MNM TEAM
MUNICH NETWORK MANAGEMENT TEAM

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Introduction to Quantum Computing – Exercise Sheet 1

The first exercise sheet is designed to help you understand the basic mathematical concepts in quantum computing. The tasks build on each other and should be worked on in sequence.

Exercise 1.1: *The Bloch-Sphere*

This task is designed to help you clarify and memorize the concept of rotating a vector about the X and Z axes in the Bloch sphere.

Take a look at the following link to a Bloch sphere simulator: [1]. In this simulator, a qubit is represented as a function of two angles θ and ϕ. Set the "Polar angle" to $\theta = \frac{3\pi}{10}$ and the "Azimuth" to $\phi = \frac{6\pi}{12}$. We call this vector v.

- **a)** Rotate the vector v by 90° around the Z axis.[2] What are the values of θ and ϕ now?
- **b)** Rotate the vector v by 90° around the X-axis. What are the values of θ and ϕ now?
- **c)** Set the angles back to the initial values so that for the state of the QuBit $|\psi\rangle \approx 0.891\,|\uparrow\rangle + 0.454i\,|\downarrow\rangle$ is shown. Instead of using the spins $|\uparrow\rangle$ and $|\downarrow\rangle$, one can also label the Bloch sphere with $|0\rangle$ (north pole) and $|1\rangle$ (south pole). Suppose $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$. To what proportion is $|\psi\rangle$ in the states $|0\rangle$ and $|1\rangle$, respectively, and with what probability does the measurement yield $|0\rangle$ and $|1\rangle$, respectively?
- **d)** Rotate the vector v by setting "azimuth" ϕ to 0 (and leaving θ at $\frac{3\pi}{10}$). How do the fractions just calculated (and thus the probabilities of measuring $|0\rangle$ and $|1\rangle$) change? What does this rotation mean for the QuBit?
- **e)** Rotate the vector v by setting "Polar angle" θ to $\frac{7\pi}{10}$. How do the fractions calculated in c) (and thus the probabilities of measuring $|0\rangle$ and $|1\rangle$) change? What does this rotation mean for the QuBit?
- **f)** *(optional)* Check out the alternative Bloch sphere simulator from Bits and electrons[3] and review the operation of the presented single QuBit gates.

Exercise 1.2: *The Tensor Product*

With the tensor product we can describe the composition of single QuBits to a so-called **quantum register**. The tensor product of two matrices $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ and $B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$ is defined as:

$$A \otimes B = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \otimes B = \begin{pmatrix} \alpha_1 B & \alpha_2 B \\ \alpha_3 B & \alpha_4 B \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_2\beta_1 & \alpha_2\beta_2 \\ \alpha_1\beta_3 & \alpha_1\beta_4 & \alpha_2\beta_3 & \alpha_2\beta_4 \\ \alpha_3\beta_1 & \alpha_3\beta_2 & \alpha_4\beta_1 & \alpha_4\beta_2 \\ \alpha_3\beta_3 & \alpha_3\beta_4 & \alpha_4\beta_3 & \alpha_4\beta_4 \end{pmatrix}$$

- **a)** Formulate the calculation rule for vectors $\alpha \otimes \beta$ with $\alpha = \binom{\alpha_1}{\alpha_2}$ and $\beta = \binom{\beta_1}{\beta_2}$.
- **b)** Calculate $|0\rangle \otimes |1\rangle$.
- **c)** Calculate $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- **d)** The following matrix is also called Hadamard matrix: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. By H_n we denote the n-fold tensor product of H with itself. Calculate $H_2 = H \otimes H$.

[1] https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html
[2] To find out how much 90° are at the angle controls, look at the "Azimuth" scale!
[3] https://bits-and-electrons.github.io/bloch-sphere-simulator/

Exercise 1.3: *Valid Qubit Representations*

a) Which of the following vectors are representations of a QuBit and how do you check it?

(i) $\frac{1}{\sqrt{2}}\binom{1}{-i}$ (ii) $\frac{1}{\sqrt{2}}\binom{0}{1}$ (iii) $\binom{0}{i}$

b) With what probability do we measure $|1\rangle$ or $|0\rangle$ at the QuBit $\frac{1}{\sqrt{3}}\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$?

Exercise 1.4: *Entanglement*

The four Bell states are:

$$|a_1a_0\rangle = |00\rangle \Rightarrow \mathbf{\Phi}^+ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |a_1a_0\rangle = |10\rangle \Rightarrow \mathbf{\Phi}^- = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|a_1a_0\rangle = |01\rangle \Rightarrow \mathbf{\Psi}^+ = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |a_1a_0\rangle = |11\rangle \Rightarrow \mathbf{\Psi}^- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

In this task you work with a 2-QuBit register.

a) Draw the circuit that puts a register in the Bell state Φ^- (Group 1), Ψ^+ (Group 2) or Ψ^- (Group 3).
b) Calculate the intermediate states $|\psi_0\rangle$, $|\psi_1\rangle$ and $|\psi_2\rangle$ of your circuit in a).
c) Show that your assigned Bell state cannot be the tensor product of two single QuBits.

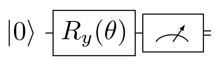
Exercise 1.5: *Teleportation*

In this task, we want to teleport a QuBit; however, $|a\rangle$ and $|b\rangle$ are not – as in the example from the slides – in the Bell state $|\beta_{00}\rangle$ at the beginning ($|\psi_1\rangle$), but in the Bell state $|\beta_{11}\rangle$ (Group 1), $|\beta_{10}\rangle$ (Group 2), or $|\beta_{01}\rangle$ (Group 3).

a) Calculate the state $|\psi_2\rangle$ of the teleportation circuit.
b) *(optional)* How do you need to adjust the circuit to perform successful teleportation based on your initial Bell state?
Note: You only need to take care of the gates after the measurement!

Exercise 1.6: *Measuring Expectation Values*

Given the following parametrized circuit:



What is the expectation value for a measurement in the Z-basis when we set:

a) $\theta = \frac{\pi}{4}$?
b) $\theta = \frac{\pi}{4}$?
c) For which values θ do we minimize, maximize the expectation value?