1. Let V be a vector space, let $\alpha \in \mathbb{R}$ be non-zero, and let $x, y \in V$ be such that $\alpha x = \alpha y$. Prove that $x = y$.

   Proof. As $\alpha \neq 0$, we have access to $1/\alpha = \alpha^{-1}$ and thus

   $$x = 1x = (\alpha^{-1}\alpha)x = \alpha^{-1}(\alpha x) = \alpha^{-1}(\alpha y) = (\alpha^{-1}\alpha)y = 1y = y$$

   Question – What if we replaced $1x = x$ with something else? How about $\gamma x = x$ for *some* $\gamma \in \mathbb{R}$?
   Answer.

   $$x = \gamma x = (\gamma.1)x = (1.\gamma)x = (1)(\gamma x) = 1.x$$

   □

2. Let $A \in \mathbb{R}^{n\times n}$, and let $\lambda \in \mathbb{R}$. Prove that $M = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$ is a subspace.

   Proof. First, we note that $A\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$ and so $\mathbf{0} \in M$.

   Now, let $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in M$. Then

   $$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} = \alpha\lambda\mathbf{x} + \beta\lambda\mathbf{y} = \lambda(\alpha\mathbf{x} + \beta\mathbf{y})$$

   and therefore, $\alpha\mathbf{x} + \beta\mathbf{y} \in M$. □

3. Determine if the following polynomials are linearly independent or linearly dependent:

   $$2 + x - 3x^2 - 8x^3, \quad 1 + x + x^2 + 5x^3, \quad 3 - 4x^2 - 7x^3$$

   Solution. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

   $$\alpha(2 + x - 3x^2 - 8x^3) + \beta(1 + x + x^2 + 5x^3) + \gamma(3 - 4x^2 - 7x^3) = 0$$

   This yields

   $$(2\alpha + \beta + 3\gamma) + (\alpha + \beta)x + (-3\alpha + \beta - 4\gamma)x^2 + (-8\alpha + 5\beta - 7\gamma)x^3 = 0$$

   and so

   $$\begin{aligned} 2\alpha + \beta + 3\gamma &= 0 \\ \alpha + \beta &= 0 \\ -3\alpha + \beta - 4\gamma &= 0 \\ -8\alpha + 5\beta - 7\gamma &= 0 \end{aligned}$$

   This yields $\alpha = \beta = \gamma = 0$, and thus we have linear independence. □

   Recall – $M = \{p \in \mathbb{R}_2[x] : p(0) = 0\}$ is a subspace of $\mathbb{R}_2[x]$ with a basis of x, x^2. What if we replace "$p(0) = 0$" with something else? Note that $p(1) = 1$ doesn't work. How about $p(1) = 0$? Can prove that $N = \{p \in \mathbb{R}_2[x] : p(1) = 0\}$ is a subspace and find a basis for it?

   (a) Let $N = \{p \in \mathbb{R}_2[x] : p(1) = 0\}$. Prove that N is a subspace of $\mathbb{R}_2[x]$

   (b) Find a basis for N.

Answer.

(a)
- Since $x^2 - 1, x - 1 \in N$, N is not empty.
- Suppose $p \in N$. Then for $\alpha \in \mathbb{R}$ we have:

$$\begin{aligned}(\alpha p)(1) &= \alpha p(1) \\ &= \alpha 0 \\ &= 0\end{aligned}$$

- Suppose $p, q \in N$. Then we have:

$$\begin{aligned}(p+q)(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

(b) Find a basis for N. We'll show that $B = \{x-1, x^2-1\}$ is a basis for N. To show such, we have to show $x-1$ and x^2-1 are linearly independent. Suppose:

$$\begin{aligned}0 &= \alpha_1(x-1) + \alpha_2(x^2-1) \\ &= (-\alpha_1 - \alpha_2) + (\alpha_1)x + \alpha_2(x^2)\end{aligned}$$

Since $1, x$, and x^2 are linearly independent, the only way for the following linear combination to be zero is

$$\begin{aligned}-\alpha_1 - \alpha_2 &= 0 \\ \alpha_1 &= 0 \\ \alpha_2 &= 0\end{aligned}$$

Which results is $\alpha_1 = 0$ and $\alpha_2 = 0$. Now we have to show $N = \text{span}(x-1, x^2-1)$. We already know $x-1, x^2-1 \in N$, hence $\text{span}(x-1, x^2-1) \subset N$. Now, to show $N \subset \text{span}(x-1, x^2-1)$, suppose $p = a + bx + cx^2 \in N$. We know:

$$p(1) = 0 \rightarrow a + b + c = 0$$

Now, in order for $p \in \text{span}(x-1, x^2-1)$, the following has to hold:

$$\begin{aligned}a + bx + cx^2 &= \alpha(x-1) + \gamma(x^2-1) \\ &= (-\alpha - \gamma) + (\alpha)x + (\gamma)x^2\end{aligned}$$

Which results in

$$\begin{aligned}-\alpha - \gamma &= a \\ \alpha &= b \\ \gamma &= c\end{aligned}$$

4. Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -8 & 6 \\ -7 & 14 & -9 \\ -2 & 4 & 0 \end{bmatrix}$$

Solution. The RREF of A is

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so a basis for the row space is

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

NOTE – the rank of A is 2 and $\dim N(A) = 1$ $\square$

5. Let $A \in \mathbb{R}^{m\times n}$.

   (a) Prove that $\dim(N(A^\top)) = m - \operatorname{rank}(A)$

   (b) Prove that if $A^\top$ is nonsingular if and only if $\operatorname{col}(A) = \mathbb{R}^m$

   Answer.

   (a) Since $A \in \mathbb{R}^{m\times n}$ then $A^\top \in \mathbb{R}^{n\times m}$, and:

$$\begin{aligned} \dim(N(A^\top)) &= m - \operatorname{rank}(A^\top) \\ &= m - \dim(\operatorname{row}(A)) \\ &= m - \operatorname{rank}(A) \quad (\text{since} \dim(\operatorname{row}(A)) = \operatorname{rank}(A)) \end{aligned}$$

   (b) If $A^\top$ is non-singular, we have $\dim(N(A^\top)) = 0$, which means we have $\operatorname{rank}(A)^\top = m$.

$$\begin{aligned} m &= \operatorname{rank}(A)^\top \\ &= \dim(\operatorname{row}(A)) \\ &= \operatorname{rank}(A) \quad (\text{since} \dim(\operatorname{row}(A)) = \operatorname{rank}(A)) \end{aligned}$$

   which leads to $\operatorname{col}(A) = \mathbb{R}^m$.

6. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

   Find the matrix A for T.

   Solution. Let

$$B = \begin{bmatrix} 8 & 3 \\ 9 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}, \quad P^{-1} = \frac{1}{-7}\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

   Then

$$A = BP^{-1} = \frac{1}{7}\begin{bmatrix} 8 & 3 \\ 9 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 27 & 2 \\ 29 & 5 \end{bmatrix}$$

   $\square$

7. Let $T\colon \mathbb{R}^n \to \mathbb{R}$ be linear. Prove that there exists a unique $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$.

Proof. Define $\mathbf{v} \in \mathbb{R}^n$ by $\mathbf{v}_k = T(\mathbf{e}_k)$. Then

$$T(\mathbf{x}) = T(\mathbf{x}_1\mathbf{e}_1 + \ldots + \mathbf{x}_n\mathbf{e}_n) = \mathbf{x}_1 T(\mathbf{e}_1) + \ldots + \mathbf{x}_n T(\mathbf{e}_n) = \mathbf{x}_1\mathbf{v}_1 + \ldots + \mathbf{x}_n\mathbf{v}_n = \mathbf{x}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{x}$$

holds for all $\mathbf{x} \in \mathbb{R}^n$. Suppose that there is a $\mathbf{w} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{w}\cdot\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{v}_k = T(\mathbf{e}_k) = \mathbf{w}\cdot\mathbf{e}_k = \mathbf{w}_1 \cdot 0 + \ldots + \mathbf{w}_k \cdot 1 + \ldots + \mathbf{w}_n \cdot 0 = \mathbf{w}_k$$

holds for all $1 \leq k \leq n$ and so $\mathbf{v} = \mathbf{w}$. □

Note – This is a special case of the *Riesz Representation Theorem.*

MOREOVER – Possible that $T(v_1), \ldots, T(v_n)$ are linearly dependent for a linear $T\colon V \to W$ even if $v_1, \ldots, v_n$ are linearly independent. What if we knew $T(v_1), \ldots, T(v_n)$ are linearly independent? What can we say about the linear independence/dependence of $v_1, \ldots, v_n$? **Answer**. We will show if $T(v_1), \ldots, T(v_n)$ are linearly independent then $v_1, \ldots, v_n$ have to be also linearly independent.
Prove by contradiction. Assume $v_1, \ldots, v_n$ are linearly dependent. Then $\exists \alpha_1, \alpha_2, \cdots \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0_V$$

with at least one of the α_is being non-zero. Now:

$$\begin{aligned} 0_W &= T(0_V) \\ &= T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \\ &= T(\alpha_1 v_1) + T(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) \\ &= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) \end{aligned}$$

Which contradicts the assumption of $T(v_1), \ldots, T(v_n)$ being linearly independent. Hence $v_1, \ldots, v_n$ are linearly independent.

8. Verify that

$$W = \left\{ \begin{bmatrix} 5r - s + 2t \\ 16t - 3r + 7s \\ t \end{bmatrix} \in \mathbb{R}^3 \colon r, s, t \in \mathbb{R}, \right\}$$

is a subspace of $\mathbb{R}^3$.
Answer. Consider the following matrix A:

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & 2 \\ -3 & 7 & 16 \\ 0 & 0 & 1 \end{bmatrix}$$

Now note that:

$$\begin{bmatrix} 5 & -1 & 2 \\ -3 & 7 & 16 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 5r - s + 2t \\ 16t - 3r + 7s \\ t \end{bmatrix}$$

Hence $W = \operatorname{col}(A)$ which is a subspace of $\mathbb{R}^3$.