Section Notes: 1

1-31-2017

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Questions?

From class or about homework?

Section Topics:

- I Larmor Formula
- II Radiative Damping
- III Optical traps (if we get there)

Some Motivation:

- Where does EM radiation come from? ultimately that is what this class is about.
- So what is the Larmor Formula?
- It's not so obvious where this $\Gamma\dot{x}$ term comes from w/ the H.O.
  - how does Γ play a role?
- Lastly some of the tools we use for these topics will be helpful for the homework (mainly 2 & 3)

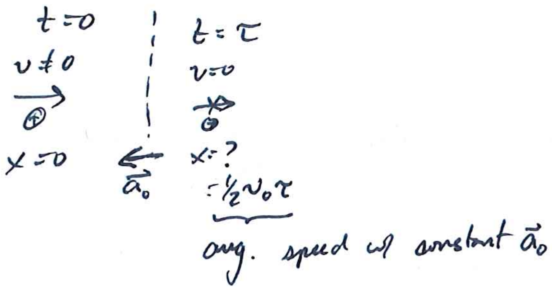
Larmor Formula (purcell derivation)

Consider the simple case of a decelerating charge. We'll take a sort of a minimal approach to the charge of the system.

Things we need: i) Conservation of Energy

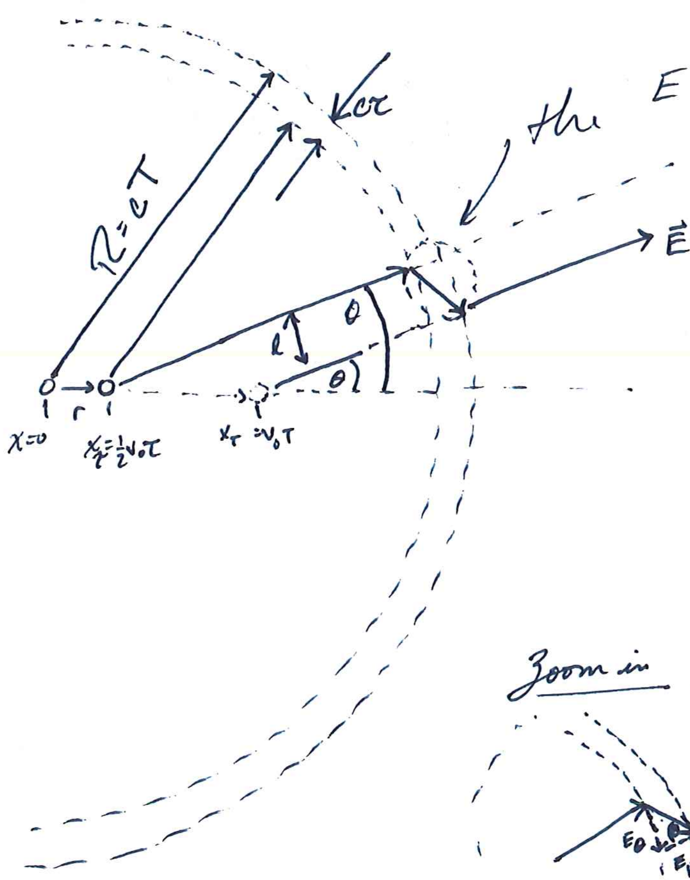
ii) Gauss's Law

iii) geometry



let's look at the fields far away from the charges! (We know how to describe this)

the E field "kinks" here



$T \gg \tau$

$R \gg r$

$r = c\tau$

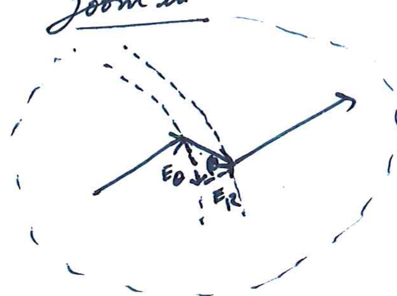
$\ell = (x_T - x_\tau)\sin\theta$

$= (v_0 T - \tfrac{1}{2}v_0\tau)\sin\theta$

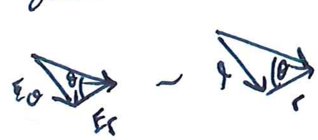
$= \underbrace{(T - \tfrac{1}{2}\tau)}_{\approx T} v_0 \sin\theta$

$\ell \sim v_0 T \sin\theta$

Zoom in



also we know



$\frac{\ell}{r} = \frac{E_\theta}{E_r} = \frac{v_0 T \sin\theta}{c\tau}$

$R = cT \Rightarrow T = R/c$

& $\tau = v_0/a$ $\quad \frac{E_\theta}{E_{r=R}} = \frac{v_0 R a}{c^2 v_0}\sin\theta \Rightarrow \boxed{\frac{E_\theta}{E_R} = \frac{aR}{c^2}\sin\theta}$

Larmor Formula cont

We learned in class these transverse waves carry away the energy we're interested in. Another consideration is that $\vec{k} \perp \hat{r}$ for such a case. So, the $\vec{E}$ we're interested in for finding the radiated energy is $\vec{E}_\theta$. Thankfully we have a relationship for $|\vec{E}_\theta|$ & $|\vec{E}_r|$ & we know $\vec{E}_{r=R}$

$$E_\theta = \frac{aR}{c^2} \sin\theta \; E_{r=R}$$

Where the $\hat{r}$ component of the E-field we know in 3-D coords at $r=R$ from Gauss's Law.

simplify

$$E_{r=R} = \frac{q}{4\pi\epsilon_0 R^2} \Rightarrow \boxed{E_\theta = \frac{aR}{c^2}\frac{q}{4\pi\epsilon_0 R^2}\sin\theta} = \boxed{E_\theta = \frac{qa}{4\pi\epsilon_0 c^2 R}\sin\theta}$$

So now to find the radiated energy we can recall the formula from class $\mathcal{E}_{TOT} = \frac{1}{2}\epsilon_0 |E|^2$

Where $\mathcal{E}(\theta) = \frac{1}{2}\epsilon_0 \left(\frac{qa}{4\pi\epsilon_0 c^2}\right)^2 \frac{1}{R^2}\sin^2\theta$

mult for B-field

$$\text{so } \mathcal{E}_{TOT} = (2)\frac{1}{2}\epsilon_0 \int dV\, \mathcal{E}(\theta) = (2)\frac{1}{2}\epsilon_0\left(\frac{qa}{4\pi\epsilon_0 c^2}\right)^2 \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{\sin\theta(1-\cos^2\theta)} \underbrace{\int_R^{R+\Delta r} r^2\, dr \frac{1}{R^2}}_{=\frac{1}{R^2}\left.\frac{r^3}{3}\right|_R^{R+\Delta r}}$$

$$= \underbrace{\int_0^\pi \sin\theta\, d\theta}_{2} - \underbrace{\int_0^\pi \sin\theta\cos^2\theta\, d\theta}_{\left.-\frac{u^3}{3}\right|_1^{-1}} = 4/3$$

$\mathcal{E}_{TOT}$ is the energy in the "shell" the place w/ $\mathcal{E}_\theta \neq 0$

$$= \epsilon_0\left(\frac{qa}{4\pi\epsilon_0 c^2}\right)^2 \frac{1}{R^2}(2\pi)(4/3)(1/3)\underbrace{\left((R+r)^3 - R^3\right)}_{\approx 3R^2 r}$$

④

Larmour Frequency cont.

$$\mathcal{E}_{TOT} = \mathcal{E}_0 \left(\frac{qa}{4\pi\epsilon_0 c^2}\right)^2 \frac{(2\pi)}{2}(4/3)\frac{1}{3\cancel{c^2}}(3\cancel{c^2}r)$$

$$= \frac{q^2a^2}{2\cancel{8}\pi\epsilon_0 c^4}\frac{1}{\cancel{2}}\frac{4}{\cancel{3}}\frac{\cancel{2}}{}\frac{\cancel{3}r}{3} = \frac{q^2a^2}{6\pi\epsilon_0 c^4}r$$

Now remember $r \equiv c\tau$

$$\mathcal{E}_{TOT} = \frac{q^2a^2}{6\pi\epsilon_0 c^3}\tau$$

where if we take $\tau \to 0$ we get the instantaneous velocity change $a(t)$

& then $\frac{E_{TOT}}{\tau} \to \frac{dE}{dt} = ? \Rightarrow P(t)$

Larmour Formula

$$\boxed{P(t) = \frac{q^2 a^2(t)}{6\pi\epsilon_0 c^3}}$$

Radiative Damping + Lorentz Oscillator

So now I want us to reconsider the Lorentz Oscillator from lecture.

Sum of all forces gives:

$$m\ddot{x} + \Gamma\dot{x} + m\omega_0^2 x + eE_0 e^{i\omega t} = 0$$

"damping" term — H.O. potential term — driving term.

But how could we have guessed $\Gamma\dot{x}$ existed and its form had we simply started from $m\ddot{x} + m\omega_0^2 x = 0$ & the Larmor Formula?

We know the form is $\Gamma\dot{x} = F_{rad}$

We also know $-\int_{\tau_1}^{\tau_2} P\,dt = \int_{\tau_1}^{\tau_2} F\cdot v\,dt$ ← more of a claim about the avg. of the two.

where $P = \dfrac{q^2 a^2}{6\pi\epsilon_0 c^3}$; $a = \ddot{x}$

for periodic motion this goes $\to 0$ (one cycle)

$$-\int_{\tau_1}^{\tau_2} \frac{q^2 a^2}{6\pi\epsilon_0}\underbrace{\frac{dv}{dt}}_{u}\cdot\underbrace{\frac{dv}{dt}}_{dv}\,dt = -\frac{q^2 a^2}{6\pi\epsilon_0 c^3}\left[\frac{dv}{dt}v\Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2}\frac{d^2v}{dt^2}v\,dt\right]$$

$$\underbrace{\frac{d\dot{x}}{dt^2}}_{du} = \underbrace{v}_{v}$$

this becomes our F from $\int_{\tau_1}^{\tau_2} F\cdot v\,dt$

so $F_{RAD} \simeq \dfrac{q^2\,\omega_0^2}{6\pi\epsilon_0 c^3}\dot{a}$ where importantly,

we can use our D.E. from H.O. $\Rightarrow$ $\ddot{x} + \omega_0^2 x = 0$, $\dddot{x} + \omega_0^2\dot{x} = 0$

$\dot{a} = -\omega_0^2\dot{x}$

$$\Rightarrow \boxed{F_{RAD} = \frac{-q^2\omega_0^2}{6\pi\epsilon_0 c^3}\dot{x}}$$

} we can think of this as a correction to the H.O. that gets us back to the Lorentz oscillator

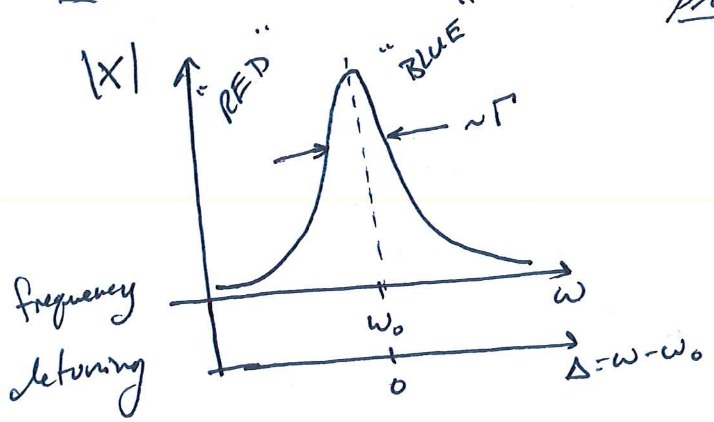
Lorentz oscillator.

You'll solve this problem more thoroughly for H.W.k with question 3 so I won't do it here. But, I will talk a bit more about the solution.

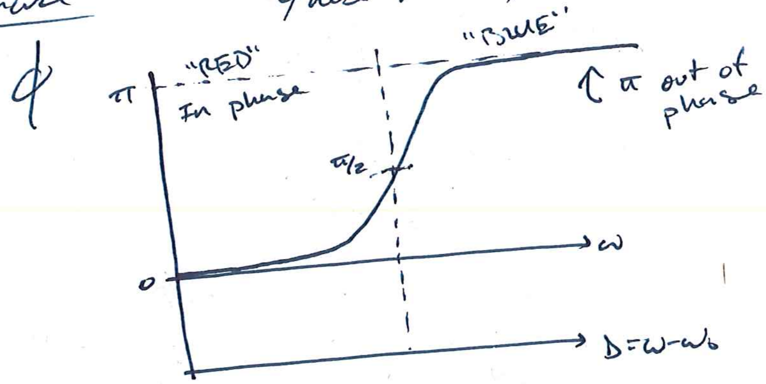
$$X = \frac{-eE_0}{(\omega_0^2-\omega^2)+i\omega\Gamma} e^{i\omega t}$$

So when we think about trapping with lasers you can think simply about the response of $X(t)$ both in magnitude and phase. (Reference the two plots scanned with these notes)

Amplitude



Phase

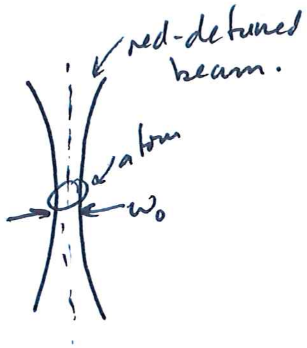


Intuition about the use of this for tweezers in atomic physics.

The atom experiences a force depending on $U = -\vec{d}\cdot\vec{E}$ where $\vec{d} \sim \vec{x}(t)$. So if we have a spatially varying beam (in intensity/strength) the atom is "attracted" to high intensity for red-detuned beams and is repelled from high intensity for blue-detuned beams. Sometimes called high-field (red) vs. low-field (blue) seeking regimes.

(7) Spatially dependent forces for tweezers. (optical)

We'll get the topic of Gaussian beams later in the semester but they're useful here for talking about these trapping light forces since they can be approximated by a H.O. too! In fact, the focal point of the Gaussian beam traps in 3-D. $U_{atom} \approx \frac{1}{2} m (\omega_r^2 \rho^2 + \omega_z^2 z^2)$



For a given Power P_0, the "trap frequencies" for the radial & axial Harmonic approximations are:

$$\boxed{\omega_r = \sqrt{\frac{8P_0}{\pi m w_0^4}}} \qquad \boxed{\omega_z = \sqrt{\frac{8P_0\lambda^2}{m\pi^3 w_0^6}}} \qquad \boxed{\frac{\omega_r}{\omega_z} = \frac{\pi w_0}{\lambda}}$$

attached will be some examples of where this is used (in Harvard Physics alone)

- Kang-Kuen Ni (tweezers)
- Markus Greiner (lattices)
- Mikhail Lukin (tweezers)

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