WS 15 KEY

1. Method: 1st order linear (integrating factors)

$$e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

$$\Rightarrow \quad xy' + y = \frac{2x}{x^2+1}$$

$$\Rightarrow \quad [xy]' = \frac{2x}{x^2+1}$$

$$\Rightarrow \quad xy = \int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$u = x^2+1$

$du = 2x\,dx$

$$= \ln(x^2+1) + C$$

$$\Rightarrow \quad y = \frac{\ln(x^2+1)}{x} + \frac{C}{x}$$

$$\ln 6 = y(1) = \ln 2 + C \quad \Rightarrow \quad C = \ln 6 - \ln 2 = \ln 3$$

$$y = \frac{\ln(x^2+1)}{x} + \frac{\ln 3}{x} = \frac{\ln(3x^2+3)}{x}$$

2. Method: Power series

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \overset{m=n-2}{=} \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m$$

$$x^2 y' = \sum_{n=1}^{\infty} n c_n x^{n+1} \overset{m=n+1}{=} \sum_{m=2}^{\infty} (m-1) c_{m-1} x^m$$

$$xy = \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{m=1}^{\infty} c_{m-1} x^m$$

$$3x = y'' + x^2 y' + 3xy$$

$$= \left(2c_2 + 3\cdot 2 c_3 x + \sum_{m=2}^{\infty} (m+2)(m+1) c_{m+2} x^m \right)$$

$$+ \sum_{m=2}^{\infty} (m-1) c_{m-1} x^m + \left(3c_0 x + \sum_{m=2}^{\infty} 3 c_{m-1} x^m \right)$$

$$= 2c_2 + (3\cdot 2 c_3 + 3c_0) x + \sum_{m=2}^{\infty} \left[(m+2)(m+1) c_{m+2} + (m+2) c_{m-1} \right] x^m$$

$$\Rightarrow \quad 2c_2 = 0 \qquad (m+2)(m+1)c_{m+2} + (m+2)c_{m-1} = 0$$

$$3 \cdot 2 c_3 + 3c_0 = 3 \qquad \text{for } m \geq 2$$

$$\Rightarrow \quad c_2 = 0 \qquad c_{m+2} = \frac{-c_{m-1}}{m+1}$$

$$c_3 = \frac{1-c_0}{2} \qquad \text{for } m \geq 2$$

$$c_0 = y(0) = 1 \qquad c_1 = y'(0) = 1$$

$$c_3 = \frac{1-1}{2} = 0 \qquad c_4 = \frac{-c_1}{3} = \frac{-1}{3}$$

$$c_6 = \frac{-c_3}{5} = 0 \qquad c_7 = \frac{-c_4}{6} = \frac{(-1)^2}{6 \cdot 3}$$

$$c_9 = \frac{-c_6}{8} = 0 \qquad c_{10} = \frac{-c_7}{9} = \frac{(-1)^3}{9 \cdot 6 \cdot 3}$$

$$\Rightarrow c_{3n} = 0 \text{ for } n \geq 1 \qquad c_{3n+1} = \frac{(-1)^n}{(3n)(3n-3)\cdots 9 \cdot 6 \cdot 3} = \frac{(-1)^n}{3^n \cdot n!}$$

$$c_0 = 1$$

$$c_2 = 0 \quad c_5 = \frac{-c_2}{4} = 0 \quad c_8 = \frac{-c_5}{7} = 0 \Rightarrow c_{3n+2} = 0$$

$$y = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \cdot n!} x^{3n+1} = 1 + xe^{-x^3/3}$$

3. Method: Undetermined coefficients

$$y'' + y' - \frac{3}{4}y = 0 \Rightarrow 0 = r^2 + r - \frac{3}{4}$$

$$\Rightarrow r = \frac{-1 \pm \sqrt{1+3}}{2} = \frac{1}{2}, \frac{-3}{2}$$

$$5\cosh\left(\tfrac{1}{2}t\right) = \frac{5}{2}e^{t/2} + \frac{5}{2}e^{-t/2}$$

$$y_p = Ate^{t/2} + Be^{-t/2}$$

$$y_p' = Ae^{t/2} + \frac{1}{2}Ate^{t/2} - \frac{1}{2}Be^{-t/2}$$

$$y_p'' = Ae^{t/2} + \frac{1}{4}Ate^{t/2} + \frac{1}{4}Be^{-t/2}$$

$$\frac{5}{2}e^{t/2} + \frac{5}{2}e^{-t/2} = y_p'' + y_p' - \frac{3}{4}y_p$$

$$= 2Ae^{t/2} - Be^{-t/2}$$

$$\Rightarrow \quad 2A = \frac{5}{2} \Rightarrow A = \frac{5}{4} \Rightarrow y_p = \frac{5}{4}te^{t/2} - \frac{5}{2}e^{-t/2}$$

$$-B = \frac{5}{2} \qquad B = \frac{-5}{2}$$

$$y = C_1e^{t/2} + C_2e^{-3t/2} + \frac{5}{4}te^{t/2} - \frac{5}{2}e^{-t/2}$$

4. (a) $$\mathcal{L}\{f(t)\} = 12 \cdot \frac{14!}{(s-2021)^{15}} + \frac{(s+4)^2-30^2}{((s+4)^2+30^2)^2}$$

(b) $$\frac{59s-248}{(2s+1)(s^2-12s+40)} = \frac{A}{2s+1} + \frac{Bs+C}{s^2-12s+40}$$

$$\Rightarrow \quad 59s-248 = A(s^2-12s+40) + (Bs+C)(2s+1)$$

when $s = -\frac{1}{2}$: $\quad -\frac{555}{2} = \frac{185}{4}A \Rightarrow A = -6$

when $s = 0$: $\quad -248 = -240 + C \Rightarrow C = -8$

when $s = 1$: $\quad -189 = -174 + 3B - 24 \Rightarrow B = 3$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{-6}{2s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{3s-8}{s^2-12s+40}\right\}$$

$$= -3\mathcal{L}^{-1}\left\{\frac{1}{s+1/2}\right\} + \mathcal{L}^{-1}\left\{\frac{3(s-6)+10}{(s-6)^2+4}\right\}$$

$$= -3e^{-t/2} + 3e^{6t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 5e^{6t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -3e^{-t/2} + 3e^{6t}\cos 2t + 5e^{6t}\sin 2t$$

5 (a). Method: Generalized eigenvectors

$$0 = \det(B - \lambda I) = \begin{vmatrix} 8-\lambda & -16 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)^2$$

$$\Rightarrow \quad \lambda = 4$$

$$B - 4I = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix} \xrightarrow{R_1 - 4R_2 \to R_1} \begin{pmatrix} 0 & 0 \\ 1 & -4 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \Rightarrow \quad v_1 = 4v_2$$

$$v = \begin{pmatrix} 4v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$(B - 4I)^2 = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector.

$$v = (B - 4I)w = \begin{pmatrix} 4 & -16 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 e^{4t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{4t}\left(t\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$= \underbrace{\begin{pmatrix} 4e^{4t} & (4t+1)e^{4t} \\ e^{4t} & te^{4t} \end{pmatrix}}_{\Phi(t)}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(b) $e^{Bt} = \Phi(t)\,\Phi(0)^{-1}$

$$\Phi(0) = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \qquad \det \Phi(0) = -1$$

$$\Phi(0)^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}$$

$$e^{Bt} = e^{4t}\begin{pmatrix} 4 & 4t+1 \\ 1 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} = e^{4t}\begin{pmatrix} 4t+1 & -16t \\ t & -4t+1 \end{pmatrix}$$

6. $\mathcal{L}\{y\} = Y$

$$s^2Y - 2s - 4 - sY + 2 - 2Y = 6\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{9}{4}} = \frac{6s+3}{s^2+s+\frac{10}{4}}$$

$$\Rightarrow (s^2 - s - 2)Y = \frac{6s+3}{s^2+s+\frac{10}{4}} + 2s + 2$$

$$\Rightarrow \quad Y = \frac{6s+3}{(s-2)(s+1)(s^2+s+\frac{10}{4})} + \frac{2s+2}{(s-2)(s+1)}$$

$$= \frac{6s+3}{(s-2)(s+1)(s^2+s+\frac{10}{4})} + \frac{2}{s-2}$$

$$\frac{6s+3}{(s-2)(s+1)(s^2+s+\frac{10}{4})} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+s+\frac{10}{4}}$$

$$\Rightarrow \quad 6s+3 = A(s+1)(s^2+s+\tfrac{10}{4}) + B(s-2)(s^2+s+\tfrac{10}{4})$$
$$+ (Cs+D)(s-2)(s+1)$$

when $s=-1$: $\quad -3 = {}^-3 \cdot \frac{10}{4} \cdot B \quad \Rightarrow \quad B = \frac{2}{5}$

when $s=2$: $\quad 15 = 3 \cdot \frac{34}{4} A \quad \Rightarrow \quad A = \frac{10}{17}$

when $s=0$: $\quad 3 = \frac{25}{17} - 2 - 2D \quad \Rightarrow \quad \frac{60}{17} = -2D$

$$\Rightarrow \quad D = -\frac{30}{17}$$

when $s=1$: $\quad 9 = \frac{90}{17} - \frac{9}{5} - 2C + \frac{60}{17}$

$$\Rightarrow \quad -2C = \frac{3}{17} - \frac{9}{5} = -\frac{138}{85} \quad \Rightarrow C = \frac{69}{85}$$

$$y = \mathcal{L}^{-1}\{Y\} = \frac{44}{17}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$
$$+ \frac{1}{85}\mathcal{L}^{-1}\left\{\frac{69s-150}{s^2+s+\frac{10}{4}}\right\}$$
$$= \frac{44}{17}e^{2t} + \frac{2}{5}e^{-t} + \frac{1}{85}\mathcal{L}^{-1}\left\{\frac{69(s+\frac{1}{2}) - \frac{369}{2}}{(s+\frac{1}{2})^2 + \frac{9}{4}}\right\}$$
$$= \frac{44}{17}e^{2t} + \frac{2}{5}e^{-t} + \frac{69}{85}e^{-t/2}\cos\left(\tfrac{3}{2}t\right) - \frac{123}{85}e^{-t/2}\sin\left(\tfrac{3}{2}t\right)$$

7. Method: Eigenvalues + eigenvectors

$$0 = \det(S - \lambda I) = \begin{vmatrix} 91-\lambda & -12 \\ -12 & 84-\lambda \end{vmatrix}$$

$$= \lambda^2 - 175\lambda + 7644 - 144$$

$$= \lambda^2 - 175\lambda + 7500$$

$$= (\lambda - 75)(\lambda - 100)$$

$$\Rightarrow \lambda = 75, 100$$

$\lambda = 75$: $\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \xrightarrow[R_2 + \frac{3}{4}R_1 \to R_2]{\frac{1}{4}R_1 \to R_1} \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow 4v_1 = 3v_2 \Rightarrow v_1 = \frac{3}{4}v_2 \underset{v_2 = 4}{\Rightarrow} v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$\lambda = 100$: $\begin{pmatrix} -9 & -12 \\ -12 & -16 \end{pmatrix} \xrightarrow[R_2 - \frac{4}{3}R_1 \to R_2]{-\frac{1}{3}R_1 \to R_1} \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow 3v_1 = -4v_2 \Rightarrow v_1 = -\frac{4}{3}v_2 \underset{v_2 = 3}{\Rightarrow} v = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$Z(t) = C_1 e^{75t}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 e^{100t}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$$