KEY

Math 267 Worksheet 5 – Fall 2021

1. If possible, find a function $F(x, y)$ such that the equation $F(x, y) = C$ implicitly solves the following differential equation. **If it is not possible to solve the equation, explain why.**

$$(yx^2e^{y^2} - 3\cos(2x - 3y))\frac{dy}{dx} + xe^{y^2} + 2\cos(2x - 3y) = 0$$

$$M = xe^{y^2} + 2\cos(2x - 3y)$$

$$N = yx^2e^{y^2} - 3\cos(2x - 3y)$$

$$\frac{\partial N}{\partial x} = 2xye^{y^2} + 6\cos(2x - 3y) = \frac{\partial M}{\partial y} \quad \checkmark$$

$$M = \frac{\partial F}{\partial x} \Rightarrow F = \int xe^{y^2} + 2\cos(2x - 3y)\,dx$$

$$= \frac{1}{2}x^2e^{y^2} + \sin(2x - 3y) + g(y)$$

$$N = \frac{\partial F}{\partial y} = yx^2e^{y^2} - 3\cos(2x - 3y) + \frac{dg}{dy}$$

$$\Rightarrow \frac{dg}{dy} = 0 \Rightarrow g = \int 0\,dy = C$$

$$F(x, y) = \frac{1}{2}x^2e^{y^2} + \sin(2x - 3y) + C$$

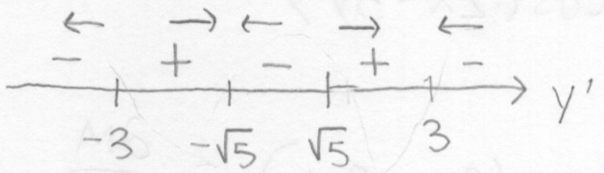
2. Consider the autonomous differential equation

$$\frac{dy}{dt} = (\alpha + 2 - y^2)(y^2 - \alpha^2)$$

that depends on a parameter α. $= (\alpha + 2 - y^2)(y - \alpha)(y + \alpha)$

(a) Find the equilibrium solutions of this differential equation when $\alpha = 3$, and identify whether each solution is stable or unstable.

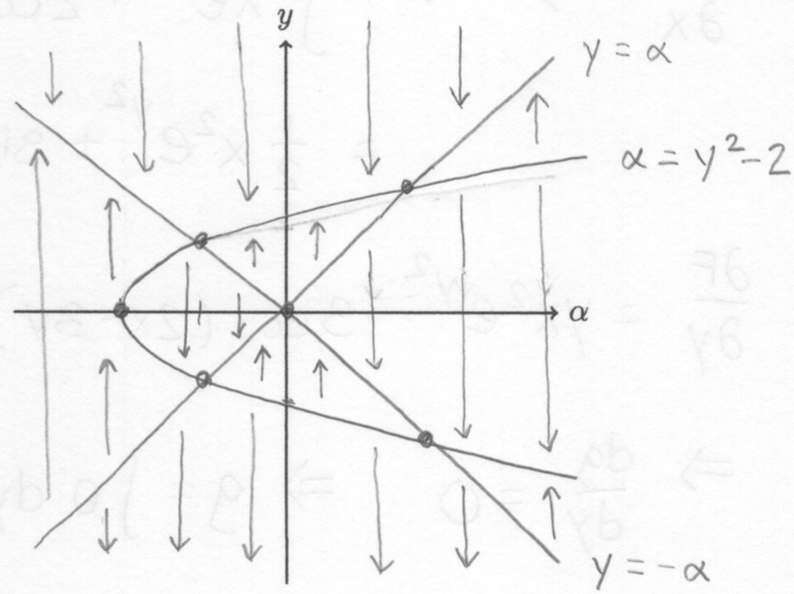
$0 = (5 - y^2)(y^2 - 9) \Rightarrow y = \pm\sqrt{5}, \pm 3$



$y = -\sqrt{5}, 3$ are stable.

$y = \sqrt{5}, -3$ are unstable.

(b) Sketch the bifurcation diagram of the differential equation on the axes below.



(c) Identify all of the bifurcation points of the differential equation.

$\alpha = -2, -1, 0, 2$

3. (a) Verify that the functions $y_1(t) = \sin 2t + \cos 2t$ and $y_2(t) = \cos t(\sin t + \cos t) - \frac{1}{2}$ both solve the following second order differential equation.

$$y'' + 4y = 0$$

$y_1' = 2\cos 2t - 2\sin 2t \qquad y_1'' = -4\sin 2t - 4\cos 2t$

$y_1'' + 4y_1 = -4\sin 2t - 4\cos 2t + 4(\sin 2t + \cos 2t) = 0$

$y_2' = -\sin t(\sin t + \cos t) + \cos t(\cos t - \sin t)$

$y_2'' = -\cos t(\sin t + \cos t) - \sin t(\cos t - \sin t)$
$\quad - \sin t(\cos t - \sin t) + \cos t(-\sin t - \cos t)$
$= -4\sin t\cos t - 2\cos^2 t + 2\sin^2 t$

$y_2'' + 4y_2 = -4\sin t\cos t - 2\cos^2 t + 2\sin^2 t$
$\quad + 4(\sin t\cos t + \cos^2 t - \frac{1}{2}) = 2\sin^2 t + 2\cos^2 t - 2 = 0$

(b) Are the functions in part (a) linearly independent? If so, write down the general solution to the above differential equation.

$$W(y_1, y_2) = \begin{vmatrix} \sin 2t + \cos 2t & \cos t\sin t + \cos^2 t - \frac{1}{2} \\ 2\cos 2t - 2\sin 2t & \cos^2 t - \sin^2 t - 2\sin t\cos t \end{vmatrix}$$

when $t = 0$: $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 1/2 \\ 2 & 1 \end{vmatrix}$

$= 1 \cdot 1 - \frac{1}{2} \cdot 2 = 0$

No, since the Wronskian vanishes when $t = 0$.

4. (a) Find the characteristic equation of the following second order linear differential equation with constant coefficients.

$$3y'' - 30y' + 75y = 0$$

$$0 = 3r^2 - 30r + 75$$

$$= 3(r^2 - 10r + 25) = 3(r-5)^2$$

(b) What is the general solution to the differential equation in part (a)?

$$y = C_1 e^{5x} + C_2 x e^{5x}$$

5. (a) Variations of the Bessel equation

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0$$

can be used to model the vibration of a beaten drum. Verify that $y_1(x) = \dfrac{\cos x}{\sqrt{x}}$ is a solution of this equation.

$$y_1' = \frac{-\sin x}{\sqrt{x}} - \frac{\cos x}{2x^{3/2}} \qquad y_1'' = \frac{-\cos x}{\sqrt{x}} + \frac{\sin x}{x^{3/2}} + \frac{3}{4}\frac{\cos x}{x^{5/2}}$$

$$x^2y_1'' + xy_1' + (x^2 - \tfrac{1}{4})y_1 = -x^{3/2}\cos x + \sqrt{x}\sin x + \frac{3}{4}\frac{\cos x}{\sqrt{x}}$$

$$- \sqrt{x}\sin x - \frac{1}{2}\frac{\cos x}{\sqrt{x}} + x^{3/2}\cos x - \frac{1}{4}\frac{\cos x}{\sqrt{x}}$$

$$= 0$$

(b) Find another solution of the Bessel equation of the form $y_2(x) = v(x)y_1(x)$ for some function $v(x)$.

$$y_2' = vy_1' + v'y_1 \qquad y_2'' = vy_1'' + 2v'y_1' + v''y_1$$

$$0 = x^2 y_2'' + x y_2' + \left(x^2 - \tfrac{1}{4}\right) y_2$$

$$= x^2 v y_1'' + 2x^2 v' y_1' + x^2 v'' y_1 + x v y_1' + x v' y_1$$

$$+ v\left(x^2 - \tfrac{1}{4}\right) y_1$$

$$= v\underbrace{\left(x^2 y_1'' + x y_1' + \left(x^2 - \tfrac{1}{4}\right) y_1\right)}_{=0} + 2x^2 v' y_1' + x^2 v'' y_1$$

$$+ x v' y_1$$

$$\Rightarrow 0 = \left(-2x^{3/2}\sin x - \cancel{\sqrt{x}\cos x}\right) v' + x^{3/2}\cos x\, v''$$

$$+ \cancel{\sqrt{x}\cos x}\, v'$$

$$\Rightarrow \quad v'' = (2\tan x)\, v' \qquad \text{set } w = v'$$

$$\Rightarrow \quad w' = (2\tan x)\, w \Rightarrow \int \frac{1}{w}\, dw = \int 2\tan x\, dx$$

$$\Rightarrow \quad \ln|w| = -2\ln|\cos x| + C \quad \Rightarrow \quad w = C\sec^2 x$$

$$\Rightarrow \quad v = \int v'\, dx = \int C\sec^2 x\, dx = C\tan x$$

$$\Rightarrow \quad y_2 = \tan x \cdot \frac{\cos x}{\sqrt{x}} = \frac{\sin x}{\sqrt{x}}$$

(c) Check whether the solutions $y_1(x)$ and $y_2(x)$ linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ -\frac{\sin x}{\sqrt{x}} - \frac{1}{2}\frac{\cos x}{x^{3/2}} & \frac{\cos x}{\sqrt{x}} - \frac{1}{2}\frac{\sin x}{x^{3/2}} \end{vmatrix}$$

$$= \frac{\cos^2 x}{x} - \cancel{\frac{1}{2}\frac{\sin x \cos x}{x^2}} - \left(-\frac{\sin^2 x}{x} - \cancel{\frac{1}{2}\frac{\sin x \cos x}{x^2}}\right)$$

$$= \frac{1}{x} \neq 0$$ Yes, they are linearly independent.

6. Find the solution of the initial value problem.

$$y'' + 2y' + 10y = 0 \qquad y(0) = 1, \quad y'(0) = -7$$

$$r^2 + 2r + 10 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

$$y = Ae^{(-1+3i)t} + Be^{(-1-3i)t} = Ae^{-t}e^{3it} + Be^{-t}e^{-3it}$$

$$= Ae^{-t}(\cos 3t + i\sin 3t) + Be^{-t}(\cos(-3t) + i\sin(-3t))$$

$$= C_1 e^{-t}\cos 3t + C_2 e^{-t}\sin 3t$$

$$1 = y(0) = C_1$$

$$y'(t) = -C_1 e^{-t}\cos 3t - 3C_1 e^{-t}\sin 3t$$
$$- C_2 e^{-t}\sin 3t + 3C_2 e^{-t}\cos 3t$$

$$-7 = y'(0) = -C_1 + 3C_2 \Rightarrow 3C_2 = -6 \Rightarrow C_2 = -2$$

$$y(t) = e^{-t}\cos 3t - 2e^{-t}\sin 3t$$