MATH 6410 Foundations of Probability Theory, Homework 1

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App. A: Problem 3

Let $I = [0, 1]$, $\Omega = \mathbb{R}$ and for $\alpha \in \mathbb{R}$,

$$A_\alpha = (\alpha - 1, \alpha + 1),$$

the open interval $\{x : \alpha - 1 < x < \alpha + 1\}$.

(a) Show that

$$\bigcup_{\alpha \in I} A_\alpha = (-1, 2) \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha = (0, 1).$$

(b) Suppose

$$J = \{x : x \in I,\, x \text{ is rational}\}.$$

Find $\bigcup_{x \in J} A_x$ and $\bigcap_{x \in J} A_x$.

Answer

(a) *Proof.* We will show $\bigcup_{\alpha \in I} A_\alpha \subset (-1, 2)$ and $(-1, 2) \subset \bigcup_{\alpha \in I} A_\alpha$.
($\subseteq$) For any $\alpha \in I$, we have $0 \leq \alpha \leq 1$, hence

$$-1 \leq \alpha - 1 < \alpha + 1 \leq 2,$$

so $(\alpha - 1, \alpha + 1) \subset (-1, 2)$. Taking the union over $\alpha \in I$ gives

$$\bigcup_{\alpha \in I} A_\alpha \subseteq (-1, 2).$$

($\supseteq$) Since $0, 1 \in I$,

$$A_0 = (-1, 1), \qquad A_1 = (0, 2),$$

and thus

$$A_0 \cup A_1 = (-1, 2) \subseteq \bigcup_{\alpha \in I} A_\alpha.$$

Therefore $(-1, 2) \subseteq \bigcup_{\alpha \in I} A_\alpha$.
Combining,

$$\bigcup_{\alpha \in [0,1]} (\alpha - 1, \alpha + 1) = (-1, 2)$$

. $\square$

Proof. We will show $\bigcap_{\alpha\in I} A_\alpha \subset (0,1)$ and $(0,1) \subset \bigcap_{\alpha\in I} A_\alpha$.
($\subseteq$) Suppose, toward a contradiction, that $\bigcap_{\alpha\in I} A_\alpha \nsubseteq (0,1)$. Then there exists $x \in \bigcap_{\alpha\in I} A_\alpha$ with $x \notin (0,1)$. Hence either $x \le 0$ or $x \ge 1$, and since $x \in \bigcap_{\alpha\in I} A_\alpha$, specifically $x \in A_1$, and $x \in A_0$.

i. If $x \le 0$, take $\alpha = 1$. Then $A_1 = (0,2)$, so $x \notin A_1$, contradicting $x \in \bigcap_{\alpha\in I} A_\alpha$.
ii. If $x \ge 1$, take $\alpha = 0$. Then $A_0 = (-1,1)$, so $x \notin A_0$, again a contradiction.

Therefore $\bigcap_{\alpha\in I} A_\alpha \subseteq (0,1)$.
($\supseteq$) Let $x \in (0,1)$. For any $\alpha \in [0,1]$ we have

$$\alpha - 1 \le 0 < x \quad \text{and} \quad x < 1 \le \alpha + 1,$$

hence $\alpha - 1 < x < \alpha + 1$, i.e. $x \in A_\alpha$. Since this holds for every $\alpha \in I$, it follows that

$$x \in \bigcap_{\alpha\in I} A_\alpha,$$

so $(0,1) \subseteq \bigcap_{\alpha\in I} A_\alpha$.
Combining,

$$\bigcap_{\alpha\in[0,1]} (\alpha - 1, \alpha + 1) = (0,1).$$

□

(b) Since $\{0,1\} \subseteq J \subseteq I$, monotonicity of unions/intersections gives

$$A_0 \cup A_1 \;\subseteq\; \bigcup_{\alpha\in J} A_\alpha \;\subseteq\; \bigcup_{\alpha\in I} A_\alpha, \qquad \bigcap_{\alpha\in I} A_\alpha \;\subseteq\; \bigcap_{\alpha\in J} A_\alpha \;\subseteq\; A_0 \cap A_1.$$

But $A_0 = (-1,1)$, $A_1 = (0,2)$, and from part (a) $\bigcup_{\alpha\in I} A_\alpha = (-1,2)$, $\bigcap_{\alpha\in I} A_\alpha = (0,1)$. Hence

$$\bigcup_{\alpha\in J} A_\alpha = (-1,2), \qquad \bigcap_{\alpha\in J} A_\alpha = (0,1).$$

App. A: Problem 5

Show that $X \equiv \{0,1\}^{\mathbb{N}}$, the set of all sequences $\{\omega_i\}_{i\in\mathbb{N}}$ where each $\omega_i \in \{0,1\}$, is *uncountable.* Conclude that $\mathcal{P}(\mathbb{N})$ is uncountable.

Answer

Assume, for contradiction, that $\{0,1\}^{\mathbb{N}}$ is countable. Then there exists a function $f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ that is surjective, so each sequence in $\{0,1\}^{\mathbb{N}}$ corresponds to some natural number. In particular, we may write

$$f(n) = \omega_n = (\omega_{n1}, \omega_{n2}, \omega_{n3}, \dots),$$

and without loss of generality, we assume this enumeration $f(n) = \omega_n$.

This would give a table:

$$
\begin{array}{c|cccc}
n & \omega_{n1} & \omega_{n2} & \omega_{n3} & \cdots \\
\hline
1 & \omega_{11} & \omega_{12} & \omega_{13} & \cdots \\
2 & \omega_{21} & \omega_{22} & \omega_{23} & \cdots \\
3 & \omega_{31} & \omega_{32} & \omega_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Now construct ω' digit by digit, where the i^{th} digit of ω' is the complement of the i^{th} digit of ω_i. Then for all $i \in \mathbb{N}$, $\omega' \neq \omega_i$ since they differ in their i^{th} digit. Hence we have found a sequence ω' not in the image of f, a contradiction.

Therefore, $\{0,1\}^{\mathbb{N}}$ is uncountable. Since $\{0,1\}^{\mathbb{N}}$ is in bijection with $\mathcal{P}(\mathbb{N})$ (each sequence represents a subset of $\mathbb{N}$ via indicator function), it follows that $\mathcal{P}(\mathbb{N})$ is also uncountable.

App. A: Problem 9

Apply the principle of induction to establish the following:

(a) For each $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

(b) For each $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_k \in \mathbb{R}$,

(i) *(The binomial formula).*

$$(x_1 + x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^r x_2^{n-r}.$$

(ii) *(The multinomial formula).*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \frac{n!}{r_1! r_2! \cdots r_k!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k},$$

where the summation extends over all $(r_1, r_2, \ldots, r_k)$ such that $r_i \in \mathbb{N}$, $0 \leq r_i \leq n$, and $\sum_{i=1}^{k} r_i = n$.

Answer

(a) We will proceed $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ using mathematical induction.

Proof. **Basis:** $n = 1$

$$\text{Left-Hand side:} \quad \sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\text{Right-Hand side:} \quad \frac{(2 \cdot 1 + 1) \cdot (1 + 1) \cdot 1}{6} = \frac{(3) \cdot (2) \cdot 1}{6} = 1$$

Therefor for $n = 1 : \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.
Induction Hypothesis:
Suppose:

$$\sum_{j=1}^{N} j^2 = \frac{(2N+1)(N+1)N}{6}.$$

holds for some $N \in \mathbb{N}$.
Inductive Step:

$$\begin{aligned}
\sum_{i=1}^{N+1} j^2 &= \left(\sum_{j=1}^{N} j^2\right) + (N+1)^2 \\
&= \underbrace{\frac{(2N+1)(N+1)N}{6}}_{\text{Induction assumption}} + (N+1)^2 \\
&= (N+1)\left(\frac{(2N+1)N}{6} + (N+1)\right) \\
&= (N+1)\left(\frac{2N^2+2N+6N+6}{6}\right) \\
&= (N+1)\left(\frac{2N^2+8N+6}{6}\right) \\
&= (N+1)\left(\frac{(2N+3)(N+2)}{6}\right) \\
&= \frac{(N+1)(2N+3)(N+2)}{6} \\
&= \frac{(2(N+1)+1)(N+2)(N+1)}{6}
\end{aligned}$$

□

(b) (i) We will proceed $(x_1+x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^r x_2^{n-r}$ using mathematical induction.

Proof. **Basis:** $n = 1$

Left-Hand side: $(x_1+x_2)^1 = x_1 + x_2$

Right-Hand side: $\sum_{r=0}^{1} \binom{n}{r} x_1^r x_2^{n-r} = \binom{1}{0} x_1^0 x_2^{1-0} + \binom{1}{1} x_1^1 x_2^{1-1} = 1.1.x_2 + 1.x_1.1 = x_1 + x_2$

Therefor for $n = 1 : (x_1+x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^r x_2^{n-r}$.
Induction Hypothesis:
Suppose:

$$(x_1+x_2)^N = \sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N-r}.$$

holds for some $N \in \mathbb{N}$.
Inductive Step:

$$
\begin{aligned}
(x_1+x_2)^{N+1} &= (x_1+x_2)^N \cdot (x_1+x_2) \\
&= \underbrace{\sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N-r}}_{\text{Induction assumption}} \cdot (x_1+x_2) \\
&= \sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N-r} \cdot x_1 + \sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N-r} \cdot x_2 \\
&= \sum_{r=0}^{N} \binom{N}{r} x_1^{r+1} x_2^{N-r} + \sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N+1-r} \\
&= \underbrace{\sum_{r=1}^{N+1} \binom{N}{r-1} x_1^r x_2^{N-(r-1)}}_{\text{Shifting Parameters by 1}} + \sum_{r=0}^{N} \binom{N}{r} x_1^r x_2^{N+1-r} \\
&= \underbrace{\binom{N}{N} x_1^{N+1} x_2^{N-(N+1-1)}}_{r=N} + \sum_{r=1}^{N} \binom{N}{r-1} x_1^r x_2^{N-(r-1)} \\
&+ \underbrace{\binom{N}{0} x_1^0 x_2^{N+1-0}}_{r=0} + \sum_{r=1}^{N} \binom{N}{r} x_1^r x_2^{N+1-r} \\
&= x_1^{N+1} + \sum_{r=1}^{N} \binom{N}{r-1} x_1^r x_2^{N+1-r} \\
&+ x_2^{N+1} + \sum_{r=1}^{N} \binom{N}{r} x_1^r x_2^{N+1-r} \\
&= x_1^{N+1} + x_2^{N+1} + \sum_{r=1}^{N} (\binom{N}{r-1} + \binom{N}{r}) x_1^r x_2^{N+1-r} \\
&= \binom{N+1}{0} x_1^{N+1} x_2^0 + \binom{N+1}{N+1} x_1^0 x_2^{N+1} + \underbrace{\sum_{r=1}^{N} \binom{N+1}{r} x_1^r x_2^{N+1-r}}_{\text{Pascal's Identity.}} \\
&= \sum_{r=0}^{N+1} \binom{N+1}{r} x_1^r x_2^{N+1-r}
\end{aligned}
$$

□

(ii) We will proceed $(x_1 + x_2 + \cdots + x_k)^n = \sum \frac{n!}{r_1! r_2! \cdots r_k!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$ using mathematical induction.

Proof. **Basis:** $k = 1$

$$\begin{aligned} \text{Left-Hand side:} \quad & (x_1)^n = x_1^n \\ \text{Right-Hand side:} \quad & \sum \frac{n!}{r_1!} x_1^{r_1} \\ & = \sum_{\substack{r_1 = n \\ r_1 \geq 0}} \frac{n!}{n!}\, x_1^n \\ & = x_1^n \end{aligned}$$

Therefor for $k = 1 : (x_1)^n = \sum_{\substack{r_1=n \\ r_1 \geq 0}} \frac{n!}{r_1!}\, x_1^{r_1}$.

Induction Hypothesis:

Suppose:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_i \geq 0}} \frac{n!}{r_1!r_2!\cdots r_k!}\, x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}.$$

holds for some $k \in \mathbb{N}$.

Inductive Step:

$$\begin{aligned} (x_1 + x_2 + \cdots + x_k + x_{k+1})^N &= (\underbrace{(x_1 + x_2 + \cdots + x_k)}_{\text{y}} + (x_{k+1}))^N \\ &= (y + x_{k+1})^N \\ &= \underbrace{\sum_{r=0}^{N} \binom{N}{r} y^r x_{k+1}^{N-r}}_{\text{Binomial formula.}} \\ &= \sum_{r=0}^{N} \binom{N}{r} (x_1 + x_2 + \cdots + x_k)^r x_{k+1}^{N-r} \\ &= \sum_{r=0}^{N} \frac{N!}{r!(N-r)!} \Big(\sum_{\substack{r_1+r_2+\cdots+r_k=r \\ r_i \geq 0}} \frac{r!}{r_1!r_2!\cdots r_k!}\, x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} \Big) x_{k+1}^{N-r} \\ &= \sum_{r=0}^{N} \frac{N!r!}{r!(N-r)!} \Big(\sum_{\substack{r_1+r_2+\cdots+r_k=r \\ r_i \geq 0}} \frac{1}{r_1!r_2!\cdots r_k!}\, x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} x_{k+1}^{N-r} \Big) \\ &= \sum_{r=0}^{N} \frac{N!}{(N-r)!} \Big(\sum_{\substack{r_1+r_2+\cdots+r_k=r \\ r_i \geq 0}} \frac{1}{r_1!r_2!\cdots r_k!}\, x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} x_{k+1}^{N-r} \Big) \\ &= \sum_{r=0}^{N} \frac{N!}{(N-r)!} \Big(\sum_{\substack{r_1+r_2+\cdots+r_k=r \\ r_i \geq 0}} \frac{x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} x_{k+1}^{N-r}}{r_1!r_2!\cdots r_k!} \Big) \\ &= \sum_{\substack{r_1+r_2+\cdots+r_k+r_{k+1}=N \\ r_i \geq 0}} \frac{N!}{r_1!r_2!\cdots r_k!r_{k+1}!} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k} x_{k+1}^{r_{k+1}} \end{aligned}$$

$\square$

App. A: Problem 11

Show that the set $A = \{r : r \in \mathbb{Q},\ r^2 < 2\}$ is bounded above in $\mathbb{Q}$ but has no l.u.b. in $\mathbb{Q}$.

Answer

Since for all $r \in A$ we have $r < \sqrt{2}$, we may take $b = 2$ as an upper bound, so A is bounded above in $\mathbb{Q}$. Suppose, for contradiction, that A has a supremum in $\mathbb{Q}$, say $\sup A$. Because $\sqrt{2}$ is an upper bound of A, we must have $\sup A \leq \sqrt{2}$. But since $\sup A \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, it follows that $\sup A < \sqrt{2}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational q such that

$$\sup A < q < \sqrt{2}.$$

Then $q^2 < 2$, so $q \in A$. But this contradicts the assumption that $\sup A$ is the least upper bound, since $\sup A < q \leq \sup A$ is impossible. Therefore, A has no supremum in $\mathbb{Q}$.

App. A: Problem 12

Show that for any two sequences $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subset \mathbb{R}$,

$$\underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \ \leq\ \underline{\lim}_{n\to\infty}(x_n + y_n) \ \leq\ \overline{\lim}_{n\to\infty}(x_n + y_n) \ \leq\ \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n.$$

Answer

Considering the sequence $\{x_n + y_n\}_{n\geq 1}$, we know:

$$\underline{\lim}_{n\to\infty}(x_n + y_n) \ \leq\ \overline{\lim}_{n\to\infty}(x_n + y_n)$$

Hence, we only have to show:

$$\underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \ \leq\ \underline{\lim}_{n\to\infty}(x_n + y_n) \quad \text{and} \quad \overline{\lim}_{n\to\infty}(x_n + y_n) \ \leq\ \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n.$$

Proof. For every $k \geq 1$:

$$\begin{aligned} &\inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq a_j + b_j, \quad \forall j \geq k \\ &\implies \inf_{n\geq k} a_n + \inf_{n\geq k} b_n \leq \inf_{n\geq k}(a_k + b_k) \end{aligned}$$

$$\begin{aligned} &a_j + b_j \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n, \quad \forall j \geq k \\ &\implies \sup_{n\geq k}(a_k + b_k) \leq \sup_{n\geq k} a_n + \sup_{n\geq k} b_n \end{aligned}$$

Now, by taking $k \to \infty$, we will have the followings:

$$\lim_{k\to\infty} \inf_{n\geq k} a_n + \lim_{k\to\infty} \inf_{n\geq k} b_n \leq \lim_{k\to\infty} \sup_{n\geq k}(a_k + b_k)$$

$$\lim_{k\to\infty} \sup_{n\geq k}(a_k + b_k) \leq \lim_{k\to\infty} \sup_{n\geq k} a_n + \lim_{k\to\infty} \sup_{n\geq k} b_n$$

Which leads to

$$\lim \inf_{n\to\infty} a_n + \lim \inf_{n\to\infty} b_n \leq \lim \sup_{n\to\infty} (a_k + b_k)$$

$$\lim \sup_{n\to\infty} (a_k + b_k) \leq \lim \sup_{n\to\infty} a_n + \lim \sup_{n\to\infty} b_n$$

$\square$

App. A: Problem 17

Find the radius of convergence, ρ, for the power series

$$A(x) \equiv \sum_{n=0}^{\infty} a_n x^n$$

where

(a) $a_n = \dfrac{n}{n+1},\ n \geq 0.$

(b) $a_n = n^p,\ n \geq 0,\ p \in \mathbb{R}.$

(c) $a_n = \dfrac{1}{n!},\ n \geq 0$ (where $0! = 1$).

Answer

Let

$$\rho \equiv \left(\overline{\lim}_{n\to\infty} |a_n|^{\frac{1}{n}} \right)^{-1}$$

(a)

$$\rho \equiv \left(\overline{\lim}_{n\to\infty} |\frac{n}{n+1}|^{\frac{1}{n}} \right)^{-1} \equiv \left(\overline{\lim}_{n\to\infty} \frac{n}{n+1}^{\frac{1}{n}} \right)^{-1}$$

We will show that $s_n = (\dfrac{n}{n+1})^{\frac{1}{n}}$ converges to 1, which is equivalent to

$$\overline{\lim}_{n\to\infty} (\frac{n}{n+1})^{\frac{1}{n}} = \underline{\lim}_{n\to\infty} (\frac{n}{n+1})^{\frac{1}{n}}$$

And we can use $\overline{\lim}_{n\to\infty} |\dfrac{n}{n+1}|^{\frac{1}{n}} = 1$. To show such, we will be using Theorem 4.1.7 in [1], which is as follows:

Theorem 0.1. *Let* $\lim_{x\to\infty} f(x) = L$*, where* $L \in \overline{\mathbb{R}}$ *(the extended reals), and suppose that* $s_n = f(n)$ *for large* n*. Then*

$$\lim_{n\to\infty} s_n = L.$$

now, let

$$s_n = (\frac{n}{n+1})^{\frac{1}{n}}, \quad f(x) = (\frac{x}{x+1})^{\frac{1}{x}}$$

$$\begin{aligned} f(x) &= (\frac{x}{x+1})^{\frac{1}{x}} \\ &= (\frac{1}{1+\frac{1}{x}})^{\frac{1}{x}} \\ &= (1+\frac{1}{x})^{-\frac{1}{x}} \end{aligned}$$

knowing that this function is uniformly continuous as $x \to \infty$:

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} (1+\frac{1}{x})^{-\frac{1}{x}} \\ &= \lim_{x\to\infty} e^{-\frac{1}{x}\log(1+\frac{1}{x})} \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} -\tfrac{1}{x}\log(1+\frac{1}{x}) &= \lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{-x} \quad \text{(L'Hopital's Rule)} \\ &= \lim_{x\to\infty} \frac{\frac{\frac{-1}{x^2}}{1+\frac{1}{x}}}{-1} \\ &= \lim_{x\to\infty} \frac{\frac{\frac{-1}{x^2}}{\frac{x+1}{x}}}{-1} \\ &= \lim_{x\to\infty} \frac{x}{x^2(x+1)} \\ &= \lim_{x\to\infty} \frac{1}{x(x+1)} \\ &= 0 \end{aligned}$$

thus,

$$\lim_{x\to\infty} e^{-\frac{1}{x}\log(1+\frac{1}{x})} = \lim_{x\to\infty} e^0 = 1$$

therefore,

$$\lim_{n\to\infty} s_n = 1.$$

and

$$\rho \equiv 1^{-1} \equiv 1$$

(b)

$$\rho \equiv \left(\overline{\lim}_{n\to\infty}|n^p|^{\frac{1}{n}}\right)^{-1} \equiv \left(\overline{\lim}_{n\to\infty} n^{\frac{P}{n}}\right)^{-1}$$

If $P = 0$, then

$$\rho \equiv \left(\overline{\lim}_{n\to\infty}|n^0|^{\frac{1}{n}}\right)^{-1} \equiv \left(\overline{\lim}_{n\to\infty} n^{\frac{0}{n}}\right)^{-1} \equiv \left(\overline{\lim}_{n\to\infty} 1\right)^{-1} \equiv 1$$

If $P < 0$, we will show that $s_n = n^{\frac{P}{n}}$ converges to 1, which is equivalent to

$$\overline{\lim}_{n\to\infty} n^{\frac{P}{n}} = \underline{\lim}_{n\to\infty} n^{\frac{P}{n}}$$

and we can use $\overline{\lim}_{n\to\infty} n^{\frac{P}{n}} = 1$. To show such, we will be using Theorem 4.1.7 in [1]. Let

$$s_n = n^{\frac{P}{n}}, \quad f(x) = x^{\frac{P}{x}}$$

$$\begin{aligned} f(x) &= x^{\frac{P}{x}} \\ &= (\frac{1}{x})^{\frac{-P}{x}} \end{aligned}$$

knowing that this function is uniformly continuous as $x \to \infty$:

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} (\frac{1}{x})^{\frac{-P}{x}} \\ &= \lim_{x\to\infty} e^{\log(\frac{1}{x})\frac{-P}{x}} \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \log(\frac{1}{x})\frac{-P}{x} &= \lim_{x\to\infty} \frac{\log(\frac{1}{x})}{\frac{x}{-P}} \quad \text{(L'Hopital's Rule)} \\ &= \lim_{x\to\infty} \frac{\frac{x}{-x^2}}{\frac{1}{-P}} \\ &= \lim_{x\to\infty} \frac{P}{x} \\ &= 0 \end{aligned}$$

thus,

$$\lim_{x\to\infty} e^{\log(\frac{1}{x})\frac{-P}{x}} = \lim_{x\to\infty} e^0 = 1$$

therefore,

$$\lim_{n\to\infty} s_n = 1, \quad P < 0.$$

and

$$\rho \equiv 1^{-1} \equiv 1, \quad P < 0$$

If $P > 0$ we will show that $s_n = n^{\frac{P}{n}}$ converges to 1, which is equivalent to

$$\overline{\lim}_{n\to\infty} n^{\frac{P}{n}} = \underline{\lim}_{n\to\infty} n^{\frac{P}{n}}$$

and we can use $\overline{\lim}_{n\to\infty} n^{\frac{P}{n}} = 1$. To show such, we will be using Theorem 4.1.7 in [1]. Let

$$s_n = n^{\frac{P}{n}}, \quad f(x) = x^{\frac{P}{x}}$$

knowing that this function is uniformly continuous as $x \to \infty$:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} x^{\frac{P}{x}} \\ &= \lim_{x\to\infty} e^{\log(x)\frac{P}{x}}\end{aligned}$$

$$\begin{aligned}\lim_{x\to\infty} \log(x)\tfrac{P}{x} &= \lim_{x\to\infty} \frac{\log(x)}{\frac{x}{P}} \quad \text{(L'Hopital's Rule)} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{P}} \\ &= \lim_{x\to\infty} \frac{P}{x} \\ &= 0\end{aligned}$$

thus,

$$\lim_{x\to\infty} e^{\log(x)\frac{P}{x}} = \lim_{x\to\infty} e^0 = 1$$

therefore,

$$\lim_{n\to\infty} s_n = 1, \quad P > 0.$$

and

$$\rho \equiv 1^{-1} \equiv 1, \quad P > 0$$

(c)

$$\rho \equiv \left(\overline{\lim}_{n\to\infty} |\frac{1}{n!}|^{\frac{1}{n}}\right)^{-1} \equiv \left(\overline{\lim}_{n\to\infty} \frac{1}{n!}^{\frac{1}{n}}\right)^{-1}$$

We will show that $s_n = \frac{1}{n!}^{\frac{1}{n}}$ converges to 0, which is equivalent to

$$\overline{\lim}_{n\to\infty} (\frac{1}{n!})^{\frac{1}{n}} = \underline{\lim}_{n\to\infty} (\frac{1}{n!})^{\frac{1}{n}}$$

and we can use $\overline{\lim}_{n\to\infty}|\frac{1}{n!}|^{\frac{1}{n}} = 0$.
considering the Taylor series of the function e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

then

$$\begin{aligned}
&\frac{x^n}{n!} \le e^x, \quad x \to n \\
&\Rightarrow \frac{1}{n!} \le \frac{e^x}{x^n} \\
&\Rightarrow \frac{1}{n!} \le \frac{e^n}{n^n} \quad (x = n) \\
&\Rightarrow (\frac{1}{n!})^{\frac{1}{n}} \le \frac{e}{n} \to 0, n \to \infty
\end{aligned}$$

therefore,

$$\lim_{n\to\infty} s_n = 0.$$

and

$$\rho \equiv 0^{-1} \equiv \infty$$

App. A: Problem 18

(a) Find the Taylor series at $a = 0$ for the function

$$f(x) = \frac{1}{1-x}$$

in $I \equiv (-1, +1)$ and show that it converges to $f(x)$ on I.

(b) Find the Taylor series of $1 + x + x^2$ in $I = (1, 3)$, centered at 2.

(c) Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } |x| < 1,\ x \ne 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(i) Show that f is infinitely differentiable at 0 and compute $f^{(j)}(0)$ for all $j \ge 1$.

(ii) Show that the Taylor series at $a = 0$ converges but not to f on $(-1, 1)$.

Answer

(a) f is n times differentiable on I, and

$$f^{(n)} = \frac{n!}{(1-x)^{n+1}}$$

Hence the Taylor series of f at $a = 0$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{\frac{n!}{(1-0)^{n+1}}}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{1}{(1-0)^{n+1}}(x-0)^n = \sum_{n=0}^{\infty} x^n$$

Now, the n^{th} partial sum of the series is

$$F_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

Since $\mid x \mid < 1$, $F_n(x)$ in closed form is:

$$F_n(x) = \frac{1 - x^{n+1}}{1 - x}$$

$\lim_{n\to\infty} x^n + 1 = 0, \quad$ for $\mid x \mid < 1$, hence $F_n(x)$ converges to $\frac{1}{1-x}$.

(b) Suppose $f = 1 + x + x^2$. Then:

$$\begin{aligned} &f^{'}(x) = 1 + 2x \\ &f^{''}(x) = 2 \\ &f^{(n)}(x) = 0, \quad \forall n \geq 3 \end{aligned}$$

Hence, the Taylor series of f at $a = 2$ is:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n &= \frac{(1+2+2^2)}{0!}(x-2)^0 + \frac{(1+2\cdot 2)}{1!}(x-2)^1 + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-0)^n \\ &= 7 + 5(x-2) + (x-2)^2 + 0 \\ &= 7 + 5x - 10 + x^2 - 4x + 4 + \\ &= x^2 + x + 1, \quad \forall x, \quad \text{including} \quad x \in I = (1,3) \end{aligned}$$

(c) (i) By calculating the first three derivative directly, we'll have:

$$\begin{aligned} f^{'}(x) &= e^{-\frac{1}{x^2}} \frac{2}{x^3}, \quad x \neq 0 \\ f^{''}(x) &= e^{-\frac{1}{x^2}} \frac{2}{x^3} \frac{2}{x^3} + e^{-\frac{1}{x^2}} \frac{-6}{x^4} = e^{-\frac{1}{x^2}} (\frac{4}{x^6} - \frac{6}{x^4}), \quad x \neq 0 \\ f^{(3)}(x) &= e^{-\frac{1}{x^2}} \frac{2}{x^3} (\frac{4}{x^6} - \frac{6}{x^4}) + e^{-\frac{1}{x^2}} (\frac{-24}{x^7} + \frac{24}{x^5}) = e^{-\frac{1}{x^2}} (\frac{8}{x^9} - \frac{36}{x^7} + \frac{24}{x^5}) \end{aligned}$$

Following this, we will show by induction $f^{(n)} = e^{-\frac{1}{x^2}} P_n(\frac{1}{x})$.

Proof. **Basis:** $k = 1$

$$f^{'}(x) = e^{-\frac{1}{x^2}} \frac{2}{x^3} = e^{-\frac{1}{x^2}} P_1(\frac{1}{x}), \quad x \neq 0$$

Induction Hypothesis:
Suppose:

$$f^{(k)}(x) = e^{-\frac{1}{x^2}} P_k(\frac{1}{x})$$

holds for some $k \in \mathbb{N}$
Inductive Step:

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} e^{-\frac{1}{x^2}} P_k(\frac{1}{x}) \\
&= e^{-\frac{1}{x^2}} \frac{2}{x^3} P_k(\frac{1}{x}) + e^{-\frac{1}{x^2}} \frac{1}{-x^2} P_k'(\frac{1}{x}) \\
&= e^{-\frac{1}{x^2}} \underbrace{(\frac{2}{x^3} P_k(\frac{1}{x}) + \frac{1}{-x^2} P_k'(\frac{1}{x})}_{P_{k+1}(\frac{1}{x})} \\
&= e^{-\frac{1}{x^2}} P_{k+1}(\frac{1}{x})
\end{aligned}$$

□

Knowing that $f^{(n)} = e^{-\frac{1}{x^2}} P_n(\frac{1}{x})$, for $x \neq 0$, we can write

$$P_n(1/x) = \sum_{m=0}^{d_n} a_{n,m}\, x^{-m}.$$

Thus

$$f^{(n)}(x) = \sum_{m=0}^{d_n} a_{n,m} \left(e^{-1/x^2} x^{-m}\right).$$

Because this is a *finite* sum, we may pass the limit inside:

$$\lim_{x \to 0} f^{(n)}(x) = \sum_{m=0}^{d_n} a_{n,m} \lim_{x \to 0} \left(e^{-1/x^2} x^{-m}\right).$$

Fix $m \geq 0$. Using $x^{-m} = e^{m \ln(1/|x|)}$, for sufficiently small $|x|$ we have

$$m \ln\bigl(1/|x|\bigr) \leq \frac{1}{2x^2} \quad \Rightarrow \quad x^{-m} \leq e^{1/(2x^2)}.$$

Hence

$$0 \leq e^{-1/x^2} x^{-m} \leq e^{-1/x^2}\, e^{1/(2x^2)} = e^{-1/(2x^2)}$$

Therefore

$$0 = \lim_{x \to 0} 0 \leq \lim_{x \to 0} e^{-1/x^2} x^{-m} \leq \lim_{x \to 0} e^{-1/x^2}\, e^{1/(2x^2)} = \lim_{x \to 0} e^{-1/(2x^2)} = 0$$

Since each term satisfies $\lim_{x \to 0} e^{-1/x^2} x^{-m} = 0$, and so

$$\lim_{x \to 0} f^{(n)}(x) = \sum_{m=0}^{d_n} a_{n,m} \cdot 0 = 0.$$

Consequently, $f^{(n)}(0) := 0$ makes $f^{(n)}$ continuous at 0.

(ii) We have shown that $f^{(n)}(0) = 0, \forall n$. Hence the Taylor series of f at $a = 0$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x-0)^n = \sum_{n=0}^{\infty} \frac{0}{n!} (x-0)^n = 0$$

However, $\forall x$, such that $\mid x \mid < 1$ and $x \neq 0$, $f(x) > 0$. So the Taylor series at $x = 0$ does not represent the function except at the single point $x = 0$.

App. A: Problem 24

(c) Draw the open unit ball $B_p \equiv \{x : x \in \mathbb{R}^2,\ d_p(x,0) < 1\}$ in $\mathbb{R}^2$ for $p = 1, 2, \infty$.

Answer

To answer this question, it's assumed that d_p is

$$d_p(x,y) = \left(\sum_{i=1}^{k} |x_i - y_i|^p \right)^{1/p},$$

Suppose $x = (x_1, x_2)$, and $0 = (0,0)$, then:

$p = 1:$

$$d_1(x,0) = \left(|x_1 - 0|^1 + |x_2 - 0|^1\right)^{1/1} = |x_1| + |x_2|$$

for $d_1(x,0) < 1$, we have to have $|x_1| + |x_2| < 1$. Fig. 1 is representing $d_1(x,0) < 1$, please note that the boarder is not considered.

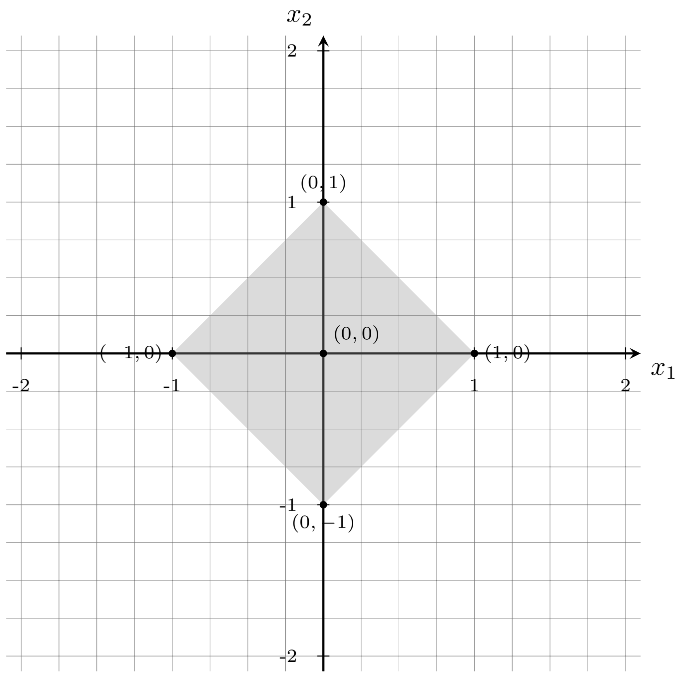

Figure 1: Shaded square (a 45°-rotated unit L^1 ball) centered at $(0,0)$ represents $d_1(x,0) < 1$, where $x = (x_1, x_2)$.

$p = 2:$

$$d_2(x,0) = \left(|x_1 - 0|^2 + |x_2 - 0|^2\right)^{1/1} = \sqrt{x_1^2 + x_2^2}$$

for $d_2(x,0) < 1$, we have to have $\sqrt{x_1^2 + x_2^2} < 1$. Fig. 2 is representing $d_2(x,0) < 1$, please note that the boarder is not considered.

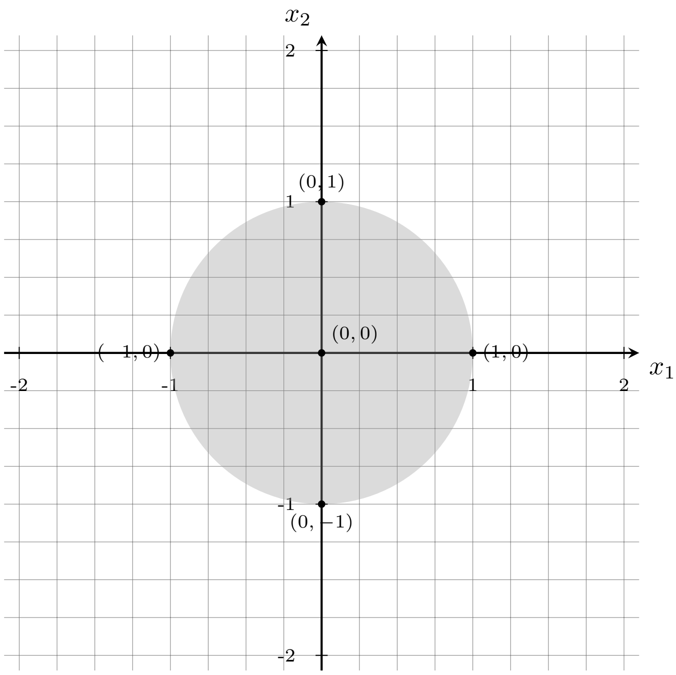

Figure 2: Shaded unit disk centered at $(0, 0)$ in $\mathbb{R}^2$ represents $d_2(x, 0) < 1$, where $x = (x_1, x_2)$.

$p = \infty$:

$$d_\infty(x, 0) = \max\{\, |x_1 - 0|, |x_2 - 0|\,\} = \max\{\, |x_1|, |x_2|\,\}.$$

for $d_\infty(x, 0) < 1$, we have to have $\max\{\, |x_1|, |x_2|\,\} < 1 \implies |x_1| < 1 \quad \text{and} \quad |x_2| < 1$. Fig. 3 is representing $d_\infty(x, 0) < 1$, please note that the boarder is not considered.

References

[1] William F Trench. *Introduction to real analysis.* 2013.

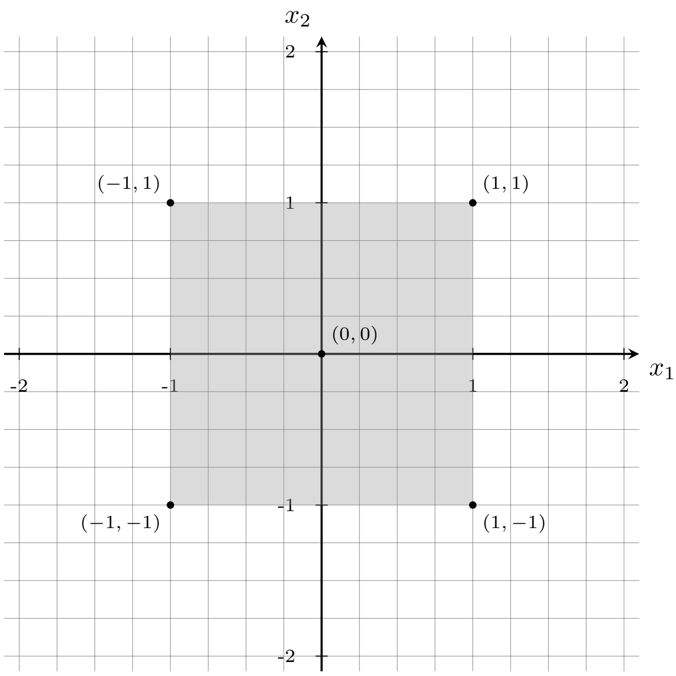

Figure 3: Shaded square in $\mathbb{R}^2$ represents $d_\infty(x, 0) < 1$, where $x = (x_1, x_2)$.