MATH 414 Analysis I,
Homework 7

Mobina Amrollahi

Deadline: 10/18/2024, 11:59 PM

2.3: Problem 2

Prove: If f is defined on a neighborhood of x_0, then f is differentiable at x_0 if and only if the discontinuity of

$$h(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

at x_0 is removable.

Answer

Removing the discontinuity of $h(x) = \frac{f(x)-f(x_0)}{x-x_0}$ at x_0 means that $h(x)$ is continuous at at x_0. Hence, we have to prove:

$$h(x) \text{ is continuous at at } x_0 \Leftrightarrow f \text{ is differentiable at } x_0$$

We will show assuming one will result in the other:

($\Rightarrow$) : suppose $h(x)$ is continuous at at x_0. This means that the following holds:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = h(x_0)$$

This means that $h(x)$ approaches a limit as x approaches x_0. The existence of this limit is equivalent to the definition of the differentiability of f at x_0, with $h(x_0)$ equal to $f'(x_0)$. Thus, by definition, f is differentiable at an interior point x_0.

($\Leftarrow$) suppose f is differentiable at x_0. This means that the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

And it's equal to some value like $f^{'}(x_0)$. Now, to show $h(x)$ is continuous at at x_0, we have to show the following:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = h(x_0)$$

We know that:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f^{'}(x_0)$$

To remove the discontinuity of $h(x)$ at x_0, we have to define $h(x_0)$ separately because clearly x_0 is not defined in the current definition of x_0. We define $h(x_0) = f^{'}(x_0)$. Then we can remove the discontinuity of $h(x)$ at x_0, in fact making it continuous at this point.

2.3: Problem 4

Suppose that p is continuous on $(a, c]$ and differentiable on (a, c), while q is continuous on $[c, b)$ and differentiable on (c, b). Let

$$f(x) = \begin{cases} p(x), & a < x \leq c, \\ q(x), & c < x < b. \end{cases}$$

(a) Show that

$$f'(x) = \begin{cases} p'(x), & a < x < c, \\ q'(x), & c < x < b. \end{cases}$$

(b) Under what additional conditions on p and q does $f'(c)$ exist? Prove that your stated conditions are necessary and sufficient.

Answer

(a) The differentiability of f is equivalent to the existence of the following limit:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad \forall x_0 \in (a, c) \cup (c, b)$$

There will be two possibilities for x:

1. $x_0 \in (a, c)$: based on the definition $f(x) = p(x)$ in this interval. So the following holds:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{p(x) - p(x_0)}{x - x_0}$$

We know p is differentiable on (a, c) which means:

$$\lim_{x \to x_0} \frac{p(x) - p(x_0)}{x - x_0}$$

approaches a limit, say $p'(x_0)$. Hence,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = p'(x_0), \quad \forall x_0 \in (a, c)$$

approaches a limit. Therefore, f is differentiable on (a, c).

2. $x_0 \in (c, b)$: based on the definition $f(x) = q(x)$ in this interval. So the following holds:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{q(x) - q(x_0)}{x - x_0}$$

We know q is differentiable on (c, b) which means:

$$\lim_{x \to x_0} \frac{q(x) - q(x_0)}{x - x_0}$$

approaches a limit, say $q'(x_0)$. Hence,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = q'(x_0), \quad \forall x_0 \in (c, b)$$

approaches a limit. Therefore, f is differentiable on (c, b).

(b) $f'(c)$ exists, if the following one-sided limits exist:

$$\lim_{x \to c^-} \frac{p(x) - p(c)}{x - c}$$

$$\lim_{x \to c^+} \frac{q(x) - q(c)}{x - c}$$

And if assuming they take values $p'(c)$ and $q'(c)$, the following holds:

$$p'_-(c) = q'_+(c)$$

Additionally,

$$p(c) = q(c)$$

$$\underbrace{p'_-(c) = q'_+(c) \text{ and } p(c) = q(c)}_{\text{(a)}} \Leftrightarrow \underbrace{f'(c) \text{ exists}}_{\text{(b)}}$$

To prove the stated conditions are necessary and sufficient, we have to show two things:

1. **Necessary**: we have to show falsity of (a) guarantees (b) doesn't exist.
   Falsity of (a) will happen under the following two possible circumstances:
   (a) $p(c) \neq q(c)$: this implies f is discontinuous at c, therefore it cannot be differentiable at c, because if that was the case, then f would also have to be continuous at c (theorem 2.3.3 [1]), which clearly is not the case.
   (b) $p'_-(c) \neq q'_+(c)$ and $p(c) = q(c)$:

$$p'_-(c) = \lim_{x \to c^-} \frac{p(x) - p(c)}{x - c}$$

$$q'_+(c) = \lim_{x \to c^+} \frac{q(x) - q(c)}{x - c}$$

   By definition $f(c) = p(c)$, thus $f(c) = q(c)$, and we can write the following:

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{p(x) - p(c)}{x - c}$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{q(x) - q(c)}{x - c}$$

Therefore:

$$f'_-(c) = p'_-(c)$$
$$f'_+(c) = q'_+(c)$$

And:

$$f'_-(c) \neq f'_+(c)$$

2. **Sufficient**: we have to show the truth of (a) guarantees (b) exists. Since the following one-sided limits exist and are equal:

$$\lim_{x \to c^-} \frac{p(x) - p(c)}{x - c}$$

$$\lim_{x \to c^+} \frac{q(x) - q(c)}{x - c}$$

We can substitute $f(x)$ in each, and have:

$$\lim_{x \to c^-} \frac{f(x) - p(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - q(c)}{x - c}$$

In addition, since $p(c) = q(c)$ and by definition $f(c) = p(c)$, thus $f(c) = q(c)$, and we can write the following:

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Therefore, the following limits exist and are equal. Thus, we can write:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Therefore $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists which we may call $f'(c)$.

2.3: Problem 6

Suppose that $f'(0)$ exists and that $f(x + y) = f(x)f(y)$ for all x and y. Prove that $f'(x)$ exists for all x.

Answer

Since $f'(0)$ exists, then we can write:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

Now, since $f(x + y) = f(x)f(y)$ for all x and y, then:

$$f(0 + 0) = f(0)f(0) \Rightarrow f(0) = 0 \text{ or } f(0) = 1$$

If $f(0)=0$, then:

$$f(x+0) = f(x)f(0) = 0, \quad \forall x$$

Which means $f(x)$ is a constant function and $f'(x) = 0\ \forall x$.
If $f(0)=1$, let $x_0 \in D_f$:

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0)f(h)-f(x_0)}{h} \\
&= f(x_0) \lim_{h\to 0} \frac{f(h)-1}{h} \\
&= f(x_0) \lim_{h\to 0} \frac{f(h)-f(0)}{h} \\
&= f(x_0) \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} \\
\text{(since } f'(0) \text{ exists)} &= f(x_0)f'(0)
\end{aligned}$$

$$\Rightarrow f'(x_0) = f(x_0)f'(0)$$

2.3: Problem 10

Prove if f and g are differentiable at x_0, then so is $f-g$ with

(b) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$,

Answer

We have to prove the following exists:

$$\lim_{x\to x_0} \frac{(f-g)(x)-(f-g)(x_0)}{x-x_0}$$

$$\begin{aligned}
\lim_{x\to x_0} \frac{(f-g)(x)-(f-g)(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{f(x)-g(x)-(f(x_0)-g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)-(g(x)-g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - \frac{g(x)-g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0) - g'(x_0)
\end{aligned}$$

2.3: Problem 14

Suppose that f is continuous and increasing on $[a, b]$. Let f be differentiable at a point $x_0 \in (a, b)$, with $f'(x_0) \neq 0$. If g is the inverse of f (Theorem 2.2.15), show that

$$g'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Answer

Based on Theorem 2.2.15, and since f is continuous and increasing on $[a, b]$, g exists and is the following:

$$g(f(x)) = x, \quad a \leq x \leq b$$

In particular,

$$g(f(x_0)) = x_0$$

In addition, since f is differentiable at a point $x_0 \in (a, b)$, with $f'(x_0) \neq 0$, then the following holds:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0, \quad \forall x_0 \in (a, b)$$

Now:

$$\begin{aligned} g'(f(x_0)) &= \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} \\ (\text{let } y = f(x)) :&= \lim_{f(x) \to f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \end{aligned}$$

f is the inverse of g, thus as y approaches $f(x_0)$, it's the same as x approaching x_0.

$$(\text{since } g(f(x)) = x) := \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

References

[1] William F Trench. *Introduction to real analysis.* 2013.