COM S 476/576 Homework 2

Problem 3.1 (5 points) Define a semi-algebraic model that removes a triangular "nose" from the region shown in Figure 3.4.

Solution Suppose that the vertices of the triangular nose in the **counterclockwise** order is are given by $v_5 = (x_5, y_5)$, $v_6 = (x_6, y_6)$, and $v_7 = (x_7, y_7)$. Define

$$
\begin{aligned}
f_5(x,y) &= a_5 x + b_5 y + c_5 \\
f_6(x,y) &= a_6 x + b_6 y + c_6 \\
f_7(x,y) &= a_7 x + b_7 y + c_7
\end{aligned}
$$

such that $f_i(x,y) < 0$ for all points to the left of the edge from v_i to v_{i+1}, $i \in \{5,6,7\}$, with $v_8 = v_5$. It can be shown that the constants are given by

$$
\begin{aligned}
a_i &= y_{i+1} - y_i \\
b_i &= x_i - x_{i+1} \\
c_i &= x_{i+1} y_i - x_i y_{i+1}
\end{aligned}
$$

For each $i \in \{5,6,7\}$, define the algebraic primitives

$$H_i = \{(x,y) \in \mathcal{W} \mid f_i(x,y) \leq 0\}. \tag{1}$$

Note that we define the vertices v_5, v_6, v_7 in the counterclockwise order because we want to get algebraic primitives for all points **inside** of the triangular nose. The triangular nose is then given by $\mathcal{O}_{nose} = H_5 \cap H_6 \cap H_7$.

Therefore, the model that removes a triangular nose from the region shown in Figure 3.4 is given by $\mathcal{O} = (H_1 \cap H_2 \cap H_3 \cap H_4) \setminus (H_5 \cap H_6 \cap H_7)$, where $H_1, \ldots, H_4$ are defined as in Example 3.1.

Problem 3.4 (5 points) An alternative to the yaw-pitch-roll formulation from Section 3.2.3 is considered here. Consider the following Euler angle representation of rotation (there are many other variants). The first rotation is $R_z(\gamma)$, which is just (3.39) with α replaced by γ. The next two rotations are identical to the yaw-pitch-roll formulation: $R_y(\beta)$ is applied, followed by $R_z(\alpha)$. This yields $R_{euler}(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$.

(a) Determine the matrix R_{euler}.

(b) Show that $R_{euler}(\alpha, \beta, \gamma) = R_{euler}(\alpha - \pi, -\beta, \gamma - \pi)$.

Solution

(a) The rotation matrices are given by

$$
R_z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}
$$

$$
R_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Thus, we get

$$
\begin{aligned}
&R_{euler}(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma) \\
&= \begin{bmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{bmatrix}
\end{aligned}
$$

(b) With

$$
\begin{aligned}
\cos(\alpha - \pi) &= -\cos(\alpha) \\
\sin(\alpha - \pi) &= -\sin(\alpha) \\
\cos(-\beta) &= \cos(\beta) \\
\sin(-\beta) &= -\sin(\beta) \\
\cos(\gamma - \pi) &= -\cos(\gamma) \\
\sin(\gamma - \pi) &= -\sin(\gamma)
\end{aligned}
$$

we get

$$
\begin{aligned}
\cos(\alpha - \pi)\cos(-\beta)\cos(\gamma - \pi) &= \cos(\alpha)\cos(\beta)\cos(\gamma) \\
\sin(\alpha - \pi)\sin(\gamma - \pi) &= \sin(\alpha)\sin(\gamma) \\
\cos(\alpha - \pi)\cos(-\beta)\sin(\gamma - \pi) &= \cos(\alpha)\cos(\beta)\sin(\gamma) \\
\sin(\alpha - \pi)\cos(\gamma - \pi) &= \sin(\alpha)\cos(\gamma) \\
\cos(\alpha - \pi)\sin(-\beta) &= \cos(\alpha)\sin(\beta) \\
\sin(\alpha - \pi)\cos(-\beta)\cos(\gamma - \pi) &= \sin(\alpha)\cos(\beta)\cos(\gamma) \\
\cos(\alpha - \pi)\sin(\gamma - \pi) &= \cos(\alpha)\sin(\gamma) \\
\sin(\alpha - \pi)\cos(-\beta)\sin(\gamma - \pi) &= \sin(\alpha)\cos(\beta)\sin(\gamma) \\
\cos(\alpha - \pi)\cos(\gamma - \pi) &= \cos(\alpha)\cos(\gamma) \\
\sin(\alpha - \pi)\sin(-\beta) &= \sin(\alpha)\sin(\beta) \\
\sin(-\beta)\cos(\gamma - \pi) &= \sin(\beta)\cos(\gamma) \\
\sin(-\beta)\sin(\gamma - \pi) &= \sin(\beta)\sin(\gamma)
\end{aligned}
$$

Thus, we obtain the result

$$
\begin{aligned}
&R_{euler}(\alpha - \pi, -\beta, \gamma - \pi) = R_z(\alpha - \pi) R_y(-\beta) R_z(\gamma - \pi) \\
&= \begin{bmatrix} \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\cos(\gamma) & \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma) & -\sin(\alpha)\cos(\beta)\sin(\gamma) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{bmatrix} \\
&= R_{euler}(\alpha, \beta, \gamma)
\end{aligned}
$$

Problem 3.7 (5 points) Consider the articulated chain of bodies shown in Figure 3.29. There are three identical rectangular bars in the plane, called $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$. Each bar has width 2 and length 12. The distance between the two points of attachment is 10. The first bar, $\mathcal{A}_1$, is attached to the origin. The second bar, $\mathcal{A}_2$, is attached to $\mathcal{A}_1$, and $\mathcal{A}_3$ is attached to $\mathcal{A}_2$. Each bar is allowed to rotate about its point of attachment. The configuration of the chain can be expressed with three angles, $(\theta_1, \theta_2, \theta_3)$. The first angle, θ_1, represents the angle between the segment drawn between the two points of attachment of $\mathcal{A}_1$ and the x-axis. The second angle, θ_2, represents the angle between $\mathcal{A}_2$ and $\mathcal{A}_1$ ($\theta_2 = 0$ when they are parallel). The third angle, θ_3, represents the angle between $\mathcal{A}_3$ and $\mathcal{A}_2$. Suppose the configuration is $(\pi/4, \pi/2, -\pi/4)$.

(a) Use the homogeneous transformation matrices to determine the locations of points a, b, and c.

(b) Characterize the set of all configurations for which the final point of attachment (near the end of $\mathcal{A}_3$) is at $(0, 0)$ (you should be able to figure this out without using the matrices).

Solution

(a) The 2D homogeneous transformation matrix is given by

$$T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_{i-1} \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With $\theta_1 = \pi/4$, $\theta_2 = \pi/2$, $\theta_3 = -\pi/4$, $a_0 = 0$, $a_1 = a_2 = 10$, we get

$$T_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & -1 & 10 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 10 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1T_2 = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 10/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 10/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1T_2T_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 20/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we obtain the locations of a, b, and c as follows.

$$\begin{aligned}
a &= T_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 1 \end{bmatrix} \\
b &= T_1 T_2 \begin{bmatrix} 11 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 21/\sqrt{2} \\ 1 \end{bmatrix} \\
c &= T_1 T_2 T_3 \begin{bmatrix} 11 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 + 20/\sqrt{2} \\ 1 \end{bmatrix}
\end{aligned}$$

(b) The configurations such that the final point of attachment is at $(0, 0)$ need to be such that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ form an equilateral triangle. (The triangle is necessarily equilateral because all the sides have the same length.) This happens when $\theta_2 = \theta_3 = 2\pi/3$ or $\theta_2 = \theta_3 = -2\pi/3$. In other words, the set of all the configurations such that the final point of attachment is at $(0, 0)$ is given by

$$\left\{(\theta_1, \theta_2, \theta_3) \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \mid \theta_2 = \theta_3 = 2\pi/3 \text{ or } \theta_2 = \theta_3 = 4\pi/3\right\}$$

Problem 3.14 (5 points) Develop and implement a kinematic model for a 2D kinematic chain $\mathcal{A}_1, \ldots, \mathcal{A}_m$. Each link has the following properties:

- W: The width of each link.
- L: The length of each link.
- D: The distance between the two points of attachment.

Write a script named `chain.py` that takes above properties and the configuration of the chain and display the arrangement of links in the plane. The script should be executed using the following command:

```
chain.py -W <width> -L <length> -D <distance> <theta_1, ..., theta_m>
```

where `<width>` $= W$, `<length>` $= L$, `<distance>` $= D$, and `<theta_1, ..., theta_m>` specifies the angles that define the chain configuration.

Solution Consider a 2D kinematic chain $\mathcal{A}_1, \ldots, \mathcal{A}_m$. Each link has width W and length L. The distance between the two points of attachment is D. Link $\mathcal{A}_1$ is attached to the origin. For each i such that $1 < i \leq m$, link $\mathcal{A}_i$ is attached to link $\mathcal{A}_{i-1}$ by a revolute joint. The configuration of the chain is expressed with m angles $(\theta_1, \ldots, \theta_m)$, where θ_1 represents the angle between $\mathcal{A}_1$ and the x-axis, and for each i such that $1 < i \leq m$, θ_i represents the angle between $\mathcal{A}_i$ and $\mathcal{A}_{i-1}$.

As in Problem 3.7, the 2D homogeneous transformation matrix is given by

$$T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & a_{i-1} \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a_0 = 0$ and $a_i = D$ for all $1 \leq i \leq m$. For each $1 \leq i \leq m$, the location of the attachment between link $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ is given by

$$T_1 \cdots T_i \begin{bmatrix} D \\ 0 \\ 1 \end{bmatrix},$$