Abstract Algebra Homework 4

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Throughout this homework, let R be a commutative ring.

Problem 1

Definition 0.1. Let M be an R-module. We say M is **divisible** if $aM = M$ for every nonzerodivisor $a \in R$.

Problem 1.a

Proposition 0.1. *Let $\varphi \colon M \twoheadrightarrow N$ be a surjective map of R-modules and suppose M is divisible. Then N is divisible.*

Proof. Let $a \in R$ be a nonzerodivisor and let $v \in N$. We must find a $v' \in N$ such that $av' = v$. It will then follow that $aN = N$, which will imply N is divisible. Since φ is surjective, we may choose a $u \in M$ such that $\varphi(u) = v$. Since M is divisible, we may choose a $u' \in M$ such that $au' = u$. Then setting $v' = \varphi(u')$, we have

$$\begin{aligned} av' &= a\varphi(u') \\ &= \varphi(au') \\ &= \varphi(u) \\ &= v. \end{aligned}$$

Thus N is divisible. □

Problem 1.b

Proposition 0.2. *Assume that R is a PID and let M be any R-module. Then M may be decomposed as $M = D \oplus N$ where D is divisible and N has no nontrivial divisible subgroups.*

Proof. We first argue using Zorn's Lemma that M contains a maximal divisible submdoule. Consider the partially ordered set $(\mathscr{F}, \subseteq)$, where $\mathscr{F}$ is the family of all divisible submodules of M:

$$\mathscr{F} = \{D \subseteq M \mid D \text{ is divisible submodule of } M\},$$

and where the partial order $\subseteq$ is set inclusion. Note that $\mathscr{F}$ is nonempty since the zero module is divisible. Let $\{D_i \mid i \in I\}$ be a totally ordered subset of $\mathscr{F}$. We claim that

$$\bigcup_{i \in I} D_i$$

is a divisible submodule of M, and hence an upper bound of $\{D_i \mid i \in I\}$.

To see this, we first show that $\bigcup_{i \in I} D_i$ is a submodule of M. Indeed, it is nonempty since $0 \in \bigcup_{i \in I} D_i$. Also, if $a \in R$ and $u, v \in \bigcup_{i \in I} D_i$, then there exists an $i \in I$ such that $u, v \in D_i$ since $\{D_i \mid i \in I\}$ is totally ordered, and so

$$au + v \in D_i \subseteq \bigcup_{i \in I} D_i.$$

Thus $\bigcup_{i \in I} D_i$ is a submodule of M.

Now we show that $\bigcup_{i \in I} D_i$ is divisible. Let a be a nonzero divisor in R and let u be an element in $\bigcup_{i \in I} D_i$. Then there exists an $i \in I$ such that $u \in D_i$, and as D_i is divisible, there exists a

$$v \in D_i \in \bigcup_{i \in I} D_i$$

such that $av = u$. It follows that $\bigcup_{i \in I} D_i$ is divisible.

Thus the conditions for Zorn's Lemma are satisfied and so there exists a maximal divisible submodule of M, say $D \subseteq M$. Since every divisible module over a PID is injective[1], we see that D is injective, and thus we have a direct sum decomposition of M say

$$M = D \oplus N$$

where N is a submodule of M. To finish the proof, assume for a contradiction that N has a nontrivial divisible submodule, say $L \subseteq N$. We claim that $D + L$ is a divisible submodule of M which properly contains D. Indeed, it is divisible since if $a \in R$ is a nonzerodivisor and $x + y \in D + L$ where $x \in D$ and $y \in L$, then we can choose $u \in D$ and $v \in L$ such that $au = x$ and $av = y$ since D and L are divisible, and so

$$\begin{aligned} a(u+v) &= au + av \\ &= x + y \end{aligned}$$

implies $D + L$ is divisible. It also properly contains D since $L \subseteq N$ is nontrivial. Thus $D + L$ is a divisible submodule of M which properly contains D. This is a contradiction as D was chosen to be a maximal divisible submodule of M. □

Problem 1.c

Proposition 0.3. *Assume that R is a PID. Then any R-module can be embedded into an R-module which is divisible.*

Proof. Any R-module can be embedded into an injective R-module and every injective R-module is divisible by Proposition (0.9) (this is proved in the Appendix). □

Problem 2

Proposition 0.4. *The sequence of R-modules*

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0 \tag{1}$$

is exact if and only if for all R-modules N the induced sequence

$$0 \longrightarrow \mathrm{Hom}_R(M_3, N) \xrightarrow{\varphi_2^*} \mathrm{Hom}_R(M_2, N) \xrightarrow{\varphi_1^*} \mathrm{Hom}_R(M_1, N) \tag{2}$$

is exact.

Proof. Suppose that (1) is exact and let N be any R-module. We first show exactness at $\mathrm{Hom}_R(M_3, N)$. Let $\psi_3 \in \ker \varphi_2^*$. Then

$$\begin{aligned} 0 &= \varphi_2^*(\psi_3) \\ &= \psi_3 \varphi_2 \\ &= \psi_3, \end{aligned}$$

where we used the fact that φ_2 is surjective to obtain the third line from the second line. Therefore φ_2^* is injective, which implies exactness at $\mathrm{Hom}_R(M_3, N)$.

Next we show exactness at $\mathrm{Hom}_R(M_2, N)$. Let $\psi_2 \in \ker \varphi_1^*$. Then

$$\begin{aligned} 0 &= \varphi_1^*(\psi_2) \\ &= \psi_2 \varphi_1 \end{aligned}$$

implies ψ_2 kills the image of φ_1. We define $\psi_3 \colon M_3 \to N$ as follows: let $u_3 \in M_3$. Choose $u_2 \in M_2$ such that $\varphi_2(u_2) = u_3$ (such a choice is possible since φ_2 is surjective). We define

$$\psi_3(u_3) = \psi_2(u_2).$$

[1]For completeness, we included proof of this in the Appendix.

The map ψ_3 is well-defined since ψ_2 kills the image of φ_1. Indeed, if $v_2 \in M_2$ was another lift of u_3 under φ_2, then

$$\begin{aligned} v_2 - u_2 &\in \ker \varphi_2 \\ &= \operatorname{im} \varphi_1. \end{aligned}$$

Thus

$$\begin{aligned} \psi_2(v_2) &= \psi_2(v_2 - u_2 + u_2) \\ &= \psi_2(v_2 - u_2) + \psi_2(u_2) \\ &= \psi_2(u_2). \end{aligned}$$

Thus the map ψ_3 is well-defined. The map ψ_3 is also R-linear. Indeed, let $a, b \in R$ and let $u_3, v_3 \in M_3$. Choose lifts of u_3, v_3 under φ_2, say $u_2, v_2 \in M_2$ (so $\varphi_2(u_2) = u_3$ and $\varphi(v_2) = v_3$). Then $au_2 + bv_2$ is easily seen to be a lift of $au_3 + bv_3$ under φ and so we have

$$\begin{aligned} \psi_3(au_3 + bv_3) &= \psi_2(au_2 + bv_2) \\ &= a\psi_2(u_2) + b\psi_2(v_2) \\ &= a\psi_3(u_3) + b\psi_3(v_3). \end{aligned}$$

Thus ψ_3 is R-linear. Finally, observe that

$$\begin{aligned} \varphi_2^*(\psi_3)(u_2) &= (\psi_3\varphi_2)(u_2) \\ &= \psi_3(\varphi_2(u_2)) \\ &= \psi_3(u_3) \\ &= \psi_2(u_2) \end{aligned}$$

for all $u_2 \in M_2$. It follows that $\psi_2 = \varphi_2^*(\psi_3)$, and hence $\psi_2 \in \operatorname{im} \varphi_2^*$. Therefore we have exactness at $\operatorname{Hom}_R(M_2, N)$.

Conversely, suppose that (1) is exact for all R-modules N. We first show φ_2 is surjective. Set $N = M_3/\operatorname{im}\varphi_2$ and let $\pi\colon M_3 \to M_3/\operatorname{im}\varphi_2$ be the quotient map. Observe that

$$\begin{aligned} \varphi_2^*(\pi) &= \pi\varphi_2 \\ &= 0 \\ &= \varphi_2^*(0). \end{aligned}$$

It follows from injectivity of φ_2^* that $\pi = 0$. In other words, $M_3 = \operatorname{im}\varphi_2$, hence φ_2 is surjective.

Next we show exactness at M_2. First set $N = M_3$. Then exactness of (1) implies

$$\begin{aligned} 0 &= (\varphi_1^*\varphi_2^*)(1_{M_3}) \\ &= (\varphi_1^*(\varphi_2^*(1_{M_3})) \\ &= \varphi_1^*(1_{M_3}\varphi_2) \\ &= 1_{M_3}\varphi_2\varphi_1 \\ &= \varphi_2\varphi_1. \end{aligned}$$

Thus $\ker\varphi_2 \supseteq \operatorname{im}\varphi_1$. For the reverse inclusion, set $N = M_2/\operatorname{im}\varphi_1$ and let $\pi\colon M_2 \to M_2/\operatorname{im}\varphi_1$ be the quotient map. Then

$$\begin{aligned} \varphi_1^*(\pi) &= \pi\varphi_1 \\ &= 0 \end{aligned}$$

implies there exists $\psi_3\colon M_3 \to M_2/\operatorname{im}\varphi_1$ such that $\pi = \varphi_2^*(\psi_3)$ by exactness of (1). Thus, if $u_2 \in \ker\varphi_2$, then

$$\begin{aligned} 0 &= \psi_3(0) \\ &= \psi_3(\varphi_2(u_2)) \\ &= (\psi_3\varphi_2)(u_2) \\ &= (\varphi_2^*(\psi_3))(u_2) \\ &= \pi(u_2) \end{aligned}$$

implies $u_2 \in \operatorname{im}\varphi_1$. Thus $\ker\varphi_2 \subseteq \operatorname{im}\varphi_1$. □

Problem 3

Problem 3.a

Proposition 0.5. *Let M be an R-module. Then*

$$\mathrm{Hom}_R(R/I, M) \cong 0 :_M I,$$

where

$$0 :_M I = \{u \in M \mid xm = 0 \text{ for all } x \in I\}.$$

Proof. We define $\Psi\colon \mathrm{Hom}_R(R/I, M) \to 0 :_M I$ by

$$\Psi(\varphi) = \varphi(\overline{1})$$

for all $\varphi \in \mathrm{Hom}_R(R/I, M)$. Note that Ψ lands in $0 :_M I$ since if $x \in I$, then

$$\begin{aligned} x\varphi(\overline{1}) &= \varphi(\overline{x}) \\ &= \varphi(\overline{0}) \\ &= 0. \end{aligned}$$

We claim that Ψ is an R-module isomorphism.

Let us first show that it is an R-linear map. Let $a, b \in R$ and let $\varphi, \psi \in \mathrm{Hom}_R(R/I, M)$. Then

$$\begin{aligned} \Psi(a\varphi + b\psi) &= (a\varphi + b\psi)(\overline{1}) \\ &= a\varphi(\overline{1}) + b\psi(\overline{1}) \\ &= a\Psi(\varphi) + b\Psi(\varphi). \end{aligned}$$

Thus Ψ is an R-linear map.

Next, we show that Ψ is bijective by constructing an inverse map. Define $\Phi\colon 0 :_M I \to \mathrm{Hom}_R(R/I, M)$ by

$$\Phi(u) = \varphi_u$$

for all $u \in 0 :_M I$, where $\varphi_u\colon R/I \to M$ is defined by

$$\varphi_u(\overline{a}) = au$$

for all $\overline{a} \in R/I$. Note that φ_u is well-defined here since if $a + x$ is another represenative of the coset $\overline{a}$ where $x \in I$, then

$$\begin{aligned} \varphi_u(\overline{a+x}) &= (a+x)u \\ &= au \\ &= \varphi_u(\overline{a}). \end{aligned}$$

Similarly, φ_u is easily checked to be R-linear. Thus Φ lands in $\mathrm{Hom}_R(R/I, M)$. Moreover, it is an inverse to Ψ since if $\varphi \in \mathrm{Hom}_R(R/I, M)$, then

$$\begin{aligned} (\Phi\Psi)(\varphi) &= \Phi(\Psi(\varphi)) \\ &= \Phi(\varphi(\overline{1})) \\ &= \varphi_{\varphi(\overline{1})} \\ &= \varphi, \end{aligned}$$

where the last equality follows from

$$\begin{aligned} \varphi_{\varphi(\overline{1})}(\overline{a}) &= \overline{a}\varphi(\overline{1}) \\ &= \varphi(\overline{a}) \end{aligned}$$

for all $\overline{a} \in R/I$. Thus $\Phi\Psi = 1$.

Similarly, if $u \in 0 :_M I$, then

$$\begin{aligned}(\Psi\Phi)(u) &= \Psi(\Phi(u)) \\ &= \Phi(\varphi_u)) \\ &= \varphi_u(\overline{1}) \\ &= u.\end{aligned}$$

Thus $\Psi\Phi = 1$. □

Corollary. *Let A be an abelian group. Then*

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong A[m]$$

where $A[m] = \{a \in A \mid ma = 0\}$.

Proof. This follows from Proposition (0.5) by taking $R = \mathbb{Z}$, $M = A$, and $I = m\mathbb{Z}$. □

Problem 3.b

Proposition 0.6. *Let $m, n \in \mathbb{N}$ and let $d = \gcd(m, n)$. Then*

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

Proof. By Corollary (), it suffices to show that $\mathbb{Z}/d\mathbb{Z} \cong 0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$. Indeed, since $0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$ is a submodule of $\mathbb{Z}/n\mathbb{Z}$, it must be equal to a module of the form $k\mathbb{Z}/n\mathbb{Z}$ where $n \mid k$. Define $\Psi \colon \mathbb{Z}/d\mathbb{Z} \to 0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$ by

$$\Psi(\overline{a}) = \overline{(n/d)a}.$$

for all $\overline{a} \in \mathbb{Z}/d\mathbb{Z}$.[2]We claim that Ψ gives the desired isomorphism. Indeed, we first need to show that Ψ is well-defined. Let $a + db$ is another representative of the coset $\overline{a}$. Then

$$\begin{aligned}\Psi(\overline{a+db}) &= \overline{(n/d)(a+db)} \\ &= \overline{(n/d)a + nb} \\ &= \overline{(n/d)a} \\ &= \Psi(\overline{a}).\end{aligned}$$

Thus Ψ is well-defined.

Next we need to show that Ψ lands in $0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$. Let $\overline{a} \in \mathbb{Z}/d\mathbb{Z}$. Then

$$\begin{aligned}m\Psi(\overline{a}) &= m\overline{(n/d)a} \\ &= \overline{m(n/d)a} \\ &= \overline{(mn/d)a} \\ &= \overline{n(m/d)a} \\ &= \overline{0}.\end{aligned}$$

Thus Ψ lands in $0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$.

Finally, we show that Ψ is an isomorphism. Note that the map Ψ is $\mathbb{Z}$-linear since it is just the "multiplication by $n/d \in \mathbb{Z}$" map. It remains to show that Ψ is bijective. We first show it is injective. Let $\overline{a} \in \ker \Psi$. Then

$$\begin{aligned}\overline{0} &= \Psi(\overline{a}) \\ &= \overline{(n/d)a}\end{aligned}$$

implies

$$(n/d)a = nb \tag{3}$$

[2]Our notation is a little ambiguous here in that we use the overline notation to denote a coset both in $\mathbb{Z}/d\mathbb{Z}$ and in $\mathbb{Z}/n\mathbb{Z}$. However we often do this in Mathematics in order to clean notation. For instance, we use the same $+$ symbol to denote addition in any abelian group. Context will always make it clear what our notation is referring to.

for some $n \in \mathbb{Z}$. Multiplying both sides of (3) by d gives us

$$\begin{aligned} dnb &= d(n/d)a \\ &= na, \end{aligned}$$

which imlpies $a = db$ since $\mathbb{Z}$ is an integral domain. Thus $\overline{a} = \overline{0}$ in $\mathbb{Z}/d\mathbb{Z}$, which implies Ψ is injective.

Now we show it is surjective. Before doing so, we first choose $x, y \in \mathbb{Z}$ such that

$$mx + ny = d.$$

Such a choice is possible since $d = \gcd(m, n)$. Now let $\overline{b} \in 0 :_{\mathbb{Z}/n\mathbb{Z}} m\mathbb{Z}$. Then $m\overline{b} = \overline{0}$ implies there exists a $c \in \mathbb{Z}$ such that

$$mb = nc$$

Then

$$\begin{aligned} b &= b((m/d)x + (n/d)y) \\ &= (bm/d)x + (n/d)by \\ &= (nc/d)x + (n/d)by \\ &= (n/d)cx + (n/d)by \\ &= (n/d)(cx + by). \end{aligned}$$

Therefore, setting $a = cx + by$, we see that

$$\begin{aligned} \Psi(\overline{a}) &= \overline{(n/d)a} \\ &= \overline{(n/d)(cx + by)} \\ &= \overline{b}. \end{aligned}$$

implies Ψ is surjective. $\square$

Problem 4

Proposition 0.7. *Let M be an R-module, let I be an index set, and let N_i be an R-module for each $i \in I$. Then*

$$Hom_R\left(\bigoplus_{i \in I} N_i, M\right) \cong \prod_{i \in I} Hom_R\left(N_i, M\right)$$

Proof. For each $i \in I$, let $\iota_i \colon N_i \to \bigoplus_{i \in I} N_i$ denote the ith inclusion map. Define a map $\Psi \colon \mathrm{Hom}_R\left(\bigoplus_{i \in I} N_i, M\right) \to \prod_{i \in I} \mathrm{Hom}_R\left(N_i, M\right)$ by

$$\Psi(\varphi) = (\varphi|_{N_i}) = (\varphi \circ \iota_i)$$

for all $\varphi \in \mathrm{Hom}_R\left(\bigoplus_{i \in I} N_i, M\right)$. The map Ψ is R-linear as it is a composition of R-linear maps in each component. To see that it is an isomorphism, we construct an inverse map. Define a map $\Phi \colon \prod_{i \in I} \mathrm{Hom}_R\left(N_i, M\right) \to \mathrm{Hom}_R\left(\bigoplus_{i \in I} N_i, M\right)$ by

$$\Phi((\varphi_i))(y_{i_1} + \cdots + y_{i_n}) = \varphi_{i_1}(y_{i_1}) + \cdots + \varphi_{i_n}(y_{i_n})$$

for all $(\varphi_i) \in \prod_{i \in I} \mathrm{Hom}_R\left(N_i, M\right)$ and $y_{i_1} + \cdots + y_{i_n} \in \bigoplus_{i \in I} N_i$.

Let us check that Ψ is indeed the inverse to Φ. Let $\varphi \in \mathrm{Hom}_R\left(\bigoplus_{i \in I} N_i, M\right)$ and let $y_{i_1} + \cdots + y_{i_n} \in \bigoplus_{i \in I} N_i$. Then

$$\begin{aligned} (\Phi\Psi)(\varphi)(y_{i_1} + \cdots + y_{i_n}) &= \Phi(\varphi|_{N_i})(y_{i_1} + \cdots + y_{i_n}) \\ &= \varphi|_{N_{i_1}}(y_{i_1}) + \cdots + \varphi|_{N_{i_n}}(y_{i_n}) \\ &= \varphi(y_{i_1}) + \cdots + \varphi(y_{i_n}) \\ &= \varphi(y_{i_1} + \cdots + y_{i_n}). \end{aligned}$$

It follows that $\Phi\Psi = 1$.

Conversely, let $(\varphi_i) \in \prod_{i \in I} \operatorname{Hom}_R(N_i, M)$. Observe that for each $i \in I$, we have

$$(\Phi(\varphi_i) \circ \iota_i)(y) = \varphi_i(y)$$

for all $y \in N_i$. It follows that $\Phi(\varphi_i) \circ \iota_i = \varphi_i$. Therefore

$$\begin{aligned}(\Psi\Phi)((\varphi_i)) &= \Psi(\Phi(\varphi_i)) \\ &= (\Phi(\varphi_i) \circ \iota_i) \\ &= (\varphi_i).\end{aligned}$$

This implies $\Psi\Phi = 1$. □

Problem 5

Example 0.1. Let R be a Noetherian integral domain, let I be a nonzero ideal, and let K be the field of fractions of R. For each $n \geq 1$, we have

$$\operatorname{Hom}_R(R/I^n, K) \cong 0.$$

To see this, we first show we cannot have $I^n = 0$ for any $n > 1$. Indeed, assume for a contradiction that $I^n = 0$ for some $n > 1$. Choose n to be minimal so that $I^{n-1} \neq 0$ and $I^n = 0$. Choose a nonzero element $x \in I$ and a nonzero element $y \in I^{n-1}$. Then $xy \in I^n = 0$ which implies $xy = 0$, contradicting the fact that R is an integral domain. Now let $m \geq 1$. Choose a nonzero element in $x \in I^m$ and suppose $\varphi \in \operatorname{Hom}_R(R/I^n, K)$. Let $\overline{a} \in R/I^n$. Then

$$\begin{aligned}x\varphi(\overline{a}) &= \varphi(x\overline{a}) \\ &= \varphi(\overline{xa}) \\ &= \varphi(0) \\ &= 0\end{aligned}$$

implies $\varphi(\overline{a}) = 0$ since $x \neq 0$ and K is the field of fractions of R. Thus $\varphi = 0$ and hence $\operatorname{Hom}_R(R/I^m, K) \cong 0$. Thus $\operatorname{Hom}_R(R/I^n, K) \cong 0$ for all $n \geq 1$, which implies

$$\prod_{n \geq 1} \operatorname{Hom}_R(R/I^m, K) \cong 0.$$

On the other hand, we claim that

$$\operatorname{Hom}_R\left(\prod_{n \geq 1} R/I^m, K\right) \ncong 0.$$

Indeed, consider the sequence element $(\overline{1}) \in \prod_{n \geq 1} R/I^m$ and let $a \in R$. Then

$$\begin{aligned}(\overline{a}) = (\overline{0}) &\iff a \in I^n \text{ for all } n \geq 1 \\ &\iff a \in \bigcap_{n \geq 1} I^n \\ &\iff a = 0\end{aligned}$$

where the last equality follows from the fact that $\bigcap_{n \geq 1} I^n = 0$ by Krull's Intersection Theorem. Therefore the map $\varphi \colon \operatorname{span}_R((\overline{1})) \to K$ given by

$$\varphi((\overline{a})) = a$$

for all $(\overline{a}) \in \operatorname{span}_R((\overline{1}))$ is a well-defined R-linear map. Since K is an injective R-module, we can extend this nonzero R-linear map to a nonzero R-linear map $\widetilde{\varphi} \in \operatorname{Hom}_R\left(\prod_{n \geq 1} R/I^m, K\right)$. Thus

$$\operatorname{Hom}_R\left(\prod_{n \geq 1} R/I^m, K\right) \ncong 0.$$

Problem 6

Proposition 0.8. *Every R-module is free if and only if R is a field.*

Proof. If R is a field, then an R-module is just an R-vector space. A standard argument using Zorn's Lemma tells us that every vector space has a basis, and hence every vector space is free.

Conversely, suppose that every R-module is free. Let I be a proper ideal in R. Then R/I is a nonzero free R-module, so there exists an $\overline{a} \in R/I$ such that

$$x\overline{a} = \overline{0}$$

implies $x = 0$ for all $x \in R$. In particular, if $x \in I$, then

$$\begin{aligned} x\overline{a} &= \overline{xa} \\ &= \overline{0} \end{aligned}$$

implies $x = 0$. Thus I must be the zero ideal. Therefore the only proper ideal of R is the zero ideal. This is equivalent to R being a field. □

Appendix

Baer's Criterion

Lemma 0.1. *Let E be an R-module. Then E is injective if and only if for every inclusion of R-modules $M \subset N$ and for every homomorphism $\psi\colon M \to E$ there exists a homomorphism $\widetilde{\psi}\colon N \to E$ such that $\widetilde{\psi}|_M = \psi$.*

Proof. One direction is obvious. To prove the other direction, let $\varphi\colon M \to N$ be an injective homomorphism of R-modules and let $\psi\colon M \to E$ be a homorphism. Since φ is injective, it induces an isomorphism $\varphi\colon M \to \varphi(M)$ of R-modules. Let φ^{-1} be the inverse homomorphism to this isomorphism. Then $\varphi(M) \subset N$ and $\psi\varphi^{-1}\colon \varphi(M) \to E$ is a homomorphism, and so by hypothesis, there exists $\widetilde{\psi}\colon N \to E$ such that $\widetilde{\psi}|_{\varphi(M)} = \psi\varphi^{-1}$. This implies

$$\begin{aligned} \widetilde{\psi}\varphi &= \widetilde{\psi}|_{\varphi(M)}\varphi \\ &= \psi\varphi^{-1}\varphi \\ &= \psi. \end{aligned}$$

Therefore E is injective. □

Theorem 0.2. *(Baer's Criterion) Let E be an R-module. Then E is injective if and only if for every ideal $I \subset R$ and for every homomorphism $\psi\colon I \to E$ there exists a morphism $\widetilde{\psi}\colon R \to E$ such that $\widetilde{\psi}|_I = \psi$.*

Proof. One direction is obvious. For the other direction, let $M \subset N$ be an inclusion of A-modules and let $\psi\colon M \to E$ be a homomorphism. Define the partially ordered set $(\mathscr{F}, \leq)$ where

$$\mathscr{F} := \{\psi'\colon M' \to N \mid M \subset M' \subset N \text{ and } \psi' \text{ extends } \psi\}.$$

and the where partial order $\leq$ is defined by

$$\psi' \leq \psi'' \text{ if and only if } \psi'' \text{ extends } \psi'.$$

If $\mathscr{T}$ is a totally ordered subset of $\mathscr{F}$, then it has an upper bound (namely we take the direct limit of a all $\psi' \in \mathscr{T}$). Therefore by Zorn's lemma, there is a homomorphism $\psi'\colon N' \to E$ with $M \subset N' \subset N$ which is maximal with respect to the property that ψ' extends ψ. We claim that $N' = N$. We will prove this by contradiciton: assume that $N' \neq N$. Choose an element $u \in N \backslash N'$ and consider the ideal

$$I = \{a \in R \mid au \in N'\}.$$

It is a nonempty proper ideal of R since $0 \in I$ and $1 \notin I$. By hypothesis, the composite

$$I \xrightarrow{\cdot u} N' \xrightarrow{\psi'} E$$

extends to a homomorphism $\widetilde{\psi}\colon R \to E$. Define $\psi''\colon N' + Ru \to E$ by the formula

$$\psi''(v + au) = \psi'(v) + \widetilde{\psi}(a)$$

for all $v + au \in N' + Rn$. To see that this is well-defined, suppose $v_1 + a_1 u$ and $v_2 + a_2 u$ represent the same element in $N' + Ru$. Then $v_2 - v_1 = (a_1 - a_2)u$ implies $a_1 - a_2 \in I$. Therefore $\widetilde{\psi}(a_1 - a_2) = \psi'((a_1 - a_2)u)$, and so

$$\begin{aligned}
\psi''(v_2 + a_2 u) &= \psi'(v_2) + \widetilde{\psi}(a_2) \\
&= \psi'(v_2 - (v_2 - v_1))) + \widetilde{\psi}(a_1 + (a_2 - a_1)) \\
&= \psi'(v_2 + (a_1 - a_2)u) + \widetilde{\psi}(a_1 + (a_2 - a_1)) \\
&= \psi'(v_1) + \psi'((a_1 - a_2)u) + \widetilde{\psi}(a_1) + \psi'((a_2 - a_1)u) \\
&= \psi'(v_1) + \widetilde{\psi}(a_1).
\end{aligned}$$

Thus ψ'' is well-defined. We also note that ψ'' extends ψ'. Since ψ' was maximal, this leads to a contradiction. So we must have $N' = N$. □

Divisible Modules Over a PID are Injective

Proposition 0.9. *Let M be an R-module. If M is injective, then M is divisible. The converse holds if R is a PID.*

Proof. Suppose M is injective and let $a \in R$ be a nonzerodivisor. Then the map $\varphi\colon M \to aM$, given by

$$\varphi(u) = au$$

for all $u \in M$ is an injective R-linear map. Thus we obtain a splitting map of φ, say $\psi\colon aM \to M$. Thus if $u \in M$, then we have

$$\begin{aligned}
u &= (\psi\varphi)(u) \\
&= \psi(\varphi(u)) \\
&= \psi(au) \\
&= a\psi(u).
\end{aligned}$$

This implies $M = aM$, that is, M is divisible.

For the converse direction, assume that R is a PID and that M is a divisible R-module. Let $\varphi\colon \langle x \rangle \to M$ be a homomorphism, where $\langle x \rangle$ is an ideal in R. Let $a \in R$ be a nonzerodivisor and set $u = \varphi(x)$. Since $M = xM$, we have $u = xv$ for some $v \in M$. Then the map $\widetilde{\varphi}\colon R \to M$, given by

$$\widetilde{\varphi}(a) = av$$

for all $a \in R$, extends φ. Indeed, it is clearly R-linear. Also

$$\begin{aligned}
\widetilde{\varphi}(bx) &= (bx)v \\
&= b(xv) \\
&= bu \\
&= b\varphi(x) \\
&= \varphi(bx)
\end{aligned}$$

for all $bx \in \langle x \rangle$. It follows from Baer's Criterion that M is injective. □