The Prime Number Theorem

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The purpose of this paper is to present a proof of the prime number theorem which is accessible to the reader who has had a course in complex analysis. Specifically, the reader should be familiar with concepts such as residues, analytic continuation, uniformly convergent sequences of analytic functions, etc. The presentation given here follows that in Titchmarsh [4], which is in turn based on the original arguments of Hadamard and de la Vallée Poussin.

1 Preliminaries

The notation $f(x) \sim g(x)$, $x \to a$ means that $\lim_{x\to a} f(x)/g(x) = 1$. Let $\pi(x)$ denote the number of primes $\leq x$.

Prime Number Theorem.

$$\pi(x) \sim \frac{x}{\log x}, \qquad x \to \infty$$

This theorem, which we will refer to as the PNT, was proved independently Hadamard and de la Vallée Poussin in 1896, extending ideas introduced by Riemann in his seminal paper *On the Number of Primes Less Than a Given Magnitude*, published in 1859. We will need several preliminary results before we can carry out the proof, and this first section is dedicated to those results.

1.1 The Riemann Zeta-function

Throughout this paper, s will denote a complex number, $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Re}(s)$.

Definition 1. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad (\mathrm{Re}(s) > 1). \tag{1}$$

Comparison of the above series with $\sum n^{-\sigma_0}$ in the halfplane $\mathrm{Re}(s) \geq \sigma_0 > 1$ shows that the convergence is uniform there and $\zeta(s)$ is analytic for $\mathrm{Re}(s) > 1$.

The connection between $\zeta(s)$ and the primes is contained in the following

Theorem 1.

$$\sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}, \qquad (\mathrm{Re}(s) > 1), \tag{2}$$

the product being taken over all primes p.

Proof. Using (1),

$$\zeta(s)(1 - 2^{-s}) = \sum n^{-s} - \sum (2n)^{-s} = \sum_{2 \nmid n} n^{-s}. \tag{3}$$

Repeating this argument,

$$\zeta(s)(1 - 2^{-s}) \cdots (1 - p_k^{-s}) = \sum_{2,\ldots,p_k \nmid n} n^{-s} \tag{4}$$

where p_k is the k^{th} prime. Given $N > 0$ there is k large enough that $p_k \geq N$ (since there are infinitely many primes), and all integers between 1 and N are products of primes less than p_k (by the fundamental theorem of arithmetic). Hence

$$|\zeta(s)(1 - 2^{-s}) \cdots (1 - p_k^{-s}) - 1| \leq \sum_{N}^{\infty} n^{-\sigma} \tag{5}$$

whenever k is large enough, and the right tends to zero as N tends to infinity. We conclude that

$$\zeta(s)\prod_p (1-p^{-s}) = 1 \tag{6}$$

when $\mathrm{Re}(s) > 1$. It is an elementary fact about infinite products that the convergence of $\sum |p^{-s}|$ implies that the product $\prod (1-p^{-s})$ converges to a nonzero value (see Ahlfors [1], Ch. 5, 2.2), so (6) gives (2). $\square$

(2) says in particular that $\zeta(s)$ has no zeros in the region $\mathrm{Re}(s) > 1$. Since this region is simply connected there is an analytic branch of $\log \zeta(s)$ defined there ([1], Ch. 4, 4.4). Throughout this paper $\log \zeta(s)$ shall refer to the unique such branch which is real for real s.

We can use (2) to express $\log \zeta(s)$ in terms of $\pi(x)$:

Theorem 2.

$$\log \zeta(s) = \int_2^\infty \frac{s\pi(x)}{x(x^s-1)}\,dx, \qquad (\mathrm{Re}(s) > 1) \tag{7}$$

Proof.

$$\begin{aligned}
\log \zeta(s) &= -\sum_p \log(1-p^{-s}) && (8)\\
&= -\sum_{n=2}^\infty [\pi(n) - \pi(n-1)] \log(1-n^{-s}) && (9)\\
&= -\sum_{n=2}^\infty \pi(n)[\log(1-n^{-s}) - \log(1-(n+1)^{-s})] && (10)\\
&= \sum_{n=2}^\infty \int_n^{n+1} \frac{s\pi(n)}{x(x^s-1)}\,dx && (11)\\
&= \int_2^\infty \frac{s\pi(x)}{x(x^s-1)}\,dx && (12)
\end{aligned}$$

The logarithms appearing in (8) are all principal. The equality of (9) and (10) follows from the fact that $\pi(n) \le n$ and $|\log(1-n^{-s})| \le 2|n^{-s}| = 2n^{-\sigma}$ for large n, so that the difference between the partial sums of the two series tends to zero.

(8) calls for some further comment: in general if $\prod a_n$ converges (to a nonzero value), and the a_n are nonzero, then we may only conclude that $\sum \log a_n = \log \prod a_n + 2\pi i k$, for some integer k, where the logarithms are principal. However, (8) holds when s is real, since in that case all the logarithms are real. Also, the series represents an analytic function of s, being uniformly convergent in any half plane $\mathrm{Re}(s) \ge \sigma_0 > 1$ (to see this, compare the series with $\sum p^{-\sigma_0}$). Thus, by analytic continuation, (8) must be valid whenever $\mathrm{Re}(s) > 1$. $\square$

The main idea behind the proof of the prime number theorem is that we can essentially invert (7) and obtain a representation of $\pi(x)$ in terms of $\zeta(s)$. Having carried out this inversion, we will need some further information about $\zeta(s)$ in order to make the necessary estimates. In particular, we need to extend the domain of definition of $\zeta(s)$ past the line $\mathrm{Re}(s) = 1$ and study the behavior of $\zeta(s)$ on this line.

Theorem 3. *$\zeta(s)$ may be extended to a meromorphic function in $\mathrm{Re}(s) > 0$ with a single pole at $s = 1$, which is simple and has residue 1.*

Proof. We use the following formula, valid for any continuously differentiable function $\phi : [a, b] \to \mathbb{C}$, where a, b are integers:

$$\sum_{a+1}^{b} \phi(n) = \int_a^b \phi(x)\, dx + \int_a^b (x - \lfloor x \rfloor)\phi'(x)\, dx. \tag{13}$$

Since the formula is additive with respect to the interval $[a, b]$, it suffices to verify (13) when $b = a + 1$, in which case it follows by integrating by parts.

Applying (13) to $\phi(x) = x^{-s}$,

$$\sum_{2}^{N} n^{-s} = \int_1^N x^{-s}\, dx - s \int_1^N \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx. \tag{14}$$

Letting $N \to \infty$ in (14) and adding 1 to both sides,

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\, dx, \qquad (\mathrm{Re}(s) > 1). \tag{15}$$

The integral in (15) is uniformly convergent in any halfplane $\mathrm{Re}(s) \geq \sigma_0 > 0$, hence represents an analytic function in $\mathrm{Re}(s) > 0$. Thus we may take the right side of (15) to be the definition of $\zeta(s)$ in this region. □

The objective of the remainder of this section is to estimate the functions $\frac{\zeta'(s)}{\zeta(s)}$ and $\log \zeta(s)$ on the line $\mathrm{Re}(s) = 1$. First of all, we must make sure these functions are defined there. The next theorem takes care of this.

Theorem 4. *$\zeta(s)$ has no zeros on the line* $\mathrm{Re}(s) = 1$.

Note that by continuity this means that $\zeta(s)$ is nonvanishing in a region containing $\{\mathrm{Re}(s) \geq 1\} - \{1\}$, which may be taken to be simply connected, so indeed $\frac{\zeta'(s)}{\zeta(s)}$ and $\log \zeta(s)$ may be extended past $\sigma = 1$.

Proof. The proof is an application of the following identity:

$$\frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta = \frac{1}{2}(1 + \cos\theta)^2 \geq 0. \tag{16}$$

By (8) we can write

$$\log \zeta(s) = \sum_p \sum_{n=1}^\infty \frac{1}{np^{ns}}, \qquad (\mathrm{Re}(s) > 1). \tag{17}$$

Taking the real part of both sides,

$$\log |\zeta(s)| = \sum_p \sum_{n=1}^\infty \frac{\cos(nt \log p)}{np^{n\sigma}}. \tag{18}$$

It follows that

$$|\zeta(\sigma)|^{\frac{3}{4}}|\zeta(\sigma+it)||\zeta(\sigma+2it)|^{\frac{1}{4}} = \exp\sum_p\sum_{n=1}^{\infty}\frac{\frac{3}{4}+\cos(nt\log p)+\frac{1}{4}\cos(2nt\log p)}{np^{n\sigma}}. \qquad (19)$$

By (16) every term in the above series is nonnegative, so

$$|\zeta(\sigma)|^{\frac{3}{4}}|\zeta(\sigma+it)||\zeta(\sigma+2it)|^{\frac{1}{4}} \geq 1, \qquad (\sigma > 1). \qquad (20)$$

Fix $t \neq 0$. As $\sigma \to 1$, $\zeta(\sigma+2it)$ tends to some finite value, while $(\sigma-1)\zeta(\sigma)$ tends to 1. Since

$$\frac{\sigma-1}{|\zeta(\sigma+it)|} \leq (\sigma-1)|\zeta(\sigma)|^{\frac{3}{4}}|\zeta(\sigma+2it)|^{\frac{1}{4}}, \qquad (21)$$

$(\sigma-1)|\zeta(\sigma+it)|^{-1}$ tends to zero as $\sigma \to 1$. This could not be the case if $\zeta(1+it)=0$, for then $\zeta(\sigma+it)=(\sigma-1)g(s)$, where $g(s)$ is analytic at $s=1$. □

Remark. (17) shows that $|\log\zeta(s)|$ is bounded by $\sum_p\sum_n n^{-1}p^{-n\sigma_0}$ in any half plane $\mathrm{Re}(s) \geq \sigma_0 > 1$. By the same reasoning, the double series is uniformly convergent in any such halfplane, so we can differentiate term by term and obtain a similar bound for $|\frac{\zeta'(s)}{\zeta(s)}|$ in $\mathrm{Re}(s) \geq \sigma_0 > 1$.

To get approximations on $\mathrm{Re}(s)=1$ we first deal with $\zeta(s)$ and $\zeta'(s)$. The notation $f(x)=O(g(x))$ as $x \to a$ is standard and means that there is $A > 0$ such that $|f(x)| \leq A|g(x)|$ for all x sufficiently close to a.

Theorem 5. *The following estimates hold uniformly in* $\sigma \geq 1$ *as* $t \to \infty$ *:*

$$\zeta(\sigma+it) = O(\log t), \qquad (22)$$

$$\zeta'(\sigma+it) = O(\log^2 t). \qquad (23)$$

Note that since $\zeta(\bar{s}) = \overline{\zeta(s)}$ and $\zeta'(\bar{s}) = \overline{\zeta'(s)}$, the same estimates hold as $t \to -\infty$.

Proof. We apply (13) again to obtain

$$\sum_{N+1}^{M} n^{-s} = \int_N^M x^{-s}\,dx - s\int_N^M \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx \tag{24}$$

for $M > N > 0$. For $\mathrm{Re}(s) > 1$, we let $M \to \infty$ to get

$$\zeta(s) - \sum_{1}^{N} n^{-s} = \frac{N^{1-s}}{s-1} - s\int_N^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx. \tag{25}$$

All functions appearing in the above formula are meromorphic in $\mathrm{Re}(s) > 0$, so (25) must be valid throughout that entire region.

If $\sigma \geq 1$ then $|n^{-s}| = n^{-\sigma} \leq n^{-1}$. The integral in (25) is bounded by $\sigma^{-1}N^{-\sigma}$, and hence

$$|\zeta(\sigma + it)| \leq \sum_{1}^{N} n^{-1} + t^{-1} + (1+t)N^{-1} \tag{26}$$

$$\leq 1 + \log N + t^{-1} + (1+t)N^{-1}. \tag{27}$$

Here N is arbitrary, so by choosing $N = \lfloor t \rfloor$ we get (22).

The proof of (23) is similar: differentiating (25) gives

$$\begin{aligned}\zeta'(s) + \sum_{1}^{N} n^{-s}\log n = &-\frac{N^{1-s}}{(s-1)^2} - \frac{N^{1-s}}{s-1}\log N \\ &- \int_N^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx + s\int_N^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\log x\,dx,\end{aligned} \tag{28}$$

from which (23) follows. □

Theorem 6. *As $t \to \infty$,*

$$\frac{\zeta'(1+it)}{\zeta(1+it)} = O(\log^9 t), \tag{29}$$

$$\log \zeta(\sigma + it) = O(\log^{\frac{9}{4}} t), \tag{30}$$

the second estimate holding uniformly in $1 \le \sigma \le 2$.

Again, the same estimates hold as $t \to -\infty$.

Proof. Write $\zeta(s) = (s-1)^{-1} g(s)$, where $g(s)$ is analytic in $\mathrm{Re}(s) > 0$. Then $|\zeta(\sigma)| \le A_1(\sigma - 1)^{-1}$ for $1 < \sigma < 2$, for some $A_1 > 0$. Hence, by (20) and (22),

$$\frac{1}{\zeta(\sigma + it)} = O\left(\frac{\log^{\frac{1}{4}} t}{(\sigma - 1)^{\frac{3}{4}}}\right) \tag{31}$$

for $1 < \sigma < 2$. Also,

$$\zeta(\sigma + it) - \zeta(1 + it) = \int_1^\sigma \zeta'(x + it)\, dx = O((\sigma - 1)\log^2 t) \tag{32}$$

for $\sigma > 1$. Combining (31) and (32),

$$|\zeta(1 + it)| \ge A_2 \frac{(\sigma - 1)^{\frac{3}{4}}}{\log^{\frac{1}{4}} t} - A_3(\sigma - 1)\log^2 t \tag{33}$$

for $1 < \sigma < 2$ and t large. Choosing $\sigma - 1 = A_4 \log^{-9} t$, for $A_4 > 0$ sufficiently small,

$$|\zeta(1 + it)| \ge A_5 \log^{-7} t \tag{34}$$

for large t, where $A_5 > 0$. (23) and (34) give (29).

To prove (30), note that

$$\log \zeta(\sigma + it) = -\int_\sigma^2 \frac{\zeta'(x + it)}{\zeta(x + it)}\, dx + \log \zeta(2 + it). \tag{35}$$

For $1 \le \sigma \le 2$ and large t the integral is bounded by

$$A_6 \log^{\frac{9}{4}} t \int_\sigma^2 \frac{dx}{(x-1)^{\frac{3}{4}}} \le 4A_6 \log^{\frac{9}{4}} t, \tag{36}$$

by (23) and (31). (30) follows, since $\log(2+it)$ is bounded (see the remark after Theorem 4). $\square$

1.2 The Fourier Transform

As mentioned earlier, the proof of the PNT will involve inverting (7). This will be done as an application of the inversion theorem for the Fourier transform, which we prove in the special case of a bounded continous function. Readers interested in learning about the Fourier transform in the proper context of L^1 and L^2 spaces are encouraged to consult Rudin [2].

A function $f(x)$ is said to be absolutely integrable on $\mathbb{R}$ if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Definition 2. If $f : \mathbb{R} \to \mathbb{C}$ is absolutely integrable then the Fourier transform of $f(x)$ is defined as the function

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} \frac{dx}{\sqrt{2\pi}} \tag{37}$$

for all real t.

Theorem 7. *If $f(x)$ is bounded, continous and absolutely integrable on $\mathbb{R}$, and if $\hat{f}(x)$ is absolutely integrable, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt} \frac{dt}{\sqrt{2\pi}}. \tag{38}$$

(38) could be restated as $f(x) = \hat{\hat{f}}(-x)$. We point out that the hypothesis that $f(x)$ be bounded is superfluous, as it actually follows from the absolute integrability of $\hat{f}(x)$. Continuity could also be dispensed with, provided we that we only require (38) to hold

almost everywhere. However, this more satisfying form of the theorem requires the theory of Lebesgue integration, and the form stated above is all we need.

Proof. Define

$$H(t) = e^{-|t|} \tag{39}$$

for real t, and

$$h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t) e^{ixt} \, \frac{dt}{\sqrt{2\pi}} \tag{40}$$

for $\lambda > 0$ and real x. By a simple calculation one finds that

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2}. \tag{41}$$

Hence

$$\int_{-\infty}^{\infty} h_\lambda(x) \, \frac{dx}{\sqrt{2\pi}} = 1. \tag{42}$$

We'll use $H(t)$ and $h_\lambda(x)$ to connect $f(x)$ with $\hat{\hat{f}}(-x)$. To do this we define the *convolution* $(f * h_\lambda)(x)$ of $f(x)$ and $h_\lambda(x)$:

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} f(x - y) h_\lambda(y) \, \frac{dy}{\sqrt{2\pi}}. \tag{43}$$

This exists for all real x since $f(x)$ is absolutely integrable and $h_\lambda(x)$ is bounded.

First we show that

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t) \hat{f}(t) e^{ixt} \, \frac{dt}{\sqrt{2\pi}}. \tag{44}$$

In fact,

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} f(x-y) \int_{-\infty}^{\infty} H(\lambda t) e^{iyt} \frac{dt}{\sqrt{2\pi}} \frac{dy}{\sqrt{2\pi}} \tag{45}$$

$$= \int_{-\infty}^{\infty} H(\lambda t) \int_{-\infty}^{\infty} f(x-y) e^{iyt} \frac{dy}{\sqrt{2\pi}} \frac{dt}{\sqrt{2\pi}} \tag{46}$$

$$= \int_{-\infty}^{\infty} H(\lambda t) \int_{-\infty}^{\infty} f(y) e^{i(x-y)t} \frac{dy}{\sqrt{2\pi}} \frac{dt}{\sqrt{2\pi}}, \tag{47}$$

and the last line is the same as (44). That we can switch the order of integration in (45) follows from Fubini's theorem. Alternatively, one could use the corresponding fact for integrals over bounded rectangles, along with some simple estimates of the remainders.

Next we show that on the one hand,

$$\lim_{\lambda \to 0} (f * h_\lambda)(x) = f(x), \tag{48}$$

while on the other hand,

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} H(\lambda t) \hat{f}(t) e^{ixt} \frac{dt}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt} \frac{dt}{\sqrt{2\pi}}, \tag{49}$$

which proves the theorem.

Let B be such that $|f(y)| \leq B$ for all y. Fix x and $\epsilon > 0$, and let $\delta > 0$ be such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then by (42),

$$|(f * h_\lambda)(x) - f(x)| \leq \int_{-\infty}^{\infty} |f(x-y) - f(x)| h_\lambda(y) \frac{dy}{\sqrt{2\pi}} \tag{50}$$

$$\leq \frac{\epsilon}{2} \int_{-\delta}^{\delta} h_\lambda(y) \frac{dy}{\sqrt{2\pi}} + \lambda \frac{2B}{\pi} \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \frac{dy}{\lambda^2 + y^2} \tag{51}$$

$$\leq \frac{\epsilon}{2} + \lambda \frac{4B}{\delta\pi} \tag{52}$$

For λ small this is less than ϵ, and (48) follows.

To prove (49), note that $H(\lambda t)\hat{f}(t) \to \hat{f}(t)$ uniformly in any finite interval $[-R, R]$, since $\hat{f}(t)$ is bounded by $\int_{-\infty}^{\infty} |f(x)|\, dx$, so that (49) holds with ∞ replaced with any $R > 0$. Hence

$$\left| \int_{-\infty}^{\infty} H(\lambda t)\hat{f}(t)e^{ixt}\,\frac{dt}{\sqrt{2\pi}} - \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}\,\frac{dt}{\sqrt{2\pi}} \right| \le \left| \int_{-R}^{R} (H(\lambda t) - 1)\hat{f}(t)e^{ixt}\,\frac{dt}{\sqrt{2\pi}} \right| + \int_{-\infty}^{-R} + \int_{R}^{\infty} |\hat{f}(t)| \frac{dt}{\sqrt{2\pi}}, \tag{53}$$

which may be made arbitrarily small by first choosing R large, then choosing λ small. $\square$

We will need another result about the Fourier transform, known as the Riemann-Lebesgue theorem.

Theorem 8. *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous execpt at a finite number of points, and suppose $f(x)$ is absolutely integrable. Then*

$$\lim_{|t|\to\infty} \hat{f}(t) = 0 \tag{54}$$

Note that $f(x)$ may be unbounded near the discontinuities. In this case $\int_{-\infty}^{\infty} |f(x)|\, dx$ is interpreted as a sum of improper integrals, and the absolute integrability of $f(x)$ means that each of these integrals converges. Once again, the theorem would hold perferctly well without the assumption about continuity.

Proof. We use the following fact: given $\epsilon > 0$, there is a continuous function $g(x)$, which vanishes outside some finite interval $[-R, R]$, such that

$$\int_{-\infty}^{\infty} |f(x) - g(x)|\, dx < \epsilon. \tag{55}$$

To see this, set $g(x) = f(x)$ in an interval $[-A, A]$, except in the neighborhood $|x - x_k| < \eta + \delta$ of each discontinuity x_k of $f(x)$. Set $g(x) = 0$ outside $[-A - \delta, A + \delta]$, and let $g(x)$

be linear in $(-A-\delta, -A)$ and $(A, A+\delta)$. Apply a similar piecewise linear construction in each neighborhood $|x - x_k| < \eta + \delta$, so that $g(x) = 0$ for $|x - x_k| < \eta$. Then by the absolute integrability of $f(x)$, we can make (55) hold by first choosing A large and η small, then choosing δ small.

The advantage of introducing $g(x)$ is that it is uniformly continuous. Since

$$\hat{f}(t) = -e^{-i\pi}\hat{f}(t) = -\int_{-\infty}^{\infty} f(x)e^{-it(x+\frac{\pi}{t})}\,\frac{dx}{\sqrt{2\pi}} \tag{56}$$

$$= -\int_{-\infty}^{\infty} f\left(x - \frac{\pi}{t}\right) e^{-itx}\,\frac{dx}{\sqrt{2\pi}}, \tag{57}$$

we have

$$2|\hat{f}(t)| \le \int_{-\infty}^{\infty} \left| f(x) - f\left(x - \frac{\pi}{t}\right) \right| \frac{dx}{\sqrt{2\pi}} \tag{58}$$

$$\le 2\int_{-\infty}^{\infty} |f(x) - g(x)|\,\frac{dx}{\sqrt{2\pi}} + \int_{-\infty}^{\infty} \left| g(x) - g\left(x - \frac{\pi}{t}\right) \right| \frac{dx}{\sqrt{2\pi}} \tag{59}$$

$$\le 2\epsilon + \int_{-R}^{R+\frac{\pi}{t}} \left| g(x) - g\left(x - \frac{\pi}{t}\right) \right| \frac{dx}{\sqrt{2\pi}}. \tag{60}$$

By uniform continuity the last integral will be less than ϵ for large enough $|t|$. (54) follows.

□

2 The Proof of the Prime Number Theorem

We now have enough background to present the proof of the PNT. We start from (7). We want to transform (7) into something resembling (37), so that we can apply Theorem 7. Define

$$\omega(s) = \int_{2}^{\infty} \frac{\pi(x)}{x^{s+1}(x^s - 1)}\,dx, \qquad \left(\operatorname{Re}(s) > \frac{1}{2}\right). \tag{61}$$

The integral is uniformly convergent in every halfplane $\mathrm{Re}(s) \geq \sigma_0 > \frac{1}{2}$, and hence analytic for $\mathrm{Re}(s) > \frac{1}{2}$.

Divde (7) by s, and subtract $\omega(s)$ from both sides to get

$$\frac{1}{s}\log\zeta(s) - \omega(s) = \int_2^\infty \frac{\pi(x)}{x^{s+1}}\,dx, \qquad (\mathrm{Re}(s) > 1). \tag{62}$$

If we make the cange of variables $x = e^y$ this is certainly of the same form as (37). However, things will actually go more smoothly if we differentiate (62) with respect to s:

$$-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s^2}\log\zeta(s) + \omega'(s) = \int_2^\infty \frac{\pi(x)\log x}{x^{s+1}}\,dx, \qquad (\mathrm{Re}(s) > 1). \tag{63}$$

Note that

$$|\omega'(s)| \leq \int_2^\infty \frac{\pi(x)\log x(1 - 2x^{\sigma_0})}{x^{\sigma_0+1}(x^{\sigma_0} - 1)}\,dx \tag{64}$$

whenever $\mathrm{Re}(s) \geq \sigma_0 > \frac{1}{2}$. If we denote the left side of (63) by $\phi(s)$ it follows from this and the remark after Theorem 4 that $\phi(s)$ is analytic in a region containing $\{\mathrm{Re}(s) \geq 1\} - \{1\}$ and bounded in any halfplane $\mathrm{Re}(s) \geq \sigma_0 > 1$.

We're almost ready to apply Theorem 7. If we tried to apply it to (63) we'd need to integrate $\phi(s)$ over a line $\mathrm{Re}(s) = c > 1$, and we only know that $\phi(s)$ is bounded there. Hence we proceed as follows. Define

$$g(x) = \int_0^x \frac{\pi(u)\log u}{u}\,du, \tag{65}$$

$$h(x) = \int_0^x \frac{g(u)}{u}\,du. \tag{66}$$

Since $\pi(x) = 0$ for $x < 2$, $g(x) = h(x) = 0$ for $x < 2$. Also,

$$g(x) \leq \int_1^x \log u\,du = x\log x - x + 1, \tag{67}$$

so that $\frac{g(x)}{x} \le \log x$ for $1 \le x$. Then, by a similar argument,

$$\frac{h(x)}{x} \le \log x, \qquad (1 \le x). \tag{68}$$

Integrating by parts,

$$\phi(s) = \int_0^\infty \frac{g'(x)}{x^s}\,dx = s\int_0^\infty \frac{g(x)}{x^{s+1}}\,dx = s\int_0^\infty \frac{h'(x)}{x^s}\,dx = s^2\int_0^\infty \frac{h(x)}{x^{s+1}}\,dx,$$

so

$$\frac{\phi(s)}{s^2} = \int_0^\infty \frac{h(x)}{x^{s+1}}\,dx, \qquad (\mathrm{Re}(s) > 1). \tag{69}$$

Now we can use Theorem 7. Fix $c > 1$, and make the change of variables $x = e^y$ in (69) to obtain

$$\frac{\phi(c+it)}{(c+it)^2} = \int_{-\infty}^\infty h(e^y)e^{-y(c+it)}\,dy = \sqrt{2\pi}\hat{f}(t), \tag{70}$$

where $f(y) = h(e^y)e^{-cy}$. $f(y)$ is continuous, and bounded because of (68) and the fact that $h(x) = 0$ for $x < 2$. $\hat{f}(t)$ is absolutely integrable since $\phi(c+it)$ is bounded. Hence, by Theorem 7,

$$h(e^y)e^{-cy} = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}\frac{\phi(c+it)}{(c+it)^2}e^{iyt}\,\frac{dt}{\sqrt{2\pi}} \tag{71}$$

Changing from y back to x,

$$\frac{h(x)}{x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi(s)}{s^2}x^{s-1}\,ds \qquad (c > 1). \tag{72}$$

We'll use this to show that

$$h(x) \sim x. \tag{73}$$

The function $-\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)}$ has a simple pole of residue 1 at $s = 1$, so we can write

$$\phi(s) = \frac{1}{s-1} + \psi(s), \tag{74}$$

where $\psi(s)$ is analytic in a region containing $\{\mathrm{Re}(s) \geq 1\} - \{1\}$. Hence

$$\frac{h(x)}{x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s^2(s-1)}\,ds + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\psi(s)}{s^2}x^{s-1}\,ds. \tag{75}$$

The first integral may be evaluated by residues. In fact, letting Q be the rectangle with vertices $c \pm iR$, $c - R \pm iR$,

$$\frac{1}{2\pi i}\int_{Q} \frac{x^{s-1}}{s^2(s-1)}\,ds = 1 - \frac{\log x}{x} - \frac{1}{x}. \tag{76}$$

When $x \geq 1$ the integrals over the top, bottom and left sides of Q tend to zero as $R \to \infty$, while the integral over the right side tends towards the first integral in (75). Thus

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s^2(s-1)}\,ds = 1 - \frac{\log x}{x} - \frac{1}{x}. \tag{77}$$

If we can show that the second integral in (75) tends to zero as $x \to \infty$ then (73) will be proved. Therem 8 would acomplish this, but only after factoring x^{c-1} out of the integral, and that term tends to ∞ since $c > 1$. To remedy this, we show that the value of the integral is unchanged if we set $c = 1$

First of all, the integral

$$\int_{1-i\infty}^{1+i\infty} \frac{\psi(s)}{s^2}x^{s-1}\,ds \tag{78}$$

is absolutely convergent. Absolute convergence at the endpoints follows from Theorem 6 and the boundedness of $\omega'(1+it)$. Near $s = 1$, $\psi(s) = \frac{1}{s^2}\log\zeta(s) + f_1(s)$ where $f_1(s)$ is bounded. $\log\zeta(1+it) = \log(-it^{-1} + f_2(1+it))$, where $f_2(1+it)$ is bounded near $t = 0$, so

$\psi(1+it) = O(-\log t)$ as $t \to 0$. Hence the integral converges absolutely at zero.

Let C be the contour obtained from the rectangle with vertices $1\pm iR$, $c\pm iR$ by replacing the segment $(1-i\epsilon, 1+i\epsilon)$ with a semicircle of radius ϵ centered at 1. By Cauchy's theorem,

$$\int_C \frac{\psi(s)}{s^2} x^{s-1}\, ds = 0. \tag{79}$$

The integrals over the horizontal sides of C tend to zero as $R \to \infty$. In fact, we can take $c < 2$, and apply (30) to get e.g.,

$$\begin{aligned}\int_{1+iR}^{c+iR} \frac{\psi(s)}{s^2} x^{s-1}\, ds &= O(R^{-2}) + O(R^{-4} log^{\frac{9}{4}} R) + O\left(R^{-3} \int_1^c \frac{\zeta'(u+iR)}{\zeta(u+iR)}\, du\right) \\ &= O(R^{-2}).\end{aligned}$$

As $\epsilon \to 0$, the integral over the semicircle tends to zero, since it has order $\epsilon \log \epsilon$. We conclude that indeed (78) is equal to the second integral in (75).

By Theoem 8,

$$\lim_{x\to\infty} \int_{1-i\infty}^{1+i\infty} \frac{\psi(s)}{s^2} x^{s-1}\, ds = \lim_{x\to\infty} i \int_{-\infty}^{\infty} \frac{\psi(1+it)}{(1+it)^2} e^{it\log x}\, dt = 0, \tag{80}$$

which proves (73).

Finally, we work back from $h(x)$ to $\pi(x)$:

Lemma. *If* $f : [1,\infty) \to [0,\infty)$ *is nondecreasing and*

$$\int_1^x \frac{f(u)}{u}\, du \sim x, \tag{81}$$

then

$$f(x) \sim x. \tag{82}$$

In light of the definitions of $g(x)$ and $h(x)$, this shows that $g(x) \sim x$ and hence that $\pi(x) \log x \sim x$, which is the PNT.

Proof. Given $\epsilon > 0$ we have

$$(1-\epsilon)x < \int_1^x \frac{f(u)}{u}\,du < (1+\epsilon)x \tag{83}$$

for x sufficiently large. Hence,

$$\begin{aligned}\int_{(1-\sqrt{\epsilon})x}^{x} \frac{f(u)}{u}\,du &= \int_1^x \frac{f(u)}{u}\,du - \int_1^{(1-\sqrt{\epsilon})x} \frac{f(u)}{u}\,du \\ &\geq (1-\epsilon)x - (1+\epsilon)(1-\sqrt{\epsilon})x = \sqrt{\epsilon}(1-\sqrt{\epsilon})^2 x.\end{aligned}$$

On the other hand,

$$\int_{(1-\sqrt{\epsilon})x}^{x} \frac{f(u)}{u}\,du \leq f(x)\int_{(1-\sqrt{\epsilon})x}^{x} \frac{du}{u} \leq \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}} f(x).$$

Combining these inequalities,

$$f(x) \geq (1-\sqrt{\epsilon})^3 x \tag{84}$$

for x sufficiently large. Since $\epsilon > 0$ can be made arbitrarily small,

$$\liminf_{x\to\infty} \frac{f(x)}{x} \geq 1. \tag{85}$$

A similar argument gives

$$\limsup_{x\to\infty} \frac{f(x)}{x} \leq 1, \tag{86}$$

which proves the lemma. $\square$

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