

Gravitational Waves (Theory)

① Einstein Field Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$\underbrace{\phantom{G_{\mu\nu}}}_{\text{Einstein tensor}} \downarrow \underbrace{\phantom{T_{\mu\nu}}}_{\text{Stress-energy tensor}}$
 (geometry) (source)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

② Spacetime Metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

\uparrow
 Minkowski ("flat")
 $(\begin{smallmatrix} -1 & +1 & 0 \\ 0 & +1 & +1 \\ 0 & 0 & +1 \end{smallmatrix})$

⇒ Important convention: since metric perturbation small, we can raise + lower tensor indices with Minkowski metric.

$$\therefore h^{\mu\nu} = n^{\mu\sigma} n^{\nu\delta} h_{\sigma\delta}$$

$$\dots \text{rather than } h^{\mu\nu} = g^{\mu\tau} g^{\nu\delta} h_{\tau\delta}$$

ONLY EXCEPTION ...

$$g^{\mu\nu} = (g_{\mu\nu})^{-1} = (n_{\mu\nu} + h_{\mu\nu})^{-1}$$
$$= n_{\mu\nu} - h_{\mu\nu}$$
$$+ O(h^2)$$

(3) Linearize the Einstein tensor

↳ tedious, but not difficult.

The Riemann tensor is defined as:

$$R_{\mu\nu\rho\gamma}^{\delta} = - \partial_{\mu}\Gamma_{\nu\gamma}^{\delta} + \partial_{\nu}\Gamma_{\mu\gamma}^{\delta} - \Gamma_{\mu\alpha}^{\delta}\Gamma_{\nu\gamma}^{\alpha} + \Gamma_{\nu\alpha}^{\delta}\Gamma_{\mu\gamma}^{\alpha}$$

NOTE : $\partial_{\mu} := \frac{\partial}{\partial x^{\mu}}$

$$\Gamma_{\mu\nu}^{\delta} = \frac{1}{2} g^{\delta\sigma} (\partial_{\mu}g_{\nu\delta} + \partial_{\nu}g_{\mu\delta} - \partial_{\delta}g_{\mu\nu})$$

⇒ LINEARIZE Γ ...

$$\Gamma_{\mu\nu}^{\delta} = \frac{1}{2} n^{\delta\sigma} (\partial_{\mu}h_{\nu\delta} + \partial_{\nu}h_{\mu\delta} - \partial_{\delta}h_{\mu\nu}) + O(h^2) ...$$

$\Gamma\Gamma$ terms are
 $\propto O(h^2)$, so we
ignore!

\Rightarrow LINEARIZE Riemann tensor...

$$\begin{aligned}
 R_{\mu\nu\gamma\delta} &= n_{\varepsilon\delta} R_{\mu\nu\gamma}{}^\varepsilon \\
 &= n_{\varepsilon\delta} \left[-2_\mu \Gamma_{\nu\gamma}^\varepsilon + 2_r \Gamma_{\mu\gamma}^\varepsilon \right] \\
 &= \frac{n_{\varepsilon\delta}}{2} \left[-n^{\varepsilon\alpha} (2_\mu 2_r h_{\gamma\alpha} + 2_\mu 2_\gamma h_{\alpha}) \right. \\
 &\quad \left. - 2_\mu 2_\alpha h_{\nu\gamma} \right] \\
 &\quad + n^{\varepsilon\alpha} \left(2_r 2_\mu h_{\gamma\alpha} + 2_r 2_\gamma h_{\mu\alpha} \right. \\
 &\quad \left. - 2_r 2_\alpha h_{\mu\gamma} \right] \\
 &= \frac{1}{2} \delta^\alpha_\mu \delta^\beta_\nu \left[-\cancel{2_\mu 2_r h_{\gamma\alpha}} + \cancel{2_r 2_\mu h_{\gamma\alpha}} \right. \\
 &\quad \left. - \cancel{2_\mu 2_\gamma h_{\alpha}} + \cancel{2_r 2_\gamma h_{\mu\alpha}} \right. \\
 &\quad \left. + 2_\mu 2_\alpha h_{\nu\gamma} - 2_r 2_\alpha h_{\mu\gamma} \right]
 \end{aligned}$$

- $R_{\mu\nu\gamma\delta} = \frac{1}{2} \left[-2_\mu 2_\gamma h_{\nu\delta} + 2_r 2_\gamma h_{\mu\delta} \right]$
 $+ 2_\mu 2_\delta h_{\nu\gamma} - 2_r 2_\delta h_{\mu\gamma}$
 $+ O(h^2) \dots$ PHEW!

OK, so that's the linearized Riemann tensor, but we now need the Ricci tensor and Ricci scalar.

$R_{\mu\nu} = R_{\mu\nu\rho}^{\rho}$ \Rightarrow take $R_{\mu\nu\rho}^{\rho}$, raise the " ρ " and make it equal to " γ ".

- $R_{\mu\nu} = \frac{1}{2} \left[-2_{\mu}{}^{\alpha} \partial_{\alpha} h_{\nu}^{\gamma} + 2_{\gamma}{}^{\alpha} \partial_{\alpha} h_{\mu}^{\gamma} + 2_{\mu}{}^{\alpha} \partial_{\alpha} h_{\nu}^{\gamma} - \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} h_{\mu\nu} \right].$

$$h = h^{\mu}_{\mu}$$

$$= \frac{1}{2} \left[-2_{\mu}{}^{\alpha} \partial_{\alpha} h + 2_{\gamma}{}^{\alpha} \partial_{\alpha} h_{\mu}^{\gamma} + 2_{\mu}{}^{\alpha} \partial_{\alpha} h_{\nu}^{\gamma} - \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} h_{\mu\nu} \right].$$

- $R = h^{\mu\nu} R_{\mu\nu}$

$$= \frac{1}{2} \left[-\eta^{\mu\nu} 2_{\mu}{}^{\alpha} \partial_{\alpha} h + 2_{\gamma}{}^{\alpha} \partial_{\alpha} h^{\gamma\nu} + 2_{\mu}{}^{\alpha} \partial_{\alpha} h^{\mu\nu} - \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} h \right]$$

$$= \partial_{\alpha} \partial_{\beta} h^{\alpha\beta} - \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta} h$$

FINALLY ---

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$= R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R$$

- $\mathcal{E}_{\mu\nu} = \frac{1}{2} \left[-2_{\mu}{}_{\nu} h + 2_{\gamma}{}_{\mu} h^{\gamma}_{\nu} + 2_{\mu}{}_{\gamma} h^{\gamma}_{\nu} - \eta^{\delta\sigma} 2_{\gamma}{}_{\delta} h_{\mu\nu} - \eta_{\mu\nu} 2_{\gamma}{}_{\delta} h^{\delta\sigma} + \eta_{\mu\nu} \eta^{\delta\sigma} 2_{\gamma}{}_{\delta} h \right]$

LINEARIZED EINSTEIN TENSOR

(4)

We can make this much more compact by defining the trace-reversed metric perturbation.

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} n_{\mu\nu} h$$

NOTE : $\bar{h} = n^{\mu\nu} \bar{h}_{\mu\nu} = h - \frac{1}{2} \underbrace{n^{\mu\nu} n_{\mu\nu}}_{=4} h$

$$= -h \quad \leftarrow \text{hence, trace reversed.}$$

- Substituting, and cancelling terms gives ...

$$\mathcal{E}_{\mu\nu} = \frac{1}{2} \left[2_{\mu}{}_{\nu} \bar{h}^{\delta} + 2_{\nu}{}_{\mu} \bar{h}^{\delta} - \eta^{\delta\sigma} 2_{\gamma}{}_{\delta} \bar{h}_{\mu\nu} - \eta_{\mu\nu} 2_{\gamma}{}_{\delta} \bar{h}^{\delta\sigma} \right].$$

The linearized Einstein field equations are :

$$-\boxed{n^{\delta\sigma} \partial_\delta \partial_\sigma \bar{h}_{\mu\nu}} - n_{\mu\nu} \partial_\delta \partial_\sigma \bar{h}^{\delta\sigma} + \partial_\mu \partial_\sigma \bar{h}^\sigma_\nu + \partial_\sigma \partial_\nu \bar{h}^\sigma_\mu = \frac{16\pi G}{c^4} T_{\mu\nu}$$

\Rightarrow The $\boxed{\square}$ term is just $\square \bar{h}_{\mu\nu}$ where \square is the **spacetime D'Alembertian operator**

$$\text{i.e. } \square = -\frac{\partial}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

\Rightarrow looks awfully like a wave equation ...
Can we make those other terms disappear?

(5) **Lorenz Gauge** \Rightarrow find a co-ordinate transformation where divergences of $T_{\mu\nu} = 0$

$$\Rightarrow x'^\mu = x^\mu + \xi^\mu$$

$$\Rightarrow g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + O(h^2)$$

$$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + O(h^2)$$

Changes to Riemann tensor are $O(h^2) \therefore \sim 0$

$$\Rightarrow h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \xi_\beta$$

Now, we want $\partial_\mu h'_\nu = 0$ in new gauge

$$\therefore \partial_\mu h'_\nu = 0 = \partial_\mu h_\nu - \square \xi_\nu$$

So, we need to find a co-ordinate transformation generated by a vector offset ξ that solves...

$$\square \xi_\nu = \partial_\mu h_\nu$$

\Rightarrow With 4 equations and 4 unknowns, we can always solve this!

\Rightarrow Change to Lorenz gauge (co-ordinate system) where all other terms on LHS of Einstein equation are 0.

$$-\square h_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}$$

wave equation sourced by RHS. ☺

⇒ In a vacuum ... $\square \bar{h}_{\mu\nu} = 0$

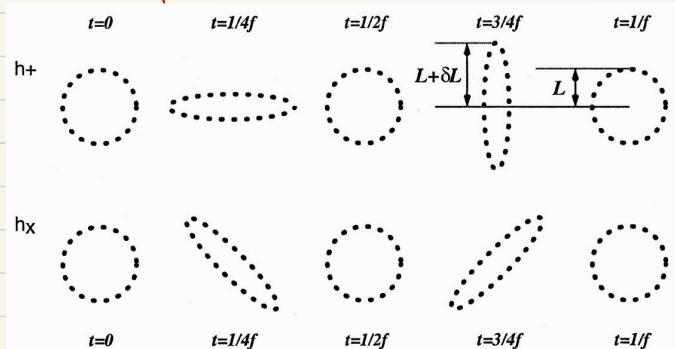
SOLUTION : $\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_0 z^5}$

We can remove first extraneous degrees of freedom by moving to the transverse traceless gauge

TT gauge : $h^{0\mu} = 0; h^i_i = 0; \partial^i \bar{h}_{ij} = 0$

$$\bar{h}_{\mu\nu}(t, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-z)]$$

⇒ transverse wave propagating along $+z$ at speed of light.



$T^{\alpha\beta} = t$	t energy density	x y z energy flux
x y z momentum density		stress tensor

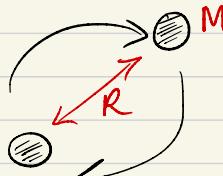
PRODUCTION OF GRAVITATIONAL WAVES

MOMENT	ELECTROMAGNETISM	GRAVITY
<u>monopole</u>	$\int \rho(\vec{r}) d^3r = q$ $\frac{dq}{dt} = 0 \quad \left. \begin{array}{l} \text{CHARGE} \\ \text{CONSERVATION} \end{array} \right.$	$\int \rho(\vec{r}) d^3r = M$ $\frac{dM}{dt} = 0 \quad \left. \begin{array}{l} \text{MASS} \\ \text{CONSERVATION} \end{array} \right.$
<u>dipole</u>	$\int \rho(\vec{r}) \vec{r} d^3r = q \vec{r}_{\text{com}}$ $\frac{d\vec{r}_{\text{com}}}{dt} = q \vec{v}_{\text{com}} = \vec{P}_e$ $\Rightarrow \text{NO CONSERVATION LAW}$ $\Rightarrow \text{DIPOLAR EM WAVES}$	$\int \rho(\vec{r}) \vec{r} d^3r = M \vec{r}_{\text{com}}$ $M \frac{d\vec{r}_{\text{com}}}{dt} = M \vec{v}_{\text{com}} = \vec{P}$ $\frac{d\vec{P}}{dt} = 0 \quad \left. \begin{array}{l} \text{MOMENTUM} \\ \text{CONSERVATION} \end{array} \right.$ $\Rightarrow \text{NEED QUADRUPOLE MASS MOMENT FOR GWs.}$

$$I^{ij}(t) = \int d^3r \cdot \rho(t, \vec{r}) r^i r^j$$

$$\boxed{\tilde{h}^{ij}(t, \vec{r}) = \frac{2}{\Gamma} \frac{d^2}{dt^2} I^{ij}(t-r)}$$

$$\frac{MR^2}{P^2}$$



$$m\omega^2 r = \frac{GMm}{r^2}$$

$$\frac{4\pi}{P^2} = \frac{M}{r^3}$$

$$P^2 = \frac{r^3}{M}$$

$$\frac{MR^2}{P^2} = \frac{M}{R^3}$$

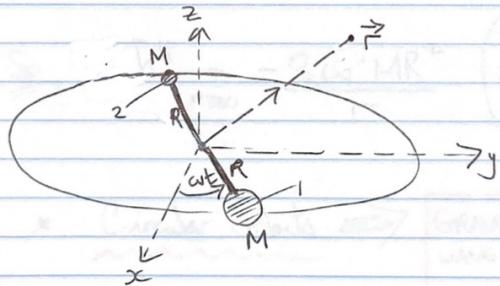
COM: $h \propto \frac{1}{r} \cdot \frac{1}{P^2} MR^2$
 USING K3 } $\propto \frac{M^{5/3} P^{-2/3}}{\Gamma}$

$$\propto \frac{M^{5/3} f_{\text{orb}}^{2/3}}{\Gamma}$$

$$r^2 = \left(\frac{MP^2}{M^{5/3} P} \right)^{2/3}$$

$$h \sim 10^{-16} \left(\frac{M}{10^9 M_\odot} \right)^{5/3} \left(\frac{f_{\text{orb}}}{10 \text{ Hz}} \right)^{2/3} \left(\frac{100 \text{ Mpc}}{r} \right)$$

• Gravitational Waves From Binaries [FULL CALCULATION]



- * Binary with equal mass components, M .
- * Circular orbit radius, R .
- * Orbital angular frequency, $\omega = 2\pi f_p$.
- * Interested in GWs emitted toward \vec{r} .

$$x_1 = R \cos \omega t ; \quad y_1 = R \sin \omega t ; \quad z_1 = 0 \\ x_2 = -R \cos \omega t ; \quad y_2 = -R \sin \omega t ; \quad z_2 = 0$$

$$\Rightarrow I^{ij} = \int d^3x \mu(t, \vec{x}) x^i x^j$$

mass-distribution here are
2 delta functions centered

$$\underline{\text{Thus}} ; \quad I^{xx} = M(x_1)^2 + M(x_2)^2 \\ = 2MR^2 \cos^2 \omega t = MR^2 [1 + \cos 2\omega t]$$

$$I^{xy} = Mx_1 y_1 + Mx_2 y_2 \\ = 2MR^2 \sin \omega t \cos \omega t = MR^2 \sin 2\omega t.$$

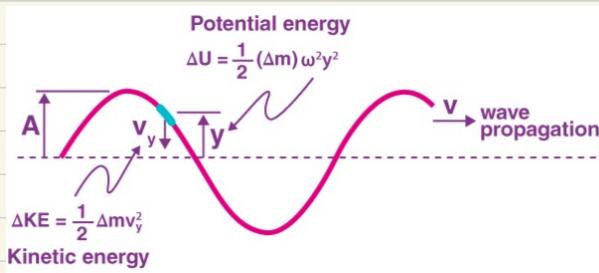
$$I^{yy} = M(y_1)^2 + M(y_2)^2 \\ = 2MR^2 \sin^2 \omega t = MR^2 [1 - \cos 2\omega t].$$

$$\Rightarrow I^{zz}, I^{zx}, I^{zy} = 0 \quad * \text{GW frequency} = 2 \times \underbrace{\text{ORBITAL FREQUENCY}}$$

$$\text{So, } \bar{h}^{ij} \underset{r \gg 0}{\rightarrow} -\frac{8\omega^2 MR^2}{r} \begin{pmatrix} \cos[2\omega(t-r)] & \sin[2\omega(t-r)] & 0 \\ \sin[2\omega(t-r)] & -\cos[2\omega(t-r)] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

* FOR CIRCULAR ORBITS

QUADRUPOLE FORMULA FOR GW EMISSION



POWER IN A WAVE

$$y = A \sin(kx - \omega t)$$

$$\langle P \rangle \propto \omega^2 A^2$$

power averaged over period.

- oof \Rightarrow (1) Energy flux quadratic in A ($\propto I^{ij}$)
(2) $\omega \propto 1/\text{time}$ implies a time derivative

$$\begin{aligned} L_{\text{GW}} &= \frac{1}{5} \left\langle \ddot{\mathcal{E}}_{ij} \ddot{\mathcal{E}}^{ij} \right\rangle = \frac{128}{5} M^2 R^4 \omega^6 \\ &= \frac{128}{5} 4^{1/3} \cdot \frac{c^5}{G} \left(\frac{\pi G M}{c^3 P} \right)^{10/3} \end{aligned}$$

- Consider total (KE + PE) in a binary orbit --

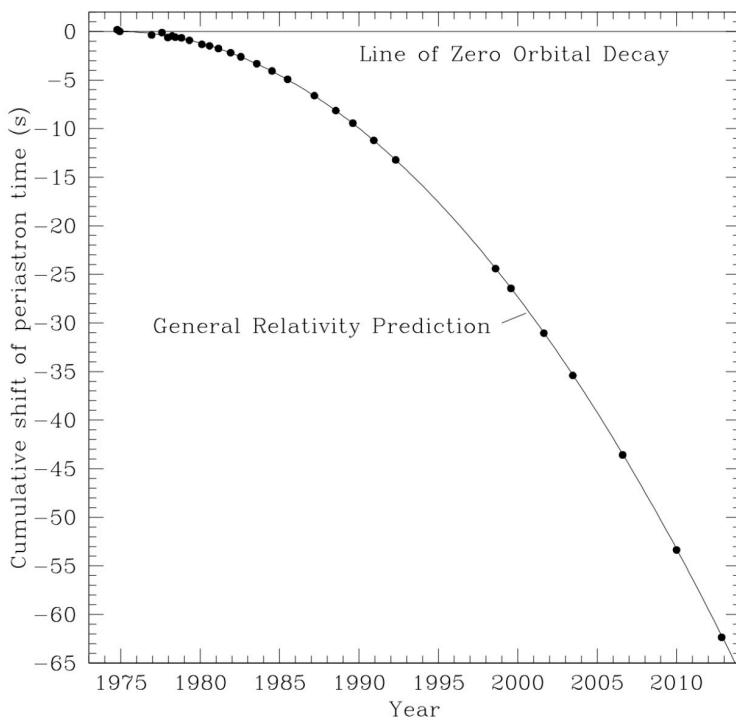
$$E_{\text{NEWTON}} = -M^2 / 4R$$

- Equate $\frac{dE_{\text{NEWTON}}}{dt}$ to $-L_{\text{GW}}$ --

$$\frac{dP}{dt} = -\frac{96}{5} \pi^4 4^{1/3} \left(\frac{2\pi M}{P} \right)^{5/3}$$

$$\frac{df_{\text{orb}}}{dt} = \frac{96}{5} \pi^4 4^{1/3} (2\pi M)^{5/3} f_{\text{orb}}^{11/3}$$

negative!
GWs cause
orbit to shrink!



CUMULATIVE
SHIFT IN
PERIASTRON
TIME OF
BINARY SYSTEM
FROM PULSAR
 $B1913+16$



1st indirect
evidence for
GWs
(1993 Nobel).

STRAIN SIGNAL
FROM LIGO
DETECTORS
SHOWING
FREQUENCY "CHIRP"
FROM GWs



1st direct
evidence of GWs
(2017 Nobel).

