# MATH 414 Analysis I,
# Homework 4

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Deadline: 09/27/2024, 11:59 PM

## 1 Section 2.1: Problem 1

Each of the following conditions fails to define a function on any domain. State why.

(a) $\sin f(x) = x$

(d) $f(x)[f(x) - 1] = x^2$

### Answer

(a) First, it's important to note that $\sin(x)$ takes value in $[-1, 1]$. Therefore, $x \in [-1, 1]$ and we have to show $f$ fails to be a function of $x$ in this domain.
Let $x \in [-1, 1]$ and $f(x) = a$, and suppose: $\sin a = x$. Now consider $a + 2\pi$. Then

$$\sin(a + 2\pi) = \sin a \cos 2\pi + \cos a \sin 2\pi = \sin a = x$$

This means that $f(x)$ can be $a$, and it can also be $a + 2\pi$ assigning more than one element to any element of the domain, which contradicts the definition of the function.

(d) Let $x = a \in D_f$, and $f(a) = y$ then:

$$f(a)[f(a) - 1] = a^2 \Rightarrow y(y - 1) = a^2 \Rightarrow y^2 - y = a^2 \Rightarrow y^2 - y - a^2 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1 + 4a^2}}{2}$$

For any $a$, there will be two possible $y$ that satisfy $y(y-1) = a^2$. This means that any $a \in D_f$, $f(a)$ can take two values which contradicts with functions assigning only one element to each element of domain.

## 2 Section 2.1: Problem 2

If

$$f(x) = \sqrt{\frac{(x-3)(x+2)}{(x-1)}} \quad \text{and} \quad g(x) = \frac{x^2 - 16}{x - 7}\sqrt{x^2 - 9}$$

find $D_f$, $D_{f\pm g}$, $D_{fg}$, and $D_{f/g}$.

## Answer

(a) $D_f$: Since $f(x) = \sqrt{\frac{(x-3)(x+2)}{(x-1)}}$, then $f(x) \geq 0$. Therefor $\frac{(x-3)(x+2)}{(x-1)} \geq 0$ which is equivalent to $(x-3)(x+2)(x-1) \geq 0$. To do this, we have to exclude the continuous subsets of real numbers that one or all three terms in the multiplication, meaning $(x-3)$, $(x+2)$, and $(x-1)$, are negative. For each individual parenthesis, if we consider the root, all the real numbers less than the root will However, if for a particular $x$, two parenthesis are negative and one positive, that can still be in $D_f$ since the multiplication result is positive. negative, and one positive, that can still be in $D_f$ since the multiplication result is positive. Therefore, it might be good to divide the real numbers as the following covering:

$$\mathbb{R} = \underbrace{(-\infty,-2)}_{\text{A}} \cup \underbrace{[-2,1)}_{\text{B}} \cup \underbrace{[1,3)}_{\text{C}} \cup \underbrace{[3,\infty)}_{\text{D}}$$

Each of -2, 1, and 3 are the roots of one parenthesis. Let's determine the sign of $(x-3)(x+2)(x-1)$ with respect to $x$ belonging to $A, B, C$, or $D$ individually.

(A) $x \in (-\infty,-2) : sign(\underbrace{(x-3)}_{-}\underbrace{(x+2)}_{-}\underbrace{(x-1)}_{-}) < 0$

(B) $x \in [-2,1] : sign(\underbrace{(x-3)}_{-}\underbrace{(x+2)}_{+}\underbrace{(x-1)}_{-}) \geq 0$

(C) $x \in (1,3) : sign(\underbrace{(x-3)}_{-}\underbrace{(x+2)}_{+}\underbrace{(x-1)}_{+}) < 0$

(D) $x \in [3,\infty) : sign(\underbrace{(x-3)}_{+}\underbrace{(x+2)}_{+}\underbrace{(x-1)}_{+}) \geq 0$

Therefore $x \in [-2,1] \cup [3,\infty)$ results in $(x-3)(x+2)(x-1) \geq 0$. However, we still need to exclude the root of $x-1$ since $f(x) = \sqrt{\frac{(x-3)(x+2)}{(x-1)}}$ is not defined there. Hence:

$$D_f = [-2,1] \cup [3,\infty) \backslash \{1\} \Rightarrow D_f = [-2,1) \cup [3,\infty)$$

(b) $D_{f\pm g}$ is $D_f \cap D_g$. Thus, we should determine $D_g$. $D_g$ has two constrains: First, $x - 7 = \neq 0$ which will result in excluding $x = 7$. Second, $x^2 - 9 \geq 0$, which means:

$$x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9 \Rightarrow |x| \geq 3 \Rightarrow x \in (-\infty,-3] \cup [3,\infty)$$

Hence:

$$D_f = (-\infty,-3] \cup [3,\infty) \backslash \{7\} \Rightarrow D_f = (-\infty,-3] \cup [3,7) \cup (7,\infty)$$

Finally:

$$D_{f\pm g} = D_f \cap D_g = [-2,1) \cup [3,\infty) \bigcap (-\infty,-3] \cup [3,7) \cup (7,\infty) = [3,7) \cup (7,\infty)$$

(c) $D_{fg}$ would be same as $D_{f\pm g}$. Hence:

$$D_{fg} = [3,7) \cup (7,\infty)$$

(d) $D_{f/g}$ is $x \in D_f \cap D_g$ such that $g(x) \neq 0$. Therefore, we will fine all $x$ such that $g(x) = 0$, and then we will exclude them from $D_f \cap D_g$.

$$g(x) = \frac{x^2 - 16}{x - 7}\sqrt{x^2 - 9} = 0 \Rightarrow x^2 - 16 = 0 \text{ or } x^2 - 9 = 0 \Rightarrow x = \pm 4 \text{ or } \pm 3$$

Hence:

$$D_{f/g} = D_f \cap D_g \backslash \{-4, -3, 3, 4\} \Rightarrow D_{f/g} = (3, 4) \cup (4, 7) \cup (7, \infty)$$

# 3 Section 2.1: Problem 4

Find $\lim_{x \to x_0} f(x)$, and justify your answer with an $\varepsilon$–$\delta$ proof.

(a) $x^2 + 2x + 1, \quad x_0 = 1$

(d) $\sqrt{x}, \quad x_0 = 4$

## Answer

(a) $\lim_{x \to 1} x^2 + 2x + 1$:
First, $f$ is defined on a deleted neighborhood of $x_0 = 1$. We will show $\lim_{x \to 1} x^2 + 2x + 1 = 4$. Let $\epsilon > 0$. We have to find $\delta > 0$ such that:

$$\begin{aligned} 0 < |x - 1| < \delta \Rightarrow\ & 0 < |x^2 + 2x + 1 - 4| < \epsilon \\ \Leftrightarrow\ & 0 < |x^2 + 2x + 1 - 4| < \epsilon \\ \Leftrightarrow\ & 0 < |x^2 - 2x + 1 + 4x - 4| < \epsilon \\ \Leftrightarrow\ & 0 < |(x^2 - 2x + 1) + 4(x - 1)| < \epsilon \\ \Leftrightarrow\ & 0 < |(x - 1)^2 + 4(x - 1)| < \epsilon \end{aligned}$$

By using the triangle inequality we have:

$$0 < |(x - 1)^2 + 4(x - 1)| < |(x - 1)^2| + |4(x - 1)| = |x - 1|^2 + 4|x - 1| < \delta^2 + 4\delta$$

If we pick $\delta = -2 + \sqrt{4 + \epsilon}$ then the following holds:

$$0 < |x - 1| < \delta \Rightarrow 0 < |x^2 + 2x + 1 - 4| < \epsilon \quad (\forall \epsilon > 0 \text{ and } \delta = -2 + \sqrt{4 + \epsilon})$$

In addition, $\delta > 0$, since:

$$0 < \epsilon \Rightarrow 4 < \epsilon + 4 \Rightarrow 2 < \sqrt{4 + \epsilon} \Rightarrow \delta = -2 + \sqrt{4 + \epsilon} > 0$$

Hence:

$$\lim_{x \to 1} x^2 + 2x + 1 = 4$$

(d) $\lim_{x \to 4} \sqrt{x}$:
First, $f$ is defined on a deleted neighborhood of $x_0 = 4$. We will show $\lim_{x \to 4} \sqrt{x} = 2$
Let $\epsilon > 0$. We have to find $\delta > 0$ such that:

$$\begin{aligned} 0 < |x - 4| < \delta \Rightarrow\ & 0 < |\sqrt{x} - 2| < \epsilon \\ \Leftrightarrow\ & 0 < |\sqrt{x} - 2||\sqrt{x} + 2| < \delta \end{aligned}$$

We can multiply $|\sqrt{x}-2|$ by $\frac{\sqrt{x}+2}{\sqrt{x}+2}$ without changing the values. Therefore:

$$\begin{aligned}
0 < |x-4| < \delta \Rightarrow 0 &< |\sqrt{x}-2| < \epsilon \\
&\Leftrightarrow 0 < |\sqrt{x}-2\frac{\sqrt{x}+2}{\sqrt{x}+2}| < \epsilon \\
&\Leftrightarrow 0 < |\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}| < \epsilon \\
&\Leftrightarrow 0 < \frac{|x-4|}{\sqrt{x}+2} < \epsilon \quad \text{since } \sqrt{x}+2 > 0 \\
&\Leftrightarrow 0 < \frac{|x-4|}{\sqrt{x}+2} < \epsilon
\end{aligned}$$

We know:

$$2 < \sqrt{x}+2 <\Rightarrow \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{x}+2}$$

Hence:

$$0 < \frac{|x-4|}{\sqrt{x}+2} < \frac{|x-4|}{2} < \delta\frac{1}{2}$$

Thus, if we take $\delta = 2\epsilon$, then the following will hold:

$$0 < |x-4| < \delta \Rightarrow 0 < |\sqrt{x}-2| < \epsilon$$

Hence:

$$\lim_{x\to 4}\sqrt{x} = 2$$

# 4 Section 2.1: Problem 6

Use Theorem 2.1.4 and the known limits $\lim_{x\to x_0} x = x_0$, $\lim_{x\to x_0} c = c$ to find the indicated limits.

(b) $\lim_{x\to 2}\left(\frac{1}{1+x} - \frac{1}{1-x}\right)$

(c) $\lim_{x\to 1}\left(\frac{x-1}{x^3+x^2-2x}\right)$

## Answer

(b) $\lim_{x\to 2}\left(\frac{1}{1+x}-\frac{1}{1-x}\right)$:
By using the theorem 2.1.4 [1] the following limit can be written as:

$$\begin{aligned}
\lim_{x\to 2}\left(\frac{1}{1+x}-\frac{1}{1-x}\right) &= \lim_{x\to 2}\left(\frac{1}{1+x}\right)-\lim_{x\to 2}\left(\frac{1}{1-x}\right)\\
&= \left(\frac{\lim_{x\to 2} 1}{\lim_{x\to 2} 1+x}\right)-\left(\frac{\lim_{x\to 2} 1}{\lim_{x\to 2} 1-x}\right)\\
&= \left(\frac{1}{\lim_{x\to 2} 1+\lim_{x\to 2} x}\right)-\left(\frac{1}{\lim_{x\to 2} 1-\lim_{x\to 2} x}\right)\\
&= \left(\frac{1}{1+2}\right)-\left(\frac{1}{1-2}\right)\\
&= \left(\frac{1}{3}\right)-\left(\frac{1}{-1}\right)\\
&= \left(\frac{1}{3}\right)+1=\frac{4}{3}
\end{aligned}$$

(c) $x^3+x^2-2x$ can be written as $(x+2)x(x-1)$. Thus, $\frac{x-1}{x^3+x^2-2x}$ can be written as $\frac{x-1}{(x+2)x(x-1)}$. Since we care about a deleted neighborhood of $x_0=1$, we can cancel out $x-1$ from both sides, guaranteed that $x-1$ won't be zero. Therefore:
Now,

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{x-1}{x^3+x^2-x}\right) &= \lim_{x\to 1}\left(\frac{1}{(x+2)x}\right)\\
&= \left(\frac{\lim_{x\to 1} 1}{\lim_{x\to 1} x^2+x}\right)\\
&= \left(\frac{1}{\lim_{x\to 1} x^2+\lim_{x\to 1} x}\right)\\
&= \left(\frac{1}{(\lim_{x\to 1} x)^2+1}\right)\\
&= \left(\frac{1}{1^2+1}\right)\\
&= \frac{1}{2}
\end{aligned}$$

## 5 Section 2.1: Problem 7

Find $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$, if they exist. Use $\epsilon$–$\delta$ proofs, where applicable, to justify your answers.

(a) $\dfrac{x+|x|}{x}, \quad x_0 = 0$

(c) $\dfrac{|x-1|}{x^2+x-2}, \quad x_0 = 1$

### Answer

(a) (1) $\lim_{x\to 0^-} \dfrac{x+|x|}{x}$: $f$ is defined in deleted neighborhood of 0 from left. We will show $\lim_{x\to 0^-} \dfrac{x+|x|}{x} = 0$.
Let $\epsilon > 0$. We have to find $\delta > 0$ such that:

$$0 - \delta < x < 0 \Rightarrow 0 < |f(x) - 0| < \epsilon \Rightarrow |f(x)| < \epsilon$$

Since $x < 0$, then $|x| = -x$. Thus:

$$\begin{aligned}
\text{showing } |f(x)| < \epsilon &\Leftrightarrow |\frac{x+|x|}{x}| < \epsilon \\
&\Leftrightarrow |\frac{x-x}{x}| < \epsilon \\
&\Leftrightarrow |\frac{0}{x}| < \epsilon \\
&\Leftrightarrow 0 < \epsilon
\end{aligned}$$

Thus, for $x \in D_f$ we only need to pick $\delta > |x|$ to satisfy the above.

(2) $\lim_{x\to 0^+} \dfrac{x+|x|}{x}$: $f$ is defined in deleted neighborhood of 0 from right. We will show $\lim_{x\to 0^+} \dfrac{x+|x|}{x} = 2$.
Let $\epsilon > 0$ We have to find $\delta > 0$ such that:

$$0 < x < 0 + \delta \Rightarrow 0 < |f(x) - 2| < \epsilon$$

Since $x > 0$, then $|x| = x$. Thus:

$$\begin{aligned}
\text{showing } |f(x) - 2| < \epsilon &\Leftrightarrow |\frac{x+|x|}{x} - 2| < \epsilon \\
&\Leftrightarrow |\frac{x+x}{x} - 2| < \epsilon \\
&\Leftrightarrow |\frac{2x}{x} - 2| < \epsilon \\
&\Leftrightarrow |2 - 2| < \epsilon \\
&\Leftrightarrow 0 < \epsilon
\end{aligned}$$

Thus, for $x \in D_f$ we only need to pick $\delta > x$ to satisfy the above.

(c) Since $x^2 + x - 2 = (x-1)(x+2)$ for convenience we will write $f$ in the following format: $f = \dfrac{|x-1|}{(x-1)(x+2)}$

(1) $\lim_{x\to 1^-} \dfrac{|x-1|}{(x-1)(x+2)}$: $f$ is defined in a deleted neighborhood of 1 from left. We will show $\lim_{x\to 1^-} \dfrac{|x-1|}{(x-1)(x+2)} = -\frac{1}{3}$.
Let $\epsilon > 0$. We have to find $\delta > 0$ such that if:

$$1 - \delta < x < 1 \Rightarrow 0 < |f(x) + \frac{1}{3}| < \epsilon$$

Let $\epsilon_0 := \min\{\epsilon, \frac{2}{3}\}$. We will show the following hold for $\epsilon_1$, which results in for holding all $\epsilon > 0$.
Since $x < 1$, then $|x-1| = -(x-1)$. Thus, $f(x) = \dfrac{|x-1|}{(x-1)(x+2)} = \dfrac{-(x-1)}{(x-1)(x+2)} = \dfrac{-1}{(x+2)}$ $(x \neq 1)$. Then:

$$\begin{aligned}
1 - \delta < x < 1 \Rightarrow 1 - x < \delta \Rightarrow 0 &< |\frac{-1}{x+2} + \frac{1}{3}| < \epsilon_0 \\
\Leftrightarrow 0 &< |\frac{x-1}{3(x+2)}| < \epsilon_0 \\
\Leftrightarrow 0 &< |\frac{x-1}{3(x+2)}| < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{1}{3}|\frac{x-1}{x+2}| < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{1}{3}|\frac{x-1}{x+2}| < \epsilon_0 \\
\Leftrightarrow 0 &< |\frac{x-1}{x+2}| < 3\epsilon_0 \\
\Leftrightarrow 0 &< \frac{|x-1|}{|x+2|} < 3\epsilon_0
\end{aligned}$$

Since we're within the close neighborhood of 1 from left:

$$|x-1| = 1 - x \text{ and } |x+2| = x+2$$

Thus:

$$\begin{aligned}
0 < \frac{|x-1|}{|x+2|} < 3\epsilon_0 \Rightarrow 0 < \frac{1-x}{x+2} < 3\epsilon_0 \\
&\Leftrightarrow 0 < 1-x < 3\epsilon_0(x+2) \\
&\Leftrightarrow 0 < 1-x < 3\epsilon_0 x + 6\epsilon_0 \\
&\Leftrightarrow -1 < -x < 3\epsilon_0 x + 6\epsilon_0 - 1 \\
&\Leftrightarrow x-1 < 0 < 3\epsilon_0 x + x + 6\epsilon_0 - 1 \\
&\Leftrightarrow 1 - 6\epsilon_0 < 3\epsilon_0 x + x \\
&\Leftrightarrow 1 - 6\epsilon_0 < x(3\epsilon_0 + 1) \\
&\Leftrightarrow \frac{1-6\epsilon_0}{3\epsilon_0+1} < x
\end{aligned}$$

For $\frac{1-6\epsilon_0}{3\epsilon_0+1} < x$ to hold, since $1-\delta < x$ we can show:

$$\begin{aligned}
&\frac{1-6\epsilon_0}{3\epsilon_0+1} < 1-\delta \\
&\Leftrightarrow \delta < 1 - \frac{1-6\epsilon_0}{3\epsilon_0+1} \\
&\Leftrightarrow \delta < 1 - \frac{-6\epsilon_0 - 2 + 3}{3\epsilon_0+1} \\
&\Leftrightarrow \delta < 1 - \frac{-2(3\epsilon_0+1)+3}{3\epsilon_0+1} \\
&\Leftrightarrow \delta < 1 + \frac{-2(3\epsilon_0+1)}{3\epsilon_0+1} + \frac{3}{3\epsilon_0+1} \\
&\Leftrightarrow \delta < 1 - 2 + \frac{3}{3\epsilon_0+1} \\
&\Leftrightarrow \delta < -1 + \frac{3}{3\epsilon_0+1}
\end{aligned}$$

Thus, as long as $\delta < -1 + \frac{3}{3\epsilon_0+1}$ then the inequality holds, and $\forall \epsilon > 0$ we need to pick a $\delta$ that satisfies the inequality. However, we need to checking whether $-1 + \frac{3}{3\epsilon_0+1}$ is going to be negative or not, and under what circumstances. Thus, suppose:

$$-1 + \frac{3}{3\epsilon_0+1} < 0 \Rightarrow \frac{3}{3\epsilon_0+1} < 1 \Rightarrow 3 < 3\epsilon_0 + 1 \Rightarrow 2 < 3\epsilon_0 \Rightarrow \epsilon_0 > \frac{2}{3}$$

This means $\forall \epsilon_0 < \frac{2}{3}$ the inequality holds.

(2) $\lim_{x\to 1^+} \dfrac{|x-1|}{(x-1)(x+2)}$: $f$ is defined in a deleted neighborhood of 1 from right. We will show $\lim_{x\to 1^+} \dfrac{|x-1|}{(x-1)(x+2)} = \frac{1}{3}$.
Let $\epsilon > 0$. We have to find $\delta > 0$ such that if

$$1 < x < 1 + \delta \Rightarrow 0 < |f(x) - \frac{1}{3}| < \epsilon$$

Let $\epsilon_0 := \min\{\epsilon, \frac{1}{3}\}$. We will show the following hold for $\epsilon_1$, which results in the proof holding for all $\epsilon > 0$.
Since $x > 1$, then $|x-1| = x - 1$. Thus, $\dfrac{|x-1|}{(x-1)(x+2)} = \dfrac{x-1}{(x-1)(x+2)} = \dfrac{1}{(x+2)}$ $(x \neq 1)$, which means:

$$\begin{aligned}
1 < x < 1+\delta \Rightarrow 0 &< |\frac{1}{x+2} - \frac{1}{3}| < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{1-x}{3(x+2)} < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{1}{3}|\frac{1-x}{x+2}| < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{1}{3}\frac{|1-x|}{|x+2|} < \epsilon_0 \\
\Leftrightarrow 0 &< \frac{|1-x|}{|x+2|} < 3\epsilon_0
\end{aligned}$$

Since we're within the close neighborhood of 1 from right:

$$|1-x| = x-1 \text{ and } |x+2| = x+2$$

$$\begin{aligned}
0 < \frac{|1-x|}{|x+2|} < 3\epsilon_0 &\Leftrightarrow 0 < \frac{x-1}{x+2} < 3\epsilon_0 \\
&\Leftrightarrow 0 < x-1 < 3\epsilon_0(x+2) \\
&\Leftrightarrow 0 < x-1 < 3\epsilon_0 x + 6\epsilon_0 \\
&\Leftrightarrow 1 < x < 3\epsilon_0 x + 6\epsilon_0 + 1 \\
&\Leftrightarrow 1 - 3\epsilon_0 x < x - 3\epsilon_0 x < 6\epsilon_0 + 1 \\
&\Leftrightarrow 1 - 3\epsilon_0 x < x(1-3\epsilon_0) < 6\epsilon_0 + 1 \\
&\Leftrightarrow 1 - 3\epsilon_0 x < x(1-3\epsilon_0) < 6\epsilon_0 + 1
\end{aligned}$$

Since $\epsilon_0 < \frac{1}{3} \Rightarrow 3\epsilon_0 < 1 \Rightarrow 0 < 1 - 3\epsilon_0$. Thus:

$$1 - 3\epsilon_0 x < x(1-3\epsilon_0) < 6\epsilon_0 + 1 \Rightarrow \frac{1-3\epsilon_0 x}{1-3\epsilon_0} < x < \frac{6\epsilon_0+1}{1-3\epsilon_0}$$

We know $x < 1+\delta$. Hence, if we show:

$$\frac{1-3\epsilon_0 x}{1-3\epsilon_0} < x \text{ and } 1+\delta < \frac{6\epsilon_0+1}{1-3\epsilon_0}$$

completes the proof.

- $\frac{1-3\epsilon_0 x}{1-3\epsilon_0} < x$. This is equivalent to showing that:

$$1 - 3\epsilon_0 x < x(1-3\epsilon_0) \Leftrightarrow 1 - 3\epsilon_0 x < x - 3\epsilon_0 x \Leftrightarrow 1 < x$$

- $1+\delta < \frac{6\epsilon_0+1}{1-3\epsilon_0}$. To show this, we have to show:

$$\begin{aligned} 1+\delta < \frac{6\epsilon_0+1}{1-3\epsilon_0} &\Leftrightarrow \delta < \frac{6\epsilon_0+1}{1-3\epsilon_0} - 1 \\ &\Leftrightarrow \delta < \frac{6\epsilon_0+1-1+3\epsilon_0}{1-3\epsilon_0} \\ &\Leftrightarrow \delta < \frac{9\epsilon_0}{1-3\epsilon_0} \end{aligned}$$

  Thus, as long as $\delta < \frac{9\epsilon_0}{1-3\epsilon_0}$ then the inequality holds.

# References


[1] William F Trench. *Introduction to real analysis.* 2013.