General Relativity and Einstein–Cartan Theory

Marc A. Nieper-Wißkirchen

March 4, 2014

Preface

Gravity is one of the fundamental interactions of nature, by which all physical bodies attract each other. The purpose of these notes is to give a physically motivated and mathematically sound introduction to Einstein's general relativity. Since its invention, it has passed all experimental tests and is the simplest theory that is able to describe all observed phenomena of gravity. In short, it is the accepted theory of gravity in modern physics. This is not to say that general relativity is set in stone. By incorporating possible intrinsic angular moment of macroscopic matter, one arrives at the mathematically equally beautiful Einstein–Cartan theory, which encloses general relativity. While general relativity is a metric theory that builds on the mathematical concept of a Lorentzian manifold, Einstein–Cartan theory is a true gauge theory of gravity. It will be described in the latter chapters of these notes.

Addressees of these notes are, on the one hand side, graduate students of physics who are willing to cope with abstract mathematical concepts and, on the other hand, graduate students of mathematics with a strong background in classical physics. These notes grew out of a lecture course the author gave in the winter term 2013/14 at the University of Augsburg, and the audience of the course was a mixture of students from the physics and the mathematical department.

The theory of general relativity and the novel effects being predicted by it — for example, gravitational time dilation or gravitational waves — have been fascinating many people, but to understand these phenomena on a quantative level, one has to delve deeply into the mathematics of general relativity. One the other hand, it is a much rewarding untertaking. One will have grasped one of the most beautiful physical theories (if not *the* most beautiful one). These notes show one of the possible routes there, a route the author would have liked to go when he learnt general relativity. Going this route also provides the reader with a solid knowledge of differential geometry.

That said, it *is* possible to formulate the heart of the theory of general relativity in one sentence in plain English, namely:

> The mass density measured by any observer is the scalar curvature of that observer's space divided by 16π.

Of course, without any further explanations of the contained mathematical terms and an accompanying physical interpretation, this statement is just as meaningful as simply stating[1] $\operatorname{div} E = 4\pi\,\rho$ without any further explanations of the terms involved. Nevertheless, the simplicity of this statement already shows the beauty of general relativity.

Augsburg, March 4, 2014 Marc Nieper-Wißkirchen

[1]Gauß's law in gaussian units

History

The theory of general relativity was being developed by Albert Einstein between 1907 and 1915. It builds upon and combines the earlier theories of Newtonian gravity and special relativity. Although the basic theory hasn't changed since then, there have been many contributions afterwards. The following timeline summarizes the history as far it is relevant to the notes at hand.

1609 Kepler pushlishes his first two laws of planetary motion.

1619 Kepler pushlishes his third law of planetary motion.

1638 Galilei's equations for a falling body.

1687 Newton publishes his law of universal gravitation.

1798 Cavendish measures Newton's gravitational constant.

1862 Maxwell's equations of electromagnetism.

1887 The Michelson-Morley experiment fails to detect a stationary luminiferous aether.

1889 FitzGerald proposes Lorentz contraction.

1905 Formulation of special relativity by Einstein.

1915 Derivation of Einstein's field equations from an action principle by Hilbert.

1915 Einstein's theory of general relativity.

1916 Schwarzschild found the first exact solution of Einstein's field equations.

1916 Einstein shows that the perihelion precession of Mercury can be fully explained by general relativity.

1919 Eddington's expedition confirms that the deflection of light by the Sun is as predicted by general relativity.

1922 Friedmann found a cosmological solution to Einstein's equations, in which the universe may expand or contract.

1922 Introduction of the cosmological constant by Einstein into his field equations.

1922 Proposal of the Einstein–Cartan theory by lie Cartan.

1929 Hubble finds evidence that the universe is expanding.

1959 Direct measurement of the gravitational redshift of light in the Pound-Rebka experiment.

1964 Discovery of the cosmis microwave background by Penzias and Wilson.

1964 Discovery of the X-ray source Cygnus X-1, now widely accepted to be a black hole.

Notations

Standard sets We use the following notations for the standard sets: The set of natural numbers (which includes 0, by definition) is denoted by $\mathbf{N}_0$, the set of integers by $\mathbf{Z}$, the set of rational numbers by $\mathbf{Q}$, the set of real numbers by $\mathbf{R}$ and the set of complex numbers by $\mathbf{C}$.

Maps The term *function* will be reserved for smooth maps with values in $\mathbf{R}$. Thus a function is always a smooth function.

Cartesian space The standard Euclidean norm an n-dimensional space $\mathbf{R}^n$ is denoted by

$$\|v\| := \sqrt{\sum_{i=1}^{n} v_i^2}$$

for $v = (v_1, \dots, v_n) \in \mathbf{R}^n$.

Contents