2025 Notebook

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I January

I.1 (1/10) Chromatic III – In the year 2025

Christmas and New Year's were short and sweet, and I've just flown back to Cambridge. Continuing my program, I would like to write something about the Morava K-theories. For continuity, let's recap the previous entry.

The derived category of abelian groups decomposes into "irreducible building blocks" in a nice way: it has minimal localizing subcategories $\mathcal{D}(\mathbb{Z})_{\mathbb{Q}}$ and $\mathcal{D}(\mathbb{Z})_p$ for p prime, and in fact any $M \in \mathcal{D}(\mathbb{Z})$ can be reconstructed from its localization data by a certain pullback. These subcategories are examples of *prime ideals in a monoidal triangulated category* (in fact this list exhausts such subcategories of $\mathcal{D}(\mathbb{Z})$) and the set of them naturally forms a topological space $\mathrm{Spc}(\mathcal{D}(\mathbb{Z}))$. Meanwhile in the category of spectra Sp, the unit S^0 has endomorphism ring $\mathrm{End}(S^0) \cong \mathbb{Z}$ and this begets a natural, continuous map

$$\mathrm{Spc}(\mathrm{Sp}) \to \mathrm{Spec}(\mathbb{Z}) \cong \mathrm{Spc}(\mathcal{D}(\mathbb{Z})).$$

This comparison map is surjective. One can ask about the fibers over each point of $\mathrm{Spec}(\mathbb{Z})$, and the beautiful fact is that each fiber admits a sequential filtration by "height." (A tad more precisely: the fiber consists of a closed point (p), a generic point (0), and a sequence of *intermediary* points which each have closure all the "above" points.)

How can we access this filtration? Let us focus on the finite p-local spectra $\mathrm{Sp}^{\omega}_{(p)}$. It turns out that the global structure of Sp remains interesting upon restriction to $\mathrm{Sp}^{\omega}_{(p)}$, and here we have more proof power. In particular, we can characterize thick subcategories as acyclics for a sequence of awesome spectra $K(p,n)$, and then remove the finiteness assumption. All this and more, coming soon!

(...)

I.2 (1/15) Spectrum of quadratic forms and $\mathbb{S}_{K(1)}$

Recall that $\mathrm{Sp}_{K(n)}$ is ∞-semiadditive, thus every $K(n)$-local spectrum has a unique higher commutative monoid structure. What can be said about that structwure of the unit $\mathbb{S}_{K(n)}$? I have been reading [CY22], whose thesis is that at $p=2$, the p-typically 1-commutative monoid structure of $\mathbb{S}_{K(1)}$ is captured by a "spectrum of symmetric bilinear forms." More precisely, in [CY22] it is exhibited that

$$L_{K(1)}\mathrm{GW}^{(-)}(\mathbb{F}_\ell) \cong \mathbb{S}^{(-)}_{K(1)}$$

as 1-precommutative monoids, i.e. in the functor category $\mathrm{Fun}(\mathrm{Span}(\mathcal{S}^{(p)}_1)^{\mathrm{op}}, \mathrm{Sp}_{K(1)})$. Here, the *Grothendieck-Witt spectrum* $\mathrm{GW}(\mathbb{F}_\ell)$ is the purported "spectrum of symmetric bilinear forms."

Remark I.2.1. To get a whole functor $\mathrm{GW}^{(-)}(\mathbb{F}_\ell)$, we must specify values on BG's and coherences. By construction, at BG this functor obtains a "spectrum of G-equivariant bilinear forms," and the coherences amount to[1] transfers and restrictions.

The spectrum GW is constructed in stages:

(1) Given a symmetric monoidal C, one considers its...

 (i) *Nondegenerate bilinear forms*, understood as the fixed points of the dualizing involution $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})$. (Understanding Φ is nontrivial because $\mathbb{D}$ lands in C^{op} and thus is not directly instantiated by a C_2-action. This is not the case for the next step.)

 (ii) *Symmetric nondegenerate bilinear forms*, understood as the fixed points $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})^{hC_2}$ of the C_2-action that precomposes with the swap map.

[1] To what extent do they literally "amount to" this?

(iii) Maximal subgroupoid of the above.

Altogether, we have a space of quadratic forms $QF(\mathsf{C}) := (\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})^{hC_2})^{\simeq}$.

(2) Next, one imagines that C is p-typically m-semiadditive symmetric monoidal. Details must be checked: since m-semiadditivity is preserved by $(-)^{\mathrm{op}}$, we may speak of the category of p-typically m-semiadditive categories with op-involution $(\mathsf{Cat}_m^{(p)})^{hC_2}$ and restrict Φ to it. This category is m-semiadditive and Φ is monoidal, whence $\Phi(\mathsf{C}^{\mathrm{dbl}}, \mathbb{D})$ is not just an object of $\mathsf{Cat}_\infty^{BC_2}$, in fact it lands in $\mathsf{CMon}_m^{(p)}(\mathsf{Cat}_\infty)^{BC_2}$. Finally, form $QF(\mathsf{C})$ by the same procedure as above. (What's changed is that $QF(\mathsf{C})$ is now a p-typically m-commutative monoid in $\mathcal{S}$.)

(3) There is a group completion functor $(-)^{\mathrm{gp}} : \mathsf{CMon}(\mathcal{S}) \to \mathrm{Sp}$. Applying this "levelwise" (for details c.f. [CY22, §3.2.3]), we compose to a functor

$$GW : \mathsf{CAlg}(\mathsf{Cat}_\infty^{(p),m}) \to \mathsf{CMon}_m^{(p)}(\mathcal{S}) \to \mathrm{PMon}_m^{(p)}(\mathrm{Sp}).$$

The GW functor does not lift to $\mathsf{CMon}_m^{(p)}(\mathrm{Sp})$.[2] However, GW turns out to be lax symmetric monoidal, whence it *does* lift to

$$\mathrm{GW} : \mathsf{CAlg}(\mathsf{Cat}_\infty^{(p),m}) \to \mathsf{CAlg}(\mathrm{PMon}_m^{(p)}(\mathrm{Sp})).$$

Of interest is the following special case (c.f. [CY22, Ex 3.2.13]). For a discete ring R, the category Mod_R is 0-semiadditive; if in addition 2 is invertible in R, then Mod_R is 2-typically 1-semiadditive. We may therefore consider the Grothendieck-Witt theory of R as a 2-typically 1-precommutative monoid (or the quadratic forms $QF(\mathsf{Mod}_R)$ as a 2-typically 1-commutative monoid)

$$\mathrm{GW}^{(-)}(R) : \mathrm{Span}(\mathcal{S}_1^{(2)})^{\mathrm{op}} \to \mathrm{Sp},$$
$$QF^{(-)}(R) : \mathrm{Span}(\mathcal{S}_1^{(2)})^{\mathrm{op}} \to \mathcal{S}.$$

In fact, we can put a CAlg on the target categories, and QF satisfies the Segal condition. (Insert or think about another day: how do we identify with $\mathbb{S}_{K(1)}$ after $K(1)$-localization, and what can we do with that equivalence?)

I.3 (1/20) Chromatic IV – Typically, cooperation is a universal good

(**Abandon hope**: this entry abruptly ends, as I outsourced it to Babytop. Will return later.)

I want to say something about formal groups, Hopf algebroids, and complex cobordism. And more importantly, I want to look inward: what can we say about *ourselves* upon candid, surrendered introspection? Vulnerable reflection is a valuable personal exercise; consider for example the complex cobordism spectrum MU, who opens up upon localization at a fixed prime. Therein we discover remarkable *p-typical* phenomena that power the chromatic machine. To this end, I want to acknowledge the p-typical story and study the height filtration.

Where to start today? As usual, I try to begin with things I know. That would be (commutative, one-dimensional) *formal group laws*. Recall that the *Lazard ring* L is the quotient of $\mathbb{Z}[a_{ij}]$ by the relations necessary for the formal sum $F(x,y) = x + y + \sum a_{ij}x^iy^j$ to be a group law, and by pushing F forward, the Lazard ring L corepresents $R \mapsto \mathrm{FGL}(R)$. Moreover, one may consider L as a graded ring with $|a_{ij}| = 2(i+j)$, and there exists a graded isomorphism $L \cong \mathbb{Z}[x_1, x_2, \dots]$ classifying a group law over $\mathbb{Z}[x_1, \dots]$ which admits an explicit description.

One step to the left, there is a theory of *complex-oriented ring spectra*. One first defines these as ring spectra admitting a theory of Chern classes, with one property relaxed: for such a spectrum E, the first

[2] In order to group complete levelwise, we must first include CMon into PMon, wherein this is possible. This does not rule out the existence of a lift, however the wording of [CY22] suggests they have a counterexample demonstrating nonexistence.

E-Chern class of the tensor product of line bundles is computed as a *formal group law* evaluated on the constituents:

$$c_1^E(\eta \otimes \zeta) = F^E(c_1^E(\eta), c_1^E(\zeta)).$$

(Crucial here is that, somehow, the map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is very cool after hitting it with E^*.) In this way, formal group laws are associated to complex-oriented spectra. The group law lives over E^*. Now, the spectrum MU is tautologically complex-oriented, whence we get a group law F^{MU} over $MU^* \cong \mathbb{Z}[x_1, x_2, \dots]$. Such a thing is classified by a map $L \to MU^*$ and Quillen proved that this is an isomorphism.

We can push this a little further and show that MU is the *universal complex oriented spectrum*. To state this precisely, recall that a *complex orientation* of a commutative ring spectrum E is a chosen class $x^E \in E^*(\mathbb{C}P^\infty)$ satisfying some properties. Something something something [Lur, Lecture 6], [Rog23, Lemma 9.5.2].

finish

Proposition I.3.1.

That concludes the utmost basics. There is ground yet to cover:

(1) A bit more formal group theory—morphisms, strictness, the moduli of formal groups, height.

(2) The stacky perspective and how it informs chromatic—cooperations, Hopf algebras and algebroids, and how to think about the height filtration.

(3) The p-typical story.

In practice, the enumerated topics are intertwined, but it is emotionally easier to sort and separate them. Maybe some goals are to understand the statement "MU_*MU is the universal spectrum with two formal group laws and an isomorphism between them," or "spectra are quasicoherent sheaves on M_{fg}," or "the height filtration on M_{fg} reflects the chromatic filtration." Let's see what exposition I can make here.

Here, maybe we can figure out why cooperations are important, motivate Hopf algebroids, and state the universal property of MU_*MU. This is all maybe most coherently stated in the language of formal groups, but this is sort of *the point*, so if we are to imagine ourselves foreign language students, it's maybe good we do not forego the work of translation.

Cooperations appear because of the Adams spectral sequence. This deserves its own entry, but for now let's be brief. Recall that cohomology operations, or generally transformations $E^*(X) \to D^*(X)$ are induced by homotopy classes of maps $f : E \to D$. The Steenrod squares arise from maps $Sq^i : H\mathbb{F}_p \to \Sigma^i H\mathbb{F}_p$, and these generate the *mod p Steenrod algebra* $\mathcal{A}$ of endomorphisms of $H\mathbb{F}_p$. This is a graded non-commutative algebra with a left action on $H^*(X; \mathbb{F}_p)$, natural in X. One notes that

$$\mathcal{A} \cong H^*(H\mathbb{F}_p; \mathbb{F}_p) \cong [H\mathbb{F}_p, H\mathbb{F}_p]_{-*}.$$

Dually, $H_*(X; \mathbb{F}_p)$ is naturally a left $\mathcal{A}^*$-comodule, and we have $\mathcal{A}^* \cong H_*(H\mathbb{F}_p; \mathbb{F}_p) \cong \pi_{-*}(H\mathbb{F}_p \otimes H\mathbb{F}_p)$. For various reasons, e.g. because $\mathcal{A}^*$ is commutative, it is easier to work with $\mathcal{A}^*$-comodules. For one reason or another, from this structure we are led to the mod p Adams spectral sequence.[3] That's why we care: this helps us compute homotopy groups.

Future entry: substantiate this claim.

(Abrupt ending: I outsourced this project to Babytop 2025.)

I.4 (1/30) Categorical Descent III — Recap

Given objects X and Y, we may be interested in geometric structures $G(X)$ and $G(Y)$ associated to them. Perhaps there is a comparison $G(X) \to G(Y)$, and especially if this map somehow simplifies $G(X)$, we should

[3]Future entry: elaborate.

like to know what data was lost, and whether we can study this data and recover $G(X)$ from it. This is *descent*, which tries to formally equate

$$G(X) = G(Y) + \text{descent data}.$$

The global sections for a sheaf of sets provide a prototypical example: a global section s is the same thing as local sections s_i which agree on overlap. But note there is no descent *data*, as agreement on overlaps is a *property* of local sections. We will come to understand this as a lack of (a choice of) *higher coherences* in a categorical problem.

Before we get on with it, here's an example of descent where we need one more level of coherence: consider a presheaf of *categories* on a space. In fact, let's just consider the whole functor

Fix this discussion. Why did I think this functor was a worthwhile subject? The point is totally obscured.

$$bun : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Cat}$$

given by $X \mapsto \mathsf{Top}_{/X}$. We ask if this is a sheaf, and then we wonder what that even means. Let's fix a base space B. AA map $Y \to B$ can be recovered as maps $f_i : Y_i \to U_i$ together with identifications $t_{ij} : f_i^{-1}U_{ij} \xrightarrow{\sim} f_j^{-1}U_{ji}$ satisfying the 1-cocycle condition $t_{jk}t_{ij} = t_{ik}$. I think this amounts to a pullback condition—more precisely, we have an equivalence

$$bun(B) \xrightarrow{\sim} \lim \left(\prod bun(U_i) \rightrightarrows \prod bun(U_i \times_B U_j) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \prod bun(U_i \times_B U_j \times U_k) \right).$$

Encoded on the right-hand side are sections $Y_i \to U_i$ and identifications t_{ij}, with the property of agreement on triple overlaps U_{ijk}. I think this limit must be formed 2-categorically for it to contain sufficient information, c.f. Stacks Project Section 003O. The need for higher categorical limits in order to "correctly" express descent is a recurring problem, and is effectively handled by working with ∞-categories. That is the merit of today's writing, in which I hope to explain how descent is expressed ∞-categorically.

Remark I.4.1. Where I have said "I think," I am truly wary. I am also skeptical of this functor *bun*, which is just the slice over functor. I had only in mind the example where the base B is fixed, and also the covering $\{U_i\}$, I am not certain in what precise sense we can or should say *bun* is a "sheaf." Perhaps it is better to consider the functor given by $X \mapsto \mathrm{Hom}(X, B)$. Or perhaps one should consider a covering $\{U_i\}$ and the map $\coprod U_i \to B$, and consider the induced functor $\mathsf{Top}_{/B} \to \mathsf{Top}_{/\coprod U_i}$.

There are two directions to develop the discussion so far: we can introduce *monads* or we can introduce some *simplicial language* (or both). Today we will make the simplicial language.

We want to understand descent as a sheaf condition. For this, we work in the general setting of a *site*. I will assume the ordinary notion as known; the ∞-categorical notion I will not bother defining for now. Given a cover $\mathcal{U} = \{U_i \to X\}_i$ of an object $X \in \mathsf{C}$, we define its *Cech nerve*

$$N_\bullet(\mathcal{U}) : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{C}$$

by $[n] \mapsto \prod U_{i_1 \ldots i_n}$ and (obvious maps; e.g. coface $\delta_i : [n] \to [n-1]$ maps to inclusion into U_J with $i \notin J$). Generalizing the above discussion, we can now take a presheaf $F \in \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$ and ask whether the canonical morphism

$$F(X) \to \lim_{\boldsymbol{\Delta}} F(N_\bullet(\mathcal{U})) \tag{I.4.2}$$

is an isomorphism.[4] If it is, then $F(X)$ determines and is determined by its local sections and gluing data (understood coherently), and we should say that F *satisfies descent for the cover* $\mathcal{U}$. And as descent is the sheaf condition, if F satisfies descent for every cover in the site's topology t, we say *F is a t-sheaf*.

[4]If F does not preserve finite products, then the right-hand side should be interpreted as the limit of the complex with vertices $\prod F(U_I)$, so that the canonical morphism exists.

Example I.4.3. Suppose that the presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is valued in an n-category.[5] I think that $\mathbf{\Delta}^{\leq n} \hookrightarrow \mathbf{\Delta}$ is n-*final* in the sense that restriction does not change m-categorical limits for $m \geq n$. Take $n = 1$ and $\mathsf{D} = \mathsf{Set}$ or $n = 2$ and $\mathsf{D} = \mathsf{Cat}$, then you recover the sheaf condition for presheaves of sets or categories (or groupoids) discussed above.

Example I.4.4. I think I will consider the case of rings in a separate writing session.

[5] If $n = 1$, this means that D is the nerve of an ordinary category. If $n = 2$, perhaps this means the Duskin nerve? And if $n > 2$, then I only mean this philosophically.

II February

II.1 (2/02) Categorical Descent IV — Faithfully flat descent and the simplicial language

I separately planned to keep with my project of thinking about forms of *descent*, but recently there's been some intersection with my project to learn chromatic. That in mind, I want to present a classical and important example of descent, that being *faithfully flat descent for modules.* There's a homotopical version of this that is sort of important in chromatic land. Let's see what fun we can distill from the ordinary case.

Consider a map of commutative rings $f : R \to S$. The basic question is: given an S-module M, can we find an R-module M^0 and an isomorphism $S \otimes_R M^0 \cong M$? We are asking about filling in the following diagram.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, dashed] & S \arrow[d] \\ M^0 \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & M \end{tikzcd}$$

The data of such a filling, which we should think of as *a descent datum*, is a pair $(M^0 \in \mathsf{Mod}_R, \phi : f^*M^0 \xrightarrow{\sim} M)$. We should like to organize these data into a category.

Here's a fact: given such an (M^0, ϕ), there is a canonical $(S \otimes_R S)$-module isomorphism

$$\tau : S \otimes_R M \cong M \otimes_R S.$$

(This can be deduced a few different ways. It is rather tricky, and I am not sure if there's a canonical reference. I found Jacob Tsimerman's lectures very helpful. Comment from the future: this is Stacks [Sta25, Section 023F].) Moreover, τ satisfies a 1-*cocycle condition* in the sense that the following diagram commutes. Read the Stacks entry.

$$\begin{tikzcd} & S \otimes_R M^0 \otimes_R S \arrow[dr, "\tau\otimes\mathrm{id}"] & \\ S \otimes_R S \otimes_R M^0 \arrow[ur, "\mathrm{id}\otimes\tau"] \arrow[rr, "\tau'"'] & & M^0 \otimes_R S \otimes_R S \end{tikzcd}$$

We define a *descent datum* as an R-module M^0 together with an isomorphism $S \otimes_R M^0 \xrightarrow{\sim} M^0 \otimes_R S$ satisfying the 1-cocycle condition. These data can be organized into a category $\mathrm{Desc}(f)$. There is an obvious functor $f^* : \mathsf{Mod}_R \to \mathrm{Desc}(f)$, and there is an obvious question: when is f^* an equivalence?

Theorem II.1.1. *If $f : R \to S$ is faithfully flat, then $f^* : \mathsf{Mod}_R \to \mathrm{Desc}(f)$ is an equivalence.*

Proof. See Tsimerman's notes. See also Section 03O6. □

Taking Spec, we can translate the above discussion into one about affine schemes: "if $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is faithfully flat and $Z \to \mathrm{Spec}(S)$ is an affine scheme over $\mathrm{Spec}(S)$, then..." We would like to realize this as an equivalence of categories, and for that, we will give another description of $\mathrm{Desc}(f)$. Observe that if f is faithfully flat, descent begets an equalizer diagram Prove this by hand.

$$M^0 \xrightarrow{\sim} \lim(M^0 \otimes_R S \rightrightarrows M^0 \otimes_R S \otimes_R S).$$

More generally, to $R \to S$ is associated a cosimplicial R-algebra $(S/R)_\bullet := R^{\otimes \bullet + 1}$, and to an R-module $R \to M^0$ is associated a cosimplicial $(S/R)_\bullet$-module $(S/R)_\bullet \otimes M^0$. I think it is important here that we are looking at not just a cosimplicial R-module, but a cosimplicial $(S/R)_\bullet$-module (why? I think this is where homotopy theory can enter the picture...)

Example II.1.2 (c.f. Section 023F)**.** One notices that $M^0 \otimes (S/R)_\bullet$ extends the equalizer diagram above. The limit over the full diagram is usually called the *totalization* of $M^0 \otimes (S/R)_\bullet$. Higher descent asks whether the totalization recovers M^0. On the one hand, this is a harder ask, since the totalization diagram is more complicated; on the other hand, since M^0 is "just" a module (as opposed to a cosimplicial one), this might degenerate to the original situation. Well, from a descent datum (M^0, τ) for $f : R \to S$ we can functorially build a degenerate cosimplicial $(S/R)_\bullet$-module $M^0_\bullet$. (How?) This describes a functor

$$\mathrm{Desc}(f) \to \mathrm{Fun}(\Delta, \mathsf{Mod}_{(S/R)_\bullet}).$$

Under this construction, the *canonical* descent datum $(M^0 \otimes_R S, \tau)$ yields $M^0 \otimes (S/R)_\bullet$. Under Dold-Kan, we can turn $M^0_\bullet$ into a cochain complex $s(M^0_\bullet)$. Now we can give an alternate description of descent data, and our intuition on the second hand was correct: if f is faithfully flat, then $\mathsf{Mod}_R \to \mathrm{Desc}(f)$ is an equivalence, with inverse given by $(M^0, \tau) \mapsto H^0(s(M^0_\bullet))$.

In the example, we asked about descent for modules and passed to simplicial junk to factor an inverse-to-be for $\mathsf{Mod}_R \to \mathrm{Desc}(f)$, and concluded that this really did yield an inverse when f is faithfully flat. Now let's do something along these same lines, but starting from $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. We can form a Cech complex

$$Z := (\mathrm{Spec} S/\mathrm{Spec} R)_\bullet := (\mathrm{Spec}(S) \to \mathrm{Spec}(S) \times_{\mathrm{Spec} R} \mathrm{Spec}(S) \rightrightarrows \dots).$$

In the category $\mathrm{Sch}^{\Delta^{\mathrm{op}}}$, there is an obvious factorization $f : \mathrm{Spec}(S) \to Z \to \mathrm{Spec}(R)$. Let $c : Z \to \mathrm{Spec}(R)$ be the latter map. This induces a pullback of quasicoherent sheaves on simplicial schemes $c^* : \mathrm{QCoh}(\mathrm{Spec} R) \to \mathrm{QCoh}(Z)$.

Remark II.1.3. Hold on, wait, what's a (quasicoherent) sheaf on a simplicial scheme $X_\bullet$? We first define a site X_{Zar} whose objects are the opens of the X_n and whose morphisms/covers are the obvious ones (c.f. [Sta25, Section 09VK]). A sheaf on X_{Zar} is equivalent data to a system of sheaves on X_n with compatible cosimplicial maps $X_m \to X_n$ for each $[m] \to [n]$. In particular, we can define the *structure sheaf* $\mathcal{O}_{X_\bullet}$ as that which specifies to $\mathcal{O}_{X_n}$ on every X_n. It is a sheaf of rings on the site X_{Zar}, so we can define its module sheaves and their children (quasicoherent, finite presentation, coherent, etcetera). But here's a hiccup: *the general definition of (say) quasicoherent sheaves does not precisely reproduce "a system of quasicoherent sheaves F_n for each X_n!"* Rather, $\mathrm{QCoh}(X_{Zar})$ is equivalent to *Cartesian* $\mathcal{O}_{X_\bullet}$-modules whose restrictions F_n are quasicoherent $\mathcal{O}_{X_n}$-modules. This is important here, because the "obvious" definition of c^* as pulling back a system of quasicoherent R-modules must be verified to produce cartesian systems. This is also important for another reason, see the final sentence of this section.

Now, I think the maneuver is to interpret $\mathrm{QCoh}(Z)$ as[6] our category of descent data and prove that c^* is an equivalence if f is faithfully flat.

Theorem II.1.4. *If $f : R \to S$ is a faithfully flat map of rings, then*

...

form an equivalence of categories.

I think the interpretation of $\mathrm{QCoh}(Z)$ as descent data is one of the important takeaways here. An important detail is that when you unwind the definition of a quasicoherent sheaf on Z, you get a system of quasicoherent modules $(S^{\otimes n} \to M^0_n)_n$, and this system comprises a descent datum *because it is cartesian* (this is basically the definition of a cartesian sheaf).

Find a reference for this. Check the details.

Remark II.1.5. How to frame this discussion so that the module and scheme discussion can be most concisely compared? In both situations, for instance, we sort of flip-flopped around between ordinary and (co)simplicial stuff...

Remark II.1.6. Recap: notice we did what we intended, that is we reinterpreted *Desc(f)*. The goal was to realize it as a category of quasicoherent sheaves, with the secret example of chromatic in mind. This required forming QCoh of a *simplicial* sheaf, though, but miraculously the abstract definition of such a thing produces a system of modules *with the added data (property?)* of being *cartesian*, which is basically what you need for a descent datum.

[6] Or should one imagine $\mathrm{QCoh}(Z)$ as the category of "higher" descent data? Tyler and I convinced ourselves that it is *not* a category of higher descent data, or rather there is no such thing for f, because f is only a map of ordinary rings/affine schemes.

That was a longer entry, and I learned a good deal writing it. Let's reflect.

- I started by defining a descent datum for rings/modules. This is rather unintuitive, only explained by the geometric/Spec story. We already hit everything with Spec later, perhaps best to start with affine schemes? But still want to develop the algebra...
- In general, I think the dual algebra/geometry stories inform one another at different times. If I were to reproduce this discussion for exposition, I may want to be more deliberate in my separation and comparison of the two situations.
- I sort of forgot that $\mathrm{QCoh}(Z)$ was supposed to be our category of descent data, so I should have emphasized that more. Also, I could explain better, and maybe make it more explicit, the way in which $\mathrm{QCoh}(Z)$ encodes descent data.
- I need to do more algebra... Let's maybe wrap up with an exercise.

Here's the basic algebra fact that one might take as a starting point to faithfully flat descent.

Proposition II.1.7. *If $f : A \to B$ is a faithfully flat ring map, then the following is an equalizer diagram.*

$$A \xrightarrow[f]{} B \underset{b\otimes 1}{\overset{1\otimes b}{\rightrightarrows}} B \otimes_A B$$

There are a few ways to rephrase the conclusion that are maybe fun to think about: it asks that $A \to B \times_{B\otimes_A B} B$ be an equivalence, or equivalently that $0 \to A \to B \oplus B \to B \otimes B \to 0$ be exact.

Unwind that last statement?

Proof. We will prove a more general statement, by a long-winded approach touching upon some details overlooked in the above discussion. Let M be an A-module. Recall we defined a cosimplicial B-module $(B/A)_\bullet$ and the $(B/A)_\bullet$-module $(B/A)_\bullet \otimes M$. Formally analogous to the case of sheaves on a site, we may call $(B/A)_\bullet$ the *Cech nerve* of $f : A \to B$, and $(B/A)_\bullet \otimes M$ is the cosimplicial object whose limit we hope to recover as M, in which case we might say $- \otimes_A M$ satisfies descent for f.

Dold-Kan gives a map $s : \mathsf{Mod}_B^\Delta \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_B)$ under which $(B/A)_\bullet$ is given maps $(B/A)_n \to (B/A)_{n+1}$ given by $\sum_i (-1)^i \delta^i_{[n]\to[n+1]}$. Now, consider

$$M \xrightarrow{m\mapsto m\otimes 1} M \otimes B \xrightarrow{m\otimes b\mapsto m\otimes b\otimes 1 - m\otimes 1\otimes b} M \otimes B \otimes B.$$

We will prove that this is exact. In fact, we can prove something even stronger: we recognize this as part of

$$Z := s(M \otimes (B/A)_\bullet) = 0 \to M \to M \otimes B \to M \otimes B \otimes B \to \dots$$

And we will show this cochain complex is exact. Since f is faithfully flat, this is equivalent to the exactness of $Z \otimes B$. Now here's a trick: $Z \otimes B$ is equivalent to $s((M \otimes B) \otimes (B \otimes_A B/B)_\bullet)$! That is to say, there is a natural equivalence

$$s(M \otimes (B/A)_\bullet) \otimes B \cong s((M \otimes B) \otimes (B \otimes B/B)_\bullet).$$

Sounds like a mouthful, and I do not have a reference, but we have reduced to checking the latter is exact, and the latter is the cochain complex obtained from $\Delta_f : B \to B \otimes_A B$ which has a section $b \otimes b' \to bb'$. The point is that we can assume f has a section, and this section allows one to witness every cocycle as a coboundary explicitly, c.f. [Alp24, §2.1.1]. □

Remark II.1.8. What is a good way to think about the role of the diagonal here?

Remark II.1.9. Writing from the future: what I am trying to say in the proof is basically what [Sta25, Section 023F] is trying to say! Let me grab a coffee and summarize this.

We are considering a homomorphism $f : A \to B$ and its induced $f^* : \mathsf{Mod}_A \to \mathrm{Desc}(f)$. We call a descent datum *effective* if it belongs to the image of f^*, and $f^*(M^0)$ is called the *canonical descent datum* associated to M^0. In an attempt to construct an inverse, we consider the functor

$$\mathrm{Desc}(f) \to \mathsf{Mod}^{\Delta}_{(B/A)_\bullet}.$$

The well-definedness of the general definition $(N, \tau) \mapsto N_\bullet$ requires a little work to verify, for which the cocycle condition is crucial. We know what this functor does to canonical descent data: it sends (M^0, can) to $(B/A)_\bullet \otimes M^0$. We also have a functor

$$s : \mathsf{Mod}^{\Delta}_{(B/A)_\bullet} \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_{(B/A)_\bullet})$$

and again, we can describe its effect on (the image of) canonical descent data (M^0, can): we have

$$s((B/A)_\bullet \otimes M^0) = (B \otimes M \to B \otimes B \otimes M \to \cdots).$$

The kernel of the first map $b \otimes m \mapsto b \otimes 1 \otimes m - 1 \otimes b \otimes m$ is M, so we can extend this cochain complex as $0 \to M \to s(...)$. (I want to say that this is an adjoint to one of the good truncations $\mathrm{CoCh} \to \mathrm{CoCh}_{\geq 0}$.) Now, here's a fact.

Proposition II.1.10. *If $f : A \to B$ admits a section, then $0 \to M \to s((B/A)_\bullet \otimes M)$ is exact.*

Proof. Generally, if a morphism $X \to Y$ in C has a section, then the constant cosimplicial object $X_\bullet$ is equivalent to $(Y/X)_\bullet$ assuming the latter can be formed. In our case, we can tensor with M (since functors preserve cosimplicial equivalences) and get $M \simeq (B/A)_\bullet \otimes M$. Now, Dold-Kan is homotopical, so observe that the constant complex on M is exact and $DK((B/A)_\bullet \otimes M)$ is what we wanted to show the exactness of. □

Now here's the trick. If $g : A \to C$ is faithfully flat and the associated complex for $C \to B \otimes_A C$ is exact, then we can conclude $0 \to M \to (B/A)_\bullet \otimes M$ is exact. (This is a definition chase, c.f. [Sta25, Lemma 023M].) The point is that when f is faithfully flat, we have an obvious candidate $g = f$, in which case we're checking $\Delta_f : B \to B \otimes_A B$. The diagonal has a natural section, so the proposition concludes $0 \to M \to s((B/A)_\bullet \otimes M)$ is exact.

The point is that when f is faithfully flat, we know where *effective* descent data goes under the composition $\mathrm{Desc}(f) \to \mathsf{Mod}^{\Delta}_{(B/A)_\bullet} \to \mathrm{CoCh}_{\geq 0}(\mathsf{Mod}_{(B/A)_\bullet})$. Namely, it goes to the complex $s((B/A)_\bullet \otimes M)$ which has $H^0 \cong M$ and trivial higher cohomology. This turns out to characterize the effective descent data:

Proposition II.1.11. *If $f : A \to B$ is faithfully flat, then a descent datum (N, τ) is effective if and only if the canonical map $B \otimes_A H^0 s(N_\bullet) \to N$ is an equivalence.*

Now one can exhibit the equivalence $\mathrm{Desc}(f) \xrightarrow{\sim} \mathsf{Mod}_R$. (...)

II.2 (2/14) Categorical Descent V — fpqc, fppf

> The art of doing mathematics consists in finding that special case which contains all the germs of generality.
>
> David Hilbert

Recall that a quasicoherent sheaf on a scheme X is an $\mathcal{O}_X$-module which is affine locally the (sheaf associated to the) module of global sections; if X is already affine then $\mathsf{Mod}_{\mathcal{O}_X} \cong \mathsf{Mod}_R$. Just as with modules, we can ask about descent for $\mathrm{QCoh}(X)$. The most efficient expression for this is adapted from a fact for modules: a (faithfully) flat ring map $f : A \to B$ is characterized by the (faithful) exactness of $f^* = (-) \otimes_A B$.

We define a *faithfully flat map of schemes* $f : X \to Y$ as one for which $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is faithfully exact. As an exercise, let's unwind this definition a little. We first return to a *ring* map $f : A \to B$ and assume f is flat. We ask what it means for f to be faithful (i.e. for f^* to be faithful, i.e. for f^* to reflect exact sequences).

Proposition II.2.1. *Let $f : R \to S$ be a flat ring map. Then f is faithful if and only if $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. [Sta25, Tag 00HP]. □

Likewise, we may suppose that $f : X \to Y$ is flat and ask for a geometric description of its faith. (Flatness is defined Zariski-locally.)

Proposition II.2.2. *Let f be a flat morphism of schemes. Then f^* is faithful if and only if f is surjective.*

Remark II.2.3. Todo: in case of modules, check that reflecting exact sequences implies faithfulness. In case of schemes, find a reference...

Now it turns out that for schemes, faithful flatness is a bit too weak for good descent phenomena. Say $f : X \to Y$ is *fpqc* if it is faithfully flat and every qc open $U' \subset Y$ occurs as $f(U)$ for a qc open $U \subset X$. These include fppf and faithfully flat, quasicompact maps. We can extend faithfully flat descent for modules to fpqc morphisms:

Proposition II.2.4. *Suppose that $f : X \to Y$ is an fpqc map of schemes.*

(1) Given $F, G \in \mathrm{QCoh}(Y)$, the following sequence is exact.

$$\mathrm{Hom}_{\mathcal{O}_Y}(F, G) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*F, f^*G) \underset{p_1^*}{\overset{p_2^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{O}_{X\times_Y X}}(q^*F, q^*G)$$

(2) Suppose as given an $H \in \mathrm{QCoh}(X)$ and an isomorphism $\alpha : p_1^*H \xrightarrow{\sim} p_2^*H$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha \cong p_{13}^*\alpha$. Then there exists a unique (M^0, ϕ) where $M^0 \in \mathrm{QCoh}(Y)$ and $\phi : f^*H \xrightarrow{\sim} M^0$ is an isomorphism such that $p_1^*\phi = p_2^*\phi \circ \alpha$.

Proof. This is [Alp24, Prop 2.1.4]. Would be a good idea to come back to this proof. Also maybe §3 in these notes. □

Come back to the proof

Remark II.2.5. It is clear from (2) how to define the category $\mathrm{Desc}(f)$. There is an obvious functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{Desc}(f)$, and conclusions (1) and (2) say that it is fully faithful and essentially surjective, respectively.

In fancier language, the proposition says that QCoh is a stack in the fpqc topology. Later we will make this our standard terminology and make a good theory for stacks; for now, however, more examples. Recall in the topological case my (crude) attempt to explain how the functor $X \mapsto \mathsf{Top}_{/X}$ demonstrates stack-like properties. Well, here's something that looks similar, at least with respect to fpqc morphisms.

Proposition II.2.6. *Let $f : X \to Y$ be an fpqc morphism of schemes. For any morphism $g : X \to Z$ equalized by the projections $p_1, p_2 : X \times_Y X \to Y$, there exists a unique h such that the following diagram commutes.*

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & & Z \end{tikzcd}$$

This translates to $\mathrm{Hom}(-, Z)$ being a stack in the fpqc topology. There is an analogous statement in the relative situation for $\mathrm{Sch}_{/S}$. (Later, we say these are statements about stackiness over the "big" and "small (over S)" fpqc site.)

How to compare to prestack sending scheme Z to morphisms of *algebraic spaces* to Z? This is "really" the stack I was alluding to earlier.

A morphism $f : X \to Y$ is called *fppf* if it is faithfully flat and locally finitely-presented. These morphisms are fpqc, hence the previous descent phenomena apply, however one can say more.

II.3 (2/26) Chromatic V — Formal groups

Two Babytalks have passed. Yesterday, Oakley reviewed "MU-theory," defined formal groups, and suggested the perspective that spectra (via their MU-homology) are quasicoherent sheaves on $\mathcal{M}_{\mathrm{fg}}$. I have neglected my chromatic notebooking in favor of organizing Babytop, first to avoid wasteful overlap, and second, to learn more before writing things down. Now that we have covered some ground (and also *not* covered some ground), I'm more confident I can say something useful and productive. Today I want to think about *formal groups*.

- Piotr's notes [Pst] are a great reference. One goal of the seminar is to get everyone comfortable with the language of these notes (or, say, Lurie's notes, or Goerss's notes).

The natural context for a formal group is the category of *formal schemes*. These are not exactly ind-schemes, although there's a relationship that's yet unclear to me—but in any case, we care about the local geometry of our formal schemes, so a study from the ground up is quite necessary.

Functor-of-points compels you to identify a commutative ring $A \in \mathsf{CRing}$ with its presheaf $\mathrm{Spec}(A) := \mathrm{Hom}_{\mathsf{CRing}}(A, -) \in \mathrm{Fun}(\mathsf{CRing}, \mathsf{Set})$. An ideal $I \subset A$ has an associated I*-adic completion* $\hat{A}_I := \lim(A/I \leftarrow A/I^2 \leftarrow \cdots)$, and dually we define the associated *formal completion* of $X := \mathrm{Spec}(A)$ along the closed subscheme $Z := \mathrm{Spec}(A/I)$ as the colimit

$$\hat{X}_Z(-) = \operatorname{colim} \mathrm{Hom}_{\mathsf{CRing}}(A/I^n, -).$$

In words, $\hat{X}_Z$ is the subfunctor of $\mathrm{Spec}(A)$ which associates to B those morphisms $A \to B$ annihilating some I^n (hence vanishing near zero). A functor $\mathsf{CRing} \to \mathsf{Set}$ of this form is called an *affine formal scheme*.

Remark II.3.1. The filtration $A \supseteq I \supseteq I^2 \supseteq \cdots$ induces an cofiltered/inverse/projective system $A/I \leftarrow A/I^2 \leftarrow \cdots$. The *limit topology* on $\hat{A}_I$ is the initial having every $\phi_n : \hat{A}_I \to A/I^n$ continuous with respect to the discrete topology on A/I^n. It is linearized by the kernels $\ker(\phi_i)$, i.e. it is the associated filtration topology. The canonical $A^{adic} \to \hat{A}_I$ is continuous, in fact it is the initial map to a *separated, Cauchy complete, linearly topologized* ring [Sin11, 8.2.4]. This includes any ring with the discrete topology; moreover observe the (0)-adic topology is discrete, so you make this a statement about Hom-sets of (complete adic rings? But what about I not finitely-generated?)

Definition II.3.2. Consider A with an I-adic topology. A continuous ring map $A^{adic} \to B^{disc}$ is a ring map $A \to B$ such that some I^n is annihilated, whence we can identify

$$\operatorname{colim} \mathrm{Hom}_{\mathsf{CRing}}(A/I^n, B) \cong \mathrm{Hom}^{cts}_{\mathsf{CRing}}(A^{adic}, B^{disc}) \cong \mathrm{Hom}_{\mathsf{CRing}}(\hat{A}_I, B).$$

Thus, the affine formal scheme arising from (A, I) depends only on the topology A^{adic} and not the specific ideal of definition I. The resulting sheaf is often called the *formal spectrum* $\mathrm{Spf}(A)$.

Example II.3.3. The *formal affine line* over R is the formal spectrum $\hat{\mathbb{A}}^1_R := \mathrm{Spf}(R[\![t]\!])$ where $R[\![t]\!]$ carries the t-adic topology. In effect, $\mathrm{Spf}(R)(S)$ consists of pairs $(R \to S, x)$ of R-algebras and a choice of nilpotent element. In particular, $\hat{\mathbb{A}}^1(S) := \mathrm{Spf}(\mathbb{Z}[\![t]\!])(S) = \mathrm{Nil}(S)$.

Definition II.3.4. Let F be a formal group law over R. Its associated *formal group* is the functor

$$G_F : \mathrm{Alg}_R \to \mathsf{Ab} \quad \text{given by} \quad S \mapsto (\mathrm{Nil}(S), x + y := F(x,y)).$$

Thus, the formal group G_F is the lift of $\mathsf{CRing}_{R/} \xrightarrow{\hat{\mathbb{A}}^1_R} \mathsf{Set}$ to Ab defined by designating F as the group operation. We can ask about the set of such lifts, i.e. the set of abelian group structures on $\hat{\mathbb{A}}^1_R$. Before specifying to studying maps $(\hat{\mathbb{A}}^1_R)^2 \to \hat{\mathbb{A}}^1_R$, we may consider maps from an (almost) arbitrary $\mathrm{Spf}(A)$.

Proposition II.3.5. *If an R-algebra A is I-adically complete, then* $\mathrm{Hom}_{/\mathrm{Spec}(R)}(\mathrm{Spf}(A), \hat{\mathbb{A}}^1_R) \cong \mathrm{Nil}^{top}(A)$.

Proof. Acknowledge the monomorphism $\hat{\mathbb{A}}^1_R \hookrightarrow \mathbb{A}^1_R$, and we can now first figure

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spf}(A), \mathbb{A}^1_R) \cong \lim \mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spec}(A/I^n), \mathbb{A}^1_R) \cong \lim \mathrm{Hom}_{\mathrm{Alg}_R}(R[t], A/I^n) \cong A.$$

The last equivalence is due to completeness. Now we ask which $a \in A$ factor through $\hat{\mathbb{A}}^1_R \hookrightarrow \mathbb{A}^1_R$, and by the magic of monomorphisms this just means computing

$$\mathrm{Hom}_{\mathrm{Spec}(R)}(\mathrm{Spf}(A), \hat{\mathbb{A}}^1_R) \cong \lim \mathrm{Hom}(\mathrm{Spec} A/I^n, \hat{\mathbb{A}}^1_R) \cong \lim \mathrm{Hom}^{cts}(R[\![t]\!], A/I^n).$$

I think the RHS just unwinds to $\mathrm{Nil}^{top}(A)$. Wait, could I have just done that to start? □ Unwind and check this...

Corollary II.3.6. *There is a bijection*

$$\mathrm{Hom}_{\acute{e}t,/\mathrm{Spec}(R)}(\hat{\mathbb{A}}^n_R, \hat{\mathbb{A}}^1_R) \cong \{F \in R[\![x_1, \ldots, x_n]\!] : F \textit{ has nilpotent constant term}\}.$$

It follows that abelian monoid structures (automatically grouplike) on $\hat{\mathbb{A}}^1_R$ for which zero is a unit are in bijection with formal group laws over R. This makes the following definition a bit easier to digest.

Definition II.3.7. A *formal group* over $\mathrm{Spec}(R)$ is an abelian group object

$$\mathbf{G} \in \mathrm{Shv}_{\acute{e}t}(\mathsf{CRing}_{R/}, \mathsf{Ab})$$

such that Zariski locally, $\mathbf{G}$ takes the form of a formal group associated to some formal group law.

Remark II.3.8. What a word salad. Let me unwind this. Before all else, $\mathbf{G}$ is an étale sheaf $\mathrm{Alg}_R \to \mathsf{Ab}$. Moreover, there exists a finite list of elements $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and for each restriction $\mathbf{G}|_i : \mathrm{Alg}_{R_{f_i}} \to \mathsf{Ab}$, there exists an isomorphism $\mathbf{G}|_i \cong G_F$ for some F.

Example II.3.9. The most important example: if E is complex-orientable, then $\mathrm{Spf}(E^*\mathbb{C}P^\infty)$ is a formal group over E^*. To explain, we will assume $E^* = \pi_* E$ is even, which avoids some unfathomable problems. By orientability, there exists an isomorphism $E^*\mathbb{C}P^\infty \cong \lim E^*\mathbb{C}P^n$ by which we can give $E^*\mathbb{C}P^\infty$ the limit topology. We may well enough form

$$\mathrm{Alg}_{E^*} \xrightarrow{\mathrm{Spf}(E^*\mathbb{C}P^\infty)} \mathsf{Ab}$$

and try to realize $\mathrm{Spf}(E^*\mathbb{C}P^\infty)$ as some G_F. A choice of orientation $t \in E^2(\mathbb{C}P^1)$ determines an isomorphism $E^*\mathbb{C}P^\infty \cong E^*[\![t]\!]$, and this isomorphism is t-adically continuous. We may write $\mathrm{Spf}(E^*\mathbb{C}P^\infty)(S) = \mathrm{Hom}^{cts}_{E^*}(E^*[\![t]\!], S) = \mathrm{Nil}(S)$. The group structure on this hom-set comes from that on $E^*[\![t]\!]$, hence from $E^*\mathbb{C}P^\infty$, hence from the tensor product of line bundles, which is the formal group law we know and love.

In the next entry, I want to collab with my Categorical Descent musings to form the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups, and explain how the geometry of $\mathcal{M}_{\mathrm{fg}}$ informs our understanding of spectra.

III March

III.1 (3/1) The Picard group of a category

Given a (commutative) monoid M, one may form its *(commutative) group of units* $M^\times$, and there is an obvious functor $M \mapsto M^\times$ which is *right* adjoint to the inclusion $\mathsf{Gp} \hookrightarrow \mathsf{Mon}$. The homotopy-coherent story asks about a right adjoint to the inclusion

$$\mathrm{Sp}_{\geq 0} \hookrightarrow \mathsf{CMon}(\mathcal{S}).$$

III.2 (3/4) Descent & Chromatic VI — A map $\mathcal{M}_{MU} : \mathrm{Sp} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$

III.3 (3/7) Spectral Sequences I

An often sufficient (but not necessary) way to recognize an algebraic topologist is look for an ability to maneuver spectral sequences. I have said before that my formal topology training was not well-formed, and I'm unfamiliar with spectral sequences as a result. Andy is giving some number of lectures introducing spectral sequences, and he's the good stuff, so I will be writing some things down related to what he says.

Here's sort of the basic germ of the way that (Serre) spectral sequences arise in algebraic topology. Consider the Hopf fibration

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2.$$

Fibrations induce long exact sequences in homotopy. What can be said about homology? The answer is more sophisticated than a long exact sequence. Homology is built out of *simplices*, so we wonder how a fibration induces some disfigurement of simplices. Fix a relative homeomorphism $\sigma : (\Delta^2, \partial\Delta^2) \xrightarrow{\sim} (S^2, *)$, which determines a generating cycle $[\sigma] \in H_2(S^2)$.[7] Since $\Delta^2 \cong I^2$, the lifting property gives us a filling

$$\begin{tikzcd} * \arrow[r] \arrow[d] & S^3 \arrow[d, "p"] \\ (\Delta^2, \partial\Delta^2) \arrow[r, "\sigma"'] \arrow[ur, dashed, "\bar{\sigma}"] & S^2 \end{tikzcd}$$

The resulting 2-simplex $\bar{\sigma}$ is *not* a cycle—this would require its boundary be trivial, but the lift has an "error" term: it need only map $\partial\Delta^2$ to a fiber circle S^1, in fact[8] it must wind nontrivially around S^1, in fact it must *generate* $H_1(S^1)$. In this sense, the generator of $H_2(S^2)$ failed to lift to $H_2(S^3)$ because it was obstructed by the nontriviality of its boundary in $H_1(S^1)$. We've learned some things:

(i) We realized the generator of $H_1(S^1)$ as a boundary in S^3, so $H_1(S^1) \to H_1(S^3)$ has trivial image.

(ii) It is impossible to lift the generator of $H_2(S^2)$ to $H_2(S^3)$, so $H_2(S^3) \to H_2(S^2)$ has trivial image.

One deduces that $H_1(S^3)$ and $H_2(S^3)$ are trivial. (How exactly does this follow?) One should view the described map $d : H_2(S^2) \to H_1(S^1)$ as controlling the situation, in particular its nontriviality has consequences. This is our first example of a *differential*.

Remark III.3.1. If we instead consider the trivial bundle $S^1 \times S^2 \to S^2$, one could compute $H_*(S^1 \times S^2)$ via Künneth; this fails for the Hopf fibration, as Künneth predicts "too many" elements of $H_1(S^3)$and $H_2(S^3)$, coming from $H_1(S^1)$ and $H_2(S^2)$, respectively. Above we described a nontrivial map $d : H_2(S^2) \to H_1(S^1)$ induced by the Hopf fibration (an isomorphism); the analogous map induced by $S^1 \times S^2 \to S^2$ is trivial. These facts are not coincidental: the map d is our first example of a *differential*, whose nontriviality is strongly related to Künneth's misprediction.

[7] Maybe you need $[\sigma - *]$?

[8] Why is this true? Does one use the nontrivial fiber bundle structure? How to phrase this using only the fact that p is a fibration?

Above, our answer to "what can we say about homology given a fibration?" was answered in the example of the Hopf fibration. We used the HLP to extract an algebraic map d which told us things. This is a simple example of an algebraic structure which is generally complicated, which we define now. The generic machinery is derived from *filtrations*, and we will specialize back to the case of spaces and fibrations via skeletal filtrations.

We start by considering a chain complex $C_\bullet$ and a *filtration* by subcomplexes

$$F_0 C_\bullet \subseteq F_1 C_\bullet \subseteq \cdots$$

Note that the boundary for $C_\bullet$ moves orthogonal to the filtration, i.e. it has bidegree $(0, -1)$. For example, if $x \in F_s C_5$, then $dx \in F_s C_4$. If we are interested in the homology of $C_\bullet$, then we are interested in dx, and our filtration coordinatizes an approximation of dx. This is the slogan which spectral sequences formalize. The first step in describing this approximation is to form the "leading term" of dx as its quotient $[dx]_s \in F_s C_4 / F_{s-1} C_4$.

If $[dx]_s = 0$, then we conclude $dx \in F_{s-1} C_4$ and repeat for $[dx]_{s-1} \in F_{s-1}/F_{s-2}$. To formalize this, we form the *associated graded* pieces

$$0 \to F_{s-1} C_\bullet \hookrightarrow F_s C_\bullet \twoheadrightarrow \mathrm{gr}_s C_\bullet \to 0 \quad \rightsquigarrow \quad \mathrm{gr}_\bullet C := \bigoplus_i \mathrm{gr}_i C.$$

The associated graded $\mathrm{gr}_\bullet C$ is a graded object in chain complexes. In our example of $x \in F_s C_5$, we sought to understand $dx \in F_s C_4$, and we proposed to interpret $[dx]_s \in \mathrm{gr}_s C_4$ as the "leading term in an approximation of dx determined by the filtration." In the general setup, we will use the homologies of the complexes in $\mathrm{gr}_\bullet C$ to track these approximations.

Definition III.3.2. We form the bigraded abelian group $E^0_{s,t} := \mathrm{gr}_s C_{s+t} = F_s C_{s+t} / F_{s-1} C_{s+t}$. The boundary from C induces a bidegree $(0, -1)$ differential $d^0 \colon E^0_{s,t} \to E^0_{s,t-1}$.

Definition III.3.3. The homology of the bicomplex $(E^0_{\bullet,\bullet}, d^0)$ is a bigraded abelian group which we denote

$$H_{s,t}(E^0_{\bullet,\bullet}) =: E^1_{s,t}.$$

Similar to $\mathrm{gr}_\bullet C$, the $E^1_{s,t}$ form a bigraded abelian group.

IV May

IV.1 (5/6) A construction of ΩS^1

But that's just $\mathbb{Z}$?! Sure, but I have a point to make. Let $M(\mathbb{R})$ denote the space of finite, unordered configurations in $\mathbb{R}$:

$$M(\mathbb{R}) := \coprod_{i\geq 0} \mathrm{UConf}_i(\mathbb{R}).$$

I would like to explain a homotopy equivalence $\Omega BM(\mathbb{R}) \xrightarrow{\sim} \Omega S^1$, in fact we will find an $\mathbb{E}_1$-equivalence. This means identifying $BM(\mathbb{R}) \simeq S^1$. You may recognize that $M \simeq \mathbb{N}$ as monoids (where M has the "juxtaposition" operation) which explains both equivalences $BM(\mathbb{R}) \simeq S^1$ and $\Omega BM(\mathbb{R}) \simeq \mathbb{Z}$. I want to carefully verify this by unwinding the bar construction, with the goal of understanding a more general fact:

Theorem IV.1.1 (The tablecloth trick)**.** *The d-fold bar construction of the $\mathbb{E}_d$-monoid of finite subsets of $\mathbb{R}^d$ is equivalent to S^d:*

$$B^d\mathrm{UConf}_{fin}(\mathbb{R}^d) \simeq S^d.$$

To start, we are going to do some point-set topology to describe the $\mathbb{A}_\infty$-monoid structure on $M(\mathbb{R})$ and to define map $\Omega BM(\mathbb{R}) \to \Omega S^1$. I'm not sure if this point-set story is the geodesic explanation, and I'm not sure I care.

Definition IV.1.2. We denote by $m : M(\mathbb{R})^2 \to M(\mathbb{R})$ the "juxtaposition" operation. Precisely, let $t : \mathbb{R} \to \mathbb{R}_+$ be any orientation-preserving homeomorphism (e.g. $t(x) = e^x$) and let $R(x) = -x$. Then we may define

$$m(\xi, \eta) := RtR(\xi) \cup t(\eta).$$

This operation is coherently associative, i.e. (M, m) admits the structure of an $\mathbb{E}_1$-monoid. However, M is definitely not grouplike: the product induces a monoid structure $\pi_0(M) \cong \mathbb{N}$. We can correct this via the group completion map $M \to \Omega BM$ which on π_0 induces $\mathbb{N} \hookrightarrow \mathbb{Z}$.

Now, three things have to happen:

(1) We want to identify ΩBM. If we pretend to understand Ω, then we want to identify BM. There are two ways to go about this: we may either unwind a coherent bar construction $B : \mathrm{Alg}_{\mathbb{E}_1}(\mathsf{Spaces}_*) \to ?$ or we can strictify M into a topological monoid M' and unwind a "classical" bar construction.

(2) We want to construct a map $M \to \Omega S^1$ that "looks like" the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$. If $M \to \Omega BM$ is to deserve its title as group completion, then some universal property should induce an $\mathbb{E}_1$-equivalence $\Omega BM \xrightarrow{\sim} \Omega S^1$.

(3) We want to understand the group completion property of $M \to \Omega BM$.

Points (1) and (2) are more specific to this situation, while (3) is general theory I do not care too much to write out. Today I want to understand the "classical" approach to (1) that strictifies M, but I am curious to work it out homotopy-coherently... I think it reduces to the same topological argument?

First, let's define the *scanning map*

$$M(\mathbb{R}) \xrightarrow{sc} \Omega S^1.$$

It's easiest to define sc in words: at the point $\xi \in C_k(\mathbb{R})$, $sc(\xi)$ is the map which at $x_0 \in \mathbb{R}$ computes $r = \min\{d(a,b) : a, b \in \xi\}$, centers a copy of $D_{x_0}(r/2) := [-\frac{r}{4}, \frac{r}{4}]$ at x_0, and maps x_0 its position in $D_{x_0}/\partial D_{x_0}$ if this copy intersects χ, and to the basepoint if not. In other words, the map $sc(\xi)$ puts little magnifying glasses up to the points of ξ and asks what they can see. Observe that if $\xi \in \mathrm{UConf}_k(\mathbb{R})$, then $sc(\xi)$ is compactly-supported as a map $\mathbb{R} \to S^1$ and it compactifies to degree k pointed map $S^1 \to S^1$. That degree fact implies

Proposition IV.1.3. *The map $\pi_0(sc)$ is the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$.*

I'll further claim monoidality. This is visually "obvious."

Proposition IV.1.4. *The scanning map $sc : M(\mathbb{R}) \to \Omega S^1$ is $\mathbb{E}_1$-monoidal.*

Cool, we've defined the map, that's (2).

Because $\pi_0 \Omega S^1$ is a group, there is an $\mathbb{E}_1$-factorization of sc through $sc^{\mathrm{gp}} : M(\mathbb{R})^{\mathrm{gp}} \to \Omega S^1$ such that $\pi_0(sc)$ induces the identity $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$. We would like to prove a stronger statement: sc^{gp} is an isomorphism, i.e. sc *is* the group completion. For this we will make a direct argument, and this relies on a concrete analysis of the group completion, in particular of the bar construction.

Definition IV.1.5. Let M denote a topological monoid. Its *bar construction* is the simplicial monoid $B_\bullet M$ defined as follows.

(1) On objects, we have $B_n M = M^n$. An element $h \in B_n M$ is denoted as a tuple of elements of M separated by "bars"

$$h = |^0 m_1 |^1 m_2 |^2 \dots |^{n-1} m_n |^n.$$

(2) The face map d_i acts by "removing the i-th bar." The degeneracy map s_i acts by adding the unit e "after the i-th bar." For example, $s_0(|m_1|m_2|) = |e|m_0|m_1|$ and $d_1(|m_1|m_2|) = |m_1 m_2|$.

Remark IV.1.6. I had no idea this was where the "bar" terminology came from!

V June

V.1 (5/4) Post-Talbot

I have just returned from this year's Talbot on "homological stability." (This explains where the subject of my last entry came from.) It was productive and rewarding, and I am very tired.

I was initially concerned that homological stability would only entertain me for the short while of Talbot, and that I would leave without substantive/material ideas to dwell on moving forward. This concern is partly explained by the intersectional character of the "area" of homological stability, wherein I struggled to discern what I am familiar with, and what I am interested in. To the contrary, however, I left the Talbot with both an unexpected interest in unfamiliar material (representation theory, certain topics in algebra, ...) and an exciting suspicion of connections to the subjects I am more familiar with (motivic, chromatic, K-theory, ...). In short, it reminded me that I enjoy math, including areas beyond what I somehow decided I am "comfortable" with.

I have many thoughts to organize. For now, I'll just write something without much explanation.

What is a *polylogarithm*?

V.2 (5/9) Pre-Eurotalbot I

I am in Chicago for the week, alone and without internet service, living and studying out of libraries and coffee shops. My hope is that this is an optimal arrangement to prepare myself for the upcoming European Talbot. (I leave early next week for the YTM, which Eurotalbot immediately follows.) I have somehow gotten into the habit of calling it *Eurotalbot*.

Maybe a good place to start is with the theory of ∞-categories, especially with the structures of stability and (symmetric) monoidality, of course with the category of spectra Sp in mind. I have previously written something about stability, but it's helpful to take things from the top, and I don't think I've ever carefully written or thought about the details of monoidal structures for ∞-categories.

Let's start with stability. One good story to tell here starts with *duality*, but I don't think I have time to tell stories. So, a proposition:

Know most of what's here, but polish for REU students?

Proposition V.2.1. *The ∞-category of pointed spaces* $\mathsf{Spaces}_* := \mathsf{Spaces}_{\Delta^0/}$ *is presentable, pointed, complete, and cocomplete.*

Proof. It is clear that Δ^0 is a zero object, and that this category is presentable. For (co)limits, first note that Spaces is complete and cocomplete: the limit of $F : I \to \mathsf{Spaces}$ may be directly exhibited as $\mathrm{Map}_{\mathsf{Spaces}^I}(\underline{pt}, F)$, and one can find colimits by standard arguments (exhibit pushouts + coequalizers, or geometric realizations). I would argue that Spaces_* is (co)complete in one of two ways: (1) appeal to the general fact that pointing a category does not change the existence of (co)limits, or (2) observe that the proof that Spaces has (co)limits works almost verbatim for pointed spaces. $\square$

This is obtained by some general argument. Track that down?

Now let's talk about suspension and looping. On objects, we may form these as $\Sigma X := * \coprod_X *$ and $\Omega X := * \times_X *$. To make these into functors, one can do Kan extension/Lurie tricks [Lur08, 4.3.2.15] to see that a suitably-defined forgetful functor $(X, \Sigma X) \mapsto X$ is a trivial Kan fibration, so there exists a section $\mathsf{Spaces}_* \to \mathrm{Fun}(\Delta^0 \times \Delta^0, \mathsf{Spaces}_*)$, and projecting gives a suspension functor; likewise for Ω. (See e.g. [**groth**].)

Proposition V.2.2. *There exists an adjunction $\Sigma \dashv \Omega$ of endofunctors of* Spaces_*.

Proof. $\square$

Remark V.2.3. All this (except for the presentability claim) goes through if we replace Spaces with a generic ∞-category that is finitely complete, finitely cocomplete, and pointed.

The (construction of the) category of spectra is greatly concerned with these functors Σ and Ω. You start out life being told that spectra are "like sequences of equivalences $\{X_n \xrightarrow{\sim} \Omega X_{n+1}\}_{n\in\mathbb{Z}}$," but then spent quite a long time being told about uglier models for spectra. If nothing else, you must appreciate that ∞-categories bring the "core" idea back to the fore of definition: an object of Sp really is a sequence of equivalences $\{X_n \xrightarrow{\sim} \Omega X_{n+1}\}_{n\in\mathbb{Z}}$. Recall some of the various implementations:

(1) Write $\mathrm{Iso}(\mathsf{Spaces}_*) \subseteq \mathrm{Fun}(\Delta^1, \mathsf{Spaces}_*)$ for the ∞-category of equivalences of pointed spaces. There are obvious maps $d_0, d_1 : \mathrm{Iso}(\mathsf{Spaces}_*) \to \mathsf{Spaces}_*$ and we may form the iterated pullback

$$\mathrm{Sp} := \lim\left(\mathrm{Iso}(\mathsf{Spaces}_*) \xrightarrow{d_0} \mathrm{Iso}(\mathsf{Spaces}_*) \xleftarrow{\Omega d_1} \mathrm{Iso}(\mathsf{Spaces}_*) \xrightarrow{d_0} \mathrm{Iso}(\mathsf{Spaces}_*) \xleftarrow{\Omega d_1} \cdots\right).$$

(2) We may also invert Σ or Ω by force, i.e. taking an appropriate (co)limit. There are at least four ways to do this: you may take

$$\mathrm{Sp} := \lim(\cdots \xrightarrow{\Omega} \mathsf{Spaces}_* \xrightarrow{\Omega} \mathsf{Spaces}_*)$$

in either Cat_∞ or in the non-full subcategory $\mathsf{Cat}_\infty^{finlim,*}$, or you may take

$$\mathrm{Sp} := \mathrm{colim}(\cdots \xleftarrow{\Sigma} \mathsf{Spaces}_* \xleftarrow{\Sigma} \mathsf{Spaces}_*)$$

in either Cat_∞ or in the non-full subcategory $\mathsf{Cat}_\infty^{fincolim,*}$.

(3) (Reduced excisive functors $\mathsf{Spaces}_*^{\mathrm{fin}} \to \mathsf{Spaces}_*$.)

Remark V.2.4. These are all equivalent. (Am I being diligent about connectivity?) If we want to replace Spaces_* here with a generic category, that category ought to be presentable, or else some things go wrong, e.g. the various definitions of (2) become nonequivalent.

There is much to say about the category of spectra. One very important point is its symmetric monoidal tensor product. Before we get to that, though, let's talk about the notion of *finiteness*. This can be considered geometrically, or by its colimit properties.

Definition V.2.5. We denote by $\mathsf{Spaces}^{\mathrm{fin}} \subseteq \mathsf{Spaces}$ the smallest full subcategory that contains $*$ and is closed under finite colimits (finite coproducts and pushouts). Equivalently, it is the full subcategory spanned by ∞-groupoids whose underlying simplicial set has finitely many nondegenerate simplices.

Proposition V.2.6. *The category* $\mathsf{Spaces}^{\mathrm{fin}}$ *is freely genrated under finite colimits by* $*$*, in the sense that if* D *has finite colimits, then an equivalence* $\mathrm{Fun}^{rex}(\mathsf{Spaces}^{\mathrm{fin}}, \mathsf{D}) \xrightarrow{\sim} \mathsf{D}$ *is given by* $F \mapsto F(*)$.

Remark V.2.7. There is a proper inclusion $\mathsf{Spaces}^{\mathrm{fin}} \hookrightarrow \mathsf{Spaces}^\omega$ into the full subcategory of compact objects. This is an idempotent completion but not an equivalence, as measured by the *Wall finiteness obstruction.* The subcategory Spaces^ω is characterized as that generated by $*$ under finite colimits *and retracts.*

One may form $\mathsf{Spaces}_*^{\mathrm{fin}}$ and identify it with a full subcategory of Spaces_*. This category behaves similarly to $\mathsf{Spaces}^{\mathrm{fin}}$ except that it is now generated by S^0.

We can ask about finiteness conditions for spectra, and what the relation is to finiteness conditions for spaces.

Definition V.2.8. We denote by $\mathrm{Sp}^{\mathrm{fin}} \subseteq \mathrm{Sp}$ the full subcategory spanned by (de)suspensions of spectra of the form $\Sigma^\infty K$ where $K \in \mathsf{Spaces}_*^{\mathrm{fin}}$.

Proposition V.2.9. *The following categories are equivalent.*

(i) $\mathrm{Sp}^{\mathrm{fin}}$,

(ii) Sp^ω,

(iii) The smallest full subcategory of Sp *that contains* $\Sigma^\infty S^0$ *and is closed under finite colimits and desuspensions;*

(iv) The [Spanier-Whitehead ∞-category]*, defined as the result of taking* $\mathsf{Spaces}_*^{\mathrm{fin}}$ *and "brute force" inverting* Σ *or* Ω*;*

(v) The stabilization *of* $\mathsf{Spaces}_*^{\mathrm{fin}}$*, i.e. the free presentable stable* ∞*-category on* $\mathsf{Spaces}_*^{\mathrm{fin}}$*;*

(vi) (Rok Gregoric writes that it is equivalent to the stabilization of $\mathsf{Spaces}_*^{\omega}$*; I believe this but can't find another reference, and this could easily be a typo. But I should be able to reason that these are equal. Well, once you stabilize, you get colimits, hence you are idempotent-complete, so if idempotent completion should commute with stabilization, then we are done).*

Double-check

Remark V.2.10. Just as $\mathsf{Ab} \cong \mathrm{Ind}(\mathsf{Ab}^{\mathrm{fg}})$, we have $\mathrm{Sp} \cong \mathrm{Ind}(\mathrm{Sp}^{\omega}) = \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})$ and $\mathsf{Mod}_R \cong \mathrm{Ind}(\mathrm{Perf}_R)$.

Something about thick subcategories?

I guess we implicitly assumed some (co)completeness properties of spectra, so let me say a bit outright.

Proposition V.2.11. *The category of spectra* Sp *has small limits, colimits, and a zero object.*

Proof. It is clear that a zero object is given by the constant sequence $0 := (*)_{\mathbb{N}} = \Sigma^\infty *$. As for the existence of (co)limits, you figure in steps. First you check that mapping spaces in Sp are given as $\mathbb{N}$-indexed limits in Spaces_* (this is true for totally formal reasons, it's basically the definition of Sp). Next note that Ω is a pullback, hence it commutes with limits, so that a "levelwise" limit is at least a spectrum. (I think you can reason similarly for filtered colimits, but I don't know off the top of my head why Ω commutes with filtered colimits, unless I'm willing to acknowledge that Ω is an equivalence...) To see its universal property, I think you use the mapping space identification. Finally, with the knowledge that Sp has limits computed "levelwise," we can compute $\Omega(X_0, X_1, \dots) \cong (\Omega X_0, X_0, X_1, \dots)$, and this is obviously an equivalence, so we conclude that Ω is an equivalence, in which case a similar argument as before finishes the proof. $\square$

Definitely worth revisiting this and writing it down more clearly and correctly...

(Limit and colimit properties of finite spectra and spaces?)

Derived categories?

The last facet of the ∞-category of spectra that I want to say something about today is its *symmetric monoidal structure*. This is one of the most important features of Sp. I guess the standard is to encode this structure as a *Grothendieck opfibration*. This approach to monoidal categories, especially in the context of ∞-categories, has gone under my radar for a while (although one of its main tenets are Segal conditions, which I know something about).

(4.1-4.2 of Groth's notes?)

V.3 (5/15) Pre-Eurotalbot II

In an ideal world, I know all about $\mathcal{D}(\mathbb{Z})$ and its homological algebra, and the previous entry contained an explanation of the stable structure on the derived ∞-category of R-modules $\mathcal{D}(R)$. I also would not have just paid seven dollars for a latte in an ideal world.

There are many basic structural similarities between (derived) abelian groups and spectra, and chromatic homotopy theory clarifies the entirety of this structure on Sp. Many of these similarities concern "global structure," which refers to structure that is sensibly "built" from p-local structure. Here are some points to formalizing this:

(1) The full subcategories $\mathcal{D}(\mathbb{Q})$ and $\mathcal{D}(\mathbb{Z})_p^\wedge$ are "irreducible" in the sense that they contain no proper, nonzero localizing subcategories (full triangulated subcategory closed under shifts and colimits);

(2) These subcategories are "building blocks" in the sense that any derived abelian group is recovered as the derived pullback of its localization data;

(3) The category of derived abelian groups is a *tensor-triangulated category*, hence it has an associated *Balmer spectrum* $\mathrm{Spc}(\mathcal{D}(\mathbb{Z}))$. The points of this space correspond to these subcategories $\mathcal{D}(\mathbb{Q})$ and $\mathcal{D}(\mathbb{Z})_p^\wedge$. The stable structure on the ∞-category of spectra induces a tensor-triangulated structure,

hence we may also form Spc(Sp). Now we consider the Hurewicz map $\mathbb{S} \to \tau_{\leq 0}\mathbb{S} = H\mathbb{Z}$, base change along constitutes a functor

$$\mathrm{Sp} \to \mathsf{Mod}_{H\mathbb{Z}}(\mathrm{Sp}) \cong \mathcal{D}(\mathbb{Z}).$$

This induces a continuous map $\mathrm{Spc}(\mathcal{D}(\mathbb{Z})) \to \mathrm{Spc}(\mathrm{Sp})$, in fact one may show that this map admits a retract. The fiber over $p \in \mathrm{Spec}(\mathbb{Z}) = \mathrm{Spc}(\mathcal{D}(\mathbb{Z}))$ consists of infinitely many points, corresponding to the so-called "intermediary primes" $K(0) = \mathbb{Q}, K(1), K(2), \ldots, K(\infty) = H\mathbb{F}_p$. These are no longer prime fields, but are rather "prime spectra."

V.4 (5/30)

VI September

VI.1 (9/1) Chromatic VI: Temperance? How Miserly...

Long absence. What happened?

Figure 1: Temperance. From *The Virtues* series. Source: Art Institute of Chicago, artwork no. 95545, `https://www.artic.edu/artworks/95545/temperance-from-the-virtues`.

I am trying to learn a bit about *transchromatic homotopy theory* and *tempered cohomology*. This is, in my opinion, a somewhat niche facet of chromatic—at least, I don't feel I heard the word "transchromatic" until well into my time learning about chromatic homotopy theory. Unfortunately I am finding it difficult and ugly, but I hope that writing down my piecemeal thoughts will somehow reflect clarity back onto myself. Let me mention that we all have a little transchromaticism within us, insofar as you believe in the *chromatic fracture squares*

$$\begin{tikzcd} L_{n-1}X \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_nX \arrow[r] & L_{n-1}L_{K(n)}X \end{tikzcd}$$

wherein we must consider the interaction between heights n and $n-1$.

My difficulty is, to my senses, due to my awkward confusions about formal groups. What precisely I don't understand, and how it obstructs my goals, is probably a waste of time to discern exactly. Let's just do some math and hopefully learn a bit about formal groups. These next few days, I want to review Lazard's theorem, deformation theory, and the Lubin-Tate theorem. This should be a pedagogically sound way warm up to formal group theory. I am specifically interested in proving Lazard's theorem (really, the symmetric cocycle lemma) using the theory of formal groups—let me advertise two references for this:

(1) Piotr's excellent 252y notes, and

(2) Dylan Wilson's pretalbot notes on a "leisurely proof of Lazard's theorem" which I only found after-the-fact thanks to a footnote in Sil Lisken's notes.

I wrote a bit about formal groups last time. This time I want to write about about their deformations, and hopefully at least state (in the classical and modern form) the very fundamental *Lubin-Tate theorem.* In his notes, Piotr actually leads into deformation theory by using it to prove Lazard's theorem. For the culture, let me begin by seeing how that goes. Let's fix some notation and recollections:

- The functor FGL : $\mathsf{CRing} \to \mathsf{Set}$ is an affine scheme represented by the Lazard ring $L = \mathbb{Z}[a_{i,j}]/I$ where I is the ideal generated by coefficients of the power series encoding associativity, commutativity, and the unitality of 0 for the tautological formal group law $F(x, y) = \sum a_{ij}x^iy^j$.
- The functor $\mathbb{G}_{inv} : \mathsf{CRing} \to \mathsf{Grp}$ given by $R \mapsto \{b_0x + b_1x^2 + \cdots \in R[[x]] : b_0 \in R^\times\}$ is an affine group scheme represented by $\mathrm{Spec}(\mathbb{Z}[b_0^{\pm 1}, b_1, b_2, \dots])$. It semi-directly decomposes as $\mathbb{G}_{inv} \cong \mathbb{G}^s_{inv} \rtimes \mathbb{G}_m$. Here, $\mathbb{G}^s_{inv}$ is the formal group of strict power series and $\mathbb{G}_m$ is the multiplicative group (which acts by $R \mapsto \{ax : a \in R^\times\}$.)

Ok, here's a little guiding fact.

Proposition VI.1.1. *For a ring R, the following data are equivalent.*

1. *A $\mathbb{G}_m$-action on the affine scheme* $\mathrm{Spec}(R)$.
2. *A grading: a choice of abelian subgroups $R_{2n} \leq R$ which make R into a graded-commutative ring.*

Proof. Suppose as given a $\mathbb{G}_m$-action on $\mathrm{Spec}(R)$, i.e. a morphism $\mathbb{G}_m \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ such that for every T, the map $\mathbb{G}_m(T) \times \mathrm{Spec}(R)(T) \to \mathrm{Spec}(R)(T)$ defines a $\mathbb{G}_m(T)$-action of sets. Note the identification $\mathbb{G}_m \cong \mathrm{Spec}(\mathbb{Z}[b, b^{-1}])$, under which the dual comultiplication $\mathbb{Z}[b^{\pm 1}] \to \mathbb{Z}[b^{\pm 1}, u^{\pm 1}]$ is given by $b \mapsto bu$. The action is dual to a homomorphism $\Delta : R \to R[b^{\pm 1}]$, in fact this defines a comodule structure.

The associativity for this comodule structure amounts to $(\mathrm{id}_{R[b^{\pm}]} \otimes \Delta) \circ \Delta = (\Delta_{\mathbb{G}_m} \otimes \mathrm{id}_R) \circ \Delta$. Let us write $\Delta(r) = \sum_{\mathbb{Z}} r_ib^i$; then the RHS acts by $r \mapsto \sum r_ib^i \mapsto \sum r_ib^it^i$ and the RHS acts by $r \mapsto \sum r_iu^i \mapsto \sum \Delta(r_i)u^i$. This implies $\Delta(r_i) = r_ib^i$. Likewise, unitality implies that $\sum r_i = r$.

Now we may form $R_i = \Delta^{-1}(R[t]_i)$. The above deductions give us a decomposition $R \cong \oplus R_i$, which decomposes r into its coefficients r_i. It's graded because $b^ib^j = b^{i+j}$.

Conversely, given a grading $R \cong \bigoplus R_i$, we can define a coaction $\delta : R \to \mathbb{Z}[b^{\pm 1}] \otimes R$ via $\delta(r) = \sum r_ib^i$. □

We prefer *even* graded-commutative rings, so we can just pretend each r_i is in degree $2i$ and/or write it r_{2i}. That in mind, recall that $\mathrm{Spec}(L)$ admits a canonical $\mathbb{G}_{inv}$-action and hence a $\mathbb{G}_m$-action.

Corollary VI.1.2. *The Lazard ring L is canonically evenly graded.*

But let's go a little further and unwind the action. Suppose we have points $\lambda \in \mathbb{G}_m(R)$ and $F \in \mathrm{Spec}(L)(R)$, so that $\lambda \in R^\times$ and $F \in FGL(R)$. Then

$$\lambda \cdot F = \lambda^{-1} \sum a_{ij}(\lambda x)^i(\lambda y)^j = \sum \lambda^{i+j-1}a_{ij}x^iy^j.$$

It follows that a_{ij} is homogeneous of degree $2(i + j - 1)$.

Corollary VI.1.3. *L is concentrated in non-negative degrees and $L_0 = \mathbb{Z}$.*

This describes somewhat the structure of L. We would like to get our hands on an isomorphism $L \cong \mathbb{Z}[b_1, b_2, \dots]$, but this turns out to be a wildly complicated thing, certainly not to be computed directly. Toward finding this isomorphism, let's get some details straight: the invertible power series $x(1 + xb_1 + x^2b_2 + \cdots) \in \mathbb{Z}[b_1, b_2, \dots][[x]]$ has an associated formal group law $\exp(\exp^{-1}(x) + \exp^{-1}(y))$ classified by a map

$$\psi : L \to \mathbb{Z}[b_1, b_2, \dots].$$

One may ask whether this ring map respects the grading (or what grading the target should even have). For this we may consider the induced

$$\mathbb{G}^s_{inv} = \mathrm{Spec}(\mathbb{Z}[b_1, b_2, \dots]) \to \mathrm{Spec}(L).$$

If this morphism of schemes is $\mathbb{G}_m$-equivariant, then ψ must respect the grading (or more accurately, induce a grading). In fact, it turns out that this morphism is $\mathbb{G}_{inv}$-equivariant, where the induced grading begets $|b_i| = 2i$.

It turns out that this graded homomorphism $\psi : L \to \mathbb{Z}[b_1, b_2, \dots]$ is *not* an isomorphism! One way to understand this is to think of $\mathbb{Z}[b_1, b_2, \dots]$ as corepresenting the set of graded formal group law *equipped with an isomorphism from the additive formal group law.* Unless every formal group law is (uniquely?) isomorphic to the additive one, the map ψ better not be an isomorphism! That being said, this suggests a corollary:

Check this by hand. Why is it obvious on the level of schemes, but hard to check on the level of rings? Am I supposed to just look at this map of schemes and say "oh yeah this is obvious $\mathbb{G}_m$-equivariant?

Corollary VI.1.4. *$\mathbb{Q} \otimes \psi : L \otimes \mathbb{Q} \to \mathbb{Q}[b_1, \dots]$ is an isomorphism.*

We forge on by studying how ψ fails to be an isomorphism, which will produce the actual isomorphism. For this, recall that if R is a non-negatively graded ring and $I = R_+$ is its ideal of positive-degree elements, then its *module of indecomposables* is the graded R-module I/I^2. Denote by I/I^2 and J/J^2 the indecomposables of L and $\mathbb{Z}[b_1, \dots]$, respectively.

Proposition VI.1.5. *$(J/J^2)_{2n} \cong \mathbb{Z}\{b_n\}$.*

It turns out that ψ is injective, and the *symmetric cocycle lemma* identifies exactly what the image looks like piece-by-piece.

Proposition VI.1.6 (Symmetric cocycle lemma)**.** *For each n, the map $(I/I^2)_{2n} \to (J/J^2)_{2n}$ is injective with image*

- *$\mathbb{Z}\{b_n\}$ when $n+1 \neq p^k$, and*
- *the subgroup generated by pb^n when $n+1 = p^k$ for some k.*

Before we prove this fact, let's record that it *does* imply Lazard's theorem.

Theorem VI.1.1. *The symmetric cocycle lemma implies Lazard's theorem.*

Proof. The symmetric cocycle lemma identifies each $(I/I^2)_{2n}$ with a free abelian group on a single generator. Choose lifts of these generators $t_n \in I$ and consider the determined graded ring map

$$\theta : \mathbb{Z}[t_1, t_2, \dots] \to L.$$

This map is surjective by a degree argument. To check injectivity, we can check that $\psi \circ \theta$ is injective because we know (by the lemma) that ψ is injective. Then it suffices to check after rationalization because the source and target are torsion-free. $\square$

Now we want to prove the symmetric coycle lemma, specifically by framing it as a formal group-theoretic deformation problem over square-zero extensions. Let's just recall where we're at:

- The functor $\operatorname{Spec}(L)$ has a $\mathbb{G}_m$-action which acts by "twisting" formal group laws: units $\lambda \in R^\times$ send a formal group law F to $F^\lambda(x, y) = \lambda^{-1} F(\lambda x, \lambda y)$. This $\mathbb{G}_m$-action induces a canonical even grading on L.
- The symmetric cocycle lemma identifies the *indecomposables of the graded pieces of L*, which we are going to redenote as (QL_{2n}), as follows: via the morphism $\psi : L \to \mathbb{Z}[b_1, \dots]$ induced by exponentiation, they are $(Q\mathbb{Z}[b_1, \dots])_{2n}$ if $n+1 \neq p^k$, and are $p(Q\mathbb{Z}[b_1, \dots])_{2n}$ otherwise.

Claim VI.1.7. I claim that we're already looking right at a sort of extension problem. For this, note we are trying to identify $(QL)_{2n}$ as some abstract copy of the *ring* $\mathbb{Z}$. As a *group*, there is a useful characterization of $\mathbb{Z}$: it is the unqiue one for which $\operatorname{Hom}_{\mathsf{Ab}}(A, \mathbb{Q}) \cong \mathbb{Q}$ and $\operatorname{Hom}_{Ab}(A, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z})$ for all p. To touch *rings*, here's a fact: for a connected and evenly-graded ring R, there is a bijection

$$\operatorname{Hom}_{\mathsf{Ab}}((QR)_{2n}, A) \cong \operatorname{Hom}_{\mathsf{GrRing}}(R, \mathbb{Z}[0] \oplus A[2n]).$$

A graded morphism $L \to \mathbb{Z} \oplus A[2n]$ is already an isomorphism on connected components, so such morphisms amount to extension problems

Check. The important fact is that $Q(\mathbb{Z} \oplus A[2n]) \cong A/A^2 \cong A$.

$$\begin{tikzcd} & L \oplus A[2n] \arrow[d] \\ L \arrow[ur, dashed] \arrow[r] & \mathbb{Z} \end{tikzcd}$$

So, we are interested in our particular group homomorphisms induced by ψ

$$QI_{2n} \to QJ_{2n},$$

and we understand that we can apply $\mathrm{Hom}_{\mathsf{Ab}}(-, A)$ and examine these maps there. By the above, we know we may translate this to the setting of graded ring theory. And by the earlier discussion, we know that graded ring theory embeds into $\mathbb{G}_m$-equivariant scheme theory. Altogether, our maps correspond to $\mathbb{G}_m$-equivariant morphisms

$$\mathbb{G}^s_{inv}(\mathbb{Z} \oplus A[2n]) \to \mathrm{FGL}(\mathbb{Z} \oplus A[2n]).$$

The left-hand size consists of strictly invertible graded power series over $\mathbb{Z} \oplus A[2n]$. The map above induced by ψ sends such a thing $\phi(t)$ to the twist of F_a it produces; this is just $\phi^{-1}(\phi(x) + \phi(y))$. In particular, the *kernel* of the above map consists of strict automorphisms of the additive formal group law F_a, i.e. those $\phi(t)$ such that $\phi'(0) = 1$ and $x + y = \phi^{-1}(\phi(x) + \phi(y))$. Crucially, the (strict) automorphisms of a formal group constitute their own formal group. In particular, there is a formal group $\mathrm{Aut}(\hat{\mathbb{G}}_a)$ given by $R \mapsto \{r(t) = t + r_2 t^2 + \cdots | x + y = r^{-1}(r(x) + r(y))\}$.

Proposition VI.1.8 (Identification of kernel)**.** *We have an identification*

$$\ker\left(\mathbb{G}^s_{inv}(\mathbb{Z} \oplus A[2n]) \to \mathrm{FGL}(\mathbb{Z} \oplus A[2n])\right) \cong \ker\left(\mathrm{Aut}(\hat{\mathbb{G}}_a)(\mathbb{Z} \oplus A[2n]) \to \mathrm{Aut}(\hat{\mathbb{G}}_a)(\mathbb{Z})\right).$$

Proof. The abelian group structures come from the earlier identification of sets of graded ring maps with sets of group homomorphisms. Suppose you are $r(t)$ in the RHS; this means you are a strictly invertible power series $r(t) \in (\mathbb{Z} \oplus A[2n])[[t]]$ such that all your coefficients $r_2, r_3, \ldots$ are elements of $A[2n]$. But that just means you were in the LHS, and the purported isomorphism is the identity. □

There's a subtlety here I am getting confused by, that uses the fact that we're thinking of $\mathbb{Z}$ and $\mathbb{Z}[b_1, \ldots]$ as graded rings. I think this is necessary simply to conclude that the r_i must be in A and not in $\mathbb{Z}$, which is implied because $\mathbb{Z}$ is concentrated in degree zero while $|b_i| = 2i$.

Proposition VI.1.9 (Identification of cokernel)**.** *The cokernel*

$$\mathrm{coker}(\mathbb{G}^s_{inv}(\mathbb{Z} \oplus A[2n]) \to \mathrm{FGL}(\mathbb{Z} \oplus A[2n]))$$

may be identified with $\pi_0\mathrm{Def}$*, where* Def *is the groupoid of deformations of the additive formal group along* $\mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z} \oplus A[2n])$.

Proof. First, let's define our things: a *deformation of* $\mathbb{G}_0$ *along* $f : S_0 \hookrightarrow S$ is a pair $(\mathbb{G}, \phi)$ where $\mathbb{G} \to Y$ is a formal group over Y and $\phi : \mathbb{G} \times_S S_0 \xrightarrow{\sim} \mathbb{G}_0$ is an isomorphism of S-schemes. These form the category Def whose morphisms are the evident commutative triangles.

Second, let's make a claim: there is a map

$$\mathrm{FGL}(\mathbb{Z} \oplus A[2n]) \to \pi_0\mathrm{Def}$$

which sends a formal group law F to (the isomorphism class of) $\mathbb{G}_F \times_{\mathrm{Spec}(\mathbb{Z} \oplus A[2n])} \mathrm{Spec}(\mathbb{Z})$, where $\mathbb{G}_F$ is the associated formal group of F. Then the claim is that this map is surjective: in effect, this means every deformation of formal groups along $\mathrm{Spec}(\mathbb{Z}) \hookrightarrow \mathrm{Spec}(\mathbb{Z} \oplus A[2n])$ comes from a formal group of an fgl. We will not prove this.

Thirdly, we can identify the kernel of this map: it is by definition the set of formal group laws F such that $\mathbb{G}_F$ is isomorphic to the additive formal group over $\mathrm{Spec}(\mathbb{Z} \oplus A[2n])$. □

VI.2 (9/5) Chromatic VII: Classism and Representation

As part of my march upon transchromatic homotopy theory, this entry should purview the introduction of Lurie's Elliptic III. (Author's warning: this entry was a mess.)

Figure 2: *The Torment of Ixion*, c. 1876, by Jules-Élie Delaunay. Source: Art Institute of Chicago, artwork no. 117221, https://www.artic.edu/artworks/117221/the-torment-of-ixion.

As usual, we can start somewhere easy. To each topological space is associated its zeroth complex K-group $KU^0(X)$ which we may identify with isomorphism classes of virtual complex vector bundles over X. More powerfully, there is a commutative ring spectrum KU.

We may harsh the vibe and seek an equivariant analog. If what we want is a cohomology theory for G-spaces, life could be as simple as setting

$$KU_{G-bor}(X) := KU^*(EG \times_G X).$$

This *G-Borelification* has some shortcomings, but nobody is perfect. It possesses the expected property that if G acts on X freely, then $KU^*_{G-bor}(X) \cong KU^*(X/G)$. (To see this, note that $X \to X/G$ is a fibration with fiber EG.) We also find immediately that $KU^*_{G-bor}(*) = KU^*(EG \times_G *) = KU^*(EG/G) = KU^*(BG)$.

Remark VI.2.1. By some stretch of the imagination, that $KU^*_{G-bor}(*) \cong KU^*(BG)$ already shows that Borelification does not produce the "correct" equivariant analog of KU. Maybe a more elementary demonstration notes that KU_{C_2-bor} does not distinguish between S^1 and S^σ, because it does not have a "full" Bott periodicity. Write down

Alternatively, you can set as a goal that an G-equivariant complex K-theory should satisfy

$$KU^0_G(X) \cong (G - \text{equivariant complex vector bundles over } X/\simeq)^{\text{gp}}.$$

There is a lot to say here, in fact too much, in which case we should say nothing except that there *is* an entire G-equivariant spectrum KU_G fitting this bill.

That established, let's pull in some basic representation theory. A G-equivariant complex bundle over $*$ amounts to a complex G-representation, so that

$$KU_G^0(*) \cong \mathrm{Rep}(G).$$

We know something about $\mathrm{Rep}(G)$: because the characters of irreducible representations form a basis for the complex vector space of class functions on G, we get an identification as follows.

Theorem VI.2.1. *For G finite, $\mathrm{Rep}(G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \{$class functions $\chi : G \to \mathbb{C}\}$.*

Claim VI.2.2. The above theorem admits a homological restatement. We have already identified $\mathrm{Rep}(G) \cong KU_G^0(*)$. So the question is, what the hell is a class function? The cop-out is to form the G-space $\coprod_G *$ having the conjugation action and consider $H^0(\coprod_G */G; \mathbb{C}) \cong Cl(G)$.

Ok, so the classical Theorem VI.2.1 seems to say something about a map from complexified KU_G^* of $X = *$ to the complexified zeroth singular homology of some discrete space. This may or may not give you deja vú: doesn't the Chern character look like that too? The punchline is that with homological language, we can make a stronger theorem where we need not restrict to $X = *$ as follows.

Theorem VI.2.2. *For G finite and any finite G-CW complex X, there is a canonical isomorphism*

$$ch_G : KU_G^0(X) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^{ev}(\left(\coprod_G X^g\right)/G; \mathbb{C}).$$

Theorem VI.2.3 (p-adic variant)**.** *For G finite, any finite G-CW complex X, and a fixed prime number p, there is a canonical isomorphism*

$$\widehat{ch}_G : \widehat{KU}^0(X_{hG}) \otimes_{\mathbb{Z}_p} \mathbb{C} \xrightarrow{\sim} H^{ev}((\coprod_{G^{(p)}} X^g)/G; \mathbb{C}).$$

Notice that in the p-adic situation, $KU_G^0(X)$ is replaced by p-adically completed $\widehat{KU}^0(X_{hG})$. This is more like the Borel situation. There is a comparison to be made: the projection $EG \times X \to X$ induces $KU_G^0(X) \to K_G^0(X \times EG) \cong KU^0(X \times_G EG) \cong KU^0_{G-bor}(X)$. Altogether we get the *Atiyah-Segal comparison map*

$$\zeta : KU_G^0(X) \to KU^0_{G-bor}(X) = KU^0(X_{hG}).$$

I'll remark that these p-adic Chern characters commute with these comparison maps.

Theorem VI.2.4. *For every finite G-CW complex X, the Atiyah-Segal comparison $\zeta : KU_G^0(X) \to KU^0(X_{hG})$ exhibits $KU^0(X_{hG})$ as the completion of $KU_G^0(X)$ at the augmentation ideal.*

In the case that $X = *$, the map ζ compares $\mathrm{Rep}(G) \to KU^0(BG)$. It would be nice to know when ζ is an isomorphism, as this would give us more control over $KU_G^0(*) = \mathrm{Rep}(G)$. One fact in this direction is the following.

Proposition VI.2.3. *If G is a p-group, then the augmentation ideal $I_G \subseteq KU_G^0(*)$ satisfies $I_G^n \subseteq p \cdot KU_G^0(*)$ for n sufficiently large.*

Proof. □

It follows that for a finite G-CW complex X and a finite p-group G, the completion theorem gives us an identification $\mathbb{Z}_p \otimes_{\mathbb{Z}} KU_G^0(*) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} KU^0(BG) \cong \widehat{KU}^0(BG)$. Thus, at the cost of specializing to the p-completed situation, we have somewhat reduced the level of G-equivariance complicating our representation ring. Now we can ask whether, in this language, there is a version of the classical identification Theorem VI.2.1. We have already identified a p-adic K-theoretic avatar for $\mathrm{Rep}(G) \otimes \mathbb{C}$; but again, the question remains: what the hell is a class function?

The earlier claim made an ad-hoc construction of a space Y such that $Cl(G; \mathbb{C}) \cong H^0(Y; \mathbb{C})$. I was hiding something better:

Proposition VI.2.4. $Cl(G;\mathbb{C}) \simeq H^0(LBG;\mathbb{C})$.

Proof. We can actually just identify the homotopy type of LBG. Maps $S^1 \to BG$ correspond to principal G-bundles over S^1, which are uniquely determined by their monodromy representation $\pi_1(S^1) \to G$ up to conjugacy, thus we have a bijection $\pi_0(LBG) \simeq G/G$. The homotopy type of each component is then $BC_G(g)$, so that $LBG \simeq \coprod_{G/G} BC_G(g) \simeq G//G$. $\square$

The classical character isomorphism then becomes

$$\widehat{KU}^0(BG) \otimes_{\mathbb{Z}_p} \mathbb{C} \cong H^0(LBG;\mathbb{C}).$$

Some points:

1. We have only meant to claim the above for finite p-groups G. In fact, we can modify the claim to work for all G at the cost of replacing $L = \mathrm{Map}(B\mathbb{Z}, -)$ with $L_p = \mathrm{Map}(B\mathbb{Z}_p, -)$. This will better handle the p-singular part $G^{(p)} \subseteq G(?)$

2. We may ask about a statement on the level of spectra. So far, we have simply taken zeroth groups and p-adically completed them. There are signs that a spectrum-level statement exists: for example, the target of the Chern character isomorphism identifying $KU^0_G(X) \otimes_{\mathbb{Z}} \mathbb{C}$ is an even singular cohomology ring, which is the zeroth part of the spectrum-level rationalization $\mathbb{Q} \otimes_{\mathbb{S}} KU$! On the level of the entire cohomology ring, the classical character isomorphism ends up becoming

$$(\star) \quad \widehat{KU}^*_p(BG) \otimes_{\mathbb{Z}_p} \mathbb{C} \cong (\mathbb{Q} \otimes_{\mathbb{S}} \widehat{KU}_p)^*(L_pBG) \otimes_{\mathbb{Q}_p} \mathbb{C}.$$

   It is this form of the character isomorphism that is amenable to seriously badass generalization. How does that work?

3. Note that the equivalence $(\star)$ relates height one ($K(1) = \widehat{KU}_p$) and height zero ($L_0K_1 = \mathbb{Q} \otimes K(1)$). Somehow, we had to freeloop and extend scalars to make it all work. These are each important points in the effort to generalize to higher heights and higher *height differences*.

Maybe we can call it here for now.

VII (9/15) Chromatic VIII: The Three of Wands

Last time, we considered the classical *character correspondence* $R(G) \xrightarrow{\sim} \mathrm{Class}(G,\mathbb{C})$ through the lens of homotopy theory to arrive at a purported equivalence

$$(\widehat{KU}_p)^*(BG) \otimes_{\mathbb{Z}_p} \mathbb{C} \xrightarrow{\sim} (\mathbb{Q} \otimes_{\mathbb{S}} \widehat{KU}_p)^*(L_pBG) \otimes_{\mathbb{Q}_p} \mathbb{C}.$$

The zero-th degree recovers the classical correspondence. Let me say now that in the classical correspondence (and thus the above) the complexifying $\mathbb{C}$ can be replaced by $\mathbb{Q}_{p^\infty} := \mathrm{colim}\, \mathbb{Q}[\zeta_{p^k}]$, the maximal ramified extension of $\mathbb{Q}_p$. Let me now label everything suggestively.

$$\underbrace{(\widehat{KU}_p)^*}_{K(1)=E(1)}(BG) \otimes_{\mathbb{Z}_p} \underbrace{\mathbb{Q}_{p^\infty}}_{\text{max ramified extension of b}L_{K(0)}E(1)^*} \xrightarrow{\sim} \underbrace{(\mathbb{Q} \otimes_{\mathbb{S}} \widehat{KU}_p)^*}_{L_{K(0)}E(1)}(\underbrace{L_p}_{L_p^{1-0}} BG) \otimes_{\mathbb{Q}_p} \underbrace{\mathbb{Q}_{p^\infty}}_{\text{max ramified ...}}.$$

In the transchromatic literature, we mean by C_t^* a certain extension of $L_{K(t)}E(n)_*$. Then the *transchromatic character correspondence* on the level of coefficients takes the form

$$(\star) \quad E(n)^*(BG) \otimes_{E(n)_*} C_t^* \xrightarrow{\sim} (L_{K(t)}E(n))^*(L_p^{n-t}BG) \otimes_{L_{K(t)}E(n)_*} C_t^*.$$

This is the Big Beautiful Thing. On the level of spectra, this is an equivalence not only of G-equivariant spectra, but actually of *global spectra*. There are several points here to explain. My goal right now is to tell you about C_t^*. This is, for better or for worse, a formal groups jumpscare—but to bravely look upon this theory until it stops being scary is the local goal in this open of my chromatic notes, so let's courageously stride forward.

Figure 3: *What Courage!* c. 1810-12, by Francisco José de Goya y Lucientes. From *The Disasters of War* plates. Source: Art Institute of Chicago, artwork no. 124845, https://api.artic.edu/api/v1/artworks/124845/manifest.json.

Actually, I think I can learn a lot by figuring out why C_0^* appears here as the maximal ramified extension of $\mathbb{Q}_p$. For this, let us take the perspective that just as classical character theory leads us to consider $E_1^0(BG)$, we should try to study $E_n^0(BG)$ or more generally $E_n^*(X)$ by way of some theory of characters. A testing ground for this is to ask about rationalizing $E_n^*(X)$, and we can say the following.

Proposition VII.0.1. *If X is finite, then the map $E_n^*(X) \to (p^{-1}E_n)^*(X)$ is an equivalence after tensoring with $(p^{-1}E_n)^*$.*

Proof. The coefficients $(p^{-1}E_n)^*$ are flat as an E_n^*-module. This makes $p^{-1}E_n^* \otimes E_n^*(-)$ a cohomology theory for finite complexes (assuming we know that E_n is so). Certainly $(p^{-1}E_n)^* \otimes E_n^* \to (p^{-1}E_n)^*$ is an isomorphism, and we can extend over *finite* cell attachments to conclude the desired isomorphism for finite X. $\square$

If X is not finite, then this proposition often fails. Our case of interest is $X = B\mathbb{Z}/p^k$. We still have a localization map

$$E_n^*(B\mathbb{Z}/p^k) \to (p^{-1}E_n)^*(B\mathbb{Z}/p^k),$$

but this gets somewhat soiled when we extend scalars along $E_n^* \to p^{-1}E_n^*$. Note that the right-hand side is unchanged, and in fact is trivial:

Proposition VII.0.2. *The cohomology of $B\mathbb{Z}/p^k$ is a trivial rank 1 module over $(p^{-1}E_n)^*$.*

Proof. The AHSS reads $E_2 = H^p(B\mathbb{Z}/p^k; (p^{-1}E_n)^*) \implies (p^{-1}E_n)^{p+q}(B\mathbb{Z}/p^k)$. However, the coefficient ring of $p^{-1}E_n$ is rational, hence $E_2^{p,q} = 0$ for $p > 0$. This collapse affects

$$(p^{-1}E_n)^*(B\mathbb{Z}/p^k) \cong (p^{-1}E_n)^*.$$

□

The left-hand side is more interesting. We can first of all compute it, and then secondly *interpret* it.

Proposition VII.0.3. $(p^{-1}E_n)^* \otimes_{E_n^*} E_n^*(B\mathbb{Z}/p^k)$ *is a free $(p^{-1}E_n)^*$-module of rank p^{kn}.*

Proof. Note that E_n is even-periodic, so that we may choose a complex orientation and identify $E_n^0(B\mathbb{Z}/p^k) \cong E_n^0[[x]]/([p^k](x))$. (For this we are using the fiber sequence $B\mathbb{Z}/p^k \to B^2\mathbb{Z} \xrightarrow{p^k} B^2\mathbb{Z}$ and the collapse of the Serre spectral sequence in the presence of an even grading. This works in general to the effect that the evenly-graded cohomology of $B\mathbb{Z}/p^k$ is formed by imposing the relation "p^k-torsion" on the formal group.) Now observe that $[p^k](x)$ is a degree p^{kn} polynomial, and Weierstrass preparation concludes that $E_n^0[[x]]/([p^k](x))$ is a free module of rank p^{kn}. □

Now let me try to explain a moral for this computation. We know that the complex orientation of E_n endows $E_n^0(\mathbb{C}P^\infty)$ with a formal group of height n. This formal group $\mathbb{G}_{E_n}$ has an associated p-divisible group

$$\mathbb{G}_{E_n}[p] \hookrightarrow \mathbb{G}_{E_n}[p^2] \hookrightarrow \cdots.$$

The cohomology of $B\mathbb{Z}/p^k$ should recover the algebra of functions on the k-th piece of this p-divisible group, i.e. $E_n^0(B\mathbb{Z}/p^k) \cong \Gamma(\mathbb{G}_{E_n}[p^k])$. It is part of the definition of a p-divisible group that since $\mathbb{G}_{E_n}$ has height n, the k-th piece has order kn over $\mathrm{Spec}E_n^0$, so its coordinate ring should be finite free of rank p^{kn}. This is the algebro-geometric interpretation that we should think of the right-hand side as "not living up to," and scrounge for a suitable modification.

Let me explain more this language. The formal group of E_n is defined as $\mathbb{G}_{E_n} := \mathrm{Spf}(E_n^0(\mathbb{C}P^\infty))$ which turns out to be the universal deformation of a height n formal group over k. We extract[9] from $\mathbb{G}_{E_n}$ its p-divisible group as the filtered diagram of its p^k-torsion, which are finite flat commutative group subschemes of $\mathbb{G}_{E_n}$. One can show that

Proposition VII.0.4. $\mathbb{G}_{E_n}[p^k] \cong \mathrm{Spec}(E_n^0(B\mathbb{Z}/p^k))$.

(We can use Spec here because $E_n^0(B\mathbb{Z}/p^k)$ is finitely-generated over E_n^0, whereas $E_n^0(\mathbb{C}P^\infty)$ was not.) We recognized

$$(p^{-1}E_n^0) \otimes E_n^0(B\mathbb{Z}/p^k) \cong \Gamma(p^{-1}E_n^0 \otimes \mathbb{G}_{E_n}[p^k]),$$

and this turns out to be an *étale* finite flat group scheme. (This is essentially because we tensored by a $\mathbb{Q}$-algebra. Actually we just needed p to be invertible in $p^{-1}E_n^0$, which it certainly is.) In this case, we can ask how to base change $p^{-1}E_n^0$ to trivialize this scheme $\mathbb{G}_{et} := p^{-1}E_n^0 \otimes \mathbb{G}_{E_n}$. That is, we would like to base-change from $p^{-1}E_n^0$ to obtain a constant scheme. There is one brutish way to do this, namely by taking the separable closure of the field of fractions of $p^{-1}E_n^0$. (We needed a field to put ourselves in correspondence with Galois representations; we took the separable closure to trivialize the Galois action, I think.) In particular there exists some $C_0 \in \mathrm{Alg}_{p^{-1}E_n^0}$ such that

$$C_0 \otimes \mathbb{G}_{et} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n.$$

[9] Actually, this is a more-or-less lossless correspondence.

This grants us $C_0 \otimes \mathbb{G}_{et}[p^k] \cong (\mathbb{Z}/p^k)^n$, and we deduce

$$C_0 \otimes E_n^0(B\mathbb{Z}/p^k) \cong \prod_{(\mathbb{Z}/p^k)^n} C_0 \cong \underbrace{C_0 \otimes_{p^{-1}E_n^0} (p^{-1}E_n)^0}_{=:C_0^0}(\coprod_{(\mathbb{Z}/p^k)^n} *).$$

I think we are meant to look at this and suspect we have found an effective way to rationalize $E_n^0(B\mathbb{Z}/p^k)$, at the cost of no longer tensoring with $p^{-1}E_n^0$ but rather some secret evil other thing C_0. This somewhat explains why we wanted to trivialize the group scheme $\mathbb{G}_{et}$:

So there exists *some* ring C_0 that helps us. We can ask for a better (in fact universal) such ring. We want two properties:

1. C_0 is an extension of $p^{-1}E_n^0$.
2. There exists an isomorphism $C_0 \otimes \mathbb{G}_{et} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$ of p-divisible groups.

These are the properties that helped us above. As the property (2) we desire is a statement about p-divisible groups, we will summon C_0 at level of individual p^k-torsion groups. For this we need a lemma.

Proposition VII.0.5. *For a finite abelian group A, denote its Pontryagin dual by $A^* := \mathrm{Hom}(A, S^1)$. Suppose that $A = A[p^k]$, in which case $A^* \cong \mathrm{Hom}(A, \mathbb{Z}/p^k)$. Then for $R \in \mathrm{Alg}_{E_n^0}$ there exists a natural isomorphism*

$$\mathrm{Hom}_{E_n^0-alg}(E_n^0(BA), R) \cong \mathrm{Hom}_{\mathrm{gpschemes}}(R \otimes A^*, R \otimes \mathbb{G}_{E_n}[p^k]).$$

VII.1 (9/27) The cohomology of $B\mathbb{Z}/p^k$

The classifying spaces $B\mathbb{Z}/p^k$ have nice homotopy groups, therefore they are homologically quite infinite and misbehaving. (This is generally true for classifying spaces of finite groups.) On the other hand, the spaces $B\mathbb{Z}/p^k$ are rather straightforwardly related to objects and phenomena we care about and maybe understand: the integers, multiplication by p, $\mathbb{C}P^\infty$, ... This makes them a useful and tractable testing ground for various sorts of things. In the previous entry, this was exactly our endeavor: we proved that the comparison map

$$E_n^*(X) \to (p^{-1}E_n^*)(X)$$

becomes an equivalence upon tensoring with $p^{-1}E_n^*$ *if X is finite* but not in general, e.g. not when $X = B\mathbb{Z}/p^k$. The problem there is that somehow, the RHS becomes utterly trivial—but the LHS remains interesting (and we can understand this because of the simple algebraic role of $B\mathbb{Z}/p^k$). Last time, we carefully dwelled upon the LHS to cook up an interesting base-change that "kind of looked like" a character map. Today, I just want to rewind and focus more on the identification of $E_n^*(B\mathbb{Z}/p^k)$, as exercise and also as part of a premonition.

First, let's acknowledge the almighty principal bundle

$$\mathbb{Z}/p^k \to S^1 \xrightarrow{[p^k]} S^1.$$

We get a fiber sequence $B\mathbb{Z}/p^k \to BS^1 \to BS^1$, and over BS^1 we have simultaneously the universal complex line bundle $L = ES^1 \times_{S^1} \mathbb{C}$ and the universal S^1-bundle $S(L)$. The fiber sequence identifies $B\mathbb{Z}/p^k \cong S(L) \times_{[p^k]} BS^1 = S(L^{\otimes p^k})$, and this powers our big move: we can take a (Thom-)Gysin sequence that looks like

$$\cdots \to 0 \to E_n^{i-2}(BS^1) \xrightarrow{e(L^{p^k})} E_n^i(BS^1) \to E_n^i(B\mathbb{Z}/p^k) \to 0 \to \cdots$$

Remark VII.1.1. This arises from the long exact sequence induced by the cofibration $S(L^{p^k}) \to D(L^{p^k}) \simeq BS^1 \to Th(BS^1; L^{p^k})$, and we're identifying $H^i(Th(BS^1))$ with $H^{i-2}(BS^1)$. This gives us the same long exact sequence as the Gysin sequence. I actually did not know that until I wrote all this out.

This identifies $E_n^*(B\mathbb{Z}/p^k) \cong E_n^*[[x]]/(e(L^{p^k}))$. What is this Euler class? Well, $e(L^{p^k}) = c_1^E(L^{p^k}) = c_1^E(L)^{p^k}$ and $c_1^E(L) = x$. It follows that $e(L^{p^k}) = [p^k](x)$, and we conclude the desired identification.

Remark VII.1.2. This was much easier than I made it out to be. The inclusion $\mathbb{Z}/m \hookrightarrow S^1$ induces a fibration $S^1/(\mathbb{Z}/m) \cong S^1 \to B\mathbb{Z}/m \to BS^1$, and the associated Gysin sequence gives us what we want here.

We also gave an algebro-geometric argument by embracing the associated formal group scheme

$$\mathbb{G}_{E_n} = \mathrm{Spf}(E_n^*(BS^1)).$$

Somehow we would like to say that $\mathbb{G}_{E_n}[p^k] \cong \mathrm{Spec}(E_n^*(B\mathbb{Z}/p^k))$, i.e. that taking the E_n-cohomology of $B\mathbb{Z}/p^k$ takes p^k-torsion on the level of formal groups. This is true, although I actually cannot produce any genuinely new algebro-geometric proof—one can just look at the previous computation and put Spf everywhere. (This works for any complex-oriented spectrum, I think.)

VIII October 2025

VIII.1 (10/25) Transchromatic categorification of representations and class functions

The higher character maps are by design *transchromatic*, so you may roughly anticipate maps of spectra $E_n \to E_t$. But as somewhat investigated previously, life is not so easy. If we think of "lowering height" as a sort of approximation, then we observed that it is already tricky to approximate the E_n-cohomology of X in this manner (e.g. even if we just want to rationalize by tensoring with $p^{-1}E_n$, this is useless for $X = B\mathbb{Z}/p^k$). Vaguely, a solution is found by changing perspective from complex-oriented ring spectra to formal groups: here it makes sense to ask for the universal $C_t \in \mathsf{Mod}_{L_{K(t)}E_n}$ that splits the etale-connected sequence (and trivializes the etale part) for the associated p-divisible groups, and this somehow manifests C_t as what we want.

Altogether, we get a "higher character map"

$$E_n^X \to C_t^{L^{n-t}X}.$$

A good question at this point, and something which I have not elaborated upon yet, is what is this a map *of*? Literally speaking, what's written is a map on cohomology. But we can upgrade this map to a much more structured and powerful context. My goal today is to shed some light on this.

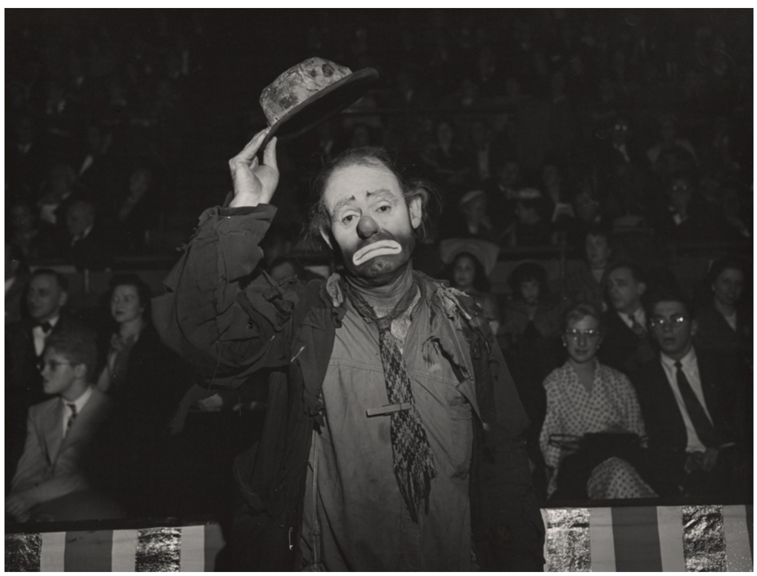

Figure 4: *Ringling Brothers and Barnum & Bailey Circus, Madison Square Garden, New York* 1943, photo by Weegee (Arthur Fellig). Source: Art Institute of Chicago, artwork ref. 2022.259, https://www.artic.edu/artworks/263380/emmett-kelly-as-weary-willie-ringling-brothers-and-barnum-bailey-circus-madison-square-garden-new-york

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To motivate the structure at hand, we can first ask what is already observable in the classical character map. (This is recovered from higher characters in the case $X = BG$, $n = 1$, and $t = 0$.) We are concerned with two functors

$$R(-) \quad \text{and} \quad \mathrm{Cl}(-).$$

But actually, we kind of have many functors here—by virtue of finiteness, in addition to the (more obvious) contravariant functors, we also get *covariant transfer*.

Example VIII.1.1 (Transfer for class functions)**.** Let $f : G \to K$ be a morphism of finite groups. Given a class function $\phi : H \to R$, we can define its *induced class function* $f_*\phi : K \to R$ by

$$(f_*\phi)(k) := \frac{1}{|f(G)|} \sum_{x \in K :\, xkx^{-1} \in f(G)} \phi(f^{-1}(xkx^{-1})).$$

This is nothing mysterious: at k, the induced class function considers those x conjugating k into $f(G)$, and averages ϕ along the fiber over those xkx^{-1}. (We average "globally" by $|f(G)|$ rather than "locally" by $|\{x : xkx^{-1} \in f(G)\}|$ so that Frobenius reciprocity holds. If this makes you feel funny, I'll remark that it also makes me feel funny.) Said differently, we are factoring f as $G \twoheadrightarrow f(G) \hookrightarrow K$, taking the standard induction of the inclusion, and then summing over the fibers of the surjection. This assembles into a *covariant* functor

$$\mathrm{Cl}(-) : \mathsf{Gp}_{\mathrm{fin}} \to \mathsf{Ab}.$$

One can similarly express transfers as $R(-) : \mathsf{Gp}_{\mathrm{fin}} \to \mathsf{Ab}$. This simultaneous covariance and contravariance wants to occur as a single functor out of *spans*. However, I think there is a problem here. I do not believe that $R(-)$ or $\mathrm{Cl}(-)$ literally refine to functors $\mathrm{Span}(\mathsf{Gp}_{\mathrm{fin}}) \to \mathsf{Ab}$. This is related to the 2-categorical nature of spans, but precisely what I believe goes wrong is rather simple: compositions go awry.

Let me explain something cute which will help suggest where to look for that misbehavior. The category Gp can be made a 2-category where for $f, g \in \mathrm{Hom}(G, H)$ we have $\mathrm{Mor}(f, f') := \{h \in H : hf(g)h^{-1} = f'(h)\}$. Now, fix a cospan

$$G \xrightarrow{f} H \xleftarrow{g} K.$$

The 1-pullback of this diagram exists and is computed as in Set_*. But, we should seek good relations with the 2-pullback. Here we must ask whether 2-pullbacks even *exist*. Since B is fully faithful, it should reflect 2-limits when they exist—this means that if we exhibit a homotopy pullback of $BG \to BH \leftarrow BK$ that is not in the essential image of B, then the cospan $G \to H \leftarrow K$ must not have a weak pullback. For this we may simply consider $Be \to BG \leftarrow Be$ for any finite group $G \neq e$, which has homotopy pullback $\Omega BG \simeq G$. Since G is not connected, it does not occur as BG', and hence $e \to G \leftarrow e$ admits no weak pullback.

Remark VIII.1.2. Let me elaborate. We should be generally interested in $\pi_0(BG \times_{BH} BK)$, for this is what we just used to detect non-existence weak pullback in $\mathsf{Gp}^{(2)}$, and this is actually all we need to check. First let me just tell you the answer:

$$\pi_0(BG \times_{BH} BK) = g(K)\backslash H/f(G).$$

To see this, regard BG and BK as one-object groupoids. The typical construction of the weak pullback then gives us a model as a groupoid: it will have objects $(*, *, h)$ where h is a morphism in BH (hence its set of objects is H), and morphisms $h \to h'$ are given by $(x, y) \in G \times K$ such that $g(y)h = h'f(x)$. (Am I unwinding the comma construction correctly?) We can define an action of $G \times K$ on H via $(x, y) \cdot h = g(x)hf(y)^{-1}$, and one deduces that π_0 is the set of orbits for this action—this unwinds to precisely the above formula for π_0. (Some of this is discussed here.)

Ok, back to the objective: does $\mathrm{Cl}(-)$ define a functor $\mathrm{Span}(\mathsf{Gp}_{\mathrm{fin}}) \to \mathsf{Ab}$? Like I said, it's compositions that go awry, and the first hint that homotopy theory is to blame is the distinction between ordinary and homotopy pullbacks of (classifying spaces of) groups. Let's consider two morphisms $\phi : G \to H$ and $\psi : K \to G$ in $\mathrm{Span}(\mathsf{Gp}_{\mathrm{fin}})$:

$$G \leftarrow H = H \quad \text{and} \quad K = K \to G.$$

This is set up with the case $H, K \leq G$ in mind. The composition $\psi \circ \phi$ is the following pullback diagram.

$$\begin{tikzcd} K \times_G H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[r, "="] \arrow[d] & H \\ K \arrow[r] \arrow[d, "="'] & G & \\ K & & \end{tikzcd}$$

If $R(-)$ or $\mathrm{Cl}(-)$ are to define functors with induction-restriction span functoriality, then we must have

$$\mathrm{Cl}(\psi \circ \phi) \stackrel{?}{=} \mathrm{Cl}(\psi) \circ \mathrm{Cl}(\phi).$$

Example VIII.1.3. Let's write out carefully what we are looking at. Denote by $\psi : K \to G$ and $\phi : G \to H$ the spans

$$K = K \hookrightarrow G \quad \text{and} \quad G \hookleftarrow H = H.$$

Then we have $R(\phi \circ \psi) = \mathrm{Ind}^H_{H\cap K}\mathrm{Res}^K_{H\cap K}$. Meanwhile, $R(\psi) \circ R(\phi) = \mathrm{Res}^G_H \mathrm{Ind}^G_K$. These two are not equal in general. Note that $R(\phi) \circ R(\psi)$ is understood by Mackey's formula:

This is quite nice. There are plenty of opportunities to go looking for homotopy theory and find it. But here I think it has brought itself to us—the natural formulation of a compatible system of restrictions and inductions for $G \mapsto R(G)$ is necessarily homotopical, and fails in a formalism lacking coherence.

That being said, we want to replace the domain Gp with something like the $(\infty, 1)$-category $\mathrm{Span}_1(\mathsf{FinAn}_{\leq 1})$. (I'm writing FinAn for the ∞-category of π-finite[10] homotopy types.) There is the small question of defining these functors on types not of the form BG; since $\mathsf{FinAn}_{\leq 1}$ is the finite coproduct-completion of finite groups, it suffices to extend $R(-)$ and $\mathrm{Cl}(-)$ to finite disjoint unions of BG's, and for this we can just take direct sums. We should like our functors of spans to preserve finite coproducts anyway, so this is forced.

There's the second question of the target for these functors. Actually, this brings us to a more preceding concern: we may think of our move from FinGp to $\mathrm{Span}(\mathsf{FinAn}_{\leq 1})$ as a "categorification," and we should subsequently ask how to categorify $R(-)$ and $\mathrm{Cl}(-)$. Let's jot down some initial observations:

- We understand that there is a *category* of representations $R(G)$, so we should already expect that the classical $R(G)$ should be replaced by some sort of category. And if we're doing that to $R(G)$, we are probably doing that to $\mathrm{Cl}(G)$ too.
- $\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathrm{Fun}(BG, \mathsf{Vect}^{\mathrm{fin}}_{\mathbb{C}})$.
- We previously made identifications

$$R(G)^\wedge_I \otimes \mathbb{Z}_p \cong KU^0_p(BG) \quad \text{and} \quad \mathrm{Cl}(G) \cong C^0_0(L_p BG).$$

So, one of our goals is to "recategorify" the ring $E^0_1(BG) = KU^0_p(BG)$ here. Lurie has the idea to parametrize the situation. Let $p_{BG} : BG \to *$ denote the terminating map. Then we may form $(p_{BG})^* E_1 \in \mathrm{Fun}(BG, \mathsf{Mod}_{E_1})$, the constant E_1-valued local system over BG. But now we can push this local system back along p_{BG} again, which amounts to taking "global sections" or forming the "cohomology with coefficients in E_1." Altogether this means

$$(p_{BG})^*(p_{BG})_* KU_p(BG) = \mathrm{Map}(BG_+, E_1).$$

Note in particular that $\pi_0 (p_{BG})^*(p_{BG})_* KU_p(BG) = KU^0_p(BG)$, which looks like what we want. The underlying idea here was to replace the association $BG \mapsto KU^0_p(BG)$ with $BG \mapsto \mathsf{Mod}^{BG}_{E_1}$, and this is how we will proceed: to categorify $G \mapsto R(G)$, we will associate to an $X \in \mathsf{FinAn}_{\leq 1}$ its category of E_1-local systems $\mathsf{Mod}^X_{E_1}$. Altogether, we are looking at the functor

$$\begin{aligned} R_{cat}(-) : \mathrm{Span}_1(\mathsf{FinAn}_{\leq 1}) &\to \mathsf{CAlg}(\mathsf{Cat}^{st}_\infty) \\ X &\mapsto \mathrm{Fun}(X, \mathsf{Mod}_{E_1}). \end{aligned}$$

The categorified class functions $\mathrm{Cl}_{cat}(-)$ is similarly formed by sending X to the category of C_0-local systems on $L_p X$.

This entry is getting long. Let me wrap up by dictating some features, niceties, and foresight:

- This framework allows us to functorially describe the free loop space shifting. The higher class functions are obtained by composing

$$X \mapsto L_p X \mapsto \mathsf{Mod}^{L_p X}_{C_0}.$$

[10]To be π-finite means you have finite and finitely many homotopy groups.

- These constructions do not require any particular amount of truncation of our spaces. Thus, we can actually just consider functions out of the category FinAn of π-finite anima.
- The category $\mathrm{Span}_1(\mathsf{FinAn})$ is the free ∞-semiadditive category on one object. We knew this, and that's sort of been the whole point in its usage here, because that semiadditivity is precisely enabling the (higher) restriction and transfers we sought to encode in R_{cat} and Cl_{cat}.
- Actually, on that note, we can recall that $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{K(t)}$ are ∞-semiadditive, and interpret E_1^X and $C_0^{L_pX}$ as occuring via functors

$$\mathrm{Span}_1(\mathsf{FinAn}) \to \mathrm{Sp}_{K(1)} \quad \text{and} \quad \mathrm{Span}_1(\mathsf{FinAn}) \to \mathrm{Sp}_{K(0)}.$$

Then we may anticipate the character correspondence as a morphism between these functors. This is in fact the case, so you may ask whether we need all that local systems business. Turns out, it really is worthwhile to make a more structured *parametrized* analysis. There are a few reasons. For one, to compare these as functors landing in $\mathrm{Sp}_{K(1)}$ and $\mathrm{Sp}_{K(0)}$, there is an implicit $L_{K(0)}$, and generally these localizations do *not* simply preserve ∞-semiadditive structure, which is bad. Secondly, we haven't explained *how* we actually get the higher character correspondences. I haven't gotten there yet either, but I think the parametrized theory is necessary to getting your hands on the height difference and free loop shifts.

- So that maybe convinces you that $R(-)$ and $\mathrm{Cl}(-)$ should, for our purposes, produce something like $\mathsf{Mod}_{E_1}^X$ and $\mathsf{Mod}_{C_0}^{L_pX}$, and so they should land in something like $\mathsf{CAlg}(\mathsf{Pr}^L_{st})$. It would be nice to have the character correspondence χ then occur as a natural transformation of functors

$$\mathrm{Span}(\mathsf{FinAn}) \to \mathsf{CAlg}(\mathsf{Pr}^L_{st}).$$

However, things are not so nice, and this is not the visage of the character maps. The precise reasons why are somewhat annoying and heavy and arcane to me currently. But here's a vibe: the target is *not* ∞-semiadditive, whereas we expect that χ should respect restriction and transfer, so there's a type-mismatch. Rather, we should step back and recall that χ is prescribing to each X a map

$$\mathsf{Mod}_{E_1}^X \to \mathsf{Mod}_{E_0}^{L_pX}$$

and I think there are *two* structural claims to make here: firstly, this map preserves the semiadditive structure (transfers/restrictions) present on both sides; secondly, as a functor of $X \in \mathsf{FinAn}$, these character maps respect the semiadditive structure *which you only get on the target if you understand it as landing in* FinAn*-parametrized categories, not* $\mathsf{CAlg}(\mathsf{Pr}^L_{st})$. That last point, I think, is important—our big move is to digest this as a *parametrized* map, i.e. we should interpret $\mathsf{Mod}_{E_1}^{(-)}$ and $\mathsf{Mod}_{C_0}^{L_p(-)}$ as parametrized categories (which turn out to be semiadditive in whatever sense you want) and χ as a parametrized family of maps between them (also as semiadditive as you want), thus χ occurs as a map

$$\mathsf{FinAn} \xrightarrow{\chi} \mathrm{Map}(\mathsf{Mod}_{E_1}^{(-)}, \mathsf{Mod}_{C_0}^{L_p(-)}).$$

I think, but I'm not sure, that this is an $(\infty,1)$-categorical way to handle the situation, and it unwinds to the "more intuitive" $(\infty,2)$-categorical approach to the original problem which we don't take because that would require higher ∞-category theory? I'm just going to move on from this point now.

Is this really an issue? Can't we make it land in say Cat_∞? I think the realer issue is that we're looking for a way to work around $(\infty,2)$-category theory, and the strategy is to use parametrized categories because they are somehow easier to deal with and "unwind" to $(\infty,2)$-categorical stuff...

- Also, secretly, I've neglected to say anything about *tempered* local systems, and some of the Mod's above should really have a hat on them. The one-liner explanation for this is that we sometimes want to restrict to (say) the subcategory of $K(n)$-local modules that are already $K(t)$-local?

Never an easy ride!

IX November 2025

IX.1 (11/14) Parametry and Prejudice

In my previous post, we reflected on the restriction and transfer functoriality of the construction $G \mapsto R_{\mathbb{C}}(G)$. Our big move was to realize this (up to completion) as the π_0 of the enhancement

$$\mathrm{Span}_1(\mathsf{FinAn}) \ni X \longmapsto \lim_{X} KU_p \cong KU_p^X \in \mathrm{Sp}_{K(1)}.$$

We similarly realized $\mathrm{Cl}(G)$ via the functor $X \mapsto C_0^X$. We were left dazed and confused as to how to similarly "categorify" the character map χ in a good way. Part of the difficulty here is that this task teases the thread of the universal fabric traversing $(\infty, 2)$-category theory, and here be dragons.

Figure 5: Chicken, Female and Serpent. Victor Brauner. Source: Art Institute of Chicago, Ref. no. 1992.202, https://www.artic.edu/artworks/117355/chicken-female-and-serpent.

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