# Singular Cohomology

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## 1 Standard Simplicies

Let $n \in \mathbb{Z}_{\geq 0}$ and let $v_0, \dots, v_n \in \mathbb{R}^{n+1}$ such that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. The **$n$-simplex** $[v_0, \dots, v_n]$ is the smallest convex set in $\mathbb{R}^m$ which contains $v_0, \dots, v_n$. The vectors $v_i$ are called the **vertices** of the simplex. The **$n$-dimensional standard simplex** is

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \in [0,1] \text{ and } \sum t_i = 1 \right\} = [e_0, \dots, e_n],$$

where the $e_i$ are the standard coordinate vectors in $\mathbb{R}^{n+1}$ (i.e. $e_i = (0, \dots, 1, \dots, 0)$ with 1 in the $i$th component and 0 everywhere else).

For purposes of homology it will be important to keep track of the order of the vertices of a simplex, so "$n$-simplex" will really mean "$n$-simplex with an ordering of its vertices". Specifying the ordering of the vertices also determines a canonical linear homeomorphism from the standard $n$-simplex $\Delta^n$ onto any other $n$-simplex $[v_0, \dots, v_n]$, preserving the order of vertices, namely, $(t_0, \dots, t_n) \mapsto \sum_i t_i v_i$. The coefficients $t_i$ are the **barycentric coordinates** of the point $\sum_i t_i v_i$ in $[v_0, \dots, v_n]$.

#### 1.0.1 Delta Complex

A **$\Delta$-complex** structure on a space $X$ is a collection of maps $\sigma_\alpha \colon \Delta^n \to X$, with $n$ depending on the index $\alpha$, such that:

1. The restriction $\sigma_\alpha|_{\Delta^n \setminus \partial \Delta^n}$ is injective, and each point of $X$ is in the image of exactly one such restriction $\sigma_\alpha|_{\Delta^n \setminus \partial \Delta^n}$.
2. Each restriction of $\sigma_\alpha$ to a face of $\Delta^n$ is one of the maps $\sigma_\beta \colon \Delta^{n-1} \to X$. Here we are identifying the face of $\Delta^n$ with $\Delta^{n-1}$ by the canonical linear homeomorphism betweem them that preserves the ordering of the vertices.
3. A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\Delta^n$ for each $\sigma_\alpha$.

Among other things, this last condition rules out trivialities like regarding all the points of $X$ are individual vertices.

### 1.1 Singular Homology

Let $X$ be a topological space. A continuous map $\sigma \colon \Delta^n \to X$ is called a **singular $n$-simplex in** $X$. A continuous map $\widetilde{\sigma} \colon [v_0, \dots, v_n] \to X$ determines a unique continuous map $\sigma \colon \Delta^n \to X$ via the canonical linear homeomorphism from $\Delta^n$ onto $[v_0, \dots, v_n]$. Thus we can think of $\widetilde{\sigma}$ as a singular $n$-simplex via $\sigma$. We frequently use this convention without comment.

The set $n$-simplices in $X$ is denoted by $\Sigma_n(X)$ and the free abelian group with basis the set of singular $n$-simplices in $X$ is denoted $S_n(X) := \mathbb{Z}[\Sigma_n(X)]$. Elements in $S_n(X)$ are called **singular $n$-chains**. We also denote $\Sigma(X) := \bigcup_n \Sigma_n(X)$.

Let $R$ a ring. For $n \in \mathbb{Z}_{\geq 0}$, let

$$S_n(X;R) := \bigoplus_{\sigma \in \Sigma_n(X)} R \quad \text{and} \quad S(X;R) := \bigoplus_{n \in \mathbb{Z}} S_n(X;R),$$

So $S(X;R)$ is a graded $R$-module whose $n$th homogeneous piece is the free $R$-module $S_n(X;R)$. Let $\partial_R \colon S(X;R) \to S(X;R)$ be the unique graded endomorphism of degree $-1$ such that if $\sigma \in \Sigma_n(X)$, then

$$\partial_R(\sigma) = \sum_i (-1)^i \sigma_{|[e_0, \dots, \widehat{e_i}, \dots, e_n]},$$

where $[e_0, \ldots, \widehat{e_i}, \ldots, e_n] = [e_0, \ldots, e_{i-1}, e_{i+1} \ldots, e_n]$. By a direct calculation, we have $\partial_R^2 = 0$: Indeed

$$\begin{aligned}
\partial_R^2(\sigma) &= \partial_R \left( \sum_{0 \le i \le n} (-1)^i \sigma_{|[e_0,\ldots,\widehat{e_i},\ldots,e_n]} \right) \\
&= \sum_{0 \le i \le n} (-1)^i \partial_R \left( \sigma_{|[e_0,\ldots,\widehat{e_i},\ldots,e_n]} \right) \\
&= \sum_{0 \le i \le n} (-1)^i \left( \sum_{0 \le j < i} (-1)^j \sigma_{|[e_0,\ldots,\widehat{e_j},\ldots\widehat{e_i},\ldots e_n]} + \sum_{i < j \le n} (-1)^{j+1} \sigma_{|[e_0,\ldots,\widehat{e_i},\ldots\widehat{e_j},\ldots e_n]} \right) \\
&= \sum_{0 \le j < i \le n} (-1)^{i+j} \sigma_{|[e_0,\ldots,\widehat{e_j},\ldots,\widehat{e_i},\ldots e_n]} + \sum_{0 \le i < j \le n} (-1)^{i+j+1} \sigma_{|[e_0,\ldots,\widehat{e_i},\ldots,\widehat{e_j},\ldots e_n]} \\
&= 0.
\end{aligned}$$

Thus $(S(X;R), \partial_R)$ forms a chain complex over $R$; it is called the **singular chain complex of $X$ over $R$**. The $n$th homology of $(S(X;R), \partial_R)$ is called the **$n$th singular homology of $X$ over $R$**, and is denoted by $H_n^{\text{sing}}(X;R)$.

Note that if $A$ is a ring and $\varphi\colon R \to A$ is a ring homomorphism, then we change our base ring $R$ to the ring $A$ by

$$\begin{aligned}
A \otimes_R S_n(X;R) &\cong A \otimes_R \left( \bigoplus_{\sigma \in \Sigma_n(X)} R \right) \\
&\cong \bigoplus_{\sigma \in \Sigma_n(X)} (A \otimes_R R) \\
&\cong \bigoplus_{\sigma \in \Sigma_n(X)} A \\
&= S_n(X;A).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
A \otimes_R S(X;R) &= A \otimes_R \left( \bigoplus_{n \in \mathbb{Z}_{\ge 0}} S_n(X;R) \right) \\
&\cong \bigoplus_{n \in \mathbb{Z}_{\ge 0}} A \otimes_R S_n(X;R) \\
&\cong S(X;A),
\end{aligned}$$

and so $A \otimes_R (S(X;R), \partial_R) \cong (S(X;A), \partial_A)$ and therefore $A \otimes_R H^n_{\text{sing}}(X;R) \cong H^n_{\text{sing}}(X;A)$. If the base ring $R$ is understood from context, then we usually simplify notation by removing the label "$R$" . For instance, we often replace $S(X;R)$ with $S(X)$, $\partial_R$ with $\partial$, and $H_n^{\text{sing}}(X;R)$ with $H_n^{\text{sing}}(X)$.

#### 1.1.1 Reduced Homology

It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the **reduced homology groups** $\widetilde{H}_n(X)$ to be the homology groups of the augmented chain complex

$$\cdots \longrightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\varepsilon} R \longrightarrow 0$$

where $R$ sits in degree $-1$ and $\varepsilon\left(\sum_i r_i \sigma_i\right) = \sum_i r_i$. Here we had better require $X$ to be nonempty in order to avoid a nontrivial homology group in dimension $-1$. Since $\varepsilon \partial_1 = 0$, $\varepsilon$ vanishes on $\text{Im}(\partial_1)$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so $H_0(X) \cong \widetilde{H}_0(X) \oplus R$. It is clear that $H_n(X) = \widetilde{H}_n(X)$ for all $n > 0$.

### 1.2 Homotopy Invariance

Let $X$ and $Y$ be topological spaces and let $f\colon X \to Y$ be a continuous map. Let $f_\#\colon S(X) \to S(Y)$ to be the unique graded homomorphism of $R$-modules such $f_\#(\sigma) = f \circ \sigma$ for all $\sigma \in \Sigma(X)$. We claim that $f_\#$ is more than just a graded homomorphism: it is a chain map. Indeed, let $\sigma \in \Sigma_n(X)$ for some $n \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
\partial f_\#(\sigma) &= \partial(f \circ \sigma) \\
&= \sum_{0 \leq i \leq n} (-1)^i (f \circ \sigma)|_{[e_0, \dots, \widehat{e}_i, \dots, e_n]} \\
&= \sum_{0 \leq i \leq n} (-1)^i \left( f \circ \sigma|_{[e_0, \dots, \widehat{e}_i, \dots, e_n]} \right) \\
&= f_\# \left( \sum_{0 \leq i \leq n} (-1)^i \sigma|_{[e_0, \dots, \widehat{e}_i, \dots, e_n]} \right) \\
&= f_\# \partial(\sigma).
\end{aligned}$$

It is easy to check that if $f \colon X \to Y$ and $g \colon Y \to Z$ are continuous maps, then $(f \circ g)_\# = f_\# \circ g_\#$. Thus we have a covariant functor $F \colon \mathbf{Top} \to \mathbf{Chain}_R$ from the category of topoligical spaces to the category of chain complexes over $R$, given by mapping a topological space $X$ to the chain complex $S(X)$ and mapping a continuous map $f \colon X \to Y$ to the chain map $f_\# \colon S(X) \to S(Y)$.

**Proposition 1.1.** *Let $f \colon X \to Y$ and $g \colon X \to Y$ be continuous functions and suppose that $f$ and $g$ are homotopically equivalent as continuous functions. Then the chain maps $f_\#$ and $g_\#$ are homotopically equivalent as chain maps.*

*Proof.* The essential ingredient is a procedure for subdividing $\Delta^n \times I$ into simplices. In $\Delta^n \times I$, let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \dots, w_n]$, where $v_i$ and $w_i$ have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of $n$-simplices, each obtained from the preceding one by moving one vertex $v_i$ up to $w_i$, starting with $v_n$ and working backwards to $v_0$. Thus the first step is to move $[v_0, \dots, v_n]$ up to $[v_0, \dots, v_{n-1}, w_n]$, then the second step is to move this up to $[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$, and so on. In the typical step $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ moves up to $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$. The region between these two $n$-simplices is exactly the $(n+1)$-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$ which has $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ as its lower face and $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ as its upper face. Altogether, $\Delta^n \times I$ is the union of $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an $n$-simplex face.

Given a homotopy $H \colon X \times I \to Y$ from $f$ to $g$ and a singular simplex $\sigma \colon \Delta^n \to X$, we can form the composition $H \circ (\sigma \times 1) \colon \Delta^n \times I \to X \times I \to Y$. Using this, we can define **prism operators** $P \colon S_n(X) \to S_{n+1}(Y)$ by the following formula:

$$P(\sigma) = \sum_{0 \leq i \leq n} (-1)^i H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}.$$

We will show that these prism operators satisfy the formula

$$\partial P = g_\# - f_\# - P\partial.$$

Geometrically, the left side of this equation represents the boundary of the prism, and the three terms on the right side represent the top $\Delta^n \times \{1\}$, the bottom $\Delta^n \times \{0\}$, and the sides $\partial \Delta^n \times I$ of the prism. To prove the relation, we calculate

$$\begin{aligned}
\partial P(\sigma) &= \partial \left( \sum_{0 \leq i \leq n} (-1)^i H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) \\
&= \sum_{0 \leq i \leq n} (-1)^i \partial \left( H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) \\
&= \sum_{0 \leq i \leq n} (-1)^i \left( \sum_{0 \leq j \leq i} (-1)^j H \circ (\sigma \times 1)|_{[v_0, \dots, \widehat{v}_j, \dots v_i, w_i, \dots, w_n]} + \sum_{i \leq j \leq n} (-1)^{j+1} H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w}_j, \dots, w_n]} \right) \\
&= \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} H \circ (\sigma \times 1)|_{[v_0, \dots, \widehat{v}_j, \dots, v_i, w_i, \dots, w_n]} + \sum_{0 \leq j \leq i \leq n} (-1)^{i+j+1} H \circ (\sigma \times 1)|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w}_j, \dots, w_n]}.
\end{aligned}$$

The terms $i = j$ in the two sums cancel except for $H \circ (\sigma \times 1)|_{[w_0, \dots, w_n]}$, which is $g \circ \sigma = g_\#(\sigma)$, and $-H \circ (\sigma \times$

$1)|_{[v_0,\dots,v_n]}$, which is $-f\circ\sigma = -f_\#(\sigma)$. The terms with $i \neq j$ are exactly $-P\partial(\sigma)$ since

$$\begin{aligned}
P\partial(\sigma) &= P\left(\sum_{0\le j\le n}(-1)^j\sigma|_{[v_0,\dots,\widehat{v}_j,\dots,v_n]}\right)\\
&= \sum_{0\le j\le n}(-1)^jP\left(\sigma|_{[v_0,\dots,\widehat{v}_j,\dots,v_n]}\right)\\
&= \sum_{0\le j\le n}(-1)^j\left(\sum_{0\le i<j}(-1)^iH\circ(\sigma\times 1)|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w}_j,\dots,w_n]} + \sum_{j<i\le n}(-1)^{i+1}H\circ(\sigma\times 1)|_{[v_0,\dots,\widehat{v}_j,\dots v_i,w_i,\dots,w_n]}\right)\\
&= \sum_{0\le i<j\le n}(-1)^{i+j}H\circ(\sigma\times 1)|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w}_j,\dots,w_n]} + \sum_{0\le j<i\le n}(-1)^{i+j+1}H\circ(\sigma\times 1)|_{[v_0,\dots,\widehat{v}_j,\dots,v_i,w_i,\dots,w_n]}.
\end{aligned}$$

Therefore $P$ is a homotopy from $f_\#$ to $g_\#$. □

**Corollary.** *If $f$ and $g$ are homotopically equivalent as continuous functions, then $f_\#$ and $g_\#$ induce the same map on homology.*

## 1.3 Exact Sequences and Excision

Let $X$ be a topological space and let $A$ a subspace of $X$. Then the inclusion map $A \hookrightarrow X$ induces a chain map $(S(A),\partial) \hookrightarrow (S(X),\partial)$. Let $(S(X,A),\overline{\partial})$ denote the cokernel of this map. Thus, $S(X,A)$ is the graded $R$-module $S(X)/S(A)$ and $\overline{\partial}$ is the boundary map induced by $\partial$. The homology of $(S(X,A),\partial)$ is called **relative homology** and is denoted $H(X,A)$. By considering the definition of the relative boundary map we see that:

- Elements of $H_n(X,A)$ are represented by **relative cycles**: $n$-chains $\alpha \in S_n(X)$ such that $\partial\alpha \in S_{n-1}(A)$.
- A relative cycle $\alpha$ is trivial in $H_n(X,A)$ if and only if it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for some $\beta \in S_{n+1}(X)$ and $\gamma \in S_n(A)$.

The quotient $S_n(X)/S_n(A)$ could also be viewed as a subgroup of $S_n(X)$, the subgroup with basis the singular $n$-simplices $\sigma\colon \Delta^n \to X$ whose image is not contained in $A$. However, the boundary map does not take this subgroup of $S_n(X)$ to the corresponding subgroup of $S_{n-1}(X)$, so it is usually better to regard $S_n(X,A)$ as a quotient rather than a subgroup of $S_n(X)$.

**Example 1.1.** In the long exact sequence of reduced homology groups for the pair $(D^n,\partial D^n)$, the maps $H_i(D^n,\partial D^n) \to \widetilde{H}_{i-1}(S^{n-1})$ are isomorphisms for all $i > 0$ since the remaining terms $\widetilde{H}_i(D^n)$ are zero for all $i$. Thus we obtain the calculation

$$H_i(D^n,\partial D^n) \cong \begin{cases}\mathbb{Z} & \text{if } i = n\\ 0 & \text{otherwise.}\end{cases}$$

**Example 1.2.** Applying the long exact sequence of reduced homology groups to a pair $(X,x_0)$ with $x_0 \in X$ yields isomorphisms $H_n(X,x_0) \cong \widetilde{H}_n(X)$ for all $n$ since $\widetilde{H}_n(x_0) \cong 0$ for all $n$.

### 1.3.1 Excision

**Theorem 1.1.** *Given subspaces $Z \subset A \subset X$ such that the closure of $Z$ is contained in the interior of $A$, then the inclusion $(X\backslash Z, A\backslash Z) \hookrightarrow (X,A)$ induces isomorphisms $H_n(X\backslash Z, A\backslash Z) \to H_n(X,A)$ for all $n$. Equivalently, for subspaces $A,B \subset X$ whose interiors cover $X$, the inclusion $(B, A\cap B) \hookrightarrow (X,A)$ induces isomorphisms $H_n(B,A\cap B) \to H_n(X,A)$ for all $n$.*

The translation between the two versions is obtained by setting $B = X\backslash Z$ and $Z = X\backslash B$. Then $A\cap B = A\backslash Z$ and the condition $\overline{Z} \subset \mathrm{int}(A)$ is equivalent to $X = \mathrm{int}(A)\cup\mathrm{int}(B)$ since $X\backslash\mathrm{int}(B) = \overline{Z}$.

For a spae $X$, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of $X$ whose interiors form an open cover of $X$, and let $S_n^{\mathcal{U}}(X)$ be the subgroup of $S_n(X)$ consisting of chains $\sum_i r_i\sigma_i$ such that each $\sigma_i$ has image contained in some set in the cover $\mathcal{U}$. The boundary map $\partial$ takes $S_n^{\mathcal{U}}(X)$ to $S_n^{\mathcal{U}}(X)$, so the groups $S_n^{\mathcal{U}}(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^{\mathcal{U}}(X)$.

**Proposition 1.2.** *The inclusion $\iota\colon S_n^{\mathcal{U}}(X) \hookrightarrow S_n(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho\colon S_n(X) \to S_n^{\mathcal{U}}(X)$ such that $\iota\rho$ and $\rho\iota$ are chain homotopic to the identity. Hence $\iota$ induces isomorphisms $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all $n$.*

*Proof.* The barycentric subdivision process will be performed at four levels, beginning with the most geometric and becoming increasingly algebraic.

**(1)** Barycentric Subdivision of Simplices: The points of a simplex $[v_0, \ldots, v_n]$ are the linear combinations $\sum_i t_i v_i$ with $\sum t_i = 1$ and $t_i \in [0,1]$ for each $i$. The **barycenter** or 'center of gravity' of the simplex $[v_0, \ldots, v_n]$ is the point $b = \sum t_i v_i$ whose barycentric coordinates $t_i$ are all equal, namely $t_i = 1/(n+1)$ for each $i$. The **barycentric subdivision** of $[v_0, \ldots, v_n]$ is the decomposition of $[v_0, \ldots, v_n]$ into the $n$-simplices $[b, w_0, \ldots, w_{n-1}]$ where, inductively, $[w_0, \ldots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \ldots, \widehat{v}_i, \ldots, v_n]$. The induction starts with the case $n = 0$ when the barycentric subdivision of $[v_0]$ is defined to be just $[v_0]$ itself. It follows from the inductive definition that the vertices of simplices in the barycentric subdivision of $[v_0, \ldots, v_n]$ are exactly the barycenters of all the $k$-dimensional faces $[v_{i_0}, \ldots, v_{i_k}]$ of $[v_0, \ldots, v_n]$ for $0 \leq k \leq n$. When $k = 0$ this gives the original vertices $v_i$ since the barycenter of 0-simplex is itself. The barycenter of $[v_{i_0}, \ldots, v_{i_k}]$ has barycentric coordinates $t_i = 1/(k+1)$ for $i = i_0, \ldots, i_k$ and $t_i = 0$ otherwise.

The $n$-simplices of the barycentric subdivision of $\Delta^n$, together with all their faces, do in fact form a $\Delta$-complex structure on $\Delta^n$, indeed a simplicial complex structure, though we shall not need to know this in what follows.

A fact we will need is that the diameter of each simplex of the barycentric subdivision of $[v_0, \ldots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \ldots, v_n]$. Here the diameter of a simplex is by definition the maximum distance between any two of its points, and we are using the metric from the ambient Euclidean space $\mathbb{R}^m$ containing $[v_0, \ldots, v_n]$. The diameter of a simplex equals the maximum distance between any of its vertices because the distance between the points $v$ and $\sum t_i v_i$ of $[v_0, \ldots, v_n]$ satisfies the inequality

$$\begin{aligned} \left| v - \sum_{i=0}^{n} t_i v_i \right| | &= \left| \sum_{i=0}^{n} t_i (v - v_i) \right| \\ &\leq \sum_{i=0}^{n} t_i |v - v_i| \\ &\leq \sum_{i=0}^{n} t_i \max_{0 \leq j \leq n} |v - v_j| \\ &= \max_{0 \leq j \leq n} |v - v_j|. \end{aligned}$$

The significance of the factor $n/(n+1)$ is that by repeated barycentric subdivision we can produce simplices of arbitrarily small diameter since $(n/(n+1))^r$ approaches 0 as $r$ goes to infinity. It is important that the bound $n/(n+1)$ does not depend on the shape of the simplex since repeated barycentric subdivision produces simplices of many different shapes.

To obtain the bound $n/(n+1)$ on the ratio of diameters, we therefore need to verify that the distance between any two vertices $w_j$ and $w_k$ of a simplex $[w_0, \ldots, w_n]$ of the barycentric subdivision of $[v_0, \ldots, v_n]$ is at most $n/(n+1)$ times the diameter of $[v_0, \ldots, v_n]$.

**(2)** Barycentric Subdivision of Linear Chains. The main part of the proof will be to construct a subdivision operator $\mathcal{S}\colon S_n(X) \to S_n(X)$ and show that this is chain homotopic to the identity map. First we will construct $\mathcal{S}$ and the chain homotopy in a more restricted linear setting.

For a convex set $Y$ in some Euclidean space, the linear maps $\Delta^n \to Y$ generate a subgroup of $S_n(Y)$ that we denote $L_n(Y)$, the **linear chains**. Note that $L(Y)$ is $\partial$-stable, so the linear chains form a subcomplex of $(S(Y), \partial)$. We can uniquely designate a linear map $\lambda\colon \Delta^n \to Y$ by $[w_0, \ldots, w_n]$ where $w_i$ is the image under $\lambda$ of the $i$th vertex of $\Delta^n$. Indeed, by linearity we have $\lambda(\sum t_i e_i) = \sum t_i \lambda(e_i)$. To avoid having to make exceptions for 0-simplices, it will be convenient to augment the complex $(L(Y), \partial)$ by setting $L_{-1}(Y) = R$ generated by the empty simplex $[\varnothing]$, with $\partial[w_0] = [\varnothing]$ for all 0-simplices $[w_0]$.

Each point $b \in Y$ determines a graded homomorphism $b\colon L(Y) \to L(Y)$ of degree 1, defined on basis elements by $b\left([w_0, \ldots, w_n]\right) = [b, w_0, \ldots, w_n]$. Geometrically, the homomorphism $b$ can be regarded as a cone operator, sending a linear chain to the cone having the linear chain as the base of the cone and the point $b$ as the tip of the cone. Applying the usual formula for $\partial$, we obtain the relation

$$\begin{aligned} \partial b([w_0, \ldots, w_n]) &= \partial[b, w_0, \ldots, w_n]) \\ &= [w_0, \ldots, w_n] - b(\partial[w_0, \ldots, w_n]). \end{aligned}$$

By linearity it follows that $\partial b(\alpha) = \alpha - b(\partial\alpha)$ for all $\alpha \in L(Y)$. This expresses algebraically the geometric fact that the boundary of a cone consists of its base together with the cone on the boundary of its base. The relation $\partial b(\alpha) = \alpha - b(\partial\alpha)$ can be rewritten as

$$\partial b + b\partial = 1,$$

so $b$ is a chain homotopy between the identity map and the zero map of the augmented chain complex $(L(Y), \partial)$.

Now we define a graded homomorphism $\mathcal{S}\colon L(Y) \to L(Y)$ by induction on $n$. Let $\lambda\colon \Delta^n \to Y$ be a generator of $L(Y)$ and let $b_\lambda$ be the image of the barycenter of $\Delta^n$ under $\lambda$. Then the inductive formula for $\mathcal{S}$ is

$$\mathcal{S}(\lambda) = b_\lambda(\mathcal{S}(\partial\lambda)),$$

where $b_\lambda\colon L(Y) \to L(Y)$ is the cone operator defined in the preceding paragraph. The induction starts with $\mathcal{S}([\varnothing]) = [\varnothing]$, so $\mathcal{S}$ is the identity on $L_{-1}(Y)$. To get a feel for the map $\mathcal{S}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned}
\mathcal{S}[w_0] &= w_0\,(\mathcal{S}(\partial[w_0])) \\
&= w_0(\mathcal{S}[\varnothing]) \\
&= w_0[\varnothing] \\
&= [w_0].
\end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter $b_{01}$. Then

$$\begin{aligned}
\mathcal{S}[w_0, w_1] &= b_{01}\,(\mathcal{S}(\partial[w_0, w_1])) \\
&= b_{01}(\mathcal{S}[w_1] - \mathcal{S}[w_0]) \\
&= b_{01}([w_1] - [w_0]) \\
&= [b_{01}, w_1] - [b_{01}, w_0].
\end{aligned}$$

Now let $[w_0, w_1, w_2] \in L_2(Y)$ with barycenter $b_{012}$. Then

$$\begin{aligned}
\mathcal{S}[w_0, w_1, w_2] &= b_{012}\,(\mathcal{S}(\partial[w_0, w_1, w_2])) \\
&= b_{012}(\mathcal{S}[w_1, w_2] - \mathcal{S}[w_0, w_2] + \mathcal{S}[w_0, w_1]) \\
&= b_{012}([b_{12}, w_2] - [b_{12}, w_1] - [b_{02}, w_2] + [b_{02}, w_0] + [b_{01}, w_1] - [b_{01}, w_0]) \\
&= [b_{012}, b_{12}, w_2] - [b_{012}, b_{12}, w_1] + [b_{012}, b_{02}, w_0] - [b_{012}, b_{02}, w_2] + [b_{012}, b_{01}, w_1] - [b_{012}, b_{01}, w_0],
\end{aligned}$$

where $b_{12}$, $b_{02}$, and $b_{01}$ are the barycenters for the simplices $[w_1, w_2]$, $[w_0, w_2]$, and $[w_0, w_1]$ respectively. In general, when $\lambda$ is an embedding, with image a genuine $n$-simplex $[w_0, \dots, w_n]$, then $\mathcal{S}(\lambda)$ is the sum of the $n$-simplices in the barycentric subdivision of $[w_0, \dots, w_n]$, with certain signs that could be computed explicitly.

Let us check that $\mathcal{S}\colon L(Y) \to L(Y)$ is a chain map, i.e. $\partial\mathcal{S} = \mathcal{S}\partial$. Since $\mathcal{S} = 1$ on $L_0(Y)$ and $L_{-1}(Y)$, we certainly have $\partial\mathcal{S} = \mathcal{S}\partial$ on $L_0(Y)$. The result for larger $n$ is given by the following calculation, in which we omit some parentheses to unclutter the formulas:

$$\begin{aligned}
\partial\mathcal{S}\lambda &= \partial b_\lambda(\mathcal{S}\partial\lambda) \\
&= (1 - b_\lambda\partial)(\mathcal{S}\partial\lambda) \\
&= \mathcal{S}\partial\lambda - b_\lambda\partial(\mathcal{S}\partial\lambda) \\
&= \mathcal{S}\partial\lambda - b_\lambda\mathcal{S}(\partial\partial\lambda) \\
&= \mathcal{S}\partial\lambda,
\end{aligned}$$

where $\partial S(\partial\lambda) = S\partial(\partial\lambda)$ follows by induction on $n$.

We next build a chain homotopy $\mathcal{T}\colon L(Y) \to L(Y)$ between $\mathcal{S}$ and the identity. We define $\mathcal{T}$ on $L_n(Y)$ inductively by setting $\mathcal{T} = 0$ for $n = -1$ and let $\mathcal{T}\lambda = b_\lambda(\lambda - \mathcal{T}\partial\lambda)$ for $n \geq 0$. The induction starts with $\mathcal{T}[\varnothing] = 0$. To get a feel for the map $\mathcal{T}$, let $[w_0] \in L_0(Y)$. Then

$$\begin{aligned}
\mathcal{T}[w_0] &= w_0\,([w_0] - \mathcal{T}\partial[w_0]) \\
&= w_0\,([w_0] - \mathcal{T}[\varnothing]) \\
&= [w_0, w_0].
\end{aligned}$$

Now let $[w_0, w_1] \in L_1(Y)$ with barycenter $b_{01}$. Then

$$\begin{aligned}
\mathcal{T}[w_0, w_1] &= b_{01}\,([w_0, w_1] - \mathcal{T}\partial[w_0, w_1]) \\
&= b_{01}\,([w_0, w_1] - \mathcal{T}[w_1] + \mathcal{T}[w_0]) \\
&= [b_{01}, w_0, w_1] - [b_{01}, w_1, w_1] + [b_{01}, w_0, w_0].
\end{aligned}$$

The geometric motivation for this formula is an inductively defined subdivision of $\Delta^n \times I$ obtained by joining all simplices in $\Delta^n \times \{0\} \cup \partial\Delta^n \times I$ to the barycenter of $\Delta^n \times \{1\}$. What $\mathcal{T}$ actually does is take the image of this subdivision under the projection $\Delta^n \times I \to \Delta^n$.

The chain homotopy formula $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ is trivial on $L_{-1}(Y)$ where $\mathcal{T} = 0$ and $\mathcal{S} = 1$. Verifying the formula on $L_n(Y)$ with $n \geq 0$ is done by the calculation

$$
\begin{aligned}
\partial\mathcal{T}\lambda &= \partial b_\lambda(\lambda - \mathcal{T}\partial\lambda) \\
&= (1 - b_\lambda\partial)(\lambda - \mathcal{T}\partial\lambda) \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\mathcal{T}\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda(1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial\lambda - b_\lambda\mathcal{S}\partial\lambda - b_\lambda\mathcal{T}\partial\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - b_\lambda\mathcal{S}\partial\lambda \\
&= \lambda - \mathcal{T}\partial\lambda - \mathcal{S}\lambda.
\end{aligned}
$$

where $\partial\mathcal{T}\partial\lambda = (1 - \mathcal{S} - \mathcal{T}\partial)\partial\lambda$ follows by induction on $n$. Now we discard $L_{-1}(Y)$ and the relation $\partial\mathcal{T} + \mathcal{T}\partial = 1 - \mathcal{S}$ still holds since $\mathcal{T}$ was zero on $L_{-1}(Y)$.

**(3)** Barycentric Subdivision of General Chains. Define $\mathcal{S}\colon S_n(X) \to S_n(X)$ by setting $\mathcal{S}\sigma = \sigma_\#\mathcal{S}\Delta^n$ for a singular $n$-simplex $\sigma\colon \Delta^n \to X$. Since $\mathcal{S}\Delta^n$ is the sum of the $n$-simplices in the barycentric subdivision of $\Delta^n$, with certain signs, $\mathcal{S}\sigma$ is the corresponding signed sum of the restrictions of $\sigma$ to the $n$-simplices of the barycentric subdivision of $\Delta^n$. For example, if $\sigma \in S_1(X)$, then

$$
\begin{aligned}
\mathcal{S}\sigma &= \sigma_\#\mathcal{S}[e_0, e_1] \\
&= \sigma \circ ([b, e_1] - [e_0, b]) \\
&= \sigma|_{[b,e_1]} - \sigma|_{[e_0,b]},
\end{aligned}
$$

where $b = (e_0 + e_1)/2$ is the barycenter of $[e_0, e_1]$.

The operator $\mathcal{S}$ is a chain map since

$$
\begin{aligned}
\partial\mathcal{S}\sigma &= \partial\sigma_\#\mathcal{S}\Delta^n \\
&= \sigma_\#\partial\mathcal{S}\Delta^n \\
&= \sigma_\#\mathcal{S}\partial\Delta^n \\
&= \sigma_\# S(\sum_i (-1)^i \Delta_i^n) \\
&= \sum_i (-1)^i \sigma_\# S\Delta_i^n \\
&= \sum_i (-1)^i S(\sigma|_{\Delta_i^n}) \\
&= S\left(\sum_i (-1)^i \sigma|_{\Delta_i^n}\right) \\
&= S(\partial\sigma).
\end{aligned}
$$

where $\Delta_i$ is the $i$th face of $\Delta^n$.

In similar fashion we define $T\colon S_n(X) \to S_n(X)$ by $T\sigma = \sigma_\# T\Delta^n$, and this gives a chain homotopy between $S$ and the identity, since the formula $\partial T + T\partial = 1 - S$ holds by the calculation

$$
\begin{aligned}
\partial T\sigma &= \partial\sigma_\# T\Delta^n \\
&= \sigma_\#\partial T\Delta^n \\
&= \sigma_\#(\Delta^n - S\Delta^n - T\partial\Delta^n) \\
&= \sigma - S\sigma - \sigma_\# T\partial\Delta^n \\
&= \sigma - S\sigma - T(\partial\sigma)
\end{aligned}
$$

where the last equality follows just as in the previous displayed calculation, with $S$ replaced by $T$.

**(4)** Iterated Barycentric Subdivision. A chain homotopy between 1 and the iterate $S^m$ is given by the operator

$D_m = \sum_{0\le i<m} TS^i$ since

$$\begin{aligned}
\partial D_m + D_m\partial &= \sum_{0\le i<m} (\partial TS^i + TS^i\partial) \\
&= \sum_{0\le i<m} (\partial TS^i + T\partial S^i) \\
&= \sum_{0\le i<m} (\partial T + T\partial) S^i \\
&= \sum_{0\le i<m} (1-S) S^i \\
&= \sum_{0\le i<m} (S^i - S^{i+1}) \\
&= 1 - S^m.
\end{aligned}$$

For each singular $n$-simplex $\sigma\colon \Delta^n \to X$ there exists an $m$ such that $S^m(\sigma)$ lies in $S_n^{\mathcal{U}}(X)$ since the diameter of the simplices of $S^m(\Delta^n)$ will be less than a Lebesgue number of the cover of $\Delta^n$ by the open sets $\sigma^{-1}(\mathrm{int}(U_j))$ if $m$ is large enough. (Recall that a Lebesgue number for an open cover of a compact metric space is a number $\varepsilon > 0$ such that every set of diameter less than $\varepsilon$ lies in some set of the cover; such a number exists by an elementary compactness argument). We cannot expect the same number $m$ to work for all $\sigma$'s, so let us define $m(\sigma)$ to be the smallest $m$ such that $S^m(\sigma)$ is in $S_n^{\mathcal{U}}(X)$.

We now define $D\colon S_n(X) \to S_{n+1}(X)$ by setting $D\sigma = D_{m(\sigma)}\sigma$ for each singular $n$-simplex $\sigma\colon \Delta^n \to X$. For this $D$ we would like to find a chain map $\rho\colon S_n(X) \to S_n(X)$ with image in $S_n^{\mathcal{U}}(X)$ satisfying the chain homotopy equation

$$\partial D + D\partial = 1 - \rho. \tag{1}$$

A quick way to do this is to simply regard this equation as defining $\rho$, so we let $\rho = 1 - \partial D - D\partial$. It follows easily that $\rho$ is a chain map since

$$\begin{aligned}
\partial\rho(\sigma) &= \partial\sigma - \partial^2 D\sigma - \partial D\partial\sigma \\
&= \partial\sigma - \partial D\partial\sigma \\
&= \partial\sigma - \partial D\partial\sigma - D\partial^2\sigma \\
&= \rho(\partial\sigma).
\end{aligned}$$

To check that $\rho$ takes $S_n(X)$ to $S_n^{\mathcal{U}}(X)$, we compute $\rho(\sigma)$ more explicitly:

$$\begin{aligned}
\rho(\sigma) &= \sigma - \partial D\sigma - D(\partial\sigma) \\
&= \sigma - \partial D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \\
&= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma).
\end{aligned}$$

The term $S^{m(\sigma)}\sigma$ lies in $S_n^{\mathcal{U}(X)}$ by the definition of $m(\sigma)$. The remaining terms $D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma)$ are linear combinations of terms $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ for $\sigma_j$ the restriction of $\sigma$ to a face of $\Delta^n$, so $m(\sigma_j) \le m(\sigma)$ and hence the difference $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ consists of terms $TS^i(\sigma_j)$ with $i \ge m(\sigma_j)$, and these terms lie in $S_n^{\mathcal{U}}(X)$ since $T$ takes $S_{n-1}^{\mathcal{U}}(X)$ to $S_n^{\mathcal{U}}(X)$.

View $\rho$ as a chain map $S_n(X) \to S_n^{\mathcal{U}}(X)$, the equation (1) says that $\partial D + D\partial = 1 - \iota\rho$ for $\iota\colon S_n^{\mathcal{U}}(X) \hookrightarrow S_n(X)$ the inclusion. Furthermore, $\rho\iota = 1$ since $D$ is identically zero on $S_n^{\mathcal{U}}(X)$, as $m(\sigma) = 0$ if $\sigma$ is in $S_n^{\mathcal{U}}(X)$, hence the summation defining $D\sigma$ is empty. Thus we have shown that $\rho$ is a chain homotopy inverse for $\iota$.

□

Let $R$ be a ring and let $M$ be an $R$-module. A derivation is a map $d\colon R \to M$ such that

## 1.4 Singular Cohomology

Let $R$ be a ring and $N$ and $R$-module. If $M$ is a graded $R$-module, then we set $\mathrm{Hom}_R(M,N)_{\mathrm{gr}}$ to be the graded $R$-module whose homogeneous component in degree $n$ is $M_n := \mathrm{Hom}_R(M_n, N)$. If $(M,d)$ is a chain complex over $R$, where $M$ is considered a graded $R$-module and $d$ is considered a graded endomorphism $d\colon M \to M$ of degree $-1$, then we obtain a cochain complex over $R$ given by $(\mathrm{Hom}_R(M,N)_{\mathrm{gr}}, d_*)$, where if $\psi \in \mathrm{Hom}_R(M_{n-1}, N)$ then $d_*(\psi) = \psi \circ d \in \mathrm{Hom}_R(M_n, N)$.

In particular, we obtain a cochain complex $(\mathrm{Hom}_R(S(X),N)_{\mathrm{gr}}, \partial_*)$ called the **singular cochain complex of** $X$ **over** $R$ **with values in** $N$. Elements in $S_n(X)^\vee$ are called **singular** $n$**-cochains** and the $n$th cohomology, called the **singular cohomology of** $X$ **over** $R$, is denoted $H^n_{\text{sing}}(X, R)$. For notational purposes, we denote $\delta := \partial_{\text{gr}}^{\vee}$ and $S^n(X, R) := S_n(X, R)^{\vee}$. We can work out $\delta$ explicitly as follows: if $\psi \in S^n(X)$, then $\delta(\psi) \in S^{n+1}(X)$ is given by

$$\delta(\psi)(\sigma) = \psi(\partial(\sigma)) = \sum_i (-1)^i \psi(\sigma_i)$$

for all $\sigma \in S_{n+1}(X)$.