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Physics 512

Assignment 2

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1

The electric field generated by a ring of radius ρ and of linear charge density λ at a distance d above the center and integrated over the polar angle ϕ can be calculated by first finding the projection of that field on the z axis, since by symmetry, only its contribution remains.

$$\cos(\theta) = \frac{d}{\sqrt{d^2 + \rho^2}}$$

Then we use Coulomb's law, integrated over the entire circle of the ring.

$$E_d = \int_0^{2\pi} \frac{\lambda \rho d\phi}{4\pi\epsilon_0} \cdot \frac{\cos(\theta)}{\rho^2 + d^2} = \frac{\lambda\rho}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \frac{d}{\sqrt{d^2+\rho^2}} \tag{1}$$

This integral can be done analytically and yields

$$E(d) = \frac{\lambda \rho d}{2\epsilon_0 (d^2 + \rho^2)^{3/2}}$$

If we consider the sphere to be made of a succession of rings, we can change slightly our variables. First, we switch to cylindrical coordinates, with the azimutal angle θ between the center of the sphere (which we set at 0) and the evaluated ring and we redefine our radius $\rho = \sqrt{R^2 - h^2}$ where R is the radius of the sphere and $h = R\sin\theta$ is the height between the center of the sphere and the evaluated ring. The distance from the ring to the point of interest becomes: $d = z - h = z - R\sin\theta$. Then all that remains is integrating over the entire azimutal range of $[0, \pi]$ and changing the linear charge density to the surface density σ:

$$E(d(z)) = \frac{\sigma \rho d}{2\epsilon_0 (d^2 + \rho^2)^{3/2}}$$

$$\implies \int_0^{\pi} \frac{\sigma \rho d d\theta}{2\epsilon_0 (d^2+\rho^2)^{3/2}} = \int_0^{\pi} \frac{\sigma\sqrt{R^2 - R^2\sin^2\theta}(z - R\sin\theta)d\theta}{2\epsilon_0((z^{-}R\sin\theta)^2 + R^2 - h^2)^{3/2}}$$

We then define $u = \cos\theta$ and $du = \sin\theta d\theta$, which will be integrated over the range $[-1, 1]$. This leaves us with

$$E(z) = \frac{\sigma \cdot R}{2\epsilon_0} \int_{-1}^{1} \frac{(z - uR)du}{(R^2 + z^2 - 2 \cdot R \cdot u)^{3/2}}$$

Which we will use to integrate numerically over u for fixed values of z. Although our own integrator breaks over the singularity at $z = R$ and at points where u is relatively big enough to break the denominator. Scipy.quad, instead of crashing, will simply output a NaN instead, which we can then interpret as zeros as in this plot:

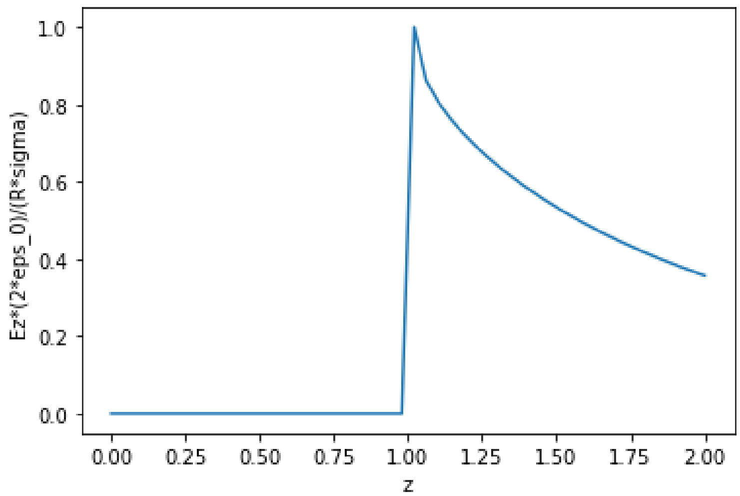

Figure 1: Plot of the normalized electric field as a function of distance to the origin.

2

For the function e^x, over the range $[-1, 1]$, we have a prediction of 2.350403 and 65 function calls, compared to the class version, which has a prediction of 2.350403 and 155 function calls. Wolframalpha provides an answer of ≈ 2.350403 for the same level of precision of 10^{-6}.

3

Every float in python can be expressed as:

$$y = x_1 \cdot 2^{x_2} \implies \log_2 y = \log_2 x_1 \cdot 2^{x_2} = \log_2 x_1 + x_2 \cdot \log_2 2 = \log_2 x1 + x_2$$

The natural logarithm expressed as a function of logarithms of base 2 is:

$$\ln x = \frac{\log_2 x}{\log_2 e}$$