MATH 414 Analysis I,
Homework 9

Mobina Amrollahi

Due: November 1^{st}, 11:59pm

2.5: Problem 2

Suppose that $f^{(n+1)}(x_0)$ exists, and let T_n be the n-th Taylor polynomial of f about x_0. Show that the function

$$E_n(x) = \begin{cases} \frac{f(x)-T_n(x)}{(x-x_0)^n}, & x \in D_f - x_0, \\ 0, & x = x_0 \end{cases}$$

is differentiable at x_0, and find $E_n'(x_0)$.

Answer

To show $E_n(x)$ is differentiable at x_0, we have to show the following limit exists:

$$\lim_{x \to x_0} \frac{E_n(x) - E_n(x_0)}{x - x_0}$$

Hence:

$$\lim_{x \to x_0} \frac{E_n(x) - E_n(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}}$$

Since $f^{(0)}(x_0)$ exists, $f(x_0) = T_n(x_0)$, and the ratio $\frac{f(x)-T_n(x)}{(x-x_0)^{n+1}}$ is indeterminate of the form $0/0$ as $x \to x_0$. Thus, L'Hospital's rule implies that:

$$\lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}} = \lim_{x \to x_0} \frac{f'(x) - T_n'(x)}{(n+1)(x - x_0)^n}$$

We will show that $f'(x_0) - T_n'(x_0) = 0$. We know:

$$T_n(x) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x-x_0)^r = \frac{f(x_0)}{0!}(x-x_0)^0 + \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{r!}(x-x_0)^r = f(x_0) + \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{r!}(x-x_0)^r$$

$$\begin{aligned}
T_n'(x) &= \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{r!} \times r \times (x-x_0)^{r-1} \\
\Rightarrow T_n'(x) &= \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x-x_0)^{r-1} \\
\Rightarrow T_n'(x_0) &= \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x_0-x_0)^{r-1}
\end{aligned}$$

If $r_1 \geq 1$, then $\frac{f^{(r)}(x_0)}{(r-1)!}(x_0-x_0)^{r-1} = 0$ and:

$$\begin{aligned}
T_n'(x_0) &= \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x_0-x_0)^{r-1} \quad \text{if } r < 2 \\
\Rightarrow T_n'(x_0) &= \frac{f^{(1)}(x_0)}{(0)!}(x_0-x_0)^0 = f'(x_0) \\
\Rightarrow f'(x_0) &- T_n'(x_0) = 0
\end{aligned}$$

Thus, the ratio $\frac{f'(x)-T_n'(x)}{(n+1)(x-x_0)^n}$ is indeterminate of the form $0/0$ as $x \to x_0$. Thus, L'Hospital's rule implies that:

$$\lim_{x \to x_0} \frac{f'(x)-T_n'(x)}{(n+1)(x-x_0)^n} = \lim_{x \to x_0} \frac{f^{(2)}(x)-T_n^{(2)}(x)}{(n+1)n(x-x_0)^{n-1}}$$

We will show that $f^{(2)}(x) - T_n^{(2)}(x) = 0$. We know:

$$\begin{aligned}
T_n'(x) &= \sum_{r=1}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x-x_0)^{r-1} \\
&= \frac{f^{(1)}(x_0)}{(1-1)!}(x-x_0)^{1-1} + \sum_{r=2}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x-x_0)^{r-1} \\
&= f'(x_0) + \sum_{r=2}^{n} \frac{f^{(r)}(x_0)}{(r-1)!}(x-x_0)^{r-1}
\end{aligned}$$

Hence:

$$\begin{aligned}
T_n^{(2)}(x) &= \sum_{r=2}^{n} \frac{f^{(r)}(x_0)}{(r-2)!}(x-x_0)^{r-2} \\
\Rightarrow T_n^{(2)}(x_0) &= f^{(2)}(x_0) + \sum_{r=3}^{n} \frac{f^{(r)}(x_0)}{(r-2)!}(x_0-x_0)^{r-2} \\
\Rightarrow T_n^{(2)}(x_0) &= f^{(2)}(x_0)
\end{aligned}$$

This process can be continued n times; each time, we will imply L'Hospital's rule, and we'll end up with an indeterminate of the form $0/0$. Finally:

$$\lim_{x \to x_0} \frac{E_n(x)-E_n(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{f^{(n)}(x)-T_n^{(n)}(x)}{(n+1)!(x-x_0)}$$

Where

$$T_n^{(n)}(x) = f^{(n)}(x_0)(x - x_0)$$

Since $\frac{f^{(n)}(x)-T_n^{(n)}(x)}{(n+1)!(x-x_0)}$ is an indeterminate of the form $0/0$, we can apply the L'Hospital's rule one more time, and we get:

$$\lim_{x\to x_0} \frac{E_n(x) - E_n(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f^{(n+1)}(x) - T_n^{(n+1)}(x)}{(n+1)!} = \lim_{x\to x_0} \frac{f^{(n+1)}(x) - f^{(n)}(x_0)}{(n+1)!}$$
$$= \frac{f^{(n+1)}(x_0) - f^{(n)}(x_0)}{(n+1)!}$$

Hence:

$$E_n'(x_0) = \frac{f^{(n+1)}(x_0) - f^{(n)}(x_0)}{(n+1)!}$$

2.5: Problem 4

(a) Prove: if $f''(x_0)$ exists, then

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0).$$

(b) Prove or give a counterexample: If the limit in (a) exists, then so does $f''(x_0)$, and they are equal.

Answer

(a) Since $f''(x_0)$ exists, then 2^{nd} Taylor polynomial of f about x_0 is:

$$T_2(x) = \sum_{r=0}^{2} \frac{f^{(r)}(x_0)}{r!}(x-x_0)^r = \frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2}(x-x_0)^2$$

Now, based on Theorem 2.5.1 [1] we know:

$$\lim_{x\to x_0} \frac{f(x) - T_n(x)}{(x-x_0)^n} = 0$$

Therefore for $n = 2$ and $x = x_0 + h$, $h > 0$:

$$\lim_{x\to x_0^+} \frac{f(x) - T_2(x)}{(x-x_0)^2} = \lim_{h\to 0} \frac{f(x_0+h) - T_2(x_0+h)}{(h)^2}$$
$$= \lim_{h\to 0} \frac{f(x_0+h) - T_2(x_0+h)}{(h)^2}$$
$$= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2}$$

Also, we can do the same procedure for $n = 2$ and $x = x_0 - h$, $h > 0$:

$$\begin{aligned}
\lim_{x\to x_0^-} \frac{f(x) - T_2(x)}{(x - x_0)^2} &= \lim_{h\to 0} \frac{f(x_0 - h) - T_2(x_0 - h)}{(h)^2} \\
&= \lim_{h\to 0} \frac{f(x_0 - h) - T_2(x_0 - h)}{(h)^2} \\
&= \lim_{h\to 0} \frac{f(x_0 - h) - f(x_0) + f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2}
\end{aligned}$$

Since $\lim\limits_{x\to x_0} \frac{f(x)-T_n(x)}{(x-x_0)^n} = 0$:

$$\lim_{x\to x_0^+} \frac{f(x) - T_n(x)}{(x - x_0)^n} = \lim_{x\to x_0^-} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0$$

Therefore:

$$\lim_{x\to x_0^+} \frac{f(x) - T_2(x)}{(x - x_0)^2} = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2} = 0$$

$$\lim_{x\to x_0^-} \frac{f(x) - T_2(x)}{(x - x_0)^2} = \lim_{h\to 0} \frac{f(x_0 - h) - f(x_0) + f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2} = 0$$

And

$$\begin{aligned}
&\lim_{x\to x_0^+} \frac{f(x) - T_2(x)}{(x - x_0)^2} - \lim_{x\to x_0^-} \frac{f(x) - T_2(x)}{(x - x_0)^2} = 0 \\
&\Rightarrow \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2} \\
&+ \lim_{-h\to 0} \frac{f(x_0 - h) - f(x_0) + f'(x_0)(h) - \frac{f^{(2)}(x_0)}{2}(h)^2}{(h)^2} = 0 \\
&\Rightarrow \lim_{h\to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h) - f^{(2)}(x_0)(h)^2}{(h)^2} = 0 \\
&\Rightarrow \lim_{h\to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{(h)^2} - \lim_{h\to 0} f^{(2)}(x_0) = 0
\end{aligned}$$

$f^{(2)}(x_0)$ is a constant, therefore:

$$\lim_{h\to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{(h)^2} = f^{(2)}(x_0)$$

(b) Consider the following counterexample:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

This function is continuous at $x_0 = 0$ and differentiable everywhere, including at x_0. However, the second derivative does not exist at x_0.

Calculating the first derivative:

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

At $x_0 = 0$:

$$f'(0) = 0.$$

Now for the second derivative, if we try to compute $f''(0)$:

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{2h \sin\left(\frac{1}{h}\right) - \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} 2 \sin\left(\frac{1}{h}\right) - \lim_{h \to 0} \frac{\cos\left(\frac{1}{h}\right)}{h}$$

This limit does not exist because $\sin\left(\frac{1}{h}\right)$ oscillates as $h \to 0$ although $\cos\left(\frac{1}{h}\right)$ takes same value in a neighborhood around 0.

However, the following limit exists:

$$\lim_{h \to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) + h^2 \sin\left(-\frac{1}{h}\right)}{h^2} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right) - \sin\left(\frac{1}{h}\right) = 0.$$

2.5: Problem 6

A function f has a double zero (or a zero of multiplicity 2) at x_0 if f is twice differentiable on a neighborhood of x_0 and $f(x_0) = f'(x_0) = 0$, while $f''(x_0) \neq 0$.

(a) Prove that f has a double zero at x_0 if and only if

$$f(x) = g(x)(x - x_0)^2,$$

where g is continuous at x_0 and twice differentiable on a deleted neighborhood of x_0, $g(x_0) \neq 0$, and

$$\lim_{x \to x_0} (x - x_0) g'(x) = 0.$$

(b) Give an example showing that g in (a) need not be differentiable at x_0.

Answer

(a) We have to prove the following:

$$f \text{ has a double zero at } x_0 \iff f(x) = g(x)(x - x_0)^2,$$

where g is continuous at x_0 and twice differentiable on a deleted neighborhood of x_0, $g(x_0) \neq 0$, and

$$\lim_{x \to x_0} (x - x_0) g'(x) = 0.$$

($\Longrightarrow$) f is twice differentiable on a neighborhood of x_0, hence $f(x_0), f^{(1)}(x_0)$ and $f^{(2)}(x_0)$ exist. Thus, we can write the 2^{nd} Taylor polynomial of f about x_0, which is:

$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2}(x - x_0)^2$$

Since $f(x_0) = f'(x_0) = 0$:

$$T_2(x) = \frac{f^{(2)}(x_0)}{2}(x - x_0)^2$$

In addition, based on Lemma 2.5.2 [1] we know If $f^{(n)}(x_0)$ exists, then:

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r + E_n(x)(x - x_0)^n,$$

where

$$E_n(x) = \begin{cases} \frac{f(x)-T_n(x)}{(x-x_0)^n}, & x \in D_f \setminus \{x_0\}; \\ 0, & x = x_0 \end{cases}$$

and

$$\lim_{x \to x_0} E_n(x) = E_n(x_0) = 0.$$

If we plug $n = 2$, we have:

$$f(x) = \sum_{r=0}^{2} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + E_2(x)(x - x_0)^2,$$

$\sum_{r=0}^{2} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ is expressing $T_2(x)$ which is $\frac{f^{(2)}(x_0)}{2}(x - x_0)^2$, hence:

$$f(x) = \frac{f^{(2)}(x_0)}{2}(x - x_0)^2 + E_2(x)(x - x_0)^2 = (\frac{f^{(2)}(x_0)}{2} + E_2(x))(x - x_0)^2$$

If we put $g(x) = \frac{f^{(2)}(x_0)}{2} + E_2(x)$. Now, we will show the following hold for g:

(a) $g(x_0) \neq 0$
We know:

$$g(x_0) = \frac{f^{(2)}(x_0)}{2} + E_2(x_0) = \frac{f^{(2)}(x_0)}{2} + 0 = \frac{f^{(2)}(x_0)}{2} \neq 0$$

(b) g is continuous at x_0
The continuity of g at x_0 depends on the continuity of $E_2(x)$ at x_0. We know:

$$\lim_{x \to x_0} E_2(x) = E_2(x_0) = 0$$

Hence $E_2(x)$ is continuous at x_0. Now:

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} \frac{f^{(2)}(x_0)}{2} + E_2(x) = \frac{f^{(2)}(x_0)}{2} + \lim_{x \to x_0} E_2(x) = \frac{f^{(2)}(x_0)}{2} = g(x_0)$$

$$\Rightarrow \lim_{x \to x_0} g(x) = g(x_0)$$

(c) g is twice differentiable on a deleted neighborhood of x_0. The differentiability of g around x_0 depends on the differentiability of $E_2(x)$ around x_0. Since $f(x), f^{(1)}(x)$ and $f^{(2)}(x)$ exist on a deleted neighborhood of x_0, hence $E_2'(x)$ and $E_2^(2)(x)$ also exist on a deleted neighborhood of x_0. Thus, $g'(x)$ and $g^{(2)}(x)$ exist for all x on a deleted neighborhood of x_0.

(d) $\lim_{x\to x_0}(x-x_0)g'(x) = 0$
Since:

$$g(x) = \frac{f^{(2)}(x_0)}{2} + E_2(x) \Rightarrow g'(x) = E_2'(x) \Rightarrow g'(x) = (\frac{f(x)-T_n(x)}{(x-x_0)^n})'$$

I don't know how to show this.

($\Longleftarrow$) since $f(x) = g(x)(x-x_0)^2$, $f(x_0) = g(x_0)(x_0-x_0)^2 = 0$. Now, to calculate $f'(x_0)$ we have:

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\ &= \lim_{x\to x_0} \frac{g(x_0)(x-x_0)^2 - 0}{x-x_0} \\ &= \lim_{x\to x_0} g(x_0)(x-x_0) \\ &\Rightarrow f'(x_0) = g(x_0)\times 0 = 0 \end{aligned}$$

Since $g(x)$ and $(x-x_0)^2$ are twice differentiable on a deleted neighborhood of x_0, considering this deleted neighborhood as I, then we can calculate $f'(x)$ for all $x \in I$:

$$\begin{aligned} &f(x) = g(x)(x-x_0)^2 \\ \Rightarrow &f'(x) = 2\times g(x)(x-x_0) + g'(x)(x-x_0)^2 \end{aligned}$$

Now, to calculate $f^{(2)}(x)$ on this deleted neighborhood, we can calculate the following:

$$\begin{aligned} f^{(2)}(x_0) &= \lim_{x\in I} \frac{f'(x)-f'(x_0)}{x-x_0} \\ &= \lim_{x\to x_0} \frac{2\times g(x)(x-x_0) + g'(x)(x-x_0)^2 - 0}{x-x_0} \\ &= \lim_{x\to x_0} 2\times g(x) + g'(x)(x-x_0) \\ &= \lim_{x\to x_0} 2\times g(x) + \lim_{x\to x_0} g'(x)(x-x_0) \\ &= 2\times g(x_0) + \lim_{x\to x_0} g'(x)(x-x_0) = 2\times g(x_0) + 0 \\ &\Rightarrow f^{(2)}(x_0) = 2\times g(x_0) \neq 0 \end{aligned}$$

Hence, f has a double zero at x_0.

(b) Consider $g(x) = |x| + 1$. Then g is continuous at $x_0 = 0$ and twice differentiable on a deleted neighborhood of $x_0 = 0$ since:

$$g(x) = \begin{cases} x, & x \geq 0; \\ -x, & x < 0 \end{cases}$$

Thus to calculate $g'(x)$ and $g^{(2)}(x)$ for all $x \neq 0$, we only need to consider the following limits in a small neighborhood where the values don't change signs:

$$\lim_{x \to x_1} \frac{g(x) - g(x_1)}{(x - x_1)} \text{ and } \lim_{x \to x_1} \frac{g'(x) - g'(x_1)}{(x - x_1)}$$

Then, based on the neighborhood we're whether dealing with x or $-x$ which are both twice differentiable functions. In addition $g(x_0) = g(0) = 1 \neq 0$. Finally to calculate $\lim_{x \to x_0}(x - x_0)g'(x)$ we will first calculate the limit from the left and then right.

$$\begin{aligned}
&\lim_{x \to x_0^+} (x - x_0)g'(x) = \lim_{x \to x_0^+} (x - x_0) \times 1 = 0 \\
&\lim_{x \to x_0^-} (x - x_0)g'(x) = \lim_{x \to x_0^-} (x - x_0) \times -1 = 0 \\
&\Rightarrow \lim_{x \to x_0^+} (x - x_0)g'(x) = \lim_{x \to x_0^-} (x - x_0)g'(x) = 0 \\
&\Rightarrow \lim_{x \to x_0} (x - x_0)g'(x) = 0
\end{aligned}$$

2.5: Problem 13

Determine whether $x_0 = 0$ is a local maximum, local minimum, or neither for each of the following functions.

(b) $f(x) = x^3 e^{x^2}$

(d) $f(x) = \frac{1+x^3}{1+x^2}$

(f) $f(x) = e^{x^2} \sin(x)$

(h) $f(x) = e^{x^2} \cos(x)$

Answer

To answer this question, we will be using Theorem 2.5.3 [1] which is the following:
Theorem 2.5.3 Suppose that f has n derivatives at x_0 and n is the smallest positive integer such that $f^{(n)}(x_0) \neq 0$:

(a) If n is odd, x_0 is not a local extreme point of f.

(b) If n is even, x_0 is a local maximum of f if $f^{(n)}(x_0) < 0$, or a local minimum of f if $f^{(n)}(x_0) > 0$.

Thus, in each part, we will be calculating the derivatives of f at $x_0 = 0$ up to the n^{th} derivative where $f^{(n)}(0) \neq 0$, and based on n we will decide about the type of $x_0 = 0$ of that function.

(b) $f(x) = x^3 e^{x^2}$

$$f^{'}(x) = 3x^2 e^{x^2} + x^3 e^{x^2} \times 2x = 3x^2 e^{x^2} + 2x^4 e^{x^2} = x^2 e^{x^2}(3 + 2x^2)$$
$$\Rightarrow f^{'}(0) = 0$$
$$f^{(2)}(x) = (2xe^{x^2} + x^2 e^{x^2} \times 2x)(3 + 2x^2) + x^2 e^{x^2}(4x) = (2xe^{x^2})(2x^4 + 5x^2 + 1)$$
$$\Rightarrow f^{(2)}(0) = 0$$
$$f^{(3)}(x) = (2e^{x^2} + 4x^2 e^{x^2})(2x^4 + 5x^2 + 1) + (2xe^{x^2})(8x^3 + 10x)$$
$$= (2e^{x^2})(4x^6 + 12x^4 + 8x^3 + 7x^2 + 10x + 1)$$
$$\Rightarrow f^{(3)}(0) = (2 \times 1)(1) = 2$$

Since $n = 3$ is an odd number, then $x_0 = 0$ is not a local extreme point of f.

(d) $f(x) = \frac{1+x^3}{1+x^2}$

$$f^{'}(x) = \frac{(3x^2)(1 + x^2) - (1 + x^3)(2x)}{(1 + x^2)^2} = \frac{x^4 + 3x^2 - 2x}{x^4 + 2x^2 + 1}$$
$$\Rightarrow f^{'}(0) = 0$$
$$f^{(2)}(x) = \frac{(4x^3 + 6x - 2)(x^4 + 2x^2 + 1) - (x^4 + 3x^2 - 2x)(4x^3 + 4x)}{(x^2 + 1)^4} = \frac{(-2) \times (1) - 0}{1}$$
$$\Rightarrow f^{(2)}(0) = -2$$

Since $n = 2$ is an even number and $f^{(n)}(0) < 0$, x_0 is a local maximum of f.

(f) $f(x) = e^{x^2} \sin(x)$

$$f^{'}(x) = e^{x^2} 2x \sin(x) + e^{x^2} \cos(x)$$
$$\Rightarrow f^{'}(0) = 0 + 1 = 1$$

Since $n = 1$ is an odd number, then $x_0 = 0$ is not a local extreme point of f.

(h) $f(x) = e^{x^2} \cos(x)$

$$f^{'}(x) = e^{x^2} 2x \cos(x) + -e^{x^2} \sin(x) = e^{x^2}(2x \cos(x) - \sin(x))$$
$$\Rightarrow f^{'}(0) = 0$$
$$f^{(2)}(x) = e^{x^2} 2x(2x \cos(x) - \sin(x)) + e^{x^2}(2 \cos(x) - 2x \sin(x) - \cos(x))$$
$$\Rightarrow f^{(2)}(0) = 0 + 1 \times (2 - 0 - 1) = 1$$

Since $n = 2$ is an even number and $f^{(n)}(0) > 0$, x_0 is a local minimum of f.

2.5: Problem 16

Find an upper bound for the magnitude of the error in the approximation.

(a) $\sin(x) \approx x$, for $|x| < \frac{\pi}{20}$.

(b) $\sqrt{1 + x} \approx 1 + \frac{x}{2}$, for $|x| < \frac{1}{8}$.

(d) $\log(x) \approx (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$, for $|x - 1| < \frac{1}{64}$.

Answer

In all sections, we are asked to approximate the function with some polynomials. Hence, we will be using Taylor polynomials for such approximation. We will propose x_0 and n, which we will use the n^{th} Taylor polynomial to approximate the given function. Finally, we will propose a remainder for such approximation, which can be written in the following form based on Theorem 2.5.4 (Taylor's Theorem) [1]:

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

Finally, to suggest an upper bound for the magnitude of such error, we will find its maximum absolute value within the given neighborhood of x_0.

(a) $n = 1, x_0 = 0$. The 1^{st} Taylor's polynomial implies:

$$\begin{aligned} T_1(x) &= f(x_0) + \frac{f^{'}(x_0)}{1!}(x - x_0) \\ \Rightarrow T_1(x) &= \sin(x_0) + \frac{\cos(x_0)}{1!}(x - x_0) \\ &= T_1(x) = x \end{aligned}$$

Hence

$$\begin{aligned} R_1(x) = f(x) - T_1(x) = \sin x - x &= \frac{f^{(2)}(c)}{(2)!}(x - x_0)^2, \\ &- \frac{\sin(c)}{(2)!}(x)^2 \end{aligned}$$

$R_1(x)$ is a quadratic function which will take its maximum value where the coefficient is the largest positive value with $c \in (-\frac{\pi}{20}, \frac{\pi}{20})$. Since $\sin(c)$ is increasing in $(-\frac{\pi}{20}, \frac{\pi}{20})$, thus $-\frac{\sin(c)}{(2)!}$ is decreasing in this interval at takes its maximum at the lower bound of the interval which is $c = -\frac{\pi}{20}$. Hence:

$$R_1(x) = -\frac{\sin(-\frac{\pi}{20})}{(2)!}(-\frac{\pi}{20})^2 = 0.00193$$

(b) $n = 1, x_0 = 0$. The 1^{st} Taylor's polynomial implies:

$$\begin{aligned} T_1(x) &= f(x_0) + \frac{f^{'}(x_0)}{1!}(x - x_0) \\ \Rightarrow T_1(x) &= \sqrt{1+x} + \frac{\frac{1}{\sqrt{1+x}}}{1!}(x) = \sqrt{1+x} + \frac{x}{\sqrt{1+x}} \end{aligned}$$

Hence

$$\begin{aligned} R_1(x) = f(x) - T_1(x) &= \frac{f^{(2)}(c)}{(2)!}(x)^2 \\ &= -\frac{\frac{1}{\sqrt{(1+c)^3}}}{4 \times (2)!}(c)^2 \\ &= -\frac{1}{8\sqrt{(1+c)^3}}(c)^2 \end{aligned}$$

Except $x_0 = 0$, $R_1(x)$ will take negative values in its domain since both $\sqrt{(1+x)^3}$ and $(x)^2$ are positive and the coefficient is negative. Hence, $R_1(x)$ takes its local maxima in $(-\frac{1}{8}, \frac{1}{8})$ at $x_0 = 0$. However, $R_1(x_0) = 0$. Therefore, the rest of the values that $R_1(x)$ takes in $(-\frac{1}{8}, \frac{1}{8})$ are negative. Thus, we have to look for the smallest value that $R_1(x)$ can take in this interval, which will result in the greatest absolute value. The general function of $-\frac{1}{8\sqrt{(1+x)^3}}(x)^2$ will go to $-\infty$ as $x \to -1$, however, this value degrades slower as $x \to \infty$. Hence, as $x \to \frac{-1}{8}$, $R_1(x)$ takes lower values than $x \to \frac{1}{8}$. Thus $R_1(x)$ takes its minimum at the lower bound of the interval, which is $c = \frac{-1}{8}$. Hence:

$$R_1(x) = -\frac{1}{8\sqrt{(1+\frac{-1}{8})^3}}(\frac{-1}{8})^2 = -0.00239$$

(d) $n = 3, x_0 = 1$. In this question, I'll assume the logarithm is with base e. The 3^{rd} Taylor's polynomial implies:

$$\begin{aligned}
&T_3(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 \\
\Rightarrow &T_3(x) = f(x_0) + \frac{1}{x_0}(x-1) + \frac{-1}{2 \times x_0^2}(x-1)^2 + \frac{1}{3 \times x_0^3}(x-1)^3 \\
\Rightarrow &T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3
\end{aligned}$$

Hence

$$\begin{aligned}
R_3(x) = f(x) - T_3(x) &= \frac{f^{(4)}(c)}{(4)!}(x-1)^4 \\
&= \frac{-6}{24 \times c^4}(x-1)^4 \\
&= -\frac{1}{4 \times c^4}(x-1)^4
\end{aligned}$$

$R_3(x)$ consists of $(x-1)^4$ and a negative coeeficient. $(x-1)^4$ Takes its global minimum at $x_0 = 1$. Hence, $R_3(x)$ takes its global maximum at $x_0 = 1$. However, $R_3(1) = 0$, which means at this point Taylor polynomial perfectly approximates f. To maximize $|\frac{1}{4\times c^4}(x-1)^4|$ we can maximize both $|\frac{1}{4\times c^4}|$ and $|(x-1)^4|$ by having $c = 1 - \frac{1}{64}$. Thus:

$$\begin{aligned}
R_3(x) &= -\frac{1}{4 \times (1 - \frac{1}{64})^4}(-\frac{1}{64})^4 \\
&= -1.587 \times 10^{-8}
\end{aligned}$$

References

[1] William F Trench. *Introduction to real analysis.* 2013.