Matthew Rispoli
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rispoli@physics.harvard.edu

Section Notes 4

- Questions?
- Homework?

Aside about Convolution:

Convolution, in words, is sometimes described as the running average, and has a seemingly funny defn.

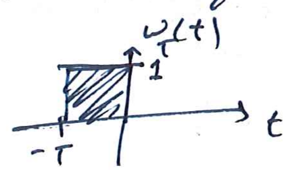
$$g * f(x) = \int_{-\infty}^{+\infty} dx' \, g(x') f(x-x')$$

This states that $g(x)$ is convolved with $f(x)$ by finding the overlap of $g(x')$ & $f(-x')$ offset by "x"

So why is this the running average?

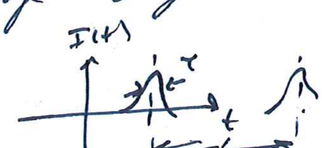
Consider a part of you that makes measurements (your eyes, ears, whatever). It doesn't make instantaneous measurements. You effectively have some window function you average over.

Consider a square.



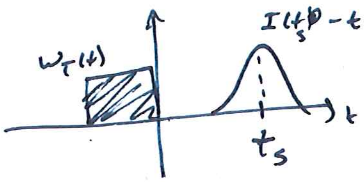
average over all things happening now & past.

So if I'm watching a police siren, or I'm at a club, I might see pulses of light coming in at period T' w/ width τ



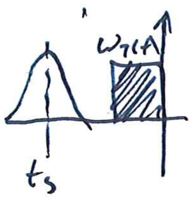
What do I see?

$t_{now} \ll t_{strobe}$



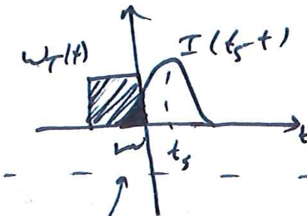
what I see: nothing

$t_{now} \gg t_{strobe}$



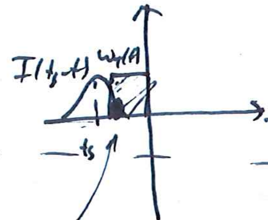
I see nothing.

$t_n \lessapprox t_s$



overlap I see!

$t_n \gtrapprox t_s$

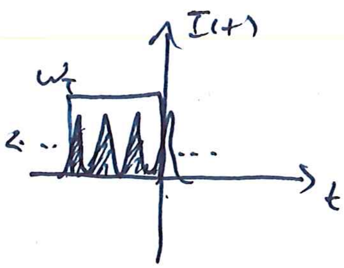


overlap!

overall

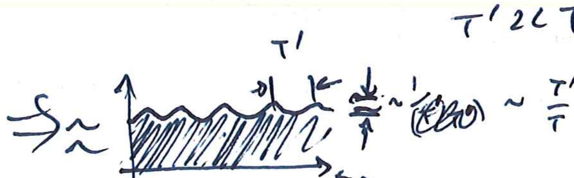
$(t_{now} - t_{strobe})$

What happens if I make these strobes close together?



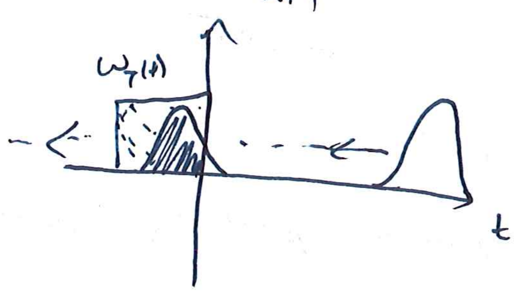
$T' \ll T$

$\Rightarrow \approx$



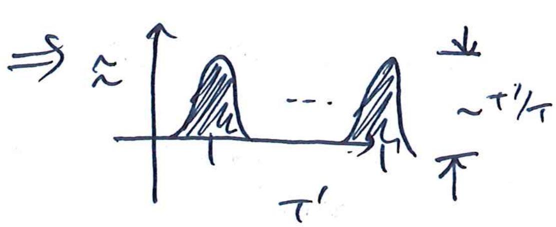
$t_{now} - t_s$

$I(t)$



$T' \gg T$

$\Rightarrow \approx$



So what does this have to do w/ Fourier Transforms??
Well, we're getting there. We can see that our window function effects which frequency components we can resolve.
So this is why we think of Fourier transforms.

Also we'll see our intuition of convolutions will be related as well.

Fourier Transform.

So what is a fourier transform?

- One way to say it in words is that it's a mapping from a spatial basis to basis of sines and cosines!

As a defn'

$$\mathcal{F}\{u(x)\} = \mathcal{U}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, u(x)\, e^{-ikx}$$

$$\mathcal{F}^{-1}\{\mathcal{U}(k)\} = u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \mathcal{U}(k)\, e^{+ikx}$$

One of the most important fourier transforms!

$$\mathcal{F}\{\delta(x-x_0)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx'\, \delta(x'-x_0)\, e^{-ikx'} = e^{-ikx_0}$$

note that $\left|\mathcal{F}\{\delta(x-x_0)\}\right| = \frac{1}{\sqrt{2\pi}}\, 1 = \text{const}$ everywhere!

$Re\left[e^{-ikx_0}\right] = \cos(kx_0)$

This is important for the defn of $\delta(x-x_0)$ actually.

$$\mathcal{F}^{-1}\{e^{-ikx_0}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{+ik(x_0-x)} = \delta(x-x_0)$$

Note, that wasn't really a proof, more of a statement.
But it's necessary if you want to prove things to yourself
like $\mathcal{F}^{-1}\{\mathcal{F}\{u(x)\}\} = u(x)$

Common pairs (missing many extra factors of π)

$x \qquad k_x$

$\delta(x) \longleftrightarrow 1$

$1 \longleftrightarrow \delta(k_x)$

$\cos(2\pi k_0 x) \longleftrightarrow \frac{1}{2}\left(\delta(k-k_0) + \delta(k+k_0)\right)$

$\sin(2\pi k_0 x) \longleftrightarrow i\left(\delta(k+k_0) - \delta(k-k_0)\right)$

$e^{-\alpha x^2} \longleftrightarrow e^{-k^2/\alpha^2}$

Properties

Linearity:

$$\mathcal{F}\{\alpha(u(x)) + \beta v(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \alpha\, u(x)\, e^{-ikx} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \beta\, v(x)\, e^{-ikx} = \boxed{\alpha U(k) + \beta (V(k))}$$

Shifting

$$\mathcal{F}\{u(x+x_0)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\; u(x+x_0)\, e^{-ikx} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\tilde{x}\; u(\tilde{x})\, e^{-ik\tilde{x}}\, e^{ikx_0}$$

$\tilde{x} = x + x_0$
$d\tilde{x} = dx$

$$= \boxed{U(k)\, e^{ikx_0}}$$

Properties cont.

Reversing x $(x \to -x)$

$$\mathcal{F}\{u(x)\} = U(k)$$

$$\boxed{\mathcal{F}\{u(-x)\} = +U(-k)}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, u(-x)\, e^{-ikx}$$

$\tilde{x} = -x$ $\quad \tilde{k} \to -k$

$d\tilde{x} = -d\tilde{x}$

$$= \frac{+1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\tilde{x}\, u(\tilde{x})\, e^{-i\tilde{k}\tilde{x}}$$

$$= U(\tilde{k}) = U(-k)$$

Scaling

$$\boxed{\mathcal{F}\{u(\alpha x)\} = \frac{1}{\alpha} U(k/\alpha)}$$

use the same trick

$\tilde{x} = \alpha x \qquad d\tilde{x} = \alpha\, dx \qquad \tilde{k} = \frac{k}{\alpha}$

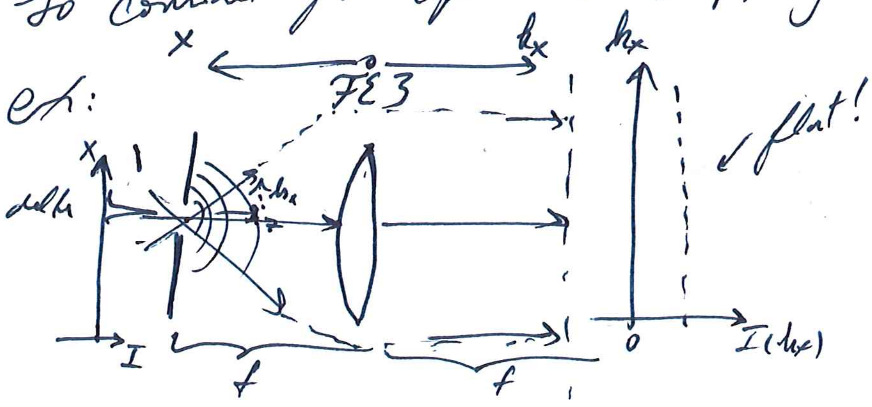
· So how do I see this w/ optics?

Optics

Remember we said this exactly maps us to a basis of sines & cosines.

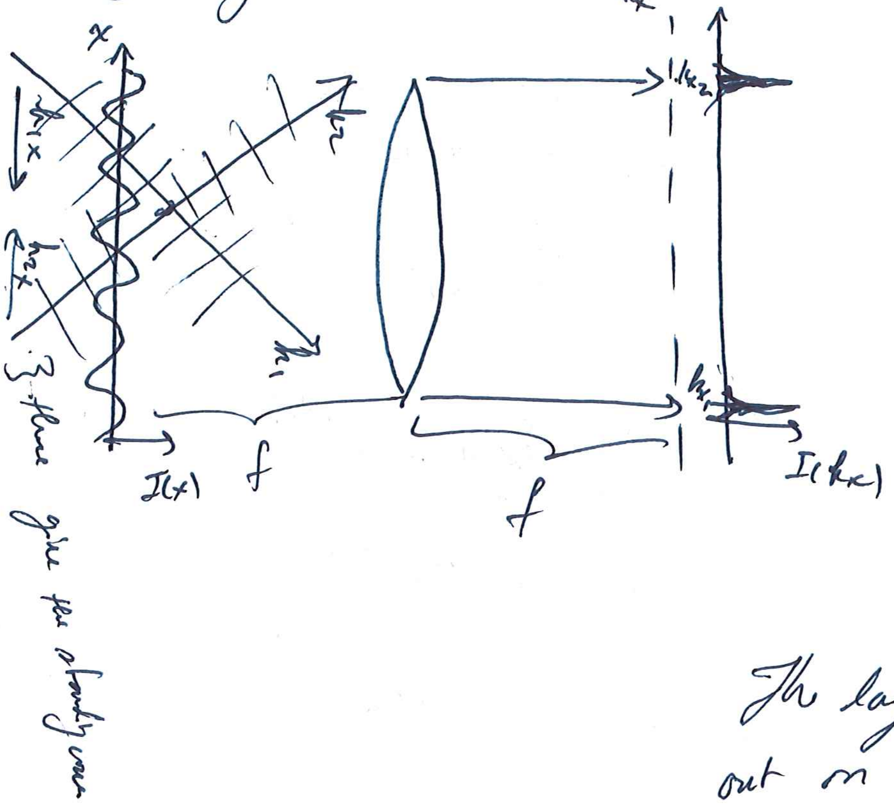
So consider for optics a mapping onto plane waves.

ex:



Consider another important example

standing wave

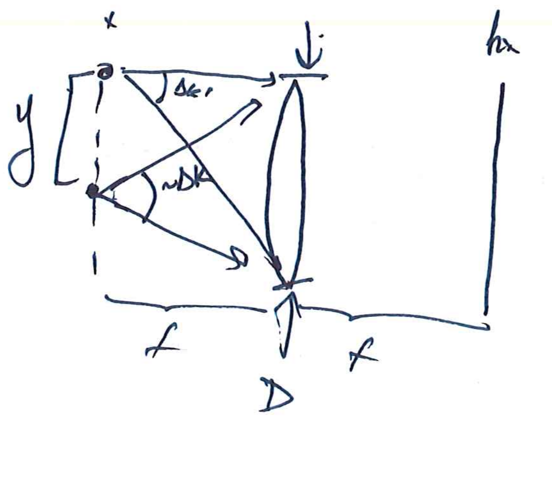


You can make a statement about the size of your aperture limiting your resolution then since features in x depend on $\cos(k_{1x} \mp k_{2x})$

The larger they are, the further out on the lens they go!

Cute one to see "field of view"

— going off axis limits my resolution.



$\Delta k' < \Delta k$

$\Rightarrow$ Resolution goes down w/ offset y.

Okay! Fourier filtering time.