Chapter 1 Vector Spaces

LEARNING OBJECTIVES

- Basic Properties of Complex Numbers
- $\mathbb{R}^n$ and $\mathbb{C}^n$
- Vector Spaces
- Subspaces
- Sums and direct sums of subspaces

1.A $\mathbb{R}^n$ and $\mathbb{C}^n$

Complex Numbers

Definition 1.1 Complex Numbers

- A **complex number** is an ordered pair (a, b) where $a, b \in \mathbb{R}$, and is written as $a + bi$.
- The set of all complex numbers is denoted by the set $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R}\}.$$

- **Addition and Multiplication** on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

  where $a, b, c, d \in \mathbb{R}$.

Definition 1.2 Properties of Complex Arithmetic

- **Commutativity**: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$
- **Associativity**: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$
- **Identities**: $(\lambda + 0) = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$
- **Additive Inverse**: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
- **Multiplicative Inverse**: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$
- **Distributive Property**: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Definition 1.3 Inverse, Subtraction, and Division of $\mathbb{C}$

Let $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ be the additive inverse of α. Thus $-\alpha$ is the unique complex number such that $\alpha + (-\alpha) = 0$

- **Subtraction** on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha)$$

- For $\alpha \neq 0$, let $1/\alpha$ be the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that $\alpha(1/\alpha) = 1$
- **Division** on $\mathbb{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha)$$

Lists

Definition 1.4 Lists

- A **list** of **length** n, where n is a nonnegative integer, is an ordered collection of n elements.
- A list of length n looks like this: $(x_1, \ldots, x_n)$
- A list can contain numbers, other lists abstract entities, etc.
- Two lists are equal *if and only if* they have the same length with the same elements in the same order.

Definition 1.5 List of $\mathbb{F}^n$

- $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$ such that

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

- We say x_j is the jth **coordinate** of $(x_1, \ldots, x_n)$.

Definition 1.6 Addition in $\mathbb{F}^n$

- **Addition** in $\mathbb{F}^n$ is defined by adding the corresponding coordinates in two lists of the same length:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Proof 1.7 Commutativity of addition in $\mathbb{F}^n$

- If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.
- **Proof:** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then,

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x. \square \end{aligned}$$

Definition 1.8 Zero (0)

- Let 0 denote the list of length n whose coordinates are all 0: $0 = (0, \ldots, 0)$.

Definition 1.9 Additive Inverse in $\mathbb{F}^n$

- For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted as $-x$, is the vector $-x \in \mathbb{F}^n$ such that $x + (-x) = 0$. In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

Definition 1.10 Scalar Multiplication in $\mathbb{F}^n$

- The **product** of a number λ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ.

$$\lambda(x_1, \ldots x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

  where $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

1.B Vector Spaces

Vector Spaces

Definition 1.11 Addition and Scalar Multiplication

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Definition 1.12 Vector Space

- A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
  - **Commutativity** (Ordering)
  - **Associativity** (Grouping)
  - **Additive Identity** (There is a $0 \in V$ such that $v + 0 = v$ for all $v \in V$)
  - **Additive Inverse** (For every $v \in V$, there exists $w \in V$ such that $v + w = 0$)
  - **Multiplicative Identity** ($1v = v$ for all $v \in V$)
  - **Distributive Properties**

Definition 1.13 Vectors and Points

- Elements of a vector space are called **vectors** or **points**.

Definition 1.14 Vector Space over $\mathbb{F}$

- A vector space over $\mathbb{R}$ is called a **real vector space**.
- A vector space over $\mathbb{C}$ is called a **complex vector space**.

- The scalar multiplication in a vector space depends on $\mathbb{F}$, so we will say that V is a vector space over $\mathbb{F}$.
  - $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$.

Definition 1.15 Set of Functions

- If S is a set, then $\mathbb{F}^S$ is the set of functions from S to $\mathbb{F}$.
- For $f, g \in \mathbb{F}^S$, the **sum** $f + g \in \mathbb{F}^S$ is the function defined by $(f+g)(x) = f(x) + g(x)$ for all $x \in S$.
- For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$.

Note 1.16 Unique Additive Properties

- A vector space has a unique additive identity.
- Every element in a vector space has a unique additive inverse.

1.C Subspaces

Subspaces

Definition 1.17 Subspace

- A subset U of V is a **subspace** of V if U is also a vector space.

Theorem 1.18 Conditions of a Subspace

- A subset U of V is a subspace of V if and only if U satisfies the following 3 conditions:
  - Additive Identity: $0 \in U$
  - Closed under Addition: $u, w \in U$ implies $u + w \in U$
  - Closed under Scalar Multiplication: $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

Sum of Subspaces

Definition 1.19 Sub of Subsets

- Suppose $U_1, \ldots, U_m$ are subsets of V. The **sum** of $U_1, \ldots U_m$, denoted by $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$.
- $U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \ : \ u_1 \in U, \ldots, u_m \in U\}$

Theorem 1.20 Sum of Subspaces is the Smallest Containing Subspace

- Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Direct Sums

Definition 1.21 Direct Sums

- Suppose $U_1, \ldots, U_m$ are subspaces of V.
  - The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
  - If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, where the $\oplus$ symbol indicates that this is a direct sum.

Theorem 1.22 Condition for a direct sum

Theorem 1.23 Direct Sum of Two Subspaces

proof