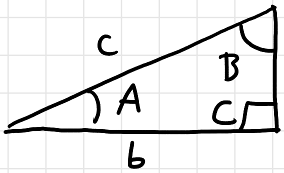
Ch4 Applications of the Spherical Triangle

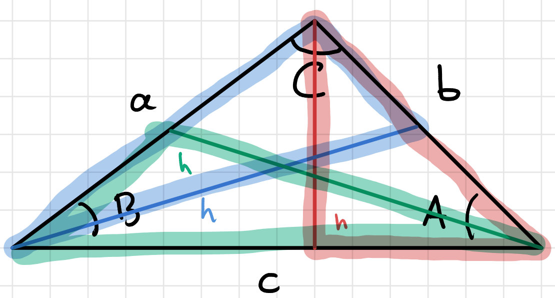
Reminder of facts   Euclidean Geometry



- we will use lower case letters $a, b, c, \ldots$ to be sides of triangle
- sum of interior angles $= 180°$

$\angle A + B + C = \pi$

Law of Sines



Area of triangle $= \frac{1}{2}$ base × height

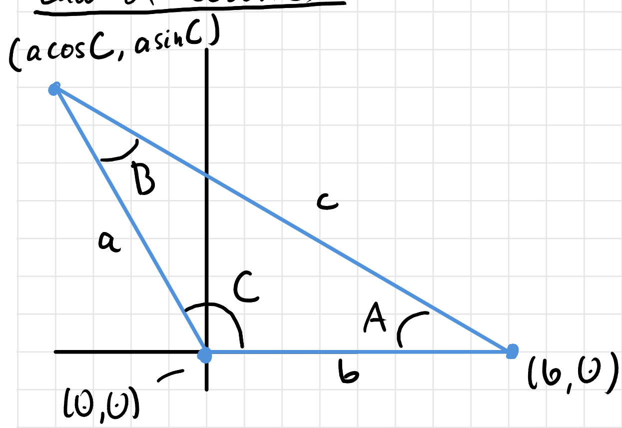
Area $= \frac{1}{2} c \cdot b \sin A$

Area $= \frac{1}{2} b \cdot a \sin C$

Area $= \frac{1}{2} a \cdot c \sin B$

$$\frac{2}{abc} \text{Area} = \boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}} \quad \text{law of sines}$$

Law of Cosines



$|c| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$|c| = \sqrt{(b - a\cos C)^2 + (-a\sin C)^2}$

$c^2 = (b - a\cos C)^2 + (-a\sin C)^2$

$c^2 = b^2 + a^2\cos^2 C - 2ab\cos C + a^2\sin^2 C$

$c^2 = a^2 + b^2 - 2ab\cos C$

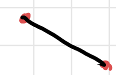
$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= c^2 + a^2 - 2ca\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}}$$

What do our trig rules look like on a spherical surface?

Spherical Geometry is a non-Euclidean geometry.

Axioms of Euclidean Plane Geometry

1) A straight line may be drawn between any two points

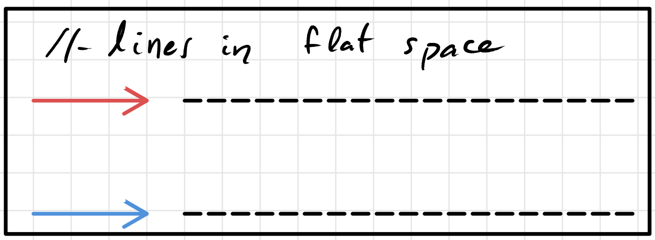


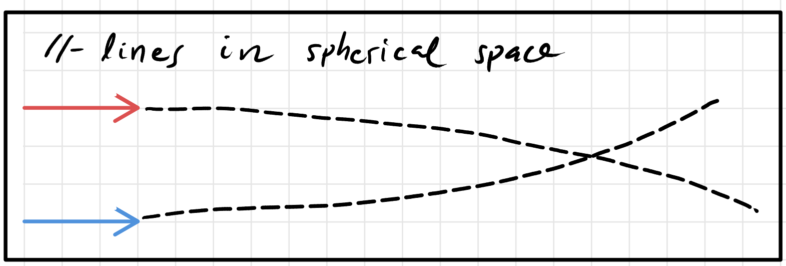
2) Any terminated straight line may be extended indefinately

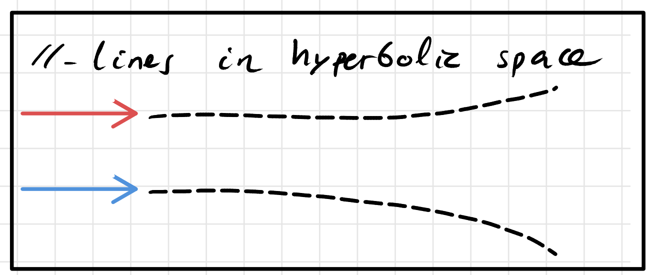
3) A circle can be drawn around any point

4) All right angles are equal

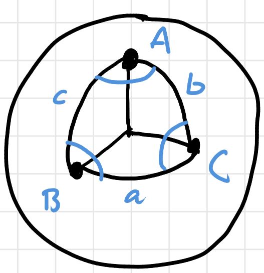
5) Parallel line postulate "parallel rays will never intersect"



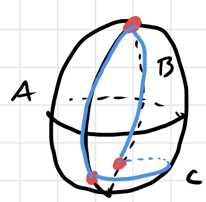




Spherical triangle



- vertices & angles denoted by upper case letters A, B, C
- sum of interior angle greater than 180°

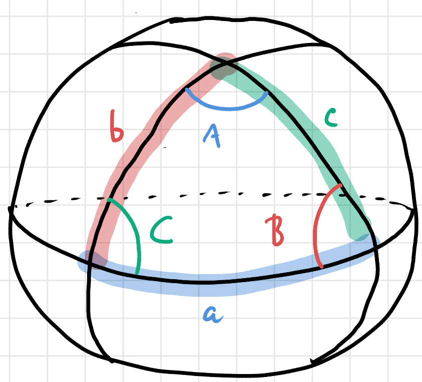


$A = \pi - \varepsilon$

$B = \pi - \varepsilon$

$C = \pi$

$$\boxed{\pi < A + B + C < 3\pi}$$



- "sides" denoted by lower case letters a, b, c and measured in <u>radians</u>

  a, b, c are angles measured from center of sphere

Class had a few questions about //-lines in spherical geometry.

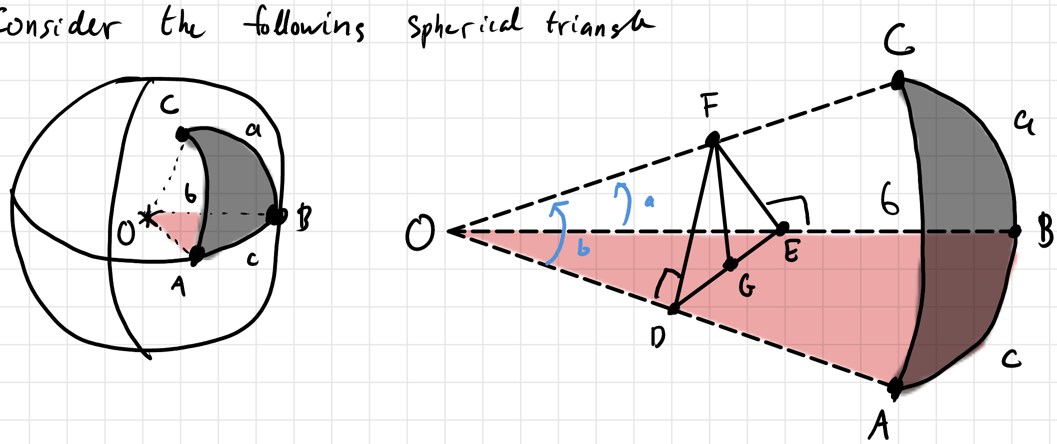
- on spherical geometry lines are defined as great circles
- all great circles intersect
- No parallel lines exist
- What about a small circle? Not a line

See wiki article on "map projections"

Projections by preservation of a metric property

- Find the law of sines & cosines for spherical triangles

Consider the following spherical triangle



- $FD \perp OA$
- $FG \perp$ plane AOB

$$\sin \angle FDG = \frac{FG}{DF} = \sin A$$

$$\sin \angle FEG = \frac{FG}{EF} = \sin B$$

$$\frac{\sin A}{\sin B} = \frac{EF}{DF}$$

$$\sin a = \frac{EF}{OF}$$

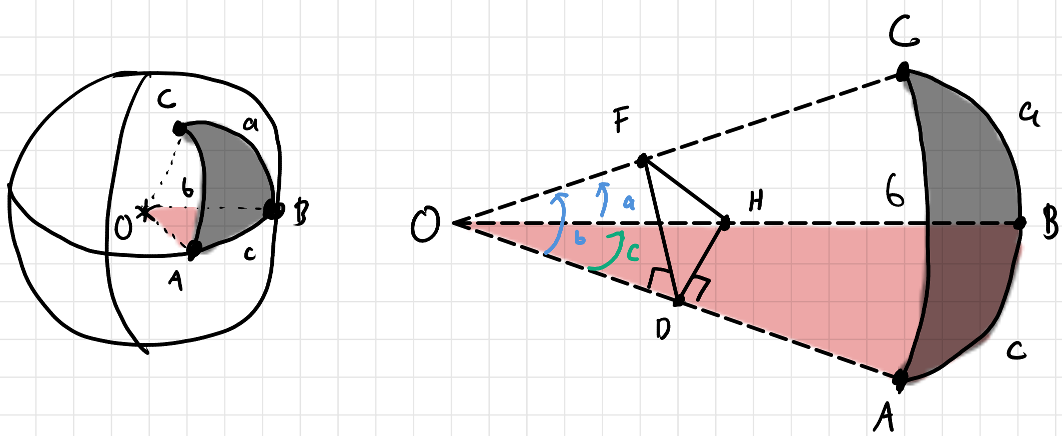
$$\sin b = \frac{DF}{OF}$$

$$\frac{\sin a}{\sin b} = \frac{EF}{DF}$$

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$$

$$\boxed{\text{Law of Sines: (spherical triangles)} \quad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}}$$

similar trick to get law of cosines



- $DF \perp OA$
- $DH \perp OA$

apply (Euclidean) law of cosines to triangles $a^2 = b^2 + c^2 - 2bc\cos A$

$$OHF: (HF)^2 = (OH)^2 + (OF)^2 - 2(OH)(OF)\cos \overbrace{\angle FOH}^{a}$$

$$DHF: (HF)^2 = (DH)^2 + (DF)^2 - 2(DH)(DF)\cos \underbrace{\angle FDH}_{A}$$

• subtract these two equations

$$0 = (OH)^2 + (OF)^2 - 2(OH)(OF)\cos a - (DH)^2 + (DF)^2 + 2(DH)(DF)\cos A$$

• right triangle DOF : $(OF)^2 = (OD)^2 + (DF)^2$

• right triangle DOH : $(OH)^2 = (DH)^2 + (OD)^2$

$$0 = (OD)^2 - 2(OH)(OF)\cos a + (OD)^2 + 2(DH)(DF)\cos A$$

$$\cos a = \frac{(OD)^2}{(OH)(OF)} + \frac{(DH)(DF)}{(OH)(OF)}\cos A$$

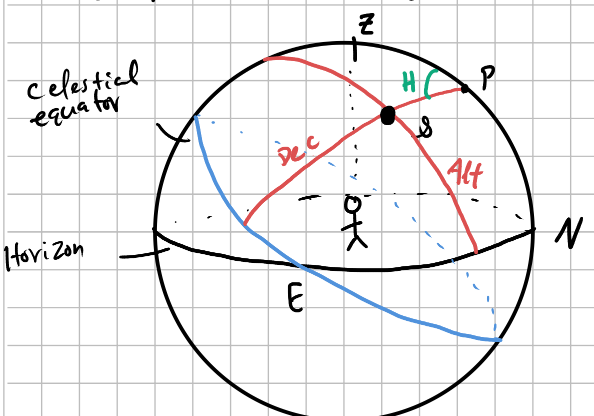
$$\cos a = \frac{(OD)}{(OF)}\frac{(OD)}{(OH)} + \frac{(DF)}{(OF)}\frac{(DH)}{(OH)}\cos A$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Law of cosines : spherical triangles	$\cos a = \cos b \cos c + \sin b \sin c \cos A$ $\cos b = \cos c \cos a + \sin c \sin a \cos B$ $\cos c = \cos a \cos b + \sin a \sin b \cos C$

Hour Angle & coordinate conversions (text fig 1.6 b)

Consider the horizon & equatorial systems simultaneously



ZPN = observer's celestial meridian

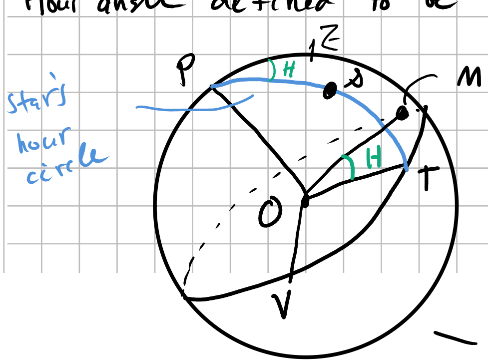
↳ divides sky into east half & west half

$\angle ZP\delta = H$ = hour angle

H negative if in east half

H positive if in west half

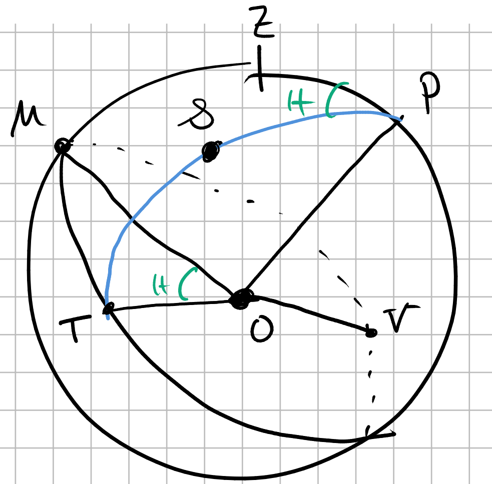
Hour angle defined to be the angle between observer's meridian & hour circle through star



— star in western half of sky

$H = \angle MOT$, M = point where your meridian crosses celestial equator

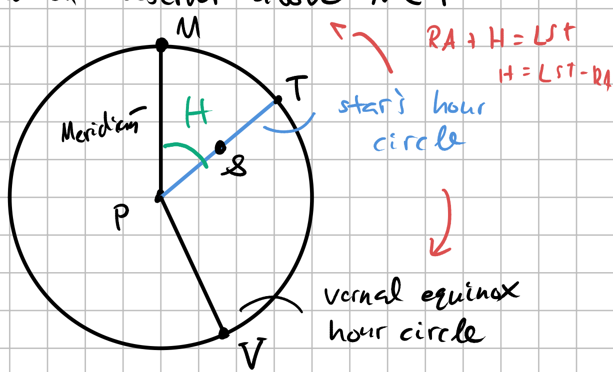
T = great circle from N-S pole crossing through star of interest



$H = \angle MOT = \angle ZPS$

Star in Eastern half of sky

View of observer above NCP



- ideal observations is when H is small (star close to your meridian)
- if $H = 12^h$, star is below the horizon or between pole & horizon

* hour angle = local sidereal time − right ascension *

Coordinate Conversion

- Determine the altitude & azimuth of a star at a specific time.
- Must know
  - (celestial) coordinates of star
  - local sidereal time
  - observer's latitude
- $H = RA - LST$ (right ascension − local sidereal time)

DEC

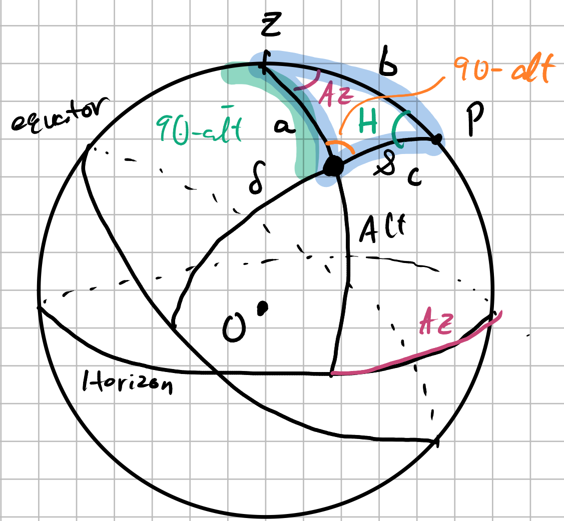
Example : Iota Herculis $RA = 17^h 40^m 5.7^s$, $\delta = 45° 59' 19.2''$

- pick time of observation to be ~ 10am (EST) error in analysis
- local sidereal time = 19.22 hr $\Rightarrow$ ~~$H = RA - LST$~~ $\approx -1.5h$ $H = LST - RA$

$H = 1^h 30^m$ east

- ϕ = our latitude = 36.84°

$= -1.5^h \cdot \frac{360°}{24^h} = -22.5°$



$H = \angle ZPS$

$PZ = 90° - \phi = 53.16°$

$Z\delta = 90° - alt$

$P\delta = 90° - \delta \approx 44°$

Find alt w/ law of cosines

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

$$\cos(90° - alt) = \cos(90° - \phi)\cos(90° - \delta) + \sin(90° - \phi)\sin(90° - \delta)\cos(H)$$

Trig ID: $\cos(90° - x) = \sin(x)$
$\sin(90° - x) = \cos(x)$

$$\boxed{\sin(alt) = \sin(\phi)\sin(\delta) + \cos(\phi)\cos(\delta)\cos(H)}$$

$$alt = \arcsin\left[\sin(36.84°)\sin(46°) + \cos(36.84°)\cos(46°)\cos(22.5)\right] = 70.9°$$

Get the azimuth w/ law of sines $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$

$$\boxed{\frac{\sin 90° - alt}{\sin H} = \frac{\sin 90° - \delta}{\sin Az}} \Rightarrow Az = \arcsin\left[\frac{\cos\delta}{\cos alt}\sin H\right] = -54.3°$$

Another example of celestial → Alt/az

Will Spica in the virgo constellation be observable at 5:30pm EST in Norfolk VA (longitude = 36.9°, latitude = −76.3°) on Feb 14th?

- Get stars celestial coordinates

Spica $RA = 13^h\, 26^{min}\, 24.8^s = 13^h + 26^{min} \cdot \frac{1^h}{60^{m}} + 24.8^s \cdot \frac{1^h}{3600^s} = 13.44^h \cdot \frac{360°}{24^h} = 201.6°$

$Dec = -11°16'\,55.2'' = -\left(11° + 16' \cdot \frac{1°}{60'} + 55.2'' \cdot \frac{1°}{3600''}\right) = -11.282°$

- Get local sidereal time: 5:30pm EST Feb 14, 2023 = 2.98^h LST = 44.7°

use online calculator

- Get hour angle

$H = LST - RA = -156.9$

- $alt = \arcsin\left[\sin(\phi)\sin(Dec) + \cos(\phi)\cos(Dec)\cos(H)\right] = -57.02°$

Not observable at this time, Spica is below the horizon

show on stellarium

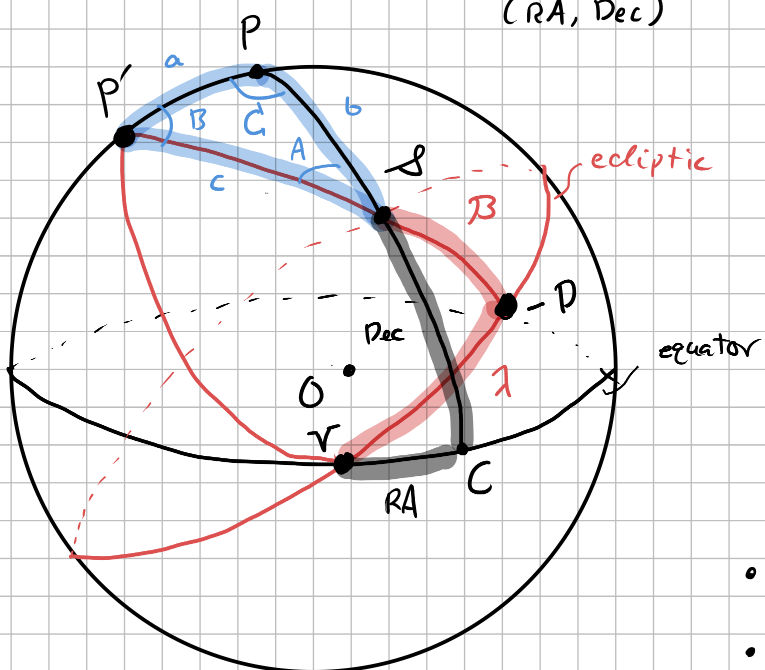
- $az = \arcsin\left[\frac{\cos(Dec)}{\cos(alt)}\sin(H)\right] = -44.97°$

to match stellarium we need +45°

Not certain what the discrepancy is

Ecliptic Coordinate transformation

Similar procedure as celestial (RA, Dec) → alt/az



RA Dec → λ, β

λ = ecliptic longitude
β = ecliptic latitude

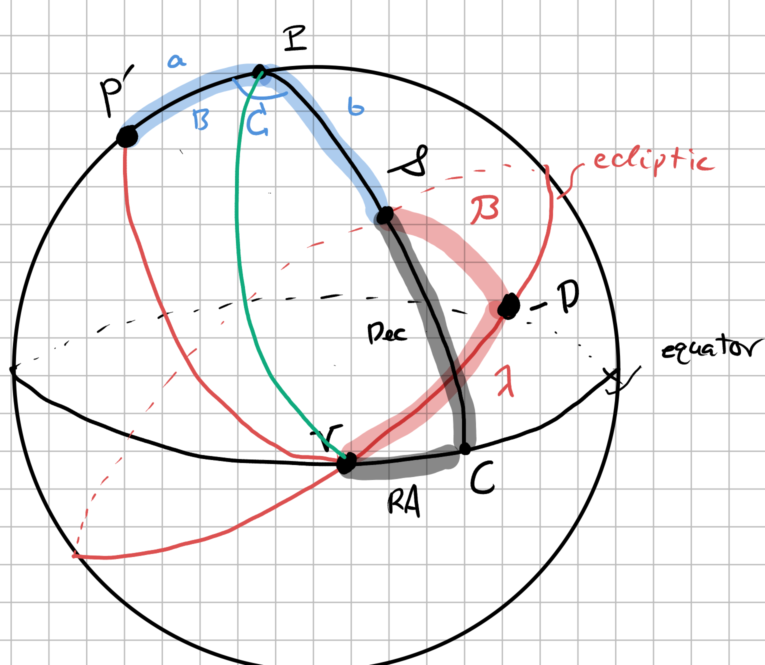
P = celestial pole
P′ = pole of ecliptic

♈ = vernal equinox

Triangle of interest: PP′♋

- $\overline{P'♋D} = 90°$ ∴ $c = 90° - \beta$
- $\overline{P♋C} = 90°$ ∴ $b = 90° - \text{Dec}$
- $\overline{PP'} = 23.4° = a$ (for now)

Need A, B, or C to be able to finish the conversions. Redraw sphere



- ♈ can be treated as pole of great circle passing through
- PP′♈ = 1/4 of great circle
  $\angle PP'♈ = 90°$
- similarly $\angle P'P♈ = 90°$
- $C = 90° + RA$

$\cos c = \cos a \cos b + \sin a \sin b \cos C$

$\cos(90° - \beta) = \cos(a)\cos(90° - \text{Dec}) + \sin(a)\sin(90° - \text{Dec})\cos(90° + RA)$

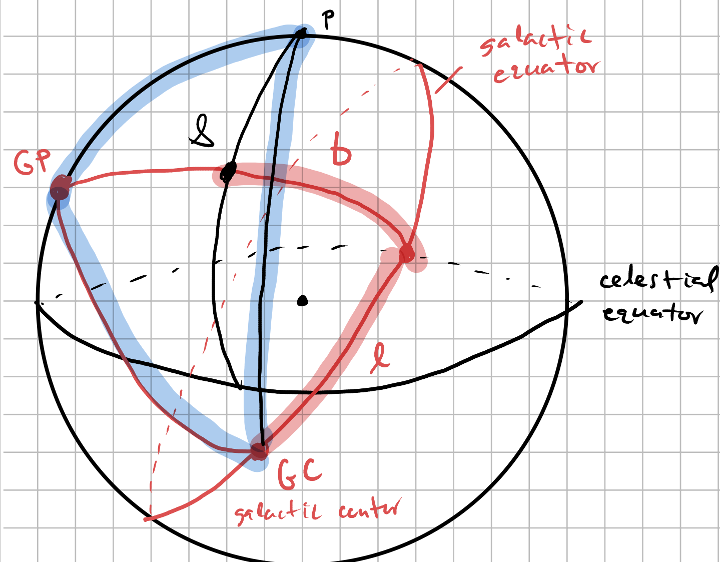
$\sin\beta = \cos a \sin \text{Dec} + \sin a \cos \text{Dec}\,(-1)\sin(RA)$

$\beta = \arcsin [\sim]$

- law of sines to get λ

Galactic Coordinates transformation

- 1st galactic coordinate system developed by William Herschel



Triangle of interest : P GP GC

- IAU adopted following values (1958)

GP:	RA = $12^h 49^m$
	Dec = $+27° 24'$
GC:	RA = $17^h 42^m$
	Dec = $-28° 55'$

from radio observations of galactic Neutral hydrogen

- Sagittarius A* located at
  RA = $17^h 45^m 40.0409^s$
  Dec = $-29° 00' 28.118''$

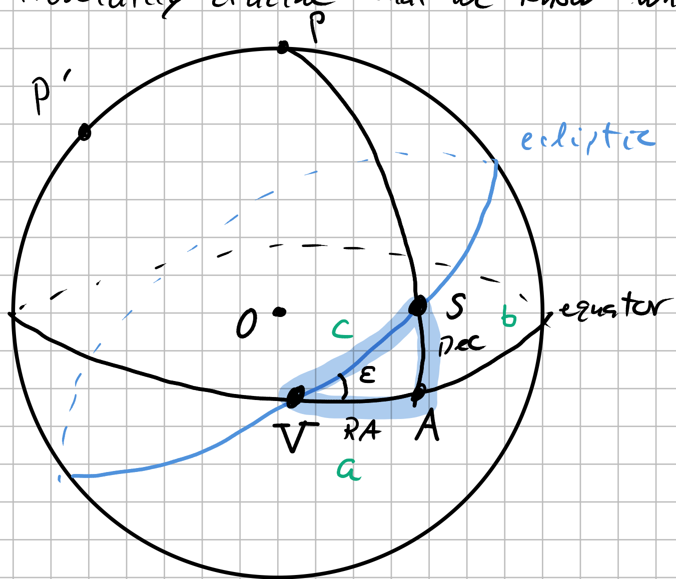
  best physical marker for galactic center

Make this a HW problem

PQ - Show that galactic longitude of the NCP is $\ell = 123°$.
Hint use cosine formula

Locating Vernal Equinox

- Absolutely crucial that we know where Υ is.



- Suppose the sun is at point S at time of observation
- ΥA = sun's right ascension
- SA = sun's declination
- $\varepsilon = \angle S\Upsilon A$ = obliquity of ecliptic

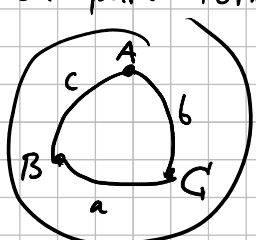
PROP DEMO

- ε found at summer solstice and recording sun's max Dec.

$$\varepsilon = 23.4^\circ$$

- triangle of interest $S\Upsilon A$

- Four-part formula (HW derive this)



$$\boxed{\cos a \cos C = \sin a \cot b - \sin C \cot B}$$

apply to $S\Upsilon A$

$$\cos(\Upsilon A)\cancel{\cos(90^\circ)} = \sin(\Upsilon A)\cot(SA) - \sin(90^\circ)\cot(\varepsilon)$$

$$0 = \sin(\Upsilon A)\cot(SA) - \cot(\varepsilon) \qquad SA = \text{Dec}$$

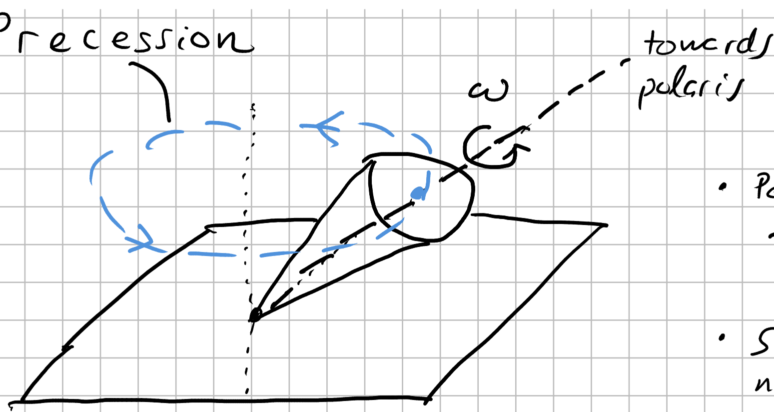
$$\sin(\Upsilon A)\cot(\text{Dec}) = \cot(\varepsilon)$$

$$\Upsilon A = \arcsin\left[\tan(\text{Dec})\cot(\varepsilon)\right]$$

* measure Dec of sun at time of meridian passage to get ΥA (sun's right ascension)

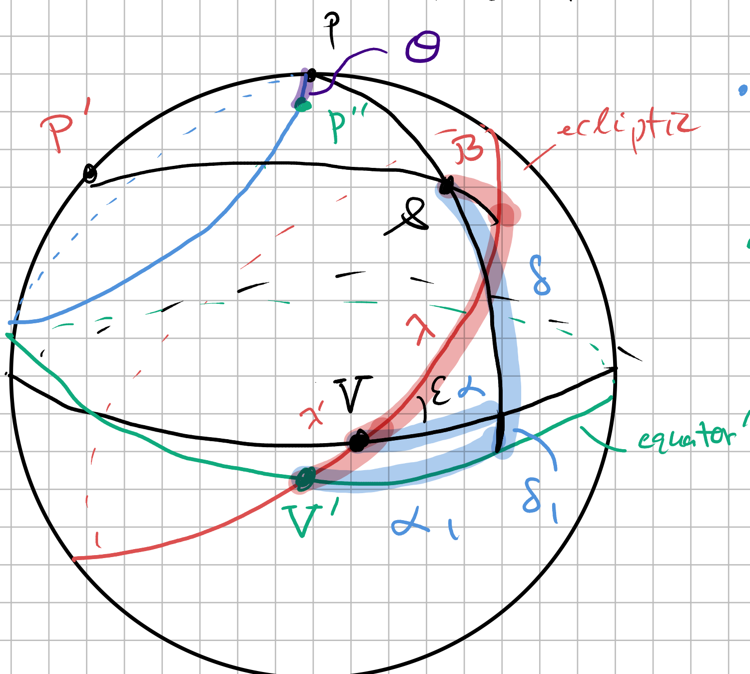
$\llcorner$ hour angle of sun $= 0$ $\therefore$ RA = LST and we have the vernal equinox!

Precession



- Period of Earth's precession $\sim 25{,}800$ years
- Seems so large that corrections not necessary on human time scales

- If your celestial coordinates from 50 years ago your telescope will not have desired object in FOV
- Even 25 year old data is inaccurate

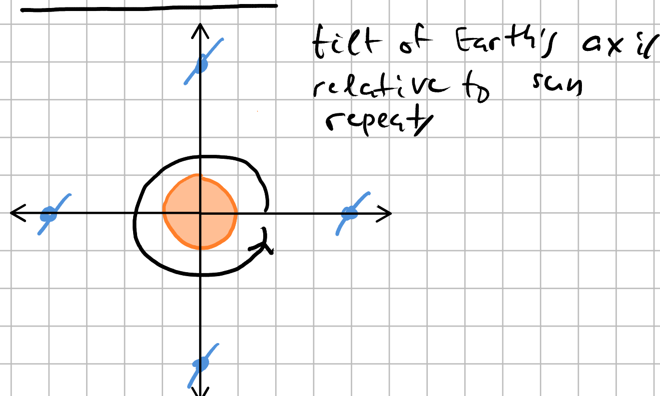


- small circle through P is path celestial pole will move due to precession
- P'' = future position of NCP
- α = RA, δ = Dec
- λ, β = initial ecliptic coordinates
- λ_1, β_1 = final ecliptic coordinates

$\ast$ $\beta_1 = \beta$ ecliptic latitude does not change from precession

- let θ = amount of <u>annual</u> precession

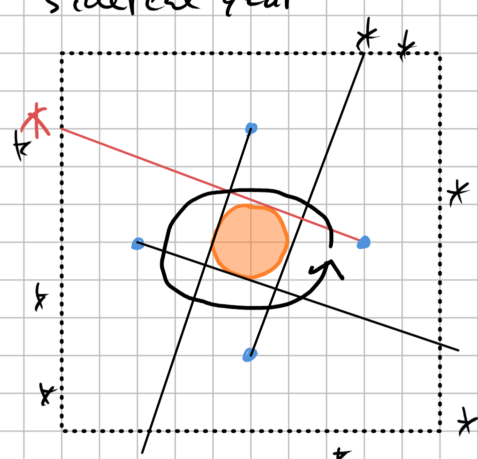
<u>tropical year</u>



vernal equinox → vernal equinox

TY = 365.242190^d

sidereal year



one orbit w/ respect to background stars

SY = 365.256363^d

• sidereal year ≈ 20 minutes longer than tropical year

$$\frac{\theta}{360^\circ} = \frac{SY - TY}{SY} = \frac{365.256363^d - 365.242190^d}{365.256363^d}$$

$$\frac{\theta}{360^\circ} = 3.88 \times 10^{-5}$$

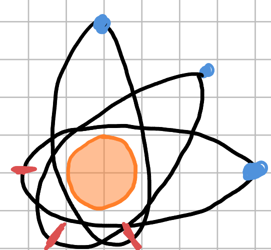
$$\theta = 0.01397^\circ = 50.29''$$

human eye resolution $\approx 30''$

- Fun Note: Another definition for a year is based on Earth's elliptical orbit

Anomalistic year : time between closest approach (perihelions) to sun

$$AY = 365.260^d$$



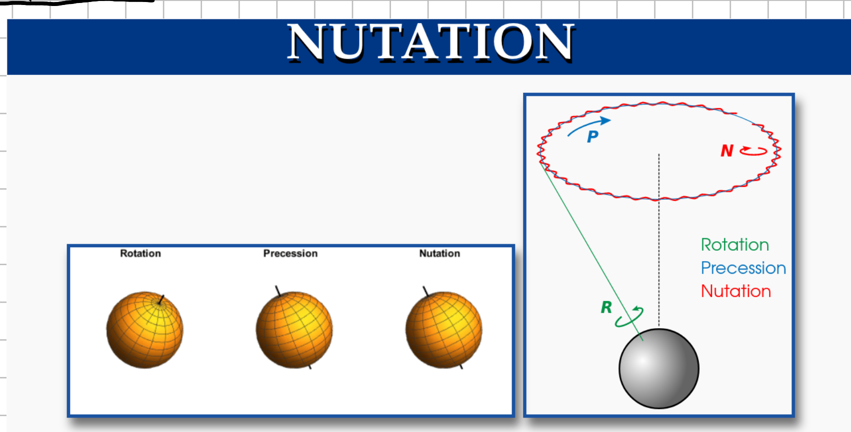
• It can be shown that

$$\Delta\alpha = \alpha_1 - \alpha = \theta(\cos\varepsilon + \sin\varepsilon \sin\alpha \tan\delta)$$
$$\Delta\delta = \delta_1 - \delta = \theta \sin\varepsilon \cos\alpha$$

gives difference between years

L see Smart's textbook for derivation

Nutation



nutation caused by moon adds up to $9''$ to Earth's precession