Understanding the Relationship Between the Topological Entropy of a Sinai Billiard Map and the Scatterer Radii Through Numerical Methods

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Introduction

The Sinai Billiard Map is a discontinuous map where a particle moves around in a flat torus, or a billiard table. There are two scatterers, one in the center and one in the corners, that are appropriately sized so that the Billiard has finite horizon i.e. the distance the particle travels between collisions is finite.

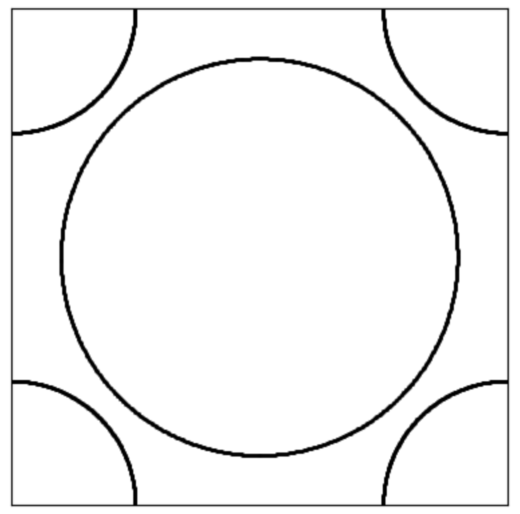

Figure 1. Billiard table with finite horizon with $r_1 = 0.4$ and $r_2 = 0.25$.

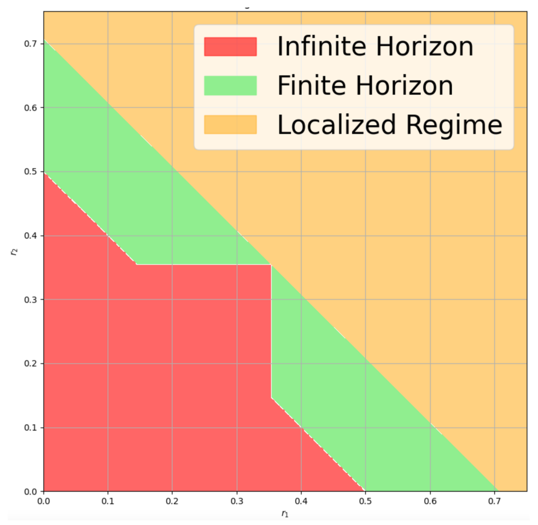

Figure 2. Parameter space for the radii of the scatterers

Problem Formulation

A Sinai Billiard is a space $Q = \mathbb{T}^2 \backslash \mathrm{B}$, where $\mathbb{T}^2$ is the two-torus and B is the space of the two scatterers. Let the scatterer with radius r_1 be the center scatterer and let the corner scatterers have a radius of r_2. A billiard table of unit size has **finite horizon** if it satisfies the following:

$½ < r_1 + r_2 < \frac{\sqrt{2}}{2}$ and $\max(\mathrm{r}_1, \mathrm{r}_2) > \frac{\sqrt{2}}{4}$.

Let $\mathcal{M} \coloneqq \partial Q \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the **phase space** of the system. Let $x = (r, \varphi) \in \mathcal{M}$, where r is the arclength along a scatterer and φ is the angle between the normal of the scatterer at r and the post-collision trajectory. We then define the **billiard map** $T: M \to M$ by $T(x)$ being the next collision of a particle at $x = (r, \varphi)$.

Let $\mathcal{S}_0 = \left\{(r, \varphi) \in \mathcal{M}: \varphi = \pm\frac{\pi}{2}\right\}$ be the set of all tangential collisions. Then we can define the **singularity set** for $T^{\pm n}$ by $\mathcal{S}_{\pm n} = \cup_{i=0}^{n} T^{\mp i} \mathcal{S}_0$ for all $n \in \mathbb{N}$. For $k, n \geq 0$ we can let $\mathcal{M}_{-k}^{n} = M \backslash (\mathcal{S}_{-k} \cup \mathcal{S}_n)$ be the partition of M into its maximally connected components.

The **Topological Entropy** of the system is the defined to be $h_* = h_*(T) \coloneqq \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n$.

We want to understand how h_* depends on the radii of the scatterers.

Methods

A particle of moves through the Billiard Table Q with unit speed and undergoes specular reflections when it collides with a scatterer. When the particle reaches a boundary, it crosses over and comes on the other side. A particle with the initial condition $x_0 = (r_0, \varphi_0)$ has its nth collision given by $T^n(x_0)$.

Because the particle moves around in a flat torus, we can record its **global grid position** as $(\mathrm{x_g}, \mathrm{y_g})$ with initial $(\mathrm{x_g}, \mathrm{y_g}) = (0,0)$, and either component is incremented by ± 1 whenever the particle crosses a boundary of Q.

Let $A = \{(0,0), (1,1), (1,0), (0,1), (½, ½)\}$ be the set of the local center of the scatterers in cartesian coordinates. If the particle has its nth collision with a scatterer with local center $(a, b) \in A$ and has a global grid position at $(\mathrm{x_g}, \mathrm{y_g})$, we say its **global position** is $(\mathrm{x}, \mathrm{y}) = (a + \mathrm{x_g}, \mathrm{b} + \mathrm{y_g})$. If $(\tilde{x}, \tilde{y})$ is the global position of the $(n-1)$th collision, then we say that the ***n*th symbol of the trajectory** with initial condition x_0 is $\mathrm{s_n} = (\mathrm{x} - \tilde{x}, \mathrm{y} - \tilde{y}, z)$, where z indicates if the particle passed by an intermediate scatterer and if it went over or under it. We say that $\mathrm{w} = \mathrm{s_1 s_2} \ldots \mathrm{s_n}$ is a **valid trajectory** of length n if there exists an $x_0 \in \mathcal{M}$ that generates it after n collisions. Let L_n be the **set of all valid trajectories** of length n.

Since each partition of the phase space corresponds to a unique trajectory, we can approximate h_* by generating N unique initial conditions and computing $h_j(N) = \frac{1}{j} \log \#L_j$, since $h_j(N) \to h_*$ as $N, j \to \infty$.

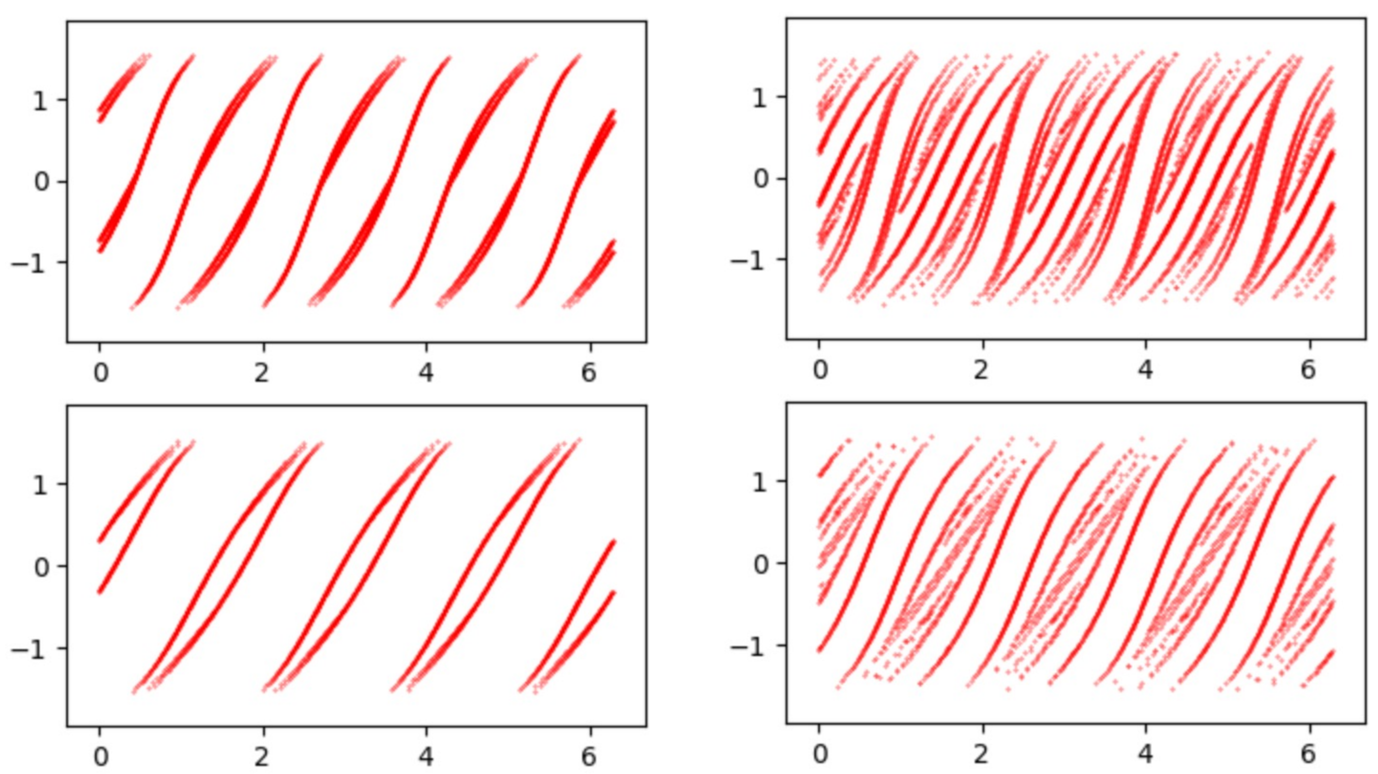

Figure 3. Singularity sets $\mathcal{S}_1$ and $\mathcal{S}_2$ with $r_1 = 0.4$ and $r_2 = 0.25$ with 12000 initial tangential collisions.

Results

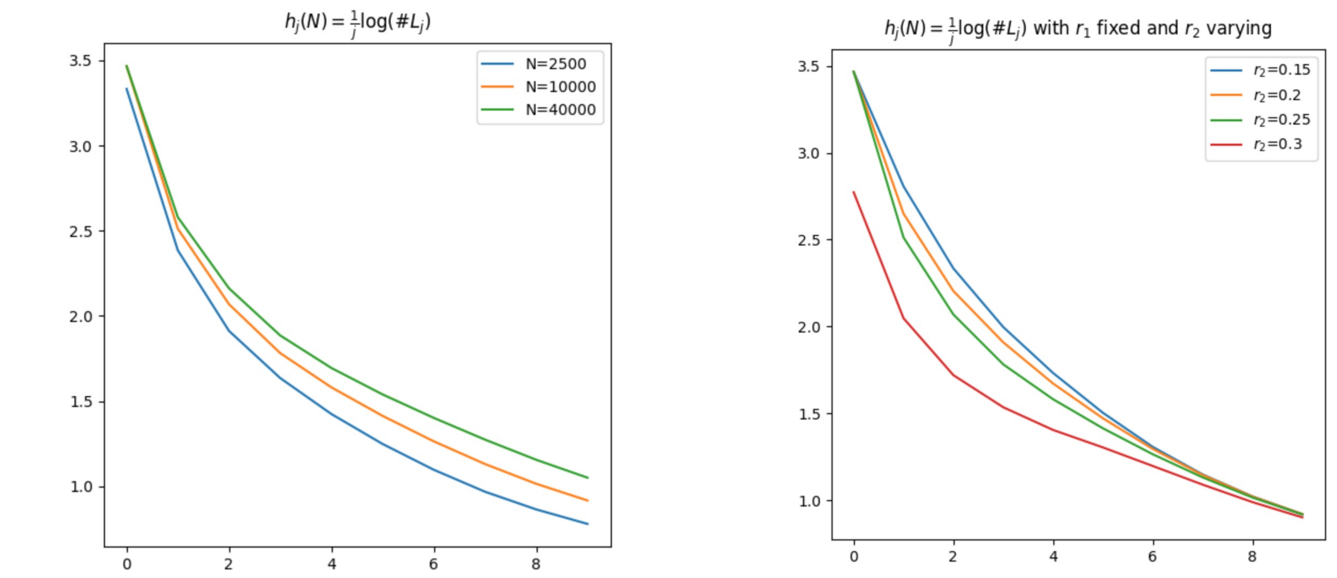

Figure 4. j vs. $h_j(N)$ for different values of N with $r_1 = 0.4$ and $r_2 = 0.25$ (left). j vs. $h_j(N)$ with r_1 fixed and r_2 varied and $N = 10000$ (right).

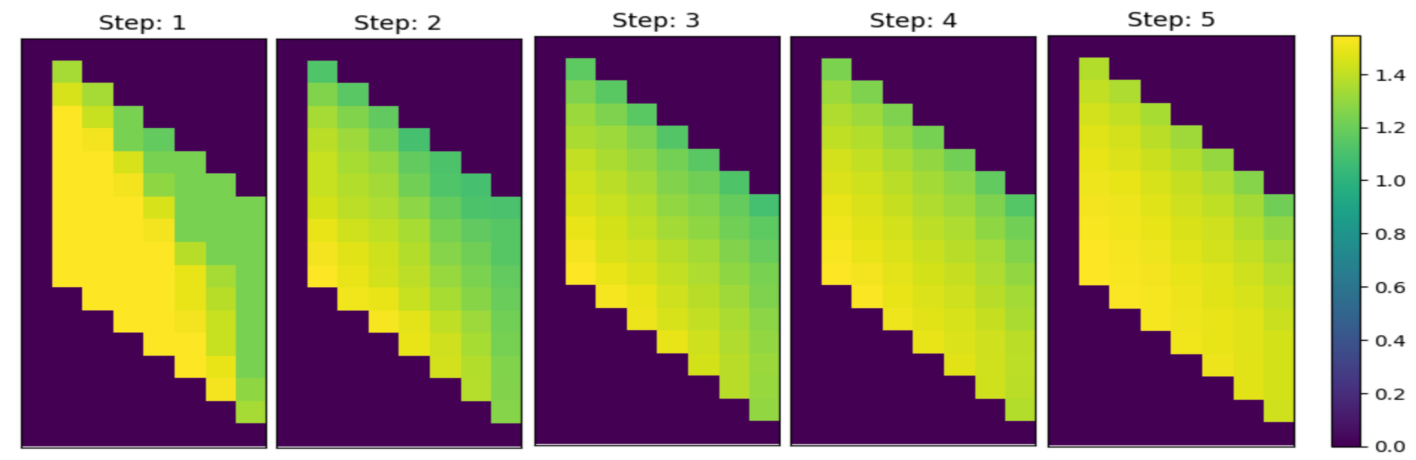

Figure 5. Heatmap of $h_j(N)$ for different values of r_1 (x-axis) and r_2 (y-axis) with $j = 1,2,3,4,5$ and $N = 2500$. We only consider $r_1 > r_2$ due to symmetry.

- Although we only check small values of j, we suspect the behavior to stay consistent as $j \to \infty$.
- There exists a clear relationship between scatterer radii and topological entropy: The smaller the combined value of the radii, the higher the entropy.
- This is due to higher curvature on the scatterer when its radius is decreased, allowing for trajectories to realize more symbols.

References

[1] Baladi and Demers, January 2020, *On the measure of maximal entropy for finite horizon Sinai billiard maps*, Journal of the American Mathematical Society
[2] Gaspard and Baras, June 1995, *Chaotic scattering and diffusion in the Lorentz gas*, Physical Review E

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