ext}^j_A(N, M) \longrightarrow \mathrm{Ext}^{j+1}_A(N/xN, M) \longrightarrow \cdots$$

The module N/xN has depth $e - 1$, so by induction $\mathrm{Ext}^{j+1}_A(N/xN, M)$ vanishes if $j + 1 > d - (e - 1)$, that is, if $j > d - e$. By Nakayama's lemma, $\mathrm{Ext}^j_A(N, M)$ vanishes if $j > d - e$.

2. From the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow M/xM \longrightarrow 0$$

we derive a long exact sequence beginning

$$0 \longrightarrow \mathrm{Hom}_A(N, M) \xrightarrow{\cdot x} \mathrm{Hom}_A(N, M) \longrightarrow \mathrm{Hom}_A(N, M/xM) \longrightarrow \mathrm{Ext}^1_A(N, M) \longrightarrow \cdots$$

Thus x is a nonzerodivisor on $\mathrm{Hom}_A(N, M)$. If N is a maximal Cohen-Macaulay module then $\mathrm{depth}(N) = d$, so $\mathrm{Ext}^1_A(N, M) = 0$ by part 1. Every homomorphism $N \to M/xM$ factors uniquely through N/xN, so $\mathrm{Hom}_A(N, M/xM) = \mathrm{Hom}_A(N/xN, M/xM)$. The short exact sequence above thus becomes

$$0 \longrightarrow \mathrm{Hom}_A(N, M) \xrightarrow{\cdot x} \mathrm{Hom}_A(N, M) \longrightarrow \mathrm{Hom}_A(N/xN, M/xM) \longrightarrow 0$$

Since $\mathrm{Hom}_A(M/xM, N/xN) = \mathrm{Hom}_{A/x}(N/xN, M/xM)$, this proves part 2.

□

Proposition 0.9. *Let A be a local ring, and let M and N be finitely generated A-modules. Suppose that x is a nonzerodivisor on M and that x is in the maximal ideal of A. If $\varphi : N \to M$ is a map and $\overline{\varphi} : N/xN \to M/xM$ is the map induced by φ, then*

1. *If $\overline{\varphi}$ is an epimorphism, then φ is an epimorhpism.*

2. *If $\overline{\varphi}$ is a monomorphism, then φ is a monomorphism.*

Proof.

1. If $\overline{\varphi}$ is an epimorphism, then φ is an epimorphism by Nakayama's lemma.

2. Suppose $\overline{\varphi}$ is a monomorphism, and let $J = \mathrm{Ker}(\varphi)$. Since J goes to zero in N/xN, we must have $J \subset xN$. On the other hand, since x is a nonzerodivisor on the image of φ, we have $J :_N x = J$. To see this, note that $n \in J :_N x$ implies $xn \in J$, thus
$$0 = \varphi(xn) = x\varphi(n).$$
Then x being a nonzerodivisor on the image of φ implies $\varphi(n) = 0$, or $n \in J$. So $J :_N x = J$ and $J \subset xN$ implies $xJ = J$, and so $J = 0$ by Nakayama's lemma.

□

Theorem 0.3. *Let A be a local Cohen-Macaulay ring of dimension d, and let W be a finitely generated A-module. Then W is a canonical module for A if and only if*

1. *$\mathrm{depth}(W) = d$.*

2. *W is a module of finite injective dimension (necessarily equal to d).*

3. *$End_A(W) = A$.*

Proof. First suppose that W is a canonical module. We do induction on the dimension of A. Suppose $\dim(A) = 0$. Then condition 1 is vacuous, since $\operatorname{depth}(W) \leq \dim(A)$. Also, condition 2 is satisfied because $W = \omega_A$ is injective. Lastly, condition 3 follows because, by duality

$$\begin{aligned}\operatorname{End}_A(\omega_A) &\cong \operatorname{End}_A(D(\omega_A)) \\ &\cong \operatorname{End}_A(A) \\ &\cong A.\end{aligned}$$

Now suppose $\dim(A) > 0$, and let x be a nonzerodivisor. By hypothesis, W/xW is a canonical module over A/x, and by induction it satisfies conditions 1,2, and 3 as an (A/x)-module. Since x is a nonzerodivisor on W and W/xW has depth $d-1$, condition 1 is satisfied. By Proposition (0.6), W has finite injective dimension, in particular

$$d - 1 = \operatorname{id}_{A/x}(W/xW) = \operatorname{id}_A(W) - 1.$$

Let $B = \operatorname{End}_A(W)$, and consider the natural map $\varphi : A \to B$ sending each element $a \in A$ to the map $m_a \in \operatorname{End}_A(W)$, where $m_a(w) = aw$ for all $w \in W$. We must show that φ is an isomorphism. By Proposition (0.8), x is a nonzerodivisor on B, and $B/xB = \operatorname{End}_{A/x}(W/xW) = A/x$. Thus by induction the map φ induces an isomorhpism $A/xA \to B/xB$. It follows from Proposition (0.6) that φ is an isomorphism.

Next suppose that W is an A-module satisfying conditions 1,2, and 3. Again, we do induction on $\dim(A)$. In case $\dim(A) = 0$ we must show that $W = \omega_A$. By Proposition (0.7), this follows from conditions 2 and 3.

Now suppose that $\dim(A) > 0$, and let x be a nonzerodivisor in A. The element x is also a nonzerodivisor on W by Proposition (0.5), so W/xW has depth $d-1$ over A/x. By Proposition (0.6), $\operatorname{id}_{A/x}(W/xW) < \infty$, and by Proposition (0.8), $\operatorname{End}_{A/x}(W/xW) = \operatorname{End}_A(W)/x\operatorname{End}_A(W) = A/x$. Thus, W/xW is a canonical module for A/x by induction, and W is a canonical module for A.

□

Uniqueness and (Often) Existence

These results imply a strong uniqueness result.

Corollary. *(Uniqueness of canonical modules). Let A be a local Cohen-Macualay ring with a canonical module W. If M is any finitely generated maximal Cohen-Macaulay A-module of finite injective dimension, then M is a direct sum of copies of W. In particular, any two canonical module of A are isomorphic.*

Proof. We do induction on $\dim(A)$, the case $\dim(A) = 0$ being Proposition (0.7). If $x \in A$ is a nonzerodivisor, then x is a nonzerodivisor on W and on M, and $M/xM \cong (W/xW)^n$ for some n by induction. By Proposition (0.9), there is an isomorphism $M \cong W^n$. □

Henceforth, we shall write ω_A for a canonical module of A (if one exists). We now come to the question of existence. We have already seen that if R is a regular local ring, then R has canonical module $\omega_R = R$. We shall now show that if A is a homomorhpic image of a local ring with a canonical module, then A has a canonical module too.

Theorem 0.4. *(Construction of canonical modules). Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring with canonical module ω_R. If A is a local R-algebra that is finitely generated as an R-module, and A is Cohen-Macaulay, then A has a canonical module. In fact, if $c = \dim(R) - \dim(A)$, then*

$$\omega_A \cong Ext^c_R(A, \omega_R)$$

Proof. We shall do induction on $\dim(A)$. First suppose that $\dim(A) = 0$. In this case, c is the dimension of R. The annihilator of A contains a power of the maximal ideal of R, say $\mathfrak{m}^n$. Since $\operatorname{depth}(\mathfrak{m}^n, R) = \operatorname{depth}(\mathfrak{m})$, we may choose a regular sequence $x_1, \ldots, x_c$ of length c in the annihilator of A. Let $R' = R/\langle x_1, \ldots, x_c\rangle$. Then R' is a local Cohen-Macaulay ring of dimension 0, and A is a finitely generated R'-module.

By definition, $\omega_R/\langle x_1, \ldots, x_c\rangle\omega_R$ is a canonical module for R', for which we shall write $\omega_{R'}$. By Proposition (0.6), applied c times,

$$\operatorname{Ext}^c_R(A, \omega_R) \cong \operatorname{Ext}^0_{R'}(A, \omega_{R'}) = \operatorname{Hom}_{R'}(A, \omega_{R'}).$$

By Proposition (0.3), this is a canonical module for A, as required.

Now suppose $\dim(A) > 0$. It suffices to show that if x is a nonzerodivisor on A, then x is a nonzerodivisor on $\mathrm{Ext}^c_R(A, \omega_R)$ and $\mathrm{Ext}^c_R(A, \omega_R)/x\mathrm{Ext}^c_R(A, \omega_R)$ is a canonical module for A/x. The short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/x \longrightarrow 0$$

gives rise to a long exact sequence in Ext of which a part is

$$\cdots \longrightarrow \mathrm{Ext}^c_R(A/x, \omega_R) \longrightarrow \mathrm{Ext}^c_R(A, \omega_R) \xrightarrow{\cdot x} \mathrm{Ext}^c_R(A, \omega_R) \longrightarrow \mathrm{Ext}^{c+1}_R(A/x, \omega_R) \longrightarrow \mathrm{Ext}^{c+1}_R(A, \omega_R) \longrightarrow \cdots$$

By induction, $\mathrm{Ext}^{c+1}_R(A/x, \omega_R)$ is a canonical module for A/x, so it suffices to show that the outer terms are 0, which we may do as follows:

Set $I = \mathrm{Ann}_R(A)$. The ring A/x is annihilated by $\langle I, x\rangle$, which has depth $c+1$ in R. Thus, $\mathrm{Ext}^c_R(A/x, \omega_R) = 0$. The ring A, being Cohen-Macaulay, has depth equal to $\dim(R) - c$, so $\mathrm{Ext}^{c+1}_R(A, \omega_R) = 0$ by Proposition (0.8).

□