Geometry

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Part I
Sheaves and Locally Ringed Spaces

1 Presheaves and Sheaves

Throughout this section let X be a topological space.

1.1 Presheaves

A **presheaf** $\mathcal{F}$ on X assigns to each open set U in X a set $\mathcal{F}(U)$, and to every pair of nested open subsets $U \subseteq V$ of X, a function $\mathrm{res}^V_U\colon \mathcal{F}(V) \to \mathcal{F}(U)$, called the **restriction map**, such that

1. $\mathcal{F}(\varnothing) = 0$,
2. res^U_U is the identity map for all open sets U in X,
3. $\mathrm{res}^V_U \circ \mathrm{res}^W_V = \mathrm{res}^W_U$ for all open sets $U \subseteq V \subseteq W$ in X.

The elements $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U; elements of $\mathcal{F}(X)$ are called **global sections**. The restriction maps res^V_U are written as $f \mapsto f|_U$. Very often we will also write $\Gamma(U, \mathcal{F})$ instead of $\mathcal{F}(U)$.

1.1.1 Morphism of Presheaves

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A **morphism** of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a family of maps $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets U of X such that for all pairs of open sets V of X such that $U \subseteq V$ the diagram

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}^V_U"'] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^V_U"] \\
\mathcal{F}(U) \arrow[r, "\varphi_U"'] & \mathcal{G}(U)
\end{tikzcd}$$

is commutative. The composite of morphisms $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ is defined in the obvious way, namely we set $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$ for all open sets U of X. We obtain the category of presheaves on X which we denote $\mathbf{Psh}(X)$. We often simplify our notation by denoting the composite of φ and ψ by $\varphi\psi$. Furthermore, we often drop U from subscript in φ_U and simply write φ whenever context is clear.

1.1.2 Category Theory

Using the language of category theory, we can define presheaves in a very concise way. We denote by $\mathbf{O}(X)$ to be the category whose objects are open subsets of X and whose morphisms are inclusion maps. Then a presheaf $\mathcal{F}$ is just a contravariant functor from $\mathbf{O}(X)$ to $\mathbf{Set}$, and morphisms of presheaves are natural transformations between functors. We can also replace the category $\mathbf{Set}$ with any other category $\mathbf{C}$ to obtain the notion of a presheaf $\mathcal{F}$ with values in $\mathbf{C}$. This signifies that $\mathcal{F}(U)$ is an object in $\mathbf{C}$ for every open subset U of X and that the restriction maps are morphisms in $\mathbf{C}$.

1.2 Sheaves

Presheaves on X are top-down constructions; we can restrict information from larger to smaller sets. However, many objects in mathematics are bottom-up constructions; they are defined locally, which we then piece together to obtain something global. Presheaves do not provide the means to deduce global properties from the properties we find locally in the open sets of X. This is where the idea of sheaves come in.

Definition 1.1. A **sheaf** on X is a presheaf $\mathcal{F}$ on X which satisfies the following **sheaf axiom**:

- Suppose $\{U_i\}_{i\in I}$ is an open covering of an open subset U and suppose that for each $i \in I$ a section $s_i \in \mathcal{F}(U_i)$ is given such that for each pair $U_{i_1}, U_{i_2} \in \{U_i\}_{i\in I}$ we have

$$s_{i_1}|_{U_{i_1}\cap U_{i_2}} = s_{i_2}|_{U_{i_1}\cap U_{i_2}}.$$

  Then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

A **morphism of sheaves** is a morphism of presheaves. We denote by $\mathbf{Sh}(X)$ to be the category whose objects are sheaves and whose morphisms are morphism of sheaves. Note that $\mathbf{Sh}(X)$ is a faithfully full subcategory of $\mathbf{Psh}(X)$.

Remark 1. Let X be a topological space, $\mathcal{F}$ a sheaf on X, $U \subseteq X$ be open, and let $\{U_i\}_{i\in I}$ be an open covering of U. Suppose for all $i \in I$, we have $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$. By the sheaf condition, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Conversely, if $s, s' \in \mathcal{F}(U)$ such that $s|_{U_i} = s'|_{U_i}$ for all $i \in I$, then $s = s'$. In particular, we can think of sections $s \in \mathcal{F}(U)$ as being a collection of compatible sections $s_i \in \mathcal{F}(U_i)$ (compatible meaning $s_i|_{U_{ij}} = s_j|_{U_{ij}}$). More formally, we have

$$\mathcal{F}(U) = \left\{ (s_i)_i \in \prod_i \mathcal{F}(U_i) \mid s_i|_{U_{ij}} = s_j|_{U_{ij}} \text{ for all } i, j \in I. \right\}$$

Proposition 1.1. *The sheaf axioms imply that any sheaf has exactly one section of the empty set.*

Proof. The empty set $\varnothing$ can be written as the union of an empty family (that is, the indexing set I is $\varnothing$). The condition given for the sheaf property is vacuously true. So there must exist a unique section in $F(\varnothing)$. $\square$

Example 1.1. Let E be a set. A presheaf of functions on X with values in E is a presheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ consists of functions from U to E for all open sets U of X. Given such a presheaf $\mathcal{F}$, note he only thing preventing $\mathcal{F}$ from being a sheaf is the *existence* of global functions since *uniqueness* is already guarenteed. Indeed, suppose $\{U_i\}_{i\in I}$ is an open covering of an open set U of X, and suppose that for all $i \in I$ we have $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_{ij}} = f_j|_{U_{ij}}$ for all $i, j \in I$ (here we use the notation $U_{ij} = U_i \cap U_j$). Then if $f, g \in \mathcal{F}(U)$ satisfy $f|_{U_i} = f_i = g|_{U_i}$ for all $i \in I$, then we must have $f = g$. This is because $f = g$ if and only if $f(x) = g(x)$ for all $x \in U$, and this is true since $x \in U_{i(x)}$ for some $i(x) \in I$ (depending on x), hence $f(x) = f_{i(x)}(x) = g(x)$.

1.2.1 Reformulating the sheaf axiom

We give a reformulation of the sheaf axiom in terms of arrows. Let $\mathcal{F}$ be a presheaf on X, let U be an open set of X, and let $\{U_i\}_{i\in I}$ be an open covering of U. We define maps

$$\begin{aligned}
\rho : \mathcal{F}(U) &\to \prod_{i\in I} \mathcal{F}(U_i), && s \mapsto (s|_{U_i})_i \\
\sigma : \prod_{i\in I} \mathcal{F}(U_i) &\to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), && (s_i)_i \mapsto (s_i|_{U_i\cap U_j})_{(i,j)} \\
\sigma' : \prod_{i\in I} \mathcal{F}(U_i) &\to \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), && (s_i)_i \mapsto (s_j|_{U_i\cap U_j})_{(i,j)}
\end{aligned}$$

The presheaf $\mathcal{F}$ is a sheaf, if it satisfies for all U and all open coverings $\{U_i\}_{i\in I}$ the following condition: The diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho"] & \prod_{i\in I} \mathcal{F}(U_i) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\sigma'"'] & \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j) \end{tikzcd}$$

is exact. This means that the map ρ is injective and that its image is the set of elements $(s_i)_i \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

For presheaves of abelian groups (or with values in any abelian category) we can reformulate the definition of a sheaf as follows: A presheaf $\mathcal{F}$ is a sheaf if and only if for all open subsets U and all coverings $\{U_i\}$ of U the sequence of abelian groups

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \prod_i \mathcal{F}(U_i) & \longrightarrow & \prod_{i,j} \mathcal{F}(U_i \cap U_j) \\
 & & s & \longmapsto & (s|_{U_i})_i & & \\
 & & & & (s_i)_i & \longmapsto & (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}
\end{array}$$

is exact.

1.3 Examples of Sheaves

1.3.1 Sheaf of Continuous Functions

Example 1.2. Let X and Y be topological spaces. For each open subset U of X, we define

$$\mathcal{C}_{X;Y}(U) := \{f : U \to Y \mid f \text{ is continuous}\}.$$

Then $\mathcal{C}_{X;Y}$ is a presheaf of Y-valued functions on X. In fact, more is true: $\mathcal{C}_{X;Y}$ is a sheaf. Indeed, let $\{U_i\}$ be an open covering of U. If $f : U \to Y$ is a continuous function, then by restriction to U_i, we get continuous maps $f_i : U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j. Conversely, if we are given continuous maps $f_i : U_i \to Y$ that agree on the overlaps (that is, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j) then there is a unique set-theoretic map $f : X \to Y$ satisfying $f|_{U_i} = f_i$ for all i and it is continuous. Indeed, for any open $V \subseteq Y$ we have that $f^{-1}(V)$ is open in U because $f^{-1}(V) \cap U_i = f_i^{-1}(V)$ is open in U_i for every i.

1.3.2 Sheaf of C^α Functions

Example 1.3. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces and let X be an open subspace of V. Let $\alpha \in \widehat{\mathbb{N}}_0$. For each open subset U of X, we define

$$\mathcal{C}^\alpha_{X;W}(U) := \{f : U \to W \mid f \text{ is } C^\alpha \text{ map}\}.$$

Then $\mathcal{C}^\alpha_{X;W}$ is a sheaf of functions on X. It is a sheaf of $\mathbb{R}$-vector spaces. If $W = \mathbb{R}$, then we simply write $\mathcal{C}^\alpha_X$.

Example 1.4. Let $\alpha \in \widehat{\mathbb{N}}$. For $X = \mathbb{R}$ and $U \subseteq \mathbb{R}$ open let $d_U : \mathcal{C}^\alpha_{\mathbb{R}}(U) \to \mathcal{C}^{\alpha-1}_{\mathbb{R}}(U)$ be the derivative $f \mapsto f'$. Then $(d_U)_U$ is a morphism of sheaves of $\mathbb{R}$-vector spaces (but not of $\mathbb{R}$-algebras because usually $(fg)' \neq f'g'$).

1.3.3 Sheaf of Holomorphic Functions

Example 1.5. Let V and W be finite-dimensional $\mathbb{C}$-vector spaces and let X be an open subspace of V. For each open subset U of X, we define

$$\mathcal{O}_{X;W}(U) := \mathcal{O}^{\text{hol}}_{X;W}(U) := \{f\colon U \to W \mid f \text{ is holomorphic}\}.$$

Then $\mathcal{O}_{X;W}$ (with the usual restriction maps) is a sheaf of $\mathbb{C}$-vector spaces.

1.3.4 Constant Sheaf

Example 1.6. Let X be a topological space and let E be a set. For each open subset U of X, we define $\mathcal{F}(U) := E$. Then $\mathcal{F}$ is a sheaf (whose restriction maps being the identity map) called the **constant sheaf with value** E.

1.4 Sheaves are determined by their values on a basis

Let $\mathcal{F}$ be a sheaf on X and let $\mathcal{B}$ be a basis of the topology on X. If we know the value $\mathcal{F}(U)$ of a sheaf on every element U of $\mathcal{B}$, then we can use the sheaf property to determine $\mathcal{F}(V)$ on an arbitrary open set V of X. We simply cover V by elements of $\mathcal{B}$. Here is a more systematic way of saying this:

$$\begin{aligned}\mathcal{F}(V) &= \left\{ (s_U)_U \in \prod_{\substack{U \in \mathcal{B} \\ U \subseteq V}} \mathcal{F}(U) \mid \text{for } U' \subseteq U \text{ both in } \mathcal{B} \text{ we have } s_U|_{U'} = s_{U'} \right\} \\ &= \lim_{\substack{U \in \mathcal{B} \\ U \subseteq V}} \mathcal{F}(U). \end{aligned} \tag{1}$$

Using this observation, we see that it suffices to define a sheaf on a basis $\mathcal{B}$ of open sets of the topology of a topological space X: Consider $\mathcal{B}$ as a full subcategory of $\mathbf{O}(X)$, then a presheaf on $\mathcal{B}$ is a contravariant functor $\mathcal{F} : \mathcal{B} \to \mathbf{Set}$. Every such presheaf $\mathcal{F}$ on $\mathcal{B}$ can be extended to a presheaf $\mathcal{F}'$ on X by using (1) as a definition.

Example 1.7. Let $\mathcal{F}$ be the presheaf of bounded continuous functions on $\mathbb{R}$ with values in $\mathbb{R}$. Then $\mathcal{F}$ is not a sheaf. Indeed, for each $i \in \mathbb{Z}$ let $U_i = (i, i+2)$ and $f_i = x|_{U_i}$. Then $\{U_i\}$ is a covering of $\mathbb{R}$ and there is no bounded continuous function f on $\mathbb{R}$ such that $f|_{U_i} = f_i$ for all i. The sheafification of $\mathcal{F}$ is isomorphic to $\mathcal{C}_{\mathbb{R};\mathbb{R}}$.

1.5 Gluing Sheaves

Proposition 1.2. *Let $\{U_i\}_{i\in I}$ be an open cover of X. For all $i \in I$, let $\mathcal{F}_i$ be a sheaf on U_i. Assume that for each pair (i, j) of indices we are given isomorphisms $\varphi_{ij}\colon \mathcal{F}_j|_{U_{ij}} \to \mathcal{F}_i|_{U_{ij}}$ satisfying for all $i, j, k \in I$ the "cocycle condition" $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ on U_{ijk}. Then there exists a sheaf $\mathcal{F}$ on X and for all $i \in I$ isomorphisms $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ such that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i, j \in I$. Moreover, $\mathcal{F}$ and ψ_i are uniquely determined up to unique isomorphism by these conditions.*

Proof. Let U be an open subset of X. We define $\mathcal{F}(U)$ to be the set of collections of sections which are locally compatible:

$$\mathcal{F}(U) = \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}_i(U_i \cap U) \mid s_i|_{U_{ij}\cap U} = \varphi_{ij}(s_j)|_{U_{ij}\cap U} \text{ for all } i, j \in I. \right\} \tag{2}$$

The restriction maps are defined pointwise. Thus if V is an open subset of U, then we set $(s_i)|_V = (s_i|_{U_i \cap V})$. The cocycle ensures that (2) is well-defined. By replacing U_i with $U_i \cap U$ if necessary, we may assume that $U_i \subseteq U$ for all $i \in I$. In this case, (2) has the slightly simpler description:

$$\mathcal{F}(U) = \left\{ (s_i) \in \prod_{i\in I} \mathcal{F}_i(U_i) \mid s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}} \text{ for all } i, j \in I. \right\}$$

Let us verify that $\mathcal{F}$ is a sheaf. Let $\{U_{i'}\}_{i'\in I'}$ be an open cover of U and for each $i' \in I'$ let $(s_{i,i'})_{i\in I} \in \mathcal{F}(U_{i'})$ such that

$$(s_{i,i'})|_{U_{ii'j'}} = (s_{i,j'})|_{U_{ii'j'}} \tag{3}$$

for all $i', j' \in I'$. We want to show that there exists a unique element $(s_i) \in \mathcal{F}(U)$ such that $(s_i)|_{U_{i'}} = (s_{i,i'})$ for all $i' \in I'$.

Note that (3) says for each $i \in I$, we have $s_{i,i'}|_{U_{ii'j'}} = s_{i,j'}|_{U_{ii'j'}}$ for all $i', j' \in I'$. Thus for each $i \in I$, since $\mathcal{F}_i$ is a sheaf and $\{U_{ii'}\}_{i'\in I'}$ is an open cover of U_i, we use the fact that $\mathcal{F}_i$ is a sheaf to obtain a unique element $s_i \in \mathcal{F}_i(U_i)$ such that $s_i|_{U_{ii'}} = s_{i,i'}$ for all $i' \in I$. Thus we obtain a unique sequence of sections $(s_i) \in \prod_{i\in I} \mathcal{F}_i(U_i)$ such that $(s_i)|_{U_{i'}} = (s_{i,i'})$ for all $i' \in I'$. This establishes uniqueness, so the only thing left to do is to check that $(s_i) \in \mathcal{F}(U)$. For each $i, j \in I$, note that $\{U_{iji'}\}_{i'\in I'}$ is an open cover of U_{ij} and

$$\begin{aligned}
s_i|_{U_{iji'}} &= s_{i,i'}|_{U_{iji'}} \\
&= \varphi_{ij}(s_{j,i'})|_{U_{iji'}} \\
&= \varphi_{ij}(s_{j,i'}|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j|_{U_{iji'}}) \\
&= \varphi_{ij}(s_j)|_{U_{iji'}}
\end{aligned}$$

for all $i' \in I'$. Thus by the uniqueness part in the sheaf axiom (for $\mathcal{F}_i$), we must have $s_i|_{U_{ij}} = \varphi_{ij}(s_j)|_{U_{ij}}$. It follows that $(s_i) \in \mathcal{F}(U)$ as claimed.

Now fix $i \in I$. We define the map $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. Let U be an open subset of U_i. Then for $s \in \mathcal{F}_i(U)$, we set $\psi_i(s) = (\varphi_{ji}(s|_{U_j\cap U}))_{j\in I}$. Conversely, if $(s_j)_{j\in I} \in \mathcal{F}|_{U_i}(U)$, then we set $\psi_i^{-1}((s_j)_{j\in I}) = s_i$. It is clear that ψ_i is a bijection with inverse ψ_i^{-1}. Furthermore, if V is an open subset of U, then $\psi_i(s|_V) = \psi_i(s)|_V$. Thus, ψ_i is an isomorphism of sheaves $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$. We repeat this construction for all $i \in I$ to get an isomorpism $\psi_i : \mathcal{F}_i \to \mathcal{F}|_{U_i}$ for all $i \in I$. Finally, that $\psi_i \circ \varphi_{ij} = \psi_j$ on U_{ij} for all $i, j \in I$ follows from a direct calculation: for $s \in \mathcal{F}_j(U_{ij})$, we have

$$\begin{aligned}
(\psi_i \circ \varphi_{ij})(s) &= \psi_i(\varphi_{ij}(s)) \\
&= (\varphi_{ki}(\varphi_{ij}(s)|_{U_{ijk}}))_{k\in I} \\
&= (\varphi_{ki}(\varphi_{ij}(s))|_{U_{ijk}})_{k\in I} \\
&= (\varphi_{kj}(s)|_{U_{ijk}})_{k\in I} \\
&= (\varphi_{kj}(s|_{U_{ijk}}))_{k\in I} \\
&= \psi_j(s).
\end{aligned}$$

$\square$

1.6 Stalks

Let $\mathcal{F}$ be a presheaf on a topological space X. Suppose that for each $x \in X$, there exists a smallest neighborhood containing x, say U_x. Then we can determine the sheaf completely by computing the values of the sheaf on these open sets. The problem of course is that the limit of the diagram which consists of all open neighborhoods of x may not exist, i.e. we may not have a smallest open neighborhood of x. However, colimits do exist in **Set**, and there's nothing stopping us from looking at colimits in the diagram of $\mathcal{F}$-images of neighborhoods of x.

Definition 1.2. Let $\mathcal{F}$ be a presheaf on a topological space X, considered as a contravariant functor $\mathcal{F} : \mathbf{O}(X) \to$ **Set**, and let $x \in X$ be a point. Let

$$\mathbf{U}(x) := \{U \subseteq X \mid U \text{ is an open neighborhood of } x\}$$

be the set of open neighborhoods of x, ordered by inclusion. We consider $\mathbf{U}(x)$ as a full subcategory of $\mathbf{O}(x)$. By restricting $\mathcal{F}$ to $\mathbf{O}(x)$, we obtain a contravariant functor $\mathcal{F} : \mathbf{U}(x) \to \mathbf{Set}$. Note that the category $\mathbf{U}(x)$ is filtered: for any two neighborhoods U_1 and U_2 of x there exists a neighborhood V of x with $V \subseteq U_1 \cap U_2$.

1. The colimit
$$\mathcal{F}_x := \operatorname{colim}_{U \in \mathbf{U}(x)} \mathcal{F}(U)$$
is called the **stalk** of $\mathcal{F}$ at x. More concretely, one has
$$\mathcal{F}_x = \{(U,s) \mid U \text{ is an open neighborhood of } x \text{ and } s \in \mathcal{F}(U)\} \,/ \sim,$$
where two pairs (U_1, s_1) and (U_2, s_2) are equivalent if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. The equivalence class corresponding to (U,s) at x will be denoted $[U,s]_x$, or even more simply by $[s]_x$. We obtain a canonical map $\mathcal{F}(U) \to \mathcal{F}_x$ given by $s \mapsto [s_x]$. We call $[s]_x$ the **germ** of s at x. Elements in $\mathcal{F}_x$ are called **germs**. If we write "let $[s]_x$ be a germ in $\mathcal{F}_x$" without first without specifying what the open neighboorhood of x is, then it will be understood that $[s]_x = [U^x, s]_x$ where U^x is some open neighborhood of x and that $s \in \mathcal{F}(U^x)$.

2. Let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. Then obtain an induced map $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ where $\varphi_x := \operatorname{colim}_{U \in \mathbf{U}(x)}$. In particular, φ_x is defined by
$$\varphi_x([s]_x) = [\varphi(s)]_x.$$
This map is well-defined since φ commutes with restriction maps. We obtain a functor $\mathcal{F} \to \mathcal{F}_x$ from the category of presheaves to the category of sets.

Remark 2. If $\mathcal{F}$ is a presheaf of functions, one should think of the stalk $\mathcal{F}_x$ as the set of functions defined in some unspecified open neighborhood of x.

Remark 3. If $\mathcal{F}$ is a presheaf on X with values in $\mathbf{C}$, where $\mathbf{C}$ is any category in which filtered colimits exist (for instance the category of groups, of rings, of R-modules, or R-algebras, etc...), then the stalk $\mathcal{F}_x$ is an object in $\mathbf{C}$ and we obtain a functor $\mathcal{F} \mapsto \mathcal{F}_x$ from the category of presheaves on X with values in $\mathbf{C}$ to the category $\mathbf{C}$. Let us make this more precise for a sheaf $\mathcal{G}$ of groups. The group law of $\mathcal{G}_x$ is defined as follows: Let $g, h \in \mathcal{G}$ be represented by (U,s) and (V,t). Choose an open neighborhood W of x with $W \subseteq U \cap V$. Then $(U,s) \sim (W, s|_W)$ and $(V,t) \sim (W, t_W)$ and the product gh is the equivalence class of $(W, (s|_W)(t|_W))$. In the same way addition and multiplication is defined on the stalk for a sheaf of rings.

1.6.1 Examples of Stalks

Example 1.8. Let $\mathcal{O}_{\mathbb{R}^n}$ be the sheaf of real analytic functions on $\mathbb{R}^n$ and let $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Let s be a germ in $\mathcal{O}_{\mathbb{R}^n,p}$ and let (U,f) be a representative of s. Since f is analytic at p, there exists an open neighborhood $V \subseteq U$ such that $f|_V$ is equal to its Taylor series at p:

$$f(x) = f(p) + \sum_i \partial_{x_i} f(p)(x_i - p_i) + \cdots + \frac{1}{k!} \sum_{i_1,\dots,i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} f(p)(x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) + \cdots$$

for all $x \in V$. Two real analytic functions f_1 and f_2 defined in open neighborhoods U_1 and U_2, respectively, of p agree on some open neighborhood $V \subseteq U_1 \cap U_2$ if and only if they have the same Taylor expansion around p. So we have a well-defined map $\mathcal{O}_{\mathbb{R}^n,p} \to$

Example 1.9. (Stalk of the sheaf of continuous functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U,f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$\mathrm{ev}_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(\mathrm{ev}_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. Let $s \in \mathcal{C}_{X,x} \backslash \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X \backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$. Therefore the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. This shows that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

Example 1.10. (Stalk of the sheaf of C^α functions) Let X be a topological space, let $\mathcal{C}_X$ be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$\mathcal{C}_{X,x} = \{(U, f) \mid U \text{ is and open neighborhood of } x \text{ and } f : U \to \mathbb{R} \text{ is continuous}\} \,/ \sim,$$

where $(U, f) \sim (V, g)$ if there exists an open subset W of $U \cap V$ such that $x \in W$ and $f|_W = g|_W$. As $\mathcal{C}_X$ is a sheaf of $\mathbb{R}$-algebras, $\mathcal{C}_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in \mathcal{C}_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$e_x : \mathcal{C}_{X,x} \to \mathbb{R}, \qquad s \mapsto s(x),$$

which is surjective because $\mathcal{C}_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x := \mathrm{Ker}(e_x)$. Then $\mathfrak{m}_x$ is a maximal ideal because $\mathcal{C}_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. We claim that this is the unique maximal ideal of $\mathcal{C}_{X,x}$, i.e. that $\mathcal{C}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$.

To prove this, we need to show that the complement of $\mathfrak{m}_x$ consists of units of $\mathcal{C}_{X,x}$. Let $s \in \mathcal{C}_{X,x} \backslash \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in U$ because f is continuous (take $(X \backslash f^{-1}\{0\}) \cap U$). Hence $1/f$ exists and hence s is a unit in $\mathcal{C}_{X,x}$.

Example 1.11. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces, let X be an open subspace of V, and let $\mathcal{O}$ denote the sheaf $\mathcal{C}^\alpha_{X;W}$. We claim that $\mathcal{O}_x$ is a local ring. Indeed, let $s \in \mathcal{O}_x$ be a germ and let (f, U) be a representative of s. By the very same argument as in the example above, we may assume that f does not vanish on U so that $1/f$ exists on U. It remains to show that $1/f$ is C^α on X. This follows from the stability of the C^α property under composition and the fact that $x \mapsto 1/x$ is a C^α map from $\mathbb{R}^\times$ to $\mathbb{R}^\times$.

1.6.2 Working With Stalks

The following result will be used very often.

Proposition 1.3. *Let X be a topological space, let $\mathcal{F}$ and $\mathcal{G}$ presheaves on X, and let $\varphi, \psi \colon \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves.*

1. *Assume that $\mathcal{F}$ is a sheaf. Then $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$ if and only if $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets U of X.*

2. *If $\mathcal{F}$ and $\mathcal{G}$ are both sheaves, then φ_x is bijective for all $x \in X$ if and only if φ_U is bijective for all open sets U of X.*

3. *If $\mathcal{G}$ is a sheaf, then the morphisms φ and ψ are equal if and only if $\varphi_x = \psi_x$ for all $x \in X$.*

Proof. 1. Suppose that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open sets U of X. Let $x \in X$ and suppose $[s]_x = [U^x, s]_x$ and $[t]_x = [V^x, t]_x$ are two germs in at x such that $\varphi_x([s]_x) = \varphi_x([t]_x)$. Then $[\varphi_{U^x}(s)]_x = [\varphi_{V^x}(t)]_x$ which implies there exists an open neighborhood W^x of x such that $W^x \subseteq U^x \cap V^x$ and $\varphi_{W^x}(s|_{W^x}) = \varphi_{W^x}(t|_{W^x})$. Since φ_{W^x} is injective, we see that $s|_{W^x} = t|_{W^x}$ which implies $[s]_x = [t]_x$. It follows that φ_x is injective for all $x \in X$.

Conversely, suppose $\varphi_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$. Let U be an open set of X and suppose s and t are two sections in $\mathcal{F}(U)$ such that $\varphi_U(s) = \varphi_U(t)$. Then for each $x \in U$, we have $\varphi_x([s]_x) = \varphi_x([t]_x)$, and since φ_x is injective, this implies $[s]_x = [t]_x$. Thus for each $x \in U$, there exists an open neighborhood U^x of x such that $s|_{U^x} = t|_{U^x}$. This implies $s = t$ since $\mathcal{F}$ is a sheaf. It follows that φ_U is injective for all open sets U of X.

2. Suppose that $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective for all open sets U of X . By 1, it suffices to show that φ_x is surjective for all $x \in X$. Let $x \in X$ and let $[t]_x = [U, t]_x$ be a germ at x. Since φ_U is surjective, there exists a

section s in $\mathcal{F}(U)$ such that that $\varphi_U(s) = t$. In particular, this implies $\varphi_x([s]_x) = \varphi_x([t]_x)$. It follows that φ_x is surjective for all $x \in X$.

Conversely, suppose that φ_x is bijective for all $x \in X$. By 1, it suffices to show that φ_U is surjective for all open sets U of X. Let U be an open set of X and let t be a section over U. For each $x \in U$, since φ_x is surjective, there exists a germ $[s^x]_x = [U^x, s^x]_x$ at x such that $\varphi_x([s^x]_x) = [t]_x$. By replacing U^x with a smaller open set if necessary, we may assume that $U^x \subseteq U$ and that $\varphi_{U^x}(s^x) = t|_{U^x}$ for each $x \in U$. For each $x, y \in U$, denote $U^{xy} = U^x \cap U^y$ and observe that

$$\begin{aligned}
\varphi_{U^{xy}}(s^x|_{U^{xy}}) &= \varphi_{U^x}(s^x)|_{U^{xy}} \\
&= (t|_{U^x})|_{U^{xy}} \\
&= t|_{U^{xy}} \\
&= (t|_{U^y})|_{U^{xy}} \\
&= \varphi_{U^y}(s^y)|_{U^{xy}} \\
&= \varphi_{U^{xy}}(s^y|_{U^{xy}}),
\end{aligned}$$

and hence $s^x|_{U^{xy}} = s^y|_{U^{xy}}$ since $\varphi_{U^{xy}}$ is injective. Since $\mathcal{F}$ is a sheaf and $\{U^x\}_{x\in U}$ is an open cover of U, this implies there exists a unique section s over U such that $s|_{U^x} = s^x$ for all $x \in U$. In particular, this implies that $\varphi_U(s)|_{U^x} = \varphi_{U^x}(s^x) = t|_{U^x}$ for all $x \in U$. Since $\mathcal{G}$ is a sheaf and $\{U^x\}_{x\in U}$ is an open cover of U, this implies $\varphi_U(s) = t$. It follows that φ_U is surjective for all $x \in X$.

3. Suppose that $\varphi = \psi$. Let $x \in X$ and let $[s]_x = [U, s]_x$ be a germ at x. Then since $\varphi_U(s) = \psi_U(s)$, we see that

$$\begin{aligned}
\varphi_x([s]_x) &= [\varphi_U(s)]_x \\
&= [\psi_U(s)]_x \\
&= \psi_x([s]_x).
\end{aligned}$$

It follows that $\varphi_x = \psi_x$ for all $x \in X$.

Conversely, suppose $\varphi_x = \psi_x$ for all $x \in X$. Let U be an open set of X and let s be a section over U. For each $x \in U$, since $[\varphi_U(s)]_x = [\psi_U(s)]_x$, there exists an open neighborhood U^x of x such that $U^x \subseteq U$ and $\varphi_U(s)|_{U^x} = \psi_U(s)|_{U^x}$. Since $\mathcal{G}$ is a sheaf and $\{U^x\}_{x\in U}$ is an open cover of U, we must have $\varphi_U(s) = \psi_U(s)$. It follows that $\varphi = \psi$. $\square$

Definition 1.3. We call a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves **injective** (respectively **surjective**, respectively **bijective**) if $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective (respectively surjective, respectively bijective) for all $x \in X$. A sequence

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

of morphisms of sheaves of groups is called **exact** if for all $x \in X$ the induced sequence of stalks

$$\mathcal{F}_x \xrightarrow{\varphi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is an exact sequence of groups.

Thus Proposition (1.3) tells us that φ is injective (respectively bijective) if and only if $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (respectively bijective) for all open subsets U of X. On ther other hand, φ is surjective if and only if for all open subsets $U \subseteq X$ and every $t \in \mathcal{G}(U)$ there exist an open cover $\{U_i\}_{i\in I}$ of U (depending on t) and sections $s_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$ for all $i \in I$. In other words, φ is surjective if locally we can find a preimage of t. In particular, surjectivity of φ does *not* imply that $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets U of X. Indeed, in the proof of Proposition (1.3), we needed injectivity of $\varphi_{U^{xy}}$ in order to patch up the various local sections. Here is an example from complex analysis which demonstrates this:

Example 1.12. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on an open set X of $\mathbb{C}$. For every open set U of X and for every $f \in \mathcal{O}_X(U)$ we let $\mathrm{D}_U(f) = f'$ be the derivative. . We obtain a morphism $\mathrm{D}\colon \mathcal{O}_X \to \mathcal{O}_X$ of sheaves of $\mathbb{C}$-vector spaces. Note that D is surjective because locally every holomorphic function has a primitive. On the other hand, there exists open sets U of X and functions f on U which have no primitive on U. For instance, take $X = \mathbb{C}$, let $U = \mathbb{C}\backslash\{0\}$, and let $f(z) = 1/z$. Assume for a contradiction that F is a primitive of f defined on U.

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the path defined by $\gamma(t) = e^{2\pi i t}$. Then observe that

$$\begin{aligned}
0 &= F(1) - F(1) \\
&= F(\gamma(1)) - F(\gamma(0)) \\
&= \int_\gamma f(z)\mathrm{d}z \\
&= \int_0^1 f(\gamma(t))\gamma'(t)\mathrm{d}t \\
&= \int_0^1 \frac{1}{e^{2\pi i t}} 2\pi i \cdot e^{2\pi i t}\mathrm{d}t \\
&= \int_0^1 2\pi i \mathrm{d}t \\
&= 2\pi i,
\end{aligned}$$

which is obviously a contradiction. The issue here comes from the fact that $\pi_1(U) \cong \mathbb{Z}$ where $\pi_1(U)$ is the fundamental group of U. More generally, by complex analysis we know that D_U is surjective if and only if every connected component of U is simply connected (meaning $\pi_1(U) = 0$). The sufficiency of this condition will also be an immediate application of cohomologial methods developed later. We obtain an exact sequence of sheaves of $\mathbb{C}$-vector spaces

$$0 \longrightarrow \mathbb{C}_X \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\mathrm{D}} \mathcal{O}_X \longrightarrow 0$$

where $\mathbb{C}_X$ denotes the sheaf of locally constant $\mathbb{C}$-valued functions on X and where ι_U is the inclusion for all $U \subseteq X$ open.

Let $\{U_i\}_{i \in I}$ be an open cover of X. A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective (respectively surjective, respectively bijective) if and only if its restriction $\varphi|_{U_i}\colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ to morphisms of sheaves on U_i is injective (respectively surjective, respectively bijective) for all $i \in I$. Indeed, this is because these notions are defined via the stalks. However note that the existence of the morphism φ is crucial: there exists sheaves $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{G}|_{U_i}$ for all i and such that $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic.

1.7 Sheafification

There is a functorial way to attach to a presheaf a sheaf:

Proposition 1.4. *Let $\mathcal{F}$ be a presheaf on a topological space X. There exists a pair $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ where $\widetilde{\mathcal{F}}$ is a sheaf on X and where $\iota_{\mathcal{F}}\colon \mathcal{F} \to \widetilde{\mathcal{F}}$ is a morphism of presheaves, such that the following holds: If $\mathcal{G}$ is a sheaf on X and $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there exists a unique morphism of sheaves $\widetilde{\varphi}' : \widetilde{\mathcal{F}} \to \mathcal{G}$ with $\widetilde{\varphi}'\iota_{\mathcal{F}} = \varphi$. The pair $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is unique up to unique isomorphism. Moreover, the following properties hold:*

1. *For all $x_0 \in X$, the map of stalks $\iota_{\mathcal{F},x_0} : \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is bijective.*

2. *For every presheaf $\mathcal{G}$ on X and every morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ making the diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\iota_{\mathcal{F}}"] \arrow[d, "\varphi"'] & \widetilde{\mathcal{F}} \arrow[d, "\widetilde{\varphi}"] \\
\mathcal{G} \arrow[r, "\iota_{\mathcal{G}}"] & \widetilde{\mathcal{F}}
\end{tikzcd} \tag{4}$$

   commutative. In particular, $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ is a functor from the category of presheaves on X to the category of sheaves on X.

*The sheaf $\widetilde{\mathcal{F}}$ is called the sheaf associated to $\mathcal{F}$ or the **sheafification** of $\mathcal{F}$. We obtain a functor $\mathcal{F} \to \widetilde{\mathcal{F}}$ from* $\mathbf{Psh}(X)$ *to* $\mathbf{Sh}(X)$ *which we call the **sheafification** functor.*

Proof. First we define $\widetilde{\mathcal{F}}$. Let U be an open set of X. We define $\widetilde{\mathcal{F}}(U)$ to be the families of elements in the stalks of $\mathcal{F}$, which locally give rise to sections of $\mathcal{F}$:

$$\widetilde{\mathcal{F}}(U) := \left\{ ([s^x]_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for all } x_0 \in U \text{ there exists an open neighborhood } U_0 \subseteq U \text{ of } x_0 \\ \text{and an } s_0 \in \mathcal{F}(U_0) \text{ such that } [s^x]_x = [U_0, s_0]_x \text{ for all } x \in U_0 \end{array} \right\}$$

If V is another open set of X such that $V \subseteq U$, then we define the restriction map $\mathrm{res}^U_V\colon \widetilde{\mathcal{F}}(U) \to \widetilde{\mathcal{F}}(V)$ pointwise, that is

$$([s^x]_x)_{x\in U}\mid_V = ([s^x]_x)_{x\in V}.$$

Let us show that $\widetilde{\mathcal{F}}$ is indeed a sheaf. Let $\{U_i\}_{i\in I}$ be an open cover of U and for each $i \in I$ let $([s^x])_{x\in U_i} \in \widetilde{\mathcal{F}}(U_i)$ such that

$$([s^x]_x)_{x\in U_i}|_{U_{ij}} = ([s^x]_x)_{x\in U_{ij}} = ([s^x]_x)_{x\in U_j}|_{U_{ij}}$$

for all $i, j \in I$. Then it is easy to see that $([s^x]_x)_{x\in U}$ is the unique element in $\mathcal{F}(U)$ such that

$$([s^x]_x)_{x\in U}|_{U_i} = ([s^x]_x)_{x\in U_i}$$

for all $i \in I$. Note that the same proof also shows that presheaf given by $U \mapsto \prod_{x\in U} \mathcal{F}_x$ is a sheaf.

Next we define the map $\iota_{\mathcal{F}}\colon \mathcal{F} \to \widetilde{\mathcal{F}}$. Let U be an open set of X. We define $\iota_{\mathcal{F},U}\colon \mathcal{F}(U) \to \widetilde{\mathcal{F}}(U)$ by

$$\iota_{\mathcal{F},U}(s) = ([s]_x)_{x\in U}$$

for all $s \in \mathcal{F}(U)$. Observe that if $x_0 \in X$ then the induced map $\iota_{\mathcal{F},x_0}\colon \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is defined by

$$\iota_{\mathcal{F},x_0}([s]_{x_0}) = [([s]_x)_{x\in U})]_{x_0}.$$

for all $[s]_{x_0} \in \mathcal{F}_{x_0}$. Now we prove the two properties stated in the proposition:

1. Let $x_0 \in X$. We want to show that $\iota_{\mathcal{F},x_0}\colon \mathcal{F}_{x_0} \to \widetilde{\mathcal{F}}_{x_0}$ is bijective. First we show that $\iota_{\mathcal{F},x_0}$ is injective. Let $[s]_{x_0} = [U, s]_{x_0}$ and $[t]_{x_0} = [V, t]_{x_0}$ be two germs in $\mathcal{F}_{x_0}$ such that

$$[([s]_x)_{x\in U})]_{x_0} = [([t]_x)_{x\in V})]_{x_0}.$$

Then there exists an open neighborhood $W \subseteq U \cap V$ of x_0 such that $([s]_x)_{x\in W} = ([t]_x)_{x\in W}$, or in other words, such that $[s]_x = [t]_x$ for all $x \in W$. In particular, $[s]_{x_0} = [t]_{x_0}$.

Next we show that $\iota_{\mathcal{F},x_0}$ is surjective. To see this, let $[([s^x]_x)_{x\in U})]_{x_0}$ be a germ in $\widetilde{\mathcal{F}}_{x_0}$; so U is an open neighborhood of x_0 and $([s^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}(U)$. In fact, using the construction of $\widetilde{\mathcal{F}}$, we can find a better representative for this germ: choose an open neighborhood U_0 of x_0 and choose a section $s_0 \in \mathcal{F}(U_0)$ such that $[s^x]_x = [s_0]_x$ for all $x \in U_0$. Then clearly we have $[([s^x]_x)_{x\in U})]_{x_0} = [([s_0]_x)_{x\in U_0})]_{x_0}$. In particular, note that

$$\iota_{\mathcal{F},x_0}([s_0]_{x_0}) = [([s_0]_x)_{x\in U_0})]_{x_0}.$$

It follows that $\iota_{\mathcal{F},x_0}$ is surjective.

2. Let $\mathcal{G}$ be a presheaf on X and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism. We define $\widetilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ as follows: let U be an open set of X and define the map $\widetilde{\varphi}_U\colon \widetilde{\mathcal{F}}(U) \to \widetilde{\mathcal{G}}(U)$ by

$$\widetilde{\varphi}_U\left(([s^x]_x)_{x\in U}\right) = ([\varphi_{U^x}(s^x)]_x)_{x\in U}$$

for all $([s^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}$. It is straightforward to check that this gives rise a morphism $\widetilde{\varphi}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ of presheaves. Furthermore, observe that

$$\begin{aligned}
\widetilde{\varphi}_U\iota_{\mathcal{F},U}(s) &= \widetilde{\varphi}_U\left(([s]_x)_{x\in U}\right)\\
&= ([\varphi_U(x)]_x)_{x\in U}\\
&= \iota_{\mathcal{G},U}(\varphi_U(s))\\
&= \iota_{\mathcal{G},U}\varphi_U(s).
\end{aligned}$$

It follows that $\widetilde{\varphi}\iota_{\mathcal{F}} = \iota_{\mathcal{G}}\varphi$, thus $\widetilde{\varphi}$ makes the diagram (4) commutative. We claim that $\widetilde{\varphi}$ is the unique morphism making the diagram (4) commutative. Indeed, if $\widetilde{\psi}\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ is another morphism making the diagram commute, then $\widetilde{\psi}\iota_{\mathcal{F}} = \iota_{\mathcal{G}}\varphi = \widetilde{\varphi}\iota_{\mathcal{F}}$ implies $\widetilde{\psi}_x\iota_{\mathcal{F},x} = \widetilde{\varphi}_x\iota_{\mathcal{F},x}$ for all $x \in X$. Since $\iota_{\mathcal{F},x}$ is a bijection, it follows that $\widetilde{\varphi}_x = \widetilde{\psi}_x$ for all $x \in X$. Therefore $\widetilde{\varphi} = \widetilde{\psi}$ by Proposition (1.3).

If we assume in addition that $\mathcal{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathcal{G}}\colon \mathcal{G} \to \widetilde{\mathcal{G}}$, which is bijective on stalks, is an isomorphism by Proposition (1.3). Thus we can define $\widetilde{\varphi}' = \iota_{\mathcal{G}}^{-1}\widetilde{\varphi}$ and it is easily seen that $\widetilde{\varphi}'$ is the unique morphism such that $\widetilde{\varphi}'\iota_{\mathcal{F}} = \varphi$. Finally, the uniqueness of $(\widetilde{\mathcal{F}}, \iota_{\mathcal{F}})$ is a formal consequence from the universal mapping property. □

1.7.1 Sheafification is left adjoint to the forgetful functor

Lemma 1.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. Then there is a bijection*

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(\widetilde{\mathcal{F}},\mathcal{G}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Psh}(X)}(\mathcal{F},\mathcal{G})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Thus the sheafification functor from $\mathbf{Psh}(X)$ to $\mathbf{Sh}(X)$ is the left adjoint to the forgetful functor from $\mathbf{Sh}(X)$ to $\mathbf{Psh}(X)$. In particular, the sheafification functor preserves all colimits whereas the forgetful functor preserves all limits.

Proof. If $\varphi\colon \mathcal{F} \to \mathcal{G}$, then there exists a unique morphism $\widetilde{\varphi}'\colon \widetilde{F} \to \mathcal{G}$ such that $\widetilde{\varphi}'\iota_{\mathcal{F}} = \varphi$. Conversely, if $\widetilde{\varphi}'\colon \widetilde{\mathcal{F}} \to \mathcal{G}$, then we *define* $\varphi\colon \mathcal{F} \to \mathcal{G}$ by $\varphi := \widetilde{\varphi}'\iota_{\mathcal{F}}$. Functoriality in $\mathcal{F}$ and $\mathcal{G}$ is an easy exercise. □

1.7.2 Sheafification of a presheaf of functions

Proposition 1.5. *Let E be a set and let $\mathcal{F}$ be a presheaf of functions on X with values in E. Define a sheaf $\mathcal{G}$ on X by*

$$\mathcal{G}(U) = \left\{g\colon U \to E \mid \text{there exists an open covering } \{U_i\}_{i\in I} \text{ of } U \text{ such that } g|_{U_i} \in \mathcal{F}(U_i) \text{ for all } i \in I\right\}.$$

for all open sets U of X with the restriction maps of $\mathcal{G}$ being the usual ones. Then $\widetilde{\mathcal{F}}$ is isomorphic to $\mathcal{G}$.

Proof. Let $([U^x, f^x]_x)_{x\in U} \in \widetilde{\mathcal{F}}(U)$. For each $x \in U$, choose an open neighborhood V^x of x together with a section $g^x \in \mathcal{F}(V^x)$ such that $V^x \subseteq U$ and $[f^y]_y = [g^x]_y$ for all $y \in V^x$. For each $y \in V^x$, choose an open neighborhood $W^{x,y}$ of y such that $W^{x,y} \subseteq U^y \cap V^x$ and $f^y|_{W^{x,y}} = g^x|_{W^{x,y}}$. Define a function $g\colon U \to E$ by setting $g(x) = g^x(x)$ for all $x \in U$. Note that the function g is independent of our choice of the triple $(W^{x,y}, V^x, g^x)$ for if $(\widetilde{W}^{x,y}, \widetilde{V}^x, \widetilde{g}^x)$ were another such triple, then we'd have $\widetilde{g}^x(x) = f^x(x) = g^x(x)$. Observe that $\{W^{x,y}\}$ forms an open cover of U as we vary $x \in X$ and $y \in V^x$. Moreover, we have $g|_{W^{x,y}} = g^x|_{W^{x,y}}$ since

$$\begin{aligned} g(z) &= g^z(z) \\ &= f^z(z) \\ &= g^x(z) \end{aligned}$$

for all $z \in W^{x,y}$. Therefore $g|_{W^{x,y}} = f^y|_{W^{x,y}} \in \mathcal{F}(W^{x,y})$ for all $x \in U$ and $y \in V^x$. It follows that $g \in \mathcal{G}(U)$. Therefore we obtain map from $\widetilde{\mathcal{F}}(U) \to \mathcal{G}(U)$ given by $([f^x]_x)_{x\in U} \mapsto g$. It is straightforward to check that this map induces an isomorphism $\widetilde{\mathcal{F}} \cong \mathcal{G}$ of sheaves on X. □

Definition 1.4. Let E be a set and let $\mathcal{F}$ be the constant presheaf with values in E. We denote by E_X to be the sheaf of locally constant functions with value E defined by

$$E_X(U) = \{f\colon U \to E \mid f \text{ is locally constant}\}.$$

for all open sets U of X. This is the sheafification of the presheaf of constant functions with values in E. The sheaf E_X is called the **constant sheaf** with values in E.

1.8 Direct and Inverse Images of Sheaves

Throughout this subsection, let $f\colon X \to Y$ be a continuous map.

1.8.1 Direct Image

Definition 1.5. Let $\mathcal{F}$ be a presheaf on X. We define a presheaf $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets V of Y. The restriction maps are given by the restriction maps for $\mathcal{F}$. We call $f_*\mathcal{F}$ the **direct image of $\mathcal{F}$ under f**.

Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X. Then we obtain a morphism $f_*(\varphi) : f_*\mathcal{F} \to f_*\mathcal{G}$ of presheaves on Y by defining $f_*(\varphi)_V = \varphi_{f^{-1}(V)}$ for all open sets V of Y. It is straightforward to check that we obtain a functor $f_*\colon \mathbf{Psh}(X) \to \mathbf{Psh}(Y)$. In fact, note that if $\mathcal{F}$ is a sheaf on X, then $f_*\mathcal{F}$ is a sheaf on Y. Indeed, let $\{V_i\}_{i\in I}$ be an open covering of an open set V of Y and let $s_i \in f_*\mathcal{F}(V_i) = \mathcal{F}(f^{-1}(V_i))$ such that

$$s_i|_{f^{-1}(V_{ij})} = s_j|_{f^{-1}(V_{ij})}$$

for all $i, j \in I$. Then $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of $f^{-1}(V)$, and so by the sheaf property of $\mathcal{F}$, there exists a unique $s \in \mathcal{F}(f^{-1}(V))$ such that $s|_{f^{-1}(V_i)} = s_i$. Thus, we obtain a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.

Example 1.13. Let $p\colon X \to Y$ be a continuous map, let E be a set, and let E_X and E_Y be the sheaf of locally constant E-valued functions on X and Y respectively. Define a morphism of sheaves $p^\star\colon E_Y \to p_* E_X$ by setting

$$p_V^\star(g) = g \circ p.$$

for all open sets V of Y and for all locally constant functions $g\colon V \to E$. Note that this definition makes sense because $g \circ p\colon p^{-1}(V) \to E$ is locally constant. Now assume that p is surjective, that Y has the induced quotient topology, and that p has connected fibers. Note that for all open sets V of Y, a locally constant map $h\colon p^{-1}(V) \to E$ is the same as a continuous map if we endow E with the discrete topology. The restriction of h to the fibers of p is constant and hence by the universal property of the quotient topology there exists a unique continuous map $g\colon V \to E$ such that $g \circ p = h$. In particular, we see that $p^\star$ is an isomorphism in htis case.

Proposition 1.6. *Assume that $f\colon X \to Y$ is a homeomorphism. Let $x \in X$ and let $\mathcal{F}$ be a presheaf on X. Then $(f_*\mathcal{F})_{f(x)} \cong \mathcal{F}_x$.*

Proof. Let $\pi_{\mathcal{F},x}\colon (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ be the map given by

$$\pi_{\mathcal{F},x}\left([V,s]_{f(x)}\right) = [f^{-1}(V), s]_x$$

where V is an open neighborhood of $f(x)$ and $s \in \mathcal{F}(f^{-1}(V))$. We need to show that this map is well-defined. Let (V', s') be another representative of the equivalence class $[V,s]_{f(x)}$. Then there exists an open neighborhood V'' of $f(x)$ such that $V'' \subseteq V \cap V'$ and $s|_{f^{-1}(V'')} = s'|_{f^{-1}(V'')}$. Since $f^{-1}(V'')$ is an open neighborhood of x such that $f^{-1}(V'') \subseteq f^{-1}(V) \cap f^{-1}(V')$, we have

$$\begin{aligned}\pi_{\mathcal{F},x}\left([V',s']_{f(x)}\right) &= [f^{-1}(V'), s']_x \\ &= [f^{-1}(V), s]_x.\end{aligned}$$

Thus this map is well-defined.

To show that $\pi_{\mathcal{F},x}$ is bijective, we simply describe the inverse map: Let $\pi_{\mathcal{F},x}^{-1}: \mathcal{F}_x \to (f_*\mathcal{F})_{f(x)}$ be the map given by $\pi_{\mathcal{F},x}^{-1}([U,s]_x) = [f(U), s]_{f(x)}$. Note that we need f to be an injective open map for this it to be well-defined (It is not enough that f is an open mapping. We also need $s \in \mathcal{F}(f^{-1}(f(U)))$, so we must have $f^{-1}(f(U)) = U$. However in general we only have $f^{-1}(f(U)) \supseteq U$). Clearly $\pi_{\mathcal{F},x}$ and $\pi_{\mathcal{F},x}^{-1}$ are inverse to each other. □

1.8.2 Inverse Image

Definition 1.6. Let $\mathcal{G}$ be a presheaf on Y. Define a presheaf $f^+\mathcal{G}$ on X by

$$f^+\mathcal{G}(U) = \varinjlim_{\substack{V \supseteq f(U) \\ V \subseteq Y \text{ open}}} \mathcal{G}(V).$$

for all open sets U of X, with the restriction maps being induced by the restriction maps on $\mathcal{G}$. We set $f^{-1}\mathcal{G}$ be the sheafification of $f^+\mathcal{G}$ and call $f^{-1}\mathcal{G}$ the **inverse image of** $\mathcal{G}$ **under** f.

Here is a more concrete description of $f^+\mathcal{G}$: for an open set U of X, we set

$$f^+\mathcal{G}(U) = \{(V,t) \mid V \text{ is open set of } Y \text{ such that } f(U) \subseteq V \text{ and } t \in \mathcal{G}(V)\} \,/\, \sim,$$

where two pairs (V,t) and (V',t') are equivalent if there exists an open neighborhood V'' of x with $f(U) \subseteq V'' \subseteq V \cap V'$ such that $t|_{V''} = t'|_{V''}$. The equivalence class corresponding to (V,t) will be denoted $[V,t]_{f(U)}$, or even more simply $[t]_{f(U)}$. If U' is an open set of X such that $U' \subseteq U$, then the restriction map is defined as

$$[t]|_{f(U)}|_{U'} = [t]|_{f(U')}.$$

Let's see why $f^+\mathcal{G}$ can fail to be a sheaf. Let $\{U_i\}_{i\in I}$ be an open cover of U and let $[V_i, t_i] \in f^+\mathcal{G}(U_i)$ such that

$$[V_i, t_i]|_{U_{ij}} = [V_j, t_j]|_{U_{ij}}$$

for all $i, j \in I$. In particular, this means for each $i, j \in I$ there exists an open set $V'_{i,j}$ of Y such that

$$f(U_{ij}) \subseteq V'_{i,j} \subseteq V_{ij} \quad \text{and} \quad t_i|_{V'_{i,j}} = t_j|_{V'_{i,j}}.$$

If there existed $[V,t] \in f^+\mathcal{G}(U)$ such that $[V,t]|_{U_i} = [V_i, t_i]$, then for all $i \in I$ there would exist an open subset V_i' of Y such that

$$f(U_i) \subseteq V_i' \subseteq V_i \quad \text{and} \quad t|_{V_i'} = t_i|_{V_i'}$$

Remark 4.

1. If f is the inclusion of a subspace X of Y, we also write $\mathcal{G}|_X$ instead of $f^{-1}\mathcal{G}$ and we write $\mathcal{G}(X) := (f^{-1}(\mathcal{G}))(X)$.
2. The construction of $f^+\mathcal{G}$ and hence of $f^{-1}\mathcal{G}$ is functorial in $\mathcal{G}$. Therefore we obtain a functor

$$f^{-1}\colon \mathbf{PSh}(Y) \to \mathbf{Sh}(X)$$

3. If $f\colon X \to Y$ is an open continuous map, then for $U \subseteq X$ open one has $f^+\mathcal{G}(U) \cong \mathcal{G}(f(U))$. Indeed, if $[t]_{f(U)} = [V,t]_{f(U)}$ and $[t']_{f(U)} = [V',t']_{f(U)}$ are two elements in $f^+\mathcal{G}(U)$ such that $[t]_{f(U)} = [t']_{f(U)}$, then

$$[t]_{f(U)} = [t|_{f(U)}]_{f(U)} = [t']_{f(U)}.$$

   Thus, the map sending the element $[V,t]_{f(U)} \in f^+\mathcal{G}(U)$ to the element $t|_{f(U)} \in \mathcal{G}(f(U))$ is well-defined and gives rise to an isomorphism of presheaves. Moreover, if $\mathcal{G}$ is a sheaf, then $f^+\mathcal{G}$ is a sheaf and hence $f^+\mathcal{G} = f^{-1}\mathcal{G}$. In particular, if f is the inclusion of an open subspace $U = X$ of Y, then for every sheaf $\mathcal{G}$ on Y and $U' \subseteq U$ open, we have

$$\mathcal{G}|_U(U') = \mathcal{G}(U').$$

Proposition 1.7. *Let $x \in X$ and let $\mathcal{G}$ be a presheaf on Y. Then $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$.*

Proof. It suffices to show that $f^+\mathcal{G}_x \cong \mathcal{G}_{f(x)}$ by Proposition (1.4). Define $\lambda_{\mathcal{G},x}\colon (f^+\mathcal{G})_x \to \mathcal{G}_{f(x)}$ as follows: let $[[t]_{f(U)}]_x = [U,[V,t]_{f(U)}]_x$ be an element in $(f^+\mathcal{G})_x$. We set

$$\lambda_{\mathcal{G},x}([[t]_{f(U)}]_x) = [t]_{f(x)}.$$

It is straightforward to check that $\lambda_{\mathcal{G},x}$ is well-defined.
Recall that

$$(f^+\mathcal{G})_x = \{(U,[V,t]) \mid U \text{ is an open neighborhood of } x \text{ and } [V,t] \in f^+\mathcal{G}(U)\} / \sim,$$

where $[V,t]$ denotes the equivalence class of $(V,t) \in f^+\mathcal{G}(U)$ and $(U,[V,t]) \sim (U',[V',t'])$ if there exists an open neighborhood U'' of x such that $U'' \subseteq U \cap U'$ and $[V,t]|_{U''} = [V',t']|_{U''}$, i.e. and there exists an open neighborhood V'' of $f(U'')$ such that $V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. Recall that $[U,[V,t]]_x$ denotes the equivalence class of $(U,[V,t]) \in f^+\mathcal{G}_x$.

Let $\lambda_{\mathcal{G},x} : f^+\mathcal{G}_x \to \mathcal{G}_{f(x)}$ be given by $\lambda_{\mathcal{G},x}([U,[V,t]]_x) = [V,t]_{f(x)}$. We need to show that this map is well-defined. Let $(U',[V',t'])$ be another representative of the equivalence class $[U,[V,t]]_x$. Then as described above, there exists an open neighborhood U'' of x such that $U'' \subseteq U \cap U'$ and there exists an open neighborhood V'' of $f(U'')$ such that $f(U'') \subseteq V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. As V'' is also an open neighborhood of $f(x)$, this implies

$$\begin{aligned}\lambda_{\mathcal{G},x}\left([U',[V',t']]_x\right) &= [V',t']_{f(x)}\\ &= [V,t]_{f(x)}.\end{aligned}$$

Thus this map is well-defined.

Next we show that $\lambda_{\mathcal{G},x}$ is injective. Suppose

$$\begin{aligned}\lambda_{\mathcal{G},x}\left([U,[V,t]]_x\right) &= [V,t]_{f(x)}\\ &= [V',t']_{f(x)}\\ &= \lambda_{\mathcal{G},x}\left([U',[V',t']]_x\right).\end{aligned}$$

Then there exists an open neighborhood V'' of $f(x)$ such that $V'' \subseteq V \cap V'$ and $t|_{V''} = t'|_{V''}$. Setting $U'' = f^{-1}(V'') \cap U \cap U'$, we see that U'' is an open neighborhood of x such that $U'' \subseteq U \cap U'$, V'' is an open neighborhood of $f(U'')$ such that $V'' \subseteq V \cap V'$, and $t|_{V''} = t|_{V''}$. Therefore $[U,[V,t]]_x = [U',[V',t']]_x$.

Finally, we that $\lambda_{\mathcal{G},x}$ is is onto (which will imply that $\lambda_{\mathcal{G},x}$ is a bijection since we've already shown it is injective). Let $[V,t]_{f(x)} \in \mathcal{G}_{f(x)}$. We just need to find an open neighborhood U of x such that $f(U) \subseteq V$. Then $\lambda_{\mathcal{G},x}\left([U,[V,t]]_x\right) = [V,t]_{f(x)}$. In fact $U = f^{-1}(V)$ does the trick, and we are done. □

1.8.3 Inverse image functor is left adjoint to the direct image functor

Direct image and inverse image are functors that are adjoint to each other. More precisely:

Proposition 1.8. *Let $\mathcal{F}$ be a presheaf on X and let $\mathcal{G}$ be a presheaf on Y. Then there is a bijection*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Psh}(X)}(f^+\mathcal{G}, \mathcal{F}) & \longleftrightarrow & \mathrm{Hom}_{\mathbf{Psh}(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ \varphi & \to & \varphi^\flat \\ \psi^\# & \leftarrow & \psi \end{array}$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$. Moreover, if $\mathcal{F}$ is a sheaf, then there is a bijection

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \longleftrightarrow \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\varphi\colon f^+\mathcal{G} \to \mathcal{F}$ be a morphism of presheaves on X. We define $\varphi^\flat\colon \mathcal{G} \to f_*\mathcal{F}$ as follows: let V be an open set of Y and let $t \in \mathcal{G}(V)$. We set

$$\varphi^\flat_V(t) = \varphi_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right).$$

Similarly, let $\psi\colon \mathcal{G} \to f_*\mathcal{F}$ be a morphism of presheaves on Y. We define $\psi^\#\colon f^+\mathcal{G} \to \mathcal{F}$ as follows: let U be an open set of X and let $[V,t]_{f(U)} \in f^+\mathcal{G}(U)$. We set

$$\psi^\#_U\left([V,t]_{f(U)}\right) = \psi_V(t)|_U.$$

Notice that $\psi_V(t) \in \mathcal{F}(f^{-1}(V))$ and $U \subseteq f^{-1}(V)$ since $f(U) \subseteq V$, so restricting $\psi_V(t)$ to U makes sense. Let us check that these two maps are inverse to each other. To see that $(\varphi^\flat)^\# = \varphi$, observe that

$$\begin{aligned} (\varphi^\flat)^\#_U\left([V,t]_{f(U)}\right) &= \varphi^\flat_V(t)|_U \\ &= \left(\varphi_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right)\right)\Big|_U \\ &= \varphi_U\left(\left([V,t]_{f\left(f^{-1}(V)\right)}\right)\Big|_U\right) \\ &= \varphi_U\left([V,t]_{f(U)}\right), \end{aligned}$$

where we used the fact that $U \subseteq f^{-1}(V)$ since $f(U) \subseteq V$ to get from the second line to the third line. To see that $(\psi^\#)^\flat = \psi$, observe that

$$\begin{aligned} (\psi^\#)^\flat_V(t) &= \psi^\#_{f^{-1}(V)}\left([V,t]_{f\left(f^{-1}(V)\right)}\right) \\ &= \psi_V(t)|_{f^{-1}(V)} \\ &= \psi_V(t), \end{aligned}$$

where we used the fact that $\psi_V(t) \in \mathcal{F}(f^{-1}(V))$ to get from the second line to the third line. Therefore these two maps are inverse to each other. Moreover, it is straightforward - albeit quite cumbersome - to check that the constructed maps are functorial in $\mathcal{F}$ and $\mathcal{G}$. The last part of the proposition follows from Lemma (1.1). □

Remark 5. We will almost never use the concrete description of $f^{-1}\mathcal{G}$ in the sequel. Very often we are given f, $\mathcal{F}$, and $\mathcal{G}$, and a morphism of sheaves $f^\flat\colon \mathcal{G} \to f_*\mathcal{F}$. Then usually it is sufficient to understand for each $x \in X$ the map

$$f^\#_x\colon \mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x \cong \mathcal{F}_x,$$

induced by $f^\#\colon f^{-1}\mathcal{G} \to \mathcal{F}$ on stalks.

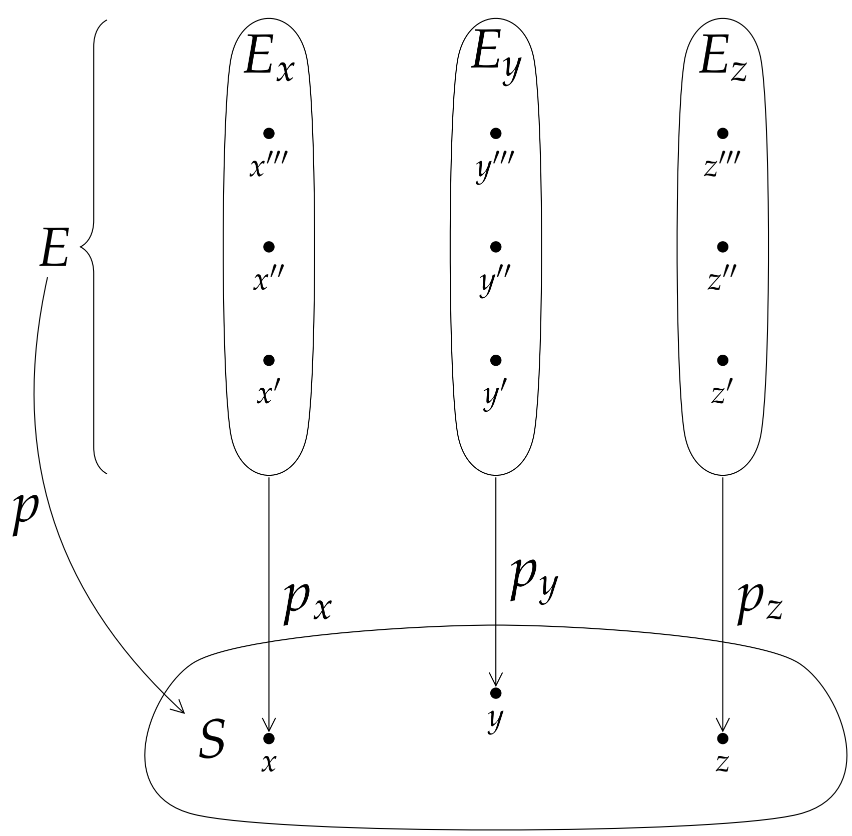
1.9 Sheaves and Etale Spaces

1.9.1 Bundles

Definition 1.7. The **slice category** $\mathbf{C}/c$ of a category $\mathbf{C}$ over an object $c \in \mathbf{C}$ is the category whose objects are morphisms $f : d \to c$ and whose morphisms from $f : d \to c$ to $f' : d' \to c$ are the morphisms $g : d \to d'$ such that $f' \circ g = f$:

<u>**Objects**</u>	<u>**Morphisms**</u>
$d \xrightarrow{f} c$	$\begin{tikzcd} d \arrow[rr, "g"] \arrow[dr, "f"'] & & d' \arrow[dl, "f'"] \\ & c & \end{tikzcd}$
object (d, f)	morphism g

Example 1.14. Let S be a set. An object $(E, p) \in \mathbf{Set}/S$ can be pictured like this:



Let's take a moment to reflect on this image, because it will serve as a nice visualization tool for other categories. For each element $s \in S$, there is an associated set E_s, which is just the inverse image of s under p, i.e. $p^{-1}(s) = E_s$. E_s is called the **fiber** of p over s. Notice that for any distinct $s, s' \in S$, $E_s \cap E_{s'} = \emptyset$, and that $\bigcup_s E_s = E$. We also have functions p_s, which is just the restriction of p to E_s. The commutativity condition for morphisms in the slice category tells us that a morphism $f : (E, p) \to (E', p')$ satisfies $f(E_s) \subseteq E'_s$ for all $s \in S$. The whole structure is called a **bundle** of sets over the **base space** S, with E being called the **total space** and p being called the **projection**.

1.9.2 Etale Spaces

Definition 1.8. Let E and X be topological spaces. A **local homeomorphism** is a continuous map $\pi : E \to X$ with the additional property that for each point $e \in E$ there exists an open neighborhood U_e in E such that $\pi(U_e)$ is open in X, and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to \pi(U_e)$.

Intuitively, a local homeomorphism preserves "local structure". For example, B is locally compact if and only if $\pi(B)$ is.

Proposition 1.9. *Let $\pi : E \to X$ be a local homeomorphism, then π is an open map.*

Proof. Let U be open in E, we need to show that $\pi(U)$ is open in X. For each $e \in U$, choose an open neighborhood U_e of e such that $\pi(U_e)$ is open in X and π restricts to a homeomorphism $\pi|_{U_e} : U_e \to p(U_e)$. Then $U \cap U_e$ is open in U_e, and since homeomorphisms are open maps, $\pi|_{U_e}(U \cap U_e)$ is open in $\pi(U_e)$ in the subspace topology. Since $\pi(U_e)$ is open in X, $\pi(U \cap U_e)$ is open in X too. Finally, since

$$\bigcup_{e \in U} \pi(U \cap U_e) = \pi(U),$$

$\pi(U)$ is open in X.

□

Example 1.15. If X is a topological space and Y is a discrete space, then the projection $Y \times X \to X$ is a local homeomorphism. On the other hand, the projection map $\mathbb{R} \times X \to X$ is never a local homeomorphism, because no product neighborhood is projected homeomorphically into X. For much the same reason, a nontrivial vector bundle is never a locally homeomorphism either.

Definition 1.9. An **etale space** over X is an object $(E, \pi) \in \mathbf{Top}/X$ such that π is a local homeomorphism. We denote by $\mathbf{Etale}(X)$ to be the full subcategory of $\mathbf{Top}/X$ whose objects are etale spaces over X and whose morphisms being the same as in $\mathbf{Top}/X$.

1.9.3 An equivalence of categories

We start with the main theorem:

Theorem 1.2. *For any topological space X there is a pair of adjoint functors*

$$\mathbf{Top}/X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set}^{O(X)^{op}}$$

where Γ assigns to each object in $(E, \pi) \in \mathbf{Top}/X$, the sheaf of all sections $\mathcal{F}_\pi$ of π, while its left adjoint Λ assigns to each preasheaf $\mathcal{F}$, the etale space $(E_\mathcal{F}, \pi_\mathcal{F})$. There are natural transformations

$$\eta_\mathcal{F} : \mathcal{F} \to \Gamma\Lambda\mathcal{F} \qquad \epsilon_E : \Lambda\Gamma E \to E$$

for a presheaf $\mathcal{F}$ and an object $(E, \pi) \in \mathbf{Top}/X$, which are unit and counit making Λ a left adjoint for Γ. If $\mathcal{F}$ is a sheaf, $\eta_\mathcal{F}$ is an isomorphism, while if (E, π) is etale, ϵ_E is an isomorphism.

1.9.4 From $\mathbf{Top}/X$ to $\mathbf{Sh}(X)$

Given an object $(E, \pi) \in \mathbf{Top}/X$, we can associate a sheaf $\mathcal{F}_\pi$ as follows: for all open subsets U of X, we define

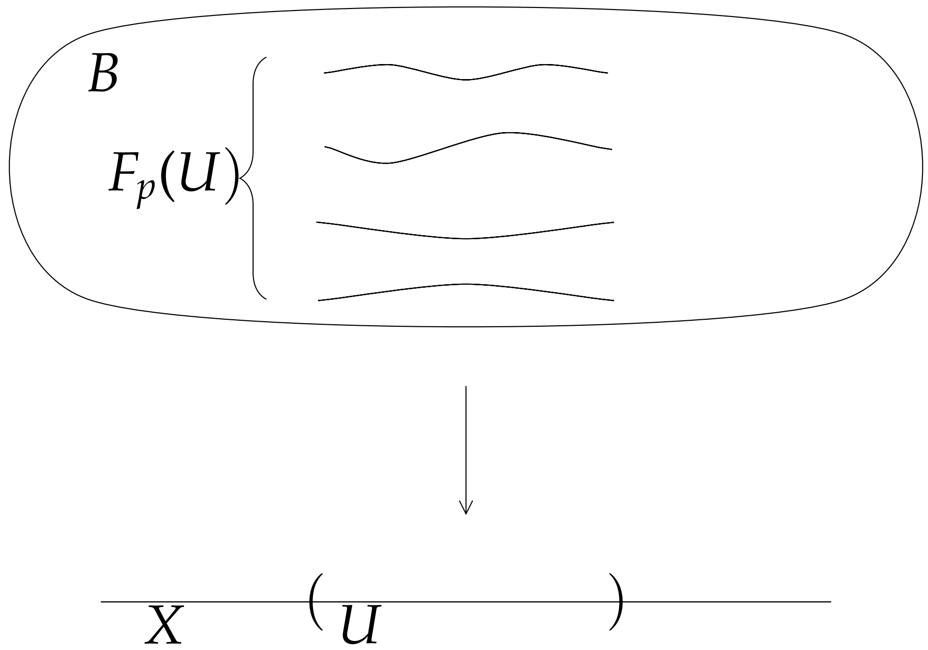
$$\mathcal{F}_\pi(U) := \{s : U \to E \mid s \text{ is continuous and } \pi|_U \circ s = \mathrm{id}_U\}.$$

For all inclusions of open sets $U \subseteq V$, we use the obvious restriction maps: if $s \in \mathcal{F}_\pi(V)$ then $s|_U \in \mathcal{F}_\pi(U)$. We claim that $\mathcal{F}_\pi$ is a sheaf (and not just a presheaf).

Indeed, let $\{U_i\}_{i \in I}$ be an open covering of an open subset U of X, and let $s_i \in \mathcal{F}_\pi(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all $i, j \in I$. We can construct an $s \in \mathcal{F}_\pi(U)$ such that $s|_{U_i} = s_i$ as follows: if $x \in U$, choose some U_i that has $x \in U_i$, and set $s(x) = s_i(x)$. We need to check that this is well-defined (i.e. independent of the choice of neighborhood of x). Suppose $x \in U_j$ for some $j \neq i$. Then because $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, we have $s(x) = s_j(x) = s_i(x)$. Thus, this construction is well-defined. Moreover, s is continous since if V is an open subset of E, then

$$s^{-1}(V) = \bigcup_{i \in I} s_i^{-1}(V)$$

is open in X. Finally, uniqueness of s is guaranteed since $\mathcal{F}_\pi$ is a presheaf of functions. We call $\mathcal{F}_\pi$ the **sheaf of sections of** π.



Let $f : (E, \pi) \to (E', \pi')$ be a morphism in $\mathbf{Top}/X$. Then for each open subset U of X, we define $f_U : \mathcal{F}_\pi(U) \to \mathcal{F}_{\pi'}(U)$ to be the function that maps a section $s \in \mathcal{F}_\pi(U)$ to $f \circ s$. The maps f_U are the components of a natural transformation from $\mathcal{F}_\pi \to \mathcal{F}_{\pi'}$. Thus, we have constructed a functor $\Gamma : \mathbf{Top}/X \to \mathbf{Sh}(X)$.

1.9.5 From $\mathbf{Psh}(X)$ to $\mathbf{Etale}(X)$

Let $\mathcal{F}$ be a presheaf on X. Define

$$E_\mathcal{F} := \coprod_{x \in X} \mathcal{F}_x = \bigcup_{x \in X} \{(x, s_0) \mid s_0 \in \mathcal{F}_x\}.$$

and let $\pi_\mathcal{F} : E_\mathcal{F} \to X$ be the obvious projection map (i.e. $(x, s) \mapsto x$). For each open subset U of X and section $s \in \mathcal{F}(U)$, let $[U, s] = \{(x, s_x) \mid x \in U\}$. Let τ be the topology on $E_\mathcal{F}$ with the collection of all $[U, s]$ as a subbasis. We claim that the collection of all $[U, s]$ is actually a basis for this topology. Indeed, let $[U, s], [V, t] \in \mathcal{B}$ and suppose $(x_0, s_{x_0}) \in [U, s] \cap [V, t]$. Then $x_0 \in U \cap V$ and $s_{x_0} = t_{x_0}$. This implies that there exists a neighborhood U_0 of x such that $U_0 \subseteq U \cap V$ and $s|_{U_0} = t|_{U_0}$. Hence $s_x = t_x$ for all $x \in U_0$. In particular, $[U_0, s|_{U_0}] \subseteq [U, s] \cap [V, t]$.

Finally, we want to show that $\pi_\mathcal{F} : E_\mathcal{F} \to X$ a local homeomorphism with respect to this topology. To see why, note that $\pi_\mathcal{F}$ maps basis elements to basis elements (i.e. $[U, s] \mapsto U$). Thus, it must be an open mapping. Also, if $(x, s_0) \in E_\mathcal{F}$, then after choosing a representative of s_0, say (U, s), we see that $(x, s_0) \in [U, s]$ and $\pi|_{[U,s]} : [U, s] \to U$ is a homeomorphism. Indeed, $\pi|_{[U,s]}$ is an open mapping and a bijection, hence its inverse must be continuous.

1.9.6 co-unit

Let $p : B \to X$ be any local homeomorphism, F_p its sheaf of sections, and $p_{F_p} : B_{F_p} \to X$ the associated sheaf of germs. Define a map $k : B \to B_{F_p}$ as follow: If $b \in B$, there exist a local section s of p through b, defined on an open set V, i.e. $b \in s(V)$ (we proved this earlier). Let $k(b) = (f(b), [s]_{f(b)})$ be the germ of s at $f(b)$. The definition of $k(b)$ does not depend on which section through b is chosen (we proved this earlier too). This gives us the following commutative diagram:

$$\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "p"'] & & B_{F_p} \arrow[dl, "p_{F_p}"] \\ & X & \end{tikzcd}$$

And k is a $\mathbf{Etale}(X)$-arrow from p to p_{F_p}, in fact, it is an iso.

1.9.7 unit

Define $\tau_U : F(U) \to F_{p_F}(U)$ by putting, for $s \in F(U)$, $\tau_U(s) = s_U$, where $s_U : U \to A_F$ is defined by putting $s_U(x) = (x, [s]_x)$ for all $x \in U$...

2 Ringed Spaces

Throughout the rest of this article, let R be a commutative ring and let $\alpha \in \widehat{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty, \omega\}$. Ringed spaces formalize the idea of giving a geometric object by specifying its underlying topological space and the "functions" on all open subsets of this space.

Definition 2.1.

1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. If $R = \mathbb{Z}$, then we simplify our notation and write "ringed space" instead of "$\mathbb{Z}$-ringed space".

2. A **locally** R**-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Usually we will denote a (locally) R-ringed space $(X, \mathcal{O}_X)$ simply by X. Also if we write "let X be a (locally) R-ringed space", then it will be understood that its structure sheaf is denoted $\mathcal{O}_X$ unless otherwise specified. For instance, we may write "let $(X, \mathcal{O})$ be a (locally) R-ringed space", and in this case the structure sheaf of X is denoted $\mathcal{O}$.

Example 2.1. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by $\mathcal{C}^\alpha_X$ the sheaf of $\mathcal{C}^\alpha$ functions: For all open subsets U of X, we have

$$\mathcal{C}^\alpha_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is } C^\alpha\}.$$

Then $\mathcal{C}^\alpha_X$ is a sheaf of $\mathbb{R}$-algebras. The same argument as for sheaves of continuous functions yields the following observation: For all $x \in X$ the stalk $C^\alpha_{X,x}$ is a local ring. In particular $(X, \mathcal{C}^\alpha_X)$ is a locally $\mathbb{R}$-ringed space.

Another example comes from Algebraic Geometry:

Example 2.2. Let k be an algebraically closed field and let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. The space X is equipped with the Zariski topology. Recall that a function $\varphi : U \to k$ from an open subset U of X to the field k is called **regular** at the point $x_0 \in X$ if there exists an open neighborhood U_0 of x_0 such that $U_0 \subseteq U$ and there are polynomials $f, g \in k[T_1, \dots, T_n]$ with $g(x) \neq 0$ and $\varphi(x) = \frac{f(x)}{g(x)}$ for all $x \in X$. With this in mind, we define the structure sheaf $\mathcal{O}_X$ of X as follows: for all open subsets U of X, we define

$$\mathcal{O}_X(U) = \{\varphi : U \to k \mid \varphi \text{ is regular}\}.$$

2.1 Morphisms of (Locally) Ringed Spaces

Definition 2.2. Let $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$ be R-ringed spaces. A **morphism of** R**-ringed spaces** $X \to Y$ is a pair $(f, f^\flat)$, where $f\colon X \to Y$ is a continuous map of the underlying topological spaces and where $f^\flat\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of R-algebras on Y.

The datum of $f^\flat$ is equivalent to the datum of a homomorphism of sheaves of R-algebras $f^\#\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X by Proposition (1.8). Usually we simply write f instead of $(f, f^\#)$ or $(f, f^\flat)$.

Morphisms of *locally* ringed spaces have to satisfy an additional property. To state this property, let $f : X \to Y$ be a morphism of R-ringed spaces and for each $x \in X$ let f_x be the composition of induced homomorphisms on the stalks

$$\mathcal{O}_{Y,f(x)} \xrightarrow{\lambda_{\mathcal{O}_Y,x}} f^{-1}(\mathcal{O}_Y)_x \xrightarrow{f^\#_x} \mathcal{O}_{X,x}$$

In particular, f_x is defined by

$$f_x([t]_{f(x)}) = [f^\flat(t)]_x$$

for all $[t]_{f(x)} \in \mathcal{O}_{Y,f(x)}$.

Definition 2.3. Let X and Y be locally R-ringed spaces. We define a **morphism of locally** R**-ringed spaces** $X \to Y$ to be a morphism $(f, f^\flat)$ of R-ringed spaces such that for all $x \in X$, the homomorphisms of local rings $f_x\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is **local**, that is

$$f_x(\mathfrak{m}_{f(x)}) \subseteq \mathfrak{m}_x$$

for all $x \in X$.

Remark 6. In the case where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of functions, the condition that f_x is local is equivalent to the condition that if $[t]_{f(x)} \in \mathcal{O}_{Y,f(x)}$ such that $t(f(x)) = 0$, then $f^\flat(t)(x) = 0$.

In general there exist locally ringed spaces and morphisms of ringed spaces between them that are not morphisms of *locally* ringed spaces. For spaces with functions of C^α functions such as the premanifolds defined below, we will see that every morphism of ringed spaces is automatically a morphism of locally ringed spaces.

Remark 7. The composition of morphisms of (locally) R-ringed spaces is defined in the obvious way using the compatibility of direct images with composition (i.e. $(g \circ f)_* = g_* \circ f_*$. We obtain the category of (locally) R-ringed spaces.

In general, $f^\flat$ (or $f^\#$) is an additional datum for a morphism. For instance it might happen that f is the identity but $f^\flat$ is not an isomorphism of sheaves. We will usually encounter the simpler case that the structure sheaf is a sheaf of functions on open subsets of X and that $f^\flat$ is given by composition with f. The following special case and its globalization is the main example.

Example 2.3. Let $X \subseteq V$ and $Y \subseteq W$ be open subsets of finite-dimensional $\mathbb{R}$-vector spaces V and W. Every C^α map $f : X \to Y$ defines by composition a morphism of locally $\mathbb{R}$-ringed spaces $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ by

$$\begin{aligned} f^\flat_U : \mathcal{C}^\alpha_Y(U) &\longrightarrow f_*(\mathcal{C}^\alpha_X)(U) = \mathcal{C}^\alpha_X(f^{-1}(U)) \\ t &\mapsto t \circ f \end{aligned}$$

for $U \subseteq Y$ open.

The induced map on stalks $f_x \colon \mathcal{C}^\alpha_{Y,f(x)} \to \mathcal{C}^\alpha_{X,x}$ is then also given by composing an $\mathbb{R}$-valued C^α function t, defined in some neighborhood of $f(x)$, with f, which yields an $\mathbb{R}$-valued C^α function $t \circ f$ defined in some neighborhood of x. Conversely, let $(f, f^\flat) : (X, \mathcal{C}^\alpha_X) \to (Y, \mathcal{C}^\alpha_Y)$ be any morphism of $\mathbb{R}$-ringed spaces. We claim:

1. $(f, f^\flat)$ is automatically a morphism of *locally* $\mathbb{R}$-ringed spaces.
2. We have $f^\flat = f^\star$.

To show 1, let $x \in X$, set $\varphi = f_x$, set $B = \mathcal{C}^\alpha_{X,x}$, and set $A = \mathcal{C}^\alpha_{Y,f(x)}$. Then $\varphi\colon A \to B$ is a homomorphism of local $\mathbb{R}$-algebras such that $A/\mathfrak{m}_A = \mathbb{R}$ and $B/\mathfrak{m}_B = \mathbb{R}$. We claim that φ is automatically local, or equivalently that $\varphi^{-1}(\mathfrak{m}_B)$ is a maximal ideal of A. Indeed, φ induces an injective homomorphism of $\mathbb{R}$-algebras

$$A/\varphi^{-1}(\mathfrak{m}_B) \hookrightarrow B/\mathfrak{m}_B = \mathbb{R},$$

and as a homomorphism of $\mathbb{R}$-algebras, it is automatically surjective (indeed 1 maps to 1), hence $A/\varphi^{-1}(\mathfrak{m}_B) \cong \mathbb{R}$ is a field and hence $\varphi^{-1}(\mathfrak{m}_B)$ is the maximal ideal of A.

Let us show 2. Let V be an open set of Y and let $x \in f^{-1}(V)$. Consider the commutative diagram of $\mathbb{R}$-algebra homomorphisms

$$\begin{tikzcd}
\mathcal{C}^\alpha_Y(V) \arrow[rrr, "f^\flat_V"] \arrow[ddd] & & & \mathcal{C}^\alpha_X(f^{-1}(V)) \arrow[ddd] \\
& t \arrow[r, mapsto] \arrow[d, mapsto] & f^\flat_V(t) \arrow[d, mapsto] & \\
& {[V,t]_{f(x)}} \arrow[r, mapsto] & {[f^{-1}(V), f^\flat_V(t)]_x} & \\
\mathcal{C}^\alpha_{Y,f(x)} \arrow[rrr, "f_x"] \arrow[ddd] & & & \mathcal{C}^\alpha_{X,x} \arrow[ddd] \\
& {[V,t]_{f(x)}} \arrow[r, mapsto] \arrow[d, mapsto] & {[f^{-1}(V), f^\flat_V(t)]_x} \arrow[d, mapsto] & \\
& t(f(x)) & f^\flat_V(t)(x) & \\
\mathbb{R} & & & \mathbb{R}
\end{tikzcd}$$

The evaluation maps are surjective. Hence there exists a homomorphism of $\mathbb{R}$-algebras $\iota\colon \mathbb{R} \to \mathbb{R}$ making the lower rectangle commutative if and only if one has $f_x(\ker(\mathrm{ev}_{f(x)})) \subseteq \ker(\mathrm{ev}_x)$, but this latter condition is satisfied because f_x is local by 1. Moreover, as a homomorphism of $\mathbb{R}$-algebras, one must have $\iota = \mathrm{id}_\mathbb{R}$. Therefore we find $f^\flat_V(t)(x) = t(f(x)) = f^\star_V(t)(x)$, which shows 2.

Remark 8. A morphism $f : X \to Y$ of R-ringed spaces is an isomorphism in the category of R-ringed spaces if and only if f is a homeomorphism and $f_x \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism of R-algebras for all $x \in X$. Indeed, $(f, f^\flat)$ is an isomorphism if and only if f is a homeomorphism and $f^\flat$ is an isomorphism of sheaves of rings. We claim that if f is a homeomorphism, then $f^\flat$ is an isomorphism if and only if f_x is an isomorphism for all $x \in X$. To see this, note that since f is a homeomorphism, we have $f_x = \pi_{\mathcal{O}_X,x} \circ f_x^\flat$, where $\pi_{\mathcal{O}_X,x}$ is the isomorphism constructed in Proposition (1.6).

2.1.1 Open embedding

Let X be a locally R-ringed space and let $U \subseteq X$ be an open set. Then $(U, \mathcal{O}_{X|U})$ is a locally R-ringed space, where $\mathcal{O}_{X|U}$ is defined by

$$\mathcal{O}_{X|U}(U') = \mathcal{O}_X(U')$$

for all open subsets U' of U and where the restrictions maps are the ones incuded by $\mathcal{O}_X$. Such a locally ringed R-space is called an **open subspace** of X. There is an $\iota \colon U \to X$ of locally R-ringed spaces, where the continuous map $\iota \colon U \to X$ is the inclusion of the underlying topological spaces and where $\iota^\flat \colon \mathcal{O}_X \to \iota_* \mathcal{O}_{X|U}$ is given by the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ for all open subsets V of X. Thus

$$\iota_V^\flat(s) = s|_{U \cap V}$$

for all $s \in \mathcal{O}_X(V)$. Notice that $\iota^\# \colon \iota^{-1}\mathcal{O}_X \to \mathcal{O}_{X|U}$ is the identity. In particular ι_x is the identity for all $x \in U$. Given any morphism $f \colon X \to Y$, we denote by $f|_U \colon U \to Y$ to be the composition $f \circ \iota$ of morphisms of locally R-ringed spaces.

Definition 2.4. Let $f \colon X \to Y$ and $i \colon Z \to X$ be morphisms of locally ringed R-spaces.

1. We say i is an **open embedding** if $i(Z)$ is an open subset of X and i induces an isomorphism $Z \cong i(Z)$ of locally ringed R-spaces.

2. We say f is a **local isomorphism** if there exists an open cover $\{U_i\}_{i \in I}$ of X such that $f|_{U_i} \colon U_i \to Y$ is an open embedding for all $i \in I$. In other words, $V_i := f(U_i)$ is an open subspace of Y and $f|_{U_i} \colon U_i \to V_i$ is an isomorphism of locally ringed R-spaces for each $i \in I$. Note that f is a local isomorphism if and only if f is a local homeomorphism and $f_x \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for all $x \in X$.

2.2 Gluing Ringed Spaces

Let X_1 and X_2 be locally R-ringed spaces, let $X_{1,2}$ be a nonempty open subset of X_1 and let $X_{2,1}$ be a nonempty open subset of X_2, and let $f \colon X_{1,2} \to X_{2,1}$ be an isomorphism of locally R-ringed spaces. We construct a locally R-ringed space X, obtained by gluing X_1 and X_2 using f as follows:

- The underlying set X is given by
$$X = X_1 \coprod X_2 / \sim$$
where $X_1 \coprod X_2$ is the disjoint union of X_1 and X_2 and where $\sim$ is the equivalence relation defined by $x \sim f(x)$ for all $x \in X_{1,2}$. We give X the structure of a topological space using the quotient topology with respect to $\sim$. Thus a set $U \subseteq X$ is open if and only if $U \cap U_1 \subseteq U_1$ and $U \cap U_2 \subseteq U_2$ are both open subsets of U_1 and U_2 respectively, where $U_1 = i_1(X_1)$ and $U_2 = i_2(X_2)$ with $i_1 \colon X_1 \to X$ and $i_2 \colon X_2 \to X$ being the obvious inclusion maps.

- We given X the structure of a locally R-ringed space by defining the structure sheaf $\mathcal{O}_X$ by
$$\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2), \text{ and } f^\flat(s_2)|_{U \cap X_{2,1}} = s_1|_{U \cap X_{1,2}})\}$$
for all open subsets U of X.

Let's go over specific examples of this construction:

Example 2.4. Let $X_1 = X_2 = \mathbb{A}^1$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $x \mapsto \frac{1}{x}$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $K = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$.

- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X.

Example 2.5. Let X be the complex affine curve

$$X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2\widetilde{x}^4 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x},\widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x,y) \mapsto (\widetilde{x},\widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right)$$
$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x},\widetilde{y}) \mapsto (x,y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^n}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

2.3 $\mathcal{O}_X$-modules

Definition 2.5. Let X be a ringed space

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. We define the product presheaf $\mathcal{F} \times \mathcal{G}$ on X with respect to $\mathcal{F}$ and $\mathcal{G}$ by setting
$$(\mathcal{F} \times \mathcal{G})(U) = \mathcal{F}(U) \times \mathcal{G}(U)$$
for all open subsets U of X where the restriction maps are the products of the restriction maps for $\mathcal{F}$ and $\mathcal{G}$. Clearly this is a sheaf if $\mathcal{F}$ and $\mathcal{G}$ are sheaves.

2. An $\mathcal{O}_X$**-module** is a sheaf $\mathcal{F}$ on X equipped with two morphisms of sheaves
$$\mathcal{F} \times \mathcal{F} \to \mathcal{F} \quad \text{and} \quad \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$$
called addition and scalar-multiplication respectively, such that for each open subset U of X, addition and scalar-multiplication by $\mathcal{O}_X(U)$ gives $\mathcal{F}(U)$ the structure of an $\mathcal{O}_X(U)$-module.

3. Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules and let $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say φ is an $\mathcal{O}_X$**-module homomorphism** if φ_U is an $\mathcal{O}_X(U)$-module homomorphism for each open subset U of X. The composition of two $\mathcal{O}_X$-module homomorphisms is again an $\mathcal{O}_X$-module homomorhism. We obtain a category of $\mathcal{O}_X$-modules and $\mathcal{O}_X$-module homomorphisms which we denote by $\mathbf{Mod}_{\mathcal{O}_X}$.

4. Assume that X is a locally ringed space. Let $x \in X$ and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Note that the $\mathcal{O}_X$-module structure on $\mathcal{F}$ induces an $\mathcal{O}_{X,x}$-module structure on $\mathcal{F}_x$. The **fiber** of $\mathcal{F}$ at x, denoted $\mathcal{F}(x)$, is the $\kappa(x)$-vector space
$$\mathcal{F}(x) := \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x).$$
If s is a section of $\mathcal{F}$ over an open neighborhood U of x, we denote by $s(x)$ the image of the germ $[s]_x \in \mathcal{F}_x$ in $\mathcal{F}(x)$.

Part II
Differential Geometry

3 Euclidean Spaces

The Euclidean space $\mathbb{R}^n$ is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. A good understanding of $\mathbb{R}^n$ is essential in generalizing differential and integral calculus to a manifold.

Definition 3.1. Let k be a nonnegative integer and U be an open subset in $\mathbb{R}^n$. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_j}} f$$

of all orders $j \leq k$ exist and are continuous at p. The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k at p for all $k \geq 0$. A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its component functions $f_1, \ldots, f_n$ are C^k at p. We say that $f : U \to \mathbb{R}^m$ is C^k on U if it is C^k at every point in U. A similar defintion holds for a C^∞ function on an open set U. We treat the terms "C^∞" and "smooth" as synonymous.

Example 3.1.

1. A C^0 function on U is a continuous function on U.

2. The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

   Thus the function f is C^0 but not C^1 at $x = 0$. On the other hand, f is C^1 on the open subset $\{x \in \mathbb{R} \mid x \neq 0\} \subseteq \mathbb{R}$. Now let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$\int_0^x f(t)dt = \int_0^x t^{1/3}dt = \frac{3}{4}x^{4/3}.$$

   Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Then f is smooth and even bijective with inverse f^{-1} given by $f^{-1}(x) = x^{1/3}$, but f^{-1} is not smooth, as shown above.

5. Continuity of a function can often be seen by inspection, but the smoothness of a function always requires a formula. The graph of $y = x^{5/3}$ looks perfectly smooth, but it is in fact not smooth at $x = 0$, since its second derivative $y'' = (10/9)x^{-1/3}$ is not defined there.

6. Consider the norm function from $\mathbb{R}^n$ to $\mathbb{R}$, given by sending $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2} \in \mathbb{R}$. We will do this in detail. First we claim that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial and $k \geq 1$. We prove this by induction on k. The base case is trivial:

$$\partial_{x_n} (\|x\|) = \frac{x_n}{\|x\|}$$

   Now suppose that $\partial_{x_n}^k (\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$ where $f(x)$ is a polynomial. Then

$$\begin{aligned} \partial_{x_n}^{k+1} (\|x\|) &= \partial_{x_n} \left(\frac{f(x)}{\|x\|^{2k-1}} \right) \\ &= \frac{(\partial_{x_n} f)(x)}{\|x\|^{2k-1}} + \frac{(1-2n)x_n f(x)}{\|x\|^{2k+1}} \\ &= \frac{(\partial_{x_1} f)(x) \left(x_1^2 + \cdots + x_n^2\right) + (1-2n)x_n f(x)}{\|x\|^{2n+1}}. \end{aligned}$$

This establishes our claim. Now given that $\partial_{x_n}^k(\|x\|)$ has the form $f(x)/\|x\|^{2k-1}$, where $f(x)$ is a polynomial, it is clear that $\partial_{x_1}^{k_1}\cdots\partial_{x_n}^{k_n}(\|x\|)$ has the form $g(x)/\|x\|^{2(k_1+\cdots+k_n)-1}$, where $g(x)$ is a polynomial (just use the same induction proof). Now since $\|x\| = 0$ if and only if $x = \mathbf{0}$, we see that the norm function is smooth in $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

Proposition 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be smooth. Then $g \circ f = \mathbb{R}^n \to \mathbb{R}$ is smooth.*

Proof. We will only sketch the proof here. By the chain rule, we have

$$\partial_{x_n}(g \circ f) = (g' \circ f) \cdot \partial_{x_n} f$$

By the product rule we have

$$\partial_{x_n}^2(g \circ f) = (g'' \circ f) \cdot (\partial_{x_n} f)^2 + (g' \circ f) \cdot \partial_{x_n}^2 f$$

Similarly, we have

$$\partial_{x_n}^3(g \circ f) = (g''' \circ f)(\partial_{x_n} f)^3 + 3(g'' \circ f)(\partial_{x_n} f)(\partial_{x_n}^2 f) + (g' \circ f)\partial_{x_n}^3 f.$$

More generally, we will have a pattern which involves stirling numbers. □

Definition 3.2. Let $p = (p_1, \ldots, p_n)$ be a point in $\mathbb{R}^n$. A **neighborhood** of p in $\mathbb{R}^n$ is an open set containing p. The function f is **real-analytic** at p if in some neighborhood of p it is equal to its Taylor series at p:

$$f(x) = f(p) + \sum_i \partial_{x_i} f(p)(x_i - p_i) + \frac{1}{2!}\sum_{i,j} \partial_{x_i}\partial_{x_j} f(p)(x_i - p_i)(x_j - p_j) + \cdots + \frac{1}{k!}\sum_{i_1,\ldots,i_k} \partial_{x_{i_1}}\cdots\partial_{x_{i_k}} f(p)(x_{i_1} - p_{i_1})\cdots(x_{i_k} - p_{i_k}) + \cdots$$

A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

then term-by-term differentiation gives

$$f'(x) = \cos x = 1 - \frac{1}{2!}x^2 - \frac{1}{4!}x^4 - \cdots$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0.

Example 3.2. (A C^∞ function very flat at 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Clearly $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(0) = 0$. Also,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-1/x}\right) = e^{-1/x}\left(\sum_{i=1}^n (-1)^{n+i}\frac{L(n,i)}{x^{n+i}}\right)$$

Where $L(n,i)$ are the Lah numbers. Both $e^{-1/x}$ and $\sum_{i=1}^n (-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ are well defined for $x > 0$ and $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-1/x}\right) \to 0$ as $x \to 0$ (since $e^{-1/x}$ approaches 0 much faster than $\sum_{i=1}^n (-1)^{n+i}\frac{L(n,i)}{x^{n+i}}$ approaches ∞), so this function is clearly C^∞ on $\mathbb{R}$. On the other hand, the Taylor series of this function at the origin is identically zero in any neighborhood of the origin since $\frac{\mathrm{d}^n f}{\mathrm{d}x^n}(0) = 0$ for all $n \geq 1$. Therefore $f(x)$ cannot be equal to its Taylor series and thus $f(x)$ is not real-analytic at 0.

3.1 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. We say that a subset S of $\mathbb{R}^n$ is **star-shaped** with respect to a point p in S if for every x in S, the line segment from p to x lies in S. The line segment from p to x is parametrized by $\gamma : [0,1] \to \mathbb{R}^n$ where $\gamma(t) = (1-t)p + tx$. S is star-shaped with respect to p if for every x in S, $(1-t)p + tx$ is in S for all $t \in (0,1)$.

Lemma 3.1. *(Taylor's theorem with remainder). Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p_1, \dots, p_n)$ in U. Then there are functions $g_1(x), \dots, g_n(x) \in C^\infty(U)$ such that*

$$f(x) = f(p) + \sum_{i=1}^{n} (x_i - p_i) g_i(x) \qquad g_i(p) = \partial_{x_i} f(p)$$

Remark 9. The idea behind this proof is to differentiate $f(p + t(x-p))$ and then integerate it.

Proof. Since U is star-shaped with respect to p, for any $x \in U$ the line segment $p + t(x-p)$, $0 \leq t \leq 1$ lies in U. So $f(p + t(x-p))$ is defined for $0 \leq t \leq 1$. By the chain rule

$$\begin{aligned}
\frac{df}{dt}(p + t(x-p)) &= \frac{df}{dt}(p_1 + t(x_1 - p_1), \dots, p_n + t(x_n - p_n)) \\
&= (\partial_{x_1} f)(p + t(x-p)) \partial_t (p_1 + t(x_1 - p_1)) + \cdots + (\partial_{x_n} f)(p + t(x-p)) \partial_t (p_n + t(x_n - p_n)) \\
&= \sum_{i=1}^{n} (x_i - p_i) \partial_{x_i} f(p + t(x-p)).
\end{aligned}$$

If we integrate both sides with respect to t from 0 to 1, we get

$$f(p + t(x-p))|_0^1 = \sum (x_i - p_i) \int_0^1 \partial_{x_i} f(p + t(x-p)) dt \tag{5}$$

Now let $g_i(x) = \int_0^1 \partial_{x_i} f(p + t(x-p)) dt$. □

Example 3.3. We want to apply this proof to the function $f(x)$ on $\mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Let $p = 0$ and let $g(x) = \int_0^1 \frac{df}{dx}(p + t(x-p)) dt$. Then

$$\begin{aligned}
g(x) &= \int_0^1 \frac{df}{dx}(tx) dt \\
&= \int_0^1 \frac{-e^{-1/tx}}{tx^2} dt \\
&= \frac{e^{-1/tx}}{x}\Big|_0^1 \\
&= \frac{e^{-1/x}}{x}.
\end{aligned}$$

Thus,

$$f(x) = f(0) + x \left(\frac{e^{-1/x}}{x} \right).$$

3.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

In elementary calculus we normally represent a vector at a point p in $\mathbb{R}^3$ algebraically as a column of numbers

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

or geometrically as an arrow emanating from p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p. Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a Euclidean space, and so would not apply to the projective plane, for example, which does not sit inside an $\mathbb{R}^n$ in any natural way.

3.2.1 The Directional Derivative

Let $p = (p_1, \ldots, p_n)$ be a point with direction $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$. The line through the point p in the direction v can be parametized by $\ell := (\ell_1, \ldots, \ell_n) : \mathbb{R} \to \mathbb{R}^n$, where

$$\ell(t) := p + tv = (p_1 + tv_1, \ldots, p_i + tv_i, \ldots, p_n + tv_n) =: (\ell_1(t), \ldots, \ell_i(t), \ldots, \ell_n(t)).$$

Now let f be a C^∞ in a neighborhood of p in $\mathbb{R}^n$. The **directional derivative** of f in the direction of v at p is defined to be

$$\begin{aligned}
D_v f &:= \lim_{t\to 0} \left(\frac{f(\ell(t)) - f(p)}{t} \right) \\
&= \partial_t f(\ell(t))|_{t=0} \\
&= \sum_{i=1}^n \partial_{x_i} f(\ell(0)) \cdot \partial_t \ell_i(0) \\
&= \sum_{i=1}^n v_i \partial_{x_i} f(p)
\end{aligned}$$

In the notation $D_v f$, it is understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v_i \partial_{x_i}|_p$$

for the map that sends a function f to the number $D_v f$. To simplify the notation we often omit the subscript p if it is clear from the context.

3.2.2 Germs of Functions

Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We introduce a relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. It is easy to check that this is an equivalence relation by showing it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(\mathbb{R}^n)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Remark 10. What happens if we weaken the relation a bit? Say $(f_1, U_1) \sim (f_2, U_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$. In this case, we no longer have an equivalence relation. The reason is because this relation is not transitive: Suppose $(f_1, U_1) \sim (f_2, U_2)$ and $(f_2, U_2) \sim (f_3, U_3)$. Then $f_1 = f_2$ on $U_1 \cap U_2$ and $f_2 = f_3$ on $U_2 \cap U_3$, but this merely implies that $f_1 = f_3$ on $U_1 \cap U_2 \cap U_3$.

Example 3.4. The functions

$$f(x) = \frac{1}{1-x}$$

with domain $\mathbb{R} \setminus \{1\}$ and

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

with domain the open interval $(-1, 1)$ have the same germ at any point p in the open interval $(-1, 1)$.

The addition and multiplication of functions induce corresponding operations on C_p^∞ making it into an $\mathbb{R}$-algebra. Indeed, let (f_1, U_1) and (f_2, U_2) be two representatives. Then multiplication is given by

$$(f_1, U_1) \cdot (f_2, U_2) = (f_1 f_2, U_1 \cap U_2).$$

We need to check that this is well-defined, so let (f_1', U_1') and (f_2', U_2') be two different represenatives respectively. Then

$$f_1 = f_1' \text{ on } W_1 \subset U_1 \cap U_1' \text{ and } f_2 = f_2' \text{ on } W_2 \subset U_2 \cap U_2'$$

This implies

$$f_1 f_2 = f_1' f_2' \text{ on } W_1 \cap W_2 \subset U_1 \cap U_2,$$

and thus

$$(f_1 f_2, U_1 \cap U_2) \sim (f_1' f_2', U_1 \cap U_2)$$

and hence this is well-defined. Similarly, addition is given by

$$(f_1, U_1) + (f_2, U_2) = (f_1 + f_2, U_1 \cap U_2).$$

Example 3.5. This example requires some knowledge of Algebraic Geometry. Let X be an affine algebraic set over an algebraically closed field K, let $R = A(X)$ be its coordinate ring, let p be a point in X, and let $\mathfrak{m}$ be the maximal ideal in R given by the set of all $f \in R$ which vanish at p. There are two equivalent ways to define the local ring $O_{X,p}$ at p.

One way is to define $O_{X,p}$ to be the local ring $R_\mathfrak{m}$. Elements in $R_\mathfrak{m}$ are equivalence classes of elements of the form f/g, where $f, g \in R$ and $g \notin \mathfrak{m}$. We say f_1/g_1 is equivalent to f_2/g_2 if there is an $h \in R$ such that $h \notin \mathfrak{m}$ and $h(g_2 f_1 - g_1 f_2) = 0$.

The other way is to define $O_{X,p}$ to be the ring of all germs of polynomial functions defined on a neighborhood of p. A "polynomial function defined on a neighborhood of p" is of the form f/g where $f, g \in R$ and $g(p) \neq 0$. We can think of f/g here as being the germ $(f/g, D(g))$, where $D(g)$ is the set of all points such that $g \neq 0$. Two such polynomial functions f_1/g_1 (or germ $(f_1/g_1, D(g_1))$) and f_2/g_2 (or germ $(f_2/g_2, D(g_2))$) represent the same germ if they agree on some small neighborhood of p. A small open neighborhood of p in the Zariski topology is simply something of the form $D(h)$ where h does not vanish at p. Thus, we need $f_1/g_1 = f_2/g_2$ on $D(h) \cap D(g_1) \cap D(g_2)$. Another way of saying this is $g_1 g_2 h(f_1/g_1 - f_2/g_2) = 0$ as a function on X; this matches precisely the criterion for f_1/g_1 and f_2/g_2 to be equal in the local ring $R_\mathfrak{m}$.

3.2.3 Derivations at a Point

We claim that D_v gives a map from C_p^∞ to $\mathbb{R}$. Indeed we just need to check that it is well-defined: suppose $(f, U) \sim (g, V)$. Then $f|_W = g|_W$ for some open set $W \subseteq U \cap V$. In particular,

$$\partial_{x_i} f(p) = \lim_{h \to 0} \frac{f(p_1, \ldots, p_i + h, \ldots, p_n)}{h} = \lim_{h \to 0} \frac{g(p_1, \ldots, p_i + h, \ldots, p_n)}{h} = \partial_{x_i} g(p).$$

for all $i = 1, \ldots, n$, which implies

$$\begin{aligned} D_v f &= \sum_{i=1}^n v_i \partial_{x_i} f(p) \\ &= \sum_{i=1}^n v_i \partial_{x_i} g(p) \\ &= D_v g. \end{aligned}$$

Thus $D_v : C_p^\infty \to \mathbb{R}$ is a well-defined map. In fact, D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = (D_v f) g(p) + f(p) D_v g, \tag{6}$$

precisely because the partial derivatives $\partial_{x_i}|_p$ have these properties.

In general, any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (6) is called a **derivation at** p or a **point-derivation** of C_p^∞. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiplication of a derivation at p are again derivations at p.

Thus far, we know that directional derivatives at p are all derivations at p, so there is a map

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n),$$

where a vector $v = (v_1, \ldots, v_n)$ in $T_p(\mathbb{R}^n)$ is mapped to the point-derivation $D_v = \sum_{i=1}^n v_i \partial_{x_i}|_p$. Since D_v is clearly linear in v, the map ϕ is a linear map of vector spaces.

Lemma 3.2. *If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.*

Proof. By $\mathbb{R}$-linearity, $D(c) = cD(1)$, so it suffices to prove that $D(1) = 0$. By the the Leibniz rule (6), we have

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Substracting $D(1)$ from both sides gives $D(1) = 0$. □

The **Kronecker delta** δ is a useful notation that we frequently call upon:

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j. \\ 0 & \text{if } i \neq j. \end{cases}$$

Theorem 3.3. *The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined above is an isomorphism of vector spaces.*

Proof. To prove injectivity, suppose $D_v = 0$ for $v = (v_1, \ldots, v_n) \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x_j gives

$$\begin{aligned} 0 &= D_v x_j \\ &= \sum_i v_i \partial_{x_i} x_j \,|_p \\ &= v_j. \end{aligned}$$

Hence $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, V) be a representative of a germ in C_p^∞. Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped. By Taylor's theorem with remainder, there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x),$$

where $g_i(p) = \partial_{x_i} f(p)$. Applying D to both sides and noting that $Df(p) = 0$ and $D(p_i) = 0$ by Lemma (3.2), we get by the Leibniz rule (6)

$$\begin{aligned} Df &= D(f(p) + \sum_{i=1}^n (x_i - p_i) g_i(x)) \\ &= D(f(p)) + \sum_{i=1}^n D\left((x_i - p_i) g_i(x)\right) \\ &= \sum_{i=1}^n D\left((x_i - p_i) g_i(x)\right) \\ &= \sum_{i=1}^n (D(x_i - p_i) g_i(p) + (p_i - p_i) D g_i) \\ &= \sum_{i=1}^n (D(x_i) - D(p_i)) g_i(p) \\ &= \sum_{i=1}^n D(x_i) g_i(p) \\ &= \sum_{i=1}^n D(x_i) \partial_{x_i} f(p) \end{aligned}$$

This proves that $D = D_v$ for $v = (Dx_1, \ldots, Dx_n)$. □

3.2.4 Vector Fields

A **vector field** $\vec{v}$ on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector $\vec{v}(p)$ in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\{\partial_{x_i}|_p\}$, the vector $\vec{v}(p)$ is a linear combination

$$\vec{v}(p) = \sum_{i=1}^n \vec{v}_i(p) \partial_{x_i}(p),$$

where $\vec{v}_i(p) \in \mathbb{R}$. Thus we may write $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$, where the $\vec{v}_i$ are now functions on U. We say that a vector field $\vec{v}$ is C^∞ on U if the coefficient functions $\vec{v}_i$ are all C^∞ on U.

Example 3.6.

1. On $\mathbb{R}^2 \backslash \{0\}$, we have the vector field
$$\vec{v} = \frac{-y}{\sqrt{x^2 + y^2}} \partial_x + \frac{x}{\sqrt{x^2 + y^2}} \partial_y.$$

2. On $\mathbb{R}^2$, we have the vector field
$$\vec{v} = x \partial_x - y \partial_y.$$

The ring of C^∞ on an open set U is commonly denoted by $C^\infty(U)$. Multiplication of vector fields by functions on U is defined pointwise:

$$(f\vec{v})(p) = f(p)\vec{v}(p).$$

Clearly if $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$ is a C^∞ vector field and f is a C^∞ function on U, then

$$f\vec{v} = \sum_{i=1}^n f\vec{v}_i \partial_{x_i}$$

is a C^∞ vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $\mathrm{Vec}(U)$, is a $C^\infty(U)$-module.

3.3 Vector Fields as Derivations

If $\vec{v}$ is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function on U by

$$(\vec{v}f)(p) = \vec{v}(p)f$$

for all $p \in U$. Writing $\vec{v} = \sum_{i=1}^n \vec{v}_i \partial_{x_i}$, we get

$$(\vec{v}f)(p) = \sum_{i=1}^n \vec{v}_i(p) \partial_{x_i} f(p)$$

or $\vec{v}f = \sum_{i=1}^n \vec{v}_i \partial_{x_i} f$, which shows that $\vec{v}f$ is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$C^\infty(U) \to C^\infty(U), \qquad f \mapsto \vec{v}f.$$

Proposition 3.2. *(Leibniz rule for a vector field) If $\vec{v}$ is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $\vec{v}(fg)$ satisfies the Leibniz rule:*

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

Proof. At each point $p \in U$, the vector $\vec{v}(p)$ satisfies the Leibniz rule:

$$\vec{v}(p)(fg) = \vec{v}(p)(f) \cdot g(p) + f(p) \cdot \vec{v}(p)(g),$$

as p varies over U, this becomes an inequality of functions:

$$\vec{v}(fg) = (\vec{v}f)g + f(\vec{v}g).$$

□

If A is an algebra over a field K, a **derivation** of A is a K-linear map $D : A \to A$ such that

$$D(ab) = (Da)b + a(Db),$$

for all $a, b \in A$. The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted by $\mathrm{Der}(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\varphi : \mathrm{Vec}(U) \to \mathrm{Der}(C^\infty(U)), \qquad \vec{v} \mapsto (f \mapsto \vec{v}f).$$

Just as the tangent vectors at a point p can be identified with the point-derivations of C_p^∞, so the vector fields on an open set U can be identified with the derivations of the algebra $C^\infty(U)$, i.e. the map φ is an isomorphism of vector spaces.

3.4 The Exterior Algebra of Multicovectors

The basic principle of manifold theory is the linearization principle, according to which every manifold can be locally approximated by its tangent space at a point, a linear object. In this way linear algebra enters into manifold theory.

Instead of working with tangent vectors, it turns out to be more fruitful to adopt the dual point of view and work with linear functions on a tangent space. After all, there is only so much that one can do with tangent vectors, which are essentially arrows, but functions, far more flexible, can be added, multiplied, and composed with other maps.

3.5 Dual Spaces

Definition 3.3. Let V and W be two $\mathbb{R}$-vector spaces. We denote by $\mathrm{Hom}_{\mathbb{R}}(V, W)$ the vector space of all linear maps $\varphi : V \to W$. Define the **dual space** $V^\vee$ of V to be the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. The elements of $V^\vee$ are called **covectors** or 1-**covectors** on V.

Let V be a finite dimensional $\mathbb{R}$-vector space with basis $\{e_1, \ldots, e_n\}$. Then every $v \in V$ can be uniquely expressed as $\sum_{i=1}^n v_i e_i$ with $v_i \in \mathbb{R}$. Let $\underline{e}_i \in V^\vee$ be the linear function that picks out the ith coordinate, $\underline{e}_i(v) = v_i$. Note that $\underline{e}_i$ is characterized by

$$\underline{e}_i(e_j) = \delta^i_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Proposition 3.3. *The functions $\underline{e}_1, \ldots, \underline{e}_n$ form a basis for $V^\vee$.*

Proof. We first show that $\underline{e}_1, \ldots, \underline{e}_n$ span $V^\vee$. Suppose $\ell \in V^\vee$. For all $v \in V$, we have

$$\begin{aligned} \ell(v) &= \ell\left(\sum_{i=1}^n v_i e_i\right) \\ &= \sum_{i=1}^n v_i \ell(e_i) \\ &= \sum_{i=1}^n \underline{e}_i(v)\ell(e_i) \\ &= \sum_{i=1}^n \ell(e_i)\underline{e}_i(v) \\ &= \left(\sum_{i=1}^n \ell(e_i)\underline{e}_i\right)(v) \end{aligned}$$

Therefore $\ell = \sum_{i=1}^n \ell(e_i)\underline{e}_i \in \mathrm{Span}(\{\underline{e}_1, \ldots, \underline{e}_n\})$. Next we show the set $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is linearly independent over $\mathbb{R}$. Suppose

$$\sum_{i=1}^n c_i \underline{e}_i = 0, \tag{7}$$

where $e_i \in \mathbb{R}$. By applying e_i to both sides of equation (7), we obtain $c_i = 0$, for all $i = 1, \ldots, n$. □

Remark 11. We say $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is the **dual basis** of $\{e_1, \ldots, e_n\}$.

Proposition 3.4. *Let V be a finite-dimensional vector space and let $\ell \in V^\vee$. Then $\mathrm{Ker}(\ell)$ is a hyperplane in V.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of V and let $\{\underline{e}_1, \ldots, \underline{e}_n\}$ be its dual basis. Write ℓ in terms of the dual basis:

$$\ell = \sum_{i=1}^n a_i \underline{e}_i,$$

where $a_i \in \mathbb{R}$. A vector $\sum_{i=1}^n x_i e_i$ belongs to the kernel of ℓ if and only if $\sum_{i=1}^n x_i a_i = 0$. Thus

$$\mathrm{Ker}(\ell) = V\left(\sum_{i=1}^n a_i X_i\right) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i x_i = 0\right\}.$$

□

Proposition 3.5. *Let V be an n-dimensional vector space and let $\ell_1, \ldots, \ell_k \in V^\vee$. Then $\{\ell_1, \ldots, \ell_k\}$ is linearly independent if and only if*

$$\dim\left(\bigcap_{1 \le i \le k} \mathrm{Ker}(\ell_i)\right) = n - k.$$

Proof. Suppose $\{\ell_1, \ldots, \ell_k\}$ is linearly independent. We may assume that we are working in $(\mathbb{R}^n)^\vee$ and that $\ell_i = \underline{e}_i$. Then

$$\begin{aligned} \dim\left(\bigcap_{1 \le i \le k} \mathrm{Ker}(\ell_i)\right) &= \dim\left(\bigcap_{1 \le i \le k} V(X_i)\right) \\ &= \dim\left(V(X_1, \ldots, X_k)\right) \\ &= n - k. \end{aligned}$$

The converse is trivial. □

3.6 Differential Forms on $\mathbb{R}^n$

The **cotangent space** to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$ is defined to be the dual space $(T_p(\mathbb{R}^n))^\vee$ of the tangent space $T_p(\mathbb{R}^n)$. In parallel with the definition of a vector field, a **covector field** or **differential** 1**-form** on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p \in U} T_p^*(\mathbb{R}^n), \qquad p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

We call a differential 1-form a 1**-form** for short.

3.7 Jacobian

Let $f = (f_1, \dots, f_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open subset U of $\mathbb{R}^n$. The **Jacobian** of f at a point $p \in U$ is the $m \times n$ matrix

$$J(f)(p) := \begin{pmatrix} (\partial_{x_1} f_1)(p) & \cdots & (\partial_{x_n} f_1)(p) \\ \vdots & \ddots & \vdots \\ (\partial_{x_1} f_m)(p) & \cdots & (\partial_{x_n} f_m)(p) \end{pmatrix}.$$

The Jacobian satisfies the following property: for all $p \in U$, we have

$$\frac{\| f(p+\varepsilon) - f(p) - J(f)_p(\varepsilon) \|}{\|\varepsilon\|} \to 0$$

as $\varepsilon \to 0$ in $\mathbb{R}^n$. One can view the Jacobian as a smooth linear map

$$J(f)(p) = (J(f)(p)_1, \dots, J(f)(p)_m) : \mathbb{R}^n \to \mathbb{R}^m,$$

where the ith component $J(f)(p)_i$ is given by

$$J(f)(p)_i(x_1, \dots, x_n) = \sum_{j=1}^n (\partial_{x_j} f_i)(p) x_j.$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$.

If $m = n$, then f is a function from $\mathbb{R}^n$ to itself and the Jacobian matrix is a square matrix. In particular, we can compute its determinant, known as the **Jacobian determinant**. The Jacobian determinant at a given point gives important information about the behavior of f near that point. For instance, the inverse function theorem tells us that f is invertible near a point $p \in \mathbb{R}^n$ if and only if the Jacobian determinant is non-zero. Furthermore, if the Jacobian determinant at p is positive, then f preserves orientation near p.

Example 3.7.

1. Consider $f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ and where $U = \{(x,y) \in \mathbb{R}^2 \mid xy \in (-\pi/2, \pi/2) \text{ and } x+y \in (0,\infty)\}$ and where $f_1(x,y) = \tan(xy)$ and $f_2(x,y) = \ln(x+y)$ for all $(x,y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in U$ is given by

$$J(f)(x_0, y_0) = \begin{pmatrix} y_0 \sec^2(x_0 y_0) & x_0 \sec^2(x_0 y_0) \\ \frac{1}{x_0+y_0} & \frac{1}{x_0+y_0} \end{pmatrix}.$$

   The Jacobian determinant is then

$$\det\left(J(f)(x_0, y_0)\right) = \frac{(y_0 - x_0)\sec^2(x_0 y_0)}{x_0 + y_0}.$$

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x,y) = x^2 + xy + y$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by

$$J(f)(x_0, y_0) = \begin{pmatrix} 2x_0 + y_0 \\ x_0 + 1 \end{pmatrix}.$$

   Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that

$$\begin{aligned} f(x_0 + \varepsilon_1, y_0 + \varepsilon_2) &= (x_0 + \varepsilon_1)^2 + (x_0 + \varepsilon_1)(y_0 + \varepsilon_2) + (y_0 + \varepsilon_2) \\ &= x_0^2 + x_0 y_0 + y_0 + (2x_0 + y_0)\varepsilon_1 + (x_0 + 1)\varepsilon_2 + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \\ &= f(x_0, y_0) + J(f)(x_0, y_0)(\varepsilon_1, \varepsilon_2) + \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 \end{aligned}$$

3. Consider $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ where $f_1(x,y) = x^2y$ and $f_2(x,y) = y^2 + x$ for all $(x,y) \in \mathbb{R}^2$. The Jacobian of f at a point $(x_0, y_0) \in \mathbb{R}^2$ is given by

$$J(f)(x_0,y_0) = \begin{pmatrix} 2x_0y_0 & x_0^2 \\ 1 & 2y_0 \end{pmatrix}.$$

Let $\varepsilon_1, \varepsilon_2 > 0$. Then observe that

$$\begin{aligned} f(x_0+\varepsilon_1, y_0+\varepsilon_2) &= ((x_0+\varepsilon_1)^2(y_0+\varepsilon_2), (y_0+\varepsilon_2^2) + (x_0+\varepsilon_1)) \\ &= (x_0^2y_0, y_0^2+x_0) + (2x_0y_0\varepsilon_1 + x_0^2\varepsilon_2, \varepsilon_1 + 2y_0\varepsilon_2) + (y_0\varepsilon_1^2 + \varepsilon_1^2\varepsilon_2, \varepsilon_2^2) \\ &= f(x_0,y_0) + J(f)(x_0,y_0)(\varepsilon_1,\varepsilon_2) + (y_0\varepsilon_1^2 + \varepsilon_1^2\varepsilon_2, \varepsilon_2^2) \end{aligned}$$

Proposition 3.6. *Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map from an open subset U of $\mathbb{R}^m$ and let p be a point in $\mathbb{R}^m$. Then*

$$f(p+\varepsilon) = f(p) + J(f)_p(\varepsilon) + \psi(\varepsilon),$$

where ψ is a smooth map such that $\|\psi(\varepsilon)\|/\|\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Proof. Define $\psi : U \to \mathbb{R}^n$ by

$$\psi(\varepsilon) := f(p+\varepsilon) - f(p) - J(f)_p(\varepsilon).$$

□

4 Manifolds

We first recall a few definitions from point-set topology. A topological space is **second countable** if it has a countable basis. A **neighborhood** of a point p in a topological space M is any open set containing p. A topological space M is **Hausdorff** if for every pair of points $x, y \in M$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. An **open cover** of M is a collection $\{U_i\}_{i \in I}$ of open sets in M whose union $\bigcup_{i \in I} U_i$ is M.

The Hausdorff condition and second countability are "hereditary properties"; they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff.

Proposition 4.1. *Let M' be a subspace of a topological space M.*

1. *If M is Hausdorff, then M' is Hausdorff.*
2. *If M is second countable, then M' is second countable.*

Proof. (1) : Suppose $x, y \in M'$. Since $x, y \in M$ and M is Hausdorff, choose a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$. Then $U' = U \cap M'$ is a neighboorhood of x in the subspace topology and $V' = V \cap M'$ is a neighborhood of y in the subspace topology and $U' \cap V' = \emptyset$. (2) : If $\{B_i\}_{i \in \mathbb{N}}$ is a countable basis for M, then $\{B_i'\}_{i \in \mathbb{N}}$ is a countable basis for M', where $B_i' = B_i \cap M'$. □

Definition 4.1. A topological space M is **locally Euclidean of dimension** n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a **chart**, U a **coordinate neighborhood** or a **coordinate open set**, and ϕ a **coordinate map** or a **coordinate system on** U. We say that a chart (U, ϕ) is **centered** at $p \in U$ if $\phi(p) = 0$.

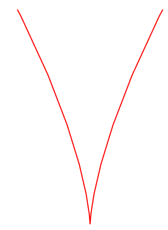
Proposition 4.2. *Let (U, ϕ) be a chart on the topological space M. If V is an open subset U, then $(V, \phi|_V)$ is a chart on M.*

Proof. This follows from the fact that if $\phi : U \to \phi(U)$ is a homeomorphism, then $\phi|_V : V \to \phi(V)$ is a homeomorphism. □

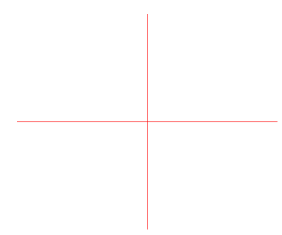
Definition 4.2. A **topological manifold** is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Example 4.1. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$, where $1_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, 1_U)$.

Example 4.2. (A cusp). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold. By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean because it is homeomorphic to $\mathbb{R}$ via the projection $(x, x^{2/3}) \mapsto x$.



Example 4.3. (A cross). The cross can be described as $\{(r,0) \mid r \in \mathbb{R}\} \cup \{(0,r) \mid r \in \mathbb{R}\}$. We show that the cross in $\mathbb{R}^2$ with the subspace topology is not locally Euclidean at the intersection $p = (0,0)$, and so cannot be a manifold. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B_\varepsilon(0) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism $U \to B$ restricts to a homemorphism $U \setminus \{p\} \to B \setminus \{0\}$. Now $B \setminus \{0\}$ is either connected if $n \geq 2$ or has two connected components of $n = 1$. Since $U \setminus \{p\}$ has four connected components, there can be no homeomorphism from $U \setminus \{p\}$ to $B \setminus \{p\}$. This contradiction proves that the cross is not locally Euclidean at p.



4.1 Compatible Charts

Suppose $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 4.3. Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞-**compatible** if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \qquad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. These two maps are called the **transition functions** between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞ compatible. To simplify this notation, we will sometimes write U_{ij} for $U_i \cap U_j$ and U_{ijk} for $U_i \cap U_j \cap U_k$. We will also sometimes write $\phi_{ij} = \phi_i \circ \phi_j^{-1}$. Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of compatible charts.

C^∞ compatibility is clearly reflexive and symmetric, but not necessarily transitive. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

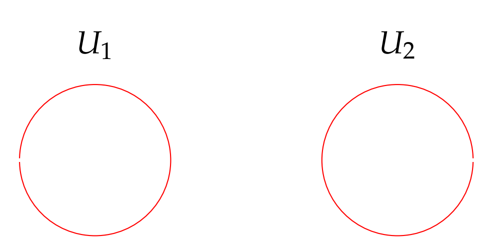
$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2)^{-1} \circ (\phi_2 \circ \phi_1^{-1})$$

is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$. A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} \setminus U_{123})$.

Definition 4.4. A C^∞ **atlas** or simply an **atlas** on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_i, \phi_i)\}_{i \in I}$ of pairwise C^∞-compatible charts that cover M, i.e. such that $M = \bigcup_{i \in I} U_i$.

Example 4.4. (A C^∞ atlas on a circle). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{2\pi i t} \in \mathbb{C} \mid 0 \leq t \leq 1\}$. Let U_1 and U_2 be the two open subsets of S^1

$$U_1 = \{e^{2\pi i t} \in \mathbb{C} \mid -\frac{1}{2} < t < \frac{1}{2}\} \qquad U_2 = \{e^{2\pi i t} \in \mathbb{C} \mid 0 < t < 1\}$$



and define $\phi_i : U_i \to \mathbb{R}$ for $i = 1, 2$ by

$$\phi_1(e^{2\pi i t}) = t \qquad \phi_2(e^{2\pi i t}) = t$$

Both ϕ_1 and ϕ_2 are branches of the complex log funciton $(1/i)\log z$ and are homeomorphisms onto their respective images. Thus (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection U_{12} consists of two connected components, the lower half A and the upper half B:

$$A = \{e^{2\pi i t} \mid -\frac{1}{2} < t < 0\} \qquad B = \{e^{2\pi i t} \mid 0 < t < \frac{1}{2}\}$$

with

$$\phi_1(U_{12}) = \phi_1(A \cup B) = \phi_1(A) \cup \phi_1(B) = \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$
$$\phi_2(U_{12}) = \phi_2(A \cup B) = \phi_2(A) \cup \phi_2(B) = \left(\frac{1}{2}, 1\right) \cup \left(0, \frac{1}{2}\right)$$

The transisition function $\phi_2 \circ \phi_1^{-1} : \phi_1(U_{12}) \to \phi_2(U_{12})$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 1 & \text{for } t \in \left(-\frac{1}{2}, 0\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Similarly,

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t - 1 & \text{for } t \in \left(\frac{1}{2}, 1\right) \\ t & \text{for } t \in \left(0, \frac{1}{2}\right) \end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1.

We say that a chart (V, ψ) is **compatible with an atlas** $\{(U_i, \phi_i)\}_{i \in I}$ if it is compatible with all the charts (U_i, ϕ_i) of the atlas.

Lemma 4.1. *Let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_i, \phi_i)\}_{i \in I}$, then they are compatible with each other.*

Proof. We want to show $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. For all $i \in I$, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is C^∞ on $\psi(V \cap W \cap U_i)$. Therefore $\sigma \circ \psi^{-1}$ is C^∞ on $\bigcup_{i \in I} \psi(V \cap W \cap U_i) = \psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. □

Remark 12. The domain of $\sigma \circ \psi^{-1}$ is $\psi(V \cap W)$ and the domain of $(\sigma \circ \phi_i^{-1}) \circ (\phi_i \circ \psi^{-1})$ is $\psi(U \cap V \cap W)$. What the equality means in the proof above is that the two maps are equal on their common domain.

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be **maximal** if it is not contained in a larger atlas; in other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 4.5. A **smooth** or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a **differentiable structure** on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a **curve**. A 2-dimensional manifold is a **surface**, and an n-dimensional manifold an n-manifold.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, because of the following proposition.

Proposition 4.3. *Any atlas $\mathfrak{U} = \{(U_i, \phi_i)\}_{i \in I}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Lemma (4.1), the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves existence. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{U}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{U}$ and so by construction must belong to $\mathfrak{M}$. This proves $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. This proves uniqueness. □

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that

1. M is Hausdorff and second countable
2. M has a C^∞ atlas.

From now on, a "manifold" will mean a C^∞ manifold. We use the terms "smooth" and "C^∞" interchangeably. In the context of manifolds, we denote the standard coordinates of $\mathbb{R}^n$ by $r^1, \dots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \dots, x^n)$ and $(U, \phi) = (U, x^1, \dots, x^n)$. Thus for $p \in U$, $(x^1(p), \dots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \dots, x^n$ are called **coordinates** or **local coordinates** on U. By abuse of notation, we sometimes omit the p. So the notations $(x^1, \dots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$.

Remark 13. A topological manifold can be endowed with different (non-compatible) differentiable structures. For instance, consider $X = \mathbb{R}$. We can give the space the structure of a C^∞-manifold using the chart $(\mathbb{R}, \varphi_1)$, where φ_1 maps $x \to x$. We can also give the space the structure of a C^∞ manifold using the chart $(\mathbb{R}, \varphi_2)$, where φ_2 maps $x \mapsto x^3$. These two charts are not C^∞-compatible since $\varphi_1 \circ \varphi_2^{-1}$ maps $x \mapsto x^{\frac{1}{3}}$, and this is *not* C^∞ on $\mathbb{R}$: $\frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ is not continuous at $x = 0$.

4.1.1 An Atlas For a Product

Proposition 4.4. *If $\mathfrak{U} = \{(U_i, \phi_i) \mid i \in I\}$ and $\mathfrak{V} = \{(V_j, \psi_j) \mid j \in J\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection*

$$\mathfrak{U} \times \mathfrak{V} = \{(U_i \times V_j, \phi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n) \mid (i, j) \in I \times J\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m + n$.

Proof. Clearly the set $\left\{U_i \times V_j \mid (i, j) \in I \times J\right\}$ covers $M \times N$, so we just need to show that any two charts in $\mathfrak{U} \times \mathfrak{V}$ are pairwise compatible. Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two charts in $\mathfrak{U} \times \mathfrak{V}$. Then $(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1}$ is C^∞, since

$$(\phi_1 \times \psi_1) \circ (\phi_2 \times \psi_2)^{-1} = \left(\phi_1 \circ \phi_2^{-1}\right) \times \left(\psi_2 \times \psi_2^{-1}\right),$$

and both $\phi_1 \circ \phi_2^{-1}$ and $\psi_2 \times \psi_2^{-1}$ are C^∞ on their respective domains. The same proof shows that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1}$ is C^∞. Thus $\mathfrak{U} \times \mathfrak{V}$ is a collection of pairwise C^∞ compatible charts that cover $M \times N$. □

Example 4.5. It follows from Proposition (4.4) that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds.

4.2 Examples of Smooth Manifolds

4.2.1 Euclidean Space

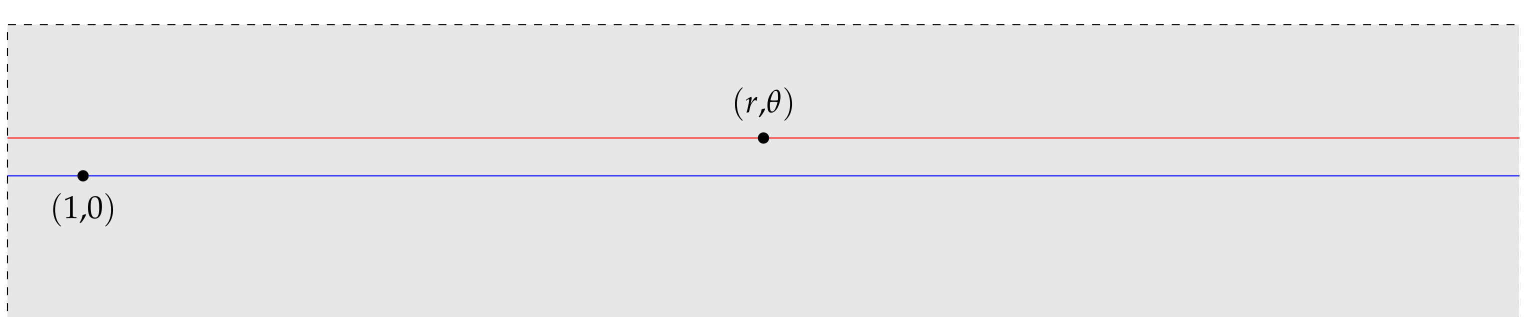
Example 4.6. (Euclidean space). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, \mathrm{id})$. We use $x_1, \dots, x_n$ to denote coordinates functions and $a_1, \dots, a_n$ to denote real numbers. Thus, if $p = (a_1, \dots, a_n)$ is a point in $\mathbb{R}^n$, we have $x_1(p) = a_1$, $x_2(p) = a_2$, and etc...

Example 4.7. The real half line $\mathbb{R}_{>0} : \{a \in \mathbb{R} \mid a > 0\}$ is also a smooth manifold, with a single chart $(\mathbb{R}_{>0}, \mathrm{id})$. In fact, $\mathbb{R}_{>0}$ is homeomorphic to $\mathbb{R}$. A homeomorphism from $\mathbb{R}_{>0}$ to $\mathbb{R}$ is given by $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

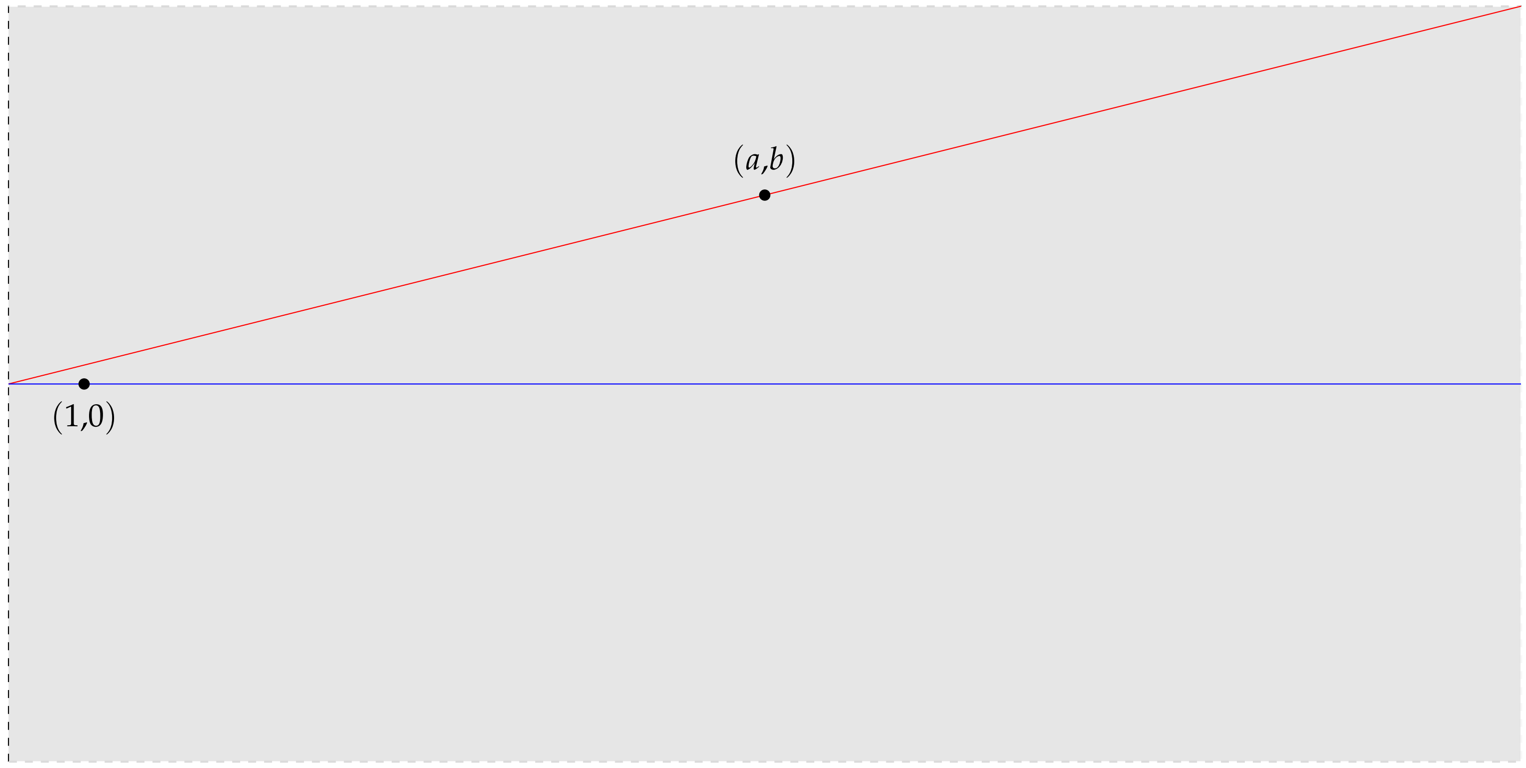
Now consider the half-open interval $(0, 2\pi)$. Open sets of the form (a, b) where and $0 \leq a < b < 2\pi$ form a basis for this topological space.

4.2.2 Right-Half Infinite Strip and the Right-Half Plane

Let $M = \mathbb{R}_{>0} \times \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. We illustrate this space below:



Now let $N = \mathbb{R}_{>0} \times \mathbb{R}$ be the right-half plane. We illustrate this space below:



We can give both M and N the structure of a smooth manifold by simply using the identity charts.
Let $\varphi : M \to N$ be given by $\varphi(r, \theta) = (\varphi_1(r, \theta), \varphi_2(r, \theta))$, where

$$\varphi_1(r, \theta) = r \sin \theta$$
$$\varphi_2(r, \theta) = r \cos \theta$$

Then φ is a diffeomorphism from M to N. The Jacobian of φ at a point $(r, \theta) \in M$:

$$J_{(r,\theta)}(\varphi) = \begin{pmatrix} \sin \theta & r \cos \theta \\ \cos \theta & -r \sin \theta \end{pmatrix}$$

The inverse to $\varphi : M \to N$ is $\psi : N \to M$, given by $\psi(a, b) = (\psi_1(a, b), \psi_2(a, b))$ where

$$\psi_1(a, b) = \sqrt{a^2 + b^2}$$
$$\psi_2(a, b) = \arctan\left(\frac{a}{b}\right)$$

The Jacobian of ψ at a point $(a, b) \in N$:

$$J_{(a,b)}(\psi) = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

4.2.3 Manifolds of Dimension Zero

Example 4.8. (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

4.2.4 Graph of a Smooth Function

Example 4.9. (Graph of a smooth function). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^n$, the **graph** of f is defined to be the subset

$$\Gamma(f) = \{(p, f(p)) \in A \times \mathbb{R}^n\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^∞, then the two maps

$$\phi : \Gamma(f) \to U \qquad (p, f(p)) \mapsto p$$

and

$$(1, f) : U \to \Gamma(f) \qquad p \mapsto (p, f(p))$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^n$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

4.2.5 Circle S^1

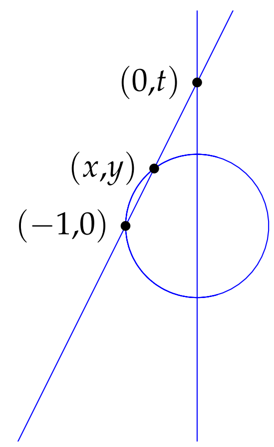
Example 4.10. (Circle) Let S^1 be the unit circle centered at the origin in $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

We shall describe an atlas on S^1 using stereographic projection. Let $U_1 = S^1 \backslash \{(-1, 0)\}$. Consider the line L which passes through the points $(-1, 0)$ and $(0, t)$ where $t \in \mathbb{R}$. The equation of this line is given by

$$Y = t(X + 1).$$

Since L passes through $(-1, 0)$ and is not tangent to $(-1, 0)$, it must pass through a unique point (x, y) in S^1. This is illustrated in the image below:



Since (x, y) lies on the line L and the unit circle, we get the relations

$$x^2 + y^2 - 1 = 0,$$
$$y - t(x + 1) = 0.$$

Using the second relation, we have $y = t(x + 1)$. Plugging in $t(x + 1)$ for y in the first relation, we get

$$t^2 = \frac{(1-x)^2}{(1+x)^2} = \frac{1-x}{1+x}.$$

Now we solve for x in terms of t, to get:

$$x = \frac{1-t^2}{1+t^2},$$
$$y = \frac{2t}{1+t^2}.$$

Now, let $\phi_1 : U_1 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1+x}.$$

This map is clearly C^∞ in its domain U_1, since $x \neq -1$, and the inverse $\phi_1^{-1} : \mathbb{R} \to U_1$ is given by

$$t \mapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

Next, let $U_2 = S^1 \backslash \{(1, 0)\}$. Following the same line of reasoning as the paragraph above, let $\phi_2 : U_2 \to \mathbb{R}$ be given by

$$(x, y) \mapsto \frac{y}{1-x}.$$

Again, this map is clearly C^∞ in its domain U_2, since $x \neq 1$, and the inverse $\phi_2^{-1} : \mathbb{R} \to U_2$ is given by

$$t \mapsto \left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2}\right).$$

Let us calculate the transition map $\phi_{12} := \phi_1 \circ \phi_2^{-1}$:

$$\begin{aligned}\phi_{12}(t) &= (\phi_1 \circ \phi_2^{-1})(t) \\ &= \phi_1\left(\frac{t^2-1}{1+t^2}, \frac{2t}{1+t^2}\right) \\ &= \frac{1}{t}.\end{aligned}$$

Remark 14. We think of t as a local coordinate of S^1 and x, y as global coordinates of S^1.

4.2.6 Projective Line

Example 4.11. Let $\mathbb{P}^1(\mathbb{R})$ be the projective line over $\mathbb{R}$. Define in $\mathbb{P}^1(\mathbb{R})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0} \in \mathbb{R},$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1} \in \mathbb{R}.$$

These maps are clearly C^∞ in their domains. The inverse maps are given by

$$\phi_0^{-1}(t) = (1 : t) \in U_0 \qquad \phi_1^{-1}(t) = (t : 1) \in U_1.$$

Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1}$:

$$\begin{aligned} \phi_0 \circ \phi_1^{-1}(t) &= \phi_0 \circ \phi_1^{-1}(t) \\ &= \phi_0 (t : 1) \\ &= \frac{1}{t}. \end{aligned}$$

Recall that this is the same transition map we calculated in Example (4.10).

4.2.7 Sphere S^2

Example 4.12. (Sphere) Let S^2 be the unit sphere

$$S^2 = \{(a,b,c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres - the front, rear, right , left, upper, and lower hemispheres

$$\begin{aligned} U_1 &= \{(a,b,c) \in S^2 \mid a > 0\} & \phi_1(a,b,c) &= (b,c) \\ U_2 &= \{(a,b,c) \in S^2 \mid a < 0\} & \phi_2(a,b,c) &= (b,c) \\ U_3 &= \{(a,b,c) \in S^2 \mid b > 0\} & \phi_3(a,b,c) &= (a,c) \\ U_4 &= \{(a,b,c) \in S^2 \mid b < 0\} & \phi_4(a,b,c) &= (a,c) \\ U_5 &= \{(a,b,c) \in S^2 \mid c > 0\} & \phi_5(a,b,c) &= (a,b) \\ U_6 &= \{(a,b,c) \in S^2 \mid c < 0\} & \phi_6(a,b,c) &= (a,b) \end{aligned}$$

The open set U_{14} is $\{(a,b,c) \in S^2 \mid b < 0 < a\}$ and $\phi_4(U_{14}) = \{(a,c) \in \mathbb{R}^2 \mid a^2 + c^2 < 1 \text{ and } a > 0\}$.

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned} \phi_{14}(a,c) &= \phi_1 \circ \phi_4^{-1}(a,c) \\ &= \phi_1 \left(a, \sqrt{1 - c^2 - a^2}, c\right) \\ &= \left(\sqrt{1 - c^2 - a^2}, c\right). \end{aligned}$$

It is easy to see that this is indeed a smooth map in its domain (since $1 - c^2 - a^2 \neq 0$). The Jacobian of ϕ_{14} at the point (a,c) is

$$J_{(a,c)}(\phi_{14}) = \begin{pmatrix} \frac{a}{\sqrt{1-c^2-a^2}} & \frac{c}{\sqrt{1-c^2-a^2}} \\ 0 & 1 \end{pmatrix}$$

Now let's compute the transition map ϕ_{45}:

$$\begin{aligned} \phi_{45}(a,b) &= \phi_4 \circ \phi_5^{-1}(a,b) \\ &= \phi_4 \left(a, b, \sqrt{1 - a^2 - b^2}\right) \\ &= \left(a, \sqrt{1 - a^2 - b^2}\right). \end{aligned}$$

4.2.8 The Sphere S^n

Example 4.13. Using stereographic projections (from the north pole and the south pole), we can define two charts on S^n and show that S^n is a smooth manifold. Let $p_N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ be the north pole and $p_S = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ be the south pole. Define the maps $\phi_N : S^n \backslash \{p_N\} \to \mathbb{R}^n$ and $\phi_S : S^n \backslash \{p_S\}$, called **stereographic projection** from the north pole (resp. south pole), by

$$\phi_N(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n) \quad \text{and} \quad \phi_S(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \ldots, x_n).$$

The inverse stereographic projections are given by

$$\phi_N^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, -1 + \sum_{i=1}^n x_i^2\right)$$

and

$$\phi_S^{-1}(x_1, \ldots, x_n) = \frac{1}{1 + \sum_{i=1}^n x_i^2}\left(2x_1, \ldots, 2x_n, 1 - \sum_{i=1}^n x_i^2\right).$$

Thus, if we let $U_N = S^n \backslash \{p_N\}$ and $U_S = S^n \backslash \{p_S\}$, we see that U_N and U_S are two open subsets convering S^n, both homeomorphic to $\mathbb{R}^n$. Furthermore, it is easily checked that on the overlap, $U_N \cap U_S$, the transition maps

$$\phi_S \circ \phi_N^{-1} = \phi_N \circ \phi_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n),$$

that is, the inversion of center $p_O = (0, \ldots, 0)$ and power 1. Clearly, this map is smooth on $\mathbb{R}^n \backslash \{O\}$, so we conclude that (U_N, ϕ_N) and (U_S, ϕ_S) form a smooth atlas for S^n.

4.2.9 Real Projective Plane

Example 4.14. (Projective Plane) Let $\mathbb{P}^2(\mathbb{R})$ be the projective plane over $\mathbb{R}$. Define in $\mathbb{P}^2(\mathbb{R})$ the three charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1 : x_2) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) =: (a, b)$$

$$U_1 = D(X_1) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) =: (c, d)$$

$$U_2 = D(X_2) = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_2 \neq 0\} \qquad \phi_2(x_0 : x_1 : x_2) = \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right) =: (e, f)$$

The reason the map ϕ_1 is a homeomorphism is because given that $x_1 \neq 0$, we use the equivalence relation to write the point $p = (x_0 : x_1 : x_2)$ as $p = \left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right)$. Now $\frac{x_0}{x_1}$ and $\frac{x_2}{x_1}$ are two real rumbers which uniquely determine the point (a, b). We think of a and b as the local coordinates in the (U_0, ϕ_0) chart.

Let U_{01} be the intersection of U_0 and U_1, that is, $U_{01} := \{(x_0 : x_1 : x_2) \in \mathbb{P}^2(\mathbb{R}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$. Then $\phi_0(U_{01}) = \left\{\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$ and $\phi_1(U_{01}) = \left\{\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \in \mathbb{R}^2 \mid x_0 \neq 0 \text{ and } x_1 \neq 0\right\}$. We can also write this in terms of local coordinates as $\phi_0(U_{01}) = \{(a, b) \in \mathbb{R}^2 \mid a \neq 0\}$ and $\phi_1(U_{01}) = \{(c, d) \in \mathbb{R}^2 \mid c \neq 0\}$. Now let's calculate the transition map $\phi_{01} := \phi_0 \circ \phi_1^{-1} : \phi_1(U_{01}) \to \phi_0(U_{01})$ using the local coordinates. We have

$$\begin{aligned}
\phi_{01}\,(c, d) &= \phi_0 \circ \phi_1^{-1}\,(c, d) \\
&= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right) \\
&= \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) \\
&= \left(\frac{1}{c}, \frac{d}{c}\right).
\end{aligned}$$

It's easy to see that ϕ_{01} is C^∞. Indeed, writing ϕ_{01}^1 and ϕ_{01}^2 for the components of ϕ_{01} (so $\phi_{01}^1(c,d) = \frac{1}{c}$ and $\phi_{01}^2(c,d) = \frac{d}{c}$), the partial derivatives $\partial_c^m \partial_d^n \phi_{01}^i$ exist and are continuous everywhere in $\phi_1(U_{01})$ for all $m,n \in \mathbb{N}$ and $i = 1,2$. This is because ϕ_{01}^1 and ϕ_{01}^2 are rational functions (i.e. ratio of two polynomials) and are they are defined everywhere since $c \neq 0$ in$\phi_1(U_{01})$.

Similarly, one can easily show that

$$\begin{aligned}
\phi_{10}(a,b) &= \left(\frac{1}{a}, \frac{b}{a}\right) \\
\phi_{20}(a,b) &= \left(\frac{1}{b}, \frac{a}{b}\right) \\
\phi_{02}(e,f) &= \left(\frac{f}{e}, \frac{1}{e}\right) \\
\phi_{12}(e,f) &= \left(\frac{e}{f}, \frac{1}{f}\right) \\
\phi_{21}(c,d) &= \left(\frac{c}{d}, \frac{1}{d}\right)
\end{aligned}$$

It is instructive to check that $\phi_{ij} \circ \phi_{ji} = 1$ and $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$.

4.2.10 Riemann Sphere

Example 4.15. (Riemann sphere) In this example we describe a **complex manifold**. A complex manifold is the complex analogue of a manifold, however in the complex manifold case, we require the transition maps to be holomorphic, and not just C^∞. Let $\mathbb{P}^1(\mathbb{C})$ be the projective line over $\mathbb{C}$ (also known as the Riemann sphere). Define in $\mathbb{P}^1(\mathbb{C})$ the two charts given by

$$U_0 = D(X_0) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0\} \qquad \phi_0(x_0 : x_1) = \frac{x_1}{x_0}$$

$$U_1 = D(X_1) = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_1 \neq 0\} \qquad \phi_1(x_0 : x_1) = \frac{x_0}{x_1}$$

This time, let $z = \frac{x_0}{x_1}$. The open set U_{01} is $\{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{C}) \mid x_0 \neq 0 \text{ and } x_1 \neq 0\}$ and $\phi_1(U_{01}) = \mathbb{C}^\times$. Now

$$\begin{aligned}
\phi_0 \circ \phi_1^{-1}(z) &= \phi_0 \circ \phi_1^{-1}\left(\frac{x_0}{x_1}\right) \\
&= \phi_0\left(\frac{x_0}{x_1} : 1\right) \\
&= \phi_0\left(1 : \frac{x_1}{x_0}\right) \\
&= \frac{x_1}{x_0} \\
&= \frac{1}{z}.
\end{aligned}$$

One can show that the map $z \mapsto \frac{1}{z}$ is holomorphic in the domain $\mathbb{C}^\times$.

4.2.11 Mobius Strip

Example 4.16. Let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold. First we consider the set of all nonvertical lines in $\mathbb{R}^2$, which we denote by U_v. A nonvertial is of the form $\ell_{a,b}^v = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell_{a,b}^v \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism. Next we consider the set of all nonhorizontal lines in $\mathbb{R}^2$, which we denote by U_h. A nonhorizontal is of the form $\ell_{c,d}^h = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell_{c,d}^h \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism. Now we have $U_v \cup U_h = \mathcal{L}$. To get a topology on $\mathcal{L}$, we glue the topologies from U_v

and U_h: a set $U \subset \mathcal{L}$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and φ_{hv}. We have

$$\begin{aligned}\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d)\\ &= \varphi_v\left(\ell^h_{c,d}\right)\\ &= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right)\\ &= \left(\frac{1}{c}, -\frac{d}{c}\right),\end{aligned}$$

and similarly,

$$\begin{aligned}\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b)\\ &= \varphi_h\left(\ell^v_{a,b}\right)\\ &= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right)\\ &= \left(\frac{1}{a}, -\frac{b}{a}\right).\end{aligned}$$

These maps are clearly C^∞. In fact, they look very similar to the transition maps for the projective plane, except they are twisted by a negative sign.

Remark 15. We can also describe $\mathcal{L}$ as $\mathbb{RP}^2 \backslash \{[0:0:1]\}$: Any line in the euclidean plane is of the form $ax + by + c = 0$, for some $a, b, c \in \mathbb{R}$. First note that these coefficients uniquely determine the line and they are homogeneous. Hence there is a well defined map $\phi : \mathcal{L} \to \mathbb{RP}^2$, given by mapping the line $\mathbf{V}(ax + by + c)$ to the point $[a : b : c]$. Now ϕ is injective, but not surjective. However if we remove the point $[0:0:1]$, then the induced map $\phi : \mathcal{L} \to \mathbb{RP}^2 \backslash \{[0:0:1]\}$ is a bijection.

4.2.12 Grassmannians

The **Grassmannian** $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $v_1, \dots, v_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the **rank** of a matrix A, denoted by $\mathrm{rk}A$, is defined to be the number of linearly independent columns of A. This matrix is called a **matrix representative** of the k-plane.

Two bases $a_1, \dots, a_k$ and $b_1, \dots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_{i=1}^{k} a_i g_{ij}$$

for all $1 \le k \le n$. In matrix notation, this says $B = Ag$. Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \text{ if and only if there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

A **real Grassmann manifold** $G(n,k)$ is defined as the space of all k-dimensional subspaces of the space $\mathbb{R}^n$. The topology in $G(n,k)$ may be described as induced by the embedding $G(n,k) \to \mathrm{End}(\mathbb{R}^n)$ which assigns to a $P \in G(n,k)$, the orthogonal projection $\mathbb{R}^n \to P$ combined with the inclusion map $P \to \mathbb{R}^n$.

In $G(4,2)$, we have

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} \sim \begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\end{pmatrix} = \begin{pmatrix} c_{11}a_{11}+c_{12}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24}\\ c_{21}a_{11}+c_{22}a_{21} & c_{11}a_{12}+c_{12}a_{22} & c_{11}a_{13}+c_{12}a_{23} & c_{11}a_{14}+c_{12}a_{24}\end{pmatrix}$$

where $c_{11}c_{22} - c_{21}c_{12} \neq 0$.

5 Smooth Maps on a Manifold

Now that we've defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined.

5.1 Smooth Functions

Definition 5.1. Let M be a smooth manifold of dimension n. A function $f\colon M \to \mathbb{R}$ is said to be C^∞ or **smooth at a point** p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$. The function f is said to be C^∞ on M if it is C^∞ at every point of M.

Observe that the definition of the smoothness of a function f at a point p is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) is any other chart about p in M, then on $\psi(U \cap V)$, we have

$$f \circ \psi^{-1}\,|_{\psi(U\cap V)} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$

which is C^∞ at $\psi(p)$. Thus $f \circ \psi^{-1}$ must be C^∞ at $\psi(p)$. Also observe that in the definition above, $f\colon M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are only interested in functions that are smooth on an open set, there is no loss of generality in assuming at the onset that f is continuous.

Proposition 5.1. *Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

1. *The function $f : M \to \mathbb{R}$ is C^∞.*
2. *The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
3. *For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

Proof. We will prove the proposition as a cyclic chain of implications.

($2 \implies 1$): This follows directly from the definition of a C^∞ function, since by (2) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

($1 \implies 3$): Let (V, ψ) be an arbitrary chart on M and let $p \in V$. By the remark above, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.

($3 \implies 2$): Obvious.

□

The smoothness conditions of Proposition (5.1) will be a recurrent motif: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on *every* chart.

Definition 5.2. Let $F : N \to M$ be a map and h a function on M. The **pullback** of h by F, denoted by F^*h, is the composite function $h \circ F$.

Remark 16. In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

Example 5.1. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counterclockwise by an angle θ and let x, y denote the standard coordinate functions on $\mathbb{R}^2$. Then

$$\begin{aligned} \phi^* x &= (\cos\theta)x - (\sin\theta)y \\ \phi^* y &= (\sin\theta)x + (\cos\theta)y. \end{aligned}$$

Indeed, let e_1, e_2 denote the standard coordinates on $\mathbb{R}^2$; so $x(e_1) =$

$$\begin{aligned} (\phi^* x)(a, b) &= x\left(\phi(a, b)\right) \\ &= x\left(\cos\theta a - \sin\theta b, \sin\theta a + \cos\theta b\right) \\ &= \cos\theta a - \sin\theta b \\ &= ((\cos\theta)x - (\sin\theta)y)(a, b). \end{aligned}$$

5.2 Smooth Maps Between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms "C^∞" and "smooth" interchangeably.

Definition 5.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ **at a point** p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V) \cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$. The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

Remark 17.

1. In the definition, we needed $F^{-1}(V)$ to be open so that $\phi(F^{-1}(V) \cap U)$ is open. Thus, C^∞ maps between manifolds are by definition continuous.

2. In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, 1_{\mathbb{R}^m})$ as a chart about $F(p)$ in $\mathbb{R}^m$. According to the definition above, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p in N such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

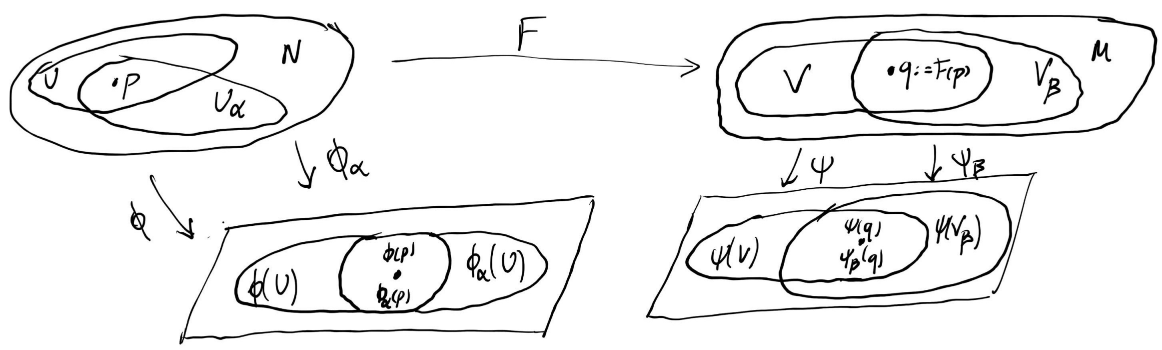
We show now that the definition of the smoothness of a map $F : N \to M$ at a point is independent of the choice of charts.

Proposition 5.2. *Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.*

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1} |_{\phi\left(F^{-1}(V \cap V_\beta) \cap U \cap U_\alpha\right)} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1}),$$

is C^∞ at $\phi(p)$. Therefore $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.



∎

Proposition 5.3. *(Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:*

1. *The map $F : N \to M$ is C^∞.*

2. *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map*

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞.

3. *For every chart (U, ϕ) on N and (V, ψ) on M, the map*

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞.

Proof.
($2 \implies 1$): Let $p \in N$. Suppose (U, ϕ) is a chart about p in $\mathfrak{U}$ and (V, ψ) is a chart about $F(p)$ in $\mathfrak{V}$. By (2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

($1 \implies 3$): Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \emptyset$. Let $p \in U \cap F^{-1}(V)$. Then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition (5.2), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitary point of $\phi(U \cap F^{-1}(V))$, the map $\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞.

(3 $\implies$ 2): Obvious. $\square$

Proposition 5.4. *(Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.*

Proof. Let (U, ϕ), (V, ψ), and (W, σ) be charts on N, M, and P respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}).$$

Since F and G are C^∞, the maps $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are also C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞, and thus $G \circ F$ is C^∞. $\square$

5.2.1 Diffeomorphisms

A **diffeomorpism** of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 5.5. *If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\left\{(\phi(U), \mathrm{id}_{\phi(U)})\right\}$ with a single chart on $\phi(U)$. Since

$$\mathrm{id}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$$

is the identity map, it is C^∞. By Proposition (5.3), ϕ is C^∞.

To test smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since

$$\phi \circ \phi^{-1} \circ \mathrm{id}_{\phi(U)} : \phi(U) \to \phi(U)$$

is the identity map, the map ϕ^{-1} is also C^∞. $\square$

Proposition 5.6. *Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition (5.5). As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U, F) is in the atlas. $\square$

5.2.2 Smoothness in Terms of Components

In this subsection, we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 5.7. *(Smoothness of a vector-valued function) Let N be a manifold and let $F : N \to \mathbb{R}^m$ be a continuous map. The following are equivalent:*

1. *The map $F : N \to \mathbb{R}^m$ is C^∞.*
2. *The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*
3. *For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.*

Proposition 5.8. *(Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F_1, \ldots, F_m : N \to \mathbb{R}$ are all C^∞.*

Proof. The $F : N \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ if and only if for every chart (U, ϕ) on N, the functions $F_i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ if and only if the functions $F_i : N \to \mathbb{R}$ are all C^∞. $\square$

5.3 Germs of C^∞ functions

Let M be an n-dimensional manifold and let p be a point in M. Consider the set of all pairs (f, U), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. Just as in the $\mathbb{R}^n$ case, we introduce an equivalence relation $\sim$ and say that $(f, U) \sim (g, V)$ if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. The equivalence class of (f, U) is called the **germ** of f at p. We write $C_p^\infty(M)$ for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Let (f, U) be represent a germ in $C_p^\infty(M)$ and suppose (U_0, ϕ) is a chart centered at p. Then $(U_0 \cap U, \phi_{|U})$ is a chart centered at p and clearly we have $(f, U) \sim (f \mid_{U_0 \cap U}, U_0 \cap U)$. Thus we may always assume that a germ can be represented by (f, U) where (U, ϕ) is a chart centered at p. In particular, we obtain an isomorphism

$$\widehat{\phi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \phi^{-1}, \phi(U))$. Of course this map depends on our choice of chart. If (V, φ) was another chart, then we'd obtain another isomorphism

$$\widehat{\varphi} : C_p^\infty(M) \to C_p^\infty(\mathbb{R}^n),$$

given by $(f, U) \mapsto (f \circ \varphi^{-1}, \varphi(U))$. We can relate these two isomorphisms via the transition function $\phi \circ \varphi^{-1}$.

Let M be a manifold and let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas on M. We describe the structure of a premanifold as follows: if U is an open subset of M, then we set

$$\mathcal{O}_M(U) := \{f : U \to \mathbb{R} \mid f|_{U \cap U_i} \circ \phi_i^{-1} : \phi_i(U \cap U_i) \to \mathbb{R} \text{ is } C^\infty\} = \{f : U \to \mathbb{R} \mid f \text{ is } C^\infty\}.$$

To see that this is a premanifold, fix $i_0 \in I$. For $U \subseteq U_{i_0}$ open let $f : U \to \mathbb{R}$ be a map such that $f \circ \phi_{i_0}^{-1} : \phi_{i_0}(U_{i_0} \cap U) \to \mathbb{R}$ is a C^∞ function. Then $f \in \mathcal{O}_M(U)$ because the change of charts between i and i_0 are C^∞-diffeomorphisms. Indeed, we have

$$f|_{U \cap U_i} \circ \phi_i^{-1} = (f \circ \phi_{i_0}^{-1}) \circ (\phi_{i_0} \circ \phi_i^{-1}).$$

Therefore ϕ_{i_0} yields an isomorphism $(U_{i_0}, \mathcal{O}_{M|U_{i_0}}) \cong (Y_{i_0}, \mathcal{O}_{i_0})$, where $\mathcal{O}_{Y_0}$ is the sheaf of C^∞ functions on Y_{i_0}. Hence, $(M, \mathcal{O}_M)$ is a ringed space that is locally isomorphic to a manifold. Hence it is a premanifold.

5.4 Examples of Smooth Maps

Example 5.2. We show that the map $F : \mathbb{R} \to S^1$ given by $F(t) = (\cos t, \sin t)$ is C^∞. For $\mathbb{R}$, we use the atlas which consists of a single chart $(\mathbb{R}, \mathrm{id})$. For S^1 we use the atlas which consists of the charts (U_1, ϕ_1), (U_2, ϕ_2), (U_3, ϕ_3) and (U_4, ϕ_4) where

$$\begin{aligned} U_1 &= \{(a, b) \in S^1 \mid a > 0\} & \phi_1(a, b) &= b \\ U_2 &= \{(a, b) \in S^2 \mid a < 0\} & \phi_2(a, b) &= b \\ U_3 &= \{(a, b) \in S^2 \mid b > 0\} & \phi_3(a, b) &= a \\ U_4 &= \{(a, b) \in S^2 \mid b < 0\} & \phi_4(a, b) &= a \end{aligned}$$

Let us do some computations. First, let's compute the transition map ϕ_{14}:

$$\begin{aligned} \phi_{14}(a) &= \phi_1 \circ \phi_4^{-1}(a) \\ &= \phi_1\left(a, \sqrt{1 - a^2}\right) \\ &= \sqrt{1 - a^2}. \end{aligned}$$

Similar computations shows that

$$\begin{aligned} \phi_{13}(a) &= \sqrt{1 - a^2} \\ \phi_{24}(a) &= \sqrt{1 - a^2} \\ \phi_{23}(a) &= \sqrt{1 - a^2}. \end{aligned}$$

Now, we need to show that $\phi_i \circ F \circ \mathrm{id}$ is C^∞ for $i = 1, 2, 3, 4$. Let's compute $\phi_1 \circ F \circ \mathrm{id}$:

$$\begin{aligned} (\phi_1 \circ F \circ \mathrm{id})(t) &= \phi_1(F(t)) \\ &= (\phi_1((\cos t, \sin t)) \\ &= \sin t. \end{aligned}$$

Similar computations shows that

$$\begin{aligned}(\phi_2 \circ F \circ \mathrm{id})(t) &= \sin t\\ (\phi_3 \circ F \circ \mathrm{id})(t) &= \cos t\\ (\phi_4 \circ F \circ \mathrm{id})(t) &= \cos t.\end{aligned}$$

These maps are all C^∞.

Example 5.3. Consider $N = \mathbb{R}$ and $M = \mathbb{R}^2$ and let $f : N \to M$ be given by $f(t) = (t^2, t^3)$.

Example 5.4. Let S^2 be the unit sphere with its smooth structure given in Example (4.12). Let $f : S^2 \to \mathbb{R}$ be given by

$$f(a, b, c) = c^2.$$

We claim that f is C^∞. To see this, we need to show that f is C^∞ at every point $p = (a, b, c)$ in S^2. First assume that $p \in U_6$. Using the chart (U_6, ϕ_6), we find that

$$\begin{aligned}(f \circ \phi_6^{-1})(a, b) &= f\left(\phi_6^{-1}(a, b)\right)\\ &= f\left(a, b, \sqrt{1 - a^2 - b^2}\right)\\ &= 1 - a^2 - b^2,\end{aligned}$$

which is clearly C^∞.

Example 5.5. Let us show that a C^∞ function $f(x, y)$ on $\mathbb{R}^2$ restricts to a C^∞-function on S^1. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a, b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a, b) = a$ and $y(a, b) = b$. Suppose that we can show that x and y restrict to C^∞-functions on S^1. Then the inclusion map $i : S^1 \to \mathbb{R}^2$, given by $i(p) = (x(p), y(p))$ is C^∞ on S^1, and so the composition $f\,|_{S^1} = f \circ i$ with be C^∞ on S^1 too.

Consider first the function x. We use the following atlas (U_i, ϕ_i) for S^1, where

$$\begin{aligned}U_1 &= \{(a, b) \in S^1 \mid b > 0\} & \phi_1(a, b) &= a\\ U_2 &= \{(a, b) \in S^1 \mid b < 0\} & \phi_2(a, b) &= a\\ U_3 &= \{(a, b) \in S^1 \mid a > 0\} & \phi_3(a, b) &= b\\ U_4 &= \{(a, b) \in S^2 \mid a < 0\} & \phi_4(a, b) &= b\end{aligned}$$

Since x is a coordinate function on U_1 and U_2, it is a coordinate function on $U_1 \cup U_2$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1} : \phi_3(U_3) \to \mathbb{R}$.

$$(x \circ \phi_3^{-1})(b) = x\left(\sqrt{1 - b^2}, b\right) = \sqrt{1 - b^2}.$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1 - b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3. On U_4, we have

$$(x \circ \phi_4^{-1})(b) = x\left(-\sqrt{1 - b^2}, b\right) = -\sqrt{1 - b^2}.$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1. The proof that y is C^∞ on S^1 is similar.

Example 5.6. Let S^2 be the unit sphere with its smooth structure given in Example (4.12). Let's construct a smooth function on S^2. First note that

$$\phi_1(U_{16}) = \{(b, c) \in \mathbb{R}^2 \mid b^2 + c^2 < 1 \text{ and } c < 0\} \qquad \text{and} \qquad \phi_6(U_{16}) = \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 < 1 \text{ and } 0 < a\}.$$

Let $f : \phi_1(U_{16}) \to \mathbb{R}^2$ be given by

$$f(b, c) = b^2 + c^2.$$

Let's pullback $f : \phi_1(U_{16}) \to \mathbb{R}^2$ to $\phi_{16}^*(f) : \phi_6(U_{16}) \to \mathbb{R}^2$ using the transition function ϕ_{16}, where

$$\begin{aligned}\phi_{16}(a, b) &= \phi_1 \circ \phi_6^{-1}(a, b)\\ &= \phi_1\left(a, b, \sqrt{1 - b^2 - a^2}\right)\\ &= \left(b, \sqrt{1 - b^2 - a^2}\right).\end{aligned}$$

We have,

$$\begin{aligned}\phi_{16}^*(f)(a, b) &= (f \circ \phi_{16})(a, b)\\ &= f\left(b, \sqrt{1 - b^2 - a^2}\right)\\ &= 1 - a^2.\end{aligned}$$

5.4.1 Diffeomorphism from $\mathbb{R}^n$ to the open unit ball $B_1(0)$

Let $\beta : \mathbb{R}^n \to B_1(0)$ be given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right) := \beta(x)$$

for all $x \in \mathbb{R}^n$. Then β is a diffeomorphism from $\mathbb{R}^n$ to $B_1(0)$ with inverse given by

$$x := (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \dots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) := \beta^{-1}(x)$$

for all $x \in B_1(0)$. Indeed, let us first check that $\beta(x) \in B_1(0)$:

$$\begin{aligned} \|\beta(x)\| &= \sqrt{\left(\frac{x_1}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2 + \cdots + \left(\frac{x_n}{\sqrt{1 + \sum_{i=1}^n x_i^2}} \right)^2} \\ &= \sqrt{\frac{\sum_{i=1}^n x_i^2}{1 + \sum_{i=1}^n x_i^2}} \\ &< \sqrt{\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2}} \\ &= 1. \end{aligned}$$

Thus $\beta(x) \in B_1(0)$. Next we check that β is smooth. This comes down to checking the component functions β_i are smooth:

$$x := (x_1, \dots, x_n) \mapsto \frac{x_i}{\sqrt{1 + \sum_{i=1}^n x_i^2}} := \beta_i(x).$$

This follows from the fact that $1 + \sum_{i=1}^n x_i^2 > 0$. That β^{-1} is smooth follows by the same reasoning. Finally, checking that $\beta(\beta^{-1}(x)) = x$ is tedious but trivial:

$$\begin{aligned} \beta(\beta^{-1}(x)) &= \frac{1}{\sqrt{1 + \sum_{i=1}^n \left(\frac{x_i}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right)^2}} \left(\frac{x_1}{\sqrt{1 - \sum_{i=1}^n x_i^2}}, \cdots, \frac{x_n}{\sqrt{1 - \sum_{i=1}^n x_i^2}} \right) \\ &= \frac{1}{\sqrt{1 - \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i^2}} (x_1, \dots, x_n) \\ &= (x_1, \dots, x_n). \end{aligned}$$

5.5 Inverse Function Theorem

We say that a C^∞ map $F : N \to M$ is **locally invertible** or a **local diffeomorphism** at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism. Given n smooth functions $F_1, \dots, F_n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F_1, \dots, F_n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 5.1. *(Inverse function theorem for $\mathbb{R}^n$). Ler $F : W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant $det(J(F)_p)$ is not zero.*

Theorem 5.2. *(Inverse function theorem for manifolds). Let $F : N \to M$ be a C^∞*

6 Tangent Spaces

By definition, the **tangent space** to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map, called its **differential**, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map. In this sense, the differential of a map between manifolds is a generalization of the derivative of a map between Euclidean spaces.

6.1 The Tangent Space at a Point

Just as for $\mathbb{R}^n$, we define a **germ** of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; which scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an $\mathbb{R}$-algebra.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a **derivation at a point** in a manifold M, or a **point-derivation** of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition 6.1. A **tangent vector** at a point p in a manifold M is a derivation at p.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by mapping (x, y) to $x^3 + y^3 + x + 1 := t$. Let $p = (x_0, y_0)$ be a point in $\mathbb{R}^2$. Then f induces a map $T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}$ by taking a derivation D in $T_p\mathbb{R}^2$ to the derivation f_*D in $T_{f(p)}\mathbb{R}$, where $(f_*D)(g) = D(g \circ f)$.

Example 6.1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^3 - x := t$. Then $\partial_x \mapsto (3x_0^2 - 1)\partial_t$.

Example 6.2. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \mapsto (f_1(x, y), f_2(x, y))$, where

$$f_1(x, y) = x$$
$$f_2(x, y) = \frac{xy^2}{y^2 + 1}$$

Then

$$J_{(x_0,y_0)}(f_1, f_2) = \begin{pmatrix} 1 & 0 \\ \frac{y_0^2}{y_0^2+1} & \frac{2x_0y_0}{(y_0^2+1)^2} \end{pmatrix}$$
$$\frac{2x_0y_0}{(y_0^2+1)^2} = 0$$

Let M be an n-dimensional manifold and let p be a point in M. We describe T_pM in another way. Let $\mathcal{P}_p$ be the set of paths through p:

$$\mathcal{P}_p := \{\gamma : (-a, a) \to M \mid \gamma \text{ is } C^\infty \text{ and } \gamma(0) = p\}.$$

We define an equivalence relation on $\mathcal{P}_p$ as follows: we say $\gamma_1 \sim \gamma_2$ if there exist a chart (U, ϕ) centered at p such that

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Here, $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are paths in $\mathbb{R}^n$.

Remark 18. This is independent of the choice of chart. If (V, ψ) is another chart centered at p, then

$$\begin{aligned}
(\psi \circ \gamma_1)' &= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_1)' \\
&= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_1)' \\
&= (\psi \circ \varphi^{-1})'(\varphi \circ \gamma_2)' \\
&= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\
&= (\psi \circ \varphi^{-1} \circ \varphi \circ \gamma_2)' \\
&= (\psi \circ \gamma_2)'
\end{aligned}$$

Here, $(\psi \circ \varphi^{-1})'$ is the Jacobian.

Definition 6.2. The tangent space at p in M is

$$T_pM := \mathcal{P}_p / \sim.$$

Example 6.3. Let $M = \{(\cos\alpha, \sin\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$ be the cylinder. Define the two charts (U, φ) and (M, ψ) where $U = M \cap \left\{(\frac{-\pi}{,2}, \frac{\pi}{2}) \times (\frac{-\pi}{,2}, \frac{\pi}{2}) \times \mathbb{R}\right\}$ and

$$\varphi(\cos\alpha, \sin\alpha, \beta) = (\alpha, \beta) \qquad \text{and} \qquad \psi(\cos\alpha, \sin\alpha, \beta) = (\cos\alpha, \beta).$$

Now let γ_1 and γ_2 be two paths in M given by

$$\gamma_1(t) = (\cos(t^2), \sin(t^2), t) \qquad \text{and} \qquad \gamma_2(t) = (0, 1, t).$$

Using the chart (U, φ), we have

$$(\varphi \circ \gamma_1)(t) = (t^2, t) \qquad \text{and} \qquad (\varphi \circ \gamma_2)(t) = \left(\frac{\pi}{2}, t\right).$$

Therefore

$$(\varphi \circ \gamma_1)'(0) = (0, 1) = (\varphi \circ \gamma_2)'(0)$$

and so $\gamma_1 \sim \gamma_2$.

6.2 Partial Derivatives

On a manifold M of dimension n, let (U, ϕ) be a chart and f a C^∞ function. As a function into $\mathbb{R}^n$, ϕ has n components $\phi_1, \ldots, \phi_n$. This means that if $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$, then $\phi_i = x_i \circ \phi$. For $p \in U$, we define the **partial derivative of f with respect to ϕ_i**, denoted $\partial_{\phi_i} f$, to be

$$\partial_{\phi_i} |_p \, f := (\partial_{\phi_i} f)(p) := \partial_{x_i}(f \circ \phi^{-1})(\phi(p)) := \partial_{x_i} |_{\phi(p)} \, (f \circ \phi^{-1}).$$

Example 6.4. Consider the projective plane $\mathbb{P}^2(\mathbb{R})$. Use the chart (U, ϕ) where

$$U = \{(a_0 : a_1 : a_2) \in \mathbb{P}^2(\mathbb{R}) \mid a_0 \neq 0\} \qquad \phi(a_0 : a_1 : a_2) = \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right).$$

Then

$$\begin{aligned} \phi_1(a_0 : a_1 : a_2) &= (x_1 \circ \phi)(a_0 : a_1 : a_2) \\ &= x_1(\phi(a_0 : a_1 : a_2)) \\ &= x_1\left(\frac{a_1}{a_0}, \frac{a_2}{a_0}\right) \\ &= \frac{a_1}{a_0}. \end{aligned}$$

Similarly, $\phi_2(a_0 : a_1 : a_2) = a_2/a_0$.

6.2.1 Polar Coordinates

Consider the following smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

Then

$$\begin{aligned} dr &= \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy \\ d\theta &= \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \end{aligned}$$

Then

$$\begin{aligned} r dr d\theta &= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy\right)\left(\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy\right) \\ &= (xdx + ydy)\left(\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy\right) \\ &= \frac{1}{x^2+y^2}(xdx + ydy)(-ydx + xdy) \\ &= \frac{1}{x^2+y^2}\left(-xydxdx + x^2 dxdy - y^2 dydx + xydydy\right) \\ &= \frac{1}{x^2+y^2}\left(x^2 dxdy - y^2 dydx\right) \\ &= \frac{1}{x^2+y^2}\left(x^2 dxdy + y^2 dxdy\right) \\ &= \frac{x^2+y^2}{x^2+y^2} dxdy \\ &= dxdy. \end{aligned}$$

Therefore, we can integrate the Gaussian as follows:

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dx = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta.$$

The inverse map is given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

6.3 Immersion, Embedding, Submersion

Let $F : N \to M$ be a C^∞ map and let p be a point in N. Then

1. F is called an **immersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is injective.
2. F is called an **immersion** if it is an immersion at every point in N.
3. F is called a **submersion** at p if the induced map $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective.
4. F is called an **submersion** if it is an submersion at every point in N.

Example 6.5. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$i(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 0, \ldots, 0).$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi(a_1, \ldots, a_m, a_{m+1}, \ldots, a_n) = (a_1, \ldots, a_m).$$

Example 6.6. If U is an open subset of a manifold M, then the inclusion $i : U \to M$ is both an immersion and submersion. This example shows in particular that a submersion need not be onto.

6.3.1 Critical Point

Definition 6.3. Let $F : N \to M$ be a C^∞ map, p a point in N, and q a point in M. Then

1. We say p is a **critical point** of F if $F_{*,p}$ is not surjective.
2. We say q is a **critical value** of F if the set $F^{-1}(q) := \{p \in N \mid F(p) = q\}$ contains a critical point.

Theorem 6.1. *Let M be a manifold and let f be a C^∞ function. Then the set of critical values of f has measure zero.*

Measure Theory on $\mathbb{R}$

Let μ be the Lebesgue measure on $\mathbb{R}$. Recall that

- If $I = (a, b)$, then $\mu(I) = b - a$.
- If I and J are disjoint intervals, then $\mu(I \cup J) = \mu(I) + \mu(J)$.
- A set $E \subset \mathbb{R}$ has measure 0 if for all $\varepsilon > 0$, you can cover E by a union of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\mu\left(\bigcup_n I_n\right) < \varepsilon$.

Example 6.7. Let $E = \{0, 1\}$. For each $n \in \mathbb{N}$, define the set

$$A_n := \left(\frac{-1}{4n}, \frac{1}{4n}\right) \cup \left(1 - \frac{1}{4n}, 1 + \frac{1}{4n}\right)$$

Then for each $n \in \mathbb{N}$, A_n is a disjoint union of intervals which covers E and

$$\begin{aligned} \mu(A_n) &= \frac{1}{4n} - \frac{-1}{4n} + 1 + \frac{1}{4n} - \left(1 - \frac{1}{4n}\right) \\ &= \frac{1}{n}. \end{aligned}$$

As $n \to \infty$, $\mu(A_n) \to 0$. Thus, E has measure 0.

Example 6.8. Let $M = \{(x, x + \sin(x)) \mid x \in \mathbb{R}\}$ and let $\pi : M \to \mathbb{R}$ be the projection onto the y-axis map, given by $(x, x + \sin(x)) \mapsto x + \sin(x)$.

Definition 6.4. A critical point is **degenerate** if the associated Hessian matrix is **singular** (i.e. has determinant equal to 0).

Example 6.9. Let $M = \{(\cos\theta, \theta, \sin\theta + 2) \mid \theta \in \mathbb{R}\}$ and $N = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Note that M is homeomorphic to $\mathbb{R}$ and N is homeomorphic to $\mathbb{R}^2$. Let $\varphi : M \to N$ be the projection map, given by $(\cos\theta, \theta, \sin\theta + 2) \mapsto (\cos\theta, \theta, 0)$. Let $\{(M, \psi_M)\}$ be an atlas on M and $\{(N, \psi_N)\}$ be an atlas on N where

$$\psi_M(\cos\theta, \theta, \sin\theta + 2) = \theta \qquad \text{and} \qquad \psi_N(x, y, 0) = (x, y)$$

What are the coordinates of $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}, 2 + \frac{\sqrt{2}}{2}\right) \in M$? Then answer is $\frac{\pi}{4}$.

Theorem 6.2. *Let f be a continuous function*

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k \right)$$

A critical point is **degenerate** if the associated Hessian matrix is **singular**

6.4 Tangent Bundle

Let M be an n-dimensional manifold. The **Tangent Bundle** of M is

$$TM := \bigcup_{p\in M} T_p M.$$

Let $\mathcal{A} := \{(U_i, \phi_i)\}$ be an atlas for M. Then an atlas for TM is given by

$$\mathcal{A}_T := \{(U_i \times \mathbb{R}^n, \phi_i \times \mathrm{id})\}$$

Thus, if we denote $\Phi_i := \phi_i \times \mathrm{id}$. Then $\Phi_i : U_i \times \mathbb{R}^n \to \mathbb{R}^{2n}$ and we think of $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ as the local coordinates of TM, where $(x_1, \ldots, x_n)$ is a point in M and $(y_1, \ldots, y_n)$ is a vector.

Example 6.10. The tangent bundle of the circle S^1 is diffeomorphic to the cylinder.

Remark 19. There exist a canonical map $\pi : TM \to M$ given by $(p, v) \mapsto v$.

Definition 6.5. A **vector field** is a smooth function ω from M to TM such that $\pi \circ \omega = \mathrm{id}$.

Remark 20. Intuitively, a vector field is the data of a vector at every point in M.

A vector field ω comes with two gadgets. The first gadget is called a 1-**parameter** flow and is denoted ω^t. The second gadget is called a **differential operator** and is denoted L_ω.

6.5 Vector Bundles

On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi : TM \to M$, given by $\pi(p, v) = p$, makes TM into a C^∞ **vector bundle** over M, which we now define.

Given any map $\pi : E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the **fiber** at p. The fiber at p is often written as E_p. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be **fiber-preserving** if $\phi(E_p) \subset E'_p$ for all $p \in M$. Equivalently, this says that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

A surjective smooth map $\pi : E \to M$ of manifolds is said to be **locally trivial of rank r** if

1. Each fiber E_p has the structure of a vector space of dimension r.

2. For each $p \in M$, there are an open neighborhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for every $q \in U$ the restriction

$$\phi\mid_{E_q}: E_q \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. Such an open set U is called a **trivializing open set** for E, and ϕ is called a **trivialization** of E over U.

The collection $\{(U, \phi)\}$, with $\{U\}$ and open cover of M, is called a **local trivialization** for E, and $\{U\}$ is called a **trivializing open cover** of M for E. A C^∞**vector bundle of rank** r is a triple (E, M, π) consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the **total space** of the vector bundle and M the **base space**. By abuse of language, we say that E is a **vector bundle over** M. Properly speaking, the tangent bundle of a manifold M is a triple (TM, M, π), and TM is the total space of the tangent bundle. In common usage, TM is often referred to as the tangent bundle.

6.5.1 Gluing

Given two local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ and $\phi_j : \pi^{-1}(U_j) \to U_j \times \mathbb{R}^k$, we obtain a smooth gluing map $\phi_j \circ \phi_i^{-1} : U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$. This map preserves images to M, and hence it sends (x, v) to $(x, g_{ji}(v))$, where g_{ji} is an invertible $k \times k$ matrix smoothly depending on x. That is, the gluing map is uniquely specified by a smooth map

$$g_{ji} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R}).$$

These are called **transition functions** of the bundle, and since they come from $\phi_j \circ \phi_i^{-1}$, they clearly satisfy $g_{ij} = g_{ji}^{-1}$, as well as the cocycle condition

$$g_{ij} g_{jk} g_{ki} = \mathrm{id}\mid_{U_i \cap U_j \cap U_k}$$

Example 6.11. To build a vector bundle, choose an open cover $\{U_i\}$ and form the pieces $\{U_i \times \mathbb{R}^k\}$. Then glue these together on the double overlaps $\{U_i \cap U_j\}$ via functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{R})$. As long as g_{ij} satisfy $g_{ij} = g_{ji}^{-1}$ as well as the cocycle condition, the resulting space has a vector bundle structure.

Example 6.12. Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection to the first factor. Then $M \times \mathbb{R}^r$ is a vector bundle of rank r, called the **product bundle** of rank r over M. The vector space structure on the fiber $\pi^{-1}(p) = \{(p, v) \mid v \in \mathbb{R}^r\}$ is the obvious one:

$$(p, u) + (p, v) = (p, u + v) \text{ and } b \cdot (p, v) = (p, bv) \text{ for } b \in \mathbb{R}.$$

A local trivialization on $M \times \mathbb{R}$ is given by the identity map $1_{M \times \mathbb{R}} : M \times \mathbb{R} \to M \times \mathbb{R}$. For example, the infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle.

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles, possibly of different ranks. A **bundle map** from E to F is a pair of maps $(f, \tilde{f})$, where $f : M \to N$ and $\tilde{f} : E \to F$ such that

1. The diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative.

2. $\tilde{f}$ is linear one each fiber; i.e. $\tilde{f} : E_p \to F_{f(p)}$ is a linear map of vector spaces for each $p \in M$.

The collection of all vector bundles together with bundle maps between them forms a category.

Example 6.13. A smooth map $f : N \to M$ of manifolds induces a bundle map $(f, \tilde{f})$, where $\tilde{f} : TN \to TM$ is given by

$$\tilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subset TM$$

for all $v \in T_pN$. This gives rise to a covariant functor T from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps: to each manifold M, we associate its tangent bundle TM, and to each C^∞ map $f : N \to M$ of manifolds, we associate the bundle map $T(f) = (f, \tilde{f})$.

If E and F are two vector bundles over the same manifold M, then a bundle map from E to F over M is a bundle map in which the base map is the identity 1_M. For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundles maps over M. In this category it makes sense to speak of an isomorphism of vector bundles over M. Any vector bundle over M is isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a **trivial bundle**.

Example 6.14. Let

$$M = \left\{ (x,y) \in \mathbb{R}^2 \mid \det \begin{pmatrix} x & 1-y \\ y & x \end{pmatrix} = 0 \right\} = \left\{ (x,y) \in \mathbb{R}^2 \mid x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4} \right\}.$$

This can be realized as the circle of radius $\frac{1}{2}$ centered at the point $(0, 1/2)$ in the plane. There is natural vector bundle associated to M. Indeed, to each point $(x,y) \in M$, let $E_p := \operatorname{Ker} \left(\begin{smallmatrix} x & 1-y \\ y & x \end{smallmatrix} \right)$. Note that E_p is nonzero since $\det \left(\begin{smallmatrix} x & 1-y \\ y & x \end{smallmatrix} \right) = 0$.

6.5.2 Smooth Sections

A **section** of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = 1_M$. This condition means precisely that for each p in M, s maps p into the fiber E_p. We say that a section is **smooth** if it is smooth as a map from M to E. A **vector field** X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point p in M. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is **smooth** if it is smooth as a map from M to TM.

Example 6.15. The formula

$$X_{(x,y)} = -y\partial_x + x\partial_y$$

defines a smooth vector field on $\mathbb{R}^2$.

6.5.3 Whitney Sum

Let (E, M, π) and (E', M, π') be two vector bundles. We can construct a new vector bundle called the **Whitney sum**, given by $(\pi, \pi') : E \oplus E' \to M$.

Example 6.16. Suppose $E = L \oplus L'$ where L and L' are line bundles. Then we can make a new bundle called $\det(E)$.

Throughout this section, let R be a commutative ring.

Definition 6.6. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally ringed** R**-space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ to be the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

Remark 21. As every ring has a unique structure as a $\mathbb{Z}$-algebra, we simply say **(locally) ringed space** instead of **(locally)** $\mathbb{Z}$**-ringed space**. Usually we will denote a (locally) R-ringed space by $(X, \mathcal{O}_X)$ simply by X.

Example 6.17. Let X be an open subset of a finite-dimensional $\mathbb{R}$-vector space. We denote by C_X^∞ the sheaf of C^∞-functions, i.e.

$$C_X^\infty(U) := \{ f : U \to \mathbb{R} \mid f \text{ is } C^\infty \text{ function} \}.$$

Then C_X^∞ is a sheaf of $\mathbb{R}$-algebras.

7 Differential Forms

7.1 Differential 1-Forms

Let M be a smooth manifold and p a point in M. The **cotangent space** of M at p, denoted by T_p^*M, is defined to be the dual space of the tangent space T_pM :

$$T_p^*M = (T_pM)^\vee = \operatorname{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}).$$

An element of the cotangent space T_p^*M is called a **covector** at p. Thus, a covector ω_p at p is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A **covector field**, also called a **differential** 1-**form** or more simply a 1-**form**, on M is a function ω that assigns to each point p in M a covector ω_p at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p. There are many reasons for the great utility of differential forms in manifold theory, among which is the fact that they can be pulled back under a map. This is in contrast to vector fields, which in general cannot be pushed forward under a map.

7.1.1 The Differential of a Function

If f is a C^∞ real-valued function on a manifold M, its **differential** is defined to be the 1-form df on M such that for any $p \in M$ and $X_p \in T_pM$, we have

$$(df)_p(X_p) = X_p f.$$

Instead of $(df)_p$ we also write $df\,|_p$ for the value of the 1-form df at p.

8 Bump Functions and Partitions of Unity

A partition of unity on a manifold is a collection of nonnegative functions that sum to 1. Usually one demands in addition that the partition of unity be **subordinate** to an open cover $\{U_i\}_{i\in I}$. What this means is that the partition of unity $\{\rho_i\}_{i\in I}$ is indexed over the same set as the open cover $\{U_i\}_{i\in I}$, and for each i in the index I, the support of ρ_i is contained in U_i.

The existence of a C^∞ partition of unity is one of the most technical tools in the theory of C^∞ manifolds. It is the single feature that makes the behavior of C^∞ manifolds so different from that of real-analytic or complex manifolds. In this section we construct C^∞ bump functions on any manifold and prove the existence of a C^∞ partition of unity on a compact manifold. The proof of the existence of a C^∞ partition of unity of a general manifold is more technical and is postponed.

A partition of unity is used in two ways:

1. to decompose a global object on a manifold into a locally finite sum of local objects on the open sets U_i of an open cover.

2. to patch together local objects on the open sets U_i into a global object on the manifold.

Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local coordinates on a manifold, there may be no global coordinates.

8.1 C^∞ Bump Functions

The **support** of a real-valued function f on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$:

$$\mathrm{supp} f := \overline{f^{-1}(\mathbb{R}\backslash\{0\})}.$$

Let q be a point in M, and U a neighborhood of q. By a **bump function at** q **supported in** U we mean any continuous nonnegative function ρ on M that is 1 in a neighborhood of q with $\mathrm{supp}\rho \subset U$.

Example 8.1. The support of the function $f : (-1,1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is the open interval $(-1,1)$, and not the closed interval $[-1,1]$, because the closure of $f^{-1}(\mathbb{R}\backslash\{0\})$ is taken in the domain $(-1,1)$ and not in $\mathbb{R}$.

Recall from Example (3.2) the smooth function f defined on $\mathbb{R}$ by the formula

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

We wish to build a smooth bump function function from f. The main challenge in building a smooth bump function from f is to construct a smooth version of a step function. We seek $g(t)$ by dividing $f(t)$ by a positive function $\ell(t)$, for the quotient $f(t)/\ell(t)$ will be zero for $t \leq 0$. The denominator $\ell(t)$ should be a positive function that agrees with $f(t)$ for $t \geq 1$, for then $f(t)/\ell(t)$ will be identically 1 for $t \geq 1$. The simplest way to construct such an $\ell(t)$ is to add to $f(t)$ a nonnegative function that vanishes for $t \geq 1$. One such nonnegative function is $f(1-t)$. This suggests that we take $\ell(t) = f(t) + f(1-t)$ and consider

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}.$$

Given two positive real numbers $a < b$, we make a linear change of variables to map $[a^2, b^2]$ to $[0, 1]$:

$$x \mapsto \left(\frac{x - a^2}{b^2 - a^2}\right):$$

Let $h : \mathbb{R} \to [0, 1]$ be given by

$$h(x) = g\left(\frac{x - a^2}{b^2 - a^2}\right).$$

Then h is a C^∞ step function such that

$$h(x) = \begin{cases} 0 & \text{if } x \le a^2 \\ 1 & \text{if } x \ge b^2. \end{cases}$$

Now replace x by x^2 to make the function symmetric in x: $k(x) = h(x^2)$. Finally, set

$$\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

This $\rho(x)$ is a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. For any $q \in \mathbb{R}$, $\rho(x - q)$ is a C^∞ bump function at q.

It is easy to extend the construction of a bump function from $\mathbb{R}$ to $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0}$ in $\mathbb{R}^n$ that is 1 on the closed balled $\overline{B_a(\mathbf{0})}$ and has support in the closed ball $\overline{B_b(\mathbf{0})}$, set

$$\sigma(x) = \rho\left(\|x\|\right) = 1 - g\left(\frac{x_1^2 + \cdots + x_n^2 - a^2}{b^2 - a^2}\right).$$

As a composition of C^∞ functions, σ is C^∞. To get a C^∞ bump function at q in $\mathbb{R}^n$, take $\sigma(x - q)$.

8.1.1 Extending C^∞ Bump Functions to M

Now suppose we have manifold M, and open subset U of M, and a point q in U. Choose a chart (ϕ_i, U_i) such that $U_i \subseteq U$ and $\phi_i(U_i) \cong B_b(\phi(q))$, for some $b > 0$, and choose an open neighborhood V_i of p such that $V_i \subseteq U_i$ and $\phi_i(V_i) \cong B_a(\phi(q))$ for some $a < b$. We've shown now to construct a bump function ρ at $\phi_i(q)$ such that $\rho(x) = 1$ for all $x \in B_a(\phi_i(q))$ and such that $\rho(x) = 0$ outside $B_b(\phi_i(q))$. Now we pull back ρ by ϕ to get a bump function on U_i:

$$(\phi_i^* \rho)(q') = \rho(\phi_i(q')) \text{ for all } q' \in U_i.$$

Finally we extend this function a bump function $\widetilde{\rho}$ on M by setting

$$\widetilde{\rho}(q') = \begin{cases} (\phi_i^* \rho)(q') & \text{if } q' \in U_i \\ 0 & \text{if } q' \notin U_i \end{cases}$$

Let us show that this function is C^∞. For $q' \in U_i$, we simply choose the chart (ϕ_i, U_i). Then

$$(\widetilde{\rho} \circ \phi_i^{-1})(x) = (\phi^* \rho)(\phi_i^{-1}(x)) = \rho(x),$$

shows that $\widetilde{\rho}$ is C^∞ in (ϕ_i, U_i). For $q' \notin U_i$, we choose a chart (ϕ_j, U_j) such that $U_i \cap U_j = \emptyset$. Then

$$(\widetilde{\rho} \circ \phi_j^{-1})(x) = \widetilde{\rho}(\phi_j^{-1}(x)) = 0.$$

Thus, the function $\widetilde{\rho}$ we constructed is C^∞ everywhere.

8.1.2 C^∞ Extension of a Function

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on M; an example is the function $\sec x$ on the open interval $(-\pi/2, \pi/2)$ in $\mathbb{R}$. However, if we require that the global function on M agree with the given function only on some neighborhood of a point in U, then a C^∞ extension is possible.

Proposition 8.1. *(C^∞ extension of a function) Suppose f is a C^∞ function defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ function $\widetilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p.*

Proof. Choose a C^∞ bump map $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p. Define

$$\widetilde{f}(q) = \begin{cases} \rho(q) f(q) & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

As the product of two C^∞ functions on U, $\widetilde{f}$ is C^∞ on U. If $q \notin U$, then $q \notin \operatorname{supp}\rho$, and so there is an open set containing q on which $\widetilde{f}$ is 0, since $\operatorname{supp}\rho$ is closed. Therefore $\widetilde{f}$ is also C^∞ at every point $q \notin U$. Finally, since $\rho = 1$ on V, the function $\widetilde{f}$ agrees with f on V. □

Remark 22. This proposition says that the natural map $C^\infty(M) \to C_p^\infty(M)$ is surjective. Thus, every germ in $C_p^\infty(M)$ can be represented by (f, M), where f is a C^∞ function on M.

8.2 Partitions of Unity

If $\{U_i\}_{i\in I}$ is a finite open cover of M, a C^∞ **partition of unity subordinate to** $\{U_i\}_{i\in I}$ is a collection of nonnegative functions $\{\rho_i : M \to \mathbb{R}\}$ such that $\operatorname{supp}\rho_i \subset U_i$ and

$$\sum_{i\in I} \rho_i = 1. \tag{8}$$

When I is an infinite set, for the sum in (8) to make sense, we will impose a **local finiteness** condition. A collection $\{A_\alpha\}$ of subsets of a topological space X is said to be **locally finite** if every point x in X has a neighborhood that meets only finitely many of the sets A_α. In particular, every point $x \in X$ is contained in only finitely many of the A_α's.

Example 8.2. Let $U_{r,n}$ be the open interval $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ on the real line $\mathbb{R}$. Then the open cover $\{U_{r,n} \mid r \in \mathbb{Q},\ n \in \mathbb{N}\}$ of $\mathbb{R}$ is not locally finite.

Definition 8.1. A C^∞ **partition of unity** on a manifold is a collection of nonnegative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i\in I}$ such that

1. The collection of supports, $\{\operatorname{supp}\rho\}_{i\in I}$, is locally finite,
2. $\sum_{i\in I} \rho_i = 1$.

Given an open cover $\{U_i\}_{i\in I}$ of M, we say that a partition of unity $\{\rho_i\}_{i\in I}$ is **subordinate to the open cover** $\{U_i\}_{i\in I}$ if $\operatorname{supp}\rho_i \subset U_i$ for every $i \in I$.

Remark 23. Since the collection of supports, $\{\operatorname{supp}\rho_i\}_{i\in I}$, is locally finite, every point q lies in only finitely many of the sets $\operatorname{supp}\rho_i$. Hence $\rho_i(q) \neq 0$ for only finitely many i. It follows that the sum $\sum_{i\in I} \rho_i(q)$ is finite.

Example 8.3. Let U and V be the open intervals $(-\infty, 2)$ and $(-1, \infty)$ in $\mathbb{R}$ respectively, and let ρ_V be a smooth step function which is equal to 0 on $(-\infty, 0)$ and equal to 1 on $(1, \infty)$. Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp}\rho_V \subset V$ and $\operatorname{supp}\rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

8.3 Existence of a Partition of Unity

In this subsection we begin a proof of the existence of a C^∞ partition of unity on a manifold. Because the case of a compact manifold is somewhat easier and already has some of the features of the general case, for pedagogical reasons we give a separate proof for the compact case.

Lemma 8.1. *If $\rho_1, \ldots, \rho_m$ are real-valued funcitons on a manifold M, then*

$$supp\left(\sum_{i=1}^m \rho_i\right) \subset \bigcup_{i=1}^m supp\rho_i.$$

Proof. Suppose $q \in \operatorname{supp}\left(\sum_{i=1}^m \rho_i\right)$. Thus $\sum_{i=1}^m \rho_i(q) \neq 0$. In particular, we must have $\rho_i(q) \neq 0$ for some $i = 1, \ldots, m$. Thus, $q \in \operatorname{supp}\rho_i \subset \bigcup_{i=1}^m \operatorname{supp}\rho_i$. □

Proposition 8.2. *Let M be a compact manifold and $\{U_i\}_{i\in I}$ an open cover of M. There exists a C^∞ partition of unity $\{\rho_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$.*

Proof. For each $q \in M$, find an open set U_i containing q from the given cover and let ψ_q be a C^∞ bump function at q supported in U_i. Because $\psi_q(q) > 0$, there is a neighborhood W_q of q on which $\psi_q > 0$. By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \dots, W_{q_m}\}$. Let $\psi_{q_1}, \dots, \psi_{q_m}$ be the corresponding bump functions. Then $\psi := \sum_{j=1}^m \psi_{q_j}$ is positive at every point q in M because $q \in W_{q_j}$ for some i. Define

$$\varphi_j = \frac{\psi_{q_j}}{\psi}, \quad j = 1, \dots, m.$$

Clearly $\sum_{j=1}^m \varphi_j = 1$. Moreover, since $\psi > 0$, $\varphi_j(q) \neq 0$ if and only if $\psi_{q_j}(q) \neq 0$, so

$$\operatorname{supp}\varphi_j = \operatorname{supp}\psi_{q_j} \subset U_i$$

for some $i \in I$. This shows that $\{\varphi_i\}$ is a partition of unity such that for every j, $\operatorname{supp}\varphi_j \subset U_i$ for some $i \in I$.

The next step is to make the index set of the partition of unity the same as that of the open cover. For each $j = 1, \dots, m$, choose $\tau(j) \in I$ to be an index such that

$$\operatorname{supp}\varphi_j \subset U_{\tau(j)}.$$

We group the collection of functions $\{\varphi_j\}$ into subcollections according to $\tau(j)$ and define for each $i \in I$,

$$\rho_i = \sum_{\tau(j)=i} \varphi_j;$$

if there is no j for which $\tau(j) = i$, the sum is empty and we define $\rho_i = 0$. Then

$$\sum_{i \in I} \rho_i = \sum_{i \in I} \sum_{\tau(j)=i} \varphi_j = \sum_{j=1}^m \varphi_j = 1.$$

Moreover by Lemma (8.1),

$$\operatorname{supp}\rho_i \subset \bigcup_{\tau(j)=i} \operatorname{supp}\varphi_j \subset U_i.$$

So $\{\rho_i\}$ is a partition of unity subordinate to $\{U_i\}$. $\square$

9 Integration on Manifolds

9.1 Riemann Integral of a Function on $\mathbb{R}^n$

A **closed rectangle** in $\mathbb{R}^n$ is a Cartesian product $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ of closed intervals in $\mathbb{R}$, where $a_i, b_i \in \mathbb{R}$. Let $f : R \to \mathbb{R}$ be a bounded function defined on a closed rectangle R. The **volume** $\mathbf{vol}(R)$ of the closed rectangle R is defined to be

$$\operatorname{vol}(R) := \prod_{i=1}^n (b_i - a_i).$$

A **partition** of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \dots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A **partition** of the rectangle R is a collection $P = \{P_1, \dots, P_n\}$, where each P_i is a partition of $[a_i, b_i]$. The partition P divides the rectangle R into closed subrectangles, which we denote by R_j.

We define the **lower sum** and the **upper sum** of f with respect to the partition P to be

$$L(f, P) := \sum_{R_j} \left(\inf_{R_j} f \right) \operatorname{vol}(R_j), \qquad U(f, P) := \sum_{R_j} \left(\sup_{R_j} f \right) \operatorname{vol}(R_j),$$

where each sum runs over all subrectangles of the partition P. For any partition P, clearly $L(f, P) \leq U(f, P)$. In fact, more is true: for any two partitions P and P' of the rectangle R,

$$L(f, P) \leq U(f, P'),$$

which we show next.

A partition $P' = \{P'_1, \ldots, P'_n\}$ is a **refinement** of the partition $P = \{P_1, \ldots, P_n\}$ if $P_i \subset P'_i$ for all $i = 1, \ldots, n$. If P' is a refinement of P, then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of P', and it is easily seen that

$$L(f, P) \leq L(f, P'),$$

because if $R'_{jk} \subset R_j$, then $\inf_{R_j} f \leq \inf_{R'_{jk}} f$. Similarly, if P' is a refinement of P, then

$$U(f, P') \leq U(f, P).$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i = P_i \cup P'_i$, and thus

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions P of R is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions of R. We define these two numbers to be the **lower integral** $\underline{\int}_R f$ and the **upper integral** $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f := \sup_P L(f, P), \qquad \overline{\int}_R f := \inf_P L(f, P).$$

Definition 9.1. Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be **Riemann integrable** if $\underline{\int}_R f = \overline{\int}_R f$; in this case, the Riemann integral of f is this common value, denoted by $\int_R f(x) dx_1 \cdots dx_n$, where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^n$.

Example 9.1. Let f be a bounded monotone increasing function on $[-1, 1]$. Then f is Riemann integrable. Indeed, consider the partition $P_n = \{p_0 < p_1 < \cdots < p_{2n-1} < p_{2n}\}$ where $p_i = -1 + i/3$. Then

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \frac{1}{n} \sum_{i=1}^{2n} f(-1 + i/3) - \frac{1}{n} \sum_{i=0}^{2n-1} f(-1 + i/3) \\ &= \frac{1}{n} \left(\sum_{i=1}^{2n} f(-1 + i/3) - \sum_{i=0}^{2n-1} f(-1 + i/3) \right) \\ &= \frac{1}{n} \left(f(1) - f(-1) \right), \end{aligned}$$

which tends to 0 as $n \to \infty$.

If $f : A \subset \mathbb{R}^n \to \mathbb{R}$, then the **extension of** f **by zero** is the function $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Now suppose $f : A \to \mathbb{R}$ is a bounded function on a bounded set A in $\mathbb{R}^n$. Enclose A in a closed rectangle R and define the Riemann integral of f over A to be

$$\int_A f(x) dx_1 \cdots dx_n = \int_R \widetilde{f}(x) dx_1 \cdots dx_n$$

if the right-hand side exists. In this way we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$. The **volume** $\mathrm{vol}(A)$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1 dx_1 \cdots dx_n$ if the integral exists.

9.2 Integrability Conditions

In this section we describe some conditions under which a function defined on an open subset of $\mathbb{R}^n$ is Riemann integrable.

Definition 9.2. A set $A \subset \mathbb{R}^n$ is said to have **measure zero** if for every $\varepsilon > 0$, there is a countable cover $\{R_i\}_{i=1}^{\infty}$ of A by closed rectangles R_i such that $\sum_{i=1}^{\infty} \mathrm{vol}(R_i) < \varepsilon$.

Theorem 9.1. *(Lebesgue's theorem) A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set $\mathrm{Disc}(\widetilde{f})$ of discontinuities of the extended function $\widetilde{f}$ has measure zero.*

Proposition 9.1. *If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.*

Proof. Being continuous on a compact set, the function f is bounded. Being compact, the set $\text{supp}(f)$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\widetilde{f}$ is continuous.

Since $\widetilde{f}$ agrees with f on U, the extended function $\widetilde{f}$ is continuous on U. It remains to show that $\widetilde{f}$ is continuous on the complement of U in $\mathbb{R}^n$ as well. If $p \notin U$, then $p \notin \text{supp}(f)$. Since $\text{supp}(f)$ is a closed subset of $\mathbb{R}^n$, there is an open ball B containing p and disjoint from $\text{supp}(f)$. On this open ball, $\widetilde{f} = 0$, which implies that $\widetilde{f}$ is continuous at $p \notin U$. Thus, $\widetilde{f}$ is continuous on $\mathbb{R}^n$. By Lebesgue's theorem, f is Riemann integrable on U. □

Example 9.2. The continuous function $f : (-1,1) \to \mathbb{R}$, given by $f(x) = \tan(\pi x/2)$, is defined on an open subset of finite length in $\mathbb{R}$, but it not bounded. The support of f is the open interval $(-1,1)$, which is not compact. Thus, the function f does not satisfy the hypotheses of either Lebesgue's theorem or Proposition (9.1). Note that it is not Riemann integrable.

Definition 9.3. A subset $A \subset \mathbb{R}^n$ is called a **domain of integration** if it is bounded and its topological boundary $\text{bd}(A)$ is a set of measure zero.

Proposition 9.2. *Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.*

Proof. Let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero. Since f is continuous on A the extension $\widetilde{f}$ is necessarily continuous at all interior points of A. Clearly, $\widetilde{f}$ is continuous at all exterior points of A also, because every exterior point has a neighborhood contained entirely in $\mathbb{R}^n \backslash A$, on which $\widetilde{f}$ is identically zero. Therefore, the set $\text{Disc}(\widetilde{f})$ of discontinuities of $\widetilde{f}$ is a subset of $\partial(A)$, a set of measure zero. By Lebesgue's theorem, f is Riemann integrable. □

9.3 The Integral of an n-Form on $\mathbb{R}^n$

Once a set of coordinates $x_1, \ldots, x_n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 9.4. Let $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ be a C^∞ n-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x_1, \ldots, x_n$. Its **integral** over a subset $A \subset U$ is defined to be the Riemann integral of $f(x)$:

$$\int_A \omega = \int_A f(x)dx_1 \wedge \cdots \wedge dx_n := \int_A f(x)dx_1 \cdots dx_n,$$

if the Riemann integral exists.

Example 9.3. If f is a bounded continuous function defined on a domain of integration A in $\mathbb{R}^n$, then the integral $\int_A f(x)dx_1 \wedge \cdots \wedge dx_n$ exists.

Let us see how the integral of an n-form $\omega = f dx_1 \wedge \cdots \wedge dx_n$ on an open subset $U \subset \mathbb{R}^n$ transforms under a change of variables. A change of variables on U is given by a diffeomorphism $T : \mathbb{R}^n \supset V \to U \subset \mathbb{R}^n$. Let $x_1, \ldots, x_n$ be the standard coordinates on U and $y_1, \ldots, y_n$ be the standard coordinates on V. Then $T_i := x_i \circ T$ is the ith component of T. We will assume that U and V are connected, and write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$dT_1 \wedge \cdots \wedge dT_n = \det(J(T))\, dy_1 \wedge \cdots \wedge dy_n.$$

Hence,

$$\begin{aligned}
\int_V T^*\omega &= \int_V (T^*f) T^*dx_1 \wedge \cdots \wedge T^*dx_n \\
&= \int_V (f \circ T) dT_1 \wedge \cdots \wedge dT_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \wedge \cdots \wedge dy_n \\
&= \int_V (f \circ T) \det(J(T)) dy_1 \cdots dy_n.
\end{aligned}$$

On the other hand, the change-of-variables formula from advanced calculus gives

$$\int_U \omega = \int_U f dx_1 \cdots dx_n = \int_V (f \circ T) |\det(J(T))| dy_1 \cdots dy_n,$$

with an absolute-value sign around the Jacobian determinant. Hence,

$$\int_V T^*\omega = \pm \int_U \omega,$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative. In particular, the integral of a differential form is not invariant under all diffeomorphisms of V with U, but only under orientation-preserving diffeomorphisms.

9.4 Integral of a Differential Form over a Manifold

Integration of an n-form on $\mathbb{R}^n$ is not so different from integration of a function. Our approach to integration over a general manifold has several distinguishing features:

1. The manifold must be oriented.
2. On a manifold of dimension n, one can integrate only n-forms, not functions.
3. The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of M. Denote by $\Omega_c^k(M)$ the vector space of C^∞ k-forms with compact support on M. Suppose (U, ϕ) is a chart in this atlas. If $\omega \in \Omega_c^n(U)$ is an n-form with compact support on U, then because $\phi : U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega = \int_{\phi(U)} (\phi^{-1})^*\omega.$$

If (U, ψ) is another chart in the oriented atlas with the same U, then $\phi \circ \psi^{-1}\psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} (\phi^{-1})^*\omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^*(\phi^{-1})^*\omega = \int_{\psi(U)} (\psi^{-1})^*\omega.$$

Thus, the integral $\int_U \omega$ on a chart U of the atlas is well defined, independent of the choice of coordinates on U. By linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha\omega$ are identically zero. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a *finite* sum. Also since $\operatorname{supp}(\rho_\alpha\omega) \subset \operatorname{supp}(\rho_\alpha) \cap \operatorname{supp}(\omega)$, $\operatorname{supp}(\rho_\alpha\omega)$ is a closed subset of the compact set $\operatorname{supp}(\omega)$. Hence, $\operatorname{supp}(\rho_\alpha\omega)$ is compact. Since $\rho_\alpha\omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha\omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega. \tag{9}$$

For this integral to be well defined, we must show that it is independent of the choices of oriented atlas and partition of unity. Let $\{V_\beta, \psi_\beta\}$ be another oriented atlas of M specifying the orientation of M, and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\{(U_\alpha \cap V_\beta, \phi_\alpha|_{U_\alpha \cap V_\beta})\}$ and $\{(U_\alpha \cap V_\beta, \psi_\beta|_{U_\alpha \cap V_\beta})\}$ are two new atlases of M specifying the orientation of M, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega && \text{(because } \textstyle\sum_\beta \chi_\beta = 1) \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega && \text{(these are finite sums)} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega,
\end{aligned}$$

where the last line follows from the fact that the support of $\rho_\alpha\chi_\beta$ is contained in $U_\alpha \cap V_\beta$. By symmetry, $\sum_\beta \int_{V_\beta} \chi_\beta\omega$ is equal to the same sum. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega,$$

proving that the integral (9) is well defined.

10 Quotients and Gluing

There are many important topological spaces (and manifolds) that are constructed by "identifying" pieces of spaces. This typically takes the form of gluing along open sets or passing to quotients by (reasonable) equivalence relations.

10.1 The Quotient Topology

Recall that an equivalence relation on a set X is a reflexive, symmetric, and transitive relation. The **equivalence class** $[x]$ of $x \in X$ is the set of all elements in X equivalent to x. An equivalence relation on X partitions X into disjoint equivalence classes. We denote the set of equivalence classes by $X/\sim$ and call this set the **quotient** of X by the equivalence relation $\sim$. There is a natural **projection map** $\pi : X \to X/\sim$ that sends $x \in X$ to its equivalence class $[x]$.

Assume now that X is a topological space. We define a topology on $X/\sim$ by declaring a set U in $X/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in X. Clearly, both the empty set $\varnothing$ and the entire quotient $X/\sim$ are open. Further, since

$$\pi^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \pi^{-1}(U_i) \text{ and } \pi^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} \pi^{-1}(U_i),$$

the collection of open sets in $X/\sim$ is closed under arbitrary unions and finite intersections, and is therefore a topology. It is called the **quotient topology** on $X/\sim$. With this topology, $X/\sim$ is called the **quotient space** of X by the equivalence relation $\sim$. The way we defined the topology on $X/\sim$ makes the projection map π continuous.

10.1.1 Continuity of a Map on a Quotient

Suppose f is a map from X to Y and is constant on each equivalence class. Then it induces a map $\overline{f} : X/\sim \to Y$, given by $\overline{f}([x]) = f(x)$ where $x \in X$.

Proposition 10.1. *The induced map $\overline{f} : X/\sim \to Y$ is continuous if and only if the map $f : X \to Y$ is continuous.*

Proof. If $\overline{f}$ is continuous, then f is continuous since $f = \overline{f} \circ \pi$ is a composition of two continuous functions. Conversely, suppose f is continuous. Let V be an open set in Y. Then $f^{-1}(V) = \pi^{-1}\left(\overline{f}^{-1}(V)\right)$ is open in X. By the definition of quotient topology, $\overline{f}^{-1}(V)$ is open in $X/\sim$. Thus $\overline{f}$ is continuous since V was arbitary. □

10.1.2 Identification of a Subset to a Point

If A is a subspace of a topological space X, we can define a relation $\sim$ on X by declaring

$$x \sim x \text{ for all } x \in X \text{ and } x \sim y \text{ for all } x, y \in A.$$

This is an equivalence relation on X. We say that the quotient space $X/\sim$ is obtained from X by **identifying** A **to a point**.

Example 10.1. Let I be the unit interval $[0,1]$ and $I/\sim$ be the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1$, given by $f(x) = e^{2\pi i x}$, assumes the same value at 0 and 1, and so induces a function $\overline{f} : I/\sim \to S^1$. Since f is continuous, $\overline{f}$ is continuous. As the continuous image of a compact set I, the quotient $I/\sim$ is compact. Thus $\overline{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. Hence it is a homeomorphism.

10.2 Open Equivalence Relations

An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection map $\pi : X \to X/\sim$ is open. In other words, the equivalence relation $\sim$ on X is open if and only if for every open set U in X, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U} [x]$$

of all points equivalent to some point of U is open.

Example 10.2. Let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 and let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map. Then π is not an open map. Indeed, let V be the open interval $(-2, 0)$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\},$$

which is not open in $\mathbb{R}$.

Given an equivalence relation $\sim$ on X, let R be the subset of $X \times X$ that defines the relation

$$R = \{(x, y) \in X \times X \mid x \sim y\}.$$

We call R the **graph** of the equivalence relation $\sim$.

Theorem 10.1. *Suppose $\sim$ is an open equivalence relation on a topological space X. Then the quotient space $X/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $X \times X$.*

Proof. There is a sequence of equivalent statements: R is closed in $X \times X$ iff $(X \times X)\backslash R$ is open in $X \times X$ iff for every $(x, y) \in (X \times X)\backslash R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \varnothing$ iff for every pair $x \not\sim y$ in X, there exist neighborhoods U of x and V of y in X such that no element of U is equivalent to an element of V iff for any two points $[x] \neq [y]$ in $X/\sim$, there exist neighborhoods U of x and V of y in X such that $\pi(U) \cap \pi(V) = \varnothing$ in $X/\sim$.

We now show that this last statement is equivalent to $X/\sim$ being Hausdorff. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $X/\sim$ containing $[x]$ and $[y]$ respectively, so $X/\sim$ is Hausdorff. Conversely, suppose $X/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $X/\sim$. Then there exist disjoint open sets A and B in $X/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$. Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $X/\sim$. $\square$

Theorem 10.2. *Let $\sim$ be an open equivalence relation on a topological space X. If $\mathcal{B} = \{B_\alpha\}$ is a basis for X, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $X/\sim$.*

Proof. Since π is an open map, $\{\pi(B(\alpha)\}$ is a collection of open sets in $X/\sim$. Let W be an open set in $X/\sim$ and $[x] \in W$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that $x \in B \subset \pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B) \subset W$, which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. $\square$

Corollary 1. *If $\sim$ is an open equivalence relation on a second-countable space X, then the quotient space is second-countable.*

10.3 Quotients by Group Actions

Many important manifolds are constructed as quotients by actions of groups on other manifolds, and this often provides a useful way to understand spaces that may have been constructed by other means. As a basic example, the Klein bottle will be defined as a quotient of $S^1 \times S^1$ by the action of a group of order 2. The circle as defined concretely in $\mathbb{R}^2$ is isomorphic to the quotient of $\mathbb{R}$ by additive translation by $\mathbb{Z}$.

Definition 10.1. Let X be a topological space and G a discrete group. A right action of G on X is **continuous** if for each $g \in G$ the action map $X \to X$ defined by $x \mapsto x.g$ is continuous (and hence a homeomorphism, as the action of g^{-1} gives an inverse). The action is **free** if for each $x \in X$ the stabilizer subgroup $\{g \in G \mid x.g = x\}$ is the trivial subgroup (in other words, $x.g = x$ implies $g = 1$). The action is **properly discontinuous** when it is continuous for the discrete topology on G and each $x \in X$ admits an open neighborhood U_x so that the G-translate $U_x.g$ meets U_x for only finitely many $g \in G$.

Proposition 10.2. *A right action of G on X is continuous if $\pi : X \times G \to X$ is continuous.*

Remark 24. Here, G has the discrete topology.

Proof. Suppose we have a right action of G on X which is continuous. Let U be an open set in X. For each $g \in G$, let $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) = \bigcup_{g \in G} U_g \times \{g\},$$

which is open. Conversely, suppose π is continuous and let $g \in G$. Let U be open in X and set $U_g := g^{-1}(U)$. Then

$$\pi^{-1}(U) \cap X \times \{g\} = U_g \times \{g\},$$

which shows that g is continuous since $\pi^{-1}(U)$ and $X \times \{g\}$ are open in $X \times G$. $\square$

Example 10.3. Suppose that X is a locally Hausdorff space, and that G acts on X on the right via a properly discontinuous action. For each $x \in X$, we get an open subset U_x such that U_x meets $U_x.g$ for only finitely many $g \in G$. This property is unaffected by replacing U_x with a smaller open subset around x, so by the locally Hausdorff property we can assume that U_x is Hausdorff. The key is that we can do better: there exists an open set $U'_x \subseteq U_x$ such that U'_x meets $U'_x.g$ if and only if $x = x.g$. Thus, if the action is also free then U'_x is disjoint from $U'_x.g$ for all $g \in G$ with $g \neq 1$.

To find U'_x, let $g_1, \ldots, g_n \in G$ be an enumeration of the finite set of elements $g \in G$ such that U_x meets $U_x.g$. For any open subset $U \subseteq U_x$ we can only have $U \cap U.g \neq \emptyset$ for g equal to one of the g_i's, so it suffices to show that for each i with $x.g_i \in U_x \backslash \{x\}$ there is an open subset $U_i \subseteq U_x$ such that $U_i \cap (U_i).g_i = \emptyset$ (and then we may take U'_x to be the intersection of the U_i's over the finitely many i such that $x.g_i \neq x$). By the Hausdorff property of U_x, when $x.g_i \in U_x \backslash \{x\}$ there exist disjoint opens $V_i, V'_i \subseteq U_x$ around x and $x.g_i$ respectively. By continuity of the action on X by $g_i \in G$ there is an open $W_i \subseteq X$ around x such that $(W_i).g_i \subseteq V'_i$. Thus $U_i = W_i \cap V_i$ is disjoint from V'_i yet satisfies $(U_i).g_i \subseteq V'_i$, so $U_i \cap (U_i).g_i = \emptyset$. This completes the construction of U'_x.

The interest in free and properly discontinuous actions is that for such actions in the locally Hausdorff case we may find an open U_x around each $x \in X$ such that U_x is disjoint from $U_x.g$ whenever $g \neq 1$. Thus, for such actions we may say that in X/G we are identifying points in the same G-orbit with this identification process not "crushing" the space X by identifying points in X that are arbitrarily close to each other. An example where things go horribly wrong is the action of $G = \mathbb{Q}$ on $\mathbb{R}$ via additive translations. This is a continuous action, but the quotient $\mathbb{R}/\mathbb{Q}$ is very bad: any two $\mathbb{Q}$-orbits in $\mathbb{R}$ contain arbitrarily close points!

Here are some examples of free and properly discontinuous actions.

Example 10.4. The antipodal map on S^n, given by $(a_1, \ldots, a_{n+1}) \mapsto (-a_1, \ldots, -a_{n+1})$, viewed as an action of the integers mod 2 is free and properly discontinuous: freeness is clear, as is continuity, and for any $x \in S^n$ the points near x all have their antipodes far away!

Example 10.5. Consider the curve $X := \mathbf{V}(x^3 + y^3 + z^3 - 1) \subseteq \mathbb{C}^3$. Then the action $(a_1, a_2, a_3) \mapsto (\zeta_3 a_1, \zeta_3 a_2, \zeta_3)$, viewed as an action of the integers mod 3 is free and properly discontinuous.

Example 10.6. Let $X = S^1 \times S^1$ be a product of two circles, where the circle

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is viewed as a topological group (using multiplication in $\mathbb{C}$, so both the group law and inversion $z \mapsto 1/z = \overline{z}$ on S^1 are continuous). The visibly continuous map $(z, w) \mapsto (1/z, -w) = (\overline{z}, -w)$ reflects through the x-axis in the first circle and rotates 180-degree in the second circle, and it is its own inverse. Thus, this give an action by the order-2 group G of integers mod 2. The associated quotient X/G will be called the (set-theoretic) **Klein bottle**.

Theorem 10.3. *Let X be a locally Hausdorff topological space with a free and properly discontinuous action by a group G. There is a unique topology on X/G such that the quotient map $\pi : X \to X/G$ is a continuous map that is a local homeomorphism (i.e. each $x \in X$ admits a neighborhood mapping homeomorphically onto an open subset of X/G). Moreover, the quotient map is open.*

A subset $S \subseteq X/G$ is open if and only if its preimage in X is open, and if $U \subseteq X$ is an open set that is disjoint from $U.g$ for all nontrivial $g \in G$ then the map $U \to X/G$ is a homeomorphism onto its open image $\overline{U}$ and the natural map $U \times G \to \pi^{-1}(\overline{U})$ over $\overline{U}$ given by $(u, g) \mapsto u.g$ is a homeomorphism when G is given the discrete topology.

Remark 25. The topology in this theorem is called the **quotient topology**, and it is locally Hausdorff since $X \to X/G$ is a local homeomorphism.

Proof. Sketch: we show that π is an open map. Let $x \in X$ and pick U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$. We first show that $\pi(U_x)$ is open. The inverse image of $\pi(U_x)$ under π is a disjoint union of open sets $\bigcup_{g \in G} U_x.g$. Therefore $\pi(U_x)$ is open. Now let U be any open subset of X. For each $x \in X$, choose U_x such that $U_x.g \cap U_x = \emptyset$ for all $g \in G \backslash \{1\}$ and $U_x \subset U$. Then

$$\pi(U) = \pi\left(\bigcup_{x \in U} U_x\right) = \bigcup_{x \in U} \pi(U_x)$$

implies $\pi(U)$ is open. $\square$

Example 10.7. (Möbius Strip) Choose $a > 0$. Let $X = (-a, a) \times S^1$, and let the group of order 2 act on it with the non-trivial element acting by $(t, w) \mapsto (-t, -w)$. This is easily checked to be a continuous action for the discrete topology of the group of order 2, and it is free and properly discontinuous. The quotient M_a is the **Möbius strip** of height $2a$.

To check that the Möbius strip M_a is Hausdorff, we use the quotient criterion: the set of points in $X \times X$ with the form $((t, w), (t', w'))$ with $(t', w') = (t, w)$ or $(t', w') = (-t, -w)$ is checked to be closed by using the sequential criterion in $X \times X$: suppose $(t_n, w_n) \sim (t'_n, w'_n)$ are sequences in $X \times X$ which converge (t, w) and (t', w') respectively. Then we need to show that $(t, w) \sim (t', w')$. Assume that $(t, w) \neq (t', w')$. Choose open neighborhoods U of (t, w) and U' of (t', w') respectively such that $U \cap U' = \emptyset$ and such that eventually $(t_n, w_n) \neq (t'_n, w'_n)$ (We can do this because they converge to different limits and our space $X \times X$ is Hausdorff). Thus, eventually we have $(t'_n, w'_n) = (-t_n, -w_n) \to (-t, -w)$.

10.4 Möbius Strip in $\mathbb{R}^3$

Recall that the Möbius strip M_a (with height $2a$) was defined as an abstract smooth manifold made as a quotient of $(-a, a) \times S^1$ by a free and properly discontinuous action by the group of order 2. Using the C^∞ isomorphism between $\mathbb{R}/2\pi\mathbb{Z}$ and the circle $S^1 \subseteq \mathbb{R}^2$ via $\theta \mapsto (\cos\theta, \sin\theta)$, we consider the standard parameter $\theta \in \mathbb{R}$ as a local coordinate on S^1. For finite $a > 0$, consider the C^∞ map $f : (-a, a) \times S^1 \to \mathbb{R}^3$ defined by

$$(t, \theta) \mapsto (2a\cos 2\theta + t\cos\theta\cos 2\theta, 2a\sin 2\theta + t\cos\theta\sin 2\theta, t\sin\theta).$$

Since $f(-t, \pi + \theta) = f(t, \theta)$ by inspection, it follows from the universal property of the quotient map $(-a, a) \times S^1 \to M_a$ that f uniquely factors through this via a C^∞ map $\overline{f} : M_a \to \mathbb{R}^3$. Our goal is to prove that $\overline{f}$ is an embedding and to use this viewpoint to understand some basic properties of the Möbius strip.

10.4.1 Embedding

Theorem 10.4. *The map $\overline{f}$ is an immersion.*

Proof. We first reduce the problem to working with f, as f is given by a simple explicit formula across its entire domain (M_a does not have global coordinates. Of course, working locally for $\overline{f}$ is "the same" as working locally for f, so the reduction step to working with f isn't really necessary if one says things a little differently. However, it seems a bit cleaner to just make the reduction step right away and so to thereby work with the map f that feel a bit more concrete than the map $\overline{f}$ at the global level.)

Let $p : (-a, a) \times S^1 \to M_a$ be the natural quotient map. Each point in M_a has the form $p(\xi_0)$ for some ξ_0 and the Chain Rule gives that the injection $df(\xi_0)$ factors as $d\overline{f}(p(\xi_0)) \circ dp(\xi_0)$ with $dp(\xi_0)$ an isomorphism (as p is a local C^∞ isomorphism, via the theory of quotients by free and properly discontinuous group actions). Hence, the tangent map for $\overline{f}$ is injective at $p(\xi_0)$ if and only if the tangent map for f is injective at ξ_0. It is therefore enough (even equivalent!) to prove that f is an immersion.

□

10.5 Construction of Manifolds From Gluing Data

The definition of a manifold assumes that the underlying set, M, is already known. However, there are situations where we only have some indirect information about the overlap of the domains U_i, of the local charts defining our manifold, M, in terms of the transition functions

$$\phi_{ji} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

but where M itself is not known. Our goal in this subsection is to try and reconstruct a manifold M by gluing open subsets of $\mathbb{R}^n$ using the transition functions ϕ_{ij}.

Definition 10.2. Let n be an integer with $n \geq 1$ and let k be either an integer with $k \geq 1$ or $k = \infty$. A set of **gluing data** is a triple

$$\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K}),$$

satisfying the following properties, where I is a (nonempty) countable set and $K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\}$:

1. For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{R}^n$ called a **parametrization domain**, for short, p**-domain**, and the Ω_i are pairwise disjoint (i.e. $\Omega_i \cap \Omega_j) = \emptyset$ for all $i \neq j$).

2. For every pair $(i,j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ij} \neq \emptyset$ if and only if $\Omega_{ji} \neq \emptyset$. Each nonempty Ω_{ij} (with $i \neq j$) is called a **gluing domain**.

3. The maps $\phi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i,j) \in K$ called a **transition function** (or **gluing function**) and the following condition holds:

   (a) The **cocycle condition** holds: for all i, j, k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ and

   $$\phi_{ki}(x) = (\phi_{kj} \circ \phi_{ji})(x)$$

   for all $x \in \phi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

4. For every pair $(i,j) \in K$ with $i \neq j$, for every $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and every $y \in \partial(\Omega_{ji}) \cap \Omega_j$, there are open balls, V_x and V_y centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}.

Remark 26.

1. In practical applications, the index set, I, is of course finite and the open subsets, Ω_i, may have special properties (for example, connected; open simplicies, etc.).

2. Observe that $\Omega_{ij} \subseteq \Omega_i$ and $\Omega_{ji} \subseteq \Omega_j$. If $i \neq j$, as Ω_i and Ω_j are disjoint, so are Ω_{ij} and Ω_{ji}.

3. The cocycle condition may seem overly complicated but it is actually needed to guarantee the transitivity of the relation, $\sim$, which we will define shortly. Since the ϕ_{ji} are bijections, the cocycle condition implies the following conditions

   (a) $\phi_{ii} = \mathrm{id}_{\Omega_i}$ for all $i \in I$. This follows by setting $i = j = k$.

   (b) $\phi_{ij} = \phi_{ji}^{-1}$ for all $(i,j) \in K$. This follows from (a) and by setting $k = i$.

4. Let M be a C^k manifold and let $\{(U_i, \phi_i)\}_{i\in I}$ be an atlas on it. Then set $\Omega_i = \phi_i(U_i)$, $\Omega_{ij} = \phi_i(U_i \cap U_j)$, and let $\phi_{ij} : \Omega_{ji} \to \Omega_{ij}$ be the corresponding transition maps. Then it's easy to check that the open sets Ω_i, Ω_{ij}, and the gluing functions ϕ_{ij}, satisfy the conditions of Definition (10.2). Indeed,

$$\begin{aligned}
\phi_{ji}^{-1}(\Omega_{jk}) &= (\phi_i \circ \phi_j^{-1})(\phi_j(U_j \cap U_k)) \\
&= \phi_i(U_j \cap U_k) \\
&= \phi_i(U_i \cap U_j \cap U_k) \\
&\subseteq \phi_i(U_i \cap U_k) \\
&= \Omega_{ik}.
\end{aligned}$$

Let us show that a set of gluing data defines a C^k manifold in a natural way.

Proposition 10.3. *For every set of gluing data $\mathcal{G} = (\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the ϕ_{ji}'s.*

Proof. Define the binary relation, $\sim$, on the disjoint union, $\Omega := \coprod_{i\in I} \Omega_i$, of the open sets, Ω_i, as follows: For all $x, y \in \Omega$,

$$x \sim y \text{ if and only if there exists } (i,j) \in K \text{ such that } x \in \Omega_{ij},\ y \in \Omega_{ji}, \text{ and } y = \phi_{ji}(x).$$

The cocycle condition ensures that this is an equivalence relation. Indeed, (a) implies reflexivity and (b) implies symmetry. The crucial step is to check transitivity. Assume that $x \sim y$ and $y \sim z$. Then there are some i, j, k such that $\phi_{ji}(x) = y$ and $\phi_{kj}(y) = z$. But then $(\phi_{kj} \circ \phi_{ji})(x) = \phi_{ki}(x) = z$. That is, $x \sim z$, as desired.

Since $\sim$ is an equivalence relation, let

$$M_{\mathcal{G}} := \Omega / \sim$$

be the quotient space by the equivalence relation $\sim$. We claim that $\sim$ is an open equivalence relation. Indeed, let $U := \coprod_{i \in I} U_i$ be an open subset of Ω, where U_i is an open subset of Ω_i for each i. Then

$$\pi^{-1}(\pi(U)) = \coprod_{i \in I} \left(\bigcup_{j \in I} \phi_{ij}(U_j \cap \Omega_{ji}) \cup U_i \right),$$

which is open in Ω since $\phi_{ij}(U_j \cap \Omega_{ji})$ is open in Ω_i for all $i \in I$. Therefore, $M_{\mathcal{G}}$ is second-countable since Ω is second-countable.

Since $\sim$ is an open equivalence relation, we can use Theorem (10.1) to show that $M_{\mathcal{G}}$ is Hausdorff by showing that the graph

$$R = \{(x,y) \in \Omega \times \Omega \mid x \sim y\}$$

is closed in $\Omega \times \Omega$. We do this by showing that if (x_n, y_n) is a sequence in R that converges to $(x,y) \in \Omega \times \Omega$, then $(x,y) \in R$. That is to say, if $x_n \sim y_n$, then $x \sim y$. Since $(x,y) \in \Omega_i \times \Omega_j$, we may assume that $(x_n, y_n) \in \Omega_i \times \Omega_j$ (since it will eventually be in there anyways). If $i = j$, then $x_n = y_n$, and hence $x = y$, so assume $i \neq j$.

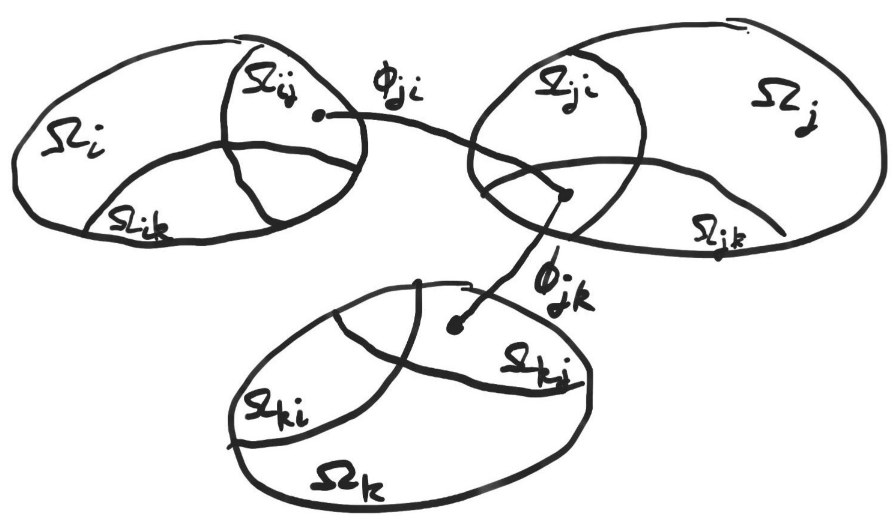
In order for us to have $x_n \sim y_n$, we must have $x_n \in \Omega_{ij}$ and $y_n \in \Omega_{ji}$. If $x \in \Omega_{ij}$, then it is easy to see that $y \in \Omega_{ji}$ and that $x \sim y$, since $x_n \sim y_n$ in arbitrarily small neighborhoods of x and y. Thus we need to show that either $x \in \Omega_{ij}$ or $y \in \Omega_{ji}$. Assume for a contradiction, that $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$. Choose open balls V_x and V_y centered at x and y so that no point in $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by ϕ_{ji}. But this implies that no point in V_x is equivalent to some point in V_y. This contradicts the fact that $x_n \to x$ and $y_n \to y$, as the sequence (x_n, y_n) must eventually be in the neighborhoods V_x and V_y. Therefore $M_{\mathcal{G}}$ is Hausdorff.

Finally, for every $i \in I$, let $\mathrm{in}_i : \Omega_i \to \coprod_{i \in I} \Omega_i$ be the natural injection and let

$$\tau_i := \pi \circ \mathrm{in}_i : \Omega_i \to M_{\mathcal{G}}.$$

Since we already noted that if $x \sim y$ and $x, y \in \Omega_i$, then $x = y$, we conclude that every τ_i is injective. If we let $U_i = \tau_i(\Omega_i)$ and $\phi_i = \tau_i^{-1}$, it is immediately verified that the (U_i, ϕ_i) are charts and this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$. □

Remark 27. Note that the condition $\phi_{ji}^{-1}(\Omega_{jk}) \subseteq \Omega_{ik}$ is needed in order for $\sim$ to be transitive. The picture below illustrates how things could go wrong:



10.5.1 Mobius Strip

Example 10.8. Let X be the set of all lines in $\mathbb{R}^2$. We want to give this set the structure of a C^∞-manifold.

Let U_v be the set of all nonvertical lines in $\mathbb{R}^2$. A nonvertical is of the form $\ell^v_{a,b} = \{(x,y) \in \mathbb{R}^2 \mid y = ax + b\}$. Each such line is uniquely determined by a point $(a,b) \in \mathbb{R}^2$. So we have bijection $\varphi_v : U_v \to \mathbb{R}^2$, given by $\ell^v_{a,b} \mapsto (a,b)$. We give U_v a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_v(U)$ is open in $\mathbb{R}^2$. This makes φ_v into a homeomorphism.

Next let U_h be the set of all nonhorizontal lines in $\mathbb{R}^2$. A nonhorizontal is of the form $\ell^h_{c,d} = \{(x,y) \in \mathbb{R}^2 \mid x = cy + d\}$. Each such line is uniquely determined by a point $(c,d) \in \mathbb{R}^2$. So we have bijection $\varphi_h : U_h \to \mathbb{R}^2$, given by $\ell^h_{c,d} \mapsto (c,d)$. We give U_h a topology using the bijection φ_v: a set $U \subset U_v$ is open if and only if $\varphi_h(U)$ is open in $\mathbb{R}^2$. This makes φ_h into a homeomorphism.

Now we have $U_v \cup U_h = X$. To get a topology on X, we glue the topologies from U_v and U_h: a set $U \subset X$ is open if and only if $U \cap U_h$ is open in U_h and $U \cap U_v$ is open in U_v. Let's calculate the transition maps φ_{vh} and

φ_{hv}. We have

$$\begin{aligned}\varphi_{vh}(c,d) &= \varphi_v \circ \varphi_h^{-1}(c,d)\\ &= \varphi_v\left(\ell^h_{c,d}\right)\\ &= \varphi_v\left(\ell^v_{\frac{1}{c},-\frac{d}{c}}\right)\\ &= \left(\frac{1}{c},-\frac{d}{c}\right),\end{aligned}$$

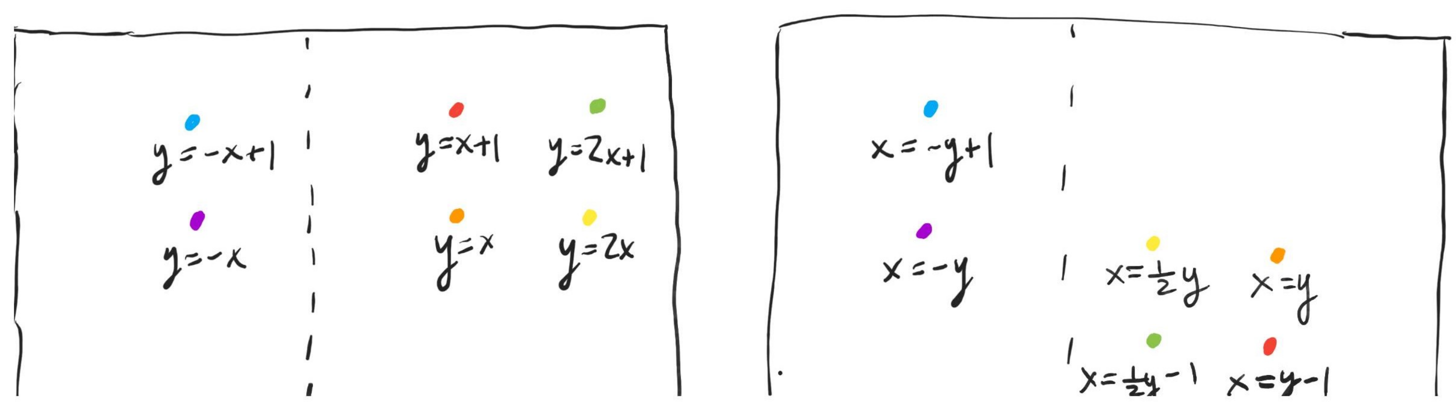
which is C^∞ whenever $c \neq 0$. Similarly,

$$\begin{aligned}\varphi_{hv}(a,b) &= \varphi_h \circ \varphi_v^{-1}(a,b)\\ &= \varphi_h\left(\ell^v_{a,b}\right)\\ &= \varphi_h\left(\ell^h_{\frac{1}{a},-\frac{b}{a}}\right)\\ &= \left(\frac{1}{a},-\frac{b}{a}\right),\end{aligned}$$

which is C^∞ whenever $a \neq 0$. Altogether, our gluing data consists of

$$\Omega_1 = \Omega_2 = \mathbb{R}^2, \qquad \Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}, \qquad \phi_{12} : (a,b) \mapsto \left(\frac{1}{a},-\frac{b}{a}\right).$$

This manifold is called the **Mobius strip**. We can visualize it as below:



Remark 28. We can describe this manifold in another way as follows: let G be the group given by

$$G := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a,b \in \mathbb{R} \right\}.$$

The group G has a natural open subgroup

$$\mathrm{Aff}(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a,b \in \mathbb{R} \text{ and } a \neq 0 \right\}.$$

Clearly G can be identified with $\Omega_1 = \Omega_2 = \mathbb{R}^2$ and $\mathrm{Aff}(\mathbb{R})$ can be identified with $\Omega_{12} = \Omega_{21} = \{(a,b) \in \mathbb{R}^2 \mid a \neq 0\}$. Using these identifications, the transition map ϕ_{12} is identified with the inverse map! Indeed, the inverse of $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1/a & -b/a \\ 0 & 1 \end{smallmatrix}\right)$.

Given a set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, it is natural to consider the collection of manifolds, M, parametrized by maps, $\theta_i : \Omega_i \to M$, whose domains are the Ω_i's and whose transition functions are given by the ϕ_{ji}'s, that is, such that

$$\phi_{ji} = \theta_j^{-1} \circ \theta_i.$$

We will say that such manifolds are **induced** by the set of gluing data $\mathcal{G}$.

The parametrization maps τ_i satisfy the property: $\tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset$ if and only if $(i,j) \in K$ and if so,

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \tau_i(\Omega_{ij}) = \tau_j(\Omega_{ji}).$$

Furthermore, they also satisfy the consistency condition:

$$\tau_i = \tau_j \circ \phi_{ji},$$

for all $(i,j) \in K$. If M is a manifold induced by the set of gluing data $\mathcal{G}$, then because the θ_i's are injective and $\phi_{ji} = \theta_j^{-1} \circ \theta_i$, the two properties stated above for the τ_i's also hold for the θ_i's. We will see that the manifold

$M_{\mathcal{G}}$ is a "universal" manifold induced by $\mathcal{G}$ in the sense that every manifold induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by some C^k map.

Interestingly, it is possible to characterize when two manifolds induced by the same set of gluing data are isomorphic in terms of a condition on their transition functions.

Proposition 10.4. *Given any set of gluing data, $\mathcal{G} = ((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\phi_{ji})_{(i,j)\in K})$, for any two manifolds M and M' induced by $\mathcal{G}$ given by families of parametrizations $(\Omega_i, \theta_i)_{i\in I}$ and $(\Omega_i, \theta'_i)_{i\in I}$, respectively, if $f : M \to M'$ is a C^k isomorphism, then there are C^k bijections, $\rho_i : W_{ij} \to W'_{ij}$, for some open subsets $W_{ij}, W'_{ij} \subseteq \Omega_i$, such that*

$$\phi'_{ji}(x) = \rho_j \circ \phi_{ji} \circ \rho_i^{-1}(x),$$

for all $x \in W_{ij}$ with $\phi_{ji} = \theta_j^{-1} \circ \theta_i$ and $\phi'_{ji} = \theta'^{-1}_j \circ \theta'_i$. Furthermore, $\rho_i = (\theta'^{-1}_i \circ f \circ \theta_i) \mid_{W_{ij}}$ and if $\theta'^{-1}_i \circ f \circ \theta_i$ is a bijection from Ω_i to itself and $\theta'^{-1}_i \circ f \circ \theta_i(\Omega_{ij}) = \Omega_{ij}$ for all i, j, then $W_{ij} = W'_{ij} = \Omega_i$.

11 Ringed Spaces

Definition 11.1. An R-**ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and where $\mathcal{O}_X$ is a sheaf of commutative R-algebras on X. The sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of $(X, \mathcal{O}_X)$. A **locally R-ringed space** is an R-ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$. We then denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$ and by $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field.

11.1 From C^p-Structures to Maximal C^p-Atlases

Let $\mathcal{O}$ be a C^p-structure on X. Let $\mathcal{A}$ be the set of all pairs (ϕ, U) where $U \subseteq X$ is a non-empty open set and $\phi : (U, \mathcal{O} \mid_U) \to \mathbb{R}^n$ is a C^p-isomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$ (with $\mathbb{R}^n$ given its usual C^p-structure). The collection $\mathcal{A}$ is a C^p-atlas because of two facts: a composite of C^p maps is C^p, and for maps between opens in finite-dimensional $\mathbb{R}$-vector spaces the "old" notion of C^p is the same as the "new" notion (in terms of structured $\mathbb{R}$-spaces). It is obvious that $\mathcal{A}$ is standardized. We want to prove that the standardized C^p-atlas $\mathcal{A}$ is maximal.

11.2 From Maximal C^p-Atlases to C^p-Structures

Let $\mathcal{A}$ be a maximal standarized C^p-atlas on X. For any non-empty open set $U_0 \subseteq X$, we define $\mathcal{O}(U_0)$ to be the set of functions $f : U_0 \to \mathbb{R}$ such that for all $(U, \phi) \in \mathcal{A}$, the composite map

$$f \circ \phi^{-1} : \phi(U \cap U_0) \to \mathbb{R}$$

is a C^p function on the open subset $\phi(U \cap U_0)$ in the Euclidean space $\mathbb{R}^n$ that is the target of ϕ. Also define $\mathcal{O}(\emptyset) = \{0\}$.

Lemma 11.1. *The correspondence $U_0 \mapsto \mathcal{O}(U_0)$ is an $\mathbb{R}$-space structure on X. For any $(U, \phi) \in \mathcal{A}$ and open $U_0 \subseteq U$, $\mathcal{O}(U_0)$ is the set of $f : U_0 \to \mathbb{R}$ such that $f \circ \phi^{-1} : \phi(U_0) \to \mathbb{R}$ is a C^p function on the open domain $\phi(U_0)$ in a Euclidean space.*

Proof. The usual notion of C^p function on an open set in a Euclidean space is preserved under restirction to smaller opens and can be checked by working on an open covering. Thus, the first claim in the lemma follows easily from the definition of $\mathcal{O}$. $\square$

12 deRham Cohomology

Suppose $F(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a smooth vector field representing a force on an open subset U of $\mathbb{R}^2$, and C is a parametrized curve $c(t) = (x(t), y(t))$ in U from a point p to a point q with $a \le t \le b$. Then the work done by the force in moving a particle from p to q along C is given by the line integral $\int_C Pdx + Qdy$.

Such a line integral is easy to compute if the vector field F is the gradient of a scalar function $f(x, y)$:

$$F = \text{grad} f = \langle \partial_x f, \partial_y f\rangle.$$

By Stoke's theorem, the line integral is simply

$$\int_C \partial_x f dx + \partial_y f dy = \int_C df = f(q) - f(p).$$

A necessary condition for the vector field $F = \langle P, Q \rangle$ to be a gradient is that

$$P_y = \partial_y \partial_x f = \partial_x \partial_y f = Q_x.$$

The question is now the following: if $P_y - Q_x = 0$, is the vector field $F = \langle P, Q \rangle$ on U the gradient of some scalar function $f(x, y)$ on U? In terms of differential forms, the question becomes the following: if the 1-form $\omega = Pdx + Qdy$ is closed on U, is it exact? The answer to this question is sometimes yes and sometimes no, depending on the topology of U.

12.1 de Rham Complex

Let M be a manifold and let R denote the ring $\Omega^0(M) := C^\infty(M)$. Then we have the following cochain complex over R

$$(\Omega(M), d) := \qquad 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \tag{10}$$

where d denotes the exterior derivative. We denote $H_{\mathrm{dR}}(M)$ to be the cohomology of $(\Omega(M), d)$ and call it the **deRham cohomology** of M. We denote by $Z(M)$ to be the cycles of $(\Omega(M), d)$ and $B(M)$ to be the boundaries of $(\Omega(M), d)$.

Proposition 12.1. *If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered- r-tuple of real numbers, each real number representing a constant function on a connected component of M.*

Proof. Since there are no nonzero exact 0-forms,

$$H^0(M) = Z^0(M).$$

Suppose f is a closed 0-form on M, i.e. f is a C^∞ function on M such that $df = 0$. On any chart $(U, x_1, \dots, x_n)$, we have

$$df = \sum_{\lambda=1}^{n} (\partial_{x_\lambda} f) dx_\lambda.$$

Thus $df = 0$ on U if and only if all the partial derivatives $\partial_{x_\lambda} f$ vanish identically on U. This in turn is equivalent to f being locally constant on U. Hence, the closed 0-forms on M are precisely the locally constant functions on M. Such a function must be constant on each connected component on M. If M has r connected components, then a locally constant function on M can be specified by an ordered set of r real numbers. Thus, $Z^0(M) = \mathbb{R}^r$. □

Proposition 12.2. *On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.*

Proof. At any point $p \in M$, then tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, the space of alternating k-linear functions on T_pM. If $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form on M is the zero form. □

12.1.1 Examples of de Rham Cohomology

Example 12.1. (De Rham cohomology of the real line) Since the real line $\mathbb{R}^1$ is connected, we have

$$H^0(\mathbb{R}^1) = \mathbb{R}.$$

For dimensional reasons, there are no nonzero 2-forms on $\mathbb{R}^1$. This implies that every 1-form on $\mathbb{R}^1$ is closed. A 1-form $f(x)dx$ on $\mathbb{R}^1$ is exact if and only if three is a C^∞ function $g(x)$ on $\mathbb{R}^1$ such that

$$f(x)dx = dg = g'(x)dx,$$

where $g'(x)$ is the calculus derivative of g with respect to x. Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example

$$g(x) = \int_0^x f(t)dt.$$

This proves that every 1-form on $\mathbb{R}^1$ is exact. Therefore, $H^1(\mathbb{R}^1) = 0$.

Example 12.2. (De Rham cohomology of the circle) Let S^1 be the unit circle in the xy-plane. Since S^1 is connected, we have $H^0(S^1) = \mathbb{R}$, and since S^1 is one-dimensional, we have $H^k(S^1) = 0$ for all $k \geq 2$. It remains to compute $H^1(S^1)$.

Let $h : \mathbb{R} \to S^1$ be given by $h(t) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$ and let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restricting the domain of h to $[0, 2\pi]$ gives a parametrization $F := h \circ i : [0, 2\pi] \to S^1$ of the circle. A nowhere-vanishing 1-form on S^1 is given by $\omega = -ydx + xdy$. Note that

$$\begin{aligned} h^*\omega &= -\sin t d(\cos t) + \cos t d(\sin t) \\ &= (\sin^2 t + \cos^2 t)dt \\ &= dt. \end{aligned}$$

Thus

$$\begin{aligned} F^*\omega &= i^*h^*\omega \\ &= i^*dt \\ &= dt, \end{aligned}$$

and so

$$\begin{aligned} \int_{S^1} \omega &= \int_{F([0,2\pi])} \omega \\ &= \int_{[0,2\pi]} F^*\omega \\ &= \int_0^{2\pi} dt \\ &= 2\pi. \end{aligned}$$

Since the cirle has dimension 1, all 1-forms on S^1 are closed, so $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1-forms on S^1 defines a linear map

$$\varphi : Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}, \quad \varphi(\alpha) = \int_{S^1} \alpha.$$

Because $\varphi(\omega) = 2\pi \neq 0$, the linear map $\varphi : \Omega^1(S^1) \to \mathbb{R}$ is onto.

By Stokes's theorem, the exact 1-forms on S^1 are in $\mathrm{Ker}(\varphi)$. Conversely, we will show that all 1-forms in $\mathrm{Ker}(\varphi)$ are exact. Suppose $\alpha = f\omega$ is a smooth 1-form on S^1 such that $\varphi(\alpha) = 0$. Let $\overline{f} = h^*f = f \circ h \in \Omega^0(\mathbb{R})$. Then $\overline{f}$ is periodic of period 2π and

$$\begin{aligned} 0 &= \int_{S^1} \alpha \\ &= \int_{F([0,2\pi])} F^*\alpha \\ &= \int_{[0,2\pi]} (i^*h^*f)(t) \cdot F^*\omega \\ &= \int_0^{2\pi} \overline{f}(t)dt. \end{aligned}$$

12.2 The C^∞ Hairy Ball Theorem

Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ with $n > 0$. If n is odd then there exists a nowhere-vanishing smooth vector field on S^n. Indeed, if $n = 2k + 1$ the consider the vector field $\vec{v}$ on $\mathbb{R}^{n+1} = \mathbb{R}^{2k+2}$ given by

$$\vec{v} = (-x_2\partial_{x_1} + x_1\partial_{x_2}) + \cdots + (-x_{2k+2}\partial_{x_{2k+1}} + x_{2k+1}\partial_{x_{2k+2}}) = \sum_{j=0}^{k} (-x_{2j+2}\partial_{x_{2j+1}} + x_{2j+1}\partial_{x_{2j+2}}).$$

For any point $p \in S^n$ it is easy to see that $\vec{v}(p) \in T_p(\mathbb{R}^{2k+2})$ is perpendicular to the line spanned by $\sum_i x_i(p)\partial_{x_i}|_p$, so it lies in the hyperplane $T_p(S^n)$ orthogonal to this line. In other words, the smooth section $\vec{v}|_{S^n}$ of the pullback bundle $(T(\mathbb{R}^{n+1}))|_{S^n}$ over S^n takes values in the subbundle $T(S^n)$, which is to say that $\vec{v}|_{S^n}$ is a smooth vector field on the manifold S^n. This is a visibly nowhere-vanishing vector field.

The above construction does not work if n is even, so there arises the question of whether there exists a nowhere-vanishing smooth vector field on S^n for even n. The answer is negative, and is called the **hairy ball** theorem.

Theorem 12.1. *A smooth vector field on S^n must vanish somewhere if n is even.*

Proof. Let $\vec{v}$ be a smooth vector field on S^n, and assume that it is nowhere-vanishing. For each $p \in S^n$, let $\gamma_p : [0, \pi/\|\vec{v}(p)\|] \to S^n$ be the smooth parametric great circle (with constant speed) going from p to $-p$ with velocity vector $\gamma_p'(0) = \vec{v}(p) \neq 0$ at $t = 0$ (This would not make sense if $\vec{v}(p) = 0$). Working in the plane spanned by $p \in \mathbb{R}^{n+1}$ and $\vec{v}(p) \in T_p(\mathbb{R}^{n+1})$ in $\mathbb{R}^{n+1}$, we get the formula

$$\gamma_p(t) = \cos(t\|\vec{v}(p)\|)\, p + \sin(t\|\vec{v}(p))\, \frac{\vec{v}(p)}{\|\vec{v}(p)\|} \in S^n \subseteq \mathbb{R}^{n+1}.$$

(These algebraic formulas would not make sense if $\vec{v}$ vanishes somewhere on S^n). Consider the "flow" mapping

$$F : S^n \times [0,1] \to S^n,$$

defined by $(p,t) \mapsto \gamma_p(\pi t/\|\vec{v}(p)\|)$. The formula for $\gamma_p(t)$ makes it clear that F is a smooth map (and is continuous if $\vec{v}$ is merely continuous and nowhere-vanishing). Now obviously $F(p,0) = p$ for all $p \in S^n$ and $F(p,1) = -p$ for all $p \in S^n$. Hence, F defines a smooth homotopy from the identity map on S^n to the antipodal map $p \mapsto -p$ on S^n (and is a continuous homotopy if $\vec{v}$ is merely continuous and nowhere-vanishing). Thus, to prove the hairy ball theorem we just have to prove that if n is even then the identity and antipodal maps $S^n \to S^n$ are not smoothly homotopic to each other; likewise to get the continuous version we just need to prove that there is no continuous homotopy deforming one of these maps into the other.

To prove the *non-existence* of such a homotopy, we shall use the (smooth) homotopy invariance of deRham cohomology. Indeed, by this homotopy-invariance we get that under the existence of such a $\vec{v}$ the antipodal map $A : S^n \to S^n$ induces the identity map $A^* : \mathrm{H}^k_{\mathrm{dR}}(S^n) \to \mathrm{H}^k_{\mathrm{dR}}(S^n)$ on the kth deRham cohomology of S^n for all $k \geq 0$. Let us focus on the case $k = n$. To get a contradiction, we just have to prove that if n is even then A^* as a self-map of $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is *not* the identity map.

Consider the n-form on $\mathbb{R}^{n+1}$ defined by

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Clearly $d\omega = (n+1)dx_1 \wedge \cdots \wedge dx_{n+1}$, so for the unit ball $B^{n+1} \subseteq \mathbb{R}^{n+1}$ with its standard orientation we have

$$\int_{B^{n+1}} d\omega = (n+1)\mathrm{vol}(B^{n+1}) \neq 0.$$

By Stokes' theorem for B^{n+1}, if we let $\eta = \omega|_{S^n}$ and we give $S^n = \partial B^{n+1}$ the induced boundary orientation, then

$$\int_{S^n} \eta = \int_{B^{n+1}} d\omega \neq 0.$$

Hence, by Stokes' theorem for the boundaryless smooth compact oriented manifold S^n we conclude that the top-degree differential form η on S^n is not exact. That is, its deRham cohomology class $[\eta] \in H^n_{\mathrm{dR}}(S^n)$ is non-zero. (Note that ω is not closed as an n-form on $\mathbb{R}^{n+1}$, but its pullback η on S^n is necessarily closed on S^n purely for elementary reasons, as S^n is n-dimensional.)

By the existence of the smooth homotopy between A and the identity map, it follows that A^* on $H^n_{\mathrm{dR}}(S^n)$ is the identity map, so $[A^*(\eta)] = A^*([\eta])$ is equal to $[\eta]$. That is, the top-degree differential forms $A^*(\eta)$ and η on S^n differ by an exact form. But the antipodal map $A : S^n \to S^n$ is induced by the negation map $N : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, and by inspection of the definition of $\omega \in \Omega^n_{\mathbb{R}^{n+1}}(\mathbb{R}^{n+1})$ we have $N^*(\omega) = (-1)^{n+1}\omega$. Hence, pulling back this equality to the sphere gives $A^*(\eta) = (-1)^{n+1}\eta$ in $\Omega^n_{S^n}(S^n)$. Thus, in $H^n_{\mathrm{dR}}(S^n)$ we have

$$[\eta] = A^*([\eta]) = [A^*(\eta)] = [(-1)^{n+1}\eta] = (-1)^{n+1}[\eta].$$

If n is even we therefore have $[\eta] = -[\eta]$, so $[\eta] = 0$. But we have already seen via Stokes' theorem for the boundaryless manifold S^n and for the manifold with boundary B^{n+1} that $[\eta]$ is nonzero. This completes the proof. □

13 Exercises

13.1 $\mathrm{SL}_2(\mathbb{R})$

Let $\mathrm{SL}_2(\mathbb{R}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid ad - bc = 1 \right\} \subset \mathbb{R}^4$. Let $\gamma : [0,1] \to \mathrm{SL}_2(\mathbb{R})$ be a path in $\mathrm{SL}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

such that $\gamma(0) = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by differentiating the identity $a(t)d(t) - b(t)c(t) = 1$ and evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= \dot{a}(0)d(0) + a(0)\dot{d}(0) - \dot{b}(0)c(0) - b(0)\dot{c}(0) \\ &= \dot{a}(0) + \dot{d}(0). \end{aligned}$$

Or $\dot{a}(0) = -\dot{d}(0)$. In particular, this means that $\mathrm{Tr}\left(\dot{\gamma}(0)\right) = 0$.

Conversely, suppose we have a matrix A such that $\mathrm{Tr}(A) = 0$. Can we find a path γ in $\mathrm{SL}_2(\mathbb{R})$ such that $\dot{\gamma}(0) = A$? Indeed, we can. The matrix exponential works:

$$e^{tA} := I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots.$$

This is because

$$\begin{aligned} \frac{d}{dt}\left(e^{tA}\right) &= \frac{d}{dt}\left(I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \cdots\right) \\ &= \frac{d}{dt}(I) + \frac{d}{dt}(tA) + \frac{1}{2}\frac{d}{dt}(t^2A^2) + \frac{1}{6}\frac{d}{dt}(t^3A^3) + \cdots \\ &= A + tA^2 + \frac{1}{2}t^2A^3 + \cdots \end{aligned}$$

Thus, $\frac{d}{dt}\left(e^{tA}\right)_{|t=0} = A$. Also we have $e^{tA} \in \mathrm{SL}_2(\mathbb{R})$ since

$$\begin{aligned} \det(e^{tA}) &= e^{\mathrm{Tr}(tA)} \\ &= e^0 \\ &= 1. \end{aligned}$$

13.2 $\mathrm{SO}_2(\mathbb{R})$

Let $\mathrm{SO}_2(\mathbb{R}) := \left\{A \in \mathrm{SL}_2(\mathbb{R}) \mid AA^t = I\right\}$. Let $\gamma : [0,1] \to \mathrm{SO}_2(\mathbb{R})$ be a path in $\mathrm{SO}_2(\mathbb{R})$, given by

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}.$$

such that $\gamma(0) = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by differentiating the identity $I = \gamma(t)\gamma(t)^t$ and evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= \dot{\gamma}(0)\gamma(0)^t + \gamma(0)\dot{\gamma}(0)^t \\ &= \dot{\gamma}(0) + \dot{\gamma}(0)^t. \end{aligned}$$

In particular, this means that $\dot{\gamma}(0)$ is a skew-symmetric matrix.

13.3 Vector Field in $\mathbb{R}^3$

Let ω be a vector field in $\mathbb{R}^3$ given by $\omega := (0, y, 0) := y\partial_y$. Let's find a path γ in $\mathbb{R}^3$ such that $\dot{\gamma} = \omega(\gamma)$. A general path γ in $\mathbb{R}^3$ has the form

$$\gamma(t) := (a(t), b(t), c(t)) \qquad \text{and} \qquad \dot{\gamma}(t) = (\dot{a}(t), \dot{b}(t), \dot{c}(t)).$$

Therefore $\omega($So we need

$$\begin{aligned} \dot{a}(t) &= 0 \\ \dot{b}(t) &= b(t) \\ \dot{c}(t) &= 0. \end{aligned}$$

In particular, $\gamma(t) = (a(0), b(0)e^t, c(0))$ works.

13.4 Lie Groups

Definition 13.1. A **Lie group** is a C^∞ manifold G that is also a group such that the two group operations, multiplication

$$G \times G \to G, \qquad (a,b) \mapsto ab,$$

and inverse

$$G \to G, \qquad a \mapsto a^{-1},$$

are C^∞.

For $a \in G$, denote by $\ell_a : G \to G$, where $\ell_a(x) = ax$, the operation of **left multiplication by** a, and by $r_a : G \to G$, where $r_a(x) = xa$, the operation of **right multiplication** by a. We also call left and right multiplications **left** and **right translations**.

Actually smoothness of invsersion can be dropped from the definition of a Lie Group.

Theorem 13.1. *Let G be a C^∞ manifold and suppose it is equipped with a group structure such that the composition law $m : G \times G \to G$ is C^∞. Then the inversion $G \to G$ is C^∞.*

Proof. Consider the "shearing transformation"

$$\Sigma : G \times G \to G \times G,$$

defined by $\Sigma(g,h) = (g, gh)$. This is bijective since we are using a group law, and it is C^∞ since the composition law m is assumed to be C^∞. (Recall that if M, M', M'' are C^∞ manifolds, a map $M \to M' \times M''$ is C^∞ if and only if its component maps $M \to M'$ and $M \to M''$ are C^∞, due to the nature of product manifold structures.)

We claim that Σ is a diffeomorphism. Granting this,

$$G = \{e\} \times G \longrightarrow G \times G \xrightarrow{\Sigma^{-1}} G \times G$$

is C^∞, but explicitely this composite map is $g \mapsto (g, g^{-1})$, so its second component $g \mapsto g^{-1}$ is C^∞ as desired. Since Σ is a C^∞ bijection, the C^∞ property for its inverse is equivalent to Σ being a **local isomorphism** (i.e. each point in its source has an open neighborhood carried diffeomorphically onto an open neighborhood in the target). By the Inverse Function Theorem, this is equivalent to the isomorphism property for the tangent map

$$d\Sigma(g,h) : T_g(G) \oplus T_h(G) = T_{(g,h)}(G \times G) \to T_{(g,gh)}(G \times G) = T_g(G) \oplus T_{gh}(G)$$

for all $g, h \in G$.

We shall now use left and right translations to reduce this latter "linear" problem to the special case $g = h = e$, and in that special case we will be able to compute the tangent map explicitely and see the isomorphism property by inspection.

$\square$

Part III
Algebraic Geometry

Throughout these notes, let K be a field and let $\overline{K}$ be an algebraic closure of K Unless otherwise specified, we let n be a positive integer. In this case, we often write $\boldsymbol{x} = (x_1, \dots, x_n)$ to denote a point in K^n whenever context is clear. Similarly we often write $K[\boldsymbol{T}] = K[T_1, \dots, T_n]$ to denote a polynomial ring in the variables $\boldsymbol{T} = T_1, \dots, T_n$ with coefficients in K whenever context is clear.

14 Affine Algebraic Sets

In this section, we will define **affine algebraic sets**. Before we do this, we first introduce the following notation: Let $\mathcal{P}$ be a set of polynomials in $K[\boldsymbol{T}]$. We denote by $\mathrm{V}(\mathcal{P})$ to be the set of common zeros of the polynomials in $\mathcal{P}$:

$$\mathrm{V}(\mathcal{P}) = \{\boldsymbol{x} \in K^n \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in \mathcal{P}\}.$$

If $\mathcal{Q}$ is another set of polynomials in $K[\boldsymbol{T}]$ such that $\mathcal{P} \subseteq \mathcal{Q}$, then we have $\mathrm{V}(\mathcal{P}) \supseteq \mathrm{V}(\mathcal{Q})$. In other words, V is **inclusion-reversing**. Now let $\mathfrak{a}$ be the ideal generated by $\mathcal{P}$. Recall that $K[\boldsymbol{T}]$ is a Noetherian ring, and thus $\mathfrak{a}$ is finitely generated as an ideal, say $\mathfrak{a} = \langle f_1, \dots, f_m \rangle$. Observe that

$$\mathrm{V}(\mathfrak{a}) = \mathrm{V}(\mathcal{P}) = \mathrm{V}(f_1, \dots, f_m),$$

where we denote $\mathrm{V}(f_1, \ldots, f_m) = \mathrm{V}(\{f_1, \ldots, f_m\})$. Indeed, it suffices to show that $\mathrm{V}(\mathfrak{a}) \supseteq \mathrm{V}(f_1, \ldots, f_m)$ since the reverse inclusion follows from the fact that V is inclusion-reversing. Given $\boldsymbol{x} \in \mathrm{V}(f_1, \ldots, f_m)$, then $f_i(\boldsymbol{x}) = 0$ for all $1 \leq i \leq m$. This implies that

$$\begin{aligned}\left(\sum_{i=1}^m g_i f_i\right)(\boldsymbol{x}) &= \sum_{i=1}^m g_i(\boldsymbol{x}) f_i(\boldsymbol{x}) \\ &= \sum_{i=1}^m g_i(\boldsymbol{x}) \cdot 0 \\ &= 0\end{aligned}$$

for all $\sum_{i=1}^m g_i f_i \in \mathfrak{a}$. Thus we have $\mathrm{V}(\mathfrak{a}) \supseteq \mathrm{V}(f_1, \ldots, f_m)$.

14.0.1 Maximal ideals defined by points

Let $\boldsymbol{x} \in K^n$ and let $\mathrm{ev}_{\boldsymbol{x}} \colon K[\boldsymbol{T}] \to K$ be the unique K-algebra homomorphism given by $\mathrm{ev}_{\boldsymbol{x}}(T_i) = x_i$ for all $i = 1, \ldots, n$. Denote by $\mathfrak{m}_{\boldsymbol{x}}$ to be the kernel of $\mathrm{ev}_{\boldsymbol{x}}$:

$$\mathfrak{m}_{\boldsymbol{x}} = \{f \in K[\boldsymbol{T}] \mid f(\boldsymbol{x}) = 0\}.$$

Then $\mathfrak{m}_{\boldsymbol{x}}$ is a maximal ideal of $K[\boldsymbol{T}]$ since $K[\boldsymbol{T}]/\mathfrak{m}_x \cong K$. Now let $\mathfrak{a}$ be another ideal of $K[\boldsymbol{T}]$. Then observe that $\boldsymbol{x} \in \mathrm{V}(\mathfrak{a})$ if and only if $\mathfrak{m}_x \supseteq \mathfrak{a}$. In particular, we can express $\mathrm{V}(\mathfrak{a})$ in terms of the maximal ideals $\mathfrak{m}_{\boldsymbol{x}}$ as follows:

$$\mathrm{V}(\mathfrak{a}) = \{\boldsymbol{x} \in K^n \mid \mathfrak{m}_{\boldsymbol{x}} \supseteq \mathfrak{a}\}$$

We will use this reformulation many times throughout this article.

14.1 The Zariski Topology

We are almost ready to define algebraic sets, but first we need to prove the following lemma:

Lemma 14.1. *The following relations hold:*

1. $\mathrm{V}(0) = K^n$ *and* $\mathrm{V}(1) = \emptyset$.
2. *For two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$, *we have*
$$\mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{V}(\mathfrak{a}\mathfrak{b}) = \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b}).$$
3. *For every family* $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ *of ideals, we have*
$$\mathrm{V}\left(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(\mathfrak{a}_\lambda).$$

Proof. 1. We have $\mathrm{V}(0) = K^n$ since $\mathfrak{m}_x \supseteq \langle 0 \rangle$ for all $x \in K^n$. Similarly we have $\mathrm{V}(1) = \emptyset$ since $\mathfrak{m}_x \not\supseteq \langle 1 \rangle$ for all $x \in K^n$.

2. Since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{b}$, it follows that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a} \cap \mathfrak{b}) \supseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$ from the inclusion-reversing property of V. It remains to show that $\mathrm{V}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$. To do this, we just need to show that $\mathfrak{m}_x \supseteq \mathfrak{a}\mathfrak{b}$ implies either $\mathfrak{m}_x \supseteq \mathfrak{a}$ or $\mathfrak{m}_x \supseteq \mathfrak{b}$ for all $x \in K^n$. But this follows from the fact that $\mathfrak{m}_x$ is a prime ideal.

3. That $\mathrm{V}\left(\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right)$ follows from the fact that $\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ is the ideal generated by $\bigcup_{\lambda \in \Lambda} \mathfrak{a}_\lambda$. That $\mathrm{V}\left(\sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda\right) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(\mathfrak{a}_\lambda)$ follows from the fact that $\mathfrak{m}_x \supseteq \sum_{\lambda \in \Lambda} \mathfrak{a}_\lambda$ if and only if $\mathfrak{m}_x \supseteq \mathfrak{a}_\lambda$ for all $\lambda \in \Lambda$ and for all $x \in K^n$.

□

Remark 29. It is very important to pay close attention to what is actually used in proofs. For example, in the proof of the second statement of this lemma, we only used the fact that $\mathfrak{m}_{\boldsymbol{x}}$ is a prime ideal (even though it is a maximal ideal). This gives us an idea for how we can generalize things. In particular, we will be replacing maximal ideals of the form $\mathfrak{m}_{\boldsymbol{x}}$ with arbitrary prime ideals. Keep this in mind!

This lemma implies that there is a unique topology on K^n for which the closed subsets are exactly those of the form $\mathrm{V}(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal of $K[\boldsymbol{T}]$. We call this topology the **Zariski topology** and write $\mathbb{A}^n(K)$ to mean the set K^n equipped with the Zariski topology. We call $\mathbb{A}^n(K)$ an n-**dimensional affine space**. Closed subspaces of $\mathbb{A}^n(K)$ are called **affine algebraic sets**. In particular, note that singletons are closed since $\{\boldsymbol{x}\} = \mathrm{V}(\mathfrak{m}_{\boldsymbol{x}})$.

14.2 Hilbert's Nullstellensatz

The connection between affine algebraic sets and commutative algebra is established by Hilbert's Nullstellensatz:

Theorem 14.2. *(Hilbert's Nullstellensatz) Let A be a finitely generated K-algebra. Then A is **Jacobson**, that is, for every prime ideal $\mathfrak{p}$ of A, we have*

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \supset \mathfrak{p} \\ \mathfrak{m} \text{ is maximal}}} \mathfrak{m}.$$

Moroever, if $\mathfrak{m}$ is a maximal ideal of A, then the field extension $K \subseteq A/\mathfrak{m}$ is finite.

We shall not prove this result here since a proof is best left for a course in Commutative Algebra (see my Algebra notes for such a proof). Instead, we will focus on the consequences of this theorem in regards to algebraic geometry. First let us consider some consequences when working over the algebraically closed field $\overline{K}$:

Proposition 14.1. *Let $\mathfrak{m}$ be a maximal ideal of $\overline{K}[\boldsymbol{T}]$. Then there exists an $\boldsymbol{x} \in \mathbb{A}^n(\overline{K})$ such that $\mathfrak{m} = \mathfrak{m}_x$.*

Proof. Observe that $\overline{K}[\boldsymbol{T}]$ is a finitely generated $\overline{K}$-algebra, thus from the Nullstellenatz, we see that $\overline{K} \hookrightarrow \overline{K}[\boldsymbol{T}]/\mathfrak{m}$ is a finite extension of fields; hence $\overline{K}[\boldsymbol{T}]/\mathfrak{m} \cong \overline{K}$ since $\overline{K}$ is algebraically closed. Now let x_i be the image of T_i by the homomorphism $\overline{K}[\boldsymbol{T}] \to \overline{K}[\boldsymbol{T}]/\mathfrak{m} \cong \overline{K}$. Then $\mathfrak{m}$ is a maximal ideal which contains the maximal ideal $\mathfrak{m}_{\boldsymbol{x}} = \langle T_1 - x_1, \dots, T_n - x_n \rangle$. Therefore both are equal. □

Thus the proposition tells us that the set of all points $\boldsymbol{x}$ in $\overline{K}^n$ are in one-to-one correspondence with the set of all maximal ideals $\mathfrak{m}$ of $\overline{K}[\boldsymbol{T}]$. Note that we really do need $\overline{K}$ to be algebraically closed for this to hold. For instance, $\langle T^2 + 1 \rangle$ is a maximal ideal of $\mathbb{R}[T]$ since $\mathbb{R}[T]/\langle T^2+1 \rangle \cong \mathbb{C}$, however $\langle T^2 + 1 \rangle$ does not come from a point in $\mathbb{R}$ since $T^2 + 1$ doesn't even vanish on $\mathbb{R}$. Thus there are more maximal ideals in $\mathbb{R}[T]$ than just the ones which correspond to points in $\mathbb{R}$ (there are more maximals ideals in $\mathbb{R}[T]$ than just those of the form $\mathfrak{m}_x = \langle T - x \rangle$ where $x \in \mathbb{R}$). On the other hand, the Nullstellensatz guarentees that for all maximal ideals $\mathfrak{m}$ of $\mathbb{R}[T]$, we will have either $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{C}$ or $\mathbb{R}[T]/\mathfrak{m} \cong \mathbb{R}$. In the second case, we will have $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in \mathbb{R}$, and in the first case, we will have $\mathfrak{m} = \mathfrak{m}_{z,\overline{z}} = \langle (T-z)(T-\overline{z}) \rangle$ where z is a complex number in the upper-half plane ($\mathrm{Im}(z) > 0$).

14.3 The Correspondence Between Radical Ideals and Affine Algebraic Sets

Now we shall focus on the consequences of Hilbert's Nullstellensatz when working over K (not necessarily algebraically closed). To do this, we introduce the following notation: let Z be a subset of $\mathbb{A}^n(K)$. We denote by $\mathrm{I}(Z)$ to be the set of all functions which vanish on Z:

$$\mathrm{I}(Z) = \{ f \in K[\boldsymbol{T}] \mid f(\boldsymbol{x}) = 0 \text{ for all } x \in Z \}.$$

It is easy to see that $\mathrm{I}(Z)$ is an ideal of $K[\boldsymbol{T}]$, thus it is also called the ideal of functions which vanish on Z. Furthemore, since $f(\boldsymbol{x}) = 0$ if and only if $f \in \mathfrak{m}_{\boldsymbol{x}}$, we have

$$\mathrm{I}(Z) = \bigcap_{x \in Z} \mathfrak{m}_x.$$

Now let A be a finitely generated K-algebra and let $\mathfrak{a}$ be an ideal of A. Then we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{a} \subseteq \mathfrak{p} \subset A \\ \mathfrak{p} \text{ prime ideal}}} \mathfrak{p} = \bigcap_{\substack{\mathfrak{a} \subseteq \mathfrak{m} \subset A \\ \mathfrak{m} \text{ maximal ideal}}} \mathfrak{m}.$$

Indeed, the first equality holds in arbitrary commutative rings and the second equality follows from Hilbert's Nullstellensatz. We shall use this fact to prove the following proposition:

Proposition 14.2. *Let $\mathfrak{a}$ be an ideal of $\overline{K}[\boldsymbol{T}]$ and let $Z \subseteq \mathbb{A}^n(\overline{K})$ be a subset. Then*

1. $\mathrm{IV}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$
2. $\mathrm{VI}(Z) = \overline{Z}$ *where $\overline{Z}$ is the closure of Z in $\mathbb{A}^n(\overline{K})$.*

Proof. 1 We have

$$\begin{aligned} \mathrm{IV}(\mathfrak{a}) &= \bigcap_{\boldsymbol{x} \in \mathrm{V}(\mathfrak{a})} \mathfrak{m}_{\boldsymbol{x}} \\ &= \bigcap_{\substack{\mathfrak{m} \supseteq \mathfrak{a} \\ \text{maximal ideal}}} \mathfrak{m} \\ &= \sqrt{\mathfrak{a}}. \end{aligned}$$

2. This is a simple assertion for which we do not need the Nullstellensatz. On the one hand we have $Z \subseteq \mathrm{VI}(Z)$ and $\mathrm{VI}(Z)$ is closed. This shows $\mathrm{VI}(Z) \supseteq \overline{Z}$. On the other hand let $V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$ be a closed subset that contains Z. Then we have $f(x) = 0$ for all $x \in Z$ and $f \in \mathfrak{a}$. This shows $\mathfrak{a} \subseteq \mathrm{I}(Z)$ and hence $\mathrm{VI}(Z) \subseteq \mathrm{V}(\mathfrak{a})$.

□

The proposition implies:

Corollary 2. *The maps*

$$\{\text{radical ideals } \mathfrak{a} \text{ of } k[T_1,\ldots,T_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\text{closed subsets } Z \text{ of } A\}$$

are mutually inverse bijections, whose restritions define a bijection

$$\{\text{maximal ideals of } k[T_1,\ldots,T_n]\} \longleftrightarrow \{\text{points of } \mathbb{A}^n(k)\}.$$

14.4 Morphisms of Affine Algebraic Sets

Having defined affine algebraic sets, we now wish to define morphisms between them.

Definition 14.1. Let X be affine algebraic subset of $\mathbb{A}^m(K)$, let Y be an affine algebraic subset of $\mathbb{A}^n(K)$, and let $f\colon X \to Y$ be a function. We say f is a **morphism** of affine algebraic sets if there exists polynomials $f_1,\ldots,f_n$ in $K[T_1,\ldots,T_m]$ such that $f(x) = (f_1(x),\ldots,f_n(x))$ for all $x \in X$. In this case, we say that the n-tuple of polynomials

$$(f_1,\ldots,f_n) \in (K[T_1,\ldots,T_m])^n$$

represents f. The f_i are called the **components** of this representation. The set of all morphisms from X to Y is denoted $\mathrm{Hom}(X,Y)$. We say $f\colon X \to Y$ is an **isomorphism** if there exists a morphim $g\colon Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Remark 30. To say that f is a morphism from $X \subseteq \mathbb{A}^m_K$ to $Y \subseteq \mathbb{A}^n_K$ represented by $(f_1,\ldots,f_n)$ means that $(f_1(x),\ldots,f_n(x))$ must satisfy the defining equations of Y for all points $x \in X$.

14.4.1 Examples of morphisms

Example 14.1. Let $X = \mathbb{A}^1(K)$, let $Y = \mathrm{V}(T_2'^2 - T_1')$, and let $f\colon X \to Y$ be defined by $f(x) = (x^2, x)$ for all $x \in X$. Then f is a morphism since it is represented by the polynomials $f_1 = T$ and $f_2 = T^2$ and since it lands in Y: for all $x \in X$ we have $f(x) \in Y$ since $(x)^2 - x^2 = 0$. Now define $g\colon Y \to X$ by $g(y) = g(y_1,y_2) = y_2$ for all $y = (y_1,y_2)$ in Y. Then g is a morphism since it is represented by the polynomial $g = T_2'$ and since it clearly lands in X. Moreover, observe that for all $y = (y_1,y_2)$ in Y, we have

$$\begin{aligned}(f \circ g)(y) &= f(g(y)) \\ &= f(y_2) \\ &= (y_2^2, y_2) \\ &= (y_1, y_2) \\ &= y \\ &= 1_Y(y).\end{aligned}$$

where we used the fact that $y \in Y$ to get from the third line to the fourth line. Similarly, for all $x \in X$, we have

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2, x) \\ &= x \\ &= 1_X(x).\end{aligned}$$

It follows that $f\colon X \to Y$ is an isomorphism of affine algebraic sets.

Example 14.2. Let $X = \mathrm{V}(T_2 - T_1^2, T_3 - T_1^3) = \mathrm{V}(p_1,p_2)$, let $Y = \mathrm{V}(T_2' - T_1' - T_1'^2) = \mathrm{V}(q)$, and let $f\colon X \to Y$ be defined by

$$f(x) = (x_1x_2, x_1^2x_2^2 + x_3) = (y_1, y_2)$$

for all $x = (x_1, x_2, x_3)$ in X. Then f is represented by the polynomials $f_1 = T_1T_2$ and $f_2 = T_1^2T_2^2 + T_3$, thus to see if f is a morphism, we only need to check that f lands in Y. To see this, we must check that the polynomial $q = T_2' - T_1' - T_1'^2$ vanishes at $y = (y_1, y_2)$: we have

$$\begin{aligned}
q(y) &= y_2 - y_1 - y_1^2 \\
&= x_1^2x_2^2 + x_3 - x_1x_2 - (x_1x_2)^2 \\
&= x_3 - x_1x_2 \\
&= x_3 - x_1^3 \\
&= 0
\end{aligned}$$

where we used $p_1(x) = 0 = p_2(x)$ to get from the third line to the fifth line. Thus f is in fact a morphism of affine algebraic sets. Note that the morphism f induces a homomorphism

$$f^* \colon K[T_1', T_2']/\langle T_2' - T_1' - T_1'^2\rangle \to K[T_1, T_2, T_3]/\langle T_2 - T_1^2, T_3 - T_1^3\rangle$$

of K-algebras, where f^* is the unique K-algebra homomorphism such that

$$\begin{aligned}
f^*(T_1') &= T_1T_2 \\
f^*(T_2') &= T_1^2T_2^2 + T_3.
\end{aligned}$$

Example 14.3. The map $\mathbb{A}^1(K) \to V(T_2^2 - T_1^2(T_1 + 1))$, given by $x \mapsto (x^2 - 1, x(x^2 - 1))$, is a morphism of affine algebraic sets. For $\mathrm{char}(K) \neq 2$, it is not bijective: 1 and -1 are both mapped to the origin $(0, 0)$. In $\mathrm{char}(K) = 2$, it is bijective but not an isomorphism.

Example 14.4. Let $X = V(1 - T_1T_2)$, let $Y = \mathbb{A}^1(K)$, and let $f \colon X \to Y$ be defined by $f(x) = x_1$ for all $x = (x_1, x_2)$ in X. Then $f(X) = \mathbb{A}^1(K)\backslash\{0\}$ is not an algebraic set. This shows that the image of an algebraic set is not necessarily an algebraic set.

The notion of an affine algebraic set is still not satisfactory. We list three problems:

- Open subsets of affine algebraic sets do not carry the structure of an affine algebraic set in a natural way. In particular, we cannot glue affine algebraic sets along open subsets (although this is a "natural operation" for geometric objects.
- Intersections of affine algebraic sets in $\mathbb{A}^n(k)$ are closed and hence again affine algebraic sets. But we cannot distinguish between $\mathrm{V}(X) \cap \mathrm{V}(Y) \subset \mathbb{A}^2(k)$ and $\mathrm{V}(Y) \cap \mathrm{V}(X^2 - Y) \subset \mathbb{A}^2(k)$ although the geometric situation seems to be different.
- Affine algebraic sets seem not to help in studying solutions of polynomial equations in more general rings than algebraically closed fields.

14.4.2 Morphisms are continuous with respect to the Zariski topology

We now want to show that morphisms are continuous with respect to the Zariski topology. We first need a lemma:

Lemma 14.3. *Let Y be an affine algebraic subset of $\mathbb{A}^n(K)$ and let $f \colon \mathbb{A}^m(K) \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Suppose $Y = \mathrm{V}(p_1, \dots, p_r)$ and let Z be a closed subset of Y. Then Z has the form

$$Z = \mathrm{V}(p_1, \dots, p_r) \cap \mathrm{V}(q_1, \dots, q_s) = \mathrm{V}(p_1, \dots, p_r, q_1, \dots, q_s).$$

where $\mathrm{V}(q_1, \dots, q_s)$ is another closed subset of $\mathbb{A}^n(K)$. In particular,

$$f^{-1}(Z) = \mathrm{V}(p_1 \circ f, \dots, p_r \circ f, q_1 \circ f, \dots, q_s \circ f)$$

is a closed subset of $\mathbb{A}^m(K)$. □

Proposition 14.3. *Let X be an affine algebraic subset of $\mathbb{A}^m(K)$, let Y be an affine algebraic subset of $\mathbb{A}^n(K)$, and let $f \colon X \to Y$ be a morphism. Then f is continuous with respect to the Zariski topology.*

Proof. Lift f to a morphism $\widetilde{f} \colon \mathbb{A}^m(K) \to Y$ such that $\widetilde{f}|_X = f$ (choosing a lift of f is equivalent to choosing a polynomial representation $(f_1, \dots, f_n)$ of f). By Lemma (14.3), $\widetilde{f}$ is continuous. Therefore its restriction $\widetilde{f}|_X = f$ must be continuous also. □

Example 14.5. Let $X \subseteq \mathbb{A}^n(k)$ be an affine algebraic sets and let $\pi_i : \mathbb{A}^n(k) \to \mathbb{A}^1(k)$ be the projection to the ith coordinate map (i.e. $\pi_i(y_1, \ldots, y_i, \ldots, y_n) = y_i$). Then $\pi_i|_X$ is continuous with respect to the Zariski topology. Let us show this directly: let $\{z_1, \ldots, z_n\}$ be a closed subset of $\mathbb{A}^1(k)$ where the z_j are distinct points in $\mathbb{A}^1(k)$ (every closed subset in $\mathbb{A}^1(k)$ is just a finite set of points). Then

$$\pi_i|_X^{-1}(\{z_1, \ldots, z_n\}) = X \cap \left(\bigcap_{j=1}^{n} V(\pi_i - z_j) \right).$$

Thus the inverse image a closed subset in $\mathbb{A}^1(k)$ is a closed subset in X.

14.4.3 Maps which are continuous with respect to the Zariski topology are not necessarily morphisms

A continuous map with respect to the Zariski topology does not have to be a morphism. Indeed, consider the complex conjugation map $\bar{\cdot} : \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$. This map is continuous with respect to the Zariski topology. To see why, let $V(p_1, \ldots, p_r)$ be a closed subset of $\mathbb{A}^1(\mathbb{C})$. Then the inverse image of $V(p_1, \ldots, p_r)$ under $\bar{\cdot}$ is the closed subset $V(\overline{p}_1, \ldots, \overline{p}_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $\overline{p}_i = \sum_{j=1}^{n_i} \overline{a}_{n_j} z^{n_j}$. On the other hand, $\bar{\cdot}$ does not have a polynomial representation: the only root of $\bar{\cdot}$ is $z = 0$, but $\bar{\cdot} \neq T^m$ for any $m \in \mathbb{N}$.

More generally, if L/K galois extension with galois group $G = \mathrm{Gal}(L/K)$. Then for all $g \in G$ the map $g\cdot : \mathbb{A}^1(L) \to \mathbb{A}^1(L)$, given by $x \mapsto g \cdot x$, is Zariski continuous because the inverse image of a closed subset $V(p_1, \ldots, p_r)$ of $\mathbb{A}^1(L)$ under $g\cdot$ is the closed subset $V(g \cdot p_1, \ldots, g \cdot p_r)$, where if $p_i = \sum_{j=1}^{n_i} a_{n_j} z^{n_j}$, then $g \cdot p_i = \sum_{j=1}^{n_i} (g \cdot a_{n_j}) z^{n_j}$. On the other hand, $g\cdot$ does not have a polynomial representation: the only root of $g\cdot$ is $z = 0$, but $g\cdot \neq T^m$ for any $m \in \mathbb{N}$. Note that $g\cdot$ gives rise to a K-algebra homomorphism $\Gamma(g\cdot) : L[T] \to L[T]$ (and not an L-algebra homomorphism).

Remark 31. One may wonder why we restrict our morphisms between affine algebraic sets in the first place. Why do we not consider $\mathrm{Hom}(X, Y)$ to be the set of all Zariski-continuous maps? The point is that the category of affine algebraic sets are naturally thought of as being objects in the category of locally ringed spaces (we will define what these are later on) rather than just in the category of topological spaces.

14.5 Affine Algebraic Sets as Reduced Finitely-Generated K-Algebras

We make the following definitions.

Definition 14.2. Let $X \subseteq \mathbb{A}^n(K)$ be an affine algebraic set.

1. The **affine coordinate ring** of X, denoted $\Gamma(X)$, is the K-algebra

$$\Gamma(X) := K[\boldsymbol{T}]/\mathrm{I}(X).$$

Notice that $\mathrm{Hom}(X, \mathbb{A}^1(K))$ has the structure of a K-algebra in the natural way, where addition and multiplication are defined pointwise, and that $\Gamma(X) \cong \mathrm{Hom}(X, \mathbb{A}^1(K))$ as K-algebras. Thus $\mathrm{Hom}(X, \mathbb{A}^1(K))$ is an equivalent description of $\Gamma(X)$ which we shall often use.

2. Let $\boldsymbol{x} \in X$. We denote by $\mathfrak{m}_{X,\boldsymbol{x}}$ to be the maximal ideal of $\Gamma(X)$ given by

$$\mathfrak{m}_{X,\boldsymbol{x}} = \{f \in \Gamma(X) \mid f(\boldsymbol{x}) = 0\}.$$

Often we simplify our notation (when context is clear) and write $\mathfrak{m}_{\boldsymbol{x}}$ instead of $\mathfrak{m}_{X,\boldsymbol{x}}$.

3. Let $\mathfrak{a}$ be an ideal of $\Gamma(X)$. We denote by $\mathrm{V}_X(\mathfrak{a})$ to be the subset of X given by

$$\mathrm{V}_X(\mathfrak{a}) = \{\boldsymbol{x} \in X \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Let $\widetilde{\mathfrak{a}}$ be an ideal of $K[\boldsymbol{T}]$ which lifts the ideal $\mathfrak{a}$ with respect to the surjective map $\pi\colon K[\boldsymbol{T}] \to \Gamma(X)$. Then observe that $\mathrm{V}_X(\mathfrak{a}) = X \cap \mathrm{V}(\widetilde{\mathfrak{a}})$. In particular, the $\mathrm{V}_X(\mathfrak{a})$ (as $\mathfrak{a}$ ranges) are the closed sets in the subspace topology X in $\mathbb{A}^n(K)$. We again call this subspace topology the **Zariski topology** of X. Often it is clear from context that we are working in X, so we will often simplify our notation and write $\mathrm{V}(\mathfrak{a})$ instead of $\mathrm{V}_X(\mathfrak{a})$.

4. For each $f \in \Gamma(X)$, we denote by $\mathrm{D}_X(f)$ to be the subset of X given by

$$\mathrm{D}_X(f) = \{\boldsymbol{x} \in X \mid f(\boldsymbol{x}) \neq 0\}.$$

These are principal open subsets of X, which we call the **principal open subsets** of X. Again we often simplify our notation by writing $\mathrm{D}(f)$ instead of $\mathrm{D}_X(f)$.

Lemma 14.4. *Let $X \subseteq \mathbb{A}^n(K)$ be an affine algebraic set. The open sets $\mathrm{D}(f)$, for $f \in \Gamma(X)$, form a basis of the topology (i.e. finite intersections of principal open subsets are again principal open subsets and for every open subset $U \subseteq X$ there exist $f_i \in \Gamma(X)$ with $U = \bigcup_i D(f_i)$.*

Proof. Clearly we have $D(f) \cap D(g) = D(fg)$ for $f, g \in \Gamma(X)$. It remains to show that every open subset U is a union of principal open subsets. We write $U = X \backslash V(\mathfrak{a})$ for some ideal $\mathfrak{a}$. For generators $f_1, \ldots, f_n$ of this ideal we find $V(\mathfrak{a}) = \bigcap_{i=1}^n V(f_i)$, and hence $U = \bigcup_{i=1}^n D(f_i)$. □

Remark 32. Let $f : X \to Y$ be a morphism of affine algebraic sets. Then f is continuous with respect to the Zariski topology. Indeed, if $D(g)$ is a basic open set in Y, then $f^{-1}(D(g)) = D(f^*g)$.

Proposition 14.4. *Let X be an affine algebraic set. The affine coordinate ring $\Gamma(X)$ is a reduced finitely generated k-alebra. Moreover, X is irreducible if and only if $\Gamma(X)$ is an integral domain.*

Proof. As $\Gamma(X) = k[T_1, \ldots, T_n]/I(X)$, it is a finitely generated k-algebra. As $I(X) = \sqrt{I(X)}$, we find that $\Gamma(X)$ is reduced. Also, X is irreducible if and only if $I(X)$ is prime if and only if $\Gamma(X)$ is an integral domain. □

Proposition 14.5. *Let $f : X \to Y$ be a morphism between affine algebraic sets. Then*

1. *$f(X)$ is dense in Y if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is injective.*
2. *$f(X) \subset Y$ is a closed subvariety and $f : X \to f(X)$ is an isomorphism if and only if $f^* : \Gamma(Y) \to \Gamma(X)$ is surjective.*

Proof.

1. First assume that $f(X)$ is dense in Y. Suppose that $f^*h = f^*g$ where $g, h \in \Gamma(Y)$. Then for all $x \in X$, we have $h(f(x)) = g(f(x))$. Or in other words, we have $(h-g)(y) = 0$ for all $y \in f(X)$. Since $f(X)$ is dense in Y and $h - g$ is continuous, we must therefore have $(h-g)(y) = 0$ for all $y \in Y$. Thus, $h = g$, which shows that f^* is injective. Conversely, assume that f^* is injective. Suppose $f(X)$ is not dense in Y. Denote $Z := \overline{f(X)}$ and pick $y \in Y$ such that $y \notin Z$. Then $Z \subset Y$ implies $I(Z) \supset I(Y)$. Thus, we can find an $g \in I(Z)$ such that $g \notin I(Y)$. This means $g(z) = 0$ for all $z \in Z$ and there exists $y \in Y$ such that $g(y) \neq 0$. But f^* is injective, $f^*g = f^*0$ implies $g = 0$, which is a contradiction.

□

Remark 33. We say $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y.

14.5.1 Equivalence of Categories Between Affine Algebraic Sets and Reduced Finitely Generated k-Algebras

Let $f : X \to Y$ be a morphism of affine algebraic sets. The map

$$\Gamma(f) : \mathrm{Hom}(Y, \mathbb{A}^1(k)) \to \mathrm{Hom}(X, \mathbb{A}^1(k)),$$

given by $g \mapsto f^*g := g \circ f$, defines a homomorphism of k-algebras. We obtain a functor

$$\Gamma : (\text{affine algebraic sets})^{\text{opp}} \to (\text{reduced finitely generated } k\text{-algebras}).$$

Proposition 14.6. *The functor Γ induces an equivalence of categories. By restriction one obtains an equivalence of categories*

$$\Gamma : \textit{(irreducible affine algebraic sets)}^{opp} \to \textit{(integral finitely generated k-algebras)}.$$

Proof. A functor induces an equivalence of categories if and only if it is fully faithful and essentially surjective. We first show that Γ is fully faithful, i.e. that for affine algebraic sets $X \subseteq \mathbb{A}^m(k)$ and $Y \subseteq \mathbb{A}^n(k)$, the map $\Gamma : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\Gamma(Y), \Gamma(X))$ is bijective. We define an inverse map. If $\varphi : \Gamma(Y) \to \Gamma(X)$ is given, there exists a k-algebra homomorphism $\widetilde{\varphi}$ that makes the following diagram commutative

$$\begin{array}{ccc} k[T_1', \ldots, T_m'] & \xrightarrow{\widetilde{\varphi}} & k[T_1, \ldots, T_n] \\ \downarrow & & \downarrow \\ \Gamma(Y) & \xrightarrow{\varphi} & \Gamma(X) \end{array}$$

We define $f : X \to Y$ by

$$f(x) := (\widetilde{\varphi}(T_1')(x), \ldots, \widetilde{\varphi}(T_n')(x))$$

and obtain the desired inverse homomorphism.

It remains to show that the functor is essentially surjective, i.e. that for every reduced finitely generated k-algebra A there exists an affine algebraic set X such that $A \cong \Gamma(X)$. By hypothesis, A is isomorphic to $k[T_1, \ldots, T_n]/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $k[T_1, \ldots, T_n]$ with $\mathfrak{a} = \sqrt{\mathfrak{a}}$. If we set $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n(k)$, we have

$$\Gamma(X) = k[T_1, \ldots, T_n]/I(V(\mathfrak{a})) = k[T_1, \ldots, T_n]/\mathfrak{a}.$$

$\square$

Remark 34. Let $X \subseteq \mathbb{A}^m(K)$ and $Y \subseteq \mathbb{A}^n(k)$ be affine algebraic sets and let $f : X \to Y$ whose components are f_i for $i = 1, \ldots, m$. Write the affine coordinate rings of X and Y as $\Gamma(X) = K[T_1, \ldots, T_m]/I(X)$ and $\Gamma(Y) = K[T_1', \ldots, T_n']/I(Y)$. Then $\Gamma(f)(T_i) := T_i \circ f = f_i$ for all $i = 1, \ldots, m$. Indeed, for all points $x \in X$, we have

$$\begin{aligned} \Gamma(f)(T_i)(x) &= T_i(f(x)) \\ &= T_i(f_1(x), \ldots, f_i(x), \ldots, f_n(x)) \\ &= f_i(x). \end{aligned}$$

Using the bijective correspondence between points of affine algebraic sets X and maximal ideals of $\Gamma(X)$, we also have the following description of morphisms.

Proposition 14.7. *Let $f : X \to Y$ be a morphism of affine algebraic sets and let $\Gamma(f) : \Gamma(Y) \to \Gamma(X)$ be the corresponding homomorphism of the affine coordinate rings. Then $\Gamma(f)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$ for all $x \in X$.*

Proof. This follows from $g(f(x)) = \Gamma(f)(g)(x)$ for all $g \in \Gamma(Y) = \mathrm{Hom}(Y, \mathbb{A}^1(k))$. $\square$

14.6 Affine Algebraic Sets as Spaces with Functions

We will now define the notion of a **space with functions**. For us this will be the prototype of a "geometric object". It is a special case of a so-called ringed space on which the notion of a scheme will be based on.

Definition 14.3.

1. A **space with functions over** K is a topological space X together with a family $\mathcal{O}_X$ of K-subalgebras $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, K)$ for every open subset $U \subseteq X$ that satisfy the following properties:

   (a) If $U' \subseteq U \subseteq X$ are open and $f \in \mathcal{O}_X(U)$, then the restriction $f|_{U'} \in \mathrm{Map}(U', K)$ is an element of $\mathcal{O}_X(U')$.

   (b) Given an open covering $\{U_i\}_{i \in I}$ of an open subset U of X and elements $f_i \in \mathcal{O}_X(U_i)$ such that

   $$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

   for all $i, j \in I$, then there exists a unique function $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

2. A **morphism** $g : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of spaces with functions is a continuous map $g : X \to Y$ such that for all open subsets V of Y and functions $f \in \mathcal{O}_Y(V)$, the function $g^* f := f \circ g|_{g^{-1}(V)} : g^{-1}(V) \to K$ lies in $\mathcal{O}_X(g^{-1}(V))$.

Clearly spaces with functions over K form a category.

Definition 14.4. Let X be a space with functions and let U be an open subset of X. We denote by $(U, \mathcal{O}_{X|U})$ the space U with functions

$$\mathcal{O}_{X|U}(V) = \mathcal{O}_X(V)$$

for $V \subseteq U$ open.

14.6.1 The Space with Functions of an Irreducible Affine Algebraic Set

Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set. It is endowed with the Zariski topology and we want to define for every open subset $U \subseteq X$ a k-algebra of functions $\mathcal{O}_X(U)$ such that $(X, \mathcal{O}_X)$ is a space with functions.

As X is irreducible, the k-algebra $\Gamma(X)$ is a domain, and by definition all the sets $\mathcal{O}_X(U)$ will be k-subalgebras of its field of fractions.

Definition 14.5. The field of fractions $K(X) := \mathrm{Frac}(\Gamma(X))$ is called the **function field** of X.

If we consider $\Gamma(X)$ as the set of morphisms $X \to \mathbb{A}^1(k)$, elements of the function field f/g, where $f, g \in \Gamma(X)$ and $g \neq 0$, usually do not define functions on X because the denominator may have zeros on X, but certainly f/g defines a function $D(g) \to \mathbb{A}^1(k)$[1] We will use functions of this kind to make X into a space with functions.

Lemma 14.5. *Let X be an irreducible affine algebraic set and let f_1/g_1 and f_2/g_2 be elements of $K(X)$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$ if and only if there exists a non-empty open subset $U \subseteq D(g_1 g_2)$ with*

$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$

for all $x \in U$. Then $f_1/g_1 = f_2/g_2$ in $K(X)$.

Proof. First suppose $f_1/g_1 = f_2/g_2$ in $K(X)$. This means $f_1 g_2 = f_2 g_1$ in $\Gamma(X)$. In particular,

$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$

for all $x \in D(g_1 g_2)$. Conversely, let $U \subseteq D(g_1 g_2)$ be a non-empty open subset such that

$$\frac{f_1(x)}{g_1(x)} = \frac{f_2(x)}{g_2(x)}$$

for all $x \in U$. Then the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. □

Proof. We have $(f_1 g_2 - f_2 g_1)(x) = 0$ for all $x \in U$. Therefore the open subset U lies in the closed subset $V(f_1 g_2 - f_2 g_1)$. As U is dense in X, this implies $V(f_1 g_2 - f_2 g_1) = X$, and hence $f_1 g_2 = f_2 g_1$ because $\Gamma(X)$ is reduced. □

[1] It might be even defined on a bigger open subset of X as there exist representations of the fraction with different denominators.

Definition 14.6. Let X be an irreducible affine algebraic set and let $U \subseteq X$ be open. We denote by $\mathfrak{m}_x$ the maximal ideal of $\Gamma(X)$ corresponding to $x \in X$ and by $\Gamma(X)_{\mathfrak{m}_x}$ the localization of the affine coordinate ring with respect to $\mathfrak{m}_x$. We define

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \Gamma(X)_{\mathfrak{m}_x} \subset K(X).$$

The localization $\Gamma(X)_{\mathfrak{m}_x}$ can be described in this situation as the union

$$\Gamma(X)_{\mathfrak{m}_x} = \bigcup_{f \in \Gamma(X) \setminus \mathfrak{m}_x} \Gamma(X)_f \subset K(X).$$

Remark 35. Note that

$$\Gamma(X)_{\mathfrak{m}_x} = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \right\}.$$

Indeed, $g(x) \neq 0$ is equivalent to $g \notin \mathfrak{m}_x$. It may be tempting to think that

$$\mathcal{O}_X(U) = \left\{ \frac{f}{g} \mid f, g \in \Gamma(X) \text{ and } g(x) \neq 0 \text{ for all } x \in U \right\},$$

but this is not necessarily the case. For instance, let $X \subset \mathbb{A}^4$ be the variety defined by the equation $T_1T_4 = T_2T_3$. Then $T_1/T_2 \in \mathcal{O}_X(D(T_2))$ and $T_3/T_4 \in \mathcal{O}_X(D(T_4))$ and by the equation of X, these two functions coincide where they are both defined;

$$\frac{T_1}{T_2}\Big|_{\mathrm{D}(T_2T_4)} = \frac{T_3}{T_4}\Big|_{\mathrm{D}(T_2T_4)}$$

So this gives rise to a regular function on $\mathrm{D}(T_2) \cup \mathrm{D}(T_4)$, but there is no representation of this function as a quotient of two polynomials in $K[T_1, T_2, T_3, T_4]$ that works on all of $\mathrm{D}(T_2) \cup \mathrm{D}(T_4)$; we have to use different representations at different points. On the other hand, it is true that

$$\mathcal{O}_{\mathbb{A}^n(K)}(U) = \left\{ \frac{f}{g} \mid f, g \in K[\boldsymbol{T}] \text{ and } g(\boldsymbol{x}) \neq 0 \text{ for all } \boldsymbol{x} \in U \right\}.$$

For instance, let $X = \mathbb{A}^2(k)$ and $U = \mathbb{A}^2(k) \setminus \{0\}$. Suppose $f \in \mathcal{O}_X(U)$ and $x \in U$. Since $f \in O_{X,p}$, we can write

$$f|_{D(g_1)} = \frac{f_1}{g_1},$$

where $g_1(x) \neq 0$. We may assume f_1 and g_1 share no common factors. If g_1 is not a constant, then there exists another point $y \in U$ such that $g_1(y) = 0$. Since $f \in \mathcal{O}_{X,y}$, we must be able to write

$$f|_{D(g_2)} = \frac{f_2}{g_2},$$

where $g_2(y) \neq 0$. This implies

$$\frac{f_1}{f_2}|_{D(g_1g_2)} = f = \frac{f_2}{g_2}|_{D(g_1g_2)}.$$

Thus, $f_1/g_1 = f_2/g_2$ in $K(X)$. But the only way we can have $f_1/g_1 = f_2/g_2$ is if $g_1 = hf_1$ and $g_2 = hf_2$, where $h \in k[T_1, T_2]$.[a] But this implies $g_2(y) = h(y)f_2(y) = 0$, which is a contradiction.

[a]This is related to the fact that $\langle g_1, g_2 \rangle$ has depth 2.

To consider $(X, \mathcal{O}_X)$ as a space with functions, we first have to explain how to identify elements $f \in \mathcal{O}_X(U)$ with functions $U \to k$. Given $x \in U$, the element f is by definition in $\Gamma(X)_{\mathfrak{m}_x}$ and we may write $f = g/h$ where $g, h \in \Gamma(X)$ and $h \notin \mathfrak{m}_x$. But then $h(x) \neq 0$ and we may set $f(x) := g(x)/h(x) \in k$. The value of $f(x)$ is well defined and Lemma (14.5) implies that this construction defines an injective map $\mathcal{O}_X(U) \to \mathrm{Map}(U, k)$.

If $V \subseteq U \subseteq X$ are open subsets we have $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V)$ by definition and this inclusion corresponds via the identification with maps $U \to k$ resp. $V \to k$ to the restriction of functions.

To show that $(X, \mathcal{O}_X)$ is a space with functions, we still have to show that we may glue functions together. But this follows immediately from the definition of $\mathcal{O}_X(U)$ as subsets of the function field $K(X)$. We call $(X, \mathcal{O}_X)$ the **space of functions associated with** X. Functions on principal open subsets $D(f)$ can be explicitely described as follows.

Proposition 14.8. *Let $(X, \mathcal{O}_X)$ be the space with functions associated to the irreducible affine algebraic set X and let $f \in \Gamma(X)$. Then there is an equality*

$$\mathcal{O}_X(D(f)) = \Gamma(X)_f$$

(as subsets of $K(X)$). In particular $\mathcal{O}_X(X) = \Gamma(X)$ (taking $f = 1$).

Proof. Clearly we have $\Gamma(X)_f \subset \mathcal{O}_X(D(f))$. Let $g \in \mathcal{O}_X(D(f))$. If we can show that $f^n g = h$, for some $n \in \mathbb{N}$ and $h \in \Gamma(X)$, then $g = h/f^n$ would show that $g \in \Gamma(X)_f$. To do this, we will work with ideals, because our argument will use Nullstellensatz which is a theorem about ideals. So set

$$\mathfrak{a} = \{q \in \Gamma(X) \mid qg \in \Gamma(X)\}.$$

Obviously $\mathfrak{a}$ is an ideal of $\Gamma(X)$ and we have to show that $f \in \operatorname{rad}\mathfrak{a}$. By Hilbert's Nullstellensatz we have $\operatorname{rad}\mathfrak{a} = I(V(\mathfrak{a}))$. Therefore it suffices to show $f(x) = 0$ for all $x \in V(\mathfrak{a})$. Let $x \in X$ be a point with $f(x) \neq 0$, i.e. $x \in D(f)$. As $g \in \mathcal{O}_X(D(f))$, we find $g_1, g_2 \in \Gamma(X)$ with $g_2 \notin \mathfrak{m}_x$ and $g = g_1/g_2$. Thus $g_2 \in \mathfrak{a}$ and as $g_2(x) \neq 0$ we have $x \notin V(\mathfrak{a})$. □

Remark 36.

1. Note that we needed to use Nullstellensatz here. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1(\mathbb{R})$, but not polynomial.

2. The proposition shows that we could have defined $(X, \mathcal{O}_X)$ also in another way, namely by setting

   $$\mathcal{O}_X(D(f)) = \Gamma(X)_f \text{ for } f \in \Gamma(X).$$

   As the $D(f)$ for $f \in \Gamma(X)$ form a basis of the topology, the axiom of gluing implies that at most one such space with functions can exist. It would remain to show the existence of such a space (i.e. that for $f, g \in \Gamma(X)$ with $D(f) = D(g)$ we have $\Gamma(X)_f = \Gamma(X)_g$ and that gluing of functions is possible). This is more or less the same as the proof of Proposition (14.8). The way we chose is more comfortable in our situation. For affine schemes we will use the other approach.

Remark 37. If A is an integral finitely generated k-algebra we may construct the space with functions $(X, \mathcal{O}_X)$ of "the" corresponding irreducible affine algebraic set directly without choosing generators of A. Namely, we obtain X as the set of maximal ideals in A. Closed subsets of X are sets of the form

$$V(\mathfrak{a}) = \{\mathfrak{m} \subset A \text{ maximal} \mid \mathfrak{m} \supseteq \mathfrak{a}\},$$

where $\mathfrak{a}$ is an ideal in A. For an open subset $U \subseteq X$ we finally define

$$\mathcal{O}_X(U) = \bigcap_{\mathfrak{m} \in U} A_{\mathfrak{m}} \subset \operatorname{Frac}(A).$$

This defines a space with functions $(X, \mathcal{O}_X)$ which coincides the space with functions of the irreducible affine algebraic set X corresponding in A. This approach is the point of departure for the definition of schemes.

14.6.2 The Functor from the Category of Irreducible Affine Algebraic Sets to the Category of Spaces with Functions

Proposition 14.9. *Let X, Y be irreducible affine algebraic sets and $f : X \to Y$ a map. The following assertions are equivalent.*

1. *The map f is a morphism of affine algebraic sets.*

2. *If $g \in \Gamma(Y)$, then $g \circ f \in \Gamma(X)$.*

3. *The map f is a morphism of spaces with functions, i.e. f is continuous and if $U \subseteq Y$ open and $g \in \mathcal{O}_Y(U)$, then $g \circ f \in \mathcal{O}_X(f^{-1}(U))$.*

Proof. The equivalence of (1) and (2) has already been proved in Proposition (14.6). Moreover, it is clear that (2) is implied by (3) by taking $U = Y$. Let us show that (2) implies (3). Let $f^* : \Gamma(Y) \to \Gamma(X)$ be the homomorphism $h \mapsto h \circ f$. For $g \in \Gamma(Y)$ we have

$$\begin{aligned} f^{-1}(D(g)) &= \{x \in X \mid f(x) \in D(g)\} \\ &= \{x \in X \mid g(f(x)) \neq 0\} \\ &= D(f^*(g)). \end{aligned}$$

As the principal open subsets form a basis of the topology, this shows that f is continuous. The homomorphism f^* induces a homomorphism of the localizations $\Gamma(Y)_g \to \Gamma(X)_{f^*(g)}$. By definition of f^* this is the map $\mathcal{O}_Y(D(g)) \to \mathcal{O}_X(D(f^*(g)))$, given by $h \mapsto h \circ f$. This shows the claim if U is principal open. As we can obtain functions on arbitrary open subsets of Y by gluing functions on principal open subsets, this proves (3). $\square$

Altogether we obtain

Theorem 14.6. *The above construction $X \mapsto (X, \mathcal{O}_X)$ defines a fully faithful functor*

$$\text{(Irreducible affine algebraic sets)} \mapsto \text{(Spaces with functions over } k).$$

15 Prevarieties

We have seen that we can embed the category of irreducible affine algebraic sets into the category of spaces with functions. Of course we do not obtain all spaces with functions in this way. We will now define prevarieties as those connected spaces with functions that can be glued together from finitely many spaces with functions attached to irreducible affine algebraic sets.

15.1 Definition of Prevarieties

We call a space with functions $(X, \mathcal{O}_X)$ **connected**, if the underlying topological space X is connected.

Definition 15.1.

1. An **affine variety** is a space with functions that is isomorphic to a space with functions associated to an irreducible affine algebraic set.
2. A **prevariety** is a connected space with functions $(X, \mathcal{O}_X)$ with the property that there exists a finite covering $X = \bigcup_{i=1}^n U_i$ such that the space with functions $(U_i, \mathcal{O}_{X|U_i})$ is an affine variety for all $i = 1, \dots, n$.
3. A **morphism** of prevarieties is a morphism of spaces with functions.

Corollary 3. *The following categories are equivalent.*

1. *The opposed category of finitely generated k-algebras without zero divisors.*
2. *The category of irreducible affine algebraic sets.*
3. *The category of affine varieties.*

We define an **open affine covering of a prevariety** X to be a family of open subspaces with functions $U_i \subseteq X$ that are affine varieties such that $X = \bigcup_i U_i$.

Proposition 15.1. *Let $(X, \mathcal{O}_X)$ be a prevariety. The topological space X is Noetherian (in particular quasi-compact) and irreducible.*

Proof. The first assertion follows from the fact that X has a finite covering of Noetherian spaces, which implies that X is Noetherian. The second assertion follows from the fact that X is connected and has a finite covering of irreducible spaces, which implies X is irreducible. $\square$

15.1.1 Open Subprevarieties

We are now able to endow open subsets of affine varieties, and more general of prevarieties with the structure of a prevariety. Note that in general open subprevarieties of affine varieties are not affine.

Lemma 15.1. *Let X be an affine variety and let $f \in \Gamma(X)$. and let $\mathrm{D}(f) \subseteq X$ be the corresponding principal open subset. Let $\Gamma(X)_f$ be the localization of $\Gamma(X)$ by f and let $(Y, \mathcal{O}_Y)$ be the affine variety corresponding to this integral finitely generated k-algebra. Then $(D(f), \mathcal{O}_{X|D(f)})$ and $(Y, \mathcal{O}_Y)$ are isomorphic spaces with functions. In particular, $(D(f), \mathcal{O}_{X|D(f)})$ is an affine variety.*

Proof. By Proposition (14.8) we have $\mathcal{O}_X(\mathrm{D}(f)) = \Gamma(X)_f$. As two affine varieties are isomorphic if and only if their coordinate rings are isomorphic, it suffices to show that $(\mathrm{D}(f), \mathcal{O}_X|_{\mathrm{D}(f)})$ is an affine variety.

Let $X \subseteq \mathbb{A}^n(k)$ and $\mathfrak{a} = I(X) \subseteq k[T_1, \dots, T_n]$ be the corresponding radical ideal. We consider $k[T_1, \dots, T_n]$ as a subring of $k[T_1, \dots, T_n, T_{n+1}]$ and denote by $\mathfrak{a}' \subseteq k[T_1, \dots, T_n, T_{n+1}]$ the ideal generated by $\mathfrak{a}$ and the polynomial $fT_{n+1} - 1$. Then the affine coordinate ring of Y is $\Gamma(Y) = \Gamma(X)_f \cong k[T_1, \dots, T_n, T_{n+1}]/\mathfrak{a}'$, and we can identify Y with $V(\mathfrak{a}') \subseteq \mathbb{A}^{n+1}(k)$.

The projection $\mathbb{A}^{n+1}(k) \to \mathbb{A}^n(k)$ to the first n coordinates induces a bijective map

$$j : Y = \{(x, x_{n+1}) \in X \times \mathbb{A}^1(k) \mid x_{n+1}f(x) = 1\} \to D(f) = \{x \in X \mid f(x) \neq 0\}.$$

We will show that j is an isomorphism of spaces with functions. As a restriction of a continuous map, j is continuous. It is also open, because for $\frac{g}{f^N} \in \Gamma(Y)$, with $g \in \Gamma(X)$, we have

$$j\left(D\left(\frac{g}{f^N}\right)\right) = j(D(gf)) = D(gf).$$

Thus j is a homeomorphism.

It remains to show that for all $g \in \Gamma(X)$ the map $\mathcal{O}_X(D(fg)) \to \Gamma(Y)_g$, given by $s \mapsto s \circ j$, is an isomorphism. But we have

$$\mathcal{O}_X(D(fg)) = \Gamma(X)_{fg} = \Gamma(Y)_g$$

and this identification corresponds to the composition with j. □

Proposition 15.2. *Let $(X, \mathcal{O}_X)$ be a prevariety and let $U \subseteq X$ be a non-empty open subset. Then $(U, \mathcal{O}_{X|U})$ is prevariety and the inclusion $U \hookrightarrow X$ is a morphism of prevarieties.*

Proof. As X is irreducible, U is connected. The previous lemma shows that U can be covered by open affine subsets of X. As X is Noetherian, U is quasi-compact. Thus a finite covering suffices. □

15.1.2 Function Field of a Prevariety

Let X be a prevariety. If $U, V \subseteq X$ are non-empty open affine subvarieties, then $U \cap V$ is open in U and non-empty. We have $\mathcal{O}_X(U) \subseteq \mathcal{O}_X(U \cap V) \subseteq K(U)$ by the definition of functions on U, and therefore $\mathrm{Frac}(\mathcal{O}_X(U \cap V)) = K(U)$. The same argument for V shows $K(U) = K(V)$. Thus the function field of a non-empty open affine subvariety U of X does not depend on U and we denote it by $K(X)$.

Definition 15.2. The field $K(X)$ is called the **function field** of X.

Remark 38. Let $f : X \to Y$ be a morphism of affine varieties. As the corresponding homomorphism $\Gamma(Y) \to \Gamma(X)$ between the affine coordinate rings is not injective in general, it does not induce a homomorphism of function fields $K(Y) \to K(X)$. Thus $K(X)$ is not functorial in X. But if $f : X \to Y$ is a morphism of prevarieties whose image contains a non-empty open (and hence dense) subset, f induces a homomorphism $K(Y) \to K(X)$. Such morphisms will be called **dominant**.

Proposition 15.3. *Let X be a prevariety and $U \subseteq X$ a non-empty open subset. Then $\mathcal{O}_X(U)$ is a k-subalgebra of the function field $K(X)$. If $U' \subseteq U$ is another open subset, the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ is the inclusion of subalgebras of $K(X)$. If $U, V \subseteq X$ are arbitrary open subsets, then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$.*

Proof. Let $f : U \to \mathbb{A}^1(k)$ be an element of $\mathcal{O}_X(U)$. Then its vanishing set $f^{-1}(0) \subseteq U$ is closed because f is continuous and $\{0\} \subseteq \mathbb{A}^1(k)$ is closed. Therefore if the restriction of f to U' is zero, then f is zero because U' is dense in U. This shows that restriction maps are injective. The axiom of gluing implies therefore $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$ for all open subsets $U, V \subseteq X$. □

15.1.3 Closed Subprevarieties

Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. We want to define on Z the structure of a prevariety. For this we have to define functions on open subsets of Z. We define:

$$\mathcal{O}'_Z(U) = \{f \in \mathrm{Map}(U, k) \mid \text{for all } x \in U, \text{there exists } V \subseteq U \text{ open and } g \in \mathcal{O}_X(V) \text{ such that } f|_{U \cap V} = g|_{U \cap V}\}.$$

The definition shows that $(Z, \mathcal{O}'_Z)$ is a space with functions and that $\mathcal{O}'_X = \mathcal{O}_X$. Once we have shown the following lemma, we will always write $\mathcal{O}_Z$ (instead of $\mathcal{O}'_Z$).

Remark 39. $\mathcal{O}'_Z$ is the sheafification of the sheaf $\mathcal{O}_{X|Z}$.

Lemma 15.2. *Let $X \subseteq \mathbb{A}^n(k)$ be an irreducible affine algebraic set and let $Z \subseteq X$ be an irreducible closed subset. Then the space with functions $(Z, \mathcal{O}_Z)$ associated to the affine algebraic set Z and the above defined space with functions $(Z, \mathcal{O}'_Z)$ coincide.*

Proof. In both case Z is endowed with the topology induced by X. As the inclusion $Z \to X$ is a morphism of affine algebraic sets it induces a morphism $(Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$. The definition of $\mathcal{O}'_Z$ shows that $\mathcal{O}'_Z(U) \subseteq \mathcal{O}_Z(U)$ for all open subsets $U \subseteq Z$.

Conversely, let $f \in \mathcal{O}_Z(U)$. For $x \in U$ there exists $h \in \Gamma(Z)$ with $x \in D(h) \subseteq U$. The restriction $f|_{D(h)} \in \mathcal{O}_Z(D(h)) = \Gamma(Z)_h$ has the form $f = g/h^n$ where $n \geq 0$ and $g \in \Gamma(Z)$. We lift g and h to elements in $\widetilde{g}, \widetilde{h} \in \Gamma(X)$, set $V := D(\widetilde{h}) \subseteq X$, and obtain $x \in V$, $\widetilde{g}/\widetilde{h}^n \in \mathcal{O}_X(D(\widetilde{h}))$ and $f|_{U \cap V} = \frac{\widetilde{g}}{\widetilde{h}^n}|_{U \cap V}$. □

As a corollary of the lemme we obtain:

Proposition 15.4. *Let X be a prevariety and let $Z \subseteq X$ be an irreducible closed subset. Let $\mathcal{O}_Z$ be the system of functions defined above. Then $(Z, \mathcal{O}_Z)$ is a prevariety.*

15.2 Gluing Prevarieties

The most general way to construct prevarieties is to take some affine varieties and patch them together:

Example 15.1. Let X_1 and X_2 be prevarieties, $U_1 \subset X_1$ and $U_2 \subset X_2$ be non-empty open subsets, and let $f : (U_1, \mathcal{O}_{X_1} |_{U_1}) \to (U_2, \mathcal{O}_{X_2} |_{U_2})$ be an isomorphism. Then we can define a prevariety X, obtained by **gluing** X_1 and X_2 along U_1 and U_2 via the isomorphism f:

- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation $x \sim f(x)$ for all $x \in U_1$.
- As a topological space, we endow X with the so-called **quotient topology** induced by the above equivalence relation, i.e. we say that a subset $U \subset X$ is open if $U \cap X_1 \subset X_1$ is open in X_1 and $U \cap X_2 \subset X_2$ is open in X_2.
- As a ringed space, we define the structure sheaf $\mathcal{O}_X$ by

  $\mathcal{O}_X(U) = \{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(U \cap X_1),\ s_2 \in \mathcal{O}_{X_2}(U \cap X_2), \text{ and } s_1 = s_2 \text{ on the overlap (i.e. } f^*(s_2 |_{U \cap U_2}) = s_1 |_{U \cap U_1})\}$

Example 15.2. Let $X_1 = X_2 = \mathbb{A}^1(k)$ and let $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $t \mapsto \frac{1}{t} := t'$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$. Of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $k = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$. Let us show that $\mathcal{O}_X(X) \cong k$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2 |_{U_2}) = b_m T^{-m} + b_{m-1} T^{1-m} + \cdots + b_0 |_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 |_{U_2}.$$

  The only way this happens is if $a_0 = b_0$ and $a_i = b_j = 0$ for all $i, j > 0$. Thus, $(s_1, s_2) = (a_0, a_0)$.
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by gluing along f is "the affine line with the zero point doubled". Obviously this is a somewhat weird place. Speaking in classical terms, if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called **Hausdorff property**. This is obviously not satisfied for our space X. Let us show that $\mathcal{O}_X(X) \cong k[T]$. Let $(s_1, s_2) \in \mathcal{O}_X(X)$. Then since $s_1 \in \mathcal{O}_{X_1}(X \cap X_1) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_1 = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0$. Similarly, since $s_2 \in \mathcal{O}_{X_2}(X \cap X_2) = \mathcal{O}_{\mathbb{A}^1(k)}(\mathbb{A}^1(k))$, we have $s_2 = b_m T'^m + b_{m-1} T'^{m-1} + \cdots + b_0$. Now

  $$f^*(s_2 |_{U_2}) = b_m T^m + b_{m-1} T^{m-1} + \cdots + b_0 |_{U_1} = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 |_{U_2}.$$

  The only way this happens is if $m = n$ and $a_i = b_i$ for all $i = 0, \ldots, n$.

Example 15.3. Let X be the complex affine curve

$$X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\}.$$

We can "compactify" X by adding two points at infinity, corresponding to the limit as $x \to \infty$ and the two possible values for y. To construct this space rigorously, we construct a prevariety as follows:

If we make the coordinate change $\widetilde{x} = \frac{1}{x}$, the equation of the curve becomes

$$y^2 \widetilde{x}^4 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

If we make an additional coordinate change $\widetilde{y} = \frac{y}{x^4}$, then this becomes

$$\widetilde{y}^2 = (1 - \widetilde{x})(1 - 2\widetilde{x})(1 - 3\widetilde{x})(1 - 4\widetilde{x}).$$

In these coordinates, we can add our two points at infinity, as they now correspond to $\widetilde{x} = 0$ (and therefore $\widetilde{y} = \pm 1$).

Summarizing, our "compactified curve" is just the prevariety obtained by gluing the two affine varieties

$$X = \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-2)(x-3)(x-4)\} \quad \text{and} \quad \widetilde{X} = \{(\widetilde{x},\widetilde{y}) \in \mathbb{C}^2 \mid \widetilde{y}^2 = (1-\widetilde{x})(1-2\widetilde{x})(1-3\widetilde{x})(1-4\widetilde{x})\}$$

along the isomorphism

$$f : U \to \widetilde{U}, \qquad (x,y) \mapsto (\widetilde{x},\widetilde{y}) = \left(\frac{1}{x}, \frac{y}{x^4}\right)$$

$$f^{-1} : \widetilde{U} \to U, \qquad (\widetilde{x},\widetilde{y}) \mapsto (x,y) = \left(\frac{1}{\widetilde{x}}, \frac{\widetilde{y}}{\widetilde{x}^4}\right)$$

where $U = \{x \neq 0\} \subset X$ and $\widetilde{U} = \{\widetilde{x} \neq 0\} \subset \widetilde{X}$.

16 Projective Varieties

By far the most important example of prevarieties are projective space $\mathbb{P}^n(k)$ and subvarieties of $\mathbb{P}^n(k)$, called (quasi-)projective varieties.

16.1 Homogeneous Polynomials

To describe the functions on projective space we start with some remarks on homogeneous polynomials. Throughout this subsection, let R be a ring. To clean our notation in what follows, we often write $R[\boldsymbol{X}]$ to denote $R[X_0,\dots,X_n]$. A monomial in $R[\boldsymbol{X}]$ is denoted by $\boldsymbol{X}^{\boldsymbol{\alpha}} = X_0^{\alpha_0}\cdots X_n^{\alpha_n}$ where $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$. We also denote $|\boldsymbol{\alpha}| = \sum_{i=0}^n \alpha_i$. The vector $(1,\cdots,1)$ in $\mathbb{Z}^n_{\geq 0}$ is denoted $\mathbf{1}$, thus $X_0\cdots X_n = \boldsymbol{X}^{\mathbf{1}}$. A point in R^{n+1} is denoted by $\boldsymbol{x} = (x_0,\dots,x_n)$. We will frequently use this notation whenever context is clear.

Definition 16.1. A polynomial $f \in R[X_0,\dots,X_n]$ is called **homogeneous** of degree $d \in \mathbb{Z}_{\geq 0}$ if f is the sum of monomials of degree d.

Lemma 16.1. *Assume R is an integral domain with infinitely many elements and let $f \in R[\boldsymbol{X}]$ be a nonzero polynomial. Then f is homogeneous of degree d if and only if*

$$f(\lambda\boldsymbol{x}) = \lambda^d f(\boldsymbol{x}) \tag{11}$$

for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$.

Proof. One direction is obvious, so we will only prove the other direction. We will prove the other direction by induction on the number of terms of a polynomial. For the base case, let f be a monomial in $R[\boldsymbol{X}]$, say $f = c\boldsymbol{X}^{\boldsymbol{\alpha}}$ where $c \neq 0$ and assume that f satisfies (11) for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$. Clearly f is homogeneous, but we still need to show that it has degree d.

Let K be the fraction field of R. Since K has infinitely many elements and since $f \neq 0$, there exists a point $\boldsymbol{a} \in \mathrm{D}(\boldsymbol{X}^{\mathbf{1}}) \cap \mathrm{D}(f)$. By clearing the denominators of $\boldsymbol{a}$ if necessary, we may assume that $\boldsymbol{a} \in R$. Then for all $\lambda \in R\backslash\{0\}$, we have

$$\lambda^d \boldsymbol{a}^{\boldsymbol{\alpha}} = \lambda^d f(\boldsymbol{a}) = f(\lambda\boldsymbol{a}) = \lambda^{|\boldsymbol{\alpha}|}\boldsymbol{a}^{\boldsymbol{\alpha}}.$$

Since R is a domain and $\boldsymbol{a}^{\boldsymbol{\alpha}} \neq 0$, it follows that $\lambda^d = \lambda^{|\boldsymbol{\alpha}|}$ for all $\lambda \in R\backslash\{0\}$. Assume without loss of generality that $d \geq |\boldsymbol{\alpha}|$ and set $r = d - |\boldsymbol{\alpha}|$. If $r > 0$, then $T^r - 1$ has infinitely many solutions in R (in fact every nonzero element of R is a solution). This is a contradiction since $T^r - 1$ can have at most r solutions in R. Thus $r = 0$, which implies $|\boldsymbol{\alpha}| = d$; hence f has degree d.

For the induction step, assume that we have proven the statement for all polynomials with k terms, where $k \geq 1$. Let f be a polynomial with $k+1$ terms such that f satisfies (11) for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$. Write f as

$$f = c\boldsymbol{X}^{\boldsymbol{\alpha}} + g,$$

where $\boldsymbol{\alpha} \in \mathbb{Z}^{n+1}_{\geq 0}$, where $c \in R\backslash\{0\}$, and where g is a nonzero polynomial in $R[\boldsymbol{X}]$ such that $\boldsymbol{X}^{\boldsymbol{\alpha}} \nmid g$. Let K be the fraction field of R and let $\boldsymbol{a} \in K^{n+1}$ be a point such that $\boldsymbol{a}^{\boldsymbol{\alpha}} \neq 0$ and $g(\boldsymbol{a}) = 0$. Note that such a point exists since $\mathrm{V}(g) \cap \mathrm{V}(\boldsymbol{X}^{\boldsymbol{\alpha}}) \neq \emptyset$. Indeed, otherwise we'd have $\mathrm{V}(g) \subseteq \mathrm{V}(\boldsymbol{X}^{\boldsymbol{\alpha}})$ which would imply $\boldsymbol{X}^{\boldsymbol{\alpha}} \mid g$, a contradiction. By clearing the denominators if necessary, we may assume that $\boldsymbol{a} \in R^{n+1}$. In particular, for all $\lambda \in R\backslash\{0\}$, we have

$$\lambda^d \boldsymbol{a}^{\boldsymbol{\alpha}} = \lambda^d f(\boldsymbol{a}) = f(\lambda\boldsymbol{a}) = \lambda^{\boldsymbol{\alpha}}\boldsymbol{a}^{\boldsymbol{\alpha}}.$$

Arguing as before, this implies $|\boldsymbol{\alpha}| = d$. Now observe that for all $\boldsymbol{x} \in R^{n+1}$ and $\lambda \in R\backslash\{0\}$, we have

$$\begin{aligned} g(\lambda \boldsymbol{x}) &= f(\lambda \boldsymbol{x}) - c(\lambda \boldsymbol{X})^{\boldsymbol{\alpha}} \\ &= \lambda^d f(\boldsymbol{x}) - c\lambda^{|\boldsymbol{\alpha}|}\boldsymbol{x}^{\boldsymbol{\alpha}} \\ &= \lambda^d f(\boldsymbol{x}) - c\lambda^d \boldsymbol{x}^{\boldsymbol{\alpha}} \\ &= \lambda^d (f(\boldsymbol{x}) - c\boldsymbol{x}^{\boldsymbol{\alpha}}) \\ &= \lambda^d g(\boldsymbol{x}). \end{aligned}$$

Thus by induction, we see that g must be homogeneous of degree d; hence f is homogeneous of degree d. □

The zero polynomial is homogeneous of degree d for all d. We denote by $R[\boldsymbol{X}]_d$ the R-submodule of all homogeneous polynomials of degree d. As we can decompose uniquely every polynomial into its homogeneous parts, we have

$$R[\boldsymbol{X}] = \bigoplus_{n \geq 0} R[\boldsymbol{X}]_d.$$

Before we proceed further, let us introduce some more notation to keep everything clean. Given $i \in \{0, \ldots, n\}$, we write $\widehat{\boldsymbol{X}}_i = (X_0, \ldots \widehat{X}_i, \ldots, X_n)$. We also denote by $R[\boldsymbol{T}]_{\leq d}$ to be the set of all polynomials f in $R[\boldsymbol{T}]$ such that $\deg f \leq d$.

Lemma 16.2. *Let $i \in \{0, \ldots, n\}$ and $d \geq 0$. There is a bijective R-linear map*

$$\Phi_i = \Phi_i^{(d)} : R[\boldsymbol{X}]_d \to R[\widehat{\boldsymbol{T}}_i]_{\leq d}$$

given by $f \mapsto f(T_0, \ldots, 1, \ldots, T_n)$.

Proof. We construct an inverse map. Let g be a polynomial in the right hand side set and let $g = \sum_{j=0}^d g_j$ be its decomposition into homogeneous parts (with respect to T_ℓ for $\ell = 0, \ldots, i-1, i+1, \ldots, n$. Define

$$\Psi_i(g) = \sum_{j=0}^{d} X_i^{d-j} g_j(X_0, \ldots, \widehat{X}_i, \ldots, X_n).$$

It is easy to see that Ψ_i and Φ_i are inverse to each other (as both maps are R-linear, it suffices to check this on monomials). □

Example 16.1. Consider $R[X_0, X_1, X_2]$. Then

$$\begin{aligned} \Psi_1(\Phi_1(X_0^2 X_2 + X_1^3 + X_1 X_2^2)) &= \Psi_1(T_0^2 T_2 + 1 + T_2^2) \\ &= X_0^2 X_2 + X_1^3 + X_1 X_2^2. \end{aligned}$$

The map Φ_i is called **dehomogenization**, and the map Ψ_i is called **homogenization** (with respect to X_i). For $f \in R[\boldsymbol{X}]_d$ and $g \in R[\boldsymbol{X}]_e$, the product fg is homogeneous of degree $d + e$ and we have

$$\Phi_i^{(d)}(f)\Phi_i^{(e)}(g) = \Phi_i^{(d+e)}(fg). \tag{12}$$

If $R = K$ is a field, we will extend homogenization and dehomogenization to fields of fractions as follows. Let $\mathcal{F}$ be the subset of $K(X_0, \ldots, X_n)$ that consists of those elements f/g, where $f, g \in K[\boldsymbol{X}]$ are homogeneous polynomials of the same degree. It is easy to check that $\mathcal{F}$ is a subfield of $K(\boldsymbol{X})$. By (12), we have a well defined isomorphism of K-extensions

$$\Phi_i : \mathcal{F} \to K(\widehat{\boldsymbol{T}}_i), \tag{13}$$

given by $f/g \mapsto \Phi_i(f)/\Phi_i(g)$. Often, we will identify $K(\widehat{\boldsymbol{T}}_i)$ with the subring $K(\boldsymbol{X}/X_i) = K\left(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right)$ of the field $K(\boldsymbol{X})$. Via this identification, the isomorphism (13) can also be described as follows. Let $f/g \in \mathcal{F}$ with $f, g \in K[\boldsymbol{X}]_d$ for some d. Set $\widetilde{f} = f/X_i^d$ and $\widetilde{g} = g/X_i^d$. Then $\widetilde{f}, \widetilde{g} \in K[\boldsymbol{X}/X_i]$ and $\Phi_i(f/g) = \widetilde{f}/\widetilde{g}$.

Example 16.2. Consider $\frac{1}{T_2^3 + T_0^2} \in K(T_0, T_2)$. Then $\frac{X_1^3}{X_2^3 + X_1 X_0^2} \in \mathcal{F}$ is its homogenization.

16.2 Definition of the Projective Space $\mathbb{P}^n(k)$

The projective space $\mathbb{P}^n(k)$ is an extremely important prevariety within algebraic geometry. Many prevarieties of interest are subprevarieties of the projective space. Moreover, the projective space is the correct environment for projective geometry which remedies the "defect" of affine geometry of missing points at infinity.

As a set, we define for every field k (not necessarily algebraically closed)

$$\mathbb{P}^n(k) := \{\text{lines through the origin in } k^{n+1}\} = (k^{n+1}\backslash\{0\})/k^\times.$$

Here a line through the origin is per definition a 1-dimensional k-subspace and we denote by $(k^{n+1}\backslash\{0\})/k^\times$ the set of equivalence classes in $k^{n+1}\backslash\{0\}$ with respect to the equivalence relation

$$(x_0,\dots,x_n) \sim (x_0',\dots,x_n') \text{ if and only if there exists } \lambda \in k^\times \text{ such that } x_i = \lambda x_i' \text{ for all } 0 \le i \le n.$$

The equivalence class of a point $\boldsymbol{x} = (x_0,\dots,x_n)$ is denoted by $[\boldsymbol{x}] = [x_0 : \cdots : x_n]$. We call the x_i the **homogeneous coordinates** on $\mathbb{P}^n(k)$.

To endow $\mathbb{P}^n(k)$ with the structure of a prevariety we will assume from now on that k is algebraically closed. The following observation is essential: For $0 \le i \le n$ we set

$$U_i := \{[\boldsymbol{x}] \in \mathbb{P}^n(k) \mid x_i \neq 0\} \subseteq \mathbb{P}^n(k).$$

This subset is well-defined and the union of the U_i is all of $\mathbb{P}^n(k)$. There are bijections

$$U_i \cong \mathbb{A}^n(k), \qquad [\boldsymbol{x}] = (x_0 : \cdots : x_n) \mapsto \left(\frac{x_0}{x_i}, \cdots \frac{\widehat{x}_i}{x_i}, \cdots, \frac{x_n}{x_i}\right) = \widehat{\boldsymbol{x}}_i/x_i.$$

Via this bijection we will endown U_i with the structure of a space with functions, isomorphic to $(\mathbb{A}^n(k), \mathcal{O}_{\mathbb{A}^n(k)})$, which we denote by $(U_i, \mathcal{O}_{U_i})$. We want to define on $\mathbb{P}^n(k)$ the structure of a space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ such that U_i becomes an open subset of $\mathbb{P}^n(k)$ and such that $\mathcal{O}_{\mathbb{P}^n(k)}|_{U_i} = \mathcal{O}_{U_i}$ for all $i = 0,\dots,n$. As $\bigcup_i U_i = \mathbb{P}^n(k)$, there's at most one way to do this:

We define the topology on $\mathbb{P}^n(k)$ by calling a subset $U \subseteq \mathbb{P}^n(k)$ open if $U \cap U_i$ is open in U_i for all i. This defines a topology on $\mathbb{P}^n(k)$ as for all $i \neq j$ the set $U_i \cap U_j = \mathrm{D}(T_j) \subseteq U_i$ is open (we use here on $U_i \cong \mathbb{A}^n(k)$ the coordinates $T_0,\dots,\widehat{T}_i,\dots,T_n$). With this definition, $\{U_i\}_{i\in\{0,\dots,n\}}$ becomes an open covering of $\mathbb{P}^n(k)$.

We still have to define functions on open subsets $U \subseteq \mathbb{P}^n(k)$. For this, we set

$$\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f \in \mathrm{Map}(U,k) \mid f|_{U\cap U_i} \in \mathcal{O}_{U_i}(U \cap U_i) \text{ for all } i = 0,\dots,n\}.$$

It is clear that this defines the structure of a space with functions on $\mathbb{P}^n(k)$, although we still have to see that $\mathcal{O}_{\mathbb{P}^n(k)}|_{U_i} = \mathcal{O}_{U_i}$ for all i. This follows from the following description of the k-algebras $\mathcal{O}_{\mathbb{P}^n(k)}(U)$ using the inverse isomorphism of the function field $k(T_0,\dots,\widehat{T}_i,\dots,T_n)$ of U_i with the subfield $\mathcal{F}$ of $k(X_0,\dots,X_n)$.

Proposition 16.1. *Let $U \subseteq \mathbb{P}^n(k)$ be open. Then*

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^n(k)}(U) = \{f : U \to k \mid \forall x \in U,\ \exists x \in V \subseteq U \text{ open and } g,h \in k[X_0,\dots,X_n]\\ \text{homogeneous of same degree such that } h(v) \neq 0 \text{ and } f(v) = g(v)/h(v) \text{ for all } v \in V\}.\end{aligned}$$

Proof. Let $f \in \mathcal{O}_{\mathbb{P}^n(k)}(U)$. As $f|_{U\cap U_i} \in \mathcal{O}_{U_i}(U\cap U_i)$, the function f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g},\widetilde{h} \in k[T_0,\dots,\widehat{T}_i,\dots,T_n]$. Applying the inverse of (13) yields the desired form of f.

Conversely, let f be an element of the right hand side. We fix $i \in \{0,\dots,n\}$. Thus locally on $U \cap U_i$ the function f has the form g/h where $g,h \in k[X_0,\dots,X_n]_d$ for some d. Once more applying the isomorphism (13) we obtain that f has locally the form $\widetilde{g}/\widetilde{h}$ where $\widetilde{g},\widetilde{h} \in k[T_0,\dots,\widehat{T}_i,\dots,T_n]$. This shows $f|_{U\cap U_i} \in \mathcal{O}_{U_i}(U\cap U_i)$. □

Example 16.3. Consider $\mathbb{P}^2(k)$ and

$$f|_{U\cap U_1} = \frac{T_2^2+1}{T_0+1}.$$

Then the inverse of (13) yields

$$\frac{X_2^2+X_1^2}{X_0^2+X_1^2}$$

Corollary 4. *Let $i \in \{0,\dots,n\}$. The bijection $U_i \cong \mathbb{A}^n(k)$ induces an isomorphism*

$$(U_i, \mathcal{O}_{\mathbb{P}^n(k)}|_{U_i}) \cong \mathbb{A}^n(k).$$

of spaces with functions. The space with functions $(\mathbb{P}^n(k), \mathcal{O}_{\mathbb{P}^n(k)})$ is a prevariety.

Proof. The first assertion follows from the proof of Proposition (16.1). This shows that $\mathbb{P}^n(k)$ is a space with functions that has a finite open covering by affine varieties. Moreover, $\mathbb{P}^n(k)$ is irreducible since it is connected and is covered by finitely many irreducible open subsets. $\square$

The function field $K(\mathbb{P}^n(k))$ of $\mathbb{P}^n(k)$ is by its very definition the function field $K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right)$ of U_i. Using the isomorphism Φ_i, we usually describe $K(\mathbb{P}^n(k))$ as the field

$$K(\mathbb{P}^n(k)) = \{f/g \mid f, g \in k[X_0, \dots, X_n] \text{ homogeneous of the same degree}\}.$$

For $0 \leq i, j \leq n$ the identification of $K(U_i) \cong K(U_j)$ is then given by $\Phi_j \circ \Phi_i^{-1}$. This can be described explicitely

$$K(U_i) = k\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \mapsto k\left(\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}\right) = K(U_j), \qquad \frac{X_\ell}{X_i} \mapsto \frac{X_\ell}{X_i}\frac{X_i}{X_j} = \frac{X_\ell}{X_j}.$$

We use these explicit descriptions to prove the following result.

Proposition 16.2. *The only global functions on $\mathbb{P}^n(k)$ are the constant functions, i.e. $\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = k$. In particular, $\mathbb{P}^n(k)$ is not an affine variety for $n \geq 1$.*

Proof. By Proposition (15.3) we have

$$\mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k)) = \bigcap_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}^n(k)}(U_i) = \bigcap_{0 \leq i \leq n} k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] = k,$$

where the intersection is taken in $K(\mathbb{P}^n(k))$. The last assertion follows because if $\mathbb{P}^n(k)$ were affine, its set of points would be in bijection to the set of maximal ideals in the ring $k = \mathcal{O}_{\mathbb{P}^n(k)}(\mathbb{P}^n(k))$. This implies that $\mathbb{P}^n(k)$ consists of only one point, so $n = 0$. $\square$

16.2.1 Gluing $\mathbb{A}^1(k)$ With $\mathbb{A}^1(k)$ to Make $\mathbb{P}^1(k)$

We now want to describe how we can glue $\mathbb{A}^1(k)$ with $\mathbb{A}^1(k)$ to make $\mathbb{P}^1(k)$ in explicit detail. First we start with the rings $k[S]$ and $k[T]$
Let X_0 and X_1 be the homogeneous coordinates of $\mathbb{P}^1(k)$ and denote $T := \frac{X_1}{X_0}$ and $S := \frac{X_0}{X_1}$.

16.3 Projective Varieties

Definition 16.2. A prevariety is called a **projective variety** if it is isomorphic to a closed subprevariety of a projective space $\mathbb{P}^n(k)$.

As in the affine case, we speak of projective varieties rather than prevarieties. Similarly, we will talk about subvarieties of projective space, instead of subprevarieties. For $[\boldsymbol{x}] \in \mathbb{P}^n(k)$ and $f \in k[\boldsymbol{X}]$ the value $f([\boldsymbol{x}])$ obviously depends on the choice of the representative of $[\boldsymbol{x}]$ and we cannot consider f as a function on $\mathbb{P}^n(k)$. But if f is homogeneous, at least the question whether the value is zero or nonzero is independent of the choice of a representative.

Let $\boldsymbol{f} = f_1, \dots, f_m$ be a finite collection of homogeneous polynomials in $k[\boldsymbol{X}]$. We define

$$\mathrm{V}_+(\boldsymbol{f}) = \{[\boldsymbol{x}] \in \mathbb{P}^n(k) \mid f_i(\boldsymbol{x}) = 0 \text{ for all } i = 1, \dots, m\}.$$

$$V_+(f_1, \dots, f_m) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid f_i(x_0 : \cdots : x_n) = 0 \text{ for all } i = 1, \dots, m\}.$$

Subsets of the form $V_+(f_1, \dots, f_m)$ are closed. More precisely we have $i = 0, \dots, n$:

$$V_+(f_1, \dots, f_m) \cap U_i = V(\Phi_i(f_1), \dots, \Phi_i(f_m)),$$

where Φ_i denotes as usual dehomogenization with respect to X_i. We will see that all closed subsets of the projective space are of this form. To do this we consider the map

$$f : \mathbb{A}^{n+1}(k) \backslash \{0\} \to \mathbb{P}^n(k), \qquad (x_0, \cdots, x_n) \mapsto (x_0 : \cdots : x_n).$$

As for all i its restriction $f\mid_{f^{-1}(U_i)}: f^{-1}(U_i) \to U_i$ is a morphism of prevarieties, this holds for f. If $Z \subseteq \mathbb{P}^n(k)$ is a closed subset, $f^{-1}(Z)$ is a closed subset of $\mathbb{A}^{n+1}(k) \backslash \{0\}$ and we denote by $C(Z)$ its closure in $\mathbb{A}^{n+1}(k)$. Affine algebraic sets $X \subseteq \mathbb{A}^{n+1}(k)$ are called **affine cones** if for all $x \in X$ we have $\lambda x \in X$ for all $\lambda \in k^\times$. Clearly $C(Z)$ is an affine cone in $\mathbb{A}^{n+1}(k)$. It is called the **affine cone of** Z.

Proposition 16.3. *Let $X \subseteq \mathbb{A}^{n+1}(k)$ be an affine algebraic set such that $X \neq \{0\}$. Then the following assertions are equivalent.*

1. *X is an affine cone.*
2. *$I(X)$ is generated by homogeneous polynomials.*
3. *There exists a closed subset $Z \subset \mathbb{P}^n(k)$ such that $X = C(Z)$.*

If in this case $I(X)$ is generated by homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$, then $Z = V_+(f_1, \ldots, f_m)$.

16.3.1 Segre Embedding

Consider $\mathbb{P}^n(k)$ with homogeneous coordinates $X_0, \ldots, X_n$ and $\mathbb{P}^m(k)$ with homogeneous coordinates $Y_0, \ldots, Y_m$. We want to find an easy description of the product $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$.

Let $\mathbb{P}^N(k) = \mathbb{P}^{(n+1)(m+1)-1}$ be projective space with homogeneous coordinates $Z_{i,j}$ where $0 \leq i \leq n$ and $0 \leq j \leq m$. There is an obviously well-defined set-theoretic map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ given by $z_{i,j} = x_i y_j$.

Lemma 16.3. *Let $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to \mathbb{P}^N(k)$ be the set-theoretic map as above. Then:*

1. *The image $X = f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ is a projective variety in $\mathbb{P}^N(k)$, with ideal generated by the homogeneous polynomials $Z_{i,j}Z_{i',j'} - Z_{i,j'}Z_{i',j}$ for all $0 \leq i, i' \leq n$ and $0 \leq j, j' \leq m$.*
2. *The map $f : \mathbb{P}^n(k) \times \mathbb{P}^m(k) \to X$ is an isomorphism. In particular, $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is a projective variety.*
3. *The closed subsets of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ are exactly those subsets that can be written as the zero locus of polynomials in $k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ that are bihomogeneous in the X_i and Y_i.*

Proof.

1. It is obvious that the points of $f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ satisfy the given equations. Conversely, let z be a point in $\mathbb{P}^N(k)$ with coordinates $z_{i,j}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to affine coordinates by setting $z_{0,0} = 1$. Then we have $z_{i,j} = z_{i,0} z_{0,j}$; so by setting $x_i = z_{i,0}$ and $y_j = z_{0,j}$ we obtain a point (x, y) in $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ that is mapped to z by f.
2. Continuing the above notation, let $z \in f(\mathbb{P}^n(k) \times \mathbb{P}^m(k))$ be a point with $z_{0,0} = 1$. If $f(x, y) = z$, it follows that $x_0 \neq 0$ and $y_0 \neq 0$, so we can assume $x_0 = 1$ and $y_0 = 1$ as the x_i and y_j are only determined up to a common scalar. But then it follows that $x_i = z_{i,0}$ and $y_j = z_{0,j}$, i.e. f is bijective. The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps; so f is an isomorphism.
3. It follows by the isomorphism of (2) that any closed subset of $\mathbb{P}^n(k) \times \mathbb{P}^m(k)$ is the zero locus of homogeneous polynomials in the $Z_{i,j}$, i.e. of bihomogeneous polynomials in the X_i and Y_j (of the same degree). Conversely, a zero locus of bihomogeneous polynomials can always be rewritten as a zero locus of bihomogeneous polynomials of the same degree in the X_i and Y_j. But such a polynomial is obviously a polynomial in the $Z_{i,j}$, so it determines an algebraic set in $X \cong \mathbb{P}^n \times \mathbb{P}^m$.

□

Example 16.4. Consider the case where $n = 1$ and $m = 2$. Then Segre embedding $f : \mathbb{P}^1(k) \times \mathbb{P}^2(k) \to \mathbb{P}^5(k)$ is given by

$$([x_0 : x_1], [y_0 : y_1 : y_2]) \mapsto [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2] := [z_{00} : z_{01} : z_{02} : z_{10} : z_{11} : z_{12}].$$

By Lemma (16.3), the vanishing ideal of $f(\mathbb{P}^1(k) \times \mathbb{P}^2(k))$ is given by

$$\langle Z_{00}Z_{11} - Z_{01}Z_{10}, Z_{00}Z_{12} - Z_{02}Z_{10}, Z_{01}Z_{12} - Z_{02}Z_{11} \rangle.$$

We can view this as the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \end{pmatrix}.$$

This is an example of a **determinantal variety**.

17 $\operatorname{Spec} A$ as a topological space

We start with the following basic definition. Let A be a ring. We set

$$\operatorname{Spec} A := \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

We will now endow $\operatorname{Spec} A$ with the structure of a topological space. For every subset S of A, we denote by $\mathrm{V}(S)$ to be the set of prime ideals of A containing S. Similarly we denote by $\mathrm{D}(S)$ to be the set of prime ideals of A which do not contain S. Clearly, if $\mathfrak{a}$ is the ideal generated by S, then $\mathrm{V}(S) = \mathrm{V}(\mathfrak{a})$ and $\mathrm{D}(S) = \mathrm{D}(\mathfrak{a})$. For any $f \in A$, we write $\mathrm{V}(f)$ and $\mathrm{D}(f)$ instead of $\mathrm{V}(\{f\})$ and $\mathrm{D}(\{f\})$ in order to simplify notation.

Lemma 17.1. *The map $\mathfrak{a} \mapsto V(\mathfrak{a})$ is an inclusion reversing map from the set of ideals of A to the set of subsets of* $\operatorname{Spec} A$*. Moreover, the following relations hold:*

1. *$V(0) = \operatorname{Spec} A$ and $V(1) = \emptyset$.*
2. *For two ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have*
$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$
3. *For every family $\{\mathfrak{a}_i\}_{i \in I}$ of ideals, we have*
$$V\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i).$$

Proof.

1. Trivial.
2. Since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a}$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{b}$, it follows that $V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a} \cap \mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b})$. It remains to show that $V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. Assume that $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$ but $\mathfrak{p} \not\supset \mathfrak{a}$ and $\mathfrak{p} \not\supset \mathfrak{b}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. Then there exists $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x, y \notin \mathfrak{p}$. But $xy \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ contradicts the fact that $\mathfrak{p}$ is prime.
3. That $V\left(\bigcup_{i \in I} \mathfrak{a}_i\right) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$ follows from the fact that $\sum_{i \in I} \mathfrak{a}_i$ is the ideal generated by $\bigcup_{i \in I} \mathfrak{a}_i$. That $V\left(\sum_{i \in I} \mathfrak{a}_i\right) = \bigcap_{i \in I} V(\mathfrak{a}_i)$ follows from the fact that $\mathfrak{p} \supset \sum_{i \in I} \mathfrak{a}_i$ if and only if $\mathfrak{p} \supset \mathfrak{a}_i$ for all $i \in I$ and for all primes $\mathfrak{p} \in \operatorname{Spec} A$.

□

The lemma shows that the subsets $V(\mathfrak{a})$ of $\operatorname{Spec} A$ form the closed sets of a topology on $\operatorname{Spec} A$. This leads us to the following definition.

Definition 17.1. Let A be a ring. The set $\operatorname{Spec} A$ of all prime ideals of A with the topology whose closed sets are the sets $V(\mathfrak{a})$, where $\mathfrak{a}$ runs through the set of ideals of A, is called the **prime spectrum** of A or simply the **spectrum** of A. The topology thus defined is called the **Zariski topology** on $\operatorname{Spec} A$.

Remark 40. If x is a point in $\operatorname{Spec} A$, we will often write $\mathfrak{p}_x$ instead of x when we think of x as a prime ideal of A.

Proposition 17.1. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Then $V(\mathfrak{a}) = V\left(\sqrt{\mathfrak{a}}\right)$.*

Proof. Since $\mathfrak{a} \subset \sqrt{\mathfrak{a}}$, we have $V(\mathfrak{a}) \supset V(\sqrt{\mathfrak{a}})$. For the reverse inclusion, let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \supset \mathfrak{a}$. Assume, for a contradiction, that $\mathfrak{p} \not\supset \sqrt{\mathfrak{a}}$. Choose $a \in \sqrt{\mathfrak{a}}$ such that $a \notin \mathfrak{p}$. Then $a^n \in \mathfrak{a} \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. But this contradicts the fact that $\mathfrak{p}$ is a prime ideal. □

For every subset Y of $\operatorname{Spec} A$, we set

$$\mathrm{I}(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

We obtain an inclusion reversing map $Y \mapsto \mathrm{I}(Y)$ from the set of subsets of $\operatorname{Spec} A$ to the set of ideals of A. Note that $\mathrm{I}(\emptyset) = A$. The maps V, D, and I are all related as follows.

Proposition 17.2. *Let $\mathfrak{a}$ an ideal in A and let Y a subset of* $\operatorname{Spec} A$*. We have*

1. $\sqrt{\mathrm{I}(Y)} = \mathrm{I}(Y)$.
2. $\mathrm{IV}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$.
3. $\mathrm{VI}(Y) = \overline{Y}$.
4. $\mathrm{ID}(\mathfrak{a}) = \sqrt{0} : \mathfrak{a}$.
5. $\mathrm{DI}(Y) = \bigcap_{\mathfrak{p} \in Y} \mathrm{D}(\mathfrak{p})$.
6. *The maps*

$$\{\textit{ideals } \mathfrak{a} \textit{ of } A \textit{ with } \mathfrak{a} = rad(\mathfrak{a})\} \underset{I}{\overset{V}{\rightleftarrows}} \{\textit{closed subsets } Y \textit{ of Spec } A\}$$

are mutually inverse bijections.

Proof. 1. The relation $\mathfrak{a} = \sqrt{\mathfrak{a}}$ means that $f^n \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ for all $f \in A$. This certainly holds for prime ideals and therefore for arbitrary intersections of prime ideals as well.

2. This follows from the fact that the radical of an ideal equals the intersection of all prime ideals containing it.

3. Let $\mathfrak{b}$ be an ideal of A. Observe that

$$\begin{aligned} \mathrm{V}(\mathfrak{b}) \supseteq Y &\iff \mathrm{IV}(\mathfrak{b}) \subseteq \mathrm{I}(Y) \\ &\iff \sqrt{\mathfrak{b}} \subseteq \mathrm{I}(Y) \\ &\iff \mathrm{V}(\sqrt{\mathfrak{b}}) \supseteq \mathrm{VI}(Y) \\ &\iff \mathrm{V}(\mathfrak{b}) \supseteq \mathrm{VI}(Y). \end{aligned}$$

Therefore $\mathrm{VI}(Y)$ is the smallest closed subset of $\operatorname{Spec} A$ which contains Y.

4. We first show that $\sqrt{0} : \mathfrak{a} \subseteq \mathrm{ID}(\mathfrak{a})$. Let $x \in \sqrt{0} : \mathfrak{a}$ and assume (to obtain a contradiction) that $x \notin \mathrm{ID}(\mathfrak{a})$. Since $x \notin \mathrm{ID}(\mathfrak{a})$, there exists a prime $\mathfrak{p}$ of A such that $\mathfrak{p} \not\supseteq \mathfrak{a}$ and $x \notin \mathfrak{p}$. Since $x \in \sqrt{0} : \mathfrak{a}$, we have $x\mathfrak{a} \subseteq \sqrt{0} \subseteq \mathfrak{p}$. In particular, either $\mathfrak{p} \supseteq \mathfrak{a}$ or $x \in \mathfrak{p}$. This is a contradiction. Thus we have $\sqrt{0} : \mathfrak{a} \subseteq \mathrm{ID}(\mathfrak{a})$.

Now we will show that $\sqrt{0} : \mathfrak{a} \supseteq \mathrm{ID}(\mathfrak{a})$. Let $x \in \mathrm{ID}(\mathfrak{a})$ (so x belongs to every prime ideal which does not contain $\mathfrak{a}$) and assume (to obtain a contradiction) that $x \notin \sqrt{0} : \mathfrak{a}$. Since $x \notin \sqrt{0} : \mathfrak{a}$, there exists $a \in \mathfrak{a}$ such that $ax \notin \sqrt{0}$. In particular, $\{(ax)^n\}_{n \in \mathbb{N}}$ forms a multiplicative set, and so we can localize at ax. Let $\mathfrak{q}$ be a prime ideal in A_{ax} and let $\mathfrak{p} := \iota_{ax}^{-1}(\mathfrak{q})$, where $\iota_{ax} \colon A \to A_{ax}$ is the canonical ring homomorphism. Then $\mathfrak{p}$ is a prime ideal in A which does not contain ax. This implies that $\mathfrak{p}$ does not contain $\mathfrak{a}$ nor x (if it did, then it'd certainly contain ax). This is a contradiction. Thus we have $\sqrt{0} : \mathfrak{a} \supseteq \mathrm{ID}(\mathfrak{a})$.

5. We have

$$\begin{aligned} \mathrm{DI}(Y) &= \mathrm{D}\left(\bigcap_{\mathfrak{p} \in Y} \mathfrak{p}\right) \\ &= \bigcap_{\mathfrak{p} \in Y} \mathrm{D}(\mathfrak{p}) \end{aligned}$$

6. This follows from part 2.

□

17.1 Properties of Spec A

Let A be a ring and let $\mathfrak{a}$ be an ideal in A. We set

$$D(\mathfrak{a}) := \operatorname{Spec}(A) \backslash V(\mathfrak{a}).$$

If $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = \langle f_1, \dots, f_n \rangle$, then we write $D(f_1, \dots, f_n)$ instead of $D(\langle f_1, \dots, f_n \rangle)$. Open subsets of $\operatorname{Spec} A$ of the form $D(f)$ are called **principal open sets** of $\operatorname{Spec} A$. Clearly, $D(0) = \emptyset$ and $D(u) = \operatorname{Spec} A$ for any unit $u \in A$. As for a prime ideal $\mathfrak{p}$ and two elements $f, g \in A$ we have $fg \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, we find

$$D(f) \cap D(g) = D(fg).$$

Lemma 17.2. *Let (f_i) be a family of elements in A and let $g \in A$. Then $D(g) \subseteq \bigcup_i D(f_i)$ if and only if there exists an integer $n > 0$ such that g^n is contained in the ideal $\mathfrak{a}$ generated by the f_i.*

Proof. Indeed, $D(g) \subseteq \bigcup_i D(f_i)$ is equivalent to $V(g) \supseteq V(\mathfrak{a})$ which is equivalent to $g \in \sqrt{\mathfrak{a}}$. □

Remark 41. Applying this to $g = 1$ it follows that $(D(f_i))_i$ is a covering of Spec$A$ if and only if the ideal generated by the f_i is equal to A.

Proposition 17.3. *Let A be a ring. The principal open subsets $D(f)$ for $f \in A$ form a basis of the topology of Spec A. For all $f \in A$ the open sets $D(f)$ are quasi-compact. In particular, the space Spec A is quasi-compact.*

Proof. Every closed subset of Spec$A$ is the intersection of closed sets of the form $V(f)$. By taking complements we see that the $D(f)$ form a basis for the topology.

Let $(g_i)_{i \in I}$ be a family of elements of A such that $D(f) \subseteq \bigcup_{i \in I} D(g_i)$. Then there exists an integer $n \geq 1$ such that $f^n = \sum_{i \in I} a_i g_i$ where $a_i \in A$ and $a_i = 0$ for all $i \notin J$, $J \subseteq I$ a suitable finite subset. Hence $D(f) \subseteq \bigcup_{j \in J} D(g_j)$. This proves that $D(f)$ is quasi-compact by the first part of the proposition. □

Proposition 17.4. *Let A be a ring and let U be an open subset of Spec A. Then U is quasi-compact if and only if it is the complement of a closed set of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is a finitely generated ideal.*

Proof. Suppose $U = D(\mathfrak{a})$ where $\mathfrak{a}$ is a finitely generated. Then $\mathfrak{a} = \langle f_1, \ldots, f_n \rangle$ for some $f_1, \ldots, f_n \in A$. In particular,

$$U = D(f_1, \ldots, f_n) = D(f_1) \cup \cdots \cup D(f_n).$$

As U is a finite union of compact spaces, it must be compact.

Conversely, suppose that U is quasi-compact. Since $\{D(g)\}_{g \in A}$ forms a basis for the topology, we can write

$$U = \bigcup_{i \in I} D(g_i)$$

for some $g_i \in A$. In particular, $\{D(g_i)\}_{i \in I}$ is an open covering of U. Since U is quasi-compact, there exists a finite subcovering, say $\{D(g_1), \ldots, D(g_n)\}$. Thus,

$$U = \bigcup_{i=1}^{n} D(g_i) = D(g_1, \ldots, g_n).$$

In particular, setting $\mathfrak{a} = \langle g_1, \ldots, g_n \rangle$, we can write $U = D(\mathfrak{a})$ where $\mathfrak{a}$ is finitely generated. □

Example 17.1. Let $A = K[x, y]$, $\mathfrak{a} = \langle x^2, y^2 \rangle$, and $\mathfrak{b} = \langle x^2, xy, y^2 \rangle$. Even though $\mathfrak{a} \subset \mathfrak{b}$ (where the inclusion is strict), we have $V(\mathfrak{a}) = V(\mathfrak{b})$, since $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.

Proposition 17.5. *Let A be a ring. A subset Y of Spec A is irreducible if and only if $\mathfrak{p} := I(Y)$ is a prime ideal. In this case $\{\mathfrak{p}\}$ is dense in $\overline{Y}$.*

Proof. Assume that Y is irreducible. Let $f, g \in A$ with $fg \in \mathfrak{p}$. Then

$$Y \subseteq V(fg) = V(f) \cup V(g).$$

As Y is irreducbile, either $Y \subseteq V(f)$ or $Y \subseteq V(g)$ which implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

Conversely let $\mathfrak{p}$ be a prime. Then by Proposition (17.2),

$$\overline{Y} = V(\mathfrak{p}) = V(I(\{\mathfrak{p}\}) = \overline{\{\mathfrak{p}\}}.$$

Therefore $\overline{Y}$ is the closure of the irreducible set $\{\mathfrak{p}\}$ and therefore irreducible. This implies that the dense subset Y is also irreducible. □

Note that for arbitrary irreducible subsets Y the prime ideal $I(Y)$ is not necessarily a point in Y. But this is clearly true if Y is closed, or more generally, if Y is locally closed.

Corollary 5. *The map $\mathfrak{p} \mapsto V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is a bijection from Spec A onto the set of closed irreducible subsets of Spec A. Via this bijection, the minimal prime ideals of A correspond to the irreducible components of Spec A.*

Definition 17.2. Let X be an arbitrary topological space.

1. A point $x \in X$ is called **closed** if the set $\{x\}$ is closed,

2. We say that a point $\eta \in X$ is a **generic point** if $\overline{\{\eta\}} = X$.

3. We say x and x' be two points of X. We say that x is a **generization** or that x' is a **specialization** of x if $x' \in \overline{\{x\}}$.

4. A point $x \in X$ is called a **maximal point** if its closure $\overline{\{x\}}$ is an irreducible component of X.

Thus a point $\eta \in X$ is generic if and only if it is a generization of every point of X. As the closure of an irreducible set is again irreducible, the existence of a generic point implies that X is irreducible.

Example 17.2. If $X = \operatorname{Spec} A$ is the spectrum of a ring, the notions introduced in Definition (17.2) have the following algebraic meaning.

1. A point $x \in X$ is closed if and only if $\mathfrak{p}_x$ is a maximal ideal.

2. A point $\eta \in X$ is a generic point of X if and only if $\mathfrak{p}_\eta$ is the unique minimal prime ideal. This exists if and only if the nilradical of A is a prime ideal.

3. A point x is a generization of a point x' (in other words, x' is a specialization of x) if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_{x'}$.

4. A point $x \in X$ is a maximal point if and only if $\mathfrak{p}_x$ is a minimal prime ideal.

17.2 The Functor $A \mapsto \operatorname{Spec} A$

We will now show that $A \mapsto \operatorname{Spec} A$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\varphi\colon A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a prime ideal of B, then $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of. Therefore we obtain a map ${}^{\mathrm{a}}\varphi = \operatorname{Spec}'$ from $\operatorname{Spec} B$ to $\operatorname{Spec} A$ given by

$$ {}^{\mathrm{a}}\varphi(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) $$

for all $\mathfrak{q} \in \operatorname{Spec} B$.

Proposition 17.6. *Let S be a subset of A and let $\mathfrak{b}$ be an ideal of B. Then*

1. $({}^{\mathrm{a}}\varphi)^{-1}(\mathrm{V}(S)) = \mathrm{V}(\varphi(S))$

2. $\mathrm{V}(\varphi^{-1}(\mathfrak{b})) = \overline{{}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))}$

Proof. 1. A prime ideal $\mathfrak{q}$ of B contains $\varphi(S)$ if and only if $\varphi^{-1}(\mathfrak{q})$ contains S.

2. By Proposition (17.2), we have

$$ \begin{aligned} \overline{{}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))} &= \mathrm{VI}({}^{\mathrm{a}}\varphi(\mathrm{V}(\mathfrak{b}))) \\ &= \mathrm{V}\left(\bigcap_{\mathfrak{q} \in \mathrm{V}(\mathfrak{b})} \varphi^{-1}(\mathfrak{q}) \right) \\ &= \mathrm{V}(\varphi^{-1}(\sqrt{\mathfrak{b}})) \\ &= \mathrm{V}\left(\sqrt{\varphi^{-1}(\mathfrak{b})} \right) \\ &= \mathrm{V}(\varphi^{-1}(\mathfrak{b})) \end{aligned} $$

□

The proposition shows in particular that ${}^{\mathrm{a}}\varphi$ is continuous. As ${}^{\mathrm{a}}(\psi \circ \varphi) ={}^{\mathrm{a}} \varphi \circ^{\mathrm{a}} \psi$ for any ring homomorphism $\psi\colon B \to C$, we obtain a contravariant functor $A \mapsto \operatorname{Spec} A$ from the category of rings to the category of topological spaces.

Corollary 6. *The map ${}^{a}\varphi$ is dominant (i.e. its image is dense in Spec A) if and only if every element of Ker(φ) is nilpotent.*

Proof. We apply (2) to $\mathfrak{b} = 0$. □

Proposition 17.7. *Let A be a ring.*

1. *Let $\varphi : A \to B$ be a surjective homomorphism of rings with kernel $\mathfrak{a}$. Then ${}^a\varphi$ is a homeomorphism of Spec B onto the closed subset $V(\mathfrak{a})$ of Spec A.*

2. *Let S be a multiplicative subset of A and let $\varphi : A \to S^{-1}A =: B$ be the canonical homomorphism. Then ${}^a\varphi$ is a homeomorphism of Spec $S^{-1}A$ onto the subspace of Spec A consisting of prime ideals $\mathfrak{p} \subset A$ with $S \cap \mathfrak{p} = \emptyset$.*

Proof. In both cases it is clear that ${}^a\varphi$ is injective with the stated image. Moreover in both cases a prime ideal $\mathfrak{q}$ of B contains an indeal $\mathfrak{b}$ of B if and only if $\varphi^{-1}(\mathfrak{q})$ contains $\varphi^{-1}(\mathfrak{b})$. This shows that ${}^a\varphi(V(\mathfrak{b})) = V(\varphi^{-1}(\mathfrak{b})) \cap \mathrm{Im}({}^a\varphi)$. Therefore ${}^a\varphi$ is a homeomorphism onto its image.

□

Remark 42. Let A be a ring and let $\mathfrak{p}, \mathfrak{q} \subset A$ be prime ideals. Proposition (17.7) shows that the passage from A to $A_\mathfrak{p}$ cuts out all prime ideals except those contained in $\mathfrak{p}$. The passage from A to $A/\mathfrak{q}$ cuts out all prime ideals except those containing $\mathfrak{q}$. Hence, if $\mathfrak{q} \subseteq \mathfrak{p}$ localizing with respect to $\mathfrak{p}$ and taking the quotient modulo $\mathfrak{q}$ (in either order as these operations commute) we obtain a ring whose prime ideals are those prime ideals of A that lie between $\mathfrak{q}$ and $\mathfrak{p}$. For $\mathfrak{q} = \mathfrak{p}$, we obtain the field

$$\kappa(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Frac}(A/\mathfrak{p}),$$

which is called the **residue field** at $\mathfrak{p}$.

18 Spectrum of a Ring as a Locally Ringed Space

Let A be a ring. We will now endow the topological space Spec A with the structure of a locally ringed space and obtain a functor $A \mapsto \operatorname{Spec} A$ from the category of rings to the category of locally ringed spaces which we will show to be fully faithful.

18.1 Structure Sheaf on Spec A

We set $X = \operatorname{Spec} A$. Recall that the principal open sets $\mathrm{D}(f)$ for $f \in A$ form a basis of the topology of X. We will define a presheaf $\mathcal{O}_X$ on this basis and then prove that the sheaf axioms are satisfied. The basic idea is this: looking back at the analogy with prevarieties, we certainly want to have $\mathcal{O}_X(X) = A$. More generally, for $f \in A$, we consider the localization A_f of A. Denote by $\iota_f : A \to A_f$ the canonical ring homomorphism $a \mapsto a/1$. By Proposition (17.7), ${}^\mathrm{a}\iota_f$ is a homeomorphism of Spec A_f onto $\mathrm{D}(f)$. So it seems reasonable to set $\mathcal{O}_X(\mathrm{D}(f)) = A_f$. Let us check that this is a sensible definition: we must check that $A_f = A_g$ whenever $\mathrm{D}(f) = \mathrm{D}(g)$, define restriction maps, and check that the sheaf axioms are satisfied.

For $f, g \in A$, we have $\mathrm{D}(f) \subseteq \mathrm{D}(g)$ if and only if there exists an integer $n \geq 1$ such that $f^n \in \langle g \rangle$ or, equivalently, $g/1 \in (A_f)^\times$. In this case we obtain a unique ring homomorphism $\rho_{f,g} \colon A_g \to A_f$ such that $\rho_{f,g} \circ \iota_g = \iota_f$ by the universal mapping property of localization. Whenever $\mathrm{D}(f) \subseteq \mathrm{D}(g) \subseteq \mathrm{D}(h)$, we have $\rho_{f,g} \circ \rho_{g,h} = \rho_{f,h}$. In particular, if $\mathrm{D}(f) = \mathrm{D}(g)$, then $\rho_{f,g}$ is an isomorphism, which we use to identify A_g and A_f. Therefore we can define

$$\mathcal{O}_X(\mathrm{D}(f)) := A_f$$

and obtain a presheaf of rings on the basis $\mathcal{B} = \{\mathrm{D}(f) \mid f \in A\}$ for the topological space Spec A. The restriction maps are the ring homomorphism $\rho_{f,g}$.

Theorem 18.1. *The presheaf $\mathcal{O}_X$ is a sheaf on $\mathcal{B}$.*

We denote the sheaf of rings on X associated to $\mathcal{O}_X$ again by $\mathcal{O}_X$. For all points $x \in X = \operatorname{Spec} A$, we have

$$\mathcal{O}_{X,x} = \lim_{\mathrm{D}(f) \ni x} \mathcal{O}_X(\mathrm{D}(f)) = \lim_{f \notin \mathfrak{p}_x} A_f = A_{\mathfrak{p}_x}.$$

In particular, $(X, \mathcal{O}_X)$ is a locally ringed space. We will often simply write Spec A instead of $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Proof. Let $\mathrm{D}(f)$ be a principal open set and let $\{\mathrm{D}(f_i)\}_{i \in I}$ be an open covering over $\mathrm{D}(f)$. We have to show the following two properties:

1. Let $s \in \mathcal{O}_X(\mathrm{D}(f))$ be such that $s|_{\mathrm{D}(f_i)} = 0$ for all $i \in I$. Then $s = 0$.

2. For $i \in I$, let $s_i \in \mathcal{O}_X(\mathrm{D}(f_i))$ be such that $s_i|_{\mathrm{D}(f_i f_j)} = s_j|_{\mathrm{D}(f_i f_j)}$ for all $i, j \in I$. Then there exists $s \in \mathcal{O}_X(\mathrm{D}(f))$ such that $s|_{\mathrm{D}(f_i)} = s_i$ for all $i \in I$.

As $\mathrm{D}(f)$ is quasi-compact, we can assume that I is finite. Restricting the presheaf $\mathcal{O}_X$ to $\mathrm{D}(f)$ and replacing A by A_f if necessary, we may assume that $f = 1$ and hence $\mathrm{D}(f) = X$ to ease the notation. The relation $X = \bigcup_{i \in I} \mathrm{D}(f_i)$ is equivalent to $\langle f_i \mid i \in I \rangle = A$ (indeed $\sqrt{\mathfrak{a}} = A$ implies $\mathfrak{a} = A$). As $\mathrm{D}(f_i) = \mathrm{D}(f_i^n)$ for all integers $n \geq 1$ there exists elements $b_i \in A$ (depending on n) such that

$$\sum_{i \in I} b_i f_i^n = 1. \tag{14}$$

Proof of 1: let $s = a \in A$ be such that the image of a in A_{f_i} is zero for all $i \in I$. As I is finite, there exists an integer $n \geq 1$, independent of i, such that $f_i^n a = 0$. By (14),

$$a = \left(\sum_{i \in I} b_i f_i^n \right) a = 0.$$

Proof of 2: as I is finite, we can write $s_i = a_i / f_i^n$ for some n independent of i. By hypothesis, the images of a_i / f_i^n and of a_j / f_j^n in $A_{f_i f_j}$ are equal for all $i, j \in I$. Therefore there exists an integer $m \geq 1$ (which again we can choose independent of i and j) such that

$$(f_i f_j)^m (f_j^n a_i - f_i^n a_j) = 0.$$

Replacing a_i by $f_i^m a_i$ and n by $n + m$ (which does not change s_i), we see that $f_j^n a_i = f_i^n a_j$ for all $i, j \in I$. We set

$$s := \sum_{j \in I} b_j a_j \in A,$$

where the b_j are the elements in (14). Then

$$\begin{aligned} f_i^n s &= f_i^n \left(\sum_{j \in I} b_j a_j \right) \\ &= \sum_{j \in I} b_j (f_i^n a_j) \\ &= \sum_{j \in I} b_j (f_j^n a_i) \\ &= \left(\sum_{j \in I} b_j f_j^n \right) a_i \\ &= a_i. \end{aligned}$$

This means that the image of s in A_{f_i} is s_i. □

Remark 43. We have just proved that the sequence

$$0 \longrightarrow A \longrightarrow \bigoplus_{i \in I} A_{f_i} \longrightarrow \bigoplus_{i,j \in I} A_{f_i f_j}$$

is exact.

18.2 The Functor $A \mapsto (\mathbf{Spec}\ A, \mathcal{O}_{\mathbf{Spec}\ A})$

Definition 18.1. A locally ringed space $(X, \mathcal{O}_X)$ is called an **affine scheme**, if there exists a ring A such that $(X, \mathcal{O}_X)$ is isomorphism to $(\mathrm{Spec}\ A, \mathcal{O}_{\mathrm{Spec}\ A})$.

19 Schemes