Thursday Notes on Semiadditivity

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I (2/12) Matthew Niemiro: Trivial Notions

Abstract: The greatest stories do not begin with ∞-category theory, and neither will ours—the germs of semiadditivity subsist willingly upon *ordinary* categories, ready for swab and cultivation. We will start our seminar by documenting some basic semiadditive phenomena (in particular: bilimits, enrichments, norms, and ambidexterity for negative one and zero semiadditivity) as it is expressible in ordinary language. Then we will anticipate higher extensions of the theory. Then we will create a *hostage situation* to canvas speakers.

I.1 Preamble to the seminar.

The theory of *(higher) semiadditivity* is firmly (higher) algebraic. Recent developments in chromatic and equivariant homotopy theory, trace methods, TQFTs, algebraic geometry, mirror symmetry, ... have motivated its closer study. Many folks seem familiar with some applications of semiadditivity, are interested in its other instantiations, and are not familiar with its machinations. Especially more recent work using semiadditivity requires a passing comprehension of the "machincations" to read, and this was one motivation to organize this seminar. That in mind, this seminar will take a category-first approach.

I.2 (Semi)additivity, biproducts, and commutative monoids.

Here is something you know.

Definition I.2.1. An ordinary category C is called *additive* under the following conditions.

(1) For each pair of objects $A, B \in \mathsf{C}$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is assigned an abelian group structure.

(2) Supposing (1), the composition of morphisms is bilinear with respect to the abelian group structure assigned to hom-sets.

(3) C admits finite biproducts (including the empty one, hence a zero object).

This is *structure*. Conditions (1) and (2) amount to an *enrichment over abelian groups*. However, some pedantic category theory dissolves the amount of choice involved. Note that such an enrichment implies that any finite product is canonically a coproduct if it exists, and vice-versa. In fact, it implies something slightly stronger: any finite (co)product is a *biproduct* $A \oplus B$ in a canonical way.[1] It likewise implies that if an initial object exists, then it is also terminal and hence a zero object. Moreover, given (1), (2), and the existence of $B \times B$ (hence the biproduct $B \oplus B$), the addition on $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is uniquely determined[2] as

$$f + g : A \xrightarrow{\Delta} A \times A \xrightarrow{f \times g} B \times B \cong B \sqcup B \to B.$$

The biproduct seems to have an important role. In fact, as soon as finite biproducts exist, you can turn most of this around. If C admits finite biproducts (including a zero object), the induced operation $(f, g) \mapsto f + g$ equips C with an enrichment over commutative monoids, and this is the unique CMon-enrichment that C may possess.[3] Therefore, the definition of an additive category can be redisplayed as follows.

Definition I.2.2. An ordinary category C is called *additive* under the following conditions.

(1) C is pointed.

(2) C admits finite products and finite coproducts.

(3) Supposing (1) and (2), for each pair of objects $A, B \in \mathsf{C}$, the *identity matrix* morphism

$$\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix} : A \times B \to A \sqcup B$$

[1] The distinction is that biproducts are unique up to unique isomorphism while coincidental (co)products need not be, among other things. A good rabbit hole is [Yua12].

[2] This is forced by bilinearity.

[3] Two discussions of this: my older notes or [CDW24, §2].

is an isomorphism. (Thus, C admits finite biproducts.)

(4) For each pair of objects $A, B \in \mathsf{C}$, the canonical commutative monoid operation $(f, g) \mapsto f + g$ defined above on $\mathrm{Hom}_{\mathsf{C}}(A, B)$ admits negatives.

It is now clear that additivity is a *property*. (This "structure that is actually a property" thing will be a recurring theme.) Moreover, it is clear that we should inspect a weaker underlying property.

Definition I.2.3. An ordinary category possessing properties (1)-(3) is called *0-semiadditive*.

Remark I.2.4 (Prescient remark). So there's this close relationship between the *property* of semiadditivity and the *structure* of a CMon-enrichment. In particular, semiadditivity induces a canonical enrichment. The converse is false: a CMon-enrichment forces finite (co)products to be biproducts, but it does not guarantee existence. The notions coincide in the category of *pointed, finitely cocomplete* categories, however. (Or pointed, finitely complete.) We sometimes call a pointed category *(-1)-semiadditive*. We point out that pointedness can be defined just like 0-semiadditivity except with finite sets replaced by just *empty* sets, and that (-1)-semiadditivity has an analogous relationship with the "structure" of Set_*-enrichments. These are our first signs of the hierarchy of higher semiadditivity. Sam will expound upon this enrichment-property game.

(Tea and cookies break?)

I.3 Biproducts and norms.

Ordinary semiadditivity concerns the existence of finite biproducts. We can use *norms* to systematize their existence. We separate the case of empty and nonempty finite biproducts.

Empty Case: A terminal (resp. initial) object in C is the same thing as as limit (resp. colimit) of the empty functor $\varnothing \to \mathsf{C}$. If C admits both an initial and terminal object, denoted $\varnothing$ and $*$, then there is a unique morphism

$$\mathrm{Nm} : \varnothing \to *.$$

In this case, the invertibility of Nm is the unique condition for $\varnothing$ and $*$ to coincide. This would occur as soon as there is *any* morphism $* \to \varnothing$, which would necessarily be the inverse Nm^{-1}.

Non-empty case: Consider a finite family of objects $F : S \to \mathsf{C}$. We may ask for a comparison map

$$\mathrm{Nm}_F : \operatorname{colim} F \to \lim F.$$

This amounts to a compatible family of morphisms $\mathrm{Nm}_F^{s_i,s_j} : F(s_i) \to F(s_j)$ indexed by ordered pairs $s_i, s_j \in S$, i.e. a compatible family of choices $\mathrm{Nm}_F^{s_i,s_j} \in \mathrm{Hom}_{\mathsf{C}}(F(s_i), F(s_j))$, i.e. a matrix

$$\begin{array}{c|cccccc}
 & s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \\
\hline
s_1 & \mathrm{id}_{F(s_1)} & ? & ? & \cdots & ? & ? \\
s_2 & ? & \mathrm{id}_{F(s_2)} & ? & \cdots & ? & ? \\
s_3 & ? & ? & \mathrm{id}_{F(s_3)} & \cdots & ? & ? \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_{n-1} & ? & ? & ? & \cdots & \mathrm{id}_{F(s_{n-1})} & ? \\
s_n & ? & ? & ? & \cdots & ? & \mathrm{id}_{F(s_n)}
\end{array}$$

Generally, there is *no* canonical Nm_F map because there are not generally distinguished (compatible) elements of the mapping sets $\mathrm{Hom}_{\mathsf{C}}(F(s_i), F(s_j))$. In other words, you do not know how to fill the ? in the matrix.

Proposition I.3.1. *The choice of a compatible family of morphisms $\varphi_{c_1,c_2} \in \mathrm{Hom}_{\mathsf{C}}(c_1, c_2)$ for each ordered pair $c_1, c_2 \in \mathsf{C}$ is equivalent to a Set_*-enrichment. Moreover, this is equivalent to "having zero morphisms." In particular, this structure is unique if it exists.*

Thus, our problem is resolved if C is pointed, because this grants zero morphisms. Then we can just fill in the off-diagonal matrix entries with zeros.

Proposition I.3.2. *If C is pointed and admits finite limits and finite colimits, then for each finite collection $F : S \to \mathsf{C}$ there is a canonical "identity matrix" morphism*

$$\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F.$$

Moreover, if Nm_F is an isomorphism, then $\lim F$ is a biproduct.

Corollary I.3.3. *For an ordinary category C, the following are equivalent.*

(1) The category C is zero semiadditive.

(2) For every finite collection $F : S \to \mathsf{C}$, the (co)limit of F exists and is a biproduct.

(3) The category C is pointed, and for every finite collection $F : S \to \mathsf{C}$, the canonical morphism $\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F$ is invertible.

We made the prior discussion to sensibly state (3). But why did we bother if we can just ask for (2)? First, let me remind you that pointedness is (-1)-semiadditivity. Next, let me pose some questions. Rather than $F : S \to \mathsf{C}$, consider a functor $F : X \to \mathsf{C}$ where now X is an ∞-groupoid and C is an ∞-category.

(1) What conditions guarantee that $\lim F$ or $\mathrm{colim}\, F$ exist?

(2) If they exist, what is the *best* way the limit and colimit may coincide? The distinction between biproducts and coincidental (co)products in an ordinary category is already substantial, so this should not be overlooked.

(3) What's all this good for?

The basic selling point of *higher semiadditivity* is its unreasonably effective, systematic, and broadly applicable answers to these questions. Our discussion of *ambidexterity and norms* will generalize the above analysis of 0- and (-1)-semiadditivity to handle these questions and define *higher semiadditivity*. Our discussion of *higher commutative monoids* will generalize the relation between zero semiadditivity and abelian monoids to characterize higher semiadditivity by its relation to $\mathbb{E}_\infty$ (and stronger!) structures.

I.4 Ambidexterity and biadjoints.

For a category C, semiadditivity is a property concerning every finite family $F : S \to \mathsf{C}$. Here, both F and S are variable. *Ambidexterity* conveniently isolates that part of semiadditivity concerning only S. For this, consider the map of sets

$$q : S \to *.$$

This induces a *pullback functor* $q^* : \mathsf{C} \to \mathsf{C}^S$.

Proposition I.4.1. *If C admits S-indexed (co)products, then the functors*

$$q_* := \lim_S(-) : \mathsf{C}^S \to \mathsf{C}$$
$$q_! := \operatorname*{colim}_S(-) : \mathsf{C}^S \to \mathsf{C}$$

are right and left adjoints to q^* respectively. In particular, they are uniquely determined as adjoints.

Proof. The interesting counit consists of diagonals $u_{*,X} : X \to \prod_S X$ and the interesting counit consists of collapse maps $c_{!,X} : \sqcup_S X \to X$. You can now check the triangle identities. $\square$ typos fix later

Corollary I.4.2. *For a set S, the category C admits S-indexed products iff q^* admits a right adjoint q_*, and it admits S-indexed coproducts iff q^* admits a left adjoint $q_!$.*

This gives us a "relative with respect to $S \to *$" way to talk about the existence of S-products and S-coproducts. We want to similarly understand S-bilimits.

Let's take a moment to dwell on what we are asking. We have an adjoint triple

$$q_! \dashv q^* \dashv q_*$$

and we just identified q_* and $q_!$ as computing S-indexed (co)products. Now, we want biproducts. We know that a biproduct is not just a coincidental product and coproduct, rather it needs compatible (co)product data. Similarly, we should not just ask for an equivalence of functors $q_* \cong q_!$, as this does not marry their roles as adjoints. This makes sense: if q_* (or $q_!$) is to properly serve as a simultaneous left and right adjoint, its four (co)units should be compatible. There are a few ways to go about this:

This is not implied by a functor isomorphism $q_* \cong q_!$, right?

(1) Explicit compatibilities: Suppose that C admits S-products and S-coproducts so that the adjoint triple in question exists. Then we may form

$$q^* q_* \xrightarrow{c_*} \mathrm{id}_{\mathsf{C}^S} \xrightarrow{u_!} q^* q_!.$$

Morally, if we want a good simultaneously left and right adjoint q_*, then this should compose to the identity and you should interpret this as a triangle identity. Technically, let's unwind. Note that given $F : S \to \mathsf{C}$, the functor $q^* q_!$ acts by $F \mapsto \coprod_S s_i$ and $q^* q_*$ acts by $F \mapsto \prod_S s_i$. The s_k-th component of this transformation at F is the map $i_k \circ \pi_k : \prod s_i \to \coprod s_i$. These are *precisely* the diagonal structure maps of a biproduct, which must assemble to an identity matrix to form a biproduct. In other words, we should call our adjunction *ambidextrous* if $u_! \circ c_* \cong \mathrm{id}$.

Improve this claim

I'm suspect of some of what I've written here

(2) Norms: To study biproducts, the appeal of norms is that one can bypass the fuss of compatibilities if you could form the identity matrix $\mathrm{Nm}_F : \mathrm{colim}\, F \to \lim F$, because its invertibility guaranteed your (co)product was a biproduct. For Nm_F to exist, we used the zero morphisms granted by pointedness. In our relative context, we may hope for a transformation

$$\mathrm{Nm}_q : q_! \to q_*$$

serving the same convenience: if we can just construct Nm_q, then its invertibility would be equivalent to the ambidexterity of our adjoint triple, i.e. the "correct" way to ask for $q_! \cong q_*$. The pause is that Nm_q may not exist: it consists of maps $\mathrm{colim}_S(-) \to \lim_S(-)$ which consist of matrices of maps, so Nm_q is only defined once you have zero morphisms for the off-diagonals, i.e. once you are (-1)-semiadditive.

(3) Wrong-way (co)units: ...

To summarize: the existence of S-(co)products amounted to the existence of adjoints $q_!$ and q_*. Then we needed pointedness to construct the norm map $\mathrm{Nm}_q : q_! \to q_*$ as the identity matrix. In this case, we say that *S is weakly C-ambidextrous*. Then if Nm_q is invertible, then C admits S-biproducts. In this case, we say that *S is C-ambidextrous*.

Proposition I.4.3. *The following are equivalent.*

(1) The set S is C-ambidextrous.

(2) The category C admits all S-indexed biproducts.

(3) The category C is pointed, and the norm $\mathrm{Nm}_q : q_! \to q_*$ associated to $q : S \to *$ is invertible.

We have delivered on our slogan: this is precisely the part of 0-semiadditivity concerning the fixed set S.

Remark I.4.4. You may replace $q : S \to *$ with an arbitrary map of finite sets $q : S \to T$. The resulting theory is as follows. The existence of adjoints is equivalent to the existence of (co)products indexed by the fibers of q; the construction of Nm_q still requires pointedness; and the invertibility of Nm_q is equivalent to the existence of biproducts indexed by the fibers of q.

Remark I.4.5. Ambidexterity, norms, and the "relative" language will become exponentially more useful in the higher setting.

Bibliography

[Yua12] Qiaochu Yuan. *A meditation on semiadditive categories*. https://qchu.wordpress.com/2012/09/14/a-meditation-on-semiadditive-categories/. Blog post on *Annoying Precision*. Sept. 2012.

[CDW24] Merlin Christ, Tobias Dyckerhoff, and Tashi Walde. *Lax Additivity*. arXiv:2402.12251v2 (17 Nov 2025). 2024. arXiv: 2402.12251 [math.CT]. URL: https://arxiv.org/abs/2402.12251.