MATH 317 **Spring 2025** **Problem Set #2**

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1. Find α and β that solve the vector equation

$$\alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Answer
Consider the given system's augmented matrix and right-hand side:

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 3 & 0 \end{array}\right]$$

We will be doing some row operations to make the augmented matrix simpler to work with:

$$\xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{cc|c} 0 & -5 & 5 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{5}R_1 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 3 \end{array}\right]$$

Hence, the solution to the system in the parametric form is the following ordered tuple:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

The entire solution set is written as:

$$S = \{(\alpha, \beta) | \alpha = 3, \beta = -1\}$$

2. Prove that if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are such that $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$, then $\mathbf{y} = \mathbf{z}$.
Answer

$$\begin{aligned}
\mathbf{y} &= \mathbf{0} + \mathbf{y} \\
&= ((\text{-}\mathbf{x}) + \mathbf{x}) + \mathbf{y} \quad \text{(Additive Inverses)} \\
&= (\text{-}\mathbf{x}) + (\mathbf{x} + \mathbf{y}) \quad \text{(Based on Additive Associativity Property)} \\
&= (\text{-}\mathbf{x}) + (\mathbf{x} + \mathbf{z}) \quad \text{(Since } \mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}\text{)} \\
&= ((\text{-}\mathbf{x}) + \mathbf{x}) + \mathbf{z} \quad \text{(Based on Additive Associativity Property)} \\
&= \mathbf{0} + \mathbf{z} \\
&= \mathbf{z}
\end{aligned}$$

3. Simplify the following span:

$$\text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \right)$$

Answer
We know:

$$\operatorname{span}\left(\begin{bmatrix}2\\-1\\2\end{bmatrix},\begin{bmatrix}3\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}5\\-1\\3\end{bmatrix}\right)=\operatorname{row}\left(\begin{bmatrix}2&-1&2\\3&0&1\\1&1&-1\\5&-1&3\end{bmatrix}\right)$$

Hence, to simplify the given span, we'll look into its equivalent row space. Specifically, we'll calculate its reduced row-echelon form (RREF):

$$\begin{aligned}
\begin{bmatrix}2&-1&2\\3&0&1\\1&1&-1\\5&-1&3\end{bmatrix}
&\xrightarrow{-R_1-R_3+R_2\to R_2}\begin{bmatrix}2&-1&2\\0&0&0\\1&1&-1\\5&-1&3\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_4}\begin{bmatrix}2&-1&2\\5&-1&3\\1&1&-1\\0&0&0\end{bmatrix}\\
&\xrightarrow{-2R_1-R_3\to R_2}\begin{bmatrix}2&-1&2\\0&0&0\\1&1&-1\\0&0&0\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}2&-1&2\\1&1&-1\\0&0&0\\0&0&0\end{bmatrix}\\
&\xrightarrow{R_1+R_2\to R_2}\begin{bmatrix}2&-1&2\\3&0&1\\0&0&0\\0&0&0\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}3&0&1\\2&-1&2\\0&0&0\\0&0&0\end{bmatrix}\\
&\xrightarrow{-R_1+R_2\to R_2}\begin{bmatrix}3&0&1\\-1&-1&1\\0&0&0\\0&0&0\end{bmatrix}\xrightarrow{\frac{1}{3}R_1\to R_1}\begin{bmatrix}1&0&\frac{1}{3}\\-1&-1&1\\0&0&0\\0&0&0\end{bmatrix}\\
&\xrightarrow{R_1+R_2\to R_2}\begin{bmatrix}1&0&\frac{1}{3}\\0&-1&\frac{4}{3}\\0&0&0\\0&0&0\end{bmatrix}\xrightarrow{-R_2\to R_2}\begin{bmatrix}1&0&\frac{1}{3}\\0&1&-\frac{4}{3}\\0&0&0\\0&0&0\end{bmatrix}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\operatorname{span}\left(\begin{bmatrix}2\\-1\\2\end{bmatrix},\begin{bmatrix}3\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}5\\-1\\3\end{bmatrix}\right)&=\operatorname{row}\left(\begin{bmatrix}2&-1&2\\3&0&1\\1&1&-1\\5&-1&3\end{bmatrix}\right)\\
&=\operatorname{row}\left(\begin{bmatrix}1&0&\frac{1}{3}\\0&1&-\frac{4}{3}\\0&0&0\\0&0&0\end{bmatrix}\right)\\
&=\operatorname{span}\left(\begin{bmatrix}1\\0\\\frac{1}{3}\end{bmatrix},\begin{bmatrix}0\\1\\-\frac{4}{3}\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right)\\
&=\operatorname{span}\left(\begin{bmatrix}1\\0\\\frac{1}{3}\end{bmatrix},\begin{bmatrix}0\\1\\-\frac{4}{3}\end{bmatrix}\right)
\end{aligned}$$

4. Let $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^n$. Prove that if $\mathbf{x}, \mathbf{y} \in \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$, then $\mathbf{x} + \mathbf{y} \in \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.
Answer

$$\mathbf{x} \in \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) \implies \quad \exists \alpha_1, \ldots, \alpha_m \in \mathbb{R} \quad \text{such that} \quad \mathbf{x} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_m \mathbf{x}_m$$

And

$$\mathbf{y} \in \operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_m) \implies \quad \exists \beta_1, \ldots, \beta_m \in \mathbb{R} \quad \text{such that} \quad \mathbf{y} = \beta_1 \mathbf{x}_1 + \cdots + \beta_m \mathbf{x}_m$$

Now

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (\alpha_1 \mathbf{x}_1 + \cdots + \alpha_m \mathbf{x}_m) + (\beta_1 \mathbf{x}_1 + \cdots + \beta_m \mathbf{x}_m) \\
&= (\alpha_1 + \beta_1)\mathbf{x}_1 + \cdots + (\alpha_m + \beta_m)\mathbf{x}_m
\end{aligned}$$

Therefore

$$\mathbf{x} + \mathbf{y} \in \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

5. Determine if the following are linearly independent/dependent:

$$\begin{bmatrix} -1 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix}$$

Answer Let A be a 4×3 matrix with the given vecotrs as its columns. We'll solve the the homogeneous system $\mathcal{LS}(A, 0)$, and look at its $\mathcal{N}(A)$. First, we'll calculate the RREF of A:

$$\begin{bmatrix} -1 & 3 & 7 \\ 2 & 3 & 3 \\ 4 & -1 & -6 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} -1 & 3 & 7 \\ 4 & -1 & -6 \\ 2 & 3 & 4 \\ 2 & 3 & 3 \end{bmatrix} \xrightarrow{R_2 - R_4 \to R_2} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & 1 \\ 4 & -1 & -6 \\ 2 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 + R_3 \to R_1} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 4 & -1 & -6 \\ 2 & 3 & 3 \end{bmatrix} \xrightarrow{-R_1 + R_3 \to R_3} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -6 \\ 2 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 + R_4 \to R_4} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -6 \\ 3 & 0 & -3 \end{bmatrix} \xrightarrow{\frac{1}{3} R_4 \to R_4} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -6 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_4 \to R_4} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -6 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -6 \\ 3 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{6R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \\ 3 & 2 & 0 \end{bmatrix} \xrightarrow{-R_1 - R_3 + R_4 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \\ 1 & 5 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_3+R_4 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \\ 0 & 8 & 0 \end{bmatrix} \xrightarrow{\frac{1}{8}R_4 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_1+3R_2 \to R_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the associated homogeneous system is:

$$\begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$$

Which means:

$$\mathcal{N}(A) = \{\mathbf{0}\}$$

Therefore, the given vectors are linearly independent.