WS 13 KEY

1. $y'' + 4y = e^t - \sin 2t$

$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$y_c = C_1 \cos 2t + C_2 \sin 2t$

$y_p = Ae^t + Bt\sin 2t + Ct\cos 2t$

$y_p' = Ae^t + B\sin 2t + 2Bt\cos 2t + C\cos 2t - 2Ct\sin 2t$

$y_p'' = Ae^t + 2B\cos 2t + 2B\cos 2t - 4Bt\sin 2t - 2C\sin 2t - 2C\sin 2t - 4Ct\cos 2t$

$$\begin{aligned} e^t - \sin 2t &= y_p'' + 4y_p \\ &= Ae^t + 4B\cos 2t - 4Bt\sin 2t - 4C\sin 2t - 4Ct\cos 2t + 4(Ae^t + Bt\sin 2t + Ct\cos 2t) \\ &= 5Ae^t + 4B\cos 2t - 4C\sin 2t \end{aligned}$$

$\Rightarrow 5A = 1, \quad 4B = 0, \quad -4C = -1$ (since e^t, $\cos 2t$, $\sin 2t$ are linearly independent)

$\Rightarrow A = \frac{1}{5}, \quad B = 0, \quad C = \frac{1}{4}$

$y = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{5}e^t + \frac{1}{4}t\cos 2t$

2. (a) $$Pz_1 = \begin{pmatrix} t-4 & 4t+6 \\ -2t-3 & 7t+5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t^2/2 + 2t}$$

$$= \begin{pmatrix} 5t+2 \\ 5t+2 \end{pmatrix} e^{5t^2/2 + 2t} = \frac{dz_1}{dt}$$

$$Pz_2 = \begin{pmatrix} t-4 & 4t+6 \\ -2t-3 & 7t+5 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{3t^2/2 - t}$$

$$= \begin{pmatrix} -6t+2 \\ -3t+1 \end{pmatrix} e^{3t^2/2 - t} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} (3t-1) e^{3t^2/2 - t}$$

$$= \frac{dz_2}{dt}$$

(b) $$z(t) = c_1 z_1(t) + c_2 z_2(t)$$

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = z(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}}_{Q} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$Q^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$z(t) = \begin{pmatrix} e^{5t^2/2 - 2t} \\ e^{5t^2/2 - 2t} \end{pmatrix} - 2 \begin{pmatrix} -2e^{3t^2/2 - t} \\ -e^{3t^2/2 - t} \end{pmatrix}$$

3. $$0 = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 2 & 4-\lambda & -2 \\ 3 & 3 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 3 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 4-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 3\lambda + 2) - 2\lambda + 4 + 3\lambda - 6$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2 - 2\lambda + 4 + 3\lambda - 6$$

$$= -\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(\lambda^2 - 4\lambda + 4) = -\lambda(\lambda-2)^2$$

$$\Rightarrow \lambda = 0, 2$$

$\lambda = 0$: $$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ 3 & 3 & -1 \end{pmatrix} \xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 6 & -4 \\ 0 & 6 & -4 \end{pmatrix} \xrightarrow[R_3 - R_2 \to R_3]{\frac{1}{6}R_2 \to R_2}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{aligned} v_1 + v_2 + v_3 &= 0 \\ v_2 - \tfrac{2}{3}v_2 &= 0 \end{aligned} \Rightarrow \begin{aligned} v_1 &= -v_2 - v_3 = \tfrac{-1}{3}v_3 \\ v_2 &= \tfrac{2}{3}v_3 \end{aligned}$$

$$\Rightarrow v = v_3\left(-\tfrac{1}{3}, \tfrac{2}{3}, 1\right)$$

$\lambda = 2$: $$\begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{pmatrix} \xrightarrow[R_3 + 3R_1 \to R_3]{\substack{-R_1 \to R_1 \\ R_2 + 2R_1 \to R_2}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow w_1 + w_2 - w_3 = 0 \Rightarrow w_1 = -w_2 + w_3$$

$$\Rightarrow w = w_2(-1, 1, 0) + w_3(1, 0, 1)$$

linearly independent eigenvectors

$$Y(t) = C_1 \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$$

4. (a) $$0 = \det(B - \lambda I_2) = \begin{vmatrix} 5-\lambda & 9 \\ -4 & -7-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda - 35 + 36 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$$

$$\Rightarrow \lambda = -1$$

$$B + I = \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} \xrightarrow[R_2 + \frac{2}{3}R_1 \to R_2]{\frac{1}{6}R_1 \to R_1} \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = -\frac{3}{2} v_2$$

$$\Rightarrow v = \begin{pmatrix} -\frac{3}{2} v_2 \\ v_2 \end{pmatrix}$$

$(B+I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector

$$u = (B + I) w = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 6 \\ 4 \end{pmatrix} e^{-t} + C_2 \left(t \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) e^{-t}$$

$$= \begin{pmatrix} 6C_1 e^{-t} + C_2(6t+1)e^{-t} \\ 4C_1 e^{-t} + C_2(4te^{-t}) \end{pmatrix} = \underbrace{\begin{pmatrix} 6e^{-t} & (6t+1)e^{-t} \\ 4e^{-t} & 4te^{-t} \end{pmatrix}}_{\Phi(t)} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

(b) $$\Phi(0) = \begin{pmatrix} 6 & 1 \\ 4 & 0 \end{pmatrix} \qquad \Phi(0)^{-1} = \frac{1}{-4} \begin{pmatrix} 0 & -1 \\ -4 & 6 \end{pmatrix}$$

$$e^{Bt} = \Phi(t)\,\Phi(0)^{-1} = -\frac{1}{4}\begin{pmatrix} 6e^{-t} & (6t+1)e^{-t} \\ 4e^{-t} & 4te^{-t} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} (6t+1)e^{-t} & -9te^{-t} \\ 4te^{-t} & (-6t+1)e^{-t} \end{pmatrix}$$

5. (a) $\mathcal{L}\{f(t)\} = 10 \cdot \dfrac{s+3}{(s+3)^2+4} + e^{-s}\dfrac{4s}{(s^2+4)^2}$

(b) $F(s) = \dfrac{-(55s+390)e^{-2s}}{s^2(s^2+4s+13)}$

$$-\frac{55s+390}{s^2(s^2+4s+13)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+4s+13}$$

$$-(55s+390) = A(s^2+4s+13) + Bs(s^2+4s+13) + (Cs+D)s^2$$

$$= (B+C)s^3 + (A+4B+D)s^2 + (4A+13B)s + 13A$$

$$\Rightarrow B+C=0, \quad A+4B+D=0, \quad 4A+13B=-55, \quad 13A=-390$$

$$\Rightarrow A=-30, \quad B=\frac{65}{13}=5, \quad C=-B=-5, \quad D=10$$

$$\mathcal{L}^{-1}\{F(s)\} = u(t-2)\,\mathcal{L}^{-1}\left\{\frac{55s+390}{s^2(s^2+4s+13)}\right\}\Bigg|_{t-2}$$

$$= u(t-2)\left[-30t+5+\mathcal{L}^{-1}\left\{\frac{-5s+10}{s^2+4s+13}\right\}\right]\Bigg|_{t-2}$$

$$= u(t-2)\left[-30t+5+\mathcal{L}^{-1}\left\{\frac{-5(s+2)+20}{(s+2)^2+9}\right\}\right]\Bigg|_{t-2}$$

$$= u(t-2)\Big[-30(t-2)+5-5e^{-2(t-2)}\cos 3(t-2)$$
$$+\frac{20}{3}e^{-2(t-2)}\sin 3(t-2)\Big]$$

6. $y'' - \frac{1}{2}y' = u(t-1)$

$$s^2Y - 2s \cancel{+1} - \frac{1}{2}sY \cancel{+1} = \frac{e^{-s}}{s}$$

$$s(s-\tfrac{1}{2})Y = \frac{e^{-s}}{s} + s$$

$$Y = \frac{e^{-s}}{s^2(s-\frac{1}{2})} + \frac{1}{s-\frac{1}{2}}$$

$$\frac{1}{s^2(s-\frac{1}{2})} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-\frac{1}{2}} \Rightarrow 1 = A(s-\tfrac{1}{2}) + Bs(s-\tfrac{1}{2}) + Cs^2$$

when $s=\frac{1}{2}$: $1 = \frac{1}{4}C \Rightarrow C = 4$

when $s=0$: $1 = -\frac{1}{2}A \Rightarrow A = -2$

when $s=1$: $1 = -1 + \frac{1}{2}B + 4 \Rightarrow B = -4$

$$y = \mathcal{L}^{-1}\{Y\} = u(t-1)\,\mathcal{L}\left\{\frac{1}{s^2(s-\frac{1}{2})}\right\}\Big|_{t-1} + e^{t/2}$$

$$= u(t-1)\left[-2t-4+4e^{t/2}\right]\Big|_{t-1} + e^{t/2}$$

$$= u(t-1)\left[-2(t-1)-4+4e^{(t-1)/2}\right] + e^{t/2}$$