KEY

Math 267 Worksheet 7 – Fall 2021

1. Suppose we want to solve the nonhomogeneous second order differential equation

$$y'' + 25y = 10xe^x \tag{*}$$

by the method of undetermined coefficients.

(a) If we guess that the particular solution of the nonhomogeneous equation has the form $y_p = Ae^x + Bxe^x$, use the method of undetermined coefficients to find y_p.

$$y_p' = Ae^x + Bxe^x + Be^x = (A+B)e^x + Bxe^x$$

$$y_p'' = (A+B)e^x + Bxe^x + Be^x = (A+2B)e^x + Bxe^x$$

$$10xe^x = (A+2B)e^x + Bxe^x + 25Ae^x + 25Bxe^x$$

$$= (26A + 2B)e^x + 26Bxe^x$$

when $x = 0$: $\quad 0 = 26A + 2B \Rightarrow A = \frac{-1}{13}B$

when $x = 1$: $\quad 10e = (26A + 2B)e + 26Be$

$$= 26Be$$

$$\Rightarrow B = \frac{5}{13}, \quad A = \frac{-5}{169}$$

$$y_p = \frac{-5}{169}e^x + \frac{5}{13}xe^x$$

(b) Write down the general solution to the *homogeneous* differential equation corresponding to (*).

$$r^2 + 25 = 0$$

$$\Rightarrow \quad r^2 = -25$$

$$\Rightarrow \quad r = \pm 5i$$

$$y = C_1 \cos 5x + C_2 \sin 5x$$

(c) Write down the general solution to the nonhomogeneous differential equation (*).

$$y = C_1 \cos 5x + C_2 \sin 5x$$

$$- \frac{5}{139} e^x + \frac{5}{13} x e^x$$

2. Suppose we want to solve the nonhomogeneous second order differential equation

$$y'' - 2y' + y = 5e^x$$

by the method of undetermined coefficients. Since the general solution to the corresponding homogeneous equation is $y = C_1e^x + C_2xe^x$, we must guess that the particular solution has the form $y_p = Ax^2e^x$. Find y_p.

$$y_p' = 2Axe^x + Ax^2e^x$$

$$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x$$

$$5e^x = 2Ae^x + 4Axe^x + Ax^2e^x - 4Axe^x - 2Ax^2e^x + Ax^2e^x$$

$$= 2Ae^x$$

$$\Rightarrow \quad A = \frac{5}{2}$$

$$y_p = \frac{5}{2}x^2e^x$$

3. In this problem, we will solve the differential equation

$$y'' + y = \csc^2 x \qquad (**)$$

by variation of parameters.

(a) First, find linearly independent functions y_1 and y_2 that solve the corresponding homogeneous equation.

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$$

$$y = C_1 \sin x + C_2 \cos x$$

$$y_1 = \sin x \qquad y_2 = \cos x$$

(b) We will then guess that the particular solution to the nonhomogeneous equation has the form $y_p = u_1y_1 + u_2y_2$ for some functions u_1 and u_2 satisfying:

$$u_1'y_1 + u_2'y_2 = 0 \qquad (***)$$

Plug y_p into the equation $(**)$ and simplify the result as much as possible using $(***)$ and the fact that y_1, y_2 solve the homogeneous equation.

$$y_p' = (u_1 \sin x + u_2 \cos x)'$$

$$= u_1' \sin x + u_1 \cos x + u_2' \cos x - u_2 \sin x$$

$$= u_1 \cos x - u_2 \sin x$$

$$y_p'' = u_1' \cos x - u_1 \sin x - u_2' \sin x - u_2 \cos x$$

$$\csc^2 x = y_p'' + y_p$$

$$= u_1' \cos x - \cancel{u_1 \sin x} - u_2' \sin x - \cancel{u_2 \cos x} + \cancel{u_1 \sin x} + \cancel{u_2 \cos x}$$

$$= u_1' \cos x - u_2' \sin x$$

(c) The simplified equation you found in the previous part together with $(***)$ form a system of equations for the unknown functions u_1' and u_2'. Solve this system of equations for u_1' and u_2'.

$$0 = u_1' \sin x + u_2' \cos x \Rightarrow u_2' = -(\tan x) u_1'$$

$$\csc^2 x = u_1' \cos x - u_2' \sin x \Rightarrow \csc^2 x = (\cos x + \sin x \tan x) u_1'$$

$$\Rightarrow u_1' = \frac{\csc^2 x}{\cos x + \sin x \tan x} = \frac{\csc^2 x}{\frac{\cos^2 x + \sin^2 x}{\cos x}} = \csc x \cot x$$

$$\Rightarrow u_2' = -\csc x$$

(d) Integrate u_1' and u_2' to find u_1 and u_2.

$$u_1 = \int \csc x \cot x \, dx = -\csc x + C$$

$$u_2 = \int -\csc x \, dx = \int -\frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$$

$$w = \csc x + \cot x \qquad = \int \frac{1}{w} dw = \ln|w| + C$$

$$dw = -\csc x \cot x - \csc^2 x \, dx$$

$$= \ln|\csc x + \cot x| + C$$

(e) What is the particular solution y_p?

$$y_p = -\csc x \cdot \sin x + \ln|\csc x + \cot x| \cdot \cos x$$

$$= -1 + \ln|\csc x + \cot x| \cdot \cos x$$