MATH 414 Analysis I,
Homework 1

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1 Problem 2

Verify that the set consists of two members, 0 and 1, with operations defined by Eqns. (1.1.1) and (1.1.2) is a field. Then show that it is impossible to define an order $<$ on this field that has properties (F), (G), and (H).

Answer

To show that $\mathbb{F} = (S = \{0, 1\}, +, \cdot)$ is a field with addition defined by

$$0 + 0 = 1 + 1 = 0, \quad 1 + 0 = 0 + 1 = 1;$$

and multiplication defined by

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

we have two show it has **(A) - (E)** properties.

(**A**) commutative laws, $a + b = b + a, ab = ba$:

- (a) If $a = 1$ and $b = 1$:
  - i. $a + b = 1 + 1 = 0$ (based on addition definition) and $b + a = 1 + 1 = 0$ (based on addition definition)$\rightarrow a + b = b + a$.
  - ii. $ab = 1 \cdot 1 = 1$ (based on multiplication definition) and $ba = 1 \cdot 1 = 1$ (based on multiplication definition)$\rightarrow ab = ba$.
- (b) If $a = 0$ and $b = 1$:
  - i. $a + b = 0 + 1 = 1$ (based on addition definition) and $b + a = 1 + 0 = 1$ (based on addition definition)$\rightarrow a + b = b + a$.
  - ii. $ab = 0 \cdot 1 = 0$ (based on multiplication definition) and $ba = 1 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow ab = ba$.
- (c) If $a = 0$ and $b = 0$:
  - i. $a + b = 0 + 0 = 0$ (based on addition definition) and $b + a = 0 + 0 = 0$ (based on addition definition)$\rightarrow a + b = b + a$.
  - ii. $ab = 0 \cdot 0 = 0$ (based on multiplication definition) and $ba = 0 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow ab = ba$.

(d) If $a = 1$ and $b = 0$:

i. $a + b = 1 + 0 = 1$ (based on addition definition) and $b + a = 0 + 1 = 1$ (based on addition definition)$\rightarrow a + b = b + a$.

ii. $ab = 1 \cdot 0 = 0$ (based on multiplication definition) and $ba = 0 \cdot 1 = 0$ (based on multiplication definition)$\rightarrow ab = ba$.

(**B**) associative laws, $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$

(a) If $a = 1$, $b = 1$ and $c = 0$:

i. $(a+b)+c = (1+1)+0 = 0+0 = 0$ (based on addition definition) and $a+(b+c) = 1 + (1 + 0) = 1 + 1 = 0$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (1\cdot1)\cdot0 = 1\cdot0 = 0$ (based on multiplication definition) and $a(bc) = 1\cdot(1\cdot0) = 1 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(b) If $a = 1$, $b = 1$ and $c = 1$:

i. $(a+b)+c = (1+1)+1 = 1+1 = 0$ (based on addition definition) and $a+(b+c) = 1 + (1 + 1) = 1 + 1 = 0$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (1\cdot1)\cdot1 = 1\cdot1 = 1$ (based on multiplication definition) and $a(bc) = 1\cdot(1\cdot1) = 1 \cdot 1 = 1$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(c) If $a = 1$, $b = 0$ and $c = 0$:

i. $(a+b)+c = (1+0)+0 = 1+0 = 1$ (based on addition definition) and $a+(b+c) = 1 + (0 + 0) = 1 + 0 = 1$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (1\cdot0)\cdot0 = 0\cdot0 = 0$ (based on multiplication definition) and $a(bc) = 1\cdot(0\cdot0) = 1 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(d) If $a = 1$, $b = 0$ and $c = 1$:

i. $(a+b)+c = (1+0)+1 = 1+1 = 0$ (based on addition definition) and $a+(b+c) = 1 + (0 + 1) = 1 + 1 = 0$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (1\cdot0)\cdot1 = 0\cdot1 = 0$ (based on multiplication definition) and $a(bc) = 1\cdot(0\cdot1) = 1 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(e) If $a = 0$, $b = 1$ and $c = 0$:

i. $(a+b)+c = (0+1)+0 = 1+0 = 1$ (based on addition definition) and $a+(b+c) = 0 + (1 + 0) = 0 + 1 = 1$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (0\cdot1)\cdot0 = 0\cdot0 = 0$ (based on multiplication definition) and $a(bc) = 0\cdot(1\cdot0) = 0 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(f) If $a = 0$, $b = 1$ and $c = 1$:

i. $(a+b)+c = (0+1)+1 = 1+1 = 0$ (based on addition definition) and $a+(b+c) = 0 + (1 + 1) = 0 + 0 = 0$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (0\cdot1)\cdot1 = 0\cdot1 = 0$ (based on multiplication definition) and $a(bc) = 0\cdot(1\cdot1) = 0 \cdot 1 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(g) If $a = 0$, $b = 0$ and $c = 0$:

i. $(a+b)+c = (0+0)+0 = 0+0 = 0$ (based on addition definition) and $a+(b+c) = 0 + (0 + 0) = 0 + 0 = 0$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

ii. $(ab)c = (0\cdot0)\cdot0 = 0\cdot0 = 0$ (based on multiplication definition) and $a(bc) = 0\cdot(0\cdot0) = 0 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(h) If $a = 0$, $b = 0$ and $c = 1$:

  i. $(a + b) + c = (0 + 0) + 1 = 0 + 1 = 1$ (based on addition definition) and $a + (b + c) = 0 + (0 + 1) = 0 + 1 = 1$ (based on addition definition)$\rightarrow (a + b) + c = a + (b + c)$.

  ii. $(ab)c = (0\cdot 0)\cdot 1 = 0\cdot 1 = 0$ (based on multiplication definition) and $a(bc) = 0\cdot(0\cdot 1) = 0 \cdot 0 = 0$ (based on multiplication definition)$\rightarrow (ab)c = a(bc)$.

(**C**) distributive law, $a(b + c) = ab + ac$

(a) If $a = 1$, $b = 1$ and $c = 0$: $a(b + c) = 1 \cdot (1 + 0) = 1 \cdot 1 = 1$ (based on addition and multiplication definition) and $ab + ac = 1 \cdot 1 + 1 \cdot 0 = 1 + 0 = 1$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(b) If $a = 1$, $b = 1$ and $c = 1$: $a(b + c) = 1 \cdot (1 + 1) = 1 \cdot 0 = 0$ (based on addition and multiplication definition) and $ab + ac = 1 \cdot 1 + 1 \cdot 1 = 1 + 1 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(c) If $a = 1$, $b = 0$ and $c = 0$: $a(b + c) = 1 \cdot (0 + 0) = 1 \cdot 0 = 0$ (based on addition and multiplication definition) and $ab + ac = 1 \cdot 0 + 1 \cdot 0 = 0 + 0 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(d) If $a = 1$, $b = 0$ and $c = 1$: $a(b + c) = 1 \cdot (0 + 1) = 1 \cdot 1 = 1$ (based on addition and multiplication definition) and $ab + ac = 1 \cdot 0 + 1 \cdot 1 = 0 + 1 = 1$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(e) If $a = 0$, $b = 1$ and $c = 0$: $a(b + c) = 0 \cdot (1 + 0) = 0 \cdot 1 = 0$ (based on addition and multiplication definition) and $ab + ac = 0 \cdot 1 + 0 \cdot 0 = 0 + 0 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(f) If $a = 0$, $b = 1$ and $c = 1$: $a(b + c) = 0 \cdot (1 + 1) = 0 \cdot 0 = 0$ (based on addition and multiplication definition) and $ab + ac = 0 \cdot 1 + 0 \cdot 1 = 0 + 0 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(g) If $a = 0$, $b = 0$ and $c = 0$: $a(b + c) = 0 \cdot (0 + 0) = 0 \cdot 0 = 0$ (based on addition and multiplication definition) and $ab + ac = 0 \cdot 0 + 0 \cdot 0 = 0 + 0 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(h) If $a = 0$, $b = 0$ and $c = 1$: $a(b + c) = 0 \cdot (0 + 1) = 0 \cdot 1 = 0$ (based on addition and multiplication definition) and $ab + ac = 0 \cdot 0 + 0 \cdot 1 = 0 + 0 = 0$ (based on addition and multiplication definition)$\rightarrow a(b + c) = ab + ac$.

(**D**) we show 1 is a unit such that $\forall a \in \mathbb{F} : a1 = a$, and 0 is additive such that $\forall a \in \mathbb{F} : a + 0 = a$:

(a) If $a = 0$: $a1 = 0\cdot 1 = 0$ (based on multiplication definition) $\rightarrow a1 = a$, and $a+0 = 0+0 = 0$ (based on addition definition) $\rightarrow a + 0 = a$.

(b) If $a = 1$: : $a1 = 1 \cdot 1 = 1$ (based on multiplication definition) $\rightarrow a1 = a$, and $a + 0 = 1 + 0 = 1$ (based on addition definition) $\rightarrow a + 0 = a$.

(**E**) we show $\forall a \in \mathbb{F} : \exists - a : a + -a = 0$

(a) If $a = 0$, and $-a = 0$: $a + (-a) = 0 + 0 = 0$ (based on addition definition) $\rightarrow a + -a = 0$.

(b) If $a = 1$, and $-a = 1$: $a + (-a) = 1 + 1 = 0$ (based on addition definition) $\rightarrow a + -a = 0$.

we also show this field is not an ordered field by showing it at least doesn't have one of **(F) - (H)** properties. To show this, we show this field doesn't have **(H)** property, meaning that $a < b$ doesn't necessarily mean $\forall c \in \mathbb{F} : a + c < b + c$
If $a = 0$, $b = 1$, and $c = 1$:

$$a < b, \quad a+c = 0+1 = 1, \quad b+c = 1+1 = 0 \text{ (based on the addition definition)} \quad \text{but} \quad a+c \not< b+c.$$

Thus, this field it's not an ordered field.

2 Problem 4

Show that $\sqrt{p}$ is irrational if p is prime.

Answer

Prove by Contradiction
For the sake of the contradiction, suppose $\sqrt{p}$ is rational. Then: $\exists a, b \in \mathbb{Z}$ such that $\sqrt{p} = \frac{a}{b}$, where $b \neq 0$. In addition, we can assume a and b are not both divisible by the same natural numbers, and if that's the case, first, we will divide them by the greatest natural number possible. This division wouldn't change the fraction result, meaning it will still be equal to $\sqrt{p}$.

$$\begin{aligned}
&\sqrt{p} = \frac{a}{b} \\
&\Rightarrow p = \frac{a^2}{b^2} \\
&\text{since } b \neq 0 \Rightarrow p \times b^2 = a^2
\end{aligned}$$

The left-hand side is divisible by p, thus has to be the right-hand side. In this case, a^2 is divisible by p, so it has to be a. Hence, consider $a = pk$, then:

$$\begin{aligned}
& p \times b^2 = (pk)^2 \\
\Leftrightarrow \quad & p \times b^2 = p^2 k^2 \\
\text{since } p \neq 0 \Leftrightarrow \quad & b^2 = p \cdot k^2
\end{aligned}$$

The right-hand side is divisible by p, thus has to be the left-hand side. In this case, b^2 is divisible by p, so it has to be b. Thus a and b are both divisible by p. However, we assumed a and b are not both divisible by the same natural numbers, which results in a contradiction. Hence, $\sqrt{p}$ is irrational if p is prime.

3 Problem 5

Find the supremum and infimum of each S. State whether they are in S.

(a) $S = \{x \mid x = -\frac{1}{n} + [1 + (-1)^n]n^2 \text{ for } n \geq 1\}$

(b) $S = \{x \mid x^2 < 9\}$

(c) $S = \{x \mid x^2 \leq 7\}$

(d) $S = \{x \mid |2x+1| < 5\}$

(e) $S = \{x \mid (x^2+1)^{-1} > \frac{1}{2}\}$

(e) $S = \{x \mid \text{x = rational and} \quad x^2 \leq 7\}$

Answer

(a) $S = \{x \mid x = -\frac{1}{n} + [1+(-1)^n]n^2 \text{ for } n \geq 1\}$: To indicate the $\sup S$ and $\inf S$, first we will write down the result of the series for the first couple of n values:

- $n = 1 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{1} + [1+(-1)^1]1^2 = -1 + [1-1] \cdot 1 = -1 + 0 = -1.$
- $n = 2 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{2} + [1+(-1)^2]2^2 = -0.5 + [1+1] \cdot 4 = -0.5 + 8 = 7.5.$
- $n = 3 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{3} + [1+(-1)^3]3^2 = -\frac{1}{3} + [1-1] \cdot 9 = -\frac{1}{3} + 0 = -\frac{1}{3}.$
- $n = 4 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{4} + [1+(-1)^4]4^2 = -\frac{1}{4} + [1+1] \cdot 16 = -0.25 + 32 = 31.75.$
- $n = 5 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{5} + [1+(-1)^5]5^2 = -\frac{1}{5} + [1-1] \cdot 25 = -0.2 + 0 = -0.2.$
- $n = 6 : -\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{6} + [1+(-1)^6]6^2 = -\frac{1}{6} + [1+1] \cdot 36 = -\frac{1}{6} + 72 = \frac{431}{6} \simeq 71.83.$
Thus, for general cases:
- n odd : $-\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{n} + [1+-1]1^2 = -\frac{1}{n} + 0 = -\frac{1}{n}.$
- n even: $-\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{n} + [1+1] \cdot n^2 = -\frac{1}{n} + 2 \cdot n^2.$

(a) $\sup S = \infty$. We show that S is not bounded above by showing for every even n, $S(n) < S(n+2)$.
n even: $-\frac{1}{n} + [1+(-1)^n]n^2 = -\frac{1}{n} + [1+1] \cdot n^2 = -\frac{1}{n} + 2 \cdot n^2.$
$n+2$ is also even: $-\frac{1}{n+2} + [1+(-1)^{n+2}](n+2)^2 = -\frac{1}{n+2} + [1+1] \cdot (n+2)^2 = -\frac{1}{n+2} + 2 \cdot (n+2)^2.$

$$-\frac{1}{n} < -\frac{1}{n+2} \quad \text{and} \quad 2 \cdot n^2 < 2 \cdot (n+2)^2 \rightarrow -\frac{1}{n} + 2 \cdot n^2 < -\frac{1}{n+2} + 2 \cdot (n+2)^2 \rightarrow S(n) < S(n+2)$$

(b) $\inf S = -1$: S is bounded below by -1, since $\forall n \in S, n$ odd: $S(n) = -\frac{1}{n}$ and $-\frac{1}{n} \geq -1$. In addition, the $\inf S$ cannot be greater than -1 because, in that case, for $n = 1 : S(n) = -1$, and it would be greater than $\inf S$.

The $\inf S$ is in S, which happens if $n = 1$ and $\sup S = \infty$.

(b) $S = \{x \mid x^2 < 9\}$: $x^2 < 9 \rightarrow \mid x \mid < 3 \rightarrow -3 < x < 3 \rightarrow S$ is bounded above and bounded below. Thus, it has a supremum and an infimum based on property **(I)**, and they're unique based on theorem 1.1.3 and theorem 1.1.8 of [1]. We show that $\sup S = 3$ and $\inf S = -3$.

(a) $\sup S = 3$: if $x \in S$, then $x < 3$. Suppose $\exists \epsilon > 0$ s.t. $\sup S = 3 - \epsilon$. Based on the density of rational numbers in $\mathbb{R}$ [1] $\exists r_0 \in \mathbb{Q}$ s.t $3 - \epsilon < r_0 < 3$ which contradicts part (a) of theorem 1.1.3 in [1]. Thus $\sup S = 3$.

(b) $\inf S = -3$: This would be proven similar to part (a). Since S is bounded below, then $-S$ is bounded above. By following the same procedure, we get $\sup(-S) = 3$ leading to $\inf S = -3$.

Finally, none of the $\sup S$ and $\inf S$ are in S.

(c) $S = \{x \mid x^2 \leq 7\}$: $x^2 \leq 7 \rightarrow \mid x \mid \leq \sqrt{7} \rightarrow -\sqrt{7} \leq x \leq \sqrt{7} \rightarrow S$ is bounded above and bounded below. Thus it has a supremum and an infimum based on property **(I)**, and they're unique based on theorem 1.1.3 and theorem 1.1.8 of [1]. We show that $\sup S = \sqrt{7}$ and $\inf S = -\sqrt{7}$.

(a) $\sup S = \sqrt{7}$: if $x \in S$, then $x \leq \sqrt{7}$. Suppose $\exists \epsilon > 0$ s.t. $\sup S = \sqrt{7} - \epsilon$. Based on the density of rational numbers in $\mathbb{R}$ [1] $\exists r_0 \in \mathbb{Q}$ s.t $\sqrt{7} - \epsilon < r_0 < \sqrt{7}$ which contradicts part (a) of theorem 1.1.3 in [1]. Thus $\sup S = \sqrt{7}$.

(b) $\inf S = -\sqrt{7}$: This would be proven similar to part (a). Since S is bounded below, then $-S$ is bounded above. By following the same procedure, we get $\sup(-S) = \sqrt{7}$ leading to $\inf S = -\sqrt{7}$.

Finally, both of the $\sup S$ and $\inf S$ are in S.

(d) $S = \{x \mid |2x + 1| < 5\}$: $-5 < 2x + 1 < 5 \rightarrow -6 < 2x < 4 \rightarrow -3 < x < 2 \rightarrow S$ is bounded above and bounded below. Thus it has a supremum and an infimum based on property **(I)**, and they're unique based on theorem 1.1.3 and theorem 1.1.8 of [1]. We show that $\sup S = 2$ and $\inf S = -3$.

(a) $\sup S = 2$: if $x \in S$, then $x < 2$. Suppose $\exists \epsilon > 0$ s.t. $\sup S = 2 - \epsilon$. Based on the density of rational numbers in $\mathbb{R}$ [1] $\exists r_0 \in \mathbb{Q}$ s.t $2 - \epsilon < r_0 < 2$ which contradicts part (a) of theorem 1.1.3 in [1]. Thus $\sup S = 2$.

(b) $\inf S = -3$: This would be proven similar to part (a). Since S is bounded below, then $-S$ is bounded above. By following the same procedure, we get $\sup(-S) = 3$ leading to $\inf S = -3$.

Finally, none of the $\sup S$ and $\inf S$ are in S.

(e) $S = \{x \mid (x^2 + 1)^{-1} > \frac{1}{2}\}$: $0 \leq x^2 \rightarrow 0 < x^2 + 1$. Thus, based on property **(E)** $x^2 + 1$ has a multiplication inverse. Based on the same argument, $\frac{1}{2}$ also has an inverse. We multiply both sides of the inequality in S by each side's multiplication inverse, and since they're both positive, the inequality side wouldn't change.

$$\frac{1}{2} < (x^2 + 1)^{-1} \rightarrow (x^2 + 1) < 2 \rightarrow x^2 < 1 \rightarrow \mid x \mid < 1 \rightarrow -1 < x < 1.$$

$\rightarrow S$ is bounded above and bounded below. Thus it has a supremum and an infimum based on property **(I)**, and they're unique based on theorem 1.1.3 and theorem 1.1.8 of [1]. We show that $\sup S = 1$ and $\inf S = -1$.

(a) $\sup S = 1$: if $x \in S$, then $x < 1$. Suppose $\exists \epsilon > 0$ s.t. $\sup S = 1 - \epsilon$. Based on the density of rational numbers in $\mathbb{R}$ [1] $\exists r_0 \in \mathbb{Q}$ s.t $1 - \epsilon < r_0 < 3$ which contradicts part (a) of theorem 1.1.3 in [1]. Thus $\sup S = 1$.

(b) $\inf S = -1$: This would be proven similar to part (a). Since S is bounded below, then $-S$ is bounded above. By following the same procedure, we get $\sup(-S) = 1$ leading to $\inf S = -1$.

Finally, none of the $\sup S$ and $\inf S$ are in S.

(f) $S = \{x \mid \text{x = rational and} \quad x^2 \leq 7\}$: $x^2 \leq 7 \rightarrow \mid x \mid \leq \sqrt{7} \rightarrow -\sqrt{7} \leq x \leq \sqrt{7} \rightarrow S$ is bounded above and bounded below. Thus it has a supremum and an infimum based on property **(I)**, and they're unique based on theorem 1.1.3 and theorem 1.1.8 of [1]. We show that $\sup S = \sqrt{7}$ and $\inf S = -\sqrt{7}$.

(a) $\sup S = \sqrt{7}$: if $x \in S$, then $x \leq \sqrt{7}$. Suppose $\exists \epsilon > 0$ s.t. $\sup S = \sqrt{7} - \epsilon$. Based on the density of rational numbers in $\mathbb{R}$ [1] $\exists r_0 \in \mathbb{Q}$ s.t $\sqrt{7} - \epsilon < r_0 < \sqrt{7}$ which contradicts part (a) of theorem 1.1.3 in [1]. Thus $\sup S = \sqrt{7}$.

(b) $\inf S = -\sqrt{7}$: This would be proven similar to part (a). Since S is bounded below, then $-S$ is bounded above. By following the same procedure, we get $\sup(-S) = \sqrt{7}$ leading to $\inf S = -\sqrt{7}$.

Finally, none of the $\sup S$ and $\inf S$ are in S. Both the $\sup S$ and $\inf S$ are irrational since 7 is a prime number, and based on the second question of homework, $\sqrt{7}$ is irrational. Since, all the elements in S are rational $\sup S$ and $\inf S$ can't be written as any element in S. Thus they're not in S.

4 Problem 7

(**A**) Show that

$$\inf S \leq \sup S \quad \text{(A)}$$

for any nonempty set S of real numbers, and give necessary and sufficient conditions for equality.

(**B**) Show that if S is unbounded, then (A) holds if it is interpreted according to Eqn. (1.1.12) and the definitions of Eqns. (1.1.13) and (1.1.14).

Answer

(**A**) Based on theorem 1.1.3 and theorem 1.1.8 of [1], we know that:

$$\forall x \in S, x \leq \sup S \quad \text{(theorem 1.1.3) and} \quad \forall x \in S, \inf S \leq x \quad \text{(theorem 1.1.8)}$$

Thus $\forall x \in S$:

$$\inf S \leq x \quad \text{and} \quad x \leq \sup S \Rightarrow \inf S \leq \sup S \quad \text{(by property } \textbf{(G)} \text{ of real number system [1])}$$

The equality happens only if S is a non-empty set with one element, e.g., x_0:

$$\underbrace{S = \{x_0\}}_{\text{(b)}} \Leftrightarrow \underbrace{\inf S = \sup S}_{\text{(a)}}$$

Then $\inf S = x_0 = \sup S$, and we will show this is a necessary and sufficient condition.

- **Necessary:** $S = \{x_0\} \subset \mathbb{R}$ is necessary for $\inf S = \sup S$, if (and only if) the falsity of $S = \{x_0\} \subset \mathbb{R}$ guarantees the falsity of $\inf S = \sup S$.
  - $\neg$a) $\Rightarrow$ $\neg$b): Suppose $S \neq \{x_0\} \subset \mathbb{R}$. Then, since $S \neq \emptyset$, then it has at least two elements. Then:

    $$\exists x_1, x_2 \in S \subset \mathbb{R} \quad \text{such that} \quad x_1 < x_2.$$

    Based on theorem 1.1.3 and theorem 1.1.8 of [1], we know that:

    $$\forall x \in S, x \leq \sup S \quad \text{(theorem 1.1.3) and} \quad \forall x \in S, \inf S \leq x \quad \text{(theorem 1.1.8)}$$

    Thus, $x_2 \leq \sup S$, and $\inf S \leq x_1$ and based on property **(F) and (G)** of real number system [1]:

    $$\rightarrow \inf S \leq x_1 < x_2 \leq \sup S \rightarrow \inf S < \sup S$$

– $\neg$b) $\Rightarrow$ $\neg$a): Suppose $\inf S \neq \sup S$. Then, by the definition, $\inf S < \sup S$. Then S cannot have only one element, suppose x_0 because this would lead to $\inf S = x_0 = \sup S$. Thus, $S \neq \{x_0\} \subset \mathbb{R}$.

- **Sufficient:** $S = \{x_0\} \subset \mathbb{R}$ is sufficient for $\inf S = \sup S$, if (and only if) the truth of $S = \{x_0\} \subset \mathbb{R}$ guarantees the truth of $\inf S = \sup S$.
  - a) $\Rightarrow$ b): Suppose, $S = \{x_0\} \subset \mathbb{R}$. Since S has only one element, x_0, then it's bounded below and above by x_0. Hence, based on property **(I)** of $\mathbb{R}$, it has a $\inf S$ and $\sup S$. Based on theorem 1.1.3 of [1], the $\sup S$ cannot be less than x_0. Now, assume $\sup S > x_0$, then we know $\forall \epsilon > 0, \exists x \in S$, such that: $x > \sup S - \epsilon$. Suppose $\epsilon = \sup S - x_0 > 0$, then $\exists x \in S$, such that $x > \sup S - (S - x_0) \to x > x_0$. However, the only element in S is x_0; thus, x should be equal to $x_0 \to x_0 > x_0$, which cannot be true. Hence, if $S = \{x_0\} \subset \mathbb{R}$, then $\inf S = \sup S$.
  - b) $\Rightarrow$ a): Suppose, $\inf S = \sup S$. Based on theorem 1.1.3 and theorem 1.1.8 of [1], we know that:

$$\forall x \in S, x \leq \sup S \quad \text{(theorem 1.1.3) and} \quad \forall x \in S, \inf S \leq x \quad \text{(theorem 1.1.8)}$$

    Thus $\forall x \in S, \inf S \leq x \leq \sup S \to \forall x \in S, x = \inf S = \sup S$. Thus, S must have only one element, which is equal to the value of $\inf S = \sup S$.

(B) 1. If S is unbounded above, then based on the Eqns. (1.1.13), $\sup S = \infty$. In this case, there will be two possibilities for $\inf S$:

   i. S is bounded below. Thus, it has a greater lower bound $\inf S \in \mathbb{R}$. Based on Eqn. (1.1.12) $\inf S < \infty$.

   ii. S is unbounded below, then based on Eqn. (1.1.14) $\inf S = -\infty$. Based on Eqn. (1.1.12) $-\infty < \infty$.

2. If S is bounded above, Then it has a least upper bound $\sup S \in \mathbb{R}$. In this case, there will be two possibilities for $\inf S$:

   i. S is bounded below. Then, it has a greater lower bound $\inf S \in \mathbb{R}$. Based on part **(A)**, this leads to $\inf S \leq \sup S$.

   ii. S is unbounded below, then based on Eqn. (1.1.14) $\inf S = -\infty$. Based on Eqn. (1.1.12) $\forall x \in \mathbb{R}, -\infty < x \to -\infty < \sup S$.

5 Problem 8

Let S and T be nonempty sets of real numbers such that every real number is in S or T, and if $s \in S$ and $t \in T$, then $s < t$. Prove that there is a unique real number β such that every real number less than β is in S and every real number greater than β is in T.

Answer

Since $\forall s \in S, \exists t_0 \in T$ such that $s < t_0$, then S is bounded above. Hence, based on the completeness axiom of $\mathbb{R}$, it has an $\sup S$. Based on a similar argument, T has an $\inf T$. First, we will show $\sup S = \inf T$.

Prove by Contradiction

Suppose this is not true, then there are two possible outcomes when comparing $\sup S$ and $\inf T$ based on property **(F)** of real numbers.

1. $\sup S < \inf T$: based on theorem 1.1.6 and 1.1.7, the rational and irrational numbers are dense in $\mathbb{R}$. Extending these two theorems results in the fact that we can find infinitely many real numbers between each of the two real numbers, such as a and b that $a < b$. We just have to repeat the procedure described in theorem 1.1.6 and 1.1.7 infinitely many times. Thus, there are infinitely many numbers between $\sup S$, $\inf T$. For convenience, we define set A, such that:

$$A = \{x \in \mathbb{R} \mid \sup S < x < \inf T\}$$

Since every real number is whether in S or T thus, $\forall x_0 \in A$, there will be two possibilities:

1.1. $x_0 \in S$: this cannot be happening, since $\sup S < x_0$, and all the elements in S should be no greater than $\sup S$.

1.2. $x_0 \in T$: this cannot be happening, since $x_0 < \inf T$, and all the elements in T should be no smaller than $\inf T$.

Thus, $\sup S \not< \inf T$.

2. $\inf T < \sup S$: In this case, $\forall \epsilon > 0$, $x = \inf T - \epsilon \in S$, since x is less than $\inf T$, it can't be in T. Thus $(-\infty, \inf T) \in S$. Based on the same argument $(\sup S, \infty) \in T$. Now, based on the same argument in (a), there are infinitely many numbers between $\inf T$ and $\sup S$. For convenience, we define set A, such that:

$$A = \{x \in \mathbb{R} \mid \inf T < x < \sup S\}$$

Since every real number is whether in S or T thus, $\forall x_0 \in A$, there will be two possibilities:

2.1. $x_0 \in S$: $\exists x_1 \in \mathbb{R}$ s.t. $\inf T < x_1 < x_0$. $x_1 \notin T$, since $x_0 \in S$, and is greater than x_1. This argument is true for all the real numbers in $(\inf T, x_0)$. Thus $(\inf T, x_0) \cup x_0 \in S \to (\inf T, x_0] \in S$. We also know that $(-\infty, \inf T) \in S$. Since $\inf T \in \mathbb{R}$, it should belong to whether S or T. However, $\inf T$ cannot belong to T because then a real number such as $x_0 \in S$ is greater than $\inf T \in T$. Thus, $\inf T \in T$. Hence, $(\inf T, x_0] \cup \inf S \cup (-\infty, \inf T) \in S \to [-\infty, x_0) \in S$. Thus, all the real numbers in T are at least greater than x_0, which can't happen since $\inf T < x_0$, and we assume that the infimum of a set is the greater real upper bound. Thus, in this case: $\inf T \not< \sup S$.

2.2. $x_0 \in T$: $\exists x_1 \in \mathbb{R}$ s.t. $x_0 < x_1 < \sup S$. $x_1 \notin S$, since $x_0 \in T$, and is less than x_1. This argument is true for all the real numbers in $(x_0, \sup S)$. Thus $(x_0, \sup S) \cup x_0 \in T \to [x_0, \sup S) \in T$. We also know that $(\sup S, \infty) \in T$. Since $\sup S \in \mathbb{R}$, it should belong to whether S or T. However, $\sup S$ cannot belong to S because then a real number such as $x_0 \in T$ is less than $\sup S \in S$. Thus, $\sup S \in T$. Hence, $[x_0, \sup S) \cup \sup S \cup (\sup S, \infty) \in T \to [x_0, \infty) \in T$. Thus, all the real numbers in S are at least less than x_0, which can't be happening since $x_0 < \sup S$, and we assume that the supremum of a set is the least real upper bound. Thus, in this case: $\inf T \not< \sup S$.

As a result of [2.1.] and [2.2.], $\inf T \not< \sup S$.

Based on the $\sup S \neq \inf T$ assumption, we lead to $\inf T \not< \sup S$ and $\sup S \not< \inf T$, which is a contradiction. Thus, $\sup S = \inf T = \beta$. Since $\beta \in \mathbb{R}$, there will be two possibilities:

1. $\beta \in S$: then $(-\infty, \beta] \in S$ and $(\beta, \infty) \in T$.
2. $\beta \in T$: then $(-\infty, \beta) \in S$ and $[\beta, \infty) \in T$.

In both cases, it is clear that every real number less than β is in S, and every real number greater than β is in T.

6 Problem 9

Using properties **(A)-(H)** of the real numbers and taking Dedekind's theorem (Exercise 1.1.8) as given, show that every nonempty set U of real numbers that is bounded above has a supremum.

Answer [1]

Let T be the set of upper bounds of U and S be the set of real numbers that are not upper bounds of U. Hence, depending on whether a real number is an upper bound of U or not, belongs to exactly one of these sets. We will show that if $s \in S$ and $t \in T$, then $s < t$.
Prove by Contradiction
Suppose $\exists s_0 \in S$, and $t_0 \in T$, such that $t_0 < s_0$. Since t_0 is an upper bound for U, hence any real number greater than t_0 is also an upper bound for U, including s_0. However, we know $s_0 \in S$, which contradicts our initial assumption, which is that S is the set of real numbers that are not upper bounds of U. Thus if $s \in S$ and $t \in T$, then $s < t$.
In this case, based on Dedekind's theorem (previous problem), there is a unique real number β such that every real number less than β is in S and every real number greater than β is in T.
We will show that $\beta = \sup U$.
Since $\beta \in \mathbb{R}$, there will be two possibilities:

1. $\beta \in S$: then $(-\infty, \beta] \in S$ and $(\beta, \infty) \in T \to \forall t_0 \in T, \beta < t_0$. Since $\beta \in S$, then β is not an upper bound for U, which means $\exists u_0 \in U$ such that $\beta < u_0$. On the other hand, $\forall t_0 \in (\beta, \infty)$, $u_0 \leq t_0$, since T is the set of all upper bounds for U. This means $\forall \epsilon > 0 : \beta < u_0 \leq \beta + \epsilon$. Consider $\epsilon = \frac{u_0 - \beta}{2}$, then:

$$\beta < u_0 \leq \beta + \frac{u_0 - \beta}{2} \to u_0 \leq \frac{u_0 + \beta}{2} \to \frac{u_0}{2} \leq \frac{\beta}{2} \to u_0 \leq \beta$$

   which cannot be true.

2. $\beta \in T$: then $(-\infty, \beta) \in S$ and $[\beta, \infty) \in T \to \forall t_0 \in T, \beta \leq t_0$. If this scenario happens, the property (I) is proven since β is an upper bound and $\forall t_0 \in T, t_0 \neq \beta : \beta \leq t_0$, indicating β is the least upper bound.

[1] To prove Dedekind's theorem for the previous problem, I have used results derived from property (I). Based on this question statement, I believe there's a proof for Dedekind's theorem using only properties (A)-(H). Since the problem statement in the previous question didn't specifically mention "without using property (I)" its proof is valid. However, for this question, I assume there's already a proof for Dedekind's cut in this way out there which only uses properties (A)-(H), so there won't be a loop in the process.

7 Problem 10

Let S and T be nonempty sets of real numbers and define

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

(a) Show that

$$\sup(S + T) = \sup S + \sup T \quad \text{(A)}$$

if S and T are bounded above, and

$$\inf(S + T) = \inf S + \inf T \quad \text{(B)}$$

if S and T are bounded below.

(b) Show that if they are properly interpreted in the extended reals, then (A) and (B) hold if S and T are arbitrary nonempty sets of real numbers.

Answer

(a) a.A. To prove the following

$$\sup(S + T) = \sup S + \sup T \quad \text{(A)}$$

We know, $\forall s \in S, s \leq \sup S$. Similarly, $\forall t \in T, t \leq \sup T$. Thus, $s+t \leq \sup S+\sup T, \forall s \in S$ and $\forall t \in T$. Hence $\sup S + \sup T$ is an upper bound for $S + T$. This will satisfy the part (a) of theorem 1.1.3. For $\sup S + \sup T$ to be supremum we also have to show $\forall \epsilon > 0, \exists s + t \in S + T$ such that $s + t > \sup S + \sup T - \epsilon$. We know:

$$\forall \frac{\epsilon}{2} > 0, \exists s \in S \text{ such that } s > \sup S - \frac{\epsilon}{2}$$

and

$$\forall \frac{\epsilon}{2} > 0, \exists t \in T \text{ such that } t > \sup T - \frac{\epsilon}{2}$$

hence, by adding both sides of inequality together:

$$\forall \epsilon > 0, \exists s \in S \text{ and } \exists t \in T \text{ such that } s + t > \sup S + \sup T - \epsilon$$

which completes the proof.

a.B. To prove the following

$$\inf(S + T) = \inf S + \inf T \quad \text{(B)}$$

We know, $\forall s \in S, \inf S \leq s$. Similarly, $\forall t \in T, \inf T \leq t$. Thus, $\inf S+\inf T \leq s+t, \forall s \in S$ and $\forall t \in T$. Hence $\inf S+\inf T$ is an lower bound for $S+T$. This will satisfy the part (a) of theorem 1.1.8. For $\inf S+\inf T$ to be infimum we also have to show $\forall \epsilon > 0, \exists s+t \in S+T$ such that $s + t < \inf S + \inf T + \epsilon$. We know:

$$\forall \frac{\epsilon}{2} > 0, \exists s \in S \text{ such that } s < \inf S + \frac{\epsilon}{2}$$

and

$$\forall \frac{\epsilon}{2} > 0, \exists t \in T \text{ such that } t < \inf T + \frac{\epsilon}{2}$$

hence, by adding both sides of inequality together:

$$\forall \epsilon > 0, \exists s \in S \text{ and } \exists t \in T \text{ such that } s + t < \sup S + \sup T + \epsilon$$

which completes the proof.

(b) b.A. **Prove by Contradiction:**
By adopting the arithmetic relationships among ∞ and $-\infty$, suppose at least on of S or T is unbounded above, e.g. S is unbounded above. Then $\sup S + \sup T = \infty$.
Assume $\sup(S+T) \neq \infty$, then $\sup(S+T) \in \mathbb{R}$, and we know $\forall s \in S$ and $\forall t \in T : s + t \leq \sup(S+T)$. Let $t_0 \in T$, then $\forall s \in S : s + t_0 \leq \sup(S+T) \to s \leq \sup(S+T) - t_0$. This means S is bounded above, which contradicts with what we assumed.

b.B. By adopting the arithmetic relationships among ∞ and $-\infty$, suppose at least on of S or T is unbounded below, e.g. S is unbounded below. Then $\inf S + \inf T = -\infty$.
Assume $\inf(S+T) \neq -\infty$, then $\inf(S+T) \in \mathbb{R}$, and we know $\forall s \in S$ and $\forall t \in T$: $\inf(S+T) < s + t$. Let $t_0 \in T$, then $\forall s \in S : \inf(S+T) < s + t_0 \to \inf(S+T) - t_0 < s$. This means S is bounded below, which contradicts with what we assumed.

References

[1] William F Trench. *Introduction to real analysis.* 2013.