This document is an exam summary that follows the script of the Discrete-time and Statistical Signal Processing lecture at ETH Zurich. The contribution to this is a short summary that includes the most important formulas, algorithms and proof ideas. This summary was created during the fall semester 2017. Due to updates to the syllabus content, some material may no longer be relevant for future versions of the lecture.

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DSSP

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1 DT Linear Systems and z-Tranform

1.1 Signals

Stable signals: A signal is stable if:
- Continuous time: $\int_{-\infty}^{\infty} |f(t)| dt < \infty$
- Discrete time: $\sum_{k=-\infty}^{\infty} |f[k]| < \infty$

Causal signals: A signal is causal if:
- Continuous time: $f(t) = 0 \,, \forall\, t < 0$
- Discrete time: $f[k] = 0 \,, \forall k < 0$

Time reversed signal: $f^r[k] = f[-k]$

Conjugate signal: $f^c[k] = \overline{f[-k]}$

→ If the signal is real $f[k] \in \mathbb{R} \Rightarrow f^c[k] = f^r[k]$

1.2 LTI Systems (page 5)

LTI Systems have the following properties:

Linearity: Let $S\{f_1(t)\} = g_1(t)$ and $S\{f_2(t)\} = g_2(t)$. Then:
- Additivity: $S\{f_1(t) + f_2(t)\} = S\{f_1(t)\} + S\{f_2(t)\} = g_1(t) + g_2(t)$
- Homogeneity: $S\{\alpha f(t)\} = \alpha S\{f(t)\}$
- Superposition principle: $S\{\alpha f_1(t) + \beta f_2(t)\} = \alpha S\{f_1(t)\} + \beta S\{f_2(t)\}$

Time invariance: $S\{f(t - t_0)\} = g(t - t_0)$

Memory: For memoryless systems, the current output only depends on the current input. For systems with memory, the output depends on past input and output values.
A system has memory if $h(t) \neq 0 \; for \; t \neq 0$

BIBO-stability: If a bounded input leads to a bounded output.
A LTI system is BIBO if if impulse response $h[.]$ (or $h(.)$) is a stable signal.

Causality: A LTI system is causal if and only if its impulse response h is a causal signal (i.e. $h(t) = 0 \; for \; t < 0 \; or \; h[k] = 0 \; for \; k < 0$)

Impulse response: If we use a Kronecker-Delta δ as an input to an LTI system, the output is its impulse response.
We get the output of a LTI system by convoluting the input signal with the impulse response.

$$u[.] \longrightarrow \boxed{h[.]} \longrightarrow y[.]$$

input signal	output signal	reason
$\delta[.]$	$h[.]$	definition of $h[.]$
$\delta[.-k]$	$h[.-k]$	time invariance
$u[k]\cdot\delta[.-k]$	$u[k]\cdot h[.-k]$	linearity (homogeneity)
$u[.] = \sum_{k\in\mathbb{Z}} u[k]\delta[.-k]$	$\sum_{k\in\mathbb{Z}} u[k]h[.-k]$	linearity (additivity)

$$u[\cdot] = \sum_{k\in\mathbb{Z}} u[k]\delta[\cdot - k] \quad , \quad y[\cdot] = \sum_{k\in\mathbb{Z}} u[k]h[\cdot - k]$$

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau \quad , \quad y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

1.3 Convolution (page 7)

Discrete time convolution

$$(f * g)[n] = \sum_{k\in\mathbb{Z}} f[k]g[n-k] = \sum_{k\in\mathbb{Z}} f[n-k]g[k]$$

Continuous time convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

1.4 The formal z-Transform (page 11)

Let f a real or complex discrete time signal. The formal z-Transform of f is

$$\boxed{F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}}$$

The z-Transform has the following properties:
- **Additivity:** $F(z) + G(z) = \sum_{k=-\infty}^{\infty} (f[k] + g[k])z^{-k}$
- **Multiplication with a scalar:** $aF(z) = \sum_{k=-\infty}^{\infty} af[k]z^{-k}$
- **Multiplication with z^m:** $z^m F(z) = \sum_{k=-\infty}^{\infty} f[k+m]z^{-k}$
- **Multiplication:** $F(z)G(z) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\sum_{m=-\infty}^{\infty} f[m]g[k-m]\right)}_{=(f*g)[k]} z^{-k}$

$\Rightarrow Convolution \sim Multiplication \; of \; formal \; z-Trafo$

Sometimes the following notation is used: $F(z) \; mod \; z = \sum_{k=0}^{\infty} f[k]z^{-k}$
→ The causal part of $F(z)$

If $h[\cdot]$ is a real signal, the z-Transform of the conjugate signal is given as:

$$\boxed{H^c(z) = H(z^{-1})}$$

The transfer function of a LTI system is the z-Transform of the impulse response.

1.5 Inverses Signals (page 13)

Let $f[\cdot]$ and $g[\cdot]$ two real or complex discrete time signals with formal z-Transform $F(z)$ and $G(z)$.
If $F(z)G(z) = 1$ (i.e. $(f * g)[\cdot] = \delta[\cdot]$) then $G(z)$ is inverse to $F(z)$ and $g[\cdot]$ an inverse signal to $f[\cdot]$.

Proposition 1.3 If $g[\cdot]$ and $h[\cdot]$ are inverse to a real or complex signal $f[\cdot]$, then $\alpha g[\cdot] + \beta h[\cdot]$ is also inverse to $f[\cdot]$ (with $\alpha + \beta = 1$)
→ Without further information, $1/F(z)$ is not a unique signal!

Theorem 1.5 If $f[\cdot]$ is right-sided and not zero everywhere, then there exists exactly one right-sided signal that is inverse to $f[\cdot]$.

1.6 The (analytical) z-Transform (page 25)

Def: The z-Transform of a real or complex Signal $f[\cdot]$ is the complex-valued signal

$$ROC(f) \longrightarrow \mathbb{C}: z \longrightarrow F(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

with the range of convergence $ROC(f) = \{z \in \mathbb{C}: r_1 < |z| < r_2\}$

Where: $\frac{1}{r_1} \triangleq \left(\limsup_{k\longrightarrow\infty} |f[k]|^{1/k}\right)^{-1}$, $r_2 \triangleq \left(\limsup_{l\longrightarrow\infty} |f[-l]|^{1/l}\right)^{-1}$

Theorem 1.8 Let $f[\cdot]$ a real or complex discrete time signal. Then:
- $ROC(f) \; contains \; the \; unit \; circle \Rightarrow \; f[\cdot]$ is stable (= abs. summable)
- $f[\cdot] \; is \; stable \Rightarrow the \; unit \; circle \; is \; in \; ROC(f) \; (i.e. r_1 \leq 1 \; and \; r_2 \geq 1)$

Properties of the ROC
- Addition: $ROC(f + g) \supseteq ROC(f) \cap ROC(g)$
- Multiplication with a scalar: $ROC(af) = aROC(f)$
- Multiplication with z^m: $H(z) = z^m F(z) \Rightarrow ROC(h) = ROC(f)$
- Multiplication: $H(z) = F(z)G(z) \Rightarrow ROC(h) \supseteq ROC(f)) \cap ROC(g)$
- Derivative: Let $h[k] = kf[k]$, then $H(z) = -z\frac{dF(z)}{dz} \Rightarrow ROC(h) = ROC(f)$
- Initial-value for causal signals: $f[0] = \lim_{|z|\longrightarrow\infty} F(z)$
- Final-value for rational right-sided signals: $\lim_{k\longrightarrow\infty} f[k] = \lim_{z\longrightarrow 1} (z-1)F(z)$

1.7 Rational z-Transform (page 29)

A rational function is a function of the form $F(z) = \frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}$

$F(z)$ is causal iff: $\deg_{numerator}(F(z)) \leq \deg_{denumerator}(F(z))$

With real or complex numbers a_i and b_i, $i = 0,1,\dots,n$, with at least one $b_i \neq 0$.

Theorem 1.10 Let $f[\cdot]$ be a real or complex discrete-time signal with rational z-Trafo.
Then: $f[\cdot] \; is \; stable \Leftrightarrow ROC(f) \; contains \; the \; unit \; circle$
If $f[\cdot]$ is right-sided, then
$f[\cdot] \; is \; stable \; \Leftrightarrow all \; poles \; of \; F(z) \; are \; (strictly) \; inside \; the \; unit \; circle$

To get a stable signal $h[\cdot]$, we need to expand all:
- Poles inside the unit circle right-sided
- Poles outside the unit circle left-sided

$F(z)$	$ROC(f)$	$f[\cdot]$
$\frac{z}{z-p}$	$\|z\| > \|p\|$	rechtsseitig: $f[k] = \begin{cases} 0 & k < 0 \\ p^k & k \geq 0 \end{cases}$
	$0 < \|z\| < \|p\|$	linksseitig: $f[k] = \begin{cases} -p^k & k < 0 \\ 0 & k \geq 0 \end{cases}$
$\frac{Az}{z-p} + \frac{\overline{A}z}{z-\overline{p}} = 2z\frac{z\,Re(A) - Re(A\overline{p})}{z^2 - 2z\,Re(p) + \|p\|^2}$	$\|z\| > \|p\|$	re: $f[k] = \begin{cases} 0 & k < 0 \\ 2\|A\|\,\|p\|^k\cos(\Omega k + \varphi) & k \geq 0 \end{cases}$
	$0 < \|z\| < \|p\|$	li: $f[k] = \begin{cases} -2\|A\|\,\|p\|^k\cos(\Omega k + \varphi) & k < 0 \\ 0 & k \geq 0 \end{cases}$

For Ω and φ: $A = |A|e^{i\varphi}$ and $p = |p|e^{i\Omega}$
To get right-sided signals: multiply with $\sigma[k]$
To get left-sided signals: multiply with $\sigma[-k-1]$

1.8 Spectrum of discrete-time signals (page 35)

Let $f[\cdot]$ be a discrete-time signal with formal z-Transform $F(z)$. The Spectrum or the discrete-time Fourier transform of $f[\cdot]$ is the function

$$\boxed{\mathbb{R} \longrightarrow \mathbb{C}: \Omega \mapsto F(e^{i\Omega}) = F(z)|_{z=e^{i\Omega}} = \sum_{k\in\mathbb{Z}} f[k]e^{-i\Omega k}}$$

The spectrum is periodic in Ω with Period 2π.
If $f[k]$ is the impulse response of a LTI system, the spectrum $F(e^{i\Omega})$ is also called the frequency response.
The spectrum of a discrete-time signal is the z-Transform on the unit circle
Proposition 1.5 If $f[k]$ is stable, then the spectrum $F(e^{i\Omega})$ is well-defined.

Special case: empty $ROC(f)$

$$F(e^{i\Omega}) = 2\pi \sum_{n\in\mathbb{Z}} \delta(\Omega - \Omega_0 - 2\pi n)$$

So $F(e^{i\Omega}) = 2\pi\delta(\Omega - \Omega_0)$ for $\Omega_0 - \pi < \Omega < \Omega_0 + \pi$
The corresponding time signal is $f[k] = e^{i\Omega_0 k}$

Spectrum of the conjugate signal $f^c[\cdot]$ (page 38)

$\boxed{F^c(z) = \overline{F(\bar{z}^{-1})} \;\; \Rightarrow \;\; F^c(e^{i\Omega}) = \overline{F(e^{i\Omega})}} \rightarrow F(z)F^c(z)|_{z=e^{i\Omega}} = |F(e^{i\Omega})|^2$

If $f[\cdot]$ is real: $F^c(z) = F(z^{-1})$

Theorem 1.12 A stable complex discrete-time signal $f[k]$ is real if $\forall\Omega \in \mathbb{R}$

$$F(e^{-i\Omega}) = \overline{F(e^{i\Omega})}$$

1.9 Fourier Series

Fourier series of a function $g: \mathbb{R} \longrightarrow \mathbb{C}$ with Period T

$$g(x) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi kxT} \quad , \quad c_k = \frac{1}{T}\int_0^T g(x)e^{-i2\pi kx/T}dx$$

If $f[k]$ is stable, then $f[-\cdot]$ is the Fourier series of $F(e^{i\Omega})$.

2 Discrete-Time to Cont. Time and Back

2.1 Laplace-Transform and Fourier-Transform

The **Laplace-Transform** of a continuous time signal is

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

Always state a ROC with a Laplace-Transform.
ROC has always the form $ROC(f) = \{s \in \mathbb{C}: r_1 < Re(s) < r_2\}$

Properties of the Laplace-Transform
- Convolution: $h(\cdot) = f(\cdot) * g(\cdot) \multimap \mathrm{H(s) = F(s)G(s) = G(s)F(s)}$
- Stability: BIBO-stable iff the imaginary axis lies in the ROC of $F(s)$
- Memoryless: If $F(s)$ is constant
- Causal, if $h(t) = 0 \; for \; t < 0$

Theorem 2.2 A right-sided signal $f(\cdot)$ with rational Laplace-Transform $F(s)$ is stable iff (i) $\deg(numerator) \leq \deg\,(denumerator)$ (ii) all poles of $F(s)$ are in the open left half plane

Theorem 2.3
- $ROC(f)$ includes the imag. Axis $\Rightarrow f(\cdot)$ is stable
- $f(\cdot)$ is stable $\Rightarrow$ the imag. Axis is in the $ROC(f)$

If $F(s)$ is **rational**: $f(\cdot)$ stable $\Leftrightarrow ROC(f)$ includes the imag. Axis

The **Fourier-Transform** is the Laplace-Transform on the imaginary axis

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \quad , \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(i\omega)e^{i\omega t}d\omega$$

The frequency response of a LTI system is the Fourier-Transform of its impulse response. A complex continuous-time signal is $f(\cdot)$ with spectrum $F(i\omega)$ is real if $\forall\omega \in \mathbb{R}$: $F(-i\omega) = \overline{F(i\omega)}$

2.2 Converting DT Signals into CT Signals

General conversion formula with $T \in \mathbb{R}$ and $h(t)$ is the impulse response of a continuous-time linear filter (should be causal & stable in praxis)

$$y(t) = \sum_{k\in\mathbb{Z}} x[k]h(t - kT)$$

Theorem 2.5 The Laplace-Transform of the signal $\tilde{f}(t) = \sum_{k\in\mathbb{Z}} f[k]\delta(t - kT)$ is $\tilde{F}(s) = F(z)|_{z=e^{sT}}$. For the spectrum holds: $\tilde{F}(i\omega) = F(e^{i\omega T})$
→ The spectrum $\tilde{F}(i\omega)$ of a CT signal $\tilde{f}(t) = \sum_{k\in\mathbb{Z}} f[k]\delta(t - kT)$ is equal to the periodic spectrum $F(e^{i\Omega})$ of the DT signal $f[\cdot]$ with $\Omega = \omega T$.

2.3 Sampling

Ideal Sampling is the creation of the DT signal $x_s[\cdot]$ from the CT signal $x(\cdot)$

$$x_s[k] = Tx(kT - \tau)$$

where $T \in \mathbb{R}$ is the sampling period. Mostly we set $\tau = 0$.
Check page 48.
Nyquist-Shannon sampling theorem see page 47
Theorem 2.7 $y(t)$ a complex-valued signal with Laplace-Trafo $Y(s)$. The z-Trafo o f the sampled signal $y_s[k] = Ty(kT)$ is

$$Y_s(z) = \sum_{s\in\mathbb{C}: e^{sT} = z} Y(s)$$

Frequency of sampled signals with $f_s = 1/T$

$$\Omega = \omega T = 2\pi f/f_s \; in \; the \; range -\pi \dots \pi \; or \; 0 \dots 2\pi$$

$$\frac{\Omega}{2\pi} = f_{norm} = f/f_s \; in \; the \; range -\frac{1}{2} \dots \frac{1}{2} \; or \; 0 \dots 1$$

2.4 Decimation/ Interpolation

Interpolation (page 51)
Interpolation means increasing the sampling frequency by an integer factor. We do this by zero-stuffing → insert $n_{up} - 1$ zeros.

$$g_z[k] = \begin{cases} g\left[\frac{k}{n}\right], if \; k \; is \; a \; multiple \; of \; n \\ 0 \,, else \end{cases}$$

Expressed in the z-Transform $G_z(z) = G(z^{n_{up}})$
Since the period got smaller, we need to filter out spectral parts outside of $|\Omega| < \pi/n_{up}$ with a discrete-time lowpass filter.
Dezimation (page 53)
Decimation means decreasing the sampling frequency by an integer factor
The decimated signal is: $g_d[k] = n_{down}g[n_{down}k]$
Through Decimation we obviously loose information. Aliasing can happen.
To prevent Aliasing, we use a Decimation filter (lowpass) with a cutoff frequency π/n_{down} .

2.5 FIR Filters & Window function

Ideal, discrete-time lowpass
- Spectrum: $H(e^{i\Omega}) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & \Omega_c \leq |\Omega| \leq \pi \end{cases}$
- Impulse response: $h[k] = \sin(\Omega_c k)\,/\pi k$

Hanning Window function (page 58)

$$\omega[k] = \begin{cases} \frac{1}{2}\left(1 + cos\left(\frac{2\pi k}{N+2}\right)\right), & |k| \leq N/2 \\ 0, & |k| > N/2 \end{cases}$$

With the Hanning window we get the impulse response $g[\cdot]$ of the causal lowpass (with $h[\cdot]$ the ideal lowpass)

$$h_W[k] = \frac{h[k]w[k]}{\sum_{n\in\mathbb{Z}} h[n]w[n]} \Rightarrow \boxed{g[k] = h_W\left[k - \frac{N}{2}\right]}$$

2.6 CT and DT IIR Filters

We often design DT IIR filters in two steps:
1. Design a CT filter
2. Transform the CT filter into a DT filter (bilinear Trafo)

Butterworth-Filter (page 60)

$$H(s) = \frac{1}{\prod_{k=1}^{N}(1 - s/p_k)} \; , \; with \; p_k = \omega_c e^{i\frac{\pi}{2N}(2k+N-1)}$$

Poles are in the left half plane and on a circle of radius ω_c
Amplitude response: $|H(i\omega)|^2 = \frac{1}{1+(\omega/\omega_c)^{2N}}$

Bilinear Transform (page 62)
$G(z) = H(s)|_{s=\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}}$, $s = \frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}$, Inversion: $z = \frac{1+sT/2}{1-sT/2}$

The imaginary axis $s = i\omega$ gets projected onto the unit circle $z = e^{i\Omega}$ according to: $\Omega = 2\tan^{-1}\left(\frac{\omega T}{2}\right)$, $\omega = \frac{2}{T}\cdot\tan\left(\frac{\Omega}{2}\right)$
The bilinear transform maps the whole imaginary axis *bijectively* (without aliasing) onto the unit circle.

3 DFT

3.1 DFT (page 72)

The DFT is defined as

$$\boxed{F[n] = F(\alpha^n) = \sum_{k=0}^{N-1} f[k]\alpha^{-kn} = \sum_{k=0}^{N-1} f[k]e^{-i2\pi kn/N}}\ , with\ \alpha = e^{i2\pi/N}$$

Therefore, it holds: $F[n] = F(e^{i\Omega})|_{\Omega=2\pi n/N}$

The DFT samples the spectrum of $f[\cdot]$

Inverse DFT

The DFT is invertible with: $f[k] = \frac{1}{N}\sum_{n=0}^{N-1} F[n]\alpha^{kn} = \frac{1}{N}\sum_{n=0}^{N-1} F[n]e^{i2\pi kn/N}$

Theorem 3.2 A complex vector $(f[0], \dots, f[N-1])^T$ is real iff its DFT satisfies

$$F[N-n] = \overline{F[n]}\ for\ n = 1, \dots, N-1\ \&\&\ F[0] = \overline{F[0]}$$

3.2 Aliasing in time (page 73)

Let $g[\cdot]$ a DT signal. For a positive integer N we form the vector $\tilde{g} = (\tilde{g}[0], \tilde{g}[1], \dots, \tilde{g}[N-1])$ with $\tilde{g}[n] = \sum_{m\in\mathbb{Z}} g[n+mN]$, $0 \le n \le N$ and the z-Trafo $\tilde{G}(z) = G(z)\ mod\ (z^N - 1) = \sum_{k\in\mathbb{Z}} g[k]z^{-(k\ mod\ N)}$
For the spectrum it follows

$$\boxed{G(e^{i\Omega})|_{\Omega=2\pi n/N} = \underbrace{G(z)\ mod(z^N-1)|_{z=e^{i2\pi n/N}}}_{DFT\ of\ \tilde{g}[n]=\sum_{m\in\mathbb{Z}} g[n+mN]}}$$

For $\Omega = \frac{2\pi n}{N}, n = 0,1, \dots, N-1$, the spectrum of a DT signal $f[\cdot]$ equals the spectrum (i.e. the DFT) of a vector with formal z-Trafo $F(z)\ mod(z^N - 1)$.

3.3 Cyclic convolution (page 74)

Proposition 3.3 The formal z-Trafo of $f \circledast g$ is $F(z)G(z)\ mod(z^N - 1)$
Theorem 3.3 Let $H[\cdot] = (H[0], \dots, H[N-1])^T \in \mathbb{C}$ be the DFT of $h[\cdot] = f \circledast g$. Then: $H[n] = F[n]\cdot G[n] = F(\alpha^n)G(\alpha^n), n = 0, \dots, N-1$
$\rightarrow$ *Cyclic convolution ~componentwise multiplication of the DFT*

4 Probability Theory

4.1 Real random variables (page 87)

- Cumulative distribution function $F_X(x) = P(X \le x)$
- Probability density function $f_X(x) = \frac{d}{dx}F_X(x)$
- Norm property: $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $P(a < X \le b) = F_X(b) - F_X(a) = \int_{a^+}^{b^+} f_X(x)dx$
- Joint distribution: $F_{X_1,..X_n}(x_1,.,x_n) = P(X_1 \le x_1\ and \dots and\ X_n \le x_n)$
  Therefore: $F_{X_1,..X_n}(x_1,.,x_n) = \int_{-\infty}^{x_1^+} \dots \int_{-\infty}^{x_n^+} f_{X_1,..,X_n}(x_1, \dots, x_n)dx_1 dx_n$
- Joint density: $f_{X_1,..X_n}(x_1, \dots, x_n) = \frac{\partial^n F_{X_1,\dots,X_n}}{\partial x_1 \dots \partial x_n}(x_1, \dots, x_n)$
  Marginal density: $f_{X_k}(x_k) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_{X_1,..X_n}(x_1, \dots, x_n)dx_1.dx_{k-1}dx_{k+1}.dx_n$

Independence

Two real random variables are independent if
- $F_{X_1,..X_n}(x_1,.,x_n) = F_{X_1}(x_1)\cdot \dots \cdot F_{X_n}(x_n)$ equivalent
- $f_{X_1,..X_n}(x_1, \dots, x_n) = f_{X_1}(x_1)\cdot \dots \cdot f_{X_n}(x_n)$

Functions of Random Variables (page 92)

Let X and W two <u>independent</u> real random variables with PDFs $f_X(x)$ and $f_W(w)$ and let $Y = X + W$. Then:

$$\boxed{f_{Y|X}(y|x) = f_W(y - x)}$$

4.2 Discrete random variables (page 85)

A discrete RV is a function $X: \Omega \longrightarrow S$ with a finite or countably infinite domain S s.t. $\forall s \in S$, the set $\{\omega \in \Omega: X(\omega) = s\}$ is an event.

Independence

Two discrete random variables are independent if

$$P(X = x\ and\ Y = y) = P(X = x)\cdot P(Y = y)$$

4.3 Expectation

Real random variables

- Expectation: $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
- Function of RVs: $E[g(X)] = \int_{-\infty}^{\infty} g(x_1, \dots, x_n)f_{X_1,..X_n}(x_1,.., x_n)dx_1..dx_n$

Discrete random variables

- Expectation: $E[X] = \sum_{x\in S} xP(X = x)$
- Functions of RVs: $E[g(X)] = \sum_{x\in S} g(x)P(X = x)$

Linearity of Expectation

$$E[aX + bY] = aE[X] + bE[Y]$$

Independence

If X and Y are independent real or complex random variables

$$E[X\cdot Y] = E[X]\cdot E[Y]\ \ and\ \ E[X\cdot\overline{Y}] = E[X]\cdot\overline{E[Y]}$$

X and Y are uncorrelated if the above holds.

X and Y independent ⇒ X and Y uncorrelated ⇔ $E[X\cdot\overline{Y}] = E[X]\cdot\overline{E[Y]}$

4.4 Variance and Correlation

Variance (page 97)

$$Var(X) = \sigma^2 = E[(X - m_X)^2] = E[X^2] - E[X]^2$$

$\rightarrow \sigma = \sqrt{Var(X)}$ standard deviation

- $Var(X + Y) = Var(X) + Var(Y) + \underbrace{2cov(X,Y)}_{=0\ if\ X\ and\ Y\ independent}$
- $Var(aX) = a^2Var(X)$
- $Z = X + iY \Rightarrow Var(Z) = Var(X) + Var(Y)$ (no i !!!)

Correlation (page 98)

The correlation of X and Y is: $E[X\cdot\overline{Y}]$
If the correlation $E[X\cdot\overline{Y}] = 0 \rightarrow X$ and Y are orthogonal

Covariance (page 98)

$$Cov(X,Y) = E\left[(X - m_X)\overline{(Y - m_Y)}\right] = E[X\overline{Y}] - m_X\overline{m_Y}$$

The Covariance is zero if $E[X\cdot\overline{Y}] = E[X]\cdot\overline{E[Y]}$, i.e. if X and Y are uncorrelated.

Correlation matrix (page 99)

The correlation matrix of a real or complex random vector $X = (X_1, \dots, X_n)^T$

$$R_X = E[XX^H]$$

$$= \begin{pmatrix} E[X_1\overline{X_1}] & E[X_1\overline{X_2}] & .. & E[X_1\overline{X_n}] \\ E[X_2\overline{X_1}] & E[X_2\overline{X_2}] & .. & E[X_2\overline{X_n}] \\ \vdots & \vdots & .. & \vdots \\ E[X_n\overline{X_1}] & E[X_n\overline{X_2}] & .. & E[X_n\overline{X_n}] \end{pmatrix}$$

The Hermitian-Transpose of A is the matrix A^H with elements $(A^H)_{i,j} = \overline{A_{j,i}}$. In general, it holds that: $R_X^H = R_X$

Covariance matrix (page 99)

The covariance matrix of a real or complex random vector $X = (X_1, \dots, X_n)^T$

$$V_X = E[(X - m_X)(X - m_X)^H] = R_X - m_X m_X^H$$

$$= \begin{pmatrix} E[(X_1 - m_{X_1})\overline{(X_1 - m_{X_1})}] & .. & E[(X_1 - m_{X_1})\overline{(X_n - m_{X_n})}] \\ \vdots & .. & \vdots \\ E[(X_n - m_{X_n})\overline{(X_1 - m_{X_1})}] & .. & E[(X_n - m_{X_n})\overline{(X_n - m_{X_n})}] \end{pmatrix}$$

$$V_X = R_X - m_X m_X^H$$

Theorem 4.9 (page 100)

Let $X = (X_1, \dots, X_n)^T$ and A a $n\times n$ matrix and let $Y = AX + b$
- Correlation matrix of Y: $R_Y = AR_XA^H + Am_Xb^H + (Am_Xb^H)^H + bb^H$
  $\rightarrow$ if b = 0: $R_Y = AR_XA^H$
- Covariance matrix of Y: $V_Y = AV_XA^H$

4.5 Discrete-time Stochastic Processes (page 102)

Def: A Discrete-time Stochastic Process is a sequence $X[k], k \in \mathbb{Z}$, of random variables.

A process is **i.i.d.** if $X[k]$ and $X[n]$ for $k \ne n$ are independent and have the same distribution.
$\rightarrow$ Autocorrelation function of a real i.i.d. stochastic process $U[\cdot]$

$$R_U[n] = E\left[U[k+n]\cdot\overline{U[k]}\right] = \begin{cases} E[U[k]^2], & n = 0 \\ E[U[k+n]]\cdot E[U[k]], & n \ne 0 \end{cases}$$

A process $X[\cdot]$ is **stationary** if, for all integers $n \ge 0$, the joint distribution of $X[k], X[k+1], \dots, X[k+n]$ does not depend on k.

stationary ⇒ weakly stationary

A process $X[\cdot]$ is **weakly stationary** if, for every $n \in \mathbb{Z}$, both $E[X[k]]$ and $E[X[k+n]\cdot\overline{X[k]}]$ exist and do not depend on k.

weakly stationary ⇒ $E[X[k+n]] = E[X[k]] = m_X$

Autocorrelation function (page 103)

$$\mathbb{Z} \rightarrow \mathbb{C}: n \rightarrow \boxed{R_X[n] = E\left[X[k+n]\cdot\overline{X[k]}\right]}$$

It holds that: $R_X[-n] = \overline{R_X[n]}$
The **average power** of a process is $R_X[0] = E[|X[k]|^2]$

Def: Two real or complex stochastic processes $X[\cdot]$ and $Y[\cdot]$ are **jointly weakly stationary** if both $X[\cdot]$ and $Y[\cdot]$ are weakly stationary and for every $n \in \mathbb{Z}$, $E\left[X[k+n]\cdot\overline{Y[k]}\right]$ exists and is independent of k.

Cross-correlation function (page 104)

The cross-correlation of two jointly weakly stationary processes is

$$R_{XY}[n] = E\left[X[k+n]\cdot\overline{Y[k]}\right]$$

In general, it holds that: $R_{XY}[-n] = \overline{R_{XY}[n]}$

White noise

A real or complex stochastic process $W[\cdot]$ is called **white noise** with power σ^2 if $W[\cdot]$ is weakly stationary and $m_W = 0$ and

$$R_W[\cdot] = \sigma^2\delta[\cdot] \Rightarrow S_W(z) = \sigma^2 \Rightarrow S_W(e^{i\Omega}) = \sigma^2$$

4.6 Linear Filtering of WS Processes (page 104)

Let $X[\cdot]$ be a weakly stationary (WS) process and $h[\cdot]$ the impulse response of a LTI-System $Y[k] = \sum_{k=-\infty}^{\infty} X[k-n]h[n]$

Theorem 4.11 $Y[\cdot]$ is weakly stationary with mean

$$m_Y = m_X\sum_{n=-\infty}^{\infty} h[n] = m_X\cdot H(z=1) = m_X\cdot H(e^{i\Omega})|_{\Omega=0}$$

$X[\cdot]$ and $Y[\cdot]$ are jointly weakly stationary with cross-correlation function

$$R_{YX}[\cdot] = (h * R_X)[\cdot]$$

And the autocorrelation function $Y[\cdot]$ is

$$R_Y[\cdot] = h^c[\cdot] * R_{YX}[\cdot] = h[\cdot] * h^c[\cdot] * R_X[\cdot]$$

For the z-Transform

$$S_X(z) = \sum_{n=-\infty}^{\infty} R_X[n]z^{-n} = S_X^c(z)$$

$$S_{YX}(z) = H(z)S_X(z)\ \Leftrightarrow S_{XY}(z) = S_{YX}^c(z) \Leftrightarrow\ S_{XY}(z) = S_X(z)H^c(z)$$
$$S_Y(z) = H^c(z)S_{YX}(z) = H(z)H^c(z)S_X(z)$$
$$\Rightarrow S_Y = S_X(e^{i\Omega})|H(e^{i\Omega})|^2$$

4.7 Power Spectral density (page 107)

The spectrum $S_X(e^{i\Omega})$ of the autocorrelation function $R_X[\cdot]$ is called the power spectral density of the process $X[\cdot]$.

$$S_X(e^{i\Omega}) = S_X(z)|_{z=e^{i\Omega}}$$

4.8 Conditional Probabilities

Bayes' rule:

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B|A)}{P(B)}$$

A finite set $\{A_1, A_2, \dots, A_n\} \subset \mathcal{E}$ of events or a countably infinite set $\{A_1, A_2, \dots\} \subset \mathcal{E}$ of events is a partition of Ω if $A_i \cap A_j = \emptyset$ for $i \ne j$ and $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$ or $A_1 \cup A_2 \cup \dots = \Omega$ respectively.

Total probability: $P(B) = P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)$
Total Expectation: $E[X] = \sum_{k=1}^{n} P(A_k)E[X|A_k]$

Conditional Density of a sum

Let X and W be independent random variables and $Y = X + W$
Then:

$$\boxed{f_{Y|X}(y|x) = f_W(y - x)}$$

4.9 Conditioning on Values of RVs (page 116)

$$p_X(x) = P(X = x)$$
$$f_{X\,|\,Y}(x\,|\,y) = f_{X\,|\,Y=y}(x) = \frac{d}{dx}F_{X\,|\,Y=y}(x)$$
$$p_{X,Y}(x,y) = P(X = x \text{ und } Y = y)$$
$$p_{X|A}(x) = P(X = x|A)$$
$$p_{X|Y}(x|y) = p_{X|Y=y}(x) = P(X = x|Y = y)$$

X, Y, Z diskret	
$P(A\|Y = y)$	$= \frac{P(A)p_{Y\|A}(y)}{p_Y(y)}$
$P(A)$	$= \sum_y p_Y(y)P(A\|Y = y)$
$p_{X\|Y}(x\|y)$	$= \frac{p_{X,Y}(x,y)}{p_Y(y)}$
$p(x,y,z)$	$= p(x)p(y\|x)p(z\|x,y)$
$E[X]$	$= \sum_y E[X\|Y = y]\, p(y)$

X, Y, Z reell	
$P(A\|Y = y)$	$= \frac{P(A)f_{Y\|A}(y)}{f_Y(y)}$
$P(A)$	$= \int_{-\infty}^{\infty} f_Y(y)P(A\|Y = y)\, dy$
$f_{X\|Y}(x\|y)$	$= \frac{f_{X,Y}(x,y)}{f_Y(y)}$
$f(x,y,z)$	$= f(x)f(y\|x)f(z\|x,y)$
$E[X]$	$= \int_{-\infty}^{\infty} E[X\|Y = y]\, f(y)\, dy$

Conditioning on $X \in S$
Theorem 4.21 Let S be a subset of the codomain of X s.t. $P(X \in S) > 0$

$$f_{X|X\in S}(x) = \begin{cases} \frac{f_X(x)}{P(X\in S)}, & x \in S \\ 0, & x \notin S \end{cases}$$

If X is a discrete random variable

$$p_{X|X\in S}(x) = P(X = x|X \in S) = \begin{cases} \frac{p_X(x)}{P(X\in S)}, & x \in S \\ 0, & x \notin S \end{cases}$$

4.10 Important Distributions

Gaussian Distribution

$$X \sim \mathcal{N}(m_X, \sigma_X^2) \rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{(x - m_X)^2}{2\sigma_X^2}\right)$$

With $E[X] = m_X$ and $Var[X] = \sigma_X^2$
Let $X_i \sim \mathcal{N}(\mu_{X_i}, \sigma_{X_i}^2)$ and all the X_i are independent. And $Y = \sum_i a_iX_i$.
Then: $Y \sim \mathcal{N}\left(\sum_i a_i\mu_{X_i}, \sum_i a_i^2\sigma_{X_i}^2\right)$

Uniform Distribution on $[a, b]$

$$X \sim uni(a,b) \rightarrow f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & else \end{cases}$$

With $E[X] = \frac{a+b}{2}$ and $Var[X] = \frac{(b-a)^2}{12}$

Exponential Distribution with Parameter $\lambda > 0$

$$X \sim \exp(\lambda) \rightarrow f_X(x) = \begin{cases} \lambda\cdot e^{-\lambda x}, & x \ge 0 \\ 0, & else \end{cases}$$

With $E[X] = \frac{1}{\lambda}$ and $Var[X] = \frac{1}{\lambda^2}$
And the cumulative distribution function: $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \ge 0 \\ 0, & else \end{cases}$

5 Detection and Estimation

Estimation: The goal is to get a function h s.t. $\hat{X} = h(Y_1,\dots,Y_n)$ is a good or optimal estimation of X, where Y_k are observations.

Mean squared error (MSE)

$$MSE = E\left[(\hat{X}-X)^2\right] = \int_{-\infty}^{\infty} E\left[(\hat{X}-X)^2 \middle| X=x\right]\cdot f_X(x)dx$$

Unbiased estimator

$\hat{X}$ is an unbiased estimator iff $E[\hat{X}|X=x]=x$

For an unbiased estimator: $E\left[(\hat{X}-X)^2|X=x\right] = Var[\hat{X}|X=x]$

5.1 Bayesian Estimation (page 119)

Bayesian estimation minimizes for each observation $Y=y$ the expectation of the cost $\hat{x} = h(y) = \underset{\hat{x}}{\operatorname{argmin}}\, E[\kappa(\hat{x},X)|Y=y]$.

Consider the cost function $\kappa(\hat{x},x) = |\hat{x}-x|^2$. The resulting estimation is called **MMSE-Estimation**.

$$Estimation: \boxed{\hat{x} = h(y) = E[X|Y=y]} = \int_{-\infty}^{\infty} x\cdot f_{X|Y}(x|y)dx$$

$$MSE = E[|\hat{x}-X|^2 \mid Y=y] = E[(X-m_X(y))^2 \mid Y=y]$$

5.2 Maximum-likelihood Estimation (page 120)

The Maximum-likelihood estimation is given by

$$\boxed{\hat{x} = h(y) = \underset{x}{\operatorname{argmax}} f_{Y|X}(y|x)}$$

Proposition 5.1 (Invariance property)

Let $X=g(U)$, where g is a bijective (i.e. invertible) function. Then $\hat{x}$ is the ML estimate of X from the observation $Y=y$ if and only if $\hat{u} \triangleq g^{-1}(\hat{x})$ is the ML estimate of U from $Y=y$.

This means that the ML estimate of X^2 or $\log(X)$ gives the same result as the ML estimate of X.

Sometimes the following ML estimation rule is also used:

$$\hat{x} = h(y) = \underset{x}{\operatorname{argmax}} P(X=x|Y=y)$$

5.3 MAP estimation

$$\boxed{\hat{x} = \underset{x}{\operatorname{argmax}} f_{X|Y}(x|y) = \underset{x}{\operatorname{argmax}} f_{Y|X}(y|x) f_X(x)}$$

Remark on argmax

If we have e.g. a ML estimation $\hat{x} = \underset{x}{\operatorname{argmax}} f_{Y|X}(y|x)$, then $\hat{x}$ is the x that maximizes $f_{Y|X}(y|x)$!!! Ant not the maximum value of $f_{Y|X}(y|x)$.

How to find $\underset{x}{\operatorname{argmax}} f_{Y|X}(y|x)$?

- $\frac{d}{dx}\left(f_{Y|X}(y|x)\right) = 0 \Leftrightarrow x = \cdots$
- Instead of maximizing $f_{Y|X}(y|x)$, we can also maximize $\ln\left(f_{Y|X}(y|x)\right)$
  This is especially useful for gaussian RVs.
  $$\Rightarrow \underset{x}{\operatorname{argmax}} f_{Y|X}(y|x) \Leftrightarrow \underset{x}{\operatorname{argmax}} \ln(f_{Y|X}(y|x))$$

5.4 Decision problems (page 125)

If X is a discrete random variable, we call the estimation problem a decision problem.

5.4.1 Bayesian Decision

The Bayesian estimation rule becomes the Bayesian decision rule

$$\hat{x} = h(y) = E[\kappa(\hat{x},X)|Y=y] = \underset{\hat{x}}{\operatorname{argmin}} \sum_{x\in S} \kappa(\hat{x},x)P(X=x) f_{Y|X}(y|x)$$

5.4.2 MAP decision rule

For cost function $\kappa(\hat{x},x) = \begin{cases}0, & \hat{x}=x\\ 1, & \hat{x}\neq x\end{cases}$ the expectation is exactly the **error probability** for the given observation $Y=y$:

$$E[\kappa(\hat{x},X)|Y=y] = \sum_{x\in S: x\neq \hat{x}} P(X=x|Y=y) = P(X\neq\hat{x}|Y=y) = 1-P(X=\hat{x}|Y=y)$$

The MAP decision rule minimizes these costs

$$\hat{x} = \underset{x}{\operatorname{argmax}} P(X=x) f_{Y|X}(y|x)$$

5.4.3 ML decision rule

Stays the same as with the estimation problem

$$\boxed{\hat{x} = h(y) = \underset{x}{\operatorname{argmax}} f_{Y|X}(y|x)}$$

ML/ MAP decision rule for 2 hypotheses A_0 and A_1 from observations of Y_1 and Y_2:

- ML: $L(y_1,y_2) = \ln\left(\frac{f(y_1,y_2|A_1)}{f(y_1,y_2|A_0)}\right) \Rightarrow \begin{cases} L(y_1,y_2)<0 \to A_0 \\ L(y_1,y_2)>0\to A_1\end{cases}$
- MAP: $M(y_1,y_2) = \ln\left(\frac{P(A_1)\cdot f(y_1,y_2|A_1)}{P(A_0)\cdot f(y_1,y_2|A_0)}\right) \Rightarrow \begin{cases} M(y_1,y_2)<0\to A_0\\ M(y_1,y_2)>0\to A_1\end{cases}$

5.5 LMMSE Estimation (page 129)

Estimation of real or complex RVs X from observations $Y_1,\dots,Y_n$ under the assumption, that the estimation function is linear:

$$\hat{X} = h(Y) = h_1Y_1+\dots+h_nY_n$$

With (real or complex) coefficients $h_1,\dots,h_n$ such that the expected squared error $E\left[|\hat{X}-X|^2\right]$ is as small as possible.

Theorem 5.2 (Orthogonality principle)

$\hat{X} = \sum_{k=1}^n h_kY_k$ is the LMMSE estimate of X from $Y_1,\dots,Y_n$ if and only if

$$E\left[(\hat{X}-X)\overline{Y_k}\right]=0$$

for $k=1,2,\dots,n$.

This principle can be represented as a matrix equation:

$$\boxed{\underbrace{\begin{pmatrix} E[Y_1\overline{Y_1}] & E[Y_2\overline{Y_1}] & \cdots & E[Y_n\overline{Y_1}]\\ E[Y_1\overline{Y_2}] & E[Y_2\overline{Y_2}] & & E[Y_n\overline{Y_2}] \\ \vdots & & \ddots & \vdots \\ E[Y_1\overline{Y_n}] & E[Y_2\overline{Y_n}] & \cdots & E[Y_n\overline{Y_n}]\end{pmatrix}}_{correlation\ matrix\ R_Y} \cdot \begin{pmatrix}h_1\\h_2\\\vdots\\h_n\end{pmatrix} = \begin{pmatrix}E[X\overline{Y_1}]\\E[X\overline{Y_2}]\\\vdots\\E[X\overline{Y_n}]\end{pmatrix}}$$

This means that the error is orthogonal to all observations.

Theorem 5.3 (LMMSE error)

Let $\hat{X}$ be the LMMSE estimate of X from $Y_1,\dots,Y_n$. Then the MSE is

$$E\left[|\hat{X}-X|^2\right] = E\left[X\overline{(X-\hat{X})}\right] = E[|X|^2] - E\left[|\hat{X}|^2\right]$$

5.6 Wiener Filter (page 132)

Let $X[\cdot]$ and $Y[\cdot]$ be real or complex stochastic processes that are jointly weakly stationary. We want to estimate $X[\cdot]$ from $Y[\cdot]$ by an LTI filter with impulse response $h[\cdot]$: $\hat{X}[\cdot] \triangleq (h*Y)[\cdot]$ s.t. the expected squared error $E\left[|\hat{X}[k]-X[k]|^2\right]$ is as small as possible.

Assume $h[n]=0\ fpr\ n<-L\ and\ for\ n>M$

The estimation then is:

$$\hat{X}[k] = \sum_{n=-L}^{M} h[n]Y[k-n]$$

For $L>0$ the filter is not causal.

The **filter order** is $N=M+L$.

By the Orthogonality principle, we get the **Wiener-Hopf equation**

$$\sum_{n=-L}^{M} h[n]R_Y[j-n] = R_{XY}[j] \qquad j=-L,\dots,M$$

For $L<\infty$ and $M<\infty$ this can be written in matrix form (see page 133).

The expected squared error of a Wiener filter is

$$E\left[|\hat{X}[k]-X[k]|^2\right] = R_X[0] - \sum_{n=-L}^{M} R_{XY}[n]\overline{h[n]}$$

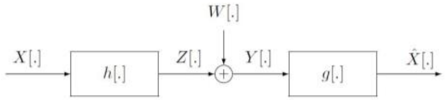
For $L=M=\infty$ the Wiener-Hopf equation becomes $(h*R_Y)[\cdot] = R_{XY}[\cdot]$

Through z-Transform, the corresponding stable filter is

$$H(z) = \frac{S_{XY}(z)}{S_Y(z)}$$

This is called the noncausal Wiener filter for the estimation of X from Y.

5.7 LMMSE Equalization (page 135)



Let $X[\cdot]$ and $W[\cdot]$ be weakly stationary, zero-mean and uncorrelated. Then the following holds:

$$R_{XY}[n] = R_{XZ}[n] \Leftrightarrow S_{XY}(z)=S_{XZ}(z)$$

$$R_Y[n] = R_Z[n]+R_W[n] \Leftrightarrow S_Y(z)=S_Z(z)+S_W(z)$$

The noncausal Wiener filter $g[\cdot]$ for the estimation of $X[\cdot]$ from $Y[\cdot]$ is

$$G(z) = \frac{S_{XY}(z)}{S_Y(z)} = \frac{S_X(z)H^c(z)}{S_X(z)H(z)H^c(z)+S_W(z)}$$

Remarks: Assume Z and W real, WS, zero-mean and uncorrelated.

$$\boxed{R_{WY}[n]} = E[W[k+n]Y[k]] = E[W[k+n](W[k]+Z[k])] = \underbrace{E[W[k+n]W[k]]}_{=R_W[n]} + \underbrace{E[W[k+n]]E[Z[k]]}_{=0,\ since\ zero-mean} \boxed{=R_W[n]}$$

6 Appendix

6.1 Power series

- $\sum_{k=0}^{\infty} aq^k = a+aq+aq^2+\ldots = \frac{a}{1-q}$ für $|q|<1$
- $\sum_{k=0}^{\infty}(k+1)q^k = 1+2q+3q^2+\ldots = \frac{1}{(1-q)^2}$ für $|q|<1$
- $\sum_{k=1}^{\infty}\frac{1}{k^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}$
- $\sum_{k=0}^{\infty}\frac{1}{k!} = 1+1+\frac{1}{2}+\frac{1}{6}+\ldots = e$

6.2 Trigonometry

6.2.1 Functions of $\alpha\pm\beta$, 2α

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1-2\sin^2\alpha$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

6.2.2 Sums and Products

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$$

6.2.3 sin/ cos/ tan values

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	$\to\infty$
cot	$\to\infty$	$\sqrt{3}$	1	$\sqrt{3}/3$	0

6.3 Decibel [dB]

Dimensionless gains are often given in decibel.

If β is a power gain:

$$\beta_{dB} = 10\cdot\log_{10}(\beta) \Leftrightarrow \beta = 10^{\beta_{dB}/10}$$

If β is a gain factor of a measure (current, voltage,…) that influences the power quadratically:

$$\beta_{dB} = 20\cdot\log_{10}(\beta) \Leftrightarrow \beta = 10^{\beta_{dB}/20}$$

Remember:

- $\log(x\cdot y) = \log(x)+\log(y)$
- $a^{x+y} = a^x\cdot a^y$ $\qquad (a^x)^y = a^{x\cdot y}$
- $\log(x^y)\ y\cdot\log(x)$
- $\log_a(x) = \log_b(x)/\log_b(a)$

6.4 Leibniz rule

$$g(t) = \frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx =$$

$$= f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t) + \int_{a(t)}^{b(t)}\frac{\partial}{\partial t}f(x,t)dx$$

6.5 Delta

$$\int_{-\infty}^{\infty} f(\tau)\delta(\tau-a)d\tau = \int_{-\infty}^{\infty} f(\tau)\delta(a-\tau)d\tau = f(a)$$

$$\Rightarrow f(t)*\delta(t-t_0) = f(t-t_0)$$

6.6 Matrix operations

- Transpose: $(A^T)_{i,j} = A_{j,i}$ $\qquad (AB)^T = B^TA^T$
- Hermitian transpose: $(A^H)_{i,j} = \overline{A_{j,i}}$ $\qquad (AB)^H = B^HA^H = \overline{B}^T\overline{A}^T$

6.7 Complex numbers

Imaginary part: $Im(z) = \frac{1}{2i}(z-z^*)$

Real part: $Re(z) = \frac{1}{2}(z+z^*)$

Euler

$$\sin(x) = \frac{1}{2i}(e^{ix}-e^{-ix}) \qquad \cos(x) = \frac{1}{2}(e^{ix}+e^{-ix})$$

$$e^{ix} = \cos(x)+i\cdot\sin(x) \qquad e^{-ix} = \cos(x)-i\cdot\sin(x)$$

Coordinate transformation $z = x+iy \Leftrightarrow z = r\cdot e^{i\varphi}$

$$r = \sqrt{x^2+y^2} \qquad \varphi = \arctan\left(\frac{y}{x}\right)\ (ev.+\pi)$$

$$x = r\cdot\cos(\varphi) \qquad y = r\cdot\sin(\varphi)$$

6.8 Differentials

$(f\cdot g)' = f'\cdot g+f\cdot g'$	$(f/g)' = (f'g-fg')/g^2$
$(a^x)' = \ln(a)\cdot a^x$	$(\ln(x))' = 1/x$
$\left(f(g(x))\right)' = f'(g(x))\cdot g'(x)$	