COMS 576 - Motion Planning for Robotics and Autonomous Systems, Midterm

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1 (10 points for COM S 5760)

To formalize this problem as a discrete planning problem, we first have to discretize the configuration space using a low-dispersion sampling method, such as the Sukharev grid, where we aim to place samples in the configuration space in a way that minimizes the largest uncovered areas, thereby ensuring a more uniform coverage of $\mathcal{C}$. This uniformity helps achieve resolution completeness. However, as we formalize the problem below, without using any low-dispersion sampling method, formalizing it into a discrete planning problem cannot be done.

1.1 Formal, mathematical definition of the free configuration space $\mathcal{C}^p_{free}$.

As stated, the objective of this problem requires that an entire path be mapped into $\mathcal{C}_{free}$. Hence, a goal configuration such as (x_G, y_G) might lie in $\mathcal{C}_{free}$, however, it may not be in $\mathcal{C}^p_{free}$ because there's no such path $\tau : [0,1] \to \mathcal{C}_{\text{free}}$ such that

$$\tau(0) \in \mathcal{C}_{free} \quad \text{and} \quad \tau(1) = q_G,$$

Intuitively, although a robot will not be in collision with any obstacle at a certain configuration, getting to that configuration can't be done while avoiding obstacles. Thus, the goal configuration is unapproachable! To define $\mathcal{C}^p_{free}$, first, we need to define $\mathcal{C}^p_{obs}$ and to define $\mathcal{C}^p_{obs}$ we're required to specify the set-up of the world as the following:

- World $\mathcal{W} = \mathbb{R}^2$
- The robot can be placed in any desired position and orientation within the $\mathcal{W}$. Thus, its configuration space is homeomorphic to $\mathcal{C} = \mathbb{R}^2 \times S^1$.
  This configuration space is associated with the "special Euclidean group" or SE(2), which represents the set of all possible translations and rotations in a 2D plane.
  Each configuration of the robot can be represented by a triplet (x, y, θ), where $(x, y) \in \mathbb{R}^2$ specifies the position of the robot's center in the world frame and $\theta \in [0, 2\pi)$ specifies the robot's orientation relative to the world frame.
- Obstacle region $\mathcal{O} \in \mathcal{W}$ can be expressed as a semi-algebraic model. Suppose we have m circular obstacles. Hence:
  $$\mathcal{O} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3 \cup \cdots \cup \mathcal{H}_m$$

Where each $\mathcal{H}_i$ is an algebraic primitive defined as follows:

$$\mathcal{H}_i = \{(x, y) \in \mathcal{W} | (x - x_{io})^2 + (y - y_{io})^2 - (r + r_{io})^2 \leq 0\} \quad 1 \leq i \leq m$$

(x_{io}, y_{io}) is the position of the obstacle o_i's center and r_{io} is its radius.
To describe the region occupied by the robot along the line segment l_i from (x_{i-1}, y_{i-1}) to (x_i, y_i), we parameterize l_i as:

$$l_i(t) = (1 - t)(x_{i-1}, y_{i-1}) + t(x_i, y_i), \quad t \in [0, 1].$$

The region occupied by the robot while being in this line segment is parameterized as f_i:

$$S_i = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = (1 - t)(x_{i-1}, y_{i-1}) + t(x_i, y_i),\ t \in [0, 1]\}$$

Now, for all configurations $q = (x, y, \theta)$ of the robot, in which $(x, y) \in S_i$, the region $\mathcal{A}_i(q)$ occupied by the robot at any point along l_i is given by:

$$\mathcal{A}_i(q) = \left\{(x', y') \in \mathcal{W} \mid (x' - x)^2 + (y' - y)^2 - r^2 \leq 0\right\},$$

where $l_{i,x}(t)$ and $l_{i,y}(t)$ are the x- and y-coordinates of the point $l_i(t)$ along the segment.
Hence, the specific obstacle region for segment l_i, denoted $\mathcal{C}_{\text{obs},o,i}$, is defined as follows:

$$\mathcal{C}_{\text{obs},o,i} = \{(x, y, \theta) \in \mathcal{C} \mid (x, y) \in S_i \text{ and } \mathcal{A}_i((x, y, \theta)) \cap \mathcal{O} \neq \emptyset\}$$

In addition, let $\mathcal{C}_{\text{obs},i,w}$ denote the set of points along the line segment l_i that lie outside the world boundary. We define it as:

$$\mathcal{C}_{\text{obs},i,w} = \{(x, y, \theta) \in \mathcal{C} \mid ((x, y) \in S_i) \wedge (x < r \text{ or } x > X_{\max} - r \text{ or } y < r \text{ or } y > Y_{\max} - r)\}.$$

Hence, the configuration space obstacle $\mathcal{C}^p_{obs}$ which considers the configurations of a single robot passing through a series of line segments in collision with obstacles or being outside the 2D world frame, is defined as:

$$\mathcal{C}^p_{obs} = \left(\bigcup_{i=1}^{n} \mathcal{C}_{\text{obs},o,i}\right) \bigcup \left(\bigcup_{i=1}^{n} \mathcal{C}_{\text{obs},i,w}\right)$$

Finally,

$$\mathcal{C}^p_{free} = C \setminus \mathcal{C}^p_{obs}$$

Based on this definition:

$$\mathcal{C}^p_{free} \subseteq \mathcal{C}_{free}$$

This means that when constructing $\mathcal{C}^p_{free}$, we might not explore all the configuration space. We only care about a part of the world that is occupied but the robot along the projected path. Hence, parts of the world might be unexplored because the path that the robots cross over will never cross those certain locations.

1.2 Specify whether $\mathcal{C}^p_{free}$ is countable, and if so, whether it is finite.

Based on the definition, $\mathcal{C}^p_{free}$ is countable. However, it's infinite.

1.3 Describe how to obtain the sequence $p = (x_0, y_0)(x_1, y_1) \dots (x_n, y_n)$ once a path $\tau^p : [0, 1] \to \mathcal{C}^p_{free}$ is found, assuming that when projecting τ^p onto the xy-plane, it corresponds to a sequence of line segments.

To verify that the path τ^p is collision-free, we apply a low-dispersion sampling technique along each line segment in the projected xy-plane.

- Discretize the configuration space using a low-dispersion sampling method, such as the Sukharev grid
- Consider Sample Points Along Path Segments:
  - For each segment l_i, consider the sampled points alongside that segment.
- Check Collision at Each Sample Point:
  - At each sample point, compute the robot's configuration $q = (x, y, \theta)$ by placing the robot's center at the sample location and applying its orientation.
  - For each configuration q, check if the robot configuration is in any obstacle in $\mathcal{C}^p_{\text{obs}}$.
- Iterate Through All Segments:
  - Apply this collision-checking process to all segments $l_1, l_2, \dots, l_n$ in the sequence. If all segments pass the check, the entire path τ^p is confirmed to be collision-free. Otherwise, adjustments to the path are required.

2 (10 points for COM S 5760) Solve the planning problem using RRT

3 (COM S 5760 only, 10 points)

3.1 Formal, mathematical definition of the configuration space $\mathcal{C}_2$

Each robot, b_i for $i = 1, 2$, is considered as a single rigid body that has its associated configuration space $\mathcal{C}^i$, its initial and goal configurations, respectively. Hence, $\mathcal{C}_2$ that considers the configurations of all robots simultaneously is defined as the Cartesian product:

$$\mathcal{C}_2 = \mathcal{C}^1 \times \mathcal{C}^2$$

In which each configuration $q \in \mathcal{C}_2$ is of the form (q^1, q^2). The dimension of $\mathcal{C}_2$ is $N = 6$, where $N = \dim(\mathcal{C}^1) + \dim(\mathcal{C}^2) = 3 + 3 = 6$.

Each of the robots can be placed in any desired position and orientation within the 2D world. Thus, its configuration space is meromorphic to $\mathcal{C}^i = \mathbb{R}^2 \times \mathbb{S}^1$, for $i = 1, 2$.

Each of the robot's configuration spaces is associated with the "special Euclidean group" or SE(2), which represents the set of all possible translations and rotations in a 2D plane.

Each configuration of each robot can be represented by a triplet (x, y, θ), where $(x, y) \in \mathbb{R}^2$ specifies the position of the robot's center in the world frame and $\theta \in [0, 2\pi)$ specifies the robot's orientation relative to the world frame.

3.2 Formal, mathematical definition of the free configuration space $\mathcal{C}_{free,2} \subseteq \mathcal{C}_2$

To calculate $\mathcal{C}_{free,2}$ we will first calculate $\mathcal{C}_{obs,2}$ which refers to the obstacle region that any of the robots is in collision at least one of the obstacles, with one another robot, or going further than the world boundaries. Thus, there are three sources of obstacle regions in the state space: 1) *robot-obstacle collisions*, 2) *robot-robot collisions*, and 3) *robot outside the world boundaries.*
The obstacle region $\mathcal{O}_o$ corresponding to a circular obstacle $o = (x_o, y_o, r_o)$ is given by:

$$\mathcal{O}_o = \{(x', y') \in \mathbb{R}^2 \mid (x' - x_o)^2 + (y' - y_o)^2 \leq r_o^2\}.$$

Given a configuration of a single robot $q = (x, y, \theta)$, the region of the world occupied by this robot is given by:

$$\mathcal{A}(q) = \{(x', y') \in \mathbb{R}^2 \mid (x' - x)^2 + (y' - y)^2 \leq r^2\}$$

Hence, for each i such that $i = 1, 2$, the subset of $\mathcal{C}_2$ that corresponds to robot b_i in collision with a circular obstacle defined by a tuple $o = (x_o, y_o, r_o) \in \mathbb{R}^2 \times \mathbb{R}_{>0}$, where (x_o, y_o) is the position of the obstacle's center and r_o is its radius is:

$$\begin{aligned}
\mathcal{C}^1_{obs,2,o} &= \{(q^1, q^2) \in \mathcal{C}_2 \mid \mathcal{A}^1(q^1) \cap \mathcal{O}_o \neq \emptyset\} \\
&= \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid (x^1 - x_o)^2 + (y^1 - y_o)^2 \leq (r + r_o)^2\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^2_{obs,2,o} &= \{(q^1, q^2) \in \mathcal{C}_2 \mid \mathcal{A}^2(q^2) \cap \mathcal{O}_o \neq \emptyset\} \\
&= \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid (x^2 - x_o)^2 + (y^2 - y_o)^2 \leq (r + r_o)^2\},
\end{aligned}$$

Hence, the configuration space obstacle corresponding to $\mathcal{O}$ and b_1 is:

$$\begin{aligned}
\mathcal{C}^1_{obs,2,\mathcal{O}} &= \{(q^1, q^2) \in \mathcal{C}_2 \mid \mathcal{A}^1(q^1) \cap \mathcal{O} \neq \emptyset\} \\
&= \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid (x^1 - x_o)^2 + (y^1 - y_o)^2 \leq (r + r_o)^2, \forall o \in \mathcal{O}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^2_{obs,2,\mathcal{O}} &= \{(q^1, q^2) \in \mathcal{C}_2 \mid \mathcal{A}^2(q^2) \cap \mathcal{O} \neq \emptyset\} \\
&= \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid (x^2 - x_o)^2 + (y^2 - y_o)^2 \leq (r + r_o)^2, \forall o \in \mathcal{O}\},
\end{aligned}$$

Please note that we're not allowing $\mathcal{A}^i$, $i = 1, 2$, to touch $\mathcal{O}$, but rather the corresponding robots to come arbitrarily close.
In addition, for the pair, b_i, $i = 1, 2$, and b_j, $j = 2, 1$, of robots, the subset of $\mathcal{C}_2$ that corresponds tob_i in collision with b_j is

$$\begin{aligned}
\mathcal{C}^{12}_{obs,2} &= \{(q^1, q^2) \in \mathcal{C}_2 \mid \mathcal{A}^1(q^1) \cap \mathcal{A}^2(q^2) \neq \emptyset\} \\
&= \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid (x^2 - x^1)^2 + (y^2 - y^1)^2 \leq (r + r)^2\},
\end{aligned}$$

And, for each i such that $i = 1, 2$, the subset of $\mathcal{C}_2$ that corresponds to robot b_i with radius r staying outside the world boundary is

$$\mathcal{C}^i_{obs,2,w} = \{(x, y, \theta) \in \mathcal{C}^i \mid x < r \text{ or } x > X_{max} - r \text{ or } y < r \text{ or } y > Y_{max} - r\}.$$

The configuration space obstacle $\mathcal{C}_{obs,2}$ which considers the configurations of all rigid bodies in collision with obstacles or with another robot simultaneously, is defined as:

$$\mathcal{C}_{obs,2} = \left(\bigcup_{i=1}^{2} \mathcal{C}^{i}_{obs,2,\mathcal{O}}\right) \bigcup (\mathcal{C}^{12}_{obs,2}) \bigcup \left(\bigcup_{i=1}^{2} \mathcal{C}^{i}_{obs,2,w}\right)$$

Finally:

$$\mathcal{C}_{free,2} = \mathcal{C}_2 \setminus \mathcal{C}_{obs,2}$$

3.3 Identify assumptions that enable the formulation of a discrete motion planning problem for this two-robot setup. Describe the resulting state space X, the set of actions U, the action space $U(x)$ for each state $x \in X$, and the state transition function $f : X \times U \to X$.

- World $\mathcal{W} = \mathbb{R}^2$
- State space X: Each robot b_i for $i = 1, 2$ has an associated configuration space $\mathcal{C}^i$. Hence, the state space X is defined as the Cartesian product product:

$$\begin{aligned} X = \{((x^1, y^1, \theta^1), (x^2, y^2, \theta^2)) \in \mathcal{C}_2 \mid x^1 \geq r \text{ and } x^1 \leq X_{\max} - r \\ \text{and } y^1 \geq r \text{ and } y^1 \leq Y_{\max} - r \text{ and} \\ (x^1 - x_o)^2 + (y^1 - y_o)^2 > (r + r_o)^2, \forall o \in O \text{ and} \\ (x^2 - x_o)^2 + (y^2 - y_o)^2 > (r + r_o)^2, \forall o \in O \text{ and} \\ (x^1 - x^2)^2 + (y^1 - y^2)^2 > (r + r)^2\} \end{aligned}$$

  In which each configuration $x \in X$ is of the form $(x_1, x_2) = ((x^1, y^1, \theta^1), (x^2, y^2, \theta^2))$.

- Set U of actions: for each robot to move to a location in the 2D world, first, it has to rotate to a certain amount θ (θ can be zero if the robot is already facing the destination), and then move forward. Thus, there are two sets of actions the robot can do. Rotating counter-clockwise by angle θ, or translating forward by a distance d. Knowing that the robot's orientation relative to the world frame θ moving forward by a distance d in this direction results in changes to the x and y coordinates of the robot's center by $d\cos(\theta)$ and $d\sin(\theta)$, respectively. Hence,

$$U = \left\{(u^1, u^2) \mid u^i \in \{(x, y, 0), (0, 0, \theta)\}, \ (x, y) \in \mathbb{R}^2, \ \theta \in [0, 2\pi)\right\}$$

- Action space $U(x)$:

$$U(x) = \{u \in U \mid f(x, u) \in X\}, \quad \forall x \in X$$

- State transition function f: Given $x = (x^1, x^2) \in X$ and $u = (u^1, u^2) \in U$, we define the state transition function as:

$$f((x_1, x_2), (u^1, u^2)) = (x_1 + u^1, x_2 + u^2), \quad \forall (x_1, x_2) \in X, \ (u^1, u^2) \in U.$$