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October 19, 2020

Numerical Methods Term Project Report

Engineering Problem: The Double Pendulum

A pendulum refers to a weight suspended from a pivot and allowed to swing freely. A double pendulum is a pendulum with another pendulum attached to its end. Its motion when left to swing freely is highly sensitive to the initial conditions: lengths of the two pendulums, the initial angles each of the bobs make with the vertical axis, and the masses of the rods if the pendulum is compound or the masses of the pendulum bobs if the pendulum is simple. The equations of motion of a double pendulum are governed by a system of ordinary differential equations which can be arrived at by analyzing the various forces acting on both masses of the double pendulum. These systems of equations cannot be solved analytically and hence numerical models are applied to solve them. The double pendulum is an important problem in physics and engineering as it displays the behavior of chaotic systems and is a good system through which chaotic motion can be studied and understood. Chaotic systems are systems in which small changes in initial conditions of a system can drastically affect the behavior of a system. The dynamics of chaotic systems are present in both natural and man-made systems and their understanding has resulted in widespread applications in modeling, control, and performance enhancements of engineering systems (Umoh and Wudil). Its understanding has resulted in breakthroughs such as the design of multimedia security, disturbance modeling in power systems, epilepsy moderation in neurology and biomedical engineering, dynamic analysis in electrical machines, performance enhancements of electronic circuits and many others.

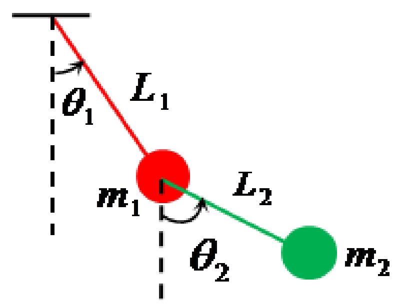

Figure 1: Diagram of a double pendulum

Description of the Inputs and Outputs

<u>Input:</u>

- L_1 length of the first bob from hanging point
- L_2 length of the second bob from first bob
- M_1 and M_2 representing the mass of each pendulum bob
- Gravitational constant: $g = 9.81\ ms^{-2}$
- Initial angle θ_1 of the first bob
- Initial angle θ_2 of the second bob

Output: (used to run the simulation and change the position of the bobs at n time interval)

- $x_1 = L_1 sin(\theta_1)$
- $y_1 = -L_1 cos(\theta_1)$
- $x_2 = x_1 + L_2 sin(\theta_2)$
- $y_2 = y_1 - L_2 cos(\theta_2)$

Mathematical Formulation of the Problem

To arrive at a function of the angle of both bobs with respect to time, the double pendulum system will first be analyzed through the sum of forces applied on each bob. First, we will formulate the equations displacement, velocity and acceleration (Neumann):

$$x_1 = L_1 sin(\theta_1)$$
$$y_1 = -L_1 cos(\theta_1)$$
$$x_2 = x_1 + L_2 sin(\theta_2)$$
$$y_2 = y_1 - L_2 cos(\theta_2)$$

Differentiating to get the *x* and *y* velocities:

$$\dot{x_1} = \dot{\theta}_1 L_1 cos(\theta_1)$$
$$\dot{y_1} = \dot{\theta}_1 L_1 sin(\theta_1)$$
$$\dot{x_2} = \dot{x_1} + \dot{\theta}_2 L_1 cos(\theta_2)$$
$$\dot{y_2} = \dot{y_1} + \dot{\theta}_2 L_1 sin(\theta_2)$$

Differentiating again to get the *x* and *y* accelerations:

$$\ddot{x_1} = -\dot{\theta}_1^2 L_1 sin(\theta_1) + \ddot{\theta}_1 L_1 cos(\theta_1)$$
$$\ddot{y_1} = \dot{\theta}_1^2 L_1 cos(\theta_1) + \ddot{\theta}_1 L_1 sin(\theta_1)$$
$$\ddot{x_2} = \ddot{x_1} - \dot{\theta}_2^2 L_2 sin(\theta_2) + \ddot{\theta}_2 L_2 cos(\theta_2)$$
$$\ddot{y_2} = \ddot{y_1} + \dot{\theta}_2^2 L_2 cos(\theta_2) + \ddot{\theta}_2 L_2 sin(\theta_2)$$

Assuming the two bobs are represented as point particles, we will then analyze the forces applied on each bob:

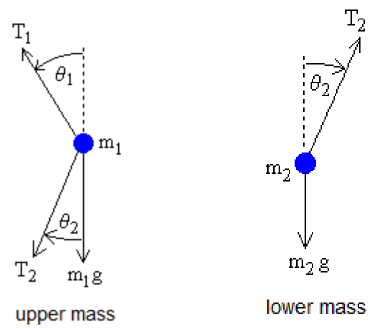

Figure 2: Free body diagrams of both the upper and lower bobs in the system (Neumann)

Using Newton's second law of motion in the *x* and *y* directions on the first bob:

$$F_{x1} = m_1 a_{x1}$$
$$m_1 \ddot{x}_1 = -T_1 sin(\theta_1) + T_2 sin(\theta_2)$$
$$F_{y1} = m_1 a_{y1}$$
$$m_1 \ddot{y}_1 = T_1 cos(\theta_1) - T_2 cos(\theta_2) - m_1 g$$

Using Newton's second law of motion in the *x* and *y* directions on the second bob:

$$F_{x2} = m_2 a_{x2}$$
$$m_2 \ddot{x}_2 = -T_2 sin(\theta_2)$$
$$F_{y2} = m_2 a_{y2}$$
$$m_2 \ddot{y}_2 = T_2 cos(\theta_2) - m_2 g$$

By solving the previous equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$, we arrive at the following:

$$\ddot{\theta}_1 = \frac{-g(2m_1 + m_2)sin(\theta_1) - m_2 g sin(\theta_1 - 2\theta_2) - 2sin(\theta_1 - \theta_2)m_2(\dot{\theta}_2^2 L_2 + \dot{\theta}_1^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))}$$

$$\ddot{\theta}_1 = \frac{2sin(\theta_1 - \theta_2)(\dot{\theta}_1^2 L_1(m_1 + m_2) + g(m_1 + m_2)cos(\theta_1) + \dot{\theta}_1^2 L_2 m_2 cos(\theta_1 - \theta_2))}{L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))}$$

The equations are currently second order differential equations, but to be solved numerically the equation has to be in the form of a first order differential equation by applying the following substitution:

$$\dot{\theta}_1 = \omega_1$$
$$\dot{\theta}_2 = \omega_2$$

We arrive at the following first order differential equations:

$$\dot{\omega}_1 = \frac{-g(2m_1 + m_2)\sin(\theta_1) - m_2 g sin(\theta_1 - 2\theta_2) - 2sin(\theta_1 - \theta_2)m_2(\omega_2^2 L_2 + \omega_1^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))}$$

$$\dot{\omega}_2 = \frac{2sin(\theta_1 - \theta_2)(\omega_1^2 L_1(m_1 + m_2) + g(m_1 + m_2)cos(\theta_1) + \omega_2^2 L_2 m_2 cos(\theta_1 - \theta_2))}{L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))}$$

Description of tentative numerical techniques to be applied

To solve the four first-order differential equations of the double pendulum system stated above, Euler's method will first be used. Euler's method works by finding the slope of the tangent line at a particular point and approximating next solution in the function (Dawkins):

$$y = y_0 + f(t_0, y_0)(t_1 - t_0)$$

By choosing a small time interval to approximate the next point in the function, we can arrive at the following general formula for approximating y_{n+1} at any point given a uniform step size, *h*:

$$y_{n+1} = y_n + hf_n$$

Given that it is a first order method, it generates a lot of error and a very small step size will be needed. However, even with a small step size, the chaotic nature of the double pendulum system will be greatly impacted even by such minor errors using a small step size. Therefore, another numerical method, the 4th-order Runge-Kutta (RK4) method, will be utilized to run the double pendulum simulation.

Euler's method was a specific first order method of the family of Runge-Kutta methods that approximate the solution of a differential equation. By using higher order terms when deriving the numerical method, we can arrive at much more accurate methods to numerically solve differential equations. One of the most widely used Runge-Kutta methods is the 4th order method, which is used as follows (Zeltkevic):

$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f(t_n + \frac{h}{2}, x_n + \frac{hk_1}{2}) \\ k_3 &= f(t_n + \frac{h}{2}, x_n + \frac{hk_2}{2}) \\ k_4 &= f(t_n + h, x_n + hk_3) \end{aligned}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

For the double pendulum system, the steps of each method will be done four times to determine the solution for each of the four first-order differential equations. As a result, the angular velocities and displacements at various times can be determined and used to calculate the *x,y*-coordinates of each pendulum bob to run the simulation.

Description of numerical experiments to be carried out

To investigate the chaotic nature of the double pendulum system and analyze its sensitivity to minute changes, several numerical experiments were performed with very slight modifications in initial conditions, such as initial angles. It was expected that the different initial conditions will cause the system to behave similarly for a short period of time but begin to greatly differ as the small changes cause greater compounded variations. The state of each system was measured as a function of time and the relative change in each system was analyzed in relation to the percentage difference of the initial conditions.

Additionally, we numerically investigated the equilibrium points of the double pendulum system. Geometrically, these equilibria points can be demonstrated as follows in Figure 3:

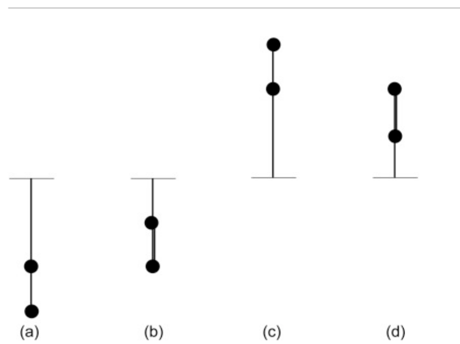

Figure 3: Possible equilibria states of the double pendulum system (Mellodge)

For both bobs, these equilibrium points lie when both bobs have an angle from the vertical of

$$\pm n\pi, n = 0, 1, 2, 3, ...$$

As such, in the simulation, the initial angles will be increased constantly and checked for equilibria by which the system's state doesn't change at such conditions after some period of time. Furthermore, after the equilibrium points are identified, they can be determined to be stable or unstable by checking the system's reaction to subtle changes away from the equilibrium conditions. A stable equilibrium point such as (a) in figure 3 would act similar to a simple linear pendulum in a non-chaotic manner while the other equilibrium points could cause greater perturbation and chaotic motion in the system.

To verify the accuracy of the numerical method utilized in the double pendulum simulation, total energy can be checked at specific intervals. Given an initial total energy for the double pendulum system, if the numerical method has zero error, then it is expected that the total energy at any time interval would remain constant. However, given that every numerical method employed will have a certain degree of error, one method to measure the error and its change over a time interval is by calculating the total energy and comparing it to the previous interval it was measured at. Since energy must always be conserved, the change in total energy would be an indicator for the relative error.

Numerical Results and Discussions

a) Results of the simulation using Euler and 4th-order Runge-Kutta method:

The lengths of both pendulums were initialized to 1m for all experiments performed. The masses of both pendulum bobs were also initialized to 1kg and acceleration due to gravity value was 9.81ms^{-2} for all experiments performed. Simulations of the positions of the double pendulum were performed for 10s with a step size of 0.01s for different initial conditions of θ_1 and θ_2. Graphs of the angles of both pendulums against time were plotted. Graphs of energy of the double pendulum system with time were plotted to show the total energy of the system with time, for the different initial conditions.

Initial conditions: θ_1=0.2π, θ_2=0.4π

Euler method: The simulation of the position of the system is shown in Figure 4. Figure 5 also shows the different angles of both pendulums at various times over the period of simulation. From Figure 6, large continuous increases in the total energy of the system over the time of the simulation was observed. This may have been due to errors in the numerical method(euler method) used.

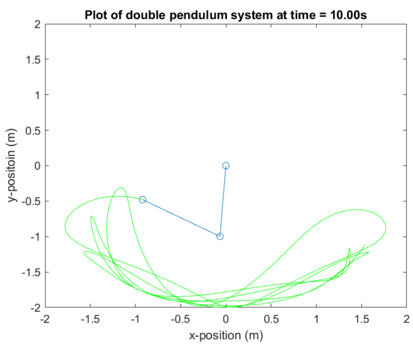

Figure 4: Simulation of positions of both bobs with time (θ_1=0.2π, θ_2=0.4π)

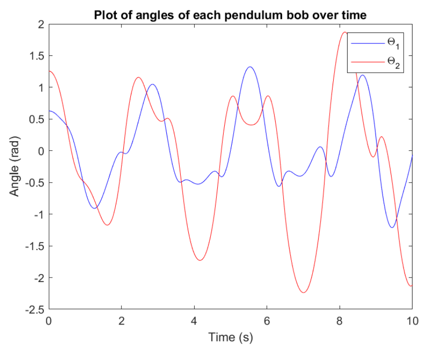

Figure 5: Graph of angles of both bobs against time (θ_1=0.2π, θ_2=0.4π)

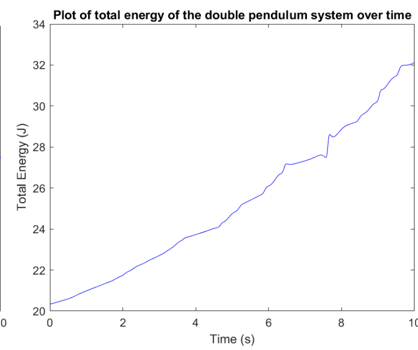

Figure 6: Graph of energy of system against time (θ_1=0.2π, θ_2=0.4π)

RK4 method: The simulation of the position of the system is shown in Figure 7. Figure 8 also shows the different angles of both pendulums at various times over the period of simulation. From Figure 9, very small fluctuations in total energy of the system over time of simulation were observed. These small fluctuations may be due to errors in the numerical method used (RK4 method).

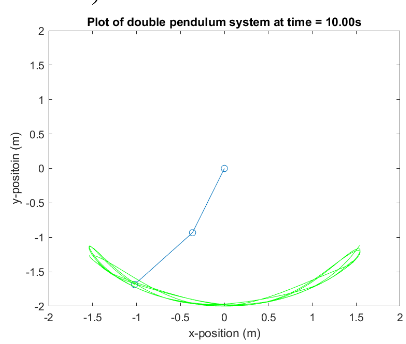

Figure 7: Simulation of positions of both bobs with time (θ_1=0.2π, θ_2=0.4π)

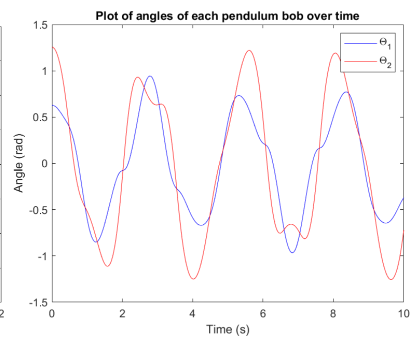

Figure 8: Graph of angles of both bobs against time (θ_1=0.2π, θ_2=0.4π)

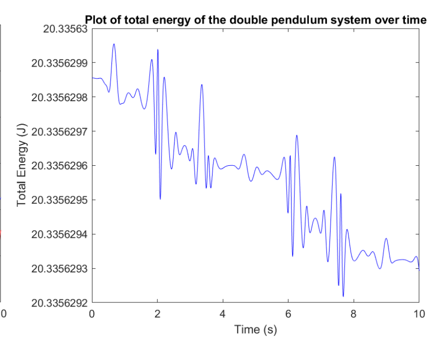

Figure 9: Graph of energy of system against time (θ_1=0.2π, θ_2=0.4π)

Initial conditions: θ_1=0.3π, θ_2=0.5π

Euler method: The simulation of the position of the system is shown in Figure 10. Figure 11 also shows the different angles of both pendulums at various times over the period of simulation.

From Figure 12, large continuous increases in the total energy of the system over the time of the simulation was observed. This may have been due to errors in the numerical method (Euler's method) used.

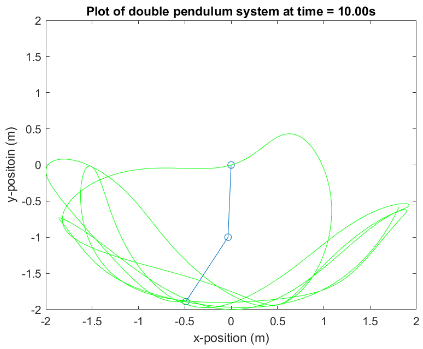

Figure 10: Simulation of positions of both bobs with time (θ_1=0.3π,θ_2=0.5π)

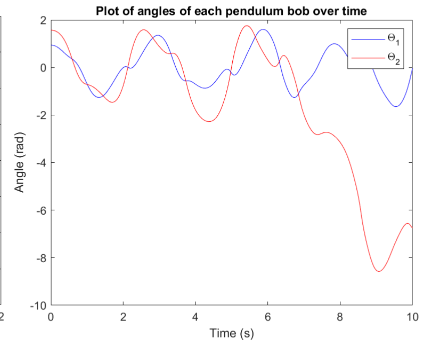

Figure 11: Graph of angles of both bobs against time (θ_1=0.3π,θ_2=0.5π)

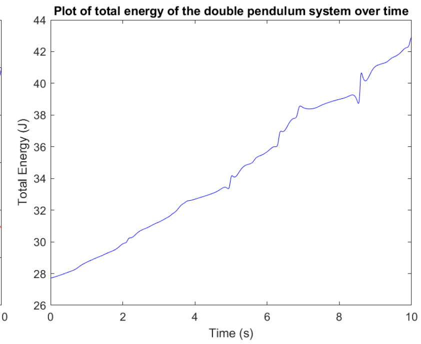

Figure 12: Graph of energy of system against time (θ_1=0.3π,θ_2=0.5π)

RK4 method: The simulation of the position of the system is shown in Figure 13. Figure 14 also shows the different angles of both pendulums at various times over the period of simulation. From Figure 15, very small fluctuations in total energy of the system over time of simulation were observed. These small fluctuations may have been due to errors in the numerical method used (RK4 method).

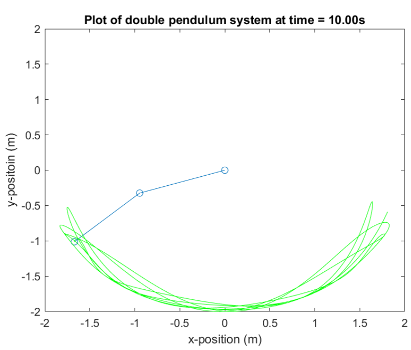

Figure 13: Simulation of positions of both bobs with time (θ_1=0.3π,θ_2=0.5π)

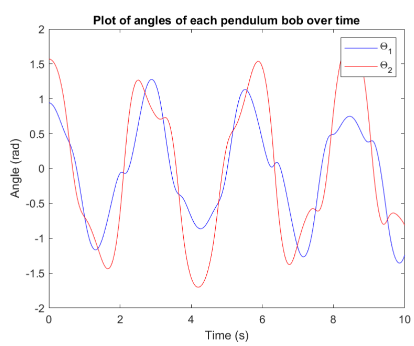

Figure 14: Graph of angles of both bobs against time (θ_1=0.3π,θ_2=0.5π)

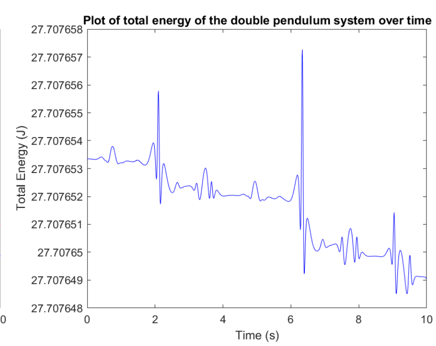

Figure 15: Graph of energy of system against time (θ_1=0.3π,θ_2=0.5π)

Initial conditions:θ_1=3.5π,θ_2=2.4π

Euler method: The simulation of the position of the system is shown in Figure 16. Figure 17 also shows the different angles of both pendulums at various times over the period of simulation.

From Figure 18, large continuous increases in the total energy of the system over the time of the simulation was observed. This may have been due to errors in the numerical method(euler method) used.

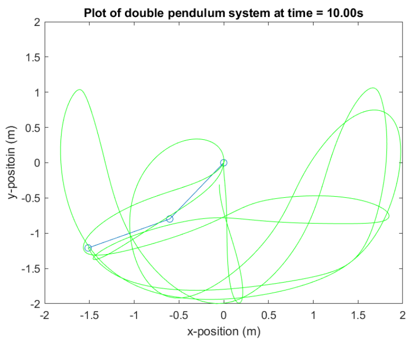

Figure 16: Simulation of positions of both bobs with time (θ_1=3.5π,θ_2=2.4π)

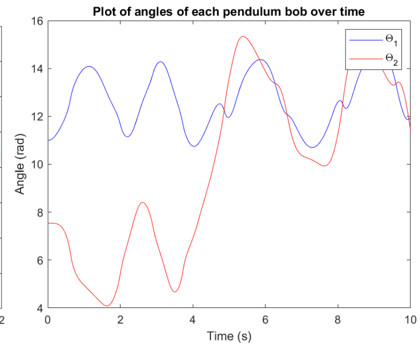

Figure 17: Graph of angles of both bobs against time (θ_1=3.5π,θ_2=2.5π)

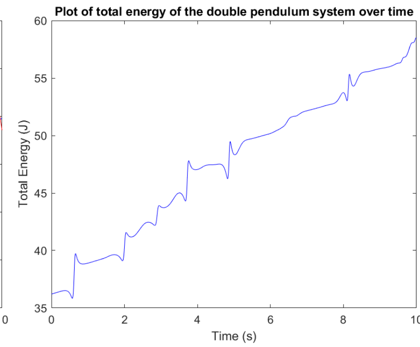

Figure 18: Graph of energy of system against time (θ_1=3.5π,θ_2=2.4π)

RK4 method: The simulation of the position of the system is shown in Figure 19. Figure 20 also shows the different angles of both pendulums at various times over the period of simulation. From Figure 21, very small fluctuations in total energy of the system over time of simulation were observed. These small fluctuations may be due to errors in the numerical method used (RK4 method).

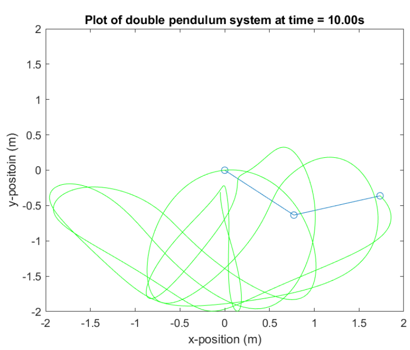

Figure 19: Simulation of positions of both bobs with time (θ_1=3.5π,θ_2=2.4π)

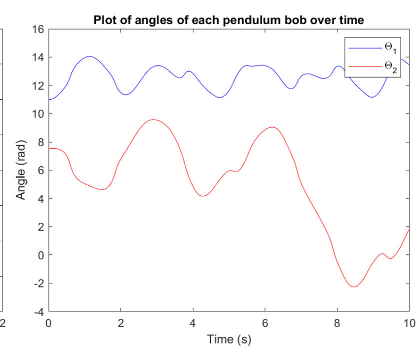

Figure 20: Graph of angles of both bobs against time (θ_1=3.5π,θ_2=2.5π)

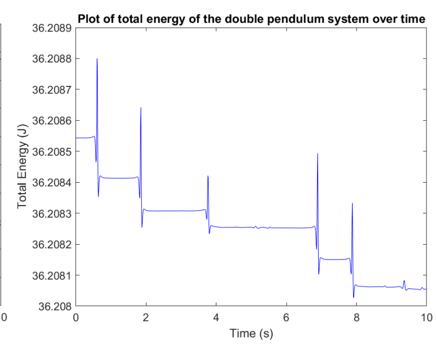

Figure 21: Graph of energy of system against time (θ_1=3.5π,θ_2=2.4π)

b) Error analysis of Euler's and 4th-order Runge-Kutta method:

The local truncation error of the Euler's method is $O(h^2)$ *(Manocha)* while that of the 4th-order Runge-Kutta method is $O(h^5)$ *(Trench)*. This means that for the Euler method, halving the step size quarters the error. For the RK4 method, halving the step size cuts the error down 0.5^5=16 times, which provides a significantly more accurate way of numerically calculating the system of differential equations.

This significantly higher accuracy of the RK4 method relative to Euler's method was demonstrated in the total energy plots of each double pendulum system. The maximum discrepancy in the total energy from the initial total energy using the RK4 method was observed in figure 21 with a value of 5×10^{-4} J; this corresponds to a percentage error difference of 1.4×10^{-3}%. On the other hand, Euler's method resulted in a maximum discrepancy of 22.28 J, which corresponds to a large percentage error difference of 38.1%. Another observation that exposes the significant error in Euler's method is the system's behavior for small angles. A double pendulum system behaves like a simple linear pendulum at small angles which was only observed using the RK4 method in figures 7 and 13. Thus, for the next numerical experiments performed, RK4 will be exclusively used to run the simulation as it provides more accurate results.

c) Effect of minor changes in initial conditions on the double pendulum system

Two double pendulum systems were initialized with almost the same conditions except for the angular velocity of the upper bob in the second system, which had an additional 0.01 added to it. The pendulum's position and each bob's angle over time were then plotted as shown below.

<u>Initial Conditions: $\theta_1=\pi$, $\theta_2=\pi$, $\omega_1=0.1$, $\omega_2=0$</u>

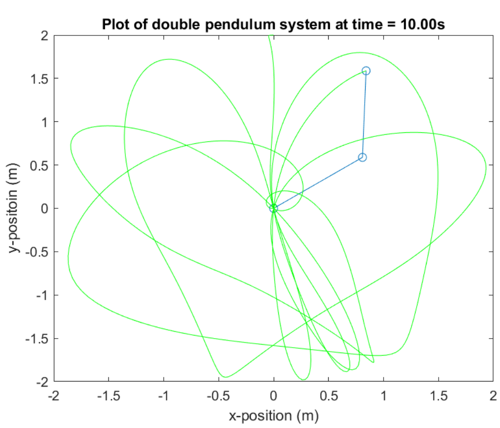

Figure 22. Simulation of both pendulums.

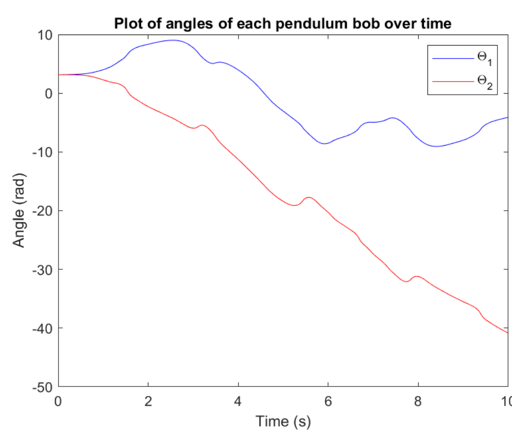

Figure 23. Angle of both bobs over time.

Initial Conditions: $\theta_1=\pi$, $\theta_2=\pi$, $\omega_1=0.11$, $\omega_2=0$

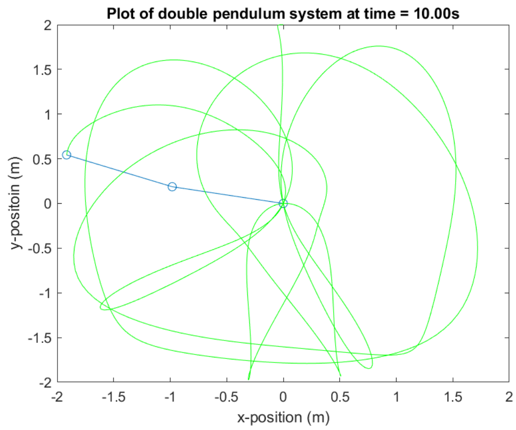

Figure 24. Simulation of both pendulums.

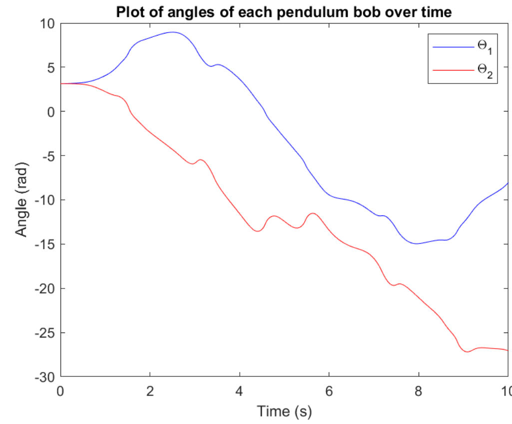

Figure 25. Angle of both bobs over time

Combined plots of each system's pendulum bobs' angles with the first system indicated by the θ_1 and θ_2 angles and the second system by the ϕ_1 and ϕ_2 angles:

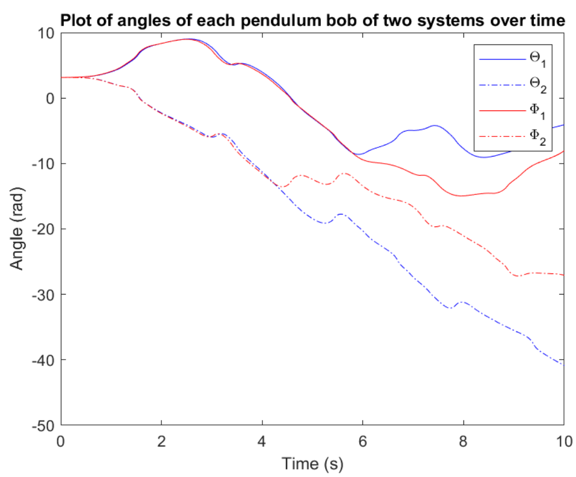

Figure 26. Graph of angles of both bobs for the two systems over time.

Given the very similar initial conditions of both systems, both followed a very similar path for the first 4 seconds of the simulation. Just after 4 seconds, the lower bob of the second system deviated from the general path of the first system's lower bob. The same behavior was also seen for the upper bob of each system; they both followed a similar path for the first 6 seconds and separated after that. This numerical test highlights the impact of very minor changes in initial conditions in a chaotic system such as that of a double pendulum system.

d) Determining equilibrium points of the double pendulum system

Equilibrium position-Initial conditions: $\theta_1=0$, $\theta_2=0$

As shown in Figure 22, the simulation showed the double pendulum system to be in equilibrium such that it remained at the same position for the entire period of the simulation. The energy-time graph also showed that the total energy of the system remained at constant 9.81J for the entire duration of the simulation(see Figure 24). The graph of the angles with time confirmed that the system was indeed in equilibrium as the angles of both pendulums remained at constant 0 for the entire duration of the simulation(see Figure 23).

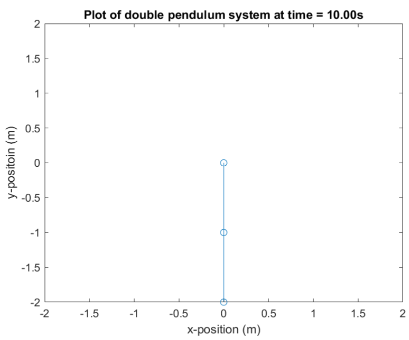

Figure 22: Simulation of positions of both pendulum with time(θ_1=0,θ_2=0)

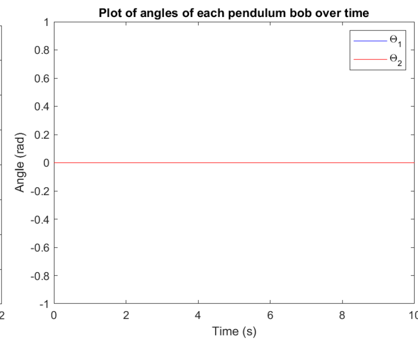

Figure 23: Graph of angles of both bobs against time(θ_1=0,θ_2=0)

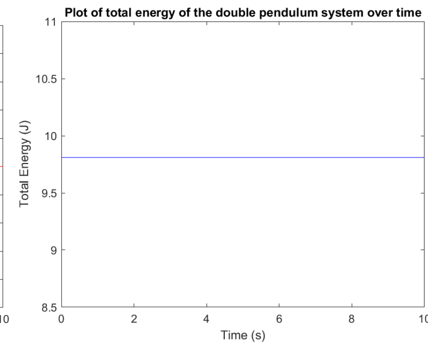

Figure 24: Graph of energy of system against time(θ_1=0,θ_2=0)

Equilibrium position-Initial conditions:θ_1=0,θ_2=π

This simulation also showed the double pendulum system to be in equilibrium such that it remained at the same position for the entire period of the simulation (see Figure 25). Energy-time graph also showed that the total energy of the system remained the same(29.43J) for the entire duration of the simulation(see Figure 27). The graph of the angles with time confirmed that the system was indeed in equilibrium as the angle of pendulum one remained at constant 0 and pendulum 2 remained at constant π rad for the entire duration of the simulation (see Figure 26).

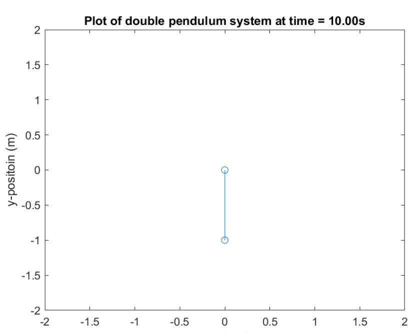

Figure 25: Simulation of positions of both pendulums with time(θ_1=0,θ_2=π)

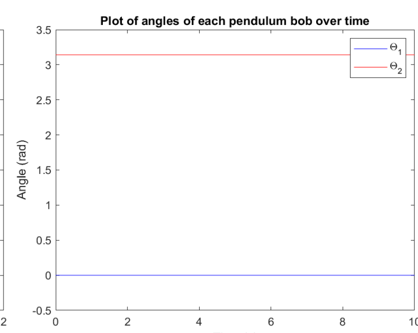

Figure 26: Graph of angles of both bobs against time (θ_1=0,θ_2=0π)

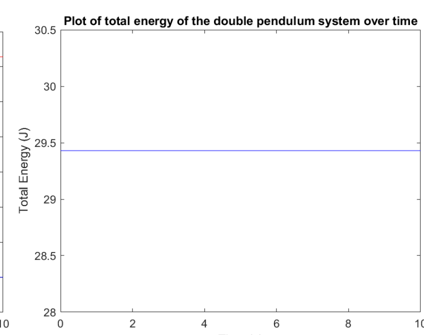

Figure 27: Graph of energy of system against time(θ_1=0,θ_2=π)

Equilibrium position-Initial conditions:θ_1=π,θ_2=0

This simulation showed the double pendulum system to be in equilibrium such that it remained at the same position for the entire period of the simulation (see Figure 28). Energy-time graph also showed that the total energy of the system remained the same(49.05J) for the entire duration of the simulation (see Figure 30). The graph of the angles with time confirmed that the system was indeed in equilibrium as the angle of pendulum one remained at constant π rad and pendulum 2 remained at constant 0 for the entire duration of the simulation (see Figure 29).

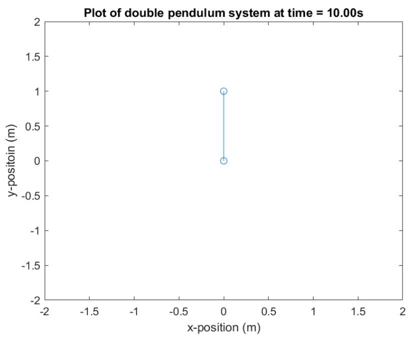

Figure 28: Simulation of positions of both pendulums with time(θ_1=π,θ_2=0)

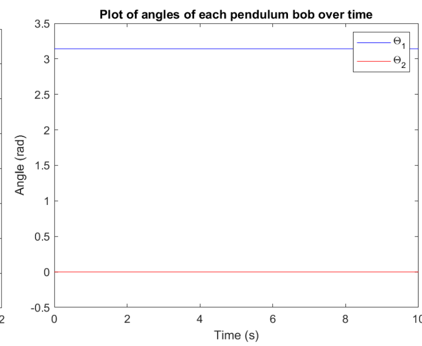

Figure 29: Graph of angles of both bobs against time(θ_1=π,θ_2=0)

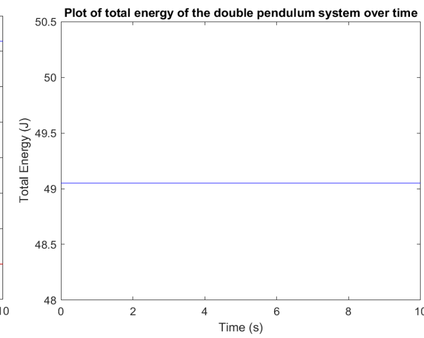

Figure 30: Graph of energy of system against time(θ_1=π,θ_2=0)

<u>Equilibrium position-Initial conditions:θ_1=π,θ_2=π</u>

This simulation showed the double pendulum system to be in equilibrium such that it remained at the same position for the entire period of the simulation (see Figure 31). Energy-time graph also showed that the total energy of the system remained the same(68.67J) for the entire duration of the simulation (see Figure 33). The graph of the angles with time confirmed that the system was indeed in equilibrium as the angle of pendulum one remained at constant π rad and pendulum 2 remained at constant π rad for the entire duration of the simulation (see Figure 32).

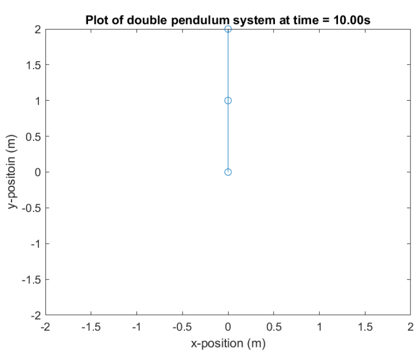

Figure 31: Simulation of positions of both pendulums with time (θ1=π,θ2=π)

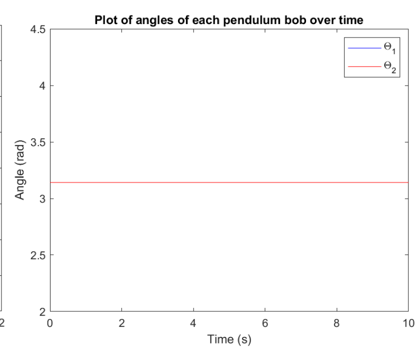

Figure 32: Graph of angles of both bobs against time(θ1=π,θ2=π)

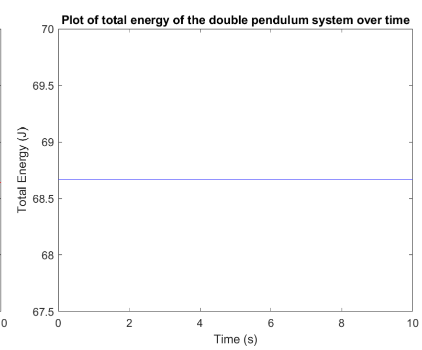

Figure 33:Graph of energy of system against time(θ1=π,θ2=π)

e) Determining stable and unstable equilibrium points

To determine whether each equilibrium point is stable or unstable, a small angular velocity of 0.01 was applied initially to the lower bob of each double pendulum system.

<u>Equilibrium position-Initial conditions:θ_1=0,θ_2=0,ω_1=0,ω_2=0.01</u>

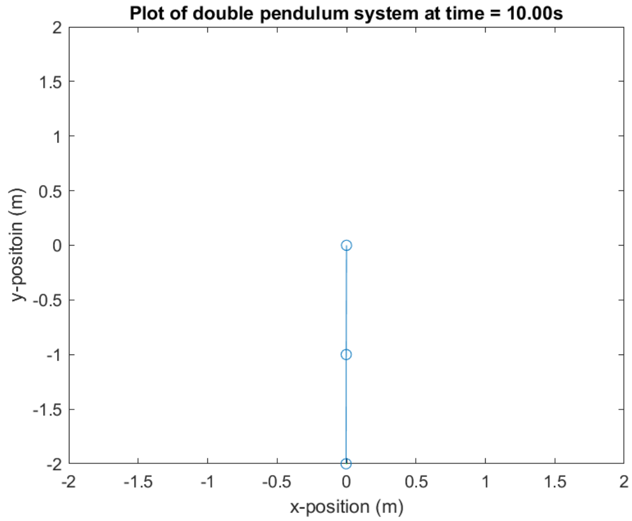

Figure 34:Simulation of positions of both bobs (θ_1=0,θ_2=0,ω_1=0,ω_2=0.01)

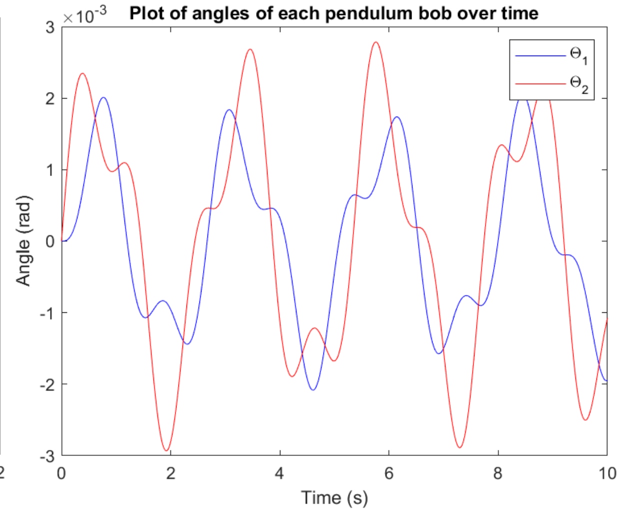

Figure 35: Angles of both bobs against time (θ_1=0,θ_2=0,ω_1=0,ω_2=0.01)

<u>Equilibrium position-Initial conditions:θ_1=0,θ_2=π,ω_1=0,ω_2=0.01</u>

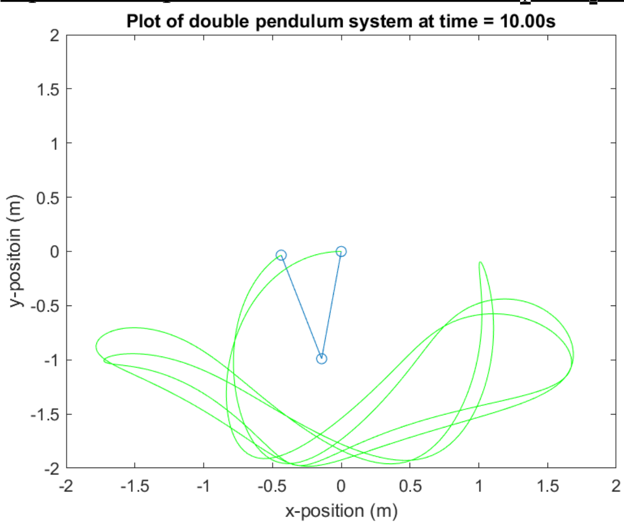

Figure 36:Simulation of positions of both bobs (θ_1=0,θ_2=π,ω_1=0,ω_2=0.01)

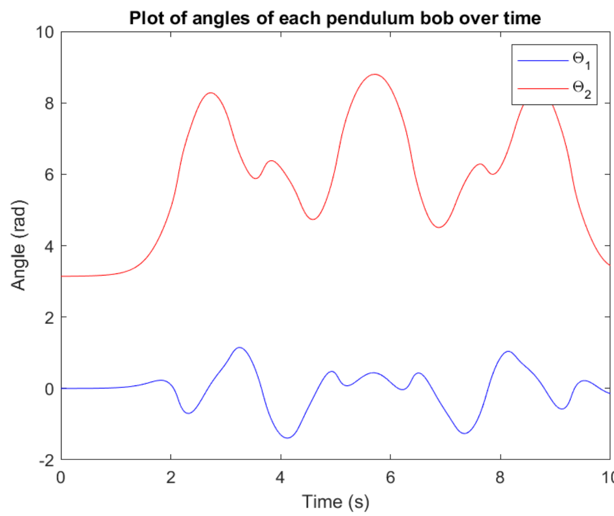

Figure 37: Graph of angles of both bobs against time (θ_1=0,θ_2=π,ω_1=0,ω_2=0.01)

Equilibrium position-Initial conditions:θ_1=π,θ_2=0,ω_1=0,ω_2=0.01

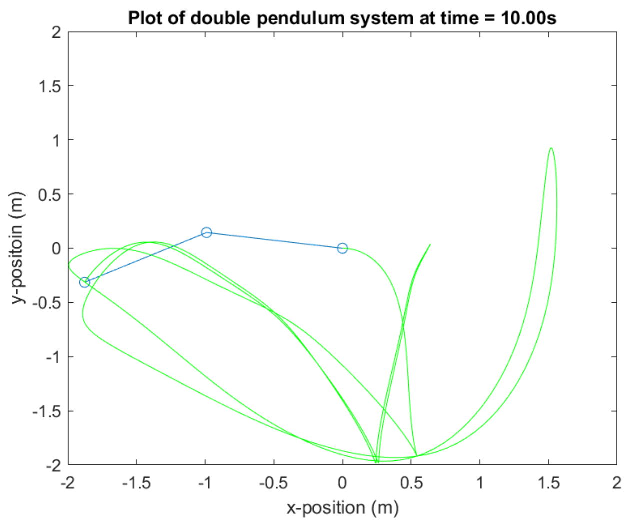

Figure 38:Simulation of positions of both pendulums with time (θ_1=π,θ_2=0,ω_1=0,ω_2=0.01)

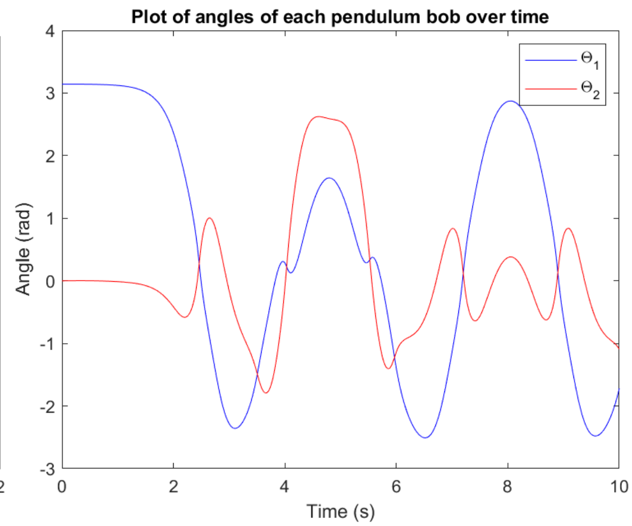

Figure 39: Graph of angles of both bobs against time (θ_1=π,θ_2=0,ω_1=0,ω_2=0.01)

Equilibrium position-Initial conditions:θ_1=π,θ_2=π,ω_1=0,ω_2=0.01

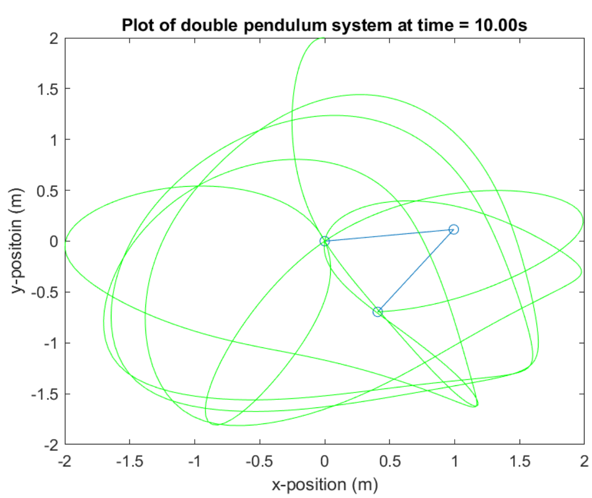

Figure 40: Simulation of positions of both pendulums with time(θ_1=π,θ_2=π,ω_1=0,ω_2=0.01)

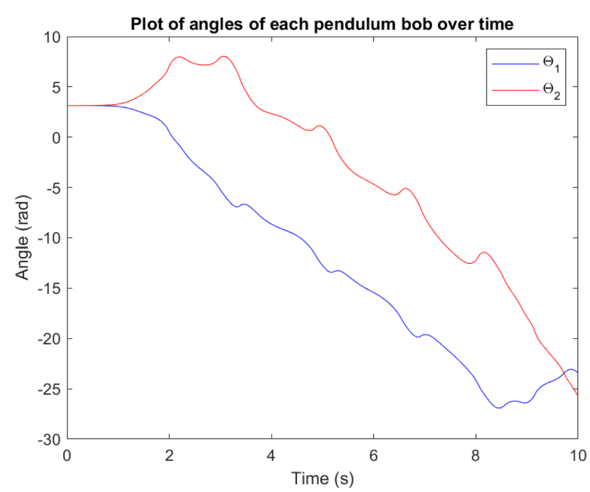

Figure 41: Graph of angles of both bobs against time (θ_1=π,θ_2=π,ω_1=0,ω_2=0.01)

As seen from the previous figures, the first equilibrium position with initial conditions of θ_1=0,θ_2=0,ω_1=0,ω_2=0.01 was the only system at equilibrium which was also observed to be stable, the small perturbation in the system by giving the lower bob an initial angular velocity of

$0.01ms^{-1}$ did not disturb the equilibrium much. The fluctuations in the angles of each bob were very minor with variations on the 10^{-3} order as seen in figure 35 and the position of the pendulum remained almost the same in figure 34. For the remaining equilibrium positions, all the systems were observed to be unstable as the small perturbation in system by adding an angular velocity of $0.01ms^{-1}$ to the lower bob caused very chaotic behavior and a rapid change in both the angles of each bob and the position of the double pendulum system.

Since an even multiple of the angle represents the same equilibrium system both mathematically and geometrically, we can generalize the conditions for a stable equilibrium system as follows:

$$\theta_1 = \pm 2k\pi,\ k = 0, 1, 2, ...$$
$$\theta_2 = \pm 2k\pi,\ k = 0, 1, 2, ...$$
$$\omega_1 = 0$$
$$\omega_2 = 0$$

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