勉強したことのまとめ

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Contents

Chapter 1

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1.1　光の測地線方程式

粒子の場を表す方程式である、クラインゴルドン方程式

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right)\phi(\boldsymbol{x}, t) = 0$$

質量のない光では

$$\begin{aligned} 0 &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi(\boldsymbol{x}, t) \\ &= \Box\phi(\boldsymbol{x}, t) \end{aligned}$$

これを一般の時空に拡張すると

$$g^{ij}\nabla_i\nabla_j\phi = 0$$

ここで、

$$\phi = Ce^{i\frac{s}{\epsilon}}$$

と書くと

$$\begin{aligned} \nabla_j\phi &= \nabla\left(Ce^{\frac{is}{\epsilon}}\right) \\ &= \left(\nabla_j C\right)e^{\frac{is}{\epsilon}} + \frac{iC}{\epsilon}e^{\frac{is}{\epsilon}}\left(\nabla_j S\right) \\ \nabla_i\nabla_j\phi &= \nabla_i\left(\left(\nabla_j C\right)e^{\frac{is}{\epsilon}} + \frac{iC}{\epsilon}e^{\frac{is}{\epsilon}}\left(\nabla_j S\right)\right) \\ &= \left(\nabla_i\nabla_j C\right)e^{\frac{is}{\epsilon}} + \left(\nabla_i S\right)\left(\nabla_j C\right)\frac{2i}{\epsilon}e^{\frac{is}{\epsilon}} + \left(\nabla_i\nabla_j S\right)\frac{iC}{\epsilon}e^{\frac{is}{\epsilon}} - \left(\nabla_i S\right)\left(\nabla_j S\right)\frac{1}{\epsilon^2}e^{\frac{2is}{\epsilon}} \\ g^{ij}\nabla_i\nabla_j\phi &= g^{ij}\left(\nabla_i\nabla_j C\right)e^{\frac{is}{\epsilon}} + g^{ij}\left(\nabla_i S\right)\left(\nabla_j C\right)\frac{2i}{\epsilon}e^{\frac{is}{\epsilon}} + g^{ij}\left(\nabla_i\nabla_j S\right)\frac{iC}{\epsilon}e^{\frac{is}{\epsilon}} - g^{ij}\left(\nabla_i S\right)\left(\nabla_j S\right)\frac{1}{\epsilon^2}e^{\frac{2is}{\epsilon}} \\ &= \left(\Box C\right)e^{\frac{is}{\epsilon}} + \left(\nabla_i S\right)\left(\nabla^i C\right)\frac{2i}{\epsilon}e^{\frac{is}{\epsilon}} + \left(\Box S\right)\frac{iC}{\epsilon}e^{\frac{is}{\epsilon}} - \left(\nabla_i S\right)\left(\nabla^i S\right)\frac{1}{\epsilon^2}e^{\frac{2is}{\epsilon}} \end{aligned}$$

$$\begin{cases} O(\epsilon^{-2}) : \left(\nabla_i S\right)\left(\nabla^i S\right) = 0 \\ O(\epsilon^{-1}) : 2\left(\nabla_i S\right)\left(\nabla^i C\right) + C\left(\Box S\right) = 0 \\ \quad O(0) : \Box C = 0 \end{cases} \tag{1.1}$$

ここで、

$$\left(\nabla_i S\right)\left(\nabla^i S\right) = 0$$

は光の測地線方程式を表している。

$$\nabla_i S = k_i$$

とおけば

$$\begin{aligned}
0 &= (\nabla_i S)\left(\nabla^i S\right) \\
&= k_i k^i \\
&= \nabla_j (k_i k^i) \\
&= \nabla_j (k_i g^{ij} k_l) \\
&= g^{il}(\nabla_j k_l)k_i + g^{il}(\nabla_j k_i)k_l \\
&= 2(\nabla_j k_i)k^i \\
&= 2g^{lj}(\nabla_j \nabla_i S)k^i \\
&= 2(\nabla_i g^{lj} k_j)k^i \\
&= 2(\nabla_i k^l)k^i \\
&= 2\left(\frac{\partial}{\partial x^i}\left(\frac{dx^l}{d\lambda}\right) + \Gamma^l{}_{im}\left(\frac{dx^m}{d\lambda}\right)\right)\frac{dx^i}{d\lambda}
\end{aligned}$$

光の測地線方程式

$$0 = \left(\frac{d^2 x^l}{d\lambda^2}\right) + \Gamma^l{}_{im}\left(\frac{dx^i}{d\lambda}\frac{dx^m}{d\lambda}\right)$$

1.2 積分

$$\begin{aligned}
\left(\frac{d\mu}{d\phi}\right)^2 &= 2MG(\mu) \\
\frac{d\mu}{d\phi} &= -\sqrt{2MG(\mu)}
\end{aligned}$$

$$\begin{cases}
\mu_1 = -\dfrac{Q+2M-P}{4MP} \\
\mu_2 = -\dfrac{1}{P} \\
\mu_3 = -\dfrac{Q-2M+P}{4MP}
\end{cases} \tag{1.2}$$

$$\begin{aligned}
\int_0^{\phi_\infty} d\phi &= \int_{\mu_2}^{0} \frac{d\mu}{-\sqrt{2MG(\mu)}} \\
\phi_\infty &= \frac{1}{\sqrt{2M}}\int_0^{\mu_2} d\mu\left((\mu-\mu_1)(\mu-\mu_2)(\mu-\mu_3)\right)^{-\frac{1}{2}}
\end{aligned}$$

ここで、

$$\begin{aligned}
\chi &= \sqrt{\mu-\mu_1} \\
d\chi &= \frac{1}{2}(\mu-\mu_1)^{-\frac{1}{2}}d\mu
\end{aligned}$$

とおくと

$$\phi_\infty = \frac{1}{\sqrt{2M}}\int_{\sqrt{-\mu_1}}^{\sqrt{\mu_2-\mu_1}} \frac{2\chi d\chi}{\sqrt{\chi^2(\chi^2+\mu_1-\mu_2)(\chi^2+\mu-\mu_3)}}$$

ここで、

$$\chi = \sqrt{\mu_2 - \mu_1}\eta$$
$$d\chi = \sqrt{\mu_2 - \mu_1}d\eta$$

とおくと

$$\phi_\infty = \frac{2}{\sqrt{2M}} \int_{\sqrt{\frac{\mu_1}{\mu_1-\mu_2}}}^{1} \frac{\sqrt{\mu_2 - \mu_1}d\eta}{\sqrt{-(\mu_2 - \mu_1)(1-\eta^2)((\mu_2 - \mu_1)\eta^2 - \mu_1 + \mu_3)}}$$

ここで、

$$\eta = \sin x$$
$$d\eta = dx \cos x$$

とおくと

$$\begin{aligned}
\phi_\infty &= \frac{2}{\sqrt{2M}} \int_{\sin^{-1}\sqrt{\frac{\mu_1}{\mu_1-\mu_2}}}^{\frac{\pi}{2}} \frac{\cos x dx}{\sqrt{\cos^2 x}\sqrt{-(\mu_2 - \mu_1)\sin^2 x((\mu_2 - \mu_1)\eta^2 - \mu_1 + \mu_3)}} \\
&= \frac{2}{\sqrt{2M}} \int_{\sin^{-1}\sqrt{\frac{\mu_1}{\mu_1-\mu_2}}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\mu_3 - \mu_1}\sqrt{-\frac{\mu_2-\mu_1}{\mu_3-\mu_1}\sin^2 x + 1)}}
\end{aligned}$$

$$\begin{cases} \mu_3 - \mu_1 = \dfrac{Q}{2MP} \\ \mu_2 - \mu_1 = \dfrac{Q + 6M - P}{4MP} \end{cases} \tag{1.3}$$

$$\begin{cases} \dfrac{\mu_2 - \mu_1}{\mu_3 - \mu_1} = \dfrac{Q + 6M - P}{2Q} \\ \dfrac{\mu_1}{\mu_1 - \mu_2} = \dfrac{Q + 2M - P}{Q + 6M - P} \end{cases} \tag{1.4}$$

$$\phi_\infty = \frac{2}{\sqrt{2M}} \int_{\sin^{-1}\sqrt{\frac{Q+2M-P}{Q+6M-P}}}^{\frac{\pi}{2}} \frac{dx}{\sqrt{\frac{Q}{2MP}}\sqrt{1 - \frac{Q+6M-P}{2Q}\sin^2 x}}$$

ここで、

$$\kappa = \sqrt{\frac{Q + 6M - P}{2Q}}$$
$$\zeta = \sin^{-1}\sqrt{\frac{Q + 2M - P}{Q + 6M - P}}$$

さらに

$$K(\kappa) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}}$$
$$F(\zeta, \kappa) = \int_0^{\zeta} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}}$$

とおくと

ϕ_∞ についての式

$$
\begin{aligned}
\phi_\infty &= 2\sqrt{\frac{P}{Q}}\int_{\zeta}^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\kappa^2\sin^2 x}} \\
&= 2\sqrt{\frac{P}{Q}}\left(K(\kappa)-F(\zeta,\kappa)\right)
\end{aligned}
$$

Chapter 2

28/08/2024

2.1 $P \gg M$

$$\begin{cases} Q \approx P\left(1+2\left(\dfrac{M}{P}\right)-8\left(\dfrac{M}{P}\right)^2\right) \\ \kappa^2 \approx 4\left(\dfrac{M}{P}\right) \\ \zeta \approx \dfrac{\pi}{4}-\dfrac{1}{2}\left(\dfrac{M}{P}\right) \end{cases}$$

以上を考慮して ϕ_∞ を計算する

$$\begin{aligned} \phi_\infty &= 2\sqrt{\frac{P}{Q}}\int_\zeta^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\kappa^2\sin^2 x}} \\ &\approx 2\sqrt{\frac{P}{Q}}\int_\zeta^{\frac{\pi}{2}} \left(1+\frac{1}{2}\kappa^2\sin^2 x\right)dx \\ &= 2\sqrt{\frac{P}{Q}}\int_\zeta^{\frac{\pi}{2}} \left(1+\frac{1}{2}\kappa^2-\frac{1}{4}\kappa^2\cos 2x\right)dx \\ &\approx 2\left(1-\frac{M}{P}\right)\int_{\frac{\pi}{4}-\frac{M}{2P}}^{\frac{\pi}{2}} \left(1+2\frac{M}{P}-\frac{M}{P}\cos 2x\right)dx \\ &= 2\left(1-\frac{M}{P}\right)\left[x+\frac{M}{P}x-\frac{M}{2P}\sin 2x\right]_{\frac{\pi}{4}-\frac{M}{2P}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}+\frac{2M}{P} \end{aligned}$$

2.2 $P \approx 3M(1+\epsilon)$

$$
\begin{cases}
Q \approx 3M + 5M\epsilon \\
\kappa \approx 1 - \dfrac{2}{3}\epsilon \\
\zeta \approx \sin^{-1}\sqrt{\dfrac{1}{3}} + \dfrac{\sqrt{3}}{9\cos\left(\sin^{-1}\sqrt{\frac{1}{3}}\right)}\epsilon \\
\sqrt{\dfrac{P}{Q}} \approx 1 - \dfrac{1}{3}\epsilon \\
\sqrt{1-\kappa^2} \approx \sqrt{\dfrac{4}{3}\epsilon} \\
b \approx b_c\left(1 + \dfrac{3}{2}\epsilon^2\right)
\end{cases}
$$

$\kappa \approx 1$ での公式

$$
K(\kappa) = \ln\frac{4}{\kappa'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2}\right)\kappa'^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\ln\frac{4}{\kappa'} - \frac{2}{1\cdot 2} - \frac{2}{3\cdot 4}\right)\kappa'^4 + ...
$$

$$
F(\zeta,\kappa) = \frac{2}{\pi}\boldsymbol{K'}\ln\tan\left(\frac{\zeta}{2}+\frac{\pi}{4}\right) - \frac{\tan\zeta}{\cos\zeta}\left(a'_0 - \frac{2}{3}a'_1\tan^2\zeta + ...\right)
$$

$$
\begin{cases}
a'_0 = \dfrac{2}{\pi}\boldsymbol{K'} - 1 \\
a'_n = a'_{n-1} - \left[\dfrac{(2n-1)!!}{2^n n!}\right]^2 \kappa'^{2n}
\end{cases}
$$

$$
\boldsymbol{K'} = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\kappa'^2 + ...\right)
$$

$$
\kappa' = \sqrt{1-\kappa^2}
$$

以上を考慮して b を計算する

$$
K(\kappa) \approx \ln\frac{4}{\kappa'}
$$

$$
F(\zeta,\kappa) \approx \ln\tan\left(\frac{\zeta}{2}+\frac{\pi}{4}\right)
$$

$$
\begin{aligned}
\phi_\infty &= 2\sqrt{\frac{P}{Q}}\left(K(\kappa) - F(\zeta,\kappa)\right) \\
&\approx 2\left(1 - \frac{\epsilon}{3}\right)\left(\ln\frac{4}{\kappa'} - \ln\tan\left(\frac{\zeta}{2} + \frac{\pi}{4}\right)\right) \\
&= \left(1 - \frac{\epsilon}{3}\right)\ln\left(\frac{\frac{12}{\epsilon}}{\tan^2\left(\frac{\sin^{-1}\sqrt{\frac{1}{3}}}{2} + \frac{\pi}{4}\right)}\right) \\
&\approx \ln\left(\frac{12}{\tan^2\left(\frac{\sin^{-1}\sqrt{\frac{1}{3}}}{2} + \frac{\pi}{4}\right)\epsilon}\right) = \ln\frac{\alpha}{\epsilon}
\end{aligned}
$$

ここで、

$$
\phi_\infty = \frac{1}{2}\pi + \frac{1}{2}\mu
$$

より、

$$
\begin{aligned}
\frac{\alpha}{\epsilon} &= \exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right) \\
\epsilon &= \frac{\alpha}{\exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right)}
\end{aligned}
$$

したがって

$$
\begin{aligned}
b &\approx b_c\left(1 + \frac{3}{2}\epsilon^2\right) \\
&= b_c\left(1 + \frac{3}{2}\left(\frac{\alpha}{\exp\left(\frac{\pi}{2} + \frac{\mu}{2}\right)}\right)^2\right) \\
&= b_c\left(1 + \frac{3}{2}\alpha^2 e^{-\mu} e^{-\pi}\right) \\
&\approx 5.19615M + 3.48228Me^{-\mu}
\end{aligned}
$$

Chapter 3

23/09/2024

3.1 3 次式の解

$$b^2 = \frac{P^3}{P - 2M}$$

この P を b について解きたい

$$\begin{cases} P = yM \\ b = xM \end{cases}$$

と置き、y を x について解く問題を考える。

$$\begin{aligned} 0 &= P^3 - Pb^2 + 2Mb^2 \\ 0 &= y^3M^3 - yx^2M^3 + 2x^2M^3 \\ 0 &= y^3 - yx^2 + 2x^2 \end{aligned}$$

$$y = u + v$$

とおくと

$$\begin{aligned} 0 &= (u+v)^3 - (u+v)x^2 + 2x^2 \\ 0 &= (u^3 + v^3 + 2x^2) + (3uv - x^2)(u+v) \end{aligned}$$

$$u + v \neq 0$$

と仮定すると

$$\begin{cases} u^3 + v^3 + 2x^2 = 0 \\ 3uv - x^2 = 0 \end{cases} \tag{3.1}$$

以上より、片方の符号を採用すると

$$\begin{cases} u^3 = -x^2 + x^2\sqrt{1 - \left(\dfrac{x}{3\sqrt{3}}\right)^2} \\ v^3 = -x^2 - x^2\sqrt{1 - \left(\dfrac{x}{3\sqrt{3}}\right)^2} \end{cases} \tag{3.2}$$

ここで (u, v) の 3 乗根の主値をそれぞれ (α, β) とおくと

$$\alpha = \left(-x^2 + x^2 \sqrt{1 - \left(\frac{x}{3\sqrt{3}} \right)^2} \right)^{\frac{1}{3}}$$

$$\beta = \left(-x^2 - x^2 \sqrt{1 - \left(\frac{x}{3\sqrt{3}} \right)^2} \right)^{\frac{1}{3}}$$

$$\omega = \exp\left(\frac{2}{3}\pi i \right)$$

(1) $D < 0$ では $(x > 3\sqrt{3})$

$$(u, v) = \begin{cases} (\alpha, \beta) \\ (\omega\alpha, \omega^2\beta) \\ (\omega^2\alpha, \omega\beta) \end{cases}$$

(2) $D = 0$ では $(x = 3\sqrt{3})$

$$(u, v) = \begin{cases} (\omega\alpha, \omega\beta) \\ (\alpha, \omega^2\beta) \\ (\omega^2\alpha, \beta) \end{cases}$$

(3) $D > 0$ では $(x < 3\sqrt{3})$

$$(u, v) = \begin{cases} (\omega\alpha, \omega\beta) \\ (\alpha, \omega^2\beta) \\ (\omega^2\alpha, \beta) \end{cases}$$

Chapter 4

02/12/2024

4.1 Buchdahl 時空における光の軌道

Buchdahl 時空の計量は以下のように与えられる。

$$g_{ij} = \mathrm{diag}\left(-\frac{(1-f(r))^2}{(1+f(r))^2}, (1+f(r))^4, r^2(1+f(r))^4, r^2(1+f(r))^4\sin^2\theta\right)$$
$$f(r) = \frac{a}{2\sqrt{1+kr^2}}$$

測地線を考えるには、赤道面 $\theta = \frac{\pi}{2}$ に注目すれば

$$\begin{aligned} S &= \int \mathcal{L} d\tau \\ &= \int \left(-\frac{(1-f(r))^2}{(1+f(r))^2}\dot{t}^2 + (1+f(r))^4\dot{r}^2 + r^2(1+f(r))^4\dot{\phi}^2\right) d\tau \end{aligned}$$

この変分を計算すれば良い。$\mathcal{L}$ は t, ϕ をあらわには含まないので、オイラーラグランジュ方程式より

$$\begin{aligned} 0 &= \left(\frac{d\mathcal{L}}{d\dot{t}}\right) = \frac{-2(1-f(r))^2}{(1+f(r))^2}\dot{t} \\ \therefore E &= \frac{(1-f(r))^2}{(1+f(r))^2}\dot{t} \\ 0 &= \left(\frac{d\mathcal{L}}{d\dot{\phi}}\right) = 2r^2(1+f(r))^4\dot{\phi} \\ \therefore L &= r^2(1+f(r))^4\dot{\phi} \end{aligned}$$

また、null の条件 $0 = g_{ij}\dot{x}^i\dot{x}^j$ より

$$\begin{aligned} 0 &= -\frac{(1-f(r))^2}{(1+f(r))^2}\dot{t}^2 + (1+f(r))^4\dot{r}^2 + r^2(1+f(r))^4\dot{\phi}^2 \\ \therefore \left(\frac{E}{L}\right)^2 &= \frac{(1-f(r))^2}{r^4(1+f(r))^6}\left(\frac{dr}{d\phi}\right)^2 + \frac{(1-f(r))^2}{r^2(1+f(r))^6} \end{aligned}$$

ここで $u = \frac{1}{r}$、$b = \frac{L}{E}$ とおくと

$$1 = r^4\left(\frac{du}{dr}\right)^2$$

より

$$\left(\frac{1}{b}\right)^2 = \frac{(1-f(1/u))^2}{(1+f(1/u))^6}\left(\frac{du}{d\phi}\right)^2 + \frac{u^2(1-f(1/u))^2}{(1+f(1/u))^6}$$

$$\therefore \left(\frac{du}{d\phi}\right)^2 = G(u) := \left(\frac{1}{b}\right)^2 \frac{(1+f(1/u))^6}{(1-f(1/u))^2} - u^2$$

4.2 パラメータの制約

$f(r)$ のとりうる範囲を確認しておく。

$$0 \leq f(r) \leq \frac{a}{2}$$

いくつかの条件から、パラメータの制約を考える

$$\rho(r) = \frac{24\pi k f(r)^5}{\pi a^4(1+f(r))^5}$$

$$p(r) = \frac{8kf(r)^6}{\pi a^4(1-f(r)^2)(1+f(r))^4}$$

1) weak energy condition (WEC)

$$0 \leq \rho(r)$$

$$0 \leq \rho(r) + p(r)$$

解析すると

$$\begin{aligned} 0 \leq & 1 + \frac{p(r)}{\rho(r)} \\ & = 1 + \frac{8kf(r)^6}{\pi a^4(1-f(r)^2)(1+f(r))^4} \times \frac{\pi a^4(1-f(r)^2)(1+f(r))^4}{8kf(r)^6} \\ & = 1 + \frac{f(r)}{3(1-f(r))} \\ \therefore & f(r) \leq 1, \frac{3}{2} \leq f(r) \end{aligned}$$

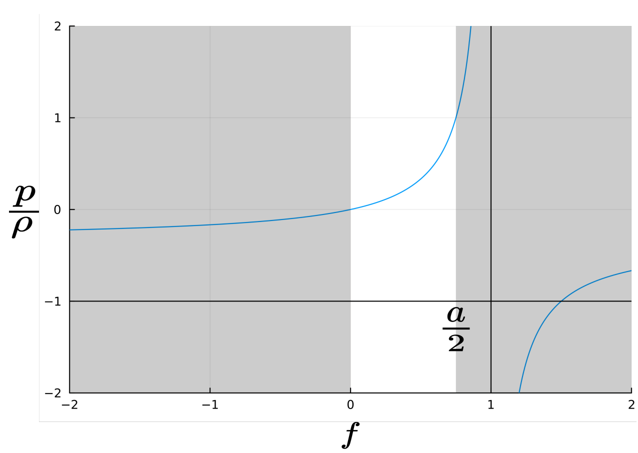

Figure 4.1: WEC

ここでの a の制約は

$$\frac{a}{2} \leq 1$$

2) strong energy condition (SEC)

$$
\begin{aligned}
0 \leq & \rho(r) \\
0 \leq & \rho(r) + 3p(r)
\end{aligned}
$$

解析すると

$$
\begin{aligned}
0 \leq & 1 + \frac{3p(r)}{\rho(r)} \\
& = 1 + \frac{f(r)}{1 - f(r)} \\
& \therefore f(r) \leq 1
\end{aligned}
$$

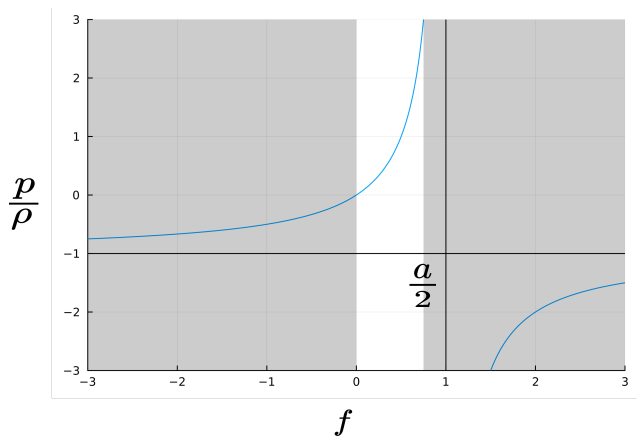

Figure 4.2: SEC

ここでの a の制約は

$$
\frac{a}{2} \leq 1
$$

3) dominant energy condition (DEC)

$$
\begin{aligned}
0 \leq & \rho(r) \\
0 \leq & \rho(r) + |p(r)|
\end{aligned}
$$

解析すると

$$
\begin{aligned}
-1 \leq & \frac{p(r)}{\rho(r)} \leq 1 \\
-1 \leq & \frac{f(r)}{3(1 - f(r))} \leq 1 \\
& \therefore f(r) \leq \frac{3}{4}, \frac{3}{2} \leq f(r)
\end{aligned}
$$

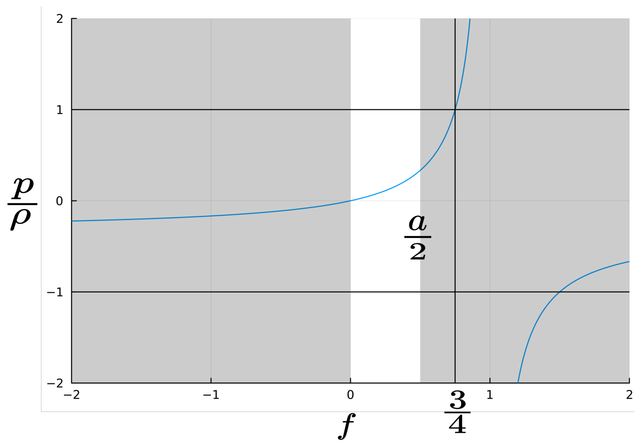

Figure 4.3: DEC

ここでの a の制約は

$$\frac{a}{2} \leq \frac{3}{4}$$

以上より、パラメータの制約は

$$0 \leq a \leq \frac{3}{2}$$

//
//

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\bar{w}_j = () \int$$

メモ用

$$1 = 0$$

$$\begin{cases} 1 = 0 \\ 1 = 0 \end{cases} \tag{4.1}$$

(1)