Koszul Almost Complete Intersections

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Abstract

Let $R = S/I$ be a quotient of a standard graded polynomial ring S by an ideal I generated by quadrics. If R is Koszul, a question of Avramov, Conca, and Iyengar asks whether the Betti numbers of R over S can be bounded above by binomial coefficients on the minimal number of generators of I. Motivated by previous results for Koszul algebras defined by three quadrics, we give a complete classification of the structure of Koszul almost complete intersections and, in the process, give an affirmative answer to the above question for all such rings.

1 Introduction

Let k be a field, S be a standard graded polynomial ring over k, $I \subseteq S$ be a graded ideal, and $R = S/I$. We say that R is a *Koszul algebra* if $k \cong R/R_+$ has a linear free resolution over R. Many rings arising from algebraic geometry are Koszul, including the coordinate rings of Grassmannians [Kem90], sets of $r \leq 2n$ points in general position in $\mathbb{P}^n$ [Kem92], and canonical embeddings of smooth curves under mild restrictions [Pol95], as well as all suitably high Veronese subrings of any standard graded algebra [Bac86]. However, the simplest examples of Koszul algebras, due to Fröberg [Fro75], are quotients by quadratic monomial ideals, and a guiding heuristic in the study of Koszul algebras has been that any reasonable property of algebras defined by quadratic monomial ideals should also hold for Koszul algebras; for example, see [ACI10], [Con13], [ACI15]. Among such properties, considering the Taylor resolution for an algebra defined by a quadratic monomial ideal leads to the following question about the Betti numbers of a Koszul algebra.

Question 1.1 ([ACI10, 6.5])**.** If R is Koszul and I is minimally generated by g elements, does the following inequality hold for all i?

$$\beta_i^S(R) \leq \binom{g}{i}$$

In particular, is $\operatorname{pd}_S R \leq g$?

The above questions are known to have affirmative answers when R is LG-quadratic (see next section) and for arbitrary Koszul algebras when $g \leq 3$ by [BHI16, 4.5]. Recall that R or I is called an *almost complete intersection* if I is minimally generated by $\operatorname{ht} I + 1$ elements.

The motivation for studying Koszul almost complete intersections comes from the fact that the above question holds trivially when I is a complete intersection or has height one so that the interesting case for Koszul algebras defined by three quadrics is precisely when I is an almost complete intersection. Our main results (4.1, 4.3, 4.4, 5.3) show that Question 1.1 has an affirmative answer for Koszul almost complete intersections generated by any number of quadrics; they are summarized in the theorem below.

Main Theorem. *Let $R = S/I$ be a Koszul almost complete intersection with I minimally generated by $g+1$ quadrics for some $g \geq 1$. Then $\beta^S_{2,3}(R) \leq 2$, and:*

(a) *If $\beta^S_{2,3}(R) = 1$, there are linear forms x, z, and w such that $I = (xz, zw, q_3, \dots, q_{g+1})$ for some regular sequence of quadrics $q_3, \dots, q_{g+1}$ on $S/(xz, zw)$.*

(b) *If $\beta^S_{2,3}(R) = 2$, there is a 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ such that $I = I_2(M) + (q_4, \dots, q_{g+1})$ for some regular sequence of quadrics $q_4, \dots, q_{g+1}$ on $S/I_2(M)$.*

Furthermore, R is LG-quadratic and, therefore, satisfies $\beta^S_i(R) \leq \binom{g+1}{i}$ for all i.

The division of the rest of the paper is as follows. In §2, we recount various properties and examples of Koszul algebras and their Betti tables which will be important in the sequel. After giving a simple characterization of the second syzygies of Koszul algebras in §3 using the technical machinery of deviations and DG algebra resolutions, §4 determines the structure of Koszul almost complete intersections with either one or two linear second syzygies. We then complete the classification of Koszul almost complete intersections in §5 by showing that every quadratic almost complete intersection has at most two linear second syzygies.

Notation. Throughout the remainder of the paper, the following notation will be in force unless specifically stated otherwise. Let k be a fixed ground field of arbitrary characteristic, S be a standard graded polynomial ring over k, $I \subseteq S$ be a proper graded ideal, and $R = S/I$. Recall that the ideal I is called *nondegenerate* if it does not contain any linear forms. We can always reduce to a presentation for R with I nondegenerate by killing a basis for the linear forms contained in I, and we will assume that this is the case throughout. We denote the irrelevant ideal of R by $R_+ = \bigoplus_{n \geq 1} R_n$.

2 Koszul Algebras and Their Betti Tables

If R is a Koszul algebra, it is well-known that its defining ideal I must be generated by quadrics, but not every ideal generated by quadrics defines a Koszul algebra. We have already noted in the introduction that every quadratic monomial ideal defines a Koszul algebra. More generally, we say that R or I is *G-quadratic* if I has a Gröbner basis consisting of quadrics and *LG-quadratic* if R is a quotient of a G-quadratic algebra A by an A-sequence of linear forms. Every G-quadratic algebra is Koszul by upper semicontinuity of the Betti numbers;

see [BC03, 3.13]. It then follows from Proposition 2.6 below that every LG-quadratic algebra is also Koszul. In particular, every complete intersection generated by quadrics is LG-quadratic. Indeed, if $R = S/(q_1, \dots, q_g)$ where $q_1, \dots, q_g$ is a regular sequence of quadrics, we can take $A = S[y_1, \dots, y_g]/(y_1^2 + q_1, \dots, y_g^2 + q_g)$ so that $A/(y_1, \dots, y_g) \cong R$. By choosing a monomial order in which the y_i are greater than every monomial in the variables of S, it follows from [Eis95, 15.15] that A is G-quadratic and that the $y_i^2 + q_i$ form a regular sequence so that $\operatorname{ht}_A(y_1, \dots, y_g) = \dim A - \dim R = \dim S - \dim R = g$ and $y_1, \dots, y_g$ is an A-sequence. In summary, we have the following implications.

$$\begin{array}{ccccc} \text{G-quadratic} & \Longrightarrow & \text{LG-quadratic} & \Longrightarrow & \text{Koszul} \\ & & \Uparrow & & \\ & & \text{Quadratic CI's} & & \end{array}$$

Each of the above implications is strict. Clearly, any quadratic monomial ideal which is not a complete intersection, such as $(xy, xz, xw) \subseteq k[x, y, z, w]$, is G-quadratic, hence also LG-quadratic.

Example 2.1. The ideal $I = (xy, (x - y)z) \subseteq k[x, y, z]$ is easily seen to be a quadratic complete intersection, hence LG-quadratic, which is not G-quadratic since either x^2z or y^2z must be a minimal generator of $\operatorname{in}_>(I)$ according to whether $x < y$ or $x > y$ respectively.

Example 2.2. Modifying the previous example slightly, consider the quadratic almost complete intersection $I = (xy, xw, (x-y)z) \subseteq k[x, y, z, w]$, and set $R = k[x, y, z, w]/I$. If we take $A = k[x, y, z, w, u]/(xy, xw, uz)$, then A is G-quadratic, $A/(u-x+y) \cong R$, and it can easily be seen by computing the primary decomposition of the defining ideal of A that $u - x + y$ is A-regular so that R is LG-quadratic. However, the same argument as above shows that R is not G-quadratic.

We will see a similar example of a Koszul algebra which is not LG-quadratic below.

Remark 2.3. If $R = S/I$ is G-quadratic and J is a quadratic initial ideal of I, then $\beta_1^S(R) = \beta_1^S(S/J)$. Indeed, we always have $\beta_1^S(R) \leq \beta_1^S(S/J)$ by upper semicontinuity of the Betti numbers, and by choosing quadrics $q_1, \dots, q_g$ whose initial monomials are the minimal generators of J, the matrix of coefficients of the quadrics $q_1, \dots, q_g$ must have rank g by construction so that they are linearly independent. Hence, we see that $\beta_1^S(S/J) = g \leq \beta_1^S(R)$. Consequently, $\beta_i(R) \leq \beta_i(S/J) \leq \binom{g}{i}$ for all i by upper semicontinuity of the Betti numbers and the Taylor resolution for S/J. Since killing a regular sequence of linear forms does not affect the Betti numbers of R, it follows that Question 1.1 holds for every LG-quadratic algebra.

We will be specifically interested in the *graded Betti numbers* of a Koszul algebra R, which are defined by $\beta_{i,j}^S(R) = \dim_k \operatorname{Tor}_i^S(k, R)_j$ and related to the usual Betti numbers by $\beta_i^S(R) = \sum_j \beta_{i,j}^S(R)$. This information is usually organized into a table, called the Betti table of R; see below for an example. As we have already pointed out in the the introduction, quadratic monomial ideals serve as a useful benchmark in the study of Koszul algebras. In particular, we note that the square-free quadratic monomial ideals are precisely edge ideals. Recall that, if G is a graph with vertex set $[n] = \{1, \dots, n\}$, the *edge ideal* of G is the ideal of the polynomial ring $S = k[x_1, \dots, x_n]$ defined by $I_G = (x_i x_j \mid ij \in E(G))$. We note that every quadratic monomial ideal can be obtained as the image of an edge ideal modulo a regular sequence of linear forms via polarization [BH93, 4.2.16], and hence, studying the Betti tables of all quadratic monomial ideals with g generators is equivalent to studying the Betti tables of edge ideals of graphs with g edges, which are reasonably simple to enumerate in practice for small values of g.

In fact, a byproduct of the proof in [BHI16] that every Koszul algebra defined by $g \leq 3$ quadrics satisfies Question 1.1 is that every such algebra has the Betti table of some edge ideal. This is no coincidence; computing all the Betti tables of edge ideals with g generators for various values of g reveals that there are only two possible Betti tables for almost complete intersections, one with a single linear syzygy and another with two, and we will see that this pattern holds more generally for all Koszul almost complete intersections. However, the mantra that Koszul algebras are similar to quotients by quadratic monomial ideals must be taken with a grain of salt.

Example 2.4. The ring $R = k[x, y, z, w]/(xy, xw, (x-y)z, z^2, x^2 + zw)$ is Koszul by a filtration argument, [Con13, 3.8]. The minimal free resolution of R over $S = k[x, y, z, w]$ can be computed via iterated mapping cones using the fact that $((xy, xw, z^2) : (x - y)z) = (xy, xw, z)$ and $((xy, xw, z^2, (x - y)z) : x^2 + zw) = (xw, y, z)$. This yields the following Betti table for R, where the entry in column i and row j is $\beta^S_{i,i+j}(R)$ and zero entries are represented by "–" for readability.

	0	1	2	3	4
0	1	–	–	–	–
1	–	5	4	–	–
2	–	–	4	6	2

From the above Betti table, we see that the Hilbert series of R is

$$H_R(t) = \frac{1 + 2t - 2t^2 - 2t^3 + 2t^4}{(1-t)^2}$$

If R were LG-quadratic, then the numerator of the Hilbert series must be the h-polynomial of a 5-generated edge ideal, since killing a regular sequence of linear forms and passing to a quadratic initial ideal do not change either the h-polynomial or the minimal number of generators of the defining ideal. As noted above, we can easily compute that the h-polynomial of R does not belong to any 5-generated edge ideal. Hence, R is not LG-quadratic, and in particular, the Betti table of R is not the Betti table of any edge ideal. This points to unexpected complications in trying to answer Question 1.1 for Koszul algebras defined by $g \geq 5$ quadrics.

Remark 2.5. We can compute the Betti tables of 5-generated edge ideals in the above example over a field of any characteristic by a result of Katzman, [Kat06, 4.1]. However, Katzman also shows that the Betti tables of edge ideals do depend on the characteristic of the ground field in general.

In the remainder of this section, we collect a few results about Koszul algebras that will be useful in the sequel. The first of these results states how the Koszul property can be passed to and from quotient rings.

Proposition 2.6 ([CDR13, §3.1, 2])**.** *Let S be a standard graded k-algebra and R be a quotient ring of S.*

(a) *If S is Koszul and $\operatorname{reg}_S(R) \leq 1$, then R is Koszul.*

(b) *If R is Koszul and $\operatorname{reg}_S(R)$ is finite, then S is Koszul.*

Compared with general quadratic algebras, the Betti tables of Koszul algebras are much more restricted. The following result, discovered in [Bac88] and [Kem90, 4], says that the Betti tables of Koszul algebras have nonzero entries only on or above the diagonal; see [Con13, 2.10] for an easier argument using regularity.

Lemma 2.7. *If $R = S/I$ is a Koszul algebra, then $\beta^S_{i,j}(R) = 0$ for all i and $j > 2i$.*

In addition, the extremal portions of the Betti table of a Koszul algebra R, namely the diagonal entries and the linear strand of I, satisfy bounds similar to those in Question 1.1.

Proposition 2.8 ([BHI16, 3.4, 4.2])**.** *Suppose that $R = S/I$ is Koszul and that I is minimally generated by g elements. Then:*

(a) *$\beta^S_{i,i+1}(R) \leq \binom{g}{i}$ for $2 \leq i \leq g$, and if equality holds for some i, then I has height one and a linear resolution of length g.*

(b) *$\beta^S_{i,2i}(R) \leq \binom{g}{i}$ for $2 \leq i \leq g$, and if equality holds for some i, then I is a complete intersection.*

Corollary 2.9. *If $R = S/I$ is a Koszul algebra which is not a complete intersection, then I has a linear syzygy.*

Proof. Suppose that I is minimally generated by $q_1, \ldots, q_g$. If the Koszul syzygies on the q_i are all minimal generators of $\operatorname{Syz}^S_1(I)$, then $\beta^S_{2,4}(R) \geq \binom{g}{2}$ contradicting the preceding proposition. Hence, some k-linear combination of the Koszul syzygies is not minimal, and therefore, it is an S-linear combination of linear syzygies. $\square$

3 Deviations of Koszul Algebras

The primary aim of this section is to prove the following proposition about the syzygies of a Koszul algebra.

Proposition 3.1. *If $R = S/I$ is a Koszul algebra, then the quadratic sygyzies on I are spanned by the Koszul syzygies on the minimal generators of I together with the quadratic syzygies which are multiples of a linear syzygy. Hence, $\operatorname{Syz}_1^S(I)$ is minimally generated by linear syzygies and Koszul syzygies.*

As motivation for the above proposition, we first consider the case of quadratic monomial ideals. Every quotient by a quadratic monomial ideal has a natural, usually non-minimal, graded free resolution called the *Taylor resolution*; see [MS05, §6.1]. If $R = S/I$ where I is a quadratic monomial ideal with minimal generators $m_1, \ldots, m_g$, then the first three terms of the Taylor resolution are

$$\bigoplus_{1\le i<j\le g} S(e_i \wedge e_j) \xrightarrow{d_2} \bigoplus_{i=1}^{g} Se_i \xrightarrow{d_1} S \longrightarrow 0$$

where $d_1(e_i) = m_i$ for all i and $d_2(e_i \wedge e_j) = \frac{m_i}{\gcd(m_i,m_j)} e_j - \frac{m_j}{\gcd(m_i,m_j)} e_i$ for all $i < j$. Each of the latter terms is either a linear syzygy or a Koszul syzygy depending on whether $\gcd(m_i, m_j)$ has degree zero or one, and so, it follows that $\operatorname{Syz}_1^S(I)_4 = [\operatorname{Ker} d_1]_4 = [\operatorname{Im} d_2]_4$ is spanned by the Koszul syzygies on the generators of I together with multiples of linear syzygies.

In order to prove the above proposition in general, we must first recall the definition of the graded deviations of R. Given a standard graded k-algebra R, the *graded Poincaré series* of k over R is the formal power series

$$\mathrm{P}_k^R(s,t) = \sum_{i,j} \beta_{i,j}^R(k) s^j t^i \in \mathbb{Z}[s][[t]]$$

Setting $s = 1$ in the above series, gives the usual Poincaré series $\mathrm{P}_k^R(t) = \sum_i \beta_i^R(k)t^i$. It is well-known that the Poincaré series of k over R can be expressed as an infinite product; see [Avr98, 7.1.1]. For the convenience of the reader, we reproduce that argument with the appropriate modifications for the graded case below.

Lemma 3.2. *Let $P(s,t) = 1 + \sum_{i=1}^{\infty} \sum_j b_{i,j} s^j t^i$ be a formal power series in $\mathbb{Z}[s][[t]]$, and for each $i \ge 1$ and $j \ge 0$, set $p_{i,j}(s,t) = (1 - s^j(-t)^i)^{(-1)^{i+1}}$. Then there exist unique $e_{i,j} \in \mathbb{Z}$ such that for each i, $e_{i,j} = 0$ for all but finitely many values of j and*

$$P(s,t) = \prod_{i=1}^{\infty} \prod_j p_{i,j}(s,t)^{e_{i,j}}$$

in the (t)-adic topology.

Proof. First, we note that $p_{i,j}(s,t) \equiv 1 + s^j t^i \pmod{t^{i+1}}$ regardless of whether i is odd or even. Given a product expansion as above, we must have $P(s,t) \equiv P_n(s,t) \pmod{t^{n+1}}$ for each n, where $P_n(s,t) = \prod_{i=1}^n \prod_j p_{i,j}(s,t)^{e_{i,j}}$ is the n-th partial product. By convention, we set $P_0(s,t) = 1$. Then for each $n \geq 1$, modulo t^{n+1} we have

$$\begin{aligned} P(s,t) - P_{n-1}(s,t) &\equiv P_n(s,t) - P_{n-1}(s,t) = P_{n-1}(s,t)(\prod_j p_{n,j}(s,t)^{e_{n,j}} - 1) \\ &\equiv P_{n-1}(s,t)(\prod_j (1 + s^j t^n)^{e_{n,j}} - 1) \\ &\equiv P_{n-1}(s,t)(\prod_j (1 + e_{n,j} s^j t^n) - 1) \\ &\equiv (\sum_j e_{n,j} s^j) t^n P_{n-1}(s,t) \equiv (\sum_j e_{n,j} s^j) t^n \end{aligned}$$

where the last equivalence follows from the fact that $P_{n-1}(s,t) \equiv 1 \pmod{t}$. This show that the $e_{n,j}$ are uniquely determined by $P(s,t)$ and the $e_{i,j}$ with $i < n$. Hence, an induction on n yields that the $e_{n,j}$ are uniquely determined by $P(s,t)$.

On the other hand, the above computation shows that we can construct the integers $e_{i,j}$ recursively. Having already found integers $e_{i,j}$ for $i < n$ such that $P_{n-1}(s,t) \equiv P(s,t) \pmod{t^n}$, where $P_{n-1}(s,t)$ is defined as above, we take $e_{n,j}$ to be the unique integers such that $P(s,t) - P_{n-1}(s,t) \equiv (\sum_j e_{n,j} s^j) t^n \pmod{t^{n+1}}$. Then modulo t^{n+1} we have

$$\begin{aligned} P_n(s,t) = P_{n-1}(s,t)(\prod_j p_{n,j}(s,t)^{e_{n,j}}) &\equiv P_{n-1}(s,t)(1 + (\sum_j e_{n,j} s^j) t^n) \\ &\equiv P_{n-1}(s,t) + (\sum_j e_{n,j} s^j) t^n \equiv P(s,t) \end{aligned}$$

as wanted, where the second to last equivalence follows from the fact that $P_{n-1}(s,t) \equiv 1 \pmod{t}$ as above. □

The exponents constructed by the preceding lemma for the graded Poincaré series of k over R are called the *graded deviations* of R and denoted by $\varepsilon_{i,j}(R)$. As with Betti numbers, the deviations of R are the sums $\varepsilon_i(R) = \sum_j \varepsilon_{i,j}(R)$.

We suspect that the following lemma may be known to experts; for example, it appears to have been discovered independently by the authors of [BDG15] for the multigraded case of edge ideals where it is asserted without proof.

Lemma 3.3. *If R is a Koszul algebra, then $\varepsilon_{i,j}(R) = 0$ for all i and $j \neq i$.*

Proof. Set $\varepsilon_{i,j} = \varepsilon_{i,j}(R)$ and $\beta_{i,j} = \beta^R_{i,j}(k)$. We argue by induction on $i \geq 1$. Using the proof of the preceding lemma and its notation, we know that the $\varepsilon_{1,j}$ are the unique integers such that $(\sum_j \varepsilon_{1,j} s^j) t \equiv \mathrm{P}^R_k(s,t) - 1 \equiv \beta_{1,1} st \pmod{t^2}$ as R is Koszul. Hence, we see that $\varepsilon_{1,j} = 0$ for $j \neq 1$. Assume now that $i \geq 2$ and that $\varepsilon_{n,j} = 0$ for all $n < i$ and $j \neq n$. Again, the $\varepsilon_{i,j}$ are the unique integers such that $(\sum_j \varepsilon_{i,j} s^j)t^i \equiv \mathrm{P}_k^R(s,t) - P_{i-1}(s,t) \pmod{t^{i+1}}$. By assumption, we know that $P_{i-1}(s,t) = \prod_{n=1}^{i-1} p_{n,n}(s,t)^{\varepsilon_{n,n}}$ is equivalent to a polynomial in st modulo t^{i+1}. Since R is Koszul, $\mathrm{P}_k^R(s,t) = 1 + \sum_{i=1}^{\infty} \beta_{i,i} s^i t^i$ is also equivalent to a polynomial in st modulo t^{i+1} so that $(\sum_j \varepsilon_{i,j} s^j)t^i$ is as well. Hence, we must have $\varepsilon_{i,j} = 0$ for $j \neq i$ as claimed. □

An alternative characterization of the graded deviations can be given in terms of minimal DG-algebra resolutions. We summarize the key features of this characterization below and refer the reader to [Avr98] for further details. A *differential graded R-algebra* (or DG R-algebra for short) is a complex A of R-modules together a product chain map $A \otimes_R A \to A$, written $a \otimes b \mapsto ab$, which has a unit $1 \in A_0$, is associative, and is graded commutative in the sense that $ab = (-1)^{|a||b|} ba$ for all $a, b \in A$ and $a^2 = 0$ if $|a|$ is odd. Here, $|a|$ denotes the homological degree of $a \in A$ so that $|a| = i$ if $a \in A_i$. The DG algebras we will be interested in will be chain complexes of graded R-modules, and so, elements $a \in A$ will also have an internal degree.

The *acyclic closure* of k over R is the unique up to isomorphism DG R-algebra $R\langle X \rangle$ admitting a surjective quasi-isomorphism $R\langle X \rangle \to k$ which is constructed as the direct union of a sequence of DG R-algebras $R\langle X_{\leq i} \rangle$ for $i \geq 1$ satisfying $H_n(R\langle X_{\leq i} \rangle) = 0$ for $0 < n < i$ in the following manner. We take $R\langle X_{\leq 1} \rangle$ to be the Koszul complex on a minimal set of generators of R_+. Having produced $R\langle X_{\leq i} \rangle$ satisfying $H_n(R\langle X_{\leq i} \rangle) = 0$ for $0 < n < i$, we let X_{i+1} be a set of indeterminates, called *free Γ-variables*, corresponding to a set of cycles in $R\langle X_{\leq i} \rangle$ whose images minimally generate $H_i(R\langle X_{\leq i} \rangle)$, and we set $R\langle X_{\leq i+1} \rangle = R\langle X_{\leq i} \rangle \otimes_R \bigotimes_{x \in X_{i+1}} R\langle x \rangle$, where $R\langle x \rangle$ with $|x| = i+1$ denotes either the exterior algebra or divided powers algebra of Rx depending on whether i is even or odd respectively. The differential of $R\langle X_{\leq i+1} \rangle$ is uniquely determined by the requirements that $R\langle X_{\leq i} \rangle$ is a subcomplex and for each $x \in X_{i+1}$ corresponding to the cycle $z \in R\langle X_{\leq i} \rangle$, we have $\partial(x^{(n)}) = z x^{(n-1)}$ for all $n \geq 1$. We set $X = \bigcup_{i=1}^{\infty} X_i$ in the direct union $R\langle X \rangle$. In our case, the set X is actually bigraded, and we let $X_{i,j}$ denote the set of free variables of homological degree i and internal degree j adjoined in the above process. The acyclic closure $R\langle X \rangle$ is a minimal graded free resolution of k over R [Avr98, 6.3.5] so that there is a bigraded isomorphism $\mathrm{Tor}_*^R(k,k) \cong R\langle X \rangle \otimes_R k = \bigotimes_{x \in X} k\langle x \rangle$. Since the generating series for the dimensions of the bigraded components of $k\langle x \rangle$ is precisely $p_{i,j}(s,t)$ for $x \in X_{i,j}$, this yields an equality

$$\mathrm{P}_k^R(s,t) = \prod_{i=1}^{\infty} \prod_j p_{i,j}(s,t)^{\mathrm{card}(X_{i,j})}$$

so that $\varepsilon_{i,j}(R) = \mathrm{card}(X_{i,j}) \geq 0$ for all i,j. In particular, we note that $\varepsilon_1(R) = \dim_k R_1 = \dim S$.

Another important and related DG-algebra resolution is the *minimal model* of R over S. It is the unique up to isomorphism DG S-algebra $S[Y]$ admitting a surjective quasi-isomorphism $S[Y] \to R$ constructed as the direct union of a sequence of DG S-algebras

$S[Y_{\le i}]$ for $i \ge 1$ in a similar manner to the acyclic closure. We take $S[Y_{\le 1}]$ to be the Koszul complex on a minimal set of generators of I. Then having produced $S[Y_{\le i}]$ for some $i \ge 1$, we let Y_{i+1} be a set of indeterminates, called *free variables*, corresponding to a set of cycles in $S[Y_{\le i}]$ whose images minimally generate $H_i(S[Y_{\le i}])$, and we set $S[Y_{\le i+1}] = S[Y_{\le i}] \otimes_S \bigotimes_{y \in Y_{i+1}} S[y]$, where $S[y]$ with $|y| = i+1$ denotes either the exterior algebra or symmetric algebra of Sy depending on whether i is even or odd respectively. It is known that $\varepsilon_{i,j}(R) = \operatorname{card}(X_{i,j}) = \operatorname{card}(Y_{i-1,j})$ for all $i \ge 2$ and all j, [Avr98, 7.2.6]. Hence, we have $\varepsilon_{2,j}(R) = \dim_k[I/S_+I]_j$ and, more importantly for us, $\varepsilon_{3,j}(R) = \dim_k[H_1(I)/S_+H_1(I)]_j$ for all j.

The minimal model and acyclic closure over R of a quotient ring R/J are isomorphic if k has characteristic zero or J is generated by a regular sequence, but in general, they may not be as the following example shows.

Example 3.4 ([Avr98, 6.1.10])**.** If k has characteristic $p > 0$ and $R = k[t]/(t^{m+1})$ for some $m \ge 1$, then $G = R[Y_{\le 2}] = R[y_1, y_2 \mid \partial(y_1) = t, \partial(y_2) = t^m y_1]$ satisfies $G_{2i} = Ry_2^i$, $G_{2i+1} = Ry_1y_2^i$, $\partial(y_2^i) = it^m y_1 y_2^{i-1}$ and $\partial(y_1 y_2^i) = ty_2^i$ for all $i \ge 0$. It follows that $H_{2ip}(G) = Ry_2^{ip}/tRy_2^{ip} \cong k$ and $H_{2ip-1}(G) = t^m Ry_1 y_2^{ip-1} \cong k$ for all $i \ge 0$. Since G is a DG-subalgebra of the minimal model $R[Y]$, it follows that y_2^{ip} is a boundary in $R[Y]$ so that $R[Y]$ cannot be a minimal resolution of k and, therefore, is not isomorphic to the acyclic closure of k.

Proof of Proposition 3.1. If $K_\bullet(I)$ denotes the Koszul complex on a minimal set of generators of I, then $Z_1(I) = \operatorname{Syz}_1^S(I)$ so that the first part of the proposition holds if and only if $Z_1(I)_4 \subseteq B_1(I) + S_+Z_1(I)$, which is equivalent to $H_1(I)_4 \subseteq S_+H_1(I)$ and $\varepsilon_{3,4}(R) = 0$ by the above discussion. Hence, the first part of the proposition follows from the preceding lemma. This implies that $Z_1(I)_4/(Z_1(I)_4 \cap S_+Z_1(I))$ is spanned by the images of the Koszul syzygies so that we can choose the minimal generators of $\operatorname{Syz}_1^S(I)$ in degree four to be Koszul syzygies. The remainder of the proposition then follows from Lemma 2.7 since $\operatorname{Syz}_1^S(I)$ does not have any minimal generators of degree greater than four. $\square$

4 Koszul Almost Complete Intersections

Recall that a standard graded k-algebra $R = S/I$ with $\operatorname{ht} I = g$ is called an *almost complete intersection* (or ACI for short) if I is minimally generated by $g+1$ elements.

Theorem 4.1. *Let $R = S/I$ be a Koszul algebra with $\beta_{2,3}^S(R) = 1$. Then there are independent linear forms x and w and a linear form z such that $I = (xz, zw, q_3, \dots, q_{g+1})$ for some regular sequence of quadrics $q_3, \dots, q_{g+1}$ on $S/(xz, zw)$, and conversely, every ideal of this form defines a Koszul algebra with $\beta_{2,3}^S(R) = 1$. Hence, R is an almost complete intersection* *with $e(R) = 2^{g-1}$ and Betti table*

	0	1	2	3	$\cdots$	$g-1$	g	$g+1$
0	1	–	–	–		–	–	–
1	–	$g+1$	1	–		–	–	–
2	–	–	$\frac{g+2}{2}\binom{g-1}{1}$	$\binom{g-1}{1}$		–	–	–
$\vdots$				$\ddots$	$\ddots$			
$g-1$	–	–	–			$\frac{2g-1}{g-1}\binom{g-1}{g-2}$	$\binom{g-1}{g-2}$	–
g	–	–	–			–	2	1

Specifically, we have $\beta^S_{i,2i}(R) = \frac{g+i}{i}\binom{g-1}{i-1}$ and $\beta^S_{i,2i-1}(R) = \binom{g-1}{i-2}$ for $i \geq 2$ so that $\beta^S_i(R) = \binom{g+1}{i}$ for all i.

Proof. Since I has a linear syzygy, it is not a complete intersection. In particular, we can write $I = (q_1, \dots, q_{g+1})$ for some linear independent quadrics q_i with $g \geq 1$. Let $U = \operatorname{Syz}^S_1(I)$, $W \subseteq U_4$ denote the k-span of the Koszul syzygies on the q_i, and $\ell \in U$ denote the unique linear syzygy up to scalar multiple. If $W \cap S_+U = 0$, then $\beta^S_{2,4}(R) \geq \binom{g+1}{2}$ so that Proposition 2.8 implies I is a complete intersection, which is a contradiction. Hence, there is a linear form z such that $z\ell \in W$ is nonzero. Write $z\ell = \sum_{1\leq i<j\leq g+1} a_{i,j}(q_je_i - q_ie_j)$ for some $a_{i,j} \in k$, where $e_1, \dots, e_{g+1}$ denotes the standard basis of $S(-2)^{g+1}$. After suitably relabeling the q_i and rescaling the equality, we may assume that $a_{1,2} = 1$. Reading off the first two coordinates of the preceding equality then gives $z\ell_1 = q_2 + \sum_{j=3}^{g+1} a_{1,j}q_j$ and $z\ell_2 = -q_1 + \sum_{j=3}^{g+1} a_{2,j}q_j$. Using these equalities, we can replace q_1 and q_2 as generators of I and assume that $q_1 = xz$ and $q_2 = zw$ for some linear forms x, z, and w. Note that x and w must be independent since the q_i are.

After making this change, we have $\ell = (w, -x, 0, \dots, 0)$ is the unique linear syzygy on the q_i, and $(q_2, -q_1, 0, \dots, 0) = z\ell$. Let $W' \subseteq U_4$ denote the k-span of the Koszul syzygies other than $(q_2, -q_1, 0, \dots, 0)$. If $W' \cap S_+U \neq 0$, then there is a linear form v such that $v\ell \in W'$ is nonzero. Write

$$v\ell = \sum_{\substack{1\leq i<j\leq g+1 \\ j\geq 3}} b_{i,j}(q_je_i - q_ie_j)$$

for some $b_{i,j} \in k$. Since $b_{i,j} \neq 0$ for some $j \geq 3$, reading off the j-th coordinate of the above equality yields a linear dependence relation on the q_i, which is a contradiction. Hence, we must have $W' \cap S_+U = 0$ so that all of the Koszul syzygies except $(q_2, -q_1, 0, \dots, 0)$ are part of a minimal set of generators for U. Because $\beta^S_{2,4}(R) \leq \binom{g+1}{2} - 1$ and $\beta^R_{2,j}(S) = 0$ for $j > 4$ by Lemma 2.7, it follows that $U = \operatorname{Syz}^S_1(I)$ is minimally generated by all the Koszul syzygies on the q_i, except $(q_2, -q_1, 0, \dots, 0)$, together with the linear syzygy ℓ.

If $f_{g+1} \in ((xz, zw, q_3, \dots, q_g) : q_{g+1})$, then we can write $f_{g+1}q_{g+1} = -\sum_{i=1}^{g} f_iq_i$ for some $f_i \in S$ so that $(f_1, \dots, f_{g+1}) \in U$. It follows from the preceding paragraph that $f_{g+1} \in (xz, zw, q_3, \dots, q_g)$ so that q_{g+1} is regular on $R' = S/(xz, zw, q_3, \dots, q_g)$. As $\operatorname{reg}_{R'} R = 1$, it follows from Corollary 2.6 that R' is also Koszul. Moreover, because we can obtain the resolution of R over S by taking the mapping cone of multiplication by q_{g+1} on the resolution of R' over S, it follows that $\beta^S_{2,3}(R') = 1$ and that $(w, -x, 0, \ldots, 0)$ is the unique linear syzygy on $xz, zw, q_3, \ldots, q_g$. Hence, induction on g implies that $q_3, \ldots, q_{g+1}$ is a regular sequence on $S/(xz, zw)$. Conversely, if $I = (xz, zw, q_3, \ldots, q_{g+1})$ for some regular sequence of quadrics $q_3, \ldots, q_{g+1}$ on $S/(xz, zw)$, it also follows from Corollary 2.6 that $R = S/I$ is Koszul since $S/(xz, zw)$ is Koszul.

From the preceding paragraph, we see that $\operatorname{ht} I = \operatorname{ht}(xz, zw) + g - 1 = g$ so that I is an almost complete intersection. If $F_\bullet$ denotes the minimal free resolution of $S/(xz, zw)$ over S, we obtain the minimal resolution of R by repeatedly taking the mapping cone of multiplication by q_{i+1} on the resolution of $S/(xz, zw, q_3, \ldots, q_i)$. Since taking the mapping cone of multiplication by q_{i+1} is the same as tensoring with the Koszul complex on q_{i+1}, we see that $F_\bullet \otimes_S K_\bullet(q_3, \ldots, q_{g+1})$ is the minimal free resolution of R over S, from which the Betti table is easily deduced. In particular, we have

$$\beta^S_i(R) = \frac{g+i}{i}\binom{g-1}{i-1} + \binom{g-1}{i-2} = \binom{g}{i} + \binom{g}{i-1} = \binom{g+1}{i}$$

Similarly, we note that the multiplicity of $S/(xz, zw, q_3, \ldots, q_{i+1})$ is twice the multiplicity of $S/(xz, zw, q_3, \ldots, q_i)$, and so, since $e(S/(xz, zw)) = 1$, we see that $e(R) = 2^{g-1}$. $\square$

Remark 4.2. In the statement of the above theorem, we can choose x and w so that $zw, q_3, \ldots, q_{g+1}$ is a maximal S-regular sequence contained in I. Indeed, since $q_3, \ldots, q_{g+1}$ is a regular sequence on $S/(xz, zw)$, we know that $q_3, \ldots, q_{g+1}$ is a regular sequence on S by Auslander's Zerodivisor Theorem, which is a consequence of the Peskine-Szpiro Intersection Theorem for arbitrary Noetherian local rings and follows from results of Serre in the regular case; see [PS73, II.0]. Each associated prime of $(q_3, \ldots, q_{g+1})$ cannot contain both xz and zw, otherwise we would have $\operatorname{ht} I \leq g - 1$ since the former ideal is unmixed. If either of xz or zw is not contained in every associated prime of $(q_3, \ldots, q_{g+1})$, we are done after possibly switching the roles of x and w. Otherwise, if xz and zw are both contained in different associated primes of $(q_3, \ldots, q_{g+1})$, then we can replace w with $w + x$.

Theorem 4.3. *Let $R = S/I$ be a Koszul almost complete intersection with $\beta^S_{2,3}(R) = 2$. Then there is a 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ such that $I = I_2(M) + (q_4, \ldots, q_{g+1})$ for some regular sequence of quadrics $q_4, \ldots, q_{g+1}$ on $S/I_2(M)$, and conversely, every ideal of this form defines a Koszul almost complete intersection with $\beta^S_{2,3}(R) = 2$.*

Hence, R has multiplicity $e(R) = 3 \cdot 2^{g-2}$ and Betti table

	0	1	2	3	$\cdots$	$g-2$	$g-1$	g
0	1	–	–	–		–	–	–
1	–	$g+1$	2	–		–	–	–
2	–	–	$3\binom{g-2}{1} + \binom{g-2}{2}$	$2\binom{g-2}{1}$		–	–	–
$\vdots$				$\ddots$	$\ddots$			
$g-2$	–	–	–			$3\binom{g-2}{g-3} + 1$	$2\binom{g-2}{g-3}$	–
$g-1$	–	–	–			–	3	2

Specifically, we have $\beta^S_{i,2i}(R) = 3\binom{g-2}{i-1} + \binom{g-2}{i}$ and $\beta^S_{i,2i-1}(R) = 2\binom{g-2}{i-2}$ for $i \geq 2$ so that $\beta^S_i(R) \leq \binom{g+1}{i}$ for all i.

Proof. Since I has a linear syzygy, it is not a complete intersection. In particular, we can write $I = (q_1, \ldots, q_{g+1})$ for some linear independent quadrics q_i with $g \geq 1$. In fact, we must have $g \geq 2$ since it is easily seen that a 2-generated graded ideal cannot have two independent linear syzygies. Let $U = \operatorname{Syz}^S_1(I)$, $W \subseteq U_4$ denote the k-span of the Koszul syzygies on the q_i, and $\ell, h \in U$ denote independent linear syzygies. Arguing as in the proof of the previous theorem, we see there are linear forms z and v such that $z\ell + vh \in W$ is nonzero. Write $z\ell + vh = \sum_{1 \leq i < j \leq g+1} a_{i,j}(q_j e_i - q_i e_j)$ for some $a_{i,j} \in k$, , where $e_1, \ldots, e_{g+1}$ denotes the standard basis of $S(-2)^{g+1}$. After suitably relabeling the q_i and rescaling the equality, we may assume that $a_{1,2} = 1$. Reading off the coordinates of the preceding equality then gives

$$\begin{aligned}
\widetilde{q_2} = z\ell_1 + vh_1 &= q_2 + \sum_{j=3}^{g+1} a_{1,j} q_j \\
-\widetilde{q_1} = z\ell_2 + vh_2 &= -q_1 + \sum_{j=3}^{g+1} a_{2,j} q_j \\
z\ell_p + vh_p &= -\sum_{i<p} a_{i,p} q_i + \sum_{i>p} a_{p,i} q_i \qquad (p \geq 3)
\end{aligned}$$

Using the above equalities, we can replace q_1 and q_2 with $\widetilde{q_1}$ and $\widetilde{q_2}$ as generators for I. As a result, we must also replace ℓ with $\widetilde{\ell} = (\ell_1, \ell_2, \ell_3 - a_{1,3}\ell_2 + a_{2,3}\ell_1, \ldots, \ell_{g+1} - a_{1,g+1}\ell_2 + a_{2,g+1}\ell_1)$ since

$$0 = \sum_{j=1}^{g+1} \ell_j q_j = \ell_1 \widetilde{q_1} + \ell_2 \widetilde{q_2} + \sum_{j=3}^{g+1} (\ell_j - a_{1,j}\ell_2 + a_{2,j}\ell_1) q_j$$

Similarly, h must be replaced with the linear syzygy $\widetilde{h}$ defined as above. It is easily seen that $\widetilde{\ell}$ and $\widetilde{h}$ must also be independent linear syzygies. Finally, setting $b_{i,j} = a_{i,j} + a_{1,j}a_{2,i} - a_{1,i}a_{2,j}$

for $3 \leq i < j \leq g+1$, we claim that

$$z\widetilde{\ell} + v\widetilde{h} = (\widetilde{q_2}, -\widetilde{q_1}, 0, \dots, 0) + \sum_{3 \leq i < j \leq g+1} b_{i,j}(q_j e_i - q_i e_j)$$

By definition of $\widetilde{q_1}$ and $\widetilde{q_2}$, it suffices to check equality in the p-th coordinate for $p \geq 3$. Using the above equalities, we see that

$$\begin{aligned} z\widetilde{\ell}_p + v\widetilde{h}_p &= z\ell_p + vh_p + a_{1,p}\widetilde{q_1} + a_{2,p}\widetilde{q_2} \\ &= -\sum_{3 \leq i < p} a_{i,p}q_i + \sum_{i > p} a_{p,i}q_i - \sum_{i=3}^{g+1} a_{1,p}a_{2,i}q_i + \sum_{i=3}^{g+1} a_{2,p}a_{1,i}q_i \\ &= -\sum_{3 \leq i < p} (a_{i,p} + a_{1,p}a_{2,i} - a_{2,p}a_{1,i})q_i + \sum_{i > p}(a_{p,i} - a_{1,p}a_{2,i} + a_{2,p}a_{1,i})q_i \\ &= -\sum_{3 \leq i < p} b_{i,p}q_i + \sum_{i > p} b_{p,i}q_i \end{aligned}$$

as required. Hence, after replacing q_1 and q_2 as above, we may assume that $q_1 = -(z\ell_2 + vh_2)$, $q_2 = z\ell_1 + vh_1$, and $a_{1,j} = a_{2,j} = 0$ for all $j \geq 3$.

If $a_{i,j} \neq 0$ for some $3 \leq i < j \leq g+1$, then after relabeling the q_i we may assume that $a_{3,4} \neq 0$. Since $a_{1,j} = a_{2,j} = 0$ for all $j \geq 3$, arguing as in the preceding paragraph shows that we can replace q_3 and q_4 with $-(z\ell_4 + vh_4)$ and $z\ell_3 + vh_3$ respectively so that $I \subseteq (z, v, q_5, \dots, q_{g+1})$ has height at most $g-1$ by Krull's Height Theorem, contradicting that I is an almost complete intersection. Therefore, $a_{i,j} = 0$ for $3 \leq i < j \leq g+1$, and we see that $(q_2, -q_1, 0, \dots, 0) = z\ell + vh$ for some linear forms z and v.

Suppose first that z and v are independent linear forms. Then $z\ell_i + vh_i = 0$ for $i > 2$ implies that $(\ell_i, h_i) = a_i(v, -z)$ for some $a_i \in k$ so that $\ell = (\ell_1, \ell_2, a_3v, \dots, a_{g+1}v)$ and $h = (h_1, h_2, -a_3z, \dots, -a_{g+1}z)$. If $a_i = 0$ for all i, then we would have two independent linear syzygies on q_1 and q_2, which we have already noted is impossible above. Hence, after relabeling, we may assume that $a_3 \neq 0$. Replacing q_3 with $a_3q_3 + \cdots + a_{g+1}q_{g+1}$, we may assume that $\ell = (\ell_1, \ell_2, v, 0, \dots, 0)$ and $h = (h_1, h_2, -z, 0, \dots, 0)$. Therefore, we have $q_1 = -(z\ell_2 + vh_2)$ and $q_2 = z\ell_1 + vh_1$, and furthermore, $zq_3 = h_1q_1 + h_2q_2 = z(\ell_1h_2 - \ell_2h_1)$ implies $q_3 = \ell_1h_2 - \ell_2h_1$ so that $I = I_2(M) + (q_4, \dots, q_{g+1})$ where M is the matrix

$$M = \begin{pmatrix} \ell_1 & h_1 \\ \ell_2 & h_2 \\ v & -z \end{pmatrix} \tag{4.1}$$

Suppose now that $v = cz$ for some $c \in k$. Then after replacing ℓ with $\ell + ch$, we may assume that $(q_2, -q_1, 0, \dots, 0) = z\ell$ so that $q_1 = -z\ell_2$, $q_2 = z\ell_1$, and $\ell = (\ell_1, \ell_2, 0, \dots, 0)$. Note that ℓ_1 and ℓ_2 must be independent linear forms or else q_1 and q_2 would not be independent. On the other hand, we know that $\sum_{i=1}^{g+1} h_iq_i = 0$ so that $(\ell_1h_2 - \ell_2h_1)z \in (q_3, \dots, q_{g+1})$. We claim that z is a nonzerodivisor modulo $(q_3, \dots, q_{g+1})$ so that $\ell_1 h_2 - \ell_2 h_1 \in (q_3, \dots, q_{g+1})$.

To see that the claim holds, we first note that $\operatorname{ht}(q_3, \dots, q_{g+1}) = g-1$ so that $(q_3, \dots, q_{g+1})$ is a complete intersection. Indeed, if this is not the case, then since $I \subseteq (z, q_3, \dots, q_{g+1})$ we would have $\operatorname{ht} I \le \operatorname{ht}(q_3, \dots, q_{g+1}) + 1 \le g - 1$ by Krull's Height Theorem and [Ser00, III, Prop. 17], contradicting that I is an almost complete intersection. If z were a zerodivisor modulo $(q_3, \dots, q_{g+1})$, then there would be an associated prime P of $(q_3, \dots, q_{g+1})$ such that $I \subseteq (z, q_3, \dots, q_{g+1}) \subseteq P$ so that $\operatorname{ht} I \le g-1$ as $(q_3, \dots, q_{g+1})$ is unmixed, again contradicting that I is an almost complete intersection. And so, we see that z must be a nonzerodivisor modulo $(q_3, \dots, q_{g+1})$ as claimed. Write $\ell_1 h_2 - \ell_2 h_1 = a_3 q_3 + \cdots + a_{g+1} q_{g+1}$ for some $a_i \in k$. If $a_i = 0$ for all i, then $\ell_1 h_2 - \ell_2 h_1 = 0$ so that $(h_2, -h_1) = b(\ell_2, -\ell_1)$ for some $b \in k$ as ℓ_1 and ℓ_2 are independent linear forms. In that case, we can replace h with $h - b\ell$ and assume that $h = (0, 0, h_3, \dots, h_{g+1})$ so that q_3 is a zerodivisor modulo $(q_4, \dots, q_{g+1})$. However, we claim that this is impossible. Indeed, by arguing as above, we see that $\operatorname{ht}(q_4, \dots, q_{g+1}) = g-2$ so that $(q_4, \dots, q_{g+1})$ is a complete intersection, and so, if q_3 were a zerodivisor modulo $(q_4, \dots, q_{g+1})$, there would be an associated prime P of $(q_4, \dots, q_{g+1})$ such that $(q_3, \dots, q_{g+1}) \subseteq P$ so that $\operatorname{ht}(q_3, \dots, q_{g+1}) \le g-2$ as $(q_4, \dots, q_{g+1})$ is unmixed, contradicting our earlier observation. Hence, after relabeling, we may assume that $a_3 \neq 0$. Replacing q_3 with $a_3 q_3 + \cdots + a_{g+1} q_{g+1} = \ell_1 h_2 - \ell_2 h_1$, we see that $h = (h_1, h_2, -z, 0, \dots, 0)$ and $I = I_2(M) + (q_4, \dots, q_{g+1})$ where M is the matrix of linear forms in (4.1) with $v = 0$.

In both of the above cases, it is easily checked that the Koszul syzygies involving any two of q_1, q_2, q_3 are non-minimal. Let $W' \subseteq U_4$ denote the k-span of the other Koszul syzygies. If $W' \cap S_+ U \neq 0$, then there are linear forms u and w such that $u\ell + wh \in W'$ is nonzero. Write

$$u\ell + wh = \sum_{j=4}^{g+1} [b_{1,j}(q_j e_1 - q_1 e_j) + b_{2,j}(q_j e_2 - q_2 e_j)] + \sum_{3 \le i < j \le g+1} b_{i,j}(q_j e_i - q_i e_j)$$

for some $b_{i,j} \in k$. Since $b_{i,j} \neq 0$ for some $j \ge 4$, reading off the j-th coordinate of the above equality yields a linear dependence relation on the q_i, which is a contradiction. Hence, we must have $W' \cap S_+ U = 0$ so that all of the Koszul syzygies involving at least one of $q_4, \dots, q_{g+1}$ are part of a minimal set of generators for U. By Proposition 3.1 and Lemma 2.7, it follows that $U = \operatorname{Syz}_1^S(I)$ is minimally generated by all the Koszul syzygies involving at least one of $q_4, \dots, q_{g+1}$ together with the linear syzygies ℓ and h.

If $f_{g+1} \in ((q_1, \dots, q_g) : q_{g+1})$, then we can write $f_{g+1} q_{g+1} = -\sum_{i=1}^{g} f_i q_i$ for some $f_i \in S$ so that $(f_1, \dots, f_{g+1}) \in \operatorname{Syz}_1^S(I)$. It follows from the preceding paragraph that $f_{g+1} \in (q_1, \dots, q_g)$ so that q_{g+1} is regular on $R' = S/(q_1, \dots, q_g)$. As $\operatorname{reg}_{R'} R = 1$, it follows from Corollary 2.6 that R' is also Koszul. Moreover, because we can obtain the resolution of R over S by taking the mapping cone of multiplication by q_{g+1} on the resolution of R' over S, it follows that $\beta_{2,3}^S(R') = 2$ and that $(\ell_1, \ell_2, v, 0, \dots, 0)$ and $(h_1, h_2, -z, 0, \dots, 0)$ are the independent linear syzygies on $q_1, \dots, q_g$. Hence, induction on g implies that $q_4, \dots, q_{g+1}$ is a regular sequence on $S/I_2(M)$. In particular, we see that $g = \operatorname{ht} I = \operatorname{ht} I_2(M) + g - 2$ so that $\operatorname{ht} I_2(M) = 2$. Conversely, if $I = I_2(M) + (q_4, \ldots, q_{g+1})$ for some 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ and some regular sequence of quadrics $q_4, \ldots, q_{g+1}$ on $S/I_2(M)$, then $S/I_2(M)$ has a Hilbert-Burch resolution by [BH93, 1.4.17] so that $\operatorname{reg}_S(S/I_2(M)) = 1$, and it follows from Corollary 2.6 that $S/I_2(M)$, and hence also $R = S/I$, is Koszul.

If $F_\bullet$ denotes the minimal free resolution of $S/I_2(M)$ over S, we obtain the minimal resolution of R by repeatedly taking the mapping cone of multiplication by q_{i+1} on the resolution of $S/(q_1, \ldots, q_i)$. Since taking the mapping cone of multiplication by q_{i+1} is the same as tensoring with the Koszul complex on q_{i+1}, we see that $F_\bullet \otimes_S K_\bullet(q_4, \ldots, q_{g+1})$ is the minimal free resolution of R over S, from which the Betti table is easily deduced. In particular, we have

$$\beta_i^S(R) = 2\binom{g-1}{i-1} + \binom{g-1}{i} = \binom{g}{i} + \binom{g-1}{i-1} \leq \binom{g}{i} + \binom{g}{i-1} = \binom{g+1}{i}$$

Similarly, the multiplicity of $S/(q_1, \ldots, q_{i+1})$ is twice the multiplicity of $S/(q_1, \ldots, q_i)$ for $i \geq 3$, and so, since $e(S/I_2(M)) = 3$, we see that $e(R) = 3 \cdot 2^{g-2}$. ☐

In the next section, we will show that every Koszul almost complete intersection has at most two linear syzygies so that the above results give a complete classification of Koszul almost complete intersections, and therefore, Question 1.1 has an affirmative answer for Koszul almost complete intersections with any number of generators. Assuming this result for the time being, we have the following corollary.

Corollary 4.4. *Koszul almost complete intersections are LG-quadratic.*

Proof. Let $R = S/I$ be a Koszul almost complete intersection, and assume first that $\beta_{2,3}^S(R) = 2$ so that $I = I_2(M) + (q_4, \ldots, q_{g+1})$ for some 3×2 matrix $M = (m_{ij})$ of linear forms and $q_4, \ldots, q_{g+1}$ a regular sequence of quadrics on $S/I_2(M)$. Set $\widetilde{S} = S[X][y_4, \ldots, y_{g+1}]$ where $X = (x_{ij})$ is a 3×2 generic matrix, $\widetilde{I} = I_2(X) + (y_4^2 + q_4, \ldots, y_{g+1}^2 + q_{g+1})$, and $A = \widetilde{S}/\widetilde{I}$. If we choose a lexicographic order on $\widetilde{S}$ with $x_{1,2} > x_{1,1} > x_{2,2} > x_{2,1} > x_{3,2} > x_{3,1}$ and y_i greater than the variables in S for all i, it follows from [Eis95, 15.15] that $\operatorname{in}_>(\widetilde{I}) = \operatorname{in}_>(I_2(X)) + (y_4^2, \ldots, y_{g+1}^2)$ so that the 2-minors of X together with the $y_i^2 + q_i$ are a Gröbner basis for $\widetilde{I}$ by [Stu90]. Hence, A is G-quadratic. Moreover, we also know that the $y_i^2 + q_i$ form a regular sequence on $\widetilde{S}/I_2(X)$, and since the latter ring is Cohen-Macaulay, we see that A is Cohen-Macaulay. If J denotes the ideal of A generated by the linear forms $x_{ij} - m_{ij}$ and y_s for $i = 1, 2, 3$, $j = 1, 2$, and $4 \leq s \leq g+1$, then $A/J \cong R$ so that R will be LG-quadratic if the linear forms generating J are a regular sequence. Since A is Cohen-Macaulay and $\widetilde{I}$ is also an almost complete intersection of height g, this follows from the fact that $\operatorname{ht} J = \dim A - \dim R = \dim \widetilde{S} - \dim S = g + 4$ is the number of generators of J.

Assume now that $\beta_{2,3}^S(R) = 1$ so that $I = (xz, zw, q_3, \ldots, q_{g+1})$ for some linear forms x, z, and w and $q_3, \ldots, q_{g+1}$ a regular sequence of quadrics on $S/(xz, zw)$. In this case, we define

$S_i = S[y_{i+1}, \dots, y_{g+1}]$ and $A_i = S_i/I_i$ for $0 \le i \le g+1$, where

$$\begin{aligned}
I_0 &= (y_1 z, y_2 z, y_3^2 + q_3, \dots, y_{g+1}^2 + q_{g+1}) \\
I_1 &= (xz, y_2 z, y_3^2 + q_3, \dots, y_{g+1}^2 + q_{g+1}) \\
I_i &= (xz, zw, q_3, \dots, q_i, y_{i+1}^2 + q_{i+1}, \dots, y_{g+1}^2 + q_{g+1}) \qquad (i \ge 2)
\end{aligned}$$

As above, we see that $\mathrm{in}_>(I_0) = (\mathrm{in}_>(z)y_1, \mathrm{in}_>(z)y_2, y_3^2, \dots, y_{g+1}^2)$ for any monomial order on S_0 in which the y_i are greater than every monomial in S so that the generators of I_0 are a Gröbner basis and A_0 is G-quadratic. In addition, we have $A_{g+1} = R$ and $A_i/(y_{i+1}) \cong A_{i+1}$ for all $i < g+1$. An initial ideal argument as above shows that $y_{i+1}^2 + q_{i+1}, \dots, y_{g+1}^2 + q_{g+1}$ is a regular sequence on $S_i/(xz, zw, q_3, \dots, q_{i+2})$ for $i \ge 2$, and similarly, the $y_j^2 + q_j$ are a regular sequence on $S_0/(y_1 z, y_2 z)$ and $S_1/(xz, y_1 z)$. Consequently, A_i is a Koszul almost complete intersection with $\beta_{2,3}^{S_i}(A_i) = 1$ and $\mathrm{ht}\, I_i = g$ for all i. It then follows from Theorem 4.1 that the A_i have the same Betti table, hence the same h-polynomial $h(t)$, over their respective polynomial rings S_i. Hence, the Hilbert series of A_i is $H_{A_i}(t) = h(t)/(1-t)^{\dim A_i}$. We then compute that $\dim A_i = \dim S_i - g = \dim S_{i+1} - g + 1 = \dim A_{i+1} + 1$ so that $(1-t)H_{A_i}(t) = H_{A_{i+1}}(t)$ for all $i < g+1$. This implies that the natural sequence $0 \to A_i(-1) \xrightarrow{y_{i+1}} A_i \to A_{i+1} \to 0$ is exact so that y_{i+1} is A_i-regular. Therefore, we see that $y_1, \dots, y_{g+1}$ is an A_0-sequence, and R is LG-quadratic. $\square$

5 Linear Syzygies of Quadratic ACI's

The following proposition is similar in spirit to Theorem 4.3. However, there are two important distinctions: We do not assume that R is Koszul, so we lose some information about the syzygies of the defining ideal I, and to make up for this loss of information, we must assume that the ground field is infinite. But first, we make a simple observation which will be useful in the proof.

Remark 5.1. If $f_1, \dots, f_n \in S$ is a regular sequence of homogeneous forms of the same degree and $f = a_1 f_1 + \cdots + a_n f_n$ for some $a_i \in k$ with $a_1 \ne 0$, then $((f_2, \dots, f_n) : f) = ((f_2, \dots, f_n) : f_1)$ so that $f, f_2, \dots, f_n$ is also a regular sequence.

Proposition 5.2. *Suppose that k is an infinite field and that $R = S/I$ is an almost complete intersection defined by quadrics with $\beta_{2,3}^S(R) \ge 2$. Then there are quadrics $q_1, \dots, q_{g+1}$ and a 3×2 matrix of linear forms M such that $I = (q_1, \dots, q_{g+1})$, $q_2, \dots, q_{g+1}$ is a regular sequence, and $I_2(M) = (q_1, q_2, q_3)$.*

Proof. Set $g = \mathrm{ht}\, I$. We note that $g \ge 2$, since otherwise we would have $I = (xz, yz)$ for some linear forms x, y, z so that $\beta_{2,3}^S(R) = 1$. First, we can find quadrics $q_1, \dots, q_{g+1}$ such that $I = (q_1, \dots, q_{g+1})$ and $q_2, \dots, q_{g+1}$ is a regular sequence. Indeed, we can take q_{g+1} to be any quadric in I. Having found quadrics $q_i, \dots, q_{g+1} \in I$ with $i > 2$ forming a regular sequence, we know that I is not contained in any associated prime of $(q_i, \dots, q_{g+1})$ since the latter ideal is unmixed of height $g-i+2<g$. Because I is generated in degree two, this implies $I_2 \nsubseteq P$ for each associated prime P of $(q_i,\dots,q_{g+1})$. Since k is infinite, I_2 is not a union of the proper subspaces $(I\cap P)_2$ for $P\in \operatorname{Ass}(S/(q_i,\dots,q_{g+1}))$. Hence, we can find a quadric $q_{i-1}\in I$ so that $q_{i-1},\dots,q_{g+1}$ is a regular sequence. So by induction we have a regular sequence of quadrics $q_2,\dots,q_{g+1}$ in I, and we can take q_1 to be any other quadric independent from $q_2,\dots,q_{g+1}$ since I is minimally generated by $g+1$ quadrics.

Let ℓ and h be be two independent linear syzygies on the q_i. Then $h_1\ell-\ell_1 h$ is a syzygy on $q_2,\dots,q_{g+1}$ and, therefore, a linear combination of Koszul syzygies. Write

$$h_1\ell-\ell_1h=\sum_{2\le i<j\le g+1} a_{i,j}(q_je_i-q_ie_j)$$

for some $a_{i,j}\in k$, where $e_1,\dots,e_{g+1}$ denotes the standard basis of $S(-2)^{g+1}$. Note that ℓ_1 and h_1 must be independent linear forms, otherwise we could find a nontrivial linear syzygy on $q_2,\dots,q_{g+1}$ since ℓ and h are independent, but that contradicts that $q_2,\dots,q_{g+1}$ is a regular sequence. If $h_1\ell-\ell_1h=0$, then $(-h_i,\ell_i)=b_i(h_1,-\ell_1)$ for some $b_i\in k$ for all $i\ge 2$ so that $h=h_1(1,-b_2,\dots,-b_{g+1})$. But then $(1,-b_2,\dots,-b_{g+1})$ must be a syzygy on the q_i, contradicting that they are independent quadrics. Hence, we see that $h_1\ell-\ell_1h\neq 0$ so that $a_{i,j}\neq 0$ for some i,j. Relabeling $q_2,\dots,q_{g+1}$ if necessary, we may assume that $a_{2,3}\neq 0$. Then by Remark 5.1, we can replace q_3 with $q=h_1\ell_2-\ell_1h_2=a_{2,3}q_3+\cdots+a_{2,g+1}q_{g+1}$. In exchanging q_3 for q, we must replace ℓ with $\widetilde{\ell}=(\ell_1,\ell_2,a_{2,3}^{-1}\ell_3,\ell_4-a_{2,3}^{-1}a_{2,4}\ell_3,\dots,\ell_{g+1}-a_{2,3}^{-1}a_{2,g+1}\ell_3)$ as

$$0=\sum_{i=1}^{g+1}\ell_iq_i=a_{2,3}^{-1}\ell_3q+\ell_1q_1+\ell_2q_2+\sum_{i=4}^{g+1}(\ell_i-a_{2,3}^{-1}a_{2,i}\ell_3)q_i$$

and we also replace h with the syzygy $\widetilde{h}$ defined as above. However, $\widetilde{\ell}$ and $\widetilde{h}$ are still independent linear syzygies since their first coordinates are independent linear forms. After making the above changes, we have $a_{2,3}=1$ and $a_{2,i}=0$ for all $i>3$. Then $h_1\ell_3-\ell_1h_3=-q_2+a_{3,4}q_4+\cdots+a_{3,g+1}q_{g+1}$, and we can we replace q_2 with $-(h_1\ell_3-\ell_1h_3)$ as above. In that case, we have $q_2=-(h_1\ell_3-\ell_1h_3)$ and $q_3=h_1\ell_2-\ell_1h_2$ so that

$$0=\sum_{i=1}^{g+1}\ell_iq_i=\ell_1(q_1+\ell_2h_3-\ell_3h_2)+\sum_{i=1}^{4}\ell_iq_i$$

implies that $q_1+\ell_2h_3-\ell_3h_2\in((q_4,\dots,q_{g+1}):\ell_1)$.

We claim that ℓ_1 is a nonzerodivisor modulo $(q_4,\dots,q_{g+1})$. If not, then ℓ_1 is contained in an associated prime of $(q_4,\dots,q_{g+1})$ so that $\operatorname{ht}(\ell_1,q_4,\dots,q_{g+1})=g-2$. But then $\operatorname{ht}(\ell_1,h_1,q_4,\dots,q_{g+1})\le \operatorname{ht}(\ell_1,q_4,\dots,q_{g+1})+1=g-1$, and since $(q_2,\dots,q_{g+1})\subseteq(\ell_1,h_1,q_4,\dots,q_{g+1})$, this contradicts $\operatorname{ht}(q_2,\dots,q_{g+1})=g$. Therefore, ℓ_1 is a nonzerodivisor modulo $(q_4,\dots,q_{g+1})$ as claimed so that $q_1+\ell_2h_3-\ell_3h_2\in(q_4,\dots,q_{g+1})$. We can then write $\ell_2h_3-\ell_3h_2=-q_1+c_4q_4+\cdots+c_{g+1}q_{g+1}$ for some $c_i\in k$. Replacing q_1 with $\ell_2h_3-\ell_3h_2$ and setting

$$M = \begin{pmatrix} \ell_1 & h_1 \\ \ell_2 & h_2 \\ \ell_3 & h_3 \end{pmatrix} \tag{5.1}$$

yields $I_2(M) = (q_1, q_2, q_3)$ as wanted. $\square$

Theorem 5.3. *If $R = S/I$ is a quadratic almost complete intersection, then $\beta^S_{2,3}(R) \leq 2$.*

Proof. Suppose that $\beta^S_{2,3}(R) \geq 3$. Let K be an infinite extension field of k, and for each k-algebra A, set $A_K = A \otimes_k K$. Then $\beta^{S_K}_{2,3}(R_K) = \beta^S_{2,3}(R)$, $\dim R_K = \dim R$, and $IS_K/I(S_K)_+ \cong I/IS_+ \otimes_S S_K \cong I/IS_+ \otimes_k K$ by faithfully flat base change so that IS_K is still a quadratic almost complete intersection, and replacing R with R_K, we may assume that the ground field k is infinite.

By the preceding proposition, there are quadrics $q_1, \ldots, q_{g+1}$ for $g = \operatorname{ht} I \geq 2$ and a 3×2 matrix of linear forms M as in (5.1) such that $I = (q_1, \ldots, q_{g+1})$, $q_2, \ldots, q_{g+1}$ is a regular sequence, and $I_2(M) = (q_1, q_2, q_3)$. We may assume that q_1, q_2, q_3 are the minors of M as in the proof of the proposition. In that case, $\ell = (\ell_1, \ell_2, \ell_3, 0, \ldots, 0)$ and $h = (h_1, h_2, h_3, 0, \ldots, 0)$ are two independent linear syzygies on the q_i. Let u be a linear syzygy independent from ℓ and h. By arguing as in the proof of the preceding proposition, we see that u_1, ℓ_1, and h_1 are independent linear forms and that $u_1\ell - \ell_1 u \neq 0$.

We claim that $u_1\ell - \ell_1 u$ is linear independent from $h_1\ell - \ell_1 h$. If not, then $u_1\ell - \ell_1 u = c(h_1\ell - \ell_1 h)$ for some nonzero $c \in k$. Setting $\widetilde{u} = u - ch$ and rearranging the preceding equality, we have that $\widetilde{u}$ is a linear syzygy independent from ℓ and h with $\widetilde{u}_1\ell - \ell_1\widetilde{u} = 0$, which is impossible as already noted in the previous paragraph. Hence, $u_1\ell - \ell_1 u$ is independent from $h_1\ell - \ell_1 h$ as claimed. In particular, we note that $g \geq 3$ since there cannot be two independent quadric syzygies on the regular sequence q_2, q_3. Write

$$u_1\ell - \ell_1 u = \sum_{2 \leq i < j \leq g+1} b_{i,j}(q_j e_i - q_i e_j)$$

for some $b_{i,j} \in k$. Since $h_1\ell - \ell_1 h = (0, q_3, -q_2, 0, \ldots, 0)$ by assumption, we must have $b_{i,j} \neq 0$ for some $(i, j) \neq (2, 3)$, otherwise we would have a contradiction to the claim. In fact, by replacing u with $u - b_{2,3}h$, we may assume that $b_{2,3} = 0$.

Next, we claim that there is a $j \geq 4$ such that $b_{2,j} \neq 0$ or $b_{3,j} \neq 0$. If not, the first three coordinates of $u_1\ell - \ell_1 u$ must be zero so that $u_1\ell - \ell_1 u = (0, 0, 0, -\ell_1 u_4, \ldots, -\ell_1 u_{g+1})$. But this implies that $(u_4, \ldots, u_{g+1})$ is a linear syzygy on $q_4, \ldots, q_{g+1}$, which is impossible since $q_4, \ldots, q_{g+1}$ is a regular sequence. Hence, relabeling if necessary, we may assume that $b_{2,4} \neq 0$ so that we can replace q_4 with $u_1\ell_2 - \ell_1 u_2 = b_{2,4}q_4 + \cdots + b_{2,g+1}q_{g+1}$.

Finally, we claim that there is a $j \geq 5$ such that $b_{3,j} \neq 0$. If not, then $u_1\ell_3 - \ell_1 u_3 = b_{3,4}q_4 = b_{3,4}(u_1\ell_2 - \ell_1 u_2)$ as $b_{2,3} = 0$. But this implies that $(b_{3,4}u_2 - u_3, \ell_3 - b_{3,4}\ell_2) = r(u_1, -\ell_1)$ for some $r \in k$ as ℓ_1 and u_1 are independent linear forms. In particular, we see that $\ell_3 \in \operatorname{span}\{\ell_1, \ell_2\}$ so that $I \subseteq (\ell_1, \ell_2, q_5, \ldots, q_{g+1})$. However, $\operatorname{ht}(\ell_1, \ell_2, q_5, \ldots, q_{g+1}) \leq g - 1$, which contradicts $\operatorname{ht} I = g$. Hence, we must have $g \geq 4$, and after relabeling, we may assume that $b_{3,5} \neq 0$ and replace q_5 with $u_1\ell_3 - \ell_1 u_3 = b_{3,4}q_4 + b_{3,5}q_5 + \cdots + b_{3,g+1}q_{g+1}$. But then $I \subseteq (\ell_1, \ell_2, \ell_3, q_6, \ldots, q_{g+1})$ and $\operatorname{ht}(\ell_1, \ell_2, \ell_3, q_6, \ldots, q_{g+1}) \leq g-1$, contradicting that $\operatorname{ht} I = g$. Therefore, we must have $\beta^S_{2,3}(R) \leq 2$. $\square$

6 Future Directions

Using our main result, we have reason to believe that Question 1.1 can be answered affirmatively for Koszul algebras defined by $g = 4$ quadrics. However, Example 2.4 points to unexpected difficulties when $g = 5$.

Our work also has unintended connections to various other conjectures of interest. For example, the Buchsbaum-Eisenbud-Horrocks Conjecture asks whether $\beta^S_i(R) \geq \binom{c}{i}$ for all i if $c = \operatorname{ht} I$. This conjecture is already known to hold for quotients by monomial ideals, and Dugger [Dug00, 2.3] established a larger lower bound $\beta^S_i(R) \geq \binom{c}{i} + \binom{c-1}{i-1}$ for almost complete intersections directly linked to a complete intersection. Among Koszul almost complete intersections $R = S/I$, those with one linear syzygy cannot be directly linked to a complete intersection since they are not Cohen-Macaulay. Nonetheless, they still satisfy this larger bound. In addition, at least when the ground field is infinite, the results of the previous section show that I is directly linked to the complete intersection $(\ell_1, h_1, q_4, \ldots, q_{g+1})$, and Theorem 4.3 yields that Dugger's bound is sharp. Boocher and Seiner have recently established a total rank version of Dugger's bound $\sum_i \beta_i(R) \geq 2^c + 2^{c-1}$ for quotients by monomial ideals which are not complete intersections, and our work affirmatively answers a couple questions they pose [BS17, 1.2, 1.3] in the Koszul ACI case.

Another consequence of our structure theorem is that the EGH Conjecture holds for Koszul almost complete intersections in a strong form similar to [CCV14, 2.1]. This conjecture asks whether, given a graded ideal $I \subseteq S = k[x_1, \ldots, x_n]$ containing a homogeneous regular sequence $f_1, \ldots, f_r$ of degrees $2 \leq d_1 \leq \cdots \leq d_r$, there is a monomial ideal J containing $x_1^{d_1}, \ldots, x_r^{d_r}$ and having the same Hilbert function as I. The EGH Conjecture is known to hold when I is a quadratic monomial ideal by the preceding paper, when I is a complete intersection of quadrics, or when I is generated by products of linear forms [Abe15], which covers quadratic ideals of height one. To this list, we add that, if I defines a Koszul almost complete intersection, we can simply take $J = (x_1^2, \ldots, x_g^2, x_g x_{g+1})$ or $J = (x_1^2, \ldots, x_g^2, x_{g-1}x_g)$ according to whether I has one or two linear syzygies respectively. Note for the former case that $g + 1 = \operatorname{pd}_S R \leq n$ by Hilbert's Syzygy Theorem.

Given the previous and new evidence for the above conjectures in the Koszul case, it is natural to ask whether they hold at the very least for Koszul algebras in general.

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