COM S 5310 Theory of Computing, Homework 1

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Due: September 5^{th}, 11:59pm on Gradescope.

Read Sipser pp. 1-25.

Problem 0.3.

Let A be the set $\{x, y, z\}$ and B be the set $\{x, y\}$.

a. Is A a subset of B? No, since $z \in A$ but $z \notin B$.

b. Is B a subset of A? Yes, since all the elements of B belong to A as well.

c. What is $A \cup B$?

$$A \cup B = \{x, y, z\} \cup \{x, y\} = \{x, y, z\} = A$$

d. What is $A \cap B$?

$$A \cap B = \{x, y, z\} \cap \{x, y\} = \{x, y\} = B$$

e. What is $A \times B$?

$$A \times B = \{(x_1, x_2) \mid x_1 \in A, x_2 \in B\} = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$$

f. What is the power set of B?

$$\mathcal{P}(B) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

Problem 0.5.

If C is a set with c elements, how many elements are in the power set of C? Explain your answer. There are 2^c elements.

Proof. Represent each subset of C using a binary string as follow:

- Let $C = \{x_1, x_2, \cdots, x_c\}$
- Each subset $S \subseteq C$ can be represented by a binary string of length c, $b_1 b_2 \ldots b_c \in \{0,1\}^c$, where

$$b_i = \begin{cases} 1 & \text{if } x_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Since there are 2^c distinct binary strings of length c, there are 2^c distinct subsets of C. Thus, $\mid \mathcal{P}(C) \mid = 2^c$.

$\square$

Problem 0.6

Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. The unary function $f : X \longrightarrow Y$ and the binary function $g : X \times Y \longrightarrow Y$ are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

a. What is the value of $f(2)$? $f(2) = 7$.

b. What are the range and domain of f? $D_f = \{1, 2, 3, 4, 5\}$

c. What is the value of $g(2, 10)$? $g(2, 10) = 6$.

d. What are the range and domain of g?

$$\begin{aligned} D_g = \{&(1,6),(1,7),(1,8),(1,9),(1,10), \\ &(2,6),(2,7),(2,8),(2,9),(2,10), \\ &(3,6),(3,7),(3,8),(3,9),(3,10), \\ &(4,6),(4,7),(4,8),(4,9),(4,10), \\ &(5,6),(5,7),(5,8),(5,9),(5,10)\} \end{aligned}$$

$$R_g = \{6, 7, 8, 9, 10\}$$

e. What is the value of $g(4, f(4))$? $g(4, f(4)) = g(4, 7) = 8$.

Problem 0.7

For each part, give a relation that satisfies the condition.

a. Reflexive and symmetric but not transitive.
Consider the relation R defined by $\forall m, n \in \mathbb{N}$, $(m, n) \in R \Leftrightarrow \mid m - n \mid \leq 1$.

R is reflexive. We have to show:

$$\forall n \in \mathbb{N}, (n, n) \in R.$$

Proof. $\forall n \in \mathbb{N}:$

$$\mid n - n \mid = 0 \leq 1$$

Therefore

$$(n, n) \in R$$

□

R is symmetric. We have to show:

$$\forall n_1, n_2 \in \mathbb{N}, (n_1, n_2) \in R \implies (n_2, n_1) \in R.$$

Proof.

$$(n_1, n_2) \in R \implies \mid n_1 - n_2 \mid \leq 1$$

Since

$$\mid n_2 - n_1 \mid = \mid n_1 - n_2 \mid$$

Thus

$$\mid n_2 - n_1 \mid = \mid n_1 - n_2 \mid \leq 1$$

And

$$(n_2, n_1) \in R$$

R not transitive. We have to show:

$$\forall n_1, n_2, n_3 \in \mathbb{N}, (n_1, n_2) \in R \wedge (n_2, n_3) \in R \not\Rightarrow (n_1, n_3) \in R.$$

Consider $n_1 = 1, n_2 = 2, n_3 = 3$

Proof.

$$(n_1, n_2) \in R, \quad \text{since} \mid 1 - 2 \mid \leq 1$$

$$(n_2, n_3) \in R, \quad \text{since} \mid 2 - 3 \mid \leq 1$$

Thus

$$n_1 \leq n_2 \leq n_3 \implies n_1 \leq n_3$$

However

$$(n_1, n_3) \notin R, \quad \text{since} \mid 1 - 3 \mid = 2 > 1$$

□

$\square$

b. Reflexive and transitive but not symmetric.
Consider the relation R defined by $\forall m, n \in \mathbb{N}$, $(m, n) \in R \Leftrightarrow (m \leq n)$.

R is reflexive. We have to show:

$$\forall n \in \mathbb{N}, (n, n) \in R.$$

Proof. $\forall n \in \mathbb{N}:$

$$n \leq n$$

Therefore

$$(n, n) \in R$$

$\square$

R is transitive. We have to show:

$$\forall n_1, n_2, n_3 \in \mathbb{N}, (n_1, n_2) \in R \wedge (n_2, n_3) \in R \implies (n_1, n_3) \in R.$$

Proof.

$$(n_1, n_2) \in R \implies n_1 \leq n_2$$

$$(n_2, n_3) \in R \implies n_2 \leq n_3$$

Thus

$$n_1 \leq n_2 \leq n_3 \implies n_1 \leq n_3$$

And

$$(n_1, n_3) \in R$$

$\square$

R is not symmetric: Consider $n_1 = 1$, $n_2 = 2$.

Proof.

$$(1, 2) \in R \implies 1 \leq 2$$

However,

$$(2, 1) \notin R, \quad \text{since} \quad 2 \not\leq 1$$

$\square$

c. Symmetric and transitive but not reflexive.
Consider the relation R defined as the empty relation on $\mathbb{N}$, i.e., $R = \varnothing$.

Proof. R is symmetric and transitive, since no two elements are in relation with each other, so the symmetry and transitivity conditions cannot be violated. More formally:
(symmetry) for all $a, b \in \mathbb{N}$, if $(a, b) \in R$ then $(b, a) \in R$ (vacuously true); (transitivity) for all $a, b, c \in \mathbb{N}$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ (also vacuously true).
However, this relation is not reflexive, since no element of the set is in relation to itself: for any $a \in \mathbb{N}$, $(a, a) \notin R$. $\square$

Problem 0.10

Find the error in the following proof that $2 = 1$. Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$ to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.
The error occurs when dividing both sides of $(a + b)(a - b) = b(a - b)$ by $(a - b)$. Since we assumed $a = b$, it follows that $a - b = 0$. Division by zero is undefined, so this step is invalid.

Problem 0.11

Let $S(n) = 1+2+\cdots+n$ be the sum of the first n natural numbers and let $C(n) = 1^3+2^3+\cdots+n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = S(n)^2$ for every n.

a. $S(n) = \frac{1}{2}n(n + 1)$. We will proceed $S(n) = \frac{1}{2}n(n + 1)$ using mathematical induction.

Proof. **Basis:** $n = 1$

$$\begin{aligned} \text{Left-Hand side:} \quad & S(1) = 1 \\ \text{Right-Hand side:} \quad & \tfrac{1}{2}1(1 + 1) = 1 \end{aligned}$$

Therefor for $n = 1 : S(n) = \frac{1}{2}n(n + 1)$.
Induction Hypothesis:
Suppose:

$$S(N) = \tfrac{1}{2}N(N + 1)$$

holds for some $N \in \mathbb{N}$.
Inductive Step:

$$\begin{aligned} S(N + 1) &= 1 + 2 + \cdots + N + (N + 1) \\ &= \underbrace{\tfrac{1}{2}N(N + 1)}_{\text{Induction assumption}} + (N + 1) \\ &= (N + 1)\left(\tfrac{1}{2}N + 1\right) \\ &= (N + 1)\left(\tfrac{N+2}{2}\right) \\ &= \left(\tfrac{(N+1)(N+2)}{2}\right). \end{aligned}$$

□

b. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n + 1)^2$. We will proceed $C(n) = \frac{1}{4}n^2(n + 1)^2$ using mathematical induction.

Proof. **Basis:** $n = 1$

$$\begin{aligned} \text{Left-Hand side:} \quad & C(1) = 1^3 = 1 \\ \text{Right-Hand side:} \quad & \tfrac{1}{4}1^2(1+1)^2 = 1 \end{aligned}$$

Therefor for $n = 1 : C(n) = \frac{1}{4}n^2(n+1)^2$.
Induction Hypothesis:
Suppose:

$$C(N) = \tfrac{1}{4}N^2(N+1)^2.$$

holds for some $N \in \mathbb{N}$.
Inductive Step:

$$\begin{aligned} C(N+1) &= 1^3 + 2^3 + \cdots + (N)^3 + (N+1)^3 \\ &= \underbrace{\tfrac{1}{4}N^2(N+1)^2}_{\text{Induction assumption}} + (N+1)^3 \\ &= (N+1)^2\left(\tfrac{1}{4}N^2 + (N+1)\right) \\ &= (N+1)^2\left(\tfrac{N^2+4N+4}{4}\right) \\ &= (N+1)^2\left(\tfrac{(N+2)^2}{4}\right) \\ &= \tfrac{1}{4}(N+1)^2(N+2)^2 \\ &= \left(\tfrac{(N+1)(N+2)}{2}\right)^2. \end{aligned}$$

□

This will lead to $C(n) = S(n)^2$.

Problem 0.12

Find the error in the following proof that all horses are the same color.

Claim: In any set of h horses, all horses are the same color.

Proof: By induction on h.

Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Induction step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. Take any set H of $k + 1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore, all the horses in H must be the same color, and the proof is complete.
The flaw lies in the induction step when moving from $h = 1$ to $h = 2$. Suppose we have two horses

h_1 and h_2 of different colors. In the proof, we remove one horse to form $H_1 = \{h_1\}$ and then remove the other to form $H_2 = \{h_2\}$. While it is true that all horses in H_1 are the same color and all horses in H_2 are the same color, there is no overlap between H_1 and H_2. Thus, we cannot conclude that h_1 and h_2 share the same color. The induction argument breaks exactly at this step.

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