KEY

Math 267 Worksheet 11 – Fall 2021

1. Solve the system of differential equations $\frac{d\mathbf{Y}}{dt} = U\mathbf{Y}$ defined by the matrix below using generalized eigenvectors.

$$U = \begin{pmatrix} -1 & 3 & 5 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

U is upper triangular $\Rightarrow \lambda = 4, -1$

$$\lambda = 4: \quad \begin{pmatrix} -5 & 3 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -5 \end{pmatrix} \xrightarrow[R_3 + \frac{5}{2}R_2 \to R_3]{R_1 + R_3 \to R_1} \begin{pmatrix} -5 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} -5v_1 + 3v_2 = 0 \\ 2v_3 = 0 \end{matrix} \Rightarrow \begin{matrix} v_1 = \frac{3}{5}v_2 \\ v_3 = 0 \end{matrix} \underset{\text{when } v_2 = 5}{\Longrightarrow} \begin{matrix} v_1 = 3 \\ v_2 = 5 \\ v_3 = 0 \end{matrix}$$

$v = (3, 5, 0)$ is an eigenvector.

$$\lambda = -1: \quad \begin{pmatrix} 0 & 3 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - \frac{5}{3}R_1 \to R_2} \begin{pmatrix} 0 & 3 & 5 \\ 0 & 0 & -\frac{19}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} 3v_2 + 5v_3 = 0 \\ -\frac{19}{3}v_3 = 0 \end{matrix} \Rightarrow \begin{matrix} v_2 = -\frac{5}{3}v_3 = 0 \\ v_3 = 0 \end{matrix} \Longrightarrow \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}$$

$$(U + I)^2 = \begin{pmatrix} 0 & 15 & 6 \\ 0 & 25 & 10 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - \frac{5}{3}R_1 \to R_2} \begin{pmatrix} 0 & 15 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$15w_2 + 6w_3 = 0 \Rightarrow w_2 = -\frac{2}{5}w_3 \Rightarrow w = (0, -2, 5)$ is a generalized eigenvector

$u = (U + I)w = (19, 0, 0)$

$$Y(t) = C_1 e^{4t}\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 19 \\ 0 \\ 0 \end{pmatrix} + C_3\left(t\begin{pmatrix} 19 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}\right)e^{-t}$$

2. Compute the Laplace transform of the function $f(t) = t^2$ using the definition of the Laplace transform as an improper integral.

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt = \lim_{R\to\infty} \int_0^R t^2 e^{-st}\,dt$$

$u = t^2$
$du = 2t\,dt$
$dv = e^{-st}\,dt$
$v = -\frac{1}{s}e^{-st}$

$$= \lim_{R\to\infty} \left[-\frac{1}{s}t^2 e^{-st}\right]_0^R + \int_0^R \frac{2}{s} t e^{-st}\,dt$$

$u = t$
$du = dt$
$dv = \frac{2}{s}e^{-st}\,dt$
$v = -\frac{2}{s^2}e^{-st}$

$$= \lim_{R\to\infty} \left[-\frac{1}{s}t^2 e^{-st} - \frac{2}{s^2} t e^{-st}\right]_0^R + \int_0^R \frac{2}{s^2} e^{-st}\,dt$$

$$= \lim_{R\to\infty} \left[-\frac{1}{s}t^2 e^{-st} - \frac{2}{s^2} t e^{-st} - \frac{2}{s^3} e^{-st}\right]_0^R$$

$$= \lim_{R\to\infty} \left(-\frac{1}{s}\frac{R^2}{e^{sR}} - \frac{2}{s^2}\frac{R}{e^{sR}} - \frac{2}{s^3}\cdot\frac{1}{e^{sR}} + \frac{2}{s^3}\right) = \frac{2}{s^3}$$

3. (a) Using the included table of Laplace transforms, find the Laplace transform $F(s)$ of the function e^{it} (where $i = \sqrt{-1}$).

$$\mathcal{L}\{e^{it}; s\} = \mathcal{L}\{1; s-i\} = \frac{1}{s-i}$$

(b) Identify the real and imaginary parts of your answer in part (a), and use Euler's formula to derive the Laplace transforms of $\sin t$ and $\cos t$. (HINT: Multiply your fraction in part (a) by $s+i$ in the numerator and denominator.)

$$\mathcal{L}\{\cos t\} + i\,\mathcal{L}\{\sin t\} = \mathcal{L}\{e^{it}\} = \frac{1}{s-i}\cdot\frac{s+i}{s+i}$$

$$= \frac{s+i}{s^2+1} = \frac{s}{s^2+1} + i\cdot\frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \quad \text{and} \quad \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

4. Find the inverse Laplace transforms of the following functions.

(a) $F(s) = \dfrac{5s}{s^2 + 2s - 24} = \dfrac{5s}{(s+6)(s-4)}$

$$\frac{5s}{s^2+2s-24} = \frac{A}{s+6} + \frac{B}{s-4}$$

when $s = 4$: $20 = 10B \Rightarrow B = 2$

$$\Rightarrow 5s = A(s-4) + B(s+6)$$

when $s = -6$: $-30 = -10A \Rightarrow A = 3$

$$\mathcal{L}^{-1}\{F(s)\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$= 3e^{-6t} + 2e^{4t}$$

(b) $G(s) = \dfrac{6s+3}{4s^2+4s+17} = \dfrac{6}{4} \cdot \dfrac{s+\frac{1}{2}}{s^2+s+\frac{17}{4}} = \dfrac{3}{2} \cdot \dfrac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+16}$

$$= \frac{3}{2} H\left(s+\frac{1}{2}\right)$$

where $H(s) = \dfrac{s}{s^2+16}$

$$\mathcal{L}^{-1}\{G(s)\} = \frac{3}{2} e^{-t/2} \mathcal{L}^{-1}\{H(s)\}$$

$$= \frac{3}{2} e^{-t/2} \cdot \cos 4t$$

5. Use Laplace transforms to solve the following initial value problems.

(a) $y'' - \cancel{6}\,4y = 3t; \quad y(0) = y'(0) = 0$ $\qquad Y = \mathcal{L}\{y\}$

$$\mathcal{L}\{y'' - 4y\} = s\mathcal{L}\{y'\} - 4Y = s^2 Y - 4Y$$

$$(s^2 - 4)Y = 3\mathcal{L}\{t\} = -3\frac{d}{ds}\mathcal{L}\{1\}$$

$$= -3\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{3}{s^2}$$

$$\Rightarrow \quad Y = \frac{3}{s^2(s^2-4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2} + \frac{D}{s-2}$$

$$\Rightarrow \quad 3 = A(s^2-4) + Bs(s^2-4) + Cs^2(s-2) + Ds^2(s+2)$$

when $s=2$: $3 = 16D \Rightarrow D = \frac{3}{16}$ $\qquad$ when $s=-2$: $3 = -16C \Rightarrow C = \frac{-3}{16}$

when $s=0$: $3 = -4A \Rightarrow A = -\frac{3}{4}$ $\qquad$ when $s=1$: $3 = \frac{9}{4} - 3B + \frac{3}{16} + \frac{9}{16} \Rightarrow B = 0$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{3}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= -\frac{3}{4}t - \frac{3}{16}e^{-2t} + \frac{3}{16}e^{2t}$$

(b) $y^{(4)} - y = 0; \quad y(0) = 1,\ y'(0) = y''(0) = y^{(3)}(0) = 0$ $\qquad Y = \mathcal{L}\{y\}$

$$\mathcal{L}\{y^{(4)} - y\} = s\mathcal{L}\{y^{(3)}\} - Y = s^2\mathcal{L}\{y''\} - Y$$

$$= s^3\mathcal{L}\{y'\} - Y = s^4 Y - s^3 - Y$$

$$s^4 Y - s^3 - Y = 0 \Rightarrow (s^4 - 1)Y = s^3$$

$$\Rightarrow Y = \frac{s^3}{s^4 - 1} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs + D}{s^2 + 1}$$

$$\Rightarrow s^3 = A(s+1)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2-1)$$

when $s = 1$: $1 = 4A \Rightarrow A = \frac{1}{4}$ $\qquad$ when $s = -1$: $-1 = -4B \Rightarrow B = \frac{1}{4}$

when $s = 0$: $0 = \frac{1}{4} - \frac{1}{4} - D \Rightarrow D = 0$ $\qquad$ when $s = 2$: $8 = \frac{15}{4} + \frac{5}{4} + 6C \Rightarrow 6C = 3 \Rightarrow C = \frac{1}{2}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \frac{1}{4}e^{t} + \frac{1}{4}e^{-t} + \frac{1}{2}\cos t$$

Laplace Transforms

Function	Laplace transform	Function	Laplace transform
1	$\frac{1}{s}$	t^n	$\frac{n!}{s^{n+1}}$
$f'(t)$	$sF(s) - f(0)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$tf(t)$	$-F'(s)$	$\cos at$	$\frac{s}{s^2+a^2}$
$e^{at}f(t)$	$F(s-a)$	$\sin at$	$\frac{a}{s^2+a^2}$
$(f*g)(t)$	$F(s)G(s)$	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$u(t-a)f(t)$	$e^{-as}\mathscr{L}\{f(t+a)\}$	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
$\delta(t-a)$	e^{-as}		