1 Introduction

My interests lie at the intersection of homological and computational commutative algebra, especially in problems for which it is possible to collect experimental evidence with the aid of computer algebra systems such as `Macaulay2`. Suppose $I = (f_1, \ldots, f_r)$ is an ideal in a polynomial ring $S = k[x_1, \ldots, x_n]$ over a field, and let $R = S/I$. Unlike in a vector space, a polynomial $f \in I$ will usually not have a unique expansion as an S-linear combination of the generators $f_1, \ldots, f_r$. The failure of such expansions to be unique is measured by the set of all relations

$$\{(g_1, \ldots, g_r) \in S^r \mid \sum_{i=1}^{r} f_i g_i = 0\}$$

called the module of *syzygies* on $f_1, \ldots, f_r$. This module has finitely many generators $h_1, \ldots, h_{r_2}$ since S is Noetherian, and we can represent these generators as the columns of a matrix $\partial_2 : S^{r_2} \to S^r$. We can iterate this process by asking for the relations on the syzygies, representing their generators as the columns of a matrix $\partial_3 : S^{r_3} \to S^{r_2}$, and so on. Hilbert's Syzygy Theorem guarantees that this process stops after at most $n - 1$ steps. We then assemble all of this data into a finite sequence of matrices

$$0 \longrightarrow S^{r_t} \xrightarrow{\partial_t} S^{r_{t-1}} \longrightarrow \cdots \longrightarrow S^{r_2} \xrightarrow{\partial_2} S^r \xrightarrow{\partial_1} S \longrightarrow R \longrightarrow 0$$

where $t \leq n$, $\partial_1 = \begin{pmatrix} f_1 & \cdots & f_r \end{pmatrix}$, and the columns of ∂_i are the generators of the relations on the columns of ∂_{i-1} for each $i > 1$. Such a sequence of maps is called a *free resolution* of the quotient ring R over the polynomial ring S.

Example 1.1. Consider the ideal $I = (x^2, xy, y^2)$ in $S = k[x, y]$. A syzygy of I is a triple of polynomials $g = (g_1, g_2, g_3) \in S^3$ such that

$$x^2 g_1 + xy g_2 + y^2 g_3 = 0 \tag{1.1}$$

Since S is a unique factorization domain and y divides the last two terms, y must also divide g_1. Hence, we can write $g_1 = yq_1$ for some $q_1 \in S$. Plugging this into (1.1) and canceling a y yields a syzygy on x and y

$$x(xq_1 + g_2) + yg_3 = 0 \tag{1.2}$$

Repeating the above divisibility argument for both x and y, we see that $xq_1 + g_2 = yq_2$ for some $q_2 \in S$ and, in fact, $g_3 = -xq_2$. Putting this all together, we see that $g = q_1(y, -x, 0) + q_2(0, y, -x)$. Furthermore, if $g = 0$, this forces $q_1 = q_2 = 0$ so that there are no nontrivial relations on the generating syzygies $(y, -x, 0)$ and $(0, y, -x)$. Therefore, the free resolution of $R = S/I$ is

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} S \longrightarrow R \longrightarrow 0$$

When $f_1, \ldots, f_r$ are k-linearly independent homogeneous polynomials of degree $d > 0$ as in the above example, we can arrange so that nonzero entries of the matrices ∂_i are also homogeneous polynomials of positive degree. In that case, the ranks $\beta_i^S(R) = r_i$ of the free modules in the resolution, called the *Betti numbers* of R, and the degrees of their generators are useful invariants for studying the ring R and any geometric or combinatorial objects associated with it. For example, if R is the coordinate ring of a smooth complex projective curve, we can extract the genus of the curve (as a compact 2-dimensional real manifold) and its degree (the number of points of intersection with a general hyperplane) from the free resolution of R. If R is the Stanley-Reisner ring of a simplicial complex Δ, Hochster [9] showed that the free resolution of R encodes the dimensions of all the reduced homology groups over k of Δ as well as its restrictions to various sets of vertices. However, computing the resolutions or even upper and lower bounds on the Betti numbers for large classes of quotient rings remains a very hard problem in general.

In §2, I describe my work on determining the Betti numbers of commutative Koszul algebras. Specifically, I was able to prove a structure theorem for the defining ideals of Koszul almost complete intersections which yields a complete description of the free resolutions of such rings. In §3, I describe an ongoing project with Hal Schenck trying to characterize which quadratic Gorenstein algebras are Koszul, while §4 details some other areas of research.

2 Betti Numbers of Koszul Algebras

Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a not necessarily commutative graded ring with $A_0 = k$ a field, and set $A_+ = \bigoplus_{i>0} A_i$. There is a natural graded k-algebra homomorphism $T(A_1) \to A$ from the tensor algebra of the vector space A_1 over k. We say that A is a *quadratic* algebra if this map is surjective and its kernel is generated elements of $A_1 \otimes_k A_1$. The ring A is called a *Koszul algebra* if the quotient ring $A/A_+ \cong k$ has a (not necessarily finite) free resolution over A in which the entries of the matrices are all homogeneous of degree one. In particular, every Koszul algebra is quadratic. For a commutative algebra $R = S/I$ as above, being Koszul implies that I is generated by homogeneous polynomials of degree two.

Koszul algebras are a very active area of research in both their commutative and noncommutative forms. On the noncommutative side, it is known that whenever the rational cohomology ring of a connected space X with finite Betti numbers is Koszul, then X is a rational $K(\pi, 1)$, and the converse holds for many spaces such as complements of complex hyperplane arrangements and compact Kähler manifolds by [14]. On the commutative side, the coordinate rings of Grassmannians [10] and canonical embeddings of general smooth curves [15] are also Koszul, and it is an open problem to determine when the coordinate rings of toric varieties are Koszul.

Although examples of commutative Koszul algebras are abundant in algebraic geometry, the simplest examples are quotients by quadratic monomial ideals. A guiding heuristic in the study of Koszul algebras has been that any reasonable property of algebras defined by quadratic monomial ideals should also hold for Koszul algebras in general. In particular, the Taylor resolution for a quotient by a quadratic monomial ideal leads to the following question about the Betti numbers of a Koszul algebra.

Question 2.1 ([1])**.** If R is Koszul and I is minimally generated by g elements, does the following inequality hold for all i?

$$\beta_i^S(R) \le \binom{g}{i}$$

In particular, is $\operatorname{pd}_S R \le g$?

The above question trivially has an affirmative answer if I is a complete intersection or has height one. It also has an affirmative answer if I has a quadratic Gröbner basis or is a deformation thereof by a regular sequence of linear forms by upper semicontinuity of the Betti numbers. In the latter case, such ideals and their quotient rings are called *LG-quadratic.* However, for general Koszul algebras, the question is known only to hold if $g \le 3$ by [2], the nontrivial part being precisely when $g = 3$ and I is an almost complete intersection. In this sense, I was able to greatly generalize this result to Koszul almost complete intersections with any number of generators by proving the following structure theorem.

Theorem 2.2 ([11])**.** *Let $R = S/I$ be a Koszul almost complete intersection with I minimally generated by $g + 1$ quadrics for some $g \ge 1$. Then $\beta_{2,3}^S(R) \le 2$, and:*

(a) *If $\beta_{2,3}^S(R) = 1$, there are linear forms x, z, and w such that $I = (xz, zw, q_3, \ldots, q_{g+1})$ for some regular sequence of quadrics $q_3, \ldots, q_{g+1}$ on $S/(xz, zw)$.*

(b) *If $\beta_{2,3}^S(R) = 2$, there is a 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ such that $I = I_2(M) + (q_4, \ldots, q_{g+1})$ for some regular sequence of quadrics $q_4, \ldots, q_{g+1}$ on $S/I_2(M)$.*

Furthermore, R is LG-quadratic and, therefore, satisfies $\beta_i^S(R) \le \binom{g+1}{i}$ for all i.

2.1 Future Directions

Even though the above theorem contributes new evidence in favor of Question 2.1 having an affirmative answer, the question remains largely open. As a concrete path towards more evidence, one might ask whether there are other classes of Koszul algebras, perhaps certain kinds of Gorenstein or Cohen-Macaulay rings, which admit a nice structure theorem as in the almost complete intersection case. My experience with almost complete intersections suggests that it may be possible to make inroads through a careful analysis of the first syzygies of the defining ideal I.

Perhaps the most surprising aspect of the above theorem is that Koszul almost complete intersections turn out to be LG-quadratic. However, there is at least one known example of a Koszul algebra defined by $g = 5$ quadrics that is not LG-quadratic [4]. This example produces graded Betti numbers that one would not expect from studying LG-quadratic algebras alone. And so, a general answer to Question 2.1 may require an investigation of the following broad question: To what extent does the class of general Koszul algebras differ from the class of LG-quadratic algebras? In fact, to my knowledge, every LG-quadratic algebra seems to have the same Betti table as some quadratic monomial ideal, which is a much stronger conclusion than one would expect from upper semicontinuity.

3 Quadratic Gorenstein Algebras

One specific class of Koszul algebras that one might hope to understand are those that are Gorenstein. Surprisingly, a recent example of Matsuda [13] shows that there is a Gorenstein toric ring defined by quadrics which is not Koszul, and so, a natural question I have been exploring with Hal Schenck is whether it is possible to characterize, perhaps by some algebro-geometric condition, which quadratic Gorenstein algebras are Koszul.

Let $R = S/I$ be a quadratic Gorenstein ring. By killing a maximal regular sequence of linear forms, one can immediately reduce the problem of whether R is Koszul to the Artinian case. In that case, R has finitely many nonzero graded components. The degree e of the last nonzero component is called the *socle degree* of R, and a natural approach is to first consider rings of small socle degree. Via Macaulay's inverse systems, the ideal I corresponds to a single homogeneous polynomial f of degree e, and whether or not R is Koszul may depend on the geometry of the hypersurface determined by f. This approach has already been exploited in [5] to show that every quadratic Gorenstein algebra of socle degree two is Koszul and those of socle degree three are Koszul for f generic.

Alternatively, since both the Gorenstein and Koszul conditions impose significant restrictions on the shape and entries of the Betti table of R, one might ask: What are the possible Betti tables of an Artinian Gorenstein Koszul algebra R with a fixed projective dimension and regularity? The regularity coincides with the socle degree e, and again, the best understood case is socle degree two, for which the Betti numbers are uniquely determined and the resolution is known by [8]. However, many other values of projective dimension and regularity seem to give Betti tables that are almost uniquely determined.

4 Other Directions

A third area of research concerns the structure of free resolutions over complete intersections. When $R = S/(f)$ is a hypersurface ring, Eisenbud [6] showed that the asymptotic behavior of the free resolutions over R of any R-module can be described compactly by a pair of squares matrices (φ, ψ) of the same size over S such that $\varphi\psi = f\,\mathrm{Id}$ and $\psi\varphi = f\,\mathrm{Id}$, called a *matrix factorization* of f. This notion was later extended to complete intersections of arbitrary codimension by Eisenbud and Peeva [7]. Within the past decade and a half, there has been substantial interest in matrix factorizations from physics ever since Orlov showed that the homotopy category of matrix factorizations over a hypersurface was equivalent to its singularity category. Orlov's theorem was subsequently generalized to complete intersections $R = S/(f_1, \dots, f_c)$ of arbitrary codimension c in [3], which showed that that the singularity category of R could be related to the homotopy category of matrix factorizations of the polynomial $W = \sum_{i=1}^{c} f_i t_i$ in $S[t_1, \dots, t_c]$. When $c = 2$, I was able to construct a faithful functor from the category of Eisenbud-Peeva matrix factorizations to the category of matrix factorizations of W [12], a first step towards reconciling the two approaches. However, it remains unclear whether these approaches can be connected in higher codimensions and how precise that connection can be made.

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