Name: KEY Section 14

Math 267 Quiz 11 – Fall 2021

Instructions: You must show all of your work, including all steps needed to solve each problem, and explain your reasoning in order to earn full credit.

1. In this problem, we will solve the differential equation

$$y' - xy = 1, \qquad y(0) = 2 \tag{$*$}$$

using power series. (Even though this is a first order linear equation, we cannot compute the necessary integrals without resorting to power series!)

(a) First, we guess that the solution has a power series expansion

$$y = \sum_{n=0}^{\infty} c_n x^n$$

for some unknown coefficients c_n. Plug y into the equation $(*)$, and simplify the left hand side by reindexing and combining terms into a single sum as much as possible.

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m$$

$$xy = \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{m=1}^{\infty} c_{m-1} x^m$$

$$1 = y' - xy = c_1 + \sum_{m=1}^{\infty} \left[(m+1) c_{m+1} - c_{m-1} \right] x^m$$

(b) By equating coefficients on the left and right sides of your equation in the previous part, write down a recurrence relation on the coefficients c_n for $n \geq 2$.

$$c_1 = 1$$

$$(m+1)c_{m+1} - c_{m-1} = 0 \Rightarrow c_{m+1} = \frac{c_{m-1}}{m+1} \quad \text{for } m \geq 1$$

$$\Rightarrow c_n = \frac{c_{n-2}}{n} \quad \text{for } n \geq 2$$

(c) Using the recurrence relation, compute the first few odd coefficients c_1, c_3, c_5, c_7, and the write down the pattern for the general odd coefficient c_{2m+1} for any $m \geq 0$.

$$c_1 = 1$$

$$c_3 = \frac{c_1}{3} = \frac{1}{3}$$

$$c_5 = \frac{c_3}{5} = \frac{1}{5\cdot 3}$$

$$c_7 = \frac{c_5}{7} = \frac{1}{7\cdot 5\cdot 3}$$

$$c_{2m+1} = \frac{1}{(2m+1)(2m-1)\cdots 7\cdot 5\cdot 3\cdot 1} = \frac{2^m \cdot m!}{(2m+1)!}$$

(d) Using the recurrence relation, compute the first few even coefficients c_0, c_2, c_4, c_6, and the write down the pattern for the general even coefficient c_{2m} for any $m \geq 0$.

$$c_0 = y(0) = 2$$

$$c_2 = \frac{c_0}{2} = \frac{2}{2}$$

$$c_4 = \frac{c_2}{4} = \frac{2}{4\cdot 2}$$

$$c_6 = \frac{c_4}{6} = \frac{2}{6\cdot 4\cdot 2}$$

$$c_{2m} = \frac{2}{(2m)(2m-2)\cdots 6\cdot 4\cdot 2} = \frac{2}{2^m \cdot m!} = \frac{1}{2^{m-1}\, m!}$$

(e) Write down the solution to differential equation.

$$y = \sum_{m=0}^{\infty} \frac{1}{2^{m-1}\, m!} x^{2m} + \sum_{m=0}^{\infty} \frac{2^m\, m!}{(2m+1)!} x^{2m+1}$$