07/09/2024

Timing Response Due to GWs

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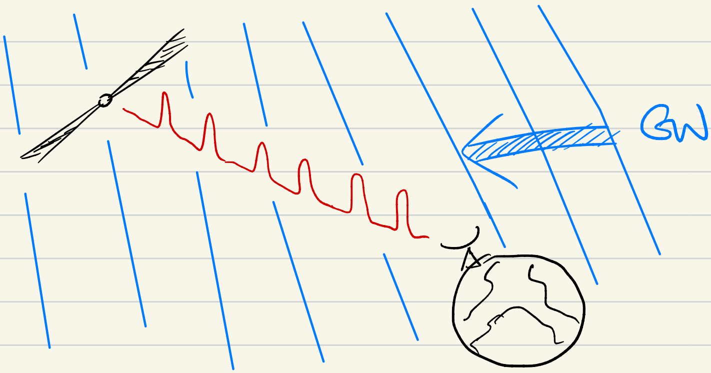

GW spacetime line element $(G=c=1)$.

$$g_{\mu\nu}dx^\mu dx^\nu = \left\{ ds^2 = -dt^2 + \left[\delta_{ij} + h^{TT}_{ij}(t,\vec{x})\right] dx^i dx^j \right.$$

$\Rightarrow$ i, j = spatial components
$\Rightarrow$ a, b = different pulsars.

Now photon path ... $ds^2 = 0$

$\hookrightarrow$ along x-axis toward observer @ origin

$$0 = -dt^2 + (1 + h^{TT}_{xx}(t,\vec{x}))dx^2$$

$$dx^2 = \left(1 + h^{TT}_{xx}(t,\vec{x})\right)^{-1} dt^2$$

$$\therefore \quad dx \approx - \left(1 - \frac{1}{2} h^{TT}_{xx}(t,\vec{x})\right) dt$$

moving towards origin

$\Rightarrow$ Consider Earth-pulsar distance, L.

$$\int_{x_p}^{x_E} dx = -L = -\int_{t_{em}}^{t_{obs}} dt' \left(1 - \frac{1}{2} h^{TT}_{xx}(t',\vec{x})\right)$$

$$L = (t_{obs} - t_{em}) - \frac{1}{2} \int_{t_{em}}^{t_{obs}} h^{TT}_{xx}(t',\vec{x})\, dt'$$

- Assumption : $h^{TT}_{xx}(t,\vec{x}) \ll 1$

$$\underbrace{\therefore \quad t_{obs} \approx t_{em} + L}_{\text{photon path} \approx \text{along unperturbed trajectory}}$$

$$\vec{x}(t) = (t_{obs} - t)\, \hat{p}$$

$\Rightarrow$ generalize to arbitrary photon origin direction

$$h^{TT}_{xx} \longmapsto p^i p^j h^{TT}_{ij}$$

Thus, $t_{obs} = t_{em} + L + \frac{p^i p^j}{2} \int_{t_{em}}^{t_{em}+L} dt' \, h^{TT}_{ij}\left[t', (t_{em}+L-t')\hat{p}\right]$

$$\underbrace{\hphantom{t_{obs} = t_{em} + L + \frac{p^i p^j}{2} \int dt' \, h^{TT}_{ij}}}_{\text{PULSE \#1}}$$

- Consider a 2^{nd} pulse, emitted one rotational period after the first.

$$\Rightarrow \quad t'_{em} = t_{em} + P$$

$$\Downarrow$$

$$t'_{obs} - t_{obs} = P + \Delta P$$

extra from GWs.

- Replace $t_{em} \longmapsto t_{em} + P$ and insert into t_{obs}

$\Rightarrow \quad t'_{obs} = t_{em} + P + L + \frac{p^i p^j}{2} \int_{t_{em}+P}^{t_{em}+P+L} dt' \, h^{TT}_{ij}\left[t', (t_{em}+P+L-t')\hat{p}\right].$

$$\boxed{t'' = t' - P \Rightarrow \text{redefine variables}}$$

$$t'_{obs} = t_{em} + P + L + \frac{p^i p^j}{2} \int_{t_{em}+P}^{t_{em}+P+L} dt' \, h^{TT}_{ij}\left[t', (t_{em}+P+L-t')\hat{p}\right].$$

$$= t_{em} + P + L + \frac{p^i p^j}{2} \int_{t_{em}}^{t_{em}+L} dt'' \cdot h^{TT}_{ij}\left[t''+P, (t_{em}+L-t'')\hat{p}\right]$$

Thus $t'_{obs} - t_{obs} = P + \frac{p^i p^j}{2} \int_{t_{em}}^{t_{em}+L} dt' \underbrace{\left\{ h^{TT}_{ij}(t'+P, \vec{x}_0(t')) - h^{TT}_{ij}(t', \vec{x}_0(t')) \right\}}_{\Delta P}$

$x_0(t') = (t_{em} + L - t')\hat{p}$

$\Rightarrow$ $P \sim 10^{-3}\,s$
$P_{GW} \sim 1/10^{-9}\,Hz \sim 10^{9}\,s$

So $\underbrace{h^{TT}_{ij}(t'+P, \vec{x}_0(t'))}_{\text{Taylor expanding}} \approx h^{TT}_{ij}(t', \vec{x}_0(t')) + \left.\frac{\partial h^{TT}_{ij}}{\partial t'}(t', \vec{x})\right|_{\vec{x}=\vec{x}_0(t')} \times P$

$$\Rightarrow \frac{\Delta P}{P} = \frac{p^i p^j}{2} \int_{t_{em}}^{t_{em}+L} dt' \cdot \left[\frac{\partial}{\partial t'} \cdot h^{TT}_{ij}(t', \vec{x})\right]_{\vec{x}=\vec{x}_0(t')}$$

$\Rightarrow$ Consider monochromatic GW propagating in direction $\hat{\Omega}$.

$$h_{ij}^{TT}(t, \vec{x}) = A_{ij}(\hat{\Omega}) \cos\left[\omega_{gw}(t - \hat{\Omega}\cdot\vec{x})\right].$$

$$\therefore \frac{\Delta P}{P} = \frac{1}{2}\frac{p^i p^j A_{ij}}{(1+\hat{\Omega}\cdot\hat{p})} \times \left\{ \cos(\omega_{gw} t_{obs}) - \cos(\omega_{gw} t_{em} - \omega_{gw}(t_{obs} - t_{em})\hat{\Omega}\cdot\hat{p}) \right\}$$

$$= \frac{1}{2}\frac{p^i p^j A_{ij}}{1+\hat{\Omega}\cdot\hat{p}} \left\{ \cos(\omega_{gw} t_{obs}) - \cos(\omega_{gw}(t_{em} - L\hat{\Omega}\cdot\hat{p})) \right\}$$

Finally $\Rightarrow$ GW-induced redshift to the arrival rate of radio pulses $\therefore z(t) = \frac{\Delta P}{P}$.

$$\therefore z(t, \hat{\Omega}) = \frac{\nu_0 - \nu(t)}{\nu_0}$$

$$= \frac{1}{2}\frac{p^i p^j}{(1+\hat{\Omega}\cdot\hat{p})}\Big[\underbrace{h_{ij}(t, \vec{x}_{earth})}_{\text{EARTH TERM}} - \underbrace{h_{ij}(t-L, \vec{x}_{pulsar})}_{\text{PULSAR TERM}}\Big]$$

$$\downarrow$$

$$z(t, \hat{\Omega}) = \frac{1}{2}\frac{p^i p^j}{(1+\hat{\Omega}\cdot\hat{p})} \cdot \Delta h_{ij}$$