Name: KEY Section 7

Math 267 Quiz 6 – Fall 2021

Instructions: You must show all of your work, including all steps needed to solve each problem, and explain your reasoning in order to earn full credit.

1. In this problem, we will solve the differential equation

$$y'' + 36y = \sin^2 6x \qquad (*)$$

by variation of parameters.

(a) First, find linearly independent functions y_1 and y_2 that solve the corresponding homogeneous equation.

$$y'' + 36y = 0 \Rightarrow r^2 + 36 = 0 \Rightarrow r = \pm 6i$$

$$y = C_1 \sin 6x + C_2 \cos 6x$$

$$y_1 = \sin 6x \qquad y_2 = \cos 6x$$

(b) We will then guess that the particular solution to the nonhomogeneous equation has the form $y_p = u_1 y_1 + u_2 y_2$ for some functions u_1 and u_2 satisfying:

$$u_1' y_1 + u_2' y_2 = 0 \qquad (**)$$

Plug y_p into the equation $(*)$ and simplify the result as much as possible using $(**)$ and the fact that y_1, y_2 solve the homogeneous equation.

$$y_p' = (u_1 \sin 6x + u_2 \cos 6x)'$$

$$= u_1' \sin 6x + 6u_1 \cos 6x + u_2' \cos 6x - 6u_2 \sin 6x$$

$$= 6u_1 \cos 6x - 6u_2 \sin 6x$$

$$y_p'' = 6u_1' \cos 6x - 36u_1 \sin 6x - 6u_2' \sin 6x - 36u_2 \cos 6x$$

$$\sin^2 6x = y_p'' + 36y_p$$

$$= 6u_1' \cos 6x - 6u_2' \sin 6x$$

(c) The simplified equation you found in the previous part together with (**) form a system of equations for the unknown functions u_1' and u_2'. Solve this system of equations for u_1' and u_2'. (HINT: $\sin^2\theta + \cos^2\theta = 1$)

$$u_1' \sin 6x + u_2' \cos 6x = 0 \Rightarrow u_2' = -(\tan 6x)\, u_1'$$

$$\sin^2 6x = 6u_1' \cos 6x - 6u_2' \sin 6x$$

$$= 6u_1' \cos 6x + 6u_1' \frac{\sin^2 6x}{\cos 6x} = \frac{6u_1'}{\cos 6x}$$

$$\Rightarrow u_1' = \frac{1}{6}\sin^2 6x \cos 6x, \quad u_2' = -\frac{1}{6}\sin^3 6x$$

(d) Integrate u_1' and u_2' to find u_1 and u_2.

(HINT: $\sin^3\theta = \sin\theta(1-\cos^2\theta) = \sin\theta - \cos^2\theta\sin\theta$)

$$u_1 = \int \frac{1}{6}\sin^2 6x \cos 6x\, dx = \int \frac{1}{36} w^2\, dw = \frac{w^3}{108}$$

$$w = \sin 6x \quad dw = 6\cos 6x\, dx \qquad = \frac{\sin^3 6x}{108}$$

$$\frac{1}{6}\, dw = \cos 6x\, dx$$

$$u_2 = \int -\frac{1}{6}\sin^3 6x\, dx = \int -\frac{1}{6}\sin 6x + \frac{1}{6}\cos^2 6x \sin 6x\, dx$$

$$w = \cos 6x \qquad = \frac{1}{36}\cos 6x - \frac{1}{36}\int w^2\, dw$$

$$dw = -6\sin 6x\, dx$$

$$-\frac{1}{6}\, dw = \sin 6x\, dx \qquad = \frac{1}{36}\cos 6x - \frac{1}{108} w^3$$

(e) What is the particular solution y_p?

$$= \frac{1}{36}\cos 6x - \frac{1}{108}\cos^3 6x$$

$$y_p = \frac{\sin^4 6x}{108} + \frac{\cos^2 6x}{36} - \frac{\cos^4 6x}{108}$$