MATH 414 Analysis I, Homework 3

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1 Section 1.3: Problem 16

(a) Prove: If S is bounded above and $\sup S = \beta$, then $\beta \in \delta S$.

(b) State the analogous result for a set bounded below.

Answer

(a) To show $\beta \in \delta S$ we have to show for $\forall \epsilon > 0$, $\epsilon-$neighborhood of β calling it N_ϵ:

$$N_\epsilon \cap S \neq \emptyset \text{ and } N_\epsilon \cap S^c \neq \emptyset$$

There will be two possible situations:

(1) $\beta \in S$: then $\forall \epsilon > 0$ if we consider the open set $(\beta, \beta + \epsilon)$, this has to be a subset of S^c because if this is not a case, and $\exists x_0 \in (\beta, \beta + \epsilon)$, also $x_0 \in S$, then:

$$x_0 \in S \text{ and } x_0 \in (\beta, \beta + \epsilon) \Rightarrow \beta < x_0 \in S$$

which is not possible because β is the $\sup S$. Therefor, for any $\epsilon-$neighborhood of β which is $(\beta - \epsilon, \beta + \epsilon)$ we know:

$$\beta \in N_\epsilon \cap S \text{ and } (\beta, \beta + \epsilon) \in N_\epsilon \cap S^c$$

Therefor β is a boundary point, and $\beta \in \delta S$

(2) $\beta \notin S$: this means $\beta \in S^c$. Therefor, to show $\beta \in \delta S$, we have to show for any $\epsilon-$neighborhood of β calling it N_ϵ:

$$N_\epsilon \cap S \neq \emptyset$$

Consider $(\beta - \epsilon, \beta)$, then $\forall \epsilon > 0$ the following must hold:

$$(\beta - \epsilon, \beta) \cap S \neq \emptyset$$

because if $\exists \epsilon_0 > 0$ such that:

$$(\beta - \epsilon_0, \beta) \cap S = \emptyset$$

then

$$\forall x_0 \in S : x_0 \leq \beta - \epsilon_0$$

which means $\beta - \epsilon_0$ is a smaller upper bound than β for S which can't be happening, since β is the $\sup S$. Therefor, for any $\epsilon-$neighborhood of β, we have:

$$\beta \in N_\epsilon \cap S^c \text{ and } N_\epsilon \cap S \neq \emptyset$$

Therefor β is a boundary point, and $\beta \in \delta S$.

(b) We have to show, if S is bounded below and $\inf S = \alpha$, then $\alpha \in \delta S$. Which means we have to show for $\forall \epsilon > 0$, $\epsilon-$neighborhood of α calling it N_ϵ:

$$N_\epsilon \cap S \neq \emptyset \text{ and } N_\epsilon \cap S^c \neq \emptyset$$

There will be two possible situations:

(1) $\alpha \in S$: then $\forall \epsilon > 0$ if we consider the open set $(\alpha - \epsilon, \alpha)$, this has to be a subset of S^c because if this is not a case, and $\exists x_0 \in (\alpha - \epsilon, \alpha)$, also $x_0 \in S$, then:

$$x_0 \in S \text{ and } x_0 \in (\alpha - \epsilon, \alpha) \Rightarrow x_0 < \alpha \in S$$

which is not possible because α is the $\inf S$. Therefor, for any $\epsilon-$neighborhood of α which is $(\alpha - \epsilon, \alpha + \epsilon)$ we know:

$$\alpha \in N_\epsilon \cap S \text{ and } (\alpha - \epsilon, \alpha) \in N_\epsilon \cap S^c$$

Therefor α is a boundary point, and $\alpha \in \delta S$

(2) $\alpha \notin S$: this means $\alpha \in S^c$. Hence, to show $\alpha \in \delta S$, we have to show for any $\epsilon-$neighborhood of α calling it N_ϵ:

$$N_\epsilon \cap S \neq \emptyset$$

Consider $(\alpha, \alpha + \epsilon)$, then $\forall \epsilon > 0$ the following must hold:

$$(\alpha, \alpha + \epsilon) \cap S \neq \emptyset$$

because if $\exists \epsilon_0 > 0$ such that:

$$(\alpha, \alpha + \epsilon_0) \cap S = \emptyset$$

then

$$\forall x \in S : \alpha + \epsilon_0 \leq x$$

which means $\alpha + \epsilon_0$ is a greater lower bound than α for S which can't be happening, since α is the $\inf S$. Therefor, for any $\epsilon-$neighborhood of α, we have:

$$\alpha \in N_\epsilon \cap S^c \text{ and } N_\epsilon \cap S \neq \emptyset$$

Therefor α is a boundary point, and $\alpha \in \delta S$.

2 Section 1.3: Problem 17

Prove: If S is closed and bounded, then $\inf S$ and $\sup S$ are both in S.

Answer

(a) If S is closed and bounded, then $\sup S$ is in S. To prove this, we will first show $\sup S$ is a limit point of S, and since S is closed, it contains all of its limit points, therefor it contains $\sup S$. Assume, $\sup S = \beta$ and $\forall \epsilon > 0$, $\epsilon-$neighborhood of β is called N_ϵ. In addition, consider the following:

$$N_\epsilon{}^- = (\beta - \epsilon, \beta)$$

If we only show:

$$N_\epsilon{}^- \cap S \neq \emptyset \quad \forall \epsilon$$

Then we have shown that any deleted neighborhood of β, contains a point in S, which means β is a limit point of S.

Prove by Contradiction

Assume the opposite, which means $\exists \epsilon_0 > 0$ such that:

$$N_{\epsilon_0}{}^- \cap S = \emptyset$$

then

$$\forall x_0 \in S : x_0 \leq \beta - \epsilon_0$$

which means $\beta - \epsilon_0$ is a smaller upper bound than β for S which can't be happening, since β is the $\sup S$. There for $\forall \epsilon > 0$:

$$N_{\epsilon_0}{}^- \cap S \neq \emptyset$$

Which means β is a limit point of S.

(b) If S is closed and bounded, then $\inf S$ is in S. Same as (a) to prove this, we will first show $\inf S$ is a limit point of S, and since S is closed, it contains all of its limit points, therefor it contains $\inf S$. Assume, $\inf S = \alpha$ and $\forall \epsilon > 0$, $\epsilon-$neighborhood of α is called N_ϵ. In addition, consider the following:

$$N_\epsilon{}^+ = (\alpha, \alpha + \epsilon)$$

If we only show:

$$N_\epsilon{}^+ \cap S \neq \emptyset \quad \forall \epsilon$$

Then we have shown that any deleted neighborhood of α, contains a point in S, which means α is a limit point of S.

Prove by Contradiction

Assume the opposite, which means $\exists \epsilon_0 > 0$ such that:

$$N_{\epsilon_0}{}^+ \cap S = \emptyset$$

then

$$\forall x_0 \in S : x_0 \geq \alpha + \epsilon_0$$

which means $\alpha + \epsilon_0$ is a greater lower bound than α for S which can't be happening, since α is the $\inf S$. There for $\forall \epsilon > 0$:

$$N_{\epsilon_0}{}^+ \cap S \neq \emptyset$$

Which means α is a limit point of S.

3 Section 1.3: Problem 18

If a nonempty subset S of $\mathbb{R}$ is both open and closed, then $S = \mathbb{R}$.

Answer

We will show that S is unbounded, and since it's not empty, the only subset of $\mathbb{R}$ that's unbounded is $\mathbb{R}$.

Prove by Contradiction

Assume the opposite. Therefore, S is bounded from at least one side, whether above or below. We will assume S is at least bounded from above, and the procedure would be the same if we assume S is bounded below. Since the S is bounded above, then it has a $\sup S$. Since S is closed, $\sup S \in S$ (we showed in problem 17 of [1]). In addition, since S is open, then $S = S^0$, which means $\forall x \in S$, there's a neighborhood of x, which is completely in S. Therefore. $\exists \epsilon_0 > 0$ such that:

$$(\sup S - \epsilon_0, \sup S + \epsilon_0) \subset S$$

In particular:

$$(\sup S, \sup S + \epsilon_0) \subset S$$

However, each point in $(\sup S, \sup S + \epsilon_0)$ is greater than $\sup S$, which contradicts the assumption of S being bounded above and having a $\sup S$.

The same argument holds with S being unbounded below. Hence, $\emptyset \neq S$ is unbounded, and $S = \mathbb{R}$.

4 Section 1.3: Problem 19

Let S be an arbitrary set. Prove:

(a) δS is closed.

(b) S^0 is open.

(c) The exterior of S is open.

(d) The limit points of S form a closed set.

(e) $\overline{(\overline{S})} = \overline{S}$.

Answer

(a) To prove δS is closed, we will show $(\delta S)^c$ is open, which is equivalent to:

$$\forall x \in (\delta S)^c : \quad \exists \epsilon > 0 \text{ such that: } (x - \epsilon, x + \epsilon) \subset (\delta S)^c$$

Let $x_0 \in (\delta S)^c$. This means x_0 is not a boundary point for S, which leads to two possibilities:

(1) $\exists \epsilon_0 > 0$ such that: $N_{\epsilon_0} = (x_0 - \epsilon_0, x_0 + \epsilon_0) \subset S$. We will show:

$$N_{\epsilon_0} \cap \delta S = \emptyset \Rightarrow N_{\epsilon_0} \subset (\delta S)^c$$

Prove by Contradiction

Suppose:

$$N_{\epsilon_0} \cap \delta S \neq \emptyset \Rightarrow \exists x_1 \in N_{\epsilon_0} \cap \delta S$$

Therefor x_1 is a boundary point which means any neighborhood of x_1 contains a point in S^c. Now consider the following:

$$\epsilon_1 := \min\{|x_1 - (x_0 - \epsilon_0)|, |x_0 + \epsilon_0 - x_1|\}$$

Then:

$$N_{\epsilon_1} = (x_1 - \epsilon_1, x_1 + \epsilon_1) \subset N_{\epsilon_0} = (x_0 - \epsilon_0, x_0 + \epsilon_0) \subset S$$

So, N_{ϵ_1} is a neighborhood of x_1 that contains no point in S^c which means x_1 cannot be a boundary point for S, which is a contradiction. Thus, $N_{\epsilon_0} \subset (\delta S)^c$.

(2) $\exists \epsilon_0 > 0$ such that: $N_{\epsilon_0} = (x_0 - \epsilon_0, x_0 + \epsilon_0) \subset S^c$. We will show:

$$N_{\epsilon_0} \cap \delta S = \emptyset \Rightarrow N_{\epsilon_0} \subset (\delta S)^c$$

Prove by Contradiction

Suppose:

$$N_{\epsilon_0} \cap \delta S \neq \emptyset \Rightarrow \exists x_1 \in N_{\epsilon_0} \cap \delta S$$

Therefor x_1 is a boundary point which means any neighborhood of x_1 contains a point in S. Now consider the following:

$$\epsilon_1 := \min\{|x_1 - (x_0 - \epsilon_0)|, |x_0 + \epsilon_0 - x_1|\}$$

Then:

$$N_{\epsilon_1} = (x_1 - \epsilon_1, x_1 + \epsilon_1) \subset N_{\epsilon_0} = (x_0 - \epsilon_0, x_0 + \epsilon_0) \subset S^c$$

So, N_{ϵ_1} is a neighborhood of x_1 that contains no point in S which means x_1 cannot be a boundary point for S, which is a contradiction. Thus, $N_{\epsilon_0} \subset (\delta S)^c$.

(b) To show S^0 is open, $\forall x \in S^0$ we will propose an $\epsilon-$neighborhood of x which is completely in S^0. Let $x_0 \in S^0$ be an arbitrary point. This means, $\exists \epsilon_0$ such that:

$$(x_0 - \epsilon_0, x_0 + \epsilon_0) \subseteq S$$

Now, take:

$$x_1 \in (x_0 - \epsilon_0, x_0 + \epsilon_0), \text{ and } \epsilon_1 := \min\{|x_1 - (x_0 - \epsilon_0)|, |x_0 + \epsilon_0 - x_1|\}$$

Then:

$$(x_1 - \epsilon_1, x_1 + \epsilon_1) \subset (x_0 - \epsilon_0, x_0 + \epsilon_0)$$

Therefore:

$$(x_1 - \epsilon_1, x_1 + \epsilon_1) \subseteq S$$

Which means x_1 is also an interior point of S, and is in S^0. Since x_1 was an arbitrary point, we can follow the same procedure for any point in $(x_0 - \epsilon_0, x_0 + \epsilon_0)$ and arguing that they're in S^0. Therefore:

$$(x_0 - \epsilon_0, x_0 + \epsilon_0) \subseteq S^0$$

Hence, x_0 is also an interior point of S^0 since there exists the ϵ_0-neighborhood of x_0 which is completely in S^0.

(c) The exterior of S is the interior of S^c. Based on (b) we have shown that for any arbitrary set such as S^c, $(S^c)^0$ is open, which is equivalent to the exterior of S being open.

(d) Assume the following [1]:

$$L := \{x \in \mathbb{R} |\text{ x is limit point of } S\}$$

To show L is closed, we have show L^c is open, which means:

$$\forall x \in L^c, \exists \epsilon > 0 \text{ such that: } (x - \epsilon, x + \epsilon) \subset L^c$$

Let $x_0 \in L^c$. Since x_0 is not a limit point, then $\exists \epsilon_0 > 0$ such that $N_{\epsilon_0} := (x_0 - \epsilon_0, x_0 + \epsilon_0) \cap S = \emptyset$. To show N_{ϵ_0} is the ϵ-neighborhood of x_0 that is fully is L^c, we have to show there can't be any other limit point of S in N_{ϵ_0}. We will prove this by contradiction.

Prove by Contradiction

Suppose:

$$N_{\epsilon_0} \cap L \neq \emptyset \Rightarrow \exists x_1 \in N_{\epsilon_0} \cap L$$

Therefore x_1 is a limit point which means any deleted-neighborhood of x_1 contains a point in S. Now consider the following:

$$\epsilon_1 := \min\{|x_1 - (x_0 - \epsilon_0)|, |x_0 + \epsilon_0 - x_1|\}$$

Then:

$$N_{\epsilon_1} = (x_1 - \epsilon_1, x_1 + \epsilon_1) \backslash x_1 \cap S \neq \emptyset$$

However:

$$N_{\epsilon_1} \subset N_{\epsilon_0}$$

Which means:

$$N_{\epsilon_1} \cap S \subset N_{\epsilon_0}$$
$$\Rightarrow N_{\epsilon_0} \cap S \neq \emptyset$$

Which is a contradiction. Thus, $N_{\epsilon_0} \subset L^c$.

(e) To show the following, we have to show:

$$\overline{(\overline{S})} \subset \overline{S} \text{ and } \overline{S} \subset \overline{(\overline{S})}$$

1 $\overline{(\overline{S})} \subset \overline{S}$

Let $x \in \overline{(\overline{S})}$. By definition, we know: $\overline{(\overline{S})} = \overline{S} + \delta\overline{S}$. Thus, at least one of the following will happen:

1.1 $x \in \overline{S}$: If this is the case then we took an element from $\overline{(\overline{S})}$ and is also in $\overline{S}$ which leads to $\overline{(\overline{S})} \subset \overline{S}$.

1.2 $x \in \delta\overline{S}$ and $x \notin \overline{S}$: Since $x \notin \overline{S}$ and it's also a boundary point, then:

$$N_\epsilon \backslash x \cap \overline{S} \neq \emptyset$$

Thus, x is a limit point for $\overline{S}$. We will show that $\overline{S}$ contains all of its limit points.

[1] The naming and the procedure might be the same with many other students because, during Friday's lecture, Dr. Gelaki could not teach due to technical issues, so he solved some homework problems with the whole class.

Proof by contradiction:

Let $\overline{S}$ be not closed, i.e. $\exists x$ such that it is a limit point of $\overline{S}$ but $x \notin \overline{S}$. Thus,

$$N_\epsilon \cap \overline{S} \neq \emptyset \quad \forall \epsilon > 0$$

Then:

$$N_\epsilon \cap \overline{S} \setminus \{x\} = (N_\epsilon \cap (S \cup \delta S)) \setminus \{x\} = ((N_\epsilon \cap S) \cup (N_\epsilon \cap \delta S)) \setminus \{x\}$$

$$= ((N_\epsilon \cap S) \setminus \{x\}) \cup (N_\epsilon \cap \delta S) \setminus \{x\}) \neq \emptyset \quad \forall \epsilon > 0$$

Two possible event might occur:
1. $N_\epsilon \cap S \setminus \{x\} \neq \emptyset$: then it would mean that x is a limit point of S, i.e. $x \in \delta S$ and therefore, $x \in \overline{S}$.
2. If $N_\epsilon \cap \delta S \setminus \{x\} \neq \emptyset$, then it would mean that x is a limit point of δS. Now, we need to show that any limit point of δS is also a limit point of S.
Let x be a point that is a limit point of δS, but not a limit point of S. Then,

$$\exists \epsilon_0 > 0 : N_{\epsilon_0} \cap \delta S \neq \emptyset \wedge (N_{\epsilon_0} \cap S = \emptyset)$$

Let $y \in (N_{\epsilon_0} \cap \delta S)$, and $0 < d(x, y) = h < q$. Then, for any $0 < \epsilon < q - h$, we must have $N_\epsilon(y) \cap S = \emptyset$, since $N_\epsilon(y) \subset N_{\epsilon_0}$. This contradicts the fact that y is a limit point of S.
Therefore, any limit point of δS is also a limit point of S, i.e. it also belongs to δS and therefore belongs to $\overline{S}$.

2 $\overline{S} \subset \overline{(\overline{S})}$: since $\overline{(\overline{S})} = \overline{S} + \delta\overline{S}$ which means $\overline{(\overline{S})} = \overline{S} \bigcup \delta\overline{S}$ it will result in $\overline{S} \subset \overline{(\overline{S})}$.

5 Section 1.3: Problem 21

Let S be a nonempty subset of $\mathbb{R}$ such that if $\mathcal{H}$ is any open covering of S, then S has an open covering $\mathcal{H}_e$ comprised of finitely many open sets from $\mathcal{H}$. Show that S is compact. [2]

Answer

To show S is compact, we have to show two things S is bounded and it's closed:

1 S is bounded: Consider the open covering of S:

$$\forall x \in S, \mathcal{H} = \bigcup_{x \in S} (x - 1, x + 1)$$

Then there exist a finite sub-covering of $\mathcal{H}$ such as $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ such that:

$$S \subset \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n$$

Consider the following:

$$\mathcal{N} := \min\{x_1 - 1, x_1 + 1, x_2 - 1, x_2 + 1, \cdots, x_n - 1, x_n + 1\}$$

[2] The naming and the procedure might be the same with many other students because, during Friday's lecture, Dr. Gelaki could not teach due to technical issues, so he solved some homework problems with the whole class.

$$\mathcal{M} := \max\{x_1-1, x_1+1, x_2-1, x_2+1, \cdots, x_n-1, x_n+1\}$$

Then:

$$S \subset \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n \subset (\mathcal{N}, \mathcal{M})$$

Since if you take any element such as y is $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n$ is belongs to at least on of $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$. Assume $y \in \mathcal{H}_t$. Then:

$$y \in (x_t - 1, x_t + 1) \subset (\mathcal{N}, \mathcal{M})$$

Since $\mathcal{N} \leq x_t - 1$ and $\mathcal{M} \geq x_t + 1$. Since $S \subset (\mathcal{N}, \mathcal{M})$. Then

$$\forall x \in S : x \leq \mathcal{M} \text{ and } x \geq \mathcal{N}$$

Therefore S is bounded.

2 S is close: To show S is closed, we can show S^c is open. To show S^c is open, we have to show:

$$\forall x \in S^c, \exists \epsilon > 0 \text{ such that: } (x-\epsilon, x+\epsilon) \subset S^c$$

Let $y \in S^c$ and consider the following open covering of S:

$$\forall x \in S, \mathcal{H} = \bigcup_{x \in S} (x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2})$$

Then there exist a finite sub-covering of $\mathcal{H}$ such as $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ such that:

$$S \subset \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_n$$

For each of $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_n$ also consider the following:

$$\mathcal{U}_i = (y - \frac{|x_i - y|}{2}, y + \frac{|x_i - y|}{2})$$

We will be showing $\mathcal{H}_i \cap \mathcal{U}_i = \emptyset$. There will be two possibilities for x_i and y:

1 $x_i < y$. Then;

$$|x_i - y| \leq y - x_i \Rightarrow \frac{|x_i-y|}{2} + \frac{|x_i-y|}{2} \leq y - x_i \Rightarrow x_i + \frac{|x_i-y|}{2} \leq y - \frac{|x_i-y|}{2}$$

Hence $\forall a_y \in \mathcal{U}_i$ and $\forall a_x \in \mathcal{H}_i$, since:

$$y - \frac{|x_i-y|}{2} < a_y \text{ and } a_x < x_i + \frac{|x_i-y|}{2} \Rightarrow a_x < a_y$$

Which means $\mathcal{H}_i \cap \mathcal{U}_i = \emptyset$.

2 $y < x_i$. Then;

$$|x_i - y| \leq x_i - y \Rightarrow \frac{|x_i-y|}{2} + \frac{|x_i-y|}{2} \leq x_i - y \Rightarrow y + \frac{|x_i-y|}{2} \leq x_i - \frac{|x_i-y|}{2}$$

Hence $\forall a_y \in \mathcal{U}_i$ and $\forall a_x \in \mathcal{H}_i$, since:

$$x_i - \frac{|x_i-y|}{2} < a_x \text{ and } a_y < y + \frac{|x_i-y|}{2} \Rightarrow a_y < a_x$$

Which means $\mathcal{H}_i \cap \mathcal{U}_i = \emptyset$.

Please note that $x_i \neq y$ since $x_i \in S$ and $y \in S^c$.
Now, consider the following:

$$\epsilon := \min\{\frac{|x_1 - y|}{2}, \frac{|x_2 - y|}{2}, \frac{|x_3 - y|}{2}, \cdots, \frac{|x_n - y|}{2}\}$$

Then:

$$\mathcal{U}_\epsilon = (y - \epsilon, y + \epsilon) \subset \mathcal{U}_i, \quad i = 1, \cdots, n$$

We will show $\mathcal{U}_\epsilon = (y - \epsilon, y + \epsilon) \cap S = \emptyset$

Prove by Contradiction

Assume the opposite, which means $\mathcal{U}_\epsilon \cap S \neq \emptyset$. Then:

$$\exists w \in \mathcal{U}_\epsilon \cap S \Rightarrow \exists i \text{ such that: } w \in \mathcal{H}_i$$

Which means:

$$w \in \mathcal{U}_\epsilon \text{ and } w \in \mathcal{H}_i \Rightarrow \mathcal{U}_\epsilon \cap \mathcal{H}_i \neq \emptyset$$

Therefore:

$$\mathcal{U}_i \cap \mathcal{H}_i \neq \emptyset, \quad i = 1, \cdots, n$$

Which is a contradiction. This means $\mathcal{U}_\epsilon = (y - \epsilon, y + \epsilon) \cap S = \emptyset$, and $\mathcal{U}_\epsilon$ is a neighborhood of y which is completely in S^c. Hence, S^c is open and S is closed.

6 Section 1.3: Problem 22

A set S is dense in a set T if $S \subset T \subset \bar{S}$.

(a) Prove: If S and T are sets of real numbers and $S \subset T$, then S is dense in T if and only if every neighborhood of each point in T contains a point from S.

(b) State how (a) shows that the definition given here is consistent with the restricted definition of a dense subset of the reals given in Section 1.1.

Answer

(a) We want to show $S \subset T \subset \bar{S} \Leftrightarrow$ every neighborhood of each point in T contains a point from S.

- $\Rightarrow$ $S \subset T \subset \bar{S}$: To show every neighborhood of each point in T contains a point from S, First, consider $x \in T \subset \bar{S}$. Since $\bar{S} = S + \delta S$ then at least two of the following possibilities could happen for x:

  1 $x \in S$: Then every neighborhood of x contains x which is in S.

  2 $x \in \delta S$. Then x is a boundary point, meaning that for every neighborhood of x there exist a point of this neighborhood in S.

  Therefore, if $S \subset T \subset \bar{S} \Rightarrow$ every neighborhood of each point in T contains a point from S.

$\Leftarrow$ $S \subset T$ and every neighborhood of each point in T contains a point from S: To show $S \subset T \subset \bar{S}$, since we already know $S \subset T$, we only need to show $T \subset \bar{S}$, meaning that every point in T is whether in S, or whether it's a boundary point of S, meaning it's in δS. Consider $x \in T$. then at least two of the following possibilities could happen for x:

1 $x \in S$: Then every neighborhood of x contains x which is in S.

2 $x \notin S$: This means $x \in S^c$. Since for every neighborhood of x, let's say N there exist a point of this neighborhood in S then:

$$N \cap S \neq \emptyset \text{ and } x \in N \cap S^c \Rightarrow N \cap S^c \neq \emptyset$$

Therefore x is a boundary point and $x \in \delta S$.

Hence, if $S \subset T$ and every neighborhood of each point in T contains a point from S $\Rightarrow S \subset T \subset \bar{S}$

(b) Based on section 1.1. definition, a set D is dense in the reals if every open interval (a, b) contains a member of D. Now, assume $D = S$ and $T = \mathbb{R}$. We know every set such as D is a subset of $\mathbb{R}$. Thus, to show the consistency with the definition provided in (a), we have to show $\mathbb{R} \subset \bar{D}$. Knowing that $\bar{D} = D + \delta D$, we have to show:

$$D^c \subset \delta D$$

Let $x \in D^c$, and $\forall \epsilon > 0$ consider $(x - \epsilon, x + \epsilon)$. Based on the definition of the density $(x-\epsilon, x+\epsilon) \cap D \neq \emptyset$. In addition, $x \in (x-\epsilon, x+\epsilon) \cap D^c \Rightarrow (x-\epsilon, x+\epsilon) \cap D^c \neq \emptyset$. Therefore, x is a boundary point for D. This procedure is true for all the elements in D^c. Hence, $D^c \subset \delta D$. This indicates that the density definition for D being $D \subset \mathbb{R} \subset \bar{D}$ is consisted with the definition of density in reals.

7 Section 1.3: Problem 23

Prove:

(a) $(S_1 \cap S_2)^0 = {S_1}^0 \cap {S_2}^0$

(b) ${S_1}^0 \cup {S_2}^0 \subset (S_1 \cup S_2)^0$

Answer

(a) To show $(S_1 \cap S_2)^0 = {S_1}^0 \cap {S_2}^0$, we have to show $(S_1 \cap S_2)^0 \subset {S_1}^0 \cap {S_2}^0$ and ${S_1}^0 \cap {S_2}^0 \subset (S_1 \cap S_2)^0$:

1. $(S_1 \cap S_2)^0 \subset {S_1}^0 \cap {S_2}^0$: Suppose $x \in (S_1 \cap S_2)^0$, then $\exists \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ, such that:

$$N_\epsilon \subset S_1 \cap S_2 \Rightarrow N_\epsilon \subset S_1 \text{ and } N_\epsilon \subset S_2$$

Thus, x is an interior point for both S_1 and S_2, meaning:

$$x \in {S_1}^0 \text{ and } x \in {S_2}^0 \Rightarrow x \in {S_1}^0 \cap {S_2}^0$$

2. ${S_1}^0 \cap {S_2}^0 \subset (S_1 \cap S_2)^0$: Suppose $x \in {S_1}^0 \cap {S_2}^0$, then x is an interior point for both S_1 and S_2 meaning:

$$\exists \epsilon_1 > 0 \text{ such that } (x - \epsilon_1, x + \epsilon_1) \subset S_1$$

And

$$\exists \epsilon_2 > 0 \text{ such that } (x-\epsilon_2, x+\epsilon_2) \subset S_2$$

Now consider:

$$\epsilon := \min\{\epsilon_1, \epsilon_2\}$$

Then:

$$(x-\epsilon, x+\epsilon) \subset (x-\epsilon_1, x+\epsilon_1) \subset S_1 \text{ and } (x-\epsilon, x+\epsilon) \subset (x-\epsilon_2, x+\epsilon_2) \subset S_2$$

Hence:

$$(x-\epsilon, x+\epsilon) \subset S_2 \cap S_2$$

which means x is an interior point for $S_2 \cap S_2$ and $x \in (S_1 \cap S_2)^0$.

(b) ${S_1}^0 \cup {S_2}^0 \subset (S_1 \cup S_2)^0$: Suppose $x \in {S_1}^0 \cup {S_2}^0$, then at least two of the followings might happen:

1. $x \in {S_1}^0$: then $\exists \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ, such that:

$$N_\epsilon \subset S_1 \Rightarrow N_\epsilon \subset S_1 \cup S_2$$

Therefore, x is an interior point for $S_1 \cup S_2$, and $x \in (S_1 \cup S_2)^0$.

2. $x \in {S_2}^0$: then $\exists \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ, such that:

$$N_\epsilon \subset S_2 \Rightarrow N_\epsilon \subset S_1 \cup S_2$$

Therefore, x is an interior point for $S_1 \cup S_2$, and $x \in (S_1 \cup S_2)^0$.

8 Section 1.3: Problem 24

(a) $\delta(S_1 \cup S_2) \subset \delta S_1 \cup \delta S_2$

(b) $\delta(S_1 \cap S_2) \subset \delta S_1 \cup \delta S_2$

(c) $\delta \bar{S} \subset \delta S$

(d) $\delta S = \delta S^c$

(e) $\delta(S - T) \subset \delta S \cup \delta T$

Answer

(a) $\delta(S_1 \cup S_2) \subset \delta S_1 \cup \delta S_2$: Suppose $x \in \delta(S_1 \cup S_2)$. Then $\forall \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ, there's at least one point in $S_1 \cup S_2$ and at least one point in $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$. There will be two possibilities for the point which is in N_ϵ and $S_1 \cup S_2$.

1. $x \in S_1$: Since $N_\epsilon \cap (S_1^c \cap S_2^c) \neq \emptyset$, then $N_\epsilon \cap S_1^c \neq \emptyset$. Which means for N_ϵ of x there's point in S_1 and one in ${S_1}^c$. Thus, x is a boundary point of S_1, and:

$$x \in \delta S_1 \Rightarrow x \in \delta S_1 \cup \delta S_2$$

2. $x \in S_2$: Since $N_\epsilon \cap (S_1^c \cap S_2^c) \neq \emptyset$, then $N_\epsilon \cap S_2^c \neq \emptyset$. Which means for N_ϵ of x there's point in S_2 and one in ${S_2}^c$. Thus, x is a boundary point of S_2, and:

$$x \in \delta S_2 \Rightarrow x \in \delta S_1 \cup \delta S_2$$

(b) $\delta(S_1 \cap S_2) \subset \delta S_1 \cup \delta S_2$: Suppose $x \in \delta(S_1 \cap S_2)$. Then $\forall \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ, there's at least one point in $S_1 \cap S_2$ and at least one point in $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$. There will be two possibilities for the point which is in N_ϵ and $S_1^c \cup S_2^c$.

1. $x \in S_1^c$: Since $N_\epsilon \cap (S_1 \cap S_2) \neq \emptyset$, then $N_\epsilon \cap S_1 \neq \emptyset$. Which means for N_ϵ of x there's point in S_1 and at least one, x, in ${S_1}^c$. Thus, x is a boundary point of S_1, and:

$$x \in \delta S_1 \Rightarrow x \in \delta S_1 \cup \delta S_2$$

2. $x \in S_2^c$: Since $N_\epsilon \cap (S_1 \cap S_2) \neq \emptyset$, then $N_\epsilon \cap S_2 \neq \emptyset$. Which means for N_ϵ of x there's point in S_2 and at least one, x, in ${S_2}^c$. Thus, x is a boundary point of S_2, and:

$$x \in \delta S_2 \Rightarrow x \in \delta S_1 \cup \delta S_2$$

(c) $\delta \bar{S} \subset \delta S$

(d) $\delta S = \delta S^c$ To show $\delta S = \delta S^c$, we have to show $\delta S \subset \delta S^c$ and $\delta S^c \subset \delta S$:

1. $\delta S \subset \delta S^c$: Suppose $x \in \delta S$, then $\forall \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ:

$$N_\epsilon \cap S \neq \emptyset \text{ and } N_\epsilon \cap S^c \neq \emptyset$$

Therefore, x is also a boundary point for S^c and $x \in \delta S^c$.

2. $\delta S^c \subset \delta S$: Suppose $x \in \delta S^c$, then $\forall \epsilon > 0$, $\epsilon-$neighborhood of x, let's say, N_ϵ:

$$N_\epsilon \cap S^c \neq \emptyset \text{ and } N_\epsilon \cap S \neq \emptyset$$

Therefore, x is also a boundary point for S and $x \in \delta S$.

(e) $\delta(S - T) \subset \delta S \cup \delta T$: Suppose $x \in \delta(S - T)$. There will be four possible situation for x based on whether $x \in (S - T)$ or $x \in (S - T)^c$

1. $x \in S \cap T$: this means $x \in (S - T)$ Then consider N_ϵ being a neighborhood of x, since x is a boundary point of $(S - T)$, then:

$$N_\epsilon \cap S - T \neq \emptyset$$

Note that, $S - T \subset T^c$. Hence:

$$x \in S \cap T \subset T \text{ and } N_\epsilon \cap S - T \neq \emptyset \Rightarrow N_\epsilon \cap T^c \neq \emptyset$$

Therefore, x is a boundary point for T, and $x \in \delta T$.

2. $x \in S \backslash S \cap T$: this means $x \in (S - T) \in S$ and $x \in (S - T) \in T^c$. Then consider N_ϵ being a neighborhood of x, since x is a boundary point of $(S - T)$, then:

$$N_\epsilon \cap (S - T)^c \neq \emptyset$$

Now, there will be two possibilities here:

2.1. $N_\epsilon \cap (S \cap T) \neq \emptyset \Rightarrow N_\epsilon \cap T \neq \emptyset$. Then,

$$x \in S \backslash S \cap T \subset T^c \text{ and } N_\epsilon \cap T \neq \emptyset$$

Therefore, x is a boundary point for T, and $x \in \delta T$.

2.2. $N_\epsilon \cap S^c \neq \emptyset$. Then,

$$x \in S \backslash S \cap T \subset S \text{ and } N_\epsilon \cap S^c \neq \emptyset$$

Therefore, x is a boundary point for S, and $x \in \delta S$.

3. $x \in T \backslash S \cap T$: this means $x \in (S-T)^c$. Then consider N_ϵ being a neighborhood of x, since x is a boundary point of $(S-T)$, then:

$$N_\epsilon \cap (S-T) \neq \emptyset$$

Note that, $S - T \subset S$. Hence:

$$x \in (S-T)^c \subset S^c \text{ and } N_\epsilon \cap (S-T) \neq \emptyset \Rightarrow N_\epsilon \cap S \neq \emptyset$$

Therefore, x is a boundary point for S, and $x \in \delta S$

4. $x \in (S \cup T)^c$: this means $x \in (S-T)^c$. Then consider N_ϵ being a neighborhood of x, since x is a boundary point of $(S-T)$, then:

$$N_\epsilon \cap (S-T) \neq \emptyset$$

Note that, $S - T \subset S$. Hence:

$$x \in (S-T)^c \subset S^c \text{ and } N_\epsilon \cap (S-T) \neq \emptyset \Rightarrow N_\epsilon \cap S \neq \emptyset$$

Therefore, x is a boundary point for S, and $x \in \delta S$

References

[1] William F Trench. *Introduction to real analysis.* 2013.