MATH 317 **Spring 2025** **Problem Set #3**

Due March 28th 2025 **Mobina Amrollahi**

1. If it exists, find the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

Answer. To detect whether A is invertible or not, we'll be calculating the RREF of the following augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$
$$\xrightarrow{R_1\leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -1 \end{array}\right]$$

which reveals that $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ is invertible and $\mathbf{A}^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$.

2. Let $\mathbf{A}$ be an $n \times n$ matrix with $\mathbf{A}^2 = \mathbf{AA} = \mathbf{0}$. Prove that $\mathbf{I} - \mathbf{A}$ is invertible and $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A}$.
Answer. We will calculate $(\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})$, and show:

$$(\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{I}$$

$$\begin{aligned} (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A}) &= \mathbf{II} + \mathbf{I}(-\mathbf{A}) + \mathbf{AI} + \mathbf{A}(-\mathbf{A}) \\ &= \mathbf{I} - \mathbf{IA} + \mathbf{A} - \mathbf{AA} \\ &= \mathbf{I} - \mathbf{A} + \mathbf{A} - \mathbf{0} \\ &= \mathbf{I} \end{aligned}$$

Hence $\mathbf{I} + \mathbf{A}$ is the unique inverse of $\mathbf{I} - \mathbf{A}$.

3. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ be an invertible $n \times n$ matrix. Prove that $\text{col}(\mathbf{A}) = \text{col}(\mathbf{AB})$.
Answer To show $\text{col}(\mathbf{A}) = \text{col}(\mathbf{AB})$, we have to show $\text{col}(\mathbf{A}) \subset \text{col}(\mathbf{AB})$ and $\text{col}(\mathbf{AB}) \subset \text{col}(\mathbf{A})$.

   (a) $\text{col}(\mathbf{A}) \subset \text{col}(\mathbf{AB})$: suppose $\mathbf{b} \in \text{col}(\mathbf{A})$, then there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$. Since

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{x}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{x} = \mathbf{A}(\mathbf{I})\mathbf{x} = \mathbf{Ax} = \mathbf{b}$$

   it follows that $\mathbf{b} \in \text{col}(\mathbf{AB})$

(b) $\text{col}(\mathbf{AB}) \subset \text{col}(\mathbf{A})$: suppose $\mathbf{b} \in \text{col}(\mathbf{AB})$, then there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{ABy} = \mathbf{b}$. Since

$$\mathbf{ABy} = \mathbf{A}(\mathbf{By}) = \mathbf{b}$$

now consider $\mathbf{x} = \mathbf{By} \in \mathbb{R}^n$, then

$$\mathbf{A}(\mathbf{By}) = \mathbf{Ax} = \mathbf{b}$$

and $\mathbf{b} \in \text{col}(\mathbf{A})$. It's important to note that $\mathbf{B} \neq \mathbf{0}_{n\times n}$ since it's invertible, hence $\mathbf{By}$ can take different values based on $\mathbf{y}$.

4. Let V be a vector space, let $x \in V$ be non-zero, and let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha x = \beta x$. Then $\alpha = \beta$.
Answer.

$$0_V = \alpha x + (-(\alpha x)) = \beta x + (-(\alpha x)) = \beta x + (-\alpha)x = (\beta - \alpha)x$$

$(\beta - \alpha)x = 0_V$, and since x is non-zero, then $\beta - \alpha = 0$ which results in $\beta = \alpha$.

5. Verify that

$$W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} \in \mathbb{R}^3 \colon 5r = s + 2t,\ 16t - 3r = 7s \right\}$$

is a subspace of $\mathbb{R}^3$.
Answer. Define

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -2 \\ -3 & -7 & 16 \end{bmatrix}$$

Now, note that

$$\mathbf{A} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5 & -1 & -2 \\ -3 & -7 & 16 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5r - s - 2t \\ -3r - 7s + 16t \end{bmatrix} = \begin{bmatrix} s + 2t - s - 2t \\ -3r - (16t - 3r) + 16t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Consequently,

$$W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} \in \mathbb{R}^3 \colon 5r = s + 2t,\ 16t - 3r = 7s \right\} = \mathcal{N}(A)$$

which is a subspace of $\mathbb{R}^3$.