Analysis Prelim Solutions

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1 Winter 2020

1.1 Problem 1

Exercise 1. Let $\mathcal{V}$ be an inner-product space.

1. Let (x_n) be a convergent sequence in $\mathcal{V}$. Then (x_n) is bounded.
2. Let (x_n) and (y_n) be two convergent sequences in $\mathcal{V}$. Prove that if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Solution 1. 1. Let (x_n) be a convergent sequence in $\mathcal{V}$. In particular, it must be a Cauchy sequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies $\|x_n - x_m\| < \varepsilon$. Set $M = \max\{\|x_1\|, \dots \|x_N\|\}$. Observe that if $n \geq N$, then we have

$$\begin{aligned}
\|x_n\| &= \|x_n - x_N + x_N\| \\
&\leq \|x_n - x_N\| + \|x_N\| \\
&< \varepsilon + \|x_N\| \\
&\leq \varepsilon + M.
\end{aligned}$$

In particular, we see that $M + \varepsilon$ is an upper bound of (x_n).

2. Choose $M \in \mathbb{N}$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\|x_n - x\| < \varepsilon/2M \text{ and } \|y_n - y\| < \varepsilon/2\|x\|.$$

Then $n \geq N$ implies

$$\begin{aligned}
|\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle| \\
&\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\
&\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \\
&\leq \|x_n - x\| M + \|x\| \|y_n - y\| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

This implies $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

1.2 Problem 2

Exercise 2. Let $\mathcal{V}$ be a normed linear space and let $\mathcal{W} \subset \mathcal{V}$ be a proper subspace. Prove that $\mathrm{Int}(\mathcal{W}) = \emptyset$.

Solution 2. Let $y \in \mathcal{V} \backslash \mathcal{W}$, let $x \in \mathcal{W}$, and let $\varepsilon > 0$. Assume for a contradiction $\mathrm{B}_\varepsilon(x) \subseteq \mathcal{W}$, where

$$\mathrm{B}_\varepsilon(x) = \{z \in \mathcal{V} \mid \|z - x\| < \varepsilon\}.$$

Then observe that $x + \frac{\varepsilon}{2\|y\|} y \in \mathrm{B}_\varepsilon(x) \subseteq \mathcal{W}$. However this implies $y \in \mathcal{W}$, which is a contradiction. Therefore $\mathrm{B}_\varepsilon(x) \nsubseteq \mathcal{W}$ for any $x \in \mathcal{W}$ and for any $\varepsilon > 0$. In particular, the only open subset of $\mathcal{V}$ which is contained in $\mathcal{W}$ is the empty set.

1.3 Problem 3

Exercise 3. Let $\ell^2(\mathbb{N})$ be the space of square summable sequences and define $T \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$T((x_n)) = (x_{n+1} - x_n)$$

for all $(x_n) \in \ell^2(\mathbb{N})$. Prove that T is bounded and find $\|T\|$.

Solution 3. Let $(x_n) \in \ell^2(\mathbb{N})$ such that $\|(x_n)\| = \sum_{n=1}^{\infty} |x_n|^2 \leq 1$. Then we have

$$\begin{aligned}
\|T(x_n)\| &= \|(x_{n+1} - x_n)\| \\
&= \sum_{n=1}^{\infty} |x_{n+1} - x_n|^2 \\
&\leq \sum_{n=1}^{\infty} ((|x_{n+1}| + |x_n|)^2) \\
&= \sum_{n=1}^{\infty} |x_{n+1}|^2 + \sum_{n=1}^{\infty} |x_n|^2 + 2\sum_{n=1}^{\infty} |x_{n+1}||x_n| \\
&\leq \sum_{n=1}^{\infty} |x_{n+1}|^2 + \sum_{n=1}^{\infty} |x_n|^2 + \sum_{n=1}^{\infty} (|x_{n+1}|^2 + |x_n|^2) \\
&\leq 4.
\end{aligned}$$

It follows that T is bounded. In fact, we claim that $\|T\| = 4$. Indeed, to see this, let $n \in 2\mathbb{N}$ and consider the sequence

$$\mathbf{x}_n = (1/\sqrt{n}, -1/\sqrt{n}, 1/\sqrt{n}, \dots, -1/\sqrt{n}, 1/\sqrt{n}, 0, \dots),$$

where the first n terms are nonzero and every term after the nth term is zero. Then note that

$$T\mathbf{x}_n = (-2/\sqrt{n}, 2/\sqrt{n}, \dots, 2/\sqrt{n}, -2/\sqrt{n}, 0, \dots),$$

where the first $n-1$ terms are nonzero and every term after the $(n-1)$th term is zero. Then we have $\|\mathbf{x}_n\| = 1$ and $\|T\mathbf{x}_n\| = 4(n-1)/n$. By taking $n \to \infty$, we obtain a sequence $(\mathbf{x}_n)$ in $\ell^2(\mathbb{N})$ where $\|\mathbf{x}_n\| = 1$ for all $n \in 2\mathbb{N}$ such that the $\|T\mathbf{x}_n\| \to 4$ as $n \to \infty$. It follows that $\|T\| = 4$.

1.4 Problem 4

Exercise 4. Let (X, d) be a compact metric space and let $f \colon X \to X$ be a continuous function. Suppose that for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $\mathrm{d}(x_\varepsilon, f(x_\varepsilon)) < \varepsilon$. Prove that there exists $x \in X$ such that $f(x) = x$.

Solution 4. Observe that the function $g \colon X \to \mathbb{R}_{\geq 0}$ given by $g(x) = \mathrm{d}(x, f(x))$ for all $x \in X$ is continuous. Indeed, it is the composite of continuous functions $X \to X \times X \to \mathbb{R}_{\geq 0}$ given by $x \mapsto (x, f(x)) \mapsto \mathrm{d}(x, f(x))$ for all $x \in X$. Since X is compact, the continuous function must attain a global minimum. Since for every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that $\mathrm{d}(x_\varepsilon, f(x_\varepsilon)) < \varepsilon$, we see that 0 is the global minimum. Thus there exists an $x \in X$ such that $\mathrm{d}(x, f(x)) = 0$. Since d is positive-definite, this implies $x = f(x)$.

1.5 Problem 5

Exercise 5. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Prove that if

$$\sup \{|\langle x, y\rangle| \mid x \in \mathcal{M},\ y \in \mathcal{N} \text{ and } \|x\| = \|y\| = 1\} < 1,$$

then $\mathcal{M} + \mathcal{N} = \{x + y \mid x \in \mathcal{M},\ y \in \mathcal{N}\}$ is a closed subspace of $\mathcal{H}$.

Solution 5. Let us check that $\mathcal{M} + \mathcal{N}$ is a subspace of $\mathcal{H}$. First note $\mathcal{M} + \mathcal{N}$ is nonempty since $0 \in \mathcal{M} + \mathcal{N}$. Next, let $\lambda, \lambda' \in \mathbb{C}$ and let $x + y, x' + y' \in \mathcal{M} + \mathcal{N}$. Then

$$\lambda(x + y) + \lambda'(x' + y') = (\lambda x + \lambda' x') + (\lambda y + \lambda' y') \in \mathcal{M} + \mathcal{N}.$$

Thus $\mathcal{M} + \mathcal{N}$ is a subspace of $\mathcal{H}$. Now we need to check that it is a *closed* subspace of $\mathcal{H}$.

We first note that for any nonzero $x \in \mathcal{M}$ and nonzero $y \in \mathcal{N}$, we have $|\langle x, y\rangle| < \|x\|\|y\|$. Indeed,

$$\begin{aligned}
1 &> \left| \langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \rangle \right| \\
&= \frac{1}{\|x\|\|y\|} |\langle x, y\rangle|.
\end{aligned}$$

In particular, this implies $\mathcal{M} \cap \mathcal{N} = 0$ (if $z \in \mathcal{M} \cap \mathcal{N}$ is nonzero, then $\|z\|^2 = |\langle z, z \rangle|$, which is a contradiction). Therefore we have a direct sum $\mathcal{M} \oplus \mathcal{N}$.

Let $(x_n + y_n)$ be a sequence in $\mathcal{M} + \mathcal{N}$ such that $x_n + y_n \to z$ where $z \in \mathcal{H}$. Using the fact that $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, write

$$z = x + w$$

where $x \in \mathcal{M}$ and $w \in \mathcal{M}^\perp$. We want to show that $x_n \to x$. Assume for a contradiction that the (x_n) does not converge to x. Then there exists $\varepsilon > 0$ and a subsequence $(x_{\pi(n)})$ of (x_n) such that

$$\|x_{\pi(n)} - x\| \geq \varepsilon$$

for all $n \in \mathbb{N}$. Now choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\|z - x_{\pi(n)} - y_{\pi(n)}\| < \varepsilon.$$

Then $n \geq N$ implies

$$\begin{aligned}
\varepsilon^2 &> \|z - x_{\pi(n)} - y_{\pi(n)}\|^2 \\
&= \|x + w - x_{\pi(n)} - y_{\pi(n)}\|^2 \\
&= \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 + \langle x - x_{\pi(n)}, w - y_{\pi(n)} \rangle + \langle w - y_{\pi(n)}, x - x_{\pi(n)} \rangle \\
&> \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 - \|x - x_{\pi(n)}\| \|w - y_{\pi(n)}\| - \|w - y_{\pi(n)}\| \|x - x_{\pi(n)}\| \\
&= \|x - x_{\pi(n)}\|^2 + \|w - y_{\pi(n)}\|^2 - 2\|x - x_{\pi(n)}\| \|w - y_{\pi(n)}\| \\
&= (\|x - x_{\pi(n)}\| - \|w - y_{\pi(n)}\|)^2.
\end{aligned}$$

which implies

$$-\varepsilon < \|x - x_{\pi(n)}\| - \|w - y_{\pi(n)}\| < \varepsilon$$

It suffices to show that (x_n) is a Cauchy sequence. Indeed, if this is case, then
Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$\|(x_n + y_n) - (x_m + y_m)\|^2 < \varepsilon.$$

Then $m, n \geq N$ implies

$$\begin{aligned}
\varepsilon^2 &> \|(x_n + y_n) - (x_m + y_m)\|^2 \\
&= \langle x_n - x_m + y_n - y_m, x_n - x_m + y_n - y_m \rangle \\
&= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 + \langle x_n - x_m, y_n - y_m \rangle + \langle y_n - y_m, x_n - x_m \rangle \\
&> \|x_n - x_m\|^2 + \|y_n - y_m\|^2 - \|x_n - x_m\| \|y_n - y_m\| - \|y_n - y_m\| \|x_n - x_m\| \\
&= \|x_n - x_m\|^2 + \|y_n - y_m\|^2 - 2\|x_n - x_m\| \|y_n - y_m\| \\
&= (\|x_n - x_m\| - \|y_n - y_m\|)^2.
\end{aligned}$$

which implies

$$-\varepsilon < \|x_n - x_m\| - \|y_n - y_m\| < \varepsilon.$$

In particular, $n \geq N$ implies

$$|\|x_N - x_n\| - \|y_N - y_n\|| < \varepsilon$$

Thus if $\|x_n - x_m\|$ is sufficiently small, then so to must $\|y_n - y_m\|$.

$$\|x_n - x_m + y_n - y_m\| < \varepsilon$$

1.6 Problem 6

Exercise 6. Let $(X, \mathcal{S})$ be a measurable space and let (E_n) be a sequence of measurable sets. Prove that the set E consisting of all points $x \in X$ that belong to infinitely many of the sets E_n is measurable.

Solution 6. We claim that

$$E = \bigcap_{N \geq 1} \bigcup_{n \geq N} E_n. \tag{1}$$

Indeed,

$$\begin{aligned} x \in \bigcap_{k \geq 1} \bigcup_{n \geq k} E_n &\iff x \in \bigcup_{n \geq k} E_n \text{ for all } k \\ &\iff x \in E_{\pi(k)} \text{ for some } \pi(k) \geq k \text{ for all } k \\ &\iff x \in E_{\pi(k)} \text{ for some sequence } (\pi(k)) \text{ of } (k) \\ &\iff x \text{ belongs to infinitely many } E_n \\ &\iff x \in E. \end{aligned}$$

Now the expression (1) shows that E is measurable.

1.7 Problem 7

Exercise 7. Let $(X, \mathcal{S}, \mu)$ be measure space and let $f \colon X \to \mathbb{R}$ be an integrable function. Suppose (E_n) is a sequence of members of $\mathcal{S}$ such that $\lim_{n \to \infty} \mu(E_n) = 0$. Prove that

$$\lim_{n \to \infty} \int_X f 1_{E_n} \mathrm{d}\mu = 0$$

Solution 7. Since $\int_X f 1_{E_n} \mathrm{d}\mu \leq \int_X |f| 1_{E_n} \mathrm{d}\mu$ for all $n \in \mathbb{N}$, it suffices to show

$$\lim_{n \to \infty} \int_X |f| 1_{E_n} \mathrm{d}\mu = 0.$$

In fact, by replacing f with $|f|$ if necessary, we may as well assume f is a nonnegative integrable function. Then $(f 1_{E_n})$ is a decreasing sequence of nonnegative measurable funtions which converges pointwise a.e. to the zero function since $\lim_{n \to \infty} \mu(E_n) = 0$. Also since f is integrable, we have $\int_X f 1_{E_1} \mathrm{d}\mu < \infty$. It follows from the decreasing monotone convergence theorem that

$$\lim_{n \to \infty} \int_X f 1_{E_n} \mathrm{d}\mu = 0.$$

For reference, we include the decreasing monotone convergence theorem below:

Proposition 1.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(f_n \colon X \to [0, \infty])$ is a decreasing sequence of nonnegative measurable functions which converges pointwise to $f \colon X \to [0, \infty]$. If $\int_X f_1 \mathrm{d}\mu < \infty$, then*

$$\lim_{n \to \infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu. \tag{2}$$

Proof. For each $n \in \mathbb{N}$, set $g_n = f_n - f_{n+1}$. Since (f_n) is a decreasing sequence, each g_n is nonnegative. Furthermore each g_n is measurable since it is a difference of two measurable functions. Set $g = \sum_{n=1}^{\infty} g_n$ and observe that

$$\begin{aligned} g &= \sum_{n=1}^{\infty} g_n \\ &= \lim_{N \to \infty} \sum_{n=1}^{N} g_n \\ &= \lim_{N \to \infty} \sum_{n=1}^{N} (f_n - f_{n+1}) \\ &= \lim_{N \to \infty} (f_1 - f_{N+1}) \\ &= f_1 - f. \end{aligned}$$

It follows from the monotone convergence theorem that

$$\begin{aligned}
\int_X f_1 \mathrm{d}\mu - \int_X f \mathrm{d}\mu &= \int_X (f_1 - f) \mathrm{d}\mu \\
&= \int_X g \mathrm{d}\mu \\
&= \sum_{n=1}^{\infty} \int_X g_n \mathrm{d}\mu && \text{MCT} \\
&= \lim_{N \to \infty} \sum_{n=1}^{N} \int_X g_n \mathrm{d}\mu \\
&= \lim_{N \to \infty} \sum_{n=1}^{N} \int_X (f_n - f_{n+1}) \mathrm{d}\mu \\
&= \lim_{N \to \infty} \int_X \sum_{n=1}^{N} (f_n - f_{n+1}) \mathrm{d}\mu \\
&= \lim_{N \to \infty} \int_X (f_1 - f_{N+1}) \mathrm{d}\mu \\
&= \int_X f_1 \mathrm{d}\mu - \lim_{N \to \infty} \int_X f_{N+1} \mathrm{d}\mu.
\end{aligned}$$

Since $\int_X f_1 \mathrm{d}\mu < 0$, we can cancel it from both sides to get (2). □

1.8 Problem 8

Exercise 8. Let $(X, \mathcal{S})$ be a measurable space and let (μ_n) be a sequence of measures on $(X, \mathcal{S})$ such that $\mu_n(X) = 1$ for all $n \in \mathbb{N}$. Prove that $\lambda \colon \mathcal{S} \to [0, \infty]$ defined by

$$\lambda(F) = \sum_{n=1}^{\infty} \frac{\mu_n(F)}{2^n}$$

for all $F \in \mathcal{S}$ is a measure on $(X, \mathcal{S})$ with $\lambda(X) = 1$.

Solution 8. First note that $\lambda(\emptyset) = 0$ since $\mu_n(\emptyset) = 0$ for all $n \in \mathbb{N}$. Next let (F_k) be a sequence of pairwise disjoint sets in $\mathcal{S}$. Then

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} F_k\right) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n \left(\bigcup_{k=1}^{\infty} F_k\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \mu_n(F_k) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_n(F_k)}{2^n} \\
&= \sum_{k=1}^{\infty} \lambda(F_k).
\end{aligned}$$

It follows that λ is a measure on $(X, \mathcal{S})$. For the last part of the problem, we have

$$\begin{aligned}
\lambda(X) &= \sum_{n=1}^{\infty} \frac{\mu_n(F)}{2^n} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= \frac{1/2}{1 - 1/2} \\
&= 1.
\end{aligned}$$

1.9 Problem 9

Exercise 9. Let $f \in L^2[0,\infty)$ and let $G\colon (0,\infty) \to \mathbb{R}$ be defined by

$$G(t) = \int_0^\infty \frac{f(x)}{1+tx} \mathrm{d}x.$$

Prove the following:

1. $\lim_{t\to\infty} G(t) = 0$;
2. G is continuous at every point of $(0,\infty)$.

Solution 9. 1. For each $t \in (0,\infty)$, we define $g_t\colon [0,\infty) \to \mathbb{R}$ by

$$g_t(x) = \frac{1}{1+tx}$$

for all $x \in [0,\infty)$. Observe that

$$\begin{aligned}\int_0^\infty |g_t(x)|^2 \mathrm{d}x &= \int_0^\infty \frac{1}{(1+tx)^2}\mathrm{d}x \\ &= \frac{-1}{t(1+tx)}\bigg|_0^\infty \\ &= 0 + 1/t \\ &= 1/t.\end{aligned}$$

Therefore $g_t \in L^2[0,\infty)$ with $\|g_t\|_2 = 1/t$. Also, note that $G(t) = \langle f, g_t \rangle$. In particlar, by Cauchy-Schwarz we have

$$\begin{aligned}|G(t)| &= |\langle f, g_t\rangle| \\ &\le \|f\|_2 \|g_t\| \\ &= \|f\|_2/t.\end{aligned}$$

So taking $t \to \infty$ gives us $|G(t)| \to 0$, which implies $\lim_{t\to\infty} G(t) = 0$.

2. Note that G is the composite of the maps $[0,\infty) \to L^2[0,\infty)$, given by $t \mapsto g_t$, with the map $L^2[0,\infty) \to \mathbb{R}$, given by $g \mapsto \langle f, g\rangle$. The latter map is continuous, so to show G is continuous, it suffices to show the former map is continuous. That is, let $t \in (0,\infty)$ and let (t_n) be a sequence in $(0,\infty)$ such that $t_n \to t$ as $n \to \infty$. Then we need to show that $g_{t_n} \to g_t$ as $n \to \infty$. Observe that

$$\begin{aligned}\|g_{t_n} - g_t\|_2^2 &= \int_0^\infty \left| \frac{1}{1+t_n x} - \frac{1}{1+tx}\right|^2 \mathrm{d}x \\ &= \int_0^\infty \left| \frac{(t-t_n)x}{(1+tx)(1+t_n x)}\right|^2 \mathrm{d}x \\ &= |t-t_n|^2 \int_0^\infty \frac{x^2}{(1+tx)^2(1+t_n x)^2}\mathrm{d}x \\ &= |t-t_n|^2 \left(\int_0^1 \frac{x^2}{(1+tx)^2(1+t_n x)^2}\mathrm{d}x + \int_1^\infty \frac{x^2}{(1+tx)^2(1+t_n x)^2}\mathrm{d}x \right) \\ &\le |t-t_n|^2 \left(\int_0^1 \frac{x^2}{(1+tx)^2(1+t_n x)^2}\mathrm{d}x + \int_1^\infty \frac{x^2}{tt_n x^4}\mathrm{d}x \right) \\ &\le |t-t_n|^2 \left(\int_0^1 x^2 \mathrm{d}x + \frac{1}{tt_n}\int_1^\infty \frac{1}{x^2}\mathrm{d}x \right) \\ &= |t-t_n|^2 \left(\frac{1}{3} + \frac{1}{tt_n} \right).\end{aligned}$$

In particular, we see that $g_{t_n} \to g_t$ as $n \to \infty$.

1.10 Problem 10

Exercise 10. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to [0,\infty)$ be a nonnegative measurable function. Suppose that for every $s > 0$ we have

$$\int_X e^{sf} \mathrm{d}\mu \leq e^{s^2}.$$

Prove that for every $t > 0$ we have

$$\mu\{f > t\} \leq e^{\frac{-t^2}{4}}.$$

Solution 10. Let $s > 0$ and $t > 0$. First note that

$$\begin{aligned} f > t &\iff sf > st \\ &\iff e^{sf} > e^{st}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \mu\{f > t\} &= \mu\{e^{sf} > e^{st}\} \\ &\leq \frac{1}{e^{st}} \int_X e^{sf} \mathrm{d}\mu \\ &\leq \frac{1}{e^{st}} e^{s^2} \\ &= e^{s(s-t)}, \end{aligned}$$

where we applied Chebyshev's inequality to get from the first line to the second line. In particular, setting $s = t/2$ gives us the desired result.

2 Winter 2019

2.1 Problem 1

Exercise 11. Let $\mathcal{X}$ be a normed linear space and let (x_n) be a sequence in $\mathcal{X}$. Suppose that every subsequence of (x_n) contains a convergent subsequence with limit $x_0 \in X$. Show that $x_n \to x_0$ as $n \to \infty$.

Solution 11. Assume for a contradiction that $x_n \not\to x_0$. Then there exists $\varepsilon > 0$ and a subsequence $(x_{\pi(n)})$ of (x_n) such that

$$\|x_{\pi(n)} - x_0\| \geq \varepsilon \tag{3}$$

for all $n \in \mathbb{N}$. In particular, (3) implies no subsequence of $(x_{\pi(n)})$ can converge to x_0, which is a contradiction.

2.2 Problem 2

Exercise 12. Let $P[0,1]$ be the collection of all polynomials with indeterminate t on $[0,1]$, namely,

$$P[0,1] = \left\{ \sum_{i=0}^{n} a_i t^i \mid a_i \in \mathbb{R} \text{ and } n \in \mathbb{N}_0 \right\}.$$

Define $\mathrm{d}\colon P[0,1] \times P[0,1] \to \mathbb{R}$ by

$$\mathrm{d}(p,q) = \int_0^1 |p(t) - q(t)| \mathrm{d}t.$$

Prove or disprove: $(P[0,1], \mathrm{d})$ is a complete metric space.

Solution 12. This is false. For each $n \in \mathbb{N}$, define $f_n \in P[0,1]$ by

$$f_n(t) = \sum_{i=0}^{n} \frac{t^i}{i!}.$$

The sequence (f_n) converges uniformly to e^t on $[0,1]$. Therefore in converges in the L^1-norm to e^t (as the measure of $[0,1]$ is finite). In particular, the sequence (f_n) is a Cauchy sequence in $P[0,1]$ which cannot converge to a polynomial. To see why this is the case, note that if it did converge to some polynomial, say $p(t)$, then $p(t)$ and e^t must agree almost everywhere. However since $p(t)$ and e^t are continuous on $(0,1)$, they in fact must agree everywhere. Indeed, if $c \in (0,1)$ such that $p(c) \neq e^c$. Then since $p(t) - e^t$ is continuous, there exists an open neighborhood of c, say

$$\mathrm{B}_\varepsilon(c) = \{x \in (0,1) \mid |x - c| < \varepsilon\},$$

such that $p(x) \neq e^x$ for all $x \in \mathrm{B}_\varepsilon(c)$. However $\mathrm{m}\,(\mathrm{B}_\varepsilon(c)) = 2\varepsilon \neq 0$, contradicting the fact that $p(t)$ and e^t agree almost everywhere.

2.3 Problem 3

Exercise 13. Let (X, d) be a metric space with the property that there are $A \subseteq X$ and $\varepsilon > 0$ such that A is uncountable for any dictinct elements $a, b \in A$ we have $\mathrm{d}(a,b) \geq \varepsilon$. Show that X is not separable.

Solution 13. Assume for a contradiction that X is separable. Choose a countable dense subset of X, say $Y \subseteq X$. For each $a \in A$, we choose $y_a \in Y$ such that $\mathrm{d}(a, y_a) < \varepsilon/2$. Observe that this gives rise to a function $y_{(-)} \colon A \to Y$, given by

$$y_{(-)}(a) = y_a$$

for all $a \in A$. We claim that $y_{(-)}$ is injective. Indeed, if $y_a = y_b$ for some distinct pair $a, b \in A$, then we have

$$\begin{aligned} \mathrm{d}(a,b) &\leq \mathrm{d}(a, y_a) + \mathrm{d}(y_b, b) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

which is a contradiction. Thus $y_{(-)}$ is an injective function, which contradicts the fact that A is uncountable. Thus X is separable.

2.4 Problem 4

Exercise 14. Recall the distance betwen two subsets A and B of a metric space (X, d) is defined as

$$\mathrm{d}(A, B) = \inf_{(a,b) \in A \times B} \mathrm{d}(a, b).$$

Show that if both A and B are compact, then there exists $x \in A$ and $y \in B$ such that

$$\mathrm{d}(x, y) = \mathrm{d}(A, B).$$

Solution 14. The function $\mathrm{d} \colon A \times B \to \mathbb{R}_{\geq 0}$ is continuous, so if A and B are both compact, then $A \times B$ is compact, which implies d attains a minimum, say at $(x, y) \in A \times B$. Thus for any $(a, b) \in A \times B$, we have $\mathrm{d}(x, y) \leq \mathrm{d}(a, b)$. This implies

$$\mathrm{d}(A, B) \leq \mathrm{d}(x, y) \leq \mathrm{d}(A, B).$$

Therefore $\mathrm{d}(x, y) = \mathrm{d}(A, B)$.

2.5 Problem 5

Exercise 15. Let $\mathcal{H}$ be a Hilbert space and let T be a nonzero linear operator on $\mathcal{H}$ such that $T^2 = T$. Show that the following are equivalent:

1. T is an orthogonal projection.
2. $\|T\| = 1$.
3. $\ker T = (\operatorname{im} T)^{\perp}$.

Solution 15. We first show 1 implies 2. Let $x \in \mathcal{H}$ such that $\|x\| \leq 1$. Then we have

$$\begin{aligned}\|Tx\|^2 &= \langle Tx, Tx\rangle \\ &= \langle x, T^2x\rangle \\ &= \|x\|^2 \\ &= 1.\end{aligned}$$

Thus T is bounded with $\|T\| \leq 1$. To see that $\|T\| = 1$, we just choose a $Ty \in \operatorname{im} T$ such that $\|Ty\| = 1$ (this can be done since $\operatorname{im} T \neq 0$). Then

$$\begin{aligned}\|T(Ty)\| &= \|T^2y\| \\ &= \|Ty\| \\ &= 1.\end{aligned}$$

Thus $\|T\| = 1$.

Now we show 2 implies 3. Let $x \in \ker T$. Then for all $Ty \in \operatorname{im} T$, we have

$$\begin{aligned}\langle x, Ty\rangle &= \langle x, T^2y\rangle \\ &=\end{aligned}$$

3 Summer 2019

3.1 Problem 1

Exercise 16. Let (X, d) be a metric space and let $A, B \subseteq X$. Prove or disprove the following statements:

1. If A and B are dense in X, then $A \cap B$ is also dense in X.
2. If A and B are open and dense in X, then $A \cap B$ is also open and dense in X.

Solution 16. 1. This is false. For instance, consider $(X, \mathrm{d}) = (\mathbb{R}, |\cdot|)$, $A = \mathbb{Q}$, and $B = \mathbb{R}\backslash\mathbb{Q}$. Then both A and B are dense in $\mathbb{R}$, but $A \cap B = \emptyset$, which is not dense in $\mathbb{R}$.

2. This is true. First note that $A \cap B$ is open since it is the intersection of two open sets, so we just need to show that it is dense in X. Let U be a nonempty open subset of X. Since A is an open dense subset of X, we see that $A \cap U$ is a nonempty open subset of X. Since B dense in X, we see that $B \cap A \cap U$ is nonempty.

3.2 Problem 2

Exercise 17. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K} \subseteq \mathcal{H}$ be a closed subspace. Then $\mathcal{K} = \mathcal{K}^{\perp\perp}$.

Solution 17. Let $x \in \mathcal{K}$. Then for any $y \in \mathcal{K}^{\perp}$, we have $\langle x, y\rangle = 0$. In particular, this implies $x \in \mathcal{K}^{\perp\perp}$. Thus $\mathcal{K} \subseteq \mathcal{K}^{\perp\perp}$. For the reverse direction, let $x \in \mathcal{K}^{\perp\perp}$. Then we have, in particular, $\langle x, x - \mathrm{P}_{\mathcal{K}}x\rangle = 0$. This implies $\|x\|^2 = \langle x, \mathrm{P}_{\mathcal{K}}x\rangle = \|\mathrm{P}_{\mathcal{K}}x\|^2$, which implies $x = \mathrm{P}_{\mathcal{K}}x \in \mathcal{K}$.

3.3 Problem 3

Exercise 18. Let $(x_n) \in l^\infty(\mathbb{N})$ be a bounded sequence. Define for each $k \in \mathbb{N}$ the simple function

$$S_k((x_n)) = \sum_{n=1}^{k} x_n \chi_{[2^{-n}, 2^{1-n}]}$$

Prove the following:

1. For each fixed $(x_n) \in l^\infty(\mathbb{N})$, the sequence $(S_k((x_n)))_{k\in\mathbb{N}}$ converges in $L^2[0,1]$.

2. Let $T\colon l^\infty(\mathbb{N}) \to L^2[0,1]$ be defined by

$$T((x_n)) = \lim_{k\to\infty} S_k((x_n)),$$

where the limit is the L^2-limit. Prove that T is bounded and find $\|T\|$.

Solution 18. 1. Let $(x_n) \in \ell^\infty(\mathbb{N})$. Let $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/M^2$. Then $m \geq k \geq N$ implies

$$\begin{aligned}
\|S_m((x_n)) - S_k((x_n))\|_2 &= \int_0^1 \left| \sum_{n=k+1}^{m} x_n \chi_{[2^{-n},2^{1-n}]} \right|^2 \mathrm{d}x \\
&= \int_0^1 \sum_{n=k+1}^{m} |x_n|^2 \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&\leq \int_0^1 \sum_{n=k+1}^{m} M^2 \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&= M^2 \int_0^1 \sum_{n=k+1}^{m} \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&= M^2 \int_0^1 \chi_{[2^{-m},2^{-k}]} \mathrm{d}x \\
&\leq M^2 2^{-k}. \\
&\leq M^2 2^{-N} \\
&< M^2 \frac{\varepsilon}{M^2} \\
&= \varepsilon.
\end{aligned}$$

This implies $(S_k((x_n)))_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2[0,1]$. In particular, it must converge since $L^2[0,1]$ is complete.

2. Let $(x_n) \in \ell^\infty(\mathbb{N})$ such that $|x_n| \leq 1$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\|T((x_n))\|_2 &= \left\| \sum_{n=1}^{\infty} x_n \chi_{[2^{-n},2^{1-n}]} \right\|_2 \\
&= \int_0^1 \sum_{n=1}^{\infty} |x_n|^2 \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&\leq \int_0^1 \sum_{n=1}^{\infty} \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&= \int_0^1 \chi_{[0,1]} \mathrm{d}x \\
&= 1.
\end{aligned}$$

This implies T is bounded with $\|T\| \leq 1$. In fact, we claim that $\|T\| = 1$. Indeed, we just take the constant sequence $(x_n) = (1)$. Then clearly in this case we have

$$\begin{aligned}
\|T((1))\|_2 &= \int_0^1 \sum_{n=1}^{\infty} \chi_{[2^{-n},2^{1-n}]} \mathrm{d}x \\
&= \int_0^1 \chi_{[0,1]} \mathrm{d}x \\
&= 1.
\end{aligned}$$

3.4 Problem 5

Exercise 19. Let (X, d) be a compact metric space and let $f\colon X \to X$ be a continuous function.

1. Prove that the set
$$f(X) = \{f(x) \mid x \in X\}$$
is compact.

2. Assume, in addition, that $f: X \to X$ is an isometry of (X, d) (that is, $\mathrm{d}(f(x), f(y)) = \mathrm{d}(x, y)$ for all $x, y \in X$). Prove that f is surjective.

Solution 19. 1. We shall a prove a more general result. Suppose X and Y are topological space and suppose $f: X \to Y$ is a surjective continuous function. We will show that if X is compact, then Y is compact. In other words, the image of a compact set under a continuous function is compact. Let $\{V_j\}_{j\in J}$ be an open covering of Y. Then $\{f^{-1}(V_j)\}_{j\in J}$ is an open covering of X. Since X is compact, there exists a finite subcovering of $\{f^{-1}(V_j)\}_{j\in J}$, say $\{f^{-1}(V_{j_1}), \dots, f^{-1}(V_{j_n})\}$. We claim that $\{V_{j_1}, \dots, V_{j_n}\}$ is a finite subcovering of $\{V_j\}_{j\in J}$. Indeed, it suffices to show that
$$Y = \bigcup_{k=1}^{n} V_{j_k}. \tag{4}$$
Indeed, this follows from the fact that f is surjective: if $y \in Y$, then we choose $x \in X$ such that $f(x) = y$, then since $\{f^{-1}(V_{j_1}), \dots, f^{-1}(V_{j_n})\}$ is an open covering of X, we see that $x \in f^{-1}(V_{j_k})$ for some k, and this implies $y \in V_{j_k}$. So $Y \subseteq \bigcup_{k=1}^{n} V_{j_k}$, and since the reverse inclusion is trivial, we have (4).

2. Let $x \in X$ and let $d = \inf\{\mathrm{d}(f(y), x) \mid y \in X\}$. We will first show that $d = 0$. Assume for a contradiction that $d > 0$. Then observe that for all $n \in \mathbb{N}$, we have
$$\mathrm{d}(f^n(x), x) \geq d.$$
In particular, since f is an isometry, this implies
$$\mathrm{d}(f^n(x), f^m(x)) \geq d$$
for all $n, m \in \mathbb{N}$. In particular, the sequence $(f^n(x))$ has no convergent subsequence since the distance between any two terms in the sequence is always greater than d. Therefore $d = 0$. Now let $g: X \to \mathbb{R}_{\geq 0}$ be the function given by
$$g(y) = \mathrm{d}(f(y), x)$$
for all $y \in X$. Note that g is continuous since it is the composite of the continuous function $X \to X \times X$, given by $y \mapsto (f(x), f(y))$, with the continuous function $X \times X \to \mathbb{R}_{\geq 0}$, given by $(x, y) \mapsto \mathrm{d}(x, y)$. Therefore it attains a minimum value, say at $x_0 \in X$. In particular, we have $\mathrm{d}(f(x_0), x) = 0$, which implies $f(x_0) = x$. Thus f is surjective.

4 Summer 2018

4.1 Problem 1

Exercise 20. Let (a_n) be a sequence of real numbers such that $a_n \to 0$. Prove that
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0. \tag{5}$$

Solution 20. Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ such that $n \geq N_\varepsilon$ implies $-\varepsilon < a_n < \varepsilon$. Then for all $k \in \mathbb{N}$, we have
$$\frac{1}{N_\varepsilon + k} \sum_{n=1}^{N_\varepsilon} a_n - \frac{k\varepsilon}{N_\varepsilon + k} \leq \frac{1}{N_\varepsilon + k} \sum_{n=1}^{N_\varepsilon + k} a_n \leq \frac{1}{N_\varepsilon + k} \sum_{n=1}^{N_\varepsilon} a_n + \frac{k\varepsilon}{N_\varepsilon + k} \tag{6}$$
Taking $k \to \infty$ in (6) gives us
$$-\varepsilon \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \leq \varepsilon$$
Since $\varepsilon > 0$ was arbitrary it follows that (5) holds.

4.2 Problem 2

Exercise 21. Let X be a normed linear subspace and $\emptyset \neq Y \subseteq X$ be a subset with the property that $X \backslash Y$ is a linear subspace. Show that Y is dense in X.

Solution 21. Since $Y \neq \emptyset$ we see that $X \backslash Y$ is a proper subspace of X. It follows that $\mathrm{int}(X \backslash Y) = \emptyset$ (see winter 2020 problem 2), or equivalently, Y is dense in X.

4.3 Problem 4

Exercise 22. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two closed linear subspaces of $\mathcal{H}$. Denote P_1 and P_2 to be the orthogonal projections onto $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Show that $\|\mathrm{P}_1 - \mathrm{P}_2\| \leq 1$.

Solution 22. Let $x \in \mathcal{H}$. We have

$$\begin{aligned}
\|(\mathrm{P}_1 - \mathrm{P}_2)(x)\|^2 &= \|\mathrm{P}_1 x - \mathrm{P}_2 x\|^2 \\
&= \|\mathrm{P}_1(\mathrm{P}_1 x - \mathrm{P}_2 x)\|^2 + \|\mathrm{P}_1 x - \mathrm{P}_2 x - \mathrm{P}_1(\mathrm{P}_1 x - \mathrm{P}_2 x)\|^2 \\
&= \|\mathrm{P}_1 x - \mathrm{P}_1\mathrm{P}_2 x\|^2 + \|\mathrm{P}_1\mathrm{P}_2 x - \mathrm{P}_2 x\|^2 \\
&= \|\mathrm{P}_1(x - \mathrm{P}_2 x)\|^2 + \|\mathrm{P}_2 x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&\leq \|x - \mathrm{P}_2 x\|^2 + \|\mathrm{P}_2 x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&= \|x\|^2 - \|\mathrm{P}_1\mathrm{P}_2 x\|^2 \\
&\leq \|x\|^2.
\end{aligned}$$

It follows that $\|P_1 - P_2\| \leq 1$.

5 Winter 2016

5.1 Problem 1

Exercise 23. Evaluate the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{7}$$

Solution 23. The series converges by the alternating series test. Recall the Mclaurin expansion for $\ln(1-x)$ is given by

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

with radius of convegence $r = 1$. Since the series (7) converges

6 Winter 2010

6.1 Problem 1

Exercise 24. Prove the following two statements that look similar but are different.

1. $E \subseteq \mathbb{R}$ is bounded and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous implies $f(E)$ is bounded.
2. $E \subseteq \mathbb{R}$ is bounded and $f\colon E \to \mathbb{R}$ is uniformly continuous implies $f(E)$ is bounded.

Find a counterexample for the following false statement: $E \subseteq \mathbb{R}$ is bounded and $f\colon E \to \mathbb{R}$ is continuous implies $f(E)$ is bounded.

Solution 24. 1. Choose $M > 0$ such that $E \subseteq [-M, M]$. Since f is continuous, the image of a compact set is a compact set. In particular, $f([-M, M])$ is compact. By the Heine-Borel theorem, $f([-M, M])$ is closed and bounded. In particular, $f(E)$ is bounded.

2. We want to show that f can be extended to a continuous function $\widetilde{f}\colon \overline{E} \to \mathbb{R}$. We define $\widetilde{f}$ as follows: let $x \in \overline{E}$. Choose a sequence (x_n) in E such that $x_n \to x$. Then we define

$$\widetilde{f}(x) = \lim_{n\to\infty} f(x_n). \tag{8}$$

We need to make sure that this definition makes since. First, note that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Choose $\delta > 0$ such that $|y - z| < \delta$ implies

$$|f(y) - f(z)| < \varepsilon$$

for all $y, z \in E$. Next, we use the fact that (x_n) is a Cauchy sequence to choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies

$$|x_n - x_m| < \delta.$$

Then $n, m \geq N$ implies

$$|f(x_n) - f(x_m)| < \varepsilon.$$

Thus $(f(x_n))$ is a Cauchy sequence, so the limit in (8) makes since. Finally we note that $\widetilde{f}$ extends f since f is continuous.

Now $\overline{E}$ is a closed and bounded subset of $\mathbb{R}$, so by the Heine-Borel theorem, it must be compact. Therefore $\widetilde{f}(\overline{E})$ is compact, and again by the Heine-Borel theorem, $\widetilde{f}(\overline{E})$ is closed and bounded. In particular, $f(E)$ is bounded.

Now let us counterexample to the last statement. Consider the function $f(x) = 1/x$ defined on the interval $E = (0, 1)$. Even though E is bounded and f is continuous on E, we see that $f(E)$ is not bounded since

$$\begin{aligned}\lim_{n\to\infty} f(1/n) &= \lim_{n\to\infty} \frac{1}{1/n} \\ &= \lim_{n\to\infty} n \\ &= \infty.\end{aligned}$$

Exercise 25. Let (X, d_X) be a compact metric space and let (Y, d_Y) be a (not necessarily complete) metric space.

1. Prove that for any continuous bijection $f\colon X \to Y$, the inverse function $f^{-1}\colon Y \to X$ is also continuous.
2. Find an example that shows (1) is not true in general if X is not compact.

Solution 25. 1. It suffices to show that f is a closed mapping (takes closed sets to closed sets). Let $E \subseteq X$ be a closed set. Since X is compact, E must also be compact. Since f is continuous, $f(E) \subseteq Y$ is also compact. Now since Y is Hausdorff, this implies $f(E)$ is closed.

2. Let (X, d) be the set of real numbers equipped with the discrete metric: that is $X = \mathbb{R}$ as sets and

$$\mathrm{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$. In particular, X is discrete and not compact. Then the identity function $f\colon X \to \mathbb{R}$, given by $f(x) = x$, is continuous (since any function out of a discrete space is continuous). However the inverse function is not continuous ($\{x\} \subseteq X$ is open in X, but $\{f(x)\}$ is not open in $\mathbb{R}$).