On random walks inside generalized quadrilaterals

Mikko Nummelin
email: mikko.nummelin@tkk.fi

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Riemann mapping theorem

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$$w(\Omega) = \mathbb{D}(0;1), \quad w(0) = 0.$$

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- Thus each Jordan domain is *conformally equivalent* to the unit disk.

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- Suppose a Jordan region Ω, $0 \in \Omega$ and a continuos time random walk $\xi(t)$ starting from the origin.
- There is a probability measure defined on $\partial\Omega$ which tells the probability that random walk $\xi(t)$ exits on particular places on $\partial\Omega$. This probability measure is called *harmonic measure* of Ω respect to the origin and usually notated

$$\omega(0, \alpha, \Omega) = \int_\alpha \rho_0(\theta)\ d\theta, \quad \alpha \subset \partial\Omega,\ \theta \in [0, 2\pi).$$

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- So, if
$$\gamma(\theta), \quad \theta \in [0, 2\pi)$$
is a parametrization of $\partial\Omega$ and
$$\mu(\theta), \quad \theta \in [0, 2\pi)$$
is the respective parametrization of $\partial\mathbb{D}(0;1)$ under a conformal mapping $w : \Omega \to \mathbb{D}(0;1)$, then the density function of harmonic measure can be computed as
$$\rho_0(\theta) = \frac{\mu'(\theta)}{2\mathrm{i}\pi\mu(\theta)}.$$

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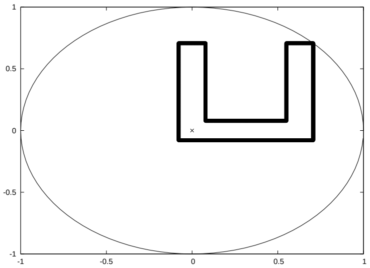
- Outward opening angles (nonconvex polygons have usually some of these), appear as singularities in the density function.

Graphical features - example



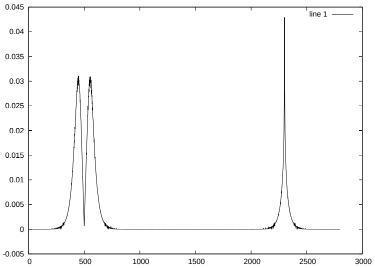


Figure: A "horseshoe"-like region with its conformal center located in the origin, pictured inside the unit disk (left) and density function of its harmonic measure (right). Note, how the only visible features in the density function are lower left corner (two smooth "hills" with steep "valley" down to zero between them) and nearby outward-pointing corner (singularity).

Graphical features - example 2

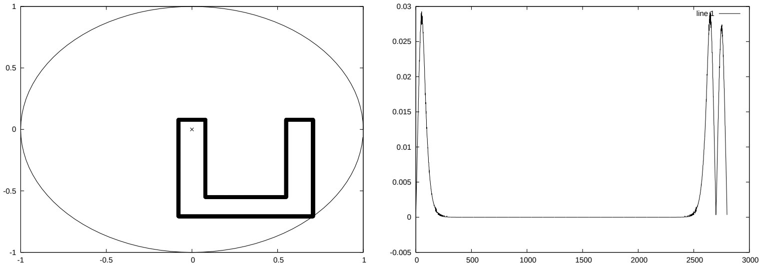


Figure: Same region with conformal center (starting point of the random walk) transformed. Note, how this changes absorption probabilities on the edges.

Osculation algorithms

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- Here we will compute it via *osculation algorithms* and concentrate on *logarithmic Koebe's algorithm*. It is not the best or most accurate algorithm for all situations, but it is quite stable.
- Osculation algorithms are, in general, designed to pick up a feature from approximation of boundary of a Jordan-region $\partial\Omega$ and decide a mapping function accordingly to find a conformal image of Ω closer to the unit disk.

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- A linear fractional transformation

$$\vec{z} \longrightarrow \frac{\vec{z} + |z_e|}{1 + |z_e|\vec{z}}$$

  is applied.

Koebe's algorithm - continued

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- The previous transformation sent all points of $\vec{z}$ nearer the edge of unit disk but not outside it.

Koebe's algorithm - fixing the origin

- Finally origin's image is fixed back to its original location by linear fractional transformation

$$\vec{z} \rightarrow \frac{\vec{z} - \sqrt{|z_e|}}{1 - \sqrt{|z_e|}\vec{z}}$$

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- Finally origin's image is fixed back to its original location by linear fractional transformation

$$\vec{z} \rightarrow \frac{\vec{z} - \sqrt{|z_e|}}{1 - \sqrt{|z_e|}\vec{z}}$$

- and optionally the rotation can also be reversed

$$\vec{z} \rightarrow \overline{z_R}\vec{z}.$$

for better visual effect.

Koebe's algorithm - summary

So, one iteration of Koebe's algorithm is:

$$
\begin{aligned}
z_e &= \min(\vec{z}) \\
z_R &= \frac{-\overline{z_e}}{|z_e|} \\
\vec{z} &\rightarrow z_R \vec{z} \\
\vec{z} &\rightarrow \frac{\vec{z} + |z_e|}{1 + |z_e| \vec{z}} \\
\vec{z} &\rightarrow \sqrt{\vec{z}} \\
\vec{z} &\rightarrow \frac{\vec{z} - \sqrt{|z_e|}}{1 - \sqrt{|z_e|} \vec{z}} \\
\vec{z} &\rightarrow \overline{z_R} \vec{z}
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holds, this suggests replacing the root with a holomorphic logarithm, taking $\vec{z}$ into left half plane.

- After taking the logarithm, image of the origin lies somewhere in left half plane. It can be transformed to -1 by scaling and then a *Cayley transformation*

$$\vec{z} \to \frac{1 + \vec{z}}{1 - \vec{z}}$$

maps the transformed area back into unit disk.

Logarithmic Koebe's algorithm - summary

So, one iteration of logarithmic Koebe's algorithm is:

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z_e &= \min(\vec{z}) \\
z_R &= \frac{-\overline{z_e}}{|z_e|} \\
\vec{z} &\rightarrow z_R \vec{z} \\
\vec{z} &\rightarrow \frac{\vec{z} + |z_e|}{1 + |z_e| \vec{z}} \\
\vec{z} &\rightarrow \ln \vec{z} \\
\vec{z} &\rightarrow \frac{-\vec{z}}{\ln(|z_e|)} \\
\vec{z} &\rightarrow \frac{1 + \vec{z}}{1 - \vec{z}} \\
\vec{z} &\rightarrow \overline{z_R} \vec{z}
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The Schwarz-Christoffel integration formula

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The Schwarz-Christoffel integration formula

- Suppose that Ω is a polygon, having n internal angles of sizes $\alpha_1, \alpha_2 \ldots \alpha_n$ respectively.
- According to the Riemann mapping theorem, Ω is conformally equivalent to the unit disk.
- The function defined by

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^{n} \alpha_k = 2\pi.$$

where A, B and each ζ_k are parameters, maps unit disk onto Ω.

Rectangle map

- The special case of Schwarz-Christoffel formula

$$w(z) = \int^z \left(1-\frac{\zeta}{\zeta_1}\right)^{-1/2}\left(1-\frac{\zeta}{\zeta_2}\right)^{-1/2}\ldots \ldots\left(1-\frac{\zeta}{\zeta_3}\right)^{-1/2}\left(1-\frac{\zeta}{\zeta_4}\right)^{-1/2} d\zeta$$

maps unit disk onto a rectangle. It can be used to construct module of a generalized quadrilateral or potential lines of a capacitor.

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&\quad \cdots \left(1-\frac{\zeta}{\zeta_3}\right)^{-1/2} \left(1-\frac{\zeta}{\zeta_4}\right)^{-1/2} d\zeta
\end{aligned}
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  maps unit disk onto a rectangle. It can be used to construct module of a generalized quadrilateral or potential lines of a capacitor.

- Here we will answer the question: if a random walk starts at some place inside Ω, where is the boundary from which it is most likely the random walk to end to either side.

Gaier quadrilateral

- Consider the so-called Gaier polygon $[0, 2, 2+\mathrm{i}, 1+\mathrm{i}, 1+2\mathrm{i}, 2\mathrm{i}]$ as a generalized quadrilateral, where $[2\mathrm{i}, 0]$ and $[2, 2+\mathrm{i}, 1+\mathrm{i}, 1+2\mathrm{i}]$ are generalized left and right sides of it, respectively.

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- Construct a rectangle map of it to $[0, 1, 1+\mathrm{i}M, \mathrm{i}M]$.

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- Construct a rectangle map of it to $[0, 1, 1 + \mathrm{i}M, \mathrm{i}M]$.
- Find the reverse image of line $[0.5, 0.5 + \mathrm{i}M]$ on the Gaier polygon and we have the middle equipotential line. A random walk starting on it will be as likely to hit the left side first than the right side.

Equipotential surface of Gaier quadrilateral

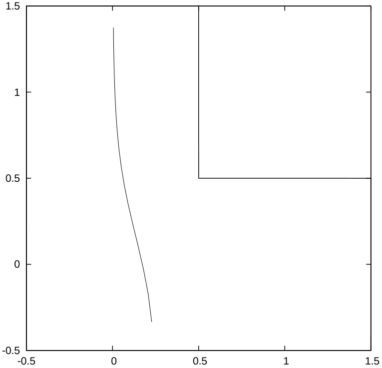

Figure: Gaier polygon as a generalized quadrilateral with middle equipotential line calculated. Note how this is to the left from geometric middle line. This is due to the lower right hand side "cave" as an attractor.

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