MATH 414 Analysis I, Homework 5

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1 Section 2.1: Problem 15

Find $\lim_{x\to\infty} f(x)$ if it exists, and justify your answer directly from Definition 2.1.7.

(a) $\frac{1}{x^2+1}$

(d) $e^{-x}\sin x$

(e) $\tan x$

Answer

(a) We will show $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$.
$f = \frac{1}{x^2+1}$ is defined on $(0, \infty)$, since $x^2 + 1 > 0\ \forall x$. Now, let $\epsilon > 0$, then we have to show $\exists\beta$ such that:

$$|f(x) - 0| < \epsilon \text{ if } x > \beta$$

Let $\epsilon_0 := \min\{\epsilon, 1\}$. We will show the following hold for ϵ_0, which results in the proof holding for all $\epsilon > 0$.

$$\begin{aligned}
|f(x)| < \epsilon_0 &\Leftrightarrow |\frac{1}{x^2+1}| < \epsilon_0 \\
(x^2 > 0 \to \frac{1}{x^2+1} > 0) &\Leftrightarrow \frac{1}{x^2+1} < \epsilon_0 \\
(x^2+1 > 0) &\Leftrightarrow 1 < \epsilon_0(x^2+1) \\
&\Leftrightarrow 1 < \epsilon_0 x^2 + \epsilon_0 \\
&\Leftrightarrow 1 - \epsilon_0 < \epsilon_0 x^2 \\
&\Leftrightarrow \frac{1-\epsilon_0}{\epsilon} < x^2
\end{aligned}$$

Since $x > 0$, we can take $\beta = \sqrt{\frac{1-\epsilon_0}{\epsilon_0}}$ $(\epsilon_0 \le 1)$. Now, if we take $\sqrt{\frac{1-\epsilon_0}{\epsilon_0}} < x$, then we have

$$\frac{1-\epsilon_0}{\epsilon_0} < x^2 \Rightarrow 1 - \epsilon_0 < \epsilon_0 x^2 \Rightarrow 1 < \epsilon_0(x^2+1) \Rightarrow \frac{1}{x^2+1} < \epsilon_0$$

Thus, $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$.

(d) We will show $\lim_{x\to\infty} e^{-x} \sin x = 0$
$f = e^{-x}\sin x$ is defined on $(0,\infty)$ both $h = e^{-x}$ and $g = \sin x$ are defined in this domains, hence $(0,\infty) \subseteq D_{hg}$. Now, let $\epsilon > 0$, then we have to show $\exists \beta$ such that:

$$|f(x) - 0| < \epsilon \text{ if } x > \beta$$

$$\begin{aligned}
|f(x) - 0| < \epsilon &\Leftrightarrow |e^{-x}\sin x| < \epsilon \\
&\Leftrightarrow |e^{-x}||\sin x| < \epsilon \\
(|\sin x| < 1) &\Leftrightarrow |e^{-x}||\sin x| < |e^{-x}| \\
(\frac{1}{e} > 0 \Rightarrow |e^{-x}| = (\frac{1}{e})^x) &\Leftrightarrow |e^{-x}||\sin x| < (\frac{1}{e})^x
\end{aligned}$$

Thus, if could find β such that $x > \beta$ the following holds:

$$(\frac{1}{e})^x < \epsilon$$

then:

$$|e^{-x}\sin x| < (\frac{1}{e})^x < \epsilon$$

$$\begin{aligned}
(\frac{1}{e})^x < \epsilon &\Leftrightarrow \ln(\frac{1}{e})^x < \ln \epsilon \\
&\Leftrightarrow \ln 1 - \ln e^x < \ln \epsilon \\
&\Leftrightarrow 0 - x < \ln \epsilon \\
&\Leftrightarrow x > -\ln \epsilon
\end{aligned}$$

This means $\forall \epsilon > 0$ if we take $\beta = -\ln \epsilon$, then for $\beta < x$, we have the following:

$$|f(x) - 0| < \epsilon$$

Thus, $\lim_{x\to\infty} e^{-x}\sin x = 0$.

(e) We will show $\lim_{x\to\infty} \tan x$ does not exist. To show this, we have to study three cases: 1) $\lim_{x\to\infty} \tan x \neq L \in \mathbb{R}$, 2) $\lim_{x\to\infty} \tan x \neq \infty$ and 3) $\lim_{x\to\infty} \tan x \neq -\infty$.

1) **Prove by Contradiction**
Suppose $\exists\ L \in \mathbb{R}$ and β, such that:

$$\forall \epsilon > 0 : |\tan x - L| < \epsilon \text{ if } x > \beta$$

Let $\epsilon = \frac{1}{2}$. Now, take $x_1 = 2\lceil|\beta|\rceil k\pi$ and $x_2 = 2\lceil|\beta|\rceil k\pi + \frac{\pi}{4}$ $(k \in \mathbb{Z}^+)$. We know x_1 and $x_2 > \beta$, thus:

$$\begin{aligned}
&|\tan x_1 - L| < \epsilon \text{ and } |\tan x_2 - L| < \epsilon \\
&\Rightarrow |\tan x_1 - L| + |\tan x_2 - L| < 2\epsilon
\end{aligned}$$

$$\Rightarrow |\tan x_1 - L| + |L - \tan x_2| < 2\epsilon$$

$$\Rightarrow |\tan x_1 - \tan x_2| = |\tan x_1 - L + L - \tan x_2| < |\tan x_1 - L| + |L - \tan x_2| < 2\epsilon$$

$$\Rightarrow |0 - 1| < 2\epsilon$$

$$\Rightarrow |0 - 1| < 1$$

Which contradicts our initial assumption. Hence $\lim_{x\to\infty} \tan x$ cannot be a real number.

2) **Prove by Contradiction**
Suppose $\forall\ M \in \mathbb{R}\ \exists\beta$ such that:

$$\forall x : \tan x > M \quad \text{if} \quad x > \beta$$

Now, take $x_0 = 2\lceil|\beta|\rceil k\pi - \frac{\pi}{4}$ $(k \in \mathbb{Z}^+)$ and $M = \frac{1}{2}$. We know $x_0 > \beta$, thus:

$$\tan x_0 > M$$

However:

$$\tan x_0 = \tan(-\frac{\pi}{4}) > M$$

$\tan(-\frac{\pi}{2}) = -1$ is less than M which contradicts our initial assumption. Hence $\lim_{x\to\infty} \tan x$ cannot be a ∞.

3) **Prove by Contradiction**
Suppose $\forall\ m \in \mathbb{R}\ \exists\beta$ such that:

$$\forall x : \tan x < m \quad \text{if} \quad x > \beta$$

Now, take $x_0 = 2\lceil|\beta|\rceil k\pi + \frac{\pi}{4}$ $(k \in \mathbb{Z}^+)$ and $m = \frac{1}{2}$. We know $x_0 > \beta$, thus:

$$\tan x_0 < m$$

However:

$$\tan x_0 = \tan(\frac{\pi}{4}) < m$$

$\tan(\frac{\pi}{4}) = 1$ is greater than m which contradicts our initial assumption. Hence $\lim_{x\to\infty} \tan x$ cannot be a $-\infty$.

Hence $\lim_{x\to\infty} \tan x$ does not exist.

2 Section 2.1: Problem 20

Find

(c) $\lim\limits_{x\to 0^+} \frac{1}{x^6}$

(d) $\lim\limits_{x\to 0^-} \frac{1}{x^6}$

Answer

(c) We will show $\lim_{x \to 0^+} \frac{1}{x^6} = \infty$.

$f = \frac{1}{x^6}$ is defined on $(0, \infty)$. We will show for each real number M, there is a $\delta > 0$ such that:

$$f > M \text{ if } 0 < x < \delta$$

Let $M_0 := \max\{1, M\}$. We will show the following holds for M_0, which results in the proof holding for all M.

$$\begin{aligned} \frac{1}{x^6} > M \text{ and } (M > 0) &\Rightarrow \frac{1}{M} > x^6 \\ &\Leftrightarrow -\sqrt[6]{\frac{1}{M}} < x < \sqrt[6]{\frac{1}{M}} \end{aligned}$$

Thus, if we pick $\delta = \sqrt[6]{\frac{1}{M}}$, then:

$$0 < x < \delta = \sqrt[6]{\frac{1}{M}}$$

(d) We will show $\lim_{x \to 0^-} \frac{1}{x^6} = \infty$.

$f = \frac{1}{x^6}$ is defined on $(-\infty, 0)$. We will show for each real number M, there is a $\delta > 0$ such that:

$$f > M \text{ if } -\delta < x < 0$$

Let $M_0 := \max\{1, M\}$. We will show the following holds for M_0, which results in the proof holding for all M.

$$\begin{aligned} \frac{1}{x^6} > M \text{ and } (M > 0) &\Rightarrow \frac{1}{M} > x^6 \\ &\Leftrightarrow -\sqrt[6]{\frac{1}{M}} < x < \sqrt[6]{\frac{1}{M}} \end{aligned}$$

Thus, if we pick $\delta = \sqrt[6]{\frac{1}{M}}$, then:

$$-\sqrt[6]{\frac{1}{M}} = -\delta < x < 0$$

3 Section 2.1: Problem 23

Define

(a) $\lim_{x \to \infty} f(x) = \infty$

Answer

We say $f(x)$ approaches ∞ as x approaches ∞ if f is defined on an interval (a, ∞), and for any real number M, there is a number β such that:

$$f(x) > M \quad \text{if} \quad x > \beta$$

4 Section 2.1: Problem 24

Find

(3) $\lim_{x\to\infty} \sqrt{x}\sin x$

Answer

We will show $\lim_{x\to\infty} \sqrt{x}\sin x$ does not exist. To show this, we have to study three cases: 1) $\lim_{x\to\infty} \sqrt{x}\sin x \neq L \in \mathbb{R}$, 2) $\lim_{x\to\infty} \sqrt{x}\sin x \neq \infty$ and 3) $\lim_{x\to\infty} \sqrt{x}\sin x \neq -\infty$.

1) **Prove by Contradiction**
Suppose $\exists\ L \in \mathbb{R}$ and β, such that:

$$\forall\epsilon > 0 : |\sqrt{x}\sin x - L| < \epsilon \quad \text{if} \quad x > \beta$$

Let $\epsilon = \frac{1}{2}$. Now, take $x_1 = 2\lceil|\beta|\rceil k\pi$ and $x_2 = 2\lceil|\beta|\rceil k\pi + \frac{\pi}{2}$ ($k \in \mathbb{Z}^+$). We know x_1 and $x_2 > \beta$, and x_1 and $x_2 > 1$ thus:

$$|\sqrt{x_1}\sin x_1 - L| < \epsilon \text{ and } |\sqrt{x_2}\sin x_2 - L| < \epsilon$$

$$\Rightarrow |\sqrt{x_1}\sin x_1 - L| + |\sqrt{x_2}\sin x_2 - L| < 2\epsilon$$

$$\Rightarrow |\sqrt{x_1}\sin x_1 - L| + |L - \sqrt{x_2}\sin x_2| < 2\epsilon$$

$$\Rightarrow |\sqrt{x_1}\sin x_1 - \sqrt{x_2}\sin x_2| = |\sqrt{x_1}\sin x_1 - L + L - \sqrt{x_2}\sin x_2| < |\sqrt{x_1}\sin x_1 - L| + |L - \sqrt{x_2}\sin x_2| < 2\epsilon$$

$$\Rightarrow |0 - \sqrt{x_2}| < 2\epsilon$$

$$\Rightarrow 1 < |\sqrt{x_2}| < 1$$

Which contradicts our initial assumption. Hence $\lim_{x\to\infty} \sqrt{x}\sin x$ cannot be a real number.

2) **Prove by Contradiction**
Suppose $\forall\ M \in \mathbb{R}\ \exists\beta$ such that:

$$\forall x : \sqrt{x}\sin x > M \quad \text{if} \quad x > \beta$$

Now, take $x_0 = 2\lceil|\beta|\rceil k\pi$ ($k \in \mathbb{Z}^+$) and $M = 1$. We know $x_0 > \beta$, thus:

$$\sqrt{x_0}\sin x_0 > M$$

However:

$$\sqrt{x_0}\sin x_0 = \sqrt{x_0}\sin 0 > M$$

$\sqrt{x_0}\sin 0 = 0$ is less than M which contradicts our initial assumption. Hence $\lim_{x\to\infty} \sqrt{x}\sin x$ cannot be a ∞.

3) **Prove by Contradiction**
Suppose $\forall\ m \in \mathbb{R}\ \exists\beta$ such that:

$$\forall x : \sqrt{x}\sin x < m \quad \text{if} \quad x > \beta$$

Now, take $x_0 = 2\lceil|\beta|\rceil k\pi + \frac{\pi}{2}$ ($k \in \mathbb{Z}^+$) and $m = 1$. We know $x_0 > \beta$, thus:

$$\sqrt{x_0}\sin x_0 < m$$

However:
$$\sqrt{x_0}\sin x_0 = \sqrt{x_0}\sin\frac{\pi}{2} = \sqrt{x_0} < m$$
$\sqrt{x_0} > \sqrt{2\pi} > 1$. This means $\sqrt{x_0}$ is greater than m which contradicts our initial assumption. Hence $\lim_{x\to\infty}\sqrt{x}\sin x$ cannot be a $-\infty$.

Hence $\lim_{x\to\infty}\sqrt{x}\sin x$ does not exist.

5 Section 2.1: Problem 31

Find

(f) $\lim\limits_{x\to\infty}\frac{x+\sqrt{x}\sin x}{2x+e^{-x}}$

Answer

We will show $\lim\limits_{x\to\infty}\frac{x+\sqrt{x}\sin x}{2x+e^{-x}} = \frac{1}{2}$

$f = \frac{x+\sqrt{x}\sin x}{2x+e^{-x}}$ is defined on $(0,\infty)$ both $g = x+\sqrt{x}\sin x$ and $h = 2x+e^{-x}$ are defined in this domains, and h is non-zero in it. Hence $(0,\infty)\subseteq D_{\frac{g}{h}}$. Now, let $\epsilon > 0$, then we have to show $\exists\beta$ such that:
$$|f(x)-\frac{1}{2}| < \epsilon \quad \text{if} \quad x > \beta$$
Let $\epsilon_0 := \min\{\epsilon, \frac{1}{4}\}$. We will show the following hold for ϵ_0, which results in the proof holding for all $\epsilon > 0$.

$$\begin{aligned}|f(x)-\frac{1}{2}| < \epsilon_0 &\Leftrightarrow |\frac{x+\sqrt{x}\sin x}{2x+e^{-x}}-\frac{1}{2}| < \epsilon_0\\ &\Leftrightarrow |\frac{2x+2\sqrt{x}\sin x-2x-e^{-x}}{2(2x+e^{-x})}| < \epsilon_0\\ &\Leftrightarrow |\frac{2\sqrt{x}\sin x-e^{-x}}{2(2x+e^{-x})}| < \epsilon_0\end{aligned}$$

We know :
$$|\frac{2\sqrt{x}\sin x-e^{-x}}{2(2x+e^{-x})}| < \frac{|2\sqrt{x}\sin x|+|e^{-x}|}{2(2x+e^{-x})} < |\frac{2\sqrt{x}}{2(2x)}| + \frac{e^{-x}}{2(2x+e^{-x})} < \frac{\sqrt{x}}{2x}+\frac{1}{2(2x)} < \frac{2\sqrt{x}+1}{4x}$$
Suppose :
$$\frac{2\sqrt{x}+1}{4x} < \epsilon_0$$
Then :
$$|\frac{x+\sqrt{x}\sin x}{2x+e^{-x}}-\frac{1}{2}| < \frac{2\sqrt{x}+1}{4x} < \epsilon_0$$

Thus, if could find β such that $x > \beta$ and the following holds:
$$\frac{2\sqrt{x}+1}{4x} < \epsilon_0$$

Then:

$$|\frac{x+\sqrt{x}\sin x}{2x+e^{-x}}-\frac{1}{2}|<\epsilon_0$$

$$\begin{aligned}(x>0):|\frac{2\sqrt{x}+1}{4x}|<\epsilon_0 &\Leftrightarrow \frac{2\sqrt{x}+1}{4x}<\epsilon_0\\ &\Leftrightarrow 2\sqrt{x}+1<4\epsilon_0 x\\ &\Leftrightarrow 0<4\epsilon_0 x-2\sqrt{x}+1\end{aligned}$$

This means $\forall\epsilon_0>0$ if we take $\beta=\frac{2+\sqrt{4-16\epsilon}}{8\epsilon}$ $(\epsilon_0<1/4)$, then for $\beta<x$, we have the following:

$$|f(x)-\frac{1}{2}|<\epsilon_0$$

Thus, $\lim\limits_{x\to\infty}\frac{x+\sqrt{x}\sin x}{2x+e^{-x}}=\frac{1}{2}$.