The interacting-multiple-model Kalman filter

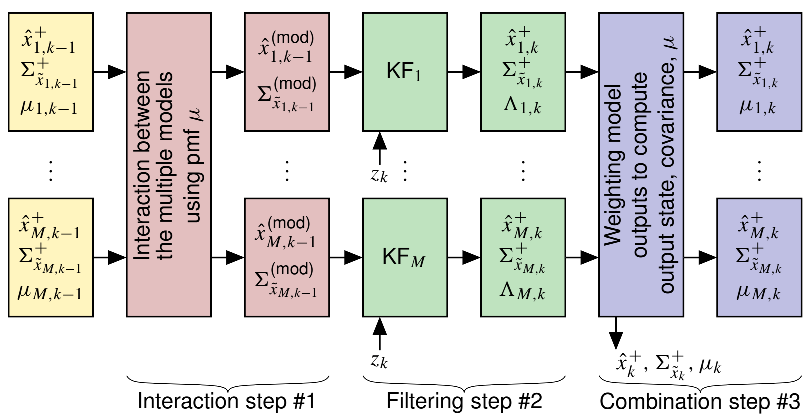
- A target that we wish to track may have multiple distinct modes of operation, where each mode has its own model.
  - For example, a target may act like NCV for a period of time, then NCP, then follow a coordinated turn, etc. In general, we consider modes $1 \dots M$.
- The interacting-multiple-model (IMM) Kalman filter operates M KFs in parallel, one for each mode, where each KF uses its mode's own dynamic model.[1]
- The IMM carefully blends state $\hat{x}^+_{j,k}$ and covariance $\Sigma^+_{\tilde{x},j,k}$ from each filter to make a composite state estimate $\hat{x}^+_k$ and covariance $\Sigma^+_{\tilde{x},k}$ as well as a pmf μ_k corresponding to the filter's belief that the target is operating in modes $1 \dots M$.

[1] A good reference is: Mazor, E., Averbuch, A., Bar-Shalom, Y., and Dayan, J., "Interacting multiple model methods in target tracking: A survey," *IEEE Trans. Aerospace and Electronic Systems,* 34(1), Jan. 1998, 103–123.

The three steps of the IMM

- The IMM repeatedly executes the following three steps per measurement interval:

1. Interaction between the filters;
2. Individual filter update; and
3. Combination of filter information to create the IMM output information.



Model requirements of the IMM

- To implement an IMM, we require M state-space models, one for each mode of operation.
  - Each model has its own A, B, C, and D matrices;
  - It also has its own $\Sigma_{\widetilde{w}}$ and $\Sigma_{\tilde{v}}$ matrices.
- An additional modeling parameter required by IMM is a matrix containing the probabilities of transitioning from mode i to mode j between timesteps (where m_k is the mode at time k):

$$p_{ij} = \Pr(m_k = j \mid m_{k-1} = i).$$

- The overall multiple-model format used by the IMM is basically a state machine, where p_{ij} is the set of transition probabilities.
- Okay! With this background, we can start deriving the steps of the IMM, beginning with the interaction step.

Interaction step: Arrival probabilities

- The prior outputs from the M independent KFs are blended to produce the inputs to the M KFs for this time step.
- We first define the conditional probability of arriving to mode j from mode i:

$$\mu_{i|j,k-1} = \Pr(m_{k-1} = i \mid m_k = j, \mathbb{Z}_{k-1}).$$

- To simplify, apply a version of Bayes' rule, $\Pr(A \mid B, C) = \frac{\Pr(B|A,C)\Pr(A|C)}{\Pr(B|C)}$:

$$\begin{aligned}\mu_{i|j,k-1} &= \Pr(m_{k-1} = i \mid m_k = j, \mathbb{Z}_{k-1}) \\ &= \frac{\Pr(m_k = j \mid m_{k-1} = i, \mathbb{Z}_{k-1})\Pr(m_{k-1} = i \mid \mathbb{Z}_{k-1})}{\Pr(m_k = j \mid \mathbb{Z}_{k-1})} \\ &= \frac{1}{\bar{c}_j} p_{ij} \Pr(m_{k-1} = i \mid \mathbb{Z}_{k-1}) = \frac{1}{\bar{c}_j} p_{ij}\mu_{i,k-1},\end{aligned}$$

where $\bar{c}_j = \Pr(m_k = j \mid \mathbb{Z}_{k-1}) = \sum_{i=1}^{M} p_{ij}\mu_{i,k-1}$.

Interaction step: Prior-step modified state estimate

- Interaction blends together all state outputs from the prior step according to the probability of their contribution to the state at this time step.
- The modified input state to filter j at time step k is

$$\begin{aligned}\hat{x}^{(\text{mod})}_{j,k-1} &= \mathbb{E}[x_{k-1} \mid m_k = j, \mathbb{Z}_{k-1}] \\ &= \sum_{i=1}^{M} \hat{x}^+_{i,k-1} \Pr(m_{k-1} = i \mid m_k = j, \mathbb{Z}_{k-1}) = \sum_{i=1}^{M} \hat{x}^+_{i,k-1}\mu_{i|j,k-1}.\end{aligned}$$

  - The second line used law of total (or iterated) expectation:

$$\mathbb{E}[X \mid A, B] = \sum_{i=1}^{M} \mathbb{E}[X \mid C, A, B]\Pr(C \mid A, B).$$

Interaction step: Prior-step modified covariance

- We now begin to derive the covariance of $\hat{x}^{(\text{mod})}_{j,k-1}$, starting with the steps:

$$\begin{aligned}\Sigma^{(\text{mod})}_{\tilde{x}_{j,k-1}} &= \mathbb{E}\Big[(x_{k-1} - \hat{x}^{(\text{mod})}_{j,k-1})(x_{k-1} - \hat{x}^{(\text{mod})}_{j,k-1})^T \mid m_k{=}j, \mathbb{Z}_{k-1}\Big] \\ &= \sum_{i=1}^{M} \mathbb{E}\Big[(x_{k-1} - \hat{x}^{(\text{mod})}_{j,k-1})(x_{k-1} - \hat{x}^{(\text{mod})}_{j,k-1})^T \mid m_{k-1}{=}i, m_k{=}j, \mathbb{Z}_{k-1}\Big]\mu_{i|j,k-1} \\ &= \sum_{i=1}^{M} \mathbb{E}\Bigg[\bigg((x_{k-1} - \hat{x}^+_{i,k-1}) + (\hat{x}^+_{i,k-1} - \hat{x}^{(\text{mod})}_{j,k-1})\bigg)\times \\ &\qquad \bigg((x_{k-1} - \hat{x}^+_{i,k-1}) + (\hat{x}^+_{i,k-1} - \hat{x}^{(\text{mod})}_{j,k-1})\bigg)^T \mid m_{k-1}{=}i, m_k{=}j, \mathbb{Z}_{k-1}\Bigg]\mu_{i|j,k-1}.\end{aligned}$$

- The second line again used the law of total expectation. The third added zero.

Interaction step covariance derivation (cont.)

- Define $\mathrm{cond} = \{m_{k-1}{=}i, m_k{=}j, \mathbb{Z}_{k-1}\}$; apply "FOIL":

$$\begin{aligned}\Sigma^{(\mathrm{mod})}_{\tilde{x}_{j,k-1}} &= \sum_{i=1}^{M} \mathbb{E}\Big[(x_{k-1}-\hat{x}^+_{i,k-1})(x_{k-1}-\hat{x}^+_{i,k-1})^T \mid \mathrm{cond}\Big]\mu_{i\mid j,k-1}\\ &\quad+\sum_{i=1}^{M}(\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1})\underbrace{\mathbb{E}\Big[x_{k-1}-\hat{x}^+_{i,k-1}\mid \mathrm{cond}\Big]}_{0}\mu_{i\mid j,k-1}\\ &\quad+\sum_{i=1}^{M}\underbrace{\mathbb{E}\Big[x_{k-1}-\hat{x}^+_{i,k-1}\mid \mathrm{cond}\Big]}_{0}(\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1})^T\mu_{i\mid j,k-1}\\ &\quad+\sum_{i=1}^{M}(\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1})(\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1})^T\mu_{i\mid j,k-1}.\end{aligned}$$

Interaction step covariance final result

- Deleting the zero terms and collapsing:

$$\begin{aligned}\Sigma^{(\mathrm{mod})}_{\tilde{x}_{j,k-1}} &= \sum_{i=1}^{M} \mathbb{E}\Big[(x_{k-1}-\hat{x}^+_{i,k-1})(x_{k-1}-\hat{x}^+_{i,k-1})^T \mid \mathrm{cond}\Big]\mu_{i\mid j,k-1}\\ &\quad+\sum_{i=1}^{M}[\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1}][\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1}]^T\mu_{i\mid j,k-1}\\ &= \sum_{i=1}^{M}\Big\{\Sigma^+_{\tilde{x}_{i,k-1}}+[\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1}][\hat{x}^+_{i,k-1}-\hat{x}^{(\mathrm{mod})}_{j,k-1}]^T\Big\}\mu_{i\mid j,k-1}.\end{aligned}$$

- The first term blends the prior covariances, and the second term is due to the "spread of the means" of the individual filters.

Filtering

- After the interaction step, one time step for each of the M Kalman filters is then executed.
- Each filter outputs:
  - The present state estimate $\hat{x}^+_{j,k}$ for that mode,
  - The estimation-error covariance $\Sigma^+_{\tilde{x}_{j,k}}$ for that mode, as well as
  - The likelihood of the present measurement given that the target is in mode j:

$$\begin{aligned}\Lambda_{j,k} &= \mathcal{N}(z_k-\hat{z}_{j,k}, \Sigma_{\tilde{z}_{j,k}})\\ &= \frac{1}{(2\pi)^{n/2}\left|\Sigma_{\tilde{z}_{j,k}}\right|^{1/2}}\exp\left(-\frac{1}{2}(z_k-\hat{z}_{j,k})^T\Sigma^{-1}_{\tilde{z}_{j,k}}(z_k-\hat{z}_{j,k})\right).\end{aligned}$$

Combination

- The final step is to combine the results of the M filters to get an overall target state estimate with corresponding estimation-error covariance, as well as a probability distribution on the mode random variable.
- We first compute the *a posteriori* probability of being in mode j at time k:

$$\begin{aligned}\mu_{j,k} &= \Pr(m_k = j \mid \mathbb{Z}_k) = \Pr(m_k = j \mid \mathbb{Z}_{k-1}, z_k)\\ &= f(z_k \mid m_k = j, \mathbb{Z}_{k-1}) \Pr(m_k = j \mid \mathbb{Z}_{k-1}) / \Pr(z_k)\\ &= \frac{1}{c}\Lambda_{j,k} \sum_{i=1}^{M} \Pr(m_k = j \mid m_{k-1} = i, \mathbb{Z}_{k-1}) \Pr(m_{k-1} = i \mid \mathbb{Z}_{k-1})\\ &= \frac{1}{c}\Lambda_{j,k} \sum_{i=1}^{M} p_{ij}\mu_{i,k-1} = \frac{1}{c}\Lambda_{j,k}\bar{c}_j,\end{aligned}$$

where c is a normalizing constant to ensure that $\mu_{j,k}$ sums to 1.

Filter output

- Then, the filter output is computed as a mixture of the individual filter estimates, according to the likelihood of the target being in the mode of that filter:

$$\hat{x}_k^+ = \sum_{j=1}^{M} \hat{x}_{j,k}^+ \mu_{j,k}$$

$$\Sigma_{\tilde{x},k}^+ = \sum_{j=1}^{M} \left\{ \Sigma_{\tilde{x}_{j,k}}^+ + [\hat{x}_{j,k}^+ - \hat{x}_k^+][\hat{x}_{j,k}^+ - \hat{x}_k^+]^T \right\} \mu_{j,k}.$$

- The overall state estimate is a weighted combination of the individual filter state estimates.
- The overall estimation-error covariance is a weighted combination of the individual filter covariance estimates, plus a "spread of means" term.

Summary

- The interacting multiple model Kalman filter is one approach to tracking a target that might have multiple different dynamic modes of operation.
- Each mode of operation has its own dynamic state-space model.
- The IMM executes one KF per model, but also intelligently combines prior IMM outputs when initializing each KF every time step, and intelligently blends the individual KF outputs when forming the overall output.
- The overall output is a combined state estimate $\hat{x}_k^+$, a combined covariance $\Sigma_{\tilde{x},k}^+$, and a pmf $\mu_{j,k}$ describing the probability that the target is operating in each mode.
- In the next lesson, you will learn how to code the IMM in Octave, and will gain some intuition regarding its performance from an example.