NON-KOSZUL QUADRATIC GORENSTEIN RINGS VIA IDEALIZATION

MATTHEW MASTROENI, HAL SCHENCK, AND MIKE STILLMAN

Abstract. Let R be a standard graded Gorenstein algebra over a field presented by quadrics. In [CRV01], Conca-Rossi-Valla show that such a ring is Koszul if $\operatorname{reg} R \le 2$ or if $\operatorname{reg} R = 3$ and $\operatorname{codim} R \le 4$, and they ask whether this is true for $\operatorname{reg} R = 3$ in general. We give a negative answer to their question by finding suitable conditions on a non-Koszul quadratic Cohen-Macaulay ring R that guarantee the Nagata idealization $\widetilde{R} = R \ltimes \omega_R(-a-1)$ is a non-Koszul quadratic Gorenstein ring.

1. Introduction

Let I be a homogeneous ideal in a standard graded polynomial ring S over a field k, and set $R = S/I$. In this paper, we study the relationship between two conditions that impose extraordinary constraints on the homological properties of R, namely the Gorenstein and Koszul properties. The ring R is *Gorenstein* if it is Cohen-Macaulay and its canonical module is isomorphic to a shift of R:

$$\omega_R = \operatorname{Ext}_S^c(R, S)(-n) \cong R(a)$$

where $\operatorname{ht} I = c$ and $\dim S = n$. This implies that the *graded Betti numbers*

$$\beta_{i,j}^S(R) = \dim_k \operatorname{Tor}_i^S(R, k)_j$$

have a symmetry

$$\beta_{i,j}^S(R) = \beta_{c-i,a+n-j}^S(R) \tag{1.1}$$

for all i, j. On the other hand, R is *Koszul* if the ground field $R/R_+ \cong k$ has a linear free resolution over R. That is, we have $\beta_{i,j}^R(k) = 0$ for all i and j with $j \neq i$. Koszul algebras have strong duality properties and appear as many rings of interest in commutative algebra, topology, and algebraic geometry; see the surveys [Frö99] and [Con14] and the references therein.

In particular, if R is the homogeneous coordinate ring of a general curve C of genus $g \ge 5$ in its canonical embedding, Vishik and Finkelberg prove that R is

---

2000 *Mathematics Subject Classification.* Primary 13D02; Secondary 14H45, 14H50.

Key words and phrases. Syzygy, Koszul algebra, Gorenstein algebra.

Schenck supported by NSF 1818646.

Stillman supported by NSF 1502294.

Koszul in [VF93]. Building on this, Polishchuk shows that R is Koszul if C is not a plane quintic, hyperelliptic, or trigonal in [Pol95]. Such rings are also quadratic and Gorenstein by [Eis05, 9.5], so a natural question is whether quadratic Gorenstein rings are always Koszul. In [Mat17], Matsuda shows that this is not the case by constructing a quadratic Gorenstein toric ring which is not Koszul. Nonetheless, there is some evidence that quadratic Gorenstein rings are Koszul, at least when R is a complete intersection or has small Castelnuovo-Mumford regularity (see §2 for a review of this concept):

- Every quadratic complete intersection is Koszul. This was first proved by Tate in [Tat57]; see [Con14, 1.19] for an easier argument due to Caviglia.
- If $\operatorname{reg} R = 2$, then R is Koszul by [CRV01, 2.12].
- If $\operatorname{reg} R = 3$ and $\operatorname{codim} R = \operatorname{ht} I \leq 5$, then R is Koszul. This follows from [CRV01, 6.15] and more recently by [EK17] when $\operatorname{codim} R = 4$ and by [Cav00] when $\operatorname{codim} R = 5$.
- If $\operatorname{reg} R = 3$, and $\dim R = 2$, then R is the canonical ring of a curve by [Eis05, 9C.2] so that R is Koszul by [VF93] and [Pol95].

Note that the symmetry (1.1) of the free resolution of a quadratic Gorenstein ring forces $\operatorname{reg} R \geq 2$ unless R is a hypersurface and that R is also Koszul in that case by the first bullet above. These results led Conca, Rossi, and Valla to pose the following question.

Question 1.1 ([CRV01, 6.13])**.** If R is a quadratic Gorenstein ring with $\operatorname{reg} R = 3$, is R Koszul?

More generally, one might ask:

Question 1.2. For which positive integers c and r is every quadratic Gorenstein ring R with $\operatorname{codim} R = c$ and $\operatorname{reg} R = r$ Koszul?

Matsuda's example in [Mat17] does not address the Conca-Rossi-Valla question since the toric ring he constructs has regularity four. We give a negative answer (Example 4.2) to Question 1.1 with codimension nine and a partial answer to Question 1.2. In fact, our main result (Theorem 3.5) provides a machine for producing lots of examples of non-Koszul quadratic Gorenstein rings by deducing conditions on a quadratic Cohen-Macaulay ring such that the idealization $\widetilde{R} = R \ltimes \omega_R(-a-1)$ is a non-Koszul quadratic Gorenstein ring.

After introducing the necessary background on Cohen-Macaulay rings in §2, we prove our main result in §3 and apply it in §4 to give many examples of non-Koszul quadratic Gorenstein rings. As a consequence, we prove the existence of non-Koszul quadratic Gorenstein rings R with $\operatorname{reg} R = 3$ and $\operatorname{codim} R = c$ for all $c \geq 9$ in characteristic zero, which is the setting originally considered in [CRV01].

Notation 1.3. Throughout the remainder of the paper, we use the following notation unless specifically stated otherwise. Let k be a fixed ground field of any characteristic, S be a standard graded polynomial ring over k, and $I \subseteq S$ be a graded ideal such that $R = S/I$ is Cohen-Macaulay. We set $\omega = \omega_R(-a-1)$, where $a = a(R)$ is the a-invariant of R, and $\widetilde{R} = R \ltimes \omega$ denotes the idealization of ω. Recall that the ideal I is called *nondegenerate* if it does not contain any linear forms. We can always reduce to a presentation for R with I nondegenerate by killing a basis for the linear forms contained in I, and we will assume that this is the case throughout. We denote the irrelevant ideal of R by $R_+ = \bigoplus_{n\geq 1} R_n$.

2. Background on Cohen-Macaulay rings

In this section, we briefly recall some invariants associated to standard graded algebras and discuss how they specifically relate to Cohen-Macaulay rings. We refer the reader to [BH93] and [BS13] for further details and any unexplained terminology.

Let $R = S/I$ be a standard graded algebra of dimension d. An important invariant of R is its (Castelnuovo-Mumford) *regularity*

$$\begin{aligned}\operatorname{reg} R &= \max\{j \mid H^i_{R_+}(R)_{j-i} \neq 0 \text{ for some } i\}\\ &= \max\{j \mid \beta^S_{i,i+j}(R) \neq 0 \text{ for some } i\}\end{aligned}$$

where $H^i_{R_+}(R)$ denotes the i-th local cohomology module of R with respect to its irrelevant ideal. Recall that the injective hull of k over R is the injective R-module $E = E_R(k) = {}^*\operatorname{Hom}_k(R,k)$. This is a graded R-module where ${}^*\operatorname{Hom}_k(R,k)_j \cong \operatorname{Hom}_k(R_{-j},k)$ is the set of k-linear maps $R \to k$ of degree j. The *canonical module* of R is the Matlis dual of its top local cohomology module

$$\omega_R = {}^*\operatorname{Hom}_R(H^d_{R_+}(R), E) \cong {}^*\operatorname{Hom}_k(H^d_{R_+}(R), k)$$

A closely related quantity is the *a-invariant* of R, which is defined by

$$a(R) = \max\{j \mid H^d_{R_+}(R)_j \neq 0\} = -\min\{j \mid (\omega_R)_j \neq 0\}$$

so that ω_R is generated in degrees at least $-a(R)$. As a consequence, we have an inequality $a(R) + \dim R \leq \operatorname{reg} R$. When R is Cohen-Macaulay, it is well known that $H^i_{R_+}(R) = 0$ for all $i \neq d$ so that the preceding becomes an equality. Moreover, we say that a Cohen-Macaulay ring R is *level* if ω_R is generated in a single degree and *Gorenstein* if ω_R is cyclic.

When R is Cohen-Macaulay, the minimal number of generators of ω_R is called the *type* of R and denoted by $\operatorname{type} R$. By Grothendieck-Serre duality, we also have $\omega_R \cong \operatorname{Ext}^c_S(R, \omega_S) \cong \operatorname{Ext}^c_S(R, S)(-n)$, where $c = \operatorname{ht} I$ and $\dim S = n$ so that applying $\operatorname{Hom}_S(-, S)$ to the minimal free resolution of R over S yields the minimal free resolution of ω_R up to shifts, and we can therefore read the type of R as the rank of last free module in the minimal free resolution of R.

We will be particularly interested in the case of Artinian rings for examples, so we elaborate on how the above definitions translate to that case. Recall that the *socle* of R is the ideal $(0 : R_+)$. When R is Artinian, it has finitely many nonzero graded components, and the degree r of the last nonzero component is called the *socle degree* of R as $R_r \subseteq (0 : R_+)$. In this case, we have $\omega_R = {}^*\operatorname{Hom}_k(R, k)$ so that

$$\operatorname{reg} R = a(R) = \max\{j \mid R_j \neq 0\}$$

is the socle degree of R. We will therefore use the terminology of regularity, a-invariant, and socle degree interchangeably in this case. Furthermore, R is level if and only if $(0 : R_+) = R_r$ and Gorenstein if and only if $(0 : R_+)$ is one-dimensional as a k-vector space.

Under appropriate conditions, all of these invariants can also be read off from the so-called *h-polynomial* $h_R(t)$ of R, which is the unique integer polynomial such that the Hilbert series of R can be expressed as a rational function $H_R(t) = h_R(t)/(1-t)^d$. The *h-vector* of R is just the vector $h(R) = (h_0, h_1, \ldots, h_r)$ of coefficients of the h-polynomial, where $h_R(t) = \sum_{i=0}^r h_i t^i$. If R is Cohen-Macaulay, then, after extending to an infinite ground field if necessary, we can reduce to the Artinian case by killing a maximal regular sequence of linear forms. Since this does not affect the Betti numbers or Hilbert series of R, we see that the length r of the h-vector is none other than regularity of R, $h_1 = \operatorname{codim} R$, and if R is level, then $h_r = \operatorname{type} R$.

We close this section with an observation from [MN13, 2.3] which is relevant to Question 1.2. For the reader's convenience, we fill in the details omitted there.

Proposition 2.1. *Suppose that $R = S/I$ is a quadratic Cohen-Macaulay ring. Then $\operatorname{reg} R \leq \operatorname{pd}_S R$, and equality holds if and only if R is a complete intersection.*

Proof. By flat base change, we may assume that the ground field k is infinite. In order prove that I is a complete intersection, we must show that $\beta^S_1(R) = \operatorname{ht} I = \operatorname{reg} R$. It suffices to check that R is a complete intersection after replacing R and S with their quotients by a maximal regular sequence of linear forms on R and S, since this will not affect the Betti numbers of R. Hence, we may assume that R is Artinian. In that case, $r = \operatorname{reg} R$ is the socle degree of R. Since we can find a quadratic complete intersection ideal $L \subseteq I$ with $\operatorname{ht} L = \operatorname{ht} I$, we have an exact sequence $0 \to I/L \to S/L \to R \to 0$. Consequently, the socle degree of R is at most the socle degree of S/L, which is $\operatorname{ht} L = \operatorname{ht} I = \operatorname{pd}_S R$ since L is generated by quadrics. If equality holds, then S/L has a 1-dimensional socle in degree r so that $(I/L)_r = 0$, and it follows that $I = L$ as every nonzero ideal of S/L must contain the socle. Thus, $R = S/L$ is a complete intersection as wanted. □

3. Non-Koszul quadratic idealizations

We now come to the central construction of this paper. Given a ring R and an R-module M, the *idealization* of M over R is the R-algebra $R \ltimes M$ whose underlying R-module is $R \oplus M$ with multiplication defined by

$$(a, x) \cdot (b, y) = (ab, ay + bx)$$

for all $a, b \in R$ and $x, y \in M$. In particular, by identifying $a \in R$ and $x \in M$ with $(a, 0)$ and $(0, x)$ in $R \ltimes M$, we view R as a subring of $R \ltimes M$, and the ideal generated by M in $R \ltimes M$ has square zero.

Remark 3.1. When R is a standard graded algebra and M is a graded R-module, the idealization has a natural $\mathbb{Z}$-grading given by $(R \ltimes M)_j = R_j \oplus M_j$ for each j. With this grading, it is clear that the idealization is standard graded if and only if M is generated in degree one. Since $\omega = \omega_R(-a-1)$ is always nonzero in degree one for $a = a(R)$, we see that the idealization $\widetilde{R} = R \ltimes \omega$ of a Cohen-Macaulay ring R is standard graded if and only if R is level.

The usefulness of idealization for our purposes is that it gives a canonical way of producing Gorenstein rings from Cohen-Macaulay rings. The following well known result, adapted here to the standard graded setting, was discovered independently by at least Foxby, Gulliksen, and Reiten; see [Rei72, 7] and the lemma preceding Theorem 3 in [Gul72].

Proposition 3.2. *If R is a standard graded level k-algebra, then $\widetilde{R} = R \ltimes \omega$ is a Gorenstein standard graded ring.*

Properties of the level algebra R often carry over to its idealization. To guarantee that $\widetilde{R}$ is still quadratic, we need to impose a slightly stronger condition on R than merely being level. We say that a standard graded algebra R is *superlevel* if it is level and ω_R has a linear presentation over R. That is, there is an exact sequence $R(a-1)^s \xrightarrow{\bar{\varphi}} R(a)^t \to \omega_R \to 0$. In particular, every Gorenstein ring is superlevel since $\omega_R \cong R(a)$ in that case. If $F_1 \xrightarrow{\varphi} F_0 \to \omega_R \to 0$ is a minimal presentation for ω_R over S, then $\bar{\varphi} = \varphi \otimes \mathrm{Id}_R$ gives a presentation for ω_R over R, which is minimal up to summands of $F_1 \otimes_S R$ that map to zero. Hence, R is superlevel if and only if the entries of the matrix of φ of degree at least two are all contained in I. However, for examples, it will suffice to find rings such that ω_R has a linear presentation over S.

Lemma 3.3. *Let $R = S/I$ be a quadratic level algebra. Then $\widetilde{R} = R \ltimes \omega$ is a quadratic if and only if R is superlevel.*

Proof. There is an obvious R-algebra isomorphism $\mathrm{Sym}_R(\omega)/(\omega)^2 \cong \widetilde{R}$. We also have $\mathrm{Sym}_R(\omega) \cong \mathrm{Sym}_S(\omega)/I\,\mathrm{Sym}_S(\omega)$, and if ω is minimally generated by t elements, then $\mathrm{Sym}_S(\omega) \cong S[y_1, \ldots, y_t]/\mathcal{L}$, where $\mathcal{L} = (\sum_{i=1}^t f_i y_t \mid (f_1, \ldots, f_t) \in \mathrm{Syz}_1^S(\omega))$. Assembling all of these facts together, we see that

$$\widetilde{R} \cong S[y_1, \ldots, y_t]/((y_1, \ldots, y_t)^2 + \mathcal{L} + (I)) \tag{3.1}$$

Moreover, since $\widetilde{R}_j = R_j \oplus \omega_j$, this isomorphism is graded if we grade $S[y_1, \ldots, y_t]$ by $(S[y_1, \ldots, y_t])_j = \bigoplus_{i \le j} \bigoplus_{|\alpha|=i} S_{j-i} y^\alpha$, where $y^\alpha = y_1^{\alpha_1} \cdots y_t^{\alpha_t}$ and $|\alpha| = \sum_i \alpha_i$ for all $\alpha \in \mathbb{N}^t$. Since I is generated by quadrics, it follows from the above presentation that $\widetilde{R}$ is quadratic if and only if the minimal first syzygies of ω whose entries are not contained in I are linear. $\square$

In order to show that non-Koszulness can be passed from R to its idealization $\widetilde{R} = R \ltimes \omega$, we make use of a technical result of Gulliksen computing the graded Poincaré series of $\widetilde{R}$ in terms of those of R and ω. The *graded Poincaré series* of a finitely generated graded R-module M is the formal power series

$$\mathrm{P}_R^M(s,t) = \sum_{i,j} \beta_{i,j}^R(M) s^j t^i \in \mathbb{Z}[s, s^{-1}][[t]]$$

When $M = k$, we omit the superscript from the notation and refer to $\mathrm{P}_R(s,t)$ as the graded Poincaré series of R. Note that R is Koszul if and only if $\mathrm{P}_R \in \mathbb{Z}[[st]]$.

Theorem 3.4 ([Gul72, Thm 2])**.** *If R is a standard graded k-algebra and M is a finitely generated graded R-module generated in degree one, then the graded Poincaré series of $\widetilde{R} = R \ltimes M$ is*

$$\mathrm{P}_{\widetilde{R}}(s,t) = \frac{\mathrm{P}_R(s,t)}{1 - t\,\mathrm{P}_R^M(s,t)}$$

Combining this result with the above observations, we have the following.

Theorem 3.5. *If R is is a non-Koszul, quadratic superlevel algebra, then $\widetilde{R} = R \ltimes \omega$ is a non-Koszul quadratic Gorenstein ring. Moreover, we have*

$$\operatorname{codim} \widetilde{R} = \operatorname{codim} R + \operatorname{type} R$$
$$\operatorname{reg} \widetilde{R} = \operatorname{reg} R + 1$$

Proof. By Lemma 3.3, it suffices to prove that $\widetilde{R}$ is not Koszul. Write

$$\mathrm{P}_{\widetilde{R}}(s,t) = \sum_i f_i(s)t^i \qquad \mathrm{P}^{\omega}_R(s,t) = \sum_i g_i(s)t^i \qquad \mathrm{P}_R(s,t) = \sum_i h_i(s)t^i$$

for some $f_i, g_i, h_i \in \mathbb{Z}[s]$ with non-negative coefficients. By the above theorem, we know that $\mathrm{P}_{\widetilde{R}}(s,t)(1 - t\,\mathrm{P}^{\omega}_R(s,t)) = \mathrm{P}_R(s,t)$ so that

$$h_i = f_i - \sum_{j=1}^{i} f_{i-j}g_{j-1}$$

for all $i \geq 1$. Since all of the polynomials in the above expression have non-negative coefficients, any monomial in the support of h_i must also belong to the support of f_i. Since R is not Koszul, there is an $i \geq 1$ such that h_i has at least two monomials in its support, hence so does f_i, and $\widetilde{R}$ is not Koszul. The statements about the codimension and regularity of $\widetilde{R}$ are immediate by considering the h-vector, which coincides with the Hilbert function of an Artinian reduction after extending to an infinite ground field if necessary and killing a maximal regular sequence of linear forms on R. ☐

Remark 3.6. The interested reader may wish to consult [CI+15, 2.3] where the above argument was also discovered in the context of retracts of rings, of which idealization is a special case. Part (1) of that result shows that Gulliksen's proof of Theorem 3.4 carries over with minimal changes to general retracts. We thank Srikanth Iyengar for bringing this paper to our attention.

4. Examples of non-Koszul superlevel algebras

4.1. **Almost complete intersections.**

Proposition 4.1. *If $R = S/I$ is a quadratic Cohen-Macaulay almost complete intersection with* $\operatorname{reg} R = 2$ *and* $\operatorname{ht} I = 4$*, then R is a non-Koszul superlevel algebra.*

Proof. Since the conclusion is preserved under flat base change and killing a regular sequence of linear forms on R, we may assume without loss of generality that the ground field k is infinite and that R is Artinian, and we can choose a quadratic complete intersection $L \subseteq I$ with $\operatorname{ht} L = \operatorname{ht} I = 4$. Set $J = (L : I)$, the ideal directly linked to I by L. Since R is an almost complete intersection, J is a Gorenstein ideal, and

$$\begin{aligned} S/J(a) &\cong \omega_{S/J} \cong \operatorname{Hom}_S(S/J, \omega_{S/L}) = \operatorname{Hom}_S(S/J, S/L(4)) \\ &\cong (0 :_{S/L} J/L)(4) = I/L(4) \end{aligned}$$

for $a = \operatorname{reg} S/J$. As $I/L \cong S/J(a-4)$ is generated in degree two, it follows $a = 2$. On the other hand, $\omega_R = J/L(4)$ is generated in degrees at least -2 so that J is generated in degrees at least 2. Combining this with the fact that S/J is Gorenstein of regularity 2, it follows that J must be generated by quadrics and S/J must have a Gorenstein linear resolution. In particular, ω_R is generated in degree exactly -2 so that R is level.

By considering the exact sequence $0 \to J/L \to S/L \to S/J \to 0$, we have an induced exact sequence

$$\operatorname{Tor}_2^S(S/J,k)_j \to \operatorname{Tor}_1^S(J/L,k)_j \to \operatorname{Tor}_1^S(S/L,k)_j = 0$$

for $j > 2$. Since S/J has a Gorenstein linear resolution, we see that $\operatorname{Tor}_2^S(S/J,k)_j = 0 = \operatorname{Tor}_1^S(J/L,k)_j$ for $j > 3$. Hence, $\operatorname{Tor}_1^S(\omega_R,k)_j = 0$ for $j > -1$, and ω_R has a linear presentation since it is generated in degree -2.

Finally, R is necessarily non-Koszul since any Cohen-Macaulay Koszul almost complete intersection of codimension 4 must have regularity 3 by [Mas18, 3.3]. $\square$

Example 4.2. As a concrete example of the above proposition, consider the ring $R = S/I$ defined by the ideal $I = (x^2, y^2, z^2, w^2, xy + zw) \subseteq k[x,y,z,w] = S$. To see that R has socle degree 2, it suffices to note that $R_+^3 = 0$. Since I contains the squares of the variables, it is enough to observe that I contains all four square-free cubic monomials by multiplying $xy + zw$ by each variable. Since type $R = h_2(R) = \binom{5}{2} - 5 = 5$, we see that $\widetilde{R} = R \ltimes \omega$ is a non-Koszul quadratic Gorenstein ring with codim $\widetilde{R} = 4 + 5 = 9$ and $\operatorname{reg} \widetilde{R} = 2 + 1 = 3$.

4.2. **Ideals of generic forms.** In this section, we assume that k is a field of characteristic zero. By a *generic* set of g quadrics, we mean a point in a Zariski-open subset of S_2^g. Five generic quadrics in four variables satisfy the conditions of Proposition 4.1, and generic quadrics in more variables provide a larger class of examples of superlevel algebras.

Theorem 4.3 ([FL02, 7.1])**.** *If $R = S/I$ is an Artinian algebra with I generated by g generic quadrics in n variables, then R is Koszul if and only if $g = n$ or $g \geq \frac{n^2+2n}{4}$.*

For an ideal I generated by g generic forms of degree d in n variables, Hochster-Laksov [HL87] prove that I has maximal growth in degree $d+1$, that is

$$\dim_k I_{d+1} = \min\left\{gn, \binom{n+d}{d+1}\right\}$$

Consequently, we see that a ring $R = S/I$ defined by g generic quadrics in n variables is non-Koszul and has socle degree 2 if and only if

$$\frac{n^2+3n+2}{6} \leq g < \frac{n^2+2n}{4} \tag{4.1}$$

The h-vector of such an algebra is simply $h(R) = (1, n, \binom{n+1}{2} - g)$.

n	g
4	5
5	$7, 8$
6	$10, 11$
7	$12, 13, 14, 15$
8	$15, 16, 17, 18, 19$

FIGURE 4.1. Numbers of generic quadrics yielding non-Koszul algebras of socle degree 2 for small n

Theorem 4.4. *Let $I \subseteq S = k[x_1, \ldots, x_n]$ be an ideal generated by g generic quadrics, where $n \geq 4$ and g satisfies (4.1). Then $R = S/I$ is non-Koszul and superlevel. Hence, $\widetilde{R} = R \ltimes \omega$ is a non-Koszul quadratic Gorenstein ring with h-vector*

$$h(\widetilde{R}) = (1, \tfrac{n^2+3n}{2} - g, \tfrac{n^2+3n}{2} - g, 1) \tag{4.2}$$

Proof. Since I is Artinian and has socle degree two, it suffices to show that $\beta^S_{n-1,n}(R) = \beta^S_{n,n+1}(R) = 0$. By upper semicontinuity of the Betti numbers (see [BC02, 3.13]), it further suffices to prove that the corresponding Betti numbers vanish for some initial ideal of I.

Let J denote the initial ideal of I in the degree reverse lexicographic order. As long as J does not contain the monomials

$$x_1x_{n-1}, \ldots, x_{n-2}x_{n-1}, x_{n-1}^2, x_1x_n, \ldots, x_1x_{n-1}, x_n^2 \tag{4.3}$$

we will have $J_2 \subseteq k[x_1, \ldots, x_{n-2}]$ so that the projective dimension of $S/(J_2)$ is at most $n-2$. In that case, the exact sequence $0 \to J/(J_2) \to S/(J_2) \to S/J \to 0$ then induces exact sequences

$$\operatorname{Tor}_i^S(J/(J_2), k)_j \to \operatorname{Tor}_i^S(S/(J_2), k)_j \to \operatorname{Tor}_i^S(S/J, k)_j \to \operatorname{Tor}_{i-1}^S(J/(J_2), k)_j$$

for all i, j. Since $J/(J_2)$ is generated in degree at least 3, we have $\beta^S_{i,j}(J/(J_2)) = 0$ for all i and all $j \leq i+2$. In particular, combining this fact with the preceding observations yields $\beta^S_{i,i+1}(S/J) = \beta^S_{i,i+1}(S/(J_2)) = 0$ for $i = n-1, n$ as wanted.

Note that the monomials (4.3) are the smallest $2n-1$ quadratic monomials in the degree reverse lex order. Since I is generated by generic forms, we may assume that the determinant of the matrix of coefficients of the g largest monomials for all the generators of I is nonzero, which is a Zariski-open condition on S_2^g. Therefore, after taking suitable k-linear combinations of generators of I, we see that J contains the g largest monomials in the degree reverse lex order, and these monomials must span J_2 as I and J have the same Hilbert function. The g largest quadratic monomials are disjoint from the $2n-1$ smallest so long as $\binom{n+1}{2} - g \geq 2n-1$. This holds for all $n \geq 7$ by the estimates

$$\binom{n+1}{2} - g \geq \binom{n+1}{2} - \left\lfloor \frac{n^2+2n}{4} \right\rfloor = \left\lceil \frac{n^2}{4} \right\rceil \geq 2n-1$$

and by an explicit check when $n = 6$ and $g = 10$.

In the remaining cases, we cannot use the above argument. However, the $n = 4$ case follows from Proposition 4.1. Additionally, when $n = 5$ and $g = 7$, we see that $(1-t)^5 h_R(t) = (1-t)^5(1+5t+8t^2)$ has no cubic term, which implies that $\beta^S_{2,3}(R) = 0$.

For the cases $(n, g) = (5, 8), (6, 11)$, we claim that $\beta^S_{3,4}(R) = 0$. Indeed, we may assume as above that the lead terms of the quadrics generating I are the g largest monomials in degree reverse lex order $x^{\alpha_1}, \dots, x^{\alpha_g}$. If $\mathcal{B}$ denotes the set of exponent vectors of the remaining degree two monomials, then we may assume each quadric has the form $x^{\alpha_i} + \sum_{\beta \in \mathcal{B}} c_{i,\beta} x^\beta$ for some $c_{i,\beta} \in k$. By Schreyer's algorithm [EM+16], we can construct a free resolution $F_\bullet$ of R from a Gröbner basis including these quadrics. Now, write $S(-4)^b$ for the number of copies of $S(-4)$ in F_3 (which does not depend on the particular coefficients of the quadrics), and consider the portion of the differential $\partial_3 : S(-4)^b \to F_2$. Since we obtain the minimal free resolution of R by pruning $F_\bullet$, we have $\beta^S_{3,4}(R) = 0$ if and only if this submatrix of scalars and linear forms splits, which occurs exactly when the scalar part of ∂_3 has rank b. Furthermore, the entries of the scalar part of this submatrix are polynomials in the $c_{i,\beta}$ so that this determines a Zariski-open condition on S_2^g for the vanishing of $\beta^S_{3,4}(R)$. Therefore, $\beta^S_{3,4}(R) = 0$ for generic sets of quadrics if we can show that there is at least one example with this property, and this is easily checked by a direct computation picking g random quadrics in Macaualay2. $\square$

Example 4.5 ([Roo93])**.** Not all non-Koszul algebras of socle degree 2 come from this construction. Roos shows that for $S = k[x, y, z, w, u, v]$ with $\mathrm{char}(k) = 0$ and

$$I = (x^2, y^2, z^2, u^2, v^2, w^2, xy, yz, uv, vw, xz + 3zw - uw, zw + xu + uw)$$

the ring $R = S/I$ is not Koszul. This ring has h-vector $(1, 6, 9)$, which cannot be realized by generic forms in 6 variables.

Roos' example is superlevel, so (4.2) does not characterize all h-vectors of non-Koszul quadratic Gorenstein rings. This raises the question of what h-vectors are possible for non-Koszul quadratic Gorenstein rings of socle degree three.

Theorem 4.6. *Over a field of characteristic zero, there exist non-Koszul quadratic Gorenstein rings with h-vector $(1, c, c, 1)$ for all $c \geq 9$.*

Proof. For each n, the value $c(n, g) = \frac{n^2+3n}{2} - g$ appearing in (4.2) is decreasing in g and takes every integer value in the range $c(n, g_{\max}(n)) \leq c \leq c(n, g_{\min}(n))$, where

$$g_{\max}(n) = \left\lceil \frac{n^2 + 2n}{4} \right\rceil - 1 \qquad g_{\min}(n) = \left\lceil \frac{n^2 + 3n + 2}{6} \right\rceil$$

Hence, there will be no gaps in the codimensions attained by $c(n, g)$ so long as $c(n, g_{\min}(n)) \geq c(n + 1, g_{\max}(n + 1)) - 1$. We claim that this holds for all $n \geq 8$. This follows from the fact that

$$\begin{aligned} c(n, g_{\min}(n)) - c(n + 1, g_{\max}(n + 1)) &> c(n, \tfrac{n^2+3n+2}{6} + 1) - c(n + 1, \tfrac{(n+1)^2+2(n+1)}{4} - 1) \\ &= \frac{n^2 - 6n - 43}{12} \geq -2 \end{aligned}$$

when $n \geq 9$ and by an explicit check when $n = 8$. Thus, the construction of Theorem 4.4 yields non-Koszul quadratic Gorenstein rings with h-vector $(1, c, c, 1)$ for all $c \geq 25$, and for all $c \geq 9$ with the exceptions of $c \in \{10, 11, 14, 15, 18, 19, 24\}$. Example 4.5 takes care of the $c = 15$ case, and slight modifications yield superlevel non-Koszul algebras of socle degree two in the remaining cases. These cases arise from ideals of the form $I = J + L$, where J is given by Figure 4.2 below and L is generated by the squares of the variables appearing in the generators of J. The quotients $R = S/I$ corresponding to these examples are easily checked in Macaulay2 to be superlevel with idealization having h-vector $(1, c, c, 1)$ for the appropriate c. ☐

Remark 4.7. We have assumed that we are working over a field of characteristic zero in this section in order to simplify the statements of our results. However, the proof of Theorem 4.4 shows that we obtain non-Koszul quadratic Gorenstein rings of regularity 3 and almost every codimension greater than or equal to 9

c	J
10	$(x_1x_5, x_1x_2, x_4x_5, x_3x_5 + x_1x_4 + x_4x_5, x_2x_4 + x_3x_5)$
11	$(x_1x_5, x_1x_2, x_3x_5 + x_1x_4 + x_4x_5, x_2x_4 + x_3x_5)$
14	$(x_1x_2, x_2x_3, x_4x_5, x_5x_6, x_1x_3 + 3x_3x_6 - x_4x_6,$ $x_3x_6 + x_1x_4 + x_4x_6, x_2x_4 + x_3x_5)$
18	$(x_1x_2 + x_2x_3, x_4x_5, x_5x_6, x_6x_7, x_1x_3 + 3x_3x_6 - x_4x_6,$ $x_3x_6 + x_1x_4 + x_4x_6, x_2x_4 + x_3x_5, x_2x_5 + x_2x_7, x_1x_7, x_3x_7)$
19	$(x_4x_5, x_5x_6, x_6x_7, x_2x_4 + x_3x_5, x_2x_5 + x_2x_7, x_1x_7, x_3x_7,$ $x_7x_8, x_1x_8, x_3x_8, x_4x_8, x_6x_8, x_3x_5, x_2x_7,$ $x_1x_2 + x_2x_3, x_1x_3 + 3x_3x_6 - x_4x_6, x_3x_6 + x_1x_4 + x_4x_6)$
24	$(x_1x_2 + x_2x_3, x_4x_5, x_5x_6, x_6x_7, x_4x_9, x_5x_7, x_7x_9, x_1x_7,$ $x_3x_7, x_7x_8, x_1x_8, x_2x_8, x_3x_8, x_5x_8, x_6x_8,$ $x_1x_3 + 3x_3x_6 - x_4x_6, x_3x_6 + x_1x_4 + x_4x_6, x_2x_4 + x_3x_5,$ $x_2x_5 + x_2x_7, x_2x_9 + x_1x_9, x_3x_9 + x_6x_9)$

FIGURE 4.2. Exceptional examples yielding non-Koszul quadratic Gorenstein rings of socle degree 3

in all characteristics. We only need to specify a particular characteristic for the exceptional cases that require a direct computation in Macaulay2.

4.3. **More examples via tensor products.** When R is a non-Koszul quadratic Gorenstein ring, the idealization $\widetilde{R}$ will again be non-Koszul, quadratic, and Gorenstein with codimension and regularity increased by one. In this case, (3.1) shows that the idealization is just the tensor product $R \otimes_k k[y]/(y^2)$. In particular, the results of the previous section show that Question 1.2 has a negative answer for all $r \geq 3$ and $c \geq r + 6$. We can produce more examples by tensoring with other Gorenstein Koszul algebras.

Proposition 4.8. *Let $R = S/I$ be a quadratic ring and B be a superlevel Koszul algebra. Then $R' = R \otimes_k B$ is Koszul (resp. level, superlevel) if and only if R is. Moreover, we have*

$$\begin{aligned} \operatorname{codim} R' &= \operatorname{codim} R + \operatorname{codim} B \\ \operatorname{type} R' &= (\operatorname{type} R)(\operatorname{type} B) \\ \operatorname{reg} R' &= \operatorname{reg} R + \operatorname{reg} B \end{aligned}$$

Proof. Since tensoring over k is exact, tensoring the minimal free resolution of k over B with R yields the minimal free resolution of R over R'. As B is Koszul, we see that $\operatorname{reg}_{R'}(R) = 0$ so that R' is Koszul if and only if R is by [CDR13, §3.1, 2]. Write $B = A/J$ for some standard graded polynomial ring A. The other parts easily follow from the fact that the minimal free resolution of R' over $S \otimes_k A$ is the tensor product of the minimal free resolutions of R over S and of B over A. In particular, when R is level, the equalities concerning the codimension, type, and regularity of R' also follow from the fact that the h-polynomial of R' is the product of the h-polynomials of R and B. □

Corollary 4.9. *Over a field of characteristic zero, there exists a non-Koszul quadratic Gorenstein ring of codimension c and regularity r for every $r \geq 6$ and $c \geq r + 3$.*

Proof. If R is a non-Koszul quadratic Gorenstein ring and B is a Gorenstein Koszul algebra, then $R' = R \otimes_k B$ is again a non-Koszul quadratic Gorenstein ring. We can therefore produce more examples of such ring by tensoring Matsuda's example R, which has $\operatorname{codim} R = 7$ and $\operatorname{reg} R = 4$, with appropriate Gorenstein Koszul algebras and combining this with our results. Specifically, if we take any quadratic Gorenstein ring B with $\operatorname{codim} B = 3$, then $\operatorname{reg} B = 2$ by the Buchsbaum-Eisenbud structure theorem for such rings so that B is Koszul, and tensoring with Matsuda's example gives a negative answer to Question 1.2 for $(c, r) = (10, 6)$. If we take $B = k[X]/I_2(X)$ where X is a 3×3 matrix of variables, then the Gulliksen-Negård resolution [BV88, 2.5, 2.26] shows that $\operatorname{codim} B = 4$ and $\operatorname{reg} B = 2$ so that we also obtain a negative answer for $(c, r) = (11, 6)$. Propagating these negative answers by tensoring with complete intersections completes the proof. □

We summarize the preceding discussion in Figure 4.3 below. Aside from the seven remaining cases of codimension $c \geq 6$, one might still hope that every quadratic Gorenstein ring R with $\operatorname{codim} R \leq \operatorname{reg} R + 2$ is Koszul, which could explain the affirmative answers in regularity three.

5. Future Directions

Matsuda's example cannot be obtained with our methods; there are no superlevel quadratic algebras with the right Hilbert function. It would be interesting to find geometric interpretations of our results. The initial example that inspired the results of this paper was a certain inverse system related to the Artinian reduction of a smooth curve of genus seven and degree eleven in $\mathbb{P}^5$ defined by five quadrics which is projectively normal but not Koszul [SS12]. We plan to investigate this

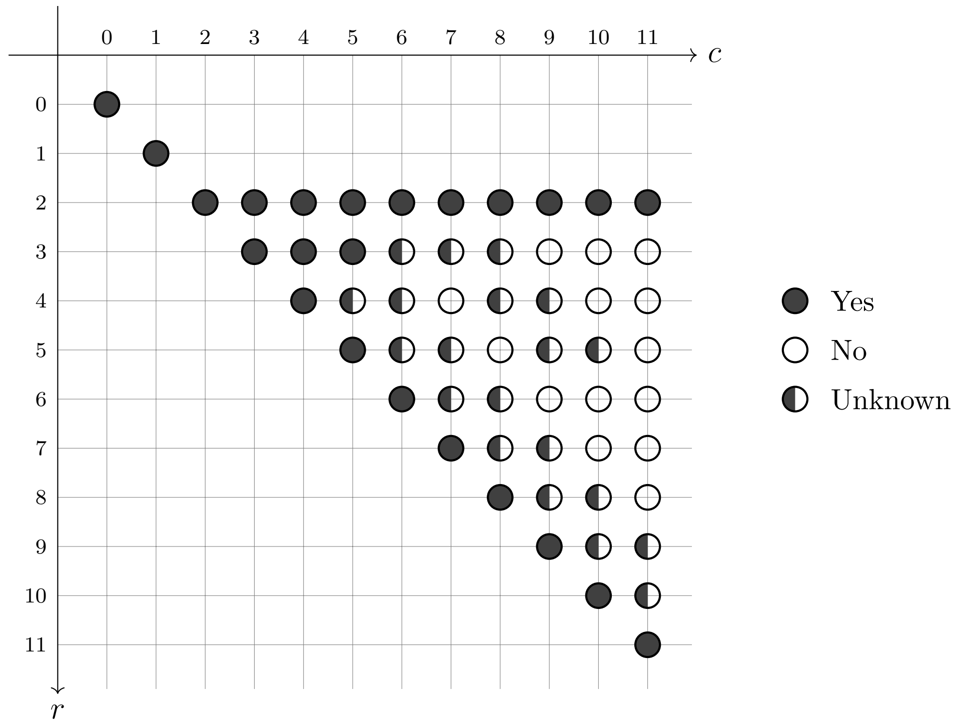

FIGURE 4.3. Is every quadratic Gorenstein ring of codimension c and regularity r Koszul?

in a followup paper, as well as studying how idealization relates to the parameter space of Gorenstein algebras and work by Iarrobino-Kanev [IK99] and Boij [Boi99].

Acknowledgements. Macaulay2 [M2] computations were essential to our work. We also thank BIRS-CMO, where we learned of Matsuda's result.

REFERENCES

[Boi99] M. Boij. Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function. *Pacific J. Math.* 187 (1999), no. 1, 1–11.

[BS13] M. Brodmann and R. Sharp. *Local cohomology.* An algebraic introduction with geometric applications. Second edition. Cambridge Studies in Advanced Mathematics, 136. Cambridge University Press, Cambridge, 2013.

[BC02] W. Bruns and A. Conca. Gröbner bases and determinantal ideals. *Commutative algebra, singularities and computer algebra (Sinaia, 2002)*, 9–66, NATO Sci. Ser. II Math. Phys. Chem., 115, Kluwer Acad. Publ., Dordrecht, 2003.

[BH93] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[BV88] W. Bruns and U. Vetter. *Determinantal rings.* Lecture Notes in Mathematics, 1327. Springer-Verlag, Berlin, 1988.

[Cav00] G. Caviglia. On a class of Gorenstein algebras that are Koszul. *Laurea in Mathematics (Masters thesis)*, Università di Genova, 2000.

[Con14] A. Conca. Koszul algebras and their syzygies. *Combinatorial algebraic geometry*, 1–31, Lecture Notes in Math., 2108, Fond. CIME/CIME Found. Subser., Springer, Cham, 2014.

[CDR13] A. Conca, E. De Negri, and M.E. Rossi. Koszul algebras and regularity. *Commutative algebra*, 285–315, Springer, New York, 2013.

[CI+15] A. Conca, S. Iyengar, H. Nguyen, and T. Römer. Absolutely Koszul algebras and the Backelin-Roos property. *Acta Math. Vietnam.* 40 (2015), no. 3, 353–374.

[CRV01] A. Conca, M. E. Rossi, G. Valla. Gröbner flags and Gorenstein algebras. *Compositio Math.* 129 (2001), no. 1, 95–121.

[Eis05] D. Eisenbud, *The geometry of syzygies.* A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

[EK17] S. El Khoury and A. Kustin. Use a Macaulay inverse system to detect an embedded deformation. Preprint.

[EM+16] B. Eröcal, O. Motsak, F.-O. Schreyer, and A. Steenpaß. Refined algorithms to compute syzygies. *J. Symbolic Comput.* 74 (2016), 308–327.

[Frö99] R. Fröberg. Koszul algebras. *Advances in commutative ring theory (Fez, 1997)*, 337–350, Lecture Notes in Pure and Appl. Math., 205, Dekker, New York, 1999.

[FL02] R. Fröberg and C. Löfwal. Koszul homology and Lie algebras with application to generic forms and points. The Roos Festschrift volume, 2. *Homology Homotopy Appl.* 4 (2002), no. 2, part 2, 227–258.

[M2] D. Grayson and M. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[Gul72] T. Gulliksen. Massey operations and the Poincaré series of certain local rings. *J. Algebra* 22 (1972), 223–232.

[HL87] M. Hochster and D. Laksov. The linear syzygies of generic forms. *Comm. Algebra* 15 (1987), no. 1-2, 227–239.

[IK99] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci.* Appendix C by Iarrobino and Steven L. Kleiman. Lecture Notes in Mathematics, 1721. Springer-Verlag, Berlin, 1999.

[Mas18] M. Mastroeni. Koszul almost complete intersections. *J. Algebra* 501 (2018), 285–302.

[Mat17] K. Matsuda. Non-Koszul quadratic Gorenstein toric rings. *Mathematica Scandinavica*, 123 (2018), no. 2, 161–173.

[MN13] J. Migliore and U. Nagel. Gorenstein algebras presented by quadrics. *Collect. Math.* 64 (2013), no. 2, 211–233.

[Pol95] A. Polishchuk. On the Koszul property of the homogeneous coordinate ring of a curve. *J. Algebra* 178 (1995), no. 1, 122–135.

[Rei72] I. Reiten. The converse to a theorem of Sharp on Gorenstein modules. *Proc. Amer. Math. Soc.* 32 (1972), 417–420.

[Roo93] J.-E. Roos. Commutative non-Koszul algebras having a linear resolution of arbitrarily high order. Applications to torsion in loop space homology. *C. R. Acad. Sci. Paris Sér. I Math.* 316 (1993), no. 11, 1123–1128.

[SS12] H. Schenck and M. Stillman. High rank linear syzygies on low rank quadrics. *Amer. J. Math.* 134 (2012), no. 2, 561–579.

[Tat57] J. Tate, Homology of local and Noetherian rings. *Illinois J. Math.* 1, (1957), 14–27.

[VF93] A. Vishik and M. Finkelberg. The coordinate ring of general curve of genus $g \geq 5$ is Koszul. *J. Algebra* 162 (1993), no. 2, 535–539.

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
E-mail address: mmastro@okstate.edu

Department of Mathematics, Iowa State University, Ames, IA 50011
E-mail address: hschenck@iastate.edu

Department of Mathematics, Cornell University, Ithaca, NY 14850
E-mail address: mike@math.cornell.edu