Name: KEY

Section 7

Math 267 Quiz 9 – Fall 2021

Instructions: You must show all of your work, including all steps needed to solve each problem, and explain your reasoning in order to earn full credit.

1. Solve the system of linear differential equations $\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$ defined by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -4 & 6 \end{pmatrix}$$

either by using eigenvalues and generalized eigenvectors or by using matrix exponentials.

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -4 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 + 4$$

$$= \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 \Rightarrow \lambda = 4$$

$$A - 4I = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow -2v_1 + v_2 = 0 \Rightarrow v_1 = \tfrac{1}{2} v_2 \Rightarrow \begin{pmatrix} \frac{1}{2} v_2 \\ v_2 \end{pmatrix}$$

$$(A - 4I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ is a generalized eigenvector

$$u = (A - 4I)w = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$Y(t) = c_1 e^{4t} \begin{pmatrix} -2 \\ -4 \end{pmatrix} + c_2 e^{4t} \left(t \begin{pmatrix} -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

$$\Phi(t) = e^{At} = e^{4It} \cdot e^{(A-4I)t}$$

$$= e^{4It} \cdot \left(I + (A-4I)t + (A-4I)^2 \frac{t^2}{2!} + \cdots \right)$$

$$= e^{4t} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

$$= e^{4t} \begin{pmatrix} -2t+1 & t \\ -4t & 2t+1 \end{pmatrix}$$

$$Y(t) = e^{4t} \begin{pmatrix} -2t+1 & t \\ -4t & 2t+1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$