Fundamentals of Linear Algebra

Spaces, Rank, and Products

Core Idea

The product of a matrix A and a vector x can be interpreted as a **linear combination** of the columns of A, with the coefficients given by the entries of x.

Let $A \in \mathbb{R}^{m\times n}$ with columns $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_n$, and let $\mathrm{x} \in \mathbb{R}^n$.

$$A = \begin{bmatrix} | & | & & | \\ \mathrm{a}_1 & \mathrm{a}_2 & \cdots & \mathrm{a}_n \\ | & | & & | \end{bmatrix}, \quad \mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The product $A\mathrm{x}$ is:

$$A\mathrm{x} = x_1\mathrm{a}_1 + x_2\mathrm{a}_2 + \cdots + x_n\mathrm{a}_n$$

Matrix-Vector Product: The Column View II

Meaning

The resulting vector Ax is a point in the vector space spanned by the columns of A.

Example of Matrix-Vector Product

Consider the matrix A and vector x:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}, \quad \mathrm{x} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Using the column view, the product is:

$$\begin{aligned} A\mathrm{x} &= 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} + \begin{bmatrix} -3 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 5+0-3 \\ 10-6+12 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \end{bmatrix} \end{aligned}$$

This is the same result as the standard "row-times-column" method.

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Column Space and Null Space

For any matrix $A \in \mathbb{R}^{m \times n}$, there are four fundamental subspaces. We focus on two:

Definition (Column Space)

The column space of A, denoted $C(A)$, is the set of all linear combinations of the columns of A. It is a subspace of $\mathbb{R}^m$.

$$C(A) = \{A\mathrm{x} \mid \mathrm{x} \in \mathbb{R}^n\}$$

Definition (Null Space)

The null space of A, denoted $N(A)$, is the set of all vectors x such that $A\mathrm{x} = 0$. It is a subspace of $\mathbb{R}^n$.

$$N(A) = \{\mathrm{x} \in \mathbb{R}^n \mid A\mathrm{x} = 0\}$$

Relationship Between Subspaces I

Theorem (Orthogonality)

The ***row space*** *of A, $C(A^T)$, is the orthogonal complement of the* ***null space*** *of A, $N(A)$.*

Relationship Between Subspaces II

Proof.

Let $\mathrm{x} \in N(A)$. This means $A\mathrm{x} = 0$. Let the rows of A be $\mathrm{r}_1^T, \ldots, \mathrm{r}_m^T$.

$$A\mathrm{x} = \begin{bmatrix} \mathrm{r}_1^T\mathrm{x} \\ \mathrm{r}_2^T\mathrm{x} \\ \vdots \\ \mathrm{r}_m^T\mathrm{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This implies that x is orthogonal to every row of A (i.e., $\mathrm{r}_i^T\mathrm{x} = 0$ for all i).
Now, take any vector $\mathrm{v} \in C(A^T)$. By definition, v is a linear combination of the rows of A:

$$\mathrm{v} = c_1\mathrm{r}_1 + \cdots + c_m\mathrm{r}_m$$

Let's compute the dot product of v and x:

$$\mathrm{v}^T\mathrm{x} = (c_1\mathrm{r}_1 + \cdots + c_m\mathrm{r}_m)^T\mathrm{x} = c_1(\mathrm{r}_1^T\mathrm{x}) + \cdots + c_m(\mathrm{r}_m^T\mathrm{x}) = c_1(0) + \cdots + c_m(0) = 0$$

Thus, every vector in $N(A)$ is orthogonal to every vector in $C(A^T)$. □

Subspaces and Solvability

The concept of column space directly relates to the solvability of a linear system $A\mathrm{x} = \mathrm{b}$.

Condition for Solvability

A linear system $A\mathrm{x} = \mathrm{b}$ has a solution if and only if the vector b is in the column space of A, i.e., $\mathrm{b} \in C(A)$.

- **Why?** The expression $A\mathrm{x}$ is, by definition, a linear combination of the columns of A.

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- Therefore, the system can only produce results (b vectors) that are already in the span of its columns.
- If b is not in $C(A)$, then there is no vector x that can combine the columns of A to produce b.

Rank of a Matrix

Definition (Rank)

The rank of a matrix A, denoted $\text{rank}(A)$, is the dimension of its column space.

$$\text{rank}(A) = \dim(C(A))$$

A Fundamental Theorem

For any matrix A, the dimension of its column space is equal to the dimension of its row space.

$$\dim(C(A)) = \dim(C(A^T))$$

- The rank tells us the number of linearly independent columns (or rows) in the matrix.
- It represents the "true dimension" of the space spanned by the columns.
- If $A \in \mathbb{R}^{m \times n}$, then $\text{rank}(A) \leq \min(m, n)$.

The Rank-Nullity Theorem

Theorem

For any matrix $A \in \mathbb{R}^{m \times n}$*:*

$$rank(A) + nullity(A) = n$$

where $nullity(A) = \dim(N(A))$ *and* n *is the number of columns.*

Interpretation

The number of dimensions in the domain ($\mathbb{R}^n$) is split between the dimensions that get mapped to the zero vector (the null space) and the dimensions that get mapped to the column space.

Proof of the Rank-Nullity Theorem I

Let A be an $m \times n$ matrix with rank r. WLOG, assume the first r columns are linearly independent. We can partition A as:

$$A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

where A_1 is $m \times r$ with linearly independent columns, and A_2 is $m \times (n-r)$. Since the columns of A_2 are in the column space of A_1, we can write $A_2 = A_1 B$ for some $r \times (n-r)$ matrix B.

$$A = \begin{bmatrix} A_1 & A_1 B \end{bmatrix}$$

Now, construct a new matrix X of size $n \times (n-r)$:

$$X = \begin{bmatrix} -B \\ I_{n-r} \end{bmatrix}$$

where I_{n-r} is the $(n-r) \times (n-r)$ identity matrix.

Proof of the Rank-Nullity Theorem II

Let's compute the product AX:

$$
\begin{aligned}
AX &= \begin{bmatrix} A_1 & A_1 B \end{bmatrix} \begin{bmatrix} -B \\ I_{n-r} \end{bmatrix} \\
&= -A_1 B + A_1 B = 0
\end{aligned}
$$

This shows that each of the $n - r$ columns of X is a solution to $A\mathrm{x} = 0$, so they are in the null space of A.

Proof of the Rank-Nullity Theorem (cont.) I

We have found $n - r$ vectors in the null space (the columns of X). We now show they form a basis for $N(A)$.

1. **Linear Independence:** The columns of X are linearly independent. If $X\mathrm{u} = 0$ for some vector u:

$$\begin{bmatrix} -B \\ I_{n-r} \end{bmatrix} \mathrm{u} = \begin{bmatrix} -B\mathrm{u} \\ \mathrm{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathrm{u} = 0$$

2. **Spanning:** Let $\mathrm{u} = \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix}$ be any solution to $A\mathrm{u} = 0$.

$$A\mathrm{u} = \begin{bmatrix} A_1 & A_1 B \end{bmatrix} \begin{bmatrix} \mathrm{u}_1 \\ \mathrm{u}_2 \end{bmatrix} = A_1\mathrm{u}_1 + A_1 B\mathrm{u}_2 = A_1(\mathrm{u}_1 + B\mathrm{u}_2) = 0$$

   Since columns of A_1 are linearly independent, this implies $\mathrm{u}_1 + B\mathrm{u}_2 = 0$, so $\mathrm{u}_1 = -B\mathrm{u}_2$.

Proof of the Rank-Nullity Theorem (cont.) II

3. Substituting this back into u:

$$\mathrm{u} = \begin{bmatrix} -B\mathrm{u}_2 \\ \mathrm{u}_2 \end{bmatrix} = \begin{bmatrix} -B \\ I_{n-r} \end{bmatrix} \mathrm{u}_2 = X\mathrm{u}_2$$

This shows any solution u is a linear combination of the columns of X.

Conclusion

The columns of X form a basis for $N(A)$. There are $n - r$ such columns, so $\text{nullity}(A) = n - r$. Therefore:

$$\text{rank}(A) + \text{nullity}(A) = r + (n - r) = n \qquad \square$$

Matrix-Matrix Product: The Outer Product View I

Core Idea

The product of two matrices A and B can be seen as the **sum of rank-1 matrices**, where each rank-1 matrix is the outer product of a column of A and a row of B.

Let $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$.

$$A = \begin{bmatrix} | & & | \\ \mathrm{a}_1 & \cdots & \mathrm{a}_k \\ | & & | \end{bmatrix}, \quad B = \begin{bmatrix} - & \mathrm{b}_1^T & - \\ & \vdots & \\ - & \mathrm{b}_k^T & - \end{bmatrix}$$

The product AB is:

$$AB = \sum_{i=1}^{k} \mathrm{a}_i \mathrm{b}_i^T = \mathrm{a}_1 \mathrm{b}_1^T + \mathrm{a}_2 \mathrm{b}_2^T + \cdots + \mathrm{a}_k \mathrm{b}_k^T$$

Matrix-Matrix Product: The Outer Product View II

Meaning

Each term $\mathrm{a}_i \mathrm{b}_i^T$ is an $m \times n$ matrix of rank 1. The full product is built by summing these fundamental components.

Example of Outer Product Expansion

Consider the matrices A and B:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$$

The columns of A are $\mathrm{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathrm{a}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. The rows of B are $\mathrm{b}_1^T = \begin{bmatrix} 4 & 5 \end{bmatrix}$, $\mathrm{b}_2^T = \begin{bmatrix} 6 & 7 \end{bmatrix}$.

$$\begin{aligned} AB &= \mathrm{a}_1\mathrm{b}_1^T + \mathrm{a}_2\mathrm{b}_2^T \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \begin{bmatrix} 6 & 7 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix}}_{\text{Rank 1}} + \underbrace{\begin{bmatrix} 0 & 0 \\ 18 & 21 \end{bmatrix}}_{\text{Rank 1}} = \begin{bmatrix} 4 & 5 \\ 26 & 31 \end{bmatrix} \end{aligned}$$