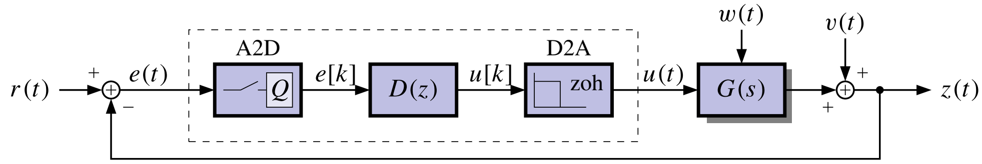
Discrete-time state-space models

- Computer monitoring of real-time systems requires analog-to-digital (A2D) and digital-to-analog (D2A) conversion.



- Linear discrete-time systems can also be represented in state-space form:

$$x_{k+1} = A_d x_k + B_d u_k + w_k$$
$$z_k = C_d x_k + D_d u_k + v_k.$$

- The subscript "d" emphasizes that, in general, the "A", "B", "C" and "D" matrices are <u>different</u> for the same discrete-time and continuous-time system.
- I will usually drop the "d" and expect you to interpret the system from its context.

1.2.5: Converting continuous-time state-space models to discrete-time

Time (dynamic) response of discrete-time models

- The full solution, via induction from $x_{k+1} = A_d x_k + B_d u_k$, is:

$$x_k = A_d^k x_0 + \underbrace{\sum_{j=0}^{k-1} A_d^{k-1-j} B_d u_j}_{\text{convolution}}.$$

- Since $z_k = C_d x_k + D_d u_k$, we also have:

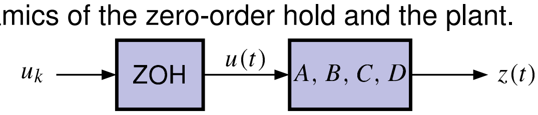
$$z_k = \underbrace{C_d A_d^k x_0}_{\text{initial resp.}} + \underbrace{\sum_{j=0}^{k-1} C_d A_d^{k-1-j} B_d u_j}_{\text{convolution}} + \underbrace{D_d u_k}_{\text{feedthrough}}.$$

- Comparing with the continuous-time solution, e^{At} has been replaced by A_d^k and integrals have been replaced by summations.

1.2.5: Converting continuous-time state-space models to discrete-time

Converting plant dynamics to discrete time (1)

- Combine the dynamics of the zero-order hold and the plant.



- Recall that the continuous-time state dynamics of the plant are:

$$\dot{x}(t) = Ax(t) + Bu(t).$$

- Evaluate $x(t)$ at discrete times. Recall also the solution for $x(t)$:[1]

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau$$
$$x_{k+1} = x((k+1)\Delta t) = \int_0^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau.$$

[1] For simplicity, we assume that $x(0) = 0$. The final solution turns out to be true even if $x(0) \neq 0$.

Converting plant dynamics to discrete time (2)

- We break up the integral into two pieces:

$$\begin{aligned} x_{k+1} &= \int_0^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau. \\ &= \int_0^{k\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau + \int_{k\Delta t}^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau \\ &= \int_0^{k\Delta t} e^{A\Delta t} e^{A(k\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau + \int_{k\Delta t}^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau \\ &= e^{A\Delta t} x(k\Delta t) + \int_{k\Delta t}^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau. \end{aligned}$$

- The first integral has become A_d times $x(k\Delta t)$.
- The second will become B_d times $u(k\Delta t)$ after a little more work.

Converting plant dynamics to discrete time (3)

- In the remaining integral, note that $u(\tau)$ is assumed to be constant from $k\Delta t$ to $(k+1)\Delta t$, and equal to $u(k\Delta t)$.
- We evaluate the integral via change of variables. Let $\sigma = (k+1)\Delta t - \tau$; $\tau = (k+1)\Delta t - \sigma$; $\mathrm{d}\tau = -\mathrm{d}\sigma$.

$$\begin{aligned} x_{k+1} &= e^{A\Delta t} x(k\Delta t) + \int_{k\Delta t}^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)} Bu(\tau)\,\mathrm{d}\tau \\ &= e^{A\Delta t} x(k\Delta t) + \left[\int_0^{\Delta t} e^{A\sigma} B\,\mathrm{d}\sigma\right] u(k\Delta t) \\ &= \underbrace{e^{A\Delta t}}_{A_d} x_k + \underbrace{\left[\int_0^{\Delta t} e^{A\sigma} B\,\mathrm{d}\sigma\right]}_{B_d} u_k. \end{aligned}$$

Computing A_d, C_d, and D_d

- Summarizing to this point, we have a discrete-time state-space representation from the continuous-time representation:

$$x_{k+1} = A_d x_k + B_d u_k,$$

  where $A_d = e^{A\Delta t}$ and $B_d = \int_0^{\Delta t} e^{A\sigma} B\,\mathrm{d}\sigma$.
- Similarly, we have:

$$z_k = Cx_k + Du_k.$$

  where $C_d = C$ and $D_d = D$.
- So, there is no conversion for the C and D matrices, and the conversion for A is straightforward via the matrix exponential $A_d = e^{A\Delta t}$. This is different from taking the exponential of each element in $A\Delta t$.
- If Octave is handy, you can type in: `Ad = expm(A*dT)`. Otherwise, you will need to compute $e^{A\Delta t} = \mathcal{L}^{-1}[(sI-A)^{-1}]\big|_{t=\Delta t}$.

Computing B_d

- Now we focus on computing B_d. Recall that

$$\begin{aligned} B_d &= \int_0^{\Delta t} e^{A\sigma} B \, d\sigma \\ &= \int_0^{\Delta t} \left(I + A\sigma + A^2 \frac{\sigma^2}{2} + \ldots \right) B \, d\sigma \\ &= \left(I \Delta t + A \frac{\Delta t^2}{2!} + A^2 \frac{\Delta t^3}{3!} + \ldots \right) B \\ &= A^{-1}(e^{A\Delta t} - I)B = A^{-1}(A_d - I)B. \end{aligned}$$

- If A is invertible, this method works nicely; otherwise, we will need to perform the integral in the first line manually.
- Also, in Octave, `[Ad,Bd]=c2d(A,B,dT)`.

Summary

- Computer monitoring of physical systems requires A2D and D2A conversion.
- The system that is "seen" by the computer is a discrete-time system, and must be represented by a discrete-time state-space model:

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k + w_k \\ z_k &= C_d x_k + D_d u_k + v_k. \end{aligned}$$

- In general, the "A", "B", "C" and "D" matrices are different for the same discrete-time and continuous-time system.
- You have learned how to convert from the original "A", "B", "C" and "D" matrices to the discrete-time "A_d", "B_d", "C_d" and "D_d" matrices.
- In the next lesson, you will see some examples of converting continuous-time to discrete-time and of simulating discrete-time state-space models.