MATH 414 Analysis I,
Homework 13

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4.1: Problem 3

Find $\lim_{n\to\infty} s_n$. Justify your answers from Definition 4.1.1.

(c) $s_n = \frac{1}{n}\sin\left(\frac{n\pi}{4}\right)$

Answer

$\{s_n\}$ **converges to a limit** $s = 0$. To show such, we will show for every $\epsilon > 0$ there is an integer N such that

$$|s_n - 0| < \epsilon \quad \text{if} \quad n \geq N.$$

Hence, we have to show exists N such that:

$$|s_n| < \epsilon \quad \text{if} \quad n \geq N.$$

$$\begin{aligned}
|s_n| < \epsilon &\Leftrightarrow |\frac{1}{n}\sin\left(\frac{n\pi}{4}\right)| < \epsilon \\
&\Leftrightarrow |\frac{1}{n}||\sin\left(\frac{n\pi}{4}\right)| < \epsilon \\
&\Leftrightarrow |\frac{1}{n}|\cancelto{<1}{|\sin\left(\frac{n\pi}{4}\right)|} < \epsilon \\
&\Leftrightarrow \cancelto{\frac{1}{n}}{|\frac{1}{n}|} < \epsilon \quad \text{since } n > 0 \\
&\Leftrightarrow \frac{1}{n} < \epsilon \\
&\Leftrightarrow \frac{1}{\epsilon} < n
\end{aligned}$$

Therefore, if we consider $\frac{1}{\epsilon} \leq N$, $\{s_n\}$ is **convergent** and

$$\lim_{n\to\infty} s_n = 0.$$

4.1: Problem 4

Find $\lim_{n\to\infty} s_n$. Justify your answers from Definition 4.1.1.

(d) $s_n = \sqrt{n^2+n} - n$

Answer

We will show $\{s_n\}$ converges to a limit $s = \frac{1}{2}$. To show such, we will be using theorem 4.1.7 [1]. Hence, we have to show $\lim_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{2}$.

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \lim_{x\to\infty} x\sqrt{1+\frac{1}{x}} - x = \lim_{x\to\infty} x(\sqrt{1+\frac{1}{x}} - 1) = \lim_{x\to\infty} x \times \lim_{x\to\infty} (\sqrt{1+\frac{1}{x}} - 1)$$
$$= \frac{\lim_{x\to\infty}(\sqrt{1+\frac{1}{x}} - 1)}{\lim_{x\to\infty} \frac{1}{x}}$$

By L'Hospital's rule,

$$\begin{aligned}
\frac{\lim_{x\to\infty} \sqrt{1+\frac{1}{x}} - 1}{\lim_{x\to\infty} \frac{1}{x}} &= \lim_{x\to\infty} \frac{\frac{1}{2} \times \frac{1}{\sqrt{1+\frac{1}{x}}} \times \frac{-1}{x^2}}{-1 \times \frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{1}{2} \times \frac{1}{\sqrt{1+\frac{1}{x}}} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, $\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$.

4.1: Problem 9

Use Theorem 4.1.6 to show that $\{s_n\}$ converges.

(b) $s_n = \frac{n!}{n^n}$

Answer

We will show $\{s_n\}$ is nonincreasing, and by therorem 4.1.6 [1] $\lim_{n\to\infty} s_n = \inf\{s_n\}$. Consider s_n and s_{n-1}. To show $\{s_n\}$ is nonincreasing, we have to show:

$$s_{n+1} \leq s_n \quad \forall n$$

Proof

$$
\begin{aligned}
s_{n+1} &= \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n)! \times (n+1)}{(n+1)^n \times (n+1)} = \frac{(n)!}{(n+1)^n} \\
s_n &= \frac{n!}{n^n} \\
&\Rightarrow \frac{s_{n+1}}{s_n} = (\frac{n}{n+1})^n < 1 \\
&\Rightarrow s_{n+1} \leq s_n
\end{aligned}
$$

Hence $\lim_{n\to\infty} s_n = \inf\{s_n\}$ and $\{s_n\}$ converges.

4.3: Problem 8

Determine convergence or divergence.

(a) $\sum \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}}$

(b) $\sum \frac{1}{n^2\left[1+\frac{1}{2}\sin\left(\frac{n\pi}{4}\right)\right]}$

(c) $\sum \frac{1-e^{-n}\log n}{n}$

(d) $\sum \cos\frac{\pi}{n^2}$

(f) $\sum \frac{1}{n}\tan\left(\frac{\pi}{n}\right)$

(g) $\sum \frac{1}{n}\cot\left(\frac{\pi}{n}\right)$

(h) $\sum \frac{\log n}{n^2}$

Answer

To answer this question, we will be using the comparison test as described in [1].

(a) Consider $a_n = \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}}$ and $b_n = \frac{\sqrt{n^2}}{\sqrt{n^5}}$:

$$\sum b_n = \sum \sqrt{\frac{n^2}{n^5}} = \sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{n^{\frac{3}{2}}}$$

Since $p = \frac{3}{2} > 1$, the p-series of $\sum \frac{1}{n^{\frac{3}{2}}}$ converges. In addition, Since $0 \leq a_n \leq b_n$, for large enough$n$, therefore, $\sum \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}}$ converges.

(b) Consider $a_n = \frac{1}{n^2\left[1+\frac{1}{2}\sin\left(\frac{n\pi}{4}\right)\right]}$ and $b_n = \frac{1}{n^2\times\frac{1}{2}}$:

$$\sum b_n = \sum \frac{2}{n^2}$$

b_n converges. In addition for large enough n:

$$\begin{aligned}
-1 \leq \sin\left(\frac{n\pi}{4}\right) &\Rightarrow -1 \times \frac{1}{2} \leq \sin\left(\frac{n\pi}{4}\right) \times \frac{1}{2} \\
&\Rightarrow 1 - \frac{1}{2} \leq 1 + \sin\left(\frac{n\pi}{4}\right) \times \frac{1}{2} \\
&\Rightarrow n^2 \times \frac{1}{2} \leq n^2 \times [1 + \sin\left(\frac{n\pi}{4}\right) \times \frac{1}{2}] \\
&\Rightarrow \frac{1}{n^2\left[1 + \frac{1}{2}\sin\left(\frac{n\pi}{4}\right)\right]} \leq \frac{1}{n^2 \times \frac{1}{2}} \\
&\Rightarrow 0 \leq a_n \leq b_n
\end{aligned}$$

Therefore, since $\sum b_n$ converges, $\sum \frac{1}{n^2\left[1+\frac{1}{2}\sin\left(\frac{n\pi}{4}\right)\right]}$ also converges.

(c) In this particular question, we will be using Theorem 4.3.11 [1]. Hence, let's recall this theorem: **Theorem 4.3.11** Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$. Then:

(a) $\sum a_n < \infty$ if $\sum b_n < \infty$ and $\overline{\lim}_{n\to\infty} \frac{a_n}{b_n} < \infty$.
(b) $\sum a_n = \infty$ if $\sum b_n = \infty$ and $\underline{\lim}_{n\to\infty} \frac{a_n}{b_n} > 0$.

Now, consider $a_n = \frac{1-e^{-n}\log n}{n}$ and $b_n = \frac{1}{n}$. It is known that $\sum b_n = \infty$. In addition:

$$\begin{aligned}
\underline{\lim}_{n\to\infty} \frac{a_n}{b_n} &= \underline{\lim}_{n\to\infty} \frac{\frac{1-e^{-n}\log n}{n}}{\frac{1}{n}} \\
&= \underline{\lim}_{n\to\infty} 1 - e^{-n}\log n
\end{aligned}$$

We will show for large n $s_n = 1 - e^{-n}\log n$ is bounded below by $\frac{1}{2}$. Then $\underline{\lim}_{n\to\infty} 1 - e^{-n}\log n$ exists and it has to be at least $\frac{1}{2}$. In that case $\underline{\lim}_{n\to\infty} 1 - e^{-n}\log n = \underline{\lim}_{n\to\infty} \frac{a_n}{b_n} > 0$, and based on Theorem 4.3.11 $\sum \frac{1-e^{-n}\log n}{n}$ diverges.

$$\begin{aligned}
1 - e^{-n}\log n > \frac{1}{2} &\Leftrightarrow \frac{1}{2} > e^{-n}\log n \\
&\Leftrightarrow \frac{1}{2} > \frac{\log n}{e^n} \\
&\Leftrightarrow e^n > 2\log n
\end{aligned}$$

We know $\log n < n$ and $e^n > n$. Hence, $e^n > 2\log n$.

(d) Consider $a_n = \cos\frac{\pi}{n^2}$ and $b_n = \frac{\pi}{n^2}$. For large enough n:

$$0 = \cos\frac{\pi}{2} < \cos\frac{\pi}{n^2} < \frac{\pi}{n^2}.$$

Therefore, since $\sum b_n$ converges, $\sum \cos\frac{\pi}{n^2}$ also converges.

(f) The equation $tan(x) = x$ holds approximately for the small values of x. In addition, by looking at the Taylor series expansion for tan(x) = x around $x = 0^+$:

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7), \quad O(x^7) > 0$$

We can say that $\tan(x) < \sqrt{x}$ for small values of x. Hence, for large enough values of n, $\frac{\pi}{n}$ will be small, and:

$$\tan\left(\frac{\pi}{n}\right) < \sqrt{\frac{\pi}{n}}$$

Therefore, considering $a_n = \frac{1}{n}\tan\left(\frac{\pi}{n}\right)$ and $b_n = \frac{\sqrt{\pi}}{n\sqrt{n}}$. For large enough n:

$$0 < a_n = \frac{1}{n}\tan\left(\frac{\pi}{n}\right) < \frac{1}{n}.\sqrt{\frac{\pi}{n}} = b_n = \frac{\sqrt{\pi}}{n^{\frac{3}{2}}}$$

Since $p = \frac{3}{2} > 1$, the p-series of $\sum \frac{\sqrt{\pi}}{n^{\frac{3}{2}}}$ converges. Therefore, since $\sum b_n$ converges, $\sum \frac{1}{n}\tan\left(\frac{\pi}{n}\right)$ also converges.

(g) Taylor series expansion for $\cot(x)$ around $x = 0^+$ is:

$$\cot(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + O(x^7).$$

Now, by replacing $x = \frac{\pi}{n}$:

$$\cot\left(\frac{\pi}{n}\right) = \frac{1}{\frac{\pi}{n}} - \frac{\frac{\pi}{n}}{3} - \frac{\left(\frac{\pi}{n}\right)^3}{45} - \frac{2\left(\frac{\pi}{n}\right)^5}{945} + O\left(\left(\frac{\pi}{n}\right)^7\right).$$

We will show $\cot\left(\frac{\pi}{n}\right) > \frac{n}{2\pi}$ as $n \to \infty$

$$\begin{aligned}\cot\left(\frac{\pi}{n}\right) - \frac{n}{2\pi} &= \left(\frac{n}{\pi} - \frac{n}{2\pi} - \frac{\pi}{3n} - \frac{\pi^3}{45n^3} - \frac{2\pi^5}{945n^5} + O\left(\frac{1}{n^7}\right)\right)\\ &= \left(\frac{n}{2\pi} - \frac{\pi}{3n} - \frac{\pi^3}{45n^3} - \frac{2\pi^5}{945n^5} + O\left(\frac{1}{n^7}\right)\right)\end{aligned}$$

Since $\frac{n}{2\pi} \gg \frac{\pi}{n}$ as $n \to \infty$, this term is positive, and thus:

$$\cot\left(\frac{\pi}{n}\right) > \frac{n}{2\pi}.$$

Therefore, considering $a_n = \frac{1}{2\pi}$ and $b_n = \frac{1}{n}\cot\left(\frac{\pi}{n}\right)$. For large enough n:

$$0 < \frac{1}{2\pi} = \frac{1}{n} \times \frac{n}{2\pi} < \frac{1}{n}\cot\left(\frac{\pi}{n}\right) = b_n.$$

Therefore, since $\sum \frac{1}{2\pi}$ diverges, $\sum \frac{1}{n}\cot\left(\frac{\pi}{n}\right)$ also diverges.

(h) We will show $\log n < \sqrt{n}$ as $n \to \infty$, and to show such, we will calculate $\lim_{n\to\infty} \frac{\sqrt{n}}{\log n}$, applying l'Hôpital's rule:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \lim_{n\to\infty} \frac{n^{1/2}}{2} = \infty.$$

Now, consider $a_n = \frac{\log n}{n^2}$, and $b_n = \frac{1}{n^{\frac{3}{2}}}$. Then, For large enough n:

$$0 < \frac{\log n}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}} = b_n$$

Since $p = \frac{3}{2} > 1$, the p-series of $\sum \frac{1}{n^{\frac{3}{2}}}$ converges. Therefore, since $\sum b_n$ converges, $\sum \frac{\log n}{n^2}$ also converges.

4.3: Problem 10

Use the integral test to find all values of p for which the series converges.

(a) $\sum \frac{n}{(n^2-1)^p}$

Answer

Let

$$c_n = f(n), \quad n \geq k, \tag{4.3.7}$$

Where

$$f(x) = \frac{x}{(x^2-1)^p}$$

if $p > \frac{1}{2}$, then, for $k > 1$, f is positive and locally integrable on $[k, \infty)$. We will show f is nonincreasing, which is equivalent to showing that the first derivative is non-positive:

$$\begin{aligned} f(x) &= \frac{x}{(x^2-1)^p} \\ \Rightarrow f'(x) &= \frac{1 \times (x^2-1)^p - x \times (p)(x^2-1)^{p-1} \times 2x}{(x^2-1)^2p} \\ &= \frac{(x^2-1)^{p-1}(x^2-1-x \times p \times 2x)}{(x^2-1)^2p} \\ &= \frac{x^2-2px^2-1}{(x^2-1)^{p+1}} \\ &= \frac{x^2(1-2p)-1}{(x^2-1)^{p+1}} \end{aligned}$$

Since $p > \frac{1}{2}$, hence $2p > 1$ and $1-2p < 0$. Additionally, $x \geq 1$, hence $x^2 \geq 1$, and $x^2(1-2p) < 0$. Therefore, in $\frac{x^2(1-2p)-1}{(x^2-1)^{p+1}}$, the upper part is negative, meanwhile the lower part is positive. Hence, the whole fraction is negative, and f is non-increasing.
Now that f is positive, locally integrable on $[1, \infty)$, and nonincreasing, we can apply the integral test [1] for $p > \frac{1}{2}$ which implies

$$\sum c_n < \infty$$

if

$$\int_k^\infty f(x)\, dx < \infty.$$

To calculate $\int_k^\infty f(x)\, dx$, we have:

$$\int_k^\infty f(x)\, dx = \lim_{a \to \infty} \int_k^a \frac{x}{(x^2-1)^p}\, dx$$

Now, to calculate the integral:

$$I = \int_k^a \frac{x}{(x^2-1)^p}\, dx,$$

Let:

$$u = x^2 - 1 \implies du = 2x\,dx.$$

Thus, the term $x\,dx$ becomes:

$$x\,dx = \frac{1}{2}\,du.$$

Hence, the integral becomes:

$$I = \int_{k^2-1}^{a^2-1} \frac{1}{u^p} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_{k^2-1}^{a^2-1} u^{-p}\,du = \frac{1}{2} \cdot \frac{u^{1-p}}{1-p}\Big|_{u=k^2-1}^{u=a^2-1}, \quad \text{for } p \neq 1.$$

Hence, by substituting the limits $u = a^2 - 1$ and $u = k^2 - 1$:

$$I = \frac{1}{2(1-p)}\left[(a^2-1)^{1-p} - (k^2-1)^{1-p}\right].$$

Therefore:

$$\int_k^a \frac{x}{(x^2-1)^p}\,dx = \frac{1}{2(1-p)}\left[(a^2-1)^{1-p} - (k^2-1)^{1-p}\right], \quad \text{for } p \neq 1.$$

And:

$$\lim_{a\to\infty}\int_k^a \frac{x}{(x^2-1)^p}\,dx = \lim_{a\to\infty}\frac{1}{2(1-p)}\left[(a^2-1)^{1-p} - (k^2-1)^{1-p}\right], \quad \text{for } p \neq 1$$

- When $\frac{1}{2} \leq p < 1$: $1 - p > 0$. Therefore, as $a \to \infty$, $(a^2-1)^{1-p} \to \infty$.
  In this case:
  $$\lim_{a\to\infty}\frac{1}{2(1-p)}\left[(a^2-1)^{1-p} - (k^2-1)^{1-p}\right] \to \infty,$$
  because $(a^2-1)^{1-p} \to \infty$, and the subtraction term $(k^2-1)^{1-p}$ is finite. The integral diverges.
- When $p > 1$: $1 - p < 0$. Therefore, as $a \to \infty$, $(a^2-1)^{1-p} \to 0$.
  In this case:
  $$\lim_{a\to\infty}\frac{1}{2(1-p)}\left[(a^2-1)^{1-p} - (k^2-1)^{1-p}\right] = \frac{1}{2(1-p)}\left[0 - (k^2-1)^{1-p}\right] = \frac{(k^2-1)^{1-p}}{2(p-1)}.$$
  Therefore, $\sum \frac{n}{(n^2-1)^p}$ coverages for $p > 1$ and diverges for $\frac{1}{2} \leq p < 1$.
- When $p < \frac{1}{2}$: suppose $a_n = \frac{n}{(n^2-1)^p}$ and $b_n = 1$. We will show $a_n > b_n$:
  $$\begin{aligned}(p < \frac{1}{2}) &\Rightarrow (n^2-1)^p < (n^2-1)^{\frac{1}{2}} < (n^2)^{\frac{1}{2}} = n \\ &\Rightarrow (n^2-1)^p < n \\ &\Rightarrow 1 < \frac{n}{(n^2-1)^p}\end{aligned}$$

  Therefore, since $\sum 1$ diverges, $\sum \frac{n}{(n^2-1)^p}$ also diverges. Hence
  $$\lim_{a\to\infty}\int_k^a \frac{x}{(x^2-1)^p}\,dx = \begin{cases} \infty, & \text{if } \frac{1}{2} \leq p, \\ \infty, & \text{if } \frac{1}{2} \leq p \leq 1, \\ \frac{(k^2-1)^{1-p}}{2(p-1)}, & \text{if } p > 1. \end{cases}$$

References

[1] William F Trench. *Introduction to real analysis*. 2013.