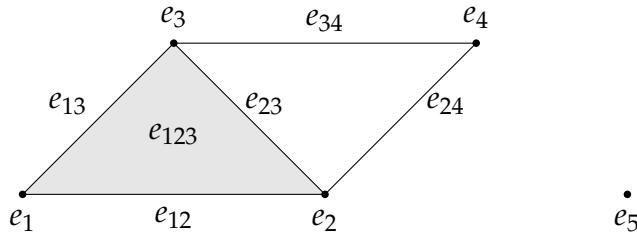


Associativity Test Using Gröbner Bases

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1 Introduction

Let Δ be a finite simplicial complex and let K be a field of characteristic 2 (we only assume characteristic 2 for simplicity in what follows). Attached to Δ is a graded K -complex F_Δ whose homogeneous component of degree $k \in \mathbb{N}$ is the K -span of all $(k - 1)$ -faces of Δ . For instance, if Δ is the simplicial complex below,



then the homogeneous components of F_Δ are given by:

$$\begin{aligned} F_{\Delta,0} &= Ke_\emptyset \\ F_{\Delta,1} &= Ke_1 + Ke_2 + Ke_3 + Ke_4 + Ke_5 \\ F_{\Delta,2} &= Ke_{12} + Ke_{13} + Ke_{23} + Ke_{24} + Ke_{34} \\ F_{\Delta,3} &= Ke_{123}. \end{aligned}$$

Note that we often write $e_\emptyset = 1 = e_0$ and we think of F_Δ as a graded K -vector space with $F_{\Delta,0} = K$. Now let us equip F_Δ with a **graded-multiplication** \star , where by a graded-multiplication, we mean that \star is a binary operator on F_Δ which satisfies the following properties:

- \star is unital with 1 being the unit;
- \star is K -bilinear;
- \star is commutative;
- \star respects the grading meaning that if α, β are homogeneous elements of F_Δ , then $\alpha \star \beta$ is homogeneous and

$$|\alpha \star \beta| = |\alpha| + |\beta|,$$

where $|\cdot|$ denote the homogeneous degree of an element in F_Δ .

Given such a graded-multiplication F_Δ , it is natural to wonder whether or not \star is associative, meaning

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$$

for all $\alpha, \beta, \gamma \in F_\Delta$. In this note, we will determine whether or not \star is associative using tools from the theory of Gröbner bases.

2 Graded K -algebras

We begin in a slightly more general context. Let F be a graded K -vector space and let \star be a graded-multiplication on F . Let $n \geq 1$ and assume that (e_0, e_1, \dots, e_n) is an ordered homogeneous basis of F such that

1. $e_0 = 1$;
2. $|e_i| \geq 1$ for all $1 \leq i \leq n$,
3. if $|e_j| > |e_i|$, then $j > i$.

For each $0 \leq i, j \leq n$, we have

$$e_i \star e_j = \sum_{k=0}^n c_{i,j}^k e_k,$$

where $c_{i,j}^k \in K$ for each k . Let S be the weighted polynomial ring $K[e_1, \dots, e_n]$ where e_i is weighted of degree $|e_i|$ for each $1 \leq i \leq n$. A monomial of S has the form $e^\alpha = e_1^{a_1} \cdots e_n^{a_n}$ where $\alpha \in \mathbb{N}^n$ where we identify the monomial $e^{(0, \dots, 0)}$ with 1 in this notation. Given a monomial e^α , we define its **degree**, denoted $\deg(e^\alpha)$, and its **weighted degree**, denoted $|e^\alpha|$, by

$$\deg(e^\alpha) = \sum_{i=1}^n a_i \quad \text{and} \quad |e^\alpha| = \sum_{i=1}^n a_i |e_i|.$$

For each $k \in \mathbb{N}$, we shall write

$$S_k = \text{span}_K \{e^\alpha \mid \deg(e^\alpha) = k\}.$$

We identify F with $S_0 + S_1 = K + \sum_{i=1}^n Ke_i$. In order to keep notation consistent, we shall write $\alpha \star \beta$ to denote the multiplication of elements $\alpha, \beta \in F$ with respect to \star , and we shall write $\alpha \beta$ to denote their multiplication with respect to \cdot in S . In particular, note that $\deg(e_i \star e_j) = 1$ and $\deg(e_i e_j) = 2$.

For each $1 \leq i, j \leq n$, let $f_{i,j}$ be the polynomial in S defined by

$$f_{i,j} = e_i e_j - \sum_k c_{i,j}^k e_k = e_i e_j - e_i \star e_j.$$

Note that since both \star and \cdot are commutative, we have $f_{i,j} = f_{j,i}$ for all $1 \leq i, j \leq n$. Let $\mathcal{F} = \{f_{i,j} \mid 1 \leq i, j \leq n\}$ and let I be the ideal of S generated by \mathcal{F} . We equip S with a weighted lexicographic ordering $>_w$ with respect to the weight vector $w = (|e_1|, \dots, |e_n|)$ which is defined as follows: given two monomials e^α and e^β in S , we say $e^\alpha >_w e^\beta$ if either

1. $|\alpha| > |\beta|$ or;
2. $|\alpha| = |\beta|$ and there exists $1 \leq i \leq n$ such that $\alpha_i > \beta_i$ and $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_{i-1} = \beta_{i-1}$.

Observe that for each $1 \leq i \leq j \leq n$, we have $\text{LT}(f_{i,j}) = e_i e_j$. Indeed, if $e_i \star e_j = 0$, then this is clear, otherwise a nonzero term in $e_i \star e_j$ has the form $c_{i,j}^k e_k$ for some k where $c_{i,j}^k \neq 0$. Since \star is graded, we must have $|e_i e_j| = |e_i| + |e_j| = |e_k|$. It follows that $|e_k| > |e_i|$ since $|e_i|, |e_j| \geq 1$. This implies $k > i$ by our assumption on (e_1, \dots, e_n) . Therefore since $|e_i e_j| = |e_k|$ and $k > i$, we see that $e_i e_j >_w e_k$.

3 Main Theorem

Before we state and prove the main theorem, let us introduce one more piece of notation. We denote $\mathcal{M} = \{e^\alpha \mid e^\alpha \notin \text{LT}(I)\}$. Since $\text{LT}(f_{i,j}) = e_i e_j$ for all $1 \leq i, j \leq n$, we see that \mathcal{M} is a subset of $\{e_1, \dots, e_n\}$. Now we are ready to state and prove the main theorem:

Theorem 3.1. *The following statements are equivalent:*

1. \star is associative.
2. \mathcal{F} is a Gröbner basis.
3. $\mathcal{M} = \{e_1, \dots, e_n\}$.

Proof. Statements 2 and 3 are easily seen to be equivalent, so we will only show that statements 1 and 2 are equivalent. Let us calculate the S-polynomial of $f_{j,k}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq n$. We have

$$\begin{aligned} S_{i,j,k} &:= S(f_{j,k}, f_{i,j}) \\ &= e_i f_{j,k} - f_{i,j} e_k \\ &= e_i (e_j e_k - e_j \star e_k) - (e_i e_j - e_i \star e_j) e_k \\ &= (e_i \star e_j) e_k - e_i (e_j \star e_k) \\ &= \left(\sum_l c_{i,j}^l e_l \right) e_k - e_i \left(\sum_l c_{j,k}^l e_l \right) \\ &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l. \end{aligned}$$

Now we divide $S_{i,j,k}$ by \mathcal{F} :

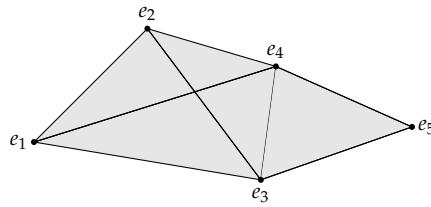
$$\begin{aligned}
S_{i,j,k} - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} &= \sum_l c_{i,j}^l e_l e_k - \sum_l c_{j,k}^l e_i e_l - \sum_l c_{i,j}^l f_{l,k} + \sum_l c_{j,k}^l f_{i,l} \\
&= \sum_l c_{i,j}^l (e_l e_k - f_{l,k}) + \sum_l c_{j,k}^l (f_{i,l} - e_i e_l) \\
&= \sum_l c_{i,j}^l (e_l e_k - e_l e_k + e_l \star e_k) + \sum_l c_{j,k}^l (e_i e_l - e_i \star e_l - e_i e_l) \\
&= \sum_l c_{i,j}^l e_l \star e_k - \sum_l c_{j,k}^l e_i \star e_l \\
&= \left(\sum_l c_{i,j}^l e_l \right) \star e_k - e_i \star \left(\sum_l c_{j,k}^l e_l \right) \\
&= (e_i \star e_j) \star e_k - e_i \star (e_j \star e_k) \\
&= [e_i, e_j, e_k].
\end{aligned}$$

Note that $\deg([e_i, e_j, e_k]) = 1$, so we cannot divide this anymore by \mathcal{F} . It follows that $S_{i,j,k}^{\mathcal{F}} = [e_i, e_j, e_k]$. A straightforward computation also shows that $S(f_{i,i}, f_{i,i})^{\mathcal{F}} = 0$ for all $1 \leq i \leq n$. Finally, let us calculate the S-polynomial of $f_{k,l}$ and $f_{i,j}$ where $1 \leq i \leq j < k \leq l \leq n$. We have

$$\begin{aligned}
S_{i,j,k,l} &:= S(f_{k,l}, f_{i,j}) \\
&= e_i e_j f_{j,k} - f_{i,j} e_k e_l \\
&= (f_{i,j} + e_i \star e_j) f_{j,k} - f_{i,j} (f_{k,l} + e_k \star e_l) \\
&= (e_i \star e_j) f_{j,k} - f_{i,j} (e_k \star e_l).
\end{aligned}$$

From this, it's easy to see that $S_{i,j,k,l}^{\mathcal{F}} = 0$. Now the equivalence of statements 1 and 2 follow immediately from Buchberger's Criterion. \square

Example 3.1. Let Δ be the simplicial complex below



and let F be the corresponding graded \mathbb{F}_2 -vector space induced by Δ . Let's write the homogeneous components of F as a graded \mathbb{F}_2 -vector space

$$\begin{aligned}
F_0 &= \mathbb{F}_2 \\
F_1 &= \mathbb{F}_2 e_1 + \mathbb{F}_2 e_2 + \mathbb{F}_2 e_3 + \mathbb{F}_2 e_4 + \mathbb{F}_2 e_5 \\
F_2 &= \mathbb{F}_2 e_{12} + \mathbb{F}_2 e_{13} + \mathbb{F}_2 e_{14} + \mathbb{F}_2 e_{23} + \mathbb{F}_2 e_{24} + \mathbb{F}_2 e_{34} + \mathbb{F}_2 e_{35} + \mathbb{F}_2 e_{45} \\
F_3 &= \mathbb{F}_2 e_{123} + \mathbb{F}_2 e_{124} + \mathbb{F}_2 e_{134} + \mathbb{F}_2 e_{234} + \mathbb{F}_2 e_{345} \\
F_4 &= \mathbb{F}_2 e_{1234}
\end{aligned}$$

Let \star be a graded-multiplication on F such that

$$\begin{aligned}
e_1 \star e_5 &= e_{14} + e_{45} \\
e_2 \star e_5 &= e_{23} + e_{35} \\
e_2 \star e_{45} &= e_{234} + e_{345} \\
e_1 \star e_{35} &= e_{134} + e_{345} \\
e_1 \star e_{23} &= e_{123} \\
e_2 \star e_{14} &= e_{124}.
\end{aligned}$$

Then \star is not associative since

$$\begin{aligned}
[e_1, e_5, e_2] &= (e_1 e_5) e_2 + e_1 (e_5 e_2) \\
&= e_{123} + e_{124} + e_{234} + e_{134} \\
&\neq 0.
\end{aligned}$$

We use Singular to determine this in the code below:

```

intvec w=(1,1,1,2,2,2,2,3,3,3,3,3);
ring A=2,(e1,e2,e5,e14,e45,e23,e35,e123,e124,e234,e134,e345),Wp(w);

poly f(1)(5) = e1*e5+e14+e45;
poly f(2)(5) = e2*e5+e23+e35;
poly f(2)(45) = e2*e45+e234+e345;
poly f(1)(35) = e1*e35+e134+e345;
poly f(1)(23) = e1*e23+e123;
poly f(2)(14) = e2*e14+e124;

ideal I = f(1)(5),f(2)(5),f(2)(45),f(1)(35),f(1)(23),f(2)(14);

poly s(1)(5)(2) = e1*f(2)(5)+e2*f(1)(5);
reduce(s(1)(5)(2),I);

// e123+e124+e234+e134

```