(1)

$$|E_1|^2 = \frac{n}{4} E_r$$

$n \in \mathbb{Z}$ just defines the lattice depth in # of E_r

$$|E_2|^2 = \frac{n}{4}(1+\epsilon) E_r$$

$\vec{E}_1 \longrightarrow \qquad \longleftarrow \vec{E}_2$

$$E_1 = \sqrt{\frac{n}{4} E_r}\, e^{ikx} \qquad \text{Beam 1}$$

$$E_2 = \sqrt{\frac{n}{4}(1+\epsilon) E_r}\, e^{-ikx} \qquad \text{Beam 2}$$

$\overbrace{\text{written in units}}$ of energy seen by atom

$$U(x) = (E_1^* + E_2^*)(E_1 + E_2)$$

$$= \left(\sqrt{\frac{n}{4} E_r}\, e^{-ikx} + \sqrt{\frac{n}{4}(1+\epsilon) E_r}\, e^{ikx}\right)\left(\sqrt{\frac{n}{4} E_r}\, e^{ikx} + \sqrt{\frac{n}{4}(1+\epsilon) E_r}\, e^{-ikx}\right)$$

$$= (E_r)(n)\left[\frac{1}{4} + \frac{1}{4}(1+\epsilon) + \frac{1}{4}\sqrt{(1+\epsilon)}\, e^{-i2kx} + \frac{1}{4}\sqrt{1+\epsilon}\, e^{i2kx}\right]$$

$$= nE_r\left[\frac{1}{4}(2+\epsilon) + \frac{1}{4}\sqrt{1+\epsilon}\,\{\underbrace{e^{-i2kx} + e^{i2kx}}_{2\cos(2kx)}\}\right]$$

for small ϵ, $\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon + \ldots$

$$= nE_r\left[\frac{1}{4}(2+\epsilon) + \frac{1}{4}(2+\epsilon)\cos(2kx)\right]$$

Note $\Rightarrow$ $1+\cos(2\alpha) = 2\cos^2\alpha$

$$= (nE_r)\left(\frac{1}{4}(2+\epsilon)\right)\underbrace{\left[1 + \cos(2kx)\right]}_{2\cos^2(kx)}$$

(2)

$$U(x) = \frac{1}{2} n E_r (2+\epsilon) \cos^2(kx) \quad : \quad \frac{n E_r (2+\epsilon)}{2} \cos^2(kx)$$

note that for this expression the function has a min @ $U(x)_{MIN} = 0$ $U_{MAX} = E_r \frac{n(2+\epsilon)}{2}$

for $\epsilon \to 0$ $U_{MAX} \Rightarrow n E_R$
# of recoil depth of lattice

So now we want to find the energy from the harmonic trap.

$$U(x) = E_r \cdot n \left(\frac{2+\epsilon}{2}\right) \underbrace{\cos^2(kx)}_{\left(1 + \frac{1}{2} k^2 x^2 \ldots\right)^2}$$

$$= \underbrace{\left(1 + k^2 x^2 + \frac{1}{4} k^4 x^4 + \ldots \right)}_{\text{keep up to } \mathcal{O}(x^2)}$$

, Drop 1 from DC, We'll add it in by hand

$$U_{H.O.} \Rightarrow n E_r \left(\frac{2+\epsilon}{\cancel{2}}\right) k^2 \cancel{x^2} = \frac{1}{\cancel{2}} m \omega^2 \cancel{x^2}$$

$$\omega^2 = \underbrace{\frac{k^2}{2m}}_{E_r} \left[2n E_r (2+\epsilon) \right] \qquad E_r \equiv \frac{k^2}{2m}$$

$$\boxed{\omega^2 = E_r^2 \sqrt{2n(2+\epsilon)}} \quad , \quad \omega = E_r \sqrt{2n(2+\epsilon)}$$

DC offsets

$$\boxed{\begin{aligned} U_{Blue} &= 0 \\ U_{RED} &= E_r n \left(\frac{2+\epsilon}{2}\right) \end{aligned}}$$

(3) So now the question is how does the ground state energy change for red or blue given additional ϵ

$$E_{G.S.,\,Red} = E_{offset} + H.O.$$

$$= U_{RED}(\epsilon) + \frac{1}{2}\omega(\epsilon)$$

$$= -E_r n\left(\frac{2+\epsilon}{2}\right) + \frac{E_r}{2}\sqrt{2n(2+\epsilon)}$$

$$= \boxed{\left(\frac{E_r}{2}\right)\left[-n(2+\epsilon) + 2\sqrt{n}\left(1+\frac{\epsilon}{2}\right)^{1/2}\right]}$$

$$E_{G.S.,\,Blue} = E_{offset} + H.O.$$

$$= U_{BLUE}(\epsilon) + \frac{1}{2}\omega(\epsilon)$$

$$= 0 + \frac{E_r}{2}\sqrt{2n(2+\epsilon)}$$

$$= \boxed{\frac{E_r}{2}\, 2\sqrt{n}\left(1+\frac{\epsilon}{2}\right)^{1/2}}$$

So for small ϵ, again...

$$\tilde{E}_R = \frac{E_{RED}}{E_r} = -n\left(1+\frac{\epsilon}{2}\right) + \sqrt{n}\left(1+\frac{1}{4}\epsilon\right) = \boxed{(-n+\sqrt{n}) + \epsilon\left(\frac{-n}{2}+\frac{\sqrt{n}}{4}\right)}$$

$$\tilde{E}_B = \frac{E_{BLUE}}{E_r} = \sqrt{n}\left(1+\frac{1}{4}\epsilon\right) = \boxed{(\sqrt{n}) + \epsilon\left(\frac{\sqrt{n}}{4}\right)}$$

(9)

So, probably what I want to know is how $\frac{\partial \tilde{E}_B}{\partial \epsilon}$ compare to $\frac{\partial \tilde{E}_R}{\partial \epsilon}$

$$\boxed{\frac{\partial \tilde{E}_B}{\partial \epsilon} = \frac{\sqrt{n}}{4}} \qquad \boxed{\frac{\partial \tilde{E}_R}{\partial \epsilon} = \frac{\sqrt{n}}{4} - \frac{n}{2}}$$

So the statement then is in both Red & Blue my ground stat energy changes as a function of ϵ linearly. But, my coefficient multiplied by ϵ depends on the depth and does not scale the same.

See plot attached.