MATH 317 **Spring 2025** **Problem Set #1**

Due Feb 10^{th} 2025 **Mobina Amrollahi**

1. Given sets A and B, we define

$$A \setminus B = \{x \in A \colon x \notin B\}$$

Prove that

$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$$

holds for all sets X, Y, Z.

Answer

We will show $X \setminus (Y \cup Z) \subseteq (X \setminus Y) \cap (X \setminus Z)$ and $(X \setminus Y) \cap (X \setminus Z) \subseteq X \setminus (Y \cup Z)$.

(a) $X \setminus (Y \cup Z) \subseteq (X \setminus Y) \cap (X \setminus Z)$

Proof. Let $x \in X \setminus (Y \cup Z)$. Then, based on the definition, $x \in X$ and $x \notin Y \cup Z$. Consequently, $x \notin Y$ and $x \notin Z$. Thus, again based on the defition, $x \in X \setminus Z$ and $x \in X \setminus Y$. Therefore, $x \in (X \setminus Z) \cap (X \setminus Y)$. $\square$

(b) $(X \setminus Y) \cap (X \setminus Z) \subseteq X \setminus (Y \cup Z)$

Proof. Let $x \in (X \setminus Y) \cap (X \setminus Z)$. Then, $x \in X \setminus Y$ and $x \in X \setminus Z$. Based on the definition, $x \in X$, $x \notin Y$, and $x \in X$, $x \notin Z$. Consequently, $x \in X$ and $x \notin (Y \cup Z)$ and thus, $x \in X \setminus (Y \cup Z)$. $\square$

2. Find a square root of i; i.e., find a complex number z such that $z^2 = i$.

Answer

Let $a, b \in \mathbb{R}$ such that $\sqrt{i} = a + bi$. Then:

$$(\sqrt{i})^2 = i = (a+bi)^2 = (a+bi)(a+bi) = a^2 + 2abi - b^2$$

Hence

$$\mathrm{Re}(i) + \mathrm{Im}(i)i = \mathrm{Re}(a^2 + 2abi - b^2) + \mathrm{Im}(a^2 + 2abi - b^2)i$$

And

$$\mathrm{Re}(i) = \mathrm{Re}(a^2 + 2abi - b^2) \quad \text{and} \quad \mathrm{Im}(i) = \mathrm{Im}(a^2 + 2abi - b^2)$$

This leads to the following:

$$\cancelto{0}{\mathrm{Re}(i)} = \cancelto{a^2-b^2}{\mathrm{Re}(a^2 + 2abi - b^2)} \quad \text{and} \quad \cancelto{1}{\mathrm{Im}(i)} = \cancelto{2ab}{\mathrm{Re}(a^2 + 2abi - b^2)}$$

$$a^2 - b^2 = 0 \implies a^2 = b^2 \implies |a| = |b| \tag{1}$$

$$2ab = 1 \implies ab = \frac{1}{2} \implies b = \frac{1}{2a} \tag{2}$$

$$\xrightarrow{(1) \text{ and } (2)} a^2 = \frac{1}{4a^2} \implies a^4 = \frac{1}{4} \implies a^2 = \frac{1}{2} \implies a = \pm\frac{\sqrt{2}}{2}$$

Therefore $\sqrt{i}$ can take each of the following two values:

(a) $a = +\frac{\sqrt{2}}{2}$ and $b = \frac{1}{2a} = +\frac{\sqrt{2}}{2} \implies \sqrt{i} = \frac{\sqrt{2}}{2}(1+i)$.

(b) $a = -\frac{\sqrt{2}}{2}$ and $b = \frac{1}{2a} = -\frac{\sqrt{2}}{2} \implies \sqrt{i} = -\frac{\sqrt{2}}{2}(1+i)$

3. Find all solutions to the system

$$\begin{aligned} 2x + 3y &= -8 \\ x - y &= 6 \end{aligned}$$

Answer
Consider the given system's augmented matrix and right-hand side:

$$\left[\begin{array}{cc|c} 2 & 3 & -8 \\ 1 & -1 & 6 \end{array}\right]$$

We will be doing some row operations to make the augmented matrix simpler to work with:

$$\xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{cc|c} 0 & 5 & -20 \\ 1 & -1 & 6 \end{array}\right] \xrightarrow{\frac{1}{5}R_1+R_2\to R_2} \left[\begin{array}{cc|c} 0 & 5 & -20 \\ 1 & 0 & 2 \end{array}\right] \xrightarrow{\frac{1}{5}R_1\to R_1} \left[\begin{array}{cc|c} 0 & 1 & -4 \\ 1 & 0 & 2 \end{array}\right]$$

Hence, the solution to the system in the parametric form is the following ordered tuple:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

The entire solution set is written as:

$$S = \{(x,y) | x = 2, y = -4\}$$

4. **By hand**, find the RREF of

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 2 & 1 & 5 & 10 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 7 & 7 & 14 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2\to R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{-4R_1+R_3\to R_3} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 10 & 10 & 20 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_3\to R_3} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-R_2+R_3\to R_3} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{3R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Find the null space of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -8 & 7 \end{bmatrix}$$

Answer
Supposing the matrix above is A, we'll solve the the homogeneous system $\mathcal{LS}(A, 0)$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & -8 & 7 & 0 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & -10 & 4 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -10 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -10 & 4 & 0 \end{array}\right] \xrightarrow{10R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 14 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{14}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2-R_3+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

All three columns on 1, 2, and 3 are pivot columns, and the variables corresponding to them, i.e., x_1, x_2, and x_3, are fixed variables. Hence:

$$\mathcal{N}(A) = \{\mathbf{0}\}$$