2.1 (c)

$$B\vec{x} = \lambda\vec{x}$$

$$\Rightarrow B^{-1}(B\vec{x}) = B^{-1}(\lambda\vec{x})$$

$$\Rightarrow \vec{x} = B^{-1}(\lambda\vec{x})$$

$$\Rightarrow \vec{x} = \lambda(B^{-1}\vec{x})$$

$$\Rightarrow B^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

$\therefore$ eigenvalues are reciprocal in between B and B^{-1} and eigenvector $(\vec{x})$ remains the same.

And, for real symmetric matrix,

$$B = B^T$$

so, by all means, B and B^T will have same igenvalues and igenvector.

$$\cancel{B = Q \Lambda Q^T}$$

$$B = Q\Lambda Q^T$$

$B^T = (Q\Lambda Q^T)^T = (Q^T)^T \Lambda^T Q^T = Q\Lambda Q^T$, showing that B^T has same igenvalues and igenvectors

Matrix $A = \begin{bmatrix} 9 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 4 & 8 \end{bmatrix}$

From definition,

$$A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$$ [I = identity matrix]

Equation has a non-zero solution if and only if

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 9-\lambda & 0 & 0 \\ 5 & 2-\lambda & 0 \\ 7 & 4 & 8-\lambda \end{vmatrix} = (9-\lambda)\{(2-\lambda)(8-\lambda) - 4 \times 0\}$$

$$= (9-\lambda)(2-\lambda)(8-\lambda)$$

$\therefore \lambda = 9, 2, 8$

$\boxed{\therefore \text{igenvalues} = 9, 2, 8}$

For, $\lambda_1 = 2$,

$$A - \lambda I = \begin{bmatrix} 7 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 4 & 6 \end{bmatrix}$$

Gaussian Elimination:

6. For $\lambda_1 = 2$

$$A - \lambda I = \begin{bmatrix} 7 & 0 & 0 \\ 5 & 0 & 0 \\ 7 & 4 & 6 \end{bmatrix} \qquad \cancel{R_1/7 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 0 & 0 \\ \cancel{7} & \cancel{4} & \cancel{6} \end{bmatrix}$$

7.

$\therefore \; 7x_1 = 0$ [from R_1]

$\Rightarrow x_1 = 0$

$5x_1 = 0$ [from R_2]

$\Rightarrow x_1 = 0$

$7x_1 + 4x_2 + 6x_3 = 0$ [from R_3]

$\Rightarrow 4x_2 + 6x_3 = 0$

$\Rightarrow \; x_2 = -\frac{3}{2} x_3$

$$\vec{x} = \begin{bmatrix} 0 \\ -\frac{3}{2} x_3 \\ x_3 \end{bmatrix} = \left\{ x_3 \begin{bmatrix} 0 \\ -\frac{3}{2} \\ 1 \end{bmatrix} \right\}$$

let, $x_3 = 1$ then, $\vec{x_1} = \begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix}$

For, $\lambda_2 = 8$

$$A - \lambda_1 I = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -6 & 0 \\ 7 & 4 & 0 \end{bmatrix}$$

$\therefore x_1 = 0$ [from R_1]

$5x_1 - 6x_2 = 0$ [from R_2]

$\Rightarrow x_1 = 6/5\, x_2$

$\Rightarrow x_2 = 0$

~~$7x_1 - 4x_2 = 0$~~ [from R_3]

General Ans. $\vec{X} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$

$= \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Let $x_3 = 1$ then $\vec{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For, $\lambda_3 = 9$

$$A - \lambda_3 I = \begin{bmatrix} 0 & 0 & 0 \\ 5 & -7 & 0 \\ 7 & 4 & -1 \end{bmatrix}$$

$5x_1 - 7x_2 = 0 \Rightarrow x_1 = \frac{7}{5} x_2$

$7x_1 + 4x_2 - x_3 = 0 \Rightarrow 7 \times \frac{7}{5} x_2 + 4x_2 - x_3 = 0$

$\Rightarrow \frac{49x_2 + 20x_2 - 5x_3}{5} = 0$

$\Rightarrow 69x_2 = 5x_3$

~~$\Rightarrow x_3 = \frac{69}{5} x_2$~~ $\Rightarrow x_2 = \frac{5}{69} x_3$

$\therefore x_1 = \frac{7}{5} \times \frac{5}{69} x_3 = \frac{7}{69} x_3$

$$\therefore \vec{x} = \left\{ \begin{bmatrix} 7/69\, x_3 \\ 5/69\, x_3 \\ x_3 \end{bmatrix} \right.$$

$$= \left\{ x_3 \begin{bmatrix} 7/69 \\ 5/69 \\ 1 \end{bmatrix} \right\}$$

Let, $x_3 = 1$, then $\vec{x} = \begin{bmatrix} 7/69 \\ 5/69 \\ 1 \end{bmatrix}$

$\boxed{\text{Ans.}}$ $\lambda_1 = 2$, $\vec{x_1} = \begin{bmatrix} 0 \\ -3/2 \\ 1 \end{bmatrix}$

$\lambda_2 = 8$ $\vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = 9$ $\vec{x_3} = \begin{bmatrix} 7/69 \\ 5/69 \\ 1 \end{bmatrix}$

Exercise 2.1

a. For a matrix $A \in \mathbb{R}^{m * n}$, it can be decomposed into $A = UDV^T$

   Now, for $A^TA = (UDV^T)^T * (UDV^T)$
   $= (U^TD^TV) * (UDV^T)$
   $= VD^2V^T$ $[D^T = D \ \& \ U^TU = I]$

   Given, D contains the eigenvalues of A^TA, so from $A^TA = VD^2V^T$, we can come to the conclusion that the singular values of A are the square root of the eigenvalues of A^TA.