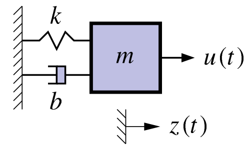
Example of a continuous-time state-space model

- Representation of the dynamics of an nth-order system as a first-order differential equation in an n-vector called the state.
  - ➠ n coupled first-order equations.
- Classic example: Second-order equation of motion.



$$m\ddot{z}(t) = u(t) - b\dot{z}(t) - kz(t)$$

$$➠ \quad \ddot{z}(t) = \frac{u(t) - b\dot{z}(t) - kz(t)}{m},$$

where $\ddot{z}(t) = \mathrm{d}^2 z(t)/\mathrm{d}t^2$, $\dot{z}(t) = \mathrm{d}z(t)/\mathrm{d}t$, etc.

- Define a (non-unique) state vector:

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \quad \text{so, } \dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} \dot{z}(t) \\ -\frac{k}{m}z(t) - \frac{b}{m}\dot{z}(t) + \frac{1}{m}u(t) \end{bmatrix}.$$

Example in continuous-time state-space form

- So far, we have:

$$\dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} \dot{z}(t) \\ -\frac{k}{m}z(t) - \frac{b}{m}\dot{z}(t) + \frac{1}{m}u(t) \end{bmatrix}.$$

- We can write this as $\dot{x}(t) = Ax(t) + Bu(t)$, where A and B are constant matrices.

$$\dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{A} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \quad \\ \quad \end{bmatrix}}_{B} u(t).$$

- Complete the model by computing $z(t) = Cx(t) + Du(t)$, where C and D are constant matrices.

$$C = \begin{bmatrix} \quad & \quad \end{bmatrix}, \qquad D = \begin{bmatrix} \quad \end{bmatrix}.$$

Standard state-space model form

- Standard form for continuous-time linear state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t),$$

where $u(t)$ is the known input, $x(t)$ is the state, A, B, C, D are constant matrices.

- Convenient way to express dynamics (matrix format great for computers).
- The first equation is called the state equation or the process equation.
  - Notice that this is the only equation that evolves over time (since it is a first-order vector ODE that integrates the right-hand side to find $x(t)$).
  - So, this equation summarizes the "dynamics" of the model.
- The second equation is called the output equation or the measurement equation.
  - It is a static linear combination of variables known at time t.

What is the system state vector?

- Standard form for continuous-time linear state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t).$$

- We think of the system state at time t_0 as a minimum set of information at t_0 that, together with input $u(t)$, $t \geq t_0$, uniquely determines system behavior for all $t \geq t_0$.
  - State variables provide access to what is going on *inside* the system.
  - The state has dimensions $x(t) \in \mathbb{R}^n$, so $A \in \mathbb{R}^{n \times n}$ and $w(t) \in \mathbb{R}^n$.
- Note that in principle we can solve the state equation numerically,

$$x(t) = x(0) + \int_0^t Ax(\tau) + Bu(\tau) + w(\tau)\,\mathrm{d}\tau,$$

but it will usually be more convenient to keep the equation in differential form.

What are the output and inputs?

- Standard form for continuous-time linear state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t).$$

- The model output is $z(t) \in \mathbb{R}^m$. This is the measurement we make.
- There are three input signals to the model: $u(t)$, $w(t)$, and $v(t)$.
- We consider $u(t) \in \mathbb{R}^r$ to be a (deterministic) input; that is, we assume that we know its value exactly at all times.
  - Based on the size of $u(t)$, we infer that $B \in \mathbb{R}^{n \times r}$.
  - The deterministic input forces $x(t)$ to evolve over time in different ways, depending on its values.
  - It also influences the output via the D term.

What are the random (stochastic) inputs?

- Standard form for continuous-time linear state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t).$$

- The model has two random input signals: $w(t)$, and $v(t)$.
- They are random in the sense that we assume that we never know their values exactly.
  - We have already seen that $w(t) \in \mathbb{R}^n$; if $z(t) \in \mathbb{R}^m$, then $v(t) \in \mathbb{R}^m$ also.
- The $w(t)$ signal is process noise. Notice that it affects the dynamics of the model by making direct changes to the evolution of $x(t)$.
- The $v(t)$ signal is sensor noise. Notice that it does not affect the dynamics of the model; it affects only the measurement $z(t)$.
- Systems having noise inputs $w(t)$ and $v(t)$ are considered in detail in week 3.

What are the matrices called?

- Standard form for continuous-time linear state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t).$$

- $A \in \mathbb{R}^{n \times n}$ is the system matrix. It models the evolution of the state in the absence of input.
- $B \in \mathbb{R}^{n \times r}$ is the input matrix. It defines how linear combinations of $u(t)$ impact the evolution of the state.
- $C \in \mathbb{R}^{m \times n}$ is the output matrix. It defines how the output depends on linear combinations of states.
- $D \in \mathbb{R}^{m \times r}$ is the feedforward (or feedthrough) matrix. It models how the output depends on linear combinations of the input (instantaneously).
- Time-varying systems have A, B, C, D that change with time.

Why do you need to know about them?

- It can be of vital interest to know the state of the system, but we usually cannot measure it directly.
- Instead, we *can* measure $u(t)$ and $z(t)$, and although we cannot measure the random signals $w(t)$ and $v(t)$, we can model some of their critical attributes.
- This enables us to make methods to estimate $x(t)$ based on what we measure and what we model.
- Kalman filters are (in some cases optimal) estimators of $x(t)$.
  - The derivation and implementation of the KF depends on modeling the system in state-space form and having knowledge of the model A, B, C, and D matrices.
  - This is why we care!

Summary

- State-space models are a compact representation of an nth-order linear system in terms of a 1st-order vector ODE.
- The standard form for linear continuous-time state-space models is:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$z(t) = Cx(t) + Du(t) + v(t),$$

- You have learned the names of the equations, the names of the matrices, and the names of the signals (and what they all mean).
- You have seen one example of a model in state-space form; in the next lesson, you will be introduced to three other important general models.