10.2 Renormalization of Yukawa Theory

Consider the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi,$$

where ϕ is a real scalar field and ψ is a Dirac fermion. Notice that this Lagrangian is invariant under the parity transformation $\psi(t,\boldsymbol{x}) \to \gamma^0\psi(t,-\boldsymbol{x})$, $\phi(t,\boldsymbol{x}) \to -\phi(t,-\boldsymbol{x})$, in which the field ϕ carries odd parity.

(a) Determine the superficially divergent amplitudes and work out the Feynman rules for renormalized perturbation theory for this Lagrangian. Include all necessary counterterm vertices. Show that the theory contains a superficially divergent 4ϕ amplitude. This means that the theory cannot be renormalized unless one includes a scalar self-interaction,

$$\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4,$$

and a counterterm of the same form. It is of course possible to set the renormalized value of this coupling to zero, but that is not a natural choice, since the counterterm will still be nonzero. Are any further interactions required?

(b) Compute the divergent part (the pole as $d \to 4$) of each counterterm, to the one-loop order of perturbation theory, implementing a sufficient set of renormalization conditions. You need not worry about finite parts of the counterterms. Since the divergent parts must have a fixed dependence on the external momenta, you can simplify this calculation by choosing the momenta in the simplest possible way.

Solution:

To begin, we calculate the superficial degree of divergence:

$$\begin{aligned} D &\equiv (\text{power of } k \text{ in the numerator}) - (\text{power of } k \text{ in the denominator}) \\ &= 4L - P_\psi - 2P_\phi, \end{aligned}$$

where L is the number of loops, P_ψ is the Dirac fermion field propagator, and P_ϕ is the scalar field propagator. The number of loops in a diagram depends on the internal lines (I) and vertices (V):

$$L = I - V + 1 = P_\phi + P_\psi - V + 1.$$

Thus, the superficial degree of divergence becomes:

$$D = 4(P_\phi + P_\psi - V + 1) - P_\psi - 2P_\phi = 3P_\psi + 2P_\phi + 4 - 4V.$$

Additionally, the number of vertices is given by:

$$V = 2P_\phi + N_\phi = \frac{1}{2}(2P_\psi + N_\psi),$$

where N represents the number of external lines for the fields. This relation arises from the ϕ and ψ interaction, where each vertex involves exactly one ϕ and two ψ fields, and propagators count twice due to their connection to vertices. Substituting this, we obtain:

$$D = 3(V - \frac{1}{2}N_\psi) + (V - N_\phi) + 4 - 4V = 4 - \frac{3}{2}N_\psi - N_\phi.$$

Using this expression, we identify the divergent diagrams, as shown in Figure 1.

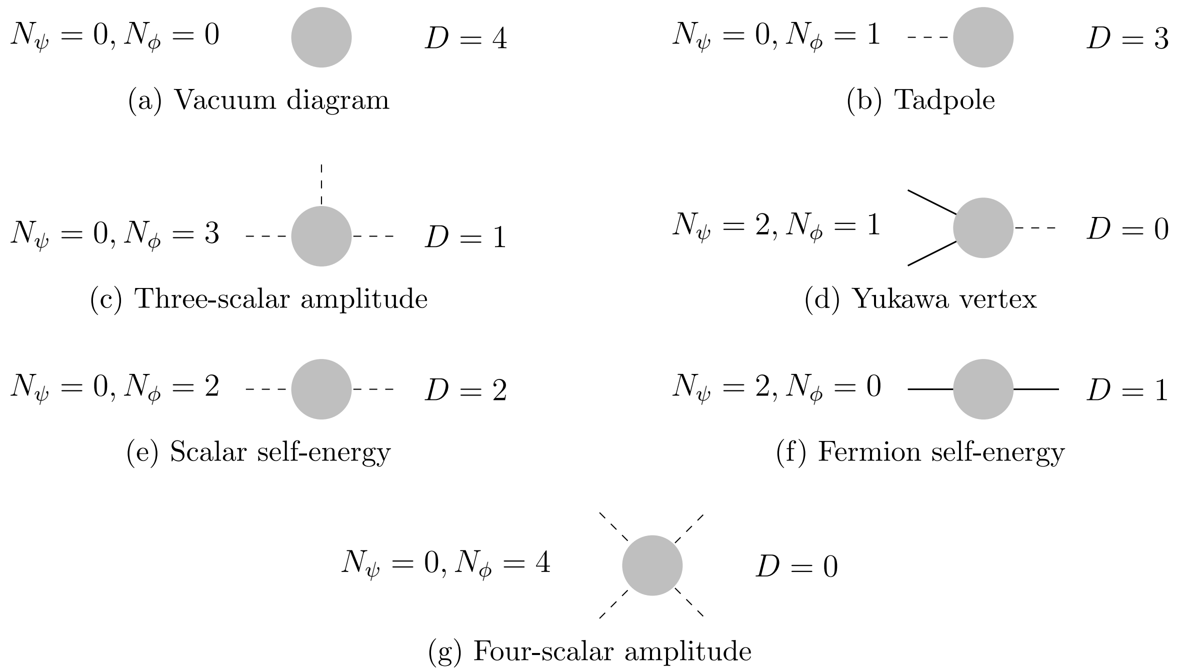


Figure 1: 1PI diagrams with their superficial degrees of divergence D. Fermion legs are solid lines and scalar legs are dashed lines.

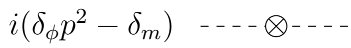
As shown in Figure 1, diagram (a) is irrelevant to scattering processes, while amplitudes (b) and (c) vanish due to symmetries. The original Lagrangian cannot be renormalized due to the divergent 4ϕ amplitude (g), necessitating the addition of the term $\delta\mathcal{L} = \frac{\lambda}{4!}\phi^4$ to the Lagrangian.
Next, we introduce the physical fields:

$$\phi_0 = Z_3^{\frac{1}{2}}\phi_r, \qquad \psi_0 = Z_2^{\frac{1}{2}}\psi_r,$$

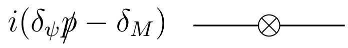
to express the Lagrangian in terms of these fields:

$$\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 + \frac{1}{2}\underbrace{(Z_3 - 1)}_{\delta_\phi}(\partial_\mu\phi_r)^2 - \frac{1}{2}\underbrace{(Z_3 m_0^2 - m_r^2)}_{\delta_m}\phi_r^2 \\
& + \bar{\psi}_r(i\partial\!\!\!/ - M_r)\psi_r + \bar{\psi}_r(i\underbrace{(Z_2 - 1)}_{\delta_\psi}\partial\!\!\!/ - \underbrace{(Z_2 M_0 - M_r)}_{\delta_M})\psi_r \\
& - i g_r\bar{\psi}_r\gamma^5\psi_r\phi_r - i\underbrace{(Z_2 Z_3^{\frac{1}{2}} g_0 - g_r)}_{\delta_g}\bar{\psi}_r\gamma^5\psi_r\phi_r \\
& - \frac{\lambda_r}{4!}\phi_r^4 - \frac{1}{4!}\underbrace{(\lambda_0 Z_3^2 - \lambda_r)}_{\delta_\lambda}\phi_r^4.
\end{aligned}$$

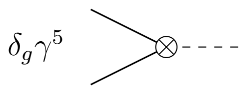
From this, we derive the Feynman rules for the counterterms, as illustrated in Figure 2.



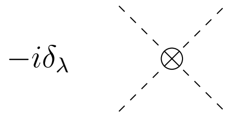
(a) Scalar self-energy counterterm



(b) Fermion self-energy counterterm



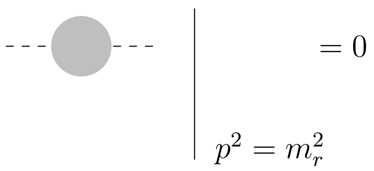
(c) Yukawa counterterm



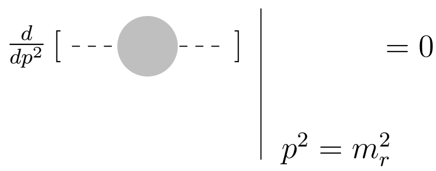
(d) Four-scalar counterterm

Figure 2: Counterterm vertices for the renormalized Yukawa theory.

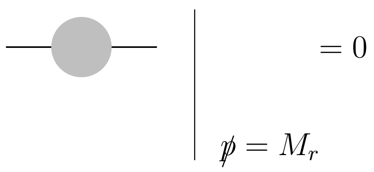
Before computing the 1PI diagrams, we establish the renormalization conditions (on-shell scheme), as shown in Figure 3.



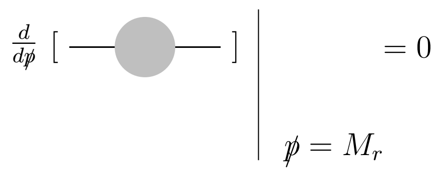
(a) Scalar self-energy



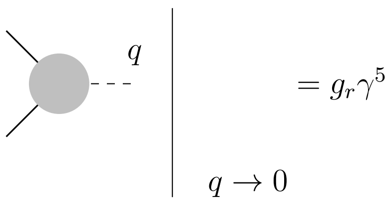
(b) Scalar self-energy derivative



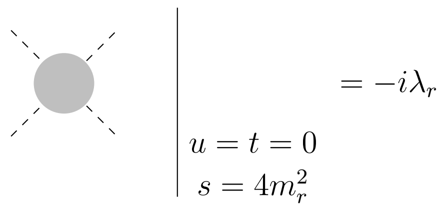
(c) Fermion self-energy



(d) Fermion self-energy derivative



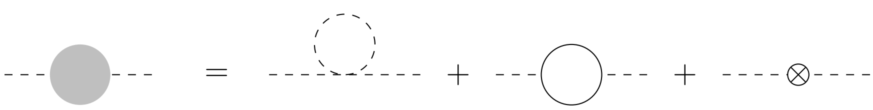
(e) Yukawa coupling

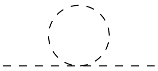


(f) Four-scalar coupling

Figure 3: Renormalization conditions for the Yukawa theory. Fermion legs are solid lines, scalar legs are dashed lines.

With these conditions, we proceed to calculate the 1PI diagrams:

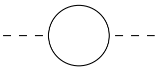




$$= \frac{1}{2}(-i\lambda_r)\int \frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m_r^2)} = \frac{1}{2}(-i\lambda_r)\int \frac{d^dl_1}{(2\pi)^d}\frac{i}{(l^2-\Delta_1)}$$

$$= \frac{\lambda_r}{2}\frac{(-1)i}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}\left(\frac{1}{\Delta_1}\right)^{1-\frac{d}{2}} = \frac{-i\lambda_r}{2}\frac{\Delta_1}{(1-d/2)}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta_1}\right)^{2-\frac{d}{2}}$$

$$\sim \frac{i\lambda_r}{2}m_r^2\frac{1}{(4\pi)^2}\frac{2}{\epsilon} = \frac{i\lambda_r m_r^2}{(4\pi)^2}\frac{1}{\epsilon}$$



$$= (-1)(g_r)^2\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{(\not{k}-M_r)}\gamma^5\frac{i}{((\not{k}-\not{p})-M_r)}\gamma^5\right]$$

$$= g_r^2\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}\left[(\not{k}+M_r)\gamma^5((\not{k}-\not{p})+M_r)\gamma^5\right]}{(k^2-M_r^2)((k-p)^2-M_r^2)}$$

$$= -g_r^2\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{\mathrm{Tr}\left[k^2-\not{k}\not{p}+M_r^2\right]}{\left((1-x)(k^2-M_r^2)+x((k-p)^2-M_r^2)\right)^2}$$

$$= -4g_r^2\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{k^2-k\cdot p+M_r^2}{\left((1-x)(k^2-M_r^2)+x((k-p)^2-M_r^2)\right)^2}$$

$$= -4g_r^2\int \frac{d^dl_2}{(2\pi)^d}\int_0^1 dx\frac{(l_2+xp)^2-(l_2+xp)\cdot p+M_r^2}{\left(l_2^2-\Delta_2\right)^2}$$

$$= -4g_r^2\int \frac{d^dl_2}{(2\pi)^d}\int_0^1 dx\frac{l_2^2+x(x-1)p^2+M_r^2}{\left(l_2^2-\Delta_2\right)^2}$$

$$= -4g_r^2\int_0^1 dx\left\{\frac{i}{\Gamma(2)}\left[-\frac{d}{2}\frac{\Delta_2}{1-d/2}+(x(x-1)p^2+M_r^2)\right]\right\}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta_2}\right)^{2-\frac{d}{2}}$$

$$\sim -4ig_r^2\int_0^1 3(x(x-1)p^2+M_r^2)\frac{1}{(4\pi)^2}\frac{2}{\epsilon}dx = \frac{4ig_r^2(p^2-6M_r^2)}{(4\pi)^2}\frac{1}{\epsilon}$$

By applying the scalar self-energy derivative condition, we find:

$$\delta_\phi \sim -\frac{4g_r^2}{(4\pi)^2}\frac{1}{\epsilon}$$

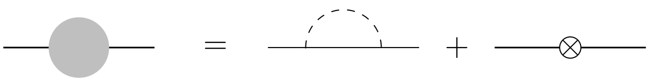
Using this result and the scalar self-energy condition, we determine δ_m:

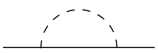
$$0 = \frac{i}{\epsilon}\left(\frac{\lambda_r m_r^2}{(4\pi)^2} + \frac{4g_r^2(m_r^2 - 6M_r^2)}{(4\pi)^2}\right) + i\left(-\frac{4g_r^2}{(4\pi)^2}m_r^2\frac{1}{\epsilon} - \delta_m\right).$$

Thus,

$$\delta_m \sim \frac{\lambda_r m_r^2 - 24g_r^2 M_r^2}{(4\pi)^2}\frac{1}{\epsilon}$$

For the fermion self-energy, we calculate:





$$= (g_r)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^5 \frac{i}{\not{k} - M_r}\gamma^5 \frac{i}{(k-p)^2 - m_r^2}$$

$$= -g_r^2 \int \frac{d^4k}{(2\pi)^4}\frac{-\not{k} + M_r}{(k^2 - M_r^2)((k-p)^2 - m_r^2)}$$

$$= g_r^2 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{\not{k} - M_r}{((1-x)(k^2 - M_r^2) + x((k-p)^2 - m_r^2))^2}$$

$$= g_r^2 \int \frac{d^dl_3}{(2\pi)^d}\int_0^1 dx \frac{x\not{p} - M_r}{(l_3^2 - \Delta_3)^2}$$

$$= g_r^2 \int_0^1 (x\not{p} - M_r)\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta_3}\right)^{2-\frac{d}{2}} dx$$

$$\sim \frac{2ig_r^2}{(4\pi)^2}\int_0^1 (x\not{p} - M_r)\frac{1}{\epsilon}dx = \frac{ig_r^2(\not{p} - 2M_r)}{(4\pi)^2}\frac{1}{\epsilon}$$

Using the fermion self-energy condition:

$$\delta_\psi \sim -\frac{g_r^2}{(4\pi)^2}\frac{1}{\epsilon},$$

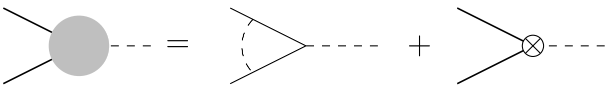
and the fermion self-energy derivative condition yields:

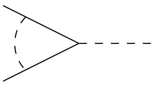
$$0 = \frac{ig_r^2(M_r - 2M_r)}{(4\pi)^2}\frac{1}{\epsilon} + i\left(-\frac{g_r^2 M_r}{(4\pi)^2}\frac{1}{\epsilon} - \delta_M\right),$$

resulting in:

$$\delta_M \sim -\frac{2g_r^2 M_r}{(4\pi)^2}\frac{1}{\epsilon}.$$

For the Yukawa and ϕ^4 vertices, since the divergent part is independent of external momenta, we set the momenta to zero and calculate:





$$= (g_r)^3 \int \frac{d^4k}{(2\pi)^4}\gamma^5 \frac{i}{(\not{k} - M_r)}\gamma^5 \frac{i}{(\not{k} - M_r)}\gamma^5 \frac{i}{(k^2 - m_r^2)}$$

$$= -ig_r^3 \int \frac{d^4k}{(2\pi)^4}\frac{\gamma^5(\not{k} + M_r)\gamma^5(\not{k} + M_r)\gamma^5}{(k^2 - M_r^2)(k^2 - M_r^2)(k^2 - m_r^2)}$$

$$= -ig_r^3\gamma^5 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dxdydz \frac{(3-1)!(\not{k} + M_r)(-\not{k} + M_r)\delta(x+y+z-1)}{(z(k^2 - M_r^2) + y(k^2 - M_r^2) + x(k^2 - m_r^2))^3}$$

$$= 2ig_r^3\gamma^5 \int \frac{d^4k}{(2\pi)^4}\int_0^1 dxdy \frac{k^2 - M_r^2}{((1-x-y)(k^2 - M_r^2) + y(k^2 - M_r^2) + x(k^2 - m_r^2))^3}$$

$$= 2ig_r^3\gamma^5 \int \frac{d^d l_4}{(2\pi)^d}\int_0^1 dx \int_0^{1-x} dy \frac{l_4^2 - M_r^2}{(l_4^2 - \Delta_4)^3}$$

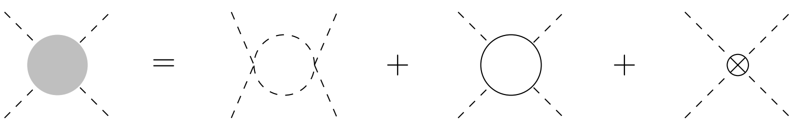
$$= -2g_r^3\gamma^5 \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Gamma(3)}\left\{\frac{d}{2} - (2 - \frac{d}{2})\frac{1}{\Delta_4}\right\}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta_4}\right)^{2-\frac{d}{2}}$$

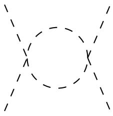
$$\sim -\frac{2g_r^3\gamma^5}{(4\pi)^2}\frac{1}{\epsilon}$$

This gives:

$$\delta_g \sim \frac{2g_r^3\gamma^5}{(4\pi)^2}\frac{1}{\epsilon}$$

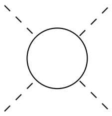
Finally, we address the remaining diagrams for the four-scalar vertex:





$$= \frac{(-i\lambda_r)^2}{2} \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{k^2 - m_r^2} \right)^2 = \frac{\lambda_r^2}{2} \int \frac{d^d l_5}{(2\pi)^d} \frac{1}{(l_5^2 - \Delta_5)^2}$$

$$= \frac{i\lambda_r^2}{2(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_5} \right)^{2-d/2} \sim \frac{i\lambda_r^2}{(4\pi)^2} \frac{1}{\epsilon}$$



$$= (-1)(g_r)^4 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left[\left(\gamma^5 \frac{i}{\not{k} - M_r^2} \right)^4 \right]$$

$$= -g_r^4 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^5(\not{k} + M_r)\gamma^5(\not{k} + M_r)\gamma^5(\not{k} + M_r)\gamma^5(\not{k} + M_r)\right]}{(k^2 - M_r^2)^4}$$

$$= -g_r^4 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}\left[(\not{k} - M_r)(\not{k} + M_r)(\not{k} - M_r)(\not{k} + M_r)\right]}{(k^2 - M_r^2)^4}$$

$$= -4g_r^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M_r^2)^2} = -4g_r^4 \int \frac{d^d l_6}{(2\pi)^d} \frac{1}{(l_6^2 - \Delta_6)^2}$$

$$= -\frac{4ig_r^4}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta_6} \right)^{2-d/2} \sim -\frac{8ig_r^4}{(4\pi)^2} \frac{1}{\epsilon}$$

Noting that the first diagram has three permutations (s, t, and u channels) and the second has six permutations, we apply the renormalization conditions to find:

$$\delta_\lambda \sim \frac{3\lambda_r^2 - 48g_r^4}{(4\pi)^2} \frac{1}{\epsilon}$$