آخر محاضرة ميكانيكا

* أوجد المؤثر التفاضلى "Nubla" بدلالة الإحداثيات القطبية

$$\vec{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

الإحداثيات الكارتيزية

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{K}$$

لو الـ Curl بساوى Zero يبقى القوة محافظة
ويمكن إيجاد دالة جهد لها

$$\vec{\nabla} = \boxed{}\, e_r + \boxed{}\, e_\theta$$

$x = r\cos(\theta)$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$y = r\sin(\theta)$

$r = \sqrt{x^2+y^2}$

$$\frac{\partial\theta}{\partial x} = \frac{\left(\frac{-y}{x^2}\right)}{1+\left(\frac{y^2}{x^2}\right)} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j}$$

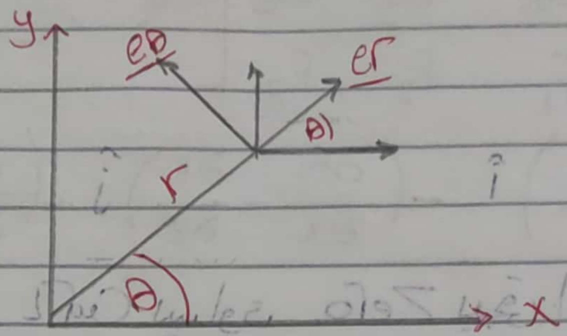
$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\cdot\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\cdot\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial x} = \frac{\cancel{2}\overset{r\cos(\theta)}{x}}{\cancel{2}\underbrace{\sqrt{x^2+y^2}}_{r}}\frac{\partial}{\partial r} + \frac{\overset{r\sin(\theta)}{-y}}{\underbrace{x^2+y^2}_{r^2}}\frac{\partial}{\partial\theta}$$

* $$\frac{\partial}{\partial x} = \cos(\theta)\,\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\,\frac{\partial}{\partial \theta}$$

* $$\frac{\partial}{\partial y} = \sin(\theta)\,\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\,\frac{\partial}{\partial \theta}$$

إيجاد متجهات الوحدة j ، i في المركبات الكارتيزية
بدلالة r,θ



$$\underline{e_r} = \cos(\theta)\,\underline{i} + \sin(\theta)\,\underline{j} \rightarrow ①$$

$$\underline{e_\theta} = -\sin(\theta)\,\underline{i} + \cos(\theta)\,\underline{j} \rightarrow ②$$

بضرب ① في $\sin(\theta)$ و ② في $\cos(\theta)$ وبالجمع

$$\sin(\theta)\,\underline{e_r} + \cos(\theta)\,\underline{e_\theta} = \overset{1}{\cancel{\left(\sin^2(\theta) + \cos^2(\theta)\right)}}\,\underline{j}$$

* $$\boxed{\underline{j} = \sin(\theta)\,\underline{e_r} + \cos(\theta)\,\underline{e_\theta}}$$

بضرب ① في $\cos(\theta)$ و ② في $-\sin(\theta)$ وبالجمع

$$\cos(\theta)\,\underline{e_r} - \sin(\theta)\,\underline{e_\theta} = \overset{1}{\cancel{\left(\cos^2(\theta) + \sin^2(\theta)\right)}}\,\underline{i}$$

* $$\boxed{\underline{i} = \cos(\theta)\,\underline{e_r} - \sin(\theta)\,\underline{e_\theta}}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\underline{i} + \frac{\partial}{\partial y}\,\underline{j}$$

$$\vec{\nabla} = \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\cdot\overbrace{\left(\sin(\theta)\,\underline{e_r} + \cos(\theta)\,\underline{e_\theta}\right)}^{\hat{i}}$$

$$+\left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\cdot\overbrace{\left(\cos(\theta)\,\underline{e_r} + \sin(\theta)\,\underline{e_\theta}\right)}^{\hat{j}}$$

$$\therefore \vec{\nabla} = \frac{\partial}{\partial r}\,\underline{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\,\underline{e_\theta} + \frac{\partial}{\partial z}\,\underline{K}$$

حيث K يعبر عن الإرتفاع

مثال ① أوجد القوة المحافظة الناشئة من دالة الجهد

$\phi = 2r\sin(\theta)$

الحل

$\therefore \vec{F}$ هي القوة المحافظة

$$\therefore \vec{F} = -\vec{\nabla}\times\phi$$

$$= -\left[\frac{\partial\phi}{\partial r}\,\underline{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\underline{e_\theta}\right]$$

$$\frac{\partial\phi}{\partial r} = 2\sin(\theta), \quad \frac{\partial\phi}{\partial\theta} = 2r\cos(\theta)$$

$$\therefore \vec{F} = -\left[2\sin(\theta)\,\underline{e_r} + \frac{1}{\cancel{r}}\,2\cancel{r}\cos(\theta)\,\underline{e_\theta}\right]$$

$$\therefore \vec{F} = -2\sin(\theta)\cos(\theta)\,\underline{i} - 2\sin^2(\theta)\,\underline{j}$$
$$+2\sin(\theta)\cos(\theta)\,\underline{i} - 2\cos^2(\theta)\,\underline{j}$$

$$\vec{F} = -2\left(\sin^2(\theta) + \cos^2(\theta)\right)\underline{j}$$

$$\vec{F} = -2\,\underline{j}$$

مثال ③ يدور كوكب حول الشمس تحت تأثير قوة مركزية جاذبة تتناسب عكسياً مع مربع المسافة بين الكوكب والشمس أثبت أن تلك القوة هي قوة محافظة ثم أوجد دالة الجهد

الحل

$$\therefore \vec{F} = \frac{-\mu}{r^2}\, \underline{e_r},$$

$$\therefore \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{e_r} & \hat{e_\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{-\mu}{r^2} & 0 & 0 \end{vmatrix} = \overrightarrow{Zero}$$

$\therefore$ القوة محافظة

$$\vec{F} = -\vec{\nabla}\phi \qquad \frac{-\mu}{r^2} = -\frac{\partial \phi}{\partial r}$$

$$\phi = \int \mu r^{-2}\, dr = \frac{-\mu}{r},$$

* وإذا كانت سرعة الكوكب عندما كان على بعد r_A تساوي V_A أوجد سرعته عند r_B

$$\phi_A + T_A = \phi_B + T_B$$ باستخدام قانون بقاء الطاقة

$$\frac{-\mu}{r_A} + \frac{1}{2} m V_A^2 = \frac{-\mu}{r_B} + \frac{1}{2} m V_B^2$$

$$V_B^2 = \frac{1}{m}\left(\frac{2\mu}{r_B} - \frac{2\mu}{r_A} + m V_A^2\right)$$

$$V_B^2 = V_A^2 + K\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \qquad \ni K = \frac{2\mu}{m}$$

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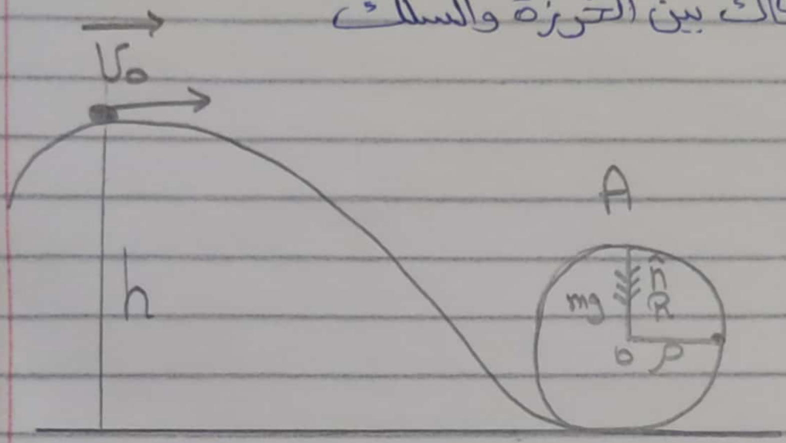
مثال ③ تنزلق خرزة على سلك أملس في مستوى رأسي كما هو موضح بالشكل . أوجد أقل سرعة إبتدائية V_0 بحيث تقوم بالكاد بعمل دورة كاملة بدون ترك السلك بفرض أنه لايوجد احتكاك بين الخرزة والسلك



الحل

مربع

$$a_n = \frac{\hat{S}}{R}$$

نصف القطر

$R = 0 \rightarrow$ عند A

$$\sum F_n = m a_n$$

$$\cancel{m} g = \cancel{m} \frac{V_A^2}{\rho}$$

$$V_A^2 = g\rho$$

$$T_0 + V_0 = T_A + V_A$$

$$\frac{1}{2}\cancel{m} V_0^2 + \cancel{m} g h = \frac{1}{2}\cancel{m} V_A^2 + \cancel{m} g (2\rho)$$

$$V_0^2 = V_A^2 + 2g(2\rho - h)$$

$$V_0^2 = g(\rho + 4\rho - 2h)$$

$$V_0^2 = g(5\rho - 2h)$$

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$$\vec{F} = -\vec{\nabla}\phi$$
$$= \left(-\frac{\partial\phi}{\partial x}, \frac{-\partial\phi}{\partial y}, \frac{-\partial\phi}{\partial z}\right)$$

٥) أوجد دالة الجهد لمجال القوة

$$\vec{F} = (ay(y^2-3z^2), 3ax(y^2-z^2) - baxyz)$$

$$\frac{-\partial\phi}{\partial x} = F_x, \quad \frac{-\partial\phi}{dy} = F_y, \quad \frac{-\partial\phi}{\partial z} = F_z$$

$$\phi = -\int ay(y^2-3z^2)\,dx = -axy(y^2-3z^2) + c_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = -3axy^2 + 3axz^2 + \frac{\partial c_1}{\partial y} = -3ax(y^2-z^2)$$

$$= -3ax(y^2-z^2) + \frac{\partial c_1}{\partial y} = -3ax(y^2-z^2)$$

$$\frac{\partial c_1}{\partial y} = 0 \Rightarrow c_1 = c_1(z)$$

$$\frac{\partial\phi}{\partial z} = \underbrace{baxyz} + \frac{\partial c_1}{\partial z} = -F_2 = \underbrace{baxyz}$$

$$\frac{\partial c_1}{\partial z} = 0 \Longrightarrow c_1 = \text{Constant}.$$

$$\phi = -axy(y^2-3z^2) + c$$

* أوجد دالة الجهد لمجال القوة :

1) $\vec{F} = (yz, xz, xy + 2z)$

2) $\vec{F} = (\sin(y), x\cos(y) + \cos(z), -y\sin(z))$

3) $\vec{F} = (y^2 z + xz^2, 2xyz, xy^2 + 2x^2 z)$

الحل

هنجيب $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ F_x & F_y & F_z \end{vmatrix} = 0$

علشان أتأكد انها قوة حافظة ولو مش قوة حافظة يبقى ملهاش دالة جهد

1)

بعد فك المحدد = $(x-x)\hat{i} - (y-y)\hat{j} + (z-z)\hat{k} = 0$

$\phi = \vec{F} = (yz, xz, xy + 2z)$

$= -\int yz\, dx = -xyz + C_1(y,z)$

$= -\int xz\, dy = -xyz + C_2(x,z)$

z^2

$= -\int xy + 2z\, dz = -(xyz + z^2) + C_3(x,y)$

zero

$\phi = -(xyz + z^2)$

2) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(y) & x\cos(y) + \cos(z) & -y\sin(z) \end{vmatrix}$

$= (-\sin(z) + \sin(z))\hat{i} - (0)\hat{j} + (\cos(y) - \cos(y))\hat{k}$

$= \vec{0}$

$\phi = (\sin(y), \; x\cos(y) + \cos(z), \; -y\sin(z))$

$= -\int \sin(y)\,dx = -x\sin(y) + C_1(y,z)$

$= -\int x\cos(y) + \cos(z)\,dy =$
$-x\sin(y) + y\cos(z) + C_2(x,z)$

$= \int y\sin(z)\,dz = -y\cos(z) + C_3(y,z)$

$\boxed{C_3 = \text{Constant}}$ $\quad \phi = -x\sin(y) - y\cos(z) + C_3$

3) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z + x z^2 & 2xyz & xy^2 + 2x^2 z \end{vmatrix}$

$= (2xy - 2xy)\hat{i} - (2xz)\hat{j} + (2yz - 2yz)\hat{k}$

$= -2xz\,\hat{j} \neq 0$

∴ القوة غير محافظة

∴ لا يوجد دالة جهد

$$\vec{F} = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial z}\right)$$

* أوجد القوة الناشئة عن دالة الجهد

1) $\phi = a x y^2 z^3$

2) $\phi = \frac{1}{2} a x^2 - \frac{1}{2} b y^2 - \frac{1}{2} c z^2$

3) $\phi = a (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

1) $\frac{\partial \phi}{\partial x} = -(a y^2 z^3)$

$\frac{\partial \phi}{\partial y} = -(2 a x y z^3)$

$\frac{\partial \phi}{\partial z} = -(3 a x y^2 z^2)$

وجمعهم في $\vec{F}$

2) $\frac{\partial \phi}{\partial x} = -a x$, $\frac{\partial \phi}{\partial y} = b y$, $\frac{\partial \phi}{\partial z} = c z$

$\vec{F} = (a x, b y, c z)$

3) $\frac{\partial \phi}{\partial x} = a x (x^2 + y^2 + z^2)^{-\frac{3}{2}}$

$\frac{\partial \phi}{\partial y} = a y (x^2 + y^2 + z^2)^{-\frac{3}{2}}$

$\frac{\partial \phi}{\partial z} = a z (x^2 + y^2 + z^2)^{-\frac{3}{2}}$

$$\vec{F} = \left(-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial z}\right)$$

finish

1) $\vec{F} = (-x^2 y z)\,\underline{i} - (x y z^2)\,\underline{k}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xyz & 0 & -xyz^2 \end{vmatrix}$$

$$(-xz^2 - 0) - (-yz^2 + xy) + (-xz) \neq \vec{0}$$

∴ القوة ليست قوة محافظة

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y - z^2 & (3xyz + xz^2) & 2x^2yz + yz^4 \end{vmatrix}$$

$$(2x^2z + z^4 - 3xy - 2xz)$$ ليست قوة محافظة

$$\vec{F} = -K r^2 \underline{r}$$ أوجد دالة الجهد

$K \rightarrow$ Constant

بالضرب $r \times$

$$\frac{\underline{r}}{r} = \underline{er}$$

$$\vec{F} = -K r^3 \underline{er}$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{er} & \hat{e\theta} & \hat{K} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ -Kr^3 & 0 & 0 \end{vmatrix} = \vec{0}$$

$$\phi = -\int -K r^3 \, dr$$

$$\phi = \frac{K}{4} r^4 + C$$

$C \rightarrow$ = Zero

$$\vec{F} = (x+2y+az)\,\underline{i} + (bx-3y-z)\,\underline{j} + (4x+cy+2z)$$

$\text{curl}\,\vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{i}^{\,+} & \hat{j}^{\,-} & \hat{k}^{\,+} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

$$(c+1)\,\hat{i} - (4-a)\,\hat{j} + (b-2)\,\hat{k} = \vec{0}$$

to $\text{curl}\,\vec{F} = \vec{0}$ we should take

$a = 4$, $b = 2$ and $c = -1$

أوجد الشغل المبذول لتحريك جسيم دورة كاملة حول دائرة في المستوى (xy) نصف قطرها 3 cm ومركزها نقطة الأصل

$$\vec{F} = (2x+y+z)\,\underline{i} + (x+y-z)\,\underline{j} + (x-2y+4z)\,\underline{k}$$

$\omega = \oint \vec{F} \cdot \underline{dr}$

$x = 3\cos(\theta)$

$y = 3\sin(\theta)$

$dx = -3\sin(\theta)\,d\theta$

$dy = 3\cos(\theta)\,d\theta$

دورة كاملة

$\theta = 0 \longrightarrow 2\pi$

$$\omega = \int_0^{2\pi} (6\cos(\theta) + 3\sin(\theta)) \cdot (-3\sin(\theta)\,d\theta)$$

$$+ \int_0^{2\pi} (3\cos(\theta) + 3\sin(\theta)) \cdot (3\cos(\theta)\,d\theta)$$

$$\omega = \int_0^{2\pi} -9\sin(\theta)\cos(\theta) + 9(\cos^2(\theta) - \sin^2(\theta))\,d\theta$$

$$\omega = \left[\frac{9}{2}(\cos\theta)^2 + \frac{9}{2}\sin(2\theta)\right]_0^{2\pi} = \text{Zero} \quad \#$$