KEY

Math 267 Worksheet 3 – Fall 2021

1. Solve the following differential equations by using separation of variables.

(a) $\dfrac{dy}{dx} = -\dfrac{y(y^2+17)}{17}$

$$\int \frac{-17}{y(y^2+17)}\,dy = \int 1\,dx$$

$$\frac{-17}{y(y^2+17)} = \frac{A}{y} + \frac{By+C}{y^2+17} \Rightarrow -17 = A(y^2+17) + (By+C)y$$

$$= (A+B)y^2 + Cy + 17A$$

$$\Rightarrow A+B=0,\quad C=0,\quad 17A=-17 \Rightarrow A=-1,\quad B=-A=1$$

$$\int \frac{-1}{y} + \frac{y}{y^2+17}\,dy = x + D \Rightarrow -\ln|y| + \frac{1}{2}\ln|y^2+17|$$

$$= x + D$$

$$\Rightarrow \frac{\sqrt{y^2+17}}{y} = De^{x} \Rightarrow \frac{y^2+17}{y^2} = De^{2x} \Rightarrow 1 + \frac{17}{y^2} = De^{2x}$$

$$\Rightarrow y = \frac{\sqrt{17}}{\sqrt{De^{2x}-1}}$$

(b) $x^2\dfrac{dy}{dx} - y\sqrt{y^2-1} = 0$

$$\int \frac{1}{y\sqrt{y^2-1}}\,dy = \int \frac{1}{x^2}\,dx$$

$y = \sec\theta$

$dy = \sec\theta\tan\theta\,d\theta$

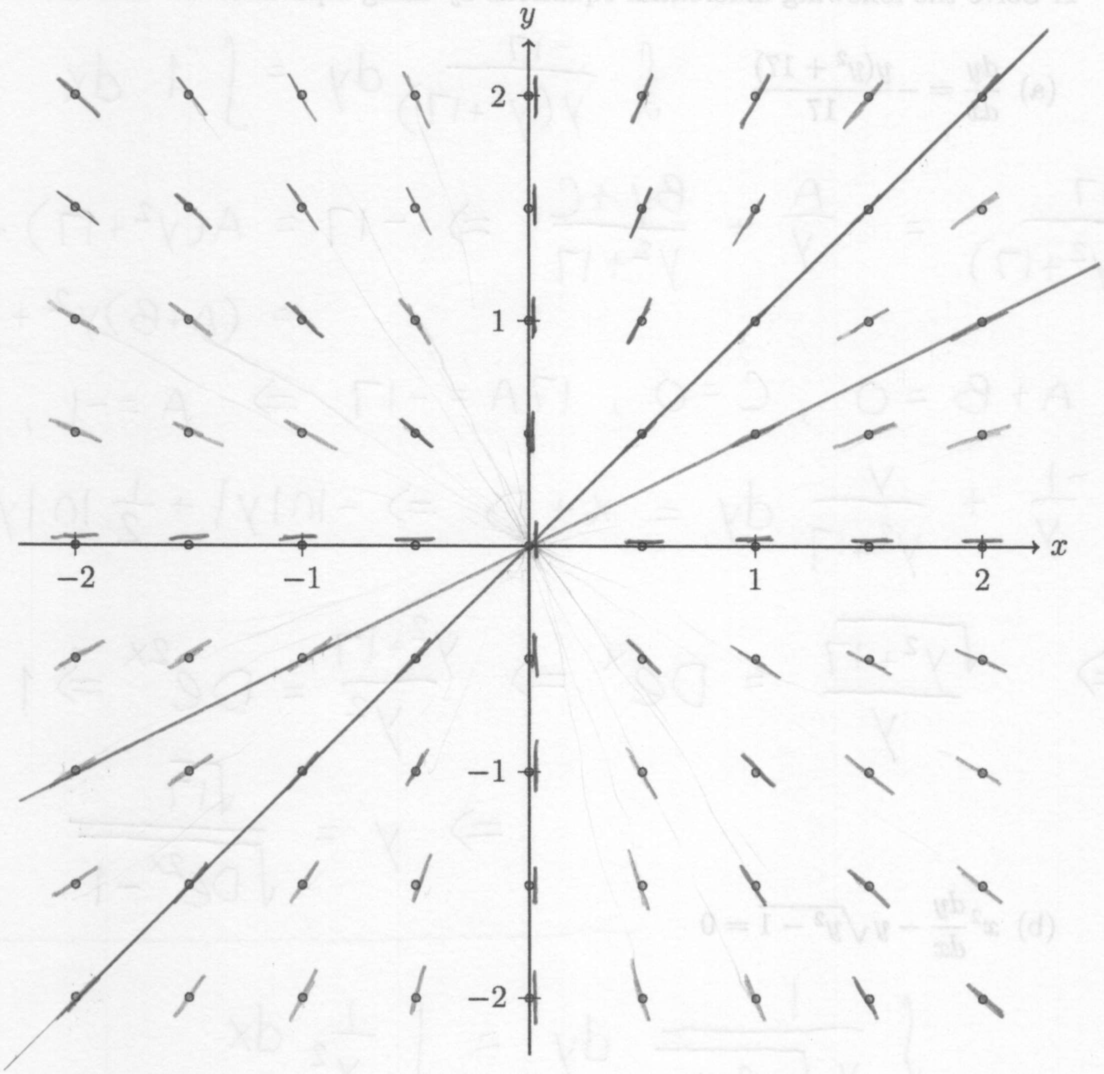
$$\int \frac{\cancel{\sec\theta}\tan\theta}{\cancel{\sec\theta}\sqrt{\sec^2\theta-1}}\,d\theta = \frac{-1}{x} + C$$

$$\int \frac{\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta = \frac{-1}{x} + C \Rightarrow \int 1\,d\theta = \frac{-1}{x} + C$$

$$\Rightarrow \theta = \frac{-1}{x} + C \Rightarrow y = \sec\left(-\frac{1}{x} + C\right)$$

KEY

2. (a) Sketch the slope field of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x}$ on the axes below.



(b) On your slope field in part (a), use the directions of the slopes to sketch an example of curve that solves the differential equation. Describe the shape of the solution curves.

The solution curves are lines through the origin.

3. Suppose we want to solve the first order linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = \tan x$ and $Q(x) = \sin x$.

(a) Compute the exponent $\int P(x)\,dx$ of the integrating factor.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\ln|u| + C$$

$$u = \cos x \quad du = -\sin x\,dx \qquad = -\ln|\cos x| + C$$

(b) After multiplying the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ by the integrating factor $e^{\int P(x)\,dx}$, solve the differential equation.

$$e^{\int P(x)\,dx} = e^{-\ln(\cos x)} = \sec x$$

$$\sec x \frac{dy}{dx} + \sec x \tan x = \tan x \Rightarrow [\sec x \cdot y]' = \tan x$$

$$\Rightarrow \sec x \cdot y = \int \tan x\,dx = -\ln|\cos x| + C$$

$$\Rightarrow \quad y = -\cos x \ln|\cos x| + C\cos x$$

4. Find the general solution to the differential equation below.

$$y' + 5y = 6x^2 \qquad e^{\int 5\,dx} = e^{5x}$$

$$e^{5x}y' + 5e^{5x}y = 6x^2e^{5x} \Rightarrow [e^{5x}y]' = 6x^2e^{5x}$$

$$e^{5x}y = \int 6x^2e^{5x}\,dx = \frac{6}{5}x^2e^{5x} - \int \frac{12}{5}xe^{5x}\,dx$$

$$= \frac{6}{5}x^2e^{5x} - \frac{12}{25}xe^{5x} + \int \frac{12}{25}e^{5x}\,dx$$

$$= \frac{6}{5}x^2e^{5x} - \frac{12}{25}xe^{5x} + \frac{12}{125}e^{5x} + C$$

$$\Rightarrow \quad y = \frac{6}{5}x^2 - \frac{12}{25}x + \frac{12}{125} + Ce^{-5x}$$

5. If possible, find functions $F(x, y)$ that implicitly solve the following differential equations. **If it is not possible to solve the equation, explain why.**

(a) $4xy\frac{dy}{dx} + 2y^2 + \sin y = 0$ $\Rightarrow 4xy\,dy + (2y^2 + \sin y)dx = 0$

$$N = 4xy \qquad M = 2y^2 + \sin y$$

$$\frac{\partial N}{\partial x} = 4y \neq 4y + \cos y = \frac{\partial M}{\partial y}$$

Since the differential equation is not exact, there is no solution.

(b) $(\frac{3}{2}x^2y^2 - y + x)\frac{dy}{dx} + xy^3 + y - \sin x = 0$

$$N = \frac{3}{2}x^2y^2 - y + x \qquad M = xy^3 + y - \sin x$$

$$\frac{\partial N}{\partial x} = 3xy^2 + 1 = \frac{\partial M}{\partial y} \checkmark$$

$$\frac{\partial F}{\partial y} = N \Rightarrow F = \int \frac{3}{2}x^2y^2 - y + x \, dy$$

$$= \frac{1}{2}x^2y^3 - \frac{1}{2}y^2 + xy + g(x)$$

$$M = \frac{\partial F}{\partial x} = xy^3 + y + \frac{dg}{dx} \Rightarrow \frac{dg}{dx} = -\sin x$$

$$g = \int -\sin x \, dx = \cos x + C$$

$$F(x,y) = \frac{1}{2}x^2y^3 - \frac{1}{2}y^2 + xy + \cos x + C$$