Section 5 Notes: Cavities

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Questions?

Let's derive some stuff about cavities!

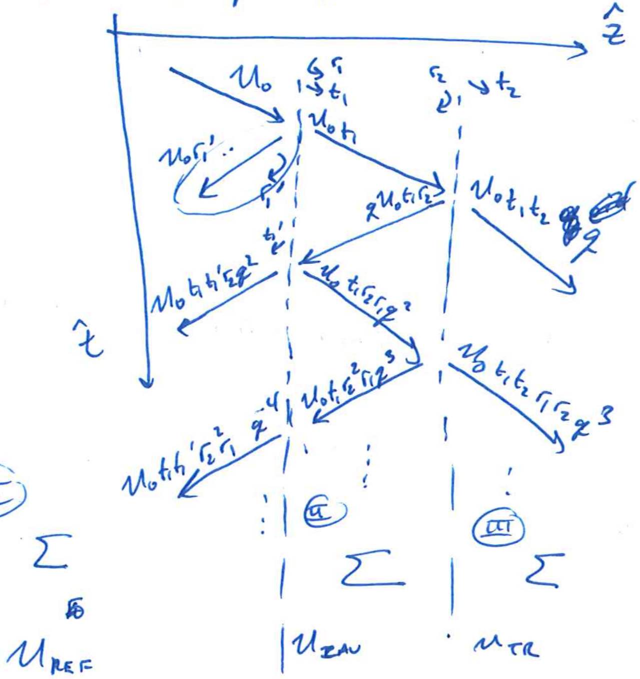
<u>Outline:</u>

- Solve cavity for, internal field, transmission, & reflection.
  aside about geometric series.
- Input vs. output spectrum.
- Some defn's
  - cooperativity, quality factor
- Modes of cavities (if we have time)

Cavity Derivation

So in painful detail.

$$q = e^{i\phi} = e^{ikd} = e^{i\frac{2\pi d}{\lambda}}$$



Let's start w/ U_{TR} since it is somewhat the easiest.

III

$$U_{TR} = \underset{n=0}{u_0 t_1 t_2 q} + \underset{n=1}{u_0 t_1 t_2 q^3 r_1 r_2} + \dots + \underset{n=n'}{u_0 t_1 t_2 q \left(r_1 r_2 q^2\right)^{n'}} + \dots$$

$$= u_0 t_1 t_2 q \left(\underbrace{\sum_{n=0}^{\infty} \left(r_1 r_2 q^2\right)^n} \right)$$

This is a geometric series!

Quick aside, the proof for the geometric series is very simple and very satisfying. (Thanks to the Bernoulli Brothers)

$$G.S. = \sum r^n = 1 + r + r^2 + \dots = S \leftarrow \text{assume}$$

$$\Rightarrow S - 1 = r + r^2 + \dots$$

$$\frac{S-1}{r} = 1 + r + r^2 + \dots \leftarrow \text{back to G.S. again!}$$

$$\frac{S-1}{r} = S \Rightarrow Sr - S = -1 \quad \boxed{S = \frac{1}{1-r}}$$

Back to III

$$U_{TR} = U_0 t_1 t_2 g \frac{1}{1 - r_1 r_2 g^2} \Rightarrow \frac{U_{TR}}{U_0} = \frac{t_1 t_2 g}{1 - r_1 r_2 g^2}$$

$g = e^{i\phi}$

$$\frac{U_{TR}}{U_0} = \frac{t_1 t_2 e^{i\phi}}{1 - r_1 r_2 e^{i2\phi}} \Rightarrow$$ Can also ask about $\frac{I_{TR}}{I_0} = \frac{|U_{TR}|^2}{|U_0|^2}$

$$\frac{U_{TR}^* U_{TR}}{U_0^* U_0} = \frac{I_{TR}}{I_0} = \frac{(t_1 t_2)^2}{(1 - r_1 r_2 e^{-i2\phi})(1 - r_1 r_2 e^{i2\phi})} = \boxed{\frac{(t_1 t_2)^2}{1 - 2 r_1 r_2 \cos(2\phi) + (r_1 r_2)^2}}$$

In the case where $r_1 = r_2$ for a symmetric cavity

& $t_1 = t_2$ where $t^2 + r^2 = 1$

i) Symmetric Mirrors!

$$= \boxed{\frac{(1-R)^2}{1 - 2R\cos(2\phi) + R^2}}$$

Max & Min? function of ϕ

$\Rightarrow$ MAX $\phi \sim \pi$ $\Rightarrow \frac{(1-R)^2}{1 - 2R + R^2} = \frac{(1-R)^2}{(1-R)^2} = \boxed{1}$

Min $\phi \sim \frac{\pi}{2} + n\pi \Rightarrow \boxed{\frac{(1-R)^2}{(1+R)^2}}$

ii) $r_2 = 0$

$\Rightarrow t_2 = 1$

$$I_{TR} = \frac{t_1^2}{1} = \boxed{T_1}$$

iii) $r_2 \to 1$

$t_2 \to 0$

$$\boxed{I_{TR} \to 0}$$

II In the cavity

$$U_{cav} = U_0 t_1 g + g U_0 t_1 r_2 r_1 g^2 + g U_0 t_1 (r_2 r_1)^2 g^4 + \dots$$

$$= U_0 t_1 g (1 + r_2 r_1 g^2 + (r_1 r_2)^2 g^4 + \dots + (r_1 r_2)^n g^{2n} + \dots)$$

$$= U_0 t_1 g \sum_{n=0}^{\infty} (r_2 r_1 g^2)^n = \frac{U_0 t_1 g}{1 - r_1 r_2 g^2}$$

$$\frac{U_{CAV}}{U_0} = \frac{t_1 g}{1 - r_1 r_2 g^2} = \frac{t_1 e^{i\phi}}{1 - r_1 r_2 e^{i2\phi}}$$

Again for $t_1 = t_2$, $r_1 = r_2$

$$\frac{I_{CAV}}{I_0} = \frac{t_1^2}{1 - 2 r_1 r_2 \cos(2\phi) + (r_1 r_2)^2} \Rightarrow \boxed{\frac{(1-R)}{1 - 2R\cos(2\phi) + R^2}}$$

ϕ_{MAX} $\frac{I}{I_0} \Rightarrow \left(\frac{1}{1-R}\right)$

ϕ_{MIN} $\frac{1-R}{(1+R)^2}$

I Easiest way to get I

Is to say for power going into the cavity or out of the cavity has to be conserved! So

$$\left|\frac{u_{REF}}{u_0}\right|^2 = 1 - \left|\frac{u_{TR}}{u_0}\right|^2$$

But perhaps not the most satisfying.

So you can start with

$$u_{REF} = u_0 r_1' + u_0 t_1 t_1' r_2 g^2 + u_0 t_1 t_1' r_2^2 r_1 g^4 + \dots$$

$$= u_0\left(r_1' + t_1 t_1' r_2 g^2 (1 + r_1 r_2 g^2 + \dots)\right)$$

$$= u_0 \left(r_1' + \frac{t_1 t_1' r_2 g^2}{1 - r_1 r_2 g^2}\right)$$

Pitfalls! Converting between $r_1 t' \Leftrightarrow r_1 t$ is very non-trivial!

Aside about unitarity:

We looked at one side of the mirror and sort of ~~willy nilly~~ flagrantly added ~~up~~ up fields and squared them without giving much thought to phase relations.

If we do this here we will get a non-normalized solution!

Unitarity requires the probability remain the same in the

$$\tilde{U}^\dagger \tilde{U} = \mathbb{1} \quad \text{where } \tilde{U} \text{ for us is}$$

$$\tilde{U}\vec{E}_{in} = \vec{E}_{out} = \begin{pmatrix} r_1' & t_1 \\ t_1' & r_1 \end{pmatrix}\begin{pmatrix} E_{Left} \\ E_{Right} \end{pmatrix} = \begin{pmatrix} \bar{E}_{Left} \\ \bar{E}_{Right} \end{pmatrix} = \begin{pmatrix} E_L r_1' + E_r t_1 \\ E_L t_1' + r_1 E_r \end{pmatrix}$$

Importantly for the squares, $r_1^* r_1 + t_1^* t_1 = 1$, $r_1^{*\prime} r_1' + t_1^{*\prime} t_1' = 1$ But! $\boxed{r_1^{*\prime} t_1 + t_1^{*\prime} r_1 = 0 ! \quad t_1^* r_1' + r_1^* t_1' = 0 !}$

Anyway, so if you work this out carefully.

and still use $t_1^* t_1 = T = t_1^{*'} t_1'$

$r_1^* r_1 = R = r_1^{*'} r_1'$

$$\Rightarrow \boxed{\left|\frac{U_{REF}}{U_0}\right|^2 = R - \frac{\left(T^2R + TR\cos(2\phi)\right)}{1+R^2-2R\cos(2\phi)}}$$

these R's all contain $R \cdot r_2$ (I think)

so it also makes sense that if we make $r_2 \to 0$ or put our hand in the way of the beam all the light gets reflected!

Frequency Response!

So what do we get out?

So, in line shape,

$$\frac{I_t}{I_0} = \frac{(1-R)^2}{1+R^2-2R\cos(2kd)}$$

$$= \frac{(1-R)^2}{1+R^2-2R\cos\left(\frac{2d}{\lambda}2\pi\right)}$$

$$\mathcal{C}(\omega) = \frac{(1-R)^2}{1+R^2-2R\cos\left(\frac{2d}{c}\left(\frac{f\,2\pi}{\omega}\right)\right)}$$

So if $I_0(\omega)$

$$\Rightarrow \boxed{I_t(\omega) = I_0(\omega)\,\mathcal{C}(\omega)}$$

Parameters then about cavities:

$$\boxed{\mathcal{F} = \frac{\pi\sqrt{|r_1||r_2|}}{1-|r_1||r_2|}}$$

Finesse is the reflectivity dependent parameter! $\mathcal{F} = \frac{\pi\sqrt{\rho}}{1-\rho}$; $\rho = |r_1||r_2|$

F.S.R

$\phi = kd$ length dependent parameter.

$= 2\pi f \frac{d}{c} = \boxed{\pi \frac{f}{FSR}}$ where $\boxed{FSR = \frac{c}{2d}}$

FSR also gives the lowest f / smallest λ supported by the cavity

Quality Factor

$\rho = |r_1||r_2|$; $FWHM = \frac{2(FSR)}{\pi} \sin^{-1}\left(\frac{\pi}{2F}\right)$

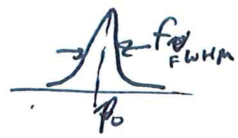
$$\boxed{f_{FWHM} = \frac{FSR}{\mathcal{F}}}$$

$Q = \frac{\nu_0}{\nu_{FWHM}} = \boxed{\frac{\mathcal{F} f_0}{FSR}}$ for $f_0 = FSR$

$\Rightarrow Q \rightarrow \mathcal{F}$

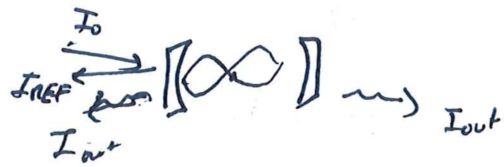
Note that since ν_{FWHM} is a finite + quantity, Q goes up w/ f_0

Another way to get to this answer of the reflection in the cavity being different is to think of how this response we saw



is just like a harmonic oscillator!

Which we know the phase response of! On resonance we need to get an i & another i somewhere for the excited cavity / atom to be out of phase w/ the driving



$I_{REF} = -I_{out}$

What happens if I put my hand into a cavity?

→ At first glance this seems very dangerous.

1W beams at a ½mm waist could already burn you for example. So if $R \sim 0.99$, then the circulating power of $\sim 100W$ certainly seems very dangerous!

- Consider case of what happens if I put my hand in and then turn on the laser (no light goes in, in the first place (well 1% does but who cares).
- Now consider doing it when at full power. How much energy does your hand absorb?

Photon Decay τ

$$\frac{1}{\tau_c} = \frac{-\ln(R_1 R_2)}{t_{RT}} \qquad t_{RT} = \frac{2d}{c}$$

in one round trip!

$$e^{\left(-\frac{t_{RT}}{\tau}\right)} = R_1 R_2$$

in two.

$$e^{-2\frac{t_{RT}}{\tau}} = (R_1 R_2)^2 \quad \ldots$$

$$\tau = \frac{2d}{c}\frac{1}{-\ln(R_1^2)}$$

$R = 0.99$

if $d = 1m$

$\Rightarrow \tau = 300ns$

$P \cdot \tau \sim E_{dumped} = (100)\,3\,(100 \times 10^{-9})$

$= 3 \cdot 10^4 \, 10^{-9} = 3 \cdot 10^{-5}\,J$

need something like $5\,J/cm^2$ to burn yourself!

Probably fine, but don't try this at home.

Modes

May not have time for this.

- Laguerre Gauss modes
- Hermite Gauss modes

Research:
- Mikhail Lukin.
- ~~[illegible]~~
- Vladan Vuletic

Gaussian Mode

$$E(r,z) = E_0 \hat{x} \left(\frac{w_0}{w(z)}\right) \cdot e^{\frac{-r^2}{w(z)^2}} e^{-i(kz)} e^{-ik\frac{r^2}{2R(z)}} e^{+i\psi(z)}$$

$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$

Hermite Gauss

$$E_{l,m}(r,z) : E_0 \frac{w_0}{w(z)} H_l\left(\frac{\sqrt{2}\,x}{w(z)}\right) H_m\left(\frac{\sqrt{2}\,y}{w(z)}\right) e^{\frac{-r^2}{w^2(z)}} e^{-ik\frac{r^2}{2R(z)}} e^{-ikz} e^{i\psi(z)}$$

$$\psi(z) = (N+1) \cdot \tan^{-1}\left(\frac{z}{z_R}\right)$$

$$N = l + m$$

Laguerre Gauss.

$$u_{lp}(r,\phi,z) = \frac{C_{lp}^{LG}}{w(z)} \left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right) \times \ldots$$

$$\psi(z) = (N+1) \tan^{-1}\left(\frac{z}{z_R}\right)$$

$$N = |l| + 2p$$