COMS 576 - Motion Planning for Robotics and Autonomous Systems,
Homework 2

Mobina Amrollahi

Deadline: 09/27/2024, 11:59 PM

Contents

Figure 1: Gingerbread face

1 Problem 3.1 (5 points)

Define a semi-algebraic model that removes a triangular "nose" from the region shown in Figure 3.4.

1.1 Answer

Consider constructing a model of the gingerbread face (See figure 1) based on [3]. The center of the outer circle has a radius of r1 and is centered at the origin. The "eyes" have radius r_2 and r_3 and are centered at (x_2, y_2) and (x_3, y_3), respectively. Let the "mouth" be an ellipse with major axis a and minor axis b and be centered at $(0, y_4)$. Then the functions are defined as:

$$\begin{aligned}
f_1 &= x^2 + y^2 - r_1^2, \\
f_2 &= -((x - x_2)^2 + (y - y_2)^2 - r_2^2), \\
f_3 &= -((x - x_3)^2 + (y - y_3)^2 - r_3^2), \\
f_4 &= -(\frac{x^2}{a^2} + \frac{(y - y_4)^2}{b^2} - 1).
\end{aligned} \tag{1}$$

First, suppose the nose of the gingerbread is an equilateral triangle at the center, and the edge of the triangle is l 1). This triangle is a point set that can be obtained by the union of three algebraic sets, therefore, by itself is a semi-algebraic set. The number "three" comes from having three vertices. By knowing each side is length l and having the above properties, we will be finding the vertices. Since it's an equilateral triangle, three kinds of cevians coincide and are equal [1]:

- The three altitudes have equal lengths.
- The three medians have equal lengths.
- The three angle bisectors have equal lengths.

1. Finding (x_5, y_5): since the triangle is at origin, then $x_5 = 0$. For y_5, we know, based on the picture, it will be x. Thus:

$$x. \sin 60^\circ = \frac{l}{2} \Rightarrow x = \frac{l}{2} \times (\sin 60^\circ)^{-1} \Rightarrow x = \frac{l}{2} \times \frac{2}{\sqrt{3}} = \frac{l}{\sqrt{3}}$$

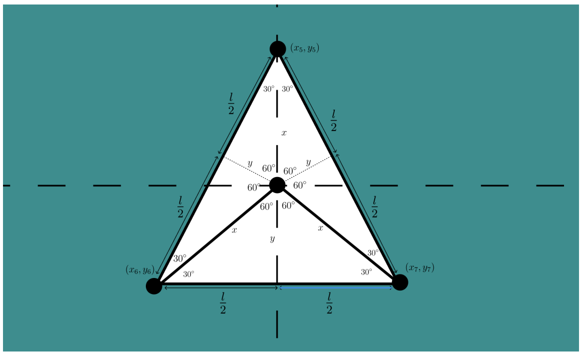

Figure 2: Gingerbread Nose

$$\Rightarrow (x_5, y_5) = (0, \frac{l\sqrt{3}}{3})$$

2. Finding (x_6, y_6): based on the picture, $x_6 = \frac{l}{2}$. For y_6, we know, based on the picture, it will be $-y$. Thus:

$$y = x.\sin 30^\circ \Rightarrow \underbrace{\frac{l}{\sqrt{3}}}_{\text{based on (1)}} .\frac{1}{2} = \frac{l\sqrt{3}}{3}.\frac{1}{2} = \frac{l\sqrt{3}}{6}$$

$$\Rightarrow (x_6, y_6) = (\frac{l}{2}, -\frac{l\sqrt{3}}{6})$$

3. Finding (x_7, y_7): based on the picture, $x_7 = -\frac{l}{2}$. In addition, $y_7 = y_6$. Thus:

$$\Rightarrow (x_7, y_7) = (-\frac{l}{2}, -\frac{l\sqrt{3}}{6})$$

Now, Suppose f_5, f_6, and f_7 are the functions of the lines passing through (x_5, y_5) and (x_6, y_6), (x_6, y_6) and (x_7, y_7) and (x_7, y_7) and (x_5, y_5). The general form of a line passing through two points (x_i, y_i) and (x_j, y_j) is:

$$\underbrace{\frac{(y_j - y_i)}{(x_j - x_i)}}_{\text{A}} = \underbrace{\frac{(y - y_i)}{(x - x_i)}}_{\text{B}}$$

We need to determine the direction of the inequality based on which side of the line contains the outlier of the triangle, since we would like to exclude the white triangle. For each line, we will pick the origin $\mathcal{P} = (0, 0)$ as the test point, plug it into the equality, and since the result should be negative we will decide whether the direction is $A - B \leq 0$ or $B - A \leq 0$.

1. Calculating f_5, function of the line passing through $(x_5, y_5) = (0, \frac{l\sqrt{3}}{3})$ and $(x_6, y_6) = (\frac{l}{2}, -\frac{l\sqrt{3}}{6})$:

$$\frac{(\frac{l\sqrt{3}}{3} - (-\frac{l\sqrt{3}}{6}))}{(0 - \frac{l}{2})} = \frac{(y - (-\frac{l\sqrt{3}}{6}))}{(x - \frac{l}{2})} \Rightarrow \frac{\frac{l\sqrt{3}}{2}}{-\frac{l}{2}} = \frac{(y + \frac{l\sqrt{3}}{6})}{(x - \frac{l}{2})} \Rightarrow -\sqrt{3} = \frac{(y + \frac{l\sqrt{3}}{6})}{(x - \frac{l}{2})} \Rightarrow -\sqrt{3}(x - \frac{l}{2}) = (y + \frac{l\sqrt{3}}{6})$$

$$\Rightarrow -\sqrt{3}x + \frac{l\sqrt{3}}{2} - y - \frac{l\sqrt{3}}{6} = 0 \Rightarrow -\sqrt{3}x + \frac{l\sqrt{3}}{3} - y = 0$$

Now, if we plug $\mathcal{P} = (0,0)$, we will be having:

$$0 + \frac{l\sqrt{3}}{3} - 0 \geq 0 \quad \text{since } l \geq 0$$

2. Calculating f_6, function of the line passing through $(x_6, y_6) = (\frac{l}{2}, -\frac{l\sqrt{3}}{6})$ and $(x_7, y_7) = (-\frac{l}{2}, -\frac{l\sqrt{3}}{6})$:

$$\frac{(-\frac{l\sqrt{3}}{6} - (-\frac{l\sqrt{3}}{6}))}{(\frac{l}{2} - (-\frac{l}{2}))} = \frac{(y - (-\frac{l\sqrt{3}}{6}))}{(x - (-\frac{l}{2}))} \Rightarrow \frac{0}{1} = \frac{(y + \frac{l\sqrt{3}}{6})}{(x + \frac{l}{2})} \Rightarrow y + \frac{l\sqrt{3}}{6} = 0$$

Now, if we plug $\mathcal{P} = (0,0)$, we will be having:

$$0 + \frac{l\sqrt{3}}{6} \geq 0 \quad \text{since } l \geq 0$$

3. Calculating f_7, function of the line passing through $(x_7, y_7) = (-\frac{l}{2}, -\frac{l\sqrt{3}}{6})$ and $(x_5, y_5) = (0, \frac{l\sqrt{3}}{3})$:

$$\frac{(-\frac{l\sqrt{3}}{6} - \frac{l\sqrt{3}}{3})}{(-\frac{l}{2} - 0)} = \frac{(y - \frac{l\sqrt{3}}{3})}{(x - 0)} \Rightarrow \frac{(-\frac{l\sqrt{3}}{2})}{(-\frac{l}{2})} = \frac{(y - \frac{l\sqrt{3}}{3})}{(x - 0)} \Rightarrow \sqrt{3} = \frac{(y - \frac{l\sqrt{3}}{3})}{(x - 0)} \Rightarrow \sqrt{3}x = (y - \frac{l\sqrt{3}}{3})$$

$$\Rightarrow \sqrt{3}x - y + \frac{l\sqrt{3}}{3} = 0$$

Now, if we plug $\mathcal{P} = (0,0)$, we will be having:

$$\Rightarrow 0 - 0 + \frac{l\sqrt{3}}{3} \geq 0 \quad \text{since } l \geq 0$$

So the region inside the triangle is described by the system of inequalities:

$$\begin{aligned} f_5 &= -(-\sqrt{3}x + \frac{l\sqrt{3}}{3} - y), \\ f_6 &= -(y + \frac{l\sqrt{3}}{6}), \\ f_7 &= -(\sqrt{3}x - y + \frac{l\sqrt{3}}{3}) \end{aligned} \tag{2}$$

Therefore, if assuming H_5, H_6, H_7 are the algebraic primitives of 2, then the shaded region (nose included) is:

$$\mathcal{O} = H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \cap H_6 \cap H_7. \blacksquare \tag{3}$$

Generalization of Triangle as a Semi-Algebraic Set Now, suppose a more general form of noses for gingerbread face, where we don't know the type of triangle 3. However, we can assume the vertices are (x_5, y_5), (x_6, y_6), and (x_7, y_7), and $\mathcal{P} = (0,0)$ is inside the triangle.

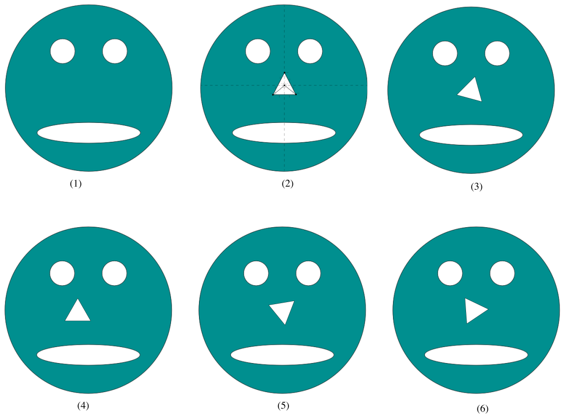

Figure 3: General form

Line Equations We'll use the general form of a line equation: $ax + by + c = 0$

For each side of the triangle, we can write this equation using each two vertices:

1. Calculating f_5, function of the line passing through (x_5, y_5) and (x_6, y_6): Line through (x_5, y_5) and (x_6, y_6):
$$a_5x + b_5y + c_5 = 0, \text{ where:}$$
$$a_5 = y_6 - y_5$$
$$b_5 = x_5 - x_6$$
$$c_5 = x_5y_5 - x_5y_6$$

2. Calculating f_6, function of the line passing through (x_6, y_6) and (x_7, y_7): Line through (x_6, y_6) and (x_7, y_7):
$$a_6x + b_6y + c_6 = 0, \text{ where:}$$
$$a_6 = y_7 - y_6$$
$$b_6 = x_6 - x_7$$
$$c_6 = x_7y_6 - x_6y_7$$

3. Calculating f_7, function of the line passing through (x_7, y_7) and (x_5, y_5): Line through (x_7, y_7) and (x_5, y_5):
$$a_7x + b_7y + c_7 = 0, \text{ where:}$$
$$a_7 = y_5 - y_7$$
$$b_7 = x_7 - x_5$$
$$c_7 = x_5y_7 - x_7y_5$$

Determining Inequalities Now, to determine whether each inequality should be $\geq$ or $\leq$, we will use the origin $(0,0)$ as a test point, since we know it's inside the triangle.

For each line i ($i = 5, 6, 7$):

- If $a_i \cdot 0 + b_i \cdot 0 + c_i > 0$, then the inequality is $a_i x + b_i y + c_i \geq 0$
- If $a_i \cdot 0 + b_i \cdot 0 + c_i < 0$, then the inequality is $a_i x + b_i y + c_i \leq 0$

The triangle is then defined as the set of all points (x, y) that satisfy:

$$(a_1 x + b_1 y + c_1 \geq 0) \wedge (a_2 x + b_2 y + c_2 \geq 0) \wedge (a_3 x + b_3 y + c_3 \geq 0)$$

or

$$(a_1 x + b_1 y + c_1 \leq 0) \wedge (a_2 x + b_2 y + c_2 \leq 0) \wedge (a_3 x + b_3 y + c_3 \leq 0)$$

depending on the results from the origin. If we consider $\mathcal{F}$ to be the set of point inside the triangle, then:

$$\mathcal{O} = H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_{\mathcal{F}}. \blacksquare \tag{4}$$

2 Problem 3.4 (5 points)

(5 points) An alternative to the yaw-pitch-roll formulation from Section 3.2.3 is considered here. Consider the following Euler angle representation of rotation (there are many other variants). The first rotation is $R_z(\gamma)$, which is just (3.39) with α replaced by γ. The next two rotations are identical to the yaw-pitch-roll formulation: $R_y(\beta)$ is applied, followed by $R_z(\alpha)$. This yields:

$$R_{\text{euler}}(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma).$$

(a) Determine the matrix R_{euler}.

(b) Show that $R_{\text{euler}}(\alpha, \beta, \gamma) = R_{\text{euler}}(\alpha - \pi, -\beta, \gamma - \pi)$.

2.1 Answer

(a)

- $R_z(\alpha)$:

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $R_y(\beta)$:

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

- $R_z(\gamma)$: since $R_z(\gamma)$, is just $R_z(\alpha)$ with α replaced by γ, we have:

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore:

$$
\begin{aligned}
R_{\text{euler}}(\alpha,\beta,\gamma) &= R_z(\alpha)R_y(\beta)R_z(\gamma) \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}
\end{aligned}
$$

(b) For this part, it's important to know that:

- $\sin(x-\pi) = -\sin(x)$ and $\cos(x-\pi) = -cos(x)$
- $\sin -x = -\sin(x)$ and $\cos(-x) = \cos(x)$

Now, in R_{euler} let's replace α with $\alpha-\pi$, β with $-\beta$, and $\gamma-\pi$ with γ. Then we will be having:

$$
\begin{pmatrix} \cos(\alpha-\pi)\cos(-\beta)\cos(\gamma-\pi) - \sin(\alpha-\pi)\sin(\gamma-\pi) & -\cos(\alpha-\pi)\cos(-\beta)\sin(\gamma-\pi) - \sin(\alpha-\pi)\cos(\gamma-\pi) & \cos(\alpha-\pi)\sin(-\beta) \\ \sin(\alpha-\pi)\cos(-\beta)\cos(\gamma-\pi) + \cos(\alpha-\pi)\sin(\gamma-\pi) & -\sin(\alpha-\pi)\cos(-\beta)\sin(\gamma-\pi) + \cos(\alpha-\pi)\cos(\gamma-\pi) & \sin(\alpha-\pi)\sin(-\beta) \\ -\sin(-\beta)\cos(\gamma-\pi) & \sin(-\beta)\sin(\gamma-\pi) & \cos(-\beta) \end{pmatrix}
$$

$$
= \begin{pmatrix} (-\cos(\alpha))\cos(\beta)(-\cos(\gamma)) - (-\sin(\alpha))(-\sin(\gamma)) & -(-\cos(\alpha))\cos(\beta)(-\sin(\gamma)) - (-\sin(\alpha))\cos(\gamma) & -(-\cos(\alpha))(-\sin(\beta)) \\ (-\sin(\alpha))\cos(\beta)(-\cos(\gamma)) + (-\cos(\alpha))(-\sin(\gamma)) & -(-\sin(\alpha))\cos(\beta)(-\sin(\gamma)) + (-\cos(\alpha))(-\cos(\gamma)) & (-\sin(\alpha))(-\sin(\beta)) \\ -(-\sin(\beta))(-\cos(\gamma)) & (-\sin(\beta))(-\sin(\gamma)) & \cos(\beta) \end{pmatrix}
$$

$$
= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}
$$

Therefore: $R_{\text{euler}}(\alpha,\beta,\gamma) = R_{\text{euler}}(\alpha-\pi,-\beta,\gamma-\pi)$

3 Problem 3.7 (5 points)

Consider the articulated chain of bodies shown in Figure 3.29 (see figure 4). There are three identical rectangular bars in the plane, called $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$. Each bar has width 2 and length 12. The distance between the two points of attachment is 10. The first bar, $\mathcal{A}_1$, is attached to the origin. The second bar, $\mathcal{A}_2$, is attached to $\mathcal{A}_1$, and $\mathcal{A}_3$ is attached to $\mathcal{A}_2$. Each bar is allowed to rotate about its point of attachment. The configuration of the chain can be expressed with three angles, $(\theta_1, \theta_2, \theta_3)$. The first angle, θ_1, represents the angle between the segment drawn between the two points of attachment of $\mathcal{A}_1$ and the x-axis. The second angle, θ_2, represents the angle between $\mathcal{A}_2$ and $\mathcal{A}_1$ ($\theta_2 = 0$ when they are parallel). The third angle, θ_3, represents the angle between $\mathcal{A}_3$ and $\mathcal{A}_2$. Suppose the configuration is $(\frac{\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{4})$.

(a) Use the homogeneous transformation matrices to determine the locations of points a, b, and c.

(b) Characterize the set of all configurations for which the final point of attachment (near the end of $\mathcal{A}_3$) is at $(0,0)$ (you should be able to figure this out without using the matrices).

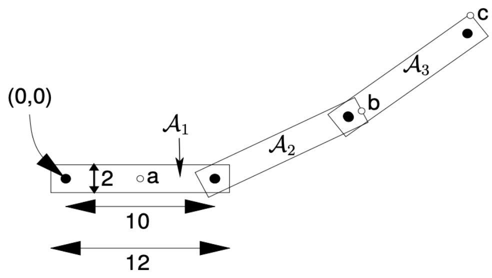

Figure 4: A chain of three bodies

3.1 Answer

(a) Let T_i representing a 3×3 homogeneous transformation matrix which determines the locations of point (x, y), where θ_i represents the angle between $\mathcal{A}_i$ and $\mathcal{A}_{i-1}$:

$$T(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation T_i expresses the difference between the body frame of A_i and the body frame of A_{i-1}. The application of T_i moves A_i from its body frame to the body frame of A_{i-1}. The application of $T_{i-1}T_i$ moves both A_i and A_{i-1} to the body frame of A_{i-2}. By following this procedure, the location in W of any point $(x, y) \in A_m$ is determined by multiplying the transformation matrices to obtain

$$T_1 T_2 \cdots T_m \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Therefore, to determine the location of a we need calculate $T_1 \begin{pmatrix} x_a \\ y_a \\ 1 \end{pmatrix}$. To calculate b we need to calculate $T_1 T_2 \begin{pmatrix} x_b \\ y_b \\ 1 \end{pmatrix}$. Finally to calculate c we need to calculate $T_1 T_2 T_3 \begin{pmatrix} x_c \\ y_c \\ 1 \end{pmatrix}$.

(a) - a

$$\begin{aligned} T_1 \begin{pmatrix} x_a \\ y_a \\ 1 \end{pmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{5\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \\ 1 \end{pmatrix} \end{aligned}$$

(a) - b

$$
\begin{aligned}
T_1T_2\begin{pmatrix} x_b \\ y_b \\ 1 \end{pmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 10 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5\sqrt{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{21\sqrt{2}}{2} \\ 1 \end{pmatrix}
\end{aligned}
$$

(a) - c

$$
\begin{aligned}
T_1T_2T_3\begin{pmatrix} x_c \\ y_c \\ 1 \end{pmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 10 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos-\frac{\pi}{4} & -\sin-\frac{\pi}{4} & 10 \\ \sin-\frac{\pi}{4} & \cos-\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5\sqrt{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 10 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 10\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 11 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 11 + 10\sqrt{2} \\ 1 \end{pmatrix}
\end{aligned}
$$

(b) To have the final point of attachment at the origin, there will be two possibilities which are shown in 5. Since the lengths of all bars are equal, then both triangles (a) in 5 are equilateral. With that said, there will be two sets of configurations, which will be determined separately.

(triangle (a)) First we will rotate $\mathcal{A}_2$ counterclockwise with $\theta_2 = \frac{2\pi}{3}$ respect to $\mathcal{A}_1$. Then, we will rotate $\mathcal{A}_3$ counterclockwise with $\theta_3 = \frac{2\pi}{3}$ respect to $\mathcal{A}_2$. Thus, the configuration set corresponding to triangle (a) is: $(0, \frac{2\pi}{3}, \frac{2\pi}{3})$

(triangle (b)) First we will rotate $\mathcal{A}_2$ counterclockwise with $\theta_2 = \frac{4\pi}{3}$ respect to $\mathcal{A}_1$. Then, we will rotate $\mathcal{A}_3$ counterclockwise with $\theta_3 = \frac{2\pi}{3}$ respect to $\mathcal{A}_2$. Thus, the configuration set corresponding to triangle (a) is: $(0, \frac{4\pi}{3}, \frac{2\pi}{3})$

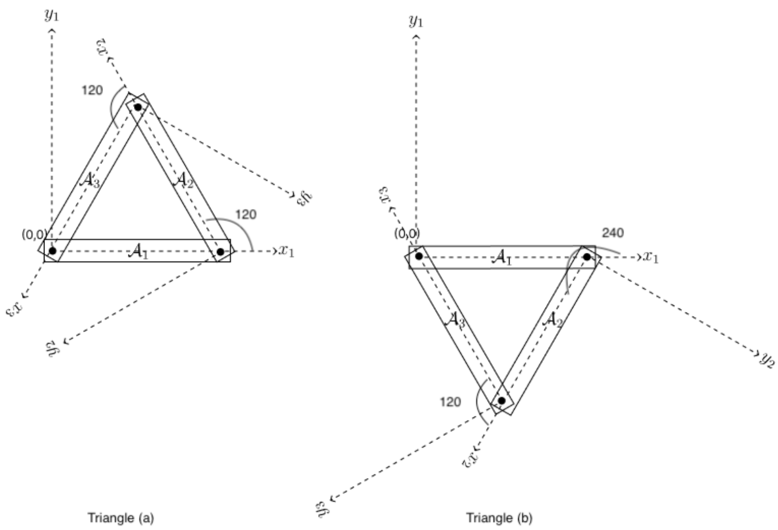

Figure 5: The set of all configurations for the final point to be at origin

4 Problem 3.14 (5 points)

4.1 Answer

We will divide the code into three main parts:

1. Script Specification and Execution Format: to be able to interpret the arguments to the command line, which in this case is:

   ```
   chain.py -W <width> -L <length> -D <distance> <theta 1, ..., theta m>
   ```

   We need to use argument parsing. There are three ways to do argument parsing in Python: 1) sys.argv, 2) get out, and 3) argparse. We will be using argparse [2].

Listing 1: Argument parser for chain.py

```python
def argument_parser():
parser = argparse.ArgumentParser(
    prog='chain.py',
    description='Visualize a 2D kinematic chain based on
        given parameters.',
    epilog='Example: %(prog)s -W 1 -L 5 -D 0.5 30 45 60'
)
parser.add_argument("-W", type=float, required=True,
   help="Width of each link")
parser.add_argument("-L", type=float, required=True,
   help="Length of each link")
```

```
        parser.add_argument("-D", type=float, required=True, help="Distance between attachment points")
        parser.add_argument("thetas", nargs='+', type=float, help="Angles defining the chain configuration")
        return parser.parse_args()
```

First, by using parser = argparse.ArgumentParser() we will be giving a description about the program. This will help the output to be more informative for the user who's running the program. Second, we will define an argument for the command-line, based on each flag. Here the difference between <width> -L <length> -D <distance> and <theta 1, ..., theta m> is that the first is a set of optional arguments and the latter is a positional largument.

2. Kinematic Model and Link Properties In this code, first the joints were extracted and then the vertices. Vertices are calculated based on the current position and the next position. It's important to note that the rotation angels have to be calculted in radians, otherwise the calculation won't work!

Listing 2: Kinematic Model Calculation

```
    def kinematic_model(W, L, D, thetas):
        joint_coordinates = [(0, 0)]  # Start position
        links = []
        current_x, current_y = 0, 0
        current_angle = 0

        for theta in thetas:
            current_angle += theta
            angle_rad = math.radians(current_angle)

            # Calculate next joint position
            dx = D * math.cos(angle_rad)
            dy = D * math.sin(angle_rad)
            next_x = current_x + dx
            next_y = current_y + dy

            # Calculate link vertices (in correct order for a rectangle)
            vertices = [
                (current_x + (-(L-D)/2) * math.cos(angle_rad) - W/2 * math.sin(angle_rad),
                 current_y + (-(L-D)/2) * math.sin(angle_rad) - (-W/2) * math.cos(angle_rad)),
                (next_x + ((L-D)/2) * math.cos(angle_rad) - W/2 * math.sin(angle_rad),
                 next_y + ((L-D)/2) * math.sin(angle_rad) - (-W/2) * math.cos(angle_rad)),
                (next_x + ((L-D)/2) * math.cos(angle_rad) + W/2 * math.sin(angle_rad),
                 next_y + ((L-D)/2) * math.sin(angle_rad) - W/2 * math.cos(angle_rad)),
```

```
                (current_x + (-(L-D)/2) * math.cos(angle_rad) +
                    W/2 * math.sin(angle_rad),
                 current_y + (-(L-D)/2) * math.sin(angle_rad) -
                    W/2 * math.cos(angle_rad)),
            ]
            links.append(vertices)

            current_x, current_y = next_x, next_y
            joint_coordinates.append((current_x, current_y))

        return joint_coordinates, links
```

3. 2D Chain Visualization Visualising chain has done in two part: 1) Joints, 2) Vertices.

Listing 3: Plotting the Kinematic Chain

```
def linksplot(links, ax):
    for i, vertices in enumerate(links):
        x = [vertex[0] for vertex in vertices]
        y = [vertex[1] for vertex in vertices]
        x.append(vertices[0][0])  # Add first point again to
            close the rectangle
        y.append(vertices[0][1])
        ax.plot(x, y, "k-", linewidth=2)

        # Label vertices
        for j, (x, y) in enumerate(vertices):
            ax.text(x, y, f'A{i+1}{j+1}', fontsize=9, ha='right'
                , va='bottom')

def joinplot(joint_coordinates, ax):
    x = [pos[0] for pos[0] in joint_coordinates]
    y = [pos[1] for pos[1] in joint_coordinates]
    ax.plot(x, y, "ro", markersize=10)

    # Label joints
    for i, (x, y) in enumerate(joint_coordinates):
        ax.text(x, y, f'J{i+1}', fontsize=9, ha='left', va='
            bottom', color='black')

def chainplot(joint_coordinates, links, W, L, D):
    fig, ax = plt.subplots(figsize=(12, 8))

    linksplot(links, ax)
    joinplot(joint_coordinates, ax)

    plt.title("2D Kinematic Chain")
    plt.xlabel("X")
    plt.ylabel("Y")
```

```
    plt.axis('equal')
    plt.grid(True)
    plt.show()
```

5 Use of Generative AI Tools

5.1 Writing Problem Statements

In order to avoid going back and forth between this LaTeXfile and homework question files, I used chatGPT to write homework question statements for LaTeXand then copied them into this file.
For question four, I used Claude Sonnet 3.5 for visualization.

References

[1] Owen Byer, Felix Lazebnik, and Deirdre L Smeltzer. *Methods for Euclidean geometry*, volume 37. American Mathematical Soc., 2010.

[2] Python Software Foundation. Cpython argparse module. `https://github.com/python/cpython/blob/3.12/Lib/argparse.py`, 2024. Accessed: 2024-09-22.

[3] SM Lavalle. Planning algorithms. *Cambridge University Press google schola*, 2:3671–3678, 2006.