ComS 331 **Spring 2025** **Name:** Mobina Amrollahi

HW 7 Due: March 14^{th} 2025

1. Consider the language

$$L = \{w \in \{a, b\}^* : \text{the longest run of } a\text{'s in } w \text{ is longer than any run of } b\text{'s in } w\}.$$

For example, $abbbbaabbbaaaaaa \in L$ because the longest run of b's in it has length four, while the longest run of a's has length six. Prove that L is not context-free.

Answer: Proof by contradiction.

Proof. Assume the contrary that L is context-free. Let p be the given pumping length. Now, consider the string $s = b^p a^{p+1} b^p$. Since the longest run of a has length of $p + 1$, and the longest run of b has length p, hence the longest run of a is longer than any run of b's in s and $s \in L$. Additionally, $\mid s \mid \geq p$, and based on pumping lemma, s may be divided into five pieces s = $uvxyz$ satisfying the conditions:

(a) for each $i \geq 0$, $uv^ixy^iz \in L$,

(b) $\mid vy \mid > 0$, and

(c) $\mid vxy \mid \leq p$.

The restriction $\mid vxy \mid \leq p$ guarantees that v and y can only span at most two out the three runs. That is, whether they span at the first run of b's and a's, and not the second run of b's or the span at the run of a's and second run of b's and not the first run of b's. Hence, no matter what, there will always we at least one of two run of b's that v and y don't span at. With this, there will be two possibilities:

(a) At least one of v or y spans at the run of a's: without the loss of generality, assume at lest v is spanning at the run of a's, and the second run of b's. Then:

$$\underbrace{b^p a^l}_{u} \underbrace{a^{p+1-l} b^p}_{vxyz}$$

In this case, consider the string uv^0xy^0z. Since v contained at least one a, $\mid uv^0xy^0z \mid_a \leq m$. However, the first run of b's has length m, which means the longest run of $a's$ in uv^0xy^0z is not longer than the longest run of b's in this string. Therefore, we arrive at a contradiction that $uv^0xy^0z \notin L$ and conclude that L is not context-free.

(b) v and y only span at a run of b's: without the loss of generality, suppose v and y only span at the second run of b's. Then:

$$\underbrace{b^p a^{p+1} b^l}_{u} \underbrace{b^i}_{v} \underbrace{b^j}_{x} \underbrace{b^k}_{y} \underbrace{b^{p-l-i-j-k}}_{z}$$

In this case, the string uv^2xy^2z is:

$$\underbrace{b^p a^{p+1} b^l}_{u} \underbrace{b^{2i}}_{v^2} \underbrace{b^j}_{x} \underbrace{b^{2k}}_{y^2} \underbrace{b^{p-l-i-j-k}}_{z} = b^p a^{p+1} b^{p+k+i}$$

If $uv^2xy^2z \in L$, then $\mid b^{p+k+i} \mid < \mid a^{p+1} \mid$. However, since $\mid vy \mid > 0$, $i + k > 0$. Consequently, $p + k + i \geq p + 1$. Which means the longest run of $a's$ in uv^2xy^2z is not longer than the longest run of b's in this string. Therefore, we arrive at a contradiction that $uv^2xy^2z \notin L$ and conclude that L is not context-free.

□

2. Consider the language $L = \{\alpha\beta\beta^R\gamma : \alpha, \beta, \gamma \in \{a, b\}^+ : |\alpha| \geq |\gamma|\}$. Is it regular? Context-free? Not even context-free? Give a proof of your answer. (This exercise requires some thought!)

Answer:

This language is context free, but not regular., We will begin by showing it's not regular.

Proof. Prove by contradiction. Assume the contrary that L is regular. Let p be the given pumping length. Now, consider $s = 0^{2p-1}100(10)^p$. There are three ways to break the s into α, β, γ and showing it's in the language:

$$\underbrace{0^{2p-1}1}_{\alpha}\underbrace{00}_{\beta\beta^R}\underbrace{(10)^p}_{\gamma} \quad \text{or} \quad \underbrace{0^{2p-1}}_{\alpha}\underbrace{1001}_{\beta\beta^R}\underbrace{0(10)^{p-1}}_{\gamma} \quad \text{or} \quad \underbrace{0^{2p-2}}_{\alpha}\underbrace{010010}_{\beta\beta^R}\underbrace{(10)^{p-1}}_{\gamma}$$

These three formats indicate that $|\alpha| = |\gamma|$ and $|\alpha|, |\gamma|, |\beta| \geq 0$. Hence, $s \in L$. Please note, any other way of breaking s while maintaining the structure $\beta\beta^R$ results in $|\alpha| < |\gamma|$. Now, since s has a length more than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, satisfying the three conditions of the lemma, where for any $k \geq 0$, the string xy^kz is in L. Since $|xy| \leq p$, y must be all 0's, and we can consider the following:

$$\underbrace{0^i}_{x}\underbrace{0^j}_{y}\underbrace{0^{2p-1-i-j}100(10)^p}_{z}$$

Knowing that $j \geq 1$. Consider $k = 0$. Consequently, xy^kz is:

$$\underbrace{0^i}_{x}\underbrace{(0^j)^0}_{y^k}\underbrace{0^{2p-1-i-j}100(10)^p}_{z} = 0^{2p-j-1}100(10)^p$$

Now, in order for this new string to be in the language there are three possibilities for $\beta\beta^R$:

(a) $\underbrace{0^{2p-j-1}1}_{\alpha}\underbrace{00}_{\beta\beta^R}\underbrace{(10)^p}_{\gamma}$: In this case, $|\alpha| = 2p - j$ and $|\gamma| = 2p$, and since $j \geq 1$, $2p - j \leq 2p - 1 < 2p$, hence $|\alpha| < |\gamma|$, and we arrive at a contradiction that $xy^0z \notin L$.

(b) $\underbrace{0^{2p-j-1}}_{\alpha}\underbrace{1001}_{\beta\beta^R}\underbrace{0(10)^{p-1}}_{\gamma}$: In this case, $|\alpha| = 2p - j - 1$ and $|\gamma| = 2p - 1$, and since $j \geq 1$, $2p - j - 1 \leq 2p - 2 < 2p - 1$, hence $|\alpha| < |\gamma|$, and we arrive at a contradiction that $xy^0z \notin L$.

(c) $\underbrace{0^{2p-j-2}}_{\alpha}\underbrace{010010}_{\beta\beta^R}\underbrace{(10)^{p-1}}_{\gamma}$: In this case, $|\alpha| = 2p - j - 2$ and $|\gamma| = 2p - 2$, and since $j \geq 1$, $2p - j - 2 \leq 2p - 3 < 2p - 2$, hence $|\alpha| < |\gamma|$, and we arrive at a contradiction that $xy^0z \notin L$.

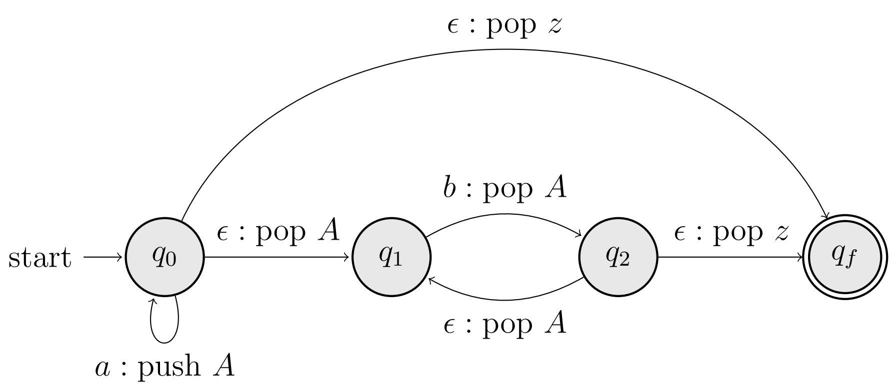
Since $xy^0z \notin L$ in any of the circumstances, we can conclude $xy^0z \notin L$, and that L is not regular. Please note that, it's not possible for $\beta\beta^R$ to appear in the $(10)^{p-1}$ portion, since this sub-string cannot have an even-length palindrome. The option of $\beta\beta^R$ appearing sooner in the string is also not possible as it make the sub-string corresponding to α even shorter than γ. □

Now, to show it's context-free, consider the following grammar: $G = (\{S, A, P\}, \{a, b\}, R, S)$ where the set of rules, R, is

$$
\begin{aligned}
S &\to ASA \mid AS \mid APA \\
P &\to aPa \mid bPb \mid aa \mid bb \\
A &\to a \mid b.
\end{aligned}
$$

3. Define an NPDA for the language $L = \{a^{2n}b^n : n \in \mathbb{N}\}$.
Answer:
Let M_1 be $(Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where

$$
\begin{aligned}
Q &= \{q_0, q_1, q_2, q_f\}, \\
\Sigma &= \{a, b\}, \\
\Gamma &= \{A, z\}, \\
F &= \{q_f\}, \text{ and}
\end{aligned}
$$



4. Define a NPDA for the language $L = \left\{uv \in \{0,1\}^* : |u| = |v| \wedge u \neq v^R\right\}$.
Answer:
Let M_1 be $(Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where

$$
\begin{aligned}
Q &= \{q_0, q_1, q_2, q_f\}, \\
\Sigma &= \{0, 1\}, \\
\Gamma &= \{A, B, z\}, \\
F &= \{q_f\}, \text{ and}
\end{aligned}
$$