Cross-correlated w_k and v_k: Coincident

- The standard KF assumes that $\mathbb{E}[w_k v_j^T] = 0$. But, sometimes we may encounter systems where this is not the case.
- This might happen if both the physical process and the measurement system are affected by the same source of disturbance.
  - Examples are changes of temperature, or inductive electrical interference.
- We now assume: $\mathbb{E}[w_k w_j^T] = \Sigma_{\widetilde{w}}\delta_{kj}$, $\mathbb{E}[v_k v_j^T] = \Sigma_{\widetilde{v}}\delta_{kj}$, and $\mathbb{E}[w_k v_j^T] = \Sigma_{\widetilde{w}\widetilde{v}}\delta_{kj}$.
- Note that the correlation between noises is memoryless: the only cross correlation between w_k and v_k is at identical time instants.
- We can modify the Kalman-filter derivation to address this case if we re-write the state equation so that it has a new process noise that is uncorrelated with the measurement noise.

Setting up the solution approach

- Using an arbitrary matrix T (to be determined), we can write:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + w_k + T\underbrace{(z_k - C_k x_k - D_k u_k - v_k)}_{\text{We added zero in a clever way!}} \\ &= (A_k - TC_k)x_k + (B_k - TD_k)u_k + w_k - Tv_k + Tz_k. \end{aligned}$$

- Denote the new state-transition matrix $\overline{A}_k = A_k - TC_k$, new input matrix as $\overline{B}_k = B_k - TD_k$, and the new process noise as $\overline{w}_k = w_k - Tv_k$.
- Also, denote the known (measured/computed) sequence as a new input $\overline{u}_k = Tz_k$.
- Then, we can write a modified state equation:

$$x_{k+1} = \overline{A}_k x_k + \overline{B}_k u_k + \overline{u}_k + \overline{w}_k.$$

Enforcing no correlation between $\overline{w}_k$ and v_k

- We can create a Kalman filter for this model, provided that the cross-correlation between the new process noise $\overline{w}_k$ and the sensor noise v_k is zero. We enforce this:

$$\mathbb{E}[\overline{w}_k v_k^T] = \mathbb{E}\left[[w_k - Tv_k]v_k^T\right] = \Sigma_{\widetilde{w}\widetilde{v}} - T\Sigma_{\widetilde{v}} = 0.$$

- This gives us that the previously unspecified matrix $T = \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}$.
- Using the above, the covariance of the new process noise may be found to be:

$$\begin{aligned} \Sigma_{\widetilde{\overline{w}}} &= \mathbb{E}[\overline{w}_k \overline{w}_k^T] \\ &= \mathbb{E}\left[\left[w_k - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}v_k\right]\left[w_k - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}v_k\right]^T\right] \\ &= \Sigma_{\widetilde{w}} - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}\Sigma_{\widetilde{w}\widetilde{v}}^T. \end{aligned}$$

The model to use in the Kalman filter

- A new Kalman filter may be generated using these definitions:

$$\begin{aligned}\bar{A}_k &= A_k - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}C_k \\ \Sigma_{\widetilde{\overline{w}}} &= \Sigma_{\widetilde{w}} - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}\Sigma_{\widetilde{w}\widetilde{v}}^T,\end{aligned}$$

which implies the following state-space model:

$$\begin{aligned}x_{k+1} &= \underbrace{(A_k - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}C_k)}_{\bar{A}_k} x_k + \underbrace{(B_k - \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}D_k)u_k + \Sigma_{\widetilde{w}\widetilde{v}}\Sigma_{\widetilde{v}}^{-1}z_k}_{\text{deterministic input}} + \overline{w}_k \\ z_k &= C_k x_k + D_k u_k + v_k.\end{aligned}$$

- The modified process noise $\overline{w}_k$ and the sensor noise v_k are not correlated, so this set of definitions satisfies the requirements of the Kalman filter.

Cross-correlated w_k and v_k: Shifted

- A close relation to the preceding analysis is when the process and sensor noise have correlation one timestep apart.
- That is, $\mathbb{E}[w_k w_j^T] = \Sigma_{\widetilde{w}}\delta_{kj}$, $\mathbb{E}[v_k v_j^T] = \Sigma_{\widetilde{v}}\delta_{kj}$, and $\mathbb{E}[w_k v_j^T] = \Sigma_{\widetilde{w}\widetilde{v}}\delta_{k,j-1}$.
  - The cross-correlation is nonzero only between w_{k-1} and v_k.
- We can re-derive the Kalman-filter equations using this assumption. We will find that the differences show up in the state-error covariance terms.
- The state prediction error is:

$$\tilde{x}_k^- = x_k - \hat{x}_k^- = A_k\tilde{x}_{k-1}^+ + w_{k-1}.$$

- With the assumptions of this section, the covariance between the state prediction error and the measurement noise is:

$$\mathbb{E}[\tilde{x}_k^- v_k^T] = \mathbb{E}\left[[A_k\tilde{x}_{k-1}^+ + w_{k-1}]v_k^T\right] = \Sigma_{\widetilde{w}\widetilde{v}}.$$

The solution to correlated shifted w_k and v_k

- The covariance between the state and measurement becomes:

$$\begin{aligned}\mathbb{E}\left[\tilde{x}_k^-\tilde{z}_k^T \mid \mathbb{Z}_{k-1}\right] &= \mathbb{E}\left[\tilde{x}_k^-\left(C_k\tilde{x}_k^- + v_k\right)^T \mid \mathbb{Z}_{k-1}\right] \\ &= \Sigma_{\tilde{x},k}^- C_k^T + \Sigma_{\widetilde{w}\widetilde{v}}.\end{aligned}$$

- The measurement prediction covariance becomes:

$$\begin{aligned}\Sigma_{\tilde{z},k} &= \mathbb{E}\left[\tilde{z}_k\tilde{z}_k^T\right] = \mathbb{E}\left[[C_k\tilde{x}_k^- + v_k][C_k\tilde{x}_k^- + v_k]^T\right] \\ &= C_k\Sigma_{\tilde{x},k}^- C_k^T + \Sigma_{\widetilde{v}} + C_k\Sigma_{\widetilde{w}\widetilde{v}} + \Sigma_{\widetilde{w}\widetilde{v}}^T C_k^T.\end{aligned}$$

- The modified Kalman-filter estimator gain then becomes:

$$L_k = \left[\Sigma_{\tilde{x},k}^- C_k^T + \Sigma_{\widetilde{w}\widetilde{v}}\right]\left(C_k\Sigma_{\tilde{x},k}^- C_k^T + \Sigma_{\widetilde{v}} + C_k\Sigma_{\widetilde{w}\widetilde{v}} + \Sigma_{\widetilde{w}\widetilde{v}}^T C_k^T\right)^{-1}.$$

- Except for the modified filter gain, all of the Kalman-filter equations are the same as in the standard case.

Summary

- You have now learned how to modify the Kalman-filter equations when process and sensor noises are correlated.
- We considered two cases: The first where $\mathbb{E}[w_k v_j^T] = \Sigma_{\widetilde{w}\tilde{v}}\delta_{kj}$, and the second where $\mathbb{E}[w_k v_j^T] = \Sigma_{\widetilde{w}\tilde{v}}\delta_{k,j-1}$.
- Note that the Kalman filter operates over the interval $[t_{k-1}, t_k]$ at iteration k:
  - w_{k-1} is the process noise at t_{k-1}, corresponding to the beginning of the interval.
  - w_k is the process noise at t_k, corresponding to the end of the interval.
  - v_j is the measurement noise at t_j, corresponding to the end of the interval.
- So, the first case considered process noise correlated with measurement noise at the beginning of the above interval, and the second case considered process noise correlated with the end of the interval.
- Both cases can be accommodated via simple modifications to the standard Kalman-filter equations.