Electromagnetics

Moazin Khatti

October 5, 2018

Contents

Chapter 1

Vector Basics

1.1 Recommended Texts

The following books are recommended for the course. The first one will be followed by the instructor.

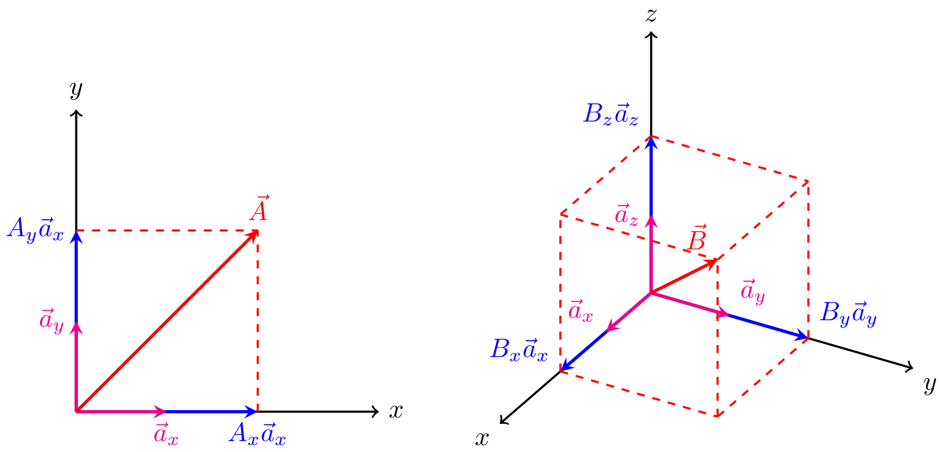
1. Engineering Electromagnetics 8th Edition by John Buck & William H. Hayt
2. Electromagnetics John D. Kraus
3. Introduction to Electrodynamics by David J. Griffiths
4. Classical Electrodynamics by John David Jackson

1.2 What is a Vector?

From a mathematical standpoint a vector is an element of a vector space. From a physical point of view a vector is a quantity that requires a magnitude as well as a direction to be represented.

1.3 Unit vectors in Rectangular Coordinate System

In a Rectangular Coordinate System (RCS), a vector can be represented as a linear sum of three unit vectors, namely $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$. In case of two dimensions, $\vec{a}_z$ is not needed. This is also depicted in figure 1.1.



(a) A vector in two dimensions (b) A vector in three dimensions

Figure 1.1

1.4 Dot Product

Suppose we have two vectors $\vec{A}$ and $\vec{B}$. $\vec{A}$ can be represented as:

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

Similarly, $\vec{B}$ can be represented as:

$$\vec{B} = B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z$$

Their dot product, $\vec{A}\cdot\vec{B}$, which is a scalar quantity, is defined as:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta \tag{1.1}$$

In case when θ becomes $90°$ the dot product automatically reduces to zero. Thus one can easily conclude that the dot product of two perpendicular vectors shall always be zero. This makes expressing the dot product in terms of its components pretty straight forward.

$$\vec{A}\cdot\vec{B} = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)\cdot(B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z)$$
$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.2}$$

A vector can also be represented in the form of column vector as:

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \vec{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

In that case the dot product, also called inner product in this context, is defined as:

$$\mathbf{A}^T\mathbf{B}$$

If the vectors $\vec{A}$ and $\vec{B}$ are of the order $n\times 1$ their inner product will have the order $1\times(n\times n)\times 1 = 1\times 1$. Thus the result will be a scalar quantity. The opposite of the inner product is known as the outer product also called the cross product which we will get to in a later topic.

1.4.1 Cauchy Bunyakovsky Schwarz Inequality

The theorem states:

$$|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$$

It's easy to see why this is the case by replacing $\vec{A} \cdot \vec{B}$ by it's value:

$$|\vec{A}||\vec{B}| \cos\theta \leq |\vec{A}||\vec{B}|$$

This inequality changes to an equality when both vectors are collinear.

1.4.2 The Triangle Inequality

$$|\vec{A}| + |\vec{B}| \geq |\vec{A} + \vec{B}|$$

It's quite easy to see why that is the case in the case of two dimensions. At this moment it should be useful to point out that:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A}$$

This shall be useful in proving the Parallelogram Equality.

1.4.3 The Parallelogram Equality

The equality states:

$$|\vec{A} + \vec{B}|^2 + |\vec{A} - \vec{B}|^2 = 2\left(|\vec{A}|^2 + \vec{B}^2\right)$$

To prove it:

$$|\vec{A} + \vec{B}|^2 = \left(\vec{A} + \vec{B}\right) \cdot \left(\vec{A} + \vec{B}\right)$$

$$= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$= |\vec{A}|^2 - 2\vec{A} \cdot \vec{B} + |\vec{B}|^2$$

Similarly:

$$|\vec{A} - \vec{B}|^2 = \left(\vec{A} - \vec{B}\right) \cdot \left(\vec{A} - \vec{B}\right)$$

$$= \vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}$$

$$= |\vec{A}|^2 + 2\vec{A} \cdot \vec{B} + |\vec{B}|^2$$

Thus:

$$|\vec{A} + \vec{B}|^2 + |\vec{A} - \vec{B}|^2 = 2|\vec{A}| + 2|\vec{B}| = 2\left(|\vec{A}| + |\vec{B}|\right)$$

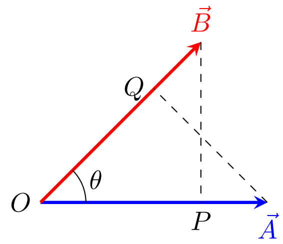


Figure 1.2: Projections of two vectors $\vec{A}$ and $\vec{B}$ on each other.

1.4.4 Scalar component of one vector in the direction of another

Dot product can be really handy when we want to figure out the projection of one vector onto the direction of another vector. Refer to figure1.2. The line $\bar{OP}$ is the projection of vector $\vec{B}$ in the direction of $\vec{A}$. Similarly, $\bar{OQ}$ is the projection of vector $\vec{A}$ in the direction of vector $\vec{B}$. Since the line from the tip of vector $\vec{B}$ to P is perpendicular to the vector $\vec{A}$ and the line from the tip of $\vec{A}$ to point Q is perpendicular to vector $\vec{B}$. We can apply trigonometry to find out these projections.

Let B_A denote the projection of vector $\vec{B}$ in the direction of $\vec{A}$ and let A_B denote the projection of $\vec{A}$ in the direction of $\vec{B}$. Applying trigonometry we get:

$$B_A = |\vec{B}| \cos\theta$$

$$A_B = |\vec{A}| \sin\theta$$

Recalling the definition of a dot product from equation 1.1 we can see:

$$B_A = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} = \vec{u}_A \cdot \vec{B}$$

$$A_B = \frac{\vec{A} \cdot \vec{B}}{|\vec{B}|} = \vec{u}_B \cdot \vec{A}$$

1.4.5 Some practical applications

Some common places in Physics where you will find the dot product being applied are:

- The formulas for work done.

$$W = \vec{F} \cdot \vec{d}$$

$$dW = \vec{F} \cdot \vec{dl}$$

$$W = \int_L \vec{F} \cdot \vec{dl}$$

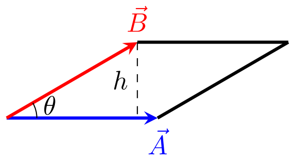


Figure 1.3: A parallelogram formed by two vectors $\vec{A}$ and $\vec{B}$

- The forumulas for Electric and Magnetic flux.

$$\phi_E = \vec{E} \cdot \vec{A}$$
$$\phi_M = \vec{B} \cdot \vec{A}$$

- A third one I don't understand yet. Need to confirm this one.

$$Q = \iint_S \vec{D} \cdot d\vec{S}$$

1.5 Cross Product

1.5.1 The Definition

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is given by:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \vec{u}_N \tag{1.3}$$

Where $\theta \in [0, \pi]$ and $\vec{u}_N$ is a vector given by the right hand rule where fingers should be curled from the direction of $\vec{A}$ to the direction of $\vec{B}$. Since the direction of the cross product is dictated by the right hand rule we can already see that it is not a commutative operation. More accurately:

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A}) \tag{1.4}$$

1.5.2 Geometric Significance

As you can see very clearly in figure 1.3 that any two vectors $\vec{A}$ and $\vec{B}$ can uniquely identify a parallelogram. The area of any parallelogram is given by the formula:

$$Area = (base) \times (height) \tag{1.5}$$

In figure 1.3 the base is the length of the vector $\vec{A}$ and the height is shown by the dotted line. But how do we figure out what h really is? By applying trigonometry here we can find out that $h = |\vec{A}| \sin\theta$. Plugging in these values in equation 1.5 we get:

$$Area = (|\vec{B}|)(|\vec{A}| \sin\theta)$$

Ahah, this is exactly the same formula that is given by $\vec{A} \cdot \vec{B}$. Thus we can conclude that the area of the parallelogram formed by two vectors $\vec{A}$ and $\vec{B}$ is given by $\vec{A} \cdot \vec{B}$.

$$Area = \vec{A} \cdot \vec{B}$$

1.5.3 Calculating Cross Product

Before we learn how to calculate the cross product of any two vectors in terms of the three basic unit vectors $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ we should think about what will be the cross production of every possible combination of two of these three unit vectors. We know from equation 1.3 that the cross product of two parallel vectors should be zero because $\sin 90° = 0$. We also know that the magnitude of a unit vector is by definition one. Using these two facts, along with the right hand rule, we can very quickly see that:

$$\vec{a}_x \times \vec{a}_y = \vec{a}_z$$

$$\vec{a}_y \times \vec{a}_z = \vec{a}_x$$

$$\vec{a}_z \times \vec{a}_x = \vec{a}_y$$

Using equation 1.4 we can easily figure out the reverses of these combinations:

$$\vec{a}_y \times \vec{a}_x = -\vec{a}_z$$

$$\vec{a}_z \times \vec{a}_y = -\vec{a}_x$$

$$\vec{a}_x \times \vec{a}_z = -\vec{a}_y$$

Now suppose we have two vectors $\vec{A}$ and $\vec{B}$, we can express them as:

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

$$\vec{B} = B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z$$

Their cross product can be written as:

$$\vec{A} \times \vec{B} = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z) \times (B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z)$$

Expanding this we get:

$$\begin{aligned}\vec{A} \times \vec{B} = {} & A_xB_x(\vec{a}_x \times \vec{a}_x) + A_xB_y(\vec{a}_x \times \vec{a}_y) + A_xB_z(\vec{a}_x \times \vec{a}_z) \\ & +A_yB_x(\vec{a}_y \times \vec{a}_x) + A_yB_y(\vec{a}_y \times \vec{a}_y) + A_yB_z(\vec{a}_y \times \vec{a}_z) \\ & +A_zB_x(\vec{a}_z \times \vec{a}_x) + A_zB_y(\vec{a}_z \times \vec{a}_y) + A_zB_z(\vec{a}_z \times \vec{a}_z)\end{aligned}$$

Firstly, we can cancel those cross products which are between collinear vectors, as we know that's going to be zero.

$$\begin{aligned}\vec{A} \times \vec{B} = {} & A_xB_y(\vec{a}_x \times \vec{a}_y) + A_xB_z(\vec{a}_x \times \vec{a}_z) \\ & +A_yB_x(\vec{a}_y \times \vec{a}_x) + A_yB_z(\vec{a}_y \times \vec{a}_z) \\ & +A_zB_x(\vec{a}_z \times \vec{a}_x) + A_zB_y(\vec{a}_z \times \vec{a}_y)\end{aligned}$$

Now, just utilizing the crossproducts of all possible combinations between unit vectors that we just derived a while ago, we can simplify this even further:

$$\begin{aligned}\vec{A} \times \vec{B} = {} & A_xB_y(\vec{a}_z) + A_xB_z(-\vec{a}_y) \\ & +A_yB_x(-\vec{a}_z) + A_yB_z(\vec{a}_x) \\ & +A_zB_x(\vec{a}_y) + A_zB_y(-\vec{a}_x)\end{aligned}$$

Grouping these terms by unit vectors we get:

$$\vec{A} \times \vec{B} = \vec{a}_x(A_y B_z - A_z B_y) + \vec{a}_y(A_z B_x - A_x B_z) + \vec{a}_z(A_x B_y - A_y B_x) \quad (1.6)$$

We can write this in an elegant way as a determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (1.7)$$

1.5.4 Properties

Cross Product does not follow the commutative law as we have already seen. However, it does follow distributive law:

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

1.5.5 Practical Applications

1. Torque:
$$\vec{\tau} = \vec{r} \times \vec{F}$$
2. Angular Momentum:
$$\vec{L} = \vec{r} \times \vec{\rho}$$
$$\vec{\rho} = m\vec{v}$$
3. Force on a moving conductor:
$$\vec{F} = I\vec{L} \times \vec{B}$$
4. Force on a moving charge:
$$\vec{F} = q\vec{v} \times \vec{B}$$
5. Lorrentz Force Formula:
$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

1.6 Scalar Triple Product

1.6.1 Definition

The Scalar Triple Product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined as:

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

We already know that:

$$\vec{B} \times \vec{C} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.8}$$

Vector $\vec{A}$ is defined as:

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

We also know from equation 1.2 that when the dot product of two vectors is taken, their respective components are mulitplied together, the unit vectors go away and the resultant values are added, right? Let $\vec{V}$ represent $\vec{B} \times \vec{C}$. So, $\vec{A} \cdot \vec{V}$ is:

$$\vec{A} \cdot \vec{V} = A_xV_x + A_yV_y + A_zV_z$$

From equation 1.8 we can see that:

$$V_x = \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix}$$

$$V_y = -\begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix}$$

$$V_z = \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

Putting these results back we get:

$$\vec{A} \cdot \vec{V} = A_x \begin{vmatrix} B_y & B_z \\ C_y & C_z \end{vmatrix} + -A_y \begin{vmatrix} B_x & B_z \\ C_x & C_z \end{vmatrix} + \begin{vmatrix} B_x & B_y \\ C_x & C_y \end{vmatrix}$$

Ahah, we have seen something similar while expanding determinants haven't we?

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1.9}$$

Exchanging two rows of a determinant two times doesn't really doesn't affect it, right? That fact, leads to the equality:

$$\vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

1.6.2 Applications

Volume of Parallelopiped

The area of a parallelopiped can be calculated by the following formula:

$$Area = (\text{area of base face}) \times (h)$$

Where h is the perpendicular distance between the base face and the top face. In figure 1.4 we can see that a parallelopiped is uniquely identified by three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. The area of the base face, as we already know, is given by

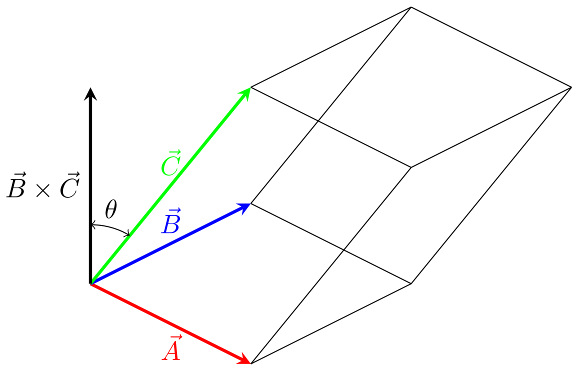


Figure 1.4: The parallelopiped formed by $\vec{A}$, $\vec{B}$ and $\vec{C}$

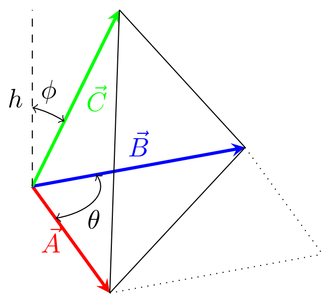


Figure 1.5: The Tetrahedron formed by $\vec{A}$, $\vec{B}$ and $\vec{C}$

the magnitude of the cross product of $\vec{A}$ and $\vec{B}$. The vector $\vec{A} \times \vec{B}$ is also shown in figure 1.4. The magnitude of this vector gives us the area and direction of this vector is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. Now we need to figure out the perpendicular distance between the base face and the top face. That distance is given by $|\vec{C}| \cos\theta$. If we take the dot product of $\vec{A} \times \vec{B}$ and $\vec{C}$ we get:

$$\begin{aligned}
\vec{C} \cdot \left(\vec{A} \times \vec{B}\right) &= \left(|\vec{C}| \cos\theta\right) \left(|\vec{A} \times \vec{B}|\right) \\
&= \left(|\vec{C}| \cos\theta\right) (\text{area of base face}) \\
&= (\text{h}) (\text{area of base face}) \\
&= \text{Volume of the parallelopiped}
\end{aligned}$$

Hence we have proven that the scalar product of three vectors gives us the volume of the parallelopiped formed by them.

Volume of a Tetrahedron

A tetrahedron is uniquely identified by three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Figure 1.5 represents one such tetrahedron. The area of a tetrahedron is given by the following formula:

$$(\text{Area of a tetrahedron}) = (\text{Area of base}) \times (h)$$

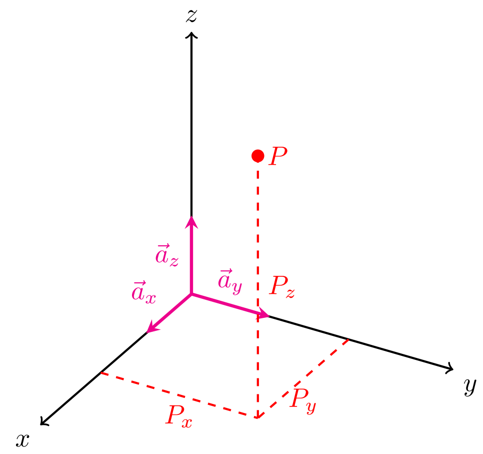


Figure 1.6: A point P in Rectangular Coordinate Systems

We can see in figure 1.5 that:

$$\text{Area of base} = \frac{1}{2} \times \text{Area of the parallelogram formed by } \vec{A}, \vec{B}$$

We also know from the previous sections that:

$$\text{Area of the parallelogram formed by } \vec{A}, \vec{B} = |\vec{A}||\vec{B}| \sin\theta$$

We can see from figure 1.5 that:

$$h = |\vec{C}| \cos\phi$$

Thus the total volume is given by:

$$\text{Volume} = \frac{1}{3}\left(\frac{1}{2}|\vec{A}||\vec{B}| \sin\theta\right)\left(|\vec{C}| \cos\phi\right)$$

$$\text{Volume} = \frac{1}{6}\left(\vec{A} \times \vec{B}\right) \cdot \vec{C}$$

1.7 Coordinate Systems

We will be dealing with three coordinate systems in this development:

1. Rectangular Coordinate System (RCS)
2. Cylindrical Coordinate System (CCS)
3. Spherical Coordinate System (SCS)

1.7.1 Rectangular Coordinate System

A point in RCS as shown in figure 1.6 is represented by three coordinates (x, y, z) each representing the distance along that particular axis as shown in the figure. RCS has three unit vectors $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$.

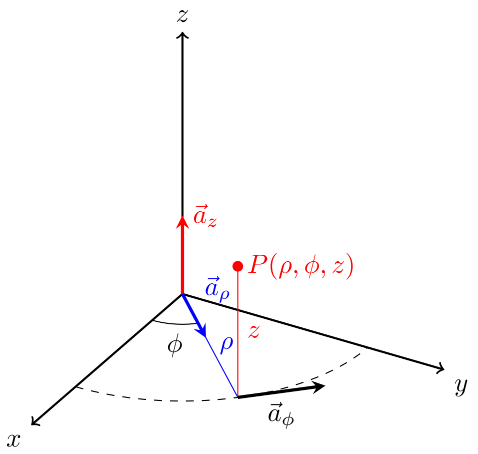


Figure 1.7: A point P represented in CCS

1.7.2 Cylindrical Coordinate System

A point in CCS, as depicted in figure 1.7, is represented by three coordinates (ρ, ϕ, z) where $\rho \in [0, \infty)$ and $\phi \in [0, 2\pi]$. ρ is the distance between the origin and the point's projection on the xy plane. ϕ is the angle between the x-axis and the ρ line. z is the distance between the point and it's projection on the xy plane. The three unit vectors in this coordinate system are $\vec{a}_\rho$, $\vec{a}_\phi$ and $\vec{a}_z$. $\vec{a}_z$ is the same as in RCS. $\vec{a}_\rho$ takes the direction along the ρ line, and $\vec{a}_\phi$ takes the direction of increasing ϕ.

Let's write out the equations to convert (x, y, z) to (ρ, ϕ, z) and back.

$$x = \rho \cos \phi \tag{1.10}$$
$$y = \rho \sin \phi \tag{1.11}$$
$$z = z \tag{1.12}$$

$$\rho = \sqrt{x^2 + y^2} \tag{1.13}$$
$$\phi = \tan^{-1} \phi \tag{1.14}$$

Let's now try and write the equations to represent the unit vectors $\vec{a}_\rho$, $\vec{a}_\phi$ in terms of $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ respectively. $\vec{a}_z$ doesn't need to be converted since its the same on both sides. We call the vector from the origin to the point's projection on xy axis $\vec{\rho}$. $\vec{\rho}$ can be written as:

$$\vec{\rho} = \rho \cos \phi \vec{a}_x + \rho \sin \phi \vec{a}_y$$

$$|\vec{\rho}| = \rho$$

Thus we can write:

$$\vec{a}_\rho = \cos \phi \vec{a}_x + \sin \phi \vec{a}_y$$

We can see that $\vec{a}_\phi$ is the cross product of $\vec{a}_z$ and $\vec{a}_\rho$:

$$\vec{a}_\phi = \vec{a}_z \times \vec{a}_\rho$$

$$\vec{a}_\rho = \cos\phi\vec{a}_x + \sin\phi\vec{a}_y + 0\vec{a}_z \tag{1.15}$$
$$\vec{a}_\phi = -\sin\phi\vec{a}_x + \cos\phi\vec{a}_y + 0\vec{a}_z \tag{1.16}$$
$$\vec{a}_z = 0\vec{a}_x + 0\vec{a}_y + \vec{a}_z \tag{1.17}$$

Writing this in an elegant form:

$$\begin{bmatrix}\vec{a}_\rho\\ \vec{a}_\phi\\ \vec{a}_z\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\vec{a}_x\\ \vec{a}_y\\ \vec{a}_z\end{bmatrix} \tag{1.18}$$

This can be summarized as:

$$\vec{u}_c = T(\phi)\,\vec{u}_R$$

Where $T(\phi)$ is the transformation written as a function of ϕ. To find the reverse transformation we can take the inverse of this transformation matrix. But we should note that this transformation is what is called an Orthogonal Transformation. In an Orthogonal Transformation the inverse transformation matrix is given by taking the transpose of the transformation matrix. Thus:

$$\begin{bmatrix}\vec{a}_x\\ \vec{a}_y\\ \vec{a}_z\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\vec{a}_\rho\\ \vec{a}_\phi\\ \vec{a}_z\end{bmatrix} \tag{1.19}$$

A vector $\vec{A}$ can be represented in RCS as:

$$\vec{A} = A_x\vec{a}_x = A_y\vec{a}_y + A_z\vec{a}_z \tag{1.20}$$

The same vector can be represented in CCS as:

$$\vec{A} = A_\rho\vec{a}_\rho + A_\phi\vec{a}_\phi + A_z\vec{a}_z$$

Let's try to write a relationship between A_ρ, A_ϕ and A_z in terms of A_x, A_y and A_z. We start with the equation:

$$\vec{A} = A_\rho\vec{a}_\rho + A_\phi\vec{a}_\phi + A_z\vec{a}_z$$

We try to replace $\vec{a}_\rho$, $\vec{a}_\phi$ with their RCS counterparts by using equation 1.18.

$$\begin{aligned}\vec{A} = A_\rho\left(\cos\phi\vec{a}_x + \sin\phi\vec{a}_y\right) +\\ A_\phi\left(-\sin\phi\vec{a}_x + \cos\phi\vec{a}_y\right) +\\ A_z\vec{a}_z\end{aligned}$$

Grouping the terms in the above equation by unit vectors we get:

$$\begin{aligned}\vec{A} = +\vec{a}_x\left(A_\rho\cos\phi - A_\phi\sin\phi\right)\\ +\vec{a}_y\left(A_\rho\sin\phi + A_\phi\cos\phi\right)\\ +A_z\vec{a}_z\end{aligned}$$

Thus comparing this with equation 1.20 we can conclude:

$$\begin{aligned}A_x &= A_\rho\cos\phi - A_\phi\sin\phi\\ A_y &= A_\rho\sin\phi + A_\phi\cos\phi\\ A_z &= A_z\end{aligned}$$

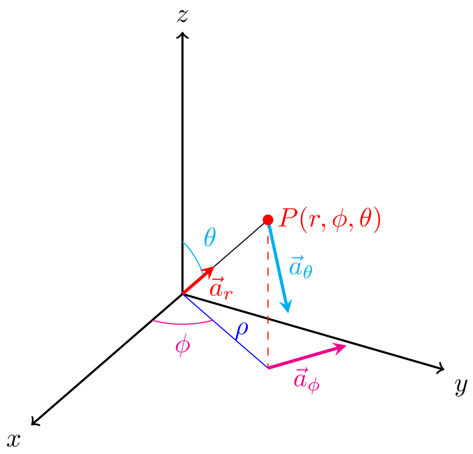


Figure 1.8: A point in Spherical Coordinate Systems

We can write this in an elegant way as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \tag{1.21}$$

Since this transformation is orthogonal too we can find the inverse relationship by simply taking the transpose of the transformation matrix:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{1.22}$$

1.7.3 Spherical Coordinate Systems

A point in spherical coordinate system can be represented as shown in figure 1.8. In spherical coordinate system, a point is represented by three coordinates (r, ϕ, θ). Here r is the distance from the origin to the point itself. ϕ is the angle the point's projection on the xy-plane makes with the positive x axis. And finally θ is the angle the line from the origin to the point itself makes with the positive z axis. Two more conditions apply here:

$$\theta \in [0, \pi]$$
$$\phi \in [0, 2\pi]$$

Let's write the equations which can let us convert between spherical coordinates and rectangular coordinates. It is very easy to see in figure 1.8 that:

$$z = r\cos\theta$$
$$\rho = r\sin\theta$$
$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$

Thus we can write:

$$z = r\cos\theta \tag{1.23}$$
$$x = r\sin\theta\cos\phi \tag{1.24}$$
$$y = r\sin\theta\sin\phi \tag{1.25}$$

We can do the reverse as:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{1.26}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \tag{1.27}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{1.28}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.29}$$

Now we need to express the unit vectors $(\vec{a}_r, \vec{a}_\phi, \vec{a}_\theta)$ in terms of $(\vec{a}_x, \vec{a}_y, \vec{a}_z)$. Suppose we have a vector $\vec{r}$, and r represents its magnitude. We can write it as:

$$\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$$

Dividing both sides by r we get:

$$\frac{\vec{r}}{r} = \frac{x}{r}\vec{a}_x + \frac{y}{r}\vec{a}_y + \frac{z}{r}\vec{a}_z$$

Doing some substitutions we get:

$$\vec{a}_r = \frac{r\sin\theta\cos\phi}{r}\vec{a}_x + \frac{r\sin\theta\sin\phi}{r}\vec{a}_y + \frac{r\cos\theta}{r}\vec{a}_z$$

Simplifying we get:

$$\vec{a}_r = \sin\theta\cos\phi\vec{a}_x + \sin\theta\sin\phi\vec{a}_y + \cos\theta\vec{a}_z$$

We already know that:

$$\vec{a}_\rho = \sin\phi\vec{a}_x + \cos\phi\vec{a}_y$$

We can see in figure 1.8 that the vector $\vec{a}_\phi$ is at an angle $\phi + 90$ from the positive x axis. Thus putting in this angle we get:

$$\vec{a}_\phi = \sin(\phi + 90)\,\vec{a}_x + \cos(\phi + 90)\,\vec{a}_y$$

Using trigonometric identities we can get:

$$\vec{a}_\phi = -\sin\phi\vec{a}_x + \cos\phi\vec{a}_y$$

We can get $\vec{a}_\theta$ by realizing that:

$$\vec{a}_\theta = \vec{a}_\phi \times \vec{a}_r$$

Evaluating this $\vec{a}_\theta$:

$$\vec{a}_\theta = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

Expanding this we get:

$$\vec{a}_\theta = \vec{a}_x \begin{vmatrix} \cos\phi & 0 \\ \sin\theta\sin\phi & \cos\theta \end{vmatrix} - \vec{a}_y \begin{vmatrix} -\sin\phi & 0 \\ \sin\theta\cos\phi & \cos\theta \end{vmatrix} + \vec{a}_z \begin{vmatrix} -\sin\phi & \cos\phi \\ \sin\theta\cos\phi & \sin\theta\sin\phi \end{vmatrix}$$

$$\vec{a}_\theta = \cos\theta\cos\phi\vec{a}_x + \sin\phi\cos\theta\vec{a}_y + \left(-sin^2\phi\sin\theta - \sin\theta\cos^2\phi\right)\vec{a}_z$$

$$\vec{a}_\theta = \cos\theta\cos\phi\vec{a}_x + \sin\phi\cos\theta\vec{a}_y - \sin\theta\vec{a}_z$$

Or more elegantly:

$$\vec{a}_\theta = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\phi\cos\theta \\ -\sin\theta \end{bmatrix}$$

Or even more elegantly we can write:

$$\begin{bmatrix} \vec{a}_r \\ \vec{a}_\theta \\ \vec{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \sin\phi\cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \vec{a}_x \\ \vec{a}_y \\ \vec{a}_z \end{bmatrix} \tag{1.30}$$

Some useful relationships between these unit vectors are:

$$\vec{a}_r \times \vec{a}_\theta = \vec{a}_\phi \tag{1.31}$$

$$\vec{a}_\theta \times \vec{a}_\phi = \vec{a}_r \tag{1.32}$$

$$\vec{a}_\phi \times \vec{a}_r = \vec{a}_\theta \tag{1.33}$$

Their reverses are easy to find. They will just have a negative sign. Suppose we have a vector $\vec{A}$. We can express it in terms of RCS as:

$$\vec{A} = \vec{A}_x\vec{a}_x + \vec{A}_y\vec{a}_y + \vec{A}_z\vec{a}_z$$

We can also express it in SCS as:

$$\vec{A} = \vec{A}_r\vec{a}_r + \vec{A}_\theta\vec{a}_\theta + \vec{A}_\phi\vec{a}_\phi$$

Let's replace $\vec{a}_r$, $\vec{a}_\theta$ and $\vec{a}_\phi$ in terms of $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ using the matrix we just derived:

$$\begin{aligned} \vec{A} =& A_r\left(\sin\theta\cos\phi\vec{a}_x + \sin\theta\sin\phi\vec{a}_y + \cos\theta\vec{a}_z\right) \\ & A_\theta\left(\cos\theta\cos\phi\vec{a}_x + \sin\phi\cos\theta\vec{a}_y - \sin\theta\vec{a}_z\right) \\ & A_\phi\left(-\sin\phi\vec{a}_x + \cos\phi\vec{a}_y\right) \end{aligned}$$

Let's group these by the unit vectors $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$

$$\begin{aligned} \vec{A} =& \vec{a}_x\left(A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi\right) \\ & \vec{a}_y\left(A_r\sin\theta\sin\phi + A_\theta\sin\phi\cos\theta + A_\phi\cos\phi\right) \\ & \vec{a}_z\left(A_r\cos\theta - A_\theta\sin\theta\right) \end{aligned}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \tag{1.34}$$

Transposing this we get the inverse transformation:

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{1.35}$$

1.7.4 A little note about Orthogonal Transformations

Orthogonal transformations have this special property that their inverse transformation matrix is just the transpose of the orignal transformation matrix. This is why we have been able to just transpose the matrix and get the reverse transformation matrix. We have done this with RCS to CCS and CCS to RCS transformations and we just did it with RCS to SCS and SCS to RCS. There are two types of orthogonal transformation matrix:

- Rotational Matrix (determinant 1)
- Reflection Matrix (determinant -1)

We can prove that the determinant of an Orthogonal Transformation matrix is always either 1 or -1 as:

$$A^{-1} = A^T$$

Let's do a right multiplication by A.

$$A^{-1}A = A^T A$$

$$I = A^T A$$

Taking determinant on both sides:

$$det(I) = det(A^T)det(A)$$

$$1 = det(A)^2$$

$$det(A) = \pm 1$$