KEY

Math 267 Worksheet 14 – Fall 2021

1. In this problem, we will solve the differential equation

$$y'' + y = x, \qquad y(0) = -1,\ y'(0) = 2 \qquad (*)$$

using power series.

(a) First, we guess that the solution has a power series expansion

$$y = \sum_{n=0}^{\infty} c_n x^n$$

for some unknown coefficients c_n. Plug y into the equation $(*)$, and simplify the left hand side by reindexing and combining terms into a single sum as much as possible.

$$y' = \sum_{n=1} n c_n x^{n-1}$$

$$y'' = \sum_{n=2} n(n-1) c_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m$$

$$x = y'' + y = \sum_{m=0}^{\infty} \left[(m+2)(m+1) c_{m+2} + c_m \right] x^m$$

(b) By equating coefficients on the left and right sides of your equation in the previous part, write down a recurrence relation on the coefficients c_n for $n \geq 2$. 4

$$0 = (m+2)(m+1) c_{m+2} + c_m \quad \text{for } m \neq 1$$

$$1 = 6c_3 + c_1$$

$$\Rightarrow \quad c_3 = \frac{1 - c_1}{6}, \quad c_{m+2} = \frac{-c_m}{(m+2)(m+1)} \Rightarrow c_n = \frac{-c_{n-2}}{n(n-1)} \quad \text{for } n \neq 3$$

(c) Using the recurrence relation, compute the first few odd coefficients c_1, c_3, c_5, c_7, and then write down the pattern for the general odd coefficient c_{2m+1} for any $m \geq 1$

$$c_1 = y'(0) = 2 \qquad c_7 = \frac{-c_5}{7 \cdot 6} = \frac{(-1)^3}{7!}$$

$$c_3 = \frac{1-2}{6} = \frac{-1}{3!}$$

$$c_5 = \frac{-c_3}{5 \cdot 4} = \frac{(-1)^2}{5!} \qquad c_{2m+1} = \frac{(-1)^m}{(2m+1)!} \quad \text{for } m \geq 1$$

(d) Using the recurrence relation, compute the first few even coefficients c_0, c_2, c_4, c_6, and then write down the pattern for the general even coefficient c_{2m} for any $m \geq 0$.

$$c_0 = y(0) = -1 \qquad c_6 = \frac{-c_4}{6 \cdot 5} = \frac{(-1)^4}{6!}$$

$$c_2 = \frac{-c_0}{2 \cdot 1} = \frac{(-1)^2}{2!} \qquad c_{2m} = \frac{(-1)^{m+1}}{(2m)!} \quad \text{for } m \geq 0$$

$$c_4 = \frac{-c_2}{4 \cdot 3} = \frac{(-1)^3}{4!}$$

(e) Write down the solution to differential equation. Do you recognize any of the parts of the power series expansion?

$$y = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(2m)!} x^{2m} + 2x + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

$$= -\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \right) + 2x + (\sin x - x)$$

$$= -\cos x + x + \sin x$$

2. (a) Solve the following initial value problem.

$$(x^2+1)y'' + 2xy' - 2y = 0, \qquad y(0) = y'(0) = 1$$

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad xy' = x\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^n$$

$$(x^2+1)y'' = (x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

$$= \sum_{m=2}^{\infty} m(m-1)c_m x^m + \sum_{m=0}^{\infty} (m+2)(m+1)c_{m+2} x^m$$

$$0 = (x^2+1)y'' + 2xy' - 2y = (2c_2 - 2c_0) + (6c_3 + \cancel{2c_1} - \cancel{2c_1})x$$

$$+ \sum_{m=2}^{\infty} \left[m(m-1)c_m + (m+2)(m+1)c_{m+2} + 2mc_m - 2c_m \right] x^m$$

$$\Rightarrow c_2 = c_0 = y(0) = 1, \quad c_3 = 0, \quad c_{m+2} = -\frac{(m-1)c_m}{m+1}$$

for $m \geq 2$

$$c_5 = -\frac{2c_3}{4} = 0, \quad c_7 = -\frac{4c_5}{6} = 0 \Rightarrow c_{2m+1} = 0 \text{ for } m \geq 1$$

$$c_4 = -\frac{c_2}{3} = -\frac{1}{3}, \quad c_6 = -\frac{3c_4}{5} = \left(-\frac{1}{\cancel{3}}\right)\left(-\frac{\cancel{3}}{5}\right) = \frac{(-1)^2}{5}$$

$$\Rightarrow c_{2m} = \frac{(-1)^{m-1}}{2m-1} \text{ for } m \geq 1$$

$$y = 1 + x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} x^{2m} = 1 + x + x\arctan x$$

(b) Find the radius of convergence of your solution.

$$\lim_{m\to\infty} \frac{\left| \frac{(-1)^m}{2m+1} x^{2m+2} \right|}{\left| \frac{(-1)^{m-1}}{2m-1} x^{2m} \right|} = |x|^2 \lim_{m\to\infty} \frac{2m-1}{2m+1} = |x|^2$$

By Ratio Test, the series converges absolutely if $|x|^2 < 1$ ($\Leftrightarrow$ $|x| < 1$) and diverges if $|x|^2 > 1$ ($\Leftrightarrow$ $|x| > 1$). So, $R = 1$.

3. Find the first 6 coefficients of the power series expansion of the solution to the initial value problem.

$$y'' + (1+x)y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

$$(1+x)y = (1+x)\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= \sum_{m=0}^{\infty} c_m x^m + \sum_{m=1}^{\infty} c_{m-1} x^m$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m$$

$$0 = y'' + (1+x)y = (2c_2 + c_0) + \sum_{m=1}^{\infty} \left[(m+2)(m+1)c_{m+2} + c_m + c_{m-1}\right] x^m$$

$$\Rightarrow c_2 = -\frac{c_0}{2}, \quad c_{m+2} = (-1)\frac{c_m + c_{m-1}}{(m+2)(m+1)} \quad \text{for } m \geq 1$$

$$c_0 = y(0) = 1, \quad c_1 = y'(0) = 0, \quad c_2 = \frac{-1}{2},$$

$$c_3 = (-1)\frac{c_1 + c_0}{3\cdot 2} = \frac{-1}{3!}, \quad c_4 = (-1)\frac{c_2 + c_1}{4\cdot 3} = \frac{(-1)^2}{4!}, \quad c_5 = (-1)^2 \cdot \frac{4}{5!}$$

$$y = 1 - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{(-1)^2}{4!}x^4 + \frac{(-1)^2}{5!}\cdot 4x^5 + \cdots$$