Auto-correlated process noise

- Another common situation that contradicts KF assumptions is when the process noise is correlated in time.
- That is, we have assumed that $x_{k+1} = A_k x_k + B_k u_k + w_k$, where w_k is a zero-mean white-noise process; yet, in our application w_k is zero-mean but is not white.
- We model the correlation of the process noise using a state-space model:

$$w_k = A_w w_{k-1} + \bar{w}_{k-1},$$

  where $\bar{w}_{k-1}$ is a zero-mean white-noise process.
- Determining A_w is beyond the scope of this lesson, but I will mention that many “system-identification” toolboxes estimate both the deterministic parts of the model (i.e., A, B, C, and D) as well as the stochastic parts (including A_w).

Handling auto-correlated process noise

- We handle this situation by estimating both the true system state x_k and also the process-noise state w_k. We have:

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix} = \begin{bmatrix} A_{k-1} & I \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x_{k-1} \\ w_{k-1} \end{bmatrix} + \begin{bmatrix} B_{k-1} \\ 0 \end{bmatrix} u_{k-1} + \begin{bmatrix} 0 \\ \bar{w}_{k-1} \end{bmatrix}$$

  (i.e., $x_k^* = A_{k-1}^* x_{k-1}^* + B_{k-1}^* u_{k-1} + w_{k-1}^*$)

$$z_k = \begin{bmatrix} C_k & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} + D_k u_k + v_k \quad \text{(i.e., } z_k = C_k^* x_k^* + D_k u_k + v_k\text{)},$$

  where the overall process noise covariance is:

$$\Sigma_{\widetilde{w}^*} = \mathbb{E}[w_{k-1}^* (w_{k-1}^*)^T] = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{E}[\bar{w}_{k-1} (\bar{w}_{k-1})^T] \end{bmatrix}.$$

- A standard Kalman filter may now be designed using the definitions of x_k^*, A_k^*, B_k^*, C_k^*, D_k, $\Sigma_{\widetilde{w}^*}$, and $\Sigma_{\widetilde{v}}$.

Auto-correlated sensor noise

- Similarly, we might encounter situations with auto-correlated sensor noise: $v_k = A_v v_{k-1} + \bar{v}_{k-1}$, where $\bar{v}_k$ is white.
- We take a similar approach. The augmented state equation is:

$$\begin{bmatrix} x_k \\ v_k \end{bmatrix} = \begin{bmatrix} A_{k-1} & 0 \\ 0 & A_v \end{bmatrix} \begin{bmatrix} x_{k-1} \\ v_{k-1} \end{bmatrix} + \begin{bmatrix} B_{k-1} \\ 0 \end{bmatrix} u_{k-1} + \begin{bmatrix} w_{k-1} \\ \bar{v}_{k-1} \end{bmatrix}$$

$$x_k^* = A_{k-1}^* x_{k-1}^* + B_{k-1}^* u_{k-1} + w_{k-1}^*$$

  with output equation:

$$\begin{aligned} z_k &= \begin{bmatrix} C_k & I \end{bmatrix} \begin{bmatrix} x_k \\ v_k \end{bmatrix} + D_k u_k + 0 \\ &= C_k^* x_k^* + D_k u_k + 0 \end{aligned}$$

Measurement differencing

- The covariance of the combined process noise is:

$$\Sigma_{\widetilde{w}^*} = \mathbb{E}\left[\left(\begin{array}{c} w_k \\ \bar{v}_k \end{array}\right)\left(\begin{array}{cc} w_k^T & \bar{v}_k^T \end{array}\right)\right] = \left[\begin{array}{cc} \Sigma_{\widetilde{w}} & 0 \\ 0 & \Sigma_{\tilde{\bar{v}}} \end{array}\right].$$

- A KF may be designed using these new definitions of of x_k^*, A_k^*, B_k^*, C_k^*, D_k, $\Sigma_{\widetilde{w}^*}$ (the placeholder for measurement noise $\Sigma_{\tilde{v}} = 0$ in the above formulations).

Measurement differencing

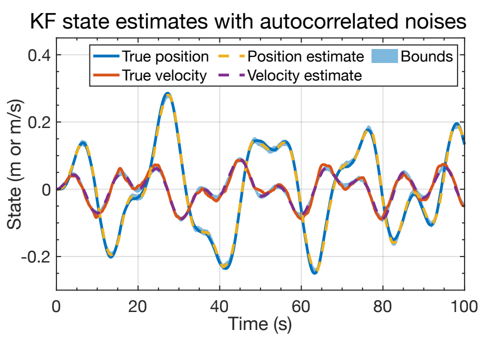
- Zero-covariance measurement noise $\Sigma_{\tilde{v}}$ can cause numerical issues in Step 2a.
- This can be corrected by redeveloping the KF equations using a synthetic “measurement” $\bar{z}_k = z_{k+1} - A_v z_k$. I will omit the (complicated) details here.[1]
- Care must be taken to deal with the “future” measurement z_{k+1} in the update equations, but it works out to a causal solution in the end.

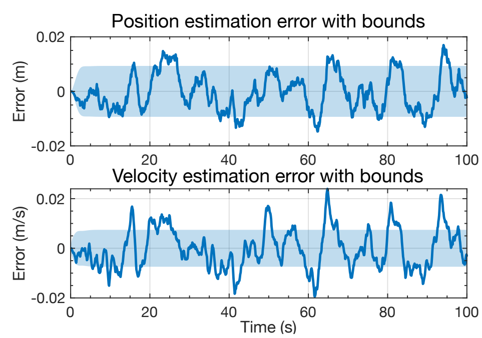
[1]Confer: Y. Bar–Shalom, X. Rong Li, and T. Kirubarajan. *Estimation with applications to tracking and navigation: theory algorithms and software*. John Wiley & Sons, 2001.

2.2.6: What about when the process and/or sensor noises are not white?

Example with correlated w_k

- In the example on slide 5 of Lesson 1.4.5, neither w_k nor v_k were white (both were autocorrelated).
- Here, I separate the effects. The example on this slide shows KF output when the dataset had autocorrelated w_k but white v_k.

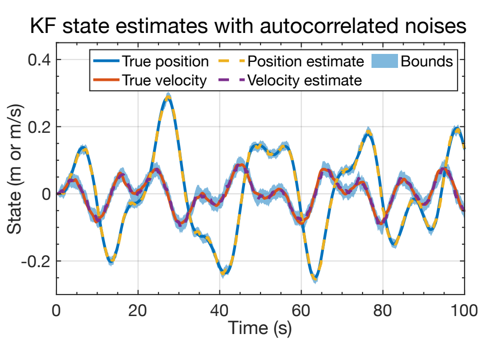


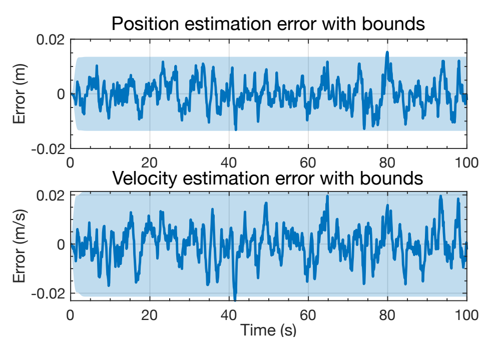


2.2.6: What about when the process and/or sensor noises are not white?

Example with correlated w_k (corrected)

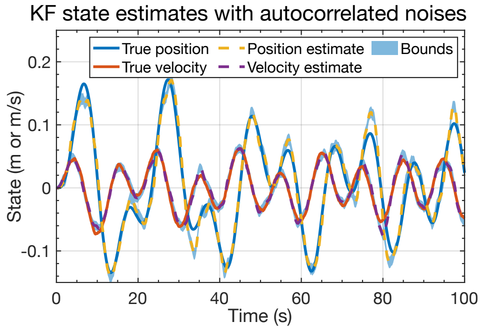
- Now, for the same dataset, I use the approach from this lesson to build an augmented A^* matrix, where $A_w = \left[\begin{array}{cc} 0.8 & 0 \\ 0 & 0.8 \end{array}\right]$.
- The output of the KF using augmented dynamics is greatly improved.

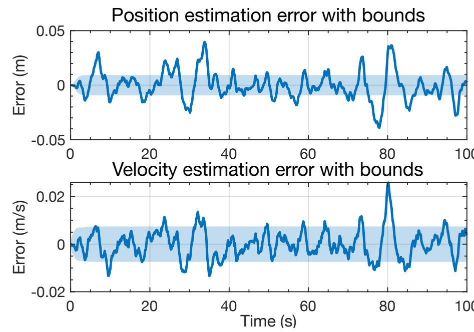




Example with correlated v_k

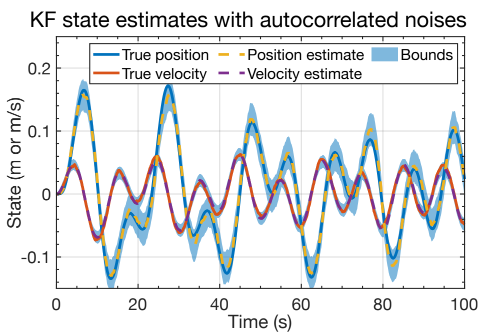
- The example on this slide shows KF output when the dataset had autocorrelated v_k but white w_k.
- Again, the unmodified KF produces poor results.

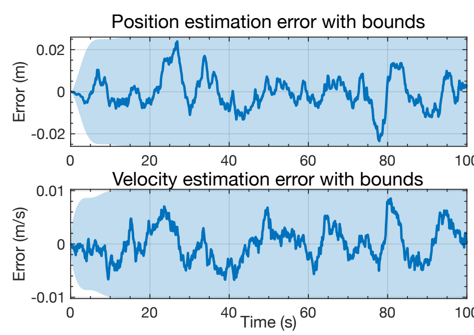




Example with correlated v_k (corrected)

- Now, for the same dataset, I use the approach from this lesson to build an augmented A^* matrix, where $A_v = 0.8$.
- The output of the KF using augmented dynamics is greatly improved.





Summary

- The derivation of the KF assumes that both the process noise and the sensor noise are white.
- In Lesson 1.4.5, you saw an example that showed that the KF estimates are poor when this assumption is violated.
- In this lesson, you learned how to augment the state-space model of the system whose state is being estimated to compensate for correlated w_k and/or v_k.
- Some examples demonstrated that the KF estimates (especially the confidence bounds) improved dramatically.