Practice Mid-Term Examination

Physics 175

March 21/23, 2017

Name: ______________________________

Instructions

You will have 90 minutes to work on the exam. Turning in your formula sheet is worth X points. Budget your time wisely, using the point values as a guide. Use empty pages for computation.

An equation sheet containing all required equations (and then some!) can be found in the back.

You are not allowed to use any material (books, lecture notes) other than *double-sided sheet with your handwritten notes*, which you're required to hand in together with your exam. The use of a calculator is permitted, but no other electronic device (cell phone, laptops, etc.) are allowed during the exam.

Extra paper is available should you require it. Please write your name on *every* sheet that you attach to your exam.

Problem	**Score**
1	___
2	___
3	___
4	___
5	___
TOTAL	___

GOOD LUCK!

1 Lenses

(a) Lens Equations

What is the lens equation?

$$\frac{1}{f} = \frac{1}{s_i} + \frac{1}{s_o}$$

(b) Imaging: Conceptual

What is the focal length of a lens?

Answer this in words or using the lens equation

There are two common answers for this:

1. The $2f \rightarrow 2f$ problem below is one common way. This defines the f as the length that images an object at a distance in front of the lens to the same length behind the lens.
2. Another common answer is that parallel waves, or objects at infinity, are imaged to the focal length away from the lens. This can be described vice versa as well.

(c) Imaging

Where is an object imaged to if it is placed $z_o = 2f$ in front of the lens at $z = 0$?

$$\frac{1}{f} = \frac{1}{2f} + \frac{1}{z}$$
$$z \rightarrow 2f$$

2 Fourier Transforms

(a) Important Examples

What is the fourier transform of f(x)?

$$f(x) = \delta(x - x_o) + \delta(x + x_o)$$

$$F(x) = \mathscr{F}[f(x)] = \frac{2}{\sqrt{2\pi}} \cos k_x x_o$$

What is the fourier transform of f(x)?

$$f(x) = \sin k_x x_o$$

$$F(x) = \mathscr{F}[f(x)] = \frac{\sqrt{2\pi}}{2i} (\delta(x + x_o) - \delta(x - x_o))$$

(b) 4F Systems

What mathematical function do lenses perform?

They perform fourier transforms from $z = -f$ to $z = f$!

(c) 4F imaging

Consider a 4F imaging system where a gaussian beam, with flat phase front at $z = -f_1$ is imaged by a lens with focal length f_1 to a plane $z = f_1$ where it impinges on an ideal grating with spacing d.

The electric field at $z = -f_1$ is given below:

$$E(z = -f_1) = G_{0,\sigma}(x)$$

The mask function $M(x')$, ideal grating in this case, is given functionally below:

$$M(x) = \text{Ш}_d(x) = \sum_{n=-\infty}^{\infty} \delta(x - nd)$$

Refer to equation sheet at end of test if confused about these functions

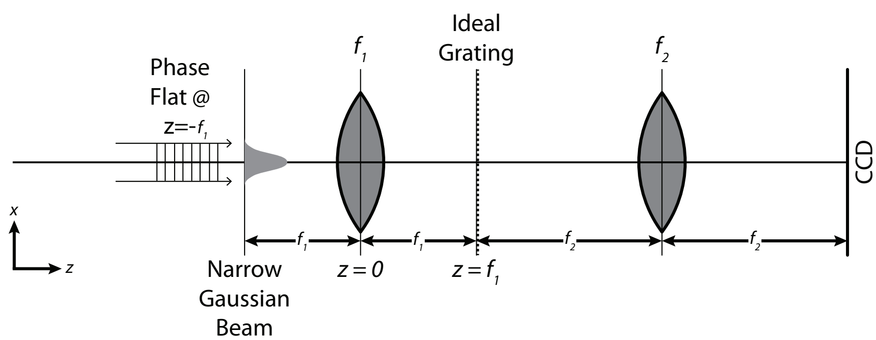

Figure 1: 4F imaging system. Problem **2a**

1. Write on the figure where the "Fourier Plane" is for this imaging system.
2. Determine what the electric field is (as a function of k_x) just before hitting the ideal grating/mask at $z = f_1 - \epsilon$, $(\epsilon \ll f_1)$.
3. Rewrite the electric field as a function of transverse coordinate x' at $z = f_1 - \epsilon$, $(\epsilon \ll f_1)$.
4. Determine the electric field just after illuminating the ideal grating, $M(x')$, as a function of transverse coordinate x' at $z = f_1 + \epsilon$, $(\epsilon \ll f_1)$.
5. Determine the electric field now at the CCD chip ($z = f_1 + 2f_2$). (For simplicity of the solution, assume that $2\pi/d \gg \sigma$.

Hint: Remember that when changing from k-space to real dimensions we use the substitution $k_x \to \frac{2\pi}{\lambda f} x'$.

Hint: The scaling theorem and convolution theorem may be *very* useful in this problem.

1. The fourier plane is exactly where the "Ideal Grating" is located.
2. We can refer to the formula sheet at the end of the midterm where we can pull out the Gaussian fourier transform formula out. Which will give us the Gaussian just before the ideal grating.

$$\mathscr{F}\left[G_{0,\sigma}(x)\right] = G_{0,1/\sigma}(k_x)$$

3. Using the hint, we can rewrite the formula above to:

$$\sigma G_{0,1/\sigma}\left(\frac{2\pi}{\lambda f_1}x'\right)$$

4. The electric field after the grating will just be the multiplication of the grating and the Gaussian beam.

$$\sigma G_{0,1/\sigma}\left(\frac{2\pi}{\lambda f_1}x'\right)\sum_{n=-\infty}^{\infty}\delta(x'-nd)$$

5. Using the convolution theorem and the scaling theorem (also our knowledge about 4f systems) we can realize that at the CCD the answer is of the form

$$\mathscr{F}\left[\sum_{n=-\infty}^{\infty}\delta(x'-nd)\right] * \mathscr{F}\left[\sigma G_{0,1/\sigma}\left(\frac{2\pi}{\lambda f_1}x'\right)\right]$$

   Be careful here to use the scaling theorem of the fourier transform to end up getting the magnification inside the gaussian correct! Also, agin going from $x' \to k'_x$ can be changed to real coordiantes with $k'_x = \frac{2\pi}{f_2\lambda}x$.

$$\left[\frac{\sqrt{2\pi}}{d}\text{Ш}_{2\pi/d}\left(\frac{2\pi}{\lambda f_2}x\right)\right] * \left[\frac{f_1\lambda}{2\pi}G_{0,\sigma}\left(\frac{f_2}{f_1}x\right)\right]$$

   With the assumption that $\sigma \ll d$, then this answers is approximately

$$\approx \sum_{n=-\infty}^{\infty} G_{0,\sigma}\left(\left(x - n\frac{f_2\lambda}{d}\right)\frac{f_2}{f_1}\right)$$

   Note: I've dropped all the scaling factors in front but in principle one could keep track of them all as well from the line above!

3 Filters by eye

(a) Filters

For the following image, guess which resultant image is a function of the input image (top left) and the fourier filter (top right)

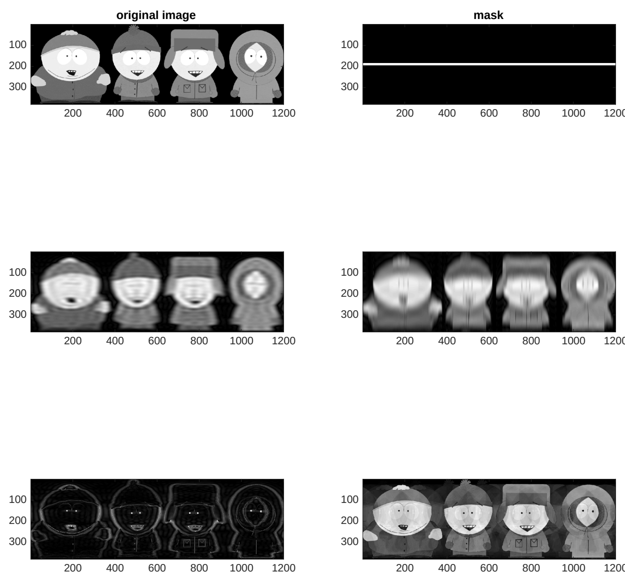

Figure 2: 4F imaging system with filtering. Top Left is the original image. The Filter in the fourier plane is the top right image.

For the following image, guess which resultant image is a function of the input image (top left) and the fourier filter (top right)

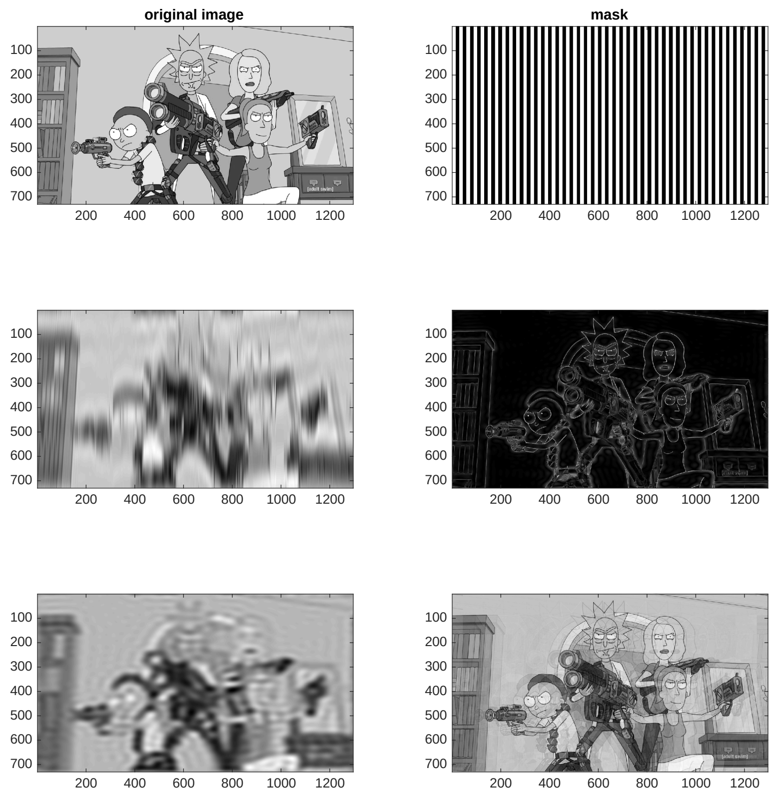

Figure 3: 4F imaging system with filtering. Top Left is the original image. The Filter in the fourier plane is the top right image.

For the following image, guess which resultant image is a function of the input image (top left) and the fourier filter (top right)

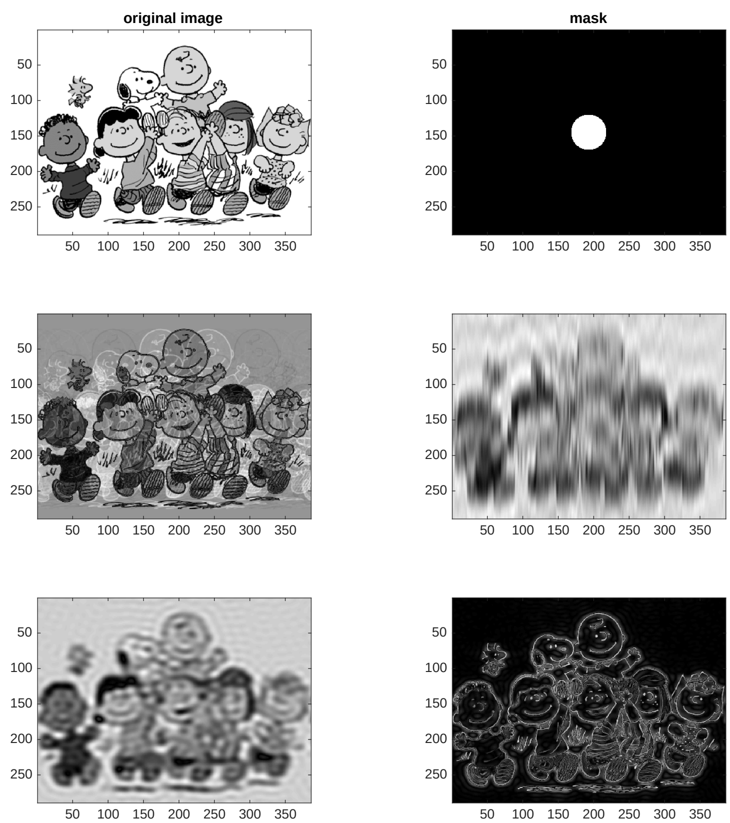

Figure 4: 4F imaging system with filtering. Top Left is the original image. The Filter in the fourier plane is the top right image.

For the following image, guess which resultant image is a function of the input image (top left) and the fourier filter (top right)

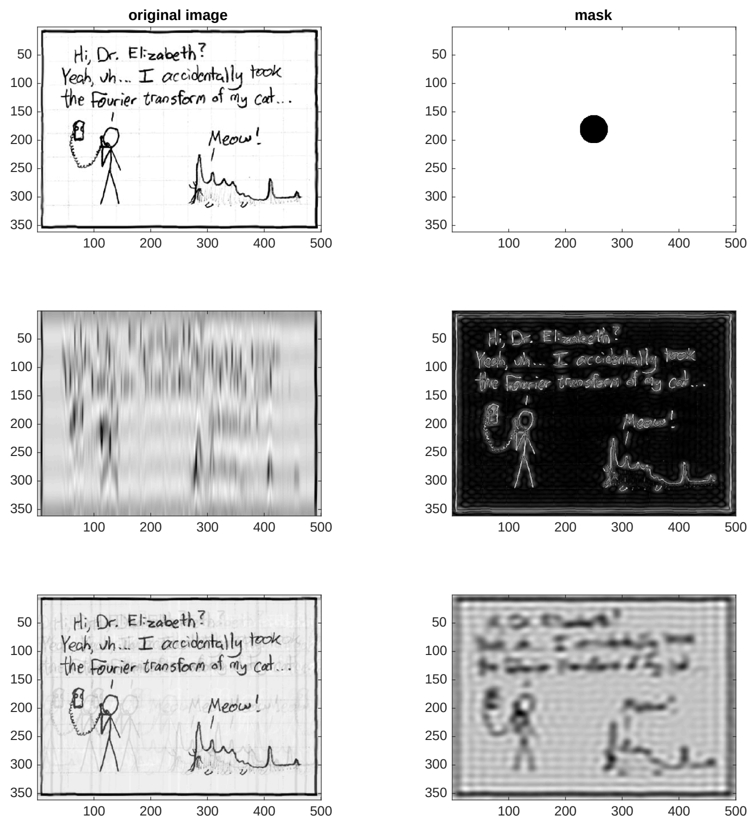

Figure 5: 4F imaging system with filtering. Top Left is the original image. The Filter in the fourier plane is the top right image.

(b) Answers

- middle right picture
- bottom right picture
- bottom left picture
- middle right picture

4 Gaussian Modes

(a) Qualitative Question 1

What equation does the “Gaussian Beam" satisfy?

Paraxial Helmholtz Equation

(b) Qualitative Question 2

What is the “rayleigh range" defined as?

So qualitatively (well, quantitatively too), it is defined as the places where the waist grows by a factor of $\sqrt{2}$ in amplitude.

As a definition it is a function of the minimum waist and the wavelength of the beam.

Remember:

$$z_o = \frac{\pi w_o^2}{\lambda}$$

(c) Less Qualitative Question 3

If I want a waist of w_o, and I place a lens of f at a location where the radius of curvature is infinite, and the beam has a waist of w':

1. Where will the minimum waist be located?
2. What f should the lens be?

Remember: The rayleigh range is determined from the minimum waist and λ.

$$z_o = \frac{\pi w_o^2}{\lambda}$$

1. The waist we start with and the waist we want detemrine the location of the minimum waist.

$$w' = w_o\sqrt{1+\left(\frac{z}{z_o}\right)^2}$$

$$\left(\frac{w'^2}{w_o^2}-1\right)z_o^2 = z^2$$

2.

$$f = R(z) = z + \frac{z_o^2}{z}$$

5 Cavities

Given a cavity with two mirrors of equal magnitude of curvature (the sign you can adjust based on placement) separated by $6z_o$, what must the radius of curvatures be?

1. Consider the case where the mirrors are placed symmetrically on either side of the minimum waist.
2. Consider the case where the mirrors are placed both on one side of the minimum waist.

1. The symmetric case is simple in the sense that the separation of $6z_o$ also fixes the two mirrors to then be at $z = -3z_o$ and $z = 3z_o$. Which then with the radius of curvature formula just gives us $R(z) = \frac{10}{3}z_o$ and $R(-z) = -\frac{10}{3}z_o$.
2. The case where the mirrors of are equal curvature and sign means they must be located both on one side of the waist. There is a maximum “curviness" that the beam can have that is located exactly at the rayleigh range. So by placing mirrors on either side of the rayleigh range with same sign and curvature we can make a stable mode. Again, solve the radius of curvature equation for one sign of z where the two radius of curvatures are the same (just solve a quadratic equation of $R(z)$ for z).

   It’s easier to use “natural units" of z_o for everything.

$$R = z + \frac{z_o^2}{z}$$

$$0 = z^2 - Rz + z_o^2 \rightarrow 0 = \frac{z}{z_o}^2 - \frac{R}{z_o}\frac{z}{z_o} + 1$$

So now absorb the z_o into all the z and R factors

$$z_{+,-} = \frac{R}{2} \pm \sqrt{\left(\frac{R}{2}\right)^2 - 1}$$

This will give a z_+ and a z_-. Then set the difference of these to the mirror separation of $6z_o$.

$$z_+ - z_- = d = 6 = 2\sqrt{\left(\frac{R}{2}\right)^2 - 1}$$

$$36/4 = \left(\frac{R}{2}\right)^2 - 1 \rightarrow R = \sqrt{40}$$

To get back into real units we can stick the z_o back in.
$R = \sqrt{40} z_o$

6 Polarization

(a) General Question 1

Using the Jones Vector notation, how do we find if another polarization is orthogonal?

Consider asking how we normalize an $\vec{E}$-field by dotting the complex conjugate of the field with itself.

$$\vec{J}^\dagger \vec{J} = 1$$

So to find when they're orthogonal would mean that :

$$\vec{J}_i^\dagger \vec{J}_j = 0$$

We know that for picking orthogonal vectors for purely real vectors we find the negative inverse of the vector. The only slight change we need to keep in mind is that for complex variables we have this conjugate transpose. So the conjugate transpose of the vector must be the negative inverse of the vector we have for it to be orthogonal.

Ex:

$$J_1 \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

So the orthogonal vector is:

$$J_2 \rightarrow \frac{-1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

We can check this by taking the dot product of the two vectors

$$J_1^\dagger J_2 = \frac{-1}{2} \left[(-i)(1) + (i)(1)\right] = 0$$

(b) General Question

What does a $\lambda/2$ wave plate do? What are the “fast" and “slow" axes in the form of its matrix representation?

Looking at the formulat sheet is very enlightening. Remember that a wave plate has birefringence such that one polarization sees a different index of refraction compared to another. This means that the effective path length for one polarization through the plate is also effectively different for each polarization! The “fast" axis we typically describe as being the one that gets no phase shift while the “slow" axis is retarded or delayed compared to the “fast" axis by a factor of $k_o \cdot d(\delta n)$.

This gives a differential phase of $exp(-i\phi)$ between the two axes where $\phi = k_o \cdot d(\delta n)$.

(c) Polarization Rotation

How can I rotate the polarization from

$$J_1 \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow J_2 \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with a $\lambda/2$ wave plate?

Use the rotation matrices!

The matrix for the $\lambda/2$ plate is just given by :

$$\begin{pmatrix} 1 & 0 \\ 0 & exp(i\phi) \end{pmatrix}$$

where $\phi = \pi$. Which gives:

$$S_\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Transforming this matrix by the rotation matrix $R(\theta)S_\pi R^\dagger(\theta)$

$$R(\theta)S_\pi R^{dagger}(\theta) = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

This acting on the initial vector produces:

$$\begin{pmatrix} \cos(2\theta) \\ -\sin(2\theta) \end{pmatrix}$$

So this means for an angle of $\theta = \pi/4$ of the wave plate, we'd completely exchange the initial vector to being y-polarized!

7 True/False (15p)

Circle **T** or **F**, as appropriate to the statement.

1. A diverging lens makes an object appear farther than it actually is. **F**
2. The focal length of a glass focusing lens underwater is typically shorter than in air. **F**
3. If a laser is collimated on earth and directed into space, the beam will eventually begin to diverge (neglect atmospheric effects). **T**
4. When a plano-convex lens is used to focus parallel rays, the curved side should face the parallel rays, to suppress aberrations. **T**
5. Two lossless mirrors can be arranged in a way such that 100% of light of two distinct frequencies is being transmitted. **T**
6. Two lossless mirrors can be arranged in a way such that 100% of white light is being transmitted. **F**
7. Two lossless mirrors can be arranged in a way such that 100% of light of a particular frequency is reflected. **F**
8. The Free Spectral Range (FSR) is purely determined by the cavity length. **T**
9. The Finesse of the cavity is determined by both the cavity length and the reflectivity.**T**
10. Metals make good lossless ideal mirrors. **F**

YOU ARE DONE!

Equations

Special Functions

$$\Pi_H(x) = \begin{cases} 1 & : |x| < 1/2 \\ 0 & : |x| \geq 1/2 \end{cases}$$

$$\text{III}_L(x) = \sum_{n=-\infty}^{\infty} \delta(x - nL)$$

$$G_{\mu,\sigma}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

δ Function Identities

$\int_{-\infty}^{\infty} dx f(x)\delta(x-\alpha) = f(\alpha)$	Definition
$\int_{-\infty}^{\infty} dx e^{\pm ikx} = 2\pi\delta(k)$	Decomposition

Fourier Transforms

$f(x)$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk g(k) e^{ikx}$	$g(k)$
$f(x)g(x)$	$f(k) * g(k)$
$f(x) * g(x)$	$f(k)g(k)$
$f(x-\alpha)$	$f(k-2\pi\alpha)$
$f(x\alpha)$	$\frac{f(k/\alpha)}{\alpha}$
$\delta(x-\alpha)$	$\frac{1}{\sqrt{2\pi}} \exp -ik\alpha$
$\exp(i\alpha x))$	$\sqrt{2\pi}\delta(k-\alpha)$
$\Pi_H(x/L)$	$\frac{L}{\sqrt{2\pi}} \frac{\sin(kL/2)}{kL/2}$
$\text{III}_L(x)$	$\frac{\sqrt{2\pi}}{L} \text{III}_{2\pi/L}(k)$
$G_{\mu,\sigma}(x)$	$\sigma e^{-ik\mu} G_{0,1/\sigma}(k)$

Maxwell's Equations

$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	Electric Gauss' Law
$\nabla \cdot \mathbf{B} = 0$	Magnetic Gauss' Law
$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$	Faraday's Law
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}$	Ampere's Law

EM Boundary Conditions

$\mathbf{D} = \epsilon\mathbf{E} + \mathbf{P}$	Electric Displacement
$\mathbf{B} = \mu\mathbf{H}$	Magnetic Field
$\mathbf{D}_{n1} = \mathbf{D}_{n2}$	Normal Electric
$\mathbf{E}_{t1} = \mathbf{E}_{t2}$	Transverse Electric
$\mathbf{B}_{n1} = \mathbf{B}_{n2}$	Normal Magnetic
$\mathbf{H}_{t1} = \mathbf{H}_{t2}$	Transverse Magnetic

Wave Optics

$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$	Speed of Light
$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$	Wave Equation
$\mathbf{E}(\mathbf{x}, t) = E_0 \mathbf{J}_0 \exp i(\omega t - \mathbf{k}\cdot\mathbf{x} + \phi)$	Plane Wave
$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$	Poynting Vector
$\langle S \rangle = \frac{1}{2\mu_0 c} \lvert\mathbf{E}_0\rvert^2$	Plane Wave Intensity (t avg)
$\nabla^2 A + \lvert\mathbf{k}\rvert^2 \frac{\partial A}{\partial z} = 0$	Helmholtz
$\nabla_T^2 A - 2i\lvert\mathbf{k}\rvert \frac{\partial A}{\partial z} = 0$	Parax. Helmholtz

Ray Optics

$P = \frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$	Lensmaker's Equation
$\frac{1}{f} = \frac{1}{s_i} + \frac{1}{s_o}$	Imaging Equation
$n_1 \sin\theta_1 = n_2 \sin\theta_2$	Snell's Law
$\tan\theta_B = \frac{n_2}{n_1}$	Brewster Angle
$r = \frac{n_1 - n_2}{n_1 + n_2}$	Normal Fresnel

Gaussian Beams

$z_0 = \frac{\pi w_0^2}{\lambda}$	Rayleigh Range
$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2}$	Waist
$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$	Curvature
$\frac{1}{f} = \frac{1}{R} + \frac{1}{R'}$	Lens

Cavities

$\mathrm{FSR} = \frac{c}{2d}$	Free Spectral Range
$\mathscr{F} = \pi \frac{\sqrt{\lvert r_1 \rvert \lvert r_2 \rvert}}{1 - \lvert r_1 \rvert \lvert r_2 \rvert}$	Finesse
$\nu_{\mathrm{FWHM}} = \frac{\mathrm{FSR}}{\mathscr{F}}$	Full Width Half Max

Polarization, Jones Calculus

$\mathbf{J} = \frac{1}{A^2+B^2} \begin{pmatrix} A \\ B \exp i\phi \end{pmatrix}$	Jones Vector
$\mathbf{J}_1 \cdot \overline{\mathbf{J}_2}$	$\mathbb{C}$ dot product
$\mathbf{J} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Horiz. Polarization
$\mathbf{J} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Vert. Polarization
$\mathbf{J} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$	Right Handed
$\mathbf{J} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$	Left Handed
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	Polarizer $\hat{\mathbf{x}}$
$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\lambda/4$ Plate (fast axis $\hat{\mathbf{x}}$)
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\lambda/2$ Plate (fast axis $\hat{\mathbf{x}}$)
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	Rotation Matrix

Quantum Optics

f $i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \hat{\mathscr{H}} \psi(t)$	Schrodinger

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$	Pythagorean
$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\beta \sin\alpha$	Sine sum
$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$	Cosine sum
$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$	Double Angle Sine
$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$	Double Angle Cosine