Measure Theory Homework 1

January 21, 2020

Throughout this homework, let X be a set and let $\mathcal{P}(X)$ denote the set of all subsets of X.

Problem 1

Proposition 0.1. *Let $A, B \in \mathcal{P}(X)$. Then*

1. $1_A = 1_B$ *if and only if* $A = B$;
2. $1_{A\cap B} = 1_A 1_B$;
3. $1_{A\cup B} = 1_A + 1_B - 1_A 1_B$;
4. $1_{A^c} = 1 - 1_A$;
5. $1_{A\setminus B} = 1_A - 1_B$ *if and only if* $B \subseteq A$;
6. $1_A + 1_B \equiv 1_{A\Delta B} \mod 2$.

Proof.
1. Suppose $1_A = 1_B$ and let $x \in A$. Then

$$\begin{aligned} 1 &= 1_A(x) \\ &= 1_B(x) \end{aligned}$$

implies $x \in B$. Thus $A \subseteq B$. Similarly, if $x \in B$, then

$$\begin{aligned} 1 &= 1_B(x) \\ &= 1_A(x) \end{aligned}$$

implies $x \in A$. Thus $B \subseteq A$.
Conversely, suppose $A = B$ and let $x \in X$. If $x \in A$, then $x \in B$, hence

$$\begin{aligned} 1_A(x) &= 1 \\ &= 1_B(x). \end{aligned}$$

If $x \notin A$, then $x \notin B$, hence

$$\begin{aligned} 1_A(x) &= 0 \\ &= 1_B(x). \end{aligned}$$

Therefore the indicator functions 1_A and 1_B agree on all of X, and hence must be equal to each other.

2. Let $x \in X$. If $x \in A \cap B$, then $x \in A$ and $x \in B$, and thus we have

$$\begin{aligned} 1_{A\cap B}(x) &= 1 \\ &= 1 \cdot 1 \\ &= 1_A(x) 1_B(x). \end{aligned}$$

If $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$. Without loss of generality, say $x \notin A$. Then we have

$$\begin{aligned} 1_{A\cap B}(x) &= 0 \\ &= 0 \cdot 1_B(x) \\ &= 1_A(x) 1_B(x). \end{aligned}$$

Therefore the functions $1_{A\cap B}$ and $1_A 1_B$ agree on all of X, and hence must be equal to each other.

3. Let $x \in X$. If $x \in A \cup B$, then either $x \in A$ or $x \in B$. Without loss of generality, say $x \in A$. Then we have

$$\begin{aligned} 1_{A\cup B}(x) &= 1 \\ &= 1 + 1_B(x) - 1_B(x) \\ &= 1 + 1_B(x) - 1 \cdot 1_B(x) \\ &= 1_A(x) + 1_B(x) - 1_A(x) 1_B(x). \end{aligned}$$

If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Therefore we have

$$\begin{aligned} 1_{A\cup B}(x) &= 0 \\ &= 0 + 0 - 0 \cdot 0 \\ &= 1_A(x) + 1_B(x) - 1_A(x) 1_B(x). \end{aligned}$$

Thus the functions $1_{A\cup B}$ and $1_A + 1_B - 1_A 1_B$ agree on all of X, and hence must be equal to each other.

4. Let $x \in X$. If $x \in A$, then $x \notin A^c$, hence

$$\begin{aligned} 1_{A^c}(x) &= 0 \\ &= 1 - 1 \\ &= 1 - 1_A(x). \end{aligned}$$

If $x \notin A$, then $x \in A^c$, hence

$$\begin{aligned} 1_{A^c}(x) &= 1 \\ &= 1 - 0 \\ &= 1 - 1_A(x). \end{aligned}$$

Therefore the functions 1_{A^c} and $1 - 1_A$ agree on all of X, and hence must be equal to each other.

5. Suppose $B \subseteq A$ and let $x \in X$. If $x \in A \backslash B$, then $x \in A$ and $x \notin B$, hence

$$\begin{aligned} 1_{A\backslash B}(x) &= 1 \\ &= 1 - 0 \\ &= 1_A(x) - 1_B(x). \end{aligned}$$

If $x \in B$, then $x \in A$ but $x \notin A \backslash B$, hence

$$\begin{aligned} 1_{A\backslash B}(x) &= 0 \\ &= 1 - 1 \\ &= 1_A(x) - 1_B(x). \end{aligned}$$

If $x \notin A$, then $x \notin B$ and $x \notin A \backslash B$, hence

$$\begin{aligned} 1_{A\backslash B}(x) &= 0 \\ &= 0 - 0 \\ &= 1_A(x) - 1_B(x). \end{aligned}$$

Therefore the functions $1_{A\backslash B}$ and $1_A - 1_B$ agree on all of X, and hence must be equal to each other.

For converse direction, we prove the contrapositive statement. Suppose $B \not\subseteq A$. Choose $b \in B$ such that $b \notin A$. Then

$$\begin{aligned} 1_{A \setminus B}(b) &= 0 \\ &\neq -1 \\ &= 0 - 1 \\ &= 1_A(b) - 1_B(b). \end{aligned}$$

Therefore $1_{A\setminus B} \neq 1_A - 1_B$.

5. Let $x \in X$. If $x \in A$, then $x \notin A^c$, hence

$$\begin{aligned} 1_{A^c}(x) &= 0 \\ &= 1 - 1 \\ &= 1 - 1_A(x). \end{aligned}$$

If $x \notin A$, then $x \in A^c$, hence

$$\begin{aligned} 1_{A^c}(x) &= 1 \\ &= 1 - 0 \\ &= 1 - 1_A(x). \end{aligned}$$

Therefore the functions 1_{A^c} and $1 - 1_A$ agree on all of X, and hence must be equal to each other.

6. We have

$$\begin{aligned} 1_{A\Delta B} &= 1_{(A\setminus B)\cup(B\setminus A)} \\ &= 1_{A\setminus B} + 1_{B\setminus A} - 1_{A\setminus B}1_{B\setminus A} \\ &= 1_{A\setminus A\cap B} + 1_{B\setminus A\cap B} - 1_{(A\setminus B)\cap(B\setminus A)} \\ &= 1_A - 1_{A\cap B} + 1_B - 1_{A\cap B} - 1_{\emptyset} \\ &= 1_A + 1_B - 2\cdot 1_{A\cap B} \\ &\equiv 1_A + 1_B \bmod 2. \end{aligned}$$

$\square$

Problem 2

Proposition 0.2. *Let I be a subinterval of $[a,b]$. Then there exists a Cauchy sequence (f_n) in $(C[a,b], \|\cdot\|_1)$ such that (f_n) converges pointwise to 1_I on $[a,b]$ and moreover*

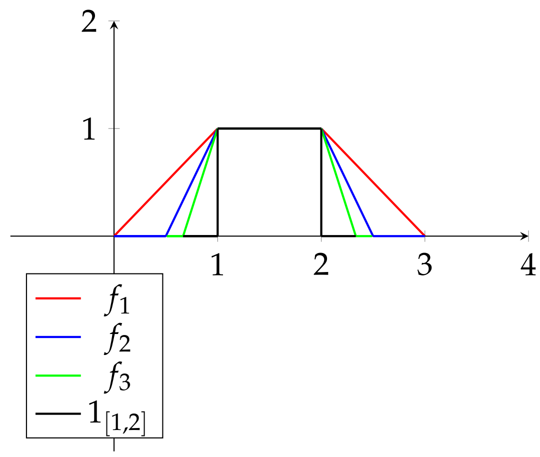
$$\lim_{n\to\infty} \|f_n\|_1 = \text{length}(I).$$

Proof. If $I = \emptyset$, then we take $f_n = 0$ for all $n \in \mathbb{N}$. Thus assume I is a nonempty subinterval of $[a,b]$. We consider two cases; namely $I = (c,d)$ and $I = [c,d]$. The other cases ($I = (c,d]$ and $I = [c,d)$) will easily be seen to be a mixture of these two cases.

Case 1: Suppose $I = [c,d]$. For each $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} 0 & \text{if } a \leq x < c - \left(\frac{c-a}{n}\right) \\ \frac{n}{c-a}(x-c)+1 & \text{if } c - \left(\frac{c-a}{n}\right) \leq x < c \\ 1 & \text{if } c \leq x \leq d \\ \frac{n}{d-b}(x-d)+1 & \text{if } d < x \leq d + \left(\frac{b-d}{n}\right) \\ 0 & \text{if } d + \left(\frac{b-d}{n}\right) < x \leq b \end{cases}$$

The image below gives the graphs for f_1, f_2, and f_3 in the case where $[a,b] = [0,3]$ and $[c,d] = [1,2]$.



For each $n \in \mathbb{N}$, the function f_n is continuous since each of its segments is continuous and are equal on their boundaries.

Let us check that (f_n) converges pointwise to 1_I: If $x \in [a, c)$, then we choose $N \in \mathbb{N}$ such that

$$x \leq c - \left(\frac{c-a}{N}\right).$$

Then $f_n(x) = 0$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} f_n(x) = 0 = 1_I(x).$$

Similarly, if $x \in (d, b]$, then we choose $N \in \mathbb{N}$ such that

$$x \geq d + \left(\frac{b-d}{N}\right).$$

Then $f_n(x) = 0$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} f_n(x) = 0 = 1_I(x).$$

Finally, if $x \in [c, d]$, then $f_n(x) = 1$ for all $n \in \mathbb{N}$ by definition and thus

$$\lim_{n\to\infty} f_n(x) = 0 = 1_I(x).$$

Let us check that (f_n) is Cauchy in $(C[a, b], \|\cdot\|_1)$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\frac{c-a+b-d}{n} < \varepsilon$$

for all $n \geq N$. Then $n \geq m \geq N$ implies

$$\begin{aligned}
\|f_n - f_m\|_1 &= \int_a^b |f_n(x) - f_m(x)| \mathrm{d}x \\
&= \int_a^b (f_n(x) - f_m(x)) \mathrm{d}x \\
&= \int_{c-\left(\frac{c-a}{m}\right)}^{c} (f_n(x) - f_m(x)) \mathrm{d}x + \int_d^{d+\left(\frac{b-d}{m}\right)} (f_n(x) - f_m(x)) \mathrm{d}x \\
&\leq \int_{c-\left(\frac{c-a}{m}\right)}^{c} \mathrm{d}x + \int_d^{d+\left(\frac{b-d}{m}\right)} \mathrm{d}x \\
&= \frac{c-a}{m} + \frac{b-d}{m} \\
&= \frac{c-a+b-d}{m} \\
&< \varepsilon.
\end{aligned}$$

Thus the sequence (f_n) is Cauchy in $(C[a, b], \|\cdot\|_1)$.

Finally, we check that $\|f_n\|_1 \to \text{length}(I)$ as $n \to \infty$. We have

$$\begin{aligned}
d - c &\leq \|f_n\|_1 \\
&= \int_a^b |f_n(x)| \mathrm{d}x \\
&= \int_a^b f_n(x) \mathrm{d}x \\
&= \int_{c-\left(\frac{c-a}{n}\right)}^{c} f_n(x)\mathrm{d}x + \int_c^d \mathrm{d}x + \int_d^{d+\left(\frac{b-d}{n}\right)} f_n(x)\mathrm{d}x \\
&\leq \int_{c-\left(\frac{c-a}{n}\right)}^{c} \mathrm{d}x + \int_c^d \mathrm{d}x + \int_d^{d+\left(\frac{b-d}{n}\right)} \mathrm{d}x \\
&= \frac{c-a}{n} + d - c + \frac{b-d}{n} \\
&\to d - c.
\end{aligned}$$

Thus for each $n \in \mathbb{N}$, we have

$$d - c \leq \|f_n\|_1 \leq d - c + \frac{c - a + b - d}{n}. \tag{1}$$

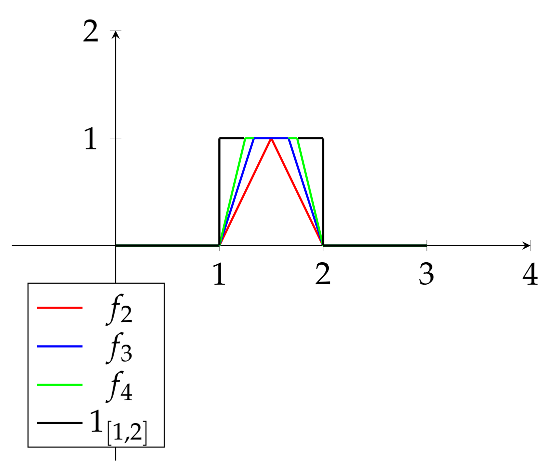
By taking $n \to \infty$ in (1), we see that

$$\begin{aligned}
\lim_{n\to\infty} \|f_n\|_1 &= d - c \\
&= \text{length}(I).
\end{aligned}$$

Case 2: Suppose $I = (c, d)$. For each $n \geq 2$, define

$$f_n(x) = \begin{cases} 0 & \text{if } a \leq x \leq c \\ \frac{n}{d-c}(x-c) & \text{if } c < x \leq c + \left(\frac{d-c}{n}\right) \\ 1 & \text{if } c + \left(\frac{d-c}{n}\right) \leq x \leq d - \left(\frac{d-c}{n}\right) \\ \frac{n}{c-d}(x-d) & \text{if } d - \left(\frac{d-c}{n}\right) \leq x \leq d \\ 0 & \text{if } d \leq x \leq b \end{cases}$$

The image below gives the graphs for f_2, f_3, and f_4 in the case where $[a, b] = [0, 3]$ and $(c, d) = (1, 2)$.



That (f_n) is a Cauchy sequence of continuous funtions in $(C[a, b], \|\cdot\|_1)$ which converges pointwise to 1_I and $\|f_n\|_1 \to \text{length}(I)$ as $n \to \infty$ follows from similar arguments used in case 1. $\square$

Problem 3

Proposition 0.3. *Let $\mathcal{A}$ be an algebra of subsets of X. Then*

1. *$\mathcal{A}$ is closed under finite unions: if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.*
2. *$\mathcal{A}$ is closed under relative compliments: if $A, B \in \mathcal{A}$, then $A \backslash B \in \mathcal{A}$.*
3. *$\mathcal{A}$ is closed under symmetric differences: if $A, B \in \mathcal{A}$, then $A \Delta B \in \mathcal{A}$.*

Proof.
1. Let $A, B \in \mathcal{A}$. Then

$$\begin{aligned} A \cup B &= ((A \cup B)^c)^c \\ &= (A^c \cap B^c)^c \\ &\in \mathcal{A}. \end{aligned}$$

2. Let $A, B \in \mathcal{A}$. Then

$$\begin{aligned} A \backslash B &= A \cap B^c \\ &\in \mathcal{A}. \end{aligned}$$

3. Let $A, B \in \mathcal{A}$. Then it follows from 1 and 2 that

$$\begin{aligned} A \Delta B &= (A \backslash B) \cup (B \backslash A) \\ &\in \mathcal{A}. \end{aligned}$$

□

Problem 4

Definition 0.1. A nonempty collection $\mathcal{E}$ of subsets of X is said to be a **semialgebra** of sets if it satisfies the following properties:

1. $\emptyset \in \mathcal{E}$;
2. If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$;
3. If $E \in \mathcal{E}$, then E^c is a finite disjoint union of sets from $\mathcal{E}$.

Problem 4.a

Proposition 0.4. *The collection of all subintervals of $[a, b]$ forms a semialgebra of sets.*

Proof. Let $\mathcal{I}$ denote the collection of all subintervals of $[a, b]$. We have $\emptyset \in \mathcal{I}$ since $\emptyset = (c, c)$ for any $c \in [a, b]$.

Now we show $\mathcal{I}$ is closed under finite intersections. Let I_1 and I_2 be subintervals of $[a, b]$. Taking the closure of I_1 and I_2 gives us closed intervals, say

$$\overline{I}_1 = [c_1, d_1] \quad \text{and} \quad \overline{I}_2 = [c_2, d_2].$$

Assume without loss of generality that $c_1 \leq c_2$. If $d_1 < c_2$, then $I_1 \cap I_2 = \emptyset$, so assume that $d_1 \geq c_2$. If $d_1 \geq d_2$, then $I_1 \cap I_2 = I_2$, so assume that $d_1 < d_2$. If $c_1 = c_2$, then $I_1 \cap I_2 = I_1$, so assume that $c_2 > c_1$. So we have reduced the case to where

$$c_1 < c_2 \leq d_1 < d_2.$$

With these assumptions in mind, we now consider four cases:

Case 1: If $I_1 = [c_1, d_1]$ or $I_1 = (c_1, d_1]$ and $I_2 = [c_2, d_2]$ or $I_2 = [c_2, d_2)$, then $I_1 \cap I_2 = [c_2, d_1]$.

Case 2: If $I_1 = [c_1, d_1)$ or $I_1 = (c_1, d_1)$ and $I_2 = [c_2, d_2]$ or $I_2 = [c_2, d_2)$, then $I_1 \cap I_2 = [c_2, d_1)$.

Case 3: If $I_1 = [c_1, d_1]$ or $I_1 = (c_1, d_1)$ and $I_2 = (c_2, d_2]$ or $I_2 = (c_2, d_2)$, then $I_1 \cap I_2 = (c_2, d_1)$.

Case 4: If $I_1 = [c_1, d_1)$ or $I_1 = (c_1, d_1)$ and $I_2 = (c_2, d_2]$ or $I_2 = (c_2, d_2)$, then $I_1 \cap I_2 = (c_2, d_1)$.

In all cases, we see that $I_1 \cap I_2$ is a subinterval of $[a, b]$.

Now we show that compliments can be expressed as finite disjoint unions. Let I be a subinterval of $[a, b]$ and write $\overline{I} = [c, d]$. We consider four cases:

Case 1: If $I = [c, d]$, then $I^c = [a, c) \cup (d, b]$.

Case 2: If $I = (c, d]$, then $I^c = [a, c] \cup (d, b]$.

Case 3: If $I = [c, d)$, then $I^c = [a, c) \cup [d, b]$.

Case 4: If $I = (c, d)$, then $I^c = [a, c] \cup [d, b]$.

Thus in all cases, we can express I^c as a disjoint union of intervals since $a \le c \le d \le b$.

□

Problem 4.b

Proposition 0.5. *Let $\mathcal{I}$ be the collection of all subintervals of $\mathbb{R} \cup \{\infty\}$ of the form $(a, b]$. Then $\mathcal{I}$ forms a semialgebra of sets.*

Proof. We have $\emptyset \in \mathcal{I}$ since $\emptyset = (c, c]$ for any $c \in \mathbb{R} \cup \{\infty\}$.

Now we show $\mathcal{I}$ is closed under finite intersections. Let $I_1 = (c_1, d_1]$ and $I_2 = (c_2, d_2]$. Assume without loss of generality that $c_1 \le c_2$. Then

$$I_1 \cap I_2 = \begin{cases} (c_2, d_1] & \text{if } c_2 \le d_1 \\ \emptyset & \text{else} \end{cases}$$

Now we show that compliments can be expressed as finite disjoint unions. Let $I = (c, d]$. Then

$$I^c = (-\infty, c] \cup (d, \infty],$$

where the union is disjoint since $c \le d$. □

Problem 4.c

Proposition 0.6. *Let $\mathcal{E}$ be a semialgebra of sets. Then the collection $\mathcal{A}$ consisting of all sets which are finite disjoint union of sets in $\mathcal{E}$ forms an algebra of sets.*

Proof. We have $\emptyset \in \mathcal{A}$ since $\emptyset \in \mathcal{E}$.

Next we show that $\mathcal{A}$ is closed under finite intersections. Let $A, A' \in \mathcal{A}$. Express A and A' as a disjoint union of members of $\mathcal{E}$, say

$$A = E_1 \cup \cdots \cup E_n \quad \text{and} \quad A' = E'_1 \cup \cdots \cup E'_{n'}.$$

Then we have

$$\begin{aligned}
A \cap A' &= \left(\bigcup_{i=1}^{n} E_i \right) \cap \left(\bigcup_{i'=1}^{n'} E'_{i'} \right) \\
&= \bigcup_{i'=1}^{n'} \left(\left(\bigcup_{i=1}^{n} E_i \right) \cap E'_{i'} \right) \\
&= \bigcup_{i'=1}^{n'} \left(\bigcup_{i=1}^{n} E_i \cap E'_{i'} \right) \\
&= \bigcup_{\substack{1 \le i \le n \\ 1 \le i' \le n'}} E_i \cap E'_{i'}
\end{aligned}$$

where the union is disjoint since the E_i and $E'_{i'}$ are disjoint from one another.

Lastly we show that $\mathcal{A}$ is closed under compliments. Let $A \in \mathcal{A}$. Express A as a disjoint union of members of $\mathcal{E}$, say

$$A = E_1 \cup \cdots \cup E_n.$$

Then we have

$$\begin{aligned}
A^c &= (E_1 \cup \cdots \cup E_n)^c \\
&= E_1^c \cap \cdots \cap E_n^c.
\end{aligned}$$

Since the E_i^c belong to $\mathcal{A}$ and $\mathcal{A}$ is closed under finite intersections, it follows that $A^c \in \mathcal{A}$. □

Problem 5

Proposition 0.7. *Let $\mathcal{A}$ be a collection of subsets of $\mathbb{Z}$ such that*

1. *X is a member of $\mathcal{A}$;*
2. *$\mathcal{A}$ is closed under relative compliments: $A \backslash B \in \mathcal{A}$ for all $A, B \in \mathcal{A}$.*

Then $\mathcal{A}$ is an algebra.

Proof. We have $\varnothing \in \mathcal{A}$ since $\varnothing = X \backslash X \in \mathcal{A}$. Clearly $\mathcal{A}$ is closed under compliments since it is closed under relative compliments, so we just need to show that $\mathcal{A}$ is closed under finite intersections. Let $A, B \in \mathcal{A}$. Then

$$\begin{aligned}
A \cap B &= A \cap (B^c)^c \\
&= A \backslash B^c \\
&\in \mathcal{A}.
\end{aligned}$$

□

Problem 6

Proposition 0.8. *Let $\mathcal{A}$ be the collection of subsets of X which satisfies the property that if $A \in \mathcal{A}$ then either A or A^c is finite. Then $\mathcal{A}$ forms an algebra.*

Proof. We have $\varnothing \in \mathcal{A}$ since $\varnothing$ is finite. Clearly $\mathcal{A}$ is closed under compliments since $A \in \mathcal{A}$ implies either A or A^c is finite which implies $A^c \in \mathcal{A}$. It remains to show that $\mathcal{A}$ is closed under finite intersections. Let $A, B \in \mathcal{A}$ and suppose that $A \cap B$ is infinite. We must show that $(A \cap B)^c = A^c \cup B^c$ is finite. In other words, we need to show that both A^c and B^c are finite. Assume for a contradiction that A^c is infinite. Then A must be finite since $A \in \mathcal{A}$. But this implies $A \cap B$ is finite, which is a contradiction. Thus A^c must be finite. Similarly, we can prove by contradiction that B^c is finite too. □

Problem 7

Proposition 0.9. *Let $(\mathcal{A}_n)$ be an ascending sequence of algebras over X, that is, $\mathcal{A}_n$ is an algebra of subsets of X and $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for all $n \in \mathbb{N}$. Then*

$$\mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$$

is an algebra.

Proof. We have $\varnothing \in \mathcal{A}$ since $\varnothing \in A_1 \subseteq \mathcal{A}$. Next we show that $\mathcal{A}$ is closed under finite intersections. Let $A, B \in \mathcal{A}$. Then $A \in A_i$ and $B \in \mathcal{A}_j$ for some $i, j \in \mathbb{N}$. Without loss of generality, assume that $i \leq j$. Then $A \in \mathcal{A}_i \subseteq \mathcal{A}_j$. Thus $A \cap B \in \mathcal{A}_j \subseteq \mathcal{A}$. Lastly we show that $\mathcal{A}$ is closed under compliments. Let $A \in \mathcal{A}$. Then $A \in A_i$ for some $i \in \mathbb{N}$. Thus $A^c \in \mathcal{A}_i \subseteq \mathcal{A}$. $\square$

Remark. The ascending condition is not necessary. Indeed, consider $X = \{a, b, c\}$ and

$$\begin{aligned} \mathcal{A} &= \{\varnothing, X, \{a\}, \{b, c\}\} \\ \mathcal{B} &= \{\varnothing, X, \{b\}, \{a, c\}\} \\ \mathcal{C} &= \{\varnothing, X, \{c\}, \{a, b\}\} \end{aligned}$$

Then $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are algebras over X, and $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \mathcal{P}(X)$ is an algebra over X, but none of the $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$ contain one another.