# Math 9853: Homework #3

Due Thursday, September 12, 2019

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(a) Note that $K[x]$ denotes the ring of polynomials over a field $K$ and $K\langle\langle x\rangle\rangle$ denotes the ring of formal power series (that is, $\sum_{i\in\mathbb{N}} c_i x^i$ where $c_i \in K$, no convergence is required for the series, and multiplication is performed just like polynomials).

(a.1) Show that, as vector spaces, $K[x]$ is isomorphic to $K^{(\mathbb{N})}$ and $K\langle\langle x\rangle\rangle$ is isomorphic to $K^{\mathbb{N}}$ (see definition in the book or in class).

(a.2) Explain why the set $\{x^i : i \in \mathbb{N}\}$ is a basis for $K[x]$ but not a basis for $K\langle\langle x\rangle\rangle$, both as vector spaces over $K$.

(a.3) Let $W$ be an arbitrary vector space over $K$. Show that, for every function $\tau : \mathbb{N} \longrightarrow K$, there is a unique linear map $f : K[x] \longrightarrow W$ so that $f(x^i) = \tau(i)$ for all $i \in \mathbb{N}$.

(b) Let $f : V \longrightarrow W$ be a linear map of vector spaces over $K$. Prove the following (if a statement is wrong, make a correct statement and prove it):

(b.1) For any $u_1, u_2, \ldots, u_n \in V$, if $f(u_1), f(u_2), \ldots, f(u_n)$ are linearly independent, then $u_1, u_2, \ldots, u_n$ are independent;

(b.2) For any $u_1, u_2, \ldots, u_n \in V$, if $u_1, u_2, \ldots, u_n$ are linearly independent, then $f(u_1), f(u_2), \ldots, f(u_n)$ are independent.

(c) Let $A \in K^{m\times n}$ over $K$. The *column space* of $A$ is the subspace of $K^m$ spanned by the columns of $A$, in short $\mathrm{Col}(A) = \mathrm{Span}(A_1, \ldots, A_n)$ where $A_i$ is the $i$th column of $A$, similarly for the *row space* $\mathrm{Row}(A)$. The *null space* of $A$ is $\mathrm{Null}(A) = \{x \in K^n \mid Ax = 0\} \subseteq K^n$.

**For this problem,** $K = \mathbb{Z}_2$. Let $f : K^5 \longrightarrow K^4$ by $f(x) = Ax$ for $x \in K^5$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \in K^{4\times 5}.$$

(c.1) Apply row operations to $A$ to get a reduced row Echelon form $A'$.

(c.2) Use Problem (b) in HW2 to claim that $\mathrm{Row}(A) = \mathrm{Row}(A')$, then get a basis for the row space $\mathrm{Row}(A)$.

(c.3) Find a basis for $\mathrm{Null}(A) = \mathrm{Null}(A')$ (you should know that the two spaces are the same).

(c.4) Using the basis in (c.3) to express each column of $A$ a combination of some independent columns of $A$, hence get a basis for the column space $\mathrm{Col}(A)$.

(c.5) Is $f$ injective or onto? What's the dimension for the kernel space and image space of $f$?

(c.6) (About dual space) Note that each vector $v \in K^5$ defines a linear functional $v^* : K^5 \longrightarrow K$ by

$$v^*(x) = \langle v, x \rangle = \sum_{i=1}^{5} v_i x_i \in K, \qquad \text{for } x \in K^5.$$

Let $W = \text{Row}(A)$ and define

$$W^\perp = \{v^* \in (K^5)^* : v^*(u) = 0 \text{ for all } u \in W\}.$$

Get a basis for $W^\perp$ using (c.3).