

MDG Algebras and Modules

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1 Basic Definitions

Throughout this document, let R be a commutative ring.

1.1 MDG R -algebras

Let (A, d) be an R -complex and let $\mu: A \otimes_R A \rightarrow A$ be a chain map. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu \left(\sum_{i=1}^m a_i \otimes b_i \right) = \sum_{i=1}^m a_i \star_{\mu} b_i,$$

furthermore, if μ is understood from context, then we will simplify our notation further by writing

$$\sum_{i=1}^m a_i \star_{\mu} b_i = \sum_{i=1}^m a_i b_i.$$

Since μ is a chain map, the Leibniz law is satisfied, which in this context says

$$d(ab) = d(a)b + (-1)^{|a|} ad(b)$$

for all $a, b \in A$ with a homogeneous, where $|a|$ denotes the homological degree of a . The Leibniz law is a condition imposed on the triple (A, d, μ) : it is an equation which involves the graded structure on A , the differential d , and the chain map μ , to clean notation however, we will often refer to the triple (A, d, μ) via its underlying graded R -module A , where we think of A as being equipped with d and μ . Here are some other conditions we will want to impose on A :

- We say A is **unital** if there exists $1 \in A$ such that

$$a1 = a = 1a$$

for all $a \in A$.

- We say A is **associative** if

$$a(bc) = (ab)c$$

for all $a, b, c \in A$.

- We say A is **graded-commutative** if

$$ab = (-1)^{|a||b|} ba$$

for all homogeneous $a, b \in A$.

- We say A is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:

$$a^2 = 0$$

for all homogeneous $a \in A$ where $|a|$ is odd.

With these definitions in place, let us now define some of the objects which we will be considering in this document.

Definition 1.1. Let (A, d) be an R -complex and let $\mu: A \otimes_R A \rightarrow A$ be a chain map.

1. We say A is an **MDG R -algebra** if A is unital. In this case, we say μ is the **multiplication** of A . The multiplication of A is sometimes denoted μ_A in case context is not clear, just like how the differential of A is sometimes denoted d_A .
2. We say A is a **DG R -algebra** if A is unital and associative. In other words, a DG R -algebra is an associative MDG R -algebra.

Throughout this document, all MDG R -algebras and DG R -algebras are also assumed to be strictly graded-commutative.

1.1.1 MDG R -algebra homomorphisms

Having defined MDG R -algebras, we now wish to define MDG R -algebras homomorphisms.

Definition 1.2. Let A and B be two MDG R -algebras and let $f: A \rightarrow B$ be a function. We say f is an MDG R -algebra **homomorphism** (or a **homomorphism** of MDG R -algebras) if it satisfies the following three properties:

1. It is a chain map of the underlying R -complexes. This means that f is a graded R -linear map which commutes with the differentials d_A and d_B .
2. It preserves the identity element. This means that $f(1) = 1$.
3. It preserves the multiplication operations. This means that $f(a_1 a_2) = f(a_1) f(a_2)$ for all $a_1, a_2 \in A$.

If, in addition, f is a bijection, then we say f is an MDG R -algebra **isomorphism**. The collection of all MDG R -algebras together with MDG R -algebra homomorphisms forms a category, which we denote by \mathbf{MDG}_R . If A and B are DG R -algebras, then a **DG R -algebra homomorphism** between them is just an MDG R -algebra homomorphism, where we view A and B as MDG R -algebras. The collection of all DG R -algebras together with DG R -algebra homomorphisms forms a category, which we denote by \mathbf{DG}_R . We have an inclusion of categories $\mathbf{DG}_R \hookrightarrow \mathbf{MDG}_R$ which is fully faithful but not essentially surjective on objects.

Remark 1. Note that property 3 in Definition (1.2) can be interpreted as saying the following diagram commutes:

$$\begin{array}{ccc} A \otimes_R A & \xrightarrow{f \otimes f} & B \otimes_R B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

Similar interpretations can be given for properties 1 and 2 as well.

Example 1.1. Assume that (R, \mathfrak{m}) is local, let $I \subseteq \mathfrak{m}$ be an ideal of R , and let $\tau: F \rightarrow R/I$ be a free resolution of R/I over R . We can view R/I as an MDG R -algebra in the obvious way, where as an R -complex, R/I is the trivial one whose only nonzero homogeneous component is in homological degree 0. The multiplication m of R/I is the one that comes from R (so R/I is even a DG R -algebra). Since F is a free resolution of R/I , we can lift $m: R/I \otimes_R R/I \rightarrow R/I$ to a chain map $\mu: F \otimes_R F \rightarrow F$. It is known that μ can be chosen such that it gives (F, d) the structure of an MDG R -algebra. Then the map $\tau: F \rightarrow R/I$ is an MDG R -algebra homomorphism and we call (F, τ) a free **MDG R -algebra resolution** of R/I over R .

1.2 MDG A -modules

Now we want to define modules over an MDG R -algebra. Essentially, these are like modules over a ring except we do not require the associative law to hold.

Definition 1.3. Let A be an MDG R -algebra.

1. A **left MDG A -module** is a triple $(X, d_X, \mu_{A,X})$ where (X, d_X) is an R -complex and where $\mu_{A,X}: A \otimes_R X \rightarrow X$ is a chain map which satisfies $1x = x$ for all $x \in X$. In this case, we say $\mu_{A,X}$ is the **left scalar-multiplication** of X .
2. A **right MDG A -module** is a triple $(X, d_X, \mu_{X,A})$ where (X, d_X) is an R -complex and where $\mu_{X,A}: X \otimes_R A \rightarrow X$ is a chain map which satisfies $x1 = x$ for all $x \in X$. In this case, we say $\mu_{X,A}$ is the **right scalar-multiplication** of X .
3. A **two-sided MDG A -module** is a triple $(X, d_X, \mu_{A,X,A})$ where (X, d_X) is an R -complex and where $\mu_{A,X,A}: (A \otimes_R X) \otimes_R A \rightarrow X$ is a chain map, called the **two-sided scalar-multiplication** of X such that
 - (a) Restricting $\mu_{A,X,A}$ to $(A \otimes_R X) \otimes_R 1$ gives a left scalar-multiplication of X , which we denote $\mu_{A,X}$, making X a left MDG A -module.
 - (b) Restricting $\mu_{A,X,A}$ to $(1 \otimes_R X) \otimes_R A$ gives a right scalar-multiplication of X , which we denote $\mu_{X,A}$, making X a right MDG A -module.
 - (c) The left scalar-multiplication of X and the right scalar-multiplication of X are compatible with the graded-commutative structure in the sense that for all homogeneous $a \in A$ and homogeneous $x \in X$ we have $ax = (-1)^{|a||x|}xa$.
4. A **left DG A -module** is a left MDG A -module X which is associative, meaning $(ab)x = a(bx)$ for all $a, b \in A$ and for all $x \in X$.
5. A **right DG A -module** is a right MDG A -module X which is associative, meaning $(xa)b = x(ab)$ for all $a, b \in A$ and for all $x \in X$.
6. A **two-sided DG A -module** is a two-sided MDG A -module X which is associative, meaning $(ab)x = a(bx)$ for all $a, b \in A$ and for all $x \in X$. Note that in this case, graded-commutativity of X then automatically implies $(ax)b = a(xb)$ and $(xa)b = x(ab)$ for all $a, b \in A$ and $x \in X$.

Here again we are using the convention that the image of a tensor $\sum_{i=1}^n a_i \otimes x_i$ in $A \otimes_R X$ under the scalar-multiplication map $\mu_{A,X}$ is denoted by

$$\mu_{A,X} \left(\sum_{i=1}^n a_i \otimes x_i \right) = \sum_{i=1}^n a_i \star_{\mu_{A,X}} x_i = \sum_{i=1}^n a_i x_i.$$

The theory of left/right/two-sided MDG A -modules are very similar for the most part, though there are some notable differences (certainly the two-sided theory requires special attention at times). We will mention some of these differences when they arise, however we will focus mostly on developing the theory left MDG A -modules. All of the tools and methods that we introduce in the theory of left MDG A -modules will have their counterparts in the theory of right/two-sided MDG A -modules as well. If we write “let X be an MDG A -module”, then it will be understood that A is an MDG R -algebra and that X is a *left* MDG A -module. In this case, we’ll simplify our notation by writing μ_X instead of $\mu_{A,X}$ to denote the scalar-multiplication of X . Note that μ_X being a chain map implies that it satisfies the Leibniz law, which in this context says that for all homogeneous $a \in A$ and $x \in X$, we have

$$d_X(ax) = d_A(a)x + (-1)^{|a|}ad_X(x).$$

Note that even in the case where A is a DG R -algebra, it still makes sense to talk about MDG A -modules: if X is an MDG A -module, then A being associative does not imply X is associative. On the other hand, it may be the case that A is non-associative, and yet X is associative. Thus it also makes sense to talk about DG A -modules when A is a non-associative MDG R -algebra.

1.2.1 MDG A -module homomorphisms

Having defined MDG A -modules, we now wish to define A -module homomorphisms between them.

Definition 1.4. Let X and Y be two MDG A -modules, let $i \in \mathbb{Z}$, and let $\varphi: X \rightarrow Y$ be a function. We say φ is an MDG A -module homomorphism of degree i if it satisfies the following two properties:

1. It is a chain map of degree i of the underlying R -complexes, meaning it is a graded R -linear map of degree i which satisfies

$$\varphi d_X = (-1)^{|\varphi|} d_Y \varphi,$$

where we denote $|\varphi| = i$.

2. It preserves the scalar-multiplications of X and Y , meaning

$$\varphi(ax) = (-1)^{|a||\varphi|} a\varphi(x)$$

for all $a \in A$ and $x \in X$.

In the case where $|\varphi| = 0$, then we will simply call φ an MDG A -module homomorphism. If, in addition, φ is a bijection, then we say φ is an MDG A -module **isomorphism**. The collection of all MDG A -modules together with MDG A -module homomorphisms forms a category, which we denote by \mathbf{MDGmod}_A . If X and Y are DG A -modules, then a **DG A -module homomorphism** between them is just an MDG A -module homomorphism, where we view X and Y as MDG A -modules. The collection of all DG A -modules together with DG A -modules homomorphisms forms a category, which we denote by \mathbf{DGmod}_A . We have an inclusion of categories $\mathbf{DGmod}_A \hookrightarrow \mathbf{MDGmod}_A$ which is fully faithful but not essentially surjective on objects.

Remark 2. If we write “let $\varphi: X \rightarrow Y$ be an MDG A -module homomorphism” without first specifying what A is, then it is understood that A is an MDG R -algebra and that X and Y are MDG A -modules.

1.2.2 Submodules

Definition 1.5. Let X and Y be two MDG A -modules. We say X is an **MDG A -submodule** of Y if $X \subseteq Y$. An MDG A -submodule of A is called an **MDG ideal** of A . Given any collection $S = \{x_\lambda \mid \lambda \in \Lambda\}$ of elements of X , we denote by $\langle\langle S \rangle\rangle$ to be the smallest PDG A -submodule of X which contains S , and we denote by $\langle S \rangle$ to be the set of all A -linear combinations of elements in S .

Proposition 1.1. Let X be an MDG A -module and let $S = \{x_\lambda \mid \lambda \in \Lambda\}$ of elements of X . Define $d_X S = \{d_X(x_\lambda) \mid \lambda \in \Lambda\}$. Then we have

$$\langle\langle S \rangle\rangle = \langle S \cup d_X S \rangle.$$

Proof. Since $\langle\langle S \rangle\rangle$ is the smallest MDG A -submodule of X which contains S , we must have $d_X(x_\lambda) \in \langle\langle S \rangle\rangle$ for all $\lambda \in \Lambda$. Thus all A -linear combinations of elements in $S \cup d_X S$ must belong to $\langle\langle S \rangle\rangle$ as well. Therefore

$$\langle S \cup d_X S \rangle \subseteq \langle\langle S \rangle\rangle.$$

For the reverse direction, notice that Leibniz law ensures that $\langle S \cup d_X S \rangle$ is d_X -stable (meaning d_X maps $\langle S \cup d_X S \rangle$ to itself). Indeed, if

$$\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j d_X(x_{\lambda_j}),$$

is a finite A -linear combination of elements in $S \cup d_X S$ where each a_i and b_j are homogeneous, then note that

$$\begin{aligned} d_X \left(\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j d_X(x_{\lambda_j}) \right) &= \sum_{i=1}^m d_X(a_i x_{\lambda_i}) + \sum_{j=1}^n d_X(b_j d_X(x_{\lambda_j})) \\ &= \sum_{i=1}^m \left(d_A(a_i) x_{\lambda_i} + (-1)^{|a_i|} a_i d_X(x_{\lambda_i}) \right) + \sum_{j=1}^n \left(d_A(b_j) x_{\lambda_j} + (-1)^{|b_j|} b_j d_X^2(x_{\lambda_j}) \right) \\ &= \sum_{i=1}^m d_A(a_i) x_{\lambda_i} + \sum_{i=1}^m (-1)^{|a_i|} a_i d_X(x_{\lambda_i}) + \sum_{j=1}^n d_A(b_j) x_{\lambda_j} \\ &\in \langle S \cup d_X S \rangle. \end{aligned}$$

In particular, we see that $\langle S \cup d_X S \rangle$ is a MDG A -submodule of X which contains S . Since $\langle\langle S \rangle\rangle$ is the *smallest* MDG A -submodule of X which contains S , it follows that

$$\langle S \cup d_X S \rangle \supseteq \langle\langle S \rangle\rangle.$$

□

1.3 Associativity

1.3.1 The associator map of an MDG A -module

Let X be an MDG A -module. To keep track of the failure for X to be associative, we introduce the following definition:

Definition 1.6. Let $a_{A,A,X}: (A \otimes_R A) \otimes_R X \rightarrow A \otimes_R (A \otimes_R X)$ denote the unique chain map defined on elementary tensors by

$$(a \otimes b) \otimes x \mapsto a \otimes (b \otimes x).$$

As usual, we simplify our notation by writing a_X instead of $a_{A,A,X}$ whenever context is clear. We define the **associator map** of X is the chain map $[\cdot, \cdot, \cdot]_X: (A \otimes_R A) \otimes_R X \rightarrow X$ defined by

$$[\cdot, \cdot, \cdot]_X := \mu_X(1 \otimes \mu_X)a_X - \mu_X(\mu_A \otimes 1).$$

In other words, the associator of X is the difference of the two maps in the diagram below both of which start at the top left part of the pentagon and end at the bottom of the pentagon:

$$\begin{array}{ccc} (A \otimes_R A) \otimes_R X & \xrightarrow{a_X} & A \otimes_R (A \otimes_R X) \\ \mu_A \otimes 1 \downarrow & & \downarrow 1 \otimes \mu_X \\ A \otimes_R X & & A \otimes_R X \\ & \searrow \mu_X \quad \swarrow \mu_X & \\ & X & \end{array}$$

In particular, $[\cdot, \cdot, \cdot]_X$ measures the failure of the diagram above to be commutative, that is, it measures the failure of X to be associative.

Remark 3. Even in the case where X is a two-sided MDG A -module, we want to define the associator of X to be the associator of $(X, d_X, \mu_{A,X})$, where we view X as a left MDG A -module. Indeed, X is associative if and only if $[\cdot, \cdot, \cdot]_{(X, d_X, \mu_{A,X})} = 0$. If X is a right MDG A -module, then obviously the associator of it should be a the flipped version of Definition (1.6).

If X is understood from context, then we will simplify our notation by dropping X from the subscript in $[\cdot, \cdot, \cdot]_X$ and simply write $[\cdot, \cdot, \cdot]$ to denote the associator of X . Observe that since $[\cdot, \cdot, \cdot]$ is a chain map, we see that $[\cdot, \cdot, \cdot]$ is a graded trilinear map which satisfies

$$d_X[a, b, x] = [d_A(a), b, x] + (-1)^{|a|}[a, d_A(b), x] + (-1)^{|a|+|b|}[a, b, d_X(x)]. \quad (1)$$

for all homogeneous $a, b \in A$ and $x \in X$. Here are some other important identities which the associator satisfies:

- For all $a, b, c \in A$ and $x \in X$ we have

$$a[b, c, x] - [ab, c, x] + [a, bc, x] - [a, b, cx] + [a, b, c]x = 0 \quad (2)$$

- If X is two-sided, then for all homogeneous $a, b \in A$ and homogeneous $x \in X$ we have a graded-commutative version of the Jacobi identity

$$(-1)^{(|a|+|x|)|b|}[a, b, x] + (-1)^{(|b|+|x|)|a|}[x, a, b] + (-1)^{(|a|+|b|)|x|}[b, x, a] = 0 \quad (3)$$

- If X is two-sided, then for all homogeneous $a, b \in A$ and homogeneous $x \in X$ we have

$$[a, b, x] + (-1)^{|a||b|+|a||x|+|b||x|}[x, b, a] = 0. \quad (4)$$

We put these identities to use in the next proposition.

Proposition 1.2. Let X be a two-sided MDG A -module. The following conditions are equivalent.

1. $[a, a, x] = 0$ for all $a \in A$ and $x \in X$;
2. $[a, x, a] = 0$ for all $a \in A$ and $x \in X$;
3. $[x, a, a] = 0$ for all $a \in A$ and $x \in X$;

Proof. Let $a \in A$ and let $x \in X$ where both a and x are homogeneous. From (4) we obtain

$$\begin{aligned} [a, a, x] &= -(-1)^{|a||a|+|a||x|+|a||x|} [x, a, a] \\ &= -(-1)^{|a|} [x, a, a]. \end{aligned}$$

Thus clearly 1 and 3 are equivalent. Combining this with (3), we obtain

$$\begin{aligned} 0 &= (-1)^{(|a|+|a|)|x|} [a, a, x] + (-1)^{(|x|+|a|)|a|} [x, a, a] + (-1)^{(|a|+|a|)|x|} [a, x, a] \\ &= [a, a, x] + (-1)^{|a||x|+|a|} [x, a, a] + [a, x, a] \\ &= -(-1)^{|a|} [x, a, a] + (-1)^{|a||x|+|a|} [x, a, a] + [a, x, a] \\ &= ((-1)^{|a||x|+|a|} - (-1)^{|a|}) [x, a, a] + [a, x, a] \\ &= (-1)^{|a|} ((-1)^{|a||x|} - 1) [x, a, a] + [a, x, a] \end{aligned}$$

In particular, we see that

$$[a, x, a] = \begin{cases} -2[x, a, a] = -2[a, a, x] = 2a(ax) & \text{if } |a| \text{ and } |x| \text{ are odd} \\ 0 & \text{else} \end{cases} \quad (5)$$

for all homogeneous $a \in A$ and homogeneous $x \in X$. Finally, the equivalence of 4 with everything else follows by definition. \square

Definition 1.7. Let X be an MDG A -module. We say X is **alternative** if any of the equivalent definitions in Definition (1.2) are satisfied.

1.3.2 The associator complex and homology of an MDG A -module

Let X be an MDG A -module. Now we want to introduce a complex associated (no pun intended) with X .

Definition 1.8. The **associator complex** of X is the R -subcomplex of X given by the image of $[\cdot, \cdot, \cdot]$. We denote this complex by $[X]$. Thus the underlying graded R -module of $[X]$ is given by

$$[X] = \text{span}_R \{ [a, b, x] \mid a, b \in A \text{ and } x \in X \}, \quad (6)$$

and the differential $d_{[X]}$ of $[X]$ is just the restriction of the differential d_X to $[X]$. The **associator homology** of X is the homology of the associator complex of X , denoted $H([X])$. We say X (or μ_X) is **homologically associative** if $H([X]) = 0$. We also say X (or μ_X) is **homologically associative in degree i** if $H_i([X]) = 0$ and we say we say X (or μ_X) is **associative in degree i** if $[X]_i = 0$.

Clearly if X is associative, then X is homologically associative. The converse is also true under certain conditions.

Proposition 1.3. Assume that (R, \mathfrak{m}) is a local ring and that $[X]$ is minimal. If X is associative in degree i , then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $[X]$ is bounded below and minimal, then X is associative if and only if X is homologically associative.

Proof. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume $H_{i+1}([X]) = 0$ and $[X]_{i+1} \neq 0$. By Nakayama's Lemma, we can find a triple (a, b, x) such that $|a| + |b| + |x| = i+1$ and such that $[a, b, x] \notin \mathfrak{m}[X]_{i+1}$. Since $[X]_i = 0$ by assumption, we have $d_X[a, b, x] = 0$. Also, since X is minimal, we have $d_X[X] \subseteq \mathfrak{m}[X]$. Thus $[a, b, x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. \square

Remark 4. Note that if both A and X are both minimal, then the Leibniz law (1) implies $[X]$ is minimal too, however the converse need not hold.

The proof of Proposition (1.3) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition:

Definition 1.9.

1. Assume that $[X]$ is bounded below. The **lower associative index** of X , denoted $\text{la}(X)$, is defined to be the smallest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\text{la}(X) = -\infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded below by setting $\text{la}(X) = -\infty$.
2. Assume that $[X]$ is bounded above. The **upper associative index** of X , denoted $\text{ua}(X)$, is defined to be the largest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\text{ua}(X) = \infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded above by setting $\text{ua}(X) = \infty$.

With the lower associative index of X defined, we see, after analyzing the proof of Proposition (1.3), that if R is local and $[X]$ is nonzero and minimal, then

$$\text{la}(X) = \inf\{i \in \mathbb{Z} \mid H_i([X]) \neq 0\}.$$

Thus, in this case, the lower associative index of X can be measured homologically.

1.3.3 The nucleus of an MDG A -module

Let X be an MDG A -module. It will be important to keep track of what elements in X associate with everything else. This leads us to the following definitions:

Definition 1.10. Let X be an MDG A -module.

1. Suppose X is a left MDG A -module. The **left nucleus** of X , denoted $N_l(X)$, is the subset of X defined by

$$N_l(X) = \{x \in X \mid [a, b, x] = 0 \text{ for all } a, b \in A\}.$$

2. Suppose X is a right MDG A -module. The **right nucleus** of X , denoted $N_r(X)$, is the subset of X defined by

$$N_r(X) = \{x \in X \mid [x, a, b] = 0 \text{ for all } a, b \in A\}.$$

3. Suppose X is two-sided MDG A -module. The **middle nucleus** of X , denoted $N_m(X)$, is the subset of X defined by

$$N_m(X) = \{x \in X \mid [a, x, b] = 0 \text{ for all } a, b \in A\},$$

and the **nucleus** of X , denoted $N(X)$, is subset of X defined by

$$N(X) = N_l(X) \cap N_m(X) \cap N_r(X),$$

where $N_l(X)$ is the left nucleus of X where we view X as a left PDG A -module, and where $N_r(X)$ is the right nucleus of X where we view X as a right PDG A -module.

Remark 5. Suppose X is a two-sided MDG A -module. Notice that the identity (4) implies $N_l(X) = N_r(X)$. Also notice that (3) implies $N_l(X) \cap N_r(X) \subseteq N_m(X)$. Combining these facts together, we see that $N(X) = N_l(X) = N_r(X)$. Thus if Y is a left (resp. right) PDG A -module, then we can define the **nucleus** of Y , denoted $N(Y)$, to be left (resp. right) nucleus of Y without any threat of conflict in our notation. With the remark above in mind, we also wish to consider the subcomplex of A given by

$$N_A(X) = \{a \in A \mid [a, b, x] = 0 \text{ for all } b \in A \text{ and } x \in X\}.$$

We call $N_A(X)$ the nucleus of A with respect to X .

Lemma 1.1. Let (A, d, μ) be an MDG R -algebra such that $A_0 = R$ and set $I = d(A_1)$. Then I kills $H(A)$.

Proof. Let $t \in I$ and let $m_t: A \rightarrow A$ be the multiplication by t map, defined by

$$m_t(a) = ta$$

for all $a \in A$. We claim that m_t is null-homotopic. Indeed, choose $e \in A$ such that $d(e) = t$, and let $m_e: A \rightarrow A$ be the multiplication by e map, defined by

$$m_e(a) = ea$$

for all $a \in A$. Note that m_e is a graded R -linear map of degree 1. Also note that for all $a \in A$, we have

$$\begin{aligned} (dm_e + m_e d)(a) &= dm_e(a) + m_e d(a) \\ &= d(ea) + ed(a) \\ &= d(e)a - ed(e) + ed(a) \\ &= d(e)a \\ &= ta \\ &= m_t(a). \end{aligned}$$

Thus m_e is a homotopy from m_t to the zero map; hence m_t is null-homotopic. It follows that t kills $H(A)$, and since $t \in I$ was arbitrary, we see that I kills $H(A)$. \square

Corollary 1. Let K be a field and let (A, d, μ) be an MDG K -algebra such that $A_0 = K$. Assume that $d(A_1) \neq 0$. Then $H(A) = 0$.

Corollary 2. Let (A, d, μ) be an MDG R -algebra such that $A_0 = R$ and set $I = d(N(A)_1)$. Then I kills $H([A])$.

Proof. Let $t \in I$ and choose $e \in N(A)$ such that $d(e) = t$. Note that the restriction of m_e lands in $[A]$ when it is restricted to $[A]$ since (4) implies $m_e[a, b, c] = [ea, b, c]$ for all $a, b, c \in A$. It follows that $m_e|_{[A]}$ is a homotopy from $[m_t]$ to the zero map; hence $[m_t]$ is null-homotopic. It follows that t kills $H([A])$, and since $t \in I$ was arbitrary, we see that I kills $H([A])$. \square

Corollary 3. Let K be a field and let (A, d, μ) be an MDG K -algebra such that $A_0 = K$. Assume that $d(N(A)_1) \neq 0$. Then $H([A]) = 0$.

1.3.4 Associator functor

Let $\varphi: X \rightarrow Y$ be an MDG A -module homomorphism. We obtain an induced chain map of R -complexes $[\varphi]: [X] \rightarrow [Y]$, where $[\varphi]$ is the unique chain map which satisfies

$$\begin{aligned} [\varphi][a, b, x] &= \varphi((ab)x - a(bx)) \\ &= \varphi((ab)x) - \varphi(a(bx)) \\ &= (ab)\varphi(x) - a\varphi(bx) \\ &= (ab)\varphi(x) - a(b\varphi(x)) \\ &= [a, b, \varphi(x)]. \end{aligned}$$

for all $a, b \in A$ and $x \in X$. In particular, the map $[\varphi]$ is just the restriction of φ to $[X]$. It is straightforward to check that the assignment $X \mapsto [X]$ and $\varphi \mapsto [\varphi]$ gives rise to a functor from \mathbf{MDGmod}_A to \mathbf{Comp}_R . We call this functor the **associator functor** over A , and we denote this functor by $[\cdot]: \mathbf{PMod}_A \rightarrow \mathbf{Comp}_R$.

The associator functor need not be exact. To see what goes wrong, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \quad (7)$$

be a short exact sequence of MDG A -modules. We obtain an induced sequence of R -complexes

$$0 \longrightarrow [X] \xrightarrow{[\varphi]} [Y] \xrightarrow{[\psi]} [Z] \longrightarrow 0 \quad (8)$$

We claim that we have exactness at $[X]$ and $[Z]$. Indeed, this is equivalent to showing $[\varphi]$ is injective and that $[\psi]$ is surjective, which follows from the fact that $[\varphi]$ (respectively $[\psi]$) is the restriction of the injective map φ (respectively the surjective map ψ). Let us see what goes wrong when trying to prove exactness at $[Y]$. Let

$$\sum_{i=1}^n [a_i, b_i, y_i] \in \ker[\psi] \subseteq \ker \psi.$$

Then by exactness of (7), there exists $x \in X$ such that $\varphi(x) = \sum_{i=1}^n [a_i, b_i, y_i]$. It is not at all clear however that $x \in [X]$. Indeed, we will see a counterexample to this later on. In order to ensure exactness of (8), we need to place a condition on (7). This leads us to consider the following definition:

Definition 1.11. Let X be an MDG A -submodule of Y . We say X is a **stable** MDG A -submodule of Y if it satisfies $[X] = X \cap [Y]$. The short exact sequence (7) is called **stable** if $\varphi(X)$ is a stable MDG A -submodule of Y .

It is easy to check that (8) is a short exact sequence of R -complexes if and only if $\varphi(X)$ is a stable MDG A -submodule of Y . Thus if $\varphi(X)$ is a stable MDG A -submodule of Y , then the short exact sequence (8) of R -complexes induces a long exact sequence in homology

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & H_{i+1}([Z]) & & \\ & & & & \downarrow & & \\ & & & & H_i([X]) & \longrightarrow & H_i([Y]) \longrightarrow H_i([Z]) \\ & & & & \downarrow & & \\ & & & & H_{i-1}([X]) & \longrightarrow & \cdots \end{array} \quad (9)$$

From this, one concludes immediately the following theorem:

Theorem 1.2. Suppose X is a stable MDG A -submodule of Y . Then μ_Y is homologically associative if and only if μ_X and $\mu_{Y/X}$ are homologically associative.

1.3.5 Application of long exact sequence

We want to discuss another application of (11). Suppose that (R, \mathfrak{m}) is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R , let F be a free MDG R -algebra resolution of R/I , and let $r \in \mathfrak{m}$ be an (R/I) -regular element. Then the mapping cone $C(r)$ is a free resolution of $R/\langle I, r \rangle$ over R . We can give $C(r)$ the structure of a free MDG R -algebra resolution of $R/\langle I, r \rangle$ as follows: first note that the underlying graded R -module of $C(r)$ has the form $F + eF$ where e is an exterior variable of degree -1 and where $\{1, e\}$ is an F -linearly independent set. In other words, every element in $F + eF$ can be expressed in the form $\alpha + e\beta$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i , then α and β are homogeneous of degrees i and $i - 1$ respectively. With this understood, we extend the multiplication μ_F on F to a multiplication $\mu_{C(r)}$ on $C(r)$ by setting

$$(\alpha + e\beta)(\gamma + e\delta) = \alpha\gamma + e(\beta\gamma + (-1)^{|\alpha|}\alpha\delta)$$

for all homogeneous $\alpha, \beta, \gamma, \delta \in F$ and extending this linearly everywhere else. Note that the mapping cone $C(r)$ inherits a natural *right* MDG F -module structure via restriction of scalars. The reason why it is more naturally viewed as a right MDG F -module rather than a left MDG F -module can be seen in the way the mapping cone differential acts on elements: given $\alpha, \beta \in F$ where α is homogeneous, we have

$$d_{C(r)}(\alpha + e\beta) = d_F(\alpha) + r\beta - ed_F(\beta).$$

Thus the mapping cone differential behaves as if e is an exterior variable of degree -1 (which is what we want).

Thus there are two associator complexes to consider: the associator complex of $C(r)$ where we view $C(r)$ as an MDG R -algebra and the associator complex of $C(r)$ where we view $C(r)$ as a right MDG F -module. In fact, these associator complexes are the same. To see this, first note that $e \in N(C(r))$ by construction of $\mu_{C(r)}$. In particular, this implies

$$\begin{aligned} [\alpha, \beta, \gamma + e\delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|+|\beta|}e[\alpha, \beta, \delta] \\ [\alpha, \beta + e\gamma, \delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|}e[\alpha, \gamma, \delta] \\ [\alpha + e\beta, \gamma, \delta] &= [\alpha, \gamma, \delta] + e[\beta, \gamma, \delta] \end{aligned}$$

Using these identities together with the fact that $e^2 = 0$ and the fact that identity $1 \in N(C(r))$, we obtain

$$\begin{aligned} [\alpha + e\beta, \gamma + e\delta, \varepsilon + e\zeta] &= [\alpha, \gamma, \varepsilon] + e[\beta, \gamma, \varepsilon] + (-1)^{|\alpha|}e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}e[\alpha, \gamma, \zeta] \\ &= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha|}e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}e[\alpha, \gamma, \zeta] \\ &= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha|}[1 + e\alpha, \delta, \varepsilon] + (-1)^{|\alpha|+|\gamma|}[1 + e\alpha, \gamma, \zeta] \end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F$. It follows that these associator complexes are the same, and thus we may denote this common associator complex by $[C(r)]$ without there being any source of confusion.

The homothety map $m_r: F \rightarrow F$ gives rise to a short exact sequence of right MDG F -modules

$$0 \longrightarrow F \xrightarrow{\iota} C(r) \xrightarrow{\pi} \Sigma F \longrightarrow 0 \quad (10)$$

where $\iota: F \rightarrow C(r)$ is the inclusion map and where $\pi: C(r) \rightarrow \Sigma F$ is the projection map given by $\pi(\alpha + e\beta) = \alpha$ for all $\alpha, \beta \in F$.

Proposition 1.4. The short exact sequence (10) is stable.

Proof. We must check that $[C(r)] \cap F \subseteq [F]$ since the reverse inclusion is trivial. Suppose $\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i]$ is an arbitrary element in $[C(r)] \cap F$ where $r_i \in R$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ for each $1 \leq i \leq m$. Observe that

$$\begin{aligned} \sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] &= \sum_{i=1}^m r_i([\alpha_i, \gamma_i, \delta_i] + e[\beta_i, \gamma_i, \delta_i]) \\ &= \sum_{i=1}^m r_i[\alpha_i, \gamma_i, \delta_i] + e \sum_{i=1}^m r_i[\beta_i, \gamma_i, \delta_i]. \end{aligned}$$

Since $\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in F$, it follows that $\sum_{i=1}^m r_i[\beta_i, \gamma_i, \delta_i] = 0$. Thus

$$\sum_{i=1}^m r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] = \sum_{i=1}^m r_i[\alpha_i, \gamma_i, \delta_i] \in [F].$$

Therefore $[C(r)] \cap F \subseteq [F]$. □

Remark 6. The same exact proof also shows that if $\varphi: X \rightarrow Y$ is an MDG A -module homomorphism, then Y is a stable MDG A -submodule of $C(\varphi) = Y + eX$ where $C(\varphi)$ is the mapping cone with respect to φ . Here we are using the fact that $C(\varphi)$ can be given the structure of a right MDG A -module by defining the right scalar-multiplication map $\mu_{C(\varphi), A}$ via the formula

$$(y + ex)a = ya + e(xa)$$

for all $y + ex \in C(\varphi)$ and $a \in A$.

Since (10) is stable, we obtain a long exact sequence in homology

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & H_i([F]) & & \\ & & & & \searrow r & & \\ & \nearrow & H_i([F]) & \longrightarrow & H_i([C(r)]) & \longrightarrow & H_{i-1}([F]) \\ & & & & \searrow r & & \\ & \nearrow & H_{i-1}([F]) & \longrightarrow & \cdots & & \end{array} \quad (11)$$

Using this long exact sequence together with Nakayama's lemma, we can deduce the following theorem:

Theorem 1.3. *Keep the notation as above.*

1. *We have*

$$\text{la}(C(r)) = \text{la}(F).$$

2. *Set $k = \text{ua}(F)$ and assume that $-\infty < k < \infty$. We have*

$$\text{ua}(C(r)) = \begin{cases} \text{ua}(F) & \text{if } r \text{ is } H_k([F])\text{-regular} \\ \text{ua}(F) + 1 & \text{else} \end{cases}$$

2 MDG R -algebras supported on monomial resolutions

Throughout this subsection, let $x = x_1, \dots, x_n$, let $R = K[x]$, and let $\mathbf{m} = m_1, \dots, m_r$ be monomials in R . For each nonempty subset $\sigma \subseteq [r]$, we set $m_\sigma := \text{lcm}(m_\lambda \mid \lambda \in \sigma)$ and we set $\mathbf{a}_\sigma \in \mathbb{N}^n$ to be the exponent vector of m_σ . For completeness, we set $m_\emptyset = 1$ and $\mathbf{a}_\emptyset = (0, \dots, 0)$. Let Re_σ be the free R -module generated by e_σ whose multidegree is \mathbf{a}_σ . Let Δ be a finite simplicial complex whose vertex set is the ordered set $[r] = \{1, \dots, r\}$. We label the vertices of Δ by m_1, \dots, m_r . More generally, if σ is a face of Δ , then we label it by m_σ . For each $\mathbf{a} \in \mathbb{N}^n$, let $\Delta_{\mathbf{a}}$ be the subcomplex of Δ defined by $\Delta_{\mathbf{a}} = \{\sigma \in \Delta \mid \mathbf{a}_\sigma \leq \mathbf{a}\}$. The differential on $\mathcal{S}(\Delta)$ is denoted ∂ , and the differential on $\mathcal{S}(\Delta_{\mathbf{a}})$ is denoted $\partial_{\mathbf{a}}$. Note that $\partial_{\mathbf{a}}$ is just the restriction of ∂ to $\mathcal{S}(\Delta_{\mathbf{a}})$.

2.1 Monomial resolution induced by a labeled simplicial complex

Definition 2.1. We define an R -complex, denoted F_Δ (or more simply denoted F if Δ is understood from context) and called **R -complex induced by Δ** (or the **R -complex of Δ over R**), as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R -module of F is given by

$$F_k := \begin{cases} \bigoplus_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d is defined on the homogeneous generators of F by $d(e_\emptyset) = 0$ and

$$d(e_\sigma) = \sum_{i \in \sigma} (-1)^{\text{pos}(i, \sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \setminus \{\emptyset\}$ where $\text{pos}(i, \sigma)$, the **position of vertex i** in σ , is the number of elements preceding i in the ordering of σ , and $\sigma \setminus i$ denotes the face obtained from σ by removing i . In the case where Δ is the r -simplex, we call F the **Taylor complex** of R/\mathbf{m} over R .

Observe that F also has the structure of an \mathbb{Z}^n -graded K -complex. In other words, we have a decomposition

of K -complexes

$$F = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} F_{\mathbf{a}},$$

where the K -complex $F_{\mathbf{a}}$ in multidegree $\mathbf{a} \in \mathbb{Z}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded K -vector space of $F_{\mathbf{a}}$ is given by

$$F_{k,\mathbf{a}} := \begin{cases} \bigoplus_{\dim \sigma = k-1} K \frac{x^{\mathbf{a}}}{m_{\sigma}} e_{\sigma} & \text{if } \sigma \in \Delta_{\mathbf{a}} \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $d_{\mathbf{a}}$ of $F_{\mathbf{a}}$ is just the restriction of d to $F_{\mathbf{a}}$. In particular, d is homogeneous with respect to that \mathbb{Z}^n -grading; hence

$$\begin{aligned} H(F) &= \ker d / \operatorname{im} d \\ &= \left(\bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \ker d_{\mathbf{a}} \right) / \left(\bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \operatorname{im} d_{\mathbf{a}} \right) \\ &\cong \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} (\ker d_{\mathbf{a}} / \operatorname{im} d_{\mathbf{a}}) \\ &= \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} H(F_{\mathbf{a}}). \end{aligned}$$

Now assume that $\Delta_{\mathbf{a}}$ contains a nonempty face, say $\sigma \in \Delta_{\mathbf{a}}$, then we have

$$\begin{aligned} d_{\mathbf{a}} \left(\frac{x^{\mathbf{a}}}{m_{\sigma}} e_{\sigma} \right) &= \frac{x^{\mathbf{a}}}{m_{\sigma}} d(e_{\sigma}) \\ &= \frac{x^{\mathbf{a}}}{m_{\sigma}} \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{m_{\sigma}}{m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{x^{\mathbf{a}} m_{\sigma}}{m_{\sigma} m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\operatorname{pos}(i,\sigma)} \frac{x^{\mathbf{a}}}{m_{\sigma \setminus i}} e_{\sigma \setminus i}. \end{aligned}$$

Thus if we define $\varphi_{\mathbf{a}}: F_{\Delta,\mathbf{a}}(1) \rightarrow \mathcal{S}(\Delta_{\mathbf{a}})$ to be the unique graded K -linear isomorphism such that $\varphi_{\mathbf{a}} \left(\frac{x^{\mathbf{a}}}{m_{\sigma}} e_{\sigma} \right) = \sigma$. Then from the computation above, we see that $d_{\mathbf{a}} \partial_{\mathbf{a}} = \partial_{\mathbf{a}} d_{\mathbf{a}}$; hence $\varphi_{\mathbf{a}}$ gives an isomorphism of K -complexes

$$\Sigma^{-1} F_{\mathbf{a}} \cong \mathcal{S}(\Delta_{\mathbf{a}}).$$

In particular, we obtain an isomorphism of homologies

$$H(F_{\mathbf{a}})(1) \cong \tilde{H}(\Delta_{\mathbf{a}}, K),$$

where $\tilde{H}(\Delta_{\mathbf{a}}, K)$ is the simplicial homology of the simplicial complex $\Delta_{\mathbf{a}}$ over K . In other words, for each $k \in \mathbb{Z}$, we have

$$H_{k+1}(F_{\mathbf{a}}) \cong \tilde{H}_k(\Delta_{\mathbf{a}}, K).$$

The following theorem follows immediately from the discussion above:

Theorem 2.1. *F is a free resolution of R/\mathbf{m} over R if and only if for all $\mathbf{a} \in \mathbb{Z}^n$ either $\Delta_{\mathbf{a}}$ is the void complex or $\Delta_{\mathbf{a}}$ is acyclic. In particular, the Taylor complex of R/\mathbf{m} over R is a free resolution. Moreover, F is minimal if and only if $m_{\sigma} \neq m_{\sigma'}$ for every proper subface σ' of a face σ .*

2.2 MDG-algebra structures on the monomial resolution induced by a labeled simplicial complex

Let (F, d) denote the R -complex induced by Δ . Let $\mu: F \otimes_R F \rightarrow F$ be a chain map such that

1. μ gives F the structure of an MDG R -algebra resolution of R/\mathbf{m} .
2. μ respects the multigrading: this means that if $\alpha \in F_{\mathbf{a}}$ and $\beta \in F_{\mathbf{b}}$, then $\alpha\beta \in F_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$.

For each $\sigma, \tau \in \Delta$ we have

$$e_\sigma e_\tau = \sum_{v \in \Delta} f_{\sigma, \tau}^v(\mu) e_v \quad (12)$$

where $f_{\sigma, \tau}^v(\mu) \in K[x]$ for each $v \in \Delta$. The $f_{\sigma, \tau}^v(\mu)$ uniquely determine μ ; they are called the **structured R -coefficients** of μ . If μ is understood from context, then we'll simplify our notation by writing $f_{\sigma, \tau}^v = f_{\sigma, \tau}^v(\mu)$. Since μ is a graded map, we must have $f_{\sigma, \tau}^v = 0$ whenever $|e_\sigma| + |e_\tau| \neq |e_v|$. In fact, since μ respects the multigrading, we must have

$$f_{\sigma, \tau}^v(\mu) = c_{\sigma, \tau}^v(\mu) \frac{m_\sigma m_\tau}{m_v}$$

where $c_{\sigma, \tau}^v(\mu) \in K$ for all $\sigma, \tau, v \in \Delta$ where $c_{\sigma, \tau}^v(\mu) = 0$ whenever $|e_\sigma| + |e_\tau| \neq |e_v|$ or $m_\sigma m_\tau \nmid m_v$. The $c_{\sigma, \tau}^v(\mu)$ also uniquely determine μ ; they are called the **structured K -coefficients** of μ . Again, if μ is understood from context, then we'll simplify our notation by writing $c_{\sigma, \tau}^v(\mu) = c_{\sigma, \tau}^v$. It would be nice if we could re-express (12) as

$$\left(\frac{e_\sigma}{m_\sigma} \right) \left(\frac{e_\tau}{m_\tau} \right) = \sum_v c_{\sigma, \tau}^v \left(\frac{e_v}{m_v} \right), \quad (13)$$

but the problem is that F does not contain terms like e_σ/m_σ . In order to make sense of (13), we need to adjoin inverses to our base ring R .

2.2.1 Base change

Let S be the multiplicatively closed set generated by $\{m_\sigma \mid \sigma \in \Delta\}$ and let T be the multiplicatively closed set generated by $x_1 \cdots x_n \in R$. The localization of R at T has a natural description in terms of the Laurent polynomial ring over K , that is $R_T = K[x, x^{-1}]$, and the localization of R at S has a natural description as an R -submodule of R_T which is generated by Laurent monomials m_σ/m_τ for all $\sigma, \tau \in \Delta$. In general, any R -submodule M of R_T which is generated by Laurent monomials x^a where $a \in \mathbb{Z}^n$ is called a **monomial module**.

We want to perform a base change from R to R_S in order to make sense of (13). The localization functor $-_S$ from the category of R -complexes to the category of R_S -complexes is an exact functor which preserves quasiisomorphisms. In particular, it takes the quasiisomorphism $F \rightarrow R/m$ to the quasiisomorphism $F_S \rightarrow 0$. Note that $F_S \rightarrow 0$ being a quasiisomorphism is equivalent to saying F_S is an exact R -complex. We can give a natural description of F_S as an MDG R_S -complex as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R_S -module of F_S is given by

$$F_{S, k} := \begin{cases} \bigoplus_{\dim \sigma = k-1} R_S e_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

The differential d_S of F_S is defined via the rule $d_S(x^a e_\sigma) = x^a d(e_\sigma)$ for all homogeneous generators e_σ and for all $a \in \mathbb{Z}^n$. Similarly, the multiplication μ_S of F_S is defined via the rule $(x^a e_\sigma)(x^b e_\tau) = x^{a+b} e_\sigma e_\tau$ for all homogeneous generators e_σ, e_τ and for all $a, b \in \mathbb{Z}^n$. Note that d_S and μ_S are just the localizations of d and μ with respect to S , so our notation here is consistent. Since $-_S$ is an exact functor, we have $[F]_S \cong [F_S]$, where $[F]_S$ is the localization (with respect to S) of the associator complex of F and where $[F_S]$ is the associator complex of F_S .

2.2.2 Determining associativity of F using \tilde{F}

To simplify our notation in what follows, let us denote $\tilde{F} = F_{S, 0}$, that is, \tilde{F} is the multidegree $\mathbf{0}$ part of F_S . The multiplication (13) makes perfect sense in \tilde{F} . Denoting $\tilde{e}_\sigma = e_\sigma/m_\sigma$ for each $\sigma \in \Delta$, we re-express (13) as

$$\tilde{e}_\sigma \tilde{e}_\tau = \sum_v c_{\sigma, \tau}^v \tilde{e}_v.$$

It is straightforward to check that \tilde{F} inherits the structure of an MDG K -algebra. We shall denote its differential by \tilde{d} and we shall denote its multiplication by $\tilde{\mu}$.

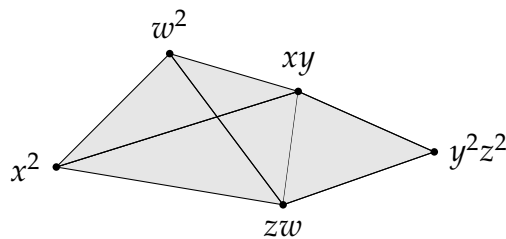
Theorem 2.2. *F is a DG R -algebra if and only if \tilde{F} is a DG K -algebra.*

Proof. A straightforward calculation gives us

$$[e_\sigma, e_\tau, e_v]_\mu = m_\sigma m_\tau m_v [\tilde{e}_\sigma, \tilde{e}_\tau, \tilde{e}_v]_{\tilde{\mu}}$$

for all $\sigma, \tau, v \in \Delta$. Thus μ is associative if and only if $\tilde{\mu}$ is associative. \square

Example 2.1. Consider the case where $R = K[x, y, z, w]$, $\mathbf{m} = x^2, w^2, zw, xy, y^2z^2$, and where Δ is the labeled simplicial complex which can be pictured below



We chose to only label the vertices in the picture above in order to keep things clean, but note that every face of the simplicial complex above should be labeled by an appropriate monomial. Since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have

$$\begin{aligned}\tilde{e}_1\tilde{e}_5 &= c_{1,5}^{14}\tilde{e}_{14} + c_{1,5}^{45}\tilde{e}_{45} \\ \tilde{e}_2\tilde{e}_5 &= c_{2,5}^{23}\tilde{e}_{23} + c_{2,5}^{35}\tilde{e}_{35} \\ \tilde{e}_2\tilde{e}_{45} &= c_{2,45}^{234}\tilde{e}_{234} + c_{2,45}^{345}\tilde{e}_{345} \\ \tilde{e}_1\tilde{e}_{35} &= c_{1,35}^{134}\tilde{e}_{134} + c_{1,35}^{345}\tilde{e}_{345} \\ \tilde{e}_2\tilde{e}_{14} &= c_{2,14}^{124}\tilde{e}_{124} \\ \tilde{e}_1\tilde{e}_{23} &= c_{1,23}^{123}\tilde{e}_{123}\end{aligned}$$

where each $c_{\sigma,\tau}^v$ above is nonzero. At this point however, $\tilde{\mu}$ is already not associative since $[\tilde{e}_1, \tilde{e}_5, \tilde{e}_2] \neq 0$. Indeed, we use Singular to calculate an appropriate Gröbner basis for us to determine this:

```
intvec w=(3,3,3,2,2,2,2,1,1,1,1,1);
ring A=(o,c14,c45,c23,c35,c234,c134,c123,c124,c345,d345),(x1,x2,x5,x14,x45,x23,x35,x123,x124,x234,x134);
ideal I =
```

```
x1*x5+c14*x14+c45*x45,
x2*x5+c23*x23+c35*x35,
x2*x45+c234*x234+c345*x345,
x1*x35+c134*x134+d345*x345,
x2*x14+c124*x124,
x1*x23+c123*x123;
```

```
std(I);
```

```
_[1]=(c23*c123)*x123+(-c14*c124)*x124+(-c45*c234)*x234+(c35*c134)*x134+(-c45*c345+c35*d345)*x345
_[2]=(-c134)*x23*x134+(-d345)*x23*x345+(c123)*x35*x123
_[3]=(-c234)*x14*x234+(-c345)*x14*x345+(c124)*x45*x124
_[4]=(-c134)*x5*x134+(-d345)*x5*x345+(c14)*x14*x35+(c45)*x45*x35
_[5]=(-c234)*x5*x234+(-c345)*x5*x345+(c23)*x45*x23+(c35)*x45*x35
_[6]=(-c124)*x5*x124+(c23)*x14*x23+(c35)*x14*x35
_[7]=x2*x45+(c234)*x234+(c345)*x345
_[8]=x2*x14+(c124)*x124
_[9]=x1*x35+(c134)*x134+(d345)*x345
_[10]=x1*x23+(c123)*x123
_[11]=x2*x5+(c23)*x23+(c35)*x35
_[12]=x1*x5+(c14)*x14+(c45)*x45
```

A quick calculation gives us

$$\begin{aligned}d[e_1, e_{45}, e_2] &= x[e_1, e_5, e_2] \\ d[e_{14}, e_5, e_2] &= y[e_1, e_5, e_2] \\ d[e_1, e_5, e_{23}] &= z[e_1, e_5, e_2] \\ d[e_1, e_{35}, e_2] &= w[e_1, e_5, e_2].\end{aligned}$$

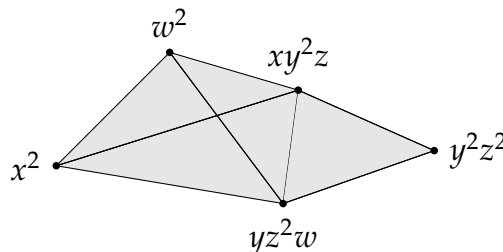
Therefore

$$H_i([F]) \cong \begin{cases} K & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

We also want to note that

$$\begin{aligned}
 [e_1, e_5, e_2] &= m_1 m_5 m_2 [\tilde{e}_1, \tilde{e}_2, \tilde{e}_5] \\
 &= m_1 m_5 m_2 \tilde{d}(\tilde{e}_{1234}) \\
 &= \frac{m_1 m_5 m_2}{m_{1234}} d(e_{1234}) \\
 &= yz d(e_{1234})
 \end{aligned}$$

Example 2.2. Consider the case where $R = K[x, y, z, w]$, $\mathbf{m} = x^2, w^2, yz^2w, xy^2z, y^2z^2$, and where Δ is the labeled simplicial complex which can be pictured below



Since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have

$$\begin{aligned}
 \tilde{e}_1 \tilde{e}_5 &= c_{1,5}^{14} \tilde{e}_{14} + c_{1,5}^{45} \tilde{e}_{45} \\
 \tilde{e}_2 \tilde{e}_5 &= c_{2,5}^{23} \tilde{e}_{23} + c_{2,5}^{35} \tilde{e}_{35} \\
 \tilde{e}_2 \tilde{e}_{45} &= c_{2,45}^{234} \tilde{e}_{234} + c_{2,45}^{345} \tilde{e}_{345} \\
 \tilde{e}_1 \tilde{e}_{35} &= c_{1,35}^{134} \tilde{e}_{134} + c_{1,35}^{345} \tilde{e}_{345} \\
 \tilde{e}_2 \tilde{e}_{14} &= c_{2,14}^{124} \tilde{e}_{124} \\
 \tilde{e}_1 \tilde{e}_{23} &= c_{1,23}^{123} \tilde{e}_{123}
 \end{aligned}$$

where each $c_{\sigma,\tau}^v$ above is nonzero. We again have $[e_1, e_5, e_2] \neq 0$ as seen in the example above. A quick calculation gives us

$$\begin{aligned}
 d[e_1, e_{45}, e_2] &= x[e_1, e_5, e_2] \\
 d[e_{14}, e_5, e_2] &= y^2 z[e_1, e_5, e_2] \\
 d[e_1, e_5, e_{23}] &= yz^2[e_1, e_5, e_2] \\
 d[e_1, e_{35}, e_2] &= w[e_1, e_5, e_2] \\
 d[e_1, e_5, e_{24}] &= xy^2 z[e_1, e_5, e_2] \\
 d[e_{13}, e_5, e_2] &= yz^2 w[e_1, e_5, e_2] \\
 d[e_{12}, e_5, e_2] &= w^2[e_1, e_5, e_2] \\
 d[e_1, e_5, e_{12}] &= x^2[e_1, e_5, e_2]
 \end{aligned}$$

In particular, the associator complex for F is different from the example above; it is given by

$$H_i([F]) \cong \begin{cases} K[y, z] / \langle y^2 z, yz^2 \rangle & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

2.2.3 \mathbb{Z}^n -graded MDG \tilde{F} -algebra structure of F_S

Observe that F_S has the structure of a \mathbb{Z}^n -graded MDG \tilde{F} -module. Indeed, we have a decomposition of F_S into MDG \tilde{F} -modules

$$F_S = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} F_{S,\mathbf{a}},$$

where the MDG \tilde{F} -module $F_{S,\mathbf{a}}$ in multidegree $\mathbf{a} \in \mathbb{Z}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded \tilde{F} -module of $F_{S,\mathbf{a}}$ is given by

$$F_{S,\mathbf{a}} := \begin{cases} \bigoplus_{\dim \sigma = k-1} K \frac{x^{\mathbf{a}}}{m_{\sigma}} e_{\sigma} & \sigma \in \Delta, 0 \leq k \leq \dim \Delta + 1, \text{ and } \frac{x^{\mathbf{a}}}{m_{\sigma}} \in R_S \\ 0 & \text{else} \end{cases}$$

The differential $d_{S,a}$ of $F_{S,a}$ is just the restriction of d_S to $d_{S,a}$, and the multiplication $\mu_{S,a}$ of $F_{S,a}$ is just the restriction of μ_S to $\mu_{S,a}$. Note that we need μ to respect the multi-grading in order for $F_{S,a}$ to be an MDG \tilde{F} -module so that the scalar-multiplication map $\tilde{F} \times F_{S,a} \rightarrow F_{S,a}$ is well-posed.

Since d_S is homogeneous with respect to the \mathbb{Z}^n -grading, we have

$$0 = H(F_S) \cong \bigoplus_{a \in \mathbb{Z}^n} H(F_{S,a}).$$

Hence each K -complex $F_{S,a}$ is an exact complex. It is also straightforward to show that

$$[F_S] = \bigoplus_{a \in \mathbb{Z}^n} [F_{S,a}],$$

where $[F_{S,a}]$ is the associator complex of $F_{S,a}$ as an MDG \tilde{F} -module. In particular,

$$[F_{S,a}] = \text{span}_{R_S} \{[\alpha, \beta, \gamma] \mid \alpha, \beta \in \tilde{F} \text{ and } \gamma \in F_{S,a}\}.$$

Thus since $-_S$ is an exact functor, we have

$$\begin{aligned} H([F])_S &\cong H([F]_S) \\ &\cong H([F_S]) \\ &\cong \bigoplus_{a \in \mathbb{Z}^n} H([F_{S,a}]). \end{aligned}$$

3 Associativity test using Gröbner bases

3.1 Setup

Let K be a field and let (F, d, μ) be an MDG K -algebra. Let $n \geq 1$ and assume that $(1, e_1, \dots, e_n)$ is an ordered homogeneous basis of F such that

1. $|e_i| \geq 1$ for all $1 \leq i \leq n$,
2. if $|e_j| > |e_i|$, then $j > i$.

In this section, we will use some tools from the theory of Gröbner bases to determine whether or not F is DG algebra, that is, whether or not F is associative. For simplicity, we will only describe how this works in the case where K has characteristic 2. Let $(c_{i,j}^k)$ be the structured K -coefficients of μ . Thus for each $0 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=0}^n c_{i,j}^k e_k,$$

where we use \star to denote multiplication with respect to μ . Let S be the weighted polynomial ring $K[e_1, \dots, e_n]$ where e_i is weighted of degree $|e_i|$ for each $1 \leq i \leq n$. A monomial of S has the form $e^a = e_1^{a_1} \cdots e_n^{a_n}$ where $a \in \mathbb{N}^n$ where we identify the monomial $e^{(0, \dots, 0)}$ with 1 in this notation. Given a monomial e^a , we define its **degree**, denoted $\deg(e^a)$, and its **weighted degree**, denoted $|e^a|$, by

$$\deg(e^a) = \sum_{i=1}^n a_i \quad \text{and} \quad |e^a| = \sum_{i=1}^n a_i |e_i|.$$

For each $k \in \mathbb{N}$, we shall write

$$S_k = \text{span}_K \{e^a \mid \deg(e^a) = k\}.$$

We identify F with $S_0 + S_1 = K + \sum_{i=1}^n K e_i$. In order to keep notation consistent, we shall write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to μ , and we shall write $\alpha \beta$ to denote their multiplication with respect to \cdot in S . In particular, note that $\deg(e_i \star e_j) = 1$ and $\deg(e_i e_j) = 2$.

For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Note that since we are working over a field of characteristic 2, we have $f_{i,j} = f_{j,i}$ for all $1 \leq i, j \leq n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let I be the ideal of S generated by \mathcal{F} . We equip S with a weighted lexicographic ordering $>_w$ with respect to the weight vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials e^a and e^b in S , we say $e^a >_w e^b$ if either

1. $|e^a| > |e^b|$ or;
2. $|e^a| = |e^b|$ and there exists $1 \leq i \leq n$ such that $\alpha_i > \beta_i$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_{i-1} = \beta_{i-1}$.

Observe that for each $1 \leq i \leq j \leq n$, we have $\text{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since μ is graded, we must have $|e_i e_j| = |e_i| + |e_j| = |e_k|$. It follows that $|e_k| > |e_i|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ by our assumption on (e_1, \dots, e_n) . Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

3.2 Main Theorem

Before we state and prove the main theorem, let us introduce one more piece of notation. We denote $\mathcal{M} = \{e^a \mid e^a \notin \text{LT}(I)\}$. Since $\text{LT}(f_{i,j}) = e_i e_j$ for all $1 \leq i, j \leq n$, we see that \mathcal{M} is a subset of $\{e_1, \dots, e_n\}$. Now we are ready to state and prove the main theorem:

Theorem 3.1. *The following statements are equivalent:*

1. F is associative.
2. \mathcal{F} is a Gröbner basis.
3. $\mathcal{M} = \{e_1, \dots, e_n\}$.

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq n$. We have

$$\begin{aligned}
 S_{i,j,k} &:= S(f_{j,k}, f_{i,j}) \\
 &= e_i f_{j,k} - f_{i,j} e_k \\
 &= e_i (e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
 &= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
 &= \left(\sum_l c_{i,j}^l e_l \right) e_k - e_i \left(\sum_l c_{j,k}^l e_l \right) \\
 &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l.
 \end{aligned}$$

Now we divide $S_{i,j,k}$ by \mathcal{F} :

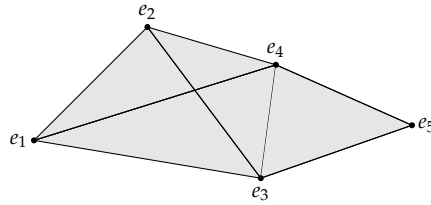
$$\begin{aligned}
 S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} \\
 &= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) + \sum_l c_{j,k}^l (f_{i,l} - e_i e_l) \\
 &= \sum_l c_{i,j}^l (e_l e_k - e_l e_k + e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l - e_i e_l) \\
 &= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
 &= \left(\sum_l c_{i,j}^l e_l \right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l \right) \\
 &= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
 &= [e_i, e_j, e_k].
 \end{aligned}$$

Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by \mathcal{F} . It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. A straightforward computation also shows that $S(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \leq i \leq n$. Finally, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq l \leq n$. We have

$$\begin{aligned}
 S_{i,j,k,l} &:= S(f_{k,l}, f_{i,j}) \\
 &= e_i e_j f_{k,l} - f_{i,j} e_k e_l \\
 &= (f_{i,j} + e_i \star e_j) f_{k,l} - f_{i,j} (f_{k,l} + e_k \star e_l) \\
 &= (e_i \star e_j) f_{k,l} - f_{i,j} (e_k \star e_l).
 \end{aligned}$$

From this, it's easy to see that $S_{i,j,k,l}^{\mathcal{F}} = 0$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. \square

Example 3.1. Let Δ be the simplicial complex below



and let (F, d) be the \mathbb{F}_2 -complex induced by Δ . Let's write the homogeneous components of F as a graded \mathbb{F}_2 -vector space

$$F_0 = \mathbb{F}_2$$

$$F_1 = \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5$$

$$F_2 = \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45}$$

$$F_3 = \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345}$$

$$F_4 = \mathbb{F}_2 e_{1234}$$

Let μ be a multiplication on F such that

$$e_1 \star_\mu e_5 = e_{14} + e_{45}$$

$$e_2 \star_\mu e_5 = e_{23} + e_{35}$$

$$e_2 \star_\mu e_{45} = e_{234} + e_{345}$$

$$e_1 \star_\mu e_{35} = e_{134} + e_{345}$$

$$e_1 \star_\mu e_{23} = e_{123}$$

$$e_2 \star_\mu e_{14} = e_{124}.$$

Then F is not associative with respect to μ since $[e_1, e_5, e_2]_\mu = e_{123} + e_{124} + e_{234} + e_{134} \neq 0$. We use Singular to determine this:

```
intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;

ideal I = f(1)(5), f(2)(5), f(2)(45), f(1)(35), f(1)(23), f(2)(14);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2), I);

// e123+e124+e234+e134 = [e1, e5, e2]
```

Let us extend μ by setting $e_5 \star_\mu e_{12} = e_{123} + e_{134} + e_{345}$. Extending the code above gives us the following code:

```

intvec w=(1,1,1,2,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e12,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(2) = e1*e2+e12;
poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;
poly f(5)(12) = e5*e12+e123+e134+e345;

ideal I = f(1)(2),f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14),f(5)(12);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2),I);

// e123+e124+e234+e134 = [e1,e5,e2]

poly s(2)(1)(5) = e2*f(1)(5)+e5*f(1)(2);
reduce(s(2)(1)(5),I);

// e123+e124+e234+e134 = [e2,e1,e5]

poly s(5)(2)(1) = e5*f(1)(2)+e1*f(2)(5);
reduce(s(5)(2)(1),I);

// o = [e5,e2,e1]

```

From this code, we see that $[e_2, e_1, e_5] \neq 0$ and $[e_5, e_2, e_1] = 0$. In fact, we have $[e_1, e_5, e_2] = [e_2, e_1, e_5]$.