MATH 414 Analysis I,
Homework 12

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3.3: Problem 13

Prove: If f is integrable and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$, with strict inequality if f is continuous and positive at some point in $[a, b]$.

Answer

We will first show if If f is integrable and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$. Based on theorem 3.3.5 [1] we know if f is integrable on $[a, b]$ then so is $|f|$ and:

$$|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$$

However, $f(x) \geq 0$ on $[a, b]$, hence $|f(x)| = f(x)$. Thus:

$$0 \leq |\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx = \int_a^b f(x)\,dx$$

And:

$$0 \leq \int_a^b f(x)\,dx.$$

Now, suppose f is continuous at $c \in [a, b]$ and $f(c) > 0$. Let $\epsilon = \frac{f(c)}{2} > 0$, then $\exists \delta > 0$ such that:

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

Whenever $|x-c| < \delta$. Hence $\forall x \in (c-\delta, c+\delta)$, we know:

$$\begin{aligned}
&|f(x) - f(c)| < \frac{f(c)}{2} \\
&\Rightarrow |f(c) - f(x)| < \frac{f(c)}{2} \\
&\Rightarrow |f(c)| - |f(x)| < \frac{f(c)}{2} \\
&\Rightarrow f(c) - |f(x)| < \frac{f(c)}{2} \quad \text{since } f(c) > 0 \\
&\Rightarrow \frac{f(c)}{2} < |f(x)| \quad \forall x \in (c-\delta, c+\delta) \\
&\Rightarrow \frac{f(c)}{2} < f(x) \quad \forall x \in (c-\delta, c+\delta) \quad \text{since } f(x) \geq 0
\end{aligned}$$

Now, define g as the following:

$$g(x) = \begin{cases} f(x), & \text{if } x \in (c-\delta, c+\delta); \\ 0, & \forall x \in [a, c-\delta] \bigcup [c+\delta, b]. \end{cases}$$

Hence:

$$g(x) \leq f(x) \quad \forall x \in [a,b] \quad \text{and} \quad 0 < g(x) \quad \forall x \in (c-\delta, c+\delta)$$

Therefore:

$$\int_a^b g(t)\,dt \leq \int_a^b f(t)\,dt \quad \text{and} \quad 0 = \int_{c-\delta}^{c+\delta} 0\,dt < \int_{c-\delta}^{c+\delta} g(t)\,dt$$

We can write $\int_a^b g(t)\,dt$ as the following:

$$\begin{aligned}
\int_a^b g(t)\,dt &= \int_a^{c-\delta} g(t)\,dt + \int_{c-\delta}^{c+\delta} g(t)\,dt + \int_{c+\delta}^{b} g(t)\,dt \\
&= \int_a^{c-\delta} 0\,dt + \int_{c-\delta}^{c+\delta} g(t)\,dt + \int_{c+\delta}^{b} 0\,dt \\
&= 0 + \int_{c-\delta}^{c+\delta} g(t)\,dt + 0
\end{aligned}$$

Thus:

$$0 < \int_a^b g(t)\,dt \Rightarrow 0 < \int_a^b f(t)\,dt$$

3.3: Problem 18

(a) Let $f^{(n+1)}$ be integrable on $[a,b]$. Show that

$$f(b) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt.$$

HINT: Integrate by parts and use induction.

(b) What is the connection between (a) and Theorem 2.5.5?

Answer

(a) Prove by Induction:

1. **Basis**, $n = 0$:

$$\begin{aligned}
\text{LHS:}\quad & f(b) \\
\text{RHS:}\quad & \sum_{r=0}^{0} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{0!}\int_a^b f^{(1)}(t)(b-t)^0\,dt. \\
& = f(a) \times 1 + 1 \times [f(t)]_a^b \\
& = f(a) + f(b) - f(a) \\
& = f(b) \\
& = \text{RHS.}
\end{aligned}$$

2. **Induction Assumption:**
Suppose $n-1$ we know that:

$$f(b) = \sum_{r=0}^{n-1} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{n-1!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt.$$

3. **Induction Step:**
Knowing that $f^{(n+1)}$ exists and is integrable on $[a,b]$ we would like to prove for n:

$$f(b) = \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt.$$

Equivalently, we have to show RHS is equal to $f(b)$.
We use integration by parts, which is defined as the following:

$$\int u\,dv = uv - \int v\,du$$

Choose:

$$u = (b-t)^n \quad \text{and} \quad dv = f^{(n+1)}(t)dt$$

Then, we have:

$$du = -n(b-t)^{n-1}dt \quad \text{and} \quad v = f^{(n)}(t)$$

By applying the integration by parts formula, we have:

$$\int_a^b f^{(n+1)}(t)(b-t)^n\,dt = \Big[f^{(n)}(t)(b-t)^n\Big]_a^b - \int_a^b f^{(n)}(t)\cdot\left(-n(b-t)^{n-1}\right)dt$$

Which can be simplified to the following:

$$\int_a^b f^{(n+1)}(t)(b-t)^n\,dt = \Big[f^{(n)}(t)(b-t)^n\Big]_a^b + n\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt$$

Hence:

$$\sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt$$
$$= \sum_{r=0}^{n-1} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\left[f^{(n)}(t)(b-t)^n\right]_a^b + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt$$
$$= \sum_{r=0}^{n-1} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\left[f^{(n)}(t)(b-t)^n\right]_a^b$$

Based on the induction hypothesis $\sum_{r=0}^{n-1} \frac{f^{(r)}(a)}{r!}(b-a)^r + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(b-t)^{n-1}\,dt$ is equal to $f(b)$. Thus, we have to show $\frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\left[f^{(n)}(t)(b-t)^n\right]_a^b$ is equal to 0.

$$\begin{aligned}\frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\left[f^{(n)}(t)(b-t)^n\right]_a^b &= \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{n!}\left(f^{(n)}(b)(b-b)^n - f^{(n)}(a)(b-a)^n\right)\\ &= \frac{f^{(n)}(a)}{n!}(b-a)^n - \frac{f^{(n)}(a)(b-a)^n}{n!}\\ &= 0\end{aligned}$$

(b) Let's recall Theorem 2.5.5:

Theorem 2.5.5 (Extended Mean Value Theorem)

Suppose that f is continuous on a finite closed interval I with endpoints a and b (that is, either $I = (a, b)$ or $I = (b, a)$), $f^{(n+1)}$ exists on the open interval I^0, and, if $n > 0$, that $f', \dots, f^{(n)}$ exist and are continuous at a. Then

$$f(b) - \sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(b-a)^r = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c in I^0. The given formula and Theorem 2.5.5 are closely related as they both expand a function f around a point a using derivatives up to order n. Both involve the idea of approximating $f(b)$ by a Taylor series-like expansion and express the remainder term in a different but equivalent way.

Both (a) and Theorem 2.5.5 include a sum of terms involving the derivatives of f at a:

$$\sum_{r=0}^{n} \frac{f^{(r)}(a)}{r!}(b-a)^r.$$

This represents the Taylor series expansion of f around a up to order n.

On the other hand, Theorem 2.5.5 represents the remainder term as:

$$R = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

where $c \in (a, b)$.

While (a) expresses the remainder term as an integral:

$$R = \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,dt.$$

This is the integral form of the remainder, which is a weighted average of $f^{(n+1)}(t)$ over the interval $[a, b]$. These two values of the remainder are equivalent. However, the integral form distributes the remainder over the interval $[a, b]$, while Theorem 2.5.5 forms it with a single evaluation at some unknown $c \in (a, b)$. Since c is generally unknown, (a) gives a way to determine this value as stated in Theorem 2.5.5.

3.4: Problem 4

Find all values of p for which the following integrals exist **(i)** as proper integrals (perhaps after defining f at the endpoints of the interval) or **(ii)** as improper integrals. **(iii)** Evaluate the integrals for the values of p for which they converge.

(c) $\int_0^\infty e^{-px} dx$

(d) $\int_0^1 x^{-p} dx$

(e) $\int_0^\infty x^{-p} dx$

Answer

(c) Fix $p \in \mathbb{R}$ and consider the function

$$f(x) = e^{-px}$$

is locally integrable on $(0, \infty)$. To see whether

$$I = \int_0^\infty e^{-px} dx$$

converges according to Definition 3.4.3, we consider the improper integrals

$$I_1 = \int_0^1 e^{-px}\, dx \quad \text{and} \quad I_2 = \int_1^\infty e^{-px}\, dx$$

separately.
We know:

$$\int_0^1 e^{-px}\, dx = \begin{cases} \frac{1-e^{-p}}{p}, & p \neq 0, \\ 1, & p = 0. \end{cases}$$

Hence, for $p \neq 0$:

$$\int_1^\infty e^{-px}\, dx = \lim_{c\to\infty} \int_1^c e^{-px} dx = \lim_{c\to\infty} \left[\frac{1 - e^{-px}}{p} \right]_1^c = \begin{cases} \frac{e^{-p}}{p}, & p > 0, \\ -\infty, & p < 0. \end{cases}$$

Therefore:

$$I = \int_0^\infty e^{-px} dx = \begin{cases} \frac{1}{p}, & p > 0, \\ 1, & p = 0, \\ -\infty, & p < 0. \end{cases}$$

(d) Fix $p \in \mathbb{R}$ and consider the function

$$f(x) = x^{-p}$$

The proper integral is when f is defined on a closed interval of $[0,1]$, and the improper integral is when f is locally integrable on $[0,1)$. First, we will be calculating the proper integral with the assumption that f is defined on a closed interval of $[0,1]$. Hence, for $p \neq 0, 1$:

$$\int_0^1 x^{-p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_0^1 = \frac{1}{1-p}$$

And for $p = 1$ and by using improper integral:

$$\int_0^1 x^{-1} dx = [\ln(x)]_0^1 = \lim_{N \to 0^+} \int_N^1 x^{-1} dx = \lim_{N \to 0^+} [\ln(x)]_N^1 = \lim_{N \to 0^+} \ln(1) - \lim_{N \to 0^+} \ln(|N|) = 0 - (-\infty) = \infty$$

Therefore:

$$\int_0^1 x^{-p} dx = \begin{cases} \frac{1}{1-p}, & p \neq 0, 1, \\ \infty, & p = 1, \\ 1, & p = 0. \end{cases}$$

Now, suppose f is locally integrable on $[0,1)$. Hence, for $p \neq 0, 1$:

$$\lim_{c \to 1^-} \int_0^c x^{-p} dx = \lim_{c \to 1^-} \left[\frac{x^{-p+1}}{-p+1} \right]_0^c = \lim_{c \to 1^-} \left(\frac{c^{-p+1}}{-p+1} - 0 \right) = \lim_{c \to 1^-} \frac{c^{-p+1}}{-p+1} = \frac{1}{-p+1}$$

And for $p = 1$:

$$\lim_{c \to 1^-} \int_0^c x^{-p} dx = \lim_{c \to 1^-} [\ln(x)]_0^c = \lim_{c \to 1^-} (\ln(c) - \ln(0)) = \lim_{c \to 1^-} \ln(c) - \lim_{c \to 1^-} \ln(0) = \infty$$

Therefore:

$$\lim_{c \to 1^-} \int_0^c x^{-p} dx = \begin{cases} \frac{1}{1-p}, & p \neq 0, 1, \\ \infty, & p = 1, \\ 1, & p = 0. \end{cases}$$

(e) $\int_0^\infty x^{-p} dx$. Fix $p \in \mathbb{R}$ and consider the function

$$f(x) = x^{-p}$$

is locally integrable on $(0, \infty)$. To see whether

$$I = \int_0^\infty x^{-p} dx$$

converges according to Definition 3.4.3, we consider the improper integrals

$$I_1 = \int_0^1 x^{-p}\, dx \quad \text{and} \quad I_2 = \int_1^\infty x^{-p}\, dx$$

separately. Based on (d), we know:

$$\lim_{c \to 1^-} \int_0^c x^{-p} dx = \begin{cases} \frac{1}{1-p}, & p \neq 0, 1, \\ \infty, & p = 1, \\ 1, & p = 0. \end{cases}$$

Now, for $p \neq 0, 1$:

$$\int_1^\infty x^{-p}\, dx = \lim_{c\to\infty} \int_1^c x^{-p} dx = \lim_{c\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^c = \begin{cases} \frac{1}{p-1}, & p > 1, \\ \infty, & p < 1. \end{cases}$$

Therefore:

$$I = \int_0^\infty x^{-p} dx = \begin{cases} \frac{1}{1-p}, & p > 1, \\ 1, & p = 0, \\ \infty, & p \leq 1. \end{cases}$$

3.4: Problem 5

5. Evaluate

(b) $\int_0^\infty e^{-x} \sin x\, dx$

(c) $\int_{-\infty}^\infty \frac{x\, dx}{x^2+1}$

Answer

(b) We use integration by parts, which is defined as the following:

$$\int u\, dv = uv - \int v\, du$$

By choosing:

$$u = e^{-x} \quad \text{and} \quad dv = \sin x\, dx$$

Then, we have:

$$du = -e^{-x} dx \quad \text{and} \quad v = -\cos x$$

Thus:

$$\begin{aligned} \int_0^\infty e^{-x} \sin x\, dx &= -e^{-x} \cos x - \int_0^\infty -e^{-x} - \cos x dx \\ &= -e^{-x} \cos x - \int_0^\infty e^{-x} \cos x dx \end{aligned}$$

We will be using integration by parts on $\int_0^\infty e^{-x} \cos x dx$ again, by choosing:

$$u = e^{-x} \quad \text{and} \quad dv = \cos x\, dx$$

Then, we have:

$$du = -e^{-x} dx \quad \text{and} \quad v = \sin x$$

Thus:

$$\begin{aligned}
\int_0^\infty e^{-x}\sin x\,dx &= -e^{-x}\cos x - \int_0^\infty -e^{-x} - \cos x dx \\
&= -e^{-x}\cos x - \int_0^\infty e^{-x}\cos x dx \\
&= -e^{-x}\cos x - \left(e^{-x}\sin x - \int_0^\infty -e^{-x}\sin x dx\right) \\
&= -e^{-x}\cos x - e^{-x}\sin x - \int_0^\infty e^{-x}\sin x dx \\
\Rightarrow 2\int_0^\infty e^{-x}\sin x\,dx &= -e^{-x}\cos x - e^{-x}\sin x
\end{aligned}$$

Hence:

$$\begin{aligned}
2\int_0^\infty e^{-x}\sin x\,dx &= 2\lim_{c\to\infty}\int_0^c e^{-x}\sin x\,dx \\
&= \lim_{c\to\infty}\left[-e^{-x}\cos x - e^{-x}\sin x\right]_0^c \\
&= \lim_{c\to\infty}\left[-e^{-c}\cos c - e^{-c}\sin c\right] - \lim_{c\to\infty}\left[-e^{-0}\cos 0 - e^{-0}\sin 0\right] \\
&= \lim_{c\to\infty} -e^{-c}(\cos c + \sin c) - \lim_{c\to\infty} -1 \\
&= \lim_{c\to\infty} -e^{-c}(\cos c + \sin c) + 1 = 0 + 1 \\
&= 1 \\
&\Rightarrow \int_0^\infty e^{-x}\sin x\,dx = \frac{1}{2}
\end{aligned}$$

(c) The function

$$f(x) = \frac{x}{x^2+1}$$

is locally integrable on $(0,\infty)$ and $(-\infty,0)$. To see whether

$$I = \int_{-\infty}^\infty \frac{x\,dx}{x^2+1}$$

converges according to Definition 3.4.3, we consider the improper integrals

$$I_1 = \int_{-\infty}^0 \frac{x\,dx}{x^2+1}\,dx \quad \text{and} \quad I_2 = \int_0^\infty \frac{x\,dx}{x^2+1}\,dx$$

separately.

For I_1, we can write:

$$\begin{aligned}
\int_{-\infty}^0 \frac{x\,dx}{x^2+1}\,dx &= \lim_{c\to-\infty}\int_c^0 \frac{x\,dx}{x^2+1}\,dx \\
&= \lim_{c\to-\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_c^0 \\
&= \frac{1}{2}\lim_{c\to-\infty}\big(\ln(1) - \ln(c^2+1)\big) \\
&= -\infty
\end{aligned}$$

For I_2, we can write:

$$\begin{aligned}
\int_0^\infty \frac{x\,dx}{x^2+1}\,dx &= \lim_{c\to\infty}\int_0^c \frac{x\,dx}{x^2+1}\,dx \\
&= \lim_{c\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_0^c \\
&= \frac{1}{2}\lim_{c\to\infty}\big(\ln(c^2+1)-\ln(1)\big) \\
&= \infty
\end{aligned}$$

Since both $\int_0^\infty \frac{x\,dx}{x^2+1}\,dx$ and $\int_{-\infty}^0 \frac{x\,dx}{x^2+1}\,dx$ diverge, so does $\int_{-\infty}^\infty \frac{x\,dx}{x^2+1}$.

3.4: Problem 6

Prove: If $\int_a^b f(x)\,dx$ exists as a proper or improper integral, then

$$\lim_{x\to b^-}\int_x^b f(t)\,dt = 0.$$

Answer

We will analyze the answer for the existence of proper and improper integral of $\int_a^b f(x)\,dx$ separately.

- Proper integral: is one where the function f is continuous on the interval $[a,b]$ except countably many points, and the limit of integration is finite. Hence, the following limit exists $\lim_{x\to b^-}\int_a^x f(t)\,dt$, and based on the definition 3.4.1 in [1]:

$$\lim_{x\to b^-}\int_a^x f(t)\,dt = \int_a^b f(t)\,dt.$$

  Hence:

$$\begin{aligned}
&\lim_{x\to b^-}\left(\int_a^x f(t)dt + \int_x^b f(t)dt\right) = \lim_{x\to b^-}\int_a^b f(t)dt \\
&\Rightarrow \lim_{x\to b^-}\int_a^x f(t)dt + \lim_{x\to b^-}\int_x^b f(t)dt = \int_a^b f(t)dt \\
&\Rightarrow \lim_{x\to b^-}\int_x^b f(t)dt = 0
\end{aligned}$$

- Improper integral: is one where the function f is locally integrable on (a,b), however whether $b=\infty$ or $b<\infty$ and f is unbounded as x approaches b from the left. Now, since $\int_a^b f(x)\,dx$ exists as an improper integral, it converges to the following:

$$\int_a^b f(t)\,dt = \lim_{x\to b^-}\int_a^x f(t)\,dt.$$

In addition, since f is locally integrable on (a, b) we define:

$$\lim_{x \to b^-} \left(\int_a^x f(t)dt + \int_x^b f(t)dt \right) = \lim_{x \to b^-} \int_a^b f(t)dt$$
$$\Rightarrow \lim_{x \to b^-} \int_a^x f(t)dt + \lim_{x \to b^-} \int_x^b f(t)dt = \int_a^b f(t)dt$$
$$\Rightarrow \int_a^b f(t)dt + \lim_{x \to b^-} \int_x^b f(t)dt = \int_a^b f(t)dt$$
$$\Rightarrow \lim_{x \to b^-} \int_x^b f(t)dt = 0$$

Please note that this question was solved with respect to the definition of the existence proper and improper integral of f stated in [1].

References

[1] William F Trench. *Introduction to real analysis.* 2013.