KEY

1. $4y'' + 24y' + 36y = 0, \quad y(0) = 2, \quad y'(0) = 0$

Category: 2nd order linear with constant coefficients

(a) $4r^2 + 24r + 36 = 0$

$\Rightarrow \quad r^2 + 6r + 9 = 0$

$\Rightarrow \quad (r+3)^2 = 0$

$\Rightarrow \quad r = -3$

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

$$2 = y(0) = C_1 + 0 \Rightarrow C_1 = 2$$

$$y'(t) = -6e^{-3t} + C_2 e^{-3t} - 3C_2 t e^{-3t}$$

$$0 = y'(0) = -6 + C_2 \Rightarrow C_2 = 6$$

$$y(t) = 2e^{-3t} + 6te^{-3t}$$

(b) critically damped

2. $\frac{dy}{dx} + \frac{y}{x} = \ln x, \quad y(1) = \frac{1}{2}$

category: 1st order linear

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$x \frac{dy}{dx} + y = x \ln x$$

$$\Rightarrow \quad [xy]' = x \ln x$$

$u = \ln x \quad du = \frac{1}{x} dx$

$dv = x \quad v = \frac{x^2}{2}$

$$\Rightarrow \quad xy = \int x \ln x \, dx$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$$\Rightarrow \quad y = \frac{x}{2} \ln x - \frac{x}{4} + \frac{C}{x}$$

$$\frac{1}{2} = y(1) = \frac{1}{2}\cancel{\ln 1}^{\,0} - \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$$

$$y(x) = \frac{x}{2}\ln x - \frac{x}{4} + \frac{3}{4x}$$

3. (a) $$x\frac{dy}{dx} = \frac{y}{\ln x}$$

category : separable

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x\ln x}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$$

$$\Rightarrow \ln|y| = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln(\ln x) + C$$

$$\Rightarrow y = Ce^{\ln(\ln x)} = C\ln x$$

(b) $$\left(2y\cos(x^2+y^2) + x\sec^2 y + y^3\right)\frac{dy}{dx} = -2x\cos(x^2+y^2) - \tan y$$

(fixed typo in problem)

category : exact

$$M = 2x\cos(x^2+y^2) + \tan y$$

$$N = 2y\cos(x^2+y^2) + x\sec^2 y + y^3$$

$$\frac{\partial N}{\partial x} = -4xy\sin(x^2+y^2) + \sec^2 y = \frac{\partial M}{\partial y} \quad \checkmark$$

$$F = \int 2x\cos(x^2+y^2) + \tan y \, dx$$

$$= \int \cos(u)\, du + x\tan y \qquad \begin{aligned} u &= x^2+y^2 \\ du &= 2x\, dx \end{aligned}$$

$$= \sin(u) + x\tan y + g(y)$$

$$= \sin(x^2+y^2) + x\tan y + g(y)$$

$$N = \frac{\partial F}{\partial y} = 2y\cos(x^2+y^2) + x\sec^2 y + \frac{dg}{dy}$$

$$\Rightarrow \frac{dg}{dy} = y^3 \Rightarrow g = \int y^3\, dy = \frac{1}{4}y^4$$

$$\sin(x^2+y^2) + x\tan y + \frac{1}{4}y^4 = C$$

(c) $x^2 \frac{dy}{dx} = x^2 + y^2 - yx$

category: homogeneous 1^{st} order

$y = vx \qquad y' = v + v'x$

$$x^2(v + v'x) = x^2 + v^2x^2 - x^2 v$$

$\Rightarrow \quad v + v'x = 1 + v^2 - v$

$\Rightarrow \quad v'x = v^2 - 2v + 1$

$\Rightarrow \quad \int \frac{1}{v^2 - 2v + 1} \, dv = \int \frac{1}{x} \, dx$

$\Rightarrow \quad \int \frac{1}{(v-1)^2} \, dv = \ln|x| + C$

$\Rightarrow \quad \frac{-1}{v-1} = \ln|x| + C$

$\Rightarrow \quad \frac{-1}{\ln|x| + C} = v - 1$

$\Rightarrow \quad v = 1 - \frac{1}{\ln|x| + C} \quad \Rightarrow y = x - \frac{x}{\ln|x| + C}$

(d) $\frac{dy}{dx} + \frac{y}{2\sqrt{x}} = y^3$ category: Bernoulli

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{2\sqrt{x}} \cdot \frac{1}{y^2} = 1 \qquad v = \frac{1}{y^2}$$

$$v' = \frac{-2}{y^3} y'$$

$$\Rightarrow -\frac{1}{2} v' + \frac{1}{2\sqrt{x}} v = 1 \qquad \Rightarrow -\frac{1}{2} v' = \frac{1}{y^3} y'$$

$$\Rightarrow v' - \frac{1}{\sqrt{x}} v = -2$$

$$e^{\int -\frac{1}{\sqrt{x}} dx} = e^{-2\sqrt{x}}$$

$$\Rightarrow e^{-2\sqrt{x}} v' - \frac{1}{\sqrt{x}} e^{-2\sqrt{x}} v = -2e^{-2\sqrt{x}}$$

$$\Rightarrow [e^{-2\sqrt{x}} v]' = -2e^{-2\sqrt{x}}$$

$$\Rightarrow e^{-2\sqrt{x}} v = \int -2e^{-2\sqrt{x}}$$

$$u = \sqrt{x}$$
$$du = \frac{1}{2\sqrt{x}} dx$$

$$= \int -4ue^{-2u} du \qquad \Rightarrow dx = 2\sqrt{x}\, du$$
$$= 2u\, du$$

$$= 2ue^{-2u} - \int 2e^{-2u} du$$

$$w = -4u \qquad dw = -4du$$
$$dv = e^{-2u} du \qquad v = \frac{-1}{2} e^{-2u}$$

$$= 2ue^{-2u} + e^{-2u} + C$$

$$= 2\sqrt{x} e^{-2\sqrt{x}} + e^{-2\sqrt{x}} + C$$

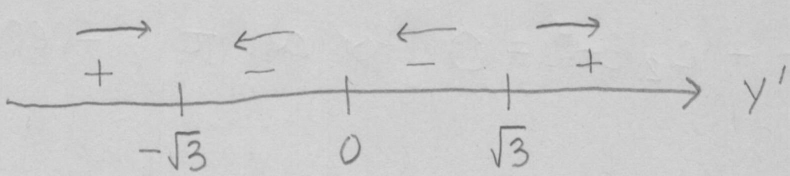
$$\Rightarrow \quad v = 2\sqrt{x} + 1 + Ce^{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{1}{y^2} = 2\sqrt{x} + 1 + Ce^{2\sqrt{x}}$$

$$\Rightarrow \quad y = \frac{\pm 1}{\sqrt{2\sqrt{x} + 1 + Ce^{2\sqrt{x}}}}$$

4. $$0 = y^6 - 9y^2 = y^2(y^4 - 9)$$
$$= y^2(y^2-3)(y^2+3)$$

$$\Rightarrow \quad y = 0 \quad , \; y = \pm\sqrt{3}$$



$y = -\sqrt{3}$ is stable.

$y = 0$ is semistable.

$y = \sqrt{3}$ is unstable.

5. (a) $y_1'(t) = 5 \quad y_1''(t) = 0 \quad y_1'''(t) = 0$

$$y_1''' - \frac{2}{t} y_1'' = 0 - 0 = 0$$

$$y_2'(t) = 12t^3 \qquad y_2''(t) = 36t^2 \qquad y_2'''(t) = 72t$$

$$y_2''' - \frac{2}{t} y_2'' = 72t - \frac{2}{t}(36t^2) = 0$$

(b) Are there nonzero numbers c_1, c_2 with

$$c_1 y_1 + c_2 y_2 = 0$$

for all t?

If $c_1 5t + c_2 3t^4 = 0$ for all t, then

$$5c_1 + 3c_2 = 0 \qquad \text{when } t = 1$$

$$-5c_1 + 3c_2 = 0 \qquad \text{when } t = -1$$

$$\Rightarrow \quad 6c_2 = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow \quad 5c_1 = 0 \Rightarrow c_1 = 0$$

Since the only solutions are $c_1 = c_2 = 0$, y_1 and y_2 are linearly independent.