

勉強したことのまとめ

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Chapter 1

07/08/2024

1.1 光の測地線方程式

粒子の場を表す方程式である、クラインゴルドン方程式

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right) \phi(\mathbf{x}, t) = 0$$

質量のない光では

$$\begin{aligned} 0 &= \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi(\mathbf{x}, t) \\ &= \square \phi(\mathbf{x}, t) \end{aligned}$$

これを一般の時空に拡張すると

$$g^{ij} \nabla_i \nabla_j \phi = 0$$

ここで、

$$\phi = C e^{\frac{is}{\epsilon}}$$

と書くと

$$\begin{aligned} \nabla_j \phi &= \nabla \left(C e^{\frac{is}{\epsilon}} \right) \\ &= (\nabla_j C) e^{\frac{is}{\epsilon}} + \frac{iC}{\epsilon} e^{\frac{is}{\epsilon}} (\nabla_j S) \\ \nabla_i \nabla_j \phi &= \nabla_i \left((\nabla_j C) e^{\frac{is}{\epsilon}} + \frac{iC}{\epsilon} e^{\frac{is}{\epsilon}} (\nabla_j S) \right) \\ &= (\nabla_i \nabla_j C) e^{\frac{is}{\epsilon}} + (\nabla_i S) (\nabla_j C) \frac{2i}{\epsilon} e^{\frac{is}{\epsilon}} + (\nabla_i \nabla_j S) \frac{iC}{\epsilon} e^{\frac{is}{\epsilon}} - (\nabla_i S) (\nabla_j S) \frac{1}{\epsilon^2} e^{\frac{2is}{\epsilon}} \\ g^{ij} \nabla_i \nabla_j \phi &= g^{ij} (\nabla_i \nabla_j C) e^{\frac{is}{\epsilon}} + g^{ij} (\nabla_i S) (\nabla_j C) \frac{2i}{\epsilon} e^{\frac{is}{\epsilon}} + g^{ij} (\nabla_i \nabla_j S) \frac{iC}{\epsilon} e^{\frac{is}{\epsilon}} - g^{ij} (\nabla_i S) (\nabla_j S) \frac{1}{\epsilon^2} e^{\frac{2is}{\epsilon}} \\ &= (\square C) e^{\frac{is}{\epsilon}} + (\nabla_i S) (\nabla^i C) \frac{2i}{\epsilon} e^{\frac{is}{\epsilon}} + (\square S) \frac{iC}{\epsilon} e^{\frac{is}{\epsilon}} - (\nabla_i S) (\nabla^i S) \frac{1}{\epsilon^2} e^{\frac{2is}{\epsilon}} \\ &\begin{cases} O(\epsilon^{-2}) : (\nabla_i S) (\nabla^i S) = 0 \\ O(\epsilon^{-1}) : 2 (\nabla_i S) (\nabla^i C) + C (\square S) = 0 \\ O(0) : \square C = 0 \end{cases} \end{aligned} \tag{1.1}$$

ここで、

$$(\nabla_i S) (\nabla^i S) = 0$$

は光の測地線方程式を表している。

$$\nabla_i S = k_i$$

とおけば

$$\begin{aligned}
0 &= (\nabla_i S) (\nabla^i S) \\
&= k_i k^i \\
&= \nabla_j (k_i k^i) \\
&= \nabla_j (k_i g^{ij} k_l) \\
&= g^{il} (\nabla_j k_l) k_i + g^{il} (\nabla_j k_i) k_l \\
&= 2 (\nabla_j k_i) k^i \\
&= 2 g^{lj} (\nabla_j \nabla_i S) k^i \\
&= 2 (\nabla_i g^{lj} k_j) k^i \\
&= 2 (\nabla_i k^l) k^i \\
&= 2 \left(\frac{\partial}{\partial x^i} \left(\frac{dx^l}{d\lambda} \right) + \Gamma^l_{im} \left(\frac{dx^m}{d\lambda} \right) \right) \frac{dx^i}{d\lambda}
\end{aligned}$$

光の測地線方程式

$$0 = \left(\frac{d^2 x^l}{d\lambda^2} \right) + \Gamma^l_{im} \left(\frac{dx^i}{d\lambda} \frac{dx^m}{d\lambda} \right)$$

1.2 積分

$$\begin{aligned}
\left(\frac{d\mu}{d\phi} \right)^2 &= 2MG(\mu) \\
\frac{d\mu}{d\phi} &= -\sqrt{2MG(\mu)} \\
\begin{cases} \mu_1 = -\frac{Q+2M-P}{4MP} \\ \mu_2 = -\frac{1}{P} \\ \mu_3 = -\frac{Q-2M+P}{4MP} \end{cases} & \quad (1.2)
\end{aligned}$$

$$\begin{aligned}
\int_0^{\phi_\infty} d\phi &= \int_{\mu_2}^0 \frac{d\mu}{-\sqrt{2MG(\mu)}} \\
\phi_\infty &= \frac{1}{\sqrt{2M}} \int_0^{\mu_2} d\mu ((\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3))^{-\frac{1}{2}}
\end{aligned}$$

ここで、

$$\begin{aligned}
\chi &= \sqrt{\mu - \mu_1} \\
d\chi &= \frac{1}{2} (\mu - \mu_1)^{-\frac{1}{2}} d\mu
\end{aligned}$$

とおくと

$$\phi_\infty = \frac{1}{\sqrt{2M}} \int_{\sqrt{-\mu_1}}^{\sqrt{\mu_2 - \mu_1}} \frac{2\chi d\chi}{\sqrt{\chi^2(\chi^2 + \mu_1 - \mu_2)(\chi^2 + \mu - \mu_3)}}$$

ここで、

$$\begin{aligned}\chi &= \sqrt{\mu_2 - \mu_1} \eta \\ d\chi &= \sqrt{\mu_2 - \mu_1} d\eta\end{aligned}$$

とおくと

$$\phi_\infty = \frac{2}{\sqrt{2M}} \int_{\sqrt{\frac{\mu_1}{\mu_1 - \mu_2}}}^1 \frac{\sqrt{\mu_2 - \mu_1} d\eta}{\sqrt{-(\mu_2 - \mu_1)(1 - \eta^2)((\mu_2 - \mu_1)\eta^2 - \mu_1 + \mu_3)}}$$

ここで、

$$\begin{aligned}\eta &= \sin x \\ d\eta &= dx \cos x\end{aligned}$$

とおくと

$$\begin{aligned}\phi_\infty &= \frac{2}{\sqrt{2M}} \int_{\sin^{-1} \sqrt{\frac{\mu_1}{\mu_1 - \mu_2}}}^{\frac{\pi}{2}} \frac{\cos x dx}{\sqrt{\cos^2 x \sqrt{-(\mu_2 - \mu_1) \sin^2 x ((\mu_2 - \mu_1)\eta^2 - \mu_1 + \mu_3)}}} \\ &= \frac{2}{\sqrt{2M}} \int_{\sin^{-1} \sqrt{\frac{\mu_1}{\mu_1 - \mu_2}}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\mu_3 - \mu_1} \sqrt{-\frac{\mu_2 - \mu_1}{\mu_3 - \mu_1} \sin^2 x + 1}} \\ &\quad \begin{cases} \mu_3 - \mu_1 = \frac{Q}{2MP} \\ \mu_2 - \mu_1 = \frac{Q + 6M - P}{4MP} \end{cases} \end{aligned} \tag{1.3}$$

$$\begin{cases} \frac{\mu_2 - \mu_1}{\mu_3 - \mu_1} = \frac{Q + 6M - P}{2Q} \\ \frac{\mu_1}{\mu_1 - \mu_2} = \frac{Q + 2M - P}{Q + 6M - P} \end{cases} \tag{1.4}$$

$$\phi_\infty = \frac{2}{\sqrt{2M}} \int_{\sin^{-1} \sqrt{\frac{Q+2M-P}{Q+6M-P}}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\frac{Q}{2MP}} \sqrt{1 - \frac{Q+6M-P}{2Q} \sin^2 x}}$$

ここで、

$$\begin{aligned}\kappa &= \sqrt{\frac{Q + 6M - P}{2Q}} \\ \zeta &= \sin^{-1} \sqrt{\frac{Q + 2M - P}{Q + 6M - P}}\end{aligned}$$

さらに

$$\begin{aligned}K(\kappa) &= \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}} \\ F(\zeta, \kappa) &= \int_0^\zeta \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}}\end{aligned}$$

とおくと

ϕ_∞ についての式

$$\begin{aligned}\phi_\infty &= 2\sqrt{\frac{P}{Q}} \int_\zeta^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}} \\ &= 2\sqrt{\frac{P}{Q}} (K(\kappa) - F(\zeta, \kappa))\end{aligned}$$

Chapter 2

28/08/2024

2.1 $P \gg M$

$$\begin{cases} Q \approx P \left(1 + 2 \left(\frac{M}{P} \right) - 8 \left(\frac{M}{P} \right)^2 \right) \\ \kappa^2 \approx 4 \left(\frac{M}{P} \right) \\ \zeta \approx \frac{\pi}{4} - \frac{1}{2} \left(\frac{M}{P} \right) \end{cases}$$

以上を考慮して ϕ_∞ を計算する

$$\begin{aligned} \phi_\infty &= 2 \sqrt{\frac{P}{Q}} \int_{\zeta}^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}} \\ &\approx 2 \sqrt{\frac{P}{Q}} \int_{\zeta}^{\frac{\pi}{2}} \left(1 + \frac{1}{2} \kappa^2 \sin^2 x \right) dx \\ &= 2 \sqrt{\frac{P}{Q}} \int_{\zeta}^{\frac{\pi}{2}} \left(1 + \frac{1}{2} \kappa^2 - \frac{1}{4} \kappa^2 \cos 2x \right) dx \\ &\approx 2 \left(1 - \frac{M}{P} \right) \int_{\frac{\pi}{4} - \frac{M}{2P}}^{\frac{\pi}{2}} \left(1 + 2 \frac{M}{P} - \frac{M}{P} \cos 2x \right) dx \\ &= 2 \left(1 - \frac{M}{P} \right) \left[x + \frac{M}{P} x - \frac{M}{2P} \sin 2x \right]_{\frac{\pi}{4} - \frac{M}{2P}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + \frac{2M}{P} \end{aligned}$$

$$2.2 \quad P \approx 3M(1 + \epsilon)$$

$$\left\{ \begin{array}{l} Q \approx 3M + 5M\epsilon \\ \kappa \approx 1 - \frac{2}{3}\epsilon \\ \zeta \approx \sin^{-1} \sqrt{\frac{1}{3}} + \frac{\sqrt{3}}{9 \cos \left(\sin^{-1} \sqrt{\frac{1}{3}} \right)} \epsilon \\ \sqrt{\frac{P}{Q}} \approx 1 - \frac{1}{3}\epsilon \\ \sqrt{1 - \kappa^2} \approx \sqrt{\frac{4}{3}}\epsilon \\ b \approx b_c \left(1 + \frac{3}{2}\epsilon^2 \right) \end{array} \right.$$

$\kappa \approx 1$ での公式

$$K(\kappa) = \ln \frac{4}{\kappa'} + \left(\frac{1}{2} \right)^2 \left(\ln \frac{4}{\kappa'} - \frac{2}{1 \cdot 2} \right) \kappa'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \left(\ln \frac{4}{\kappa'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} \right) \kappa'^4 + \dots$$

$$F(\zeta, \kappa) = \frac{2}{\pi} \mathbf{K}' \ln \tan \left(\frac{\zeta}{2} + \frac{\pi}{4} \right) - \frac{\tan \zeta}{\cos \zeta} \left(a'_0 - \frac{2}{3} a'_1 \tan^2 \zeta + \dots \right)$$

$$\left\{ \begin{array}{l} a'_0 = \frac{2}{\pi} \mathbf{K}' - 1 \\ a'_n = a'_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 \kappa'^{2n} \end{array} \right.$$

$$\mathbf{K}' = \frac{\pi}{2} \left(1 + \left(\frac{1}{2} \right)^2 \kappa'^2 + \dots \right)$$

$$\kappa' = \sqrt{1 - \kappa^2}$$

以上を考慮して b を計算する

$$K(\kappa) \approx \ln \frac{4}{\kappa'}$$

$$F(\zeta, \kappa) \approx \ln \tan \left(\frac{\zeta}{2} + \frac{\pi}{4} \right)$$

$$\begin{aligned}
\phi_\infty &= 2\sqrt{\frac{P}{Q}}(K(\kappa) - F(\zeta, \kappa)) \\
&\approx 2\left(1 - \frac{\epsilon}{3}\right)\left(\ln \frac{4}{\kappa'} - \ln \tan\left(\frac{\zeta}{2} + \frac{\pi}{4}\right)\right) \\
&= \left(1 - \frac{\epsilon}{3}\right)\ln\left(\frac{\frac{12}{\epsilon}}{\tan^2\left(\frac{\sin^{-1}\sqrt{\frac{1}{3}}}{2} + \frac{\pi}{4}\right)}\right) \\
&\approx \ln\left(\frac{12}{\tan^2\left(\frac{\sin^{-1}\sqrt{\frac{1}{3}}}{2} + \frac{\pi}{4}\right)\epsilon}\right) = \ln \frac{\alpha}{\epsilon}
\end{aligned}$$

ここで、

$$\phi_\infty = \frac{1}{2}\pi + \frac{1}{2}\mu$$

より、

$$\begin{aligned}
\frac{\alpha}{\epsilon} &= \exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right) \\
\epsilon &= \frac{\alpha}{\exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right)}
\end{aligned}$$

したがって

$$\begin{aligned}
b &\approx b_c\left(1 + \frac{3}{2}\epsilon^2\right) \\
&= b_c\left(1 + \frac{3}{2}\left(\frac{\alpha}{\exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right)}\right)^2\right) \\
&= b_c\left(1 + \frac{3}{2}\alpha^2 e^{-\mu} e^{-\pi}\right) \\
&\approx 5.19615M + 3.48228M e^{-\mu}
\end{aligned}$$

Chapter 3

18/09/2024

3.1 降着円盤から発せられた光

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$$\bar{w}_j = () \int$$

メモ用

$$1 = 0$$