MATH 414 Analysis I,
Homework 2

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1 Section 1.2: Problem 5

Prove by if $a_i \geq 0$, $i \geq 1$, then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1+a_1+a_2+\cdots+a_n.$$

1.1 Answer: Prove by Induction

Suppose the left-hand side (LHS) and right-hand side (RHS) are the following parts of the inequality in our proof:

$$\underbrace{(1+a_1)(1+a_2)\cdots(1+a_n)}_{\text{(LHS)}} \geq \underbrace{1+a_1+a_2+\cdots+a_n}_{\text{(RHS)}}, \quad a_i \geq 0 \quad \forall i \in \{1,2,\cdots,n\}.$$

Thus, in the basis and the induction step, we will calculate each side individually, showing by considering the induction assumption they inequality holds.

1. **Basis**, $n = 1$:

$$\begin{aligned} \text{LHS:} &\quad 1+a_1 \\ \text{RHS:} &\quad 1+a_1 \\ \Rightarrow &\quad \text{LHS} = \text{RHS} \end{aligned}$$

2. **Induction Assumption:**
   Suppose for $n \geq 1$ we know that:

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1+a_1+a_2+\cdots+a_n$$

3. **Induction Step:**
   We would like to prove for $n+1$:

$$(1+a_1)(1+a_2)\cdots(1+a_n)\,(1+a_{n+1}) \geq 1+a_1+a_2+\cdots+a_n+a_{n+1}.$$

To begin, we would multiply $(1+a_1)(1+a_2)\cdots(1+a_n)$ by each element in $(1+a_{n+1})$ (which are 1 and a_{n+1})

$$\Rightarrow \text{LHS} = \underbrace{(1+a_1)(1+a_2)\cdots(1+a_n)\times 1}_{\text{(a)}} + \underbrace{(1+a_1)(1+a_2)\cdots(1+a_n)\times a_{n+1}}_{\text{(b)}}$$

$$\text{(a):}\quad \underbrace{(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1+a_1+a_2+\cdots+a_n}_{\text{(Based on the induction assumption)}}$$

$$\text{(b):}\quad \underbrace{(1+a_1)(1+a_2)\cdots(1+a_n)\times a_{n+1} \geq (1+a_1+a_2+\cdots+a_n)\times a_{n+1}}_{\text{(Based on the induction assumption)}}$$

$$\underbrace{(1+a_1+a_2+\cdots+a_n)\times a_{n+1} \geq (1)\times a_{n+1}}_{\text{(Since } a_i \geq 0\text{)}}$$

$$\Rightarrow \quad (1+a_1)(1+a_2)\cdots(1+a_n)\times a_{n+1} \geq a_{n+1}$$

$$\text{(a) + (b):}\quad (1+a_1)(1+a_2)\cdots(1+a_n) + (1+a_1)(1+a_2)\cdots(1+a_n)\times a_{n+1}$$

$$\geq 1+a_1+a_2+\cdots+a_n+a_{n+1}$$

$$\Rightarrow \quad (1+a_1)(1+a_2)\cdots(1+a_n)(1+a_{n+1}) \geq \underbrace{1+a_1+a_2+\cdots+a_n+a_{n+1}}_{\text{(RHS)}}$$

This completes the proof.

2 Section 1.2: Problem 7

Suppose that $s_0 > 0$ and $s_n = 1 - e^{-s_{n-1}}$, $n \geq 1$. Show that $0 < s_n < 1$, $n \geq 1$.

2.1 Answer: Prove by Induction

(*) The proof is based on the fact that if $x \in \mathbb{R}$ and $0 < x < 1$, the $\forall \epsilon > 0$: $0 < x^\epsilon < 1$ [1].

1. **Basis**, $n = 1$: Then $s_1 = 1 - e^{-s_0} \rightarrow s_1 = 1 - \frac{1}{e^{s_0}}$. Since $0 < \frac{1}{e} < 1$, and $S_0 > 0$, then $0 < \frac{1}{e^{s_0}} < 1$. Thus, for $n = 1$, $0 < S_1 < 1$

2. **Induction Assumption:**
   Suppose for $n \leq 1 : 0 < s_n < 1$, where $s_n = 1 - e^{-s_{n-1}}$.

3. **Induction Step:**
   We would like to prove for $n + 1$:

   $$\text{if } s_{n+1} = 1 - e^{-s_{n+1}} \Rightarrow 0 < s_{n+1} < 1$$

   We know:

   $$s_{n+1} = 1 - e^{-s_{n+1}} \Rightarrow s_{n+1} = 1 - (\frac{1}{e})^{s_n}$$

   Since $0 < s_n < 1$ (Based on the induction assumption) and $0 < \frac{1}{e} < 1$, then based on (*) $0 < (\frac{1}{e})^{s_n} < 1$.
   Thus:

   $$0 > -(\frac{1}{e})^{s_n} > -1 \Rightarrow 0 + 1 > -(\frac{1}{e})^{s_n} + 1 > -1 + 1 \Rightarrow 1 > 1 - (\frac{1}{e})^{s_n} > 0$$

[1] During a discussion with both the TA and Dr. Gelaki, I'd be using this fact without being required to prove it.

$$\Rightarrow 0 < s_{n+1} < 1$$

Which completes the proof.

3 Section 1.2: Problem 17

The Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$ are defined by $F_1 = F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 2$$

Prove by induction that

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \quad n \geq 1.$$

3.1 Answer: Prove by Induction

Suppose the left-hand side (LHS) and right-hand side (RHS) are the following parts of the equality in our proof:

$$\underbrace{F_n}_{\text{(LHS)}} = \underbrace{\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},}_{\text{(RHS)}} \quad n \geq 1.$$

Thus, in the basis and the induction step, we will calculate each side individually, showing by considering the induction assumption they will be equal.
For $n = 1$ and $n = 2$ we would first show:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

By directly calculating the right hand side, and then we prove the basis of the induction for $n = 3$. This is because, we would like to use the $F_{n+1} = F_n + F_{n-1}, \quad n \geq 2$ property of Fibonacci numbers in our proof by induction, by assuming the relationship holds for n and $n - 1$

n=1

$$\begin{aligned}
\text{LHS:}\quad & F_1 = 1 \\
\text{RHS:}\quad & \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1 \\
\Rightarrow\quad & \text{LHS} = \text{RHS}
\end{aligned}$$

n=2

$$\begin{aligned}
\text{LHS:}\quad & F_2 = 1 \\
\text{RHS:}\quad & \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} = \frac{(1+2\sqrt{5}+5)-(1-2\sqrt{5}+5)}{4\sqrt{5}} = \frac{4\sqrt{5}}{4\sqrt{5}} = 1 \\
\Rightarrow\quad & \text{LHS} = \text{RHS}
\end{aligned}$$

1. **Basis**, $n = 3$:

$$\begin{aligned}
\text{LHS:} \quad & F_3 = F_2 + F_1 \Rightarrow F_3 = 1 + 1 = 2 \\
\text{RHS:} \quad & \frac{(1+\sqrt{5})^3 - (1-\sqrt{5})^3}{2^3\sqrt{5}} \\
\text{LHS: } F_3 &= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} + \frac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2^1\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} + \frac{2(1+\sqrt{5})^1 - 2(1-\sqrt{5})^1}{2^2\sqrt{5}} \\
&= \frac{2(1+\sqrt{5})^1 - 2(1-\sqrt{5})^1 + (1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^2 + 2(1+\sqrt{5})^1 - (1-\sqrt{5})^2 - 2(1-\sqrt{5})^1}{2^2\sqrt{5}}
\end{aligned}$$

We will multiplying F_3 which is so far $\frac{(1+\sqrt{5})^2+2(1+\sqrt{5})-(1-\sqrt{5})^2-2(1-\sqrt{5})}{2^2\sqrt{5}}$ by $\frac{2}{2}$ which wouldn't change the fraction.

$$\begin{aligned}
\Rightarrow F_3 \times \frac{2}{2} &= \frac{(1+\sqrt{5})^2 + 2(1+\sqrt{5})^1 - (1-\sqrt{5})^2 - 2(1-\sqrt{5})^1}{2^2\sqrt{5}} \times \frac{2}{2} \\
\Rightarrow F_3 &= \frac{2(1+\sqrt{5})^2 + 4(1+\sqrt{5})^1 - 2(1-\sqrt{5})^2 - 4(1-\sqrt{5})^1}{2^3\sqrt{5}} \\
&= \frac{(1+\sqrt{5})(2+2\sqrt{5}+4) - (1-\sqrt{5})(2-2\sqrt{5}+4)}{2^3\sqrt{5}} \\
&= \frac{(1+\sqrt{5})\,(5+2\sqrt{5}+1) - (1-\sqrt{5})\,(5-2\sqrt{5}+1)}{2^3\sqrt{5}}
\end{aligned}$$

$(5+2\sqrt{5}+1)$ and $(5-2\sqrt{5}+1)$ can be written as $(1+\sqrt{5})^2$ and $(1-\sqrt{5})^2$. Thus,

$$\begin{aligned}
F_3 &= \frac{(1+\sqrt{5})(1+\sqrt{5})^2 - (1-\sqrt{5})(1-\sqrt{5})^2}{2^3\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^3 - (1-\sqrt{5})^3}{2^3\sqrt{5}} \\
&= \text{RHS}
\end{aligned}$$

2. **Induction Assumption:**
Suppose $\forall i = 2 \cdots n$ we know that:

$$F_i = F_{i-1} + F_{i-2},$$

and

$$F_i = \frac{(1+\sqrt{5})^i - (1-\sqrt{5})^i}{2^i\sqrt{5}}$$

3. **Induction Step:**
   We would like to prove for F_{n+1} :

$$F_{n+1} = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

$$\begin{aligned}
\text{LHS:} &\quad F_{n+1} = F_n + F_{n-1} \\
\text{RHS:} &\quad \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}
\end{aligned}$$

We would like to show LHS and RHS are equal.
By using the induction hypothesis:

$$\begin{aligned}
\text{LHS: } F_{n+1} &= \frac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{2^{n}\sqrt{5}} + \frac{(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n-1}}{2^{n-1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n} - (1-\sqrt{5})^{n}}{2^{n}\sqrt{5}} + \frac{2(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n-1}}{2^{n}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n} + 2(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n} - 2(1-\sqrt{5})^{n-1}}{2^{n}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n} + 2(1+\sqrt{5})^{n-1} - (1-\sqrt{5})^{n} - 2(1-\sqrt{5})^{n-1}}{2^{n}\sqrt{5}} \times \frac{2}{2} \\
&= \frac{2(1+\sqrt{5})^{n} + 4(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n} - 4(1-\sqrt{5})^{n-1}}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(2(1+\sqrt{5})+4) - (1-\sqrt{5})^{n-1}(2(1-\sqrt{5})+4)}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(2+2\sqrt{5}+4) - (1-\sqrt{5})^{n-1}(2-2\sqrt{5}+4)}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(5+2\sqrt{5}+1) - (1-\sqrt{5})^{n-1}(5-2\sqrt{5}+1)}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n-1}(1+\sqrt{5})^2 - (1-\sqrt{5})^{n-1}(1-\sqrt{5})^2}{2^{n+1}\sqrt{5}} \\
&= \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \\
&= \text{RHS}
\end{aligned}$$

Which completes the proof.

4 Section 1.3: Problem 2

Let $S_k = (1 - \frac{1}{k}, 2 + \frac{1}{k}],\ k \geq 1$. Find

(a) $\bigcup_{k=1}^{\infty} S_k$

(b) $\bigcap_{k=1}^{\infty} S_k$

(c) $\bigcup_{k=1}^{\infty} {S_k}^c$

(d) $\bigcap_{k=1}^{\infty} {S_k}^c$

Answer

(a) $\bigcup_{k=1}^{\infty} S_k = S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4 \cdots = (0,3] \bigcup (\frac{1}{2}, \frac{5}{2}] \bigcup (\frac{2}{3}, \frac{7}{3}] \bigcup (\frac{3}{4}, \frac{9}{4}] \cdots = (0,3]$

(b) $\bigcap_{k=1}^{\infty} S_k = S_1 \bigcap S_2 \bigcap S_3 \bigcap S_4 \cdots = (0,3] \bigcap (\frac{1}{2}, \frac{5}{2}] \bigcap (\frac{2}{3}, \frac{7}{3}] \bigcap (\frac{3}{4}, \frac{9}{4}] \cdots = [1,2]$

(c) $\bigcup_{k=1}^{\infty} {S_k}^c = (\bigcap_{k=1}^{\infty} S_k)^c = \mathbb{R} - \bigcap_{k=1}^{\infty} S_k = \mathbb{R} - [1,2] = (-\infty, 1) \bigcup (2, \infty)$

(d) $\bigcap_{k=1}^{\infty} {S_k}^c = (\bigcup_{k=1}^{\infty} S_k)^c = \mathbb{R} - \bigcup_{k=1}^{\infty} S_k = \mathbb{R} - (0,3] = (-\infty, 0] \bigcup (3, \infty)$

5 Section 1.3: Problem 4

Find the largest ϵ such that S contains an ϵ-neighborhood of x_0.

(a) $x_0 = \frac{3}{4}$, $S = [\frac{1}{2}, 1)$

(b) $x_0 = \frac{2}{3}$, $S = [\frac{1}{2}, \frac{3}{2}]$

(c) $x_0 = 5$, $S = (-1, \infty)$

(d) $x_0 = 1$, $S = (0, 2)$

Answer

The general idea to pick the largest ϵ for x_0 considering $S = (a, b)$ is to first make sure $x_0 \in S$, then pick the $\epsilon = \min\{x_0 - a, b - x_0\}$. This will assure that $(x_0 - \epsilon, x_0 + \epsilon) \in S$, and ϵ is the largest possible without $(x_0 - \epsilon, x_0 + \epsilon)$ falling out of the S.

(a) $x_0 = \frac{3}{4}$, $S = [\frac{1}{2}, 1)$

1. $x_0 \in S$ $\surd$
2. $\min\{x_0 - a, b - x_0\} = \min\{\frac{3}{4} - \frac{1}{2}, 1 - \frac{3}{4}\} = \min\{\frac{1}{4}, \frac{1}{4}\} = \frac{1}{4} \to \epsilon = \frac{1}{4}$

$\Rightarrow (x_0 - \epsilon, x_0 + \epsilon) = (\frac{3}{4} - \frac{1}{4}, \frac{3}{4} + \frac{1}{4}) = (\frac{1}{2}, 1)$

(b) $x_0 = \frac{2}{3}$, $S = [\frac{1}{2}, \frac{3}{2}]$

1. $x_0 \in S$ $\surd$
2. $\min\{x_0 - a, b - x_0\} = \min\{\frac{2}{3} - \frac{1}{2}, \frac{3}{2} - \frac{2}{3}\} = \min\{\frac{1}{6}, \frac{5}{6}\} = \frac{1}{6} \to \epsilon = \frac{1}{6}$

$\Rightarrow (x_0 - \epsilon, x_0 + \epsilon) = (\frac{2}{3} - \frac{1}{6}, \frac{2}{3} + \frac{1}{6}) = (\frac{1}{2}, \frac{5}{6})$

(c) $x_0 = 5$, $S = (-1, \infty)$

1. $x_0 \in S$ $\surd$
2. $\min\{x_0 - a, b - x_0\} = \min\{5 - (-1), \infty - 5\} = \min\{6, \infty\} = 6 \to \epsilon = 6$

$\Rightarrow (x_0 - \epsilon, x_0 + \epsilon) = (5-6, 5+6) = (-1, 11)$

(d) $x_0 = 1$, $S = (0, 2)$

1. $x_0 \in S$ ✓
2. $\min\{x_0 - a, b - x_0\} = \min\{1-0, 2-1\} = \min\{1, 1\} = 1 \to \epsilon = 1$

$\Rightarrow (x_0 - \epsilon, x_0 + \epsilon) = (1-1, 1+1) = (0, 2)$

6 Section 1.3: Problem 5

Describe the following sets as open, closed, or neither, and find S^0, $(S^c)^0$, and $\left(S^0\right)^c$.

(a) $S = (-1, 2) \cup [3, \infty)$

(b) $S = (-\infty, -1) \cup (2, \infty)$

(c) $S = [-3, -2] \cup [7, 8]$

(d) $S = \{x \mid x = \text{integer}\}$

Answer

To find $(S^c)^0$ we have to find S^c, and the find the set of its interior points. Thus, in number 4 for each set, we will find S^c.

(a) $S = (-1, 2) \cup [3, \infty)$

1. Description: S is neither closed or open. It's not open, because $3 \in S$ doesn't belong to S^0. It's not closed because then it means S^c is open, which is not true since $2 \in S^c$ doesn't belong to $(S^c)^0$.
2. $S^0 = (-1, 2) \cup (3, \infty)$
3. $\left(S^0\right)^c = \mathbb{R} - S^0 = (-\infty, -1] \cup [2, 3)$
4. $S^c = \mathbb{R} - S = (-\infty, -1] \cup [2, 3)$
5. $(S^c)^0 = (-\infty, -1) \cup (2, 3)$

(b) $S = (-\infty, 1) \cup (2, \infty)$

1. Description: S is the union of open sets, thus it's open.
2. $S^0 = (-\infty, 1) \cup (2, \infty)$
3. $\left(S^0\right)^c = \mathbb{R} - S^0 = [1, 2]$
4. $S^c = \mathbb{R} - S = [1, 2]$
5. $(S^c)^0 = (1, 2)$

(c) $S = [-3, -2] \cup [7, 8]$

1. Description: S is the union of closed sets, thus it's closed.
2. $S^0 = (-3, -2) \cup (7, 8)$
3. $\left(S^0\right)^c = \mathbb{R} - S^0 = (-\infty, -3] \cup [-2, 7] \cup [8, \infty)$

4. $S^c = \mathbb{R} - S = (-\infty, -3) \cup (-2, 7) \cup (8, \infty)$
5. $(S^c)^0 = (-\infty, -3) \cup (-2, 7) \cup (8, \infty)$

(d) $S = \{x \mid x = \text{integer}\}$

1. Description: S is closed, and to show that, we will show $S^c = \mathbb{R} - \mathbb{Z}$, is open by showing $\forall x \in S^c$ there's an $\epsilon-$neighborhood of x in S^c, meaning that x is an interior point in S. Consider n_M as the smallest integer that is greater than x. In addition, consider n_m as the greatest integer that's smaller than x. Suppose $\epsilon = \min\{x - n_m, n_M - x\}$. Then, this $\epsilon-$neighborhood of x in completely in S^c. Thus, S^c is open, and $(S^c)^c$ which is S is closed.
2. $S^0 = \emptyset$
3. $\left(S^0\right)^c = \mathbb{R} - S^0 = \mathbb{R}$
4. $S^c = \mathbb{R} - S = \{x \mid x = \text{not integer}\}$
5. $(S^c)^0 = S^c$

7 Section 1.3: Problem 8

(a) Show that the intersection of finitely many open sets is open.

(b) Give an example showing that the intersection of infinitely many open sets may fail to be open.

Answer

(a) Suppose $S = \bigcap_{i=1}^{n} S_i$. We will show $\forall x \in S$, there's an $\epsilon-$neighborhood of x, N such that $N \subseteq S$.
Let $x_0 \in S$. Then:

$$x_0 \in \bigcap_{i=1}^{n} S_i \to x_0 \in S_i \quad \forall i$$

Since S_is are open, Then $S_i = (S_i)^0$, meaning that $\forall x \in S_i$ there's an ϵ_i-neighborhood of x, N_i such that $N_i \subseteq S_i$. Thus, $\exists \epsilon_1, \epsilon_2, \cdots, \epsilon_n > 0$ such that:

$$(x - \epsilon_1, x + \epsilon_1) \subseteq S_1, (x - \epsilon_2, x + \epsilon_2) \subseteq S_2, \cdots, (x - \epsilon_n, x + \epsilon_n) \subseteq S_n$$

Suppose $\epsilon := \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$, then, if we consider $N = (x - \epsilon, x + \epsilon)$:

$$N \subseteq N_1, N \subseteq N_2, \cdots, N \subseteq N_n \Rightarrow N \subseteq S_1, N \subseteq S_2, \cdots, N \subseteq S_n \Rightarrow N \subseteq \bigcap_{i=1}^{n} S_i \Rightarrow N \subseteq S$$

Which completes the proof.
Please note that the finite assumption is used when considering $\epsilon := \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. We can't assume such when we have infinitely many open sets!

(b) Suppose $S_n = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n})\ \forall n \in \mathbb{Z}^+$, and $S = \bigcap_{i=1}^{\infty} S_i$. Then:

$$S = \bigcap_{i=1}^{\infty} S_i \Rightarrow S = \{\frac{1}{2}\}$$

Consider $S^c = \mathbb{R} - \{\frac{1}{2}\}$. Since, for any element in S^c there's a neighborhood of it that it's completely in S^c, this set is open. Therefore, $\{\frac{1}{2}\}$ is closed.

8 Section 1.3: Problem 9

(a) Show that the union of finitely many closed sets is close.

(b) Give an example showing that the union of infinitely many closed sets may fail to be close.

Answer

(a) Suppose $S = \bigcup_{i=1}^{n} S_i$. Then $S^c = (\bigcup_{i=1}^{n} S_i)^c = \bigcap_{i=1}^{n} {S_i}^c$. Since S_is are closed, ${S_i}^c$s are open. Based on the previous problem's part (a) the intersection of finite open sets is open. Thus $\bigcap_{i=1}^{n} {S_i}^c$ is open. Therefore its complete is closed. Hence S^c is open, and $(S^c)^c$ is closed, which means S is closed.

(b) Suppose $S_n = [\frac{1}{n}, 1]\ \forall n \in \mathbb{Z}^+$, and $S = \bigcup_{i=1}^{\infty} S_i$. Then we will show:

$$S = \bigcup_{i=1}^{\infty} S_i \Rightarrow S = (0, 1]$$

$\forall 0 < \epsilon \leq 1$ and by using the Archimedean property of real numbers [1], we can consider n begin the first positive integer number greater than $\frac{1}{\epsilon}$ (consider $\epsilon = 1, \rho = \frac{1}{\epsilon}$ in the Archimedean theorem) then:

$$\frac{1}{\epsilon} \leq n \rightarrow \frac{1}{n} \leq \epsilon \text{ and } \epsilon \leq 1 \rightarrow \epsilon \in S_n \Rightarrow \epsilon \in S$$

Thus,

$$S = (0, 1]$$

As you can see S is neither open or closed.

9 Section 1.3: Problem 11

Find the set of limit points of S, ∂S, $\bar{S}$, the set of isolated points of S, and the exterior of S.

(a) $S = (-\infty, -2) \cup (2, 3) \cup \{4\} \cup (7, \infty)$

(b) $S = \{\text{all integers}\}$

(c) $S = \cup\{(n, n+1) \mid n = \text{ integer}\}$

(d) $S = \left\{x \mid x = \frac{1}{n},\ n = 1, 2, 3, \ldots\right\}$

Answer

(a) $S = (-\infty, -2) \cup (2, 3) \cup \{4\} \cup (7, \infty)$

1. Set of limit points of $S = (-\infty, -2] \cup [2, 3] \cup [7, \infty)$
2. $\delta S = \{-2, 2, 3, 4, 7\}$
3. $\bar{S} = S \cup \delta S = (-\infty, -2] \cup [2, 3] \cup \{4\} \cup [7, \infty)$
4. Set of isolated points of $S = \{4\}$
5. Exterior of $S = (S^c)^0$:

$$S^c = \mathbb{R} - S = [-2, 2] \cup [3, 4) \cup (4, 7]$$
$$\Rightarrow S = (S^c)^0 = (-2, 2) \cup (3, 4) \cup (4, 7)$$

(b) $S = \{\text{all integers}\}$

1. Set of limit points of $S = \emptyset$
2. $\delta S = S$: For every $\epsilon-$neighborhood of element $x \in S$, the element itself is in S, and the deleted neighborhood is in S^c. For $x \notin S$, we can find an $\epsilon-$neighborhood that doesn't contain any point in S.
3. $\bar{S} = S \cup \delta S = S$
4. Set of isolated points of $S = S$
5. Exterior of $S = (S^c)^0$:

$$S^c = \mathbb{R} - \mathbb{Z} = \cdots (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup \cdots$$
$$\Rightarrow S = (S^c)^0 = \cdots (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup \cdots$$

(c) $S = \cup\{(n, n+1) \mid n = \text{ integer}\}$

1. Set of limit points of $S = \cup\{[n, n+1] \mid n = \text{ integer}\} = \mathbb{R}$
2. $\delta S = \{n \mid n = \text{ integer}\}$
3. $\bar{S} = S \cup \delta S = \cup\{[n, n+1] \mid n = \text{ integer}\} = \mathbb{R}$
4. Set of isolated points of $S = \emptyset$
5. Exterior of $S = (S^c)^0$:

$$S^c = \mathbb{R} - S = \{n \mid n = \text{ integer}\}$$
$$\Rightarrow S = (S^c)^0 = \emptyset$$

(d) $S = \left\{x \mid x = \frac{1}{n}, \, n = 1, 2, 3, \dots \right\}$

1. Set of limit points of $S = \emptyset$. Suppose $x \in S$. Any $\epsilon-$neighborhood of x contains some irrational numbers that can't be in S based on its definition.
2. $\delta S = S$
3. $\bar{S} = S \cup \delta S = S$
4. Set of isolated points of $S = S$
5. Exterior of $S = (S^c)^0$:

$$S^c = \mathbb{R} - S = (-\infty, 0] \cup [1, \infty) \cup (1, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{4}) \cup (1, 2) \cup \cdots$$
$$\Rightarrow S = (S^c)^0 = (-\infty, 0) \cup (1, \infty) \cup (1, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{4}) \cup (1, 2) \cup \cdots$$

10 Section 1.3: Problem 12

Prove: A limit point of a set S is either an interior point or a boundary point of S.

Answer

Let $x_0 \in \mathbb{R}$ be a limit point of S. In addition, consider $\forall \epsilon > 0$, $\epsilon-$neighborhood of x_0 is called N_ϵ. Since x_0 is a limit point of S, then:

$$N_\epsilon \backslash \{x_0\} \cap S \neq \emptyset$$

Consider two following scenarios:

1. $x_0 \in S$: then $x_0 \in N_\epsilon \cap S$, $\forall \epsilon > 0$. There are two possible situations that might occur:
   (a) $\forall \epsilon > 0$:
   $$N_\epsilon \cap S^c \neq \emptyset$$
   Then, every $\epsilon-$neighborhood of x_0 contains at least one point in S^c, and at least one point in S (which is x_0). Thus, x_0 is a **boundary point**.
   (b) $\exists \epsilon_0 > 0$ such that:
   $$N_{\epsilon_0} \cap S^c = \emptyset$$
   Which means S contains ϵ_0-neighborhood of x_0. Therefore x_0 is an **interior point** of S.
2. $x_0 \notin S$: then $x_0 \in S^c$, and since $N_\epsilon \backslash \{x_0\} \cap S \neq \emptyset$ $\forall \epsilon > 0$, every $\epsilon-$neighborhood of x_0 contains at least one point in S, and at least one point in S^c (which is x_0). Thus, x_0 is a **boundary point**.

11 Section 1.3: Problem 13

Prove: An isolated point of S is a boundary point of S^c.

Answer

Suppose $\forall \epsilon > 0$, $\epsilon-$neighborhood of x_0 is called N_ϵ. To prove x_0 is a boundary point of S^c, we need to prove:

$$N_\epsilon \cap S^c \neq \emptyset \text{ and } N_\epsilon \cap S \neq \emptyset, \quad \forall \epsilon > 0$$

Since, x_0 is an isolated point of S, $x_0 \in S$. Therefore $x_0 \in N_\epsilon \cap S$, $\forall \epsilon > 0$. Thus, $N_\epsilon \cap S \neq 0$. Additionally, since x_0 is an isolated point of S, there exists an $\epsilon-$neighborhood of x_0 that contains no other point in S. Meaning, $\exists \epsilon_0 > 0$ such that:

$$N_{\epsilon_0} - \{x_0\} \subseteq S^c$$

Let $\epsilon_0 \neq \epsilon_1 > 0$ be another $\epsilon-$neighborhood of x_0. There will be two possible outcomes:

1. $\epsilon_1 > \epsilon_0$, then:
   $$N_{\epsilon_0} \subseteq N_{\epsilon_1}$$
   Therefore, every member in N_{ϵ_0} is in N_{ϵ_1}, including x_0, and many others. Hence, if we consider $N_{\epsilon_0} \backslash \{x_0\}$:
   $$N_{\epsilon_0} \backslash \{x_0\} \cap N_{\epsilon_1} \neq \emptyset$$

By considering $N_{\epsilon_0}\backslash\{x_0\} \cap N_{\epsilon_1}$, since every element of this set is in $N_{\epsilon_0}\backslash\{x_0\}$, and S^c contains $N_{\epsilon_0}\backslash\{x_0\}$, therefore every element of this set is also in S^c, which leads to

$$N_{\epsilon_1} \cap S^c \neq \emptyset.$$

2. $\epsilon_1 < \epsilon_0$, then:

$$N_{\epsilon_1} \subseteq N_{\epsilon_n}$$

This means every member in N_{ϵ_1} is in N_{ϵ_0}, including x_0, and many others. Hence, if we consider $N_{\epsilon_1} - \{x_0\}$:

$$N_{\epsilon_1} \cap N_{\epsilon_0}\backslash\{x_0\} \neq \emptyset$$

By considering $N_{\epsilon_1} \cap N_{\epsilon_0}\backslash\{x_0\}$, since every element of this set is in $N_{\epsilon_0}\backslash\{x_0\}$, and S^c contains $N_{\epsilon_0}\backslash\{x_0\}$, therefore every element of this set is also in S^c, which leads to

$$N_{\epsilon_1} \cap S^c \neq \emptyset.$$

This will show:

$$N_\epsilon \cap S^c \neq 0, \quad \forall \epsilon > 0$$

Which completes the proof.

12 Section 1.3: Problem 14

Prove:

(a) A boundary point of a set S is either a limit point or an isolated point of S.

(b) A set S is closed if and only if $S = \bar{S}$.

Answer

(a) Suppose $\forall \epsilon > 0$, $\epsilon-$neighborhood of x_0 is called N_ϵ. Let x_0 be a boundary point of S, then

$$N_\epsilon \cap S^c \neq \emptyset \text{ and } N_\epsilon \cap S \neq \emptyset, \quad \forall \epsilon > 0$$

Consider two following scenarios:

1. $x_0 \in S$: then $x_0 \in N_\epsilon \cap S$, $\forall \epsilon > 0$. In addition, since $N_\epsilon \cap S^c \neq \emptyset$ and the fact that $x_0 \notin S^c$, leads to $N_{\epsilon_0}\backslash\{x_0\} \bigcap S^c \neq \emptyset$. This will lead to two possible situations that might occur:

   1.1. $\exists \epsilon_0 > 0$ such that:

   $$N_{\epsilon_0}\backslash\{x_0\} \cap S^c = N_{\epsilon_0}\backslash\{x_0\}$$

   Then, we found a ϵ_0-neighborhood of x_0 that contains no other point than x_0 in S. Therefore, x_0 is an **isolated point**.

   1.2. $\forall \epsilon > 0$:

   $$N_\epsilon\backslash\{x_0\} \cap S^c \subset N_\epsilon\backslash\{x_0\}$$

   Which means every deleted $\epsilon-$neighborhood of x_0 contains at least one point in S. Thus, x_0 is a **limit point**.

2. $x_0 \notin S$: then $x_0 \in S^c$, and since $N_\epsilon \cap S \neq \emptyset$ $\forall \epsilon > 0$, in addition to the fact that $x_0 \notin S$ leads to:

$$N_\epsilon \backslash \{x_0\} \cap S \neq \emptyset \quad \forall \epsilon > 0$$

Therefore x_0 is a **limit point** of S.

(b) We want to show S is closed $\Leftrightarrow S = \bar{S}$.

$\Rightarrow$ S is closed: To show $S = \bar{S}$, we have to show two things:

$$S \subseteq \bar{S} \text{ and } \bar{S} \subseteq S$$

Since $\bar{S} = S \bigcup \delta S$, therefore $S \subseteq \bar{S}$.
We will prove $\bar{S} \subseteq S$ by contradiction.
Suppose $\bar{S} \nsubseteq S$. Then, $\exists x_0 \in \bar{S}$ such that $x_0 \notin S$. Thus, $x_0 \in \delta S$. Meaning that every $\epsilon-$neighborhood of x_0 contains at least one point in S, and at least one point in S^c. Consider $\forall \epsilon > 0$, $\epsilon-$neighborhood of x_0 is called N_ϵ.

$$N_\epsilon \cap S \neq \emptyset \text{ and } x_0 \notin S \Rightarrow N_\epsilon \backslash x_0 \bigcap S \neq \emptyset, \quad \forall \epsilon > 0$$

Then, any deleted neighborhood of x_0 contains a point in S. Thus, x_0 is a limit point. Since S is closed, it contains all of its limit point. Which is a contradiction. Therefor $\bar{S} \subseteq S$.

$\Leftarrow$ $S = \bar{S}$: To show S is closed, we have to show it contains all of its limit points.
Prove by Contradiction: Suppose the opposite is true. Meaning that the $\exists x_0 \notin S$ which is a limit point of S. Then in that case, $x_0 \in S^c$. In addition, since x_0 is a limit point, it means any deleted neighborhood of x_0 contains a point in S. Then x_0 is a boundary point, since for all $\epsilon-$neighborhood of x_0 contains a point in S, and contains x_0 in S^c. We know $S = \bar{S}$. Also, we know $\bar{S} = S \bigcup \delta S$, where δS is the set of boundary points in S including x_0. Thus:

$$x_0 \in \delta S \in\Rightarrow x \in \bar{S} \Rightarrow x_0 \in S$$

Which is a contradiction. Therefore S contains all of its limit point, hence it's closed.

References

[1] William F Trench. *Introduction to real analysis.* 2013.