Solving Schwarz-Christoffel parameter problem by osculation algorithms

Mikko Nummelin
email: mikko.nummelin@tkk.fi

Helsinki Analysis seminar, 2007-10-15

Riemann mapping theorem

- Riemann mapping theorem states that there always exists a conformal map from a Jordan-domain Ω, $0 \in \Omega$ onto the unit disk, fixing the origin:

$$w(\Omega) = \mathbb{D}(0;1), \quad w(0) = 0.$$

Riemann mapping theorem

- Riemann mapping theorem states that there always exists a conformal map from a Jordan-domain Ω, $0 \in \Omega$ onto the unit disk, fixing the origin:

$$w(\Omega) = \mathbb{D}(0;1), \quad w(0) = 0.$$

- Thus each Jordan domain is *conformally equivalent* to the unit disk.

The Schwarz-Christoffel integration formula

- Suppose that Ω is a polygon, having n internal angles of sizes $\alpha_1, \alpha_2 \ldots \alpha_n$ respectively.

The Schwarz-Christoffel integration formula

- Suppose that Ω is a polygon, having n internal angles of sizes $\alpha_1, \alpha_2 \ldots \alpha_n$ respectively.
- According to the Riemann mapping theorem, Ω is conformally equivalent to the unit disk.

The Schwarz-Christoffel integration formula

- Suppose that Ω is a polygon, having n internal angles of sizes $\alpha_1, \alpha_2 \ldots \alpha_n$ respectively.
- According to the Riemann mapping theorem, Ω is conformally equivalent to the unit disk.
- The function defined by

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^{n} \alpha_k = 2\pi.$$

where A, B and each ζ_k are parameters, maps unit disk onto Ω.

The parameter problem

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^{n} \alpha_k = 2\pi.$$

- Parameters $A, B \in \mathbb{C}$ correspond to a simple translation and rotation and can be obtained by solving two linear equations.

The parameter problem

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^n \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^n \alpha_k = 2\pi.$$

- Parameters $A, B \in \mathbb{C}$ correspond to a simple translation and rotation and can be obtained by solving two linear equations.
- Parameters $\zeta_k \in \partial\mathbb{D}(0;1)$, which have $n-3$ degrees of freedom, correspond to respective side lengths of Ω, but the dependence is of nonlinear type.

Example: Triangle

- A triangle has a simple Schwarz-Christoffel representation

$$w(z) = A + B \cdot \int_0^z (1-\zeta)^{\frac{-\alpha_1}{\pi}} (1-\mathrm{i}\zeta)^{\frac{-\alpha_2}{\pi}} (1+\mathrm{i}\zeta)^{\frac{-\alpha_3}{\pi}} \, d\zeta.$$

Example: Triangle

- A triangle has a simple Schwarz-Christoffel representation

$$w(z) = A + B \cdot \int_0^z (1-\zeta)^{\frac{-\alpha_1}{\pi}} (1-\mathrm{i}\zeta)^{\frac{-\alpha_2}{\pi}} (1+\mathrm{i}\zeta)^{\frac{-\alpha_3}{\pi}} \, d\zeta.$$

- It has only linear parameters A and B as the three prevertices 1, i and $-\mathrm{i}$ can be chosen arbitrarily.

Example: Triangle

- A triangle has a simple Schwarz-Christoffel representation

$$w(z) = A + B \cdot \int_0^z (1-\zeta)^{\frac{-\alpha_1}{\pi}} (1-\mathrm{i}\zeta)^{\frac{-\alpha_2}{\pi}} (1+\mathrm{i}\zeta)^{\frac{-\alpha_3}{\pi}} \, d\zeta.$$

- It has only linear parameters A and B as the three prevertices 1, i and $-\mathrm{i}$ can be chosen arbitrarily.
- Three points can be mapped to another three points by a linear fractional transformation (Möbius-transformation). This is also precisely the reason, why three prevertices can be arbitrarily chosen in any Schwarz-Christoffel mapping.

Quadrilateral

- A quadrilateral has one nonlinear parameter, which cannot be arbitrarily chosen, in its Schwarz-Christoffel formula.

Quadrilateral

- A quadrilateral has one nonlinear parameter, which cannot be arbitrarily chosen, in its Schwarz-Christoffel formula.
- Consider the following Schwarz-Christoffel map from upper half plane $\mathbb{H}$ onto a rectangle:

$$w(z) = \int_0^z (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta.$$

Quadrilateral

- A quadrilateral has one nonlinear parameter, which cannot be arbitrarily chosen, in its Schwarz-Christoffel formula.
- Consider the following Schwarz-Christoffel map from upper half plane $\mathbb{H}$ onto a rectangle:

$$w(z) = \int_0^z (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta.$$

- The upper half plane equipped with prevertices -1, 0, s and ∞ is called *Teichmüller quadrilateral* and the nonlinear parameter s is called *modular s*.

Jacobi elliptic integral of first kind

- Complete Jacobi elliptic integral of the first kind is defined by

$$\mathrm{K}(k) = \mathrm{K} = \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}},$$

where k is called *modular parameter*.

Jacobi elliptic integral of first kind

- Complete Jacobi elliptic integral of the first kind is defined by

$$\mathrm{K}(k) = \mathrm{K} = \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}},$$

where k is called *modular parameter*.

- $k' = \sqrt{1-k^2}$ is called *complementary modular parameter* and $\mathrm{K}'(k) = \mathrm{K}(k') = \mathrm{K}'$ is called *complementary elliptic integral* of the first kind.

Jacobi elliptic integral of first kind

- Complete Jacobi elliptic integral of the first kind is defined by

$$\mathrm{K}(k) = \mathrm{K} = \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}},$$

  where k is called *modular parameter*.
- $k' = \sqrt{1-k^2}$ is called *complementary modular parameter* and $\mathrm{K}'(k) = \mathrm{K}(k') = \mathrm{K}'$ is called *complementary elliptic integral* of the first kind.
- The shorthand notation

$$\frac{\mathrm{K}'}{\mathrm{K}}(k) = \frac{\mathrm{K}'(k)}{\mathrm{K}(k)}$$

  will also be used.

Module of a quadrilateral

- The ratio

$$\mathrm{qm}(s) = \frac{\int_0^s (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta}{\mathrm{i}\int_{-1}^0 (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta}$$

which corresponds to rectangle's heigth divided by width is called *shape factor*, *capacitance* or *module of a quadrilateral*.

Module of a quadrilateral

- The ratio

$$\mathrm{qm}(s) = \frac{\int_0^s (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta}{\mathrm{i}\int_{-1}^0 (-\zeta)^{-1/2}(1+\zeta)^{-1/2}(s-\zeta)^{-1/2}\, d\zeta}$$

  which corresponds to rectangle's heigth divided by width is called *shape factor*, *capacitance* or *module of a quadrilateral*.

- A simple substitution $\zeta = u^2 - 1$ shows that qm can be expressed in terms of complete Jacobi elliptic integrals of the first kind:

$$\mathrm{qm}(s) = \frac{\mathrm{K}'}{\mathrm{K}}\left(\frac{1}{\sqrt{s+1}}\right).$$

Module of arbitrary quadrilateral

- Suppose that there is a *conformal mapping* w from Jordan domain Ω onto a rectangle S whose shape factor is M, w is continuous on $\partial\Omega$ and there are prevertices z_1, z_2, z_3, z_4 located counterclockwise on $\partial\Omega$.

Module of arbitrary quadrilateral

- Suppose that there is a *conformal mapping* w from Jordan domain Ω onto a rectangle S whose shape factor is M, w is continuous on $\partial\Omega$ and there are prevertices z_1, z_2, z_3, z_4 located counterclockwise on $\partial\Omega$.
- If images of these prevertices $w(z_1)$, $w(z_2)$,$w(z_3)$ and $w(z_4)$ correspond to corners of rectangle from lower left hand corner in counterclockwise order, then we define $(\Omega, z_1, z_2, z_3, z_4)$ as a *generalized quadrilateral* having shape factor M.

Module of arbitrary quadrilateral

- Suppose that there is a *conformal mapping* w from Jordan domain Ω onto a rectangle S whose shape factor is M, w is continuous on $\partial\Omega$ and there are prevertices z_1, z_2, z_3, z_4 located counterclockwise on $\partial\Omega$.
- If images of these prevertices $w(z_1)$, $w(z_2)$,$w(z_3)$ and $w(z_4)$ correspond to corners of rectangle from lower left hand corner in counterclockwise order, then we define $(\Omega, z_1, z_2, z_3, z_4)$ as a *generalized quadrilateral* having shape factor M.
- We will sometimes use the "overloaded" notation

$$\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right) = M.$$

Switching of prevertices

- If
$$\operatorname{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right) = M,$$
then
$$\operatorname{qm}\left((\Omega, z_2, z_3, z_4, z_1)\right) = 1/M.$$

Switching of prevertices

- If
$$\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right) = M,$$
then
$$\mathrm{qm}\left((\Omega, z_2, z_3, z_4, z_1)\right) = 1/M.$$
- This is due to the fact that if R is a rectangle whose height is M and width is 1 then $\mathrm{i}R$ is a rectangle whose width and height are reversed.

Switching of prevertices - application

- By translation and scaling

$$
\begin{aligned}
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,1,z_1,z_2,0)\right) &= \\
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,0,z_1-1,z_2-1,-1)\right) &= \\
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,0,1,\frac{z_2-1}{z_1-1},\frac{1}{1-z_1})\right) &= 1.
\end{aligned}
$$

Switching of prevertices - application

- By translation and scaling

$$
\begin{aligned}
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,1,z_1,z_2,0)\right) &= \\
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,0,z_1-1,z_2-1,-1)\right) &= \\
\mathrm{qm}\left((\Omega,0,1,z_1,z_2)\right)\mathrm{qm}\left((\Omega,0,1,\frac{z_2-1}{z_1-1},\frac{1}{1-z_1})\right) &= 1.
\end{aligned}
$$

- This property can be exploited in testing conformal mapping algorithms. Choose z_1 and z_2 at random so that a convex quadrilateral is formed and try to verify the above equation numerically.

Module of a quadrilateral - example

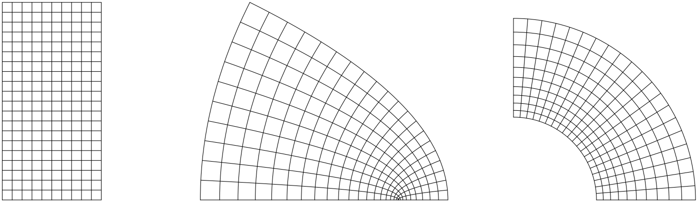

Figure: All these quadrilaterals have shape factor of 2. Note how angles are preserved and small squares mapped to approximate squares.

Computing qm from conformal map

- Consider computing $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$.

Computing qm from conformal map

- Consider computing $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$.
- First construct a conformal mapping from Ω onto unit disk $\mathbb{D}(0;1)$ and keep track on images of prevertices on $\partial\Omega$.

Computing qm from conformal map

- Consider computing $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$.
- First construct a conformal mapping from Ω onto unit disk $\mathbb{D}(0;1)$ and keep track on images of prevertices on $\partial\Omega$.
- Label these prevertices ζ_1, ζ_2, ζ_3 and ζ_4.

Computing qm from conformal map

- Consider computing $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$.
- First construct a conformal mapping from Ω onto unit disk $\mathbb{D}(0;1)$ and keep track on images of prevertices on $\partial\Omega$.
- Label these prevertices ζ_1, ζ_2, ζ_3 and ζ_4.
- Compute Schwarz-Christoffel-map

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{4} \left(1 - \frac{\zeta}{\zeta_k}\right)^{-1/2} d\zeta,$$

and choose A and B so that $w(\zeta_1) = 0$ and $w(\zeta_2) = 1$.

Computing qm from conformal map

- Consider computing $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$.
- First construct a conformal mapping from Ω onto unit disk $\mathbb{D}(0;1)$ and keep track on images of prevertices on $\partial\Omega$.
- Label these prevertices ζ_1, ζ_2, ζ_3 and ζ_4.
- Compute Schwarz-Christoffel-map

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{4} \left(1 - \frac{\zeta}{\zeta_k}\right)^{-1/2} d\zeta,$$

  and choose A and B so that $w(\zeta_1) = 0$ and $w(\zeta_2) = 1$.
- $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$ now equals height of the resulting rectangle.

Computing qm - alternative

- If we are interested only of the numerical value of $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$, not the whole rectangle map, we have an easier alternative.

Computing qm - alternative

- If we are interested only of the numerical value of $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$, not the whole rectangle map, we have an easier alternative.
- Suppose we have computed the prevertices ζ_1, ζ_2, ζ_3 and ζ_4 on $\partial\mathbb{D}(0;1)$.

Computing qm - alternative

- If we are interested only of the numerical value of $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$, not the whole rectangle map, we have an easier alternative.
- Suppose we have computed the prevertices ζ_1, ζ_2, ζ_3 and ζ_4 on $\partial\mathbb{D}(0;1)$.
- Find a linear fractional transformation that sends ζ_1 to -1, ζ_2 to 0 and ζ_4 to ∞. Image of ζ_3 is now the modular $s \in \mathbb{R}_+$.

Computing qm - alternative

- If we are interested only of the numerical value of $\mathrm{qm}\left((\Omega, z_1, z_2, z_3, z_4)\right)$, not the whole rectangle map, we have an easier alternative.
- Suppose we have computed the prevertices ζ_1, ζ_2, ζ_3 and ζ_4 on $\partial\mathbb{D}(0;1)$.
- Find a linear fractional transformation that sends ζ_1 to -1, ζ_2 to 0 and ζ_4 to ∞. Image of ζ_3 is now the modular $s \in \mathbb{R}_+$.
- A cross-ratio formula

$$s = \frac{(\zeta_1 - \zeta_4)(\zeta_2 - \zeta_3)}{(\zeta_2 - \zeta_1)(\zeta_4 - \zeta_3)}$$

will suffice and now we recall the earlier formula

$$\mathrm{qm}(s) = \frac{\mathrm{K}'}{\mathrm{K}}\left(\frac{1}{\sqrt{s+1}}\right).$$

Schwarz-Christoffel parameters from disk map

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^n \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^n \alpha_k = 2\pi.$$

- Recall the parameter problem, all prevertices ζ_k except three of them cannot be arbitrarily chosen but are nonlinearly dependent of sidelengths of polygon.

Schwarz-Christoffel parameters from disk map

$$w(z) = A + B \cdot \int_0^z \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{\frac{-\alpha_k}{\pi}} d\zeta, \quad \sum_{k=1}^{n} \alpha_k = 2\pi.$$

- Recall the parameter problem, all prevertices ζ_k except three of them cannot be arbitrarily chosen but are nonlinearly dependent of sidelengths of polygon.
- If we have a mapping $z((\Omega, (w_k)))$ which sends Ω conformally onto the unit disk and all prevertices w_k to $\partial\mathbb{D}(0;1)$, the Schwarz-Christoffel-parameters are exactly coordinates of $z(w_1)$, $z(w_2)$, etc!

Conformal map onto unit disk

- We now turn to our main task: how to find an approximation for conformal mapping $w : \Omega \to \mathbb{D}(0;1)$ where Ω is an arbitrary Jordan region?

Conformal map onto unit disk

- We now turn to our main task: how to find an approximation for conformal mapping $w : \Omega \to \mathbb{D}(0; 1)$ where Ω is an arbitrary Jordan region?
- Actually we will make an assumption that Ω is also *bounded*. For unbounded regions it is often possible to transform conformally to bounded region via linear fractional transformation or other elementary functions.

Conformal map onto unit disk

- We now turn to our main task: how to find an approximation for conformal mapping $w : \Omega \rightarrow \mathbb{D}(0;1)$ where Ω is an arbitrary Jordan region?
- Actually we will make an assumption that Ω is also *bounded*. For unbounded regions it is often possible to transform conformally to bounded region via linear fractional transformation or other elementary functions.
- There are many methods, for example solving the Schwarz-Christoffel parameters by nonlinear solvers, polynomial approximations, variational methods etc. but here we will concentrate on *osculation algorithms*.

Approximating Jordan region Ω

- We will approximate region Ω by a complex vector $\vec{z}$, which contains sequence of boundary points of Jordan curve $\partial\Omega$ in counterclockwise order.

Approximating Jordan region Ω

- We will approximate region Ω by a complex vector $\vec{z}$, which contains sequence of boundary points of Jordan curve $\partial\Omega$ in counterclockwise order.
- Region Ω should contain the origin, $0 \in \Omega$.

Approximating Jordan region Ω

- We will approximate region Ω by a complex vector $\vec{z}$, which contains sequence of boundary points of Jordan curve $\partial\Omega$ in counterclockwise order.
- Region Ω should contain the origin, $0 \in \Omega$.
- Region Ω should be completely inside the unit disk. If it is not, then we should find the maximum element of $\vec{z}$ (the one having largest absolute value) and scaling

$$\vec{z} \to \frac{\vec{z}}{\|\vec{z}\|_\infty}.$$

General idea of osculation algorithms

- Find an extremal point of $\vec{z}$ and call it z_e. It varies from osculation algorithm to another, which point should be considered "extremal". Usually it is the point which has smallest absolute value but sometimes it is the point having sharpest outward or inward angle.

General idea of osculation algorithms

- Find an extremal point of $\vec{z}$ and call it z_e. It varies from osculation algorithm to another, which point should be considered "extremal". Usually it is the point which has smallest absolute value but sometimes it is the point having sharpest outward or inward angle.
- Compute a conformal map

$$\vec{z} \longrightarrow u(\vec{z}, z_e)$$

where u depends on z_e. u should be chosen so that the resulting domain has more properties resembling $\mathbb{D}(0;1)$, i.e. points of $\vec{z}$ closer to $\partial\mathbb{D}(0;1)$, less sharp angles, more appropriate curvature compared to $\partial\mathbb{D}(0;1)$ etc.

Koebe's algorithm

- In Koebe's algorithm the extremal point is taken as $z_e = \min(\vec{z})$.

Koebe's algorithm

- In Koebe's algorithm the extremal point is taken as $z_e = \min(\vec{z})$.
- After the minimum is found, a rotation coefficient is computed:

$$z_R = \frac{-\overline{z_e}}{|z_e|}$$

Koebe's algorithm

- In Koebe's algorithm the extremal point is taken as $z_e = \min(\vec{z})$.
- After the minimum is found, a rotation coefficient is computed:

$$z_R = \frac{-\overline{z_e}}{|z_e|}$$

- Pointwise multiplication

$$\vec{z} \longrightarrow z_R \vec{z}, \;\; z_e \longrightarrow z_R z_e$$

which rotates the extremal (minimum) point to negative real axis, is applied.

Koebe's algorithm

- In Koebe's algorithm the extremal point is taken as $z_e = \min(\vec{z})$.
- After the minimum is found, a rotation coefficient is computed:

$$z_R = \frac{-\overline{z_e}}{|z_e|}$$

- Pointwise multiplication

$$\vec{z} \to z_R\vec{z},\ \ z_e \to z_R z_e$$

  which rotates the extremal (minimum) point to negative real axis, is applied.
- A linear fractional transformation

$$\vec{z} \to \frac{\vec{z} + |z_e|}{1 + |z_e|\vec{z}}$$

  is applied.

Koebe's algorithm - continued

- The latter transformation ensured that the Jordan-curve approximated by $\vec{z}$ passes through the origin but does not enclose it inside. This guarantees existence of continuous, holomorphic square root.

Koebe's algorithm - continued

- The latter transformation ensured that the Jordan-curve approximated by $\vec{z}$ passes through the origin but does not enclose it inside. This guarantees existence of continuous, holomorphic square root.
- Holomorphic square root

$$\vec{z} \longrightarrow \sqrt{\vec{z}}$$

is applied so that first nonzero element in $\vec{z}$ is transformed into lower half plane and for following points the branch of square root minimizing the distance to previous point's image is chosen.

Koebe's algorithm - continued

- The latter transformation ensured that the Jordan-curve approximated by $\vec{z}$ passes through the origin but does not enclose it inside. This guarantees existence of continuous, holomorphic square root.
- Holomorphic square root

$$\vec{z} \longrightarrow \sqrt{\vec{z}}$$

  is applied so that first nonzero element in $\vec{z}$ is transformed into lower half plane and for following points the branch of square root minimizing the distance to previous point's image is chosen.
- The previous transformation sent all points of $\vec{z}$ nearer the edge of unit disk but not outside it.

Koebe's algorithm - fixing the origin

- Finally origin's image is fixed back to its original location by linear fractional transformation

$$\vec{z} \to \frac{\vec{z} - \sqrt{|z_e|}}{1 - \sqrt{|z_e|}\vec{z}}$$

Koebe's algorithm - fixing the origin

- Finally origin's image is fixed back to its original location by linear fractional transformation

$$\vec{z} \longrightarrow \frac{\vec{z} - \sqrt{|z_e|}}{1 - \sqrt{|z_e|}\vec{z}}$$

- and optionally the rotation can also be reversed

$$\vec{z} \longrightarrow \overline{z_R}\vec{z}.$$

for better visual effect.

Logarithmic Koebe algorithm

- If, instead of taking square root in Koebe algorithm, a cube root or higher is taken, it accelerates convergence.

Logarithmic Koebe algorithm

- If, instead of taking square root in Koebe algorithm, a cube root or higher is taken, it accelerates convergence.
- As a limit formula

$$\ln(z) = \lim_{n \to \infty} n \left(\sqrt[n]{z} - 1 \right)$$

holds, this suggests replacing the root with a logarithm, taking $\vec{z}$ into left half plane.

Logarithmic Koebe algorithm

- If, instead of taking square root in Koebe algorithm, a cube root or higher is taken, it accelerates convergence.
- As a limit formula

$$\ln(z) = \lim_{n\to\infty} n\left(\sqrt[n]{z} - 1\right)$$

holds, this suggests replacing the root with a logarithm, taking $\vec{z}$ into left half plane.

- After taking the logarithm, image of the origin lies somewhere in left half plane. It can be transformed to -1 by scaling and then a *Cayley transformation*

$$\vec{z} \to \frac{1+\vec{z}}{1-\vec{z}}$$

maps the transformed area back into unit disk.

sinh-ln-algorithm

- If, after the logarithm has been applied, image of the Jordan domain is scaled and translated into a half strip

$$\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) \in (-\pi/2, \pi/2)\}$$

and this image is mapped into left half plane by hyperbolic sine, this seems to accelerate convergence.

sinh-ln-algorithm

- If, after the logarithm has been applied, image of the Jordan domain is scaled and translated into a half strip

$$\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) \in (-\pi/2, \pi/2)\}$$

  and this image is mapped into left half plane by hyperbolic sine, this seems to accelerate convergence.

- However, as

$$\sinh(z) = \frac{e^z - e^{-z}}{2},$$

  then points far away in left half plane are mapped approximately to distance $e^{|z|}/2$ from origin.

sinh-ln-algorithm

- If, after the logarithm has been applied, image of the Jordan domain is scaled and translated into a half strip

$$\{z : \mathrm{Re}(z) < 0, \mathrm{Im}(z) \in (-\pi/2, \pi/2)\}$$

  and this image is mapped into left half plane by hyperbolic sine, this seems to accelerate convergence.
- However, as

$$\sinh(z) = \frac{e^z - e^{-z}}{2},$$

  then points far away in left half plane are mapped approximately to distance $e^{|z|}/2$ from origin.
- In practice, this means *exponential cumulation of errors*.

Grassman's method

- For nonconvex regions, there often exist sharp outward angles.

Grassman's method

- For nonconvex regions, there often exist sharp outward angles.
- Consider the location of this angle on $\partial\Omega$ as the extremal point z_e and apply Koebe mapping on it. Applying this a few times often results in convex or starlike region where other algoritms converge better.

Grassman's method

- For nonconvex regions, there often exist sharp outward angles.
- Consider the location of this angle on $\partial\Omega$ as the extremal point z_e and apply Koebe mapping on it. Applying this a few times often results in convex or starlike region where other algoritms converge better.
- There exists a similar method on inward angles of convex hull of Ω but here this is omitted.

The Joukowski mapping

- Joukowski mapping is

$$w_J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

in its basic form. We will analyze its basic properties and develop an osculation algorithm based on it.

The Joukowski mapping

- Joukowski mapping is

$$w_J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

in its basic form. We will analyze its basic properties and develop an osculation algorithm based on it.

- Complex derivative of Joukowski mapping is

$$w_J'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right).$$

It has two zeros at $z = \pm 1$.

Properties of the Joukowski mapping

- Joukowski mapping decreases conformal density near its pole

$$\lim_{z\to 0}\left|\frac{1}{2}\left(1-\frac{1}{z^2}\right)\right| = \infty$$

and increases conformal density near its turnpoints

$$\lim_{z\to \pm 1}\left|\frac{1}{2}\left(1-\frac{1}{z^2}\right)\right| = 0.$$

Properties of the Joukowski mapping

- Joukowski mapping decreases conformal density near its pole

$$\lim_{z \to 0} \left| \frac{1}{2} \left(1 - \frac{1}{z^2} \right) \right| = \infty$$

and increases conformal density near its turnpoints

$$\lim_{z \to \pm 1} \left| \frac{1}{2} \left(1 - \frac{1}{z^2} \right) \right| = 0.$$

- Joukowski mapping approaches linear map far away when moving towards complex infinity:

$$\lim_{z \to \infty} \frac{1}{2} \left(1 - \frac{1}{z^2} \right) = \frac{1}{2}.$$

Locally decreasing conformal density

- Joukowski mapping is the simplest complex function having properties of locally decreasing conformal density and approaching linear map towards complex infinity.

Locally decreasing conformal density

- Joukowski mapping is the simplest complex function having properties of locally decreasing conformal density and approaching linear map towards complex infinity.
- This suggests a class of osculation methods based on weighted Joukowski transformation where the pole is placed on the edge of the unit disk or just outside it and turnpoints further away on the sides, also outside the unit disk.

Joukowski mapping osculation algorithm

- Define $z_e = \min(\vec{z})$ as in Koebe's algorithm.

Joukowski mapping osculation algorithm

- Define $z_e = \min(\vec{z})$ as in Koebe's algorithm.
- Find rotation complex number $z_R = z_e/|z_e|$ and apply

$$\vec{z} \longrightarrow z_R\vec{z}$$

to get the extremal point to positive real axis.

Joukowski mapping osculation algorithm

- Define $z_e = \min(\vec{z})$ as in Koebe's algorithm.
- Find rotation complex number $z_R = z_e/|z_e|$ and apply

$$\vec{z} \longrightarrow z_R \vec{z}$$

  to get the extremal point to positive real axis.
- Apply weighted Joukowski transformation

$$\vec{z} \longrightarrow \vec{z} - \frac{(|z_e| - 1)^2}{2(\vec{z} - 1)}$$

  which brings image of z_e and points near it half way closer to the edge of the unit disk and repeat.

Roots-of-unity method

- A more time-consuming but often more accurate way to choose weight factors of Joukowski transformation can be achieved by examining edge of unit disk

Roots-of-unity method

- A more time-consuming but often more accurate way to choose weight factors of Joukowski transformation can be achieved by examining edge of unit disk
- Construct all m:th roots of unity

$$\vec{u} = \{1, e^{\frac{2i\pi}{m}}, e^{\frac{2\cdot 2i\pi}{m}}, .., e^{\frac{2(m-1)i\pi}{m}}\}$$

and find

$$\max_k \min_\ell |u_k - z_\ell|,$$

i.e. approximately the point on $\partial\mathbb{D}$ furthest away from Ω.

Roots-of-unity method

- A more time-consuming but often more accurate way to choose weight factors of Joukowski transformation can be achieved by examining edge of unit disk
- Construct all m:th roots of unity

$$\vec{u} = \{1, e^{\frac{2i\pi}{m}}, e^{\frac{2\cdot 2i\pi}{m}}, .., e^{\frac{2(m-1)i\pi}{m}}\}$$

and find

$$\max_k \min_\ell |u_k - z_\ell|,$$

i.e. approximately the point on $\partial\mathbb{D}$ furthest away from Ω.

- Define this maximum distance as R and apply a rotation like before and

$$\vec{z} \to \vec{z} - \frac{R^2}{\vec{z}+1}.$$

Convergence of osculation algorithms

- It can be proved that Koebe's algorithm, both quadratic and logarithmic, has universal convergence. It is actually used in proving Riemann mapping theorem. It is, however quite slow and increases the number of sharp inward corners, therefore losing accuracy when applied on smooth boundaries.

Convergence of osculation algorithms

- It can be proved that Koebe's algorithm, both quadratic and logarithmic, has universal convergence. It is actually used in proving Riemann mapping theorem. It is, however quite slow and increases the number of sharp inward corners, therefore losing accuracy when applied on smooth boundaries.
- Experimental data shows that Joukowski mapping algorithm converges only on convex and very regular starlike regions which should not be too elongated. If these regions have smooth boundaries, it appears that Joukowski mapping algorithm converges more rapidly and is more accurate than Koebe's algorithm.

Nonconvergence of Joukowski mapping algorithm

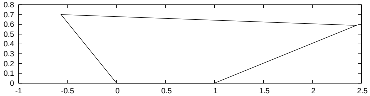

Figure: A convex quadrilateral for which straight application of Joukowski mapping algorithm diverges. The polygon is too elongated. The situation can be remedied by applying either Koebe's algorithm, Grassmans algorithm or roots-of-unity algorithm first at least a few times.

Osculation algorithms - example

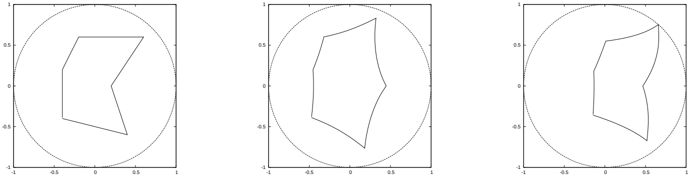

Figure: The original polygon (left), one step of Koebe's algorithm (middle) and one step of Joukowski mapping algorithm (right).

Osculation algorithms - example 2

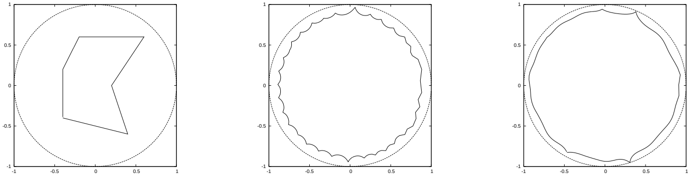

Figure: The original polygon (left), 30 steps of Koebe's algorithm (middle) and 30 steps of Joukowski mapping algorithm (right). Note how Joukowski mapping algorithm creates much more smoother boundary than Koebe's algorithm.

Harmonic measure

- Suppose a Jordan region Ω, $0 \in \Omega$ and a continuos time random walk $\xi(t)$ starting from the origin.

Harmonic measure

- Suppose a Jordan region Ω, $0 \in \Omega$ and a continuos time random walk $\xi(t)$ starting from the origin.
- There is a probability measure defined on $\partial\Omega$ which tells the probability that random walk $\xi(t)$ exits on particular places on $\partial\Omega$. This probability measure is called *harmonic measure* of Ω respect to the origin and usually notated

$$\omega(0, \alpha, \Omega) = \int_\alpha \rho_0(\theta)\ d\theta, \quad \alpha \subset \partial\Omega,\ \theta \in [0, 2\pi).$$

Computing harmonic measure

- It can be proved via Kolmogorov's law of total probability and first step analysis that harmonic measure is conformally invariant.

Computing harmonic measure

- It can be proved via Kolmogorov's law of total probability and first step analysis that harmonic measure is conformally invariant.
- So, if

$$\gamma(\theta), \quad \theta \in [0, 2\pi)$$

is a parametrization of $\partial\Omega$ and

$$\mu(\theta), \quad \theta \in [0, 2\pi)$$

is the respective parametrization of $\partial\mathbb{D}(0;1)$ under a conformal mapping $w : \Omega \to \mathbb{D}(0;1)$, then the density function of harmonic measure can be computed as

$$\rho_0(\theta) = \frac{\mu'(\theta)}{2\mathrm{i}\pi\mu(\theta)}.$$

Harmonic measure and SC points - example

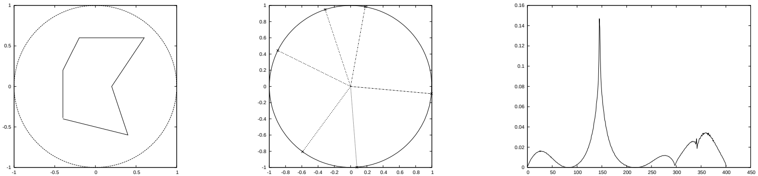

Figure: The original polygon (left), approximate conformal image on unit disk computed by 1 step of Koebe's algorithm and 1000 steps of Joukowski mapping algorithm with coordinates of Schwarz-Christoffel parameters (middle) and approximated harmonic measure (right).

Internal points

- Suppose that we have a conformal map described by $\partial\Omega$ in the physical plane and its image $w(\partial\Omega)$ in the model plane. How do we compute images of arbitrary points of Ω under the mapping w?

Internal points

- Suppose that we have a conformal map described by $\partial\Omega$ in the physical plane and its image $w(\partial\Omega)$ in the model plane. How do we compute images of arbitrary points of Ω under the mapping w?
- The answer is *Cauchy integration formula*, defined by

$$w(z) = \frac{1}{2i\pi} \int_{\partial\Omega} \frac{w(\zeta)\ d\zeta}{\zeta - z}, \quad z \in \Omega.$$

Internal points

- Suppose that we have a conformal map described by $\partial\Omega$ in the physical plane and its image $w(\partial\Omega)$ in the model plane. How do we compute images of arbitrary points of Ω under the mapping w?
- The answer is *Cauchy integration formula*, defined by

$$w(z) = \frac{1}{2\mathrm{i}\pi} \int_{\partial\Omega} \frac{w(\zeta)\, d\zeta}{\zeta - z}, \quad z \in \Omega.$$

- Cauchy integration formula can be approximated by trapezoidal rule (w_k and z_k are of modulo n):

$$w(z) \approx \frac{1}{2\mathrm{i}n\pi} \sum_{k=0}^{n-1} \frac{(w_{k+1} + w_k)(z_{k+1} - z_k)}{z_k - z}.$$

- Milton Abramowitz, Irene A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964 `http://www.math.sfu.ca/~cbm/aands/`
- Lars V. Ahlfors: *Complex analysis, An Introduction to the Theory of Analytic Functions of One Complex Variable*, New York, Toronto, London, McGraw-Hill Book Company, Inc., 1953
- Glen D. Anderson, Mavina K. Vamanamurthy, Matti K. Vuorinen: *Conformal invariants, inequalities, and quasiconformal maps*, John Wiley & Sons, Inc., New York, 1997
- Lehel Banjai: *Revisiting the crowding phenomenon in Schwarz-Christoffel mapping*, Institut für Mathematik, Universität Zürich, 2007

- Daniel J. Bernstein: *Computing logarithm intervals using arithmetic-geometric mean iteration*, National Science Foundation, Preprint, 2003
  `http://cr.yp.to/papers.html#logagm`
- Christopher J. Bishop: *Conformal welding and Koebe's theorem*, Preprint, 2003
- Tobin A. Driscoll: *Schwarz-Christoffel Toolbox*
  `http://www.math.udel.edu/~driscoll/software/SC`
- Tobin A. Driscoll, Lloyd N. Trefethen: *Schwarz-Christoffel mapping*, Cambridge monographs on applied and computational mathematics, 2002
- Dieter Gaier: *Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden*, Numer. Math. 19, 179-194, Springer-Verlag, 1972.

- Theodore W. Gamelin: *Complex analysis*, Springer-Verlag, New York, Inc., 2001
- John B. Garnett, Donald E. Marshall: *Harmonic measure*, Cambridge University Press, 2005
- Ville Heikkala, Mavina K. Vamanamurthy, Matti Vuorinen: *Generalized elliptic integrals*, Preprint, 2007. `arXiv:math/0701436`
- Louis Hill Howell: *Computation of conformal maps by modified Schwarz-Christoffel transformations*, PhD Thesis, Massachusetts Institute Of Technology, 1985.
- Stig Larsson, Vidar Thomée: *Partial Differential Equations with Numerical Methods*, Springer-Verlag, 2003.
- PlanetMath: *Automorphisms of unit disk* `http://planetmath.org/encyclopedia/AutomorphismsOfU`

- PlanetMath: *Schwarz-Christoffel transformation (circular version)*
  http://planetmath.org/encyclopedia/SchwarzChristoffelTransformationCircularVersion.htm
- R. Michael Porter: *History and Recent Developments in Techniques for Numerical Conformal Mapping*, Preprint, 2005.
  http://www.math.utu.fi/projects/madras/w_proc_mike.
- Roland A. Schinzinger, Patricio A. A. Laura: *Conformal Mapping - Methods And Applications*, Dover Publications Inc, Mineola, New York, 2003
- Theodore Theodorsen: *Theory of wing sections of arbitrary shape*, NACA Report No. 411, 1933
  http://naca.central.cranfield.ac.uk/reports/1933/na

Lloyd N. Trefethen, Tobin A. Driscoll: *Schwarz-Christoffel mapping in the computer era*, Proceedings of the International Congress of Mathematicians, Vol. III, 533-542.

Guangyu Zou, Jing Hua, Xianfeng Gu, Otto Muzik: *An approach for intersubject analysis of 3D brain images based on conformal geometry* http://www.cs.sunysb.edu/~vislab/papers/ICIP-2873.p