KEY

Math 267 Worksheet 8 – Fall 2021

1. In this problem, we will solve the differential equation

$$y'' + 9y = 3\sec 3x \qquad (*)$$

by variation of parameters.

(a) First, find linearly independent functions y_1 and y_2 that solve the corresponding homogeneous equation.

$$y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

$$y = C_1 \sin 3x + C_2 \cos 3x$$

$$y_1 = \sin 3x \qquad y_2 = \cos 3x$$

(b) We will then guess that the particular solution to the nonhomogeneous equation has the form $y_p = u_1 y_1 + u_2 y_2$ for some functions u_1 and u_2 satisfying:

$$u_1' y_1 + u_2' y_2 = 0 \qquad (**)$$

Plug y_p into the equation $(*)$ and simplify the result as much as possible using $(**)$ and the fact that y_1, y_2 solve the homogeneous equation.

$$y_p' = (u_1 \sin 3x + u_2 \cos 3x)'$$

$$= u_1' \sin 3x + 3u_1 \cos 3x + u_2' \cos 3x - 3u_2 \sin 3x$$

$$= 3u_1 \cos 3x - 3u_2 \sin 3x$$

$$y_p'' = 3u_1' \cos 3x - 9u_1 \cos 3x - 3u_2' \sin 3x - 9u_2 \cos 3x$$

$$3\sec 3x = y_p'' + 9y_p = 3u_1' \cos 3x - 3u_2' \sin 3x$$

(c) The simplified equation you found in the previous part together with (**) form a system of equations for the unknown functions u_1' and u_2'. Solve this system of equations for u_1' and u_2'.

$$u_1' \sin 3x + u_2' \cos 3x = 0 \Rightarrow u_2' = (-\tan 3x) u_1'$$

$$3u_1' \cos 3x - 3u_2' \sin 3x = 3\sec 3x$$

$$\Rightarrow 3\sec 3x = (3\cos 3x) u_1' + \frac{3\sin^2 3x}{\cos 3x} u_1' = \frac{3}{\cos 3x} u_1'$$

$$\Rightarrow u_1' = 1, \quad u_2' = -\tan 3x$$

(d) Integrate u_1' and u_2' to find u_1 and u_2.

$$u_1 = \int 1 \, dx = x$$

$$u_2 = \int -\tan 3x \, dx = \int -\frac{\sin 3x}{\cos 3x} dx \qquad \begin{aligned} u &= \cos 3x \\ du &= -3\sin 3x \, dx \\ -\tfrac{1}{3} du &= \sin 3x \, dx \end{aligned}$$

$$= \int \frac{1}{3} \cdot \frac{1}{u} du = \frac{1}{3} \ln|u| = \frac{1}{3} \ln|\cos 3x|$$

(e) What is the particular solution y_p?

$$y_p = x \sin 3x + \frac{1}{3} \ln|\cos 3x| \cdot \cos 3x$$

2. Consider the following vectors and matrices.

$$A = \begin{pmatrix} \overset{1}{\cancel{-2}} & -3 & 5 \\ 0 & -7 & 2 \\ 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 & 1 \\ -2 & \frac{1}{2} & 0 \\ 0 & -3 & -4 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} +1 \\ +10 \\ -7 \end{pmatrix}$$

Compute the following using vector/matrix operations or write "undefined" if the operation is not well-defined. (Recall that I_3 denotes the 3×3 identity matrix.)

(a) $-\frac{5}{2}\mathbf{u} + \frac{1}{2}\mathbf{v} + \mathbf{w} =$ $\begin{pmatrix} -5/2 \\ -10 \\ 15/2 \end{pmatrix} + \begin{pmatrix} 3/2 \\ 0 \\ -1/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 10 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(b) $A + I_3 =$ $\begin{pmatrix} 1 & -3 & 5 \\ 0 & -7 & 2 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 5 \\ 0 & -6 & 2 \\ 1 & 0 & 2 \end{pmatrix}$

(c) $A + 2B =$ $\begin{pmatrix} 1 & -3 & 5 \\ 0 & -7 & 2 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 2 \\ -4 & 1 & 0 \\ 0 & -6 & -8 \end{pmatrix} = \begin{pmatrix} 7 & -3 & 7 \\ -4 & -6 & 2 \\ 1 & -6 & -7 \end{pmatrix}$

(d) $\mathbf{v}A =$ undefined

number of columns of v $\neq$ number of rows of A

(e) $B\overset{u}{\mathbf{w}} = \begin{pmatrix} 3 & 0 & 1 \\ -2 & \frac{1}{2} & 0 \\ 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(f) $AB = \begin{pmatrix} 1 & -3 & 5 \\ 0 & -7 & 2 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ -2 & \frac{1}{2} & 0 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 9 & -33/2 & -19 \\ 14 & -19/2 & -8 \\ 3 & -3 & -3 \end{pmatrix}$

(g) $BA = \begin{pmatrix} 3 & 0 & 1 \\ -2 & \frac{1}{2} & 0 \\ 0 & -3 & -4 \end{pmatrix}\begin{pmatrix} 1 & -3 & 5 \\ 0 & -7 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -9 & 16 \\ -2 & -19/2 & -9 \\ -4 & 21 & -10 \end{pmatrix}$

3. Consider the system of 1st order linear differential equations:

$$\frac{dy}{dt} = 3y + 5v$$

$$\frac{dv}{dt} = 6y + 2v$$

(a) If $\mathbf{Y}(t)$ denotes the vector-valued function $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$, write down the coefficient matrix A such that:

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

$A = \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}$

(b) Verify that the vector-valued functions $\mathbf{Y}_1(t) = \begin{pmatrix} e^{8t} \\ e^{8t} \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} -5e^{-3t} \\ 6e^{-3t} \end{pmatrix}$ both solve the given system of differential equations.

$$Y_1' = \begin{pmatrix} 8e^{8t} \\ 8e^{8t} \end{pmatrix} \quad AY_1 = \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} e^{8t} \\ e^{8t} \end{pmatrix} = \begin{pmatrix} 8e^{8t} \\ 8e^{8t} \end{pmatrix} \checkmark$$

$$Y_2' = \begin{pmatrix} 15e^{-3t} \\ -18e^{-3t} \end{pmatrix} \quad AY_2 = \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} -5e^{-3t} \\ 6e^{-3t} \end{pmatrix} = \begin{pmatrix} 15e^{-3t} \\ -18e^{-3t} \end{pmatrix} \checkmark$$

(c) What is the form of the general solution to the system of differential equations?

$$Y = C_1Y_1 + C_2Y_2 = \begin{pmatrix} C_1e^{8t} - 5C_2e^{-3t} \\ C_1e^{8t} + 6C_2e^{-3t} \end{pmatrix}$$

(d) Find the specific function $\mathbf{Y}(t)$ that solves the system of differential equations subject to the initial condition $\mathbf{Y}(0) = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

$$\begin{pmatrix} -1 \\ 5 \end{pmatrix} = Y(0) = \begin{pmatrix} C_1 - 5C_2 \\ C_1 + 6C_2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -5 & -1 \\ 1 & 6 & 5 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -5 & -1 \\ 0 & 11 & 6 \end{array}\right] \xrightarrow{\frac{1}{11}R_2 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & -5 & -1 \\ 0 & 1 & \frac{6}{11} \end{array}\right] \xrightarrow{R_1 + 5R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{19}{11} \\ 0 & 1 & \frac{6}{11} \end{array}\right] \Rightarrow \begin{array}{l} C_1 = \frac{19}{11} \\ C_2 = \frac{6}{11} \end{array}$$

$$Y = \begin{pmatrix} \frac{19}{11}e^{8t} - \frac{30}{11}e^{-3t} \\ \frac{19}{11}e^{8t} + \frac{36}{11}e^{-3t} \end{pmatrix}$$

4. Consider the 2nd order linear differential equation:

$$y'' + 4y' - 5y = 0$$

(a) If we set $v = y'$, write down a system of 1st order linear differential equations involving y and v which is equivalent to solving the above 2nd order equation.

$$v' = y'' = -4v + 5y$$

$$y' = v$$

(b) Write down the coefficient matrix A that expresses your system of equations in part (a) as a matrix equation

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

where $\mathbf{Y} = \begin{pmatrix} y \\ v \end{pmatrix}$.

$$A = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}$$

(c) Compute the determinant of the matrix $A - rI_2$ where r is a variable. Do you recognize this determinant?

$$\det(A - rI_2) = \begin{vmatrix} -r & 1 \\ 5 & -4-r \end{vmatrix}$$

$$= r^2 + 4r - 5$$