Deconstructing the matrix exponential

- Have seen the key role of e^{At} in the solution for $x(t)$.
- Impacts the system response, but need more insight.
- Consider what happens if the matrix A is *diagonalizable,* that is, there exists a matrix T such that $T^{-1}AT = \Lambda$ =diagonal. Then,

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ &= I + T\Lambda T^{-1}t + \frac{T\Lambda T^{-1}T\Lambda T^{-1}t^2}{2!} + \frac{T\Lambda T^{-1}T\Lambda T^{-1}T\Lambda T^{-1}t^3}{3!} + \cdots \\ &= T\left[I + \Lambda t + \frac{\Lambda^2t^2}{2!} + \frac{\Lambda^3t^3}{3!} + \cdots\right]T^{-1} = Te^{\Lambda t}T^{-1}, \end{aligned}$$

and

$$e^{\Lambda t} = \operatorname{diag}\left(\; e^{\lambda_1 t}, \quad e^{\lambda_2 t}, \quad \ldots \quad e^{\lambda_n t} \;\right).$$

How to find the transformation matrix

- Much simpler form for the exponential, but how to find T, Λ?
- Write $T^{-1}AT = \Lambda$ as $T^{-1}A = \Lambda T^{-1}$ with

$$T^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}, \qquad i.e., w_j^T \text{ are the rows of } T^{-1}.$$

  - We have: $w_i^T A = \lambda_i w_i^T$, so w_i is a *left eigenvector* of A.
- Can also write $T^{-1}AT = \Lambda$ as $AT = \Lambda T$ with

$$T = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}, \qquad i.e., v_j \text{ are the columns of } T.$$

  - We have $Av_i = \lambda_i v_i$, so v_i is a *right eigenvector of* A.
- Also, since $T^{-1}T = I$, we have that $w_i^T v_j = \delta_{i,j}$.

How the deconstruction simplifies things

- So, T comprises the eigenvectors of A. How does this help?

$$\begin{aligned} e^{At} &= Te^{\Lambda t}T^{-1} \\ &= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_{n^T} \end{bmatrix} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T. \end{aligned}$$

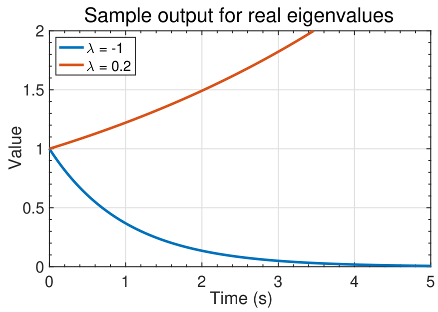
- Very simple form, which can be used to develop intuition about dynamic response:

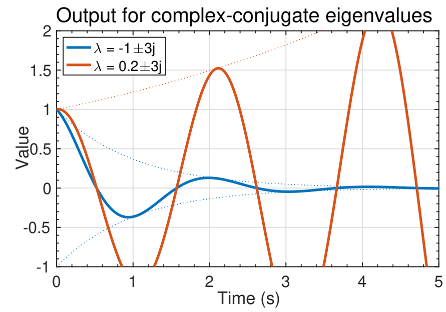
$$x(t) = e^{At}x(0) = Te^{\Lambda t}T^{-1}x(0) = \sum_{i=1}^{n} e^{\lambda_i t} v_i (w_i^T x(0)).$$

- Notice that the only time-varying components are $e^{\lambda_i t}$, which are then multiplied by constant matrices so that the states are linear combinations of $e^{\lambda_i t}$.

Visualizing $e^{\lambda_i t}$

- The signal $e^{\lambda_i t}$ can take on several different characteristics, as shown in the figures below.
  - If λ_i is real, we observe a smooth monotonically growing or decaying signal.
  - If λ_i occur in complex-conjugate pairs, the output is of the pair is of the form $e^{\mathbb{R}(\lambda_i)t}\cos(\mathbb{I}(\lambda_i)t)$: a cosine with an exponential "envelope."

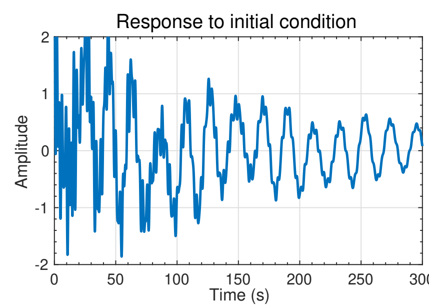




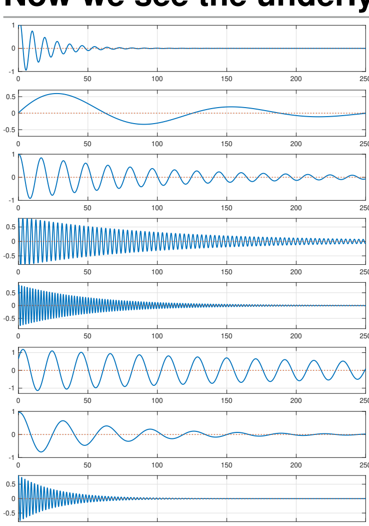
An example to illustrate the complexity

- Consider again: $x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i (w_i^T x(0))$.
- Can be expressed as a linear combination of modes: $v_i e^{\lambda_i t}$.
- Left eigenvectors decompose $x(0)$ into modal coordinates $w_i^T x(0)$.
- $e^{\lambda_i t}$ propagates mode forward in time.

- **EXAMPLE:** Consider a specific system with $x(t) \in \mathbb{R}^{16\times 1}$, $z(t) \in \mathbb{R}$ (16-state, 1-output):
$$\dot{x}(t) = Ax(t)$$
$$z(t) = Cx(t).$$
- A lightly damped system, for which typical output to initial conditions is shown.
- Waveform is complicated. Appears random.



Now we see the underlying simplicity



- However, the solution can be decomposed into much simpler modal components.
- The waveforms to the left show $e^{-\lambda_i t}$ for complex-conjugate pairs $\{\lambda_i, \lambda_i^*\}$ summed together.
- All are decaying sinusoids with different magnitudes, frequencies, and phase shifts.
- When summed together, the result appears complicated; but the individual components of the response are simple.

Summary

- The matrix exponential is key to understanding the response of a state-space model.
- Since it is likely an unfamiliar concept to you until this point, it is valuable to develop intuition regarding what it looks like.
- You have learned that we can (almost always)[1] write $e^{At} = Te^{\Lambda t}T^{-1}$, where the columns of T are the eigenvectors of A.
- All the dynamics are contained in $e^{\Lambda t}$, which turn out to be the standard scalar exponentials of (possibly complex) $\lambda_i t$ terms.
- So, the matrix exponential is a linear sum of exponentially-enveloped sinusoids.
- The frequencies of the sinusoids are the eigenvalues of A. The eigenvectors specify the relative weighting of each component to the overall output.

[1] If we cannot diagonalize A, there is a more general "Jordan form" for which the ideas presented in this lesson apply with a few modifications... basically multiplying the sinusoids by polynomials.