MATH 317 **Spring 2025** **Problem Set #5**

Due May 6^{th} 2025 **Mobina Amrollahi**

1. Let $T\colon V \to W$ be a linear transform, and let $\dim V = \dim W$

   (a) If T is one-to-one, then prove that T is a linear isomorphism.

   (b) If T is onto, then prove that T is a linear isomorphism.

   Answer. In both parts, we'd like to show that the linear transformation T is invertible. To do so, we have to show that T is both one-to-one and onto. Part (a) gives us the one-to-one condition, hence we have to show it's also onto. Part (b) gives us the onto condition, hence we have to show it's one-to-one.

   (a) Fix a basis B for V. Then, $T[B]$ is a basis for $T[V]$ since T is a linear mapping. Also,

   $$\dim T[V] = \dim V = \dim W,$$

   and thus T must be onto.

   (b) Fix a basis $w_1, \cdots, w_n$ for W. As T is onto, there exist $v_1, \cdots, v_n \in V$ such that $T(v_k) = w_k$ for each $1 \le k \le n$. Since $w_1, \cdots, w_n$ are linearly independent, it follows that $v_1, \cdots, v_n$ are also linearly independent. Furthermore, as $\dim V = \dim W$, we have $v_1, \cdots, v_n$ as a basis for V.
   Now, let $x \in \ker(T)$. Then $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$ holds for some $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$. Then,

   $$0_W = T(x) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = \alpha_1 w_1 + \cdots + \alpha_n w_n.$$

   Since $w_1, \cdots, w_n$ are linearly independent, it follows that $\alpha_1, \cdots, \alpha_n = 0$, which means:

   $$x = \alpha_1 v_1 + \cdots + \alpha_n v_n = 0_V,$$

   and

   $$\ker(T) = \{0_V\}.$$

   Hence, T is one-to-one.

2. Doing the computation **by hand**, find the determinant of the matrix

   $$\begin{bmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

   Answer. Supposing the given matrix is $\mathbf{A}$, then:

   $$\begin{aligned} \det(\mathbf{A}) &= -2 \det \begin{bmatrix} -2 & 1 \\ 4 & 2 \end{bmatrix} - (3) \det \begin{bmatrix} -4 & 1 \\ 2 & 2 \end{bmatrix} - 2 \det \begin{bmatrix} -4 & -2 \\ 2 & 4 \end{bmatrix} \\ &= -2(-4-4) - (3)(-8-2) - 2(-16+4) \\ &= 16 + 30 + 24 \\ &= 70 \end{aligned}$$

3. (a) Let a be a non-zero real number. Verify that the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix}$$

satisfies $\mathbf{A}^2 = -\mathbf{I}$

(b) Prove that there is no 3×3 real matrix $\mathbf{A}$ such that $\mathbf{A}^2 = -\mathbf{I}$

Hint – Use determinants

Answer.

(a)

$$\begin{aligned}\mathbf{A}^2 &= \begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -\mathbf{I}\end{aligned}$$

(b) Suppose the contrary is true; then:

$$(\det(\mathbf{A}))^2 = \det(\mathbf{A}^2) = \det(-\mathbf{I}) = \det(-1\mathbf{I}) = (-1)^3 \det(\mathbf{I}) = -1$$

Which results in $(\det(\mathbf{A}))^2 = -1$ which cannot happen since $\mathbf{A}$ is a real matrix. Hence the assumption is false.

4. Find the eigenvalues and corresponding eigenspaces for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Answer. The characteristic polynomial of $\mathbf{A}$ is

$$\lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

and thus $\lambda \in \{0, 2\}$.
For $\lambda = 0$, any $\mathbf{v}$ in the eigen-space of $\lambda = 0$ will satisfy:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0.$$

The solution set is:

$$E_0 = \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

For $\lambda = 2$, any $\mathbf{v}$ in the eigen-space of $\lambda = 2$ will satisfy:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2.$$

The solution set is:

$$E_2 = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

5. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be such that $\mathbf{A}^2 = \mathbf{A}$. Prove that the only possible eigenvalues of $\mathbf{A}$ are $\lambda = 0$ or $\lambda = 1$
Answer. Let λ be an eigenvalue for $\mathbf{A}$. Then there exist a non-zero $\mathbf{v} \in \mathbb{R}^n$ with

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Consequently

$$\mathbf{A}^2\mathbf{v} = \mathbf{A}(\mathbf{A}\mathbf{v}) = \mathbf{A}(\lambda\mathbf{v}) = \lambda(\mathbf{A}\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}$$

On the other hand

$$\mathbf{A}^2\mathbf{v} = \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

Therefore

$$\lambda^2\mathbf{v} = \lambda\mathbf{v}$$

Since $\mathbf{v}$ is non-zero, then $\lambda^2 = \lambda$ and $\lambda \in \{0, 1\}$.