GNSS Signal Processing Lab : GPS Receiver Case Study

Lorenzo Ortega

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1 GPS Receiver Position Computation

1.1 Trilateration : theoretical principle

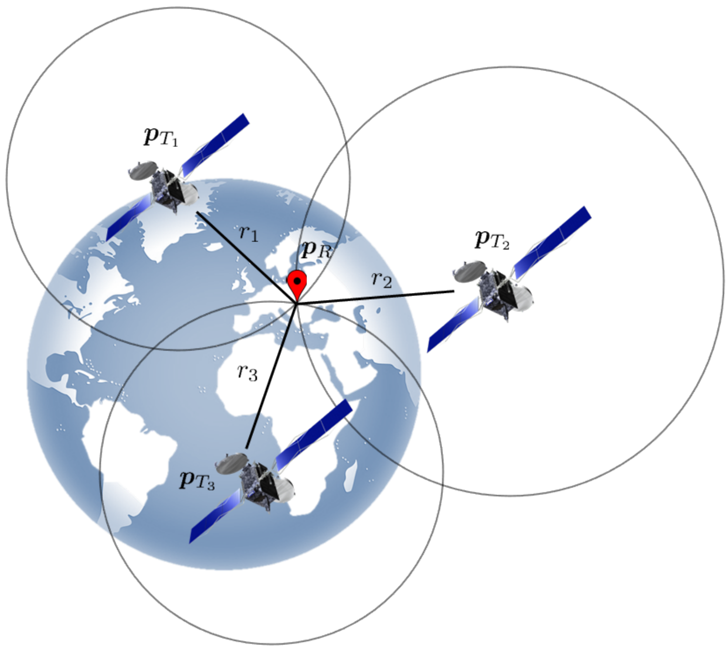

FIGURE 1 – GNSS positioning working principle : trilateration

Positioning via a constellation of satellites is the implementation of a known geometric property since antiquity, namely that the intersection of 3 spheres defines two points in the 3-dimensional space. This is the principle of trilateration : 3 satellites $\{T_1, T_2, T_3\}$ at known positions $\{\mathbf{p}_{T_1}, \mathbf{p}_{T_2}, \mathbf{p}_{T_3}\}$ define the center of 3 spheres, and the position $\mathbf{p}_R$ of an observer on Earth is one of the two intersection points of these 3 spheres (the other one on space), which results in the following system of 3 (nonlinear) equations with 3 unknowns ($\mathbf{p}_R$) :

$$\begin{cases} r_1 = \|\mathbf{p}_{T_1} - \mathbf{p}_R\| \\ r_2 = \|\mathbf{p}_{T_2} - \mathbf{p}_R\| \\ r_3 = \|\mathbf{p}_{T_3} - \mathbf{p}_R\| \end{cases}, \tag{1}$$

which can be solved if the 3 distances $\{r_1, r_2, r_3\}$ are also known (see Figure 1). Then, we must be able to obtain the 3 distances $\{r_1, r_2, r_3\}$, which translates to measure the 3 travelling times

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$\{\tau_1, \tau_2, \tau_3\} = \left\{\frac{r_1}{c}, \frac{r_2}{c}, \frac{r_3}{c}\right\}$ of an electromagnetic (EM) signal between each satellite and the observer that has a receiver adapted to perform this task (c is the speed of the EM waves = speed of light in the void).

Indeed, the propagation of an EM signal can be modeled (at first order) as the combination of an amplitude attenuation β (propagation budget) and a propagation time delay $\tau = \frac{r}{c}$, which leads to the following received signal model,

$$x(t) = \beta e(t - \tau) = \beta e\left(t - \frac{r}{c}\right), \quad |\beta| \ll 1,\ \tau, r > 0. \tag{2}$$

The measurement of the propagation time between a transmitter T and a receiver R is obtained by calculating the cross-correlation function between the received signal $x(t)$ and the transmitted signal $e(t)$,

$$\breve{R}_{x,e}(\tau') = \int_{-\infty}^{+\infty} x(t)\, e\left(t - \tau'\right)^* dt. \tag{3}$$

Question 1

Show that the maximum of $\left|\breve{R}_{x,e}(\tau')\right|$ is obtained for $\tau' = \tau$.

We therefore deduce that the propagation delay is identified by

$$\tau = \arg\max_{\tau'} \left\{\left|\breve{R}_{x,e}(\tau')\right|\right\}. \tag{4}$$

But from (1), we need to measure 3 traveling times $\{\tau_1, \tau_2, \tau_3\} = \left\{\frac{r_1}{c}, \frac{r_2}{c}, \frac{r_3}{c}\right\}$. This is made possible by the simultaneous reception of *quasi-orthogonal* signals transmitted by the 3 satellites $\{T_1, T_2, T_3\}$,

$$x(t) = \beta_1 e_1(t - \tau_1) + \beta_2 e_2(t - \tau_2) + \beta_3 e_3(t - \tau_3) = \sum_{k=1}^{3} \beta_k e_k(t - \tau_k), \tag{5}$$

which verify

$$\begin{aligned}
\breve{R}_{x,e_1}(\tau') &= \int_{-\infty}^{+\infty} x(t)\, e_1\left(t - \tau'\right)^* dt = \left|\begin{array}{l}
\beta_1 \int_{-\infty}^{+\infty} e_1(t - \tau_1)\, e_1\left(t - \tau'\right)^* dt + \\
\beta_2 \int_{-\infty}^{+\infty} e_2(t - \tau_2)\, e_1\left(t - \tau'\right)^* dt + \\
\beta_3 \int_{-\infty}^{+\infty} e_3(t - \tau_3)\, e_1\left(t - \tau'\right)^* dt
\end{array}\right. \\
\breve{R}_{x,e_1}(\tau') &\simeq \beta_1 \int_{-\infty}^{+\infty} e_1(t - \tau_1)\, e_1\left(t - \tau'\right)^* dt,
\end{aligned}$$

as well as

$$\breve{R}_{x,e_2}(\tau') \simeq \beta_2 \int_{-\infty}^{+\infty} e_2(t - \tau_2)\, e_2\left(t - \tau'\right)^* dt, \quad \breve{R}_{x,e_3}(\tau') \simeq \beta_3 \int_{-\infty}^{+\infty} e_3(t - \tau_3)\, e_3\left(t - \tau'\right)^* dt,$$

which makes possible to obtain 3 decoupled ("independent") searches of the 3 propagation times. For each transmitter, the function

$$\begin{aligned}\int_{-\infty}^{+\infty} e_k\left(t-\tau_k\right) e_k\left(t-\tau'\right)^* dt &= \int_{-\infty}^{+\infty} e_k\left(t-\tau_k\right) e_k\left(t-\tau_k-\left(\tau'-\tau_k\right)\right)^* dt \\ &\downarrow u = t-\tau_k, du = dt \\ &= \int_{-\infty}^{+\infty} e_k\left(u\right) e_k\left(u-\left(\tau'-\tau_k\right)\right)^* du = \int_{-\infty}^{+\infty} e_k\left(t\right) e_k\left(t-\left(\tau'-\tau_k\right)\right)^* dt,\end{aligned}$$

only depends on the delay difference $\left(\tau'-\tau_k\right)$, that is,

$$\int_{-\infty}^{+\infty} e_k\left(t-\tau_k\right) e_k\left(t-\tau'\right)^* dt = \overset{\smile}{R}_{e_k}\left(\tau\right)\Big|_{\tau=\tau'-\tau_k}, \quad \overset{\smile}{R}_{e_k}\left(\tau\right) = \int_{-\infty}^{+\infty} e_k\left(t\right) e_k\left(t-\tau\right)^* dt, \tag{6}$$

where $\overset{\smile}{R}_{e_k}\left(\tau\right)$ is the so-called autocorrelation function of the signal $e_k\left(t\right)$.

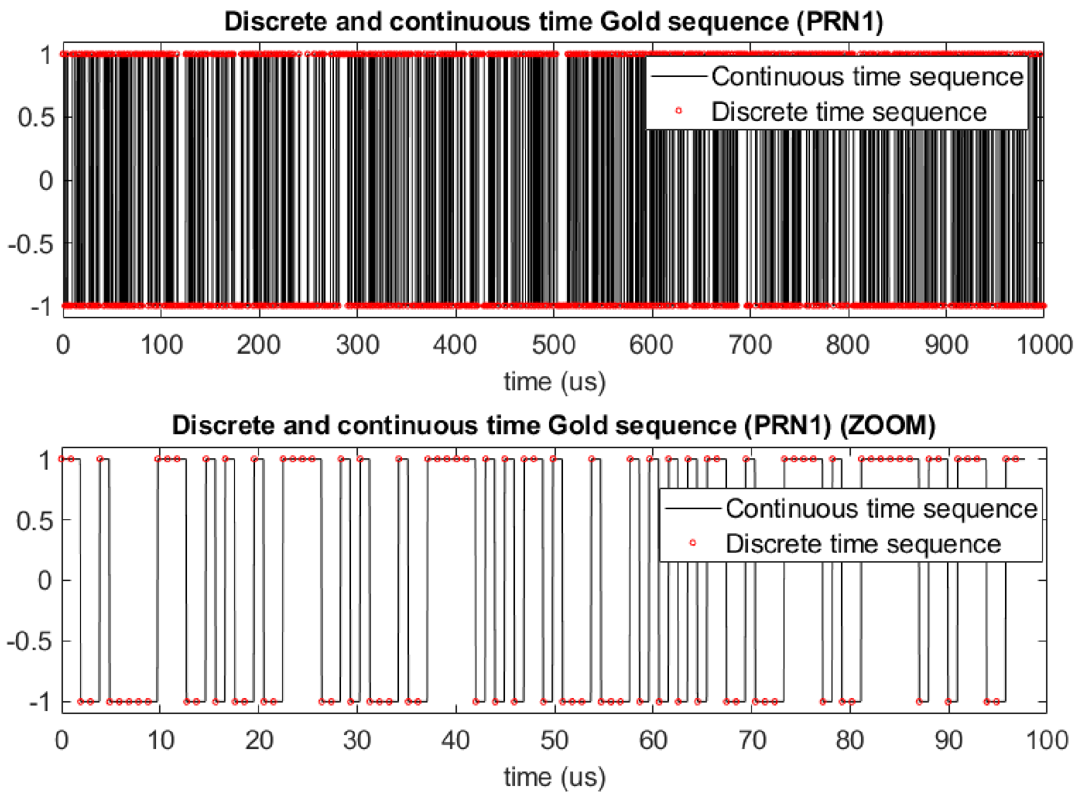


FIGURE 2 – Example of the digital-to-analog conversion (DAC) for the Gold code #1

A possible solution, being the one adopted in the legacy GPS L1 C/A signal, is to use Gold sequences (binary sequences) $\left\{c_k\left(n\right)\right\}_{n\in\mathbb{Z}}$ (see Figures 1-3 Matlab) converted into a continuous signal by a digital-to-analog converter (DAC),

$$\left\{c_k\left(n\right)\right\}_{n\in\mathbb{Z}} - \boxed{\text{DAC}} \rightarrow e_k\left(t\right) = \sum_{n\in\mathbb{Z}} c_k\left(n\right) \pi_{T_c}\left(t-\left(n-1\right)T_c\right), \pi_{T_c}\left(t\right) = \left| \begin{array}{l} 1 \text{ if } 0 \leq t \leq T_c \\ 0 \text{ otherwise} \end{array}\right. \tag{7}$$

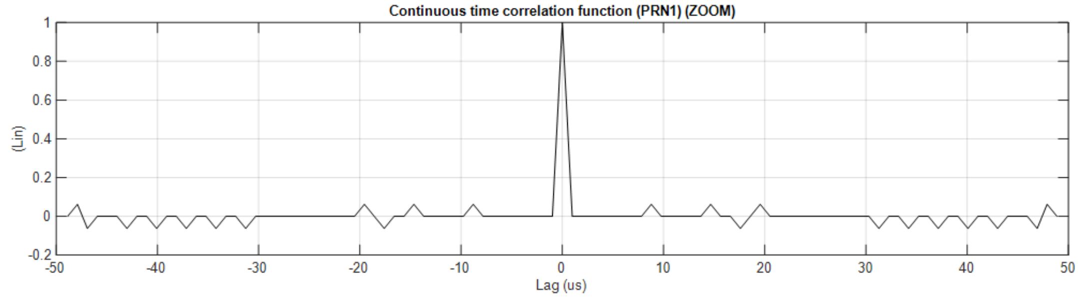
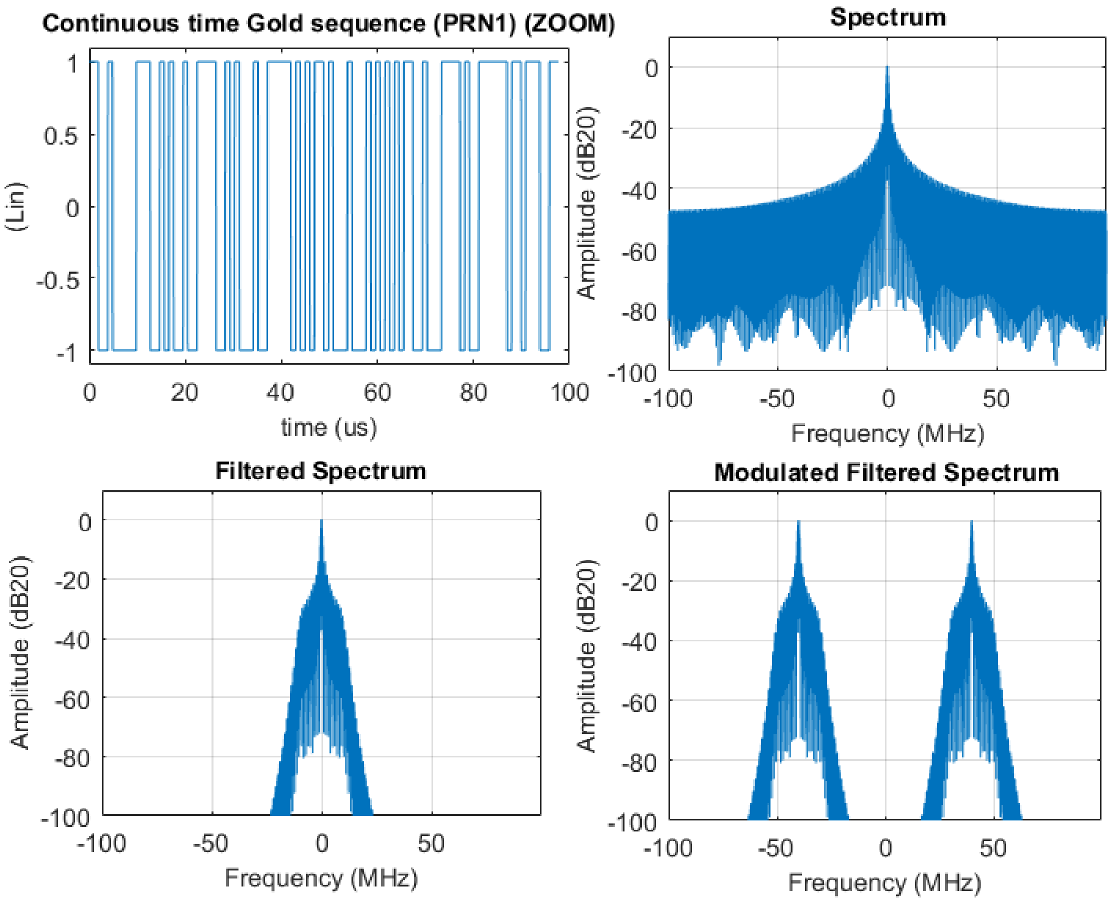


FIGURE 3 – Autocorrelation function $\breve{R}_{e_k}(\tau)$ (6) for $e_k(t) = e_1(t)$, as a function of τ (Lag).

1.2 Trilateration : practical implementation

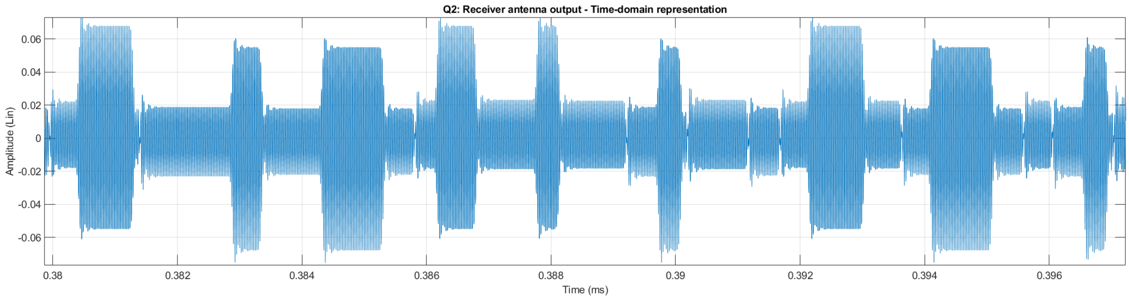
The propagation of an EM signal with "minimal" attenuation requires the modulation by a carrier frequency which makes it possible to transpose the spectrum of the signal $x(t)$, known as baseband signal (without carrier), to higher frequencies where the attenuation of the atmosphere (and other EM characteristics) is more favorable. The transmission electronics also contain a "shaping" filter (with impulse response $h_T(t)$) which makes it possible to limit the bandwidth (frequency support) of the transmitted signal in order to comply with international standards for sharing the frequency spectrum :

$$e_k(t) \xrightarrow{\boxed{\begin{array}{c}\text{filter}\\ h_T(t)\end{array}}} \tilde{e}_k(t) = e_k(t) * h_T(t) \simeq e_k(t) \xrightarrow{\boxed{\begin{array}{c}\text{modulation}\\ f_c\end{array}}} \tilde{e}_k(t)\cos(2\pi f_c t)\,. \quad (8)$$



Finally, the signal at the reception antenna is written as (see Figure 4 Matlab) :

$$\boxed{x_A(t) = \sum_{k=1}^{3} \beta_k \tilde{e}_k (t - \tau_k) \cos(2\pi f_c (t - \tau_k)) = \sum_{k=1}^{3} \beta_k \tilde{e}_k (t - \tau_k) \operatorname{Re}\left\{e^{-j2\pi f_c \tau_k} e^{j2\pi f_c t}\right\}.} \quad (9)$$



2 Radio frequency (RF) front-end architecture

From a real band-limited signal (BLS) of bandwidth B modeling the receiver antenna output (9) (see Figure (4) and Figure 4 Matlab) and an ADC operating at F_e (sampling frequency), **we are trying to reconstruct the baseband signal** (5) :

$$x(t) = \sum_{k=1}^{3} \beta'_k \tilde{e}_k (t - \tau_k) \simeq \sum_{k=1}^{3} \beta'_k e_k (t - \tau_k), \quad (10)$$

in order to be able to determine the propagation delays allowing to solve (1).

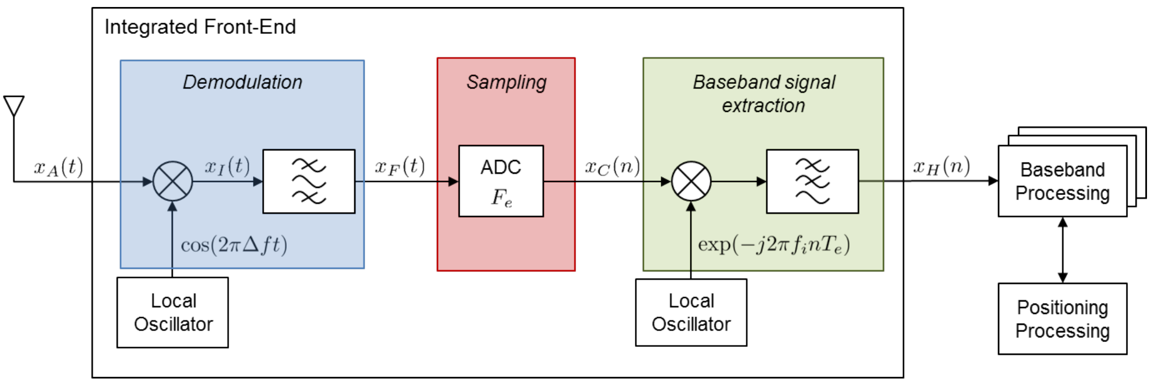


FIGURE 4 – GPS receiver RF front-end architecture.

Signal processing concepts

Nyquist-Shannon Theorem : real band-limited signal representation

If $x(t)$ is a real band-limited signal (**BLS**) in $\left[-\frac{B}{2}, \frac{B}{2}\right]$, then $\forall F_s \in \mathbb{R} \mid F_s > B$:

$$
\begin{array}{rcl}
X(f) & = & \int\limits_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt = \sum\limits_{n\in\mathbb{Z}} x(nT_s)\, e^{-j2\pi fnT_s} T_s \\
 & \downharpoonleft \; \upharpoonleft & \\
x(t) & = & \sum\limits_{n\in\mathbb{Z}} x(nT_s) \sin c\left(\pi F_s (t - nT_s)\right).
\end{array}
$$

What you must remember...

Any BLS $x(t)$ with bandwitdh B can be decomposed on the **Shannon's orthogonal basis** (not orthonormal) (Shannon o.b.) $\{g_n(t) = \sin c(\pi F_s (t - nT_s))\}_{n\in\mathbb{Z}}$ iff $T_s < \frac{1}{B}$. In this basis, the coordinates of $x(t)$ are $x(nT_s)$.
The Fourier transform (FT) of a BLS is equal to its approximation by the rectangle mid point rule if the integration domain discretization is carried out with a sufficiently small constant step $\left(T_s < \frac{1}{B}\right)$.
If $x(t)$ is not a BLS, then the projection of $x(t)$ over the Shannon o.b. is only an approximation of $x(t)$, where the approximation arises because of spectral aliasing,

$$
f \in \left[-\frac{F_s}{2}, \frac{F_s}{2}\right] : \sum_{n\in\mathbb{Z}} x(nT_s)\, e^{-j2\pi fnT_s} T_s = X(f) + \sum_{n\in\mathbb{Z}^*} X(f + nF_s)
$$

$$
\downharpoonleft \upharpoonleft
$$

$$
\sum_{n\in\mathbb{Z}} x(nT_s) \sin c\left(\pi F_s (t - nT_s)\right)
$$

2.1 Demodulation

We want to perform a demodulation, i.e., move the useful part of the spectrum from the carrier frequency f_c to an intermediate frequency $f_i < f_c$ ($f_i = 15$ Mhz). This demodulation is carried out in two stages : a first stage translates the spectrum to the intermediate frequency, which is followed by the second state aiming to filter unwanted frequencies.

Question 2

→ Figure 4 Matlab

(a) Based on the graphical representation of the amplitude of the signal $x_A(t)$ spectrum in decibels (dB), what is the value, in MHz, of the carrier frequency f_c ?

(b) What is the frequency F_s used for the Shannon basis $\{g_n(t) = \sin c(\pi F_s (t - nT_s))\}_{n\in\mathbb{Z}}$?

(c) Is the frequency F_s sufficient to represent the signals at the antenna output $x_A(t)$ (9) ?

Question 3

We want to transpose the received signal to an intermediate frequency $f_i < f_c$ (→ Figure 5 Matlab).

(a) Mathematically describe the transformation to be performed on $x_A(t)$ in order to obtain the transposed signal $x_I(t)$.

(b) Give the mathematical expression of $X_I(f)$, the spectrum of $x_I(t)$, as a function of $X_A(f)$, the spectrum of $x_A(t)$.

Question 4

We want to filter the high frequencies of the signal $x_I(t)$, since the useful signal is at the low frequencies.

Signal processing concepts

Linear time-invariant (LTI) systems

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \quad \xrightarrow{\boxed{L}} \quad \begin{aligned} y(t) &= \int_{-\infty}^{\infty} X(f)\, H(f)\, e^{j2\pi ft} df \\ y(t) &= \int_{-\infty}^{\infty} x(u)\, h(t-u)\, du = x(t) * h(t) \end{aligned}$$

where $H(f)$ is the "transfer function" (in the frequency domain), and its inverse FT $h(t) \rightleftharpoons H(f)$ is the "impulse response" (in the time domain).
If $x(t)$ is a **BLS** in $\left[-\frac{B}{2}, \frac{B}{2}\right]$, then $\forall F_s \in \mathbb{R} \mid F_s > B$:

$$\begin{aligned} (x*h)(t) &= \sum_{n\in\mathbb{Z}} (x*h)(nT_s) \sin c\left(\pi F_s (t - nT_s)\right) \\ (x*h)(nT_s) &= T_s \sum_{l\in\mathbb{Z}} x(lT_s)\, h((n-l)T_s) \end{aligned}$$

(a) Express $x_I(t)$ from the expression of $x_A(t)$ (9) in such a way as to reveal the low frequency components $x_I^-(t)$ and high frequency components $x_I^+(t)$ ($\rightarrow$ Figure 5 Matlab) such as :

$$x_I(t) = x_I^-(t) + x_I^+(t)$$

(b) What is the filtering operation performed by a LTI system ?

(c) Express the signal $x_F(t)$ obtained at the output of the filter as a function of $x_I(t)$ and $h_I(t)$.

(d) Deduce the expression of the FT of $x_F(t)$.

(e) What properties should an analog filter verify ? Does the considered filter ($\rightarrow$ Figure 6 Matlab) verify them ?

(f) By observing the spectrum of the filtered signal ($\rightarrow$ Figure 7 Matlab), what can you say from a qualitative point of view about this demodulation operation ?

(g) If the demodulation operation was ideal, what would be $x_F(t)$?

What you must remember...

For any BLS $x(t)$ with bandwidth B, any transformation by a LTI system (filter,....) can be expressed from the transformation of its coordinates $\{x(nT_s)\}_{n\in\mathbb{Z}}$ in the **Shannon o.b.**

2.2 Analogique-to-Digital Conversion

We want to simulate the ADC operation (see Figure 4) with sampling frequency $F_e = 50$ MHz.

Signal processing concepts

Nyquist-Shannon Theorem : sampling real signals

Any sequence of time samples $\{x(nT_e)\}_{n\in\mathbb{Z}}$ obtained from a real signal $x(t) \rightleftharpoons X(f)$, is associated with a continuous real signal with band-limited bandwidth $F_e = \frac{1}{T_e}$:

$$x_e(t) = \sum_{n\in\mathbb{Z}} x(nT_e)\sin c\left(\pi F_e (t - nT_e)\right) \rightleftharpoons \begin{array}{l} X_e(f) = \sum\limits_{n\in\mathbb{Z}} x(nT_e)\, e^{-j2\pi f nT_e} T_e \\ X_e(f) = \sum\limits_{l\in\mathbb{Z}} X(f + lF_e) \end{array}$$

What you must remember...

If the continuous signal $x(t)$ at the input of the ADC is of bandwidth B, and if the sampling frequency of the ADC verifies $F_e < B$, then there is spectral aliasing.

Question 5

(a) We note $x_C(n)$ the samples at the output of the ADC ($\rightarrow$ Figure 8 Matlab). Express the sampled FT $X_C(f)$ associated to the samples $x_C(n)$ as a function of $X_F(f)$.

(b) What is the sampling effect on the spectrum of $X_F(f)$?

(c) If the demodulation had been ideal, what would be $X_C(f)$?

(d) What would a priori be the best choice for f_i ? In this case what would "ideally" be $x_C(t) \rightleftharpoons X_C(f)$?

(e) Are the frequencies f_i and F_e well suited one to the other ?

Question 6

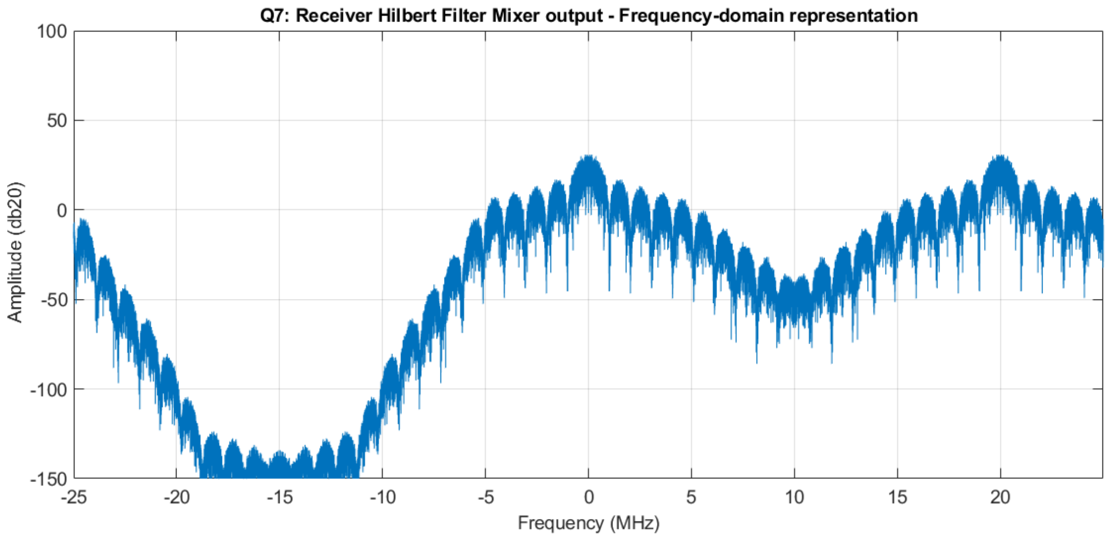
Show that there are two sequences $x_C^+(n)$ et $x_C^-(n)$ such that :

$$X_C(f) = \sum_{n\in\mathbb{Z}} x_C(n)\, e^{-j2\pi f nT_e} T_e = X_C^+(f - f_i) + X_C^-(f + f_i)\,, \quad \left| \begin{array}{l} X_C^+(f) = \sum\limits_{n\in\mathbb{Z}} x_C^+(n)\, e^{-j2\pi f nT_e} T_e \\ X_C^-(f) = \sum\limits_{n\in\mathbb{Z}} x_C^-(n)\, e^{-j2\pi f nT_e} T_e \end{array} \right.$$

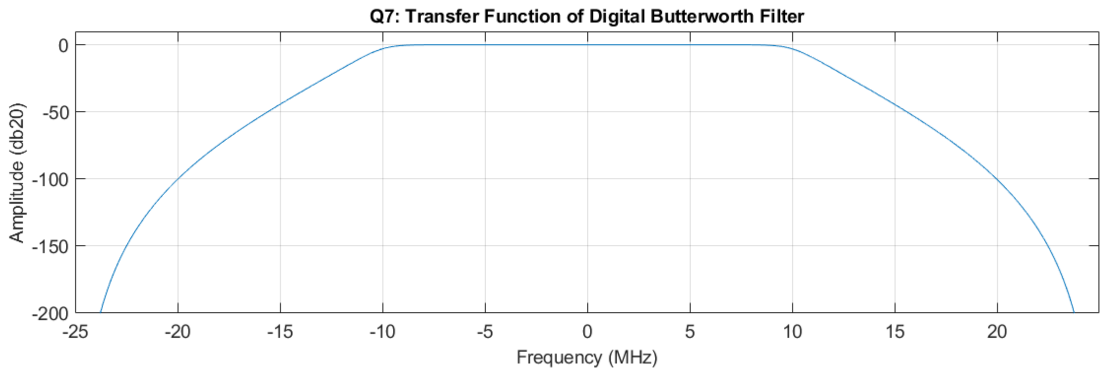
2.3 Digital Filtering

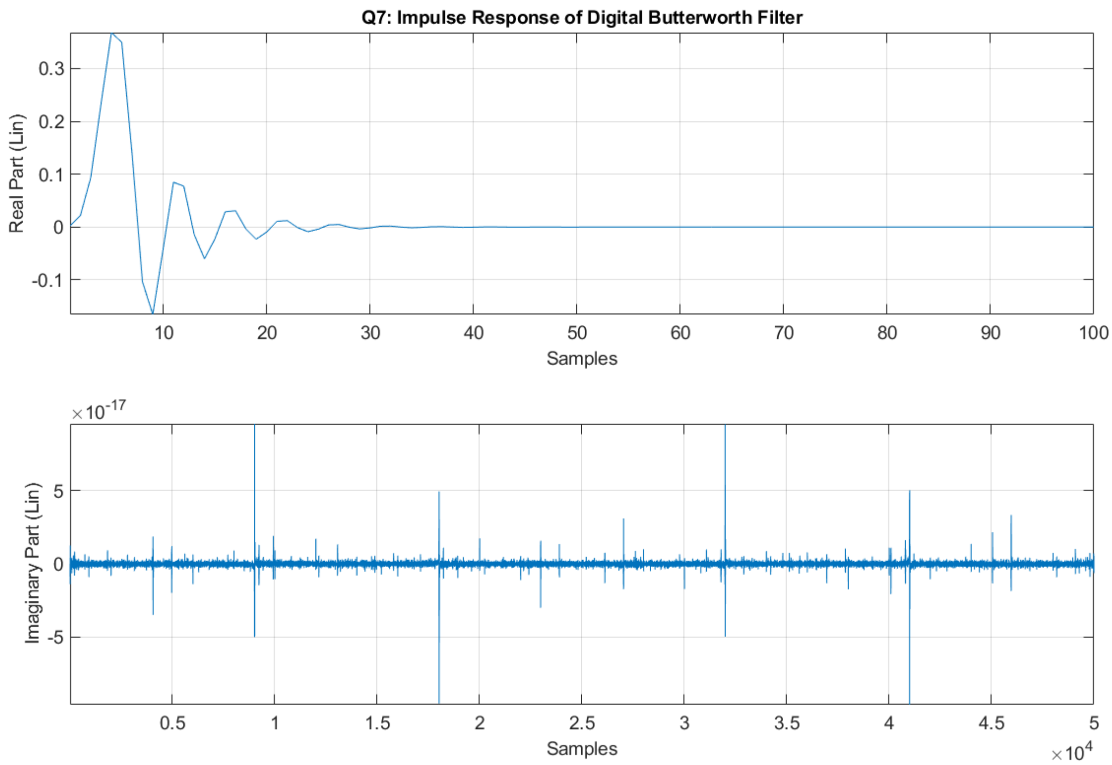
We want to preserve only $x_C^+(n)$. For this we use a digital Hilbert filter ($\rightarrow$ Figure 10 Matlab) : frequency shift $-f_i$

$$\begin{array}{rcl} x_C(n) & = & x_C^+(n)\, e^{j2\pi f_i nT_e} + x_C^-(n)\, e^{j2\pi(-f_i)nT_e} \\ & \downarrow & \\ x_C(n)\, e^{-j2\pi f_i nT_e} & = & x_C^+(n) + x_C^-(n)\, e^{j2\pi(-2f_i)nT_e} \Leftrightarrow \left| \begin{array}{l} X_C^+(f - f_i) \rightarrow X_C^+(f) \\ X_C^-(f + f_i) \rightarrow X_C^-(f + 2f_i) \end{array} \right. \end{array}$$



followed by a digital low-pass filter :





Generally speaking, the Hilbert filter is a LTI system which allows, from a real passband signal $x(t)$, to construct the complex signal $z(t)$ (so-called complex analytic signal) whose spectrum $Z(f)$ coincides with $X(f)$ for positive frequencies :

$$x\left(t\right)=\mathrm{Re}\left\{z\left(t\right)\right\}=\frac{z\left(t\right)+z^{*}\left(t\right)}{2}\Leftrightarrow X\left(f\right)=\frac{Z\left(f\right)}{2}+\frac{Z\left(-f\right)^{*}}{2}\ /\ \forall f<0:Z\left(f\right)=0. \tag{11}$$

Question 7

By observing the spectrum of the filtered signal ($\rightarrow$ Figure 10 Matlab), what can you say qualitatively about this Hilbert filtering operation ?

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3 Exploiting the baseband signal to determine the propagation delays

At the output of the Hilbert filter, ideally we obtained :

$$x_{H}\left(n\right)\simeq x_{C}^{+}\left(n\right)=\sum_{k=1}^{3}\beta_{k}'\widetilde{e}_{k}\left(nT_{e}-\tau_{k}\right)\simeq\sum_{k=1}^{3}\beta_{k}'e_{k}\left(nT_{e}-\tau_{k}\right),\ \beta_{k}'\in\mathbb{C} \tag{12}$$

$$e_{k}\left(n\right)=\sum_{l=0}^{1022}c_{k}\left(l\right)\pi\left(n-lM\right),\quad \pi\left(n\right):\left|\begin{array}{l}1\text{ if }0\leq n\leq M-1\\ 0\text{ otherwise}\end{array}\right.,\quad c_{k}\left(l\right)\in\left\{-1,1\right\} \tag{13}$$

where $MT_e = T_c$ and T_c is the "chip" duration, that is, of a GPS code symbol. Each GPS code is a Gold sequence with 1023 symbols and a total duration equal to $10^{-3}s$. Then, $M=\frac{10^{-3}}{1023}F_e\simeq 49$ (see Figure 5).

Recall that the samples $\left\{x_H\left(n\right)\right\}_{n\in\mathbb{Z}}$ and $\left\{e_k\left(n\right)\right\}_{n\in\mathbb{Z}}$ are the coordinates of the BLS $x_H\left(t\right)$ and $e_k\left(t\right)$ on the Shannon's o.b. with bandwidth F_e :

$$x_{H}\left(t\right)=\sum_{n\in\mathbb{Z}}x_{H}\left(n\right)\sin c\left(\pi F_{e}\left(t-nT_{e}\right)\right),\quad e_{k}\left(t\right)=\sum_{n\in\mathbb{Z}}e_{k}\left(n\right)\sin c\left(\pi F_{e}\left(t-nT_{e}\right)\right) \tag{14}$$

Question 8

Compare the "measured" cross-correlation functions (obtained from the measured $x_H\left(t\right)$ (14), $\rightarrow$ Figures 201-203 Matlab)

$$\breve{R}_{x_{H},e_{k}}\left(\tau'\right)=\int_{-\infty}^{+\infty}x_{H}\left(t\right)e_{k}\left(t-\tau'\right)^{*}dt$$

and the "ideal" autocorrelation functions ($\rightarrow$ Figures 101-103 Matlab)

$$\breve{R}_{e_{k}}\left(\tau'\right)=\int_{-\infty}^{+\infty}e_{k}\left(t\right)e_{k}\left(t-\tau'\right)^{*}dt$$

Question 9

Do the results obtained for $\breve{R}_{x_H,e_1}\left(\tau\right)$,...,$\breve{R}_{x_H,e_3}\left(\tau\right)$ validate the quasi-orthogonality hypothesis of the signals $e_k\left(t\right)$?

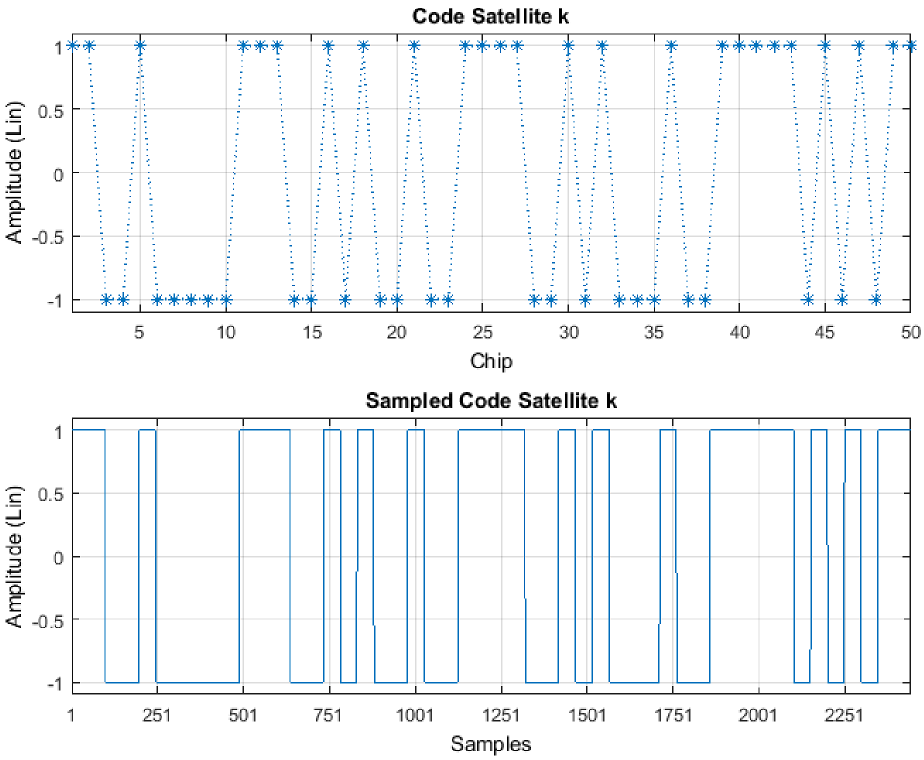

FIGURE 5 – GPS code $\{c_k\,(l)\}_{l\in[0,1022]}$ for satellite k and the corresponding baseband signal sampled at F_e : $\{e_k\,(n)\}_{n\in[0,1023\times 49-1]}$.