Electromagnetics

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Chapter 1

Vector Basics

1.1 Recommended Texts

The following books are recommended for the course. The first one will be followed by the instructor.

1. Engineering Electromagnetics 8th Edition by John Buck & William H. Hayt

2. Electromagnetics John D. Kraus

3. Introduction to Electrodynamics by David J. Griffiths

4. Classical Electrodynamics by John David Jackson

1.2 What is a Vector?

From a mathematical standpoint a vector is an element of a vector space. From a physical point of view a vector is a quantity that requires a magnitude as well as a direction to be represented.

1.3 Unit vectors in Rectangular Coordinate System

In a Rectangular Coordinate System (RCS), a vector can be represented as a linear sum of three unit vectors, namely $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$. In case of two dimensions, $\vec{a}_z$ is not needed. This is also depicted in figure 1.1.

1.4 Dot Product

Suppose we have two vectors $\vec{A}$ and $\vec{B}$. $\vec{A}$ can be represented as:

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

(a) A vector in two dimensions   (b) A vector in three dimensions

Figure 1.1

Similarly, $\vec{B}$ can be represented as:

$$\vec{B} = B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z$$

Their dot product, $\vec{A} \cdot \vec{B}$, which is a scalar quantity, is defined as:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta \tag{1.1}$$

In case when θ becomes $90°$ the dot product automatically reduces to zero. Thus one can easily conclude that the dot product of two perpendicular vectors shall always be zero. This makes expressing the dot product in terms of its components pretty straight forward.

$$\vec{A} \cdot \vec{B} = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z) \cdot (B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z)$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.2}$$

A vector can also be represented in the form of column vector as:

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad \vec{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

In that case the dot product, also called inner product in this context, is defined as:

$$\mathbf{A}^T\mathbf{B}$$

If the vectors $\vec{A}$ and $\vec{B}$ are of the order $n \times 1$ their inner product will have the order $1 \times (n \times n) \times 1 = 1 \times 1$. Thus the result will be a scalar quantity. The opposite of the inner product is known as the outer product also called the cross product which we will get to in a later topic.

1.4.1 Cauchy Bunyakovsky Schwarz Inequality

The theorem states:

$$|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$$

It's easy to see why this is the case by replacing $\vec{A} \cdot \vec{B}$ by it's value:

$$|\vec{A}||\vec{B}| \cos \theta \leq |\vec{A}||\vec{B}|$$

This inequality changes to an equality when both vectors are collinear.

1.4.2 The Triangle Inequality

$$|\vec{A}| + |\vec{B}| \geq |\vec{A} + \vec{B}|$$

It's quite easy to see why that is the case in the case of two dimensions. At this moment it should be useful to point out that:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A}$$

This shall be useful in proving the Parallelogram Equality.

1.4.3 The Parallelogram Equality

The equality states:

$$|\vec{A} + \vec{B}|^2 + |\vec{A} - \vec{B}|^2 = 2\left(|\vec{A}|^2 + \vec{B}^2\right)$$

To prove it:

$$|\vec{A} + \vec{B}|^2 = \left(\vec{A} + \vec{B}\right) \cdot \left(\vec{A} + \vec{B}\right)$$

$$= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$= |\vec{A}|^2 - 2\vec{A} \cdot \vec{B} + |\vec{B}|^2$$

Similarly:

$$|\vec{A} - \vec{B}|^2 = \left(\vec{A} - \vec{B}\right) \cdot \left(\vec{A} - \vec{B}\right)$$

$$= \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$= |\vec{A}|^2 + 2\vec{A} \cdot \vec{B} + |\vec{B}|^2$$

Thus:

$$|\vec{A} + \vec{B}|^2 + |\vec{A} - \vec{B}|^2 = 2|\vec{A}| + 2|\vec{B}| = 2\left(|\vec{A}| + |\vec{B}|\right)$$

Figure 1.2: Projections of two vectors $\vec{A}$ and $\vec{B}$ on each other.

1.4.4   Scalar component of one vector in the direction of another

Dot product can be really handy when we want to figure out the projection of one vector onto the direction of another vector. Refer to figure1.2. The line $\bar{OP}$ is the projection of vector $\vec{B}$ in the direction of $\vec{A}$. Similarly, $\bar{OQ}$ is the projection of vector $\vec{A}$ in the direction of vector $\vec{B}$. Since the line from the tip of vector $\vec{B}$ to P is perpendicular to the vector $\vec{A}$ and the line from the tip of $\vec{A}$ to point Q is perpendicular to vector $\vec{B}$. We can apply trigonometry to find out these projections.

Let B_A denote the projection of vector $\vec{B}$ in the direction of $\vec{A}$ and let A_B denote the projection of $\vec{A}$ in the direction of $\vec{B}$. Applying trigonometry we get:

$$B_A = |\vec{B}| \cos\theta$$

$$A_B = |\vec{A}| \sin\theta$$

Recalling the definition of a dot product from equation 1.1 we can see:

$$B_A = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} = \vec{u}_A \cdot \vec{B}$$

$$A_B = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|} = \vec{u}_B \cdot \vec{A}$$

1.4.5   Some practical applications

Some common places in Physics where you will find the dot product being applied are:

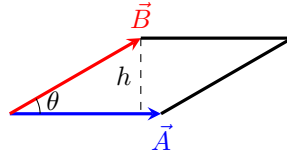- The formulas for work done.

$$W = \vec{F} \cdot \vec{d}$$

$$dW = \vec{F} \cdot \vec{dl}$$

$$W = \int_L \vec{F} \cdot \vec{dl}$$

- The forumulas for Electric and Magnetic flux.

$$\phi_E = \vec{E} \cdot \vec{A}$$

$$\phi_M = \vec{B} \cdot \vec{A}$$

Figure 1.3: A parallelogram formed by two vectors $\vec{A}$ and $\vec{B}$

- A third one I don't understand yet. Need to confirm this one.

$$Q = \iint\limits_{S} \vec{D} \cdot \vec{dS}$$

1.5 Cross Product

1.5.1 The Definition

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is given by:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \vec{u}_N \tag{1.3}$$

Where $\theta \in [0, \pi]$ and $\vec{u}_N$ is a vector given by the right hand rule where fingers should be curled from the direction of $\vec{A}$ to the direction of $\vec{B}$. Since the direction of the cross product is dictated by the right hand rule we can already see that it is not a commutative operation. More accurately:

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \tag{1.4}$$

1.5.2 Geometric Significance

As you can see very clearly in figure 1.3 that any two vectors $\vec{A}$ and $\vec{B}$ can uniquely identify a parallelogram. The area of any parallelogram is given by the formula:

$$Area = (base) \times (height) \tag{1.5}$$

In figure 1.3 the base is the length of the vector $\vec{A}$ and the height is shown by the dotted line. But how do we figure out what h really is? By applying trigonometry here we can find out that $h = |\vec{A}| \sin\theta$. Plugging in these values in equation 1.5 we get:

$$Area = (|\vec{B}|)(|\vec{A}| \sin\theta)$$

Ahah, this is exactly the same formula that is given by $\vec{A} \cdot \vec{B}$. Thus we can conclude that the area of the parallelogram formed by two vectors $\vec{A}$ and $\vec{B}$ is given by $\vec{A} \cdot \vec{B}$.

$$Area = \vec{A} \cdot \vec{B}$$

1.5.3 Calculating Cross Product

Before we learn how to calculate the cross product of any two vectors in terms of the three basic unit vectors $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ we should think about what will

be the cross production of every possible combination of two of these three unit vectors. We know from equation 1.3 that the cross product of two parallel vectors should be zero because $\sin 90° = 0$. We also know that the magnitude of a unit vector is by definition one. Using these two facts, along with the right hand rule, we can very quickly see that:

$$\vec{a}_x \times \vec{a}_y = \vec{a}_z$$

$$\vec{a}_y \times \vec{a}_z = \vec{a}_x$$

$$\vec{a}_z \times \vec{a}_x = \vec{a}_y$$

Using equation 1.4 we can easily figure out the reverses of these combinations:

$$\vec{a}_y \times \vec{a}_x = -\vec{a}_z$$

$$\vec{a}_z \times \vec{a}_y = -\vec{a}_x$$

$$\vec{a}_x \times \vec{a}_z = -\vec{a}_y$$

Now suppose we have two vectors $\vec{A}$ and $\vec{B}$, we can express them as:

$$\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

$$\vec{B} = B_x \vec{a}_x + B_y \vec{a}_y + B_z \vec{a}_z$$

Their cross product can be written as:

$$\vec{A} \times \vec{B} = (A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z) \times (B_x \vec{a}_x + B_y \vec{a}_y + B_z \vec{a}_z)$$

Expanding this we get:

$$\vec{A} \times \vec{B} = A_x B_x (\vec{a}_x \times \vec{a}_x) + A_x B_y (\vec{a}_x \times \vec{a}_y) + A_x B_z (\vec{a}_x \times \vec{a}_z)$$
$$+ A_y B_x (\vec{a}_y \times \vec{a}_x) + A_y B_y (\vec{a}_y \times \vec{a}_y) + A_y B_z (\vec{a}_y \times \vec{a}_z)$$
$$+ A_z B_x (\vec{a}_z \times \vec{a}_x) + A_z B_y (\vec{a}_z \times \vec{a}_y) + A_z B_z (\vec{a}_z \times \vec{a}_z)$$

Firstly, we can cancel those cross products which are between collinear vectors, as we know that's going to be zero.

$$\vec{A} \times \vec{B} = A_x B_y (\vec{a}_x \times \vec{a}_y) + A_x B_z (\vec{a}_x \times \vec{a}_z)$$
$$+ A_y B_x (\vec{a}_y \times \vec{a}_x) + A_y B_z (\vec{a}_y \times \vec{a}_z)$$
$$+ A_z B_x (\vec{a}_z \times \vec{a}_x) + A_z B_y (\vec{a}_z \times \vec{a}_y)$$

Now, just utilizing the crossproducts of all possible combinations between unit vectors that we just derived a while ago, we can simplify this even further:

$$\vec{A} \times \vec{B} = A_x B_y (\vec{a}_z) + A_x B_z (-\vec{a}_y)$$
$$+ A_y B_x (-\vec{a}_z) + A_y B_z (\vec{a}_x)$$
$$+ A_z B_x (\vec{a}_y) + A_z B_y (-\vec{a}_x)$$

Grouping these terms by unit vectors we get:

$$\vec{A} \times \vec{B} = \vec{a}_x (A_y B_z - A_z B_y) + \vec{a}_y (A_z B_x - A_x B_z) + \vec{a}_z (A_x B_y - A_y B_x) \quad (1.6)$$

We can write this in an elegant way as a determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (1.7)$$

1.5.4 Properties

Cross Product does not follow the commutative law as we have already seen. However, it does follow distributive law:

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

1.5.5 Practical Applications

1. Torque:
$$\vec{\tau} = \vec{r} \times \vec{F}$$

2. Angular Momentum:
$$\vec{L} = \vec{r} \times \vec{\rho}$$
$$\vec{\rho} = m\vec{v}$$

3. Force on a moving conductor:
$$\vec{F} = I\vec{L} \times \vec{B}$$

4. Force on a moving charge:
$$\vec{F} = q\vec{v} \times \vec{B}$$

5. Lorrentz Force Formula:
$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

1.6 Scalar Triple Product

1.6.1 Definition

The Scalar Triple Product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined as:

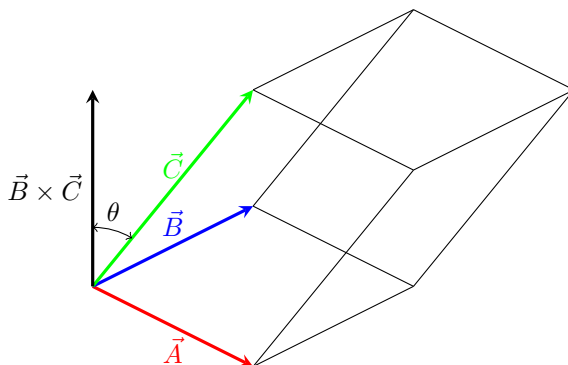$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

We already know that:

$$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.8}$$

Vector $\vec{A}$ is defined as:

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

We also know from equation 1.2 that when the dot product of two vectors is taken, their respective components are mulitplied together, the unit vectors go away and the resultant values are added, right? Let $\vec{V}$ represent $\vec{B} \times \vec{C}$. So, $\vec{A} \cdot \vec{V}$ is:

$$\vec{A} \cdot \vec{V} = A_xV_x + A_yV_y + A_zV_z$$

Figure 1.4: The parallelopiped formed by $\vec{A}$, $\vec{B}$ and $\vec{C}$

From equation 1.8 we can see that:

$$V_x = \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix}$$

$$V_y = -\begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix}$$

$$V_z = \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

Putting these results back we get:

$$\vec{A} \cdot \vec{V} = A_x \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + -A_y \begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix} + \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

Ahah, we have seen something similar while expanding determinants haven't we?

$$\vec{A} \cdot \left(\vec{B} \times \vec{C} \right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.9}$$

Exchanging two rows of a determinant two times doesn't really doesn't affect it, right? That fact, leads to the equality:

$$\vec{B} \cdot \left(\vec{C} \times \vec{A} \right) = \vec{C} \cdot \left(\vec{A} \times \vec{B} \right)$$

1.6.2   Applications

The area of a parallelopiped can be calculated by the following formula:

$$Area = (\text{area of base face}) \times (h)$$

Where h is the perpendicular distance between the base face and the top face. In figure 1.4 we can see that a parallelopiped is uniquely identified by three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. The area of the base face, as we already know, is given by the magnitude of the cross product of $\vec{A}$ and $\vec{B}$. The vector $\vec{A} \times \vec{B}$ is also shown in figure 1.4. The magnitude of this vector gives us the area and direction of

this vector is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. Now we need to figure out the perpendicular distance between the base face and the top face. That distance is given by $|\vec{C}|\cos\theta$. If we take the dot product of $\vec{A} \times \vec{B}$ and $\vec{C}$ we get:

$$\vec{C} \cdot \left(\vec{A} \times \vec{B}\right) = \left(|\vec{C}|\cos\theta\right)\left(|\vec{A} \times \vec{B}|\right)$$
$$= \left(|\vec{C}|\cos\theta\right)(\text{area of base face})$$
$$= (\text{h})(\text{area of base face})$$
$$= \text{Volume of the parallelopiped}$$

Hence we have proven that the scalar product of three vectors gives us the volume of the parallelopiped formed by them.

1.7 Curl and Divergence

Before we discuss Curl and Divergence, we define the ∇ operator as:

$$\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

1.7.1 Divergence

Divergence of a vector field $\vec{F}$ is defined as:

$$div(\vec{F}) = \nabla \cdot \vec{F}$$

$$div(\vec{F}) = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right) \cdot \left(\vec{F}_x\vec{a}_x + \vec{F}_y\vec{a}_y + \vec{F}_z\vec{a}_z\right)$$

Solving that we get:

$$div(\vec{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.10}$$

It would be good here to mention here that one of the Maxwell's equation uses divergence:

$$\nabla \cdot \vec{B} = 0$$

Another interpretation of this equation is "No magnetic monopoles exist".