KEY

Math 267 Worksheet 2 – Fall 2021

1. (a) Show that for any number C, the function $y = Ce^{x^4-x}$ is a solution to the differential equation

$$\frac{dy}{dx} = 4x^3y - y$$

$$\frac{dy}{dx} = Ce^{x^4-x} \cdot (4x^3-1) = 4x^3Ce^{x^4-x} - Ce^{x^4-x}$$

$$= 4x^3y - y$$

(b) What is the function that solves the differential equation in part (a) subject to the initial condition $y(0) = 2e$?

$$2e = Ce^0 = C$$

$$\Rightarrow y = 2e \cdot e^{x^4-x} = 2e^{x^4-x+1}$$

2. Find the form of the general solution to the differential equation

$$\frac{d^2y}{dx^2} = -\frac{3x}{(x^2+4)^2}$$

$$\frac{dy}{dx} = \int \frac{-3x}{(x^2+4)^2}\,dx = \int -\frac{3}{2}\cdot\frac{1}{u^2}\,du = \frac{3}{2}\cdot\frac{1}{u} + C_1$$

$$u = x^2+4 \quad du = 2x\,dx \qquad = \frac{3}{2}\cdot\frac{1}{x^2+4} + C_1$$

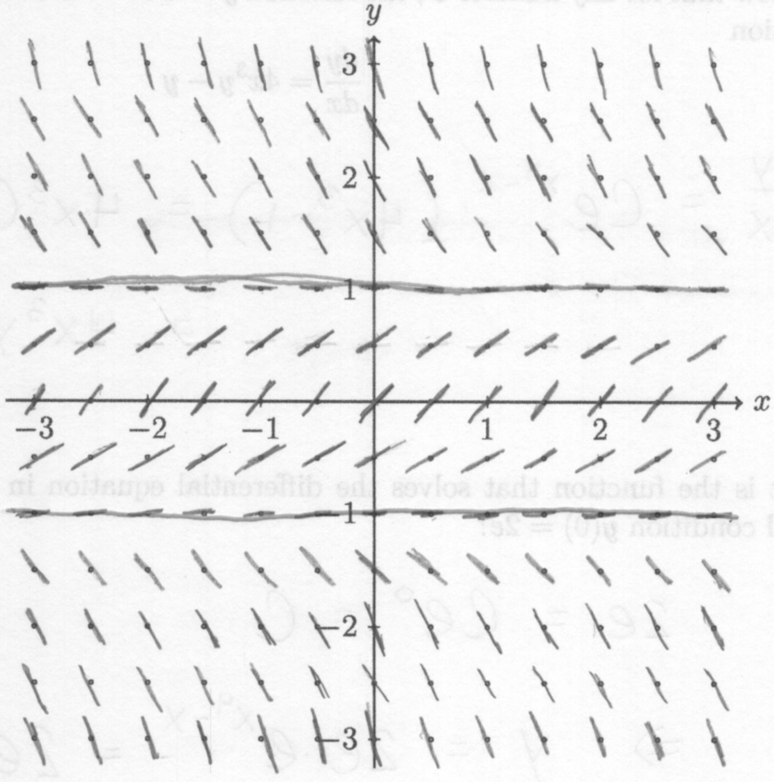
$$\Rightarrow \frac{1}{2}\,du = x\,dx$$

$$y = \int \frac{3}{2}\cdot\frac{1}{x^2+4} + C_1\,dx = \int \frac{3}{8}\cdot\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx + \int C_1 x$$

$$u = \frac{x}{2} \qquad = \int \frac{3}{4}\cdot\frac{1}{u^2+1}\,du + C_1x = \frac{3}{4}\arctan u + C_1x + C_2$$

$$du = \frac{1}{2}\,dx \qquad = \frac{3}{4}\arctan\left(\frac{x}{2}\right) + C_1x + C_2$$

3. (a) Sketch the slope field of the differential equation $\frac{dy}{dx} = 1 - y^2$ on the axes below.



(b) A curve where $\frac{dy}{dx} = 0$ on the entire curve is called a **steady state** of the differential equation. Sketch all of the steady state solutions on the slope field diagram, and write down their equations below.

$y = 1$, $y = -1$

4. Solve the following differential equations by using separation of variables.

(a) $\frac{dy}{dx} = \frac{y^2}{x}$ $\Rightarrow \int \frac{1}{y^2}\, dy = \int \frac{1}{x}\, dx$

$\Rightarrow -\frac{1}{y} = \ln|x| + C$

$\Rightarrow y = \frac{-1}{\ln|x| + C}$

(b) $\frac{dy}{dx} = y(1+y) \quad \Rightarrow \quad \int \frac{1}{y(1+y)}\,dy = \int 1\,dx$

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \qquad \Rightarrow \int \frac{1}{y} - \frac{1}{y+1}\,dy = x + C$$

$$1 = A(y+1) + By$$

$$\Rightarrow \quad \ln|y| - \ln|y+1| = x + C$$

when $y=0$: $A = 1$

$$\Rightarrow \quad \ln\left|\frac{y}{1+y}\right| = x + C$$

when $y=-1$: $1 = -B$

$$\Rightarrow \quad \frac{y}{1+y} = Ce^{x} \Rightarrow Ce^{-x} = 1 + \frac{1}{y}$$

$\Rightarrow B = -1$

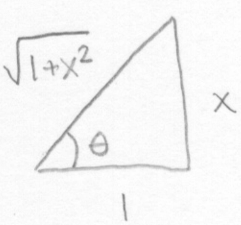
$$\Rightarrow \quad \frac{1}{y} = Ce^{-x} - 1 \Rightarrow y = \frac{1}{Ce^{-x}-1}$$

(c) $\frac{dy}{dx} = \frac{y^2}{(1+x^2)^2}$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int \frac{1}{(1+x^2)^2}\,dx \qquad \begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta\,d\theta \end{aligned}$$

$$\Rightarrow -\frac{1}{y} = \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta = \int \frac{1}{(\sec^2\theta)^{\cancel{2}}}\cancel{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta = \int \frac{1}{2} + \frac{\cos 2\theta}{2}\,d\theta$$



$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C = \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\,\frac{1}{\sqrt{1+x^2}}\cdot\frac{x}{\sqrt{1+x^2}} + C$$

$$= \frac{(1+x^2)\arctan x + x + C(1+x^2)}{1+x^2}$$

$$\Rightarrow \quad y = -\frac{1+x^2}{(1+x^2)\arctan x + x + C(1+x^2)}$$