Math 267 Worksheet 4 – Fall 2021

KEY

1. Find the general solution to the linear differential equation below.

$$\frac{dy}{dt} + \frac{2y}{t} = 10 \sin t$$

$$e^{\int P(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = t^2$$

$$t^2 \frac{dy}{dt} + 2ty = 10t^2 \sin t$$

$$\Rightarrow \left[t^2 y \right]' = 10t^2 \sin t$$

$u = t^2 \quad dv = \sin t\,dt$

$du = 2t\,dt \quad v = -\cos t$

$$\Rightarrow t^2 y = \int 10t^2 \sin t\,dt$$

$$= 10\left(-t^2 \cos t + \int 2t \cos t\,dt\right)$$

$u = 2t \quad dv = \cos t\,dt$

$du = 2\,dt \quad v = \sin t$

$$= 10\left(-t^2 \cos t + 2t \sin t - \int 2 \sin t\,dt\right)$$

$$= -10t^2 \cos t + 20t \sin t + 2\cos t + C$$

$$\Rightarrow y = -10\cos t + 20\frac{\sin t}{t} + 2\frac{\cos t}{t^2} + \frac{C}{t^2}$$

2. Find the general solution to the homogeneous differential equation below.

$$x^2y' + xy + y^2 = 0$$

$y = xv \qquad y' = v + xv'$

$x^2(v + xv') + x^2v + x^2v^2 = 0$

$\Rightarrow \quad \cancel{x^3}v' + 2\cancel{x^2}v + \cancel{x^2}v^2 = 0$

$\Rightarrow \quad -xv' = 2v + v^2$

$\Rightarrow \quad \displaystyle\int \frac{-1}{v(2+v)}\,dv = \int \frac{1}{x}\,dx$

$\dfrac{-1}{v(v+2)} = \dfrac{A}{v} + \dfrac{B}{2+v} \Rightarrow -1 = A(2+v) + Bv$

when $v = 0$: $-1 = 2A \Rightarrow A = \frac{-1}{2}$

when $v = -2$: $-1 = -2B \Rightarrow B = +\frac{1}{2}$

$\Rightarrow \displaystyle\int \frac{-1}{2}\cdot\frac{1}{v} + \frac{1}{2}\cdot\frac{+1}{2+v}\,dv = \ln|x| + C$

$\Rightarrow -\frac{1}{2}\ln|v| + \frac{1}{2}\ln|2+v| = \ln|x| + C$

$\Rightarrow \dfrac{2+v}{v} = Cx^2 \Rightarrow v = \dfrac{2}{Cx^2-1} \qquad \Rightarrow y = \dfrac{2x}{Cx^2-1}$

3. If possible, find a function $F(x, y)$ that implicitly solves the following differential equation. **If it is not possible to solve the equation, explain why.**

$$(xye^y - \sin x \sin y + \sec^2 y)\frac{dy}{dx} + \cos x \cos y + ye^y - y = 0$$

$$M = \cos x \cos y + ye^y - y$$

$$N = xye^y - \sin x \sin y + \sec^2 y$$

$$\frac{\partial N}{\partial x} = ye^y - \cos x \sin y$$

$$\frac{\partial M}{\partial y} = -\cos x \sin y + ye^y + e^y - e^y = \frac{\partial N}{\partial x} \quad \checkmark$$

$$\frac{\partial F}{\partial x} = M \Rightarrow F = \int \cos x \cos y + ye^y - e^y \, dx$$

$$= \sin x \cos y + xye^y - xe^y + g(y)$$

$$N = \frac{\partial F}{\partial y} = -\sin x \sin y + xye^y + \cancel{xe^y} - \cancel{xe^y} + \frac{dg}{dy}$$

$$\Rightarrow \frac{dg}{dy} = \sec^2 y \Rightarrow g = \int \sec^2 y \, dy$$

$$= \tan y + C$$

$$\Rightarrow F = \sin x \cos y + xye^y - xe^y + \tan y + C$$

4. A community contains a population of 15,000 people susceptible to being infected with Michaud's syndrome. At time $t = 0$, the number of people $N(t)$ who have been infected is 5000 and is increasing by 500 cases per day. Assume $N'(t)$ is proportional to the product of the numbers of infected and uninfected people.

(a) Find the function $N(t)$ that models the number of infected people after t days.

$$\frac{dN}{dt} = kN(15000 - N)$$

when $t = 0$:

$$\cancel{500} = k \cdot 5\cancel{000}(15000 - 5000)$$

$$= k \cdot 50 \cdot 10000$$

$$\Rightarrow k = \frac{1}{100,000}$$

$$\int \frac{1}{N(15000-N)}\,dN = \int k\,dt$$

$$\frac{1}{N(15000-N)} = \frac{A}{N} + \frac{B}{15000-N}$$

$$\Rightarrow 1 = A(15000 - N) + BN$$

when $N = 0$: $A = \frac{1}{15,000}$

when $N = 15000$: $B = \frac{1}{15000}$

$$\frac{1}{15000}\int \frac{1}{N} + \frac{1}{15000-N}\,dN = kt + C$$

$$\Rightarrow \ln|N| - \ln|15000 - N| = \frac{3}{20}t + C$$

$$\Rightarrow \frac{N}{15000-N} = Ce^{\frac{3}{20}t} \Rightarrow N = \frac{15000}{1+Ce^{-3t/20}}$$

when $t = 0$:

$$5000 = \frac{15000}{1+C}$$

$$\Rightarrow C = 2$$

(b) How many people can we expect to be infected in the long run?

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{15000}{1+2e^{-3t/20}} = \frac{15000}{1+0}$$

$$= 15,000$$

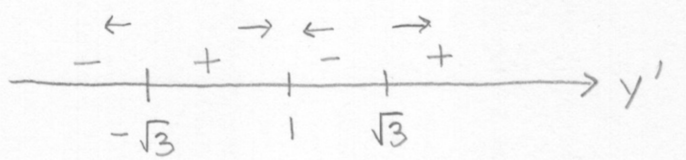
5. Consider the autonomous differential equation

$$\frac{dy}{dt} = (y-1)(y^2-\alpha)$$

that depends on a parameter α.

(a) Find the equilibrium solutions of this differential equation when $\alpha = 3$, and identify whether each solution is stable or unstable.

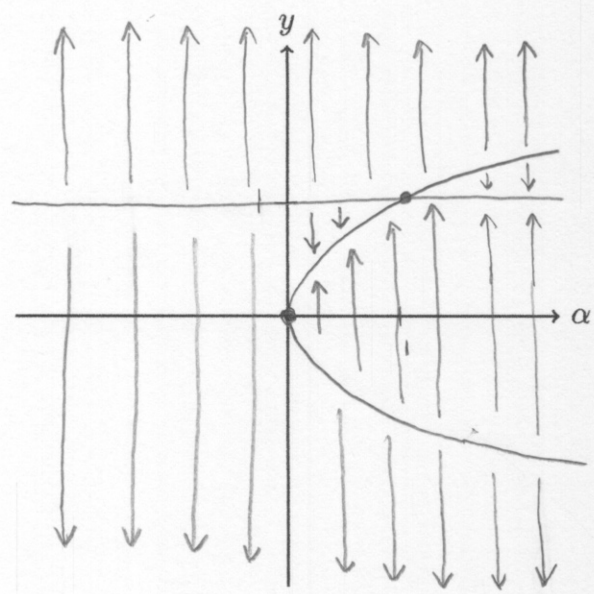
$0 = (y-1)(y^2-3) \Rightarrow y = 1$ or $y = \pm\sqrt{3}$



$y = 1$ is stable.

$y = \pm\sqrt{3}$ is unstable.

(b) Sketch the bifurcatuon diagram of the differential equation on the axes below.



(c) Identify all of the bifurcation points of the differential equation.

$\alpha = 1$ and $\alpha = 0$