Intro to continuous-time state-space models

- You have seen that simulation is a valuable tool for understanding state-space models.
- But, we would like to develop more insight into the system response by more general analytic means; specifically, by looking at the time-domain solution for $x(t)$.
- We solve first for the homogeneous solution ($u(t) = 0$):
  - Start with $\dot{x}(t) = Ax(t)$ and some initial state $x(0)$.
  - Take Laplace transform:

$$
\begin{aligned}
sX(s) - x(0) &= AX(s) && \text{Initial value theorem} \\
(sI - A)X(s) &= x(0) && \text{Be careful with matrix dimensions!} \\
X(s) &= (sI - A)^{-1}x(0). && \text{Assume invertible}
\end{aligned}
$$

  (*Detailed* Laplace-transform knowledge is helpful, but is not required on quizzes.)

  - So, to this point we have: $x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0)$.

The state-transition matrix

- Recapping, we have: $x(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]x(0)$. But,

$$(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

  so,

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \triangleq e^{At} \qquad \text{(matrix exponential).}$$

- Therefore, we write $x(t) = e^{At}x(0)$.
  - e^{At} is called the "transition matrix" or "state-transition matrix."
  - Note that e^{At} is <u>not</u> the same as taking the exponential of every element of At.
  - $e^{(A+B)t} = e^{At}e^{Bt}$ iff $AB = BA$. (that is, not in general).
  - In Octave, can compute $x(t)$ as: `x = expm(A*t)*x0;`
- Will say more about e^{At} when we discuss the structure of A.

Computing the state-transition matrix by hand

- Computing $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$ is straightforward for 2×2.
- **EXAMPLE:** Find e^{At} when $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.
  - Solve:

$$
\begin{aligned}
(sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix} \\
e^{At} &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} 1(t).
\end{aligned}
$$

- This is probably the best way to find e^{At} if the A matrix is 2×2.

Forced state response

- Solving for the forced solution ($u(t) \neq 0$), we find:

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau}_{\text{convolution}}.$$

- Where did this come from?

1. $\dot{x}(t) - Ax(t) = Bu(t)$, simply by rearranging the state equation.
2. Multiply both sides by e^{-At} and rewrite the LHS as the middle expression:

$$e^{-At}[\dot{x}(t) - Ax(t)] = \frac{\mathrm{d}}{\mathrm{d}t}[e^{-At}x(t)] = e^{-At}Bu(t).$$

3. Integrate the middle and the RHS; rearrange and keep new middle and RHS:

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}[e^{-A\tau}x(\tau)]\,\mathrm{d}\tau = e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,\mathrm{d}\tau.$$

Forced output response

- So, we have established the following state dynamics:

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau}_{\text{convolution}}.$$

- Since $z(t) = Cx(t) + Du(t)$, the system output is then comprised of three parts:

$$z(t) = \underbrace{Ce^{At}x(0)}_{\text{initial resp.}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau}_{\text{convolution}} + \underbrace{Du(t)}_{\text{feedthrough}}.$$

Summary

- You have learned how to compute the homogeneous and forced response of a state-space model.
- The output $z(t)$ comprises three parts:
  - The response due to the initial conditions: $Ce^{At}x(0)$.
  - The dynamic response caused by the forcing input: $\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$.
  - The instantaneous response due to the model's feedthrough term: $Du(t)$.
- These terms rely on the matrix exponential e^{At}, which may be a new concept to you.
  - Be careful! e^{At} is not the same thing as taking the exponential of every element of At. It is not correct to compute it as: `exp(A*t);`
  - One way to compute it (e.g., by hand) is via: $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$.
  - Or, in Octave, it is correct to compute: `expm(A*t);`
- You will learn more about the matrix exponential in the next lesson.