MATH 414 Analysis I, Homework 8

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2.4: Problem 8, 16, 20, 36, 40

Find the indicated limits.

(8) $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

(16) $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

(20) $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$

(36) $\lim_{x \to \infty} \left(\frac{x^x}{x \log x}\right)$

(40) $\lim_{x \to 0} \left(\frac{e^{-1/x^2}}{x^n}\right) = 0 \quad (n = integer)$

Answer

(8) The multiplication $x \sin\left(\frac{1}{x}\right)$ is in form of $0.\infty$ as $x \to \infty$. Converting it to an $0/0$ form and applying L'Hospital's rule yields:

$$
\begin{aligned}
\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) &= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad \left(\sin\left(\frac{1}{x}\right) \text{ and } \frac{1}{x} \text{ are differential on a } [a, \infty)\right) \\
&= \lim_{x \to \infty} \frac{-\cos\left(\frac{1}{x}\right)\frac{1}{x^2}}{-\frac{1}{x^2}} \quad \left(-\cos\left(\frac{1}{x}\right)\frac{1}{x^2} \text{ and } -\frac{1}{x^2} \text{ are differential on the } [a, \infty)\right) \\
&= \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x \to \infty} \cos\left(\frac{1}{x}\right) \\
&= 1 \\
&\Rightarrow \lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) = 1
\end{aligned}
$$

(16) The difference

$$\frac{1}{\sin x} - \frac{1}{x}$$

is of the form $\infty - \infty$ as $x \to 0$, but it can be rewritten as the 0/0 form

$$\frac{x - \sin x}{x \sin x}$$

Hence,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \quad (x - \sin x \text{ and } x\sin x \text{ are differential on a } (-\delta,\delta)) \\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} \quad (1-\cos x \text{ and } \sin x + x\cos x \text{ are differential on the } (-\delta,\delta)) \\
&= \lim_{x\to 0}\frac{\sin x}{\cos x - x\sin x + \cos x} \\
&= \lim_{x\to 0}\frac{0}{1-0+1} \\
&= 0 \\
&\Rightarrow \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0
\end{aligned}$$

(20) The function $(1+x)^{\frac{1}{x}}$ is of the form 1^∞ as $x \to 0$. Since

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)} = \exp(\frac{\ln(1+x)}{x})$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\ln(1+x)}{x} &= \lim_{x\to 0}\frac{1}{1+x} \quad (\ln(1+x) \text{ and } x \text{ are differential on a } (-\delta,\delta)) \\
&= 1 \\
&\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \exp(\frac{1}{1}) = e
\end{aligned}$$

(36) The ratio $\frac{x^x}{x\log x}$ is of the form ∞/∞ as $x \to \infty$, and applying L'Hospital's rule yields:

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^x}{x\log x} &= \lim_{x\to\infty}\frac{x^x + \ln(x)x^x}{1+\log x} \quad (x^x \text{ and } x\log x \text{ are differential on a } [a,\infty)) \\
&= \lim_{x\to\infty}\frac{x^x + \ln(x)x^x + \ln(x)(x^x + \ln(x)x^x) + (1/x)(x^x)}{1/x} \\
&\quad (x^x + \ln(x)x^x \text{ and } 1+\log x \text{ are differential on the } [a,\infty)) \\
&= \lim_{x\to\infty} x^{x+1} + \ln(x)x^{x+1} + \ln(x)(x^{x+1} + \ln(x)x^{x+1}) + x^x \\
&= \infty \\
&\Rightarrow \lim_{x\to\infty}\left(\frac{x^x}{x\log x}\right) = \infty
\end{aligned}$$

(40) The ratio $\frac{e^{-1/x^2}}{x^n}$ is of the form 0/0 as $x \to 0$, and applying L'Hospital's rule yields:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} &= \lim_{x\to 0} \frac{\frac{2e^{-1/x^2}}{x^3}}{nx^{n-1}} \\
&= \lim_{x\to 0} \frac{2e^{-1/x^2}}{nx^{n+2}} \quad (2e^{-1/x^2} \text{ and } nx^{n+2} \text{ are differential on a } (-\delta,\delta)) \\
&= \lim_{x\to 0} \frac{4e^{-1/x^2}}{n(n+2)x^{n+4}} \quad (4e^{-1/x^2} \text{ and } n(n+2)x^{n+4} \text{ are differential on a } (-\delta,\delta)) \\
&= \lim_{x\to 0} \frac{8e^{-1/x^2}}{n(n+2)(n+4)x^{n+6}}
\end{aligned}$$

After k applications of L'Hopital's Rule:

$$\lim_{x\to 0} \left(\frac{e^{-1/x^2}}{x^n}\right) = \lim_{x\to 0^+} \frac{2^k e^{-1/x^2}}{n(n+2)(n+4)(n+6)\dots(n+2k)x^{n+2k}}.$$

As $x \to 0$ x^{n+2k} the approaches 0 significantly faster than any polynomial growth in e^{-1/x^2}. Thus, for all n:

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = 0.$$