MATH 317 **Spring 2025** **Problem Set #4**

Due April 14^{th} 2025 **Mobina Amrollahi**

1. Determine if the polynomials

$$x^2 - x + 5, \quad 4x^3 - x^2 + 5x, \quad 3x + 2$$

are linearly independent or linearly dependent.
Answer. Suppose we have a relation of linear dependence on this set,

$$\begin{aligned}\mathbf{0} &= 0x^3 + 0x^2 + 0x + 0 \\ &= \alpha_1(x^2 - x + 5) + \alpha_2(4x^3 - x^2 + 5x) + \alpha_3(3x + 2) \\ &= (4\alpha_2)x^3 + (\alpha_1 - \alpha_2)x^2 + (-\alpha_1 + 5\alpha_2 + 3\alpha_3)x + (5\alpha_1 + 2\alpha_3)\end{aligned}$$

Using the definitions of vector addition and scalar multiplication in P3, we arrive at:

$$0x^3 + 0x^2 + 0x + 0 = (4\alpha_2)x^3 + (\alpha_1 - \alpha_2)x^2 + (-\alpha_1 + 5\alpha_2 + 3\alpha_3)x + (5\alpha_1 + 2\alpha_3)$$

Equating coefficients, we arrive at the homogeneous system of equations,

$$\begin{aligned} 4\alpha_2 &= 0 \\ \alpha_1 - \alpha_2 &= 0 \\ -\alpha_1 + 5\alpha_2 + 3\alpha_3 &= 0 \\ 5\alpha_1 + 2\alpha_3 &= 0 \end{aligned}$$

We form the coefficient matrix of this homogeneous system of equations

$$\left[\begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 5 & 3 & 0 \\ 5 & 0 & 2 & 0 \end{array}\right]$$

and row-reduced it

$$\left[\begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 5 & 3 & 0 \\ 5 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 5 & 3 & 0 \\ 5 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 5 & 3 & 0 \\ 5 & 0 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 \\ 5 & 0 & 2 & 0 \end{array}\right] \xrightarrow{-5R_2 + R_4 \to R_4} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_4 \to R_4} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-5R_1 - 3R_4 + R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, the only solution to this homogeneous system is the trivial one where $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$. Hence, the polynomials are linearly independent.

2. Determine if the matrices

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

are linearly independent or linearly dependent.

Answer. We will build the generic relation of linear dependence which is,

$$\alpha_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \mathbf{0}$$

Based on the definition of vector addition and scaler multiplication in M_{22} we have

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 & 2\alpha_1 + \alpha_2 + \alpha_3 \\ 2\alpha_1 - \alpha_2 + \alpha_3 & \alpha_1 + 2\alpha_2 + 2\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equating entries we get the homogeneous system of four equations with four variables

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ 2\alpha_1 - \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 + 2\alpha_3 &= 0 \end{aligned}$$

We form the coefficient matrix of this homogeneous system of equations

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

and row-reduced it

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 4 & 0 & 2 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1+R_4\to R_4} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_4\to R_4} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_3+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_4+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3\leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The only solution to this homogeneous system is the trivial one where $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$. Therefore, these matrices are linearly independent.

3. Recall that $\mathbb{R}_2[x]$ is the vector space of polynomials of the form $a + bx + cx^2$, where $c \neq 0$.

   (a) Let $Q = \{p \in \mathbb{R}_2[x] \colon p(0) = 0\}$. Prove that Q is a subspace of $\mathbb{R}_2[x]$

   (b) Find a basis for Q.

   Answer.

   (a)
   - Q is non-empty since $x + x^2 \in Q$.
   - for all $\alpha \in \mathbb{R}$ and $p \in Q$:

     $$(\alpha p)(0) = \alpha p(0) = \alpha 0 = 0$$

     Hence $\alpha p \in Q$
   - for all $p, q \in Q$

     $$\begin{aligned}(p+q)(0) &= p(0) + q(0)\\ &= 0 + 0\\ &= 0\end{aligned}$$

     Hence $p + q \in Q$

   We have shown that Q meets the three conditions of being a sub-space of $\mathbb{R}_2[x]$.

   (b) For $p \in Q$ such that $p = a + bx + cx^2$ we know

   $$0 = p(0) = a + b0 + c0 = a$$

   This implies that the polynomials in Q have the form:

   $$p(x) = bx + cx^2$$

   In other words, $Q = \text{span}(x, x^2)$ and since x and x^2 are linearly independent $B = \{x, x^2\}$ is a basis for Q.

4. Let V be a vector space with a basis $u, v, w \in V$. Prove that

   $$u - 2v, \quad 3u - w, \quad 2w - u$$

   is also a basis for V.
   Answer. Since V has a basis $\{u, v, w\}$, we know that $dim(V) = 3$. So any three linearly independent vectors in V will form a basis. Hence, we only need to make sure $u - 2v, 3u - w, 2w - u$ are linearly independent. Suppose

   $$\begin{aligned}0_V &= \alpha_1(u - 2v) + \alpha_2(3u - w) + \alpha_3(2w - u)\\ &= (\alpha_1 + 3\alpha_2 - \alpha_3)u + (-2\alpha_1)v + (-\alpha_2 + 2\alpha_3)w\end{aligned}$$

   Since u, v, w are linearly independent the following holds:

   $$\begin{aligned}\alpha_1 + 3\alpha_2 - \alpha_3 &= 0\\ -2\alpha_1 &= 0\\ -\alpha_2 + 2\alpha_3 &= 0\end{aligned}$$

We form the coefficient matrix of this homogeneous system of equations

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

and row-reduced it

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\frac{-1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{-R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 3 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{3R_3+R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R_1 \to R_1} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The only solution to this homogeneous system is the trivial one where $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$. Therefore, $u - 2v, 3u - w, 2w - u$ are linearly independent.

5. Let n be odd. Prove that there is no $n \times n$ matrix $\mathbf{A}$ such that $N(\mathbf{A}) = \text{col}(A)$
Answer. Proof by contradiction. Assume, for the sake of contradiction, that

$$N(\mathbf{A}) = \text{col}(A)$$

We know from the rank-nullity theorem that:

$$N(\mathbf{A}) + \text{col}(A) = n$$

Substituting the assumption into the equation gives:

$$2 \cdot \text{col}(A) = n$$

This implies that n is even. However, we are given that n is odd, which leads to a contradiction.

Therefore, our assumption must be false, and we conclude that:

$$N(\mathbf{A}) \neq \text{col}(A)$$