KEY

Math 267 Worksheet 12 – Fall 2021

1. Use Laplace transforms to solve the following initial value problem.

$$y'' - 6y' + 10y = t; \quad y(0) = y'(0) = 0$$

$$\frac{1}{s^2} = \mathcal{L}\{t\} = \mathcal{L}\{y'' - 6y' + 10y\}$$

$$= s\,\mathcal{L}\{y'\} - 6sY + 10Y$$

$$= s^2 Y - 6sY + 10Y$$

$$\Rightarrow \quad Y = \frac{1}{s^2(s^2 - 6s + 10)} \qquad \left(\text{note } s^2 - 6s + 10 \text{ does not factor over } \mathbb{R}\right)$$

$$= \frac{A}{s^2} + \frac{B}{s} + \frac{Cs + D}{s^2 - 6s + 10}$$

$$1 = A(s^2 - 6s + 10) + Bs(s^2 - 6s + 10) + (Cs + D)s^2$$

$$= (B + C)s^3 + (A - 6B + D)s^2 + (-6A + 10B)s + 10A$$

$$\Rightarrow \quad B + C = 0, \quad A - 6B + D = 0, \quad -6A + 10B = 0, \quad 10A = 1$$

$$\Rightarrow \quad A = \frac{1}{10}, \quad B = \frac{6}{100}, \quad C = \frac{-6}{100}, \quad D = \frac{26}{100}$$

$$y(t) = \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{6}{100}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{100}\mathcal{L}^{-1}\left\{\frac{6s - 26}{s^2 - 6s + 10}\right\}$$

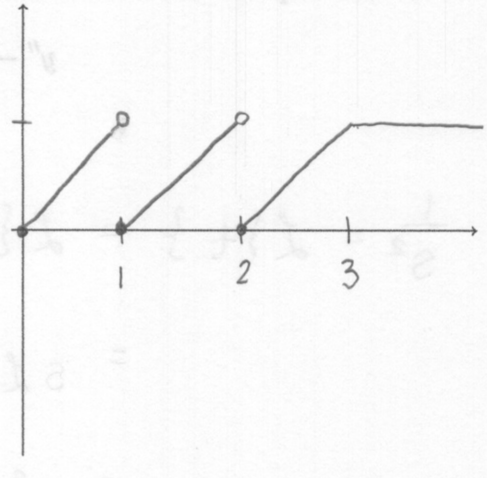
$$= \frac{1}{10}t + \frac{6}{100} - \frac{1}{100}\mathcal{L}^{-1}\left\{\frac{6(s-3) - 8}{(s-3)^2 + 1}\right\}$$

$$= \frac{1}{10}t + \frac{6}{100} - \frac{1}{100}e^{3t}\mathcal{L}^{-1}\left\{\frac{6s}{s^2+1}\right\} - \frac{1}{100}e^{3t}\mathcal{L}^{-1}\left\{\frac{-8}{s^2+1}\right\}$$

$$= \frac{1}{10}t + \frac{6}{100} - \frac{6}{100}e^{3t}\cos t + \frac{8}{100}e^{3t}\sin t$$

2. Find the inverse Laplace transforms of the following functions. Sketch the graphs of the inverse transforms for $t \geq 0$ on the axes provided.

(a) $F(s) = \dfrac{1 - s e^{-s} - s e^{-2s} - e^{-3s}}{s^2}$



$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\}$$

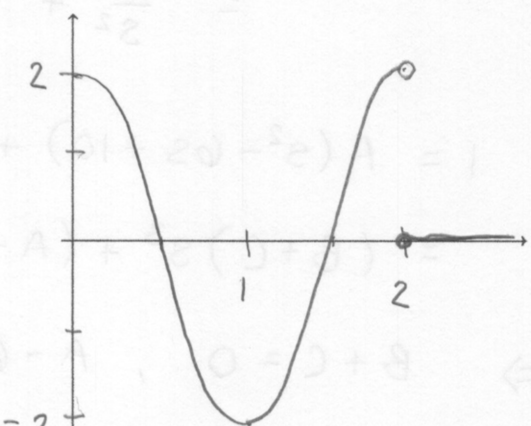
$$- \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$$

$$= t - u(t-1)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$- u(t-2)\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - u(t-3)\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Big|_{t-3}$$

$$= t - u(t-1) - u(t-2) - u(t-3)\cdot(t-3)$$

(b) $G(s) = \dfrac{2 s - 2 s e^{-2s}}{s^2 + \pi^2}$



$$\mathcal{L}^{-1}\{G(s)\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}$$

$$- 2\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+\pi^2}\right\}$$

$$= 2\cos(\pi t) - 2u(t-2)\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}\Big|_{t-2}$$

$$= 2\cos(\pi t) - 2u(t-2)\cos(\pi(t-2))$$

$$= 2\cos(\pi t) - 2u(t-2)\cos(\pi t - 2\pi)$$

$$= 2\cos(\pi t)\left[1 - u(t-2)\right]$$

3. Use Laplace transforms to solve the following initial value problems.

(a) $y'' + 16y = \delta(t-1)$; $\quad y(0) = y'(0) = 0$

$$e^{-s} = \mathcal{L}\{\delta(t-1)\} = \mathcal{L}\{y'' + 16y\}$$

$$= s\mathcal{L}\{y'\} + 16Y$$

$$= s^2 Y + 16Y$$

$$\Rightarrow Y = \frac{e^{-s}}{s^2+16}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+16}\right\} = u(t-1)\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}\Big|_{t-1}$$

$$= \frac{1}{4}u(t-1)\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\}\Big|_{t-1}$$

$$= \frac{1}{4}u(t-1)\sin(4(t-1))$$

(b) $y'' + 2y' + y = -\delta(t-2); \quad y(0) = y'(0) = 2$

$$-e^{-2s} = \mathcal{L}\{-\delta(t-2)\}$$

$$= \mathcal{L}\{y'' + 2y' + y\}$$

$$= s\mathcal{L}\{y'\} - 2 + 2(sY - 2) + Y$$

$$= s^2 Y - 2s + 2sY - 6 + Y$$

$$\Rightarrow \quad Y = \frac{2s + 6 - e^{-2s}}{s^2 + 2s + 1} = \frac{2(s+1) + 4 - e^{-2s}}{(s+1)^2}$$

$$= \frac{2}{s+1} + \frac{4}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{(s+1)^2}\right\}$$

$$= 2e^{-t} + 4te^{-t} - u(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\Big|_{t-2}$$

$$= 2e^{-t} + 4te^{-t} - u(t-2)(t-2)e^{-(t-2)}$$