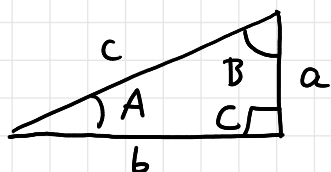


Ch4 Applications of the Spherical Triangle

Reminder of facts Euclidean Geometry

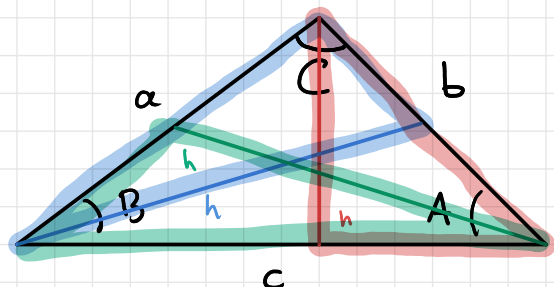


- we will use lower case letters a, b, c, \dots to be sides of triangle

- sum of interior angles = 180°

$$\angle A + \angle B + \angle C = \pi$$

Law of Sines



Area of triangle = $\frac{1}{2} \text{base} \times \text{height}$

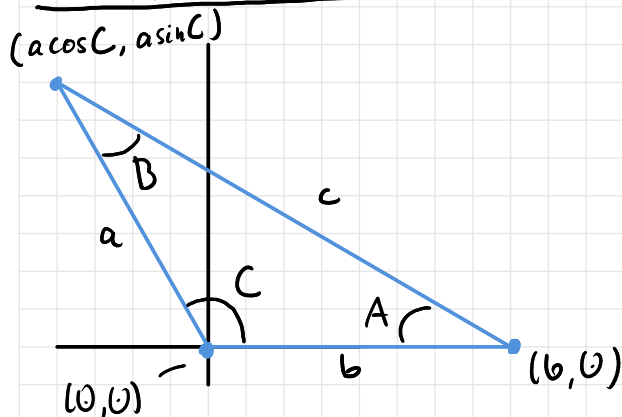
$$\text{Area} = \frac{1}{2} c \cdot b \sin A$$

$$\text{Area} = \frac{1}{2} b \cdot a \sin C$$

$$\text{Area} = \frac{1}{2} a \cdot c \sin B$$

$$\frac{2}{abc} \text{Area} = \boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}} \quad \text{law of sines}$$

Law of Cosines



$$|c| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$|c| = \sqrt{(b - a \cos C)^2 + (-a \sin C)^2}$$

$$c^2 = (b - a \cos C)^2 + (-a \sin C)^2$$

$$c^2 = b^2 + a^2 \cos^2 C - 2ab \cos C + a^2 \sin^2 C$$



$$\boxed{c^2 = a^2 + b^2 - 2ab \cos C}$$

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= c^2 + a^2 - 2ca \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

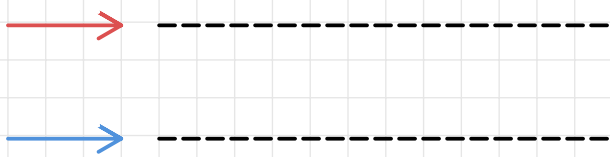
What do our trig ruler look like on a spherical surface?

Spherical Geometry is a non-Euclidean geometry.

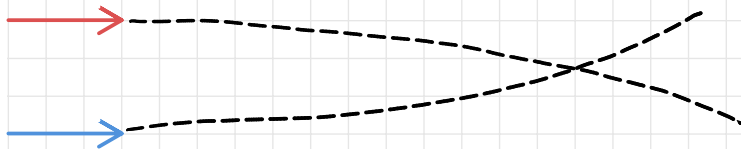
Axioms of Euclidean Plane Geometry

- 1) A straight line may be drawn between any two points 
- 2) Any terminated straight line may be extended indefinitely 
- 3) A circle can be drawn around any point
- 4) All right angles are equal
- 5) Parallel line postulate "parallel rays will never intersect"

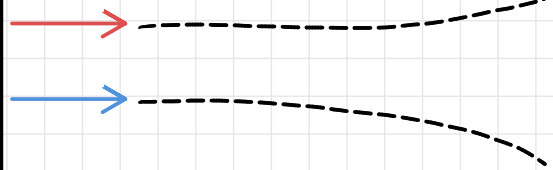
//- lines in flat space



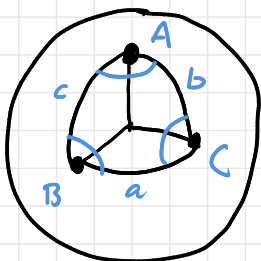
//- lines in spherical space



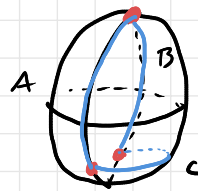
//- lines in hyperbolic space



Spherical triangle



- vertices & angles denoted by upper case letters A, B, C
- sum of interior angle greater than 180°



$$A = \pi - \epsilon$$

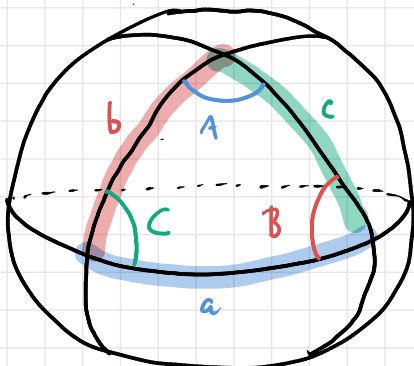
$$B = \pi - \epsilon$$

$$C = \pi$$

$$\pi < A + B + C < 3\pi$$

- "sides" denoted by lower case letters a, b, c and measured in radians

a, b, c are angles measured from center of sphere



Class had a few questions about // - lines in spherical geometry.

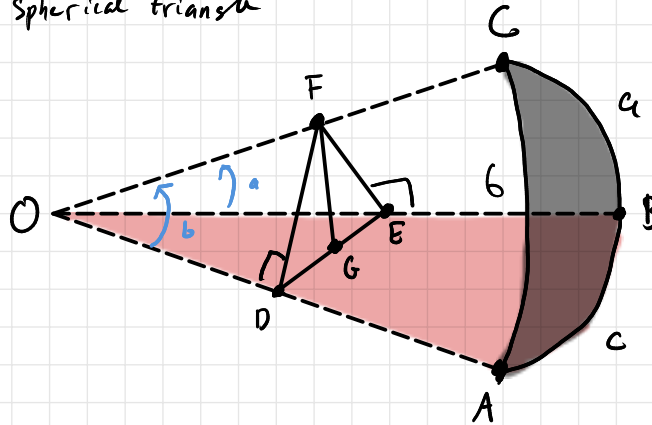
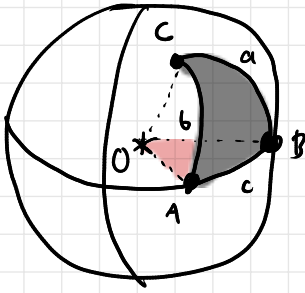
- on spherical geometry lines are defined as great circles
- all great circles intersect
- No parallel lines exist
- What about a small circle? Not a line

See wiki article on "map projections"

Projections by preservation of a metric property

- Find the law of sines & cosines for spherical triangles

Consider the following spherical triangle



$$\bullet FD \perp OA$$

$$\bullet FG \perp \text{plane } AOB$$

$$\sin \angle FDG = \frac{FG}{DF} = \sin A$$

$$\sin \angle FEG = \frac{FG}{EF} = \sin B$$

$$> \frac{\sin A}{\sin B} = \frac{EF}{DF}$$

$$\sin a = \frac{EF}{OF}$$

$$\sin b = \frac{DF}{OF}$$

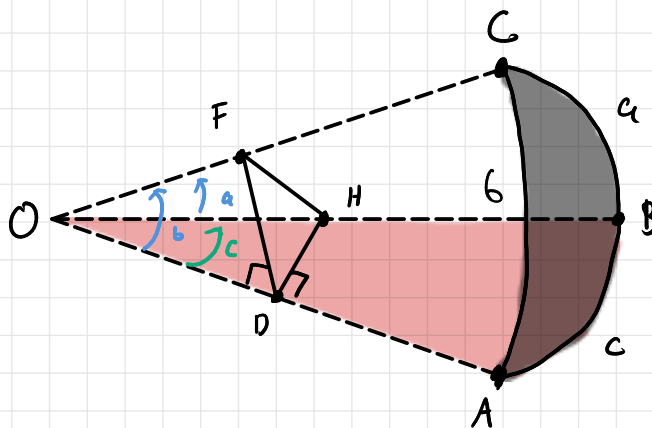
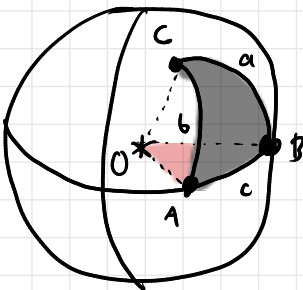
$$\frac{\sin a}{\sin b} = \frac{EF}{DF}$$

$$\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$$

$$\text{Law of Sines: } \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

(spherical triangles)

similar trick to get law of cosines



$$\bullet DF \perp OA$$

$$\bullet DH \perp OA$$

apply (Euclidean) law of cosines to triangles $a^2 = b^2 + c^2 - 2bc \cos A$

$$OHF: (HF)^2 = (OH)^2 + (OF)^2 - 2(OH)(OF) \cos \angle FOH$$

$$DHF: (HF)^2 = (DH)^2 + (DF)^2 - 2(DH)(DF) \cos \angle FDH$$

- subtract these two equations

$$0 = (OH)^2 + (OF)^2 - 2(OH)(OF)\cos\alpha - (DH)^2 + (DF)^2 + 2(DH)(DF)\cos A$$

- right triangle DOF: $(OF)^2 = (OD)^2 + (DF)^2$

- right triangle DOH: $(OH)^2 = (DH)^2 + (OD)^2$

$$0 = (OD)^2 - 2(OH)(OF)\cos\alpha + (OD)^2 + 2(DH)(DF)\cos A$$

$$\cos\alpha = \frac{(OD)^2}{(OH)(OF)} + \frac{(DH)(DF)\cos A}{(OH)(OF)}$$

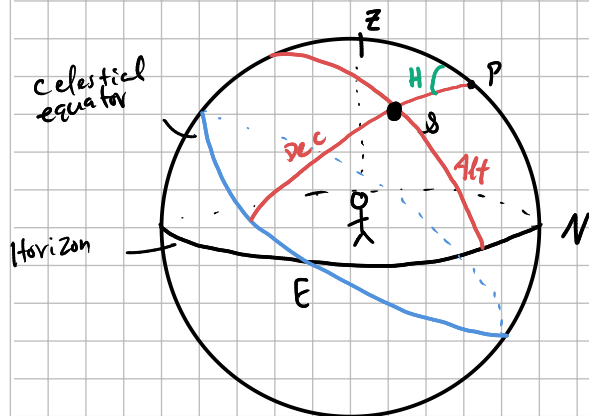
$$\cos\alpha = \frac{(OD)}{(OF)} \frac{(OD)}{(OH)} + \frac{(DF)}{(OF)} \frac{(DH)}{(OH)} \cos A$$

$$\cos\alpha = \cos b \cos c + \sin b \sin c \cos A$$

Law of cosines :	$\cos\alpha = \cos b \cos c + \sin b \sin c \cos A$
spherical triangle,	$\cos b = \cos c \cos a + \sin c \sin a \cos B$
	$\cos c = \cos a \cos b + \sin a \sin b \cos C$

Hour Angle & coordinate conversions (text fig 1.6 b)

Consider the horizon & equatorial systems simultaneously



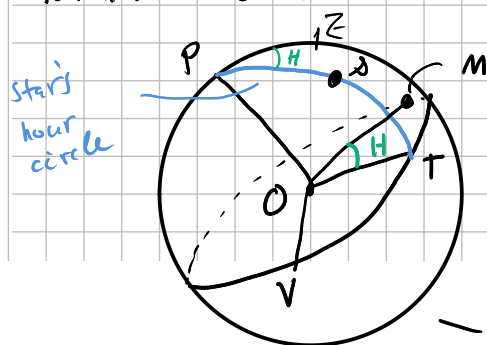
ZPN = observer's celestial meridian

L divides sky into east half & west half

$\angle ZPS = H = \text{hour angle}$

H negative if in east half
H positive if in west half

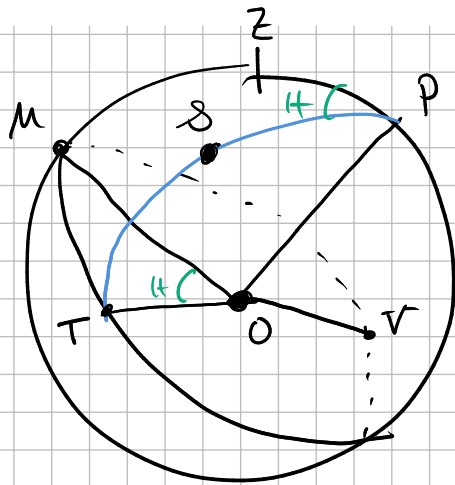
Hour angle defined to be the angle between observer's meridian & hour circle through star



$H = \angle MOT$, M = point where your meridian crosses celestial equator

T = great circle from N-S pole crossing through star of interest

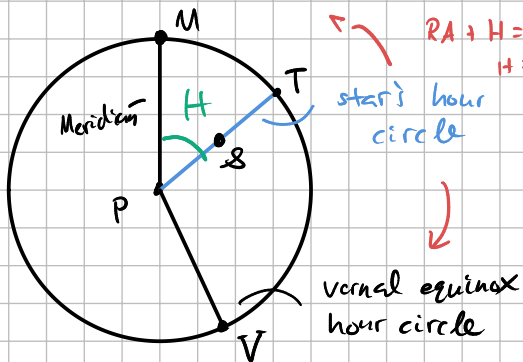
— star in western half of sky



$$H = \angle MOT = \angle ZPS$$

Star in Eastern half of sky

View of observer above NCP



$$RA \rightarrow H = LST$$

$$H = LST - RA$$

• ideal observations is when H is small (star close to your meridian)

• if $H = 12^h$, star is below the horizon or between pole & horizon

* hour angle = local sidereal time - right ascension *

Coordinate Conversion

- Determine the altitude & azimuth of a star at a specific time.
- Must know
 - (celestial) coordinates of star
 - local sidereal time
 - observer's latitude
- $H = RA - LST$ (right ascension - local sidereal time)

DEC

Example: Iota Herculis RA = $17^h 40^m 5.7^s$, $\delta = 45^\circ 59' 19.2''$

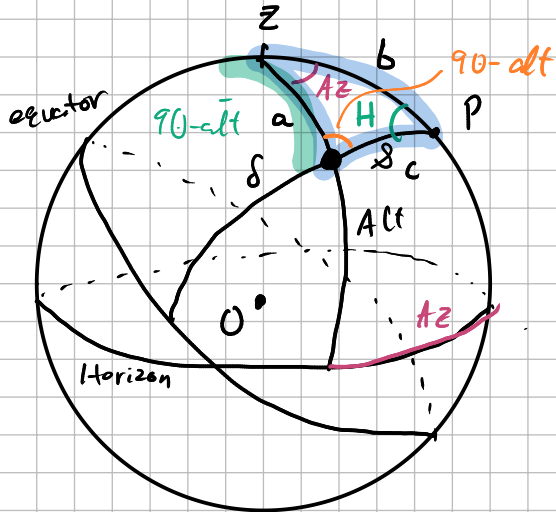
- pick time of observation to be $\sim 10\text{am}$ (EST) error in calculation

- local sidereal time = $19.22\text{ hr} \Rightarrow H = RA - LST \approx -1.5\text{h}$ $H = LST - RA$

$$H = 1^h 30^m \text{ east}$$

- ϕ = our latitude = 36.84°

$$= -1.5^h \cdot \frac{360^\circ}{24\text{h}} = -22.5^\circ$$



$$H = \angle ZPS$$

$$PZ = 90^\circ - \phi = 53.16^\circ$$

$$ZO = 90^\circ - \text{alt}$$

$$PO = 90^\circ - \delta \approx 44^\circ$$

Find alt w/ law of cosines

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos(90^\circ - \text{alt}) = \cos(90^\circ - \phi) \cos(90^\circ - \delta) + \sin(90^\circ - \phi) \sin(90^\circ - \delta) \cos(H)$$

$$\text{Trig ID: } \cos(90^\circ - x) = \sin(x)$$

$$\sin(90^\circ - x) = \cos(x)$$

$$\sin(\text{alt}) = \sin(\phi) \sin(\delta) + \cos(\phi) \cos(\delta) \cos(H)$$

$$\text{alt} = \arcsin[\sin(36.84^\circ) \sin(46^\circ) + \cos(36.84^\circ) \cos(46^\circ) \cos(22.5)] = 70.9^\circ$$

Get the azimuth w/ law of sines

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\frac{\sin 90^\circ - \text{alt}}{\sin H} = \frac{\sin 90^\circ - \delta}{\sin Az} \Rightarrow Az = \arcsin\left[\frac{\cos \delta \sin H}{\cos \text{alt}}\right] = -54.3^\circ$$

Another example of celestial \rightarrow Alt/Az

Will Spica in the Virgo constellation be observable at 5:30pm EST in Norfolk VA (longitude = 76.9° , latitude = -76.3°) on Feb 14th?

• Get stars celestial coordinates

$$\text{Spica RA} = 13^h 26^m 24.8^s = 13^h + 26^m \cdot \frac{1^h}{60^m} + 24.8^s \cdot \frac{1^h}{3600^s} = 13.44^h \cdot \frac{360^\circ}{24^h} = \boxed{201.6^\circ}$$

$$\text{Dec} = -11^\circ 16' 55.2'' = -\left(11^\circ + 16' \cdot \frac{1^\circ}{60'} + 55.2'' \cdot \frac{1^\circ}{3600''}\right) = \boxed{-11.282^\circ}$$

• Get local sidereal time: 5:30pm EST = $\boxed{2.98^h \text{ LST}}$ = 44.7°
Feb 14, 2023 , use online calculator

• Get hour angle

$$H = \text{LST} - \text{RA} = \boxed{-156.9^\circ}$$

$$\text{alt} = \arcsin[\sin(\phi) \sin(\text{Dec}) + \cos(\phi) \cos(\text{Dec}) \cos(H)] = \boxed{-57.02^\circ}$$

↳ Not observable at this time, Spica is below the horizon
↳ show on stellation

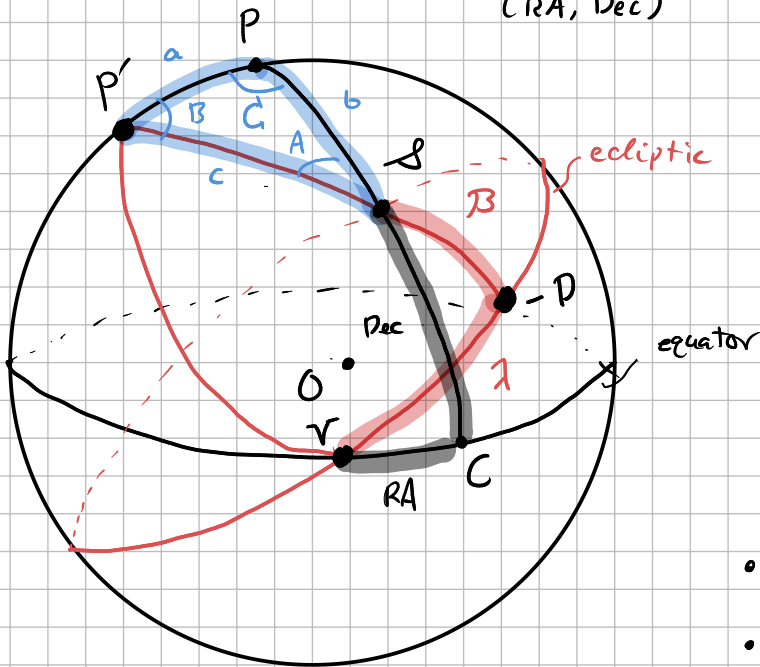
$$\text{az} = \arcsin\left[\frac{\cos(\text{Dec})}{\cos(\text{alt})} \sin(H)\right] = -44.97^\circ$$

↳ to match stellation we need $+45^\circ$

Not certain what the discrepancy is

Ecliptic Coordinate Transformation

Similar procedure as celestial \rightarrow alt/az
(RA, Dec)



RA Dec \rightarrow λ, β

λ = ecliptic longitude
 β = ecliptic latitude

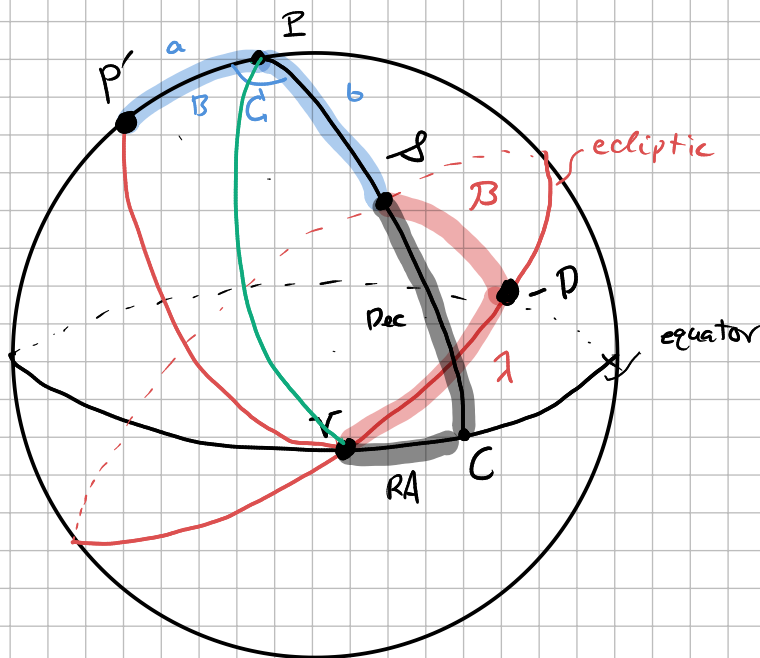
P = celestial pole
P' = pole of ecliptic

V = vernal equinox

Triangle of interest: PP'S

- $\overline{P'SD} = 90^\circ \therefore c = 90^\circ - \beta$
- $\overline{PSC} = 90^\circ \therefore b = 90^\circ - \text{Dec}$
- $\overline{PP'} = 23.4^\circ = a$ (for now)

Need A, B, or C to be able to finish the conversions. Redraw sphere



- V can be treated as pole of great circle passing through S
- $PP'V = 1/4$ of great circle
 $\angle PP'V = 90^\circ$
- similarly $\angle P'PV = 90^\circ$
- $C = 90^\circ + RA$

$$\cos C = \cos a \cos b + \sin a \sin b \cos C$$

$$\cos 90^\circ - \beta = \cos(a) \cos(90^\circ - \text{Dec}) + \sin(a) \sin(90^\circ - \text{Dec}) \cos(90^\circ + RA)$$

$$\sin \beta = \cos a \sin \text{Dec} + \sin a \cos \text{Dec} (-1) \sin(RA)$$

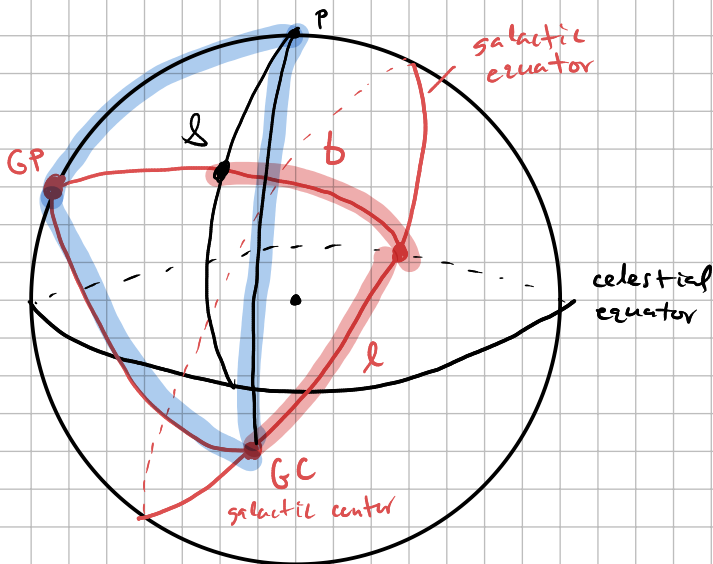
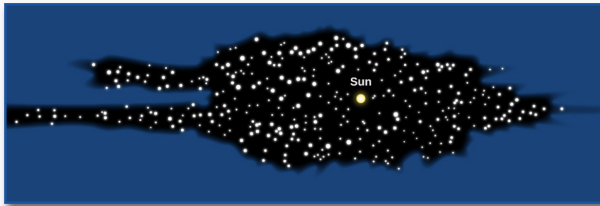
$$B = \arcsin [\sim]$$

- law of sines to get λ

Galactic Coordinates transformation

- 1st galactic coordinate system developed by William Herschel

HERSCHEL MILKY WAY MAP



Triangle of interest: P G C

- IAU adopted following values (1958)

$$\begin{aligned} \text{GP: } RA &= 12^h 49^m \\ Dec &= +27^\circ 24' \end{aligned}$$

$$\begin{aligned} \text{GC: } RA &= 17^h 42^m \\ Dec &= -28^\circ 55' \end{aligned}$$

from radio observations of
Galactic Neutral hydrogen

- Sagittarius A* located at
 $RA = 17^h 45^m 40.0409^s$
 $Dec = -29^\circ 00' 28.18''$

best physical marker for galactic center

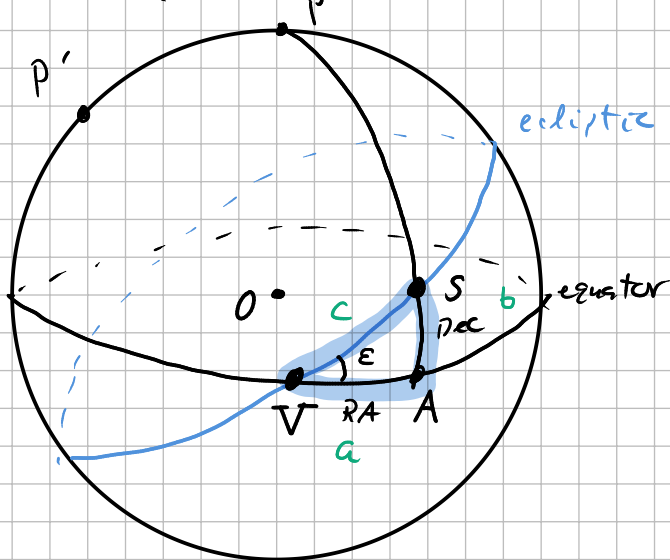
Make this a HW problem

- Show that galactic longitude at the NCP is $l = 123^\circ$.
 Hint use cosine formula

PQ

Locating Vernal Equinox

- Absolutely crucial that we know where V is.



- Suppose the sun is at point S at time of observation
- VA = sun's right ascension
- SA = sun's declination
- $E = \angle SVA$ = obliquity of ecliptic

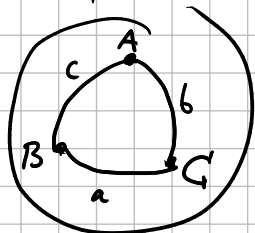
PROP DEMO

- E found at summer solstice and recording sun's max Dec.

$$E = 23.4^\circ$$

- triangle of interest SVA

- Four-part formula (HW derive this)



$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

apply to SVA

$$\cos(VA) \cos(90^\circ) = \sin(VA) \cot(SA) - \sin(90^\circ) \cot(E)$$

$$\textcircled{1} = \sin(VA) \cot(SA) - \cot(E) \quad \downarrow \quad SA = \text{Dec}$$

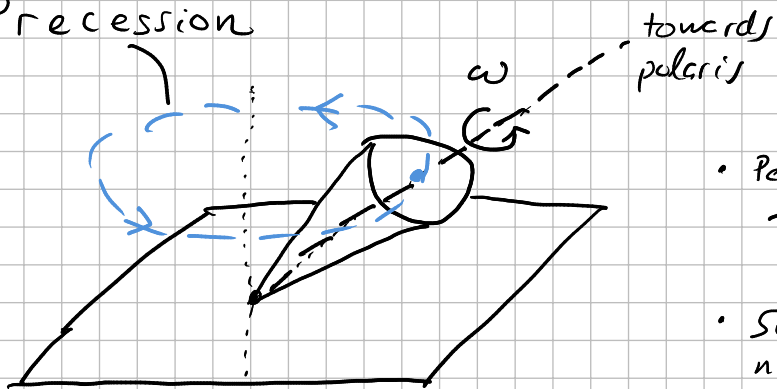
$$\sin(VA) \cot(\text{Dec}) = \cot(E)$$

$$VA = \arcsin[\tan(\text{Dec}) \cot(E)]$$

* measure Dec of sun at time of meridian passage to get VA (sun's right ascension)

L hour angle of sun = $\textcircled{1}$ $\therefore RA = LST$ and we have the vernal equinox!

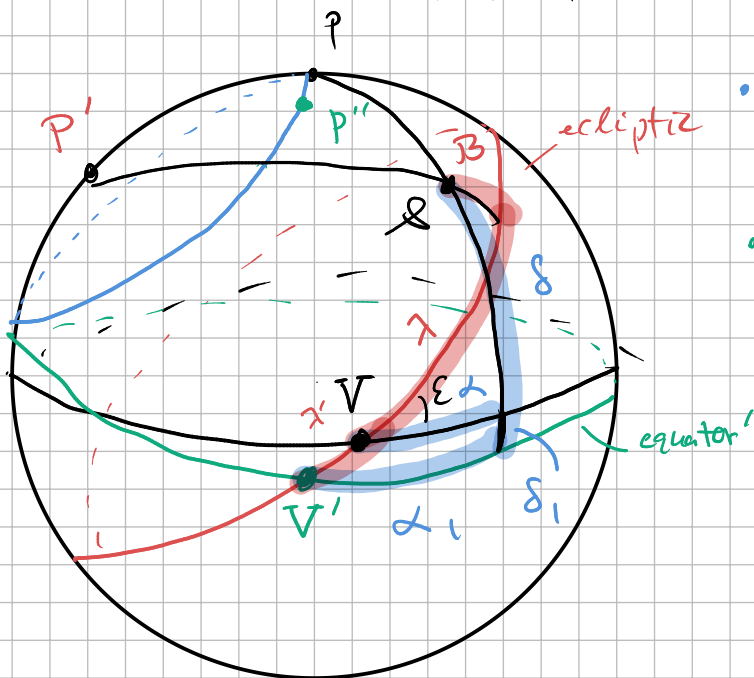
Precession



- Period of Earth's precession
~ 25,800 years

- Seems so large that corrections not necessary on human time scales

- If your celestial coordinates from 50 years ago your telescope will not have desired object in FOV
- Even 25 year old data is inaccurate



- small circle through P is path celestial pole will move due to precession

- P'' = future position of NCP

- α = RA, δ = Dec

- λ, β = initial ecliptic coordinate

- λ_1, β_1 = final ecliptic coordinate

$$\beta_1 = \beta$$

ecliptic latitude does not change from precession

- let θ = amount of annual precession.

$$\frac{\text{sidereal year} - \text{tropical year}}{\text{sidereal year}} = \frac{\theta}{360^\circ} = \frac{365.256363^d - 365.242190^d}{365.256363^d}$$

$$\theta = 50.29''$$

- It can be shown that

$$\Delta \alpha = \alpha_1 - \alpha = \theta (\cos \epsilon + \sin \epsilon \sin \alpha \tan \delta)$$

$$\Delta \delta = \delta_1 - \delta = \theta \sin \epsilon \cos \alpha$$