COM S 4760/5760 Homework 3: C-Space and C-Space Obstacles

Consider a mobile robot operating in a 2D world. The robot is modeled as a rigid body with a circular base of radius r. The robot can move in two ways:

- Translation $T(d)$: The robot can move along the direction specified by its current orientation, where $d \in \mathbb{R}$ is the distance parameter. A positive distance $d > 0$ indicates forward movement, while a negative distance $d < 0$ indicates backward movement.

- Rotation $R(\theta)$: The robot can rotate in place by an angle θ around its center, where $\theta \in \mathbb{R}$ is the rotation parameter.

1. (10 points for COM S 4760, 7 points for COM S 5760) Configuration Space:

   (a) Define the placement of the robot's body frame and use it to derive a semi-algebraic model of the robot.

   Solution Place the origin of the robot's body frame at its center with the x-axis aligned with its heading. The robot can then be modeled by a single primitive

   $$H = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$$

   (b) Suppose the robot starts with its center at position $(x_i, y_i) \in \mathbb{R}^2$ in the world frame, with orientation $\theta_i \in [0, 2\pi)$, measured with respect to the world frame. It then rotates counterclockwise by an angle θ using $R(\theta)$ and moves forward along the resulting orientation by distance d using $T(d)$. Determine the region in the world occupied by the robot after this transformation.

   Solution After applying the rotation $R(\theta)$, the robot's orientation relative to the world frame becomes $\theta_i + \theta$. Moving forward by a distance d in this direction results in changes to the x and y coordinates of the robot's center by $d\cos(\theta + \theta_i)$ and $d\sin(\theta + \theta_i)$, respectively. Therefore, after applying the translation $T(d)$, the robot's center will be at the position $(x_i + d\cos(\theta + \theta_i), y_i + d\sin(\theta + \theta_i))$. The region in the world occupied by the robot is therefore

   $$\{(x, y) \in \mathbb{R}^2 \mid (x - x_i - d\cos(\theta + \theta_i))^2 + (y - y_i - d\sin(\theta + \theta_i))^2 \leq r^2\}.$$

   (c) Suppose the robot starts with its center at position $(x_i, y_i) \in \mathbb{R}^2$ in the world frame, with orientation $\theta_i \in [0, 2\pi)$, measured with respect to the world frame. Now, suppose we want to move the robot so that its center is at

$(x_g, y_g) \in \mathbb{R}^2$ with an orientation $\theta_g \in [0, 2\pi)$ relative to the world frame. Describe a sequence of transformations $T(d)$ and $R(\theta)$ that the robot can perform to achieve this, determining the values for the parameters d and θ in terms of $x_i, y_i, \theta_i, x_g, y_g, \theta_g$ for each translation and rotation. Note that multiple translations and/or rotations may be required.

Solution First, let's address the special case where $x_g = x_i$ and $y_g = y_i$. In this case, the robot only needs to applied $R(\theta_g - \theta_i)$ to adjust its orientation to θ_g. Note that this case must be handled separately, as $\text{atan2}(y_g - y_i, x_g - x_i)$, which is needed in other cases, would not be defined.

Now let's consider the case where $x_g \neq x_i$ or $y_g \neq y_i$. Let:

$$\begin{aligned} d &= \sqrt{(x_g - x_i)^2 + (y_g - y_i)^2} \\ \theta &= \text{atan2}(y_g - y_i, x_g - x_i) \end{aligned} \tag{1}$$

The robot can then perform the following sequence of transformation

- Apply $R(\theta - \theta_i)$ to adjust its orientation to θ relative to the world frame.
- Apply $T(d)$ to move its center to (x_g, y_g).
- Apply $R(\theta_g - \theta)$ to change its orientation to θ_g.

In summary, the sequence of the transformations is $R(\theta - \theta_i)T(d)R(\theta_g - \theta)$.

(d) Describe the configuration space $\mathcal{C}$ of the robot and identify the mathematical structure (group) associated with $\mathcal{C}$. Describe how each configuration is represented, i.e., identify the key parameters that uniquely define a configuration of the robot.

Solution From part (b), we can conclude that the robot can be placed in any desired position and orientation within the 2D world. Thus, its configuration space is homeomorphic to $\mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$.

This configuration space is associated with the "special Euclidean group" or $SE(2)$, which represents the set of all possible translations and rotations in a 2D plane.

Each configuration of the robot can be represented by a triplet (x, y, θ), where $(x, y) \in \mathbb{R}^2$ specifies the position of the robot's center in the world frame and $\theta \in [0, 2\pi)$ specifies the robot's orientation relative to the world frame.

2. (10 points for COM S 4760, 7 points for COM S 5760) C-Space Obstacles:

The workspace, where the robot operates, is a rectangular area defined by the set $[0, X_{max}] \times [0, Y_{max}]$, with the bottom-left corner at $(0, 0)$ and the top-right corner at (X_{max}, Y_{max}). The world contains a set O of circular obstacles, each defined by a tuple $o = (x_o, y_o, r_o) \in \mathbb{R}^2 \times \mathbb{R}_{>0}$, where (x_o, y_o) is the position of the obstacle's center and r_o is its radius.

(a) Describe the configuration space obstacle $\mathcal{C}_{obs,o}$ corresponding to each circular obstacle $o = (x_o, y_o, r_o)$ by identifying a closed form expression for a function $f : \mathcal{C} \times \mathbb{R}^2 \times \mathbb{R}_{>0} \to \mathbb{R}$ such that $\mathcal{C}_{obs,o}$ can be expressed as

$$\mathcal{C}_{obs,o} = \{q \in \mathcal{C} \mid f(q, x_o, y_o, r_o) \leq 0\}.$$

Solution The obstacle region $\mathcal{O}_o$ corresponding to a circular obstacle $o = (x_o, y_o, r_o)$ is given by

$$\mathcal{O}_o = \{(x', y') \in \mathbb{R}^2 \mid (x' - x_o)^2 + (y' - y_o)^2 \leq r_o^2\}.$$

Given a configuration $q = (x, y, \theta)$ of the robot, the region of the world occupied by the robot is given by

$$\mathcal{A}(q) = \{(x', y') \in \mathbb{R}^2 \mid (x' - x)^2 + (y' - y)^2 \leq r)^2\}$$

The configuration space obstacle corresponding to o is therefore

$$\begin{aligned} \mathcal{C}_{obs,o} &= \{(x, y, \theta) \in \mathbb{R}^2 \times [0, 2\pi) \mid \mathcal{A}(x, y, \theta) \cap \mathcal{O} \neq \emptyset\} \\ &= \{(x, y, \theta) \in \mathbb{R}^2 \times [0, 2\pi) \mid (x - x_o)^2 + (y - y_o)^2 \leq (r + r_o)^2\} \end{aligned} \tag{2}$$

The function $f : \mathcal{C} \times \mathbb{R}^2 \times \mathbb{R}_{>0} \to \mathbb{R}$ is, therefore, given by

$$f(x, y, \theta, x_o, y_o, r_o) = (x - x_o)^2 + (y - y_o)^2 - (r + r_o)^2$$

(b) Consider the area outside the workspace boundaries as an obstacle. Determine the complete configuration space obstacle $\mathcal{C}_{obs}$, incorporating both the workspace boundaries and the circular obstacles within the workspace.

Solution The configuration space obstacle corresponding to the area outside the workspace is given by

$$\mathcal{C}_{obs,w} = \{(x, y, \theta) \in \mathbb{R}^2 \times [0, 2\pi) \mid x < r \text{ or } x > X_{max} - r \text{ or } y < r \text{ or } y > Y_{max} - r\}$$

Note the strict inequality above because the workspace is closed.

The complete configuration space obstacle region is therefore

$$\mathcal{C}_{obs} = \mathcal{C}_{obs,w} \cup \bigcup_{o \in O} \mathcal{C}_{obs,o},$$

where $\mathcal{C}_{obs,o}$ is defined in (2).

3. (COM S 5760 only, 6 points) Discretized Workspace: Suppose X_{max} and Y_{max} are natural numbers, and the workspace is discretized into a finite number of grid points, defined as

$$W = \{(x, y) \mid x \in \{0, 1, \dots, X_{max}\}, y \in \{0, 1, \dots, Y_{max}\}\}.$$

The robot's initial position and goal are always located at one of these grid points. Furthermore, the robot can only move between these points, transitioning only to directly neighboring points in the grid—either horizontally or vertically (but not diagonally)–using a combination of translation $T(d)$ and rotation $R(\theta)$, where $d, \theta \in \mathbb{R}$.

For example, suppose the robot is located at $(1, 1)$ with orientation 0 (i.e., its heading aligns with the x-axis), it can move to $(2, 1)$ or $(1, 2)$, using the sequence of transformations derived in Problem 1c.

For simplicity, when computing the configuration space $\mathcal{C}$ and the configuration space obstacle $\mathcal{C}_{obs}$, consider only the configurations of the robot when its center is at a grid point. Although the robot moves continuously between these points, we will disregard its intermediate configurations and assume that collisions—whether with the workspace boundaries or circular obstacles—can only occur when the robot's center is precisely at a grid point.

(a) Assume that the robot is considered to have reached the goal as soon as its center is located at the goal grid point, regardless of its orientation. Given the described setup, is it possible to formulate a discrete motion planning problem for the robot? If so, describe the resulting state space X, the set of actions U, the action space $U(x)$ for each state $x \in X$, and the state transition function $f : X \times U \to X$.

Solution Under this setup, the robot's orientation can be neglected because, as established in Problem 1c, the robot's orientation does not affect the feasibility of moving from one grid point to another. Additionally, from Problem 2, whether the robot collides with obstacles is independent of its orientation.

As a result, we can formulate a discrete motion planning problem for the robot by defining

$$\begin{aligned}
X &= \{(x, y) \in W \mid x \geq r \text{ and } x \leq X_{max} - r \text{ and } \\
&\qquad y \geq r \text{ and } y \geq Y_{max} - r \text{ and } \\
&\qquad ((x - x_o)^2 + (y - y_o)^2 > (r + r_o)^2), \forall o \in O\}. \\
U &= \{(0, 1), (0, -1), (-1, 0), (1, 0)\} \\
U(x) &= \{u \in U \mid f(x, u) \in X\}, \forall x \in X \\
f((x, y), (u_x, u_y)) &= (x + u_x, y + u_y), \forall (x, y) \in X, (u_x, u_y) \in U.
\end{aligned}$$

Note, however, that the actual sequence of transformations that corresponds to each of the above actions depends on the orientation of the robot. For example, if the robot's orientation is 0, then action $(1,0)$ corresponds to $T(1)$. On the other hand, if the robot's orientation is $\frac{\pi}{2}$, then action $(1,0)$ requires applying $R(-\frac{\pi}{2})T(1)$.

(b) Given the following parameters:

- Workspace dimensions: $X_{max} = 6$ and $Y_{max} = 5$.
- Robot radius: $r = 0.5$.
- Set of circular obstacles: $O = \{o_1, o_2\}$, where
  - $o_1 = (3.0, 1.5, 1.12)$ represents an obstacle centered at $(3.0, 1.5)$ with a radius of 1.12.
  - $o_2 = (2.5, 3.0, 0.5)$ represents an obstacle centered at $(2.5, 3.0)$ with a radius of 0.5.

List all the grid points in the workspace that the robot can safely occupy without causing a collision with either the workspace boundaries or circular obstacles.

Solution [(1, 1), (1, 2), (1, 3), (1, 4), (2, 4), (3, 4), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)]

(c) Consider the same setup as in part (b). The robot's initial position is $(1,1)$ with an orientation of $\frac{\pi}{4}$ (i.e., 45 degree relative to the x-axis).

Write a script `main.py` to compute a sequence of grid points for the robot to reach the goal located at $(5,3)$. You should be able to run the script with the following command:

```
python main.py
```

This should print out the sequence of grid points visited by the robot, starting at $(1,1)$ and ending at $(5,3)$.

Solution See `main.py`. One possible path is
[(1, 1), (1, 2), (1, 3), (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (5, 3)]