Matrix Factorizations and Singularity Categories in Codimension Two

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Abstract

A theorem of Orlov from 2004 states that the homotopy category of matrix factorizations on an affine hypersurface Y is equivalent to a quotient of the bounded derived category of coherent sheaves on Y called the singularity category. This result was subsequently generalized to complete intersections of higher codimension by Burke and Walker. In 2013, Eisenbud and Peeva introduced the notion of matrix factorizations in arbitrary codimension. As a first step towards reconciling these two approaches, this note describes how to construct a functor from codimension two matrix factorizations to the singularity category of the corresponding complete intersection.

1 Introduction

Matrix factorizations were invented by Eisenbud in [Eis80] as a means of compactly describing the minimal free resolutions of stable maximal Cohen-Macaulay modules over a local hypersurface ring. In the past decade, matrix factorizations have taken on a greater significance in the physics of B-branes on a Landau-Ginzburg model. More specifically, for a Landau-Ginzberg model whose target space is a smooth affine variety X over $\mathbb{C}$ with superpotential $W : X \to \mathbb{C}$, Kapustin and Li [KL03] argued that the category of B-branes with critical value $\lambda \in \mathbb{C}$ should be given by matrix factorizations of $W - \lambda$ up to a suitable notion of homotopy. It was subsequently shown by Orlov [Orl04] that the homotopy category of matrix factorizations of $W - \lambda$ is equivalent to the triangulated category of singularities $\mathsf{D}_{\mathrm{sg}}(X_\lambda)$ on the fiber X_λ of W. The latter category is just the Verdier quotient of the bounded derived category of coherent sheaves on X_λ by the thick subcategory of perfect complexes and, therefore, provides a potential definition of B-branes on nonaffine varieties as well.

In [EP16], Eisenbud and Peeva identify a particularly nice class of modules over local complete intersection rings and construct the minimal free resolutions of such modules by defining matrix factorizations of arbitrary codimension. Ideally, one might be able to define a notion of homotopy for these higher codimension matrix factorizations and extend Orlov's result almost verbatim. This seems out of reach for the time being, but as a first step towards this goal in codimension two, we construct a functor from the category of matrix factorizations to the singularity category of the corresponding complete intersection.

Notation 1.1. Let Q be a regular ring of finite Krull dimension, f_1, f_2 be a Q-regular sequence, and $R = Q/(f_1, f_2)$. We also consider graded matrix factorizations over the polynomial ring $S = Q[T_1, T_2]$ of the element $W = f_1T_1 + f_2T_2$, and we set $Y = \operatorname{Proj} S/(W)$.

Burke and Walker [BW15, 2.10, 6.8] have already shown that $\mathsf{D}_{\mathrm{sg}}(Y) \simeq \mathsf{D}_{\mathrm{sg}}(R)$ is equivalent to a quotient of the homotopy category of graded matrix factorizations of W over S. Thus, it will be sufficient for our purposes to construct a functor from codimension two matrix factorizations of f_1, f_2 over Q to graded matrix factorizations of W over S. The strategy is simple: Burke and Walker show how to obtain a graded matrix factorization of W from a finite Q-free resolution of an R-module together with a chosen system of higher homotopies. Additionally, Eisenbud and Peeva show how to construct a finite Q-free resolution of a matrix factorization module coming from a codimension two matrix factorization. This note describes how every codimension two matrix factorization encodes a canonical choice of higher homotopies on its induced Q-free resolution and how morphisms of matrix factorizations induce chain maps on Q-free resolutions and morphisms on the corresponding graded matrix factorizations.

2 Matrix Factorizations

In this section, we recall the definitions of the various kinds of matrix factorizations that will be relevant below. Because higher codimension matrix factorizations are defined recursively, we must first understand matrix factorizations in codimension one.

Definition 2.1. A (codimension one) *matrix factorization* of f_1 consists of a pair of finitely generated free Q-modules B_{01} and B_{11} together with homomorphisms

$$B_{01} \xrightarrow{h_1} B_{11} \xrightarrow{b_1} B_{01}$$

such that $b_1h_1 = f_1 \cdot \mathrm{Id}_{B_{01}}$ and $h_1b_1 = f_1 \cdot \mathrm{Id}_{B_{11}}$. We frequently omit the modules from the notation and write (b_1, h_1) for the matrix factorization. A *morphism* of matrix factorizations is a pair of homomorphisms $\alpha_1 : B_{01} \to B'_{01}$ and $\beta_1 : B_{11} \to B'_{11}$ such that the following diagram commutes.

$$\begin{tikzcd}
B_{01} \arrow[r, "h_1"] \arrow[d, "\alpha_1"'] & B_{11} \arrow[r, "b_1"] \arrow[d, "\beta_1"] & B_{01} \arrow[d, "\alpha_1"] \\
B'_{01} \arrow[r, "h'_1"] & B'_{11} \arrow[r, "b'_1"] & B'_{01}
\end{tikzcd}$$

Because f_1 is regular on Q, the multiplication by f_1 map on B_{m1} is injective for $m = 0, 1$. A simple consequence of this fact is that b_1 and h_1 are both injective maps. In addition, to check that (α_1, β_1) is a morphism, it suffices to check that only the right square above commutes. We will use these observations freely in the proofs below and refer the reader to [Eis80], [LW12], or [Yos90] for more on codimension one matrix factorizations.

Definition 2.2. A *codimension two matrix factorization* of f_1, f_2 consists of four finitely generated free Q-modules B_{01}, B_{02}, B_{11} and B_{12} together with homomorphisms as shown below

$$B_{01} \xrightarrow{h_1} B_{11} \xrightarrow{b_1} B_{01} \qquad \begin{tikzcd} B_{01} \arrow[r, "\rho_1"] \arrow[dr, "\theta_1"'] & B_{11} \arrow[r, "b_1"] & B_{01} \\ \oplus & \oplus & \oplus \\ B_{02} \arrow[r, "\theta_2"] \arrow[ur, "\rho_2"] & B_{12} \arrow[r, "b_2"] \arrow[ur, "\psi"] & B_{02} \end{tikzcd} \tag{2.1}$$

such that (b_1, h_1) is a codimension one matrix factorization of f_1 and if we set

$$d = \begin{pmatrix} b_1 & \psi \\ 0 & b_2 \end{pmatrix} : B_{11} \oplus B_{12} \longrightarrow B_{01} \oplus B_{02} \qquad h_2 = \begin{pmatrix} \rho_1 & \rho_2 \\ \theta_1 & \theta_2 \end{pmatrix} : B_{01} \oplus B_{02} \to B_{11} \oplus B_{12}$$

then we have the following equalities modulo f_1

$$dh_2 \equiv f_2 \cdot \mathrm{Id} \qquad h_2 d \equiv \begin{pmatrix} * & * \\ 0 & f_2 \end{pmatrix}$$

Notation 2.3. For $m = 0, 1$, we set $A_m = B_{m1} \oplus B_{m2}$. We can think of A_m as being filtered by the submodules $A_m(p) = \bigoplus_{q \le p} B_{mq}$ for $p = 1, 2$, and we define

$$h = \begin{pmatrix} h_1 & \rho_1 & \rho_2 \\ 0 & \theta_1 & \theta_2 \end{pmatrix} : A_0(1) \oplus A_0(2) \to A_1$$

As in the codimension one case, we frequently omit the modules from the notation and write (d, h) for the matrix factorization. The R-module $M = \mathrm{Coker}(d \otimes \mathrm{Id}_R)$ is called the *matrix factorization module* of (d, h) and will sometimes be denoted by $\mathrm{Coker}(d, h)$.

The following simple lemma will be well-known to experts, but we state it explicitly because it plays such a key role in all our constructions.

Lemma 2.4. *Suppose that F and G are free Q-modules and that $\varphi : F \to G$ is a homomorphism such that $\varphi \otimes \mathrm{Id}_{Q/(f_1)} = 0$. Then there exists a unique homomorphism $t : F \to G$ such that $\varphi = f_1 t$.*

As a consequence of the preceding lemma, we see that there exist homomorphisms that

$$dh_2 = \begin{pmatrix} f_2 + f_1\lambda_1 & f_1\lambda_2 \\ f_1\epsilon_1 & f_2 + f_1\epsilon_2 \end{pmatrix} \qquad h_2 d = \begin{pmatrix} * & * \\ f_1\omega_1 & f_2 + f_1\omega_2 \end{pmatrix}$$

By replacing h_2 with the matrix

$$\widetilde{h}_2 = \begin{pmatrix} \rho_1 - h_1\lambda_1 & \rho_2 - h_2\lambda_2 \\ \theta_1 & \theta_2 \end{pmatrix}$$

it is easily checked that we get a matrix factorization $(d, \widetilde{h})$ with the same matrix factorization module such that

$$d\widetilde{h}_2 = \begin{pmatrix} f_2 & 0 \\ f_1\epsilon_1 & f_2 + f_1\epsilon_2 \end{pmatrix} \qquad \widetilde{h}_2 d = \begin{pmatrix} * & * \\ f_1\omega_1 & f_2 + f_1\omega_2 \end{pmatrix} \tag{2.2}$$

We will henceforth assume without loss of generality that (2.2) holds for every matrix factorization; this will simplify certain expressions later on.

Definition 2.5. A *morphism* of codimension two matrix factorizations is a triple of homomorphisms

$$\alpha = \begin{pmatrix} \alpha_1 & \gamma \\ 0 & \alpha_2 \end{pmatrix} : B_{01} \oplus B_{02} \to B'_{01} \oplus B'_{02} \qquad \beta = \begin{pmatrix} \beta_1 & \delta \\ 0 & \beta_2 \end{pmatrix} : B_{11} \oplus B_{12} \to B'_{11} \oplus B'_{12}$$

$$\widetilde{\alpha} = \begin{pmatrix} \alpha_1 & \chi_1 & \chi_2 \\ 0 & \alpha_1 & \gamma \\ 0 & 0 & \alpha_2 \end{pmatrix} : A_0(1) \oplus A_0(2) \to A'_0(1) \oplus A'_0(2)$$

such that (α_1, β_1) is a morphism of codimension one matrix factorizations and the following diagram commutes modulo f_1

$$\begin{tikzcd} A_0(1) \oplus A_0(2) \arrow[r, "h"] \arrow[d, "\widetilde{\alpha}"'] & A_1 \arrow[r, "d"] \arrow[d, "\beta"] & A_0 \arrow[d, "\alpha"] \\ A'_0(1) \oplus A'_0(2) \arrow[r, "h'"] & A'_1 \arrow[r, "d'"] & A'_0 \end{tikzcd}$$

The category of codimension two matrix factorizations and morphisms of f_1, f_2 over Q will be denoted by $\mathsf{MF}(Q, f_1, f_2)$.

We will abuse notation slightly and denote a morphism of matrix factorizations simply by the pair (α, β). Using Lemma 2.4 once again, we see that there exist homomorphisms such that

$$\alpha d - d'\beta = \begin{pmatrix} 0 & f_1 u \\ 0 & f_1 v \end{pmatrix} \qquad \beta h - h'\widetilde{\alpha} = \begin{pmatrix} 0 & f_1 s_1 & f_1 s_2 \\ 0 & f_1 r_1 & f_1 r_2 \end{pmatrix}$$

By replacing the maps δ, χ_1 and χ_2 with $\delta + h'_1 u$, $\chi_1 + u\theta_1 + b'_1 s_1$, and $\chi_2 + u\theta_2 + b'_1 s_2$ respectively, we may assume without loss of generality that we have equalities of the form

$$\alpha d - d'\beta = \begin{pmatrix} 0 & 0 \\ 0 & f_1 v \end{pmatrix} \qquad \beta h - h'\widetilde{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_1 r_1 & f_1 r_2 \end{pmatrix} \tag{2.3}$$

The following is a simple but technical lemma relating the matrix factorization maps defined above that we will need later.

Lemma 2.6. *Let $(\alpha,\beta):(d,h)\to(d',h')$ and $(\alpha',\beta'):(d',h')\to(d'',h'')$ be morphisms of matrix factorizations. Set $(\alpha'',\beta'')=(\alpha'\alpha,\beta'\beta)$. Then we have the following equalities:*

$$\theta_1=\omega_1 h_1 \tag{2.4}$$

$$\chi_1=\gamma\epsilon_1-\psi' r_1 \qquad \chi_2=\gamma\epsilon_2-\psi' r_2 \tag{2.5}$$

$$\gamma' v=0 \qquad v''=v'\beta_2+\alpha_2' v \tag{2.6}$$

$$r_1''=\beta_2' r_1+r_1'\alpha_1 \qquad r_2''=\beta_2' r_2+r_1'\gamma+r_2'\alpha_2 \tag{2.7}$$

Proof. The basic idea of the proof is to multiply by f_1, do some formal manipulations using known commutativity relations (2.2) and (2.3), and then cancel f_1 to obtain the desired relations. To make the computation less opaque to the reader, we underline the terms being replaced in each step. For example, we have

$$\begin{aligned} f_1\chi_1 = b_1'\underline{h_1'\chi_1} &= \underline{b_1'\beta_1}\rho_1+b_1'\delta\theta_1-\underline{b_1'\rho_1'}\alpha_1=\alpha_1\underline{b_1\rho_1}+b_1'\delta\theta_1+\psi'\theta_1\alpha_1-f_2\alpha_1 \\ &= \psi'\theta_1'\alpha_1+\underline{(-\alpha_1\psi+b_1'\delta)}\theta_1=\psi'\underline{(\theta_1'\alpha_1-\beta_2\theta_1)}+\gamma b_2\theta_1 \\ &= -f_1\psi' r_1+\gamma\underline{b_2\theta_1}=f_1(\gamma\epsilon_1-\psi' r_1) \end{aligned}$$

Canceling f_1, we see that equality for χ_1 holds. The other equalities are easily checked in a similar fashion. ∎

Remark 2.7. Because r_1 and r_2 are the unique maps satisfying $f_1 r_1=\beta_2\theta_1-\theta_1'\alpha_1$ and $f_2 r_2=\beta_2\theta_2-\theta_1'\gamma-\theta_2'\alpha_2$, the preceding lemma shows that the maps χ_1 and χ_2 in the definition of a morphism of matrix factorizations are actually uniquely determined by the maps (α,β), justifying our earlier abuse of notation.

Recall that we set $S=Q[T_1,T_2]$ and $W=f_1T_1+f_2T_2$. We view S as a graded ring with the standard $\mathbb{Z}$-grading by total degree.

Definition 2.8. A *graded matrix factorization* E of W is a pair of finitely generated free graded S-modules E_0 and E_1 together with graded homomorphisms

$$E_1\xrightarrow{e_1}E_0\xrightarrow{e_0}E_1(1)$$

such that $e_0e_1=e_1(1)e_0=W$. A *morphism* of graded matrix factorizations $\alpha:E\to E'$ is a pair of graded homomorphisms φ_0 and φ_1 such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "e_1"] \arrow[d, "\varphi_1"'] & E_0 \arrow[r, "e_0"] \arrow[d, "\varphi_0"] & E_1(1) \arrow[d, "\varphi_1(1)"] \\ E_1' \arrow[r, "e_1'"] & E_0' \arrow[r, "e_0'"] & E_1'(1) \end{tikzcd}$$

The category of all graded matrix factorizations of W over S will be denoted by $\mathsf{MF}^{\mathrm{gr}}(S,W)$.

3 Construction of the Functor

As noted in the introduction, systems of higher homotopies also play an important role in constructing the desired functor on codimension two matrix factorizations. The description of higher homotopies involves multi-indices $J = (J_1, J_2) \in \mathbb{Z}^2$. We write e_i for the i-th standard basis vector over $\mathbb{Z}$, and set $|J| = J_1 + J_2$. In this section, a graded Q-module F will also refer to an externally graded collection $\{F_n\}_{n\in\mathbb{Z}}$ of Q-modules, and a graded map $f : F \to G$ between graded modules will be a family of homomorphisms $f_n : F_n \to G_n$ for every $n \in \mathbb{Z}$. We denote by $F[i]$ the graded module such that $F[i]_n = F_{n+i}$. In particular, if F and G are chain complexes, then a nullhomotopy for a chain map $f : F \to G$ is an example of a graded map $F \to G[1]$.

Definition 3.1. Let F be a complex of Q-modules. A *system of higher homotopies* for f_1, f_2 on F is a family of homotopies $\sigma_J : F \to F[2|J| - 1]$ for each $J \in \mathbb{N}^2 \setminus \{0\}$ such that:

(i) σ_{e_i} is a nullhomotopy for the multiplication by f_i map on F for each i.

(ii) σ_K is a nullhomotopy for the chain map $-\sum_{I+J=K} \sigma_I\sigma_J$ for $|K| \geq 2$.

Let (d, h) be a matrix factorization of codimension two. Eisenbud and Peeva prove that the following complex is a Q-free resolution of its MF module, [EP16, 3.1.3-3.1.6].

$$\mathbf{L}(d,h) : \qquad B_{12} \xrightarrow{\begin{pmatrix} h_1\psi \\ -f_1 \\ b_2 \end{pmatrix}} B_{11} \oplus B_{12} \oplus B_{02} \xrightarrow{\begin{pmatrix} b_1 & \psi & 0 \\ 0 & b_2 & f_1 \end{pmatrix}} B_{01} \oplus B_{02} \tag{3.1}$$

Construction 3.2. Suppose we are given a morphism $(\alpha, \beta) : (d, h) \to (d', h')$. Then we can construct a chain map $\mathbf{L}(\alpha, \beta) : \mathbf{L}(d, h) \to \mathbf{L}(d', h')$ with $\mathbf{L}(\alpha, \beta)_2 = \beta_2$, $\mathbf{L}(\alpha, \beta)_0 = \alpha$ and

$$\mathbf{L}(\alpha,\beta)_1 = \begin{pmatrix} \beta_1 & \delta & h_1'\gamma \\ 0 & \beta_2 & 0 \\ 0 & v & \alpha_2 \end{pmatrix}$$

In addition, there is canonical choice of higher homotopies $\sigma(d, h)$ on $\mathbf{L}(d, h)$ coming from the data of the matrix factorization as given below.

$$\sigma(d,h)_{e_1} : \qquad \begin{pmatrix} 0 & -\mathrm{Id}_{B_{12}} & 0 \end{pmatrix} \qquad \begin{pmatrix} h_1 & 0 \\ 0 & 0 \\ 0 & \mathrm{Id}_{B_{02}} \end{pmatrix}$$

$$\sigma(d,h)_{e_2} : \qquad \begin{pmatrix} \omega_1 & \omega_2 & \theta_2 \end{pmatrix} \qquad \begin{pmatrix} \rho_1 & \rho_2 \\ \theta_1 & \theta_2 \\ -\epsilon_1 & -\epsilon_2 \end{pmatrix}$$

and $\sigma(d,h)_J = 0$ for all $J \in \mathbb{N}^2 \setminus \{0\}$ with $|J| \geq 2$. Following [BW15, §6.3], we can then construct a graded matrix factorization $\mathbf{E}(d,h)$ of W as shown below

$$(B_{11} \oplus B_{12} \oplus B_{02}) \otimes S \xrightarrow{e_1(d,h)} \begin{matrix} B_{12} \otimes S(1) \\ \oplus \\ (B_{01} \oplus B_{02}) \otimes S \end{matrix} \xrightarrow{e_0(d,h)} (B_{11} \oplus B_{12} \oplus B_{02}) \otimes S(1)$$

where we have

$$e_1(d,h) = \begin{pmatrix} T_2\omega_1 & -T_1 + T_2\omega_2 & T_2\theta_2 \\ b_1 & \psi & 0 \\ 0 & b_2 & f_1 \end{pmatrix} \qquad e_0(d,h) = \begin{pmatrix} h_1\psi & T_1h_1 + T_2\rho_1 & T_2\rho_2 \\ -f_1 & T_2\theta_1 & T_2\theta_2 \\ b_2 & -T_2\epsilon_1 & T_1 - T_2\epsilon_2 \end{pmatrix}$$

Here, we abuse notation slightly and do not distinguish between the matrices over Q and the matrices tensored over S. Every morphism (α,β) of codimension two matrix factorizations induces a morphism $\mathbf{E}(\alpha,\beta)$ of graded matrix factorizations via $\mathbf{E}(\alpha,\beta)_1 = \mathbf{L}(\alpha,\beta)_1$ and

$$\mathbf{E}(\alpha,\beta)_0 = \begin{pmatrix} \beta_2 & -T_2r_1 & -T_2r_2 \\ 0 & \alpha_1 & \gamma \\ 0 & 0 & \alpha_2 \end{pmatrix}$$

Theorem 3.3. *With the notation as above:*

(a) *There is a faithful, additive functor $\mathbf{L} : \mathsf{MF}(Q, f_1, f_2) \longrightarrow \mathsf{Perf}(Q)$ to the category of perfect complexes of Q-modules given by the assignments $(d,h) \mapsto \mathbf{L}(d,h)$ and $(\alpha,\beta) \mapsto \mathbf{L}(\alpha,\beta)$.*

(b) *The maps $\sigma(d,h)$ form a system of higher homotopies on $\mathbf{L}(d,h)$ for every matrix factorization (d,h).*

(c) *There is a faithful, additive functor $\mathbf{E} : \mathsf{MF}(Q, f_1, f_2) \longrightarrow \mathsf{MF}^{\mathrm{gr}}(S,W)$ given by the assignments $(d,h) \mapsto \mathbf{E}(d,h)$ and $(\alpha,\beta) \mapsto \mathbf{E}(\alpha,\beta)$.*

Proof. (a) Let $(\alpha,\beta) : (d,h) \to (d',h')$ be a morphism of codimension two matrix factorizations. First, we check that $\mathbf{L}(\alpha,\beta)$ is a chain map as claimed above. We denote by $\partial_i : \mathbf{L}(d,h)_{i+1} \to \mathbf{L}(d,h)_i$ the i-th differential of the complex $\mathbf{L}(d,h)$, and similarly, we denote the differentials of $\mathbf{L}(d',h')$ by ∂'_i. Simple calculations using the relations (2.3) show

$$\mathbf{L}(\alpha,\beta)_1\partial_1 = \begin{pmatrix} \beta_1h_1\psi - f_1\delta + h'_1\gamma b_2 \\ -f_1\beta_2 \\ -f_1v + \alpha_2b_2 \end{pmatrix} = \begin{pmatrix} h'_1(\alpha_1\psi + \gamma b_2 - b'_1\delta) \\ -f_1\beta_2 \\ b'_2\beta_2 \end{pmatrix} = \begin{pmatrix} h'_1\psi'\beta_2 \\ -f_1\beta_2 \\ b'_2\beta_2 \end{pmatrix} = \partial'_1\mathbf{L}(\alpha,\beta)_2$$

$$\mathbf{L}(\alpha,\beta)_0\partial_0 = \begin{pmatrix} \alpha_1b_1 & \alpha_1\psi + \gamma b_2 & f_1\gamma \\ 0 & \alpha_2b_2 & f_1\alpha_2 \end{pmatrix}$$

$$\partial_0'\mathbf{L}(\alpha,\beta)_1 = \begin{pmatrix} b_1'\beta_1 & b_1'\delta + \psi'\beta_2 & b_1'h_1'\gamma \\ 0 & b_2'\beta_2 + f_1 v & f_1\alpha_2 \end{pmatrix}$$

Comparing the entries of the latter two matrices using the relations, it is clear that they agree. Hence, $\mathbf{L}(\alpha,\beta)$ is a chain map as claimed.

If $(\alpha,\beta) = (\mathrm{Id},\mathrm{Id})$ is the identity morphism, it is easily checked that $\mathbf{L}(\alpha,\beta)$ is the identity map on $\mathbf{L}(d,h)$, since the maps γ, δ, and v must all be zero in this case. Suppose (α',β') is another morphism, and set $(\alpha'',\beta'') = (\alpha'\alpha,\beta'\beta)$. Because we have

$$\alpha'' = \begin{pmatrix} \alpha_1'\alpha_1 & \alpha_1'\gamma + \gamma'\alpha_2 \\ 0 & \alpha_2'\alpha_2 \end{pmatrix} \qquad \beta'' = \begin{pmatrix} \beta_1'\beta_1 & \beta_1'\delta + \delta'\beta_2 \\ 0 & \beta_2'\beta_2 \end{pmatrix} \tag{3.2}$$

it is immediate that $\mathbf{L}(\alpha'',\beta'')_i = \mathbf{L}(\alpha',\beta')_i\mathbf{L}(\alpha,\beta)_i$ for $i = 0,2$. In addition, we calculate

$$\mathbf{L}(\alpha',\beta')_1\mathbf{L}(\alpha,\beta)_1 = \begin{pmatrix} \beta_1'\beta_1 & \beta_1'\delta + \delta'\beta_2 + h_1''\gamma' v & \beta_1'h_1'\gamma + h_1''\gamma'\alpha_2 \\ 0 & \beta_2'\beta_2 & 0 \\ 0 & v'\beta_2 + \alpha_2' v & \alpha_2'\alpha_2 \end{pmatrix}$$

Since $\beta_1'h_1'\gamma + h_1''\gamma'\alpha_2 = h_1''(\alpha_1'\gamma + \gamma'\alpha_2)$, we need only show that $h_1''\gamma' v = 0$ and $v'' = v'\beta_2 + \alpha_2' v$ to prove that $\mathbf{L}(\alpha'',\beta'')_1 = \mathbf{L}(\alpha',\beta')_1\mathbf{L}(\alpha,\beta)_1$. This is the content of Lemma 2.6. Thus, the assignment $(\alpha,\beta) \mapsto \mathbf{L}(\alpha,\beta)$ is functorial. It is also easily seen that the functor $\mathbf{L}$ is additive. If $\mathbf{L}(\alpha,\beta) = 0$, then the maps α and β must be zero, and Lemma 2.6 implies that $\chi_1 = 0 = \chi_2$. Hence, $\mathbf{L}$ is also faithful.

(b) Since the resolution $\mathbf{L}(d,h)$ has length two, it is clear that we must have $\sigma(d,h)_J = 0$ for $|J| \geq 2$, so we need only check that $\sigma(d,h)_{e_i}$ is a nullhomotopy for f_i for $i = 1,2$. Checking that $\sigma(d,h)_{e_1}$ is a nullhomotopy for f_1 is a straightforward calculation which we leave to the reader. To see that $\sigma(d,h)_{e_2}$ is a nullhomotopy for f_2, we note that $\partial_0[\sigma(d,h)_{e_2}]_0 = f_2\,\mathrm{Id}_{\mathbf{L}(d,h)_0}$ follows immediately from (2.2) and (2.3). We also have

$$[\sigma(d,h)_{e_2}]_1\partial_1 = \begin{pmatrix} \omega_1 & \omega_2 & \theta_2 \end{pmatrix}\begin{pmatrix} h_1\psi \\ -f_1 \\ b_2 \end{pmatrix} = \omega_1 h_1\psi \underline{- f_1\omega_2 + \theta_2 b_2} = \omega_1 h_1\psi + f_2\,\mathrm{Id}_{B_{01}} - \underline{\theta_1}\psi = f_2\,\mathrm{Id}_{\mathbf{L}(d,h)_2}$$

where the last equality follows from Lemma 2.6 and we have marked the terms being replaced at each step as in the proof of the lemma. Finally, using (2.2) and (2.3), we compute that

$$\partial_1[\sigma(d,h)_{e_2}]_1 + [\sigma(d,h)_{e_2}]_0\partial_0 = \begin{pmatrix} h_1\psi\omega_1 + \rho_1 b_1 & h_1\psi\omega_2 + \rho_1\psi + \rho_2 b_2 & h_1\psi\theta_2 + f_1\rho_2 \\ 0 & f_2 & 0 \\ b_2\omega_1 - \epsilon_1 b_1 & b_2\omega_2 - \epsilon_1\psi - \epsilon_2 b_2 & f_2 \end{pmatrix}$$

We must show that the above map is equal to $f_2\,\mathrm{Id}_{\mathbf{L}(d,h)_1}$. We note that $h_1\psi\theta_2 + f_1\rho_2 =$

$h_1(\psi\theta_2 + b_1\rho_2) = 0$ by (2.2), and

$$\underline{f_1}(b_2\underline{\omega_1} - \epsilon_1 b_1) = \underline{(b_2\theta_1 - f_1\epsilon_1)}b_1 = 0$$

so that canceling f_1 yields $b_2\omega_1 - \epsilon_1 b_1 = 0$. That $b_2\omega_2 - \epsilon_1\psi - \epsilon_2 b_2 = 0$ is proved in a similar manner. Furthermore, we see that

$$b_1(h_1\psi\omega_1 + \rho_1 b_1) = \underline{f_1}\psi\underline{\omega_1} + b_1\rho_1 b_1 = \underline{(\psi\theta_1 + b_1\rho_1)}b_1 = f_2 b_1$$

Since b_1 is also an injective map, we can cancel b_1 from the preceding equality to obtain $h_1\psi\omega_1 + \rho_1 b_1 = f_2 \operatorname{Id}_{B_{11}}$, and that $h_1\psi\omega_2 + \rho_1\psi + \rho_1 b_2 = 0$ is similarly proved. Hence, we have $\partial_1[\sigma(d,h)_{e_2}]_1 + [\sigma(d,h)_{e_2}]_0\partial_0 = f_2 \operatorname{Id}_{\mathbf{L}(d,h)_1}$ so that $\sigma(d,h)_{e_2}$ is a nullhomotopy for f_2 as claimed.

(c) It is immediate from part (b) and [BW15, §6.3] that $\mathbf{E}(d,h)$ is a graded matrix factorization of W. Given a morphism $(\alpha,\beta) : (d,h) \to (d',h')$, we check that $\mathbf{E}(\alpha,\beta)$ is a morphism of graded matrix factorizations. The homomorphisms $\mathbf{E}(\alpha,\beta)_i$ for $i = 0,1$ are graded by construction. We set $e_i = e_i(d,h)$ and $e'_i = e_i(d',h')$ for $i = 0,1$. By our earlier remarks about morphisms of codimension one matrix factorizations, it is enough to prove that $\mathbf{E}(\alpha,\beta)_0 e_1 = e'_1\mathbf{E}(\alpha,\beta)_1$. A simple calculation shows

$$\mathbf{E}(\alpha,\beta)_0 e_1 = \begin{pmatrix} T_2(\beta_2\omega_1 - r_1 b_1) & -T_1\beta_2 + T_2(\beta_2\omega_2 - r_1\psi - r_2 b_2) & T_2(\beta_2\theta_2 - f_1 r_2) \\ \alpha_1 b_1 & \alpha_1\psi + \gamma b_2 & f_1\gamma \\ 0 & \alpha_2 b_2 & f_1\alpha_2 \end{pmatrix}$$

$$e'_1\mathbf{E}(\alpha,\beta)_1 = \begin{pmatrix} T_2\omega'_1\beta_1 & -T_1\beta_2 + T_2(\omega'_1\delta + \omega'_2\beta_2 + \theta'_2 v) & T_2(\omega'_1 h'_1\gamma + \theta'_2\alpha_2) \\ b'_1\beta_1 & b'_1\delta + \psi'\beta_2 & b'_1 h'_1\gamma \\ 0 & b'_2\beta_2 + f_1 v & f_1\alpha_2 \end{pmatrix}$$

Comparing entries, it is immediately clear from the relations (2.3) that all of the entries in the second and third rows agree. That the 1,3-entries agree also follows from this and the equality $\theta'_1 = \omega'_1 h'_1$ of Lemma 2.6, so we need only show that the following equalities hold:

$$\beta_2\omega_1 - r_1 b_1 = \omega'_1\beta_1 \tag{3.3}$$

$$\beta_2\omega_2 - r_1\psi - r_2 b_2 = \omega'_1\delta + \omega'_2\beta_2 + \theta'_2 v \tag{3.4}$$

Multiplying the left side of (3.3) by f_1, we obtain

$$\underline{f_1}(\beta_2\underline{\omega_1} - \underline{r_1}b_1) = \beta_2\theta_1 b_1 - \beta_2\theta_1 b_1 + \theta'_1\underline{\alpha_1 b_1} = \underline{\theta'_1 b'_1}\beta_1 = f_1\omega'_1\beta_1$$

so that canceling the f_1 yields the desired equality. The equality (3.4) is proved similarly so that $\mathbf{E}(\alpha,\beta)$ is a morphism of graded matrix factorizations as claimed.

If $(\alpha,\beta) = (\operatorname{Id},\operatorname{Id})$ is the identity morphism, it is easily checked that $\mathbf{E}(\alpha,\beta)$ is the identity morphism on $\mathbf{E}(d,h)$, since the maps γ, δ, v and r_i must all be zero in this case.

Suppose (α', β') is another morphism, and set $(\alpha'', \beta'') = (\alpha'\alpha, \beta'\beta)$. We already know that $\mathbf{E}(\alpha'', \beta'')_1 = \mathbf{E}(\alpha', \beta')_1\mathbf{E}(\alpha, \beta)_1$ from part (a). On the other hand, we have

$$\mathbf{E}(\alpha', \beta')_0\mathbf{E}(\alpha, \beta)_0 = \begin{pmatrix} \beta_2'\beta_2 & -T_2(\beta_2' r_1 + r_1'\alpha_1) & -T_2(\beta_2' r_2 + r_1'\gamma + r_2'\alpha_2) \\ 0 & \alpha_1'\alpha_1 & \alpha_1'\gamma + \gamma'\alpha_2 \\ 0 & 0 & \alpha_2'\alpha_2 \end{pmatrix}$$

And so, using (3.2), it is enough to show that $r_1'' = \beta_2' r_1 + r_1'\alpha_1$ and $r_2'' = \beta_2' r_2 + r_1'\gamma + r_2'\alpha_2$ to prove that $\mathbf{E}(\alpha'', \beta'')_0 = \mathbf{E}(\alpha', \beta')_0\mathbf{E}(\alpha, \beta)_0$. This was already established in Lemma 2.6. Thus, the assignment $(\alpha, \beta) \mapsto \mathbf{E}(\alpha, \beta)$ is functorial. It is also easily seen that the functor $\mathbf{E}$ is additive. If $\mathbf{E}(\alpha, \beta) = 0$, then recalling the fact that we suppressed all of the tensor with Id_S signs in our formulation of $\mathbf{E}(\alpha, \beta)$ but that S is faithfully flat over Q, it follows immediately that α and β must be zero so that Lemma 2.6 implies that $\chi_1 = 0 = \chi_2$. Hence, $\mathbf{E}$ is also faithful. □

The above recipe can be used to produce graded matrix factorizations even in higher codimension since the the resolution $\mathbf{L}(d, h)$ of the matrix factorization module always admits a system of higher homotopies by [EP16, 3.4.2]. The main point in codimension two is that there is a canonical choice for each matrix factorization which makes everything functorial.

To condense what may have admittedly seemed a bit like alphabet soup up to this point, we give a concrete example.

Example 3.4. Consider the regular sequence $f_1 = xz$, $f_2 = y^2$ in $Q = \mathbb{Q}[x, y, z]$. We can construct nontrivial matrix factorizations of f_1, f_2 over Q by using the package `Complete IntersectionResolutions` for Macaulay2.

```
i1 : loadPackage "CompleteIntersectionResolutions";

i2 : Q = QQ[x,y,z]; f = matrix {{x*z, y^2}}; R = Q/ideal(f);
     M = highSyzygy coker matrix {{x,y}};

             1       2
o3 : Matrix Q  <--- Q

i6 : mf = matrixFactorization(f, M, Check => true)

o6 = {{2} | 0  z y  0 |, {3} | y -z 0 0  0 |}
      {2} | -x y 0  0 |  {3} | x 0  0 y  0 |
      {2} | 0  0 -x y |  {3} | 0 0  y -z 0 |
                         {3} | 0 0  x 0  y |

o6 : List
```

The above matrices are the maps d and h of the matrix factorization respectively. Breaking these maps into their respective blocks, we have the following codimension two matrix factorization.

$$Q^2 \xrightarrow{\begin{pmatrix} y & -z \\ x & 0 \end{pmatrix}} Q^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -x & y \end{pmatrix}} Q^2 \qquad \begin{tikzcd} Q^2 \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}"] \arrow[ddr, bend left=15, "\theta_1"] & Q^2 \arrow[r, "\begin{pmatrix} x & 0 \\ y & z \end{pmatrix}"] & Q^2 \\ \oplus & \oplus & \oplus \\ Q \arrow[r, "\begin{pmatrix} 0 \\ y \end{pmatrix}"'] & Q^2 \arrow[uur, "\psi"] \arrow[r, "\begin{pmatrix} -x & y \end{pmatrix}"'] & Q \end{tikzcd}$$

where $\psi = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$ and $\theta_1 = h_1 = \begin{pmatrix} y & -z \\ x & 0 \end{pmatrix}$. In this case, the resolution $\mathbf{L}(d, h)$ is

```
i7 : L = (makeFiniteResolutionCodim2(mf, f))#"resolution"; L.dd

          3                             5
o8 = 0 : Q  <------------------------- Q  : 1
               {2} | 0  z y  0 0  |
               {2} | -x y 0  0 0  |
               {2} | 0  0 -x y xz |

          5                        2
     1 : Q  <-------------------- Q  : 2
               {3} | y2  0   |
               {3} | xy  0   |
               {3} | -xz 0   |
               {3} | 0   -xz |
               {4} | -x  y   |

o8 : ChainComplexMap
```

In order to streamline the remainder of the example, we use supplementary functions not included in the above package, which are available at `https://github.com/mnmastro/Codim-2-Matrix-Factorizations`. To construct the system of higher homotopies, we need the maps ϵ_i and ω_i for $i = 1, 2$.

```
i9 : epsilonMaps mf

o9 = {{2} | 0 1 |, 0}

o9 : List

i10 : omegaMaps mf

o10 = {{3} | 1 0 |, 0}
       {3} | 0 1 |

o10 : List
```

We can then construct the system of higher homotopies and check that they are actually nullhomotopies for multiplication by f_i for $i = 1, 2$.

```
i11 : sigma = sigmaMaps mf

            5                      3               5                      3
o11 = {0 : Q  <------------------ Q  : 0     , 0 : Q  <------------------ Q  : 0   }
                 {3} | y -z 0 |                          {3} | 0 0  0 |
                 {3} | x 0  0 |                          {3} | 0 y  0 |
                 {3} | 0 0  0 |                          {3} | y -z 0 |
                 {3} | 0 0  0 |                          {3} | x 0  y |
                 {4} | 0 0  1 |                          {4} | 0 -1 0 |

            2                           5           2                          5
       1 : Q  <----------------------- Q  : 1  1 : Q  <--------------------- Q  : 1
                 {5} | 0 0 -1 0  0 |                     {5} | 1 0 0 0 0 |
                 {5} | 0 0 0  -1 0 |                     {5} | 0 1 0 0 y |

o11 : List


i12 : dL = map(L[-1],L, i -> L.dd_i);

i13 : {dL[1]*(sigma#0) + (sigma#0)[-1]*dL == f_0_0*id_L,
      dL[1]*(sigma#1) + (sigma#1)[-1]*dL == f_1_0*id_L}

o13 = {true, true}
```

For this particular example, the above system of higher homotopies agrees with the one constructed by the `makeHomotopies` function in the complete intersections package. It is then easy to build the corresponding graded matrix factorization of $W = xzT_1 + y^2T_2$ over $S = Q[T_1, T_2]$.

```
i14 : E = toGradedMF mf

o14 = {{0, 3} | y2  0   T_1y -T_1z 0    |, {0, 5} | T_2 0   -T_1 0    0    |}
       {0, 3} | xy  0   T_1x T_2y  0    |  {0, 5} | 0   T_2 0    -T_1 T_2y |
       {0, 3} | -xz 0   T_2y -T_2z 0    |  {0, 2} | 0   z   y    0    0    |
       {0, 3} | 0   -xz T_2x 0     T_2y |  {0, 2} | -x  y   0    0    0    |
       {0, 4} | -x  y   0    -T_2  T_1  |  {0, 2} | 0   0   -x   y    xz   |

o14 : List

i15 : S = ring E#0; W = (f*(transpose vars S))_0_0;

i17 : {E#0*E#1 == W*id_(source E#1), E#1*E#0 == W*id_(source E#0)}

o17 = {true, true}

o17 : List
```

Having constructed the desired functor $\mathbf{E} : \mathsf{MF}(Q, f_1, f_2) \to \mathsf{MF}^{\mathrm{gr}}(S, W)$, we can compose it with the functor $\Psi : \mathsf{MF}^{\mathrm{gr}}(S, W) \to \mathsf{D}_{\mathrm{sg}}(R)$ of Burke and Walker to get a functor from codimension two matrix factorizations to the singularity category.[1] On the other hand, there is a natural functor $\mathrm{Coker} : \mathsf{MF}(Q, f_1, f_2) \to \underline{\mathsf{MCM}}(R)$ taking a matrix factorization to its MF module; if $(\alpha, \beta) : (d, h) \to (d', h')$ is a morphism of matrix factorizations, we get an induced map $\mathrm{Coker}(\alpha, \beta) : \mathrm{Coker}(d, h) \to \mathrm{Coker}(d', h')$ which is clearly functorial since $d \otimes \mathrm{Id}_R$ is a presentation for the MF module of (d, h). Alternatively, Coker is just the composition of the zeroth homology functor with the functor $\mathbf{L}$ above. Because $R = Q/(f_1, f_2)$ is Gorenstein, a result of Buchweitz [Buc86, 4.4.1] shows that $\mathsf{D}_{\mathrm{sg}}(R)$ is equivalent as a triangulated category to the stable category $\underline{\mathsf{MCM}}(R)$ of maximal Cohen-Macaulay R-modules via the natural functor sending a maximal Cohen-Macaulay R-module to itself viewed as a stalk complex in $\mathsf{D}_{\mathrm{sg}}(R)$. Hence, we have a diagram

$$\begin{tikzcd}
\mathsf{MF}(Q, f_1, f_2) \arrow[r, "\mathbf{E}"] \arrow[d, "\mathrm{Coker}"'] & \mathsf{MF}^{\mathrm{gr}}(S, W) \arrow[d, "\Psi"] \\
\underline{\mathsf{MCM}}(R) \arrow[r, "\simeq"] & \mathsf{D}_{\mathrm{sg}}(R)
\end{tikzcd}$$

which commutes up to isomorphisms in $\mathsf{D}_{\mathrm{sg}}(R)$ by [BW15, 6.7]. Furthermore, the functor on the right becomes an equivalence after passing to a suitable quotient of the homotopy category of graded matrix factorizations, and so, a natural question is whether there is a suitable notion of homotopy for codimension two matrix factorizations so that the functor on the left also becomes an equivalence.

4 Potential Future Work

At this point, there is still much to be done. Here, we work only with codimension two matrix factorizations; the next obvious step would be to extend the constructions to arbitrary codimension. It is also possible to suggest definitions of nullhomotopies between codimension two matrix factorizations so that the functor $\mathbf{E}$ preserves and reflects nullhomotopic morphisms. A much larger problem is whether any such definition yields a notion of a homotopy category which is triangulated and such that $\mathbf{E}$ induces an exact functor of homotopy categories, but this may be overly optimistic.

[1] We are sweeping a few details under the rug. Technically, what we have called Ψ here involves first sheafifying the graded matrix factorization E to obtain a matrix factorization $\widetilde{E}$ of W between locally free sheaves on $\mathbb{P}^1_Q$ and then applying the functor Ψ of Burke and Walker; see [BW15, 2.11].

Acknowledgments

The author wishes to thank Hal Schenck for reading various drafts of this paper and making helpful comments as well as Sheldon Katz for originally bringing the work of Eisenbud and Peeva to his attention. He also acknowledges support from National Science Foundation grant DMS 08-38434 "EMSW21-MCTP: Research Experience for Graduate Students".

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