General Relativity and Einstein–Cartan Theory

Marc A. Nieper-Wißkirchen

March 10, 2014

Preface

Gravity is one of the fundamental interactions of nature, by which all physical bodies attract each other. The purpose of these notes is to give a physically motivated and mathematically sound introduction to Einstein's general relativity. Since its invention, it has passed all experimental tests and is the simplest theory that is able to describe all observed phenomena of gravity. In short, it is the accepted theory of gravity in modern physics. This is not to say that general relativity is set in stone. By incorporating possible intrinsic angular moment of macroscopic matter, one arrives at the mathematically equally beautiful Einstein–Cartan theory, which encloses general relativity. While general relativity is a metric theory that builds on the mathematical concept of a Lorentzian manifold, Einstein–Cartan theory is a true gauge theory of gravity. It will be described in the latter chapters of these notes.

Addressees of these notes are, on the one hand side, graduate students of physics who are willing to cope with abstract mathematical concepts and, on the other hand, graduate students of mathematics with a strong background in classical physics. These notes grew out of a lecture course the author gave in the winter term 2013/14 at the University of Augsburg, and the audience of the course was a mixture of students from the physics and the mathematical department.

The theory of general relativity and the novel effects being predicted by it — for example, gravitational time dilation or gravitational waves — have been fascinating many people, but to understand these phenomena on a quantative level, one has to delve deeply into the mathematics of general relativity. One the other hand, it is a much rewarding untertaking. One will have grasped one of the most beautiful physical theories (if not *the* most beautiful one). These notes show one of the possible routes there, a route the author would have liked to go when he learnt general relativity. Going this route also provides the reader with a solid knowledge of differential geometry.

That said, it *is* possible to formulate the heart of the theory of general relativity in one sentence in plain English, namely:

> The mass density measured by any observer is the scalar curva-

ture of that observer's space divided by 16π.

Of course, without any further explanations of the contained mathematical terms and an accompanying physical interpretation, this statement is just as meaningful as simply stating[1] $\operatorname{div} E = 4\pi\,\rho$ without any further explanations of the terms involved. Nevertheless, the simplicity of this statement already shows the beauty of general relativity.

Augsburg, March 10, 2014 Marc Nieper-Wißkirchen

[1] Gauß's law in gaussian units

Special thanks

Special thanks go to Caren Schinko for providing a written transcript of the course on which these notes are based, and to Tim Baumann for careful proofreading of preliminary versions of them.

History

The theory of general relativity was being developed by Albert Einstein between 1907 and 1915. It builds upon and combines the earlier theories of Newtonian gravity and special relativity. Although the basic theory hasn't changed since then, there have been many contributions afterwards. The following timeline summarizes the history as far it is relevant to the notes at hand.

1609 Kepler pushlishes his first two laws of planetary motion.

1619 Kepler pushlishes his third law of planetary motion.

1638 Galilei's equations for a falling body.

1687 Newton publishes his law of universal gravitation.

1798 Cavendish measures Newton's gravitational constant.

1862 Maxwell's equations of electromagnetism.

1887 The Michelson-Morley experiment fails to detect a stationary luminiferous aether.

1889 FitzGerald proposes Lorentz contraction.

1905 Formulation of special relativity by Einstein.

1915 Derivation of Einstein's field equations from an action principle by Hilbert.

1915 Einstein's theory of general relativity.

1916 Schwarzschild found the first exact solution of Einstein's field equations.

1916 Einstein shows that the perihelion precession of Mercury can be fully explained by general relativity.

1919 Eddington's expedition confirms that the deflection of light by the Sun is as predicted by general relativity.

1922 Friedmann found a cosmological solution to Einstein's equations, in which the universe may expand or contract.

1922 Introduction of the cosmological constant by Einstein into his field equations.

1922 Proposal of the Einstein–Cartan theory by lie Cartan.

1929 Hubble finds evidence that the universe is expanding.

1959 Direct measurement of the gravitational redshift of light in the Pound-Rebka experiment.

1964 Discovery of the cosmis microwave background by Penzias and Wilson.

1964 Discovery of the X-ray source Cygnus X-1, now widely accepted to be a black hole.

Notations

Standard sets We use the following notations for the standard sets: The set of natural numbers (which includes 0, by definition) is denoted by $\mathbf{N}_0$, the set of integers by $\mathbf{Z}$, the set of rational numbers by $\mathbf{Q}$, the set of real numbers by $\mathbf{R}$ and the set of complex numbers by $\mathbf{C}$.

The set of the positive real numbers is denoted by a subscript: $\mathbf{R}_+$.

Real numbers For any interval $J \subseteq \mathbf{R}$ and any $s \in \mathbf{R}$, one defines the interval

$$-s + J \coloneqq \{t \in \mathbf{R} : s + t \in J\}.$$

Linear algebra In an n-dimensional vector space, the Kronecker symbol is a scalar defined by

$$\delta_{ij} \coloneqq \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for all i, $j \in \{1, \ldots, n\}$.

Given a vector space V, its *dual space* V^* is the set $\mathrm{Hom}(V, \mathbf{R})$ of linear maps from V to the scalars $\mathbf{R}$, endowed with a vector space structure by defining addition and scalar multiplication point-wise.

Maps By a *differential map* we will always mean of map of class $\mathcal{C}^\infty$ that is a smooth map. *Differentiability* thus means the existence of continuous derivates to all orders.

The term *function* will be reserved for smooth maps with values in $\mathbf{R}$. Thus a function is always a smooth function.

Cartesian space The standard basis formed by $e_1, \ldots, e_n$ of n-dimensional space $\mathbf{R}^n$ is a basis of the underlying vector space such that $v = \sum_{i=1}^n v_i \, e_i$ for each $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$.

The standard Euclidean norm of n-dimensional space $\mathbf{R}^n$ is denoted by

$$\|v\| \coloneqq \sqrt{\sum_{i=1}^{n} v_i^2}$$

for $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$.

The partial derivative of a function ϕ defined on an open subset of $\mathbf{R}^n$ in direction i is denoted by ∂_i, that is

$$\partial_i \phi \colon v \mapsto \frac{\partial \phi(v + t\, e_i)}{\partial t}\Big|_{t=0}$$

for $i = 1, \ldots, n$. In case of $n = 1$, we set $\partial \coloneqq \partial_1$.

Contents

Chapter 1

Spacetime

1.1 Introduction

The fundamental notion of special relativity is that of an *event*. In pre-relativity, things located at a specific point in space and time, for example the start of the Saturn V rocket launching the Apollo 11 spaceflight, are described by a unique time in Newton's absolute time and a unique place in Euclidean absolute space. The notion of an event combines these two qualities into one, which is essential for special relativity as space and time by themselves lose their absoluteness.

Spacetime is the set of all possible events. Any event that can be imagined to happen is an element or a *point* in spacetime. Mathematically, spacetime has more structure than simply being a set: A point in this set, that is an event, is usually described by four scalars, one time and three space coordinates, making spacetime four-dimensional. For a general set, however, there is no well-defined notion of coordinates or dimension. The theory of manifolds, which will be presented in the next section, is the correct mathematical setting in which notions like coordinates and dimension make sense.

1.2 Manifolds

An *n-dimensional chart* $x = (x, U)$ on a set M, whose elements we call *points*, is an injective map $x = (x_1, \ldots, x_n)\colon U \to \mathbf{R}^n$ onto an open subset of $\mathbf{R}^n$ defined on a subset U of M, the *domain of definition of* x. Given a point p lying in the domain of definition U, the n scalars $x_1(p)$, …, $x_n(p)$ are the *coordinates of* p *with respect to the chart* x. A family $\{(x_i, U_i)\}_{i\in I}$ of charts on M *covers* M if $M \subseteq \bigcup_{i\in I} U_i$ that is if every point p of M lies in at least domain of definition of the charts x_i. In order to be able to describe events in spacetime by four coordinates, we postulate that *there is a distinguished family of four-dimensional charts that cover spacetime.*

Given two n-dimensional charts (x, U) and (y, V) of a set M, the map

$$y \circ x^{-1} | x(U \cap V) \colon x(U \cap V) \to y(U \cap V)$$

is called the *coordinate transformation from x to y*. If this map is a diffeomorphism between open subsets of $\mathbf{R}^n$, the charts x and y are said to be *compatible*. An *n-dimensional atlas of M* is a family of pairwise compatible n-dimensional charts of M that covers M. In order to be able to employ analytic methods, we extend our postulate above by postulating that *there is a distinguished four-dimensional atlas of spacetime.*

Let $\mathfrak{A}$ be any n-dimensional atlas of a set M, and let x and y be two arbitrary n-dimensional charts of M. If x and y are each compatible with each chart in $\mathfrak{A}$, they are compatible with each other. Therefore, every atlas $\mathfrak{A}$ can be uniquely enlarged to a *maximal atlas* in which it is contained by adding all n-dimensional charts to $\mathfrak{A}$ that are compatible with each in chart in $\mathfrak{A}$. We also say that the maximal atlas is *generated* by the charts contained in $\mathfrak{A}$.

An *n-dimensional premanifold M* is a set M together with a maximal n-dimensional atlas $\mathfrak{U}^\infty(M)$ of M. The set of all charts (x, U) in $\mathfrak{U}^\infty(M)$ that contains a given point $p \in M$ is denoted by $\mathfrak{U}^\infty(p, M)$. A *chart of M* is a chart in the maximal atlas $\mathfrak{U}^\infty(M)$. An *atlas of the premanifold M* is any atlas of M which is contained in the maximal atlas $\mathfrak{U}^\infty(M)$. With these terms, we can say that *spacetime is a four-dimensional premanifold.*

Using the atlas of a premanifold M, one can define the notion of neighborhoods of points on M: A subset G of the underlying set of M is an *open subset of M* if $x(U \cap G)$ is an open subset for each chart (x, U) of M. The system of these open subsets of M is a topology (see Section A.1) on the underlying set of M, the *canonical topology of M*, so that M is a topological space in a canonical way.

For any chart (x, U) of the n-dimensional premanifold, a subset V of U is open in M if and only if $x(V)$ is open in $\mathbf{R}^n$. In particular, U is open. The map $x \colon U \to \mathbf{R}^n$ is continuous in the sense of maps between topological spaces.

Generally, the so-defined topology is ill-behaved, however; points may not be distinguishable by the topology and one may need infinitely many charts to cover even small neighborhoods of points. Excluding these cases by adding technical conditions on the underlying topological space leads to the final definition:

Definition 1.1. An *n-dimensional manifold M* is an n-dimensional premanifold M whose underlying topological space is a paracompact Hausdorff space (see Section A.7).

We extend our postulate from above by postulating that *spacetime is a four-dimensional manifold.*

The most basic example of an n-dimensional manifold is the n-dimensional cartesian space $\mathbf{R}^n$. Its canonical atlas is the maximal atlas which is generated by the identity map $\mathrm{id}_{\mathbf{R}^n}\colon \mathbf{R}^n \to \mathbf{R}^n$ viewed as an n-dimensional chart. The topology defined by this atlas is, of course, the canonical topology of $\mathbf{R}^n$, which is the topology of a paracompact Hausdorff space.

Requiring that the underlying topological space of spacetime is a paracompact Hausdorff space has a number of pleasant consequences: For example, the underlying topological spaces of manifolds are normal, which follows from Proposition A.2. Furthermore, we have:

Proposition 1.1. *The underlying space of an n-dimensional manifold M is locally compact.*

(For the definition of local compactness, see Section A.6.)

Proof. For $p \in M$ choose a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$. By openness of $x(U)$ in $\mathbf{R}^n$, there exists an $\epsilon > 0$ such that $U_\epsilon(x(p)) \subseteq x(U)$. We claim that

$$K \coloneqq x^{-1}(\overline{U_{\frac{\epsilon}{2}}(x(p))})$$

is a compact neighborhood of p:

The subset K is a neighborhood of p as $p \in x^{-1}(U_{\frac{\epsilon}{2}}(x(p))) \subseteq K$ and because $x^{-1}(U_{\frac{1}{2}\epsilon}(x(p)))$ is an open subset of M.

To show that K is compact, let $(U_i)_{i\in I}$ be a family of open subsets of U that cover K. By definition of the topology of M, the images $(x(U_i))_{i\in I}$ form an open cover of the compact space $\overline{U_{\frac{\epsilon}{2}}(x(p))}$. Thus there is a finite subset $J \subset I$ such that $(x(U_i))_{i\in J}$ is an open cover of $\overline{U_{\frac{\epsilon}{2}}(x(p)}$. It follows that $(U_i)_{i\in J}$ is a finite open cover of K. □

Often, we have to restrict our attention to small pieces (that is, open subsets) of a given n-dimensional manifold M, for example spacetime. This can be done as follows: Let G be an open subspace of M. Let $\mathfrak{U}^\infty(G)$ be the unique maximal n-dimensional atlas of G that contains the atlas

$$\{(x|U \cap G, U \cap G) : (x, U) \in \mathfrak{U}^\infty(M)\}.$$

Then G becomes an n-dimensional manifold itself, whose underlying topological space is the subspace G of M. An n-dimensional manifold of this form is called an *open submanifold of M*. The domains of definitions of the charts of M are open submanifolds of M.

1.3 Functions on a manifold

As we have postulated, we can "measure" each event in spacetime by giving the scalar values of four coordinates (after choosing a chart). Each coordinate can be thought of as a scalar field that assigns to each point (in the domain of definition of its chart) a scalar value. This notion is generalised as follows:

Definition 1.2. A *(smooth) function* $\phi\colon M \to \mathbf{R}$ on an n-dimensional manifold M is a mapping ϕ from the underlying set of M to the reals such that for each chart (x, U) of M the map

$$\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$$

is smooth.

In order to show that a map $\phi\colon M \to \mathbf{R}$ is a function in the above sense, it suffices that for each point $p \in M$ there exists a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$ such that $\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$ is smooth.

Every function is continuous with respect to the underlying topologies of M and $\mathbf{R}$, respectively. Let ϕ be a function on M and G an open submanifold. The *restriction $\phi|G$ of ϕ to G*, given by

$$\phi|G\colon G \to M, p \mapsto f(p)$$

is a function on the manifold G. For any chart (x, U) of M, the coordinate functions $x_1, \ldots, x_n\colon U \to M$ are smooth functions on the open submanifold U.

All functions on M form an algebra (over the reals), denoted by $\mathcal{C}^\infty(M)$, where addition and multiplication are defined point-wise. The function with constant value $c \in \mathbf{R}$ is often denoted by $\underline{c}$. The sets of functions of the open submanifolds of M fulfill the *sheaf condition*, that is for every open cover $(U_i)_{i\in I}$ of M and functions $\phi_i \in \mathcal{C}^\infty(U_i), i \in I$ one has

$$(\forall i, j \in I : \phi_i|U_i \cap U_j = \phi_j|U_i \cap U_j) \implies \exists! \phi \in \mathcal{C}^\infty(M)\, \forall i \in I : \phi|U_i = \phi_i.$$

In other words, we can uniquely glue functions along open submanifolds.

If $\Phi\colon \mathbf{R} \to \mathbf{R}$ is any smooth function between the reals, the composition $\Phi \circ \phi\colon M \to \mathbf{R}$ is a function on M whenever $\phi\colon M \to \mathbf{R}$ is a function.

A manifold possesses many functions in a sense made precise by the following theorem, which relies essentially on the paracompactness of the manifold (for the notion of the support supp of a function, see Section A.2):

Theorem 1.1. *Every open cover $(U_i)_{i\in I}$ of a manifold M has a* subordinate partition $(\lambda_i)_{i\in I}$ of unity*, which is a family of functions $\lambda_i\colon M \to \mathbf{R}$ with the following properties:*

Range *For all $i \in I$ and $p \in M$, one has* $0 \leq \lambda_i(p) \leq 1$.

Support *For all $i \in I$, one has* $\operatorname{supp} \lambda_i \subseteq U_i$.

Local finiteness *Every $p \in M$ posesses a neighborhood $G \in \mathfrak{U}^0(p, M)$ such that there are only finitely many $i \in I$ with* $\operatorname{supp} \lambda_i \cap G \neq \emptyset$.

Normalization *For every $p \in M$, the equality $\sum_{i \in I} \lambda_i(p) \equiv 1$ holds.*

The proof relies on the following lemma, which is also of independent interest:

Lemma 1.1. *Let K be a compact subspace of an n-dimensional manifold M. For any open neighborhood G of K in M, there exists a function $\phi \colon M \to \mathbf{R}$ with*

$$\forall p \in M : \phi(p) \geq 0, \qquad \forall p \in K : \phi(p) > 0, \qquad \operatorname{supp} \phi \subseteq G. \tag{1.1}$$

Proof of Lemma 1.1. For any $p \in K$ choose a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$. By the openness of $x(U)$ in $\mathbf{R}^n$, there exists an $\epsilon > 0$ with $U_\epsilon(x(p)) \subseteq x(U \cap G)$. Choose a bump function $\psi \colon \mathbf{R}^n \to \mathbf{R}$ (see Section B.1) such that $\psi(u) \geq 0$ for all $u \in \mathbf{R}^n$, $\psi(u) = 1$ for $\|u\| \leq \frac{\epsilon}{3}$ and $\psi(u) = 0$ for $u \geq \frac{2\epsilon}{3}$. By the sheaf condition,

$$\phi_p \colon M \to \mathbf{R}, q \mapsto \begin{cases} \psi(x(q) - x(p)) & \text{if } q \in U, \\ 0 & \text{if } q \in M \setminus x^{-1}(\overline{U_{\frac{2\epsilon}{3}}(x(p))}) \end{cases}$$

defines a function on M as U and $M \setminus x^{-1}(\overline{U_{\frac{2\epsilon}{3}}(x(p))})$ form an open cover of M. This function has the properties

$$\forall q \in M : \phi_p(q) \geq 0, \qquad \phi_p(p) > 0, \qquad \operatorname{supp} \phi_p \subseteq G.$$

The open subsets $U_p \coloneqq \{q \in M : \phi_p(q) > 0\}$ with $p \in K$ cover K. By compactness of K, there exists a finite subset $A \subseteq K$ such that K is covered by $(U_p)_{p \in K}$. By construction, the function $\phi = \sum_{p \in A} \phi_p$ fulfills (1.1). □

Proof of Theorem 1.1. For every $p \in M$ choose by local compactness of M a relatively compact neighborhood $G_p \in \mathfrak{U}(p, M)$. By the covering property, there exists an $i \in I$ with $p \in U_i$. The intersection $G_p \cap U_i$ is again relatively compact in M, so we may assume that already $G_p \subseteq U_i$. So $(G_p)_{p \in M}$ is a refinement of the cover $(U_i)_{i \in I}$. Let $(V_j)_{j \in J}$ be a locally finite refinement of $(G_p)_{p \in M}$. In particular, $(V_j)_{j \in J}$ is a locally finite refinement of $(U_i)_{i \in I}$ and each V_j is relatively compact in M. By the shrinking lemma, Proposition A.1, and the normality of the underlying topological space of M, there exists another open cover $(V'_j)_{j\in J}$ of M with $\overline{V'_j} \subseteq V_j$ for all $j \in J$.

For every $j \in J$, choose by compactness of $\overline{V'_j}$ and Lemma 1.1 a function $\phi_j \in \mathcal{C}^\infty(M)$ with

$$\forall p \in M : \phi_j(p) \geq 0, \qquad \forall p \in V'_j : \phi_j(p) > 0, \qquad \operatorname{supp} \phi_j \subseteq V_j.$$

By the local finiteness of the open cover $(V_j)_{j\in J}$, the sum $\phi := \sum_{j\in J} \phi_j$ is locally a finite sum and thus defines a function on M by the sheaf condition. By the covering property of $(V'_j)_{j\in J}$, one has $f(p) > 0$ for all $p \in M$.

As $(V_j)_{j\in J}$ is a refinement of the cover $(U_i)_{i\in I}$, there exists a map $\alpha\colon J \to I$ with $V_j \subseteq U_{\alpha(i)}$ for all $j \in J$. For each $i \in I$ set $J_i := \alpha^{-1}(i)$, so $(J_i)_{i\in I}$ becomes a partition of I. For all $i \in I$, set $U'_i := \bigcup_{j\in J_i} V_j \subseteq U_i$ and finally

$$\lambda_i = \sum_{j\in J_i} \frac{\phi_i}{\phi}.$$

By a similar argument as above, the sum on the right hand side is locally finite and, thus, λ_i is a function on M with $\lambda_i(p) \geq 0$ for all $p \in M$.

By construction, $\operatorname{supp} \lambda_i \subseteq \bigcup_{j\in J_i} V_j = U'_j \subseteq U_i$, which proves the support axiom of a partition of unity. The covering $(U'_i)_{i\in I}$ is locally finite as the covering $(V_j)_{j\in J}$ is locally finite; this uses the disjointness of the J_i. From $\operatorname{supp} \lambda_i \subseteq U'_i$ for all $i \in I$, the local finiteness axiom follows. By construction, $\sum_{i\in I} \lambda_i \equiv 1$, which is the normalization axiom of a partition of unity. From this, the range axiom follows as we already know that $\lambda_i(p) \geq 0$ for all $p \in M$. $\square$

The existence of partitions of unity on manifolds implies that functions can be extended in the following sense:

Corollary 1.1. *Let ϕ be a function defined on an open neighborhood G of a point p in a manifold M. Then there exists a function $\widehat{\phi} \in \mathcal{C}^\infty(M)$ with $\operatorname{supp} \widehat{\phi} \subseteq G$ and such that $\widehat{\phi}$ coincides with ϕ on a neighboorhood $U \subseteq G$ of p in M.*

The function $\widehat{\phi}$ is called an *extension of ϕ by zero away from p.*

Proof. By local compactness (see Proposition A.1), there exists a compact neighborhood $K \in \mathfrak{U}(p, G)$. Choose a partition (λ, μ) of unity subordinate to the open cover $(G, M \setminus K)$ of M. By the sheaf condition,

$$\hat{\phi}\colon M \to \mathbf{R}, \begin{cases} \lambda(p) \cdot \phi(p) & \text{if } p \in G \\ 0 & \text{if } p \in M \setminus \operatorname{supp} \lambda \end{cases}$$

is a well-defined function on M with $\operatorname{supp} \hat{\phi} \subseteq G$ and which coincides with ϕ on K. $\square$

For any two points p, $q \in M$ with $p \neq q$, there exists an open neighborhoods $G \in \mathfrak{U}^0(p, M)$ with $q \notin G$. Extending the constant function $\underline{1}|G$ by zero away from p yields a function, which is 1 on p and 0 on q. Thus, we have

$$\forall p, q \in M : p \neq q \implies \exists \lambda \in \mathcal{C}^\infty(M) : \lambda(p) = 1, \lambda(q) = 0. \tag{1.2}$$

If we denote by p^* for all $p \in M$ the algebra homomorphism

$$p^* \colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto \phi(p), \tag{1.3}$$

we can reformulate (1.2) by saying that the map $p \mapsto p^*$ is injective, that is the algebra of functions *separate points*. The algebra of functions on spacetime is therefore a full set of observables: for any two distinct events there is a function that takes different values on both events.

1.4 Morphisms

A physical body traces out a curve of events in spacetime M, namely those events where an observer meets the physical body, its *world line*. If the physical body carries a clock with it, each point of its world line is parametrized by a scalar, the clock's time measured at that event. In other words, the path of the physical body in spacetime together with its clock defines a map $J \to M$, where J is an (open) interval. Both the domain and the target of this map is a manifold, where J is viewed as an open submanifold of the reals. A physical body does not jump through spacetime, so the map will be continuous. In order to employ analytical methods, it is sensible to assume moreover that this map is a morphism according to the following definition:

Definition 1.3. A *morphism* $f \colon M \to N$ between two manifolds M and N is a continuous map $f \colon M \to N$ such that for each pair of charts $(x, U) \in \mathfrak{U}^\infty(M)$ and $(y, V) \in \mathfrak{U}^\infty(N)$ the composition

$$y \circ f \circ x^{-1} | x(f^{-1}(V) \cap U) \colon x(f^{-1}(V) \cap U) \to y(V)$$

is a smooth map (between open subsets of cartesian spaces).

In accordance with our above wording, a *curve* $\alpha \colon J \to M$ is a morphism where J is an open interval viewed as a submanifold of $\mathbf{R}$. Thus a world line of a physical body becomes a curve in this sense by endowing it with a clock.

A map $f\colon M \to N$ between manifolds is a morphism if and only if for all functions $\psi \in \mathcal{C}^\infty(N)$ the *pullback of ψ by f*, given by

$$f^{-1}\psi\colon M \to \mathbf{R}, p \mapsto \psi(f(p)), \tag{1.4}$$

is a function on M. (The pullback itself is a map

$$f^{-1}\colon \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M) \tag{1.5}$$

of algebras.)

Thus, a map $\phi\colon M \to \mathbf{R}$ defined on a manifold M is a morphism if and only if it is a function. Further, a map $f\colon G \to H$ between open subsets of cartesian spaces is a morphism if and only if it is a smooth map in the sense of calculus.

The identity $\mathrm{id}_M\colon M \to M$ of M is a morphism. The composition $g \circ f$ of two morphisms $f\colon M \to N$ and $g\colon N \to P$ between manifolds is again a morphism. The manifolds together with the morphisms between them thus form a *category*. A morphism $f\colon M \to N$ between manifolds that is bijective and whose inverse $f^{-1}\colon N \to M$ is again a morphism is called a *diffeomorphism between M and N*. In a categorical sense, diffeomorphisms are exactly the isomorphisms.

The *inclusion* $i\colon U \to M, p \mapsto p$ of an open submanifold U of M is a morphism. Thus, the restriction $f|U = f \circ i$ of a morphism $f\colon M \to N$ to U is again a morphism.

1.5 Product manifolds

When investigating possible universe, that is possible spacetimes, we will have to construct manifolds. One important construction, which constructs a manifold out of simpler ones is the product. Let M and N be two manifolds of dimensions m and n, respectively. Given any chart $(x, U) \in \mathfrak{U}^\infty(M)$ and any chart $(y, V) \in \mathfrak{U}^\infty(N)$, the map

$$x \times y\colon U \times V \to \mathbf{R}^{m+n}, (p, q) \mapsto (x(p), y(q))$$

defines an $(m+n)$-dimensional chart on the cartesian product

$$M \times N \coloneqq \{(p, q) : p \in M, q \in N\}.$$

Let $\mathfrak{U}^\infty(M \times N)$ be the unique maximal atlas of $M \times N$ that contains the atlas

$$\{(x \times y, U \times V) : (x, U) \in \mathfrak{U}^\infty(M), (y, V) \in \mathfrak{U}^\infty(N)\}.$$

Then $M \times N$ becomes an $(m+n)$-manifold, whose underlying topological space is the product topological space of M and N.

Definition 1.4. The manifold $M \times N$ is called the *product of the manifolds M and N*.

The two canonical projection maps $\mathrm{pr}_1\colon M \times N \to M, (p,q) \mapsto p$ and $\mathrm{pr}_2\colon M \times N \to N, (p,q) \mapsto q$ are morphisms.

As simple example of a product manifold is given by $\mathbf{R}^n$. More precisely, for any p, q with $p+q=n$, the map

$$\mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n, ((u_1,\ldots,u_p),(v_1,\ldots,v_q)) \mapsto (u_1,\ldots,u_p,v_1,\ldots,v_q)$$

is a diffeomorphism. (Here, the left hand side is, of course, endowed with the structure of a product manifold.)

One example of a product manifold in classical physics is given by Galilean spacetime; it is the product $L \times Q$ of the one-dimensional manifold L of absolute time and the three-dimensional manifold Q of Galilean space.

1.6 Vectors

By an *observer*, we mean a curve $\alpha\colon J \to M$ in spacetime M, which might be traced out by a physical body, and relative to which one may measure events. First and foremost, an observer defines a set of events, namely the set $\alpha(J)$ of events traced out by the observer. This defines the relation of being at the same spatial location as the observer: an event p is *at the same spatial location as the oberver* if and only if $p \in \alpha(J)$. The curve α, however, is not determined by the set $\alpha(J)$ of events alone; it does not capture the dynamics of α. For example, if we move infinitesimally from t to $t + \mathrm{d}t$ on J, we expect that $\alpha(t)$ moves infinitesimally to $\alpha(t) + d\alpha(t)$. In other words, we expect $\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}$ to be a vector on M tangent to the curve $\alpha(t)$. At the moment, howoever, we have no notion of such a thing as a vector, which with we could describe dynamics, on a manifold M. The purpose of this section is to remedy this.

The basic idea of defining vectors on a manifold M is the following: Any point p assigns a value $p^*\phi = \phi(p)$ to each function $\phi \in \mathcal{C}^\infty(M)$, and we can reconstruct the point from these values as the map $p \mapsto p^*$ is injective. Assume that we already have the notion of a vector v at a point p. It would assign a change of values along v, a derivative, written $v \cdot \phi$, to each function f. Thus such a vector would give rise to a function

$$v\colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto v \cdot \phi.$$

A vector should be reconstructable from this map, so that the functions form a complete set of observables for vectors. This map should have the properties of a first-order derivative operator, so we end up with the following definition:

Definition 1.5. A *vector v at a point $p \in M$* is a map

$$v\colon \mathcal{C}^\infty(M) \to \mathbf{R}, \phi \mapsto v \cdot \phi,$$

such that the following axioms hold:

Constants For all $c \in \mathbf{R}$, one has $v \cdot \underline{c} = 0$.

Linearity For all a, $b \in \mathbf{R}$ and all ϕ, $\psi \in \mathcal{C}^\infty(M)$, one has $v \cdot (a\,\phi + b\,\psi) = a\,v \cdot \phi + b\,v \cdot \psi$.

Leibniz rule For all ϕ, $\psi \in \mathcal{C}^\infty(M)$, one has $v \cdot (\phi\,\psi) = (v \cdot \phi)\,\psi(p) + \phi(p)\,(v \cdot \psi)$.

The real number $v \cdot \phi$ is the *derivative of ϕ along v*. For example, the constants axiom says that constants have vanishing derivative along any vector.

The set of all vectors at a point $p \in M$ is called the *tangent space of M at p* and is denoted by $\mathrm{T}_p M$. The tangent space $\mathrm{T}_p M$ is canonically a (real) vector space, where addition and scalar multiplication are defined "function-wise":

$$0 \cdot \phi \coloneqq 0, \qquad (a\,v) \cdot \phi \coloneqq a\,(v \cdot \phi), \qquad (v + w) \cdot \phi \coloneqq v \cdot \phi + w \cdot \phi$$

for all v, $w \in \mathrm{T}_p M$, $a \in \mathbf{R}$ and $\phi \in \mathcal{C}^\infty(M)$.

The first basic result, which partially shows that the above definition of a vector in fact coincides with the intuitive notion, is that the derivative of a function ϕ along a vector at a point p depends only on the values of ϕ in a neighborhood of p:

Lemma 1.2. *Let G be an open neighborhood of a point $p \in M$ such that $\phi|G = \psi|G$ for two functions ϕ and $\psi \in \mathcal{C}^\infty(M)$. Then*

$$\forall v \in \mathrm{T}_p M : v \cdot \phi = v \cdot \psi.$$

Proof. By Lemma 1.1, choose a smooth function $\lambda \in \mathcal{C}^\infty$ with $\lambda(p) = 1$ and $\operatorname{supp} \lambda \subseteq G$. Then

$$\begin{aligned} v \cdot \phi - v \cdot \psi &= v \cdot (\phi - \psi) = v \cdot ((1 - \lambda)\,(\phi - \psi)) \\ &= (v \cdot (1 - \lambda))\,(\phi(p) - \psi(p)) + (1 - \lambda(p))\,(v \cdot (\phi - \psi)) = 0 + 0 \\ &= 0. \end{aligned}$$

$\square$

Given a vector v at p and a function ϕ defined in a neighborhood G of p, we can therefore speak of the derivative $v \cdot \phi$ of ϕ along v: Let $\widehat{\phi}$ be any extension of ϕ by zero away from p. As $\widehat{\phi}$ and ϕ coincide in a neighborhood of p, the derivative

$$v \cdot \phi \coloneqq v \cdot \widehat{\phi}$$

does not depend on the choice of $\widehat{\phi}$.

With this result we can completely determine the structure of the tangent space T_pM. In fact, we have:

Theorem 1.2. *Let M be an n-dimensional manifold. For each point $p \in M$, the tangent space T_pM is an n-dimensional vector space. More precisely, given a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$, a basis of T_pM is given by the vectors*

$$\frac{\partial}{\partial x_i}|_p \colon \mathcal{C}^\infty(M) \to \mathbf{R}, f \mapsto \partial_i((f|U) \circ x^{-1})(x(p)) =: \frac{\partial f}{\partial x_i}(p), \tag{1.6}$$

where $i = 1, \ldots, n$. If $(y, V) \in \mathfrak{U}^\infty(p, M)$ is another chart around p, the change of basis is given by

$$\frac{\partial}{\partial x_i}|_p = \sum_{j=1}^{n} \partial_i((y_j|U) \circ x^{-1})(x(p)) \frac{\partial}{\partial y_j}|_p = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i}(p) \frac{\partial}{\partial y_j}|_p. \tag{1.7}$$

The basis $(\frac{\partial}{\partial x_i}|_p)_i$ is called the *chart's x induced basis of* T_pM.

Proof. To show that (1.6) defines a vector at p is straight-forward and follows from the linearity and product rule of the partial differentiation in calculus.

The linear independence of the n vectors defined by (1.6) follows directly from the following observation: For any n-tupel $(a_1, \ldots, a_n)$ of scalars with $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}|_p = 0$, one calculates

$$\begin{aligned} 0 &= \left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}|_p \right) \cdot x_j = \sum_{i=1}^{n} a_i \, \partial_i(x_j \circ x^{-1})(x(p)) \\ &= \sum_{i=1}^{n} a_i \, \delta_{ij} = a_j \end{aligned}$$

for all $i = 1, \ldots, n$.

To show that the n vectors defined by (1.6) span the tangent space T_pM at p, let ϕ be any function on M. By Theorem B.2, there exist functions g_1, $\ldots$, g_n defined in a neighborhood of p in U such that

$$\phi = \phi(p) + \sum_{i=1}^{n} (x_i - x_i(p)) \, g_i(u)$$

in a neighborhood of p in $\mathbf{R}^n$. Given any vector $v \in \mathrm{T}_p M$, we thus have

$$
\begin{aligned}
v \cdot \phi &= v \cdot (\phi(p) + \sum_{i=1}^{n} (x_i - x_i(p))\, g_i) \\
&= \sum_{i=1}^{n} ((v \cdot (x_i - x_i(p)))\, g_i(p) + (x_i(p) - x_i(p))\,(v \cdot g_i)) = \sum_{i=1}^{n} (v \cdot x_i)\, g_i(p).
\end{aligned}
$$

In other words, $v \cdot \phi = \sum_{j=1}^{n} (v \cdot x_j) \frac{\partial}{\partial x_j}|_p \cdot \phi$ for any vector $v \in \mathrm{T}_p M$. Thus

$$\forall v \in \mathrm{T}_p M : v = \sum_{i=1}^{n} (v \cdot x_j) \frac{\partial}{\partial x_i}|_p,$$

which is a linear combination of the induced basis.

As this formula also works for the coordinate system (y, V), we get (1.7) for $v = \frac{\partial}{\partial x_i}|_p$. □

The theorem immediately gives the tangent space at a point p of an open submanifold U of n-dimensional space: It is spanned by the n linearly independent vectors $\partial_1(p), \ldots, \partial_n(p)$ where

$$\partial_i(p) \cdot \phi \coloneqq (\partial_i \phi)(p)$$

for all $i \in \{1, \ldots, n\}$. Thus, there is a canonical isomorphism

$$\mathbf{R}^n \to \mathrm{T}_p M, (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} u_i\, \partial_i(p).$$

whose inverse is denoted by

$$\mathrm{d}_p x \colon \mathrm{T}_p M \to \mathbf{R}^n.$$

In case of $n = 1$, we write $\partial_t \coloneqq \partial(t) \coloneqq \partial_1(t)$, where $t \in \mathbf{R}$.

By the very definition of a vector, we can differentiate functions on a manifold, that is morphisms to the manifold $\mathbf{R}$ along a vector. What is yet missing is the notion of a derivate of an arbitrary morphism along a vector. In other words, we are missing a definition of how a morphism maps a vector on the domain manifold to a vector on the target manifold. This definition is straight-forward:

Let $f \colon M \to N$ be a morphism of manifolds. For a point $p \in M$, let $v \in \mathrm{T}_p M$. The map

$$f_* v \colon \mathcal{C}^\infty(N) \to \mathbf{R}, \psi \mapsto v \cdot (\psi \circ f)$$

fulfills the linearity and constant axioms and a Leibniz rule as follows

$$(f_*v) \cdot (\psi\,\psi') = (v \cdot \phi)\,\psi(f(p)) + \phi(f(p))\,(v \cdot \psi).$$

In other words, f_*v defines a vector at $f(p)$ defined by the equation

$$\forall \psi \in \mathcal{C}^\infty(N) : (f_*v) \cdot \psi = v \cdot (f^{-1}\psi).$$

The vector f_*v is called the *push-forward of v along f*. The induced linear map on the tangent spaces is denoted by

$$\mathrm{T}_p f \colon \mathrm{T}_p M \to \mathrm{T}_p N, v \mapsto f_*v$$

and called the *tangent map of f at p*. That the push-forward indeed generalizes the derivative of functions follows from the observation that

$$\phi_*v = (v \cdot \phi)\,\delta(\phi(p))$$

for any function $\phi \in \mathcal{C}^\infty(M)$ viewed as a morphism $\phi \colon M \to \mathbf{R}$.

The tangent map of the identity morphism is the identity map, the tangent map of a composition is the composition of the tangent maps. More precisely, we have

$$\forall v \in \mathrm{T}_p M : \quad (\mathrm{id}_M)_* v = v, \quad (g \circ f)_* v = g_*(f_*v),$$

where $f \colon M \to N$ and $g \colon N \to P$ are two composable morphisms between manifolds. This observation can be subsumed by saying that the tangent space is a functor from the category of pointed manifolds to the category of finite-dimensional vector spaces.

Given a curve $\alpha \colon J \to M$ and a point $t \in J$, which we may view as a coordinate of time, we can differentiate the curve along $\partial(t)$. The vector

$$\dot\alpha(t) \coloneqq \alpha_*\partial(t)$$

is called the *derivative* or *velocity of the curve α at t*. The statement about the functoriality of the tangent space can be viewed as the chain-rule in disguise: in the one-dimensional case we have

$$(\alpha \circ c)^{\cdot}(s) = \alpha_* c_* \partial(s) = \alpha_*(c'(s)\,\delta(c(s))) = c'(s)\,\alpha_*\delta(c(s)) = c'(s)\dot\alpha(c(s)),$$

where $c \colon I \to J$ is any differential map and s a point in the open subset I of $\mathbf{R}$.

Given an observer $\alpha \colon J \to M$ in space-time M and a clock reading $t \in J$, we get two quantities, namely the event $p \coloneqq \alpha(t)$ and the derivative $u \coloneqq \dot\alpha(t)$, which is a vector at p. In fact, u is all what we need because a vector knows at which point it is attached to. In the reference frame of the observer α, the observer does not move spatially. In other words, u is a vector pointing in the direction of the observers future at the event p. The length of the vector u can be seen as defining the observer's unit of time. We thus define an *instantaneous observer* as just a non-vanishing vector u at any point in space-time.

1.7 Tensors

A vector v at a point p on a manifold M can be seen as a quantity attached to that point p. For example, v could be the velocity of a fluid modelled on M at the point p. Some scalar quantities can also be seen as attached to a point; for example, the temperature of the fluid at the point p. However, not all quantities attached to points are to be modelled by a scalar or a vector.

As an example, consider a (non-conservative) force field in space, which is given by a manifold M, the set of points in space. The force field is defined by the work done when a test particle moves through space. If the test particle is moved infinitesimally from p to $p + \mathrm{d}p$, an infinitesimal amount of work $\mathrm{d}W(p)$ is done, which is a scalar quantity after we have fixed a unit of work. In other words, the force field (or, equivalently, the work done by the force field) at a point $p \in M$ is given by a map

$$\mathrm{d}W(p)\colon \mathrm{T}_pM \to \mathbf{R},$$

which associates to each vector v at p the work done when the particle moves along this vector. By basic principles, this map is linear. In other words, (after choice of a unit of work) $\mathrm{d}W(p)$ is an element of the dual space

$$\mathrm{T}_p^*M := (\mathrm{T}_pM)^*$$

of the vector space T_pM. For any manifold M and a point $p \in M$, the vector space T_p^*M is called the *cotangent space of M at p*. An element of T_p^*M is called a *covector* or *linear form at p*. At a given point p, a force field is thus modelled by a covector.

An important source of covectors is given by the derivation: Let $\phi \in \mathcal{C}^\infty(M)$ be a function on an n-dimensional manifold M. By definition of the vector space structure of a tangent space T_pM at a point $p \in M$, the map

$$\mathrm{d}_p\phi\colon \mathrm{T}_pM \to \mathbf{R}, v \mapsto v \cdot \phi$$

is a linear one and thus an element in the cotangent space. The linear form $\mathrm{d}_p\phi \in \mathrm{T}_p^*M$ is the *(total) differential of ϕ at p*. Given a function ϕ that is only defined in a neighborhood of p, the differential $\mathrm{d}_p\widehat{\phi}$ does not depend on the extension $\widehat{\phi}$ of ϕ by zero away from p. Thus we can define the *differential* $\mathrm{d}_p\phi := \mathrm{d}_p\widehat{\phi}$ *of* ϕ *at* p. In particular, every chart $(x, U) \in \mathfrak{U}^\infty(p, M)$ defines n differentials $\mathrm{d}_p x_1, \ldots, \mathrm{d}_p x_n$. These differentials form a basis of $\mathrm{T}_p^* M$ dual to the basis $\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p$ of $\mathrm{T}_p M$, that is

$$\forall i, j \in \{1, \ldots, n\} : \mathrm{d}_p x_i(\frac{\partial}{\partial x_i}|_p) = \delta_{ij}.$$

Besides scalars, vectors and covectors (linear forms), there are even more complicated objects we need to model physical quantities attached to a point $p \in M$. Assume that we have a way to define lengths of vectors and angles between vectors on a manifold M. (This is, for example, true if M is Galilean space where classical mechanics, Newtonian gravity or electrostatics take place.) After choice of a unit of length, the (square of the) length of a vector $v \in \mathrm{T}_p M$ is given by

$$\|v\|^2 := \langle v, v \rangle_p,$$

where $\langle \cdot, \cdot \rangle_p$ is a positive-definite symmetric bilinear form

$$\mathrm{T}_p M \times \mathrm{T}_p M \to \mathbf{R},$$

in other words an inner product on $\mathrm{T}_p M$. (Recall that a bilinear map is a map that is linear in each of its two arguments. In general, a *multilinear map* is linear in all of its arguments.) Thus also symmetric bilinear forms can be attached to a point of space.

As a final example we consider on a manifold M a crystalline solid with possible dislocations, which are one-dimensional lattice defects. An infinitesimal parallelogram at a point $p \in M$ is given by two vectors u and $v \in \mathrm{T}_p M$ that span the two edges of the parallelogram at p. If one goes around the parallelogram-shaped small loop in the crystalline solid at p, one can roll along the way an idealized perfect version of the crystalline solid. By this we mean that whenever we move from one constituent of the actual crystalline solid to the next one while going around the loop, we take the corresponding move in the idealized crystalline solid. At the end of the closed loop in the actual crystalline solid, we may not end at the starting point in the idealized crystalline solid. In general the end point in the idealized crystalline solid differs from the starting point by a vector w, which we can identify via the crystalline structure with a vector w at p in M. The non-vanishing of w is a measurement of the dislocations of the crystalline solid along the surface element spanned by u and v. All dislocations at p are therefore modelled by a map

$$\mathrm{T}_p M \times \mathrm{T}_p M \to \mathrm{T}_p M$$

that maps (u, v) to w. Again by basic principles, we can assume that this map is linear. This map is an even more general quantity than a scalar, vector, covector or a bilinear form; it is a bilinear map on T_pM taking values in T_pM.

The general notion that subsumes all these types of quantities is that of a tensor:

Definition 1.6. A *tensor of type* (k, ℓ) *at a point* p *on a manifold* M is a multilinear map

$$(\mathrm{T}_p^*M)^k \times (\mathrm{T}_pM)^\ell \to \mathbf{R}.$$

This means, given a tensor A of type (k, ℓ) and k covectors $\alpha_1, \ldots, \alpha_k$ and ℓ vectors $v_1, \ldots, v_\ell$, one gets a scalar $A(\alpha_1, \ldots, \alpha_k, v_1, \ldots, v_\ell)$, and this scalar depends linearly on each α_i and on each v_j. The set of tensors at a point $p \in M$ is denoted by $\mathrm{T}_p^{(k,\ell)}M$. As the sum of two multilinear maps and the multiplication of a multinear map with a scalar are again multilinear maps, the set $\mathrm{T}_p^{(k,\ell)}M$ becomes a vector space by point-wise addition and scalar multiplication, the *space of tensors of type* (k, ℓ) *at* p.

For example, the work done by a force field at a point p is a tensor of type $(0, 1)$ (which is nothing but a linear form or covector) and the inner product $\langle \cdot, \cdot \rangle_p$ from above is a special tensor (namely a symmetric one) of type $(0, 2)$ at p A scalar, like the temperature of a fluid at a point p, is also a tensor, namely a tensor of type $(0, 0)$. (By convention, a multilinear map of no arguments is just an element of the target space, which is given by the reals in this case.) The velocity v of a fluid at a point p, however, is not immediately seen to be a tensor of any type as defined above as v is firstly a vector and not a (multi-)linear map on any space. Nevertheless, this vector induces a linear map, namely

$$v^* \colon \mathrm{T}_p^*M \to \mathbf{R}, \alpha \mapsto \alpha(v),$$

which is a tensor of type $(1, 0)$. From v^* we can reconstruct the vector v as

$$v = \sum_{i=1}^n v^*(\theta_i)\, u_i$$

where $u_1, \ldots, u_n$ is any basis of T_pM with dual basis $\theta_1, \ldots, \theta_n$. (This observation is nothing but the well-known fact that canonical linear map

$$V \to V^{**}, v \mapsto (v^* \colon \alpha \mapsto \alpha(v))$$

from a vector space to its double dual is an isomorphism for finite-dimensional vector spaces.) Therefore, when modelling physical quantities, we won't distinguish between a vector quantity v and the corresponding tensor quantity v^* of type $(1, 0)$.

More generally, any $(k+\ell-1)$-linear map of the form

$$L\colon (\mathrm{T}_p^*M)^{k-1} \times (\mathrm{T}_pM)^\ell \to \mathrm{T}_pM$$

induces a $(k+\ell)$-linear map

$$\begin{aligned} L^*\colon & (\mathrm{T}_p^*M)^k \times (\mathrm{T}_pM)^\ell \to \mathrm{T}_pM, \\ & (\alpha_0,\dots,\alpha_{k-1},v_1,\dots,v_\ell) \mapsto \alpha_0(L(\alpha_1,\dots,\alpha_{k-1},v_1,\dots,v_\ell), \end{aligned}$$

that is a tensor if type (k,ℓ), and L can be recovered by the formula

$$L(\alpha_1,\dots,\alpha_{k-1},v_1,\dots,v_\ell) = \sum_{i=1}^n L^*(\theta_i,\alpha_1,\dots,\alpha_{k-1},v_1,\dots,v_\ell)\,u_i.$$

Thus the dislocation of a crystalline solid at a point p, which above was modelled by a vector-valued bilinear form, is nothing but a tensor of type $(1,2)$.

1.8 Vector fields

As a basic example of a vector on an n-dimensional manifold M we gave the velocity vector of a fluid modelled on M at a point p. Often, however, one is not interested in a single velocity vector but at the totality of all velocity vectors at all points of M. This yields to the following definition:

Definition 1.7. A *vector field* X on a manifold M is a family $(X_p)_{p\in M}$ of vectors on M such that $X_p \in \mathrm{T}_pM$ for all $p \in M$ and such that

$$X\cdot\phi\colon p \mapsto X_p \cdot \phi$$

is a (differentiable) function on M for any function $\phi \in \mathcal{C}^\infty(M)$.

The latter condition ensures that the vectors X_p vary smoothly when p is varied. In general, the velocitites at all points of a fluid are thus modelled by such a vector field. The set of vector fields is denoted by $\mathfrak{X}(M)$. This set becomes an $\mathcal{C}^\infty(M)$-module by defining addition and scalar multiplication point-wise.

Every chart (x,U) on M induces n vector fields on the open submanifold U, namely

$$\frac{\partial}{\partial x_i}\colon p \mapsto \frac{\partial}{\partial x_i}|_p.$$

for $i = 1, \dots, n$, which form a basis of T_pM for all $p \in U$. Given any system $\phi_1, \dots, \phi_n \in \mathcal{C}^\infty(U)$ of functions on U, the linear combination $\sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}$

is a vector field on U. On the other hand, every vector field on U is of this form.

If a test particle is put into the fluid on M, it will move with the fluid. The test particle will trace out a curve $\alpha\colon J \to M$ on M such that the particles velocity will coincide with the fluids velocity at the point of the particle at all times:

$$\forall t \in J : \dot{\alpha}(t) = X_{\alpha(t)}$$

where X is the vector field describing the fluid's velocity with respect to a chosen unit of time. A curve for which $\dot{\alpha} = X \circ \alpha$ holds for a given vector field X on a manifold M is called an *integral curve of* X. It is the curve that is traced out by following the "flow" of the vector field. The question of existence of integral curves is handled by the following theorem:

Theorem 1.3. *Let M be a manifold and $X \in \mathfrak{X}(M)$.*

Existence of maximal integral curves *For each pair $(t_0, p) \in \mathbf{R} \times M$ there exists an integral curve $\alpha\colon J \to M$ with $\alpha(t_0) = p$ that is maximal and unique in the following sense: Is $\widetilde{\alpha}\colon \widetilde{J} \to M$ any integral curve on X with $\widetilde{\alpha}(t_0) = p$, then $\widetilde{J} \subseteq J$ and $\alpha|\widetilde{J} = \widetilde{\alpha}$. The integral curve α is the* maximal integral curve of X to the initial condition (t_0, p). *The maximal integral curve of X to the initial condition $(0, p)$ is denoted by $\alpha_p := \alpha_p^X\colon J_p \to M$.*

Translation invariance of integral curves *If $\alpha\colon J \to M$ is any integral curve on M and $s \in \mathbf{R}$, then $\beta\colon -s + J \to M, t \mapsto \alpha(t+s)$ is an integral curve.*

Maximal flow *The set*

$$D := D^X := \bigcup_{p \in M} (J_p \times \{p\})$$

is an open neighborhood of $\{0\} \times M$ in $\mathbf{R} \times M$ and

$$\Phi := \Phi^X\colon D \to M, (t, p) \mapsto \alpha_p(t)$$

is a morphism, which is called the maximal flow of X.

Local 1-parameter group *For $t \in \mathbf{R}$, let D_t be the open subset*

$$D_t := \{p \in M : (t, p) \in D\} \subseteq M.$$

The morphism

$$\Phi_t \coloneqq \Phi_t^X \colon D_t \to M, p \mapsto \Phi(t,p)$$

is a diffeomorphism onto D_{-t}. (In particular $D_{-t} \neq \emptyset$ whenever $D_t \neq \emptyset$.) One has $D_0 = M$ and $\Phi_0 = \mathrm{id}_M$. Moreover one has

$$\forall t, s \in \mathbf{R} : 0 \le t < s \implies D_s \subseteq D_t,$$
$$M = \bigcup_{t \in \mathbf{R}_+} D_t,$$

and

$$\forall t, s \in \mathbf{R}, p \in D_s : \begin{cases} \Phi_s(p) \in D_t \iff p \in D_{t+s}, \\ p \in D_{t+s} \implies \Phi_t \circ \Phi_s(p) = \Phi_{t+s}(p). \end{cases}$$

The family $(\Phi_t^X)_{t \in \mathbf{R}}$ is the dynamical system generated by X.

Proof. We first show that unique integral curves exist locally: Let (x, U) be a chart on M and $G \coloneqq x(U) \subseteq \mathbf{R}^n$. There are n functions $\phi_1, \ldots, \phi_n$ on U such that

$$X|U \coloneqq (X_p)_{p \in U} = \sum_{i=1}^n \phi_i \frac{\partial}{\partial x_i}.$$

We set $f \coloneqq (\phi_1 \circ x^{-1}, \ldots, \phi_n \circ x^{-1}) \colon G \to \mathbf{R}^n$. If $p_0 \in U$ and $\alpha \colon J \to M$ a curve with $t_0 \in J$ and $\alpha(t_0) = p_0$, then by continuity of α there exists an $\epsilon > 0$ such that $I \coloneqq U_\epsilon(t_0) \subseteq J$ and $\alpha(I) \subseteq U$. The map $x \circ \alpha \coloneqq x \circ \alpha|I \colon I \to \mathbf{R}^n$ fulfills $(x \circ \alpha)(I) \subseteq G$ and

$$\forall t \in I : (x \circ \alpha)'(t) = \mathrm{d}_{\alpha(t)} x(\dot{\alpha}(t)).$$

On the other hand,

$$\forall t \in I : f((x \circ \alpha)(t)) = \mathrm{d}_{\alpha(t)}(X_{\alpha(t)}).$$

It follows that $\alpha|I$ is an integral curve of X with $\alpha(t_0) = p_0$ if and only if the map $x \circ \alpha$ is a solution of the system of ordinary differential equations $y'(t) = f(y(t))$ with initial value $y(t_0) = x(p_0)$.

We use this result to show the existence of maximal integral curves. For given $(t_0, p_0) \in \mathbf{R} \times M$ let $\Sigma(t_0, p_0)$ be the set of all integral curves $\beta \colon J_\beta \to M$, where $J_\beta \subseteq \mathbf{R}$ is an open interval with $t_0 \in J_\beta$ and $\beta(t_0) = p_0$. By the local existence of integral curves, $\Sigma(t_0, p_0) \neq \emptyset$. For any pair $\beta, \gamma \in \Sigma(t_0, p_0)$, the set $J_\beta \cap J_\gamma$ is an open interval containing t_0. We claim that $\beta|J_\beta \cap J_\gamma = \gamma|J_\beta \cap J_\gamma$: Let

$$A \coloneqq \{t \in J_\beta \cap J_\gamma : \beta(t) = \gamma(t)\}.$$

To prove our claim, we have to show that $A = J_\beta \cap J_\gamma$. First of all, $A \neq \emptyset$ because $t_0 \in A$. Furthermore, A is a closed subset of $J_\beta \cap J_\gamma$ because it is defined by an equation in a Hausdorff space. Finally, A is open because of the local uniqueness of integral curves. Thus $A = J_\beta \cap J_\gamma$ by the connectedness of intervals in $\mathbf{R}$. □

1.9 Tensor fields

1.10 Problems

Problem 1.1. Prove that every n-dimensional atlas of a set M is contained in a unique maximal n-dimensional atlas of M.

Problem 1.2. Prove that the system of open subsets of a premanifold M is a topology on the underlying set of M.

Problem 1.3. Let (x, U) be a chart of an n-dimensional premanifold M. Show that a subset V of U is open in M if and only if $x(V)$ is open in $\mathbf{R}^n$. Conclude that $x\colon U \to \mathbf{R}^n$ is a continuous map.

Problem 1.4. Prove that an open submanifold of a manifold is in fact a manifold.

Problem 1.5. Prove the following: Let M be a manifold and $\phi\colon M \to \mathbf{R}$ a map. Assume that for each point $p \in M$ there exists a chart $(x, U) \in \mathfrak{U}^\infty(p, M)$ such that $\phi \circ x^{-1}\colon x(U) \to \mathbf{R}$ is smooth. Then ϕ is a (smooth) function.

Problem 1.6. Prove that every function f on an n-dimensional manifold M is a continuous map $f\colon M \to \mathbf{R}$ for the underlying topologies of M and $\mathbf{R}$.

Problem 1.7. Prove that the functions on a manifold M form a *sheaf of algebras*, that is prove that $\mathcal{C}^\infty(U)$ is closed under constants, addition and multiplication for every open subset U of M, that the restrictions $\mathcal{C}^\infty(U) \to \mathcal{C}^\infty(V)$ are homomorphisms of algebras for every inclusion $V \subseteq U$ of open subsets of M, and that for every open cover $(U_i)_{i \in I}$ of M and functions $\phi_i \in \mathcal{C}^\infty(M)$ one has

$$(\forall i, j \in I : \phi_i | U_i \cap U_j = \phi_j | U_i \cap U_j) \implies \exists! \phi \in \mathcal{C}^\infty(M)\, \forall i \in I : \phi | U_i = \phi_i.$$

Problem 1.8. Prove that a map $f\colon M \to N$ between manifolds is a morphism if and only if

$$\forall \psi \in \mathcal{C}^\infty(M) : f^{-1}\psi \in \mathcal{C}^\infty(N).$$

Problem 1.9. Prove that the composition $g \circ f$ or two morphisms $f\colon M \to N$ and $g\colon N \to P$ between manifolds is again a morphism.

Problem 1.10. Prove that the inclusion morphism $i\colon U \to M$ for any open submanifold U of a manifold M is in fact a morphism.

Problem 1.11. Let M and N be two manifolds. Show that

$$\{(x \times y, U \times V) : (x, U) \in \mathfrak{U}^{\infty}(M), (y, V) \in \mathfrak{U}^{\infty}(N)\}.$$

is an atlas of the cartesian product $M \times N$.

Problem 1.12. Show that the underlying topological space of the product $M \times N$ of two manifolds M and N coincides of the topological product of the topological spaces underlying M and N.

Problem 1.13. Prove that the projections $\mathrm{pr}_1\colon M \times N \to M$ and $\mathrm{pr}_2\colon M \times N \to N$ defined on a product manifold $M \times N$ are in fact morphisms.

Problem 1.14. Let $f\colon M \to N$ be a morphism between manifolds and $p \in M$ a point. Prove that

$$f_* v\colon \mathcal{C}^{\infty}(N) \to \mathcal{C}^{\infty}(N), \psi \mapsto v \cdot (\psi \circ f)$$

in fact defines a vector at $f(p)$.

Problem 1.15. Let $f\colon M \to N$ be a morphism between manifolds and $p \in M$ a point. Proof that the tangent map at p is a linear map

$$\mathrm{T}_f M\colon \mathrm{T}_p M \to \mathrm{T}_{f(p)} N, v \mapsto f_* v.$$

Problem 1.16. Let M be a manifold and $p \in M$ be a point. Show that

$$\forall \phi \in \mathcal{C}^{\infty}(M) : \phi_* v = (v \cdot \phi)\, \delta(\phi(p)).$$

for all $v \in \mathrm{T}_p M$.

Problem 1.17. Let M be a manifold and $p \in M$ be a point. Show that

$$\forall v \in \mathrm{T}_p M : \quad (\mathrm{id}_M)_* v v, \quad (g \circ f)_* v = g_*(f_* v),$$

where $f\colon M \to N$ and $g\colon N \to P$ are two (composable) morphisms between manifolds.

Problem 1.18. Let ϕ and ψ be two functions on a manifold M that coincides in a neighborhood of a point $p \in M$. Why is $\mathrm{d}_p \phi = \mathrm{d}_p \psi$?

Problem 1.19. Let M be a manifold and $p \in M$ be a point. Show that for any chart $(x, U) \in \mathfrak{A}^\infty(p, M)$, the system

$$\mathrm{d}_p x_1, \dots, \mathrm{d}_p x_n$$

is a dual basis to the basis

$$\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p.$$

Problem 1.20. Let $v \in \mathrm{T}_p M$ be a vector at a point p on a manifold M. Let $(x, U) \in \mathfrak{A}^\infty(p, M)$ be any chart near p. Let $v^* \colon \alpha \mapsto \alpha(v)$ be the tensor of type $(1, 0)$ at p induced by p. Show that

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}|_p,$$

where $v_i \coloneqq v^*(\mathrm{d}_p x_i)$ for $i \in \{1, \dots, n\}$.

Problem 1.21. Let (x, U) be a chart of an n-dimensional manifold M. Show that every vector field $X \in \mathfrak{X}(U)$ uniquely defines a system $\phi_1, \dots, \phi_n \in \mathcal{C}^\infty(U)$ such that

$$X = \sum_{i=1}^{n} \phi_i \frac{\partial}{\partial x_i}.$$

Chapter 2

Equations of motion in spacetime

2.1 Introduction

2.2 Affine manifolds

2.3 Parallel displacement

2.4 Torsion

2.5 Autoparallels

2.6 The exponential map

2.7 Problems

Chapter 3

Tidal forces

3.1 Introduction

3.2 Curvature

3.3 Bianchi identities

3.4 Jacobi fields

3.5 Taylor expansion of the exponential map

3.6 Problems

Chapter 4

Vacuum structure and causality

4.1 Introduction

4.2 Pseudo-riemannian manifolds

4.3 Levi–Civita connection

4.4 Geodesics

4.5 Riemannian curvature

4.6 Geodesic discs and balls

4.7 Ricci curvature

4.8 Problems

Chapter 5

General relativity

5.1 Introduction

5.2 Einstein tensor

5.3 Mass density

5.4 Einstein's field equations

5.5 Newton's theory of gravity

5.6 Equations of motion

5.7 Killing vector fields

5.8 Schwarzschild metric

5.9 Schwarzschild potential

5.10 Problems

Chapter 6

Electromagnetism

6.1 Introduction

6.2 Differential forms

6.3 Hodge dual

6.4 Maxwell's theory

6.5 Energy-momentum tensor

6.6 Charged black holes

6.7 Problems

Chapter 7

Action principles

7.1 Introduction

7.2 Euler-Lagrange equation

7.3 Conserved currents

7.4 Maxwell's theory

7.5 Hilbert's action principle

7.6 Problems

Chapter 8

Einstein–Cartan theory

8.1 Introduction

8.2 Lie groups

8.3 Fibre bundles

8.4 Ehresmann connections

8.5 Electromagnetism

8.6 Cartan connections

8.7 Einstein–Cartan theory

8.8 Spin

8.9 Equations of motion

8.10 Problems

Appendix A

Topology

A.1 Topological spaces

Definition A.1. Let X be a set. A *topology on* X is a subset of the power set of X, whose elements are called *open subsets of the topology* and which fulfills the following axioms:

Union Let $(U_i)_{i \in I}$ be a family of open subsets of the topology. Then the union $\bigcup_{i \in I} U_i$ is again an open subset of the topology.

Intersection Let $U_1, \ldots, U_n$ be finitely many open subsets of the topology. Then their intersection $U_1 \cap \cdots \cap U_n$ is again an open subset of the topology.

A *topological space* X is a set X together with a topology $\mathfrak{U}^0(X)$ on X. An *open set of a topological space* X is an open set of the topology of X. A *point* $p \in X$ *of a topological space* X is an element of the underlying set of X.

As the union of an empty family of subsets of a set X is the empty subset, the empty subset is open with respect to any topology on X. As the intersection of an empty family of subsets of X is the whole set X, the whole set X is open with respect to any topology on X.

Let X be a topological space. A *neighborhood of a point* p *in* X is a subset G of the underlying set of X such that there exists an open subset U of X with $p \in U \subseteq G$. The set of neighborhoods of p is denoted by $\mathfrak{U}(p, X)$. A neighborhood that is at the same time an open subset is an *open neighborhood*. The set of open neighborhoods of p is denoted by $\mathfrak{U}^0(p, X)$. A subset U of the underlying set of the topological space X is open if and only if it is a neighborhood of every point $p \in U$. A *neighborhood of a subset* A

of X is a subset G that is a neigborhood of every point $p \in A$. The set of neighborhoods of A is denoted by $\mathfrak{U}(A, X)$.

An *closed subset* $Z \subseteq X$ *of* X is a subset of the underlying set of X such that its complement $X \setminus Z$ is open. Let Y be an arbitrary subset of X. The *closure* $\overline{Y}$ *of* Y *in* X is the smallest closed subset of X containing Y, that is

$$\overline{Y} = \bigcap_{Y \subseteq Z \subseteq X \text{ is a closed subset}} Z \tag{A.1}$$

Let Y be any subset of the underlying set of X. Set

$$\mathfrak{U}^0(Y) \coloneqq \{Y \cap U : U \in \mathfrak{U}^0(X)\}$$

This is a topology, the *subspace topology*. Endowed with this topology, Y canonically becomes a topological space itself. Topological spaces of this form are called *subspaces of* X. If Y is a closed subset of X, a subset of Y is closed in Y if and only if it is closed in X. In particular, the closure of a subset of Y in Y is the same as its closure in X.

The cartesian space $\mathbf{R}^n$ is canonically a topological space with the topology defined by

$$\mathfrak{U}^0(\mathbf{R}^n) = \{U \subseteq \mathbf{R}^n : \forall p \in U \, \exists \epsilon \in \mathbf{R}^+ : U_\epsilon(p) \subseteq U\}, \tag{A.2}$$

where

$$U_\epsilon(p) \coloneqq \{q \in \mathbf{R}^n : \|q - p\| < \epsilon\}$$

for an $\epsilon > 0$ is the standard ϵ*-neighborhood*. Its closure under the so-defined topology is given by

$$\overline{U}_{\epsilon(p)} = B_\epsilon(p) \coloneqq \{q \in \mathbf{R}^n : \|q - p\| \leq \epsilon\}.$$

Given a pair of topological space X and Y, the product $X \times Y = \{(x, y) : x \in X, y \in Y\}$ carries a canonical topology defined by

$$\mathfrak{U}^0(X \times Y) = \{W \subseteq X \times Y \forall (x, y) \in W \, \exists U \in \mathfrak{U}^0(x, X), V \in \mathfrak{U}^0(y, Y) : U \times V \subseteq W\}. \tag{A.3}$$

Endowed with this topology, $X \times Y$ is called the *product space* or *topological product of* X *and* Y. If A is a subspace of X and B is a subspace of Y, the subspace $A \times B$ of $X \times Y$ carries the product topology of the topological product between A and B.

A.2 Continuous maps

Definition A.2. A *continuous map* $f\colon X \to Y$ *from an topological space* X *to a topological space* Y is a mapping $f\colon X \to Y$ such that

$$\forall V \in \mathfrak{U}^0(Y) : f^{-1}(V) \in \mathfrak{U}^0(X).$$

Let X be a topological space and let $\phi\colon X \to \mathbf{R}$ be a continuous map. The *support* $\operatorname{supp}\phi$ *of* ϕ is the closed subset

$$\{p \in X : \forall G \in \mathfrak{U}(p, X) : \phi|G \not\equiv 0\}.$$

It is given by

$$\operatorname{supp}\phi = \overline{\{p \in X : \phi(p) \neq 0\}}. \tag{A.4}$$

If X and Y are two topological spaces, the two projection maps $\mathrm{pr}_1\colon X \times Y \to X, (x, y) \mapsto x$ and $\mathrm{pr}_2\colon X \times Y \to Y, (x, y) \mapsto y$ are continuous.

A.3 Hausdorff spaces

Definition A.3. A *Hausdorff space* X is a topological space X such that for every pair p and q of points of X with $p \neq q$, there are neighborhoods G of p and H of q, respectively, such that $G \cap H = \emptyset$.

A subspace of a Hausdorff space is again a Hausdorff space. Any finite subset of a Hausdorff space is a closed subset. The topological product of two Hausdorff spaces is again a Hausdorff space.

Cartesian space $\mathbf{R}^n$ is an example for a Hausdorff space.

A.4 Normal spaces

Definition A.4. A Hausdorff space X is *normal* such that for any disjoint closed subsets A and B of X there exists neighborhoods $U \in \mathfrak{U}(A, X)$ and $V \in \mathfrak{U}(B, X)$ with $U \cap V = \emptyset$.

An open cover $(U_i)_{i \in I}$ of a topological space X is *locally finite* if every point of X possesses a neighborhood G such that there only finitely many $i \in I$ with $G \cap U_i \neq \emptyset$.

Theorem A.1 (Shrinking lemma)**.** *For any locally finite open cover* $(U_i)_{i \in I}$ *of a normal Hausdorff space* X*, there exists a locally finite open cover* $(V_i)_{i \in I}$ *with*

$$\forall i \in I : V_i \subseteq \overline{V_i} \subseteq U_i.$$

A.5 Compact spaces

A family $(U_i)_{i\in I}$ of open subsets of a topological space X is an *open cover of X* if

$$X = \bigcup_{i\in I} U_i.$$

A *subcover* of an open cover $(U_i)_{i\in I}$ of X is an open cover of the form $(U_i)_{i\in I'}$ for $I' \subseteq I$. An open cover $(U_i)_{\in I}$ of X is *finite* if the index set I is finite.

If Y is a subspace of a topological space X and $(U_i)_{i\in I}$ is a family of open subsets of X such that $(Y \cap U_i)_{i\in I}$ is an open cover of Y, we often say for simplicity that $(U_i)_{i\in I}$ is an open cover of Y.

Definition A.5. A *compact space* is a Hausdorff space X such that every open cover of X has a finite subcover.

A subset K of the underlying set of a Hausdorff space X is *compact* if it is a compact space when endowed with the subspace topology. It is *relatively compact in* X if its closure in X is compact.

A compact subspace of a Hausdorff space X is always a closed subset of X. A closed subset of a compact space is again compact. The product of two compact spaces is again a compact space.

Theorem A.2. *A subset K of cartesian space $\mathbf{R}^n$ is compact if and only if it is closed and there exists an $R \in \mathbf{R}_+$ such that $K \subseteq \{p \in \mathbf{R}^n : \|p\| \leq R\}$.*

The latter condition on K says that K is *bounded*.

A.6 Locally compact spaces

Definition A.6. A Hausdorff space X is *locally compact* if each $p \in X$ has a neighborhood $U \in \mathfrak{U}(p, X)$, which is compact.

The condition of being locally compact for a Hausdorff space X is equivalent to requiring that every point $p \in X$ possesses an open neighborhood $U \in \mathfrak{U}(p, X)$, which is relatively compact in X.

Proposition A.1. *On a locally compact Hausdorff space X, every neighborhood G of a point $p \in X$ contains a compact neighborhood $K \in \mathfrak{U}(p, X)$.*

The product of two locally compact Hausdorff spaces is again a locally compact Hausdorff space.

A.7 Paracompact Hausdorff spaces

A *refinement* of an open cover $(U_i)_{i\in I}$ of a topological space X is an open cover $(V_j)_{j\in J}$ of X such that

$$\forall j \in J\, \exists i \in I : V_j \subseteq U_i.$$

Definition A.7. A *paracompact Hausdorff space* is a Hausdorff space X such that every open cover of X has a locally finite refinement.

Proposition A.2. *A paracompact Hausdorff space is normal.*

Theorem A.3. *The topological product of two locally compact paracompact Hausdorff spaces is again a (locally compact) paracompact Hausdorff space.*

Cartesian space $\mathbf{R}^n$ is an example of a paracompact Hausdorff space.

A.8 Problems

Problem A.1. Let X be a topological space. Prove that a subset U of the underlying set of X is open if and only if it is a neighborhood of every point $p \in U$.

Problem A.2. Prove that the subspace topology of a subset of the underlying set of a topological space is in fact a topology.

Problem A.3. Let Y be a closed subset of a topological space X. Let Z be a subset of Y. Show that Z is closed in Y if and only if it is closed in X.

Problem A.4. Show that (A.2) defines a topology on $\mathbf{R}^n$.

Problem A.5. Show that (A.3) defines a topology on the cartesian product of the topological spaces X and Y.

Problem A.6. Show that the closure of a standard ϵ-neighborhood in $\mathbf{R}^n$ for the canonical topology is given by

$$\overline{U}_{\epsilon(p)} = \{q \in \mathbf{R}^n : \|q - p\| \leq \epsilon\}. \tag{A.5}$$

Problem A.7. Let A and B be subspaces of the topological spaces X and Y, respectively. Show that the subspace $A \times B$ of $X \times Y$ carries the product topology of the topological product between A and B.

Problem A.8. Let X be a topological space. Prove that the support of a continuous map $\phi\colon X \to \mathbf{R}$ is indeed closed and that the formula (A.4) holds.

Problem A.9. Let X and Y be two topological spaces. Show that the two projection maps pr_1 and pr_2 defined on the topological product $X \times Y$ are continuous.

Problem A.10. Prove that a subspace of a Hausdorff space is again a Hausdorff space.

Problem A.11. Prove that a finite subset of a Hausdorff space is a closed subset.

Problem A.12. Prove that a product of two Hausdorff spaces is again a Hausdorff space.

Problem A.13. Prove that a compact subspace K of a Hausdorff space X is always a closed subset of X.

Problem A.14. Prove that a closed subset Z of a compact space K is again compact.

Problem A.15. Prove that the product of two compact spaces is again a compact space.

Problem A.16. Prove that a Hausdorff space X is locally compact if and only if every point $p \in X$ possesses an open neighborhood $U \in \mathfrak{U}(p, X)$ such that U is relatively compact in X:

Problem A.17. Prove that the product of two locally compact Hausdorff spaces is again a locally compact Hausdorff space.

Problem A.18. Show that the canonical topology on $\mathbf{R}^n$ makes cartesian space into a paracompact Hausdorff space.

Appendix B

Analysis

B.1 Bump functions

Theorem B.1. *For any two real numbers* $0 < r < R$ *there exists a smooth function* $\phi\colon \mathbf{R}^n \to \mathbf{R}$ *with*

$$\forall p \in \mathbf{R}^n : \phi(p) \geq 0, \quad \forall p \in \mathbf{R}^n : \|p\| \leq r \implies \phi(p) = 1, \quad \forall p \in \mathbf{R}^n : \|p\| \geq R \implies \phi(p) = 0.$$

A function as in the theorem is called a *bump function*.

B.2 Hadamard's lemma

A subset U of n-dimensional Euclidean space $\mathbf{R}^n$ is *star-shaped with respect to a point* $p \in U$ if

$$\forall x \in U \forall 0 \leq t \leq 1 : (1-t)\, a + t\, x \in U.$$

Any neighborhood of a point $p \in \mathbf{R}^n$ contains an open neighborhood star-shaped with respect to the point p as the standard ϵ-neighborhoods are star-shaped with respect to their center (in fact, with respect to any point of their interior).

Theorem B.2. *Let* ϕ *be a smooth function defined on an open subset* U *of* n*-dimensional Euclidean space* $\mathbf{R}^n$ *that is star-shaped with respect to a point* $a \in U$*. Then there exist smooth functions* $g_1, \ldots, g_n$ *on* U *such that*

$$\forall x \in U : \phi(x) = \phi(a) + \sum_{i=1}^{n} (x_i - a_i)\, g_i(x).$$