WS 10 KEY

1. $A = \begin{pmatrix} 6 & -4 \\ 13 & -6 \end{pmatrix}$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -4 \\ 13 & -6-\lambda \end{vmatrix}$$

$$= (6-\lambda)(-6-\lambda) + 52 = \lambda^2 + 16 \Rightarrow \lambda = \pm 4i$$

$\lambda = 4i$:

$$\begin{pmatrix} 6-4i & -4 \\ 13 & -6-4i \end{pmatrix} \xrightarrow{(6+4i)R_1 \to R_1} \begin{pmatrix} 52 & -24-16i \\ 13 & -6-4i \end{pmatrix}$$

$$\xrightarrow{R_1 - 4R_2 \to R_1} \begin{pmatrix} 0 & 0 \\ 13 & -6-4i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 13 & -6-4i \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow 13v_1 + (-6-4i)v_2 = 0 \Rightarrow v_1 = \left(\frac{6}{13} + \frac{4}{13}i\right)v_2$$

$v = \begin{pmatrix} 6+4i \\ 13 \end{pmatrix}$ is an eigenvector for $\lambda = 4i$.

$\lambda = -4i$: $\quad Av = 4iv \underset{\text{take conjugates}}{\Rightarrow} A\bar{v} = \overline{Av} = \overline{4iv} = -4i\bar{v}$

$\bar{v} = \begin{pmatrix} 6-4i \\ 13 \end{pmatrix}$ is an eigenvector for $\lambda = -4i$.

$$\begin{pmatrix} y \\ v \end{pmatrix} = c_1 e^{4it}\begin{pmatrix} 6+4i \\ 13 \end{pmatrix} + c_2 e^{-4it}\begin{pmatrix} 6-4i \\ 13 \end{pmatrix}$$

$$= c_1(\cos 4t + i\sin 4t)\begin{pmatrix} 6+4i \\ 13 \end{pmatrix}$$

$$+ c_2(\cos 4t - i\sin 4t)\begin{pmatrix} 6-4i \\ 13 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{c}_1(6\cos 4t - 4\sin 4t) + \tilde{c}_2(4\cos 4t + 6\sin 4t) \\ 13\tilde{c}_1\cos 4t + 13\tilde{c}_2 \sin 4t \end{pmatrix}$$

where $\tilde{c}_1 = c_1 + c_2$, $\tilde{c}_2 = i(c_1 - c_2)$

2.(a)

$$A - 3I_6 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix} \Rightarrow \begin{matrix} v_3 = 0 \\ v_5 = 0 \\ v_6 = 0 \end{matrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \\ v_4 \\ 0 \\ 0 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$r_1 = 3$

(b)

$$(A - 3I_6)^2 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \Rightarrow v_6 = 0$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$+ v_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$r_2 = 5$$

$$(A - 3I_6)^3 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$$

All vectors are solutions to $(A - 3I_6)^3 v = 0$,

so $r_3 = 6$.

(c)

$$A = \left(\begin{array}{c|cc|ccc} 3 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

1 block of size 1

1 block of size 2

1 block of size 3

$s_1 = r_1 = 3 =$ blocks of size ≥ 1 ✓

$s_2 = r_2 - r_1 = 2 =$ blocks of size ≥ 2 ✓

$s_3 = r_3 - r_2 = 1 =$ blocks of size ≥ 3 ✓

(d) $m_1 = s_1 - s_2 = 1 =$ blocks of size 1 ✓

$m_2 = s_2 - s_3 = 1 =$ blocks of size 2 ✓

$m_3 = s_3 = 1 =$ blocks of size 3 ✓

(e) Pick any vector v with $(A-3I_6)^3 v = 0$ but $(A-3I_6)^2 v \neq 0$.

Since any v with $v_6 = 0$ is a solution to $(A-3I_6)^2 v = 0$ by part (b), we need $v_6 \neq 0$.

$$v = (0,0,0,0,0,1) \quad \text{works!}$$

The chain of generalized eigenvectors is:

$$v = (0,0,0,0,0,1)$$

$$(A-3I_6)v = (0,0,0,0,1,0)$$

$$(A-3I_6)^2 v = (0,0,0,1,0,0)$$

3. (a)

$$A - 2I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 \\ -1/2 & -5/2 & -3 & 1/2 \\ 5 & 1 & 6 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -1/2 & -5/2 & -3 & 1/2 \\ 5 & 1 & 6 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 - 5R_2 \to R_4]{R_3 + \frac{1}{2}R_2 \to R_3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 1/2 \\ 0 & -4 & -4 & 1 \end{pmatrix} \xrightarrow{R_4 - 2R_3 \to R_4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & -2 & -2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_3 \to R_3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - R_3 \to R_2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1/4 \\ 0 & 1 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad v_1 = -v_3 - \tfrac{1}{4}v_4 \,, \quad v_2 = -v_3 + \tfrac{1}{4}v_4$$

$$v = \begin{pmatrix} -v_3 - \frac{1}{4}v_4 \\ -v_3 + \frac{1}{4}v_4 \\ v_3 \\ v_4 \end{pmatrix} = v_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} -1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}$$

$$r_1 = 2$$

$$(A - 2I_4)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -6 & -4 & 2 \\ -1 & 3 & 2 & -1 \\ 4 & -12 & -8 & 4 \end{pmatrix} \xrightarrow[R_4 + 4R_3 \to R_4]{R_2 + 2R_3 \to R_2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad v_1 = 3v_2 + 2v_3 - v_4$$

$$v = \begin{pmatrix} 3v_2 + 2v_3 - v_4 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$r_2 = 3$$

$$(A - 2I_4)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

All vectors are solutions

$$r_3 = 4$$

(b)

$s_1 = r_1 = 2$ $\qquad$ $m_1 = s_1 - s_2 = 1$

$s_2 = r_2 - r_1 = 1$ $\qquad$ $m_2 = s_2 - s_3 = 0$

$s_3 = r_3 - r_2 = 1$ $\qquad$ $m_3 = s_3 = 1$

(c) Pick a vector v_3 with $(A - 2I_4)^3 v_3 = 0$ but $(A-2I)^2 v \neq 0$.

By part (a), any vector with $v_1 \neq 3v_2 + 2v_3 - v_4$ will do.

$$v_3 = (0, 0, 0, 1)$$

$$v_2 = (A - 2I_4) v_3 = (0, 0, \tfrac{1}{2}, 1)$$

$$v_1 = (A - 2I_4)^2 v_3 = (0, 2, -1, 4)$$

(d) Since all of v_1, v_2, v_3 have zero first coordinate, any eigenvector with nonzero first coordinate will do.

By part (a), we can use $v_4 = (-1, -1, 1, 0)$.

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(e)

$$\left[\overbrace{\begin{matrix} 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & -1 \\ -1 & \frac{1}{2} & 0 & 1 \\ 4 & 1 & 1 & 0 \end{matrix}}^{P}\;\middle|\;\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right] \xrightarrow[R_3+R_1 \to R_3]{\substack{-R_1 \to R_1 \\ R_2-R_1 \to R_2}}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4-2R_2 \to R_4]{\substack{\frac{1}{2}R_2 \to R_2 \\ R_3+\frac{1}{2}R_2 \to R_3}}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1/2 & 1/2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow[R_4-2R_3 \to R_3]{2R_3 \to R_3}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & -2 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_4]{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 1 & 0 & 1 & -3 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3}$$

$$\left[\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\;\middle|\;\underbrace{\begin{matrix} -1/2 & 1/2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -3 & -2 & 1 \\ -1 & 0 & 0 & 0 \end{matrix}}_{P^{-1}}\right]$$

(f) $$\begin{pmatrix} -1/2 & 1/2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -3 & -2 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 2 & 4 & 4 & 0 \\ -1/2 & -5/2 & -1 & 1/2 \\ 5 & 1 & 6 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & -1 \\ -1 & \frac{1}{2} & 0 & 1 \\ 4 & 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1/2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & -3 & -2 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -2 \\ 4 & 2 & 0 & -2 \\ -2 & 0 & 1/2 & 2 \\ 8 & 6 & 3 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right)$$

1 Jordan block of size 1

1 Jordan block of size 3

4. B is upper triangular $\Rightarrow$ eigenvalues on the diagonal

$$\Rightarrow \lambda = 3, 2$$

$\lambda = 3$:

$$B - 3I_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow[R_3+R_4 \to R_3]{\substack{R_1+R_4\to R_1 \\ R_2-R_4 \to R_2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1+R_2+R_3 \to R_1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow v_2 = v_3 = v_4 = 0$$

$(1,0,0,0)$ is an eigenvector.

$\lambda = 2$:

$$B - 2I_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 + R_3 \to R_2]{R_1 - R_3 \to R_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad v_1 = -v_2 - v_3, \quad v_4 = 0$$

$$v = \begin{pmatrix} -v_2 - v_3 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$r_1 = 2$$

$$(B - 2I_4)^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = -v_2 - v_3 - v_4$$

$r_2 = 3$

$$v = \begin{pmatrix} -v_2 - v_3 - v_4 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$w = (-1, 0, 0, 1)$ is a generalized eigenvector

$$(B - 2I_4)w = (0,-1,1,0)$$

$(-1,1,0,0)$ is another independent eigenvector

$$Y(t) = c_1 e^{3t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + c_3 e^{2t}\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}$$

$$+ c_4 \left(t\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right) e^{2t}$$

$$= \begin{pmatrix} c_1 e^{3t} - c_2 e^{2t} - c_4 e^{2t} \\ c_2 e^{2t} - c_3 e^{2t} - c_4 t e^{2t} \\ c_3 e^{2t} + c_4 t e^{2t} \\ c_4 e^{2t} \end{pmatrix}$$

5. L is lower triangular $\Rightarrow \lambda = 5$

$$x(t) = e^{Lt} = e^{5I_3 t} e^{(L-5I_3)t}$$

$$L - 5I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & 1 & 0 \end{pmatrix} \qquad (L-5I_3)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$(L-5I_3)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{(L-5I_3)t} = I_3 + (L-5I_3)t + (L-5I_3)^2 \cdot \frac{t^2}{2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2t & 0 & 0 \\ -3t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2-3t & t & 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{5t} & 0 & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ t^2-3t & t & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{5t} & 0 & 0 \\ 2te^{5t} & e^{5t} & 0 \\ (t^2-3t)e^{5t} & te^{5t} & e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} = x(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 4e^{5t} \\ 8te^{5t} - e^{5t} \\ 4t^2e^{5t} - 13te^{5t} \end{pmatrix}$$