Section 10 : Phys 175

4-13-2017

Atom & Light field quantized

- Brief discussion about "quantum" & the problems of the semi-classical approach.
- Jaynes - Cummings Model
  - quantized atom & quantized ~~[illegible]~~ light
- Spontaneous Emission.

Questions?

Some Comments: (mostly covered in Scully & Zubairy's "Quantum Optics")

- Up till now we've either 1) quantized the atom (Rabi flopping w/ classical field, Ω) or

2) quantized the electric field with no atom at all (Coherent states and a, $a^\dagger$ operators)

However, we never put the two together. Doing this is called the Jaynes-Cummings Model.

- Aside about the "photon."

We typically learn the concept of the photon was justified by the photo-electric effect and so we give credit to Einstein for the use of the "photon." (There is some subtlety to why we don't give this to Plank, who quantized the modes in the cavity).

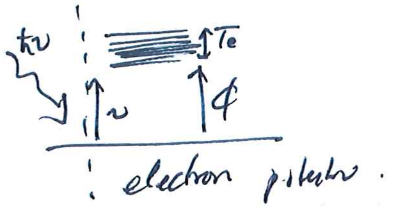
What we know from photo-electric effect:

1) $h\nu = \phi + T_e$; photon energy $> \phi$ gives e^- a K.E. of T_e

2) # of e^-'s is $\propto$ # of photons (well it's proportional to $|\Omega|^2$)

3) There is no time delay between turning on the light field & getting out e^-'s

Interestingly, the first 2 points don't require photons. In fact, from solving the 2-level atom and observing Rabi flopping we see that 1) comes from a quantized atom but classical field (for the ν arguement)



2) also ~~requires~~ the rate / # of e's we get out is proportional to $|\Omega|^2$ similar to what we saw from the Rabi-flopping that the rate is related ~~th~~ to the $|\Omega|^2$ intensity (you can get this from the ρ's equations of motion)

3) The time delay (well, the lack of one) is what can't be classical. The field always takes some time $\sim \Omega_L$ to excite the e^- which means there is always some delay as the e^- is coherently coupled out. But in the photon case, there is no delay, even if the flux of photons is low, the small # of e^-'s emitted will happen immediately! This implies the light is truly "chunky" at some microscopic level.

Total hamiltonian.

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_e + \mathcal{H}_i$$

Energy of atom

Energy of light field.

energy of atom-light interaction.

In general

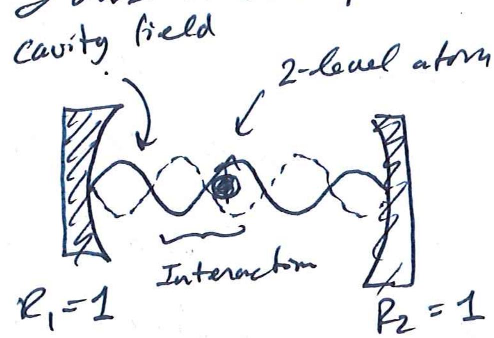
$$\mathcal{H}_a = \hbar \sum_i \omega_i |i\rangle\langle i|$$

$$\mathcal{H}_e = \hbar \sum_k \nu_k \left(\hat{a}_k^{+} \hat{a}_k + 1/2\right)$$

$$\mathcal{H}_i = -e\vec{r}\cdot\vec{E}$$

Graphically

Thus in our problem



where $\quad e\vec{r} = e\sum_{i,j} |i\rangle\langle i|\, r\, |j\rangle\langle j| = \sum_{i,j} \mu_{ij}\sigma_{ij}$

& $\quad \sigma_{ij} = |i\rangle\langle j|$

$$\vec{E} = \sum_k \hat{\epsilon}_k \mathcal{E}_k (a_k + a_k^{+})$$

where $\quad \mathcal{E}_k = \left(\frac{\hbar \nu_k}{2\epsilon_0 V}\right)^{1/2}$

& we will define a $\quad g_k^{ij} = \frac{-\mu_{ij}\mathcal{E}_k}{\hbar}$

$$\mathcal{H} = \sum \hbar \nu_k a_k^{+} a + \sum_i \hbar\omega_i \sigma_{ii} + \hbar \sum_{ij}\sum_k g_k^{ij}\sigma_{ij}(a_k + a_k^{+})$$

we dropped the DC offset of 1/2

So now the terms we have are

~~$\mathcal{H}$~~ $\mathcal{H}_a$ = energy of atom

$\mathcal{H}_\ell$ = energy in light field.

$\mathcal{H}_i$ = interaction of the two.

Side Note!

g_k^{ij} is ≥ 0 & $\in \mathbb{R}$

for our purposes.

we can always do this

by saying $g_k^{ij} = |g_k^{ij}| e^{i\phi_0}$

since g_k^{ij} is accompanied by $|j\rangle\langle i|$

$$g_k^{ij} |j\rangle\langle i| \rightarrow |\tilde{j}\rangle\langle i| \, |g_k^{ij}|$$

$$|\tilde{j}\rangle = |j\rangle e^{i\phi_0^{ij}{}_k}$$

We will now restrict ourselves to a 2-level atom & one light frequency ν

$$g_k^{ij} \rightarrow g^{21} = g^{12} \rightarrow g \quad ;$$

also remember

$$a_k^\dagger |n_k\rangle = \sqrt{n_k+1}\, |n_k+1\rangle$$
$$a_k |n_k\rangle = \sqrt{n_k}\, |n_k-1\rangle$$

$$\mathcal{H} = \underbrace{\sum_k \hbar\omega_k a_k^\dagger a_k}_{\mathcal{H}_\ell}$$

$$+ \underbrace{\hbar\omega_2 \sigma_{22} + \hbar\omega_1 \sigma_{11}}_{\mathcal{H}_a}$$

$$+ \underbrace{g\left(\sigma_{12} a_k + \sigma_{12} a_k^\dagger + \sigma_{21} a_k + \sigma_{21} a_k^\dagger\right)}_{\mathcal{H}_i}$$

Not all of these terms are physical, consider

$a_k |1\rangle\langle 2| \Rightarrow$ a removes a photon; $|1\rangle\langle 2|$ removes atom excitation. } Both! energy cons??

$a_k^\dagger |2\rangle\langle 1| \Rightarrow$ $a^\dagger$ adds a photon; $|2\rangle\langle 1|$ adds atomic excitation.

* Note: One should be very careful when dropping these terms, it requires $g \ll \omega+\nu$ & $\omega+\nu \gg \omega-\nu$; which is not always valid. See: arXiv:0912.3261

Scully & Zubairy's soln take this problem by writing it in a convenient way to solve w/ the interaction picture.

$\exists$

$$\mathcal{H} = \sum_k \hbar\nu_k a_k^\dagger a_k + \hbar\omega_{12}\sigma_z + \hbar\sum_k g_k \sigma_+ a_k + g_k \sigma_- a_k^\dagger$$

$$\omega_{12} = \omega_2 - \omega_1$$

$$\sigma_z = |2\rangle\langle 2| - |1\rangle\langle 1| \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ = |2\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = |1\rangle\langle 2| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For the purposes of this section I will jump to just writing down what we expect from the hamiltonian. (for one ν, $k=1$)

$$\mathcal{H} = \left[\hbar\nu a^\dagger a + \hbar\omega |2\rangle\langle 2|\right] \quad \text{self-energy terms}$$

$$+\left[g a^\dagger \underbrace{|1\rangle\langle 2|}_{\sigma_-} + g a \underbrace{|2\rangle\langle 1|}_{\sigma_+}\right] \quad \text{interaction term}$$

We will use a short hand for states since we have both photon # states $|n\rangle$ & atom level states $|1\rangle$ & $|2\rangle$ & for example

$$|1\rangle|n\rangle = |1,n\rangle \qquad a^\dagger|1,n\rangle = \sqrt{n+1}\,|1,n+1\rangle$$

$$|2\rangle|n\rangle = |2,n\rangle \qquad \sigma_+|1,n\rangle = |2,n\rangle$$

We can always write our operators as projectors.

like $\sigma_+ \Rightarrow |2\rangle\langle 1|$; $a^+ \rightarrow |n+1\rangle\langle n| \, (\sqrt{n+1})$

$\sigma_- \Rightarrow |1\rangle\langle 2|$ $\quad a \rightarrow |n\rangle\langle n+1| \, (\sqrt{n+1})$

So,

$$\mathcal{H} = \sum_{n=0}^{\infty} \hbar\nu \, (\sqrt{n+1})^2 \left(|n+1\rangle\langle n| \cdot |n\rangle\langle n+1| \right)$$

$$+ \; \hbar\omega \, |2\rangle\langle 2|$$

$$+ \; g\,(|1\rangle\langle 2|) \otimes (|n+1\rangle\langle n| \sqrt{n+1}) + g\,(|2\rangle\langle 1|) \otimes (|n\rangle\langle n+1| \sqrt{n+1})$$

now in our combined notation

$$\boxed{\begin{aligned} \mathcal{H} = & \sum_{n=0}^{\infty} \hbar\nu \,(n+1) \left(|1,n+1\rangle\langle 1,n+1| + |2,n+1\rangle\langle 2,n+1| \right) \\ & + \hbar\omega \sum_{n=0}^{\infty} |2,n\rangle\langle 2,n| \\ & + \sum_{n=0}^{\infty} \left[g\sqrt{n+1} \, |1,n+1\rangle\langle 2,n| + g\sqrt{n+1} \, |2,n\rangle\langle 1,n+1| \right] \end{aligned}}$$

So this is our entire hamiltonian!! The Hilbert space is spanned by $[\text{fock states } |n=0\rangle \rightarrow |\infty\rangle] \otimes [\text{atomic states } |1\rangle \,\&\, |2\rangle]$

so it has ∞ dimension but can be solved in general.

For the purpose of section let's say we will start w/ a state of $|2,0\rangle$, the atom is excited and there is no photon in the cavity.

Since all states of $|i,n\rangle$ are eigenstates of $\hat{\mathcal{H}}_a$ & $\hat{\mathcal{H}}_e$ we only need to see how $\hat{\mathcal{H}}_i$ couples them.

$$\left.\begin{aligned} \mathcal{H}_i\ |2,0\rangle &\to g\,|1,1\rangle \\ \mathcal{H}_i\ |1,1\rangle &\to g\,|2,0\rangle \end{aligned}\right\}$$

great! Closed sub-space!

effective hamiltonian.

$$\Rightarrow \tilde{\mathcal{H}} = \hbar\nu\,|1,1\rangle\langle 1,1| + \hbar\omega\,|2,0\rangle\langle 2,0| + g\,|2,0\rangle\langle 1,1| + g\,|1,1\rangle\langle 2,0|$$

And our general $|\psi\rangle = C_{1,1}\,|1,1\rangle + C_{2,0}\,|2,0\rangle$

we first go to a rotating frame of

$$\tilde{C}_{2,0} = C_{2,0}\,e^{-i\omega t}$$

$$\tilde{C}_{1,1} = C_{1,1}\,e^{-i\nu t}$$

The rotating frame; it we find from.

$$\langle 2,0| \, i\hbar \tfrac{\partial}{\partial t}|\psi\rangle = \langle 2,0| H |\psi\rangle$$

$$(1) \quad \dot{\tilde{C}}_{2,0} = -i\left(g \, e^{-i\Delta t} \, \tilde{C}_{1,1}\right)$$

$$\langle 1,1| \, i\hbar \tfrac{\partial}{\partial t}|\psi\rangle = \langle 1,1| H |\psi\rangle$$

$$(2) \quad \dot{\tilde{C}}_{1,1} = -i\left(g \, e^{i\Delta t} \, \tilde{C}_{2,0}\right)$$

(1) & (2) are our specific equations of motion now that for the initial state of $|\psi(t=0)\rangle = |2,0\rangle$

$$\Rightarrow \boxed{\begin{aligned} \tilde{C}_{2,0}(t) &= \left[\cos\left(\frac{\Omega' t}{2}\right) - \frac{i\Delta}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right)\right] e^{i\Delta t/2} \\ \tilde{C}_{1,1}(t) &= \left[\frac{-2ig}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right)\right] e^{-i\Delta t/2} \end{aligned}}$$

$$\boxed{\Omega'^2 = 4g^2 + \Delta^2}$$

Importantly now we've found coherent dynamics that allow an atom to decay!

So what happens if our cavity is on resonant?

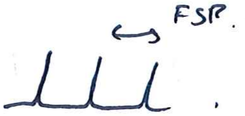
$$\Delta = 0 \Rightarrow \Omega'^2 = g^2$$

$$C_{2,0}(t) = \cos(gt)$$

$$C_{1,1}(t) = -i\sin(gt)$$

So for this problem we considered only one ν_1 w/ perfectly reflecting mirrors (this also implies $(\delta\nu = 0)$). This is super important for the concept of "Wigner-Weisskopf theory" when using quantized atoms & fields we can calculate the γ of the atom.

Remember that $\left(FSR = \frac{2c}{d}\right)$ gives the frequency difference between frequency teeth in the cavity



. As we take our $R=1$ cavity length $\to \infty$, we get a cavity that supports every frequency but could be analyzed as a $\delta(\nu - \nu_0)$ at any frequency. Finding the atom's overlap with all the ν's around ω can be used to calculate the effective lifetime then of the cavity.

Continued notes: (Not in section)

Background and more general soln to the vacuum Rabi flopping problem.

Let's go back to

$$\mathcal{H} = \sum_{n=0}^{\infty} \hbar\nu(n+1)\left\{ |1,n+1\rangle\langle 1,n+1| + |2,n+1\rangle\langle 2,n+1| \right\}$$

$$+ \hbar\omega \sum_{n=0}^{\infty} |2,n\rangle\langle 2,n|$$

$$+ \sum_{n=0}^{\infty}\left\{ g\sqrt{n+1}\,|1,n+1\rangle\langle 2,n| + g\sqrt{n+1}\,|2,n\rangle\langle 1,n+1| \right\}$$

where in general $|\psi\rangle = \sum_{n=0}^{\infty} c_{1,n}|1,n\rangle + c_{2,n}|2,n\rangle$

We want know from our previous work that the atom only gives an exchange of $n \pm 1$

So we will work w/ coefficients $c_{1,n+1}$ & $c_{2,n}$

$$\langle 2,n|\, i\hbar\frac{\partial}{\partial t}\,|\psi\rangle = \langle 2,n|\,H\,|\psi\rangle$$

(1) $\quad i\hbar\,\dot{c}_{2,n} = \hbar\nu n\, c_{2,n} + \hbar\omega\, c_{2,n} + \hbar g\sqrt{n+1}\, c_{1,n+1}$

go to the rotating frame of $\quad \tilde{c}_{2,n} = c_{2,n}\, e^{i(\frac{\hbar\nu}{\hbar} n t)}\, e^{-i(\hbar\omega)\frac{t}{\hbar}}$

& $\tilde{c}_{1,n+1} = c_{1,n}\, e^{+\frac{it}{\hbar}(\hbar\nu(n+1))}$.

So, $\dot{\tilde{C}}_{2,n} = \dot{C}_{2,n}\, e^{-\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\omega)} + C_{2,n}\, e^{\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\omega)}\, (i\nu - i\omega)$

sub into (1)

(2) $i\hbar\left[\dot{\tilde{C}}_{2,n}\, e^{-\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\omega)} - (i\nu + i\omega)\, C_{2,n}\right] = i\hbar\, \dot{C}_{2,n}$

$$= C_{2,n}(\hbar\nu n) + C_{2,n}\,\hbar\omega + \hbar g\sqrt{n+1}\, C_{1,n+1}$$

(3) $i\hbar\, \dot{\tilde{C}}_{2,n}\, e^{\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\omega)} + \cancel{(+\hbar\nu n + \hbar\omega)\, C_{2,n}} = \cancel{C_{2,n}(\hbar\nu n) + C_{2,n}\,\hbar\omega} + g\sqrt{n+1}\, C_{1,n+1}$

$$i\hbar\, \dot{\tilde{C}}_{2,n} = \hbar g\sqrt{n+1}\, e^{\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\omega)}\, C_{1,n+1}$$

remember $C_{1,n+1} = \tilde{C}_{1,n+1}\, e^{-\frac{it}{\hbar}[\hbar\nu(n+1)]}$

$$\tilde{C}_{1,n+1} = C_{1,n+1}\, e^{\frac{it}{\hbar}(\hbar\nu n)}\, e^{\frac{it}{\hbar}(\hbar\nu)}$$

$$i\hbar\, \dot{\tilde{C}}_{2,n} = \hbar g\sqrt{n+1}\, e^{-\frac{it}{\hbar}(\hbar\nu)}\, e^{\frac{it}{\hbar}(\hbar\omega)}\, \tilde{C}_{1,n+1}$$

$$\delta = \nu - \omega$$

(4) $\boxed{i\hbar\, \dot{\tilde{C}}_{2,n} = \hbar g\sqrt{n+1}\, e^{i\delta t}\, \tilde{C}_{1,n+1}}$

for $\langle 1,n+1|\, i\hbar\frac{\partial}{\partial t}\,|\psi\rangle = \langle 1,n+1|\, H\, |\psi\rangle$ you get (5)

(5) $\boxed{i\hbar\, \dot{\tilde{C}}_{1,n+1} = \hbar g\sqrt{n+1}\, e^{-i\delta t}\, \tilde{C}_{2,n}}$

From (4) & (5) we can create 2, uncoupled, second-order differential equations. One thing we do need though are $\tilde{C}_{2,n}(0)$ & $\tilde{C}_{1,n+1}(0)$

In general, these will solve to:

(6) $$\tilde{C}_{2,n}(t) = \left\{ \tilde{C}_{2,n}(0) \left[\cos\left(\frac{\Omega_n' t}{2}\right) - \frac{i\Delta}{\Omega_n'} \sin\left(\frac{\Omega_n' t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n'} \tilde{C}_{1,n+1}(0) \sin\left(\frac{\Omega_n' t}{2}\right) \right\} e^{i\Delta t/2}$$

(7) $$\tilde{C}_{1,n+1} = \left\{ \tilde{C}_{1,n+1}(0) \left[\cos\left(\frac{\Omega_n' t}{2}\right) + \frac{i\Delta}{\Omega_n'} \sin\left(\frac{\Omega_n' t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n'} \tilde{C}_{2,n}(0) \sin\left(\frac{\Omega_n' t}{2}\right) \right\} e^{-i\Delta t/2}$$

where $(\Omega_n')^2 = (2g\sqrt{n+1})^2 + \Delta^2$

The take-away from this is that Ω_n', the analogue to the generalized Rabi frequency from the semi-classical model $\Omega'^2 = \Omega^2 + \Delta^2$ now explicitly depends on photon #. Classically getting rid of all the photons would set $\Omega \to 0$.

However, we see from treating the light field as actual photons, that the $\Omega = g\sqrt{n+1}$ and is bounded on the bottom by "g". This means $|0\rangle$, which is $n=0$, or the vacuum state, <u>still</u> coherently couples the states $|2\rangle$ & $|1\rangle$ in the cavity!! By taking an atom now in the excited state $\Rightarrow |2,0\rangle$, when the atom is just in vacuum, free space, the cavity can be enormous, the size of the universe, the $|2\rangle$ & $|1\rangle$ are coherently coupled by the vacuum. Importantly though, once the photon leaves, it doesn't come back. From our quantum treatment of light we see the mechanism for the spontaneous decay γ.