Polar measurements and a Cartesian state

- It is often most convenient to express the state dynamics of a target in Cartesian coordinates.
- However, in active sonar and radar systems, the measurement is returned in polar coordinates.
- To handle this case, we can either:
  1. Use a nonlinear KF (with output equation in polar coordinates), or
  2. Compute an "equivalent" measurement in Cartesian coordinates from the true measurement in polar coordinates.
- Here, we handle the second case, which would be trivial except for the noises involved.[1]
- We denote true range to target by r and true bearing angle by θ.

[1] From: Lerro, D. and Bar-Shalom, Y., "Tracking with debiased consistent converted measurements versus EKF" *IEEE Trans. Aerospace and Electronic Systems,* 29(3), July 1993, 1015–22.

Modeling a noisy polar measurement

- We measure $\{r_m, \theta_m\}$ with noises $\{\tilde{r}, \tilde{\theta}\}$:

$$r_m = r + \tilde{r} \qquad \text{and} \qquad \theta_m = \theta + \tilde{\theta}.$$

- A standard conversion to Cartesian coordinates gives

$$x_m = r_m \cos\theta_m \qquad \text{and} \qquad y_m = r_m \sin\theta_m.$$

- The errors in each coordinate can be found by expanding:

$$\begin{aligned} x_m &= x + \tilde{x} = (r + \tilde{r})\cos(\theta + \tilde{\theta}) \\ y_m &= y + \tilde{y} = (r + \tilde{r})\sin(\theta + \tilde{\theta}), \end{aligned}$$

  which we will do on the next slide using the trigonometric identities:

$$\begin{aligned} \cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a + b) &= \sin(a)\cos(b) + \cos(a)\sin(b), \end{aligned}$$

The Cartesian error in a noisy polar measurement

- Applying the trigonometric identities, we obtain:

$$\begin{aligned} \tilde{x} &= r\cos\theta(\cos\tilde{\theta} - 1) - \tilde{r}\sin\theta\sin\tilde{\theta} - r\sin\theta\sin\tilde{\theta} \\ &\qquad + \tilde{r}\cos\theta\cos\tilde{\theta} \\ \tilde{y} &= r\sin\theta(\cos\tilde{\theta} - 1) + \tilde{r}\cos\theta\sin\tilde{\theta} + r\cos\theta\sin\tilde{\theta} + \tilde{r}\sin\theta\cos\tilde{\theta}, \end{aligned}$$

  which are not independent, and each coordinate error depends on the true range and bearing as well as the errors in range and bearing.
- The mean and variance of the errors can be found by assuming that $\tilde{\theta}$ is zero-mean and Gaussian (we also assume $\mathbb{E}[\tilde{r}] = 0$ and that $\tilde{r}$ and $\tilde{\theta}$ are uncorrelated). Then:

$$\mathbb{E}[\cos\tilde{\theta}] = e^{-\sigma_\theta^2/2}, \qquad \mathbb{E}[\sin\tilde{\theta}] = 0, \qquad \mathbb{E}[\sin\tilde{\theta}\cos\tilde{\theta}] = 0;$$

$$\mathbb{E}[\cos^2\tilde{\theta}] = \frac{1 + e^{-2\sigma_\theta^2}}{2}, \qquad \mathbb{E}[\sin^2\tilde{\theta}] = \frac{1 - e^{-2\sigma_\theta^2}}{2}.$$

The true mean error and true mean covariance

- The true mean error of the (x_m, y_m) conversion is then:

$$\bar{v}(r,\theta) = \begin{bmatrix} \mathbb{E}[\tilde{x} \mid r,\theta] \\ \mathbb{E}[\tilde{y} \mid r,\theta] \end{bmatrix} = \begin{bmatrix} r\cos\theta(e^{-\sigma_\theta^2/2} - 1) \\ r\sin\theta(e^{-\sigma_\theta^2/2} - 1) \end{bmatrix}.$$

- The true values of the elements of the converted measurement covariance are (I have omitted a lot of detailed trigonometric manipulation):

$$\begin{aligned} \Sigma_{\tilde{v},t}^{11} &= \mathrm{var}(\tilde{x} \mid r,\theta) = r^2 e^{-\sigma_\theta^2}[\cos^2\theta(\cosh(\sigma_\theta^2) - 1) + \sin^2\theta\sinh(\sigma_\theta^2)] \\ &\quad + \sigma_r^2 e^{-\sigma_\theta^2}[\cos^2\theta\cosh(\sigma_\theta^2) + \sin^2\theta\sinh(\sigma_\theta^2)] \\ \Sigma_{\tilde{v},t}^{22} &= \mathrm{var}(\tilde{y} \mid r,\theta) = r^2 e^{-\sigma_\theta^2}[\sin^2\theta(\cosh(\sigma_\theta^2) - 1) + \cos^2\theta\sinh(\sigma_\theta^2)] \\ &\quad + \sigma_r^2 e^{-\sigma_\theta^2}[\sin^2\theta\cosh(\sigma_\theta^2) + \cos^2\theta\sinh(\sigma_\theta^2)] \\ \Sigma_{\tilde{v},t}^{12} &= \mathrm{cov}(\tilde{x},\tilde{y} \mid r,\theta) = \sin\theta\cos\theta e^{-2\sigma_\theta^2}[\sigma_r^2 + r^2(1 - e^{\sigma_\theta^2})] = \Sigma_{\tilde{v},t}^{21}. \end{aligned}$$

Debiasing the synthetic measurement

- Ideally, we would subtract this true mean error from the conversion, and use $\Sigma_{\tilde{v},t}$ in the measurement update.
- However, these two quantities are uncomputable since we do not know the true r and θ (we know only the noisy measurements r_m and θ_m).
- We can, however, compute the average true bias and the average true covariance.
- The average true bias is (a lot of trigonometric operations again omitted):

$$\mathbb{E}[\bar{v}(r,\theta) \mid r_m,\theta_m] = \begin{bmatrix} r_m\cos\theta_m(e^{-\sigma_\theta^2} - e^{-\sigma_\theta^2/2}) \\ r_m\sin\theta_m(e^{-\sigma_\theta^2} - e^{-\sigma_\theta^2/2}) \end{bmatrix}.$$

- So, we compute our debiased synthetic Cartesian measurement as:

$$z = \begin{bmatrix} r_m\cos\theta_m(1 - e^{-\sigma_\theta^2} + e^{-\sigma_\theta^2/2}) \\ r_m\sin\theta_m(1 - e^{-\sigma_\theta^2} + e^{-\sigma_\theta^2/2}) \end{bmatrix}.$$

Debiasing the synthetic covariance

- The average true covariance matrix has elements:

$$\begin{aligned} \Sigma_{\tilde{v},a}^{11} &= r_m^2 e^{-2\sigma_\theta^2}[\cos^2\theta_m(\cosh 2\sigma_\theta^2 - \cosh\sigma_\theta^2) \\ &\quad + \sin^2\theta_m(\sinh 2\sigma_\theta^2 - \sinh\sigma_\theta^2)] + \sigma_r^2 e^{-2\sigma_\theta^2}[\cos^2\theta_m(2\cosh 2\sigma_\theta^2 - \cosh\sigma_\theta^2) \\ &\quad + \sin^2\theta_m(2\sinh 2\sigma_\theta^2 - \sinh\sigma_\theta^2)] \\ \Sigma_{\tilde{v},a}^{22} &= r_m^2 e^{-2\sigma_\theta^2}[\sin^2\theta_m(\cosh 2\sigma_\theta^2 - \cosh\sigma_\theta^2) + \cos^2\theta_m(\sinh 2\sigma_\theta^2 - \sinh\sigma_\theta^2)] \\ &\quad + \sigma_r^2 e^{-2\sigma_\theta^2}[\sin^2\theta_m(2\cosh 2\sigma_\theta^2 - \cosh\sigma_\theta^2) + \cos^2\theta_m(2\sinh 2\sigma_\theta^2 - \sinh\sigma_\theta^2)] \\ \Sigma_{\tilde{v},a}^{12} &= \sin\theta_m\cos\theta_m e^{-4\sigma_\theta^2}[\sigma_r^2 + (r_m^2 + \sigma_r^2)(1 - e^{\sigma_\theta^2})] = \Sigma_{\tilde{v},a}^{21}. \end{aligned}$$

- Average covariance is *larger* than true covariance conditioned on the exact position; takes into account additional errors incurred by evaluating it at the measured posn.

Summarizing the conversion

- Thus, the final polar-to-Cartesian unbiased consistent conversion which corrects for the average bias is:

$$z = \begin{bmatrix} r_m \cos\theta_m (1 - e^{-\sigma_\theta^2} + e^{-\sigma_\theta^2/2}) \\ r_m \sin\theta_m (1 - e^{-\sigma_\theta^2} + e^{-\sigma_\theta^2/2}) \end{bmatrix}$$

$$\Sigma_{\tilde{v}} = \Sigma_{\tilde{v},a} = \begin{bmatrix} \Sigma_{\tilde{v},a}^{11} & \Sigma_{\tilde{v},a}^{12} \\ \Sigma_{\tilde{v},a}^{12} & \Sigma_{\tilde{v},a}^{22} \end{bmatrix}.$$

  - Note that the corrections to $\Sigma_{\tilde{v}}$ are time varying, so must be computed for every measurement.
- Ironically, a nonlinear Kalman filter may be less complex computationally than applying these corrections to a linear Kalman filter!

Example

- Consider an example where $r = 1000$ m, $\theta = \pi/4$, $\sigma_r = 1$ m, and $\sigma_\theta = \pi/32$. I simulated 100,000 random measurements.
- The figure plots the pdfs of the “naive estimate” errors and the “corrected” errors.
- For reference, the true target location is:

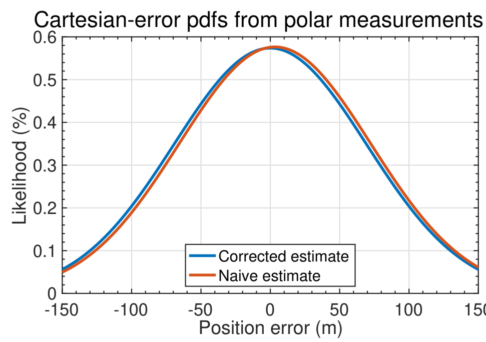
$$z = \begin{bmatrix} 707.11 \\ 707.11 \end{bmatrix}.$$

- The mean of the “naive” estimates is:

$$\hat{z}_{\text{naive}} = \begin{bmatrix} r_m \cos\theta_m \\ r_m \sin\theta_m \end{bmatrix} = \begin{bmatrix} 703.77 \\ 703.65 \end{bmatrix}.$$

- The mean of the corrected estimates is:

$$\hat{z} = \begin{bmatrix} 707.13 \\ 707.02 \end{bmatrix}.$$



Summary

- Many target-tracking applications use sensors that return measurements in polar coordinates.
- Naively converting these noisy measurements to Cartesian coordinates (e.g., via $\hat{x} = r_m \cos(\theta_m)$) biases the measurement due to the nonlinear conversion.
  - The bias is especially large for measurements having large cross-range errors (long ranges and large bearing errors).
- You have learned how to process these measurements to correct their values, producing compensated position and position-error-covariance estimates in Cartesian coordinates that are correct on average.
- An example demonstrated a small but noticeable improvement.
  - And remember, the KF is sensitive to bias errors and so this procedure will improve KF performance more than might be anticipated by the example because it eliminated the measurement bias that is introduced by the naive method.