Ch5 Lecture Notes

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Contents

1 Units and Light

(Not textbook notes but stuff I added)

Below is a list of words we associate with light, but can we give these proper *physical* definitions?

Brightness, flux, intensity, luminosity, lux, lumens, magnitude, polarization, ...

1.1 Lumens, Luminosity, Lux

We have many **many** ways to quantify light. Example flashlight with adjustable end that concentrates beam or disperses it.

- **Luminosity**: absolute measure of electromagnetic power.

$$[L] = \frac{\text{Joules}}{\text{second}} = \text{Watt}$$

$$L\text{sol} = 3.828 \times 10^{26}\text{W}$$

- **Flux**: How bright does a light source appear? Depends on luminosity, **distance**, and amount of dust between us and the light source.

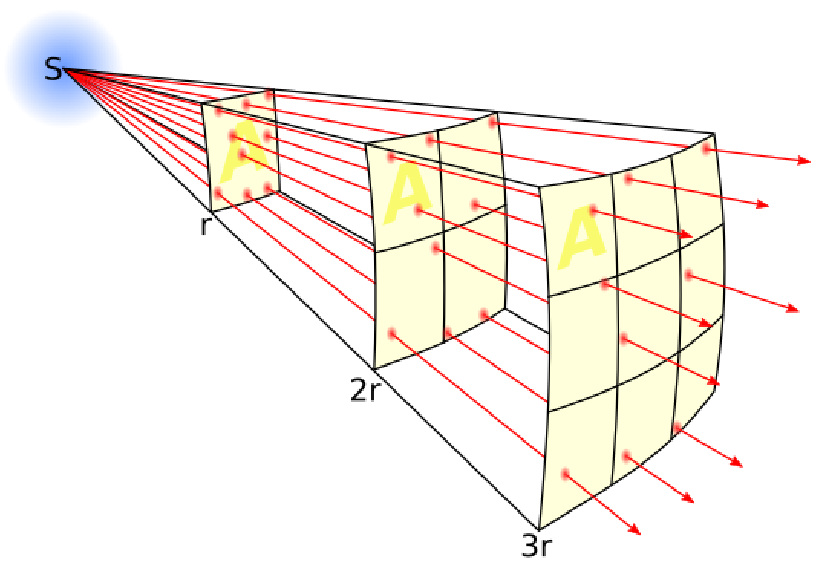

Figure 1: Flux from a point light source.

$$\text{Flux} = \frac{\text{Luminosity}}{\text{Surface Area}}$$

- **Candella**: SI unit for *luminous intensity* (cd) Specifically a measure of luminous power per solid angle, **but** wavelengths are weighted. Weights given by *luminosity function* (model of human eyes sensitivity to different wavelengths)

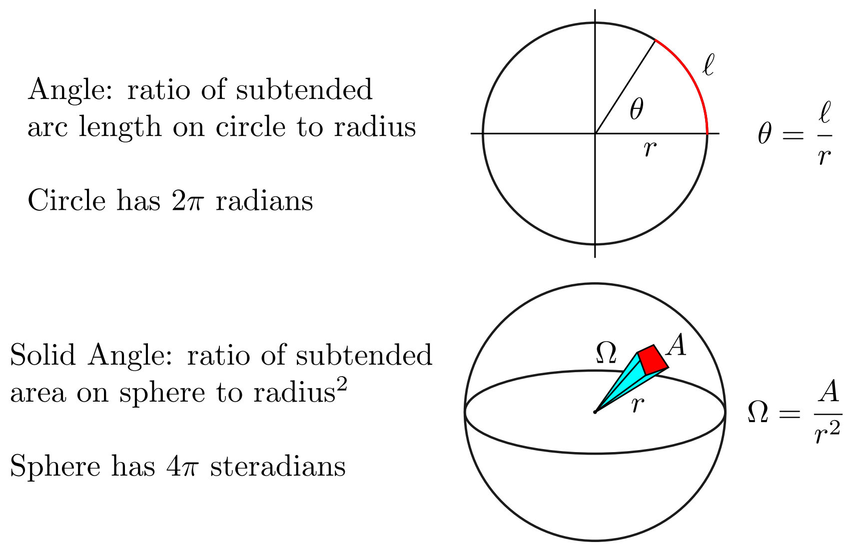

Figure 2: Explanation of what a solid angle is

A wax candle has a luminous intensity of about 1cd.

- **Lumen**: unit of luminous flux (ℓm), a measure of **percieved** power of light. Luminous flux weights the power of different wavelengths based on the human eye sensitivity.

$$1\ell m = 1\text{cd} \times 1\text{sr}$$

- **Lux**: unit of luminous flux per unit area. Example: flashlight same distance from wall but is the beam concentrated or spread out.

$$1\ell x = 1\ell m/m^2$$

2 Random facts about light

- $c = \lambda\nu = 2.998e8\text{m/s}$
- Energy of light given by Plank's constant times the frequency

$$E = h\nu \quad h = 6.626\text{e-34Js}$$

- Information we can get from studying light
  - apparent brightness
  - spectral energy distribution (find example)
  - Doppler shift
  - spectral line broadening (find example)
  - Zeeman line splitting (find example)
  - temporal variations
  - polarization

  Applying physics principles we can also determine
  - light source's **distance**
  - luminosity
  - temperature
  - chemical composition
  - size
  - rotation
  - magnetic fields
  - radial and transverse velocity
  - **intervening absorption by gas and dust**
- Astronomical sources categorized as *point* and *extended*
  - **Point**: most stars
  - **Extended**: sol, nebulae, *resolved* galaxies, diffuse synchotron emission, **CMB**, *IR dust emission in the Solar system.*

  Light measured from the two sources has to be handled differently

3 The Magnitude Scale

A star's **apparent brightness** is referred to as **magnitude**.

Higher number equals fainter object

- "Cumbersome" system inherited from antiquity and still widely used.
- Originally based on appearance of stars between **sunset** and **astronomical twilight** (see figure 3)

  Sunset → end of twilight broken into 6 segments. Stars that appear in the first segment were magnitude 1 (the brightest), stars appearing in the second time segment were magnitude 2, . . .
- Human eye the only tool to quantify magnitudes for centuries.
- Invention of photometers revealed two facts
  1. Magnitude 1 was too broad. Sirius *much* brighter than Regulus but both are mag1.
  2. Ratios of magnitude brightness ≈ 2.5.

$$\frac{B_3}{B_4} \approx 2.5, \quad \frac{B_3}{B_5} \approx 2.5^2$$

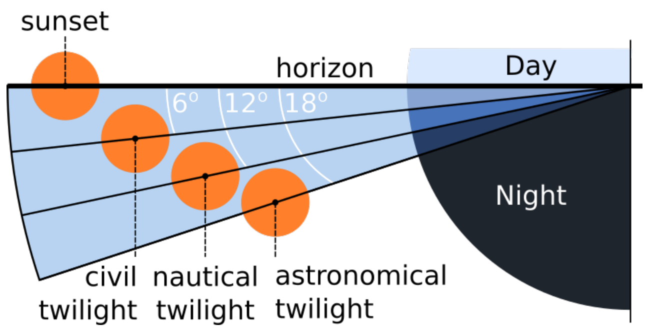

Figure 3: What is twilight

Humans see *equal brightness* ratios as equal steps in *magnitude* $\therefore$ **human vision is logarithmic**

m = magnitude, F = apparent brightness (flux)

$$\Delta m \propto \log\left(\frac{F_2}{F_1}\right)$$

- 1856 Pogson proposes modern definition of magnitude scale.

$$\boxed{\frac{F_1}{F_2} \equiv \left(\sqrt[5]{100}\right)^{m_2-m_1}} \approx 2.5119^{m_2-m_1}$$

Mag difference of 5 exactly equals ratio of 100 to 1.

$$\log\left[\frac{F_1}{F_2}\right] = \log\left[\left(\sqrt[5]{100}\right)^{m_2-m_1}\right]$$

$$\log\left[\frac{F_1}{F_2}\right] = \log(100^{\frac{m_2-m_1}{5}})$$

$$\log\left[\frac{F_1}{F_2}\right] = \frac{m_2-m_1}{5}\log(10^2)$$

$$-\log\left[\frac{F_2}{F_1}\right] = \frac{m_2-m_1}{5}2$$

$$\boxed{m_2 - m_1 = -2.5\log\left[\frac{F_2}{F_1}\right]}$$

1. Cannot identify magnitude of a single star by itself. Must compare stars through difference in magnitudes.
2. No zero from log $\therefore$

$$\boxed{\text{Pogson Equation: } m_i = -2.5 \log F_i + C}$$

where C is the *zero-point offset*. Astronomers have to agree a specific star has a specified magnitude (see Bolometric magnitude and IAU 2015 resolution B2 for examples).

$$[F] = \text{photons s}^{-1} \text{ cm}^{-2}$$

- What difference in magnitude results from a *small* difference in apparent brightness?

$$\Delta m = -2.5 \log \left[\frac{F_2}{F_1} \right] \quad \rightarrow \quad f(x) = -2.5 \log(x)$$

Use a Taylor series to expand the log function

$$f(x) = f(a) + \frac{1}{1!} f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \ldots$$

Need to change base to compute derivatives correctly

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$$

$$f(x) = -2.5 \frac{\ln(x)}{\ln(10)} = -1.086 \ln(x)$$

$$f(a) = -1.086 \ln(x)$$

$$f'(x) = -1.086 \frac{1}{x}, \quad f'(a) = -1.086 \frac{1}{a}$$

$$f''(x) = +1.086 \frac{1}{x^2}, \quad f''(a) = +1.086 \frac{1}{a^2}$$

Let $a = 1$ (i.e. $F_2 = F_1$)

$$f(x) = -1.086(x-1) + 1.086 \frac{(x-1)^2}{2} + \ldots$$

Evaluate this function at $x = F_2/F_1$ under the condition that $F_2 = F_1 + \epsilon$. $\therefore x \approx 1$ and $(x-1)^n \approx 0$ for $n > 1$.

$$f(x = F_2/F_1) = -1.086 \left(\frac{F_2}{F_1} - 1 \right) + O(x^2)$$

$$f(x) \approx 1.086 \left(\frac{F_2}{F_1} - \frac{F_1}{F_1} \right)$$

$$\boxed{\Delta m \approx -\frac{\Delta F}{F_1}}$$

Suppose star 1 has a magnitude of $m_1 = 3.5$ and star 2 $m_2 = 3.6$. Than

$$\Delta m = 0.1$$

and star 2 is about 10% brighter.

$$0.1 \approx \frac{\Delta F}{F_1}$$

$$0.1F_1 \approx (F_2 - F_1)$$

$$F_2 \approx 1.1F_1$$

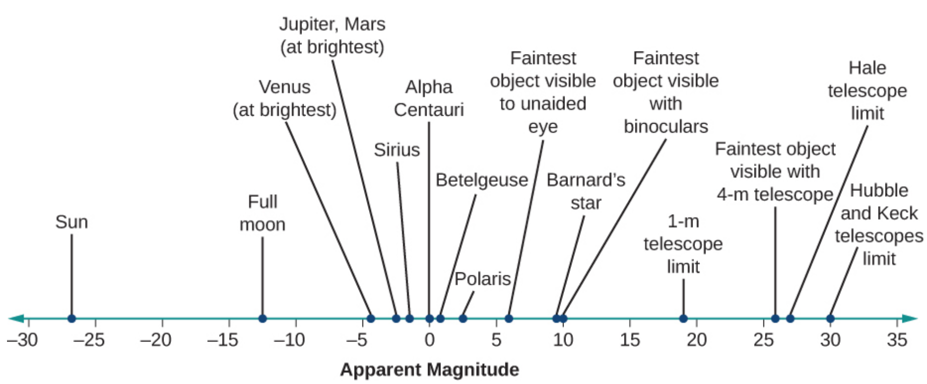

Figure 4: Modern Magnitude system

3.1 Magnitude and Wavelength Dependence

The human eye's sensitivity to to different wavelengths means sources can appear to have different magnitudes, even if they are the same luminosity. Compare a flashlight to the IR beam from a TV remote.

The Pogson equation is an example of the **visual magnitude**.

$$\boxed{\text{Monochromatic Pogson Equation: } m_\lambda = -2.5 \log F_\lambda + C}$$

Bolometric magnitude is the opposite of monochromatic, and includes ALL EM radiation emitted by the source.

$$\text{Bolometric correction: } BC_{\text{band}} = m_{\text{bol}} - m_{\text{band}}$$

m_{band} is the magnitude in some passband.

Example: The bolometric correction to the visual magnitude, BC_{V}, for the sun is $BC_{\text{V}} = -0.07$ magnitudes.

3.2 Absolute Magnitude

Absolute Magnitude: The apparent magnitude a star would be *if* it was 10 parsecs away.

Must know the stars apparent magnitude and distance.

$$\textbf{Distance Modulus}\text{: } m - M = 5\log\left(\frac{d}{10}\right)$$

Distance Modulus	Distance (parsec)
1	15.8
5	100
10	1000
15	10000

If $d > 10$pc must consider **interstellar absorption**!

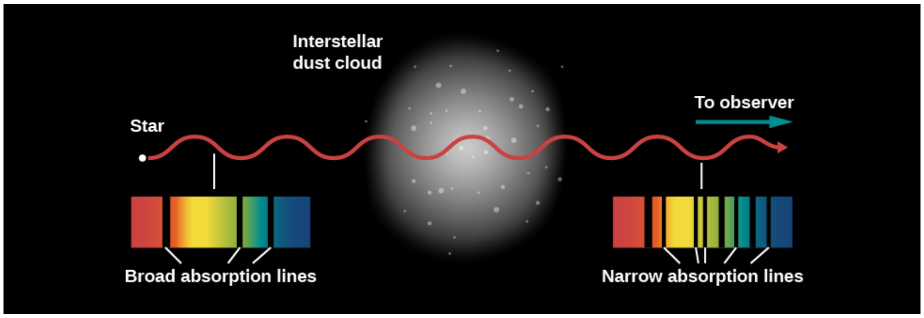

Figure 5: Interstellar absorption

$$\text{Apparent Distance modulus: } (m - M)_\lambda = (m - M)_0 + A_\lambda$$

A_λ is absorption (in magnitude) at that passband (wavelength λ).

Interstellar absorption always makes an object appear further.

4 Color Index

Advances in photography in near end of 19th century allowed **quantitative** measurements of a star's color.

Earliest photographic plates more sensitive to blue light.

- Blue stars appear brighter
- Red stars appear dimmer

Color index is the difference between magnitudes of a star

5 Flux

The energy flux (F) or just flux, describes the *apparent brightness* in physical units.

F is the amount of light energy per unit area (ΔA) per unit time (Δt) at given bandpass

$$F = \frac{E_{\text{band}}}{\Delta A \, \Delta t}$$

$$[F] = \text{erg cm}^{-2}\ \text{s}^{-1}\ \text{or Wcm}^{-2}$$

$$1\text{erg} = 100\text{nJ (an erg is a cgs unit of energy)}$$

In practice, we report the **monochromatic flux**, flux at a specific λ or ν.

$$F_\lambda = \frac{E_\lambda}{\Delta A \, \Delta t \, \Delta\lambda} \quad , \quad F_\nu = \frac{E_\lambda}{\Delta A \, \Delta t \, \Delta\nu}$$

$$\nu F_\nu = \lambda F_\lambda$$

$[F_\lambda]$=erg s^{-1} cm^{-2} Å^{-1} , $[F_\nu]$=erg s^{-1} cm^{-2} Hz^{-1}

A popular unit for F_ν (for radio astronomy) is the *jansky*

$$1\ \text{jansky} = 10^{-26}\,\text{W m}^{-2}\ \text{Hz}^{-1}$$

Note that most optical astronomical detectors do not detect energy, they detect photons. F from Pogson equation is measured in photons per s per cm^2.

Converting flux units from photon flux to energy flux is

$$\left[F \text{ in } \frac{\text{photons}}{\text{s cm}^2}\right] = \left[F \text{ in } \frac{\text{energy}}{\text{s cm}^2}\right] h\nu$$

$h\nu$ is the energy per photon.

There are *many* ways to present the **spectral energy distribution** (SED) of an astronomical source.

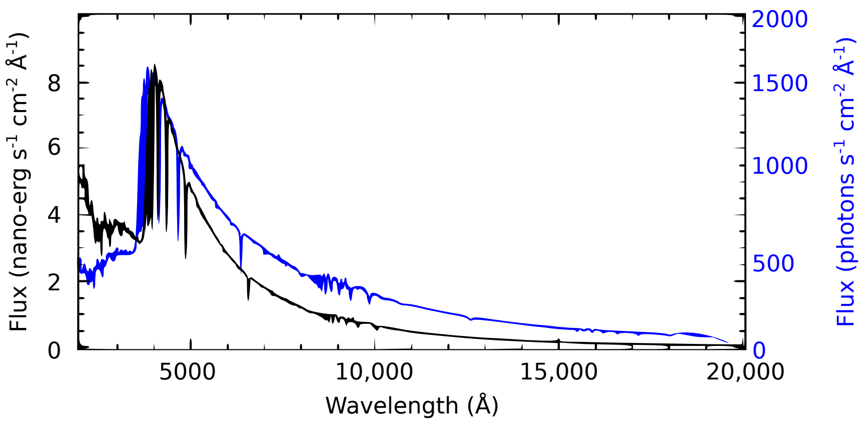

Figure 6: Energy Flux vs Photon Flux for Vega. Notice slope is steeper for energy flux

6 Blackbody Radiation

(Start with video about object "blacker" than VANTA black)

https://youtu.be/JoLEIiza9B://youtu.be/JoLEIiza9Bc

An object that absorbs EM-radiation at **all** wavelengths is a **blackbody**.

- idealized object (no real blackbodies but still a useful model)
- opaque
- non-reflective

Blackbody radiation is the *thermal* EM radiation emitted by a blackbody that is in **thermal equilibrium** with its environment.

The SED of blackbody depends *only* on its temperature.

- composition
- size
- shape

do not determine a blackbody's SED.

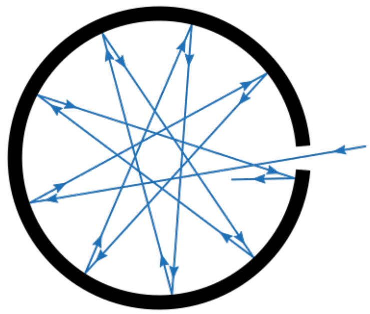

Figure 7: A cavity with a hole a useful model of a blackbody

6.1 Model Blackbody Radiation: Rayleigh-Jeans Law

We can "easily" derive an expression for the EM flux of a blackbody[1]. A blackbody is a perfect absorber **and** therefore also a perfect emitter.

We'll model blackbody radiation as the EM radiation that is in thermal equilibrium with the walls of the blackbody. The walls of the blackbody are in thermal equilibrium with the "temperature" of radiation (energy in = energy out).

- Consider our blackbody cavity to be a cube of length L (it can be shown that shape does not matter but the analysis is easier with Cartesian symmetry).
- Walls of cavity are oscillating charge particles.
- Charge particles oscillation frequency = EM radiation frequency
- At thermal equilibrium, average energy of oscillating charge = to average energy EM field.

Oscillating particle has energy

$$H = \frac{p^2}{2m} + \tfrac{1}{2}aq^2$$
$$H = H_{\text{kin}} + H_{\text{pot}}$$

Equipartition theorem: At thermal equilibrium, energy is shared evenly among all possible forms.

$$\langle H \rangle = \langle H_{\text{kin}} \rangle + \langle H_{\text{pot}} \rangle = (\,\tfrac{1}{2}k_B T\,) + (\,\tfrac{1}{2}k_B T\,)$$

$$\boxed{\langle H \rangle = k_B T}$$

where k_B =Boltzmann constant = 1.381e-23 J/K. **The outgoing EM waves of a blackbody have energy $k_B T$!**

[1]Derivation based on work found here [https://elliptigon.com/raleigh-jeans/]

- We have found the energy one "thermal" wave has.
- **Find the total number of waves in the cavity.**
  - Solve wave equation for standing waves
  - Count waves in k-space
  - Convert to wavelength space

$$\text{Wave Equation: } \left(\nabla^2 - c^2 \frac{\partial^2}{\partial t^2}\right)\Psi(\vec{r},t) = 0$$

Ansatz: $\Psi(\vec{r},t) = \psi(\vec{r}\,)\,\phi(t)$ (for separation of variables).

$$\left(\nabla^2 - c^2 \frac{\partial^2}{\partial t^2}\right)\psi\phi = 0$$
$$\phi\nabla^2\psi - c^2\psi\frac{\partial^2\phi}{\partial t^2} = 0$$
$$\frac{1}{\psi}\nabla^2\psi = \frac{1}{\phi}c^2\frac{\partial^2\phi}{\partial t^2}$$

Left hand side depends only on position. Right hand side depends only on time, meaning this equation must equal a constant! Call that constant k^2.

Focus on just the position dependent ψ equation.

$$\nabla^2\psi = k^2\psi$$

Ansatz: $\psi(\vec{r}\,) = E_x(x)\,E_y(y)\,E_z(z)$

$$\frac{d^2E_x}{dx^2}\,E_y\,E_z + E_x\,\frac{d^2E_y}{dy^2}\,E_z + E_x\,E_y\,\frac{d^2E_z}{dz^2} = k^2E_x\,E_y\,E_z$$

Divide whole equation by $E_x\,E_y\,E_z$

$$\underbrace{\frac{d^2E_x}{dx^2}\frac{1}{E_x}}_{\text{only x dependence}} + \underbrace{\frac{d^2E_y}{dy^2}\frac{1}{E_y}}_{\text{only y dependence}} + \underbrace{\frac{d^2E_z}{dz^2}\frac{1}{E_z}}_{\text{only z dependence}} = k^2$$

$$\therefore \quad \frac{d^2E_x}{dx^2}\frac{1}{E_x} = \text{ constant } = (k_x)^2$$

$$(k_x)^2 + (k_y)^2 + (k_z)^2 = k^2$$

k here is called the *wave-number*. For EM waves $k = 2\pi/\lambda$.

$$\frac{d^2E_x}{dx^2}\frac{1}{E_x} = (k_x)^2 \quad \Rightarrow \quad \frac{d^2E_x}{dx^2} = (k_x)^2E_x$$

$$E_x = A\cos(k_x x) + B\sin(k_x x)$$

Apply boundary condition for standing waves

$$E_x(x=0) = 0, \quad E_x(x=L) = 0$$

This forces the A constant on cosine to be zero and we have

$$E_x(x=L) = 0 = B\sin(k_x L)$$

This condition is only valid when

$$k_x L = n_x \pi \qquad \underbrace{n_x = 1, 2, 3, \ldots}_{\text{ask class why no n=0}}$$

Apply this to the y and z equations to find

$$\boxed{\psi(\vec{r}) = B\sin(k_x x)\sin(k_y y)\sin(k_z z)}$$

and

$$k^2 = \frac{\pi^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$$

Look at k-space to count the number of standing waves.

- Points in k space are always separated by steps of π/L.
- Every $(\pi/L)^3$ cube contains one standing wave.

The number of standing waves, $N(k)$, in the spherical shell shown in the diagram is the volume between k and $k + dk$, divided by $(\pi/L)^3$.

$$N(k) = \frac{\text{volume of } dk \text{ shell}}{(\pi/L)^3}$$

$$\text{volume} = \frac{1}{8}\left(\frac{4}{3}\pi(k+dk)^3 - \frac{4}{3}\pi k^3\right)$$

$$\text{volume} = \frac{1}{8}\frac{4}{3}\pi\Big((k^2 + dk^2 + 2k\,dk)(k+dk) - k^3\Big)$$

$$\text{volume} = \frac{1}{8}\frac{4}{3}\pi\Big(k^3 + \underbrace{k^2 dk} + dk^2\,k + dk^3 + \underbrace{2k^2 dk} + 2k\,dk^2 - k^3\Big)$$

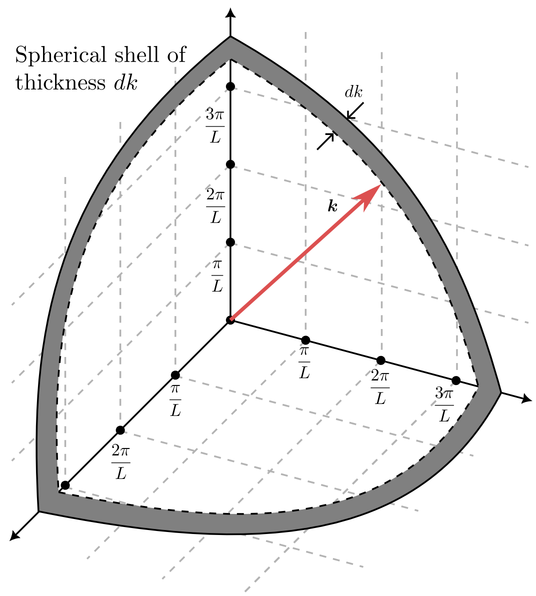

Figure 8: k space is positive definite, therefore only 1/8 of a sphere present.

$$\text{volume} = \frac{1}{8}\frac{4}{3}\pi\left(3k^2dk\right)$$

$$\text{volume} = \tfrac{1}{2}\pi\, k^2 dk$$

$$N(k) = \frac{\frac{1}{2}\pi\, k^2 dk}{(\pi/L)^3} = \frac{V k^2 dk}{2\pi^2}$$

where $V = L^3$ is the volume of the whole cavity.

For any EM wave, there are 2 perpendicular polarizations for each mode $\therefore$

$$N(k) = \frac{V k^2 dk}{\pi^2}$$

We'll convert from wave-number to wavelength.

$$k = \frac{2\pi}{\lambda} \quad \rightarrow \quad dk = (-1)\frac{2\pi}{\lambda^2}d\lambda$$

$$N(\lambda) = V\frac{k^2 dk}{\pi^2} = V\frac{1}{\pi^2}\left(\frac{2\pi}{\lambda}\right)^2\left[(-1)\frac{2\pi}{\lambda^2}d\lambda\right]$$

$$\boxed{N(\lambda) = (-V)\frac{8\pi}{\lambda^4}d\lambda}$$

We now have the number of standing EM waves in the cavity at thermal equilibrium[2].

Each wave carries energy $k_B T$, thus the energy density of the EM fields of a blackbody at thermal equilibrium is

$$u(\lambda)d\lambda = k_B T\frac{N(\lambda)}{V} = \frac{8\pi k_B T}{\lambda^4}d\lambda$$

$$\boxed{\text{Rayleigh-Jeans Law: } u(\lambda) = \frac{8\pi k_B T}{\lambda^4}}$$

[2]Interpret the minus sign as waves leaving the system

The first model for the blackbody SED is the Rayleigh-Jeans Law

$$B_\lambda = \frac{2ck_BT}{\lambda^4} \quad \text{or} \quad B_\nu = \frac{2\nu^2 k_BT}{c^2}$$

where

quantity	**symbol**	**value**
Speed of light	c	2.998e8 m/s
Boltzmann constant	k_b	1.381e-23 J/K

Plank functions gives a blackbody's flux

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}$$

|Plank's constant | h | 6.626e-34 Js |

$[B_{\nu,\lambda}]$ = power emitted from surface, per unit area, per unit solid angle, per spectral unit (frequency, wavelength).

ex: B_λ is the amount of energy emitted each second over a wavelength interval of 1 unit length (cm, nm, Å, ...) by a surface area of 1 m^2 into a solid angle

- Low frequency limit $h\nu << k_B T$
- High frequency limit $h\nu >> k_B T$

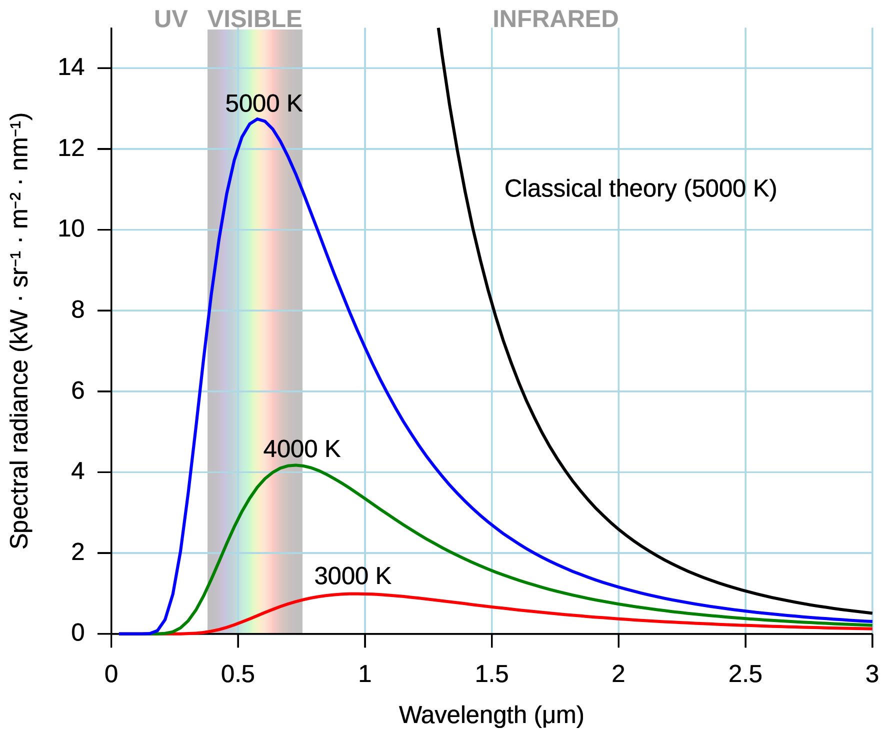

Figure 9: Plank's law for a variety of temperatures and wavelengths