# MATH 414 Analysis I,<br>Homework 6

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## 2.2: Problem 6

Let

$$f(x) = \begin{cases} -1, & \text{if } x \text{ is irrational,} \\ 1, & \text{if } x \text{ is rational.} \end{cases}$$

Show that $f$ is not continuous anywhere.

### Answer

Based on the definition $D_f = \mathbb{R}$. To show $f$ is not continuous anywhere we will show $\forall x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) \neq f(x_0)$. In particular, we will show $\lim_{x \to x_0} f(x)$ does not exist.

#### Prove by Contradiction

For the sake of the contradiction, let $x_0 \in \mathbb{R}$ and $\lim_{x \to x_0} f(x) = L$. There will be three possibilities for $L$:

1. $L \in \mathbb{R}$: this means $\forall \epsilon > 0\ \exists \delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |f(x) - L| < \epsilon$$

Take $\epsilon = 1$. Based on the density property of rational and irrational numbers in $\mathbb{R}$ [1], we can pick $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}^c$ such that $|x_1 - x_0| < \delta$ and $|x_2 - x_0| < \delta$. Hence:

$$\begin{aligned} & |f(x_1) - L| < \epsilon = 1 \\ & |f(x_2) - L| < \epsilon = 1 \Rightarrow |L - f(x_2)| < \epsilon = 1 \\ \text{By the triangle inequality: } & |f(x_1) - L + L - f(x_2)| < |f(x_1) - L| + |L - f(x_2)| < 2\epsilon = 2 \\ \Rightarrow & |1 + 1| = 2 < 2 \end{aligned}$$

Which is a contradict the definition of $\lim_{x \to x_0} f(x) = L$. Hence $\forall x_0 \in \mathbb{R}\ \lim_{x \to x_0} f(x)$ cannot be a real number.

2. $L = \infty$: suppose $\forall\ M \in \mathbb{R}\ \exists \delta > 0$ such that:

$$\forall x: \text{ if } |x_1 - x_0| < \delta \text{ then } f(x) > M$$

Take $M = 1$. Based on the density property of rational and irrational numbers in $\mathbb{R}$ [1] we can pick $x_1 \in \mathbb{Q}^c$ such that $|x_1 - x_0| < \delta$. Hence:

$$f(x_1) > M = 1 \Rightarrow -1 > 1$$

Which contradicts our initial assumption. Hence $\lim_{x \to x_0} f(x)$ cannot be a $\infty$.

3. $L = -\infty$: suppose $\forall\ M \in \mathbb{R}\ \exists \delta > 0$ such that:

$$\forall x: \text{ if } |x_1 - x_0| < \delta \text{ then } f(x) < M$$

Take $M = 0$. Based on the density property of rational and irrational numbers in $\mathbb{R}$ [1] we can pick $x_1 \in \mathbb{Q}$ such that $|x_1 - x_0| < \delta$. Hence:

$$f(x_1) < M = 0 \Rightarrow 1 < 0$$

Which contradicts our initial assumption. Hence $\lim_{x \to x_0} f(x)$ cannot be a $-\infty$.

Therefore, $\forall x \in \mathbb{R}$ $\lim_{x \to x_0} f(x)$ does not exist which means $f$ is not continuous on any point of its domain.

## 2.2: Problem 9

The characteristic function $\psi_T$ of a set $T$ is defined by

$$\psi_T(x) = \begin{cases} 1, & x \in T, \\ 0, & x \notin T. \end{cases}$$

Show that $\psi_T$ is continuous at a point $x_0$ if and only if $x_0 \in T^0 \cup (T^C)^0$.

### Answer

We have to show the following:

$$\psi_T \text{ is continuous at a point } x_0 \Leftrightarrow x_0 \in T^0 \cup (T^C)^0$$

($\Rightarrow$) $\psi_T$ is continuous at a point $x_0$: we have to show:

$$x_0 \in T^0 \cup (T^C)^0$$

Since $\psi_T$ is continuous at point $x_0$, hence:

$$\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$$

Now, there will be two possibilities for $x_0$:

(a) $x \in T$: this means:

$$\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0) = 1$$

Based on the definition of limit, it further means $\forall \epsilon > 0 \; \exists \delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - 1| < \epsilon$$

Which is equivalent to:

$$\text{if } x \in (x_0 - \delta, x_0 + \delta) \text{ then } |\psi_T(x) - 1| < \epsilon$$

If we show $\exists \delta_0 > 0$ such that $(x_0 - \delta_0, x_0 + \delta_0) \subseteq T$, then we found a neighborhood of $x_0$ that is completely in $T$, hence $x_0 \in T^0$, and $x_0 \in T^0 \cup (T^C)^0$.

**Prove by Contradiction** for the sake of contradiction, suppose any neighborhood of $x_0$ contains at least one point in $T^c$. Now, take $\epsilon = \frac{1}{2}$, thus $\exists \delta_1 > 0$ such that:

$$\text{if } x \in (x_0 - \delta_1, x_0 + \delta_1) \text{ then } |\psi_T(x) - 1| < \epsilon$$

Since any neighborhood of $x_0$ contains at least on point in $T^c$, we can take $x_1 \in (x_0 - \delta_1, x_0 + \delta_1)$ such that $x_1 \in T^c$ and $\psi_T(x_1) = 0$. Now:

$$\begin{aligned} \forall x \in (x_0 - \delta_1, x_0 + \delta_1) :& |\psi_T(x) - 1| < \epsilon \\ \Rightarrow :& |\psi_T(x_1) - 1| < \frac{1}{2} \\ \Rightarrow :& 1 < \frac{1}{2} \end{aligned}$$

Which contradicts with $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$.

(b) $x \notin T \to x \in T^c$: this means:

$$\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0) = 0$$

Based on the definition of limit, it further means $\forall \epsilon > 0 \; \exists \delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - 0| < \epsilon$$

Which is equivalent to:

$$\text{if } x \in (x_0 - \delta, x_0 + \delta) \text{ then } |\psi_T(x)| < \epsilon$$

If we show $\exists \delta_0 > 0$ such that $(x_0 - \delta_0, x_0 + \delta_0) \subseteq T^c$, then we found a neighborhood of $x_0$ that is completely in $T^c$, hence $x_0 \in (T^c)^0$, and $x_0 \in T^0 \cup (T^C)^0$.

**Prove by Contradiction** for the sake of contradiction, suppose any neighborhood of $x_0$ contains at least one point in $T$. Now, take $\epsilon = \frac{1}{2}$, thus $\exists \delta_1 > 0$ such that:

$$\text{if } x \in (x_0 - \delta_1, x_0 + \delta_1) \text{ then } |\psi_T(x)| < \epsilon$$

Since any neighborhood of $x_0$ contains at least on point in $T$, we can take $x_1 \in (x_0 - \delta_1, x_0 + \delta_1)$ such that $x_1 \in T$ and $\psi_T(x_1) = 1$. Now:

$$\begin{aligned}
\forall x \in (x_0 - \delta_1, x_0 + \delta_1) :& |\psi_T(x)| < \epsilon \\
\Rightarrow :& |\psi_T(x_1)| < \frac{1}{2} \\
\Rightarrow :& 1 < \frac{1}{2}
\end{aligned}$$

Which contradicts with $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$.

($\Leftarrow$) $x_0 \in T^0 \cup (T^C)^0$: we have to show:

$$\psi_T \text{ is continuous at a point } x_0$$

Since $x_0 \in T^0 \cup (T^C)^0$ at least one of the followings has to happen:

$$x_0 \in T^0 \text{ or } x_0 \in (T^C)^0$$

Hence, there will be two possibilities for $x_0$:

(a) $x_0 \in T^0$: this means $\exists \delta_0 > 0$ such that:

$$(x_0 - \delta_0, x_0 + \delta_0) \in T^0$$

We will show on this neighborhood $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$, which leads to $\psi_T$ being continuous at a point $x_0$.
$\psi_T(x)$ is defined on $(x_0 - \delta_0, x_0 + \delta_0)$ since $\psi_T(x)$ is defined on $\mathbb{R}$. Now, to show $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$ we have to show $\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$$\begin{aligned}
&\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - \psi_T(x_0)| < \epsilon \\
\Rightarrow\ &\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - 0| < \epsilon
\end{aligned}$$

Take $\delta = \delta_0$, then $\forall x \in (x_0 - \delta_0, x_0 + \delta_0)$ $\psi_T(x) = \psi_T(x_0) = 0$, and:

$$\text{if } |x - x_0| < \delta_0 \text{ then } |\psi_T(x) - \psi_T(x_0)| = 0 < \epsilon$$

Hence, $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$ and $\psi_T$ is continuous at a point $x_0$.

(b) $x_0 \in (T^C)^0$: this means $\exists \delta_0 > 0$ such that:

$$(x_0 - \delta_0, x_0 + \delta_0) \in (T^C)^0$$

We will show on this neighborhood $\lim_{x \to x_0} \psi_T(x) = \psi_T(x_0)$, which leads to $\psi_T$ being continuous at a point $x_0$.

$\psi_T(x)$ is defined on $(x_0 - \delta_0, x_0 + \delta_0)$ since $\psi_T(x)$ is defined on $\mathbb{R}$. Now, to show $\lim_{x\to x_0} \psi_T(x) = \psi_T(x_0)$ we have to show $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

$$\begin{aligned}
&\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - \psi_T(x_0)| < \epsilon \\
\Rightarrow\ &\text{if } |x - x_0| < \delta \text{ then } |\psi_T(x) - 1| < \epsilon
\end{aligned}$$

Take $\delta = \delta_0$, then $\forall x \in (x_0 - \delta_0, x_0 + \delta_0)$ $\psi_T(x) = \psi_T(x_0) = 1$, and:

$$\text{if } |x - x_0| < \delta_0 \text{ then } |\psi_T(x) - \psi_T(x_0)| = 0 < \epsilon$$

Hence, $\lim_{x\to x_0} \psi_T(x) = \psi_T(x_0)$ and $\psi_T$ is continuous at a point $x_0$.

## 2.2: Problem 12

Prove that the function $f(x) = e^{ax}$ is continuous on $(-\infty, \infty)$. Take the following properties as given:

1. $\lim_{x\to 0} f(x) = 1$.
2. $f(x_1 + x_2) = f(x_1)f(x_2), \quad -\infty < x_1, x_2 < \infty$.

### Answer

We will first show $f(x) = e^{ax}$ is continuous on $x_0 = 0$ and then we will show is continuous on $(-\infty, \infty)$.

1. $x_0 = 0$: $f(x_0) = e^0 = 1$ Hence based on property (a):

   $$\lim_{x\to 0} f(x) = f(0)$$

   Thus, $f(x) = e^{ax}$ is continuous on $x_0 = 0$
2. $x \in \mathbb{R}$ and $f(x) = e^{ax}$ is continuous on $x_0 = 0$: let $x_0 \in \mathbb{R}\bigcup\{\pm\infty\}$, to show $\lim_{x\to x_0} f(x) = f(x_0)$, we have to show $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

   $$\text{if } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \epsilon$$

   $$\begin{aligned}
   &\text{if } |f(x) - f(x_0)| < \epsilon \\
   \Leftrightarrow\ &|e^{ax} - e^{ax_0}| < \epsilon \\
   \Leftrightarrow\ &|e^{ax_0}||e^{ax-ax_0} - 1| < \epsilon \\
   \Leftrightarrow\ &|e^{ax-ax_0} - 1| < \frac{\epsilon}{e^{ax_0}}
   \end{aligned}$$

Thus, if only $x - x_0$ approaches 0, then since $\lim_{h\to 0} f(h) = 1$, hence $\lim_{x-x_0\to 0} f(x - x_0) = 1$, which means: $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |f(x - x_0) - 1| = |e^{ax-ax_0} - 1| < \epsilon$$

Thus, $\forall\epsilon > 0$ if we pick $\epsilon_0 = \frac{\epsilon}{e^{ax_0}}$, $\exists\delta > 0$ such that:

$$\begin{aligned}&\text{if } |x - x_0| < \delta \text{ then } |e^{ax-ax_0} - 1| < \frac{\epsilon}{e^{ax_0}}\\ &\Rightarrow |e^{ax} - e^{ax_0}| < \epsilon\end{aligned}$$

To show $x - x_0$ approaches 0, we have to show $\lim_{x\to x_0} x - x_0 = 0$. Hence, we have to show $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |x - x_0 - 0| < \epsilon$$

If we take $\delta = \epsilon$, then:

$$|x - x_0| < \delta \to |x - x_0 - 0| < \epsilon$$

Please note that to show $f$ is continuous, we have to show $\forall x \in \mathbb{R}$, $\lim_{x\to x_0} f(x) = f(x_0)$, and by definition $f(x_0) = e^{ax_0}$, which is always a real number. Hence, the continuity of $x = \pm\infty$ is not defined for such a function.

## 2.2: Problem 15

(a) Prove: If $f$ is continuous at $x_0$ and $f(x_0) > \mu$, then $f(x) > \mu$ for all $x$ in some neighborhood of $x_0$.

(b) State a result analogous to (a) for the case where $f(x_0) < \mu$.

(c) Prove: If $f(x) \le \mu$ for all $x$ in $S$ and $x_0$ is a limit point of $S$ at which $f$ is continuous, then $f(x_0) \le \mu$.

(d) State results analogous to (a), (b), and (c) for the case where $f$ is continuous from the right or left at $x_0$.

### Answer

(a) **Prove by Contradiction:** suppose there's no such neighborhood of $x_0$ that for all $x$ in this neighborhood $f(x) > \mu$. This means any neighborhood of $x_0$ will at least have one point, such as $x_1$ that $f(x_1) < \mu$. Since $f$ is continuous at $x_0$, then $\lim_{x\to x_0} f(x) = f(x_0)$, which by definition means $\forall\epsilon > 0$, $\exists\delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \epsilon$$

Now, take $\epsilon = \frac{1}{2}(f(x_0) - \mu)$. Since $f(x_0) > \mu \Rightarrow f(x_0) - \mu > 0 \Rightarrow \frac{1}{2}(f(x_0) - \mu) > 0 \Rightarrow \epsilon > 0$. Then $\exists \delta_0 > 0$ such that:

$$\text{if } |x - x_0| < \delta_0 \text{ then } |f(x) - f(x_0)| < \epsilon$$

Since for all neighborhoods of $x_0$ there exists at least one point, such as $x$ that $f(x) < \mu$, thus $\exists x_1 \in (x_0 - \delta_0, x_0 + \delta_0)$ such that:

$$f(x_1) < \mu \text{ and } |f(x_1) - f(x_0)| < \epsilon$$

Hence:

$$\begin{aligned}
&|f(x_1) - f(x_0)| < \epsilon \\
&\Rightarrow |f(x_1) - f(x_0)| < \frac{1}{2}(f(x_0) - \mu) \\
|x| = |-x| \; &\Rightarrow |f(x_0) - f(x_1)| < \frac{1}{2}(f(x_0) - \mu)
\end{aligned}$$

$$f(x_1) < \mu < f(x_0) \Rightarrow f(x_0) - f(x_1) > 0$$

$$\begin{aligned}
&\Rightarrow f(x_0) - f(x_1) < \frac{1}{2}(f(x_0) - \mu) \\
&\Rightarrow \frac{1}{2}(f(x_0) + \mu) < f(x_1) < \mu \\
&\Rightarrow \frac{1}{2} f(x_0) < \frac{1}{2}\mu \\
&\Rightarrow f(x_0) < \mu
\end{aligned}$$

Which contradicts our initial assumption of $f(x_0) > \mu$. Therefore $\forall x \in (x_0 - \delta_0, x_0 + \delta_0) : f(x) > \mu$.

(b) **Prove by Contradiction:** suppose there's no such neighborhood of $x_0$ that for all $x$ in this neighborhood $f(x) < \mu$. This means any neighborhood of $x_0$ will at least have one point, such as $x_1$ that $f(x_1) > \mu$. Since $f$ is continuous at $x_0$, then $\lim_{x \to x_0} f(x) = f(x_0)$, which by definition means $\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$$\text{if } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \epsilon$$

Now, take $\epsilon = \frac{1}{2}(\mu - f(x_0))$. Since $f(x_0) < \mu \Rightarrow \mu - f(x_0) > 0 \Rightarrow \frac{1}{2}(\mu - f(x_0)) > 0 \Rightarrow \epsilon > 0$. Then $\exists \delta_0 > 0$ such that:

$$\text{if } |x - x_0| < \delta_0 \text{ then } |f(x) - f(x_0)| < \epsilon$$

Since for all neighborhoods of $x_0$ there exists at least one point, such as $x$ that $f(x) > \mu$, thus $\exists x_1 \in (x_0 - \delta_0, x_0 + \delta_0)$ such that:

$$f(x_1) > \mu \text{ and } |f(x_1) - f(x_0)| < \epsilon$$

Hence:

$$\begin{aligned} &|f(x_1) - f(x_0)| < \epsilon \\ \Rightarrow &|f(x_1) - f(x_0)| < \frac{1}{2}(\mu - f(x)) \end{aligned}$$

$$\begin{aligned} f(x_0) < \mu < f(x_1) \Rightarrow & f(x_1) - f(x_0) > 0 \\ \Rightarrow & f(x_1) - f(x_0) < \frac{1}{2}(\mu - f(x_0)) \\ \Rightarrow & f(x_1) < \frac{1}{2}(f(x_0) + \mu) \\ \Rightarrow & \mu < \frac{1}{2}(f(x_0) + \mu) \\ \Rightarrow & \frac{1}{2}\mu < \frac{1}{2}f(x_0) \\ \Rightarrow & \mu < f(x_0) \end{aligned}$$

Which contradicts our initial assumption of $f(x_0) < \mu$. Therefore $\forall x \in (x_0 - \delta_0, x_0 + \delta_0)$ : $f(x) < \mu$.

(c) **Prove by Contradiction:** suppose $f(x_0) > \mu$. Since $f$ is continuous at $x_0$, then $\lim_{x \to x_0} f(x) = f(x_0)$, which by definition means $\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$$\text{if } x \in (x - \delta, x + \delta) \text{ then } |f(x) - f(x_0)| < \epsilon$$

Now, take $\epsilon = \frac{1}{2}(f(x_0) - \mu)$. Since $f(x_0) > \mu \Rightarrow f(x_0) - \mu > 0 \Rightarrow \frac{1}{2}(f(x_0) - \mu) > 0 \Rightarrow \epsilon > 0$. Then $\exists \delta_0 > 0$ such that:

$$\text{if } x \in (x_0 - \delta_0, x_0 + \delta_0) \text{ then } |f(x) - f(x_0)| < \epsilon$$

Since $x_0$ is a limit point of $S$, that means for any deleted neighborhood of $x_0$, there exists at least one point that belongs to $S$. Hence there exists at least one point, such as $x_1$, such that:

$$x_1 \in (x_0 - \delta, x_0 + \delta) \backslash \{x_0\} \cap S \text{ and } f(x_1) \leq \mu$$

Hence:

$$\begin{aligned} &|f(x_1) - f(x_0)| < \epsilon \\ \Rightarrow &|f(x_1) - f(x_0)| < \frac{1}{2}(f(x_0) - \mu) \\ |x| = |-x| \Rightarrow &|f(x_0) - f(x_1)| < \frac{1}{2}(f(x_0) - \mu) \end{aligned}$$

$$
\begin{aligned}
f(x_1) \le \mu < f(x_0) \Rightarrow f(x_0) - f(x_1) > 0 \\
\Rightarrow & f(x_0) - f(x_1) < \frac{1}{2}(f(x_0) - \mu) \\
\Rightarrow & \frac{1}{2}(f(x_0) + \mu) < f(x_1) \le \mu \\
\Rightarrow & \frac{1}{2} f(x_0) < \frac{1}{2}\mu \\
\Rightarrow & f(x_0) < \mu
\end{aligned}
$$

Which contradicts our initial assumption of $f(x_0) > \mu$.

(d) - $f$ is continuous from the right at $x_0$:
    (a) If $f(x_0) > \mu$ then $\exists\delta > 0$ in which $f(x) > \mu$ for all $x$ in $(x_0, x_0 + \delta)$.
    (b) If $f(x_0) < \mu$ then $\exists\delta > 0$ in which $f(x) < \mu$ for all $x$ in $(x_0, x_0 + \delta)$.
    (c) If $f(x) \le \mu$ for all $x$ in $S$ and $x_0$ is a limit point of $S$ and $\forall\delta > 0$, $(x_0, x_0 + \delta_0) \cap S \ne \emptyset$, then $f(x_0) \le \mu$.
  - $f$ is continuous from the left at $x_0$:
    (a) If $f(x_0) > \mu$ then $\exists\delta > 0$ in which $f(x) > \mu$ for all $x$ in $(x_0 - \delta, x_0)$.
    (b) If $f(x_0) < \mu$ then $\exists\delta > 0$ in which $f(x) < \mu$ for all $x$ in $(x_0 - \delta, x_0)$.
    (c) If $f(x) \le \mu$ for all $x$ in $S$ and $x_0$ is a limit point of $S$ and $\forall\delta > 0$, $(x_0 - \delta_0, x_0) \cap S \ne \emptyset$, then $f(x_0) \le \mu$.

## 2.2: Problem 21(d)

Find the domains of $fog$ and $gof$.

(d) $f(x) = \sqrt{x}, \quad g(x) = \sin 2x$

### Answer

(1) $D_{fog}$: $fog = \sqrt{g(x)} = \sqrt{\sin 2x}$. Thus $D_{fog}$ is all $x \in D_g$ such that $g(x) \ge 0$. We know $R_{g(x)} = [-1, 1]$. Hence, $D_{fog}$ is all $x \in D_g$ such that $\sin 2x \in [0, 1]$. Therefore:

$$
\begin{aligned}
2x &\in \bigcup [2k\pi, (2k+1)\pi] \quad k \in \mathbb{Z} \\
\Rightarrow x &\in \bigcup [k\pi, k\pi + \frac{\pi}{2}] \quad k \in \mathbb{Z} \\
D_{fog} &= \bigcup [k\pi, k\pi + \frac{\pi}{2}] \quad k \in \mathbb{Z}
\end{aligned}
$$

(2) $D_{gof}$: $gof = \sin 2f(x) = \sin 2\sqrt{x}$. Thus $D_{gof}$ could be all $x \in D_f$, since $D_g = \mathbb{R}$. Hence $D_{gof} = D_f$ which is $[0, \infty)$.

## References

[1] William F Trench. *Introduction to real analysis.* 2013.