Fourier series optimization opportunity

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FOURIER SERIES OPTIMIZATION OPPORTUNITY

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ABSTRACT: We discuss the introduction of Fourier series as an immediate application of optimization of a function of more than one variable. Specifically, we show how the study of Fourier series can be motivated to enrich a multivariable calculus class. We do this through discovery learning and use of technology wherein students build the sine Fourier series for the simple function $f(x) = x$ and then generalize to the n^{th} term sine Fourier series for a general function, $f(x)$. We show how the students can then explore the power of the Fourier series to represent functions.

KEYWORDS: Discovery learning, Fourier series, optimization, *Mathematica*.

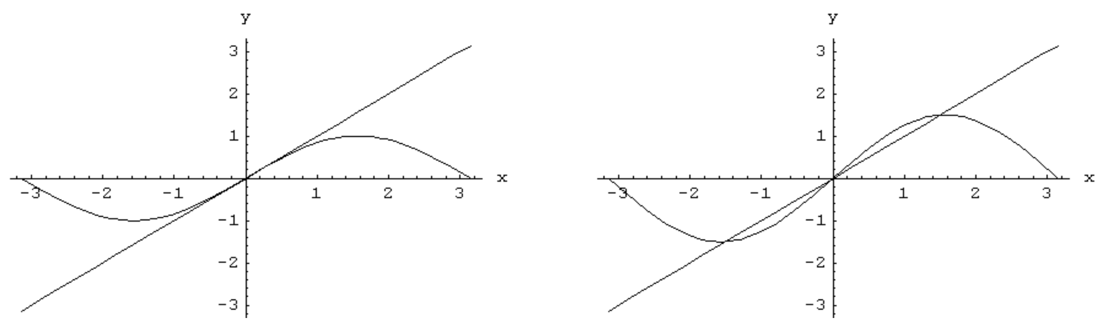
INTRODUCTION

We have introduced the study of Fourier series as an immediate application of optimization of a function of several (many!) variables in a number of calculus settings. Minimizing the sum of square errors is an obvious application of the optimization strategy that students naturally see and can develop on their own with a bit of guidance.

There are a few details we need to cover and we can be off to the fun world of approximating functions, wild functions with the classical trigonometric family. We first need to be sure that students are aware of what an odd function is as we shall confine our introductions to using sine functions to represent our sample function. Indeed, we intend to guide the students into discovering the "best" coefficients $b_1, b_2, b_3, \ldots, b_n$ to put into a linear combination $f_n(x) = b_1 \sin(x)+b_2 \sin(2x)+b_3 \sin(3x)+ \ldots + b_n \sin(nx)$ so that $f_n(x)$ best approximates the straight line $f(x) = x$ over the interval $[-\pi, \pi]$. This includes letting students decide what is "best" naturally.

Often we have some scenario to motivate our students. At The United States Military Academy we have told them that some mathematics faculty member at the United States Naval Academy (our arch gridiron rival!) has found a bunch of sinusoidal signal generators which generate $\sin(x), \sin(2x), \sin(3x), \ldots, \sin(100x)$ and she does not know what to do with them! It seems that she did not study any mathematics relevant to this issue in preparation for teaching at said Naval Academy. Could we at our prestigious Military Academy, where Army officers are educated, offer some help? We discuss a number of things from the students' backgrounds and see what they know. They know things about musical instruments, about frequencies and sound synthesizers, about waves, about vibrations, and about spectrometers. We get them excited about the prospect of learning how these ideas may be related. We take it easy at first and suppose that we want to produce the "signal" $f(x) = x$, say on the interval $[-\pi, \pi]$. At each stage of our development we tell the students that we are trying to keep it simple at first and that we hope to do more exciting things by the time we finish our studies.

We use *Mathematica* and so it is quite easy to play with plots and produce comparisons. However, we first send our students to the boards or paper and pencil and ask them to sketch several "one term" approximations, i.e. $f_1(x) = b_1 \sin(x)$, where they pick the b_1 value to get a feel for what values produce good approximations and what values produce bad ones. For example, $b_1 = 1$ is the obvious one to start, but $b_1 = -1$ is another. The latter is not as good as the former, as the latter goes "opposite" they would say, while the former at least stays in the right direction. Then playing with the amplitude of the one term approximation $b_1 \sin(x)$ permits them to see that when $b_1 = 1.5$ we are better, although we have not suggested a technical definition of better just yet. It is all just visual judgments. See Figure 1. Which looks better?



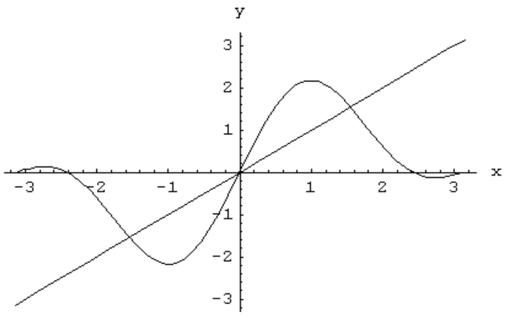
(a) $f_1(x) = \sin(x)$ approximation of $f(x) = x$.

b) $f_1(x) = 1.5\sin(x)$ approximation of $f(x) = x$.

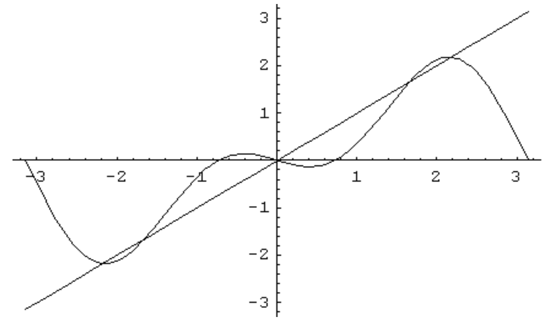
Figure 1. Two initial attempts at using just one sine generator $\sin(x)$ to approximate $f(x) = x$ on the interval $[-\pi, \pi]$. (a) uses $f_1(x) = \sin(x)$ and (b) uses $f_1(x) = 1.5\sin(x)$.

After playing with several values of b_1, still playing by hand, we see just how weak using only one sine term is, so we move on to two terms. We offer up plots of two terms in Figure 2. In class we keep students on the board or paper and pencil and ask them to plot by hand by adding multiples b_1 and b_2 of the two curves in $f_2(x) = b_1 \sin(x) + b_2 \sin(2x)$. We do a few and tire of the time it takes to sketch, add, and resketch the sums. We turn to technology, in our case, *Mathematica*.

We realize as a class that guessing will not do. We need to develop some criteria for best and the class discusses this in small groups and ALWAYS suggests that the area between the curves in the interval $[-\pi, \pi]$ has to be minimum for best to occur, i.e. we have to pick the coefficients b_1 and b_2 so that $\int_{-\pi}^{\pi} (f(x) - (b_1 \sin(x) + b_2 \sin(2x))\, dx$ is minimum. "Wait," one of the students says – this always happens. "These areas could cancel each other out," and so we need to make sure this does not happen. Take the absolute value or square the difference comes up. They do not like or trust absolute value, so squaring the difference it is. We remind them of words like "least squares" from their science lab past and present and we discuss the difference between pointwise differences and differences over the entire interval, hence the integral. Thus their criteria, rather quickly arrived at, is to pick the coefficients b_1 and b_2 so that the integral of the difference squared between the function $f(x)$ and the approximation $f_2(x) = b_1 \sin(x) + b_2 \sin(2x)$ over the interval $[-\pi, \pi]$ is minimal, i.e. minimize the following.



(a) $f_2(x) = 1.5\sin(x) + \sin(2x)$ approximation of $f(x) = x$.



b) $f_2(x) = 1.5\sin(x) - \sin(2x)$ approximation of $f(x) = x$.

Figure 2. Two initial attempts at using two sine generator $\sin(x)$ to approximate $f(x) = x$ on the interval $[-\pi, \pi]$. (a) uses $f_2(x) = 1.5\sin(x) + \sin(2x)$ and (b) uses $f_2(x) = 1.5\sin(x) - \sin(2x)$.

$$\int_{-\pi}^{\pi} (f(x) - (b_1 \sin(x) + b_2 \sin(2x))^2\, dx\,.$$

It is a small leap, but the students make it quite easily, to recognize this integral as a function of b_1 and b_2, indeed for our function $f(x) = x$ we actually compute, in *Mathematica*, this objective function:

$$\begin{aligned} S(b_1, b_2) &= \int_{-\pi}^{\pi} (f(x) - (b_1 \sin(x) + b_2 \sin(2x))^2\, dx \\ &= \pi b_1^2 - 4\pi b_1 + \pi b_2^2 - 2b_2\pi + \frac{2\pi^3}{3}. \end{aligned}$$

Students are amazed that this will be so simple to minimize as a function of the two variables, b_1 and b_2. Indeed the two partials with respect to b_1 and b_2 are respectively:

$$\frac{\partial S}{\partial b_1} = 2\pi b_1 - 4\pi \quad \text{and} \quad \frac{\partial S}{\partial b_2} = 2\pi b_2 - 2\pi\,.$$

Solving, for when both partials are 0 (they earlier learned that this is a viable strategy for finding minima, i.e. taking the partials and setting them equal to 0) we obtain these values: $b_1 = 2$ and $b_2 = -1$. The *Mathematica* command for this is:

```
Solve[{D[S[b1, b2], b1] == 0, D[S[b1, b2], b2] == 0}, {b1, b2}]
```

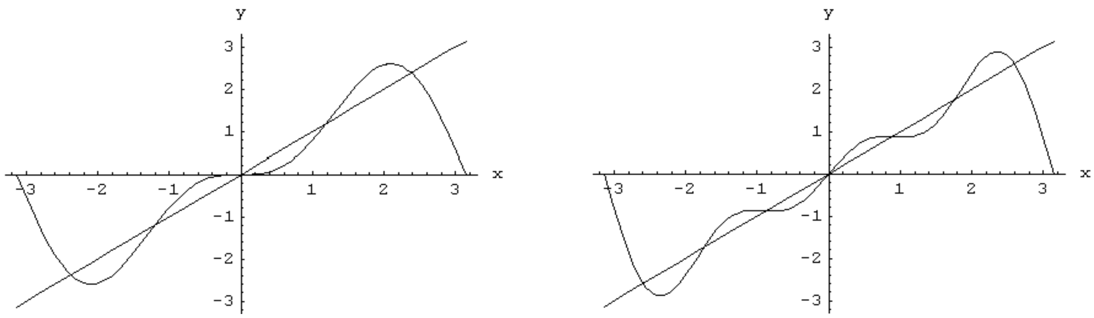
with output

```
{{b1 -> 2, b2 -> -1}}
```

so we can use the technology rather effectively to do our calculations. We now know the best fitting combination of $\sin(x)$ and $\sin(2x)$ terms is

$$f_2(x) = 2\sin(x) - 1\sin(2x)$$

which we immediately plot (see Figure 3(a)) and confirm visually as being better than those guessed in Figure 2.



(a) $f_2(x) = 2\sin(x) - 1\sin(2x)$, best two term approximation of $f(x) = x$.

b) $f_2(x) = 2\sin(x) - 1\sin(2x) + \frac{2}{3}\sin(3x)$, best three term approximation of $f(x) = x$.

Figure 3. Best two (a) and three (b) term approximations of $f(x) = x$ on the interval $[-\pi, \pi]$.

Immediately several students begin the necessary typing to get three coefficients b_1, b_2, and b_3 which minimize the integral of the difference squared using three sine terms:

$$\int_{-\pi}^{\pi} (f(x) - (b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x))^2 \, dx \, .$$

In a similar manner as with two terms they find these are $b_1 = 2$, $b_2 = -1$, and $b_3 = \frac{2}{3}$. The students notice that as we add more and more terms the initial best coefficients stay the same, they do not change. They think that is nice to see, for it says that they can push to more and more terms and be reasonably confident that the first terms they found stay the same. This conjecture is confirmed by every step we take in approximation and in developing the formulae for the Fourier coefficients later in the development.

A typing frenzy begins as students try to discover more and more terms. A typical class will try for 6 to 10 terms; this means 6 to 10 partial derivatives have to be set to 0 and all such equations have to be solved. With *Mathematica* on their side the students do not hesitate to go after this many coefficients. Indeed, they usually have no trouble replicating syntax and producing the following set of coefficients:

b_1	b_2	b_3	b_4	b_5	b_6	b_7
2	-1	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{2}{5}$	-$\frac{1}{3}$	$\frac{2}{7}$

Students surmise the pattern and others who have typed ahead to 10 terms confirm this pattern:

$$b_n = \begin{cases} -\frac{2}{n}, & \text{if } n \text{ is even} \\ \frac{2}{n}, & \text{if } n \text{ is odd.} \end{cases}$$

OR in one general formula $b_n = (-1)^{n+1}\left(\frac{2}{n}\right)$ for all n. This is, of course, a conjecture based on a few observations, but we praise the conjecture and ask them to recognize this is NOT a proof for all n. We have convinced ourselves that the pattern looks good. However, a picture is worth a lot to students at this stage in their mathematical development and we present them with the summation formula commands in *Mathematica* from which they can compute an approximation with a good many terms.

```
f[n_, x_] := Sum[(-1)^(k + 1) 2/k Sin[k*x], {k, 1, n}]
```

We use this to compute the 20 term approximation, `f[20,x]`.

$$\begin{aligned} f_{20}(x) &= 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) - \frac{1}{3}\sin(6x) + \frac{2}{7}\sin(7x) \\ &\quad -\frac{1}{4}\sin(8x) + \frac{2}{9}\sin(9x) - \frac{1}{5}\sin(10x) + \frac{2}{11}\sin(11x) - \frac{1}{6}\sin(12x) + \frac{2}{13}\sin(13x) \\ &\quad -\frac{1}{7}\sin(14x) + \frac{2}{15}\sin(15x) - \frac{1}{8}\sin(16x) + \frac{2}{17}\sin(17x) - \frac{1}{9}\sin(18x) + \frac{2}{19}\sin(19x) \\ &\quad -\frac{1}{10}\sin(20x) \end{aligned}$$

and then plot (Figure 4) this 20 term approximation:

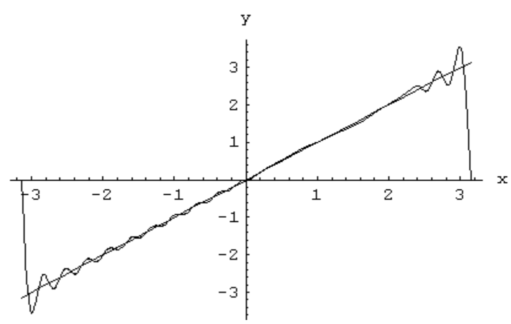


Figure 4. Best twenty term approximations of $f(x) = x$ on the interval $[-\pi, \pi]$.

GENERAL FOURIER SERIES

Up to this point we have not used the word Fourier or even series because we are confining ourselves to approximations with a finite number of terms, indeed, we have confined our exploration to finding sums of a "few" sine functions to approximate just one function $f(x) = x$ over the specified interval $[-\pi, \pi]$. Students are asking in class if this sort of thing, i.e. better and better approximations with more and more terms – at least that is what it looks like to them – can be done for other functions $f(x)$. So we ask them what would have to happen. They are quite quick to write down a long expression to minimize:

$$\begin{aligned} f_n(b_1, b_2, \ldots, b_n) &= \int_{-\pi}^{\pi} (f(x) - (b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \ldots + b_n\sin(nx))^2\,dx \\ &= \int_{-\pi}^{\pi} (f(x) - b_1\sin(x) - b_2\sin(2x) - b_3\sin(3x) - \ldots - b_n\sin(nx))^2\,dx \qquad (1) \end{aligned}$$

This poses a rather daunting task for there are lots of cross terms from the squaring of such a large expression and then it is inside the integral and we still would have to integrate. Mathematics offers the development and practice of a number of problem solving strategies. One of them is to break down a problem into simple pieces. So we do a group thrashing about at the board, filling it with equations, but in reasoned and orderly fashion, to see just what this messy integral of the differences squared will give us. We give some flavor of this activity here. We first write out, as in "long multiplication", the product inside the integral:

$$\begin{array}{rl} & f(x) - b_1\sin(x) - b_2\sin(2x) - b_3\sin(3x) - \ldots - b_n\sin(nx) \\ \times & f(x) - b_1\sin(x) - b_2\sin(2x) - b_3\sin(3x) - \ldots - b_n\sin(nx) \\ \hline \end{array}$$

planning our strategy to actually do the multiplication.

Now, systematically looking into all the possible products we see we have 4 types of terms:

Type 1	Type 2	Type 3	Type 4
$f(x)f(x)$	$b_k f(x)\sin(kx)$	$b_i b_j \sin(ix)\sin(jx)$ $i \neq j$	$b_i b_j \sin(ix)\sin(jx) = b_i^2\sin^2(ix)$ $i = j$

Upon examination of these terms, in light of the fact that we wish to find the b_k's which minimize the sum of squared differences integral in Equation (1), we see that the Type 1 term (actually only one such term) has no bearing on the minimization and can be ignored. Type 2 terms are clearly of interest as they contain our variables, b_k, and we see that these terms' contribution to the sum, upon integration are

$$\int_{-\pi}^{\pi} b_k f(x)\sin(kx)\,dx = b_k\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

Type 3 and 4 are similar and we investigate two integrals:

$$b_i b_j \int_{-\pi}^{\pi}\sin(ix)\sin(jx)\,dx \ \text{ for } \ i\neq j \quad \text{and} \quad b_i^2\int_{-\pi}^{\pi}\sin^2(ix)\,dx$$

The first cases of Type 3 (where $i \neq j$) yield $b_i b_j \frac{2j\cos(j\pi)\sin(i\pi)-2i\cos(i\pi)\sin(j\pi)}{i^2-j^2}$ in *Mathematica*. Clearly as i and j are integers then the terms $\sin(ix)$ and $\sin(jx)$ are all 0. This means all these integrals are 0. So they contribute nothing to our minimization problem. Terrific! Now, the second cases of Type 3 (where $i = j$) yield $b_i^2\pi - \frac{\sin(2i\pi)}{2i}$. These terms are all just $b_i^2\pi$ as $\sin(2i\pi)$ is 0 for integers i. So the class gathers this information and it appears that Equation (1), the function of n variables, $b_1, b_2, \ldots, b_n$, we seek to minimize, is simply:

$$f_n(b_1, b_2, \ldots, b_n) = \int_{-\pi}^{\pi}(f(x))^2\,dx + \sum_{j=1}^{n} 2b_j\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx + \sum_{j=1}^{n} b_j^2\pi. \tag{2}$$

Note that we get two of the terms when $i \neq j$. From Equation (2) and our understanding of optimization of a function of more than one variable we see that if we are to minimize $f_n(b_1, b_2, \ldots, b_n)$ then each of the partial derivatives with respect to b_1, b_2, $\ldots$, b_n, respectively, are derivatives of a quadratic in that respective variable. So, for example, taking the partial derivative with respect to b_j and setting this derivative equal to 0 yields:

$$\frac{\partial f_n(b_1, b_2, \ldots, b_n)}{\partial b_j} = 2\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx + 2b_j\pi = 0$$

which tells us that

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx\,, \quad j = 1, 2, \ldots, n. \tag{3}$$

There it is, for a given function $f(x)$ over the interval $[-\pi, \pi]$ pick the coefficients b_j for the $\sin(jx)$ term according to Equation (3) in order to minimize the integral of the squared difference between our sample function and the sine approximation candidate. We immediately try it for our original case, $f(x) = x$, for which we "guessed" $b_j = (-1)^{j+1}\frac{2}{j}$ based on a few terms.

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(jx)\,dx = \frac{2\sin(j\pi) - 2j\pi\cos(j\pi)}{2j^2\pi}\,. \tag{4}$$

In Equation (4) since j is an integer the $\sin(j\pi)$ term is 0 and this leaves us with

$$b_j = \frac{2\sin(j\pi) - 2j\pi\cos(j\pi)}{j^2\pi} = \frac{-2\cos(j\pi)}{j}$$

in which $\cos(j\pi)$ oscillates between $+1$ and -1 as j is even or odd, respectively, i.e. $b_j = (-1)^{j+1}\frac{2}{j}$ as we had conjectured before. This helps validate a derivation for students.

As a class we summarize our accomplishment.

> Given $f(x)$ defined on the interval $[-\pi, \pi]$, if we select $b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)\,dx$ for $j = 1, 2, \ldots, n$ then the sum $\sum_{j=1}^{n} b_j \sin(jx)$ is a reasonable approximation for the function $f(x)$ on the interval $[-\pi, \pi]$ and as n increases this approximation appears to get better.

With this accomplishment we use some modest *Mathematica* code to play with some functions. We try out these new formulae with our original $f(x) = x$ and see the results of our labor:

```
f[x_] = x;
b[j_] := 1/Pi Integrate[f[x] Sin[j x],{x,-Pi, Pi}]
g[n_,x_] :=Sum[b[j] Sin[j x],{j,1,n}]
```

We obtain our 20 term approximation `fa[x_] = g[20,x]` (it is the same as when we built it by conjecture earlier) and our plot of the approximation and the function is identical to that demonstrated in Figure 3. We are good to go! We have developed a general process. We now share with the students the fact that this is the sine *Fourier* series (where $n \to \infty$ to be a full series and there can be issues with convergence) and *they* built it. Congratulations! We go over some history about Fourier's work as time permits.

At this point we return to a text (or provide further notes) in which the full sine - cosine Fourier series over any interval is developed and we discuss issues of convergence, but the key we stress to the students is that they used their calculus and *Mathematica* to develop a powerful concept to tell them how to approximate a function as a best linear combination of trigonometric, periodic functions. They also developed the criteria for best. We point out that we developed an approach for odd functions using sine functions, but general functions will need odd and even functions, hence the full sine and cosine Fourier series and more general intervals.

As an example of what they accomplish consider the stepwise defined odd function.

```
f[x_] = If[x > 0, 1, -1]
```

and the first 20 terms of the sine-cosine Fourier series they develop. See Figure 5.

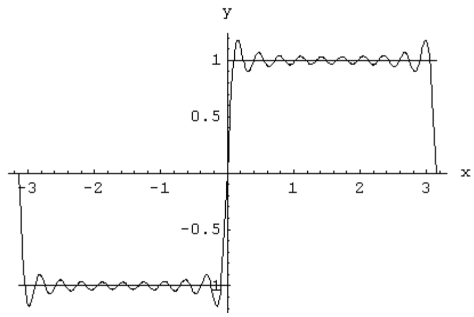


Figure 5. Best twenty term approximations of $f(x)$ defined to be 1 on the interval $(0, \pi]$ and -1 on the interval $[-\pi, 0]$.

Such plots get us into discussions of convergence issues at points of discontinuity and the Gibbs' phenomena of overjumping at these points, clearly visible on the plots.

Once let loose, the students go for their own functions. Here you see one such result using the full sine-cosine Fourier series (see Figure 6) which can be accomplished only with technology.

```
f[x_] = If[x > 0, 1 + x + .1 x^2 + 8Log[1 + 3x], -5 - .5x^2 - .6x^3]
```

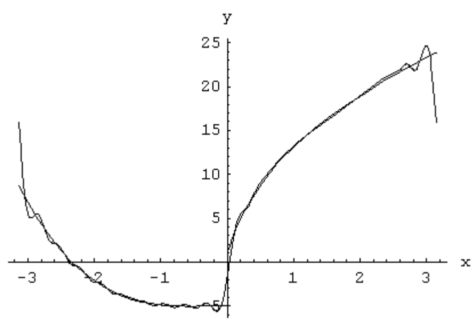


Figure 6. Best twenty term approximations of $f(x)$ defined by a student on the interval $[-\pi, \pi]$.

Once students have internalized the idea of a series of trigonometric functions to approximate ANY function over any interval then either through assigned homework problems or student generated projects we investigate Fourier series further. We almost always develop the notion of the spectrum to fingerprint functions and other signals, such as those used in spectrometers in their chemistry class, sound synthesizers used in popular music, signal analyzers to detect irregularities in motion of machines through motion detector signals, and voice recognition units for security. *Mathematica* is capable of graphically presenting us with images that reinforce our thinking and learning in all these investigations.

CONCLUSION

We have taken the reader on the same trip we take our students to show how students develop the Fourier series from rather elementary optimization techniques applied to the integral of the squared difference between our given function and a finite number of terms in an approximation. This is a deeply technology enabled trip and once at an early destination in the journey our students can see way past that point to other representations and applications. We heartily recommend you try this route with your students.

BIOGRAPHICAL SKETCH

Brian Winkel is Professor of Mathematical Sciences at the United States Military Academy where he teaches, writes, and mentors faculty. While his doctoral work was in Noetherian ring theory he has moved to an inquiry-based career in which applications of mathematics to such diverse areas as biology, economics, physics, and engineering, motivate his own learning and his teaching. He has founded and edited three scholarly journals, *Cryptologia*, *Collegiate Microcomputer*, and *PRIMUS - Problems, Resources, and Issues in Mathematics Undergraduate Studies* and is still the Editor-in-Chief of *PRIMUS*.