11.3 The Gross-Neveu Model

The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2} g^2 (\bar{\psi}_i \psi_i)^2$$

with $i = 1, \dots, N$. The kinetic term of two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \qquad \gamma^1 = i\sigma^1,$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0 \gamma^1 = \sigma^3;$$

this matrix anticommutes with the γ^μ.

(a) Show that this theory is invariant with respect to

$$\psi_i \to \gamma^5 \psi_i,$$

and that this symmetry forbids the appearance of a fermion mass.

(b) Show that this theory is renormalizable in 2 dimensions (at the level of dimensional analysis).

(c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{i \int d^2x \, \mathcal{L}} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\sigma \, \exp\left[i \int d^2x \left\{ \bar{\psi}_i i\partial\!\!\!/ \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{1}{2g^2} \sigma^2 \right\} \right],$$

where $\sigma(x)$ (not be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

(d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant, diagonalize the operator by first going to Fourier components and then diagonalizing the 2×2 Pauli matrix associated with each Fourier mode. (Alternatively, you might just take the determinant of this 2×2 matrix.) This 1-loop contribution requires a renormalization proportional to σ^2 (that is, a renormalization of g^2). Renormalize by minimal subtraction.

(e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of part (a). Convince yourself that this result does not depend on the particular renormalization condition chosen.

(f) Note that the effective potential derived in part (e) depends on g and N according to the form

$$V_{\text{eff}}(\sigma_{\text{cl}}) = N \cdot f(g^2 N).$$

(The overall factor of N is expected in a theory with N fields.) Construct a few of the higher-order contributions to the effective potential and show that they contain additional factors of N^{-1} which suppress them if we take the limit $N \to \infty$, $(g^2 N)$ fixed. In this limit, the result of part (e) is unambiguous.

Solution:

To verify the chiral symmetry of the Lagrangian, we apply the transformation $\psi_i \to \gamma^5 \psi_i$:

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2} g_0^2 (\bar{\psi}_i \psi_i)^2 \\
&\to (\gamma^5 \psi_i)^\dagger \gamma^0 i\partial\!\!\!/ \gamma^5 \psi_i + \frac{1}{2} g_0^2 ((\gamma^5 \psi_i)^\dagger \gamma^0 \gamma^5 \psi_i)^2 \\
&= -\psi_i^\dagger \gamma^0 \gamma^5 i\partial\!\!\!/ \gamma^5 \psi_i + \frac{1}{2} g_0^2 (\psi_i^\dagger \gamma^5 \gamma^0 \gamma^5 \psi_i)^2 \\
&= \psi_i^\dagger \gamma^0 \gamma^5 \gamma^5 i\partial\!\!\!/ \psi_i + \frac{1}{2} g_0^2 (-\psi_i^\dagger \gamma^0 \gamma^5 \gamma^5 \psi_i)^2 \\
&= \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2} g_0^2 (\bar{\psi}_i \psi_i)^2 = \mathcal{L}.
\end{aligned}$$

This transformation leaves the Lagrangian invariant, confirming the discrete chiral symmetry. However, a mass term $m\bar{\psi}_i\psi_i$ transforms as

$$m\bar{\psi}_i\psi_i \to -m\bar{\psi}_i\psi_i.$$

This sign change violates the chiral symmetry, prohibiting the presence of a fermion mass term in the Lagrangian.

To show that the theory is renormalizable in two dimensions via dimensional analysis, we note that the fermion kinetic term implies $[\psi_i] = 1/2$. Thus, the interaction term $\frac{1}{2} g^2 (\bar{\psi}_i \psi_i)^2$ requires $[g] = 0$, indicating that the coupling is dimensionless, a hallmark of a renormalizable theory in two dimensions.

The model can be reformulated equivalently by introducing an auxiliary scalar field $\sigma(x)$ and integrating it out:

$$\begin{aligned}
Z &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \, \exp\left[i \int d^2x \left\{ \bar{\psi}_i i\partial\!\!\!/ \psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g_0^2}\sigma^2 \right\}\right] \\
&= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \, \exp\left[i \int d^2x \left\{ \bar{\psi}_i i\partial\!\!\!/ \psi_i - \frac{1}{2g_0^2}\left(\sigma^2 + 2g_0^2 \sigma\bar{\psi}_i\psi_i\right) \right\}\right] \\
&= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \, \exp\left[i \int d^2x \left\{ \bar{\psi}_i i\partial\!\!\!/ \psi_i - \frac{1}{2g_0^2}\left(\sigma + g_0^2\left(\bar{\psi}_i\psi_i\right)\right)^2 + \frac{1}{2g_0^2}\left(\bar{\psi}_i\psi_i\right) \right\}\right] \\
&= C \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{i\int d^2x \, \mathcal{L}},
\end{aligned}$$

where $C = (\det [1/g_0^2])^{-1/2}$ is a constant that is irrelevant for expectation values, as it cancels out in physical observables. Integrating out the fermion fields yields:

$$\begin{aligned}
Z &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[i \int d^2x \left\{ \bar{\psi}_i i\partial\!\!\!/ \psi_i - \mu^\epsilon \sigma\bar{\psi}_i\psi_i - \frac{1}{2g_0^2}\sigma^2 \right\}\right] \\
&= \left[\det\left(i\partial\!\!\!/ - m\right)\right]^N e^{-\frac{1}{2g^2}\sigma^2} = \exp\left\{ N\mathrm{Tr}\left[\ln\left(i\partial\!\!\!/ - m\right)\right] - \frac{1}{2g_0^2}\sigma^2 \right\},
\end{aligned}$$

where we have used

$$\det A = e^{\mathrm{Tr}\ln A},$$

and defined $m = \mu^\epsilon\sigma$. The renormalization scale μ is introduced due to dimensional regularization. To keep the coupling g dimensionless in $d = 2-2\epsilon$ dimensions, the scalar field σ has dimension $[\sigma] = 1-\epsilon$, and the fermion field has $[\psi] = 1/2-\epsilon$. Thus, the interaction term $\sigma\bar{\psi}_i\psi_i$ has dimension $2-3\epsilon$, requiring the factor μ^ϵ to ensure a dimensionless action in $\int d^{2-2\epsilon}x$. Since we seek the effective potential, we treat $\sigma(x) = \sigma$ as a spacetime constant, corresponding to the determinant for N fermions with mass $m = \mu^\epsilon\sigma$.
Going to the Fourier space and using the same relation, yields:

$$\mathrm{Tr}\left[\ln\left(i\partial\!\!\!/ - m\right)\right] = \int \frac{d^2x\, d^2k}{(2\pi)^2}\mathrm{Tr}\left[\ln\left(k\!\!\!/ - m\right)\right] = (LT)\int \frac{d^2k}{(2\pi)^2}\ln\left[\det\left(k\!\!\!/ - m\right)\right].$$

We can find the determinant by either explicitly taking the determinant:

$$\begin{aligned}\det\left(k\!\!\!/ - m\right) &= \det\left[\begin{pmatrix}0 & -ik^0\\ ik^0 & 0\end{pmatrix} + \begin{pmatrix}0 & ik^1\\ ik^1 & 0\end{pmatrix} - \begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}\right]\\ &= \det\begin{pmatrix}-m & -ik^0+ik^1\\ ik^0+ik^1 & -m\end{pmatrix} = m^2 - ((k^0)^2-(k^1)^2) = -k^2+m^2.\end{aligned}$$

Or by diagonalizing the $k\!\!\!/$ matrix by using the facts:

$$k\!\!\!/ k\!\!\!/ = k^2, \qquad \mathrm{Tr}k\!\!\!/ = 0.$$

Then we'll have (in 2 dimensions):

$$\det(k\!\!\!/ - m) = \det\begin{pmatrix}-m+\sqrt{k^2} & 0\\ 0 & -m-\sqrt{k^2}\end{pmatrix} = -k^2+m^2.$$

Now we can get back to the integral and apply the Wick rotation and dimensional regularization:

$$\begin{aligned}\int \frac{d^2k}{(2\pi)^2}\ln\left[\det\left(k\!\!\!/ - m\right)\right] &= \int \frac{d^dk}{(2\pi)^d}\ln\left[-k^2+m^2\right] = i\int \frac{d^dk_E}{(2\pi)^d}\ln\left[k_E^2+m^2\right]\\ &= -i\frac{\partial}{\partial\alpha}\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2+m^2)^\alpha}\bigg|_{\alpha=0}\\ &= -i\frac{\partial}{\partial\alpha}\left(\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\frac{1}{(m^2)^{\alpha-d/2}}\right)\bigg|_{\alpha=0}\\ &= -i\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{d/2}}(m^2)^{d/2}.\end{aligned}$$

Thus in $d \to 2-2\epsilon$, we get:

$$\frac{1}{(LT)}\mathrm{Tr}\left[\ln\left(i\partial\!\!\!/ - m\right)\right] = -i\frac{\Gamma(\epsilon-1)}{(4\pi)^{1-\epsilon}}(\mu^{2\epsilon}\sigma^2)^{1-\epsilon}$$

$$= i\frac{\sigma^2}{4\pi}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln\frac{\sigma^2}{\mu^2} + 1 + \mathcal{O}(\epsilon)\right).$$

Therefore,

$$\begin{aligned} e^{-i\int d^2x\, V_{\text{eff}}(\sigma)} &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp\left[i\int d^2x\left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \mu^\epsilon\sigma\bar{\psi}_i\psi_i - \frac{1}{2g_0^2}\sigma^2\right\}\right] \\ &= \exp\left\{-i\left[\frac{1}{2g_0^2}\sigma^2 + \frac{N}{4\pi}\left(-\frac{1}{\epsilon} + \gamma - \ln 4\pi\right)\sigma^2 + \frac{N}{4\pi}\left(\sigma^2\ln\frac{\sigma^2}{\mu^2} - 1\right)\right]\right\}. \end{aligned}$$

In the $\overline{\text{MS}}$ scheme, we define:

$$\frac{1}{2g^2} = \frac{1}{2g_0^2} - \frac{N}{4\pi}\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right),$$

yielding the effective potential:

$$V_{\text{eff}}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\left(\sigma^2\ln\frac{\sigma^2}{\mu^2} - 1\right).$$

To find the vacuum expectation value, we minimize the effective potential:

$$0 = \frac{\partial V(\sigma)}{\partial\sigma} = \frac{1}{g^2}\sigma + \frac{N}{2\pi}\sigma\ln\frac{\sigma^2}{\mu^2},$$

This yields nonzero solutions $\langle\sigma\rangle = \pm\mu e^{-\pi/g^2N}$. At this minimum, the fermions acquire a mass $m = \mu^\epsilon\langle\sigma\rangle$, dynamically breaking the chiral symmetry identified in part (a). The result is independent of the renormalization scheme, because although it would change the value of c in $1/2g^2 = 1/2g_0^2 + c$ and therefore changes the expression by a factor $e^{-\pi c/N}$ but the discrete chiral symmetry would be still broken.

The effective potential derived above does not yet account for contributions from σ-loop corrections, as the σ field has not been integrated out. To evaluate the significance of these higher-order terms, we consider the large-N limit with g_0^2N held fixed. In this limit, the effective action for the σ field takes the form:

$$\Gamma[\sigma] = N\left[-\mathrm{Tr}\ln\left(i\partial\!\!\!/ - \sigma\right) + \frac{1}{2g_0^2N}\sigma^2\right].$$

Here, the factor of N amplifies the contribution of the leading terms, causing the effective action to be dominated by its stationary point, σ_{cl}. This allows us to apply the saddle-point approximation (also known as the steepest descent method), where only the stationary point contributes significantly to the path integral. Consequently, the effective potential calculated in part (e) becomes exact in the large-N limit.