MC Project - Phase I

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1-

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g + \frac{c}{M}\frac{{x_3}^2}{0.1-x_1} - \frac{f_v x_2}{M} \\ \dot{x}_3 = \frac{1}{L}\left(-Rx_3 + u\right) \end{cases} \qquad \{y = x_1$$

Equilibrium Points :

$$\begin{cases} \dot{x}_1 = 0 \to x_2 = 0 \\ \\ \dot{x}_2 = 0 \to -g + \frac{c}{M}\frac{{x_3}^2}{0.1-x_1} - \frac{f_v x_2}{M} = 0 \to {x_3}^2 = \frac{Mg(0.1-x_1)}{c} \\ \\ \dot{x}_3 = 0 \to \frac{1}{L}\left(-Rx_3 + u\right) = 0 \to x_3 = \frac{u}{R} \end{cases}$$

$$y = x_1 \ \&\ y^* = 0.06 \to x_1 = 0.06m \to {x_3}^2 = \frac{(0.425)(9.8)(0.04)}{0.3} \to x_3 = \pm 0.745$$

$$u = x_3 R \to u = 50 \times 0.745 = 37.259$$

$$\Rightarrow x_{eq} = \begin{bmatrix} 0.06 \\ 0 \\ \pm 0.745 \end{bmatrix}$$

2-

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{c}{M}\frac{{x_3}^2}{(0.1-x_1)^2} & -\frac{f_v}{M} & \frac{2c}{M}\frac{x_3}{(0.1-x_1)} \\ 0 & 0 & \frac{-R}{L} \end{bmatrix}$$

$$A_1|_{(0.06,0,0.745)} = \begin{bmatrix} 0 & 1 & 0 \\ 244.86 & -0.094 & 26.29 \\ 0 & 0 & -250 \end{bmatrix} \quad , \quad A_2|_{(0.06,0,-0.745)} = \begin{bmatrix} 0 & 1 & 0 \\ 244.86 & -0.094 & -26.29 \\ 0 & 0 & -250 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} \to B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\dot{x} = A\left(x - x_0\right) + B\left(u - u_0\right)$$

$$\to \dot{x} = A_1 \begin{bmatrix} x_1 - 0.06 \\ x_2 \\ x_3 - 0.745 \end{bmatrix} + B\left(u - 37.259\right) \quad , \quad \dot{x} = A_1 \begin{bmatrix} x_1 - 0.06 \\ x_2 \\ x_3 + 0.745 \end{bmatrix} + B\left(u - 37.259\right)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

3-

$$\det\left(A_1 - \lambda I\right) = 0 \rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 244.86 & -0.094-\lambda & 26.29 \\ 0 & 0 & -250-\lambda \end{vmatrix} = 0 \rightarrow \lambda_1 = -250, \lambda_2 = -15.7, \lambda_3 = 15.6$$

$$\det\left(A_2 - \lambda I\right) = 0 \rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 244.86 & -0.094-\lambda & -26.29 \\ 0 & 0 & -250-\lambda \end{vmatrix} = 0 \rightarrow \lambda_1 = -250, \lambda_2 = -15.7, \lambda_3 = 15.6$$

A positive eigenvalue is found in the linearized system, showing that it is unstable. As a result, the original non-linear system is unstable at equilibrium points.

4-

Controllability and observability of the formulated state-space representation are examined at one of the equilibrium points. Considering the similar system characteristics around both equilibrium points, it can be deduced that the same conclusions regarding controllability and observability will be observed at the other equilibrium point.

$$C = \begin{bmatrix} B & BA & A^2B \end{bmatrix} \rightarrow C = \begin{bmatrix} 0 & 0 & 131.45 \\ 0 & 131.45 & -32874.85 \\ 5 & -1250 & 312500 \end{bmatrix} \rightarrow \text{Rank}\left(C\right) = 3$$

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 244.86 & -0.094 & 26.29 \end{bmatrix} \rightarrow \text{Rank}\left(O\right) = 3$$

The system is both controllable and observable. It is established that an observable and controllable system is a minimal form.

Here's a step-by-step approach to examine controllability and observability of the system using MATLAB:

```
%state-space representation
A = [0 1 0; 244.86 -0.094 +26.29; 0 0 -250];
B = [0;0;5];
C = [1 0 0];
sys = ss(A, B, C, D);
%controllability
Co = ctrb(sys)
rank(Co)
%observability
Oo = obsv(sys)
rank(Oo)
>> p4

Co =

   1.0e+05 *

```

```
         0         0    0.0013
         0    0.0013   -0.3287
    0.0001   -0.0125    3.1250


ans =

     3


Oo =

    1.0000         0         0
         0    1.0000         0
  244.8600   -0.0940   26.2900


ans =

     3
```

MATLAB Code 1: Controllability and Observability of The System

5-

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 244.86 & -0.094 & 26.29 \\ 0 & 0 & -250 \end{bmatrix}$$

$$\lambda_1 = -250 \rightarrow v_1 = \begin{bmatrix} \frac{2629}{6223164} \\ \\ -\frac{328625}{3111582} \\ \\ 1 \end{bmatrix} = \begin{bmatrix} 0.00042 \\ -0.11 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -15.7 \rightarrow v_2 = \begin{bmatrix} \frac{-\sqrt{244862209}+47}{244860} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.0637 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 15.6 \rightarrow v_3 = \begin{bmatrix} \frac{\sqrt{244862209}+47}{244860} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0641 \\ 1 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \rightarrow J = Q^{-1} A_1 Q \rightarrow J = \begin{bmatrix} -250 & 0 & 0 \\ 0 & -15.7 & 0 \\ 0 & 0 & 15.6 \end{bmatrix} \rightarrow e^{Jt} = \begin{bmatrix} e^{-250t} & 0 & 0 \\ 0 & e^{-15.7t} & 0 \\ 0 & 0 & e^{15.6t} \end{bmatrix}$$

$$\rightarrow \varphi(t) = e^{A_1 t} = Q e^{Jt} Q^{-1}$$

$$\rightarrow \varphi(t) = \begin{bmatrix} 0.498e^{-15.7t} + 0.502e^{15.6t} & 0.032\left(e^{15.6t} - e^{-15.7t}\right) & 0.0004e^{-250t} - 0.004e^{-15.7t} + 0.003e^{15.6t} \\ 7.825\left(e^{15.6t} - e^{-15.7t}\right) & 0.498e^{15.6t} + 0.502e^{-15.7t} & -0.11e^{-250.t} + 0.058e^{-15.7t} + 0.052e^{15.6t} \\ 0 & 0 & e^{-250t} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 244.86 & -0.094 & -26.29 \\ 0 & 0 & -250 \end{bmatrix}$$

$$\tilde{\lambda}_1 = -250 \rightarrow \tilde{v}_1 = \begin{bmatrix} \frac{-2629}{6223164} \\ \frac{328625}{3111582} \\ 1 \end{bmatrix} = \begin{bmatrix} -0.00042 \\ 0.11 \\ 1 \end{bmatrix}$$

$$\tilde{\lambda}_2 = -15.7 \rightarrow \tilde{v}_2 = \begin{bmatrix} \frac{-\sqrt{244862209}+47}{244860} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.0637 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{\lambda}_3 = 15.6 \rightarrow \tilde{v}_3 = \begin{bmatrix} \frac{\sqrt{244862209}+47}{244860} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0641 \\ 1 \\ 0 \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 \end{bmatrix} \rightarrow J = \tilde{Q}^{-1} A_1 \tilde{Q} \rightarrow J = \begin{bmatrix} -250 & 0 & 0 \\ 0 & -15.7 & 0 \\ 0 & 0 & 15.6 \end{bmatrix} \rightarrow e^{Jt} = \begin{bmatrix} e^{-250t} & 0 & 0 \\ 0 & e^{-15.7t} & 0 \\ 0 & 0 & e^{15.6t} \end{bmatrix}$$

$$\rightarrow \tilde{\varphi}(t) = e^{A_2 t} = \tilde{Q} e^{Jt} \tilde{Q}^{-1}$$

$$\rightarrow \tilde{\varphi}(t) = \begin{bmatrix} 0.498e^{-15.7t} + 0.502e^{15.6t} & 0.032\left(e^{15.6t} - e^{-15.7t}\right) & -0.0004e^{-250t} + 0.004e^{-15.7t} - 0.003e^{15.6t} \\ 7.825\left(e^{15.6t} - e^{-15.7t}\right) & 0.498e^{15.6t} + 0.502e^{-15.7t} & 0.11e^{-250.t} - 0.058e^{-15.7t} - 0.052e^{15.6t} \\ 0 & 0 & e^{-250t} \end{bmatrix}$$

To verify the accuracy of the obtained state transition matrices, a step-by-step approach is implemented in MATLAB as follows:

```
% Define the matrix A1
A1 = [0 1 0;
      244.86 -0.094 26.29;
      0 0 -250];

% Symbolic time variable
syms t;

% Compute the Jordan decomposition
[P, J] = jordan(A1);
```

```

% Compute the symbolic matrix exponential of J*t
e_Jt = expm(J * t);

% Compute the state transition matrix symbolically
Phi_t_symbolic = P * e_Jt * inv(P);

% Simplify the symbolic expression
Phi_t_simplified = simplify(Phi_t_symbolic);

% Convert the result to a decimal form for a specific value of t (e.g., t = 1 second)
t_value = 1; % Set the desired time value here
Phi_t_decimal = vpa(subs(Phi_t_simplified, t, t_value), 6); % Decimal form with 6 significant digits

% Display the results
disp('The symbolic state transition matrix phi(t) is:');
disp(Phi_t_simplified);

disp('The state transition matrix phi(t) in decimal form at t = 1 is:');
disp(Phi_t_decimal);
>> p5
The state transition matrix phi(t) in decimal form at t = 1 is:
[ 2990420.0,  190533.0,      18859.5]
[4.66538e+7, 2972510.0,     294228.0]
[         0,         0, 2.66919e-109]
```

MATLAB Code 2: State Transition Matrix Verification

$\varphi\,(t=1)$ is consistent with the manually obtained results.

6-

$$G_1\,(s) = C(sI - A_1)^{-1}B + D \rightarrow G_1\,(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ -244.86 & s+0.094 & 26.29 \\ 0 & 0 & s+250 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$\rightarrow G_1\,(s) = \frac{131.45}{s^3+250.094s^2-221.36s-61215}$

Poles :

$$s^3 + 250.094s^2 - 221.36s - 61215 = 0 \rightarrow p_1 = -250, p_2 = -15.7, p_3 = 15.6$$

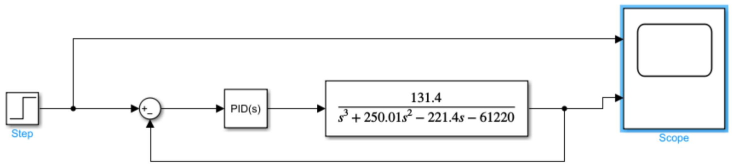
Zeros: No zeros exist in the transfer function.

$$G_2\,(s) = \frac{-131.45}{s^3 + 250.094s^2 - 221.36s - 61215}$$

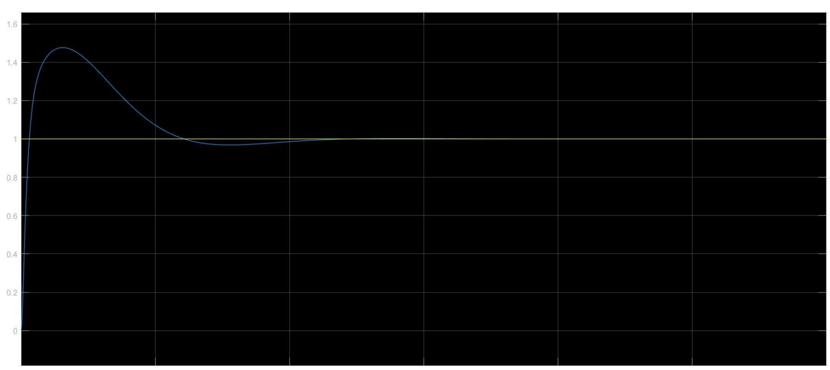
The poles and zeros are identical to those of the previous transfer function, as expected.

7.

A PID controller is designed to stabilize the linear system in the Simulink environment. The designed system is illustrated below.

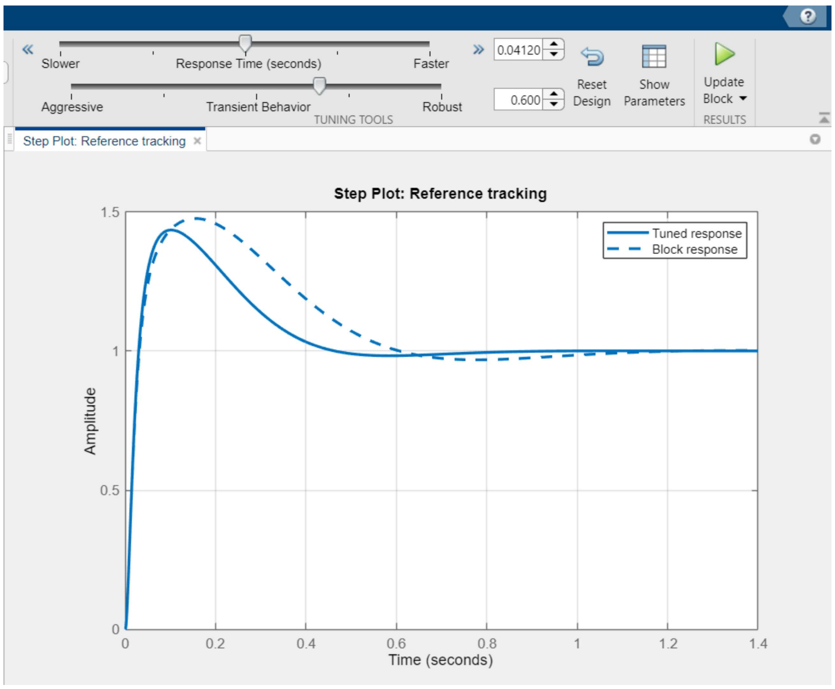


The provided Simulink diagram represents a feedback control system.
The system starts with a step input signal. The difference between the reference and the feedback is the error signal, which is the input to the controller. The error signal passes through a PID controller. The output of the PID controller drives the plant, which is represented by the transfer function. The system's output is connected to a scope (the block on the far right), which visualizes the system's response over time.



The step response is shown above. It demonstrates that the error signal converges to zero as time approaches infinity, indicating that the designed system is stable.

In this stage, the designed system is tuned, and a step plot is displayed. The response time and transient behavior can be adjusted to meet the desired performance criteria, ensuring logical and consistent outcomes, as shown in the figure below.



The PID parameters for the adjusted system are determined as follows:

$$k_P = 1278.860486 \quad , \quad \mathrm{k}_I = 3736.5702745 \quad , \quad k_D = 101.7786163$$

Let's analyze potential problems associated with the designed PID controller:

1. A high integral gain can cause the system to respond aggressively to eliminate steady-state error, which may lead to instability.

2. The PID controller might work well for specific operating conditions but fail to maintain performance under changes in system parameters, disturbances, or nonlinearities in the plant.

3 The derivative term amplifies high-frequency noise in the system. If the system includes significant measurement noise, this could lead to anomolous or unstable behavior.

8.

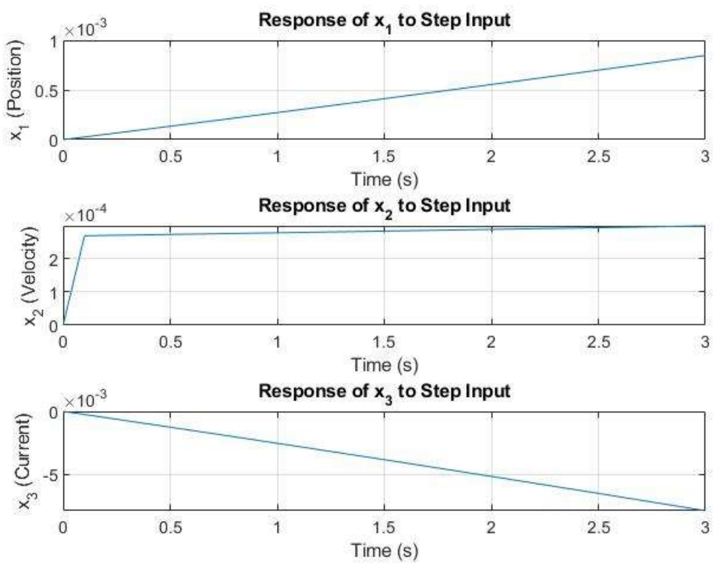
The MATLAB code below is presented to plot the response of the state variables to a step input:

```
% System matrices (from linearization or given parameters)
A = [0 1 0; 244.86 -0.094 26.29; 0 0 -250];
B = [0;0;5];
C = [1 0 0];
D = 0;

% PID controller parameters (from previous step)
Kp = 1278.8604861075;  % Proportional gain
Ki = 3736.57027457271;  % Integral gain
Kd = 101.778616391159; % Derivative gain
C_pid = pid(Kp, Ki, Kd);

% Convert the system to transfer function
[numerator, denominator] = ss2tf(A, B, C, D);
G = tf(numerator, denominator);

% Closed-loop transfer function
T = feedback(C_pid * G, 1);

% Simulate the state-space system response with the PID controller
sys_cl = ss(A - B * [Kp Ki Kd], B, eye(3), 0); % Closed-loop
    state-space system

% Time vector
t = 0:0.1:3;

% Step input
u = ones(size(t));

% Response of the state variables
[y, t, x] = lsim(sys_cl, u, t);

% Plot state variable responses
figure;
subplot(3, 1, 1);
plot(t, x(:, 1));
xlabel('Time (s)');
ylabel('x_1 (Position)');
title('Response of x_1 to Step Input');
grid on;

subplot(3, 1, 2);
plot(t, x(:, 2));
xlabel('Time (s)');
ylabel('x_2 (Velocity)');
title('Response of x_2 to Step Input');
grid on;

subplot(3, 1, 3);
plot(t, x(:, 3));
xlabel('Time (s)');
```

```
ylabel('x_3 (Current)');
title('Response of x_3 to Step Input');
grid on;
```

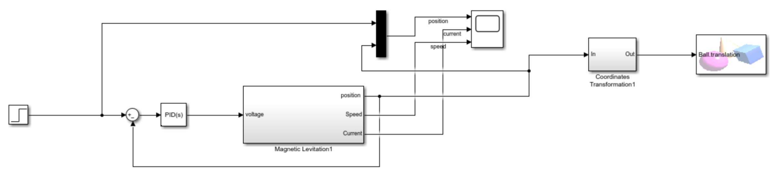
MATLAB Code 3: Response of The State Variables To Step Input

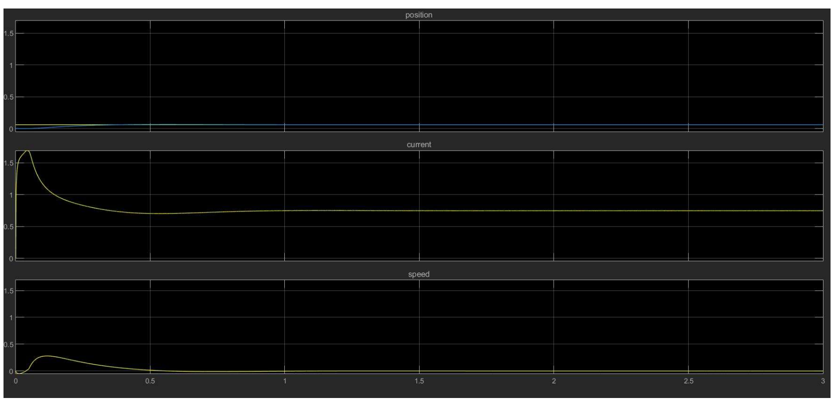
The graphs are illustrated as follows :



9-

In this section, the designed controller is connected to the introduced nonlinear system, and the response of the state variables to a step input is illustrated below.





In the designed PID controller, the position variable is used as feedback, as it is the variable expected to be controlled. As shown in the lower figure, the time response of the position variable converges to a constant value, as expected. The same behavior is observed in the remaining time response graphs, indicating that the nonlinear system is successfully controlled.
Additionally, the time response of the speed variable converges to zero, demonstrating that the ball remains stationary and the system is effectively stabilized.

Now, let us delve into how the ball remains stationary after connecting the designed PID controller to the nonlinear system by presenting figures that illustrate the stabilization process as follows: