MATH 414 Analysis I,
Homework 11

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3.2: Problem 7

A function f is of *bounded variation on* $[a, b]$ if there is a number K such that

$$\sum_{j=1}^{n} |f(a_j) - f(a_{j-1})| \le K$$

whenever $a = a_0 < a_1 < \cdots < a_n = b$. (The smallest number with this property is the *total variation of f on* $[a, b]$.)

(a) Prove: If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$.

(b) Prove: If f is of bounded variation on $[a, b]$, then f is integrable on $[a, b]$.
Hint: Use Theorems 3.1.4 and 3.2.7.

Answer

(a) To show f is bounded, we have to show $\exists C \in \mathbb{R}$ such that for every $x \in [a, b]$:

$$|f(x)| < C$$

Consider a partition where $x = a_i, 0 \le x \le n$. Then we know:

$$\begin{aligned}
&\sum_{j=1}^{n} |f(a_j) - f(a_{j-1})| \le K \\
&\Rightarrow |f(a_1) - f(a)| + \cdots + |f(x) - f(a_{i-1})| + |f(a_{i+1}) - f(x)| + \cdots + |f(b) - f(a_{n-1})| \le K \\
&\Rightarrow |f(x) - f(a)| + |f(b) - f(x)| \le K \\
&\Rightarrow |f(x) - f(a)| + |f(x) - f(b)| \le K \\
&\Rightarrow |f(x)| - |f(a)| + |f(x)| - |f(b)| \le K \\
&= 2\,|f(x)| - |f(a)| - |f(b)| \le K \\
&\Rightarrow |f(x)| \le \frac{K + |f(b)| + |f(a)|}{2}
\end{aligned}$$

Hence, by putting $C = \frac{K+|f(b)|+|f(a)|}{2}$, then $\forall x \in [a, b]$:

$$|f(x)| < C$$

(b) Since f is of bounded variation on $[a, b]$, thus, the total variation of f on $[a, b]$ is bounded:

$$V(f, [a, b]) = \sup_P \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq K.$$

Now, consider the following:

$$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x)$$

and

$$m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x).$$

Hence,

$$S(P) - s(P) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}).$$

Since f is of bounded variation, for any partition P, we have:

$$M_j - m_j \leq |f(x_j) - f(x_{j-1})| \leq K.$$

In addition, we know $0 < x_j - x_{j-1} \leq \|P\|$. Hence:

$$S(P) - s(P) \leq \sum_{j=1}^{n} K(x_j - x_{j-1}) = K\|P\|.$$

For any $\epsilon > 0$, we can choose a partition P such that $\|P\| < \frac{\epsilon}{K}$, and this gives:

$$S(P) - s(P) < \epsilon.$$

Therefore, f is integrable on $[a, b]$, by Theorem 3.2.7.

3.3: Problem 2

Show if $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $c_1, c_2, \ldots, c_n$ are constants, then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is integrable on $[a, b]$ and

$$\int_a^b (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx + \cdots + c_n \int_a^b f_n(x)\, dx.$$

Answer: Proof by Induction

Base Case: For $n = 2$, we have:

$$\int_a^b (c_1 f_1(x) + c_2 f_2(x))\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx.$$

This follows from Theorem 3.3.1[1] and Theorem 3.3.2 [1].
Inductive Hypothesis: Assume the result holds for n, i.e.,

$$\int_a^b (c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x))\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx + \cdots + c_k \int_a^b f_k(x)\, dx.$$

Inductive Step: We have to prove the following holds for $n+1$. By applying Theorem 3.3.1, we get:

$$\int_a^b ((c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)) + c_{k+1} f_{k+1}(x))\ dx =$$
$$\int_a^b (c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x))\ dx + \int_a^b c_{k+1} f_{k+1}(x)\, dx.$$

Using the inductive hypothesis, we get:

$$\begin{aligned}
&\int_a^b (c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x))\ dx + \int_a^b c_{k+1} f_{k+1}(x)\, dx \\
&= c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx + \cdots + c_k \int_a^b f_k(x)\, dx + c_{k+1} \int_a^b f_{k+1}(x)\, dx
\end{aligned}$$

Thus, the result holds for $n+1$.

3.3: Problem 3

Can $|f|$ be integrable on $[a, b]$ if f is not?

Answer

Yes. Consider the following example:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational and} x \in [0,1]; \\ -x, & \text{if } x \text{ is irrational and} x \in [0,1]. \end{cases}$$

f is not integrable on $[0, 1]$ since:

$$\overline{\int_0^1} f(x)\, dx = \frac{1}{2} \quad \text{and} \quad \underline{\int_0^1} f(x)\, dx = -\frac{1}{2}$$

However, $|f|$ is the same function as $f(x) = x$ on $\mathbb{R}^+$, which is integrable in this domain. Hence $|f|$ is integrable on a subset of $\mathbb{R}^+$ including $[0, 1]$.

3.3: Problem 5

Prove: If f is integrable on $[a, b]$ and $|f(x)| \geq \rho > 0$ for $a \leq x \leq b$, then $\frac{1}{f}$ is integrable on $[a, b]$.

Answer

We know:

$$\begin{aligned}
&|f(x)| \geq \rho > 0 \quad \text{for} \quad a \leq x \leq b \\
&\Rightarrow 0 < |\frac{1}{f(x)}| \leq \frac{1}{\rho} \quad \text{for} \quad a \leq x \leq b
\end{aligned}$$

If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, then

$$S_{\frac{1}{f}}(P) - s_{\frac{1}{f}}(P) = \sum_{j=1}^{n} \left(M_{\frac{1}{f},j} - m_{\frac{1}{f},j} \right) (x_j - x_{j-1}).$$

$M_{\frac{1}{f},j} \leq \frac{1}{\rho}$ and $m_{\frac{1}{f},j} \geq -\frac{1}{\rho}$. Hence:

$$M_{\frac{1}{f},j} - m_{\frac{1}{f},j} \leq \frac{1}{\rho} - (-\frac{1}{\rho}) = \frac{2}{\rho}$$

Since $0 < x_j - x_{j-1} \leq \|P\|$ and from the last inequality and the last inequality,

$$S_{\frac{1}{f}}(P) - s_{\frac{1}{f}}(P) \leq \frac{2}{\rho}\|P\|$$

To make this difference less than ϵ, we choose a partition P such that:

$$\|P\| < \frac{\rho\epsilon}{2}.$$

This ensures that:

$$S_{\frac{1}{f}}(P) - s_{\frac{1}{f}}(P) < \epsilon.$$

Since we can make

$$S_{\frac{1}{f}}(P) - s_{\frac{1}{f}}(P)$$

arbitrarily small by refining the partition, $\frac{1}{f}$ is integrable on $[a, b]$, by Theorem 3.2.7 [1].

3.3: Problem 11

Suppose that f is continuous on $[a, b]$ and $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$. Show that there is a Riemann sum of f over P that equals $\int_a^b f(x)\,dx$.

Answer

Suppose $P = \{x_0, x_1\}$. Then, based on the first mean value theorem [1] and knowing that f is continuous on $[a, b]$, $\exists c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c) \int_a^b dx = f(c)(b - a)$$

which is a Riemann sum of f over P.
Now, based on theorem 3.2.8 [1], since f is continuous on $[a, b]$, then it's also integrable on this interval. In addition, based on theorem 3.3.8 [1] since f is integrable on $[a, b]$ and $a \leq a_1 < b_1 \leq b$, then f is integrable on $[a_1, b_1]$. Hence, for any partition $P = \{x_0, x_1, \dots, x_n\}$, we can consider $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$, and f will integrable on all of them. Additionally, based on the first mean value theorem $\forall x_{j-1}, x_j : \exists c_j \in [x_{j-1}, x_j]$ such that:

$$\begin{aligned} \int_{x_{j-1}}^{x_j} f(x)\,dx &= f(c_j) \int_{x_{j-1}}^{x_j} dx \\ &= f(c_j)(x_j - x_{j-1}) \end{aligned}$$

Based on theorem 3.3.9 [1] we know:

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx \\ &= \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})\end{aligned}$$

which the right hand side is a Riemann sum of f over P.

References

[1] William F Trench. *Introduction to real analysis*. 2013.