Sensor tilt via conic sections

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Abstract

Distortions due to a misalignment between the lens elements and imaging sensor of a camera, are modelled in this article using conic sections. Light enters the camera lens at a *field angle* θ relative to the optical axis, and exits the lens pupil at the *lens distorted angle* β. For a given lens distortion angle, a cone of possible light paths exist which are intersected by a tilted imaging sensor creating an ellipse of *sensor tilt shifted points*. Conversely a sensor at right angles to the optical axis intersects this cone to create a circle of possible points. We derive the mapping between the circle and ellipse, as a function of the unit normal to the tilted sensor, and outline how this formalism for sensor tilt correction can be integrated into a generic camera calibration algorithm.

Keywords: ccd sensor tilt, tangential distortion, camera calibration, conic sections.

Camera calibration is the process of determining the geometric properties of a camera, and is a prerequisite for making accurate geometric measurements from image data [6]. A calibration algorithm requires both the forward and backward mapping of the camera model. In this way the expected 2D image of a 3D object can be easily generated, and given the 2D image the process is invertible as the 3D point coordinates of the imaged object can be calculated [4, 5, 7]. Calibration is generally performed by photographing a calibration target whose geometric properties are known. Typically the target is a one or three plane checkerboard, and a detector is used to extract the pixel coordinates of the checkerboard corners. Given the camera model, and a set of images with detected corners, the camera parameters are optimized until the errors between the reprojected points from the model, and the detected points in the image, are minimized.

In this article we outline a typical camera calibration algorithm with one modification - we model the distortions caused by misalignments between the lens and imaging sensor using conic sections and non-linear maps [9], in favour of the thin prism model [2] currently in use. The model is illustrated in figure 1.

1 Camera model

Following refs [1]-[11] a generic camera calibration algorithm can be represented by the sequence of mappings:

$$\hat{X} \xmapsto{\text{extrinsics}} \hat{P} \xmapsto{\text{lens}} \hat{p} \xmapsto{\text{sensor}} \hat{p}' \xmapsto{\text{intrinsics}} \hat{x}$$

World coordinates $\hat{X} \mapsto \hat{P}$: The matrix $\hat{X} = [\vec{X}_1\ \vec{X}_2\ \vec{X}_3 \cdots \vec{X}_i \cdots]$ are the 3D homogeneous coordinates of the test target corners $\vec{X}_i = (X_i\ \ Y_i\ \ Z_i\ \ 1)^\dagger$, expressed in the coordinates system of the target. $\hat{P}$ are the world coordinates of the target corners after rotation and translation by the extrinsic matrix, $\hat{P} = [\,\hat{R}\mid\vec{t}\,]\,\hat{X}$.

Lens distorted coordinates $\hat{P} \mapsto \hat{p}$: The world points $\hat{P}$ are projected through the lens according to the lens' radial distortion function

$$r(\theta_i) = \theta_i + \kappa_1\theta_i^3 + \kappa_2\theta_i^5 + \kappa_3\theta_i^7 + \cdots$$

where $\kappa_1, \kappa_2, \ldots$ are the Taylor series coefficients, and the field angle θ_i is the arctangent of the length of the world point $\vec{P}_i$ in the xy plane, divided by the z component. The azimuthal angle is the arctangent of the x and y coordinates. We drop the use of the point index i for brevity $\theta_i \to \theta$, $\phi_i \to \phi$, and $\vec{P}_i = (P_x, P_y, P_z)$.

$$\theta = \tan^{-1}\left[\sqrt{P_x^2 + P_y^2}/P_z\right] \qquad\qquad \phi = \tan^{-1}\left[P_y/P_x\right]$$

These parameters are then used to describe the lens distorted points in homogeneous camera coordinates:

$$\vec{p} = \begin{pmatrix} r(\theta)\cos(\phi) & r(\theta)\sin(\phi) & 1 \end{pmatrix}^\dagger \qquad \text{and} \qquad \hat{p} = \left[\ \vec{p}_1\ \vec{p}_2 \cdots \vec{p}_i \cdots \right]$$

The points $\hat{p}$ are projected onto the tilted imaging sensor as illustrated in figure $1(a)$.

The next mapping $\hat{p} \xmapsto{\text{sensor}} \hat{p}'$, takes account of distortions that arise due to the misalignment of the lens elements and imaging sensor. Typically these are referred to as tangential distortions and are accounted for using the Brown-Conrady model [1, 2]. At this point in the discussion we deviate from tradition and propose to model pixel distortions from a tilted ccd imaging sensor, using conic sections and non-linear maps.

Illustrated in figure $1(c)$ is a side view of the camera model. The world points are projected through the lens element onto the imaging sensor(s). The ccd sensor has 2 positions. In green is the ideal position - perpendicular to the optical axis, and in red is the tilted sensor. The perpendicular and tilted sensors are labelled S_0 and S_1, and have unit normals $-\vec{z}$, and $\vec{n}$, respectively. Photons emanating from the world point(s) $\vec{P}$ are projected through the lens elements to impact the perpendicular sensor S_0 at the point(s) $\vec{p}$.

$$\vec{z} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\dagger \qquad \vec{n} = \begin{pmatrix} n_x & n_y & n_z \end{pmatrix}^\dagger \qquad \vec{p} = \begin{pmatrix} p_x & p_y & 1 \end{pmatrix}^\dagger$$

Tilt shifted coordinates $\hat{p} \mapsto \hat{p}'$: The projected point forms a vector $\lambda\vec{p}$ touching the tilted plane S_1, and connecting with the vector $\vec{q}$ in the local coordinates of S_1, see figure $1(c)$. Taking the dot product of both sides of this vector equation, with the unit normal $\vec{n}$ to the tilted sensor S_1, we arrive at an expression for the scaling factor λ, since $\vec{n}\cdot\vec{q} = 0$.

$$\vec{q} = \lambda\vec{p} - \vec{z} \quad \to \quad \lambda = \vec{n}\cdot\vec{z}\left(\vec{n}\cdot\vec{p}\right)^{-1} \quad \to \quad \lambda = n_z\left(n_x p_x + n_y p_y + n_z\right)^{-1} \tag{1}$$

In order to express $\vec{q}$ in the local co-ordinate system of S_0, the S_1 plane is rotated to align with the (camera) coordinate system of S_0. As the lens is rotationally symmetric about the optical axis, the roll angle is ambiguous, and S_1 maps to S_0 as the unit normal $\vec{n}$ is rotated to $-\vec{z}$. The Rodrigues rotation matrix with rotation axis $\vec{k} = -\vec{n}\times\vec{z}$, and rotation angle $\vartheta = \cos^{-1}(-\vec{n}\cdot\vec{z})$, rotates the vector $\vec{n}$ to $-\vec{z}$:

$$\vec{k} = \begin{pmatrix} -n_y & n_x & 0 \end{pmatrix}^\dagger \qquad\qquad \cos(\vartheta) = -n_z$$

and the rotation matrix simplifies to

$$\hat{R} = \hat{\sigma}_1 + \hat{k} + \hat{k}^2\frac{1}{1+\cos(\vartheta)} = \begin{pmatrix} 1+\frac{n_x^2}{n_z-1} & \frac{n_x n_y}{n_z-1} & n_x \\ \frac{n_x n_y}{n_z-1} & 1+\frac{n_y^2}{n_z-1} & n_y \\ -n_x & -n_y & -n_z \end{pmatrix} \tag{2}$$

where $\hat{\sigma}_1$ is the identity matrix, and $\hat{k}$ is a skew symmetric matrix composed of the elements of $\vec{k}$.

To perform the rotation around the projective plane origin, the points are first shifted to the coordinate centre $\vec{o}$, then rotated, then shifted back to the original location. The tilt shifted points in the coordinate system of the S_0 sensor plane are defined:

$$\vec{p}' = \hat{R}\left(\lambda\vec{p} - \vec{z}\right) + \vec{z} \tag{3}$$

The tilt shifted map $\vec{p} \mapsto \vec{p}'$:

$$p_x{}' = \left(\left(n_x^2 + n_z(n_z-1)\right)p_x + n_x n_y p_y\right)\left(n_x p_x + n_y p_y + n_z\right)^{-1}(n_z-1)^{-1} \tag{4a}$$

$$p_y{}' = \left(\left(n_y^2 + n_z(n_z-1)\right)p_y + n_x n_y p_x\right)\left(n_x p_x + n_y p_y + n_z\right)^{-1}(n_z-1)^{-1} \tag{4b}$$

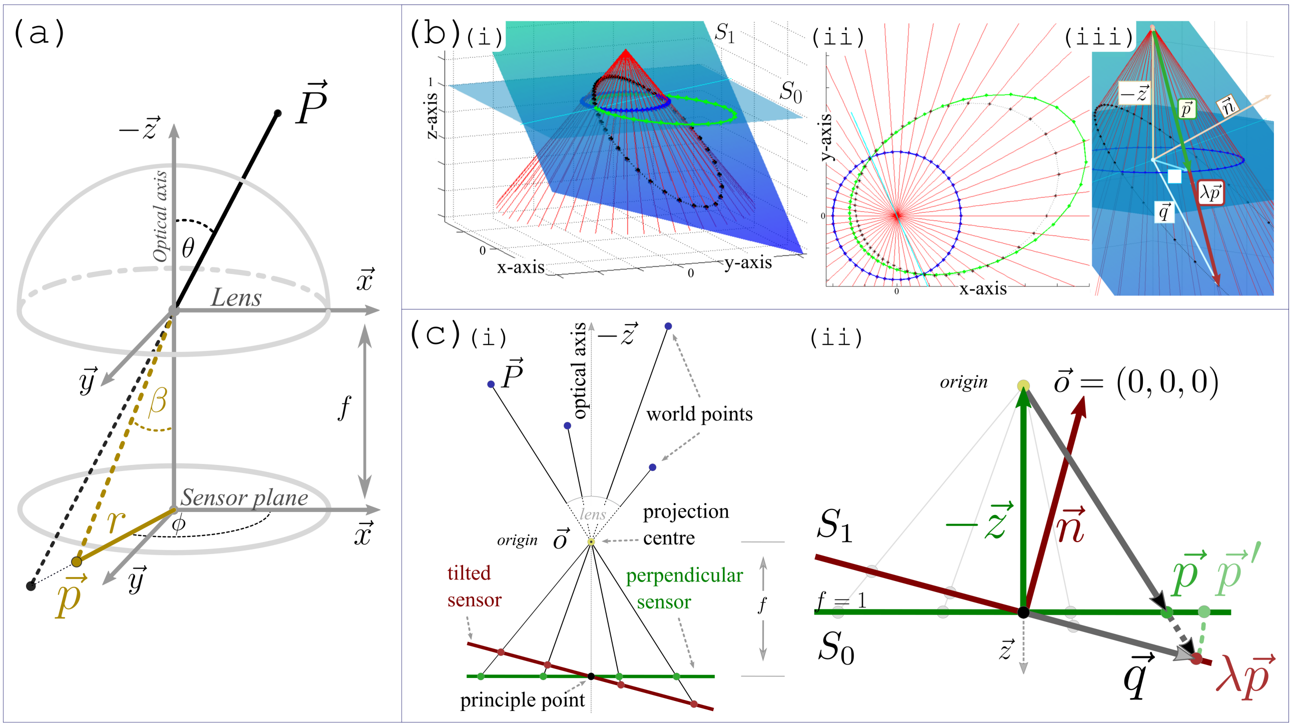

Figure 1: (a) Camera model. (b) Conic sections (i) intersection of the cone with the sensors S_0 and S_1, (ii) top view of the cone-plane intersection showing the circle to ellipse transformation, (iii) illustration of the vector map. (c) Sensor tilt (i) schematic, (ii) vector map.

Pixel coordinates $\hat{p}' \mapsto \hat{x}$: The mapping from homogeneous camera coordinates to pixel coordinates is the linear transformation $\hat{x} = \hat{K}\hat{p}'$, where $\hat{K}$ is the intrinsic camera matrix containing the xy-focal lengths (f_x, f_y) and the optical centre (x_c, y_c). This concludes our outline of the generic camera model.

2 Inverse camera model

For the purposes of camera calibration it is equally important that the inverse mappings of the camera model are known. The inverse camera model begins with the imaged 2D target corners $\hat{x}$, unfolding until the registered 3D target corners $\hat{X}$.

$$\hat{x} \xmapsto{\text{intrinsics}} \hat{p}' \xmapsto{\text{sensor}} \hat{p} \xmapsto{\text{lens}} \hat{P} \xmapsto{\text{extrinsics}} \hat{X}$$

The inverse mappings are specifically used in the calculation of the Jacobian, and non-linear optimization of the camera parameters. As sensor tilt via conic sections is the only new consideration to the calibration protocol, here we confine our discussion to the inverted map of the tilt shifted points.

Tilt compensated coordinates $\hat{p}' \mapsto \hat{p}$: Given the tilt distorted points $\vec{p}'$ generated by a sensor of known tilt $\vec{n}$, the tilt compensated coordinates $\vec{p}$ are obtained by first shifting the points to $\vec{o}$, performing the inverse rotation $\hat{R}^{\dagger}$, shifting back, and then rescaling through division by the z component.

$$\vec{p} = \lambda' \left(\hat{R}^{\dagger} \left(\vec{p}' - \vec{z} \right) + \vec{z} \right) \tag{5}$$

where the scaling factor λ' is the pinhole projection,

$$\lambda' = \left(n_x p'_x + n_y p'_y + 1 \right)^{-1} \tag{6}$$

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The tilt corrected map $\vec{p}\,' \mapsto \vec{p}$:

$$p_x = \left(\left(n_x^2 + n_z - 1\right) p_x{}' + n_x n_y p_y{}'\right) \left(n_x p_x{}' + n_y p_y{}' + 1\right)^{-1} (n_z - 1)^{-1} \tag{7a}$$

$$p_y = \left(\left(n_y^2 + n_z - 1\right) p_y{}' + n_x n_y p_x{}'\right) \left(n_x p_x{}' + n_y p_y{}' + 1\right)^{-1} (n_z - 1)^{-1} \tag{7b}$$

3 Summary and outlook

We have shown that conic sections and non-linear maps are a geometric transformation that describes pixel distortions due to a tilted ccd sensor, in a natural way. We believe that the conic sections model for sensor-lens misalignment is of great interest for camera calibration algorithms when and where precision is paramount.

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