ComS 331 **Spring 2024** **Name: Mobina Amrollahi**

HW 0 Due: 28 Jan 2024

1. Learn the wonderful world of LaTeX! For drawings, you can use `tgif` (downloadable from the web), your favorite drawing tool, or TikZ. Reproduce the text and figure in the rectangular box below, as it appears, including the rectangular border.

This is an inline equation: $x + y = 3$.
This is a displayed equation:

$$x + \frac{y}{z - \sqrt{3}} = 2.$$

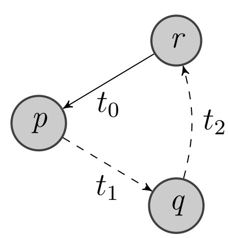
This is how you can define a piece-wise linear function:

$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ 7x + 2 & \text{if } x \geq 0 \text{ and } x < 10 \\ 5x + 22 & \text{otherwise.} \end{cases}$$

This is a matrix:

9	8	7	9
6	6	6	
3		3	3

This is a graph with two types (solid and dashed) of labeled edges:



Files `Notation.tex` and `Figure.tex` should give you enough LaTeX hints to get you started.

Points will be subtracted if your font, font size, spacing, or alignment substantially differ from the one shown, but not if your figure is slightly different, since different tools may draw slightly different figures.

As for all other assignments, you must turn in: (1) a single source LaTeX file, plus one eps or pdf file for each figure you need to include (just one for this homework; of course, if you are using TikZ, you should have the TikZ figure directly in your source latex file, so no additional file is needed), and (2) a single pdf file obtained by running `pdflatex` on your source latex file.

2. Prove that $\mathbb{N}$ (natural numbers) and $\mathbb{Z}$ (integer numbers) are equinumerous.
Answer To prove such, we will prove there is a bijection $f : \mathbb{N} \to \mathbb{Z}$.
First, we will try to construct such a function, and then we will show that it's a bijection. The

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	4	$\cdots$

Table 1: Mapping of $\mathbb{N}$ (natural numbers) to $\mathbb{Z}$ (integers).

Table 1 corresponds to the following piece-wise linear function:

$$f(n) = \begin{cases} -\left(\frac{n+1}{2}\right) & \text{if } n \in \mathbb{N} \text{ and is odd,} \\ \frac{n}{2} & \text{if } n \in \mathbb{N} \text{ and is even,} \\ 0 & \text{if } n = 0. \end{cases}$$

To show this function is a bijection, we have to show it's one-to-one and onto. We will proceed with proving each of them separately.

(a) one-to-one: To show the function $f : \mathbb{N} \to \mathbb{Z}$ is *one-to-one* we have to show:

$$\forall n_1, n_2 \in \mathbb{N}, f(n_1) = f(n_2) \implies n_1 = n_2, .$$

Proof. By contradiction: Let $N_1, N_2 \in \mathbb{N}$ such that $f(N_1) = f(N_2)$, and assume $N_1 \neq N_2$. Since $N_1 \neq N_2$ they cannot both be zero, both even, or both odd. Therefore differences fall into the following cases about N_1 and N_2:

i. N_1 even and N_2 odd (or vice versa) and both non-zero. Then, since $f(N_1) = f(N_2)$ we have:

$$\frac{N_1}{2} = -\left(\frac{N_2 + 1}{2}\right)$$

However, since $N_1 \in \mathbb{N}$, non-zero, and even, $\frac{N_1}{2}$ is non-negative, and $-\left(\frac{N_2+1}{2}\right)$ is negative, which contradicts $f(N_1) = f(N_2)$.

ii. consider one of N_1 or N_2 being zero. Since N_1 and N_2 are arbitrary, we can assume $N_1 = 0$. Then, since $f(N_1) = f(N_2)$, based on whether N_2 is odd or even we have one of the following:

A. N_2 is even. Then:

$$0 = \frac{N_2}{2} \Rightarrow N_2 = 0$$

which is a contradiction since N_2 is non-zero.

B. N_2 is odd. Then:

$$0 = -\left(\frac{N_2 + 1}{2}\right) \Rightarrow 0 = \frac{N_2 + 1}{2} \Rightarrow 0 = N_2 + 1 \Rightarrow N_2 = -1$$

which is a contradiction since $N_2 \in \mathbb{N}$.

Hence, our initial opposed assumption of $N_1 \neq N_2$ leads to a contradiction, and $f : \mathbb{N} \to \mathbb{Z}$ has to be *one-to-one*.

□

(b) onto: To show the function $f : \mathbb{N} \to \mathbb{Z}$ is *onto* we have to show

$$\forall z \in \mathbb{Z}, \exists n \in \mathbb{N}, f(n) = z,$$

Proof. Let $Z \in \mathbb{Z}$, based on whether $Z > 0$ or $Z = 0$ or $Z < 0$ we will show $\exists N \in \mathbb{N}$ such that:

$$f(N) = Z$$

.

i. $Z > 0$: consider $N = 2 \cdot Z$. Since $Z > 0$, $N \in \mathbb{N}$ and it's even. Now, if we look at $f(N)$, we have:

$$f(N) = \frac{N}{2} = \frac{2Z}{2} = Z$$

ii. $Z = 0$: consider $N = Z = 0$. Now, if we look at $f(N)$, we have:

$$f(N) = f(0) = 0 = Z$$

iii. $Z < 0$: consider $N = -2 \cdot Z - 1$. Since $Z < 0$, we have:

$$Z \leq -1 \Rightarrow -Z \geq 1 \Rightarrow -2 \cdot Z \geq 2 \Rightarrow -2 \cdot Z - 1 \geq 1.$$

Hence N is a positive integer. Thus, $N \in \mathbb{N}$ and it's odd. Now, if we look at $f(N)$, we have:

$$f(N) = -\left(\frac{N+1}{2}\right) = -\left(\frac{-2 \cdot Z - 1 + 1}{2}\right) = -\left(\frac{-2 \cdot Z}{2}\right) = \left(\frac{2 \cdot Z}{2}\right) = Z$$

Hence, $f : \mathbb{N} \to \mathbb{Z}$ has to be *onto*. □

3. Prove that the relation R defined by $\forall m, n \in \mathbb{N}$, $(m, n) \in R \Leftrightarrow (m - n) \mod 3 = 0$ is an equivalence relation, and describe its equivalence classes.
Answer To show R is an *equivalence relation* we have to show it is 1) reflexive, 2) symmetric, and 3) transitive. We will proceed with proving each of them separately.

(a) R is reflexive. We have to show:

$$\forall n \in \mathbb{N}, (n, n) \in R.$$

Proof. $\forall n \in \mathbb{N}$:

$$n - n = 0$$

Since 0 is divisible by 3, $n - n$ is also divisible by 3. Therefore

$$(n - n) \mod 3 = 0$$

And

$$(n, n) \in R$$

□

(b) R is symmetric. We have to show:

$$\forall n_1, n_2 \in \mathbb{N}, (n_1, n_2) \in R \implies (n_2, n_1) \in R.$$

Proof.

$$(n_1, n_2) \in R \Rightarrow (n_1 - n_2) \mod 3 = 0$$

Now, $\exists k \in \mathbb{Z}$ such that:

$$n_1 - n_2 = 3 \cdot k$$

Hence

$$n_2 - n_1 = -3 \cdot k = 3 \cdot (-k)$$

And $n_2 - n_1$ is also divisible by 3, which implies:

$$(n_2 - n_1) \mod 3 = 0$$

And

$$(n_2, n_1) \in R$$

$\square$

(c) R is transitive. We have to show:

$$\forall n_1, n_2, n_3 \in \mathbb{N}, (n_1, n_2) \in R \wedge (n_2, n_3) \in R \implies (n_1, n_3) \in R.$$

Proof.

$$(n_1, n_2) \in R \implies (n_1 - n_2) \mod 3 = 0 \implies \exists k_1 \in \mathbb{Z} \quad \text{such that} \quad n_1 - n_2 = 3 \cdot k_1$$

$$(n_2, n_3) \in R \implies (n_2 - n_3) \mod 3 = 0 \implies \exists k_2 \in \mathbb{Z} \quad \text{such that} \quad n_2 - n_3 = 3 \cdot k_2$$

We can write

$$n_1 - n_3 = (n_1 - n_2) + (n_2 - n_3) = 3 \cdot k_1 + 3 \cdot k_2 = 3 \cdot (k_1 + k_2)$$

Hence $n_1 - n_3$ is also divisible by 3, which implies:

$$(n_1 - n_3) \mod 3 = 0$$

And

$$(n_1, n_3) \in R$$

$\square$

4. Show that $\sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$.
   Answer We will proceed $\sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$ using mathematical induction.

   Proof. **Basis:** $n = 0$

$$\text{Left-Hand side:} \quad \sum_{i=1}^{0} i^2 = 0^2 = 0$$

$$\text{Right-Hand side:} \quad \frac{(2 \cdot 0 + 1) \cdot (0 + 1) \cdot 0}{6} = \frac{(1) \cdot (1) \cdot 0}{6} = 0$$

Therefor for $n = 0 : \sum_{i=1}^{n} i^2 = (2n+1)(n+1)n/6$.
Induction Hypothesis:
Suppose:

$$\sum_{i=1}^{N} i^2 = (2N+1)(N+1)N/6.$$

holds for some $N \in \mathbb{N}$.
Inductive Step:

$$\begin{aligned}
\sum_{i=1}^{N+1} i^2 &= \left(\sum_{i=1}^{N} i^2\right) + (N+1)^2 \\
&= \underbrace{(2N+1)(N+1)N/6}_{\text{Induction assumption}} + (N+1)^2 \\
&= (N+1)\left((2N+1)N/6 + (N+1)\right) \\
&= (N+1)\left(\frac{2N^2+2N+6N+6}{6}\right) \\
&= (N+1)\left(\frac{2N^2+8N+6}{6}\right) \\
&= (N+1)\left(\frac{(2N+3)(N+2)}{6}\right) \\
&= \frac{(N+1)(2N+3)(N+2)}{6} \\
&= \frac{(2(N+1)+1)(N+2)(N+1)}{6}
\end{aligned}$$

□

5. Show that, for $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$.
   Answer We will proceed $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ using mathematical induction.

   Proof. **Base Case:** $n = 1$

$$\text{Left-Hand side:} \quad \sum_{i=1}^{1} \frac{1}{1^2} = 1$$

$$\text{Right-Hand side:} \quad 2 - \frac{1}{n} = 2 - \frac{1}{1} = 1$$

   Since $1 \leq 1$, therefore for $\sum_{i=1}^{1} \frac{1}{i^2} \leq 2 - \frac{1}{1}$.
   Induction Hypothesis:
   Suppose:

$$\sum_{i=1}^{N} \frac{1}{i^2} \leq 2 - \frac{1}{N}.$$

   holds for some $N \in \mathbb{N}$.

Inductive Step:

$$\begin{aligned}
\sum_{i=1}^{N+1} \frac{1}{i^2} &= \left(\sum_{i=1}^{N} \frac{1}{i^2}\right) + \frac{1}{(N+1)^2} \\
&\leq \underbrace{2 - \frac{1}{N}}_{\text{Induction assumption}} + \frac{1}{(N+1)^2}
\end{aligned}$$

Now, if we show $\forall N \in \mathbb{N}$:

$$2 - \frac{1}{N} + \frac{1}{(N+1)^2} \leq 2 - \frac{1}{N+1}$$

Then, since

$$\sum_{i=1}^{N+1} \frac{1}{i^2} \leq 2 - \frac{1}{N} + \frac{1}{(N+1)^2}$$

Additionally

$$\sum_{i=1}^{N+1} \frac{1}{i^2} \leq 2 - \frac{1}{N+1}$$

And the proof will be complete. This is equivalent to showing that:

$$\begin{aligned}
2 - \frac{1}{N} + \frac{1}{(N+1)^2} \leq 2 - \frac{1}{N+1} &\iff \frac{1}{(N+1)^2} + \frac{1}{N+1} \leq \frac{1}{N} \\
&\iff \frac{1+N+1}{(N+1)^2} \leq \frac{1}{N} \\
&\iff \frac{N+2}{(N+1)^2} \leq \frac{1}{N} \\
&\iff \frac{N+2}{(N+1)^2} \leq \frac{1}{N} \\
&\iff (N+2)(N) \leq (N+1)^2 \\
&\iff N^2 + 2N \leq N^2 + 2N + 1 \\
&\iff 0 \leq 1
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{N+1} \frac{1}{i^2} &= \left(\sum_{i=1}^{N} \frac{1}{i^2}\right) + \frac{1}{(N+1)^2} \\
&\leq 2 - \frac{1}{N} + \frac{1}{(N+1)^2} \\
&\leq 2 - \frac{1}{N+1}
\end{aligned}$$

$\square$

6. Using a formal proof by induction, show that

$$\forall a \in \mathbb{R}, a \neq 1, \sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}.$$

We will proceed $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$ using mathematical induction for any arbitary a such that $a \in \mathbb{R}, a \neq 1$.

Proof. **Base Case:** $n = 0$

$$\text{Left-Hand side:} \quad \sum_{i=0}^{n} a^i = \sum_{i=0}^{0} a^i = a^0 = 1$$
$$\text{Right-Hand side:} \quad \frac{1-a^{n+1}}{1-a} = \frac{1-a^{0+1}}{1-a} = \frac{1-a^1}{1-a} = 1$$

Therefore for $n = 0$, $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$.

Induction Hypothesis:

Suppose:

$$\sum_{i=0}^{N} a^i = \frac{1-a^{N+1}}{1-a}.$$

holds for some $N \in \mathbb{N}$.

Inductive Step:

$$\begin{aligned}
\sum_{i=0}^{N+1} a^i &= \left(\sum_{i=1}^{N} \frac{1}{i^2} \right) + a^{N+1} \\
&= \underbrace{\frac{1-a^{N+1}}{1-a}}_{\text{Induction assumption}} + a^{N+1} \\
&= \frac{1-a^{N+1}}{1-a} + \frac{(1-a)(a^{N+1})}{1-a} \\
&= \frac{1-a^{N+1}+a^{N+1}-a^{N+2}}{1-a} \\
&= \frac{1-a^{N+2}}{1-a}
\end{aligned}$$

□

Using the previous result, what is the sum of the first 30 powers of 2 (starting from 2^0)?

Answer Since we're starting from 0, the first 30 powers goes all the way to 29. By applying $n = 29$ and $a = 2$ to the previous result, we will get:

$$\sum_{i=0}^{29} 2^i = \frac{1-2^{29+1}}{1-2} = \frac{1-2^{30}}{-1} = 2^{30} - 1 = 1073741824 - 1 = 1073741823$$