A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

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A Numerically Stable Fourier Continuation Approximation for the Solution of Partial Differential Equations

Melanie Vining

Integrated Applied Mathematics
Department of Mathematics & Statistics
University of New Hampshire

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A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Overview

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Motivation

The Fourier Continuation combined with Alternating Direction
(FC-AD) method splits a PDE into a series of one-dimensional BVPs
which are then solved using Fourier Continuation approximations.
This research develops and shows new approach to Fourier
Continuation approximation that offers greater stability and accuracy.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

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FC-AD Overview

- Based on Alternating Direction Implicit (ADI) methods
- Splits the PDE into a series of (the same) BVP
- Results in stability as long as the solution to the BVP is stable under that operation

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

FC-AD Computation

Given the 2-D Heat Equation

$$u_t = k(u_{xx} + u_{yy}) + Q(x, y, t), \quad (x, y, t) \in \Omega \times (0, T],$$
$$u(x, y, t) = G(x, y, t) \quad (x, y) \in \partial\Omega, t \in (0, T] \quad (2.1)$$
$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega,$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

FC-AD

- Discretize in time, with $t^n = n\Delta t$ and use central difference centered on $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$.

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{k}{2}\frac{\partial^2}{\partial x^2}(u^{n+1} + u^n) + \frac{k}{2}\frac{\partial^2}{\partial y^2}(u^{n+1} + u^n) \\ + Q^{n+\frac{1}{2}} + E_1(x, y, \Delta t) \quad (2.2)$$

- Solve for u^{n+1}:

$$\left(1 - \frac{k\Delta t}{2}\frac{\partial^2}{\partial x^2} - \frac{k\Delta t}{2}\frac{\partial^2}{\partial y^2}\right) u^{n+1} \\ = \left(1 + \frac{k\Delta t}{2}\frac{\partial^2}{\partial x^2} + \frac{k\Delta t}{2}\frac{\partial^2}{\partial y^2}\right) u^n + Q^{n+\frac{1}{2}} + E_1(x, y, \Delta t) \quad (2.3)$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

FC-AD

Expanding and arranging terms gives

$$\left(1 - \frac{k\Delta t}{2}\frac{\partial^2}{\partial x^2}\right)\left(1 - \frac{k\Delta t}{2}\frac{\partial^2}{\partial y^2}\right)u^{n+1}$$

$$= \left(1 + \frac{k\Delta t}{2}\frac{\partial^2}{\partial x^2}\right)\left(1 + \frac{k\Delta t}{2}\frac{\partial^2}{\partial y^2}\right)u^n + \frac{k^2\Delta t^2}{4}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}(u^{n+1} - u^n)$$

$$+ \Delta t Q^{n+\frac{1}{2}} + \Delta t E_1(x, y, \Delta t)$$

(2.4)

Here, inverting the operators $\left(1 - \frac{k\Delta t}{2}\frac{\partial^2}{\partial x^2}\right)$ and $\left(1 - \frac{k\Delta t}{2}\frac{\partial^2}{\partial y^2}\right)$ gives an expression for u^{n+1} in terms of some f^{n+1}.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

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Inverting the operators on the left hand side results in equations of the form

$$-\alpha u'' + u = f, \quad u(x_\ell) = B_\ell, \quad u(x_r) = B_r \qquad (2.5)$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Motivation for using FC

- Fourier Series are known to represent periodic functions well, with cheap computational cost

- Fourier Series used to represent non-periodic functions experience Gibbs' Phenomenon

Gibbs Phenomenon for periodic extension of f(x)=x on [-1,1]
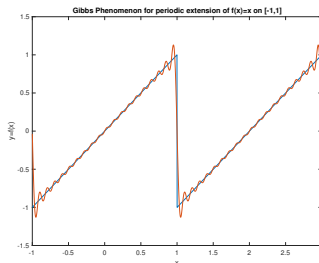
- Goal of Fourier Continuation is to get the similar accuracy, convergence, and computational cost with non-periodic functions

- As a result of using Fourier Continuation approximations, we are given periodic extensions of smooth functions that can be used in conjunction with FFTs to compute numerical solutions to differential equations.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
**Fourier
Continuation**
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

Construction

Given a smooth function $y = f(x)$ on $[0, 1]$ where $f \in C^k[0, 1]$, $k > 0$. Discretize the interval onto N points and let x_j be the jth point on the interval with corresponding function value $y_j = f(x_j)$. Choose period $b > 1$, and expand in terms of M Fourier Modes, $M < N$, and the problem to solve becomes

$$y_j = \sum_{k \in t(M)} a_k e^{\dfrac{2\pi i}{b} k x_j}, \quad j = 1 \dots N \tag{2.6}$$

As an example, consider $f(x) = x$ on $[0, 1]$. We discretize on 16 points, let $b = 2$, and $M = 7$. A resulting Fourier Continuation approximation for f is

Fourier Continuation approximation for f(x)=x on [0,1] with 16 points

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining
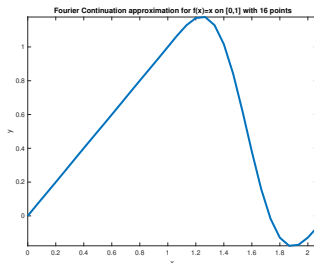
Fourier Continuation Conclusion

The previous figure is just **one** of the possible Fourier Continuation approximations for $f(x) = x$ on $[0, 1]$. Using different methods for solution of the least-squares system can result in different forms.

Is there a "best" Fourier Continuation approximation for any given problem?

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

FC-Gram

- Begin with a basis of polynomials on n points, $f_0, f_1, \ldots, f_{n-1}$

- Create and store the Fourier Continuation approximations for each of the polynomials

- Given a function f on any interval with any number of points m, sample the first n points and the last n points of f. If $m = n$, use the same n points in order.

- Fit the first n points to the basis and the last n points to the basis separately, giving us a_1 and a_2.

- An appropriate combination of a_1 and a_2 is taken to get a smooth, periodic extension of f

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
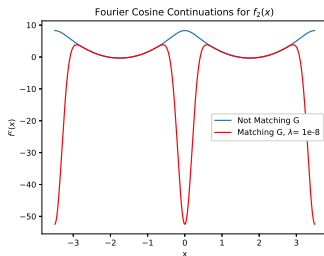Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

FC-Gram Example
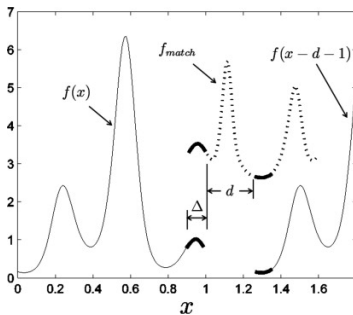
Figure: A comprehensive picture of the FC-Gram method as applied to $f(x) = e^{\sin(5.4\pi x - 2.7\pi) - \cos(2\pi x)}$[1]

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

FC-Gram Conclusion

Using FC-Gram to create periodic extensions of functions allows for
the use of FFTs and the other "nice" properties of Fourier Series to
be used in solving differential equations.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Overview

1. Introduction
   Motivation
   FC-AD
   Fourier Continuation
   FC-Gram

2. Current Work
   Green's Function Solution
   Minimization Problem
   Results

3. Future Work
   Computational Work
   Analytical Work

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Multiple Continuation approximations

As explained in the Fourier Continuation section, there are several possible Fourier Continuation approximations for the same function.

Fourier Cosine Continuations for $f_2(x)$

By adding constraints to the system of equations, we can control the shape of the continuation and ultimately create a stable approximation.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Green's Function Solutions

Matching the differential operator to the Green's Function solutions for the given BVP is a natural choice. The Green's Function solutions are inherently stable under the ∞-norm and thus provide the beginning steps for the new form the continuations will take.

- Compute the Green's Function for the differential equation $-\alpha u'' + u = f$, $u(x_\ell) = 0$, $u(x_r) = 0$. This yields $G(x, a)$.

- Compute the Green's Function solution as $\displaystyle\int_{x_\ell}^{x_r} G(x, a) f(a) da$.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

The calculated Green's Function for $x_\ell = -1$, $x_r = b$ is

$$G(x, a) = \begin{cases} -\dfrac{1}{2} \dfrac{\sqrt{\alpha}\left(-e^{\frac{-1-2b+a}{\sqrt{\alpha}}} + e^{-\frac{a+1}{\sqrt{\alpha}}}\right)\left(-e^{\frac{2(2+b+x)}{\sqrt{\alpha}}} + e^{\frac{2(x+1)}{\sqrt{\alpha}}} + e^{\frac{2(1+b)}{\sqrt{\alpha}}} - 1\right)e^{\frac{-x+1+2b}{\sqrt{\alpha}}}}{\left(e^{\frac{2(1+b)}{\sqrt{\alpha}}} - 1\right)^2} & x < a \\[4ex] \dfrac{1}{2} \dfrac{\sqrt{\alpha}\left(e^{\frac{2+b+a}{\sqrt{\alpha}}} - e^{\frac{a-b}{\sqrt{\alpha}}}\right)\left(e^{\frac{2(-x+b)}{\sqrt{\alpha}}} + e^{\frac{2(1+b)}{\sqrt{\alpha}}} - e^{\frac{2+4b-2x}{\sqrt{\alpha}}} - 1\right)e^{\frac{2+b+x}{\sqrt{\alpha}}}}{\left(e^{\frac{2(1+p)}{\sqrt{\alpha}}} - 1\right)^2} & x \geq a \end{cases}$$

(3.1)

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

- Given n points on an interval, we construct the Gram polynomials $f_0, f_1, \ldots, f_{n-1}$. Then a sine or cosine continuation matrix is created on a grid, and the functions are evaluated on the same grid.

- Next, the function $\tilde{f}$ and its cosine (or sine) continuation C_u are matched, which is the equivalent of solving

$$\min ||C_u f^c - \tilde{f}||^2. \tag{3.2}$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

Minimization Problem 2

- Using symbolic representations, the Green's function solutions for each of the polynomials are computed and evaluated on the grid. Then a numerical differential operator $I - \alpha \dfrac{\partial^2}{\partial x^2}$ is created for the sine or cosine continuation.

- We constrain the original minimization problem using the Green's function and the inverted differential operator applied to the continuation. This yields

$$\min ||C_u f^c - \tilde{f}||^2 + ||C_u D_c^\dagger f^c - \tilde{G}||^2 \qquad (3.3)$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Outline

Introduction
Motivation
FC-AD
Fourier
Continuation
FC-Gram

Current Work
Green's Function
Solution
Minimization
Problem
Results

Future Work
Computational
Work
Analytical Work

Minimization Problem 3

In computation, solving this minimization problem does not achieve great accuracy nor stability. The homogeneous solutions are not well-represented in the Fourier basis, but the Green's function solution is created from them. An additional constraint is needed in order to allow the system to accurately represent the function while separating out the homogeneous solutions. Including this constrain modifies the system to

$$\min ||C_u f^c - \tilde{f}||^2 + ||C_u D_c^\dagger f^c + c_1 \tilde{h_1} + c_2 \tilde{h_2} - \tilde{G}||^2 \qquad (3.4)$$

where c_1 and c_2 will be chosen by the computational method to achieve greatest accuracy in the least-squares sense.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Minimization Problem 4

Lastly, two other parameters λ and μ which force the continuations f^c to be stable. The first, λ, is applied to control the magnitude of c_1 and c_2 relative to the homogeneous solutions. The second, μ is applied to the coarse-grid point values of the differential operator to control the magnitude of the continuation under the derivative. The final minimization problem we have is

$$
\begin{aligned}
\min ||C_u f^c - \tilde{f}||^2 + ||C_u D_c^\dagger f^c + c_1 \tilde{h}_1 + c_2 \tilde{h}_2 - \tilde{G}||^2 \\
+ \lambda^2 ||c_1 \tilde{h}_1 + c_2 \tilde{h}_2||^2 + \mu^2 ||C_u D_c^\dagger|_{\text{coarse}}||^2
\end{aligned}
\tag{3.5}
$$

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Implementation

We implement this solver in Julia, using 200 bits of precision in
BigFloat. For 200 bits, $\epsilon \approx 10^{-77}$. With the conditioning of the
system, having 77 digits of precision mitigates great error
propagation. The Julia packages GenericSVD, SymPy, and PyPlot
were used. To find the least-squares solution to our minimization
problem, we used an SVD and found the pseudoinverse with a
tolerance of 10^{-40}.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Some Results

Letting $n = 10$, $b = 3.5$, setting the unit interval from 1.25 to 2.25 and shifting polynomial and Green's Function solution evaluation accordingly, stability was achieved with great accuracy. Accuracy was computed on a 10 times finer grid, while stability was calculated on the coarse grid of 10 points.

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Some Results

Here are two cosine continuations for f_0 and f_1 with $\alpha = 0.001$:

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations
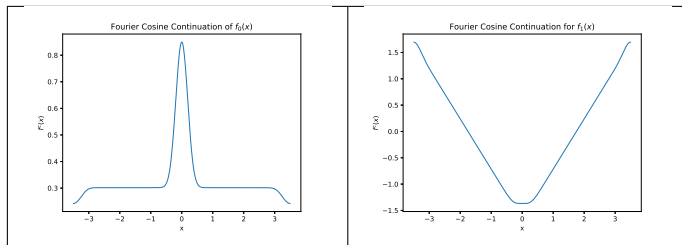
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Some Results

Here is a cosine continuation for f_1, accompanied by the computed solution u, for $\alpha = 0.01$
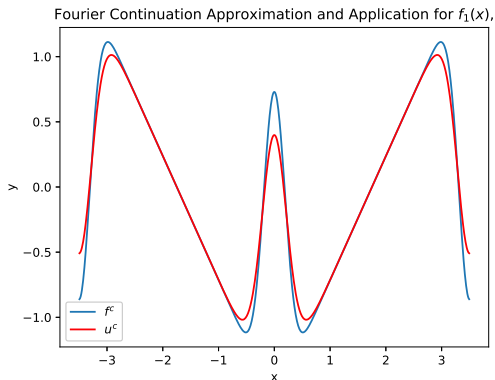
Fourier Continuation Approximation and Application for $f_1(x)$,

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

Overview

A Numerically
Stable Fourier
Continuation
Approximation
for the Solution
of Partial
Differential
Equations

Vining

📄 Oscar P. Bruno and Mark Lyon.
High-order unconditionally stable fc-ad solvers for general
smooth domains i. basic elements.
Journal of Computational Physics, 229:2009–2033, 2010.