General Relativity and Einstein–Cartan Theory

Marc A. Nieper-Wißkirchen

March 6, 2014

Preface

Gravity is one of the fundamental interactions of nature, by which all physical bodies attract each other. The purpose of these notes is to give a physically motivated and mathematically sound introduction to Einstein's general relativity. Since its invention, it has passed all experimental tests and is the simplest theory that is able to describe all observed phenomena of gravity. In short, it is the accepted theory of gravity in modern physics. This is not to say that general relativity is set in stone. By incorporating possible intrinsic angular moment of macroscopic matter, one arrives at the mathematically equally beautiful Einstein–Cartan theory, which encloses general relativity. While general relativity is a metric theory that builds on the mathematical concept of a Lorentzian manifold, Einstein–Cartan theory is a true gauge theory of gravity. It will be described in the latter chapters of these notes.

Addressees of these notes are, on the one hand side, graduate students of physics who are willing to cope with abstract mathematical concepts and, on the other hand, graduate students of mathematics with a strong background in classical physics. These notes grew out of a lecture course the author gave in the winter term 2013/14 at the University of Augsburg, and the audience of the course was a mixture of students from the physics and the mathematical department.

The theory of general relativity and the novel effects being predicted by it — for example, gravitational time dilation or gravitational waves — have been fascinating many people, but to understand these phenomena on a quantative level, one has to delve deeply into the mathematics of general relativity. One the other hand, it is a much rewarding untertaking. One will have grasped one of the most beautiful physical theories (if not *the* most beautiful one). These notes show one of the possible routes there, a route the author would have liked to go when he learnt general relativity. Going this route also provides the reader with a solid knowledge of differential geometry.

That said, it *is* possible to formulate the heart of the theory of general relativity in one sentence in plain English, namely:

> The mass density measured by any observer is the scalar curva-

ture of that observer's space divided by 16π.

Of course, without any further explanations of the contained mathematical terms and an accompanying physical interpretation, this statement is just as meaningful as simply stating[1] $\operatorname{div} E = 4\pi\,\rho$ without any further explanations of the terms involved. Nevertheless, the simplicity of this statement already shows the beauty of general relativity.

Augsburg, March 6, 2014 — Marc Nieper-Wißkirchen

[1] Gauß's law in gaussian units

History

The theory of general relativity was being developed by Albert Einstein between 1907 and 1915. It builds upon and combines the earlier theories of Newtonian gravity and special relativity. Although the basic theory hasn't changed since then, there have been many contributions afterwards. The following timeline summarizes the history as far it is relevant to the notes at hand.

1609 Kepler pushlishes his first two laws of planetary motion.

1619 Kepler pushlishes his third law of planetary motion.

1638 Galilei's equations for a falling body.

1687 Newton publishes his law of universal gravitation.

1798 Cavendish measures Newton's gravitational constant.

1862 Maxwell's equations of electromagnetism.

1887 The Michelson-Morley experiment fails to detect a stationary luminiferous aether.

1889 FitzGerald proposes Lorentz contraction.

1905 Formulation of special relativity by Einstein.

1915 Derivation of Einstein's field equations from an action principle by Hilbert.

1915 Einstein's theory of general relativity.

1916 Schwarzschild found the first exact solution of Einstein's field equations.

1916 Einstein shows that the perihelion precession of Mercury can be fully explained by general relativity.

1919 Eddington's expedition confirms that the deflection of light by the Sun is as predicted by general relativity.

1922 Friedmann found a cosmological solution to Einstein's equations, in which the universe may expand or contract.

1922 Introduction of the cosmological constant by Einstein into his field equations.

1922 Proposal of the Einstein–Cartan theory by lie Cartan.

1929 Hubble finds evidence that the universe is expanding.

1959 Direct measurement of the gravitational redshift of light in the Pound-Rebka experiment.

1964 Discovery of the cosmis microwave background by Penzias and Wilson.

1964 Discovery of the X-ray source Cygnus X-1, now widely accepted to be a black hole.

Notations

Standard sets We use the following notations for the standard sets: The set of natural numbers (which includes 0, by definition) is denoted by $\mathbf{N}_0$, the set of integers by $\mathbf{Z}$, the set of rational numbers by $\mathbf{Q}$, the set of real numbers by $\mathbf{R}$ and the set of complex numbers by $\mathbf{C}$.

Linear algebra In an n-dimensional vector space, the Kronecker symbol is a scalar defined by

$$\delta_{ij} \coloneqq \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for all i, $j \in \{1, \ldots, n\}$.

Maps By a *differential map* we will always mean of map of class $\mathcal{C}^\infty$ that is a smooth map. *Differentiability* thus means the existence of continuous derivates to all orders.

The term *function* will be reserved for smooth maps with values in $\mathbf{R}$. Thus a function is always a smooth function.

Cartesian space The standard basis formed by $e_1, \ldots, e_n$ of n-dimensional space $\mathbf{R}^n$ is a basis of the underlying vector space such that $v = \sum_{i=1}^n v_i\, e_i$ for each $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$.

The standard Euclidean norm of n-dimensional space $\mathbf{R}^n$ is denoted by

$$\|v\| \coloneqq \sqrt{\sum_{i=1}^n v_i^2}$$

for $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$.

The partial derivative of a function ϕ defined on an open subset of $\mathbf{R}^n$ in direction i is denoted by ∂_i, that is

$$\partial_i \phi \colon v \mapsto \frac{\partial \phi(v + t\, e_i)}{\partial t}\Big|_{t=0}$$

for $i = 1, \ldots, n$. In case of $n = 1$, we set $\partial \coloneqq \partial_1$.

Contents

Chapter 1

Spacetime

1.1 Introduction

The fundamental notion of special relativity is that of an *event*. In pre-relativity, things located at a specific point in space and time, for example the start of the Saturn V rocket launching the Apollo 11 spaceflight, are described by a unique time in Newton's absolute time and a unique place in Euclidean absolute space. The notion of an event combines these two qualities into one, which is essential for special relativity as space and time by themselves lose their absoluteness.

Spacetime is the set of all possible events. Any event that can be imagined to happen is an element or a *point* in spacetime. Mathematically, spacetime has more structure than simply being a set: A point in this set, that is an event, is usually described by four scalars, one time and three space coordinates, making spacetime four-dimensional. For a general set, however, there is no well-defined notion of coordinates or dimension. The theory of manifolds, which will be presented in the next section, is the correct mathematical setting in which notions like coordinates and dimension make sense.

1.2 Manifolds

An *n-dimensional chart* $x = (x, U)$ on a set M, whose elements we call *points*, is an injective map $x = (x_1, \ldots, x_n) \colon U \to \mathbf{R}^n$ onto an open subset of $\mathbf{R}^n$ defined on a subset U of M, the *domain of definition of* x. Given a point p lying in the domain of definition U, the n scalars $x_1(p), \ldots, x_n(p)$ are the *coordinates of p with respect to the chart x*. A family $\{(x_i, U_i)\}_{i \in I}$ of charts on M *covers* M if $M \subseteq \bigcup_{i \in I} U_i$ that is if every point p of M lies in at least domain of definition of the charts x_i. In order to be able to describe events in spacetime by four coordinates, we postulate that *there is a distinguished family of four-dimensional charts that cover spacetime.*

Given two n-dimensional charts (x, U) and (y, V) of a set M, the map

$$y \circ x^{-1}|x(U \cap V) \colon x(U \cap V) \to y(U \cap V)$$

is called the *coordinate transformation from x to y.* If this map is a diffeomorphism between open subsets of $\mathbf{R}^n$, the charts x and y are said to be *compatible.* An *n-dimensional atlas of M* is a family of pairwise compatible n-dimensional charts of M that covers M. In order to be able to employ analytic methods, we extend our postulate above by postulating that *there is a distinguished four-dimensional atlas of spacetime.*

Let $\mathfrak{A}$ be any n-dimensional atlas of a set M, and let x and y be two arbitrary n-dimensional charts of M. If x and y are each compatible with each chart in $\mathfrak{A}$, they are compatible with each other. Therefore, every atlas $\mathfrak{A}$ can be uniquely enlarged to a *maximal atlas* in which it is contained by adding all n-dimensional charts to $\mathfrak{A}$ that are compatible with each in chart in $\mathfrak{A}$. We also say that the maximal atlas is *generated* by the charts contained in $\mathfrak{A}$.

An *n-dimensional premanifold M* is set M together with a maximal n-dimensional atlas $\mathfrak{U}^\infty(M)$ of M. The set of all charts (x, U) in $\mathfrak{U}^\infty(M)$ that contains a given point $p \in M$ is denoted by $\mathfrak{U}^\infty(p, M)$. A *chart of M* is a chart in the maximal atlas $\mathfrak{U}^\infty(M)$. An *atlas of the premanifold M* is any atlas of M which is contained in the maximal atlas $\mathfrak{U}^\infty(M)$. With these terms, we can say that *spacetime is a four-dimensional premanifold.*

Using the atlas of a premanifold M, one can define the notion of neighborhoods of points on M: A subset G of the underlying set of M is an *open subset of M* if $x(U \cap G)$ is an open subset for each chart (x, U) of M. The system of these open subsets of M is a topology (see Section A.1) on the underlying set of M, the *canonical topology of M*, so that M is a topological space in a canonical way.

For any chart (x, U) of the n-dimensional premanifold, a subset V of U is open in M if and only if $x(V)$ is open in $\mathbf{R}^n$. In particular, U is open. The map $x \colon U \to \mathbf{R}^n$ is continuous in the sense of maps between topological spaces.

Generally, the so-defined topology is ill-behaved, however; points may not be distinguishable by the topology and one may need infinitely many charts to cover even small neighborhoods of points. Excluding these cases by adding technical conditions on the underlying topological space leads to the final definition:

Definition 1.1. An *n-dimensional manifold M* is an n-dimensional premanifold M whose underlying topological space is a paracompact Hausdorff space (see Section A.7).

We extend our postulate from above by postulating that *spacetime is a four-dimensional manifold.*

The most basic example of an n-dimensional manifold is the n-dimensional cartesian space $\mathbf{R}^n$. Its canonical atlas is the maximal atlas which is generated by the identity map $\mathrm{id}_{\mathbf{R}^n} \colon \mathbf{R}^n \to \mathbf{R}^n$ viewed as an n-dimensional chart. The topology defined by this atlas is, of course, the canonical topology of $\mathbf{R}^n$, which is the topology of a paracompact Hausdorff space.

Requiring that the underlying topological space of spacetime is a paracompact Hausdorff space has a number of pleasant consequences: For example, the underlying topological spaces of manifolds are normal, which follows from Proposition A.2. Furthermore, we have:

Proposition 1.1. *The underlying space of an n-dimensional manifold M is locally compact.*

(For the definition of local compactness, see Section A.6.)

Proof. For $p \in M$ choose a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$. By openness of $x(U)$ in $\mathbf{R}^n$, there exists an $\epsilon > 0$ such that $U_\epsilon(x(p)) \subseteq x(U)$. We claim that

$$K \coloneqq x^{-1}(\overline{U_{\frac{\epsilon}{2}}(x(p))})$$

is a compact neighborhood of p:

The subset K is a neighborhood of p as $p \in x^{-1}(U_{\frac{\epsilon}{2}}(x(p))) \subseteq K$ and because $x^{-1}(U_{\frac{1}{2}\epsilon}(x(p)))$ is an open subset of M.

To show that K is compact, let $(U_i)_{i \in I}$ be a family of open subsets of U that cover K. By definition of the topology of M, the images $(x(U_i))_{i \in I}$ form an open cover of the compact space $\overline{U_{\frac{\epsilon}{2}}(x(p))}$. Thus there is a finite subset $J \subset I$ such that $(x(U_i))_{i \in J}$ is an open cover of $\overline{U_{\frac{\epsilon}{2}}(x(p)}$. It follows that $(U_i)_{i \in J}$ is a finite open cover of K. $\square$

Often, we have to restrict our attention to small pieces (that is, open subsets) of a given n-dimensional manifold M, for example spacetime. This can be done as follows: Let G be an open subspace of M. Let $\mathfrak{U}^\infty(G)$ be the unique maximal n-dimensional atlas of G that contains the atlas

$$\{(x|U \cap G, U \cap G) : (x, U) \in \mathfrak{U}^\infty(M)\}.$$

Then G becomes an n-dimensional manifold itself, whose underlying topological space is the subspace G of M. An n-dimensional manifold of this form is called an *open submanifold of M*. The domains of definitions of the charts of M are open submanifolds of M.

1.3 Functions on a manifold

As we have postulated, we can "measure" each event in spacetime by giving the scalar values of four coordinates (after choosing a chart). Each coordinate can be thought of as a scalar field that assigns to each point (in the domain of definition of its chart) a scalar value. This notion is generalised as follows:

Definition 1.2. A *(smooth) function* $\phi\colon M \to \mathbf{R}$ on an n-dimensional manifold M is a mapping ϕ from the underlying set of M to the reals such that for each chart (x, U) of M the map

$$\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$$

is smooth.

In order to show that a map $\phi\colon M \to \mathbf{R}$ is a function in the above sense, it suffices that for each point $p \in M$ there exists a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$ such that $\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$ is smooth.

Every function is continuous with respect to the underlying topologies of M and $\mathbf{R}$, respectively. Let ϕ be a function on M and G an open submanifold. The *restriction $\phi|G$ of ϕ to G*, given by

$$\phi|G\colon G \to M, p \mapsto f(p)$$

is a function on the manifold G. For any chart (x, U) of M, the coordinate functions $x_1, \ldots, x_n\colon U \to M$ are smooth functions on the open submanifold U.

All functions on M form an algebra (over the reals), denoted by $\mathcal{C}^\infty(M)$, where addition and multiplication are defined point-wise. The function with constant value $c \in \mathbf{R}$ is often denoted by $\underline{c}$. The sets of functions of the open submanifolds of M fulfill the *sheaf condition*, that is for every open cover $(U_i)_{i\in I}$ of M and functions $\phi_i \in \mathcal{C}^\infty(U_i), i \in I$ one has

$$(\forall i, j \in I : \phi_i|U_i \cap U_j = \phi_j|U_i \cap U_j) \implies \exists! \phi \in \mathcal{C}^\infty(M)\, \forall i \in I : \phi|U_i = \phi_i.$$

In other words, we can uniquely glue functions along open submanifolds.

If $\Phi\colon \mathbf{R} \to \mathbf{R}$ is any smooth function between the reals, the composition $\Phi \circ \phi\colon M \to \mathbf{R}$ is a function on M whenever $\phi\colon M \to \mathbf{R}$ is a function.

A manifold possesses many functions in a sense made precise by the following theorem, which relies essentially on the paracompactness of the manifold (for the notion of the support supp of a function, see Section A.2):

Theorem 1.1. *Every open cover $(U_i)_{i\in I}$ of a manifold M has a* subordinate partition $(\lambda_i)_{i\in I}$ of unity*, which is a family of functions $\lambda_i\colon M \to \mathbf{R}$ with the following properties:*

Range *For all $i \in I$ and $p \in M$, one has $0 \leq \lambda_i(p) \leq 1$.*

Support *For all $i \in I$, one has $\operatorname{supp} \lambda_i \subseteq U_i$.*

Local finiteness *Every $p \in M$ posesses a neighborhood $G \in \mathfrak{U}^0(p, M)$ such that there are only finitely many $i \in I$ with $\operatorname{supp} \lambda_i \cap G \neq \emptyset$.*

Normalization *For every $p \in M$, the equality $\sum_{i \in I} \lambda_i(p) \equiv 1$ holds.*

The proof relies on the following lemma, which is also of independent interest:

Lemma 1.1. *Let K be a compact subspace of an n-dimensional manifold M. For any open neighborhood G of K in M, there exists a function $\phi\colon M \to \mathbf{R}$ with*

$$\forall p \in M : \phi(p) \geq 0, \qquad \forall p \in K : \phi(p) > 0, \qquad \operatorname{supp} \phi \subseteq G. \tag{1.1}$$

Proof of Lemma 1.1. For any $p \in K$ choose a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$. By the openness of $x(U)$ in $\mathbf{R}^n$, there exists an $\epsilon > 0$ with $U_\epsilon(x(p)) \subseteq x(U \cap G)$. Choose a bump function $\psi\colon \mathbf{R}^n \to \mathbf{R}$ (see Section B.1) such that $\psi(u) \geq 0$ for all $u \in \mathbf{R}^n$, $\psi(u) = 1$ for $\|u\| \leq \frac{\epsilon}{3}$ and $\psi(u) = 0$ for $u \geq \frac{2\epsilon}{3}$. By the sheaf condition,

$$\phi_p\colon M \to \mathbf{R}, q \mapsto \begin{cases} \psi(x(q) - x(p)) & \text{if } q \in U, \\ 0 & \text{if } q \in M \setminus x^{-1}(\overline{U_{\frac{2\epsilon}{3}}(x(p))}) \end{cases}$$

defines a function on M as U and $M \setminus x^{-1}(\overline{U_{\frac{2\epsilon}{3}}(x(p))})$ form an open cover of M. This function has the properties

$$\forall q \in M : \phi_p(q) \geq 0, \qquad \phi_p(p) > 0, \qquad \operatorname{supp} \phi_p \subseteq G.$$

The open subsets $U_p \coloneqq \{q \in M : \phi_p(q) > 0\}$ with $p \in K$ cover K. By compactness of K, there exists a finite subset $A \subseteq K$ such that K is covered by $(U_p)_{p \in K}$. By construction, the function $\phi = \sum_{p \in A} \phi_p$ fulfills (1.1). $\square$

Proof of Theorem 1.1. For every $p \in M$ choose by local compactness of M a relatively compact neighborhood $G_p \in \mathfrak{U}(p, M)$. By the covering property, there exists an $i \in I$ with $p \in U_i$. The intersection $G_p \cap U_i$ is again relatively compact in M, so we may assume that already $G_p \subseteq U_i$. So $(G_p)_{p \in M}$ is a refinement of the cover $(U_i)_{i \in I}$. Let $(V_j)_{j \in J}$ be a locally finite refinement of $(G_p)_{p \in M}$. In particular, $(V_j)_{j \in J}$ is a locally finite refinement of $(U_i)_{i \in I}$ and each V_j is relatively compact in M. By the shrinking lemma, Proposition A.1, and the normality of the underlying topological space of M, there exists another open cover $(V'_j)_{j\in J}$ of M with $\overline{V'_j} \subseteq V_j$ for all $j \in J$.

For every $j \in J$, choose by compactness of $\overline{V'_j}$ and Lemma 1.1 a function $\phi_j \in \mathcal{C}^\infty(M)$ with

$$\forall p \in M : \phi_j(p) \geq 0, \qquad \forall p \in V'_j : \phi_j(p) > 0, \qquad \operatorname{supp} \phi_j \subseteq V_j.$$

By the local finiteness of the open cover $(V_j)_{j\in J}$, the sum $\phi := \sum_{j\in J} \phi_j$ is locally a finite sum and thus defines a function on M by the sheaf condition. By the covering property of $(V'_j)_{j\in J}$, one has $f(p) > 0$ for all $p \in M$.

As $(V_j)_{j\in J}$ is a refinement of the cover $(U_i)_{i\in I}$, there exists a map $\alpha\colon J \to I$ with $V_j \subseteq U_{\alpha(i)}$ for all $j \in J$. For each $i \in I$ set $J_i := \alpha^{-1}(i)$, so $(J_i)_{i\in I}$ becomes a partition of I. For all $i \in I$, set $U'_i := \bigcup_{j\in J_i} V_j \subseteq U_i$ and finally

$$\lambda_i = \sum_{j\in J_i} \frac{\phi_i}{\phi}.$$

By a similar argument as above, the sum on the right hand side is locally finite and, thus, λ_i is a function on M with $\lambda_i(p) \geq 0$ for all $p \in M$.

By construction, $\operatorname{supp} \lambda_i \subseteq \bigcup_{j\in J_i} V_j = U'_j \subseteq U_i$, which proves the support axiom of a partition of unity. The covering $(U'_i)_{i\in I}$ is locally finite as the covering $(V_j)_{j\in J}$ is locally finite; this uses the disjointness of the J_i. From $\operatorname{supp} \lambda_i \subseteq U'_i$ for all $i \in I$, the local finiteness axiom follows. By construction, $\sum_{i\in I} \lambda_i \equiv 1$, which is the normalization axiom of a partition of unity. From this, the range axiom follows as we already know that $\lambda_i(p) \geq 0$ for all $p \in M$. $\square$

The existence of partitions of unity on manifolds implies that functions can be extended in the following sense:

Corollary 1.1. *Let ϕ be a function defined on an open neighborhood G of a point p in a manifold M. Then there exists a function $\widehat{\phi} \in \mathcal{C}^\infty(M)$ with $\operatorname{supp} \widehat{\phi} \subseteq G$ and such that $\widehat{\phi}$ coincides with ϕ on a neighboorhood $U \subseteq G$ of p in M.*

The function $\widehat{\phi}$ is called an *extension of ϕ by zero away from p.*

Proof. By local compactness (see Proposition A.1), there exists a compact neighborhood $K \in \mathfrak{U}(p, G)$. Choose a partition (λ, μ) of unity subordinate to the open cover $(G, M \setminus K)$ of M. By the sheaf condition,

$$\hat{\phi}\colon M \to \mathbf{R}, \begin{cases} \lambda(p) \cdot \phi(p) & \text{if } p \in G \\ 0 & \text{if } p \in M \setminus \operatorname{supp} \lambda \end{cases}$$

is a well-defined function on M with $\operatorname{supp}\hat{\phi} \subseteq G$ and which coincides with ϕ on K. $\square$

For any two points p, $q \in M$ with $p \neq q$, there exists an open neighborhoods $G \in \mathfrak{U}^0(p, M)$ with $q \notin G$. Extending the constant function $\underline{1}|G$ by zero away from p yields a function, which is 1 on p and 0 on q. Thus, we have

$$\forall p, q \in M : p \neq q \implies \exists \lambda \in \mathcal{C}^\infty(M) : \lambda(p) = 1, \lambda(q) = 0. \tag{1.2}$$

If we denote by p^* for all $p \in M$ the algebra homomorphism

$$p^* \colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto \phi(p), \tag{1.3}$$

we can reformulate (1.2) by saying that the map $p \mapsto p^*$ is injective, that is the algebra of functions *separate points*. The algebra of functions on spacetime is therefore a full set of observables: for any two distinct events there is a function that takes different values on both events.

1.4 Morphisms

A physical body traces out a curve of events in spacetime M, namely those events where an observer meets the physical body, its *world line*. If the physical body carries a clock with it, each point of its world line is parametrized by a scalar, the clock's time measured at that event. In other words, the path of the physical body in spacetime together with its clock defines a map $J \to M$, where J is an (open) interval. Both the domain and the target of this map is a manifold, where J is viewed as an open submanifold of the reals. A physical body does not jump through spacetime, so the map will be continuous. In order to employ analytical methods, it is sensible to assume moreover that this map is a morphism according to the following definition:

Definition 1.3. A *morphism* $f \colon M \to N$ between two manifolds M and N is a continuous map $f \colon M \to N$ such that for each pair of charts $(x, U) \in \mathfrak{U}^\infty(M)$ and $(y, V) \in \mathfrak{U}^\infty(N)$ the composition

$$y \circ f \circ x^{-1}|x(f^{-1}(V) \cap U) \colon x(f^{-1}(V) \cap U) \to y(V)$$

is a smooth map (between open subsets of cartesian spaces).

In accordance with our above wording, a *curve* $\alpha \colon J \to M$ is a morphism where J is an open interval viewed as a submanifold of $\mathbf{R}$. Thus a world line of a physical body becomes a curve in this sense by endowing it with a clock.

A map $f\colon M \to N$ between manifolds is a morphism if and only if for all functions $\psi \in \mathcal{C}^\infty(N)$ the *pullback of ψ by f*, given by

$$f^{-1}\psi\colon M \to \mathbf{R}, p \mapsto \psi(f(p)), \tag{1.4}$$

is a function on M. (The pullback itself is a map

$$f^{-1}\colon \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M) \tag{1.5}$$

of algebras.)

Thus, a map $\phi\colon M \to \mathbf{R}$ defined on a manifold M is a morphism if and only if it is a function. Further, a map $f\colon G \to H$ between open subsets of cartesian spaces is a morphism if and only if it is a smooth map in the sense of calculus.

The identity $\mathrm{id}_M\colon M \to M$ of M is a morphism. The composition $g \circ f$ of two morphisms $f\colon M \to N$ and $g\colon N \to P$ between manifolds is again a morphism. The manifolds together with the morphisms between them thus form a *category*.

The *inclusion* $i\colon U \to M, p \mapsto p$ of an open submanifold U of M is a morphism. Thus, the restriction $f|U = f \circ i$ of a morphism $f\colon M \to N$ to U is again a morphism.

1.5 Vectors

By an *observer*, we mean a curve $\alpha\colon J \to M$ in spacetime M, which might be traced out by a physical body, and relative to which one may measure events. First and foremost, an observer defines a set of events, namely the set $\alpha(J)$ of events traced out by the observer. This defines the relation of being at the same spatial location as the observer: an event p is *at the same spatial location as the oberver* if and only if $p \in \alpha(J)$. The curve α, however, is not determined by the set $\alpha(J)$ of events alone; it does not capture the dynamics of α. For example, if we move infinitesimally from t to $t + \mathrm{d}t$ on J, we expect that $\alpha(t)$ moves infinitesimally to $\alpha(t) + d\alpha(t)$. In other words, we expect $\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}$ to be a vector on M tangent to the curve $\alpha(t)$. At the moment, howoever, we have no notion of such a thing as a vector, which with we could describe dynamics, on a manifold M. The purpose of this section is to remedy this.

The basic idea of defining vectors on a manifold M is the following: Any point p assigns a value $p^*\phi = \phi(p)$ to each function $\phi \in \mathcal{C}^\infty(M)$, and we can reconstruct the point from these values as the map $p \mapsto p^*$ is injective. Assume that we already have the notion of a vector v at a point p. It would assign a change of values along v, a derivative, written $v \cdot \phi$, to each function f. Thus such a vector would give rise to a function

$$v\colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto v \cdot \phi.$$

A vector should be reconstructable from this map, so that the functions form a complete set of observables for vectors. This map should have the properties of a first-order derivative operator, so we end up with the following definition:

Definition 1.4. A *vector v at a point $p \in M$* is a map

$$v\colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto v \cdot \phi,$$

such that the following axioms hold:

Constants For all $c \in \mathbf{R}$, one has $v \cdot \underline{c} = 0$.

Linearity For all a, $b \in \mathbf{R}$ and all ϕ, $\psi \in \mathcal{C}^\infty(M)$, one has $v \cdot (a\,\phi + b\,\psi) = a\,v \cdot \phi + b\,v \cdot \psi$.

Leibniz rule For all ϕ, $\psi \in \mathcal{C}^\infty(M)$, one has $v \cdot (\phi\,\psi) = (v \cdot \phi)\,\psi(p) + \phi(p)\,(v \cdot \psi)$.

The real number $v \cdot \phi$ is the *derivative of ϕ along v*. For example, the constants axiom says that constants have vanishing derivative along any vector.

The set of all vectors at a point $p \in M$ is called the *tangent space of M at p* and is denoted by $\mathrm{T}_p M$. The tangent space $\mathrm{T}_p M$ is canonically a (real) vector space, where addition and scalar multiplication are defined “function-wise”:

$$0 \cdot \phi := 0, \qquad (a\,v) \cdot \phi := a\,(v \cdot \phi), \qquad (v + w) \cdot \phi := v \cdot \phi + w \cdot \phi$$

for all v, $w \in \mathrm{T}_p M$, $a \in \mathbf{R}$ and $\phi \in \mathcal{C}^\infty(M)$.

The first basic result, which partially shows that the above definition of a vector in fact coincides with the intuitive notion, is that the derivative of a function ϕ along a vector at a point p depends only on the values of ϕ in a neighborhood of p:

Lemma 1.2. *Let G be an open neighborhood of a point $p \in M$ such that $\phi|G = \psi|G$ for two functions ϕ and $\psi \in \mathcal{C}^\infty(M)$. Then*

$$\forall v \in \mathrm{T}_p M : v \cdot \phi = v \cdot \psi.$$

Proof. By Lemma 1.1, choose a smooth function $\lambda \in \mathcal{C}^\infty$ with $\lambda(p) = 1$ and $\operatorname{supp} \lambda \subseteq G$. Then

$$\begin{aligned} v \cdot \phi - v \cdot \psi &= v \cdot (\phi - \psi) = v \cdot ((1 - \lambda)\,(\phi - \psi)) \\ &= (v \cdot (1 - \lambda))\ (\phi(p) - \psi(p)) + (1 - \lambda(p))\ (v \cdot (\phi - \psi)) = 0 + 0 \\ &= 0. \end{aligned}$$

□

Given a vector v at p and a function ϕ defined in a neighborhood G of p, we can therefore speak of the derivative $v \cdot \phi$ of ϕ along v: Let $\widehat{\phi}$ be any extension of ϕ by zero away from p. As $\widehat{\phi}$ and ϕ coincide in a neighborhood of p, the derivative

$$v \cdot \phi \coloneqq v \cdot \widehat{\phi}$$

does not depend on the choice of $\widehat{\phi}$.

With this result we can completely determine the structure of the tangent space T_pM. In fact, we have:

Theorem 1.2. *Let M be an n-dimensional manifold. For each point $p \in M$, the tangent space T_pM is an n-dimensional vector space. More precisely, given a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$, a basis of T_pM is given by the vectors*

$$\frac{\partial}{\partial x_i}|_p \colon \mathcal{C}^\infty(M) \to \mathbf{R}, f \mapsto \delta_i((f|U) \circ x^{-1})(x(p)) =: \frac{\partial f}{\partial x_i}(p), \tag{1.6}$$

where $i = 1, \ldots, n$. If $(y, V) \in \mathfrak{U}^\infty(p, M)$ is another chart around p, the change of basis is given by

$$\frac{\partial}{\partial x_i}|_p = \sum_{j=1}^n \partial_i((y_j|U) \circ x^{-1})(x(p))\, \frac{\partial}{\partial y_j}|_p = \sum_{j=1}^n \frac{\partial y_j}{\partial x_i}(p)\, \frac{\partial}{\partial y_j}|_p. \tag{1.7}$$

The basis $(\frac{\partial}{\partial x_i}|_p)_i$ is called the *chart's x induced basis of* T_pM.

Proof. To show that (1.6) defines a vector at p is straight-forward and follows from the linearity and product rule of the partial differentiation in calculus.

The linear independence of the n vectors defined by (1.6) follows directly from the following observation: For any n-tupel $(a_1, \ldots, a_n)$ of scalars with $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p = 0$, one calculates

$$\begin{aligned} 0 &= \left(\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p\right) \cdot x_j = \sum_{i=1}^n a_i\, \partial_i(x_j \circ x^{-1})(x(p)) \\ &= \sum_{i=1}^n a_i\, \delta_{ij} = a_j \end{aligned}$$

for all $i = 1, \ldots, n$.

To show that the n vectors defined by (1.6) span the tangent space $\mathrm{T}_p M$ at p, let ϕ be any function on M. By Theorem B.2, there exist functions g_1, $\ldots$, g_n defined in a neighborhood of p in U such that

$$\phi = \phi(p) + \sum_{i=1}^{n} (x_i - x_i(p))\, g_i(u)$$

in a neighborhood of p in $\mathbf{R}^n$. Given any vector $v \in \mathrm{T}_p M$, we thus have

$$\begin{aligned} v \cdot \phi &= v \cdot (\phi(p) + \sum_{i=1}^{n} (x_i - x_i(p))\, g_i) \\ &= \sum_{i=1}^{n} ((v \cdot (x_i - x_i(p)))\, g_i(p) + (x_i(p) - x_i(p))\, (v \cdot g_i)) = \sum_{i=1}^{n} (v \cdot x_i)\, g_i(p). \end{aligned}$$

In other words, $v \cdot \phi = \sum_{j=1}^{n} (v \cdot x_j)\, \frac{\partial}{\partial x_j}|_p \cdot \phi$ for any vector $v \in \mathrm{T}_p M$. Thus

$$\forall v \in \mathrm{T}_p M : v = \sum_{i=1}^{n} (v \cdot x_j)\, \frac{\partial}{\partial x_i}|_p,$$

which is a linear combination of the induced basis.

As this formula also works for the coordinate system (y, V), we get (1.7) for $v = \frac{\partial}{\partial x_i}|_p$. □

The theorem immediately gives the tangent space at a point p of an open submanifold U of n-dimensional space: It is spanned by the n linearly independent vectors $\partial_1(p)$, $\ldots$, $\partial_n(p)$ where

$$\partial_i(p) \cdot \phi \coloneqq (\partial_i \phi)(p)$$

for all $i \in \{1, \ldots, n\}$. Thus, there is a canonical isomorphism

$$\mathbf{R}^n \to \mathrm{T}_p M, (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} u_i\, \partial_i(p).$$

whose inverse is denoted by

$$\mathrm{d}_p x \colon \mathrm{T}_p M \to \mathbf{R}^n.$$

By the very definition of a vector, we can differentiate functions on a manifold, that is morphisms to the manifold $\mathbf{R}$ along a vector. What is yet missing is the notion of a derivate of an arbitrary morphism along a vector.

In other words, we are missing a definition of how a morphism maps a vector on the domain manifold to a vector on the target manifold. This definition is straight-forward:

Let $f\colon M \to N$ be a morphism of manifolds. For a point $p \in M$, let $v \in \mathrm{T}_p M$. The map

$$f_* v\colon \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(N), \psi \mapsto v \cdot (\psi \circ f)$$

fulfills the linearity and constant axioms and a Leibniz rule as follows

$$(f_* v) \cdot (\psi\, \psi') = (v \cdot \phi)\, \psi(f(p)) + \phi(f(p))\, (v \cdot \psi).$$

In other words, $f_* v$ defines a vector at $f(p)$ defined by the equation

$$\forall \psi \in \mathcal{C}^\infty(N) : (f_* v) \cdot \psi = v \cdot (f^{-1} \psi).$$

The vector $f_* v$ is called the *push-forward of v along f*. The induced linear map on the tangent spaces is denoted by

$$\mathrm{T}_p f\colon \mathrm{T}_p M \to \mathrm{T}_p N, v \mapsto f_* v$$

and called the *tangent map of f at p*. That the push-forward indeed generalizes the derivative of functions follows from the observation that

$$\phi_* v = (v \cdot \phi)\, \delta(\phi(p))$$

for any function $\phi \in \mathcal{C}^\infty(M)$ viewed as a morphism $\phi\colon M \to \mathbf{R}$.

The tangent map of the identity morphism is the identity map, the tangent map of a composition is the composition of the tangent maps. More precisely, we have

$$\forall v \in \mathrm{T}_p M : \quad (\mathrm{id}_M)_* v v, \quad (g \circ f)_* v = g_*(f_* v),$$

where $f\colon M \to N$ and $g\colon N \to P$ are two composable morphisms between manifolds. This observation can be subsumed by saying that the tangent space is a function from the category of pointed manifolds to the category of finite-dimensional vector spaces.

Given a curve $\alpha\colon J \to M$ and a point $t \in J$, which we may view as a coordinate of time, we can differentiate the curve along $\partial(t)$. The vector

$$\dot{\alpha}(t) \coloneqq \alpha_* \partial(t)$$

is called the *derivative* or *velocity of the curve α at t*. The statement about the functoriality of the tangent space can be viewed as the chain-rule in disguise: in the one-dimensional case we have

$$(\alpha \circ c)^{\cdot}(s) = \alpha_* c_* \partial(s) = \alpha_*(c'(s)\, \delta(c(s))) = c'(s)\, \alpha_* \delta(c(s)) = c'(s) \dot{\alpha}(c(s)),$$

where $c\colon I \to J$ is any differential map and s a point in the open subset I of $\mathbf{R}$.

Given an observer $\alpha\colon J \to M$ in space-time M and a clock reading $t \in J$, we get two quantities, namely the event $p := \alpha(t)$ and the derivative $u := \dot{\alpha}(t)$, which is a vector at p. In fact, u is all what we need because a vector knows at which point it is attached to. In the reference frame of the observer α, the observer does not move spatially. In other words, u is a vector pointing in the direction of the observers future at the event p. The length of the vector u can be seen as defining the observer's unit of time. We thus define an *instantaneous observer* as just a non-vanishing vector u at any point in space-time.

1.6 Tensors

1.7 Vector fields

1.8 Tensor fields

1.9 Problems

Problem 1.1. Prove that every n-dimensional atlas of a set M is contained in a unique maximal n-dimensional atlas of M.

Problem 1.2. Prove that the system of open subsets of a premanifold M is a topology on the underlying set of M.

Problem 1.3. Let (x, U) be a chart of an n-dimensional premanifold M. Show that a subset V of U is open in M if and only if $x(V)$ is open in $\mathbf{R}^n$. Conclude that $x\colon U \to \mathbf{R}^n$ is a continuous map.

Problem 1.4. Prove that an open submanifold of a manifold is in fact a manifold.

Problem 1.5. Prove the following: Let M be a manifold and $\phi\colon M \to \mathbf{R}$ a map. Assume that for each point $p \in M$ there exists a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$ such that $\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$ is smooth. Then ϕ is a (smooth) function.

Problem 1.6. Prove that every function f on an n-dimensional manifold M is a continuous map $f\colon M \to \mathbf{R}$ for the underlying topologies of M and $\mathbf{R}$.

Problem 1.7. Prove that the functions on a manifold M form a *sheaf of algebras*, that is prove that $\mathcal{C}^\infty(U)$ is closed under constants, addition and multiplication for every open subset U of M, that the restrictions $\mathcal{C}^\infty(U) \to \mathcal{C}^\infty(V)$ are homomorphisms of algebras for every inclusion $V \subseteq U$ of open subsets of M, and that for every open cover $(U_i)_{i\in I}$ of M and functions $\phi_i \in \mathcal{C}^\infty(M)$ one has

$$(\forall i, j \in I : \phi_i|U_i \cap U_j = \phi_j|U_i \cap U_j) \implies \exists! \phi \in \mathcal{C}^\infty(M)\, \forall i \in I : \phi|U_i = \phi_i.$$

Problem 1.8. Prove that a map $f\colon M \to N$ between manifolds is a morphism if and only if

$$\forall \psi \in \mathcal{C}^\infty(M) : f^{-1}\psi \in \mathcal{C}^\infty(N).$$

Problem 1.9. Prove that the composition $g \circ f$ or two morphisms $f\colon M \to N$ and $g\colon N \to P$ between manifolds is again a morphism.

Problem 1.10. Proof that the inclusion morphism $i\colon U \to M$ for any open submanifold U of a manifold M is in fact a morphism.

Problem 1.11. Let $f\colon M \to N$ be a morphism between manifolds and $p \in M$ a point. Prove that

$$f_* v\colon \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(N), \psi \mapsto v \cdot (\psi \circ f)$$

in fact defines a vector at $f(p)$.

Problem 1.12. Let $f\colon M \to N$ be a morphism between manifolds and $p \in M$ a point. Proof that the tangent map at p is a linear map

$$\mathrm{T}_f M\colon \mathrm{T}_p M \to \mathrm{T}_{f(p)} N, v \mapsto f_* v.$$

Problem 1.13. Let M be a manifold and $p \in M$ be a point. Show that

$$\forall \phi \in \mathcal{C}^\infty(M) : \phi_* v = (v \cdot \phi)\, \delta(\phi(p)).$$

for all $v \in \mathrm{T}_p M$.

Problem 1.14. Let M be a manifold and $p \in M$ be a point. Show that

$$\forall v \in \mathrm{T}_p M : \quad (\mathrm{id}_M)_* v v, \quad (g \circ f)_* v = g_*(f_* v),$$

where $f\colon M \to N$ and $g\colon N \to P$ are two (composable) morphisms between manifolds.

Chapter 2

Equations of motion in spacetime

2.1 Introduction

2.2 Affine manifolds

2.3 Parallel displacement

2.4 Torsion

2.5 Autoparallels

2.6 The exponential map

2.7 Problems

Chapter 3

Tidal forces

3.1 Introduction

3.2 Curvature

3.3 Bianchi identities

3.4 Jacobi fields

3.5 Taylor expansion of the exponential map

3.6 Problems

Chapter 4

Vacuum structure and causality

4.1 Introduction

4.2 Pseudo-riemannian manifolds

4.3 Levi–Civita connection

4.4 Geodesics

4.5 Riemannian curvature

4.6 Geodesic discs and balls

4.7 Ricci curvature

4.8 Problems

Chapter 5

General relativity

5.1 Introduction

5.2 Einstein tensor

5.3 Mass density

5.4 Einstein's field equations

5.5 Newton's theory of gravity

5.6 Equations of motion

5.7 Killing vector fields

5.8 Schwarzschild metric

5.9 Schwarzschild potential

5.10 Problems

Chapter 6

Electromagnetism

6.1 Introduction

6.2 Differential forms

6.3 Hodge dual

6.4 Maxwell's theory

6.5 Energy-momentum tensor

6.6 Charged black holes

6.7 Problems

Chapter 7

Action principles

7.1 Introduction

7.2 Euler-Lagrange equation

7.3 Conserved currents

7.4 Maxwell's theory

7.5 Hilbert's action principle

7.6 Problems

Chapter 8

Einstein–Cartan theory

8.1 Introduction

8.2 Lie groups

8.3 Fibre bundles

8.4 Ehresmann connections

8.5 Electromagnetism

8.6 Cartan connections

8.7 Einstein–Cartan theory

8.8 Spin

8.9 Equations of motion

8.10 Problems

Appendix A

Topology

A.1 Topological spaces

Definition A.1. Let X be a set. A *topology on* X is a subset of the power set of X, whose elements are called *open subsets of the topology* and which fulfills the following axioms:

Union Let $(U_i)_{i \in I}$ be a family of open subsets of the topology. Then the union $\bigcup_{i \in I} U_i$ is again an open subset of the topology.

Intersection Let $U_1, \ldots, U_n$ be finitely many open subsets of the topology. Then their intersection $U_1 \cap \cdots \cap U_n$ is again an open subset of the topology.

A *topological space* X is a set X together with a topology $\mathfrak{U}^0(X)$ on X. An *open set of a topological space* X is an open set of the topology of X. A *point* $p \in X$ *of a topological space* X is an element of the underlying set of X.

As the union of an empty family of subsets of a set X is the empty subset, the empty subset is open with respect to any topology on X. As the intersection of an empty family of subsets of X is the whole set X, the whole set X is open with respect to any topology on X.

Let X be a topological space. A *neighborhood of a point* p *in* X is a subset G of the underlying set of X such that there exists an open subset U of X with $p \in U \subseteq G$. The set of neighborhoods of p is denoted by $\mathfrak{U}(p, X)$. A neighborhood that is at the same time an open subset is an *open neighborhood*. The set of open neighborhoods of p is denoted by $\mathfrak{U}^0(p, X)$. A subset U of the underlying set of the topological space X is open if and only if it is a neighborhood of every point $p \in U$. A *neighborhood of a subset* A

of X is a subset G that is a neigborhood of every point $p \in A$. The set of neighborhoods of A is denoted by $\mathfrak{U}(A, X)$.

An *closed subset* $Z \subseteq X$ *of* X is a subset of the underlying set of X such that its complement $X \setminus Z$ is open. Let Y be an arbitrary subset of X. The *closure* $\overline{Y}$ *of* Y *in* X is the smallest closed subset of X containing Y, that is

$$\overline{Y} = \bigcap_{Z \subseteq X \text{ is a closed subset}} Z \tag{A.1}$$

Let Y be any subset of the underlying set of X. Set

$$\mathfrak{U}^0(Y) \coloneqq \{Y \cap U : U \in \mathfrak{U}^0(X)\}$$

This is a topology, the *subspace topology*. Endowed with this topology, Y canonically becomes a topological space itself. Topological spaces of this form are called *subspaces of* X. If Y is a closed subset of X, a subset of Y is closed in Y if and only if it is closed in X. In particular, the closure of a subset of Y in Y is the same as its closure in X.

The cartesian space $\mathbf{R}^n$ is canonically a topological space with the topology defined by

$$\mathfrak{U}^0(\mathbf{R}^n) = \{U \subseteq \mathbf{R}^n : \forall p \in U\, \exists \epsilon \in \mathbf{R}^+ : U_\epsilon(p) \subseteq U\}, \tag{A.2}$$

where

$$U_\epsilon(p) \coloneqq \{q \in \mathbf{R}^n : \|q - p\| < \epsilon\} \tag{A.3}$$

for an $\epsilon > 0$ is the standard ϵ*-neighborhood*. Its closure under the so-defined topology is given by

$$\overline{U}_{\epsilon(p)} = B_\epsilon(p) \coloneqq \{q \in \mathbf{R}^n : \|q - p\| \leq \epsilon\}. \tag{A.4}$$

A.2 Continuous maps

Definition A.2. A *continuous map* $f\colon X \to Y$ *from an topological space* X *to a topological space* Y is a mapping $f\colon X \to Y$ such that

$$\forall V \in \mathfrak{U}^0(Y) : f^{-1}(V) \in \mathfrak{U}^0(X).$$

Let X be a topological space and let $\phi\colon X \to \mathbf{R}$ be a continuous map. The *support* $\operatorname{supp}\phi$ *of* ϕ is the closed subset

$$\{p \in X : \forall G \in \mathfrak{U}(p, X) : \phi|G \not\equiv 0\}.$$

It is given by

$$\operatorname{supp}\phi = \overline{\{p \in X : \phi(p) \neq 0\}}. \tag{A.5}$$

A.3 Hausdorff spaces

Definition A.3. A *Hausdorff space* X is a topological space X such that for every pair p and q of points of X with $p \neq q$, there are neighborhoods G of p and H of q, respectively, such that $G \cap H = \emptyset$.

A subspace of a Hausdorff space is again a Hausdorff space. Any finite subset of a Hausdorff space is a closed subset.

Cartesian space $\mathbf{R}^n$ is an example for a Hausdorff space.

A.4 Normal spaces

Definition A.4. A Hausdorff space X is *normal* such that for any disjoint closed subsets A and B of X there exists neighborhoods $U \in \mathfrak{U}(A, X)$ and $V \in \mathfrak{U}(B, X)$ with $U \cap V$.

An open cover $(U_i)_{i \in I}$ of a topological space X is *locally finite* if every point of X possesses a neighborhood G such that there only finitely many $i \in I$ with $G \cap U_i \neq \emptyset$.

Theorem A.1 (Shrinking lemma)**.** *For any locally finite open cover $(U_i)_{i \in I}$ of a normal Hausdorff space X, there exists a locally finite open cover $(V_i)_{i \in I}$ with*

$$\forall i \in I : V_i \subseteq \overline{V_i} \subseteq U_i.$$

A.5 Compact spaces

A family $(U_i)_{i \in I}$ of open subsets of a topological space X is an *open cover of* X if

$$X = \bigcup_{i \in I} U_i.$$

A *subcover* of an open cover $(U_i)_{i \in I}$ of X is an open cover of the form $(U_i)_{i \in I'}$ for $I' \subseteq I$. An open cover $(U_i)_{\in I}$ of X is *finite* if the index set I is finite.

If Y is a subspace of a topological space X and $(U_i)_{i \in I}$ is a family of open subsets of X such that $(Y \cap U_i)_{i \in I}$ is an open cover of Y, we often say for simplicity that $(U_i)_{i \in I}$ is an open cover of Y.

Definition A.5. A *compact space* is a Hausdorff space X such that every open cover of X has a finite subcover.

A subset K of the underlying set of a Hausdorff space X is *compact* if it is a compact space when endowed with the subspace topology. It is *relatively compact in* X if its closure in X is compact.

A compact subspace of a Hausdorff space X is always a closed subset of X. A closed subset of a compact space is again compact.

Theorem A.2. *A subset K of cartesian space $\mathbf{R}^n$ is compact if and only if it is closed and there exists an $R \in \mathbf{R}^+$ such that $K \subseteq \{p \in \mathbf{R}^n : \|p\| \leq R\}$.*

The latter condition on K says that K is *bounded*.

A.6 Locally compact spaces

Definition A.6. A Hausdorff space X is *locally compact* if each $p \in X$ has a neighborhood $U \in \mathfrak{U}(p, X)$, which is compact.

The condition of being locally compact for a Hausdorff space X is equivalent to requiring that every point $p \in X$ possesses an open neighborhood $U \in \mathfrak{U}(p, X)$, which is relatively compact in X.

Proposition A.1. *On a locally compact Hausdorff space X, every neighborhood G of a point $p \in X$ contains a compact neighborhood $K \in \mathfrak{U}(p, X)$.*

A.7 Paracompact Hausdorff spaces

A *refinement* of an open cover $(U_i)_{i \in I}$ of a topological space X is an open cover $(V_j)_{j \in J}$ of X such that

$$\forall j \in J \, \exists i \in I : V_j \subseteq U_i.$$

Definition A.7. A *paracompact Hausdorff space* is a Hausdorff space X such that every open cover of X has a locally finite refinement.

Proposition A.2. *A paracompact Hausdorff space is normal.*

Cartesian space $\mathbf{R}^n$ is an example of a paracompact Hausdorff space.

A.8 Problems

Problem A.1. Let X be a topological space. Prove that a subset U of the underlying set of X is open if and only if it is a neighborhood of every point $p \in U$.

Problem A.2. Prove that the subspace topology of a subset of the underlying set of a topological space is in fact a topology.

Problem A.3. Let Y be a closed subset of a topological space X. Let Z be a subset of Y. Show that Z is closed in Y if and only if it is closed in X.

Problem A.4. Show that (A.2) defines a topology on $\mathbf{R}^n$.

Problem A.5. Show that the closure of a standard ϵ-neighborhood in $\mathbf{R}^n$ for the canonical topology is given by

$$\overline{U}_{\epsilon(p)} = \{q \in \mathbf{R}^n : \|q - p\| \leq \epsilon\}. \tag{A.6}$$

Problem A.6. Let X be a topological space. Prove that the support of a continuous map $\phi\colon X \to \mathbf{R}$ is indeed closed and that the formula (A.5) holds.

Problem A.7. Prove that a subspace of a Hausdorff space is again a Hausdorff space.

Problem A.8. Prove that a finite subset of a Hausdorff space is a closed subset.

Problem A.9. Prove that a compact subspace K of a Hausdorff space X is always a closed subset of X.

Problem A.10. Prove that a closed subset Z of a compact space K is again compact.

Problem A.11. Prove that a Hausdorff space X is locally compact if and only if every point $p \in X$ possesses an open neighborhood $U \in \mathfrak{U}(p, X)$ such that U is relatively compact in X:

Problem A.12. Show that the canonical topology on $\mathbf{R}^n$ makes cartesian space into a paracompact Hausdorff space.

Appendix B

Analysis

B.1 Bump functions

Theorem B.1. *For any two real numbers* $0 < r < R$ *there exists a smooth function* $\phi\colon \mathbf{R}^n \to \mathbf{R}$ *with*

$$\forall p \in \mathbf{R} : \phi(p) \geq 0, \quad \forall p \in \mathbf{R} : \|p\| \leq r \implies \phi(p) = 1, \quad \forall p \in \mathbf{R} : \|p\| \geq R \implies \phi(p) = 0.$$

A function as in the theorem is called a *bump function*.

B.2 Hadamard's lemma

A subset U of n-dimensional Euclidean space $\mathbf{R}^n$ is *star-shaped with respect to a point* $p \in U$ if

$$\forall x \in U \forall 0 \leq t \leq 1 : (1-t)\, a + t\, x \in U.$$

Any neighborhood of a point $p \in \mathbf{R}^n$ contains an open neighborhood star-shaped with respect to the point p as the standard ϵ-neighborhoods are star-shaped with respect to their center (in fact, with respect to any point of their interior).

Theorem B.2. *Let* ϕ *be a smooth function defined on an open subset* U *of* n*-dimensional Euclidean space* $\mathbf{R}^n$ *that is star-shaped with respect to a point* $a \in U$*. Then there exist smooth functions* $g_1, \ldots, g_n$ *on* U *such that*

$$\forall x \in U : \phi(x) = \phi(a) + \sum_{i=1}^{n} (x_i - a_i)\, g_i(x).$$