11.2 A Zeroth-Order Natural Relation

This problem studies an $N = 2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}((\phi^i)^2)^2 + \bar{\psi}(i\partial\!\!\!/)\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi,$$

where ϕ^i is a two-component field, $i = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$\begin{aligned}
\phi^1 &\to \cos\alpha\phi^1 - \sin\alpha\phi^2, \\
\phi^2 &\to \sin\alpha\phi^1 + \cos\alpha\phi^2, \\
\psi &\to e^{-i\alpha\gamma^5/2}\psi.
\end{aligned}$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)).$$

Write out the Lagrangian in these new variables, and show that the fermion acquires a mass given by

$$m_f = g \cdot v. \tag{1}$$

(c) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the $\psi\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation 1 receives nonzero corrections but that these corrections are *finite*. This is in accord with our general discussion in Section 11.6.

Solution:

We first demonstrate that the Lagrangian exhibits the specified global symmetry under the given transformations:

$$\begin{aligned}
\frac{1}{2}((\partial_\mu\phi^1)^2 + (\partial_\mu\phi^2)^2) \to &\frac{1}{2}((\partial_\mu(\cos\alpha\phi^1 - \sin\alpha\phi^2))^2 + (\partial_\mu(\sin\alpha\phi^1 + \cos\alpha\phi^2))^2) \\
&= \frac{1}{2}(\cos^2\alpha(\partial_\mu\phi^1)^2 - 2\cos\alpha\sin\alpha\partial_\mu\phi^1\partial^\mu\phi^2 + \sin^2\alpha(\partial_\mu\phi^2)^2) \\
&+ \frac{1}{2}(\sin^2\alpha(\partial_\mu\phi^1)^2 + 2\cos\alpha\sin\alpha\partial_\mu\phi^1\partial^\mu\phi^2 + \cos^2\alpha(\partial_\mu\phi^2)^2) \\
&= \frac{1}{2}((\partial_\mu\phi^1)^2 + (\partial_\mu\phi^2)^2).
\end{aligned}$$

$$\frac{1}{2}\mu^2((\phi^1)^2 + (\phi^2)^2) \to \frac{1}{2}\mu^2((\cos\alpha\phi^1 - \sin\alpha\phi^2)^2 + (\sin\alpha\phi^1 + \cos\alpha\phi^2)^2) = \frac{1}{2}\mu^2((\phi^1)^2 + (\phi^2)^2).$$

$$-\frac{\lambda}{4}((\phi^1)^2 + (\phi^2)^2)^2 \to -\frac{\lambda}{4}((\cos\alpha\phi^1 - \sin\alpha\phi^2)^2 + (\sin\alpha\phi^1 + \cos\alpha\phi^2)^2)^2 = -\frac{\lambda}{4}((\phi^1)^2 + (\phi^2)^2)^2.$$

$$\bar{\psi}(i\partial\!\!\!/)\psi \to (e^{-i\alpha\gamma^5/2}\psi)^\dagger\gamma^0(i\partial\!\!\!/)e^{-i\alpha\gamma^5/2}\psi = \psi^\dagger e^{i\alpha\gamma^5/2}\gamma^0 e^{i\alpha\gamma^5/2}(i\partial\!\!\!/)\psi = \bar{\psi}(i\partial\!\!\!/)\psi$$

$$\begin{aligned}-g\bar{\psi}(\phi^1+i\gamma^5\phi^2)\psi \to &-g(e^{-i\alpha\gamma^5/2}\psi)^\dagger\gamma^0(\cos\alpha\phi^1-\sin\alpha\phi^2+i\gamma^5(\sin\alpha\phi^1+\cos\alpha\phi^2))e^{-i\alpha\gamma^5/2}\psi\\ &=-g\psi^\dagger e^{i\alpha\gamma^5/2}\gamma^0((\cos\alpha+i\gamma^5\sin\alpha)\phi^1+i\gamma^5(\cos\alpha+i\gamma^5\sin\alpha)\phi^2)e^{-i\alpha\gamma^5/2}\psi\\ &=-g\psi^\dagger\gamma^0 e^{-i\alpha\gamma^5/2}e^{i\alpha\gamma^5}(\phi^1+i\gamma^5\phi^2)e^{-i\alpha\gamma^5/2}\psi=-g\bar{\psi}(\phi^1+i\gamma^5\phi^2)\psi.\end{aligned}$$

These transformations leave the Lagrangian invariant, confirming the global symmetry. Additionally, the classical equations of motion yield a minimum energy solution where ϕ^i acquires a vacuum expectation value, breaking this symmetry spontaneously.
Following spontaneous symmetry breaking, the field ϕ^i acquires a vacuum expectation value v. Using the new variables $\phi^i = (v+\sigma(x),\pi(x))$, the Lagrangian becomes:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu\sigma)^2+\frac{1}{2}(\partial_\mu\pi)^2+\frac{1}{2}\mu^2(\sigma^2+2v\sigma+\pi^2)-\frac{\lambda}{4}(v^2+\sigma^2+2v\sigma+\pi^2)^2\\ &+\bar{\psi}(i\partial\!\!\!/)\psi-g\bar{\psi}(v+\sigma+i\gamma^5\pi)\psi\\ &=\frac{1}{2}(\partial_\mu\sigma)^2+\frac{1}{2}(\partial_\mu\pi)^2-\frac{1}{2}2\mu^2\sigma^2-\frac{1}{4}\lambda(\sigma^4+\pi^4)-\frac{1}{2}\lambda\sigma^2\pi^2-\lambda v\sigma^3-\lambda v\sigma\pi^2\\ &+\bar{\psi}(i\partial\!\!\!/-gv)\psi-g\bar{\psi}(\sigma+i\gamma^5\pi)\psi,\end{aligned}$$

where $v=\sqrt{\mu^2/\lambda}$ at the classical level. From the term $-g\bar{\psi}(v+\sigma+i\gamma^5\pi)\psi$, the fermion acquires a mass $m_f=g\cdot v$, as seen in the mass term $-gv\bar{\psi}\psi$.
To compute the one-loop radiative correction to the fermion mass m_f, we first express the Lagrangian in terms of renormalized fields, substituting $\phi_0^i=\sqrt{Z_{\phi^i}}\phi_r^i$ and $\psi_0=\sqrt{Z_\psi}\psi_r$:

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}(\partial_\mu\phi_0^i)^2+\frac{1}{2}\mu_0^2(\phi_0^i)^2-\frac{\lambda_0}{4}((\phi_0^i)^2)^2+\bar{\psi}_0(i\partial\!\!\!/)\psi_0-g_0\bar{\psi}_0(\phi_0^1+i\gamma^5\phi_0^2)\psi_0\\ =&\frac{1}{2}Z_\phi(\partial_\mu\phi_r^i)^2+\frac{1}{2}Z_\phi\mu_0^2(\phi_r^i)^2-\frac{\lambda_0}{4}Z_\phi^2((\phi_r^i)^2)^2+Z_\psi\bar{\psi}_r(i\partial\!\!\!/)\psi_r-Z_\psi\sqrt{Z_\phi}g_0\bar{\psi}_r(\phi_r^1+i\gamma^5\phi_r^2)\psi_r\\ =&\frac{1}{2}(\partial_\mu\phi_r^i)^2+\frac{1}{2}\underbrace{(Z_\phi-1)}_{\delta_\phi}(\partial_\mu\phi_r^i)^2+\frac{1}{2}\mu_r^2(\phi_r^i)^2+\frac{1}{2}\underbrace{(Z_\phi\mu_0^2-\mu_r^2)}_{\delta_\mu}(\phi_r^i)^2\\ &-\frac{\lambda_r}{4}((\phi_r^i)^2)^2-\frac{1}{4}\underbrace{(Z_\phi^2\lambda_0-\lambda_r)}_{\delta_\lambda}((\phi_r^i)^2)^2+\bar{\psi}_r(i\partial\!\!\!/)\psi_r+\underbrace{(Z_\psi-1)}_{\delta_\psi}\bar{\psi}_r(i\partial\!\!\!/)\psi_r\\ &-g_r\bar{\psi}_r(\phi_r^1+i\gamma^5\phi_r^2)\psi_r-\underbrace{(Z_\psi\sqrt{Z_\phi}g_0-g_r)}_{\delta_g}\bar{\psi}_r(\phi_r^1+i\gamma^5\phi_r^2)\psi_r.\end{aligned}$$

After spontaneous symmetry breaking, we would have:

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2}(\partial_\mu \sigma_r)^2 + \frac{1}{2}(\partial_\mu \pi_r)^2 + \frac{1}{2}\delta_\phi(\partial_\mu \sigma_r)^2 + \frac{1}{2}\delta_\phi(\partial_\mu \pi_r)^2 + \frac{1}{2}\mu_r^2(\sigma_r^2 + 2v\sigma_r + \pi_r^2) \\
& + \frac{1}{2}\delta_\mu(\sigma_r^2 + 2v\sigma_r + \pi_r^2) + \bar{\psi}_r(i\partial\!\!\!/)\psi_r + \delta_\psi \bar{\psi}_r(i\partial\!\!\!/)\psi_r \\
& - g_r\bar{\psi}_r(\sigma_r + v + i\gamma^5\pi_r)\psi_r - \delta_g\bar{\psi}_r(\sigma_r + v + i\gamma^5\pi_r)\psi_r \\
& - \frac{\lambda_r}{4}(\pi_r^4 + 2v^2\pi_r^2 + 4v\sigma_r\pi_r^2 + 4v^3\sigma_r + 2\pi_r^2\sigma_r^2 + 6v^2\sigma_r^2 + 4v\sigma_r^3 + \sigma_r^4) \\
& - \frac{1}{4}\delta_\lambda(\pi_r^4 + 2v^2\pi_r^2 + 4v\sigma_r\pi_r^2 + 4v^3\sigma_r + 2\pi_r^2\sigma_r^2 + 6v^2\sigma_r^2 + 4v\sigma_r^3 + \sigma_r^4)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{1}{2}(\partial_\mu \sigma_r)^2 + \frac{1}{2}(\partial_\mu \pi_r)^2 + \frac{1}{2}\delta_\phi(\partial_\mu \sigma_r)^2 + \frac{1}{2}\delta_\phi(\partial_\mu \pi_r)^2 + \frac{1}{2}(\mu_r^2 - 3v^2)\sigma_r^2 \\
& + \frac{1}{2}(\delta_\mu - 3v^2\delta_\lambda)\sigma_r^2 + \frac{1}{2}(\mu_r^2 - \lambda_r v^2)\pi_r^2 + \frac{1}{2}(\delta_\mu - \lambda_r v^2)\pi_r^2 - \lambda_r v\sigma_r^3 - \delta_\lambda v\sigma_r^3 \\
& + (v\mu_r^2 - v^3\lambda_r)\sigma_r + (v\delta_\mu - v^3\delta_\lambda)\sigma_r - v\lambda_r\sigma_r\pi_r^2 - v\delta_\lambda\sigma_r\pi_r^2 + \bar{\psi}_r(i\partial\!\!\!/ - vg_r)\psi_r \\
& + \bar{\psi}_r(i\delta_\psi\partial\!\!\!/ - v\delta_g)\psi_r - g_r\bar{\psi}_r(\sigma_r + i\gamma^5\pi_r)\psi_r - \delta_g\bar{\psi}_r(\sigma_r + i\gamma^5\pi_r)\psi_r \\
& - \frac{\lambda_r}{4}(\pi_r^2 + \sigma_r^2)^2 - \frac{1}{4}\delta_\lambda(\pi_r^2 + \sigma_r^2)^2.
\end{aligned}
$$

In order to keep v and g intact, we choose the following renormalization conditions (Fig. 1):

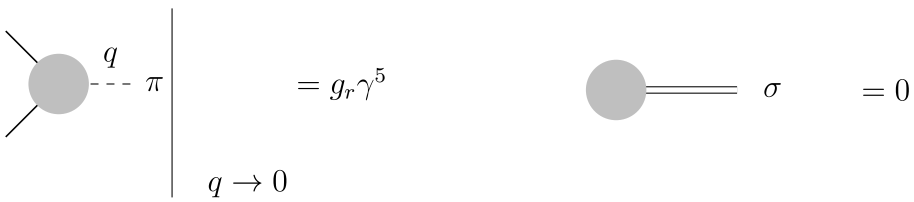
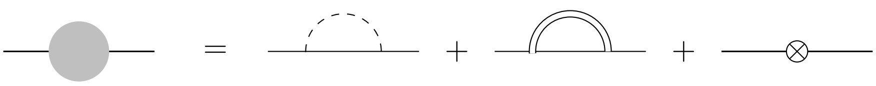
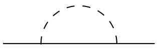


Figure 1: Renormalization conditions chosen specifically to keep v and g unchanged.

To demonstrate that the radiative correction to the fermion mass m_f is finite, we calculate the one-loop self-energy diagrams shown below:





$$= (g_r)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^5 \frac{i}{\not{k} - m_f} \gamma^5 \frac{i}{(k-p)^2}$$

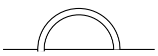
$$= -g_r^2 \int \frac{d^4k}{(2\pi)^4} \frac{-\not{k} + m_f}{(k^2 - m_f^2)((k-p)^2)}$$

$$= g_r^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\not{k} - m_f}{((1-x)(k^2 - m_f^2) + x((k-p)^2))^2}$$

$$= g_r^2 \int \frac{d^d l_\pi}{(2\pi)^d} \int_0^1 dx \frac{x\not{p} - m_f}{(l_\pi^2 - \Delta_\pi)^2}$$

$$= g_r^2 \int_0^1 dx (x\not{p} - m_f) \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_\pi}\right)^{2 - \frac{d}{2}}$$

$$= \frac{i g_r^2}{(4\pi)^2} \int_0^1 dx (x\not{p} - m_f)(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(\Delta_\pi)).$$



$$= (-i g_r)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} - m_f} \frac{i}{(k-p)^2 - 2\mu^2}$$

$$= g_r^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + m_f}{(k^2 - m_f^2)((k-p)^2 - 2\mu^2)}$$

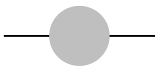
$$= g_r^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\not{k} + m_f}{((1-x)(k^2 - m_f^2) + x((k-p)^2 - 2\mu^2))^2}$$

$$= g_r^2 \int \frac{d^d l_\sigma}{(2\pi)^d} \int_0^1 dx \frac{x\not{p} + m_f}{(l_\sigma^2 - \Delta_\sigma)^2}$$

$$= g_r^2 \int_0^1 dx (x\not{p} + m_f) \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_\sigma}\right)^{2 - \frac{d}{2}}$$

$$= \frac{i g_r^2}{(4\pi)^2} \int_0^1 dx (x\not{p} + m_f) \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(\Delta_\sigma)\right).$$

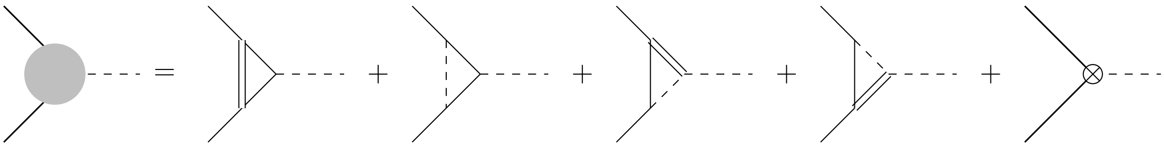
Hence,

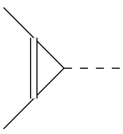


$$=i\not{p}\left(\delta_\psi+\frac{g_r^2}{(4\pi)^2}\int_0^1 dx\left\{2x\left[\frac{2}{\epsilon}-\gamma+\log(4\pi)-\frac{1}{2}\log(\Delta_\pi\Delta_\sigma)\right]\right\}\right)$$

$$-i\left(v\delta_g+\frac{g_r^2}{(4\pi)^2}\int_0^1 dx\left[m_f\log\frac{\Delta_\sigma}{\Delta_\pi}\right]\right).$$

The one-loop corrections to m_f from the π and σ loop diagrams are finite, as their divergent terms cancel. To ensure the total radiative correction to m_f (including the counterterm) is finite, we must verify that the counterterm δ_g is finite.





$$=(-ig_r)^2 g_r\int\frac{d^4k}{(2\pi)^4}\frac{i}{(\not{k}-m_f)}\gamma^5\frac{i}{(\not{k}-m_f)}\frac{i}{((k-p)^2-2\mu^2)}$$

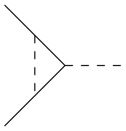
$$=ig_r^3\int\frac{d^4k}{(2\pi)^4}\frac{(\not{k}+m_f)\gamma^5(\not{k}+m_f)}{(k^2-m_f^2)(k^2-m_f^2)((k-p)^2-2\mu^2)}$$

$$=-ig_r^3\gamma^5\int\frac{d^4k}{(2\pi)^4}\frac{(k^2-m_f^2)}{(k^2-m_f^2)(k^2-m_f^2)((k-p)^2-2\mu^2)}$$

$$=-ig_r^3\gamma^5\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{((1-x)(k^2-m_f^2)+x((k-p)^2-2\mu^2))^2}$$

$$=-ig_r^3\gamma^5\int\frac{d^dl_1}{(2\pi)^d}\int_0^1 dx\frac{1}{(l_1^2-\Delta_1)^2}=-ig_r^3\gamma^5\int_0^1 dx\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta_1}\right)^{2-\frac{d}{2}}$$

$$=\frac{g_r^3\gamma^5}{(4\pi)^2}\int_0^1 dx\left(\frac{2}{\epsilon}-\gamma+\log(4\pi)-\log(\Delta_1)\right).$$



$$= (g_r)^3 \int \frac{d^4k}{(2\pi)^4} \gamma^5 \frac{i}{(\not{k} - m_f)} \gamma^5 \frac{i}{(\not{k} - m_f)} \gamma^5 \frac{i}{(k-p)^2}$$

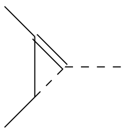
$$= -ig_r^3\gamma^5 \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} + m_f)\gamma^5(\not{k} + m_f)\gamma^5}{(k^2 - m_f^2)(k^2 - m_f^2)(k-p)^2}$$

$$= ig_r^3\gamma^5 \int \frac{d^4k}{(2\pi)^4} \frac{(k^2 - m_f^2)}{(k^2 - m_f^2)(k^2 - m_f^2)(k-p)^2}$$

$$= ig_r^3\gamma^5 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{1}{((1-x)(k^2 - m_f^2) + x(k-p)^2)^2}$$

$$= ig_r^3\gamma^5 \int \frac{d^dl_2}{(2\pi)^d} \int_0^1 dx \frac{1}{(l_2^2 - \Delta_2)^2} = ig_r^3\gamma^5 \int_0^1 dx \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_2}\right)^{2-\frac{d}{2}}$$

$$= -\frac{g_r^3\gamma^5}{(4\pi)^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(\Delta_2)\right).$$



$$= (-ig_r)(g_r)(-iv\lambda_r) \int \frac{d^4k}{(2\pi)^4} \gamma^5 \frac{i}{(\not{k} - m_f)} \frac{i}{((k-p)^2 - 2\mu^2)} \frac{i}{(k-p)^2}$$

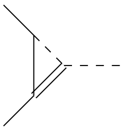
$$= ig_r^2\lambda_r v\gamma^5 \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} + m_f)}{(k^2 - m_f^2)((k-p)^2 - 2\mu^2)(k-p)^2}$$

$$= ig_r^2\lambda_r v\gamma^5 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdydz \frac{(3-1)!\delta(x+y+z-1)(\not{k} + m_f)}{(z(k^2 - m_f^2) + y((k-p)^2 - 2\mu^2) + x(k-p)^2)^3}$$

$$= 2ig_r^2\lambda_r v\gamma^5 \int \frac{d^dl_3}{(2\pi)^d} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)\not{p} + m_f}{(l_3^2 - \Delta_3)^3}$$

$$= 2ig_r^2\lambda_r v\gamma^5 \int_0^1 dx \int_0^{1-x} dy [(x+y)\not{p} + m_f] \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \left(\frac{1}{\Delta_3}\right)^{3-\frac{d}{2}}$$

$$= \frac{g_r^2\lambda_r v\gamma^5}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)\not{p} + m_f}{\Delta_3}.$$



$$= (-ig_r)(g_r)(-iv\lambda_r) \int \frac{d^4k}{(2\pi)^4} \frac{i}{(\not{k} - m_f)} \gamma^5 \frac{i}{((k-p)^2 - 2\mu^2)} \frac{i}{(k-p)^2}$$

$$= -ig_r^2 \lambda_r v \gamma^5 \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} - m_f)}{(k^2 - m_f^2)((k-p)^2 - 2\mu^2)(k-p)^2}$$

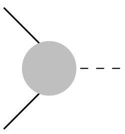
$$= -ig_r^2 \lambda_r v \gamma^5 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dxdydz \frac{(3-1)!\delta(x+y+z-1)(\not{k} - m_f)}{(z(k^2 - m_f^2) + y((k-p)^2 - 2\mu^2) + x(k-p)^2)^3}$$

$$= -2ig_r^2 \lambda_r v \gamma^5 \int \frac{d^d l_4}{(2\pi)^d} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)\not{p} - m_f}{(l_4^2 - \Delta_3)^3}$$

$$= -2ig_r^2 \lambda_r v \gamma^5 \int_0^1 dx \int_0^{1-x} dy[(x+y)\not{p} - m_f] \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3 - \frac{d}{2})}{\Gamma(3)} \left(\frac{1}{\Delta_3}\right)^{3-\frac{d}{2}}$$

$$= -\frac{g_r^2 \lambda_r v \gamma^5}{(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)\not{p} - m_f}{\Delta_3}.$$

Using the renormalization condition:



$$= \gamma^5 \left(\delta_g + \frac{g_r^2}{(4\pi)^2} \int_0^1 dx \left\{ g_r \log \frac{\Delta_2}{\Delta_1} + 2\lambda_r v \int_0^{1-x} dy \frac{m_f}{\Delta_3} \right\} \right) = 0.$$

The calculation shows that δ_g is finite, confirming that the total radiative correction to m_f is finite. This result aligns with the discussion in Section 11.6, which states that loop corrections to classically valid quantities, such as the fermion mass in this model, are finite and provide predictive results in quantum field theory.