MDG Algebras and Modules

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1 Basic Definitions

Throughout this document, let R be a commutative ring.

1.1 MDG R-algebras

Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ be a chain map. If $\sum_{i=1}^n a_i \otimes b_i$ is a tensor in $A \otimes_R A$, then we often denote its image under μ by

$$\mu\left(\sum_{i=1}^m a_i \otimes b_i\right) = \sum_{i=1}^m a_i \star_\mu b_i,$$

furthermore, if μ is understood from context, then we will simplify our notation further by writing

$$\sum_{i=1}^m a_i \star_\mu b_i = \sum_{i=1}^m a_i b_i.$$

Since μ is a chain map, the Leibniz law is satisfied, which in this context says

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\mathrm{d}(b)$$

for all $a, b \in A$ with a homogeneous, where $|a|$ denotes the homological degree of a. The Leibniz law is a condition imposed on the triple (A, d, μ): it is an equation which involves the graded structure on A, the differential d, and the chain map μ, to clean notation however, we will often refer to the triple (A, d, μ) via its underlying graded R-module A, where we think of A as being equipped with d and μ. Here are some other conditions we will want to impose on A:

- We say A is **unital** if there exists $1 \in A$ such that
$$a1 = a = 1a$$
for all $a \in A$.
- We say A is **associative** if
$$a(bc) = (ab)c$$
for all $a, b, c \in A$.
- We say A is **graded-commutative** if
$$ab = (-1)^{|a||b|} ba$$
for all homogeneous $a, b \in A$.
- We say A is **strictly graded-commutative** if it is graded-commutative and satisfies the following extra property:
$$a^2 = 0$$
for all homogeneous $a \in A$ where $|a|$ is odd.

With these defintions in place, let us now define some of the objects which we will be considering in this document.

Definition 1.1. Let (A, d) be an R-complex and let $\mu\colon A \otimes_R A \to A$ be a chain map.

1. We say A is an **MDG R-algebra** if A is unital. In this case, we say μ is the **multiplication** of A. The multiplication of A is sometimes denoted μ_A in case context is not clear, just like how the differential of A is sometimes denoted d_A.
2. We say A is a **DG R-algebra** if A is unital and associative. In other words, a DG R-algebra is an associative MDG R-algebra.

Throughout this document, all MDG R-algebras and DG R-algebras are also assumed to be strictly graded-commutative.

1.1.1 MDG R-algebra homomorphisms

Having defined MDG R-algebras, we now wish to define MDG R-algebras homomorphisms.

Definition 1.2. Let A and B be two MDG R-algebras and let $f\colon A \to B$ be a function. We say f is an MDG R-algebra **homomorphism** (or a **homomorphism** of MDG R-algebras) if it satisfies the following three properties:

1. It is a chain map of the underlying R-complexes. This means that f is a graded R-linear map which commutes with the differentials d_A and d_B.

2. It preserves the identity element. This means that $f(1) = 1$.

3. It preserves the multiplication operations. This means that $f(a_1 a_2) = f(a_1)f(a_2)$ for all $a_1, a_2 \in A$.

If, in addition, f is a bijection, then we say f is an MDG R-algebra **isomorphism**. The collection of all MDG R-algebras algebras together with MDG R-algebra homomorphisms forms a category, which we denote by $\mathbf{MDG}_R$. If A and B are DG R-algebras, then a **DG R-algebra homomorphism** between them is just an MDG R-algebra homomorphism, where we view A and B as MDG R-algebras. The collection of all DG R-algebras algebras together with DG R-algebra homomorphisms forms a category, which we denote by $\mathbf{DG}_R$. We have an inclusion of categories $\mathbf{DG}_R \hookrightarrow \mathbf{NDG}_R$ which is fully faithful but not essentially surjective on objects.

Remark. Note that property 3 in Definition (1.2) can be interpreted as saying the following diagram commutes:

$$\begin{tikzcd}
A \otimes_R A \arrow[r, "f\otimes f"] \arrow[d, "\mu_A"'] & B \otimes_R B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Similar interpretations can be given for properties 1 and 2 as well.

Example 1.1. Assume that $(R, \mathfrak{m})$ is local, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let $\tau\colon F \to R/I$ be a free resolution of R/I over R. We can view R/I as an MDG R-algebra is the obvious way, where as an R-complex, R/I is the trivial one who only nonzero homogeneous component is in homological degree 0. The multiplication m of R/I is the one that comes from R (so R/I is even a DG R-algebra). Since F is a free resolution of R/I, we can lift $\mathrm{m}\colon R/I \otimes_R R/I \to R/I$ to a chain map $\mu\colon F \otimes_R F \to F$. It is known that μ can be chosen such that it gives (F, d) the structure of an MDG R-algebra. Then the map $\tau\colon F \to R/I$ is an MDG R-algebra homomorphism and we call (F, τ) a free **MDG R-algebra resolution** of R/I over R.

1.2 MDG A-modules

Now we want to define modules over an MDG R-algebra. Essentially, these are like modules over a ring except we do not require the associative law to hold.

Definition 1.3. Let A be an MDG R-algebra.

1. A **left MDG A-module** is a triple $(X, \mathrm{d}_X, \mu_{A,X})$ where (X, d_X) is an R-complex and where $\mu_{A,X}\colon A \otimes_R X \to X$ is a chain map which satisfies $1x = x$ for all $x \in X$. In this case, we say $\mu_{A,X}$ is the **left scalar-multiplication** of X.

2. A **right MDG A-module** is a triple $(X, \mathrm{d}_X, \mu_{X,A})$ where (X, d_X) is an R-complex and where $\mu_{X,A}\colon X \otimes_R A \to X$ is a chain map which satisfies $x1 = x$ for all $x \in X$. In this case, we say $\mu_{X,A}$ is the **right scalar-multiplication** of X.

3. A **two-sided MDG A-module** is a triple $(X, \mathrm{d}_X, \mu_{A,X,A})$ where (X, d_X) is an R-complex and where $\mu_{A,X,A}\colon (A \otimes_R X) \otimes_R A \to X$ is a chain map, called the **two-sided scalar-multiplication** of X such that
   (a) Restricting $\mu_{A,X,A}$ to $(A \otimes_R X) \otimes_R 1$ gives a left scalar-multiplication of X, which we denote $\mu_{A,X}$, making X a left MDG A-module.
   (b) Restricting $\mu_{A,X,A}$ to $(1 \otimes_R X) \otimes_R A$ gives a right scalar-multiplication of X, which we denote $\mu_{X,A}$, making X a right MDG A-module.
   (c) The left scalar-multiplication of X and the right scalar-multiplication of X are compatible with the graded-commutative structure in the sense that for all homogeneous $a \in A$ and homogeneous $x \in X$ we have $ax = (-1)^{|a||x|}xa$.

4. A **left DG** A-module is a left MDG A-module X which is associative, meaning $(ab)x = a(bx)$ for all $a, b \in A$ and for all $x \in X$.

5. A **right DG** A-module is a right MDG A-module X which is associative, meaning $(xa)b = x(ab)$ for all $a, b \in A$ and for all $x \in X$.

6. A **two-sided DG** A-module is a two-sided MDG A-module X which is associative, meaning $(ab)x = a(bx)$ for all $a, b \in A$ and for all $x \in X$. Note that in this case, graded-commutativity of X then automatically implies $(ax)b = a(xb)$ and $(xa)b = x(ab)$ for all $a, b \in A$ and $x \in X$.

Here again we are using the convention that the image of a tensor $\sum_{i=1}^n a_i \otimes x_i$ in $A \otimes_R X$ under the scalar-multiplication map $\mu_{A,X}$ is denoted by

$$\mu_{A,X}\left(\sum_{i=1}^n a_i \otimes x_i\right) = \sum_{i=1}^n a_i \star_{\mu_{A,X}} x_i = \sum_{i=1}^n a_i x_i.$$

The theory of left/right/two-sided MDG A-modules are very similar for the most part, though there are some notable differences (certainly the two-sided theory requires special attention at times). We will mention some of these differences when they arise, however we will focus mostly on developing the theory left MDG A-modules. All of the tools and methods that we introduce in the theory of left MDG A-modules will have their counterparts in the theory of right/two-sided MDG A-modules as well. If we write "let X be an MDG A-module", then it will be understood that A is an MDG R-algebra and that X is a *left* MDG A-module. In this case, we'll simplify our notation by writing μ_X instead of $\mu_{A,X}$ to denote the scalar-multiplication of X. Note that μ_X being a chain map implies that it satisfies the Leibniz law, which in this context says that for all homogeneous $a \in A$ and $x \in X$, we have

$$\mathrm{d}_X(ax) = \mathrm{d}_A(a)x + (-1)^{|a|}a\mathrm{d}_X(x).$$

Note that even in the case where A is a DG R-algebra, it still makes sense to talk about MDG A-modules: if X is an MDG A-module, then A being associative does not imply X is associative. On the other hand, it may be the case that A is non-associative, and yet X is associative. Thus it also makes sense to talk about DG A-modules when A is a non-associative MDG R-algebra.

1.2.1 MDG A-module homomorphisms

Having defined MDG A-modules, we now wish to define A-module homomorphisms between them.

Definition 1.4. Let X and Y be two MDG A-modules, let $i \in \mathbb{Z}$, and let $\varphi\colon X \to Y$ be a function. We say φ is an MDG A**-module homomorphism of degree** i if it satisfies the following two properties:

1. It is a chain map of degree i of the underlying R-complexes, meaning it is a graded R-linear map of degree i which satisfies
$$\varphi \mathrm{d}_X = (-1)^{|\varphi|} \mathrm{d}_Y \varphi,$$
where we denote $|\varphi| = i$.

2. It preserves the scalar-multiplications of X in and Y, meaning
$$\varphi(ax) = (-1)^{|a||\varphi|} a\varphi(x)$$
for all $a \in A$ and $x \in X$.

In the case where $|\varphi| = 0$, then we will simply call φ an MDG A-module homomorphism. If, in addition, φ is a bijection, then we say φ is an MDG A-module **isomorphism**. The collection of all MDG A-modules together with MDG A-module homomorphisms forms a category, which we denote by $\mathbf{MDGmod}_A$. If X and Y are DG A-modules, then a **DG** A**-module homomorphism** between them is just an MDG A-module homomorphism, where we view X and Y as MDG A-modules. The collection of all DG A-modules together with DG A-modules homomorphisms forms a category, which we denote by $\mathbf{DGmod}_A$. We have an inclusion of categories $\mathbf{DGmod}_A \hookrightarrow \mathbf{MDGmod}_A$ which is fully faithful but not essentially surjective on objects.

Remark. If we write "let $\varphi\colon X \to Y$ be an NDG A-module homomorphism" without first specifying what A is, then it is understood that A is an NDG R-algebra and that X and Y are NDG A-modules.

1.2.2 Submodules

Definition 1.5. Let X and Y be two MDG A-modules. We say X is an **MDG** A**-submodule** of Y if $X \subseteq Y$. An MDG A-submodule of A is called an **MDG ideal** of A. Given any collection $\mathcal{S} = \{x_\lambda \mid \lambda \in \Lambda\}$ of elements of X, we denote by $\langle\langle \mathcal{S} \rangle\rangle$ to be the smallest PDG A-submodule of X which contains $\mathcal{S}$, and we denote by $\langle \mathcal{S} \rangle$ to be the set of all A-linear combinations of elements in $\mathcal{S}$.

Proposition 1.1. *Let X be a PDG A-module and let $\mathcal{S} = \{x_\lambda \mid \lambda \in \Lambda\}$ of elements of X. Define $\mathrm{d}_X\mathcal{S} = \{\mathrm{d}_X(x_\lambda) \mid \lambda \in \Lambda\}$. Then we have*
$$\langle\langle \mathcal{S} \rangle\rangle = \langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle.$$

Proof. Since $\langle\langle \mathcal{S} \rangle\rangle$ is the smallest PDG A-submodule of X which contains $\mathcal{S}$, we must have $\mathrm{d}_X(x_\lambda) \in \langle\langle \mathcal{S} \rangle\rangle$ for all $\lambda \in \Lambda$. Thus all A-linear combinations of elements in $\mathcal{S} \cup \mathrm{d}_X\mathcal{S}$ must belong to $\langle\langle \mathcal{S} \rangle\rangle$ as well. Therefore
$$\langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle \subseteq \langle\langle \mathcal{S} \rangle\rangle.$$

For the reverse direction, notice that Leibniz law ensures that $\langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle$ is d_X-stable (meaning d_X maps $\langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle$ to itself). Indeed, if
$$\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j \mathrm{d}_X(x_{\lambda_j}),$$
is a finite A-linear combination of elements in $\mathcal{S} \cup \mathrm{d}_X\mathcal{S}$ where each a_i and b_j are homogeneous, then note that
$$\begin{aligned}
\mathrm{d}_X\left(\sum_{i=1}^m a_i x_{\lambda_i} + \sum_{j=1}^n b_j \mathrm{d}_X(x_{\lambda_j})\right) &= \sum_{i=1}^m \mathrm{d}_X(a_i x_{\lambda_i}) + \sum_{j=1}^n \mathrm{d}_X(b_j \mathrm{d}(x_{\lambda_j})) \\
&= \sum_{i=1}^m \left(\mathrm{d}_A(a_i) x_{\lambda_i} + (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i})\right) + \sum_{j=1}^n \left(\mathrm{d}_A(b_j) x_{\lambda_j} + (-1)^{|b_j|} b_j \mathrm{d}_X^2(x_{\lambda_j})\right) \\
&= \sum_{i=1}^m \mathrm{d}_A(a_i) x_{\lambda_i} + \sum_{i=1}^m (-1)^{|a_i|} a_i \mathrm{d}_X(x_{\lambda_i}) + \sum_{j=1}^n \mathrm{d}_A(b_j) x_{\lambda_j} \\
&\in \langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle.
\end{aligned}$$

In particular, we see that $\langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle$ is a PDG A-submodule of X which contains $\mathcal{S}$. Since $\langle\langle \mathcal{S} \rangle\rangle$ is the *smallest* PDG A-submodule of X which contains $\mathcal{S}$, it follows that
$$\langle \mathcal{S} \cup \mathrm{d}_X\mathcal{S} \rangle \supseteq \langle\langle \mathcal{S} \rangle\rangle.$$

□

1.2.3 Kernels

Proposition 1.2. *Let $\varphi \colon X \to Y$ be an MDG A-module homomorphism and set $W = \ker \varphi$. Then W has the structure of an MDG A-submodule of X, where $\mathrm{d}_W = \mathrm{d}_X|_W$ and where $\mu_W = \mu_X|_{A \otimes_R W}$.*

Proof. Since both d_W and μ_W are restrictions of d_X and μ_X respectively, we just need to check that d_W and μ_W land in W, since in this case, all of the properties needed in order for W to be an MDG A-submodule of X will be inherited from X. First we show d_W lands in W. Let $w \in W$. Then

$$\begin{aligned} \varphi \mathrm{d}_W(w) &= \mathrm{d}_W \varphi(w) \\ &= \mathrm{d}_W(0) \\ &= 0 \end{aligned}$$

implies $\mathrm{d}_W(w) \in W$. It follows that d_W lands in W. Now we show μ_W lands in W. Let $a \otimes w$ be an elementary tensor in $A \otimes_R W$. Then

$$\begin{aligned} \varphi \mu_W(a \otimes w) &= \varphi(aw) \\ &= a\varphi(w) \\ &= a \cdot 0 \\ &= 0. \end{aligned}$$

It follows that μ_W lands in W. □

1.2.4 Images

Proposition 1.3. *Let $\varphi \colon X \to Y$ be a PDG A-module homomorphism and set $Z = \operatorname{im} \varphi$. Then Z has the structure of a PDG A-submodule of Y, where $\mathrm{d}_Z = \mathrm{d}_Y|_Z$ and where $\mu_Z = \mu_Y|_{A \otimes_R Z}$.*

Proof. It suffices to check that d_Z and μ_Z land in Z. First we show d_Z lands in Z. Let $\varphi(x) \in Z$ where $x \in X$. Then

$$\begin{aligned} \mathrm{d}_Z(\varphi(x)) &= \mathrm{d}_Z \varphi(x) \\ &= \mathrm{d}_Y \varphi(x) \\ &= \varphi \mathrm{d}_X(x) \\ &= \varphi(\mathrm{d}_X(x)). \end{aligned}$$

It follows that d_Z lands in Z. Now we show μ_Z lands in Z. Let $a \otimes \varphi(x)$ be an elementary tensor in $A \otimes_R Z$ where $x \in X$. Then

$$\begin{aligned} \mu_Z((a \otimes \varphi(x)) &= \mu_Y((a \otimes \varphi(x)) \\ &= a\varphi(x) \\ &= \varphi(ax). \end{aligned}$$

It follows that μ_Z lands in Z. □

1.3 Associativity

1.3.1 The associator of an MDG A-module

Let X be an MDG A-module. To keep track of the failure for of X to be associative, we introduce the following definition:

Definition 1.6. Let $\mathrm{a}_{A,A,X}\colon (A\otimes_R A)\otimes_R X \to A\otimes_R (A\otimes_R X)$ denote the unique chain map defined on elementary tensors by

$$(a\otimes b)\otimes x \mapsto a\otimes (b\otimes x).$$

As usual, we simplify our notation by writing a_X instead of $\mathrm{a}_{A,A,X}$ whenever context is clear. We define the **associator** of X is the chain map $[\cdot,\cdot,\cdot]_X\colon (A\otimes_R A)\otimes_R X \to X$ defined by

$$[\cdot,\cdot,\cdot]_X := \mu_X(1\otimes\mu_X)\mathrm{a}_X - \mu_X(\mu_A\otimes 1).$$

In other words, the associator of X is the difference of the two maps in the diagram below both of which start at the top left part of the pentagon and end at the bottom of the pentagon:

$$\begin{tikzcd}
(A\otimes_R A)\otimes_R X \arrow[rr, "\mathrm{a}_X"] \arrow[d, "\mu_A\otimes 1"'] & & A\otimes_R (A\otimes_R X) \arrow[d, "1\otimes\mu_X"] \\
A\otimes_R X \arrow[dr, "\mu_X"'] & & A\otimes_R X \arrow[dl, "\mu_X"] \\
& X &
\end{tikzcd}$$

In particular, $[\cdot,\cdot,\cdot]_X$ measures the failure of the diagram above to be commutative, that is, it measures the failure of X to be associative.

Remark. Even in the case where X is a two-sided MDG A-module, we want to define the associator of X to be the associator of $(X,\mathrm{d}_X,\mu_{A,X})$, where we view X as a left MDG A-module. Indeed, X is associative if and only if $[\cdot,\cdot,\cdot]_{(X,\mathrm{d}_X,\mu_{A,X})}=0$. If X is a right NDG A-module, then obviously the associator of it should be a the flipped version of Definition (1.6).

If X is understood from context, then we will simplify our notation by dropping X from the subscript in $[\cdot,\cdot,\cdot]_X$ and simply write $[\cdot,\cdot,\cdot]$ to denote the associator of X. Observe that since $[\cdot,\cdot,\cdot]$ is a chain map, we see that $[\cdot,\cdot,\cdot]$ is a graded trilinear map which satisfies

$$\mathrm{d}_X[a,b,x] = [\mathrm{d}_A(a),b,x] + (-1)^{|a|}[a,\mathrm{d}_A(b),x] + (-1)^{|a|+|b|}[a,b,\mathrm{d}_X(x)]. \tag{1}$$

for all homogeneous $a,b\in A$ and $x\in X$. Here are some other important identities which the associator satisfies:

- For all $a,b,c\in A$ and $x\in X$ we have

$$a[b,c,x]-[ab,c,x]+[a,bc,x]-[a,b,cx]+[a,b,c]x=0 \tag{2}$$

- If X is two-sided, then for all homogeneous $a,b\in A$ and homogeneous $x\in X$ we have a graded-commutative version of the Jacobi identity

$$(-1)^{(|a|+|x|)|b|}[a,b,x]+(-1)^{(|b|+|x|)|a|}[x,a,b]+(-1)^{(|a|+|b|)|x|}[b,x,a]=0 \tag{3}$$

- If X is two-sided, then for all homogeneous $a,b\in A$ and homogeneous $x\in X$ we have

$$[a,b,x]+(-1)^{|a||b|+|a||x|+|b||x|}[x,b,a]=0. \tag{4}$$

We put these identities to use in the next proposition.

Proposition 1.4. *Let X be a two-sided PDG A-module. The following conditions are equivalent.*

1. $[a,a,x]=0$ *for all* $a\in A$ *and* $x\in X$;
2. $[a,x,a]=0$ *for all* $a\in A$ *and* $x\in X$;
3. $[x,a,a]=0$ *for all* $a\in A$ *and* $x\in X$;

Proof. Let $a\in A$ and let $x\in X$ where both a and x are homogeneous. From (4) we obtain

$$\begin{aligned}[a,a,x] &= -(-1)^{|a||a|+|a||x|+|a||x|}[x,a,a]\\ &= -(-1)^{|a|}[x,a,a].\end{aligned}$$

Thus clearly 1 and 3 are equivalent. Combining this with (3), we obtain

$$\begin{aligned}
0 &= (-1)^{(|a|+|a|)|x|}[a,a,x] + (-1)^{(|x|+|a|)|a|}[x,a,a] + (-1)^{(|a|+|a|)|x|}[a,x,a] \\
&= [a,a,x] + (-1)^{|a||x|+|a|}[x,a,a] + [a,x,a] \\
&= -(-1)^{|a|}[x,a,a] + (-1)^{|a||x|+|a|}[x,a,a] + [a,x,a] \\
&= ((-1)^{|a||x|+|a|} - (-1)^{|a|})[x,a,a] + [a,x,a] \\
&= (-1)^{|a|}((-1)^{|a||x|} - 1)[x,a,a] + [a,x,a]
\end{aligned}$$

In particular, we see that

$$[a,x,a] = \begin{cases} -2[x,a,a] = -2[a,a,x] = 2a(ax) & \text{if } |a| \text{ and } |x| \text{ are odd} \\ 0 & \text{else} \end{cases} \tag{5}$$

for all homogeneous $a \in A$ and homogeneous $x \in X$. Finally, the equivalence of 4 with everything else follows by definition. □

Definition 1.7. Let X be an NDG A-module. We say X is **alternative** if any of the equivalent definitions in Definition (1.4) are satisfied.

1.3.2 The associator complex and homology of an MDG A-module

Let X be an NDG A-module. Now we want to introduce a complex associated (no pun intended) with X.

Definition 1.8. The **associator complex** of X is the R-subcomplex of X given by the image of $[\cdot,\cdot,\cdot]$. We denote this complex by $[X]$. Thus the underlying graded R-module of $[X]$ is given by

$$[X] = \mathrm{span}_R\{[a,b,x] \mid a,b \in A \text{ and } x \in X\}, \tag{6}$$

and the differential $\mathrm{d}_{[X]}$ of $[X]$ is just the restriction of the differential d_X to $[X]$. The **associator homology** of X is the homology of the associator complex of X, denoted $\mathrm{H}([X])$. We say X is **homologically associative** if $\mathrm{H}([X]) = 0$. We also say X is **homologically associative in degree** i if $\mathrm{H}_i([X]) = 0$. For completeness, we say X is **associative in degree** i if $[X]_i = 0$.

Clearly if X is associative, then X is homologically associative. The converse is also true under certain conditions.

Proposition 1.5. *Assume that $(R,\mathfrak{m})$ is a local ring and that $[X]$ is minimal. If X is associative in degree i, then X is associative in degree $i+1$ if and only if X is homologically associative in degree $i+1$. In particular, if $[X]$ is bounded below and minimal, then X is associative if and only if X is homologically associative.*

Proof. Clearly if X is associative in degree $i+1$, then it is homologically associative in degree $i+1$. To show the converse, assume for a contradiction that X is homologically associative in degree $i+1$ but that it is not associative in degree $i+1$. In other words, assume $\mathrm{H}_{i+1}([X]) = 0$ and $[X]_{i+1} \neq 0$. By Nakayama's Lemma, we can find a triple (a,b,x) such that $|a|+|b|+|x| = i+1$ and such that $[a,b,x] \notin \mathfrak{m}[X]_{i+1}$. Since $[X]_i = 0$ by assumption, we have $\mathrm{d}_X[a,b,x] = 0$. Also, since X is minimal, we have $\mathrm{d}_X[X] \subseteq \mathfrak{m}[X]$. Thus $[a,b,x]$ represents a nontrivial element in homology in degree $i+1$. This is a contradiction. □

Remark. Note that if both A and X are both minimal, then the Leibniz law (1) implies $[X]$ is minimal too, however the converse need not hold.

The proof of Proposition (1.5) tells us something a bit more than what was stated in the proposition. To see this, we first need a definition:

Definition 1.9.

1. Assume that $[X]$ is bounded below. The **lower associative index** of X, denoted $\mathrm{la}(X)$, is defined to be the smallest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\mathrm{la}(X) = -\infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded below by setting $\mathrm{la}(X) = -\infty$.

2. Assume that $[X]$ is bounded above. The **upper associative index** of X, denoted $\mathrm{ua}(X)$, is defined to be the largest $i \in \mathbb{Z} \cup \{\infty\}$ such that $[X]_i \neq 0$ where we set $\mathrm{ua}(X) = \infty$ if X is associative. We extend this definition to case where $[X]$ is not bounded above by setting $\mathrm{ua}(X) = \infty$.

With the lower associative index of X defined, we see, after analyzing the proof of Proposition (1.5), that if R is local and $[X]$ is nonzero and minimal, then

$$\mathrm{la}(X) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i([X]) \neq 0\}.$$

Thus, in this case, the lower associative index of X can be measured homologically.

1.3.3 The lower associative index of a free resolution

Assume that $(R, \mathfrak{m})$ is local, let $I \subseteq \mathfrak{m}$ be an ideal of R, and let $\tau \colon F \to R/I$ be a free resolution of R/I over R. iewing R/I as a (, it is clear that $\mathrm{la}(R/I) = -\infty$ for trivial reasons. However
The **lower associative index** of R/I, denoted $\mathrm{la}(R/I)$, is defined to be

$$\mathrm{la}(R/I) = \sup_{\mu}\{\mathrm{la}(\mu)\},$$

where the supremum is taken over all multiplications $\mu \colon F \otimes F \to F$ which lifts the multiplication map $R/I \otimes R/I \to R/I$ and which gives F the structure of a PDG R-algebra.

1.3.4 Nuclei

In the category of R-modules, we have the concept of annihilators. In particular, suppose M is an R-module and let $m \in M$. We define the **annihilator** with respect to u to be the subset of R given by

$$0 :_R m = \{r \in R \mid rm = 0\}$$

In fact, $0 :_R m$ is in an ideal of R. Indeed, it is easy to check that it is an abelian group. We can also show that it is closed under R-scaling, but notice that we need the associative law to get this: if $r \in R$ and $s \in 0 : m$, then $(rs)m = r(sm) = 0$ implies $rs \in 0 : m$. More generally, if $\mathcal{S}$ is a collection of elements of M, then we can define the **annihilator** with respect to $\mathcal{S}$ to be the subset of R given by

$$0 :_R \mathcal{S} = \{r \in R \mid rm = 0 \text{ for all } m \in \mathcal{S}\}.$$

Again, it is easy to check that this an ideal of R. Now let us consider the case where X is a PDG A-module and let $\mathcal{S}$ be a collection of elements in X. We can still tentatively define the annihilator $0 :_A \mathcal{S}$ with respect to $\mathcal{S}$ as a subset of A as before:

$$0 :_R \mathcal{S} = \{r \in R \mid rx = 0 \text{ for all } x \in \mathcal{S}\}.$$

It is easy to show that $0 :_A \mathcal{S}$ is an abelian group, but certainly it need not be the case that $0 :_A \mathcal{S}$ is a PDG ideal of A. Indeed, first of all, $0 :_A \mathcal{S}$ need not be a subcomplex of A. To see why, note that if $a \in 0 :_A \mathcal{S}$ and $x \in \mathcal{S}$, then

$$\begin{aligned} 0 &= \mathrm{d}_A(0) \\ &= \mathrm{d}_A(ax) \\ &= \mathrm{d}_A(a)x + (-1)^{|a|} a \mathrm{d}_X(x) \end{aligned}$$

implies $\mathrm{d}_A(a)x = (-1)^{|a|+1} a \mathrm{d}_X(x)$. In particular, we need not have $\mathrm{d}_A(a) \in 0 :_A \mathcal{S}$. However if we replace $\mathcal{S}$ with $\mathcal{S}' := \mathcal{S} \cup \mathrm{d}_X \mathcal{S}$, then it easy to see that $0 :_A \mathcal{S}'$ is a subcomplex of A . However even in this case, it still not need be true that $0 :_A \mathcal{S}'$ be a PDG ideal of A because it's possible that $0 :_A \mathcal{S}'$ is not closed under A-scaling. Our proof that $0 :_R m$ was closed under R-scaling used associativity! In particular, if $a \in A$, $\alpha \in 0 :_A \mathcal{S}'$ and $x \in \mathcal{S}'$, then

$$\begin{aligned} (a\alpha)x &= a(\alpha x) + [a, \alpha, x] \\ &= a \cdot 0 + [a, \alpha, x] \\ &= [a, \alpha, x]. \end{aligned}$$

In particular, we need not have $a\alpha \in 0 :_A \mathcal{S}'$. We want to replace $\mathcal{S}'$ with another set $\mathcal{S}''$ such that $0 :_A \mathcal{S}''$ will become a PDG A-ideal. Before doing so, we need to define the nucleus of X:

Definition 1.10. Let A be a PDG R-algebra.

1. Suppose X is a left PDG A-module. The **left nucleus** of X, denoted $\mathrm{N}_l(X)$, is the subset of X defined by
$$\mathrm{N}_l(X) = \{x \in X \mid [a,b,x] = 0 \text{ for all } a,b \in A\}.$$

2. Suppose X is a right PDG A-module. The **right nucleus** of X, denoted $\mathrm{N}_r(X)$, is the subset of X defined by
$$\mathrm{N}_r(X) = \{x \in X \mid [x,a,b] = 0 \text{ for all } a,b \in A\}.$$

3. Suppose X is two-sided PDG A-module. The **middle nucleus** of X, denoted $\mathrm{N}_m(X)$, is the subset of X defined by
$$\mathrm{N}_m(X) = \{x \in X \mid [a,x,b] = 0 \text{ for all } a,b \in A\},$$
and the **nucleus** of X, denoted $\mathrm{N}(X)$, is subset of X defined by
$$\mathrm{N}(X) = \mathrm{N}_l(X) \cap \mathrm{N}_m(X) \cap \mathrm{N}_r(X),$$
where $\mathrm{N}_l(X)$ is the left nucleus of X where we view X as a left PDG A-module, and where $\mathrm{N}_r(X)$ is the right nucleus of X where we view X as a right PDG A-module.

Remark. Suppose X is a two-sided PDG A-module. Notice that the identity (4) implies $\mathrm{N}_l(X) = \mathrm{N}_r(X)$. Also notice that (3) implies $\mathrm{N}_l(X) \cap \mathrm{N}_r(X) \subseteq \mathrm{N}_m(X)$. Combining these facts together, we see that $\mathrm{N}(X) = \mathrm{N}_l(X) = \mathrm{N}_r(X)$. Thus if Y is a left (resp. right) PDG A-module, then we can define the **nucleus** of Y, denoted $\mathrm{N}(Y)$, to be left (resp. right) nucleus of Y without any threat of conflict in our notation.

Continuing our conversation as before, we claim that if we replace $\mathcal{S}'$ with $\mathcal{S}'' =: \mathcal{S}' \cap \mathrm{N}(X)$, then $0 :_A \mathcal{S}''$ will become a PDG A-ideal. Indeed, we will now have A-scaling, but we need to check that $0 :_A \mathcal{S}''$ is d_X-stable. In fact, this follows from the fact that $\mathrm{N}(X)$ is d_X-stable: if $x \in \mathrm{N}(X)$, then
$$\begin{aligned} 0 &= \mathrm{d}_X(0) \\ &= \mathrm{d}_X([a,b,x]) \\ &= [\mathrm{d}_A(a),b,x] + (-1)^{|a|}[a,\mathrm{d}_A(b),x] + (-1)^{|a|+|b|}[a,b,\mathrm{d}_X(x)] \\ &= (-1)^{|a|+|b|}[a,b,\mathrm{d}_X(x)] \end{aligned}$$
for all homogeneous $a,b \in A$.

Lemma 1.1. *Let (A,d,μ) be a PDG R-algebra such that $A_0 = R$ and set $I = \mathrm{d}(A_1)$. Then I kills $\mathrm{H}(A)$.*

Proof. Let $t \in I$ and let $\mathrm{m}_t \colon A \to A$ be the multiplication by t map, defined by
$$\mathrm{m}_t(a) = ta$$
for all $a \in A$. We claim that m_t is null-homotopic. Indeed, choose $e \in A$ such that $\mathrm{d}(e) = t$, and let $\mathrm{m}_e \colon A \to A$ be the multiplication by e map, defined by
$$\mathrm{m}_e(a) = ea$$
for all $a \in A$. Note that m_e is a graded R-linear map of degree 1. Also note that for all $a \in A$, we have
$$\begin{aligned} (\mathrm{d}\mathrm{m}_e + \mathrm{m}_e\mathrm{d})(a) &= \mathrm{d}\mathrm{m}_e(a) + \mathrm{m}_e\mathrm{d}(a) \\ &= \mathrm{d}(ea) + e\mathrm{d}(a) \\ &= \mathrm{d}(e)a - e\mathrm{d}(e) + e\mathrm{d}(a) \\ &= \mathrm{d}(e)a \\ &= ta \\ &= \mathrm{m}_t(a). \end{aligned}$$
Thus m_e is a homotopy from m_t to the zero map; hence m_t is null-homotopic. It follows that t kills $\mathrm{H}(A)$, and since $t \in I$ was arbitrary, we see that I kills $\mathrm{H}(A)$. □

Corollary. *Let K be a field and let (A,d,μ) be an MDG K-algebra such that $A_0 = K$. Assume that $\mathrm{d}(A_1) \neq 0$. Then $\mathrm{H}(A) = 0$.*

Corollary. *Let (A,d,μ) be an MDG R-algebra such that $A_0 = R$ and set $I = \mathrm{d}(\mathrm{N}(A)_1)$. Then I kills $\mathrm{H}([A])$.*

Proof. Let $t \in I$ and choose $e \in \mathrm{N}(A)$ such that $\mathrm{d}(e) = t$. Note that the restriction of m_e lands in $[A]$ when it is restricted to $[A]$ since (4) implies $\mathrm{m}_e[a,b,c] = [ea,b,c]$ for all $a,b,c \in A$. It follows that $\mathrm{m}_e|_{[A]}$ is a homotopy from $[\mathrm{m}_t]$ to the zero map; hence $[\mathrm{m}_t]$ is null-homotopic. It follows that t kills $\mathrm{H}([A])$, and since $t \in I$ was arbitrary, we see that I kills $\mathrm{H}([A])$. □

Corollary. *Let K be a field and let (A, d, μ) be an MDG K-algebra such that $A_0 = K$. Assume that $\mathrm{d}(\mathrm{N}(A)_1) \neq 0$. Then $\mathrm{H}([A]) = 0$.*

1.3.5 Hom

Let X and Y be two PDG A-modules. We denote by $\mathrm{Hom}_A(X,Y)$ to be the set of all A-module homomorphisms from X to Y and we denote by $\mathrm{Hom}_A^\star(X,Y)$ to be the R-complex whose underlying graded R-module in degree i is given by

$$\mathrm{Hom}_A^\star(X,Y)_i = \{\varphi\colon X \to Y \mid \varphi \text{ is an } A\text{-module homomorphism of degree } i\},$$

and whose differential, denoted $\mathrm{d}^\star_{X,Y}$, is defined by

$$\mathrm{d}^\star_{X,Y}(\varphi) = \mathrm{d}_Y\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}_X$$

for all homogeneous $\varphi \in \mathrm{Hom}_A^\star(X,Y)$. The computation below shows that $\mathrm{d}^\star_{X,Y}(\varphi)$ respects A-scaling, and thus really does land in $\mathrm{Hom}_A^\star(X,Y)$:

$$\begin{aligned}
\mathrm{d}^\star_{(X,Y)}(\varphi)(ax) &= (\mathrm{d}_Y\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}_X)(ax) \\
&= \mathrm{d}_Y\varphi(ax) - (-1)^{|\varphi|}\varphi\mathrm{d}_X(ax) \\
&= (-1)^{|a||\varphi|}\mathrm{d}_Y(a\varphi(x)) - (-1)^{|\varphi|}\varphi(\mathrm{d}_A(a)x + (-1)^{|a|}a\mathrm{d}_X(x)) \\
&= (-1)^{|a||\varphi|}\mathrm{d}_A(a)\varphi(x) + (-1)^{|a||\varphi|+|a|}a\mathrm{d}_Y\varphi(x) - (-1)^{|\varphi|+|\varphi|(|a|-1)}\mathrm{d}_A(a)\varphi(x) - (-1)^{|\varphi|+|a|+|\varphi||a|}a\varphi\mathrm{d}_X(x) \\
&= \Big((-1)^{|a||\varphi|}\mathrm{d}_A(a)\varphi(x) - (-1)^{|\varphi||a|}\mathrm{d}_A(a)\varphi(x)\Big) + \Big((-1)^{|a||\varphi|+|a|}a\mathrm{d}_Y\varphi(x) - (-1)^{|\varphi|+|a|+|\varphi||a|}a\varphi\mathrm{d}_X(x)\Big) \\
&= (-1)^{|a||\varphi|+|a|}a\mathrm{d}_Y\varphi(x) - (-1)^{|\varphi|+|a|+|\varphi||a|}a\varphi\mathrm{d}_X(x) \\
&= (-1)^{|\varphi||a|+|a|}a\mathrm{d}_Y\varphi(x) - (-1)^{|\varphi||a|+|a|+|\varphi|}\varphi\mathrm{d}_X(x) \\
&= (-1)^{(|\varphi|+1)|a|}a(\mathrm{d}_Y\varphi - (-1)^{|\varphi|}\varphi\mathrm{d}_X)(x) \\
&= (-1)^{(|\varphi|-1)|a|}a\mathrm{d}^\star_{(X,Y)}(\varphi)(x)
\end{aligned}$$

Note that it is not necessarily true that $\mathrm{Hom}_A^\star(X,Y)$ has the structure of a PDG A-module. To see why, suppose we attempted to defined a left scalar-multiplication on $\mathrm{Hom}_A^\star(X,Y)$ by

$$(a \cdot \varphi)(x) = (-1)^{|a||\varphi|}\varphi(ax) \tag{7}$$

for all homogeneous $a \in A$ and homogeneous $\varphi \in \mathrm{Hom}_A^\star(X,Y)$. Then $a \cdot \varphi$ is not necessarily respect A-scaling; indeed suppose b is another homogeneous element in A. Then

$$\begin{aligned}
((ab)\cdot\varphi)(x) &= (-1)^{|a||\varphi|+|b||\varphi|}\varphi((ab)x) \\
&= (-1)^{|a||\varphi|+|b||\varphi|+|a||b|}\varphi((ba)x) \\
&= (-1)^{|a||\varphi|+|b||\varphi|+|a||b|}\varphi(b(ax) + [b,a,x]) \\
&= (-1)^{|a||\varphi|+|b||\varphi|+|a||b|}\varphi(b(ax)) + (-1)^{|a||\varphi|+|b||\varphi|+|a||b|}\varphi([b,a,x]) \\
&= (-1)^{|a||\varphi|+|a||b|}(b\cdot\varphi)(ax) + (-1)^{|a||b|}[b,a,\varphi(x)] \\
&= (a\cdot(b\cdot\varphi))(x) + (-1)^{|a||b|}[b,a,\varphi(x)].
\end{aligned}$$

Thus $((ab)\cdot\varphi)(x) = (a\cdot(b\cdot\varphi))(x)$ if and only if $[b,a,\varphi(x)] = 0$. However not all is lost, as we still have:

Proposition 1.6. $\mathrm{Hom}_A^\star(X, \mathrm{N}_l(Y))$ *has the structure of a PDG A-module with the scalar-multiplication map defined as in (7).*

1.4 Associator functor

Let X and Y be two PDG A-modules and let $\varphi\colon X \to Y$ be an A-linear map. We obtain an induced chain map of R-complexes $[\varphi]\colon [X] \to [Y]$, where $[\varphi]$ is the unique chain map which satisfies

$$\begin{aligned}[\varphi][a,b,x] &= \varphi((ab)x - a(bx)) \\ &= \varphi((ab)x) - \varphi(a(bx)) \\ &= (ab)\varphi(x) - a\varphi(bx) \\ &= (ab)\varphi(x) - a(b\varphi(x)) \\ &= [a,b,\varphi(x)].\end{aligned}$$

for all $a, b \in A$ and $x \in X$. In particular, the map $[\varphi]$ is just the restriction of φ to $[X]$. It is straightforward to check that the assignment $X \mapsto [X]$ and $\varphi \mapsto [\varphi]$ gives rise to a functor from the category of PDG A-modules to the category of R-complexes. We call this functor the **associator functor** over A, and we denote this functor by $[\cdot]\colon \mathbf{PMod}_A \to \mathbf{Comp}_R$.

1.4.1 Stable PDG A-Submodules

The associator functor $[\cdot]\colon \mathbf{PMod}_A \to \mathbf{Mod}_R$ need not be exact. To see what goes wrong, let

$$0 \longrightarrow X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \longrightarrow 0 \tag{8}$$

be a short exact sequence of PDG A-modules. We obtain an induced sequence of R-complexes

$$0 \longrightarrow [X] \xrightarrow{[\varphi]} [Y] \xrightarrow{[\psi]} [Z] \longrightarrow 0 \tag{9}$$

We claim that we have exactness at $[X]$ and $[Z]$. Indeed, this is equivalent to showing $[\varphi]$ is injective and $[\psi]$ is surjective, which follows from the fact that $[\varphi]$ (respectively $[\psi]$) is the restriction of the injective map φ (respectively the surjective map ψ). Let us see what goes wrong when trying to prove exactness at $[Y]$. Let

$$\sum_{i=1}^{n}[a_i, b_i, y_i] \in \ker[\psi] \subseteq \ker\psi.$$

Then by exactness of (8), there exists $x \in X$ such that $\varphi(x) = \sum_{i=1}^{n}[a_i, b_i, y_i]$. It is not at all clear however that $x \in [X]$. Indeed, we will see a counterexample to this later on. This leads us to consider the following definition:

Definition 1.11. Let X be a PDG A-submodule of Y. We say X is a **stable** PDG A-submodule of Y if it satisfies $[X] = X \cap [Y]$.

Now it is easy to check that (9) is a short exact sequence of R-complexes if and only if $\varphi(X)$ is a stable PDG A-submodule of Y. Thus if $\varphi(X)$ is a stable PDG A-submodule of Y, then the short exact sequence (9) of R-complexes induces a long exact sequence in homology

$$\begin{tikzcd} & \cdots \arrow[r] & \mathrm{H}_{i+1}([Z]) \arrow[dll, rounded corners] \\ \mathrm{H}_i([X]) \arrow[r] & \mathrm{H}_i([Y]) \arrow[r] & \mathrm{H}_i([Z]) \arrow[dll, rounded corners] \\ \mathrm{H}_{i-1}([X]) \arrow[r] & \cdots & \end{tikzcd} \tag{10}$$

From this, one concludes immediately the following theorem:

Theorem 1.2. *Suppose X is a PDG A-submodule of Y. Then μ_Y is homologically associative if and only if μ_X and $\mu_{Y/X}$ are homologically associative.*

2 Invariant

Suppose that $(R, \mathfrak{m})$ is a local ring. Let $I \subseteq \mathfrak{m}$ be an ideal of R, let F be a free MDG R-algebra resolution of R/I, and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $\mathrm{C}(r)$ is a free resolution of $R/\langle I, r\rangle$

over R. We can give $\mathrm{C}(r)$ the structure of a free MDG R-algebra resolution of $R/\langle I, r\rangle$ as follows: first note that the underlying graded R-module of $\mathrm{C}(r)$ has the form $F + eF$ where e is an exterior variable of degree -1 and where $\{1, e\}$ is an F-linearly independent set. In other words, every element in $F + eF$ can be expressed in the form $\alpha + e\beta$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i, then α and β are homogeneous of degrees i and $i-1$ respectively. With this understood, we define a multiplication on $\mathrm{C}(r)$ by

$$(\alpha + e\beta)(\gamma + e\delta) = \alpha\gamma + e(\beta\gamma + (-1)^{|\alpha|}\alpha\delta)$$

for all homogeneous $\alpha, \beta, \gamma, \delta \in F$ and extend this everywhere else. By restricting scalars, we see that the mapping cone $\mathrm{C}(r)$ inherits a natural *right* MDG F-module structure via restriction of scalars. The reason why it is more naturally viewed as a right MDG F-module rather than a left MDG F-module can be seen in the way the mapping cone differential acts on elements: given $\alpha, \beta \in F$ where α is homogeneous, we have

$$\mathrm{d}_{\mathrm{C}(r)}(\alpha + e\beta) = \mathrm{d}_F(\alpha) + r\beta - e\mathrm{d}_F(\beta).$$

Thus the mapping cone differential behaves as if e is an exterior variable of degree -1.

2.1 Mapping Cone Construction

Let I be an ideal of R and let F be a free resolution of R/I over R. Choose a multiplication μ_F on F which lifts the multiplication map $R/I \otimes_R R/I \to R$ and let $r \in \mathfrak{m}$ be an (R/I)-regular element. Then the mapping cone $\mathrm{C}(r)$ is a free resolution of $R/\langle I, x\rangle$ over R. The multiplication μ_F on F induces a multiplication $\mu_{\mathrm{C}(r)}$ on $\mathrm{C}(r)$ as follows: first note that $F \oplus F(-1)$ is the underlying graded R-module of $\mathrm{C}(r)$. Express this graded R-module in the form $F + eF$ where e is an exterior variable of degree -1 so that $\{1, e\}$ is an F-linearly independent set. Thus an element in $F + eF$ can be expressed in the form $\alpha + e\beta$ for unique $\alpha, \beta \in F$. If this element is homogeneous of degree i, then α and β are homogeneous of degrees i and $i-1$ respectively. With this understood, the multilpication $\mu_{\mathrm{C}(r)}$ is defined on homogeneous elements $\alpha, \beta, \gamma, \delta \in F$ by

$$(\alpha + e\beta)(\gamma + e\delta) = \alpha\gamma + e(\beta\gamma + (-1)^{|\alpha|}\alpha\delta)$$

and extended R-linearly everywhere else. The mapping cone $\mathrm{C}(r)$ inherits a natural *right* PDG F-module structure via restriction of scalars. The reason why it is more naturally viewed as a right PDG F-module rather than a left PDG F-module can be seen in the way the mapping cone differential acts on elements: given $\alpha, \beta \in F$ where α is homogeneous, we have

$$\mathrm{d}_{\mathrm{C}(r)}(\alpha + e\beta) = \mathrm{d}_F(\alpha) + r\beta - e\mathrm{d}_F(\beta).$$

Thus the mapping cone differential behaves as if e is an exterior variable of degree -1.

Thus there are two associator complexes to consider: the associator complex of $\mathrm{C}(r)$ where we view $\mathrm{C}(r)$ an MDG R-algebra and the associator complex of $\mathrm{C}(r)$ where we view $\mathrm{C}(r)$ as a right MDG F-module. In fact, these associator complexes are the same. Indeed, first note that a quick calculation gives us

$$\begin{aligned}
[\alpha, \beta, \gamma + e\delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha| + |\beta|} e[\alpha, \beta, \delta] \\
[\alpha, \beta + e\gamma, \delta] &= [\alpha, \beta, \gamma] + (-1)^{|\alpha|} e[\alpha, \gamma, \delta] \\
[\alpha + e\beta, \gamma, \delta] &= [\alpha, \gamma, \delta] + e[\beta, \gamma, \delta]
\end{aligned}$$

Using these identities together with the fact that $e^2 = 0$ and the fact that identity 1 associates with everything, we obtain

$$\begin{aligned}
[\alpha + e\beta, \gamma + e\delta, \varepsilon + e\zeta] &= [\alpha, \gamma, \varepsilon] + e[\beta, \gamma, \varepsilon] + (-1)^{|\alpha||} e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha| + |\gamma|} e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||} e[\alpha, \delta, \varepsilon] + (-1)^{|\alpha| + |\gamma|} e[\alpha, \gamma, \zeta] \\
&= [\alpha + e\beta, \gamma, \varepsilon] + (-1)^{|\alpha||} [1 + e\alpha, \delta, \varepsilon] + (-1)^{|\alpha| + |\gamma|} [1 + e\alpha, \gamma, \zeta]
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in F$. Thus we may denote this common associator complex by $[\mathrm{C}(r)]$. Now the homothety map $F \xrightarrow{r} F$ gives rise to a short exact sequence of right MDG F-modules

$$0 \longrightarrow F \xrightarrow{\iota} \mathrm{C}(r) \xrightarrow{\pi} \Sigma F \longrightarrow 0 \tag{11}$$

where $\iota \colon F \to \mathrm{C}(r)$ is the inclusion map and where $\pi \colon \mathrm{C}(r) \to \Sigma F$ is the projection map given by $\pi(\alpha + e\beta) = \alpha$ for all $\alpha, \beta \in F$.

Proposition 2.1. *The short exact sequence (11) is stable. In other words, F is a stable MDG F-submodule of* $\mathrm{C}(r)$.

Proof. We must check that $[\mathrm{C}(r)] \cap F \subseteq [F]$ since the reverse inclusion is trivial. Suppose $\sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in [\mathrm{C}(r)] \cap F$ where $r_i \in R$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ for each $1 \leq i \leq m$. Observe that

$$\begin{aligned} \sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] &= \sum_{i=1}^{m} r_i([\alpha_i, \gamma_i, \delta_i] + e[\beta_i, \gamma_i, \delta_i]) \\ &= \sum_{i=1}^{m} r_i[\alpha_i, \gamma_i, \delta_i] + e\sum_{i=1}^{m} r_i[\beta_i, \gamma_i, \delta_i]. \end{aligned}$$

Since $\sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] \in F$, it follows that $\sum_{i=1}^{m} r_i[\beta_i, \gamma_i, \delta_i] = 0$. Thus

$$\sum_{i=1}^{m} r_i[\alpha_i + e\beta_i, \gamma_i, \delta_i] = \sum_{i=1}^{m} r_i[\alpha_i, \gamma_i, \delta_i] \in [F].$$

Therefore $[\mathrm{C}(r)] \cap F \subseteq [F]$. $\square$

Since (11) is stable, we obtain a long exact sequence in homology

$$\begin{tikzcd} & & \cdots \arrow[r] & \mathrm{H}_i([F]) \arrow[dll, rounded corners, "r"'] \\ & \mathrm{H}_i([F]) \arrow[r] & \mathrm{H}_i([\mathrm{C}(r)]) \arrow[r] & \mathrm{H}_{i-1}([F]) \arrow[dll, rounded corners, "r"'] \\ & \mathrm{H}_{i-1}([F]) \arrow[r] & \cdots & \end{tikzcd} \tag{12}$$

In particular, we have

Theorem 2.1. *We have*

1. $\mathrm{ua}(\mathrm{C}(r)) = \mathrm{ua}(F) + 1$ *if and only if r is not* $\mathrm{H}([F])$*-regular.*

Proof. Let $i = \mathrm{ua}(\mathrm{C}(r))$. From (12) we obtain an exact sequence

$$0 \to \mathrm{H}_{i+1}([\mathrm{C}(r)]) \to \mathrm{H}_i([F]) \xrightarrow{r} \mathrm{H}_i([F]) \to \mathrm{H}_i([\mathrm{C}(r)]).$$

If $\mathrm{H}_i([\mathrm{C}(r)]) = 0$, then Nakayama's lemma would imply $\mathrm{H}_i([F]) = 0$ which is a contradiction, thus $\mathrm{H}_i([\mathrm{C}(r)]) \neq 0$.
oo . would imply Thus $\mathrm{H}_i([\mathrm{C}(r)]) \neq 0$ and H $\square$

if $\mathrm{H}_i([\mathrm{C}(r)]) = 0$, then Nakayama's lemma implies $\mathrm{H}_i([F]) = 0$. Thus, we have

Theorem 2.2. *With the notation above, if $\mu_{\mathrm{C}(r)}$ is homologically associative in degree i, then μ_F is homologically associative in degree i. Moreover, we have*

$$\mathrm{index}(\mu_F) = \mathrm{index}(\mu_{\mathrm{C}(r)}).$$

Corollary. *With the notation above, we have*

$$\mathrm{index}(R/\langle I, r\rangle) \geq \mathrm{index}(R/I).$$

3 PDG-algebras on monomomial resolutions

Throughout this subsection, let $\boldsymbol{x} = x_1, \dots, x_n$, let $R = K[\boldsymbol{x}]$, and let $\boldsymbol{m} = m_1, \dots, m_r$ be monomials in R. For each nonempty subset $\sigma \subseteq [r]$, we set $m_\sigma := \mathrm{lcm}(m_\lambda \mid \lambda \in \sigma)$ and we set $\boldsymbol{a}_\sigma \in \mathbb{N}^n$ to be the exponent vector of m_σ. For completeness, we set $m_\varnothing = 1$ and $a_\varnothing = (0, \dots, 0)$. Let Re_σ be the free R-module generated by e_σ whose multidegree is $\boldsymbol{a}_\sigma$. Let Δ be a finite simplical complex whose vertex set is the ordered set $[r] = \{1, \dots, r\}$. We label the vertices of Δ by $m_1, \dots, m_r$. More generally, if σ is a face of Δ, then we label it by m_σ. For each $\boldsymbol{a} \in \mathbb{N}^n$, let $\Delta_{\boldsymbol{a}}$ be the subcomplex of Δ defined by $\Delta_{\boldsymbol{a}} = \{\sigma \in \Delta \mid \boldsymbol{a}_\sigma \leq \boldsymbol{a}\}$. The differential on $\mathcal{S}(\Delta)$ is denoted ∂, and the differential on $\mathcal{S}(\Delta_{\boldsymbol{a}})$ is denoted $\partial_{\boldsymbol{a}}$. Note that $\partial_{\boldsymbol{a}}$ is just the restriction of ∂ to $\mathcal{S}(\Delta_{\boldsymbol{a}})$.

3.1 Monomial resolution induced by a labeled simplicial complex

Definition 3.1. We define an R-complex, denoted F_Δ and called R**-complex induced by** Δ (or the R**-complex of** Δ **over** R), as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R-module of F_Δ is given by

$$F_{\Delta,k} := \begin{cases} \bigoplus_{\dim \sigma = k-1} Re_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential d_Δ is defined on the homogeneous generators of F_Δ by $\mathrm{d}_\Delta(e_\varnothing) = 0$ and

$$\mathrm{d}_\Delta(e_\sigma) = \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i}$$

for all $\sigma \in \Delta \setminus \{\varnothing\}$ where $\mathrm{pos}(i,\sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma \setminus i$ denotes the face obtained from σ by removing i. In the case where Δ is the r-simplex, we call F_Δ the **Taylor complex** of $R/\boldsymbol{m}$ over R.

Observe that F_Δ also has the structure of an $\mathbb{N}^n$-graded K-complex. In other words, we have a decomposition of K-complexes

$$F_\Delta = \bigoplus_{\boldsymbol{a} \in \mathbb{N}^n} F_{\Delta,\boldsymbol{a}},$$

where the K-complex $F_{\Delta,\boldsymbol{a}}$ in multidegree $\boldsymbol{a} \in \mathbb{N}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded K-vector space of $F_{\Delta,\boldsymbol{a}}$ is given by

$$F_{\Delta,k,\boldsymbol{a}} := \begin{cases} \bigoplus_{\dim \sigma = k-1} K \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma & \text{if } \sigma \in \Delta_{\boldsymbol{a}} \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

and the differential $\mathrm{d}_{\Delta,\boldsymbol{a}}$ of $F_{\Delta,\boldsymbol{a}}$ is just the restriction of d_Δ to $F_{\Delta,\boldsymbol{a}}$. In particular, d_Δ is homogeneous with respect to that $\mathbb{N}^n$-grading; hence

$$\begin{aligned} \mathrm{H}(F_\Delta, \mathrm{d}_\Delta) &= \ker \mathrm{d}_\Delta / \operatorname{im} \mathrm{d}_\Delta \\ &= \left(\bigoplus_{\boldsymbol{a} \in \mathbb{N}^n} \ker \mathrm{d}_{\Delta,\boldsymbol{a}} \right) / \left(\bigoplus_{\boldsymbol{a} \in \mathbb{N}^n} \operatorname{im} \mathrm{d}_{\Delta,\boldsymbol{a}} \right) \\ &\cong \bigoplus_{\boldsymbol{a} \in \mathbb{N}^n} (\ker \mathrm{d}_{\Delta,\boldsymbol{a}} / \operatorname{im} \mathrm{d}_{\Delta,\boldsymbol{a}}) \\ &= \bigoplus_{\boldsymbol{a} \in \mathbb{N}^n} \mathrm{H}(F_{\Delta,\boldsymbol{a}}, \mathrm{d}_{\Delta,\boldsymbol{a}}). \end{aligned}$$

Now assume that $\Delta_{\boldsymbol{a}}$ contains a nonempty face, say $\sigma \in \Delta_{\boldsymbol{a}}$, then we have

$$\begin{aligned} \mathrm{d}_{\Delta,\boldsymbol{a}} \left(\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma \right) &= \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} \mathrm{d}_\Delta(e_\sigma) \\ &= \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{m_\sigma}{m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{a}} m_\sigma}{m_\sigma m_{\sigma \setminus i}} e_{\sigma \setminus i} \\ &= \sum_{i \in \sigma} (-1)^{\mathrm{pos}(i,\sigma)} \frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_{\sigma \setminus i}} e_{\sigma \setminus i}. \end{aligned}$$

Thus if we define $\varphi_{\boldsymbol{a}} \colon F_{\Delta,\boldsymbol{a}}(1) \to \mathcal{S}(\Delta_{\boldsymbol{a}})$ be the unique graded K-linear isomorphism such that $\varphi_{\boldsymbol{a}} \left(\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma} e_\sigma \right) = \sigma$. Then from the computation above, we see that $\mathrm{d}_{\Delta,\boldsymbol{a}} \partial_{\boldsymbol{a}} = \partial_{\boldsymbol{a}} \mathrm{d}_{\Delta,\boldsymbol{a}}$; hence $\varphi_{\boldsymbol{a}}$ gives an isomorphism of K-complexes

$$\Sigma^{-1} F_{\Delta,\boldsymbol{a}} \cong \mathcal{S}(\Delta_{\boldsymbol{a}}).$$

In particular, we obtain an isomorphism of homologies

$$\mathrm{H}(F_{\Delta,\boldsymbol{a}}, \mathrm{d}_{\Delta,\boldsymbol{a}})(1) \cong \widetilde{\mathrm{H}}(\Delta_{\boldsymbol{a}}, K),$$

where $\widetilde{\mathrm{H}}(\Delta_{\boldsymbol{a}}, K)$ is the simplicial homology of the simplicial complex $\Delta_{\boldsymbol{a}}$ over K. In other words, for each $k \in \mathbb{Z}$, we have

$$\mathrm{H}_{k+1}(F_{\Delta,\boldsymbol{a}}, \mathrm{d}_{\Delta,\boldsymbol{a}}) \cong \widetilde{\mathrm{H}}_k(\Delta_{\boldsymbol{a}}, K).$$

The following theorem follows immediately from the discussion above:

Theorem 3.1. *F_Δ is a free resolution of $R/\boldsymbol{m}$ over R if and only if for all $\boldsymbol{a} \in \mathbb{N}^n$ either $\Delta_{\boldsymbol{a}}$ is the void complex or $\Delta_{\boldsymbol{a}}$ is acyclic. In particular, the Taylor complex of $R/\boldsymbol{m}$ over R is a free resolution of $R/\boldsymbol{m}$ over R. Moreover, F_Δ is minimal if and only if $m_\sigma \neq m_{\sigma'}$ for every proper subface σ' of a face σ.*

3.2 MDG-algebra structures on the monomial resolution induced by a labeled simplicial complex

Let (F, d) denote the R-complex induced by Δ. Let $\mu\colon F \otimes_R F \to F$ be a chain map such that

1. μ gives F the structure of an MDG R-algebra resolution of $R/\boldsymbol{m}$.
2. μ respects the multigrading: this means that if $\alpha \in F_{\boldsymbol{a}}$ and $\beta \in F_{\boldsymbol{b}}$, then $\alpha\beta \in F_{\boldsymbol{a}+\boldsymbol{b}}$ for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^n$.

For each $\sigma, \tau \in \Delta$ we have

$$e_\sigma e_\tau = \sum_{v \in \Delta} f^v_{\sigma,\tau}(\mu) e_v \tag{13}$$

where $f^v_{\sigma,\tau}(\mu) \in K[\boldsymbol{x}]$ for each $v \in \Delta$. The $f^v_{\sigma,\tau}(\mu)$ uniquely determine μ; they are called the **structured R-coefficients** of μ. If μ is understood from context, this we'll simplify our notation by writing $f^v_{\sigma,\tau} = f^v_{\sigma,\tau}(\mu)$. Since μ is a graded map, we must have $f^v_{\sigma,\tau} = 0$ whenever $|e_\sigma| + |e_\tau| \neq |e_v|$. In fact, since μ respects the multigrading, we must have

$$f^v_{\sigma,\tau}(\mu) = c^v_{\sigma,\tau}(\mu) \frac{m_\sigma m_\tau}{m_v}$$

where $c^v_{\sigma,\tau}(\mu) \in K$ for all $\sigma, \tau, v \in \Delta$ where $c^v_{\sigma,\tau}(\mu) = 0$ whenever whenever $|e_\sigma| + |e_\tau| \neq |e_v|$ or $m_\sigma m_\tau \nmid m_v$. The $c^v_{\sigma,\tau}(\mu)$ also uniquely determine μ; they are called the **structured K-coefficients** of μ. If μ is understood from context, this we'll simplify our notation by writing $c^v_{\sigma,\tau}(\mu) = c^v_{\sigma,\tau}$. It would be nice if we could re-express (13) as

$$\left(\frac{e_\sigma}{m_\sigma}\right)\left(\frac{e_\tau}{m_\tau}\right) = \sum_v c^v_{\sigma,\tau} \left(\frac{e_v}{m_v}\right), \tag{14}$$

but the problem is that F does not contain terms like e_σ/m_σ. In order to make sense of (14), we need to adjoin inverves to our base ring R.

3.2.1 Base change

Let S be the multiplicatively closed set generated by $\{m_\sigma \mid \sigma \in \Delta\}$ and let T be the multiplicatively closed set generated by $x_1 \cdots x_n \in R$. The localization of R at T has a natural description in terms of the Laurent polynomial ring over K, that is $R_T = K[\boldsymbol{x}, \boldsymbol{x}^{-1}]$, and the localization of R at S has a natural description as an R-submodule of R_T which is generated by Laurent monomials m_σ/m_τ for all $\sigma, \tau \in \Delta$. In general, any R-submodule M of R_T which is generated by Laurent monomials $\boldsymbol{x}^{\boldsymbol{a}}$ where $\boldsymbol{a} \in \mathbb{Z}^n$ is called a **monomial module**.

We want to perform a base change from R to R_S in order to make sense of (14). The localization functor $-_S$ from the category of R-complexes to the category of R_S-complexes is an exact functor which preserves quasiisomorphisms. In particular, it takes the quasiisomorphism $F \to R/\boldsymbol{m}$ to the quasiisomorphim $F_S \to 0$. Note that $F_S \to 0$ being a quasiisomorphism is equivalent to saying F_S is an exact R-complex. We can give a natural description of F_S as an MDG R_S-complex as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded R_S-module of F_S is given by

$$F_{S,k} := \begin{cases} \bigoplus\limits_{\dim \sigma = k-1} R_S e_\sigma & \text{if } \sigma \in \Delta \text{ and } 0 \leq k \leq \dim \Delta + 1 \\ 0 & \text{else} \end{cases}$$

The differential d_S of F_S is defined via the rule $\mathrm{d}_S(\boldsymbol{x}^{\boldsymbol{a}} e_\sigma) = \boldsymbol{x}^{\boldsymbol{a}} \mathrm{d}(e_\sigma)$ for all homogeneous generators e_σ and for all $\boldsymbol{a} \in \mathbb{Z}^n$. Similarly, the multiplication μ_S of F_S is defined via the rule $(\boldsymbol{x}^{\boldsymbol{a}} e_\sigma)(\boldsymbol{x}^{\boldsymbol{b}} e_\tau) = \boldsymbol{x}^{\boldsymbol{a}+\boldsymbol{b}} e_\sigma e_\tau$ for all homogeneous generators e_σ, e_τ and for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$. Note that d_S and μ_S are just the localizations of d and μ with respect to S, so our notation here is consistent. Since $-_S$ is an exact functor, we have $[F]_S \cong [F_S]$, where $[F]_S$ is the localization (with repsect to S) of the associator complex of F and where $[F_S]$ is the associator complex of F_S.

Observe that F_S has the structure of a $\mathbb{Z}^n$-graded MDG K-algebra. Indeed, we have a decomposition of F_S into MDG K-algebras

$$F_S = \bigoplus_{\boldsymbol{a}\in\mathbb{N}^n} F_{S,\boldsymbol{a}},$$

where the MDG K-algebra $F_{S,\boldsymbol{a}}$ in multidegree $\boldsymbol{a} \in \mathbb{Z}^n$ is defined as follows: the homogeneous component in homological degree $k \in \mathbb{Z}$ of the underlying graded K-vector space of $F_{S,\boldsymbol{a}}$ is given by

$$F_{S,k,\boldsymbol{a}} := \begin{cases} \bigoplus_{\dim\sigma=k-1} K\boldsymbol{x}^{\boldsymbol{a}}\frac{e_\sigma}{m_\sigma} & \text{if } \sigma\in\Delta,\ 0\leq k\leq \dim\Delta+1, \text{ and } \boldsymbol{x}^{\boldsymbol{a}}\in R_S \\ 0 & \text{else} \end{cases}$$

Notice the difference between $F_{S,k,\boldsymbol{a}}$ and $F_{k,\boldsymbol{a}}$: in $F_{S,k,\boldsymbol{a}}$, we do not need to have $\sigma\in\Delta_{\boldsymbol{a}}$ since it is okay if $m_\sigma \nmid \boldsymbol{x}^{\boldsymbol{a}}$. The differential $\mathrm{d}_{S,\boldsymbol{a}}$ of $F_{S,\boldsymbol{a}}$ is just the restriction of d_S to $\mathrm{d}_{S,\boldsymbol{a}}$, and the multiplication $\mu_{S,\boldsymbol{a}}$ of $F_{S,\boldsymbol{a}}$ is just the restriction of μ_S to $\mu_{S,\boldsymbol{a}}$. In particular, d_S homogeneous with respect to the $\mathbb{Z}^n$-grading which implies

$$0 = \mathrm{H}(F_S) \cong \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n} \mathrm{H}(F_{S,\boldsymbol{a}}).$$

Hence each K-complex $F_{S,\boldsymbol{a}}$ is an exact complex. Furthermore, since μ_S homogeneous with respect to the $\mathbb{Z}^n$-multrigrading, the associator complex $[F_S]$ with respect to μ_S has the structure of a $\mathbb{Z}^n$-graded K-complex:

$$[F_S] = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^n} [F_{S,\boldsymbol{a}}],$$

where $[F_{S,\boldsymbol{a}}]$ is the associator complex of $F_{S,\boldsymbol{a}}$. Thus since $-_S$ is an exact functor, we have

$$\begin{aligned} \mathrm{H}([F])_S &\cong \mathrm{H}([F]_S) \\ &\cong \mathrm{H}([F_S]) \\ &\cong \bigoplus_{\boldsymbol{a}\in\mathbb{N}^n} \mathrm{H}([F_{S,\boldsymbol{a}}]). \end{aligned}$$

It follows that

$$\mathrm{H}([F])_S \cong \mathrm{H}([F_S]) \cong \bigoplus_{\boldsymbol{a}\in\mathbb{N}^n} \mathrm{H}([F_{S,\boldsymbol{a}}]).$$

Hence μ_S is homologically associative if and only if

To simplifiy our notation in what follows, let us denote $\widetilde{F} = F_{S,\mathbf{0}}$, that is, $\widetilde{F}$ is the multidegree $\mathbf{0}$ part of F_S. It follows that

$$\mathrm{H}$$

The multiplication (14) makes perfect sense in $\widetilde{F}$. Denoting $\widetilde{e}_\sigma = e_\sigma/m_\sigma$ for each $\sigma\in\Delta$, we re-express (14) as

$$\widetilde{e}_\sigma\widetilde{e}_\tau = \sum_{v} c^{v}_{\sigma,\tau}\widetilde{e}_v.$$

Now let $\widetilde{F}$ denote the

Indeed, we have a decomposition of F_S into graded K-complethe homogeneous component in homological degree $k\in\mathbb{Z}$ of the underlying graded

Ton the homogeneous generators of F_S by $\mathrm{d}_\Delta(e_\varnothing) = 0$ and

$$\mathrm{d}_\Delta(e_\sigma) = \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{m_\sigma}{m_{\sigma\setminus i}}e_{\sigma\setminus i}$$

for all $\sigma\in\Delta\backslash\{\varnothing\}$ where $\mathrm{pos}(i,\sigma)$, the **position of vertex** i in σ, is the number of elements preceding i in the ordering of σ, and $\sigma\backslash i$ denotes the face obtained from σ by removing i. In the case where Δ is the r-simplex, we call F_Δ the **Taylor complex** of $R/\boldsymbol{m}$ over R.

has a natural descThe $\mathbb{Z}^n$-graded K-vector space structure of R_S gives rise to a $\mathbb{Z}^n$-graded K-complex structure of F_S. Indeed, we have a decomposition of F_S into K-complexes

$$F_S = \bigoplus_{\boldsymbol{a}\in\mathbb{N}^n} F_{S,\boldsymbol{a}},$$

where the K-complex $F_{S,\boldsymbol{a}}$ in multidegree $\boldsymbol{a}\in\mathbb{N}^n$ is defined as follows: the homogeneous component in homological degree $k\in\mathbb{Z}$ of the underlying graded K-vector space of $F_{S,\boldsymbol{a}}$ is given by

$$F_{S,k,\boldsymbol{a}} := \begin{cases} \bigoplus_{\dim\sigma=k-1} K\frac{\boldsymbol{x}^{\boldsymbol{a}}}{m_\sigma}e_\sigma & \text{if } \sigma\in\Delta \text{ and } 0\leq k\leq\dim\Delta+1 \\ 0 & \text{else} \end{cases}$$

to give F_S the structure
which means it takes exact

$$S_{\boldsymbol{a}} =$$

Just at R has a natural The $\mathbb{N}^n$-grading on the R-complex F induces a $\mathbb{Z}^n$-grading on the $R_{x_1\cdots x_n}$-complex $F \otimes_R R_{x_1\cdots x_n}$ in the obvious way. Indeed,
Let $\widetilde{F}$ be the K-complex given by the multidegree $\mathbf{0} = (0,\dots,0)$ part of $F \otimes_R R_{x_1\cdots x_n}$ and denote $\widetilde{e}_\sigma = e_\sigma / m_\sigma$ for each $\sigma \subseteq \Delta$. Then in $\widetilde{F}$, we can reexpress (13) as

$$\widetilde{e}_\sigma \widetilde{e}_\tau = \sum_v c^v_{\sigma,\tau} \widetilde{e}_v.$$

Let $\widetilde{\mu}$ denote this multilplication. Note that the differential d gives rise to a differential $\widetilde{\mathrm{d}}$ on $\widetilde{F}$ defined by:

$$\begin{aligned}
\widetilde{\mathrm{d}}(\widetilde{e}_\sigma) &= \frac{1}{m_\sigma}\mathrm{d}(e_\sigma) \\
&= \frac{1}{m_\sigma}\sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{m_\sigma}{m_{\sigma\setminus i}}e_{\sigma\setminus i} \\
&= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{m_\sigma}{m_\sigma m_{\sigma\setminus i}}e_{\sigma\setminus i} \\
&= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\frac{e_{\sigma\setminus i}}{m_{\sigma\setminus i}} \\
&= \sum_{i\in\sigma}(-1)^{\mathrm{pos}(i,\sigma)}\widetilde{e}_{\sigma\setminus i}.
\end{aligned}$$

Thus the PDG R-algebra (F,μ,d) induces a PDG K-algebra $(\widetilde{F},\widetilde{\mu},\widetilde{\mathrm{d}})$.

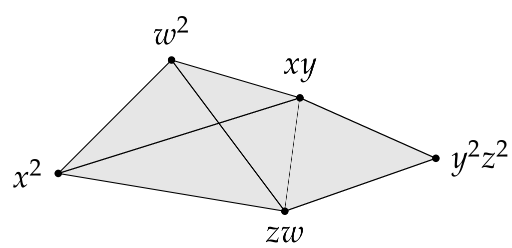
Theorem 3.2. *F is a DG R-algebra if and only if $\widetilde{F}$ is a DG K-algebra.*

Proof. A straightforward calculation gives us

$$[e_\sigma, e_\tau, e_v]_\mu = m_\sigma m_\tau m_v [\widetilde{e}_\sigma, \widetilde{e}_\tau, \widetilde{e}_v]_{\widetilde{\mu}}$$

for all $\sigma,\tau,v \in \Delta$. Thus μ is associative if and only if $\widetilde{\mu}$ is associative. □

Example 3.1. Consider the case where $R = K[x,y,z,w]$, $\boldsymbol{m} = x^2, w^2, zw, xy, y^2z^2$, and where Δ is the labeled simplicial complex which can be pictured below



We chose to only label the vertices in the picture above in order to keep things clean, but note that every face of the simplicial complex above should be labeled by an appropriate monomial. Since μ needs to respect the multigrading and needs to satisfy Leibniz law, we are forced to have

$$\begin{aligned}
\widetilde{e}_1\widetilde{e}_5 &= c^{14}_{1,5}\widetilde{e}_{14} + c^{45}_{1,5}\widetilde{e}_{45} \\
\widetilde{e}_2\widetilde{e}_5 &= c^{23}_{2,5}\widetilde{e}_{23} + c^{35}_{2,5}\widetilde{e}_{35} \\
\widetilde{e}_2\widetilde{e}_{45} &= c^{234}_{2,45}\widetilde{e}_{234} + c^{345}_{2,45}\widetilde{e}_{345} \\
\widetilde{e}_1\widetilde{e}_{35} &= c^{134}_{1,35}\widetilde{e}_{134} + c^{345}_{1,35}\widetilde{e}_{345} \\
\widetilde{e}_2\widetilde{e}_{14} &= c^{124}_{2,14}\widetilde{e}_{124} \\
\widetilde{e}_1\widetilde{e}_{23} &= c^{123}_{1,23}\widetilde{e}_{123}
\end{aligned}$$

where each $c^v_{\sigma,\tau}$ above is nonzero. At this point however, $\widetilde{\mu}$ is already not associative since $[\widetilde{e}_1, \widetilde{e}_5, \widetilde{e}_2] \neq 0$. Indeed, we use Singular to calculate an appropriate Gröbner basis for us to determine this:

```
intvec w=(3,3,3,2,2,2,2,1,1,1,1,1);
ring A=(0,c14,c45,c23,c35,c234,c134,c123,c124,c345,d345),(x1,x2,x5,x14,x45,x23,x35,x123,x124,x234,x13
ideal I =

x1*x5+c14*x14+c45*x45,
x2*x5+c23*x23+c35*x35,
x2*x45+c234*x234+c345*x345,
x1*x35+c134*x134+d345*x345,
x2*x14+c124*x124,
x1*x23+c123*x123;

std(I);

_[1]=(c23*c123)*x123+(-c14*c124)*x124+(-c45*c234)*x234+(c35*c134)*x134+(-c45*c345+c35*d345)*x345
_[2]=(-c134)*x23*x134+(-d345)*x23*x345+(c123)*x35*x123
_[3]=(-c234)*x14*x234+(-c345)*x14*x345+(c124)*x45*x124
_[4]=(-c134)*x5*x134+(-d345)*x5*x345+(c14)*x14*x35+(c45)*x45*x35
_[5]=(-c234)*x5*x234+(-c345)*x5*x345+(c23)*x45*x23+(c35)*x45*x35
_[6]=(-c124)*x5*x124+(c23)*x14*x23+(c35)*x14*x35
_[7]=x2*x45+(c234)*x234+(c345)*x345
_[8]=x2*x14+(c124)*x124
_[9]=x1*x35+(c134)*x134+(d345)*x345
_[10]=x1*x23+(c123)*x123
_[11]=x2*x5+(c23)*x23+(c35)*x35
_[12]=x1*x5+(c14)*x14+(c45)*x45
```

A quick calculation gives us

$$\begin{aligned} \mathrm{d}[e_1, e_{45}, e_2] &= x[e_1, e_5, e_2] \\ \mathrm{d}[e_{14}, e_5, e_2] &= y[e_1, e_5, e_2] \\ \mathrm{d}[e_1, e_5, e_{23}] &= z[e_1, e_5, e_2] \\ \mathrm{d}[e_1, e_{35}, e_2] &= w[e_1, e_5, e_2]. \end{aligned}$$

Therefore

$$\mathrm{H}_i([F]) \cong \begin{cases} K & \text{if } i = 3 \\ 0 & \text{else} \end{cases}$$

4 Associativity Test using Gröbner Bases

Let K be a field and let (F, d, μ) be an MDG K-algebra. Let $n \geq 1$ and assume that $(1, e_1, \dots, e_n)$ is an ordered homogeneous basis of F such that

1. $|e_i| \geq 1$ for all $1 \leq i \leq n$,
2. if $|e_j| > |e_i|$, then $j > i$.

In this section, we will use some tools from the theory of Gröbner bases to determine whether or not F is DG algebra, that is, whether or not F is associative. For simplicity, we will only describe how this works in the case where K has characteristic 2. Let $(c^k_{i,j})$ be the structured K-coefficients of μ. Thus for each $0 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=0}^{n} c^k_{i,j} e_k,$$

where we use $\star$ to denote multiplication with respect to μ. Let S be the weighted polynomial ring $K[e_1, \dots, e_n]$ where e_i is weighted of degree $|e_i|$ for each $1 \leq i \leq n$. A monomial of S has the form $\boldsymbol{e}^{\boldsymbol{a}} = e_1^{a_1} \cdots e_n^{a_n}$ where $\boldsymbol{a} \in \mathbb{N}^n$ where we identify the monomial $\boldsymbol{e}^{(0,\dots,0)}$ with 1 in this notation. Given a monomial $\boldsymbol{e}^{\boldsymbol{a}}$, we define its **degree**, denoted $\deg(\boldsymbol{e}^{\boldsymbol{a}})$, and its **weighted degree**, denoted $|\boldsymbol{e}^{\boldsymbol{a}}|$, by

$$\deg(\boldsymbol{e}^{\boldsymbol{a}}) = \sum_{i=1}^{n} a_i \quad \text{and} \quad |\boldsymbol{e}^{\boldsymbol{a}}| = \sum_{i=1}^{n} a_i |e_i|.$$

For each $k \in \mathbb{N}$, we shall write

$$S_k = \text{span}_K\{e^a \mid \deg(e^a) = k\}.$$

We identity F with $S_0 + S_1 = K + \sum_{i=1}^n Ke_i$. In order to keep notation consistent, we shall write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to μ, and we shall write $\alpha\beta$ to denote their multiplication with respect to $\cdot$ in S. In particular, note that $\deg(e_i \star e_j) = 1$ and $\deg(e_i e_j) = 2$.

For each $1 \le i, j \le n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Note that since we are working over a field of characteristic 2, we have $f_{i,j} = f_{j,i}$ for all $1 \le i, j \le n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \le i, j \le n\}$ and let I be the ideal of S generated by $\mathcal{F}$. We equip S with a weighted lexicographic ordering $>_w$ with respect to the weight vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials e^a and e^b in S, we say $e^a >_w e^b$ if either

1. $|e^a| > |e^a|$ or;
2. $|e^a| = |e^b|$ and there exists $1 \le i \le n$ such that $\alpha_i > \beta_i$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_{i-1} = \beta_{i-1}$.

Observe that for each $1 \le i \le j \le n$, we have $\mathrm{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since μ is graded, we must have $|e_i e_j| = |e_i| + |e_j| = |e_k|$. It follows that $|e_k| > |e_i|$ since $|e_i|, |e_j| \ge 1$. This implies $k > i$ by our assumption on $(e_1, \dots, e_n)$. Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

4.1 Main Theorem

Before we state and prove the main theorem, let us introduce one more piece of notation. We denote $\mathcal{M} = \{e^a \mid e^a \notin \mathrm{LT}(I)\}$. Since $\mathrm{LT}(f_{i,j}) = e_i e_j$ for all $1 \le i, j \le n$, we see that $\mathcal{M}$ is a subset of $\{e_1, \dots, e_n\}$. Now we are ready to state and prove the main theorem:

Theorem 4.1. *The following statements are equivalent:*

1. *F is associative.*
2. *$\mathcal{F}$ is a Gröbner basis.*
3. *$\mathcal{M} = \{e_1, \dots, e_n\}$.*

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $1 \le i \le j < k \le n$. We have

$$\begin{aligned}
S_{i,j,k} &:= S(f_{j,k}, f_{i,j}) \\
&= e_i f_{j,k} - f_{i,j} e_k \\
&= e_i(e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\
&= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\
&= \left(\sum_l c_{i,j}^l e_l\right) e_k - e_i \left(\sum_l c_{j,k}^l e_l\right) \\
&= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l.
\end{aligned}$$

Now we divide $S_{i,j,k}$ by $\mathcal{F}$:

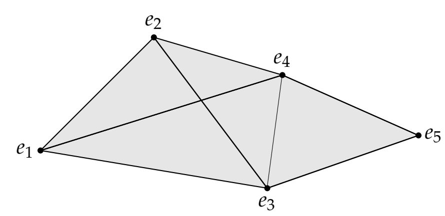
$$\begin{aligned}
S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} \\
&= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) + \sum_l c_{j,k}^l (f_{i,l} - e_i e_l) \\
&= \sum_l c_{i,j}^l (e_l e_k - e_l e_k + e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l - e_i e_l) \\
&= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
&= \left(\sum_l c_{i,j}^l e_l\right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l\right) \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k].
\end{aligned}$$

Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by $\mathcal{F}$. It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. A straightforward comptutation also shows that $S(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \leq i \leq n$. Finally, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq l \leq n$. We have

$$\begin{aligned} S_{i,j,k,l} &:= S(f_{k,l}, f_{i,j}) \\ &= e_i e_j f_{j,k} - f_{i,j} e_k e_l \\ &= (f_{i,j} + e_i \star e_j) f_{j,k} - f_{i,j}(f_{k,l} + e_k \star e_l) \\ &= (e_i \star e_j) f_{j,k} - f_{i,j}(e_k \star e_l). \end{aligned}$$

From this, it's easy to see that $S_{i,j,k,l}^{\mathcal{F}} = 0$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. □

Example 4.1. Let Δ be the simplicial complex below



and let (F, d) be the $\mathbb{F}_2$-complex induced by Δ. Let's write the homogeneous components of F as a graded $\mathbb{F}_2$-vector space

$$\begin{aligned} F_0 &= \mathbb{F}_2 \\ F_1 &= \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5 \\ F_2 &= \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45} \\ F_3 &= \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345} \\ F_4 &= \mathbb{F}_2 e_{1234} \end{aligned}$$

Let μ be a multiplication on F such that

$$\begin{aligned} e_1 \star_\mu e_5 &= e_{14} + e_{45} \\ e_2 \star_\mu e_5 &= e_{23} + e_{35} \\ e_2 \star_\mu e_{45} &= e_{234} + e_{345} \\ e_1 \star_\mu e_{35} &= e_{134} + e_{345} \\ e_1 \star_\mu e_{23} &= e_{123} \\ e_2 \star_\mu e_{14} &= e_{124}. \end{aligned}$$

Then F is not associative with respect to μ since $[e_1, e_5, e_2]_\mu = e_{123} + e_{124} + e_{234} + e_{134} \neq 0$. We use Singular to determine this:

```
intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2),I);

// e123+e124+e234+e134
```