KEY

Math 267 Worksheet 9 – Fall 2021

1. Consider the system of 1st order linear differential equations:

$$\frac{dz}{dt} = 5z + 4w \qquad \frac{dw}{dt} = z + 2w$$

(a) If $\mathbf{Z}(t)$ denotes the vector-valued function $\mathbf{Z} = \begin{pmatrix} z \\ w \end{pmatrix}$, write down the coefficient matrix A such that $\frac{d\mathbf{Z}}{dt} = A\mathbf{Z}$.

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

(b) Verify that the vector-valued functions $\mathbf{Z}_1(t) = \begin{pmatrix} 4e^{6t} \\ e^{6t} \end{pmatrix}$ and $\mathbf{Z}_2(t) = \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix}$ both solve the given system of differential equations.

$$AZ_1 = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4e^{6t} \\ e^{6t} \end{pmatrix} = \begin{pmatrix} 24e^{6t} \\ 6e^{6t} \end{pmatrix} = \frac{dZ_1}{dt}$$

$$AZ_2 = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix} = \begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix} = \frac{dZ_2}{dt}$$

(c) What is the form of the general solution to the system of differential equations?

$$Z(t) = C_1\begin{pmatrix} 4e^{6t} \\ e^{6t} \end{pmatrix} + C_2\begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix}$$

(d) Find the specific function $\mathbf{Z}(t)$ that solves the system of differential equations subject to the initial condition $\mathbf{Z}(0) = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$.

$$\begin{pmatrix} 6 \\ -3 \end{pmatrix} = Z(0) = C_1\begin{pmatrix} 4 \\ 6 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \overbrace{\begin{pmatrix} 4 & 1 \\ 6 & -1 \end{pmatrix}}^{Q}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$Q^{-1} = \frac{-1}{10}\begin{pmatrix} -1 & -1 \\ -6 & 4 \end{pmatrix} \qquad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} -1 & -1 \\ -6 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 3/10 \\ 48/10 \end{pmatrix}$$

$$Z(t) = \frac{3}{10}\begin{pmatrix} 4e^{6t} \\ e^{6t} \end{pmatrix} + \frac{48}{10}\begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix}$$

2. Consider the matrix

$$A = \begin{pmatrix} 4 & 4 \\ 6 & -1 \end{pmatrix}.$$

(a) Find the eigenvalues of A.

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 4-\lambda & 4 \\ 6 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 24$$

$$= \lambda^2 - 3\lambda - 28 = (\lambda - 7)(\lambda + 4)$$

$$\lambda = 7, -4$$

(b) Find an eigenvector corresponding to each eigenvalue of A.

$$\lambda = 7: \begin{pmatrix} -3 & 4 \\ 6 & -8 \end{pmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{pmatrix} -3 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow -3v_1 + 4v_2 = 0 \Rightarrow v_1 = \frac{4}{3} v_2$$

$$\begin{pmatrix} \frac{4}{3} v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 4/3 \\ 1 \end{pmatrix} \underset{v_2 = 3}{\rightsquigarrow} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ is an eigenvector for } \lambda = 7$$

$$\lambda = -4: \begin{pmatrix} 8 & 4 \\ 6 & 3 \end{pmatrix} \xrightarrow[\frac{1}{3}R_2 \to R_2]{\frac{1}{4}R_1 \to R_1} \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_2 - R_1 \to R_1} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow 2v_1 + v_2 = 0 \Rightarrow v_2 = -2v_1$$

$$\begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \underset{v_1 = 1}{\rightsquigarrow} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is an eigenvector for } \lambda = -4$$

3. Consider the system of 1st order linear differential equations:

$$\frac{dy}{dt} = -y + 2v \qquad \frac{dv}{dt} = -2y - v$$

(a) If $\mathbf{Y}(t)$ denotes the vector-valued function $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$, write down the coefficient matrix A such that $\dfrac{d\mathbf{Y}}{dt} = A\mathbf{Y}$.

$$A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$$

(b) Find the eigenvalues of A.

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 4$$

$$= \lambda^2 + 2\lambda + 5 \implies \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

(c) Find an eigenvector corresponding to each eigenvalue of A.

$$\lambda = -1 + 2i: \quad \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \xrightarrow{R_2 + iR_1 \to R_2} \begin{pmatrix} -2i & 2 \\ 0 & 0 \end{pmatrix}$$

$$-2i v_1 + 2v_2 = 0 \implies v_2 = i v_1$$

$$\begin{pmatrix} v_1 \\ iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \underset{v_1 = 1}{\leadsto} v = \begin{pmatrix} 1 \\ i \end{pmatrix}$$ is an eigenvector for $\lambda = -1 + 2i$

$\bar{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector for $\bar{\lambda} = -1 - 2i$

(d) What is the form of the general solution to the system of differential equations?

$$Y(t) = C_1 e^{(-1+2i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{(-1-2i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{C}_1 e^{-t} \cos 2t + \tilde{C}_2 e^{-t} \sin 2t \\ -\tilde{C}_1 e^{-t} \sin 2t + \tilde{C}_2 e^{-t} \cos 2t \end{pmatrix} \qquad \begin{aligned} \tilde{C}_1 &= C_1 + C_2 \\ \tilde{C}_2 &= i(C_1 - C_2) \end{aligned}$$

4. Consider system of differential equations $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ defined by the matrix:

$$A = \begin{pmatrix} -12 & -12 & -2 \\ 13 & 13 & 2 \\ \frac{13}{4} & 3 & \frac{3}{2} \end{pmatrix}$$

This matrix has only two eigenvalues $\lambda = 1, \frac{1}{2}$.

(a) Check that $\mathbf{v}_1 = (-4, 4, 1)$ is an eigevector for A with eigenvalue $\lambda = \frac{1}{2}$.

$$Av_1 = \begin{pmatrix} -12 & -12 & -2 \\ 13 & 13 & 2 \\ 13/4 & 3 & 3/2 \end{pmatrix}\begin{pmatrix} -4 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1/2 \end{pmatrix} = \frac{1}{2} v_1$$

(b) Find two linearly independent eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$ with eigenvalue $\lambda = 1$. (Recall $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent if the only solution to the equation $c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ is $c_2 = c_3 = 0$.)

$$\begin{pmatrix} -13 & -12 & -2 \\ 13 & 12 & 2 \\ 13/4 & 3 & 1/2 \end{pmatrix} \xrightarrow[R_3 + \frac{1}{4}R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{pmatrix} -13 & -12 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-13a - 12b - 2c = 0 \Rightarrow c = -\frac{13}{2}a - 6b$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ -\frac{13}{2}a - 6b \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 \\ 0 \\ -13/2 \end{pmatrix}}_{v_1} + b\underbrace{\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}}_{v_2}$$

If $av_1 + bv_2 = (0,0,0)$, the above shows $a = b = 0$.

(c) Show that $\mathbf{Y}(t) = C_1e^{t/2}\mathbf{v}_1 + C_2e^t\mathbf{v}_2 + C_3e^t\mathbf{v}_3$ solves the system of differential equations.

$$Y(t) = \begin{pmatrix} -4C_1e^{t/2} + C_2e^t \\ 4C_1e^{t/2} + C_3e^t \\ C_1e^{t/2} - \frac{13}{2}C_2e^t - 6C_3e^t \end{pmatrix}$$

$$AY = \begin{pmatrix} -2C_1e^{t/2} + C_2e^t \\ 2C_1e^{t/2} + C_3e^t \\ \frac{1}{2}C_1e^{t/2} - \frac{13}{2}C_2e^t - 6C_3e^t \end{pmatrix} = \frac{dY}{dt}$$