Semiadditive Thursdays

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I (2/12, Matthew Niemiro) Trivial Notions

Abstract: The greatest stories do not begin with ∞-category theory, and neither will ours—the germs of semiadditivity subsist willingly upon *ordinary* categories, ready for swab and cultivation. We will start our seminar by documenting some basic semiadditive phenomena (in particular: bilimits, enrichments, norms, and ambidexterity for negative one and zero semiadditivity) as it is expressible in ordinary language. Then we will anticipate higher extensions of the theory. Then we will create a *hostage situation* to canvas speakers.

I.1 Preamble to the seminar.

The theory of *(higher) semiadditivity* is firmly (higher) algebraic. Recent developments in chromatic and equivariant homotopy theory, trace methods, TQFTs, algebraic geometry, geometric Langlands, mirror symmetry, ... have motivated its closer study. Many folks seem familiar with some applications of semiadditivity, are interested in its other instantiations, and are not familiar with its machinations. Especially more recent work using semiadditivity requires a passing comprehension of the details to read, and this was one motivation to organize this seminar. That in mind, this seminar will take a category-first approach.

I.2 (Semi)additivity, biproducts, and commutative monoids.

Here is something you know.

Definition I.2.1. An ordinary category C is called *additive* under the following conditions.

(1) For each pair of objects $A, B \in \mathsf{C}$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is assigned an abelian group structure.

(2) Supposing (1), the composition of morphisms is bilinear with respect to the abelian group structure assigned to hom-sets.

(3) C admits finite biproducts (including the empty one, hence a zero object).

This is *structure*. Conditions (1) and (2) amount to an *enrichment over abelian groups*. However, some pedantic category theory dissolves the amount of choice involved. Note that such an enrichment implies that any finite product is canonically a coproduct if it exists, and vice-versa. In fact, it implies something slightly stronger: any finite (co)product is a *biproduct* $A \oplus B$ in a canonical way.[1] It likewise implies that if an initial object exists, then it is also terminal and hence a zero object. Moreover, given (1), (2), and the existence of $B \times B$ (hence the biproduct $B \oplus B$), the addition on $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is uniquely determined[2] as

$$f + g : A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \cong B \sqcup B \to B.$$

The biproduct seems to have an important role. In fact, as soon as finite biproducts exist, you can turn most of this around. If C admits finite biproducts (including a zero object), the induced operation $(f, g) \mapsto f + g$ equips C with an enrichment over commutative monoids, and this is the unique CMon-enrichment that C may possess.[3] Therefore, the definition of an additive category can be redisplayed as follows.

Definition I.2.2. An ordinary category C is called *additive* under the following conditions.

(1) C is pointed.

(2) C admits finite products and finite coproducts.

(3) Supposing (1) and (2), for each pair of objects $A, B \in \mathsf{C}$, the *identity matrix* morphism

$$\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix} : A \times B \to A \sqcup B$$

[1] The distinction is that biproducts are unique up to unique isomorphism while coincidental (co)products need not be, among other things. A good rabbit hole is [Yua12].

[2] This is forced by bilinearity.

[3] Two discussions of this: my older notes or [CDW24, §2].

is an isomorphism. (Thus, C admits finite biproducts.)

(4) For each pair of objects $A, B \in \mathsf{C}$, the canonical commutative monoid operation $(f, g) \mapsto f + g$ defined above on $\mathrm{Hom}_{\mathsf{C}}(A, B)$ admits negatives.

It is now clear that additivity is a *property*. (This "structure that is actually a property" thing will be a recurring theme.) Moreover, it is clear that we should inspect a weaker underlying property.

Definition I.2.3. An ordinary category possessing properties (1)-(3) is called *0-semiadditive*.

Remark I.2.4 (Prescient remark). So there's this close relationship between the *property* of semiadditivity and the *structure* of a CMon-enrichment. In particular, semiadditivity induces a canonical enrichment. The converse is false: a CMon-enrichment forces finite (co)products to be biproducts, but it does not guarantee existence. The notions coincide in the category of *pointed, finitely cocomplete* categories, however. (Or pointed, finitely complete.) We sometimes call a pointed category *(-1)-semiadditive*. We point out that pointedness can be defined just like 0-semiadditivity except with finite sets replaced by just *empty* sets, and that (-1)-semiadditivity has an analogous relationship with the "structure" of Set_*-enrichments. These are our first signs of the hierarchy of higher semiadditivity. Sam will expound upon this enrichment-property game.

(Tea and cookies break?)

I.3 Biproducts and norms.

Ordinary semiadditivity concerns the existence of finite biproducts. We can use *norms* to systematize their existence. We separate the case of empty and nonempty finite biproducts.

Empty Case: A terminal (resp. initial) object in C is the same thing as as limit (resp. colimit) of the empty functor $\varnothing \to \mathsf{C}$. If C admits both an initial and terminal object, denoted $\varnothing$ and $*$, then there is a unique morphism

$$\mathrm{Nm} : \varnothing \to *.$$

In this case, the invertibility of Nm is the unique condition for $\varnothing$ and $*$ to coincide. This would occur as soon as there is *any* morphism $* \to \varnothing$, which would necessarily be the inverse Nm^{-1}.

Non-empty case: Consider a finite family of objects $F : S \to \mathsf{C}$. We may ask for a comparison map

$$\mathrm{Nm}_F : \operatorname{colim} F \to \lim F.$$

This amounts to a family of morphisms $\mathrm{Nm}_F^{s_i, s_j} : F(s_i) \to F(s_j)$ indexed by ordered pairs $s_i, s_j \in S$, i.e. a family of choices $\mathrm{Nm}_F^{s_i, s_j} \in \mathrm{Hom}_{\mathsf{C}}(F(s_i), F(s_j))$, i.e. a matrix

$$\begin{array}{c|cccccc}
 & s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \\
\hline
s_1 & \mathrm{id}_{F(s_1)} & ? & ? & \cdots & ? & ? \\
s_2 & ? & \mathrm{id}_{F(s_2)} & ? & \cdots & ? & ? \\
s_3 & ? & ? & \mathrm{id}_{F(s_3)} & \cdots & ? & ? \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_{n-1} & ? & ? & ? & \cdots & \mathrm{id}_{F(s_{n-1})} & ? \\
s_n & ? & ? & ? & \cdots & ? & \mathrm{id}_{F(s_n)}
\end{array}$$

Generally, there is *no* canonical Nm_F map because there are not generally distinguished elements of the mapping sets $\mathrm{Hom}_{\mathsf{C}}(F(s_i), F(s_j))$. In other words, you do not know how to fill the ? in the matrix. An easy fix is to stipulate that C is pointed, because then we can insert zero morphisms.

Remark I.3.1. The choice of a compatible family of morphisms $\varphi_{c_1,c_2} \in \mathrm{Hom}_{\mathsf{C}}(c_1, c_2)$ for each ordered pair $c_1, c_2 \in \mathsf{C}$ is equivalent to a Set_*-enrichment. Moreover, this is equivalent to "having zero morphisms." In particular, this structure is unique if it exists.

Proposition I.3.2. *If C is pointed and admits finite limits and finite colimits, then for each finite collection $F : S \to \mathsf{C}$ there is a canonical "identity matrix" morphism*

$$\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F.$$

Moreover, if Nm_F is an isomorphism, then $\lim F$ is a biproduct.

Corollary I.3.3. *For an ordinary category C, the following are equivalent.*

(1) The category C is zero semiadditive.

(2) For every finite collection $F : S \to \mathsf{C}$, the (co)limit of F exists and is a biproduct.

(3) The category C is pointed, and for every finite collection $F : S \to \mathsf{C}$, the canonical morphism $\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F$ is invertible.

We made the prior discussion to sensibly state (3). But why did we bother if we can just ask for (2)? First, let me remind you that pointedness is (-1)-semiadditivity. Next, let me pose some questions. Rather than $F : S \to \mathsf{C}$, consider a functor $F : X \to \mathsf{C}$ where now X is an anima and C is an ∞-category.

(1) What conditions guarantee that $\lim F$ or $\mathrm{colim}\, F$ exist?

(2) If they exist, what is the *best* way the limit and colimit may coincide? The distinction between biproducts and coincidental (co)products in an ordinary category is already substantial, so this should not be overlooked.

(3) What's all this good for?

The basic selling point of *higher semiadditivity* is its unreasonably effective, systematic, and broadly applicable answers to these questions. Our discussion of *ambidexterity and norms* will generalize the above analysis of 0- and (-1)-semiadditivity to handle these questions and define *higher semiadditivity*. Our discussion of *higher commutative monoids* will generalize the relation between zero semiadditivity and abelian monoids to characterize higher semiadditivity by its relation to $\mathbb{E}_\infty$ (and stronger!) structures.

I.4 Ambidexterity and biadjoints.

For a category C, semiadditivity is a property concerning every finite family $F : S \to \mathsf{C}$. Here, both F and S are variable. *Ambidexterity* conveniently isolates that part of semiadditivity concerning only S. For this, consider the map of sets

$$q : S \to *.$$

This induces a *pullback functor* $q^* : \mathsf{C} \to \mathsf{C}^S$.

Proposition I.4.1. *If C admits S-indexed (co)products, then the functors*

$$q_* := \lim_S(-) : \mathsf{C}^S \to \mathsf{C}$$
$$q_! := \mathrm{colim}_S(-) : \mathsf{C}^S \to \mathsf{C}$$

are right and left adjoints to q^* respectively. In particular, they are uniquely determined as adjoints.

Proof. The interesting counit consists of diagonals $u_{*,X} : X \to \prod_S X$ and the interesting counit consists of collapse maps $c_{!,X} : \sqcup_S X \to X$. You can now check the triangle identities. □ typos fix later

Corollary I.4.2. *For a set S, the category C admits S-indexed products iff q^* admits a right adjoint q_*, and it admits S-indexed coproducts iff q^* admits a left adjoint $q_!$.*

This gives us a "relative with respect to $S \to *$" way to talk about the existence of S-products and S-coproducts. We want to similarly understand S-bilimits.

Let's take a moment to dwell on what we are asking. We have an adjoint triple

$$q_! \dashv q^* \dashv q_*$$

and we just identified q_* and $q_!$ as computing S-indexed (co)products. Now, we want biproducts. We know that a biproduct is not just a coincidental product and coproduct, rather it needs compatible (co)product data. Similarly, we should not just ask for an equivalence of functors $q_* \cong q_!$, as this does not marry their roles as adjoints. This makes sense: if q_* (or $q_!$) is to properly serve as a simultaneous left and right adjoint, its four (co)units should be compatible. There are a few ways to go about this:

This is not implied by a functor isomorphism $q_* \cong q_!$, right?

(1) Explicit compatibilities: Suppose that C admits S-products and S-coproducts so that the adjoint triple in question exists. Then we may form

$$q^* q_* \xrightarrow{c_*} \mathrm{id}_{\mathsf{C}^S} \xrightarrow{u_!} q^* q_!.$$

Morally, if we want a good simultaneously left and right adjoint q_*, then this should compose to the identity and you should interpret this as a triangle identity. Technically, let's unwind. Note that given $F : S \to \mathsf{C}$, the functor $q^* q_!$ acts by $F \mapsto \coprod_S s_i$ and $q^* q_*$ acts by $F \mapsto \prod_S s_i$. The s_k-th component of this transformation at F is the map $i_k \circ \pi_k : \prod s_i \to \coprod s_i$. These are *precisely* the diagonal structure maps of a biproduct, which must assemble to an identity matrix to form a biproduct. In other words, we should call our adjunction *ambidextrous* if $u_! \circ c_* \cong \mathrm{id}$.

Improve this claim

I'm suspect of some of what I've written here

(2) Norms: To study biproducts, the appeal of norms is that one can bypass the fuss of compatibilities if you could form the identity matrix $\mathrm{Nm}_F : \operatorname{colim} F \to \lim F$, because its invertibility guaranteed your (co)product was a biproduct. For Nm_F to exist, we used the zero morphisms granted by pointedness. In our relative context, we may hope for a transformation

$$\mathrm{Nm}_q : q_! \to q_*$$

serving the same convenience: if we can just construct Nm_q, then its invertibility would be equivalent to the ambidexterity of our adjoint triple, i.e. the "correct" way to ask for $q_! \cong q_*$. The pause is that Nm_q may not exist: it consists of maps $\operatorname{colim}_S(-) \to \lim_S(-)$ which consist of matrices of maps, so Nm_q is only defined once you have zero morphisms for the off-diagonals, i.e. once you are (-1)-semiadditive.

(3) Wrong-way (co)units: ...

To summarize: the existence of S-(co)products amounted to the existence of adjoints $q_!$ and q_*. Then we needed pointedness to construct the norm map $\mathrm{Nm}_q : q_! \to q_*$ as the identity matrix. In this case, we say that *S is weakly C-ambidextrous*. Then if Nm_q is invertible, then C admits S-biproducts. In this case, we say that *S is C-ambidextrous*.

Proposition I.4.3. *The following are equivalent.*

(1) The set S is C-ambidextrous.

(2) The category C admits all S-indexed biproducts.

(3) The category C is pointed, and the norm $\mathrm{Nm}_q : q_! \to q_*$ associated to $q : S \to *$ is invertible.

We have delivered on our slogan: this is precisely the part of 0-semiadditivity concerning the fixed set S.

Remark I.4.4. You may replace $q : S \to *$ with an arbitrary map of finite sets $q : S \to T$. The resulting theory is as follows. The existence of adjoints is equivalent to the existence of (co)products indexed by the fibers of q; the construction of Nm_q still requires pointedness; and the invertibility of Nm_q is equivalent to the existence of biproducts indexed by the fibers of q.

Remark I.4.5. Ambidexterity, norms, and the "relative" language will become exponentially more useful in the higher setting.

II (2/19, Matthew Niemiro) Higher semiadditivity and norms

Abstract: In this session, we will make the first definition of semiadditivity using *ambidexterity and norms*. The basic idea is that semiadditivity amounts to the existence of certain bilimits, hence it is a property concerning certain families $F : X \to C$ where F and X are variable. This, in turn, may be understood inductively by analyzing the existence and invertibility of norm maps. But there's still a lot going on. The role of ambidexterity is to isolate that part of semiadditivity concerning X in a functorial and controllable manner, and this will help us run the machine. In particular, we will use ambidexterity to construct relative integration, a "higher" form of addition that will clarify the existence problem for norms. We will demonstrate this by constructing 1-finite norms for finite G-actions in the presence of ordinary semiadditivity.

II.1 Preamble to tea and cookies.

First, let's summarize parts of the previous talk which will be important today.

(1) A *biproduct* is an object serving as a product and coproduct of a family such that the canonical (co)product morphisms are compatible. If one exists, then it is unique up to unique isomorphism. This is generally distinct from an abstractly coincidental (co)product.[4] There are various equivalent definitions of biproducts, and various conditions implying their existence. If C is pointed, then biproducts admit a convenient characterization: for a set-indexed family $F : S \to \mathsf{C}$, you can use zero morphisms to define the *identity matrix* morphism

$$\mathrm{Nm}_F : \operatorname{colim} F \to \lim F,$$

and if Nm_F is invertible, then the (co)product is automatically a biproduct.

(2) The simplest definition of *ordinary semiadditivity* for a category C just asks that C admits finite biproducts. This is equivalent to demanding pointedness together with the invertibility of Nm_F for every finite-set indexed family $F : S \to \mathsf{C}$. This reformulation models the inductive definition of higher semiadditivity.

(3) Semiadditivity concerns all finite sets S and all functors $F : S \to \mathsf{C}$. *Ambidexterity* isolates the part of this property concerning only the fixed set S: just say that S is C-*ambidextrous* if C admits all S-bilimits. This may again be characterized with norms: we say that S is *weakly* C-*amibdextrous* if every $F : S \to \mathsf{C}$ admits its norm map

$$\mathrm{Nm}_F : \operatorname*{colim}_S F \to \lim_S F,$$

and if S is weakly C-ambidextrous, then it is C-ambidextrous if and only if every Nm_F is invertible. Altogether, C is 0-semiadditive if and only if every finite set S is C-ambidextrous.

(4) There is a more general and very useful way to handle ambidexterity and norms. For this, consider

$$q : S \to *.$$

The existence of S-indexed (co)limits is equivalent to the existence of adjoints $q_! \dashv q^* \dashv q_*$ to the pullback functor q^*. If C is pointed, then the norm maps for S-families assemble into a transformation $\mathrm{Nm}_q : q_! \to q_*$. Moreover, C admits S-biproducts if and only if Nm_q is invertible. We say that q is *weakly* C-*amibidextrous* if Nm_q exists, and C-*ambidextrous* if Nm_q is an isomorphism.[5] More generally, we may consider

$$q : S \to T.$$

[4] Natalie asked for an example last time. I gave the example of *the full subcategory of* Set *spanned by sets isomorphic to* $\mathbb{Z}$. Here, $A \times B$ and $A \sqcup B$ are both countably infinite, hence there are Σ_∞ isomorphisms between them as sets. However, there is no good definition of a biproduct which their (co)product can be given, and affirmingly, this category does not possess many of the good properties expected of one admitting (finite) biproducts.

[5] Notice that norms continue to serve as a convenient means for detecting biproducts: it would not be sufficient that $q_!$ and q_* are abstractly isomorphic, but if specifically $\mathrm{Nm}_q : q_! \to q_*$ exists and is invertible, then C admits S-biproducts.

Now, we say that q is *weakly* C-*ambidextrous* if $\mathrm{Nm}_q : q_! \to q_*$ exists (which requires that C is pointed and that C admits all (co)limits indexed by all the fibers of q), and we say that q is C-*ambidextrous* if Nm_q is invertible (which amounts to C being pointed and admitting all biproducts indexed by the fibers of q).

Second, let's demo the formation of *integration* using addition.

Slogan II.1.1. *The main feature of a C-ambidextrous function $f : S \to *$ is the ability to naturally integrate an S-family of morphisms $F : S \to \mathrm{Map}_{\mathsf{C}}(A, B)$ into a single morphism $\int_S F \in \mathrm{Map}_{\mathsf{C}}(A, B)$. More generally, the main feature of a C-ambidextrous function $f : S \to T$ is the ability to naturally integrate S-families of T-local systems.*

Example II.1.2 (Ordinary integration)**.** The ordinary category of abelian groups Ab admits finite bilimits, hence it is zero semiadditive. Thus, it is enriched in commutative monoids: given a finite family $f : S \to \mathrm{Hom}_{\mathsf{Ab}}(A, B)$ we may form

$$\left(\int_S f\right)(a) := \sum_{s \in S} f_s(a).$$

We can describe this categorically. First, identify f with its mate $\overline{f} : \coprod_S A \to B$. We would like to define $\int_S f$ by postcomposing this with a "diagonal into the coproduct" $A \to \coprod_S A$. We always have a morphism $\Delta : A \to \prod_S A$, and the invertibility of $\mathrm{Nm}_\varnothing : \varnothing \xrightarrow{\sim} *$ induces $\mathrm{Nm}_S : \coprod_S A \to \prod_S A$. Fortunately, the semiadditivity of Ab means that Nm_S is invertible, and we may form

$$\int_S f \quad \text{as the morphism} \quad A \xrightarrow{\Delta} \prod_S A \xrightarrow{\mathrm{Nm}_S^{-1}} \coprod_S A \xrightarrow{\overline{f}} B.$$

Now, let's frame this as a replicable and *functorial* consequence of the ambidexterity of $q : S \to *$. For this, here is a little dictionary.

Absolute notation	**Relative notation for $q : S \to *$**
$\coprod_S X$	$q_! q^* X$
$\prod_S X$	$q_* q^* X$
$\Delta_S : X \to \prod_S X$	$u_* : X \to q_* q^* X$
$f : S \to \mathrm{Map}_C(X, Y)$	$f : q^* X \to q^* Y$
$\overline{f} : \coprod_S X \to Y$	$\overline{f} : q_! q^* X \xrightarrow{q_! f} q_! q^* Y \xrightarrow{c_{!,Y}} Y$

We define *ordinary integration* as the function

$$\begin{array}{ccccccccc} \int_q : & \mathrm{Map}_{\mathsf{Ab}^S}(q^*A, q^*B) & \xrightarrow{q_!} & \mathrm{Map}_{\mathsf{Ab}}(q_!q^*A, q_!q^*B) & \xrightarrow{c_{!,B}\circ -} & \mathrm{Map}_{\mathsf{Ab}}(q_!q^*A, B) & \xrightarrow{-\circ \mathrm{Nm}_q^{-1} \circ u_*} & \mathrm{Map}_{\mathsf{Ab}}(A, B) \\ & f & \longmapsto & q_! f & \longmapsto & \overline{f} & \longmapsto & \overline{f} \circ \mathrm{Nm}_q^{-1} \circ u_{*,A} \end{array}$$

What's nice is that this is obviously functorial and easy to generalize.

II.2 Higher semiadditivity, the norm problem, and ambidexterity.

A finite set is a space X such that $\pi_0(X, x_0)$ is finite and all higher $\pi_n(X, x_0)$ vanish for all $x_0 \in X$. You may be inclined to consider "higher finite sets" as follows.

Definition II.2.1. A space X is called *π-finite* if it has finite and finitely many nonzero homotopy groups. A π-finite space which is m-truncated is called *m-finite*. A space is called *(-1)-finite* if it is empty or contractible, and *(-2)-finite* if it is contractible. A map of spaces $q : X \to Y$ is called *m-finite* if its homotopy fibers are m-finite.

Remark II.2.2. There are interesting reasons to think of π-finite spaces as "higher finite sets" which we unfortunately could not get a speaker for, c.f. **[anel]**.

Definition II.2.3. Just as ordinary semiadditivity asked for finite set-indexed biproducts, we say that an ∞-category C is *m-semiadditive* if every m-finite diagram $F : X \to \mathsf{C}$ admits a bilimit.

Our understanding of set-indexed bilimits readily generalizes to space-indexed bilimits. In particular,

Claim II.2.4. Let $F : X \to \mathsf{C}$ denote an m-finite diagram. Under certain hypotheses (spoiler: if X is weakly ambidextrous), there exists a natural morphism $\mathrm{Nm}_F : \operatorname{colim} F \to \lim F$ whose invertibility detects that F admits a bilimit. In particular, C is m-semiadditive if and only if it is $(m-1)$-semiadditive and the norm Nm_F associated to every m-finite diagram is invertible (i.e. every m-finite space X is ambidextrous).

What I will call "the norm problem" is the issue that Nm_F does not exist *a priori*, and when it does, its construction is inductive and a bit involved. As before, it will be convenient to take a functor calculus approach and work relative to a map $q : X \to Y$ of m-finite spaces. In this situation, we envision the norm as a transformation $\mathrm{Nm}_q : q_! \to q_*$ whose invertibility detects that *all* diagrams indexed by the fibers of q are bilimits. Toward making that precise, we should first ask:

Question II.2.5. *When do $q_!$ and q_* exist?*

We should suspect that the existence of $q_!$ and q_* depends on C admitting certain (co)limits.

Lemma II.2.6. *Given a functor between ∞-categories $q : X \to Y$, if the pullback $q^* : \mathsf{C}^Y \to \mathsf{C}^X$ admits a left adjoint $q_!$, then $q_!$ is essentially unique and is computed by the fiberwise left Kan extension* Cite somethng in HTT

$$q_!(F : X \to \mathsf{C}) = \left(y \mapsto \operatorname{co}\lim_{X \times_Y y} F\right).$$

Dually for the right adjoint q_* of q^*.

Corollary II.2.7. *If X and Y are anima, then the pullback $X \times_Y y$ is the homotopy fiber of q over $y \in Y$. Thus, the pullback q^* admits a left (resp. right) adjoint if and only if C admits all colimits (resp. limits) indexed by the homotopy fibers of q. In this case we say that C admits q-(co)limits.*

Example II.2.8 (Absolute case)**.** Let $q : X \to *$ denote the terminating map. An ∞-category C admits q-limits if and only if it admits all X-indexed limits $\lim(F : X \to \mathsf{C})$. Dually for q-colimits.

Now, suppose that C admits q-limits and q-colimits so that the adjoints exist.

Question II.2.9. *What does it take for $\mathrm{Nm}_q : q_! \to q_*$ to exist?*

Vague Definition II.2.10. Generalizing the ordinary case, we want to make the following language. We fix an ∞-category C and a map of spaces $q : X \to Y$.

- If Nm_q exists, then we say q is *weakly C-ambidextrous*. This map should detect q-bilimits, in the sense that if Nm_q is invertible, then C should admit all bilimits indexed by the fibers of q.
- If q is weakly ambidextrous and Nm_q is invertible, then we say q is *C-ambidextrous*.
- If $Y = *$, then we refer to the *space* X as (weakly) ambidextrous rather than the map $q : X \to *$.

Am I hallucinating, or is this actually *not* a vague definition because of some "essentially unique if it exists" fact about norms as bilimit detectors? Did I read this somewhere?

Of course, we still have to figure out when/how we can construct Nm_q. There are two equivalent ways forward, both of which will end up relying on $(m-1)$-semiadditivity. (Or at least some amount of ambidexterity.)

(1) We may construct Nm_q as a matrix whose components do not resemble an identity matrix, but rather are *integrals over certain path spaces.*

(2) We may construct the mate $\overline{\mathrm{Nm}}_q$ utilizing some special properties of the category of spaces.

Approach (1) will provide exposure therapy for integration. Approach (2) will provide exposure therapy for Beck-Chevalley conditions. We will survey both.

II.3 Norms, integration, and the definition of ambidexterity.

Let us momentarily consider the consequences of having an invertible norm map, i.e. the property of ambidexterity. We will demonstrate how this permits integration, and then we will identify how integration can be used to construct higher norm maps, starting the inductive machine.

Slogan II.3.1. *The main feature of a C-ambidextrous morphism $q : X \to *$ is that it enables the natural integration of an X-family of morphisms $F : X \to \mathrm{Map}_{\mathsf{C}}(A, B)$ into a single morphism $\int_X F \in \mathrm{Map}_{\mathsf{C}}(A, B)$. In particular, an m-semiadditive category is equipped with a natural way to integrate m-finite families*

$$\int_X : \mathrm{Map}(X, \mathrm{Map}_{\mathsf{C}}(A, B)) \to \mathrm{Map}_{\mathsf{C}}(A, B).$$

Remark II.3.2. More generally, the main feature of a C-ambidextrous map $q : X \to Y$ is the ability to naturally integrate an X-family of morphisms of Y-local systems.

Definition II.3.3. If $q : X \to Y$ is C-ambidextrous so that $\mathrm{Nm}_q : q_! \xrightarrow{\sim} q_*$ exists and is invertible, then we define *integration* relative to q as the map

$$\begin{array}{llllllll} \int_q : & \mathrm{Map}_{\mathsf{C}^X}(q^*A, q^*B) & \xrightarrow{q_!} & \mathrm{Map}_{\mathsf{C}^Y}(q_!q^*A, q_!q^*B) & \xrightarrow{c_{!,B}\circ -} & \mathrm{Map}_{\mathsf{C}^Y}(q_!q^*A, B) & \xrightarrow{-\circ \mathrm{Nm}_q^{-1} \circ u_*} & \mathrm{Map}_{\mathsf{C}^Y}(A, B) \\ & f & \longmapsto & q_! f & \longmapsto & \overline{f} & \longmapsto & \overline{f} \circ \mathrm{Nm}_q^{-1} \circ u_{*,A} \end{array}$$

Now let us return to our original question and identify where integration provides an answer.

Question II.3.4. *For an m-finite space $q : X \to *$ and an ∞-category C, when does $\mathrm{Nm}_q : q_! \to q_*$ exist?*

Informally, a morphism $\mathrm{Nm}_q : q_! \to q_*$ consists of a morphism $\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F$ for each m-finite diagram $F : X \to \mathsf{C}$. Such a map, in turn, we understand as an "$X \times X$ matrix." More precisely, Nm_F is the data of a compatible family of morphisms

$$\mathrm{Nm}_F^{x_0,x_1} \in \mathrm{Map}_{\mathsf{C}}(F(x_0), F(x_1)) \quad \text{for every } x_0, x_1 \in X.$$

A priori, there is no canonical such map. But the diagram *does* give us a family of such maps, namely

$$F_{x_0,x_1} : \mathrm{Map}_X(x_0, x_1) \to \mathrm{Map}_{\mathsf{C}}(F(x_0), F(x_1)).$$

The big move is to observe that the path space $\mathrm{Map}_X(x_0, x_1)$ is $(m-1)$-finite. Thus, if we can integrate these $(m-1)$-finite families, then we can define Nm_q by taking $\mathrm{Nm}_F^{x_0,x_1} := \int_{\mathrm{Map}(x_0,x_1)} F_{x_0,x_1}$. This we understand to be possible if $\mathrm{Map}_X(x_0, x_1)$-families can be integrated, i.e. if $\mathrm{Map}_X(x_0, x_1)$ has an invertible associated norm, i.e. if we can call it "C-ambidextrous."

Remark II.3.5. For a general map $q : X \to Y$, we may similarly regard a morphism $\mathrm{Nm}_q : q_! \to q_*$ as a *family* of matrices varying over the base. Its component at $F : X \to \mathsf{C}$ is a morphism $\mathrm{Nm}_F : q_!F \to q_*F$ of Y-local systems, which at y specify to a morphism $\mathrm{colim}_{X_y} F \to \lim_{X_y} F$, which itself specifies an $X_y \times X_y$ matrix of morphisms $\mathrm{Nm}_{F,y}^{x_0,x_1} : F_y(x_0) \to F_y(x_1)$. We again lack a canonical choice of such a morphism, but we have a *relative path space*'s worth of choices. Informally, we should get such a morphism $\mathrm{Nm}_{F,y}^{x_0,x_1}$ for every path in X_y from x_0 to x_1, i.e. we get a *family indexed by the relative path space.*

Remark II.3.6. Each time we implicitly claim that a morphism "amounts to" the components it specifies, we depend upon an amount of naturality between the entries with respect to morphisms between their indices. This naturality exists for different reasons. Let me spitball probably incorrect justificiations:

- The assembly of $\mathrm{Nm}_{F,y}$ from $\mathrm{Nm}_{F,y}^{x_0,x_1}$ uses the universal property of (co)limits and Fubini/base-change properties of integration (which are reflections of certain Beck-Chevalley compatibilities).

- The assembly of Nm_F out of $\mathrm{Nm}_{F,y}$'s uses the functoriality of composition for integration (again, Beck-Chevalley compatibilities) and naturality of Kan extensions.

- The assembly of Nm_q out of Nm_F's uses that all the previous stuff is already functorial.

Definition II.3.7. Let C denote an ∞-category and let $q : X \to Y$ denote a map of spaces.

- For $m \geq -2$, say that q is *weakly* C*-ambidextrous* if q is m-truncated, C admits q-(co)limits, and

  (1) $m = -2$, in which case q is an equivalence and Nm_q is the essentially unique inverse of q^*, or

  (2) $m \geq -1$ and every $(m-1)$-finite relative path space $\mathrm{Map}_{\mathsf{C}^Y}(y_0, y_1)$ is C-ambidextrous. In this case, the norm map $\mathrm{Nm}_q : q_! \to q_*$ is the one assembled as above.

- For $m \geq -2$, say that q is C*-ambidextrous* if it is weakly C-ambidextrous and the associated Nm_q is an equivalence.

Corollary II.3.8. *A map of spaces $q : X \to Y$ is C-ambidextrous if and only if its homotopy fibers are C-ambidextrous [CSY21, Remark 2.1.2]. In particular, the definition of semiadditivity gains no generality by considering the ambidexterity of arbitrary maps $q : X \to Y$ rather than just spaces. Though, we maintain our point that this generality is useful for handling the naturality properties of norms and integration.*

Proposition II.3.9. *An ∞-category C is m-semiadditive $\iff$ every m-finite space is C-ambidextrous.*

Example II.3.10. Suppose that C is zero semiadditive. Then for every discrete space S, the map $q : S \to *$ is ambidextrous, hence every diagram $F : S \to \mathsf{C}$ admits an *invertible* norm and hence a bilimit. Moreover, for every finite group G the map $q : BG \to *$ is weakly ambidextrous, hence every object with G-action $F : BG \to \mathsf{C}$ admits a norm $\mathrm{Nm}_F : F(*)_{hG} \to F(*)^{hG}$ which may or may not be invertible.

II.4 Norms, diagonals, and Beck-Chevalley.

You may object to the amount of wishful thinking in our construction of norms (and thus, our definition of ambidexterity) using matrices. There is a more concise approach that better evidences naturality.

Given a map $q : X \to Y$, we previously imagined $\mathrm{Nm}_q : q_! \to q_*$ as an assemblage of integrals over the relative path spaces of $X \to Y$. This was useful to motivate the inductive definition of ambidexterity and its relationship to integration, but depended upon a deluded amount of hope. This was evident in our attempt to justify naturality—we could not avoid mentioning *Beck-Chevalley conditions* but did not explain them.

Let's temporarily zoom out to a simplifying level of generality.

Question II.4.1. *Suppose as given a functor*[6] $C : S^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *and a commutative square in* S

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "q'"] & A \arrow[d, "q"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

such that the functors q^* *and* $(q')^*$ *admit right adjoints* q_* *and* q'_**. What data does this situation present?*

The commutativity of the square induces an invertible 2-morphism $\alpha : g^*q^* \xrightarrow{\sim} q'^*f^*$. Our right adjoints q_* and q'_* are equivalent to their (co)unit data (u^q_*, c^q_*) and $(u^{q'}_*, c^{q'}_*)$, with which we can cook up another 2-morphism:

$$f^*q_*X \xrightarrow{u^{q'}_{f^*q_*X}} q'_*q'^*f^*q_X \xrightarrow{q'_*(\alpha^{-1})_{q_*X}} q'_*g^*q^*q_*X \xrightarrow{q'_*g^*c^q_X} q'_*g^*X.$$

This *wants* to express commutativity of the new square obtained by applying C and considering right adjoints. You should not think of this as expressing meaningful *compositional coherence* for C, rather it expresses a base-change property. Therefore, while you would expect compositional coherence, you definitely should not expect Beck-Chevalley conditions to hold in general!

Example II.4.2. The pseudofunctor $\mathsf{Set}^{/-} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$ fails to satisfy the Beck-Chevalley conditions for most non-pullback squares in Set. For an explicit example, see VIII.2.14 in my 2025 notebook. (I think this is an if and only if statement.)

Definition II.4.3. Given a commutative square $\square$ in S and a functor $C : S^{\mathrm{op}} \to \mathsf{Cat}_\infty$ as above, the *(right) Beck-Chevalley transform* associated to $\square$ is the 2-cell just defined componentwise:

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "q'"] & A \arrow[d, "q"] \\ B' \arrow[r, "f"] & B \end{tikzcd} \quad \rightsquigarrow \quad BC_* : f^*q_* \to q'_*g^*.$$

Definition II.4.4. Given C, suppose that the image q^* of every morphism $q : X \to Y$ in S admits a left adjoint (thus we might say C is *fiberwise complete*), thus a BC_* transformation. We say that C *satisfies the right Beck-Chevalley condition* if every BC_* transformation associated to a pullback square is invertible.

Definition II.4.5. Analogously, there are *left Beck-Chevalley transformations* $BC_! : f^*q_! \to q'_!g^*$.

Question II.4.6. *What did that have to do with norms?*

Proposition II.4.7 (Height 3.1.2, HL 4.3.3)**.** *Suppose as given the following pullback diagram of spaces.*

$$\begin{tikzcd} A' \arrow[r, "f"] \arrow[d, "q'"] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "q"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

As soon as C *admits* q*-limits (resp.* q*-colimits), the local systems functor* $\mathrm{Map}(-, \mathsf{C}) : \mathsf{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *satisfies the right (resp. left) Beck-Chevalley condition for this square.*

[6] If you wanted to demo this situation in ordinary category land, you would consider a *pseudofunctor* C.

For a map $q : X \to Y$ of m-finite spaces, consider the following diagram.

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta_q"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "q"] \\ & X \arrow[r, "q"] & Y \end{tikzcd}$$

Observe that the homotopy fiber of the relative diagonal $\Delta_q : X \to X_{\times Y} X$ over $(x_0, x_1) \in X \times_Y X$ is equivalent to the space of paths identifying x_0 and x_1 within $X_{q(x_0)} = X_{q(x_1)}$. These were precisely the spaces we needed to be C-ambidextrous in order to construct Nm_q and hence define what it means for q to be *weakly* C*-ambidextrous*.

Remark II.4.8. We previously remarked that (weak) ambidexterity can be checked fiberwise. Hence, we could immediately make a (re)definition: the map $q : X \to Y$ is weakly ambidextrous if and only if Δ_q is ambidextrous, for the latter amounts to the ambidexterity of the very spaces necessary to construct Nm_q. However, this fiberwise criterion secretly relies on the Beck-Chevalley properties we are currently trying to explain.

Definition II.4.9. Let $q : X \to Y$ denote a map of spaces and suppose that C admits q-limits and q-colimits. Define the morphism

$$T : \mathrm{Map}(\mathrm{id}_{\mathsf{C}^Y}, (\Delta_q)_!(\Delta_q)^*) \to \mathrm{Map}(q_!, q_*)$$

which associates to $f : \mathrm{id}_{\mathsf{C}^Y} \to (\Delta_q)_!(\Delta_q)^*$ the composite

$$q^* q_! \xrightarrow{BC_!^{-1}} (\pi_2)_!(\pi_1)^* \xrightarrow{(\pi_2)_! f_{\pi_1^*}} (\pi_2)_!(\Delta_q)_!(\Delta_q)^*(\pi_1)^* \xrightarrow{\sim} \mathrm{id}_{\mathsf{C}^X}.$$

Definition II.4.10 (Compare Defn. 3.1.5 of [CSY22])**.** Consider a map of spaces $q : X \to Y$, an ∞-category C, and an integer $m \geq -2$.

- Say that q is *weakly* C*-ambidextrous* if C admits q-(co)limits and...
  - $m = -2$, in which case q^* is an equivalence, the essentially unique inverse is a left and right adjoint, and we define Nm_q as the identity on that inverse; or
  - $m \geq -1$ and the relative diagonal Δ_q is C-ambidextrous, in which case $\mathrm{Nm}_{\Delta_q} : (\Delta_q)_! \xrightarrow{\sim} (\Delta_q)_*$ is invertible and we define $\mathrm{Nm}_q : q_! \to q_*$ as the construction T applied to the mate of Nm_q^{-1}.
- Say that q is C*-ambidextrous* if it is weakly C-ambidextrous and the associated Nm_q is invertible.

Proposition II.4.11. *Suppose that $q : X \to Y$ is a map of m-finite spaces and that q is weakly* C*-ambidextrous. Then Nm_q detects q-bilimits.*

Remark II.4.12. To define weak ambidexterity, it was important for the norm Nm_{Δ_q} to be invertible, so that mating would produce a morphism $\mathrm{id} \to q_! q^*$ rather than to $q^* q_!$.

Proposition II.4.13. *This definition is equivalent to our previous one.*

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