KEY

Math 267 Worksheet 1 – Fall 2021

1. (a) Find the most general form of all solution to the differential equation $\frac{d^2y}{dx^2} = \sec^2 x$.

$$\frac{dy}{dx} = \int \sec^2 x \, dx = \tan x + C_1$$

$$y = \int \tan x + C_1 \, dx = \int \frac{\sin x}{\cos x} \, dx + C_1 x + C_2$$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= \int -\frac{1}{u} \, du + C_1 x + C_2$$

$$= -\ln|u| + C_1 x + C_2$$

$$= -\ln|\cos x| + C_1 x + C_2$$

(b) What is the function that solves the differential equation in part (a) subject to the initial conditions $y(0) = 5$ and $y'(\frac{\pi}{4}) = 0$?

$$0 = y'\left(\tfrac{\pi}{4}\right) = \tan\tfrac{\pi}{4} + C_1 = 1 + C_1 \Rightarrow C_1 = -1$$

$$5 = y(0) = -\ln|\cos 0| - 0 + C_2 = C_2$$

$$y = -\ln|\cos x| - x + 5$$

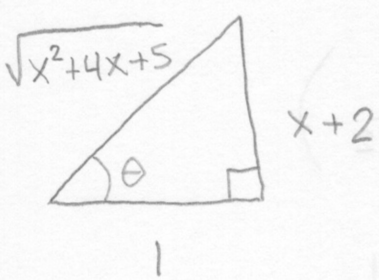
2. Compute the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+1}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}} \sec^2\theta \, d\theta$

$x + 2 = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} \, d\theta = \int \sec\theta \, d\theta$$



$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{x^2+4x+5} + x + 2\right| + C$$

(b) $\displaystyle\int \frac{3x^2+x}{(x-12)(x^2+4)}\,dx$ $\qquad \dfrac{3x^2+x}{(x-12)(x^2+4)} = \dfrac{A}{x-12} + \dfrac{Bx+C}{x^2+4}$

$$3x^2+x = A(x^2+4) + (Bx+C)(x-12)$$

when $x=12$:

$444 = 148A$

$\Rightarrow A = 3$

when $x=0$:

$0 = 12 - 12C$

$\Rightarrow C = 1$

when $x=2$:

$14 = 24 - 10B - 10 \Rightarrow B = 0$

$$\int \frac{3x^2+x}{(x-12)(x^2+4)}\,dx = \int \frac{3}{x-12} + \frac{1}{x^2+4}\,dx$$

$u = \frac{x}{2}$

$du = \frac{1}{2}\,dx$

$$= 3\ln|x-12| + \int \frac{1}{4}\cdot\frac{1}{(\frac{x}{2})^2+1}\,dx$$

$$= 3\ln|x-12| + \frac{1}{2}\int \frac{1}{u^2+1}\,du$$

$$= 3\ln|x-12| + \frac{1}{2}\arctan u + C$$

$$= 3\ln|x-12| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

(c) $\displaystyle\int_0^\infty te^{-st}\,dt \quad (s>0)$

$$\int te^{-st}\,dt = -\frac{1}{s}te^{-st} + \int \frac{1}{s}e^{-st}\,dt$$

$u = t \qquad du = dt$

$dv = e^{-st}\,dt \qquad v = -\frac{1}{s}e^{-st}$

$$= -\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st} + C$$

$$\int_0^\infty te^{-st}\,dt = \lim_{R\to\infty}\int_0^R te^{-st}\,dt = \lim_{R\to\infty}\left[-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right]_0^R$$

$$= \lim_{R\to\infty}\left(-\frac{R}{se^{+sR}} - \frac{1}{s^2e^{+sR}} + \frac{1}{s^2}\right)$$

$$= \left(\lim_{R\to\infty}\frac{-1}{s^2e^{sR}}\right) + 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

3. Let $P(x) = \dfrac{2x}{1+x^2}$ and $Q(x) = x$.

(a) Find $\int P(x)\,dx$. $\quad = \int \dfrac{2x}{1+x^2}\,dx = \int \dfrac{1}{u}\,du$

$u = 1+x^2$

$= \ln|u| + C$

$du = 2x\,dx$

$= \ln(1+x^2) + C$

(b) Compute an antiderivative $f(x)$ for the function $Q(x)e^{\int P(x)\,dx}$.

$$f(x) = \int x e^{\ln(1+x^2)}\,dx = \int x(1+x^2)\,dx$$

$$= \int x + x^3\,dx = \frac{1}{2}x^2 + \frac{1}{4}x^4 + C$$

(c) Show that the function $y(x) = f(x)e^{-\int P(x)\,dx}$ solves the differential equation below.

$$y' + P(x)y = Q(x)$$

$$y = \left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)e^{-\ln(1+x^2)} = \frac{1}{4}\cdot\frac{2x^2+x^4}{1+x^2}$$

$$y' + Py = \frac{1}{4}\,\frac{(1+x^2)(4x+4x^3) - \cancel{(2x^2+x^4)(2x)}}{(1+x^2)^2}$$

$$+\ \frac{2x}{1+x^2}\cdot\cancel{\frac{1}{4}}\cdot\frac{\cancel{2x^2+x^4}}{1+x^2}$$

$$= \frac{1}{4}\cdot\frac{4x\cancel{(1+x^2)^2}}{\cancel{(1+x^2)^2}} = x = Q$$

4. Consider the differential equation $xy' = 4y$.

(a) Show that $y_1(x) = x^4$ is a solution to the above differential equation.

$$xy_1' = x(4x^3) = 4x^4 = 4y_1$$

(b) Show that the function

$$y_2(x) = \begin{cases} x^4 & , x \geq 0 \\ -x^4 & , x < 0 \end{cases}$$

is also a solution to the above differential equation.

Already checked for $x \geqslant 0$ by part (a).

For $x \leq 0$:

$$xy_2' = x(-4x^3) = -4x^4 = 4y_2$$

(c) More generally, show that any function of the form $y(x) = C_1y_1(x) + C_2y_2(x)$ also solves the differential equation.

$$xy' = x(C_1y_1' + C_2y_2') = C_1xy_1' + C_2xy_2'$$

by (a) and (b) $= C_1(4y_1) + C_2(4y_2) = 4y$

(d) Even though the given differential equation is a first-order equation, the initial value problem solving the differential equation subject to the initial condition $y(1) = -3$ has infinitely many solutions. **Explain why.**

The initial condition imposes only one constraint on the two parameters C_1 and C_2, namely $-3 = y(1) = C_1 + C_2$.

We need 2 constraints to specify both C_1 and C_2.