MATH 414 Analysis I,
Homework 10

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3.1: Problem 1

Show that there cannot be more than one number L that satisfies Definition 3.1.1.

Answer: Proof by Contradiction

For the sake of the contradiction suppose there's more than one number L. Hence, suppose there are at least two numbers $L_1 \neq L_2$, $L_1, L_2 \in \mathbb{R}$ where:

$$\int_a^b f(x)\,dx = L_1 \text{ and } \int_a^b f(x)\,dx = L_2.$$

Meaning, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|\sigma - L_1| < \epsilon \text{ and } |\sigma - L_2| < \epsilon$$

if σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$. Consider $\frac{\epsilon}{2} > 0$. We know $\exists \delta_1$, such that:

$$|\sigma - L_1| < \frac{\epsilon}{2} \text{ if } \|P\| < \delta_1$$

In addition, $\exists \delta_2$, such that:

$$|\sigma - L_2| < \frac{\epsilon}{2} \text{ if } \|P\| < \delta_2$$

Suppose $\delta = \min\{\delta_1, \delta_2\}$, then $\exists \|P\| < \delta$, hence:

$$\begin{aligned}
&|\sigma - L_1| < \frac{\epsilon}{2} \quad \text{and} \quad |\sigma - L_2| < \frac{\epsilon}{2} \\
&\Rightarrow |\sigma - L_1| + |\sigma - L_2| < \epsilon \\
&= |L_1 - \sigma| + |\sigma - L_2| < \epsilon \\
&\Rightarrow |L_1 - L_2| < \epsilon \quad \text{By Triangle Inequality}
\end{aligned}$$

Since ϵ can be relatively small, we may pick $\epsilon = \frac{|L_1 - L_2|}{2}$ which will lead to a contradiction.

3.1: Problem 3

Suppose that $\int_a^b f(x)\,dx$ exists and there is a number A such that, for every $\epsilon > 0$ and $\delta > 0$, there is a partition P of $[a, b]$ with $\|P\| < \delta$ and a Riemann sum σ of f over P that satisfies the inequality $|\sigma - A| < \epsilon$. Show that $\int_a^b f(x)\,dx = A$.

Answer: Proof by Contradiction

Suppose $\int_a^b f(x)\,dx = L$ and $L \neq A$.
Consider $\frac{\epsilon}{2} > 0$. Then we know by the definition of $\int_a^b f(x)\,dx = L$, $\exists \delta > 0$ such that:

$$\text{if} \quad \|P\| < \delta \quad \text{then} \quad |\sigma - L| < \frac{\epsilon}{2}$$

Now, for the same $\frac{\epsilon}{2} > 0$ and $\delta > 0$, we know:

$$\exists \|P_1\| \text{ and } \sigma_1 \text{ such that: } \|P_1\| < \delta \quad \text{and} \quad |\sigma_1 - A| < \frac{\epsilon}{2}$$

The above term is also held for L, hence:

$$\|P_1\| < \delta \quad \text{and} \quad |\sigma_1 - L| < \frac{\epsilon}{2}$$

$$\begin{aligned}
&|\sigma_1 - L| < \frac{\epsilon}{2} \quad \text{and} \quad |\sigma_1 - A| < \frac{\epsilon}{2} \\
&\Rightarrow |\sigma_1 - L| + |\sigma_1 - A| < \epsilon \\
&= |L - \sigma_1| + |\sigma - A| < \epsilon \\
&\Rightarrow |L - A| < \epsilon \quad \text{By Triangle Inequality}
\end{aligned}$$

Since ϵ can be relatively small, we may pick $\epsilon = \frac{|L-A|}{2}$ which will lead to a contradiction.

3.1: Problem 9

Find $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$ if

(a)

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ -x, & \text{if } x \text{ is irrational.} \end{cases}$$

(b)

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

Answer

Since we're calculating $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$ on $[0,1]$, we can assume for any partition $\|P\| = \max_{1 \leq j \leq n}(x_i - x_{i-1})$ we can assume $x_{i-1} \leq x_i$.

(a)

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational;} \\ -x, & \text{if } x \text{ is irrational.} \end{cases}$$

f is bounded on $[0,1]$ with having the lower bound as -1, and upper bound as 1 and suppose $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[0,1]$, let

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

and

$$m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

Since both $\mathcal{Q}$ and $\mathcal{Q}^c$ are dense in $[0,1]$ we can say:

$$\sup_{x_{j-1} \le x \le x_j} f(x) = x_j \quad \text{and} \quad \inf_{x_{j-1} \le x \le x_j} f(x) = -x_j$$

Hence, the *upper sum* of f over P is

$$S(P) = \sum_{j=1}^{n} x_j(x_j - x_{j-1}) = \sum_{j=1}^{n} x_j^2 - x_j x_{j-1}$$

By writing

$$x_j = \frac{x_j + x_{j-1}}{2} + \frac{x_j - x_{j-1}}{2}$$

we see that

$$S(P) = \frac{1}{2}\sum_{j=1}^{n}(x_j^2 - x_{j-1}^2) + \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

$$= \frac{1}{2}(1^2 - 0^2) + \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

Since

$$0 < \sum_{j=1}^{n}(x_j - x_{j-1})^2 \le \|P\| \sum_{j=1}^{n}(x_j - x_{j-1}) = \|P\|(1-0)$$

it implies that

$$\frac{1}{2} < S(P) \le \frac{1}{2} + \frac{\|P\|}{2}$$

Since $\|P\|$ can be made as small as we please, hence:

$$\overline{\int_0^1} f(x)\,dx = \frac{1}{2}$$

A similar argument starting from shows that:

$$s(P) = -\frac{1}{2}\sum_{j=1}^{n}(x_j^2 - x_{j-1}^2) - \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

$$= -\frac{1}{2}(1^2 - 0^2) - \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

Since

$$0 < \sum_{j=1}^{n}(x_j - x_{j-1})^2 \le \|P\| \sum_{j=1}^{n}(x_j - x_{j-1}) = \|P\|(1-0)$$

$$-\frac{1}{2} - \frac{\|P\|}{2} \le s(P) < -\frac{1}{2}$$

Since $\|P\|$ can be made as small as we please, hence:

$$\underline{\int_0^1} f(x)\,dx = -\frac{1}{2}$$

(b)

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational;} \\ x, & \text{if } x \text{ is irrational.} \end{cases}$$

f is bounded on $[0,1]$ with having the lower bound as 0, and upper bound as 1 and suppose $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[0,1]$, let

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

and

$$m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

Since both $\mathcal{Q}$ and $\mathcal{Q}^c$ are dense in $[0,1]$ we can say:

$$\sup_{x_{j-1} \le x \le x_j} f(x) = 1 \quad \text{and} \quad \inf_{x_{j-1} \le x \le x_j} f(x) = x_{j-1}$$

Hence, the *upper sum* of f over P is:

$$S(P) = \sum_{j=1}^{n} 1(x_j - x_{j-1}) = 1 - 0 = 1$$

And:

$$\overline{\int_0^1} f(x)\,dx = 1$$

Additionally, the *lower sum* of f over P is

$$s(P) = \sum_{j=1}^{n} x_{j-1}(x_j - x_{j-1}) = \sum_{j=1}^{n} x_j x_{j-1} - x_{j-1}^2$$

By writing

$$x_{j-1} = \frac{x_j + x_{j-1}}{2} - \frac{x_j - x_{j-1}}{2}$$

we see that

$$s(P) = \frac{1}{2}\sum_{j=1}^{n}(x_j^2 - x_{j-1}^2) - \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

$$= \frac{1}{2}(1^2 - 0^2) - \frac{1}{2}\sum_{j=1}^{n}(x_j - x_{j-1})^2$$

Since

$$0 < \sum_{j=1}^{n}(x_j - x_{j-1})^2 \leq \|P\| \sum_{j=1}^{n}(x_j - x_{j-1}) = \|P\|(1 - 0)$$

it implies that

$$\frac{1}{2} - \frac{\|P\|}{2} \leq s(P) < \frac{1}{2}$$

Since $\|P\|$ can be made as small as we please, hence:

$$\underline{\int_0^1} f(x)\,dx = \frac{1}{2}$$

3.2: Problem 2

Show that if f is integrable on $[a, b]$, then

$$\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Answer

Suppose P is any partition of $[a, b]$ and σ is a Riemann sum of f over P. We have:

$$\int_a^b f(x)\,dx - \underline{\int_a^b} f(x)\,dx = s(P) - \underline{\int_a^b} f(x)\,dx + \sigma - s(P) + \int_a^b f(x)\,dx - \sigma$$

$$\Rightarrow \left| \int_a^b f(x)\,dx - \underline{\int_a^b} f(x)\,dx \right| = \left| s(P) - \underline{\int_a^b} f(x)\,dx + \sigma - s(P) + \int_a^b f(x)\,dx - \sigma \right|$$

$$\Rightarrow \left| \int_a^b f(x)\,dx - \underline{\int_a^b} f(x)\,dx \right| \leq \left| s(P) - \underline{\int_a^b} f(x)\,dx \right| + |\sigma - s(P)| + \left| \int_a^b f(x)\,dx - \sigma \right|$$

$$= \left| \underline{\int_a^b} f(x)\,dx - s(P) \right| + |s(P) - \sigma| + \left| \sigma - \int_a^b f(x)\,dx \right| \qquad (*)$$

Let $\epsilon > 0$. By definition, there's a partition P_0 such that:

$$\underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{3} \leq s(P_0) \leq \underline{\int_a^b} f(x)\,dx \qquad (1)$$

Now, for the same $\epsilon > 0$, $\exists \delta > 0$ such that:

$$\left| \sigma - \int_a^b f(x)\,dx \right| < \frac{\epsilon}{3} \quad \text{if} \quad \|P\| < \delta \qquad (2)$$

In addition, $\forall$ partition P with $\|P\| < \delta$ which is a refinement of P_0:

$$\left| \underline{\int_a^b} f(x)\,dx - s(P) \right| < \frac{\epsilon}{3} \qquad (3)$$

Therefore by (*), (2), and (3) implies:

$$|\int_a^b f(x)\,dx - \underline{\int_a^b} f(x)\,dx| < \frac{2\epsilon}{3} + |\sigma - s(P)| \tag{4}$$

for every Riemann sum σ over a refinement P of P_0 with $\|P\| < \delta$.
Since $s(P)$ is the infimum of Riemann sums, we may choose σ so that $|\sigma - s(P)| < \frac{\epsilon}{3}$. Then by (4):

$$|\int_a^b f(x)\,dx - \underline{\int_a^b} f(x)\,dx| < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

$\epsilon > 0$ is arbitrary and can be sufficiently small which implies: $\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$.

3.2: Problem 4

Prove: If f is integrable on $[a, b]$ and $\epsilon > 0$, then $S(P) - s(P) < \epsilon$ if $\|P\|$ is sufficiently small.

Answer

Since

$$\underline{\int_a^b} f(x)\,dx - \epsilon < s(P) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq S(P) < \overline{\int_a^b} f(x)\,dx + \epsilon \quad \forall P$$

Since f is integrable on $[a, b]$:

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = L \in \mathbb{R}$$

By lemma 3.2.4 [1] for $\frac{\epsilon}{2} > 0$, $\exists \delta > 0$:

$$L - \frac{\epsilon}{2} < s(P) \leq L \leq S(P) < L + \frac{\epsilon}{2} \quad \text{if} \quad \|P\| < \delta$$

Hence:

$$L - \frac{\epsilon}{2} - s(P) < 0 \leq L - s(P) \leq S(P) - s(P) < L + \frac{\epsilon}{2} - s(P) \quad \text{if} \quad \|P\| < \delta$$

Since $L - \frac{\epsilon}{2} < s(P) \quad \forall \frac{\epsilon}{2} > 0$:

$$L + \frac{\epsilon}{2} - s(P) < L + \frac{\epsilon}{2} - (L - \frac{\epsilon}{2}) = 2(\frac{\epsilon}{2}) = \epsilon$$

Therefore:

$$L - \frac{\epsilon}{2} - s(P) < 0 \leq L - s(P) \leq S(P) - s(P) < L + \frac{\epsilon}{2} - s(P) < \epsilon \quad \text{if} \quad \|P\| < \delta$$

References

[1] William F Trench. *Introduction to real analysis.* 2013.