# Adiabatic Growth of Black Holes in a Plummer Sphere

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## 1 Spherical Collisionless Systems

In a collisionless system, the constituent particles (e.g., stars or dark matter particles) move under the influence of the smooth gravitational potential of the system $\Phi(\mathbf{r}, t)$ (created by all other particles); that is, particles don't react to other particles, but rather the entire system itself. In this case, the state of the system can be described by a distribution function (DF) $f(\mathbf{r}, \mathbf{v}, t)$; the quantity $f(\mathbf{r}, \mathbf{v}, t)\, d^3\mathbf{r} d^3\mathbf{v}$ is the number (or mass) of particles having positions in the volume $d^3\mathbf{r}$ centred on $\mathbf{r}$ and having the range of velocities $d^3\mathbf{v}$ centred on $\mathbf{v}$.

Any physical quantity can be calculated from the DF; for example, the density is

$$\rho(\mathbf{r}) = \int f d^3\mathbf{v},$$

and various moments of the velocity components $v_i$ can be computed from

$$\langle v_i^m v_j^n \rangle = \frac{1}{\rho} \int v_i^m v_j^n f \, d^3\mathbf{v},$$

Given the density, the gravitational potential can be found from Poisson's equation,

$$\nabla^2 \Phi = 4\pi G \rho(\mathbf{r}),$$

and once we have the potential, the motion of particles is given by Newton's second law.

Now, in the case that the DF is *steady-state* (no time dependence) and spherically symmetric in all its properties, the system will admit four integrals of motion: the energy (per unit mass) $E = v^2/2 + \Phi$, and the three components of the angular momentum $\mathbf{J}$. Furthermore, the DF will depend only on the energy and the magnitude of the angular momentum: $f = f(E, J)$ only.

Let's write the velocity in terms of a *radial* component and a *tangential* (or *transverse*) component:

$$v^2 = v_r^2 + v_t^2.$$

The transverse component is related to the angular momentum, since from

$$\mathbf{J} = \mathbf{r} \times \mathbf{v}$$

we have

$$J = r v_t, \tag{1}$$

so the energy can be written

$$E = \frac{v_r^2}{2} + \frac{J^2}{2r^2} + \Phi,$$

or, rearranged for the radial velocity, we have

$$v_r = \sqrt{2(E - \Phi) - J^2/r^2}. \tag{2}$$

Note that since the radial velocity can't be imaginary, the angular momentum has a maximum possible value

$$J_{\max} = \sqrt{2r^2(E - \Phi)}. \tag{3}$$

Finally, we're ready to transform from $(\mathbf{r}, \mathbf{v})$ to $(E, J)$ space. In terms of the energy and angular momentum, the density becomes

$$\rho(r) = 4\pi \int_{\Phi(r)}^{\Phi(\infty)} dE \int_0^{J_{\max}} \frac{J\,dJ}{r^2 v_r} f(E, J), \tag{4}$$

and the velocity moments are

$$\langle v_r^m v_t^n \rangle = \frac{4\pi}{\rho} \int_{\Phi(r)}^{\Phi(\infty)} dE \int_0^{J_{\max}} \frac{J\,dJ}{r^2 v_r} v_r^m v_t^n f(E, J). \tag{5}$$

Poisson's equation, for spherical symmetry, becomes

$$\Phi(r) = 4\pi G \int_0^r \rho(s) s\, ds - \frac{GM(r)}{r}, \tag{6}$$

where $M(r)$ is the total mass within a radius $r$,

$$M(r) = 4\pi \int_0^r \rho(s) s^2\, ds. \tag{7}$$

# 2 The Plummer Sphere

We're investigating the growth of supermassive black holes in dark matter halos, and to start we need a simple model of the halo. Although it's not a *great* model, the Plummer sphere will do for now; it's defined by the DF

$$f(E, J) = \frac{24\sqrt{2}}{7\pi^3} \frac{a^2}{G^5 M^4} (-E)^{7/2}, \tag{8}$$

where $a$ is a scale radius and $M$ is the total mass of the system. Normally we'll take $G = M = a$. The gravitational potential of the Plummer sphere is

$$\Phi(r) = -\frac{GM}{a}\frac{1}{\sqrt{1 + r^2/a^2}} \tag{9}$$

and the density is

$$\rho(r) = \frac{3}{4\pi}\frac{M}{a^3}\frac{1}{(1 + r^2/a^2)^{5/2}}. \tag{10}$$

From the above equations, we can calculate

$$\langle v_r^2 \rangle = \frac{GM}{6a}\frac{1}{\sqrt{1 + r^2/a^2}} \tag{11}$$

and

$$\langle v_t^2 \rangle = \frac{GM}{3a}\frac{1}{\sqrt{1 + r^2/a^2}}. \tag{12}$$

# 3 Adiabatic Growth of Black Holes

See thesis for details, too lazy to write it out here. The Jupyter notebook also has details. Just the plot: