MATH 6410 Foundations of Probability Theory, Homework 5

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Ch. 2: Problem 15(c)

Consider the probability space $((0,1), \mathcal{B}((0,1)), m)$, where $m(\cdot)$ is the Lebesgue measure. Let $F : \mathbb{R} \to \mathbb{R}$ be a cdf. For $0 < x < 1$, define

$$\begin{aligned} F_1^{-1}(x) &= \inf\{y \in \mathbb{R} : F(y) \geq x\}, \\ F_2^{-1}(x) &= \sup\{y \in \mathbb{R} : F(y) \leq x\}. \end{aligned}$$

Let Z_i be the random variable defined by

$$Z_i = F_i^{-1}(x), \quad 0 < x < 1, \quad i = 1, 2.$$

(i) Find the cdf of Z_i, $i = 1, 2$.

(**Hint.** Verify using the right–continuity of F that for any $0 < x < 1$ and $t \in \mathbb{R}$, $F(t) \geq x \iff F_1^{-1}(x) \leq t$.)

(ii) Show that $F_1^{-1}(\cdot)$ is left–continuous and $F_2^{-1}(\cdot)$ is right–continuous.

Answer

(i) ($\Rightarrow$) Suppose $F(t) < x$. By right–continuity, there exists $\delta > 0$ such that $F(u) < x$ for all $u \in [t, t+\delta)$. Hence no point in $[t, t+\delta)$ belongs to $A_x = \{y : F(y) \geq x\}$, so $\inf A_x \geq t + \delta > t$, i.e. $F_1^{-1}(x) > t$.

($\Leftarrow$) Suppose $F_1^{-1}(x) > t$. Then $t \notin A_x$, i.e. $F(t) \ngeq x$, so $F(t) < x$.

Therefore $F(t) < x \iff F_1^{-1}(x) > t$, which is equivalent to $F(t) \geq x \iff F_1^{-1}(x) \leq t$.

Now, for $i = 1$,

$$\mathbb{P}(Z_1 \leq t) = m\big(\{x \in (0,1) : F_1^{-1}(x) \leq t\}\big) = m\big(\{x \in (0,1) : x \leq F(t)\}\big) = F(t).$$

For $i = 2$, note that

$$F_2^{-1}(x) \leq t \iff x < F(t),$$

hence

$$\mathbb{P}(Z_2 \leq t) = m\big(\{x \in (0,1) : x < F(t)\}\big) = F(t),$$

since the boundary $\{x = F(t)\}$ has Lebesgue measure 0. Thus, for $i = 1, 2$,

$$\mathbb{P}(Z_i \leq t) = F(t), \qquad t \in \mathbb{R}.$$

(ii) From part (i),

$$\{x \in (0,1) : F_1^{-1}(x) \le t\} = (0, F(t)] \quad \text{and} \quad \{x \in (0,1) : F_2^{-1}(x) \le t\} = (0, F(t)).$$

Fix $x \in (0,1)$ and let $x_n \uparrow x$. Then for every t,

$$F_1^{-1}(x_n) \le t \ \forall n \iff x_n \le F(t) \ \forall n \iff x \le F(t) \iff F_1^{-1}(x) \le t,$$

which implies $\lim_{n\uparrow\infty} F_1^{-1}(x_n) = F_1^{-1}(x)$. Hence F_1^{-1} is left–continuous.
Similarly, let $x_n \downarrow x$. Then for every t,

$$F_2^{-1}(x_n) \le t \ \forall n \iff x_n < F(t) \ \forall n \iff x \le F(t) \iff F_2^{-1}(x) \le t,$$

which results in $\lim_{n\downarrow\infty} F_2^{-1}(x_n) = F_2^{-1}(x)$. Hence F_2^{-1} is right–continuous.

Ch. 2: Problem 20

Apply Corollary 2.3.5 to show that for any collection $\{a_{ij} : i, j \in \mathbb{N}\}$ of nonnegative numbers,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

Answer

Recall Corollary 2.3.5 [1]:
Let $\{h_n\}_{n\ge1}$ be a sequence of nonnegative measurable functions on a measure space $(\Omega, \mathcal{F}, \mu)$. Then

$$\int\left(\sum_{n=1}^{\infty} h_n\right) d\mu \;=\; \sum_{n=1}^{\infty}\int h_n\, d\mu.$$

Consider the measure space

$$(\Omega, \mathcal{F}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \#),$$

where $\#$ is the counting measure. For each $i \in \mathbb{N}$, define the measurable function

$$h_i(j) = a_{ij}, \qquad j \in \mathbb{N}.$$

Since under the counting measure

$$\int_{\mathbb{N}} f\, d\# = \sum_{n=1}^{\infty} f(n),$$

we have

$$\text{LHS:} \quad \int\left(\sum_{i=1}^{\infty} h_i\right) d\# = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} h_i(j)\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right),$$

$$\text{RHS:} \quad \sum_{i=1}^{\infty}\int h_i\, d\# = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} h_i(j)\right) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right).$$

Hence,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right).$$

Ch. 2: Problem 25

If $f(x) = I_{\mathbb{Q}_1}(x)$ where $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1]$, $\mathbb{Q}$ being the set of all rationals, then show that for any partition P,

$$U(P, f) = 1 \quad \text{and} \quad L(P, f) = 0.$$

Answer

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a finite partition of $[0,1]$ with $x_0 = 0$ and $x_n = 1$. For each subinterval $[x_i, x_{i+1}]$, define

$$M_i = \sup\{f(x) : x_i \le x \le x_{i+1}\}, \qquad m_i = \inf\{f(x) : x_i \le x \le x_{i+1}\}.$$

Then, by definition,

$$U(P, f) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i), \qquad L(P, f) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i).$$

Since both the rationals $\mathbb{Q}$ and irrationals $\mathbb{Q}^c$ are dense in $[0,1]$, every interval $[x_i, x_{i+1}]$ contains at least one rational and one irrational point. Therefore, within each subinterval:

$$\exists x_r \in [x_i, x_{i+1}] \text{ such that } f(x_r) = 1, \quad \text{and} \quad \exists x_{ir} \in [x_i, x_{i+1}] \text{ such that } f(x_{ir}) = 0.$$

Hence $M_i = 1$ and $m_i = 0$ for all i. Replacing these values,

$$U(P, f) = \sum_{i=0}^{n-1} 1 \cdot (x_{i+1} - x_i) = x_n - x_0 = 1,$$
$$L(P, f) = \sum_{i=0}^{n-1} 0 \cdot (x_{i+1} - x_i) = 0.$$

Therefore, for any partition P, we have

$$U(P, f) = 1 \quad \text{and} \quad L(P, f) = 0.$$

Extra Problem A

Let $f : \Omega \to \mathbb{R}$ be a $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable nonnegative function. Define for $n \ge 1$, and $x \in \Omega$,

$$f_n(x) = \begin{cases} \dfrac{(i-1)}{2^n}, & \text{if } \dfrac{(i-1)}{2^n} \le f(x) < \dfrac{i}{2^n} \text{ for } i = 1, \ldots, n2^n, \\ n, & \text{if } f(x) \ge n. \end{cases}$$

Then prove the following:

(i) For each $n \ge 1$, $f_n(x)$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable.

(ii) For each $x \in \Omega$, $f_n(x)$ is increasing with n (i.e. $\{f_n(\cdot)\}$ is an increasing sequence of functions).

(iii) Prove that $f_n(x) \to f(x)$ pointwise (i.e. for each fixed x), as $n \to \infty$.

Answer

(i) We can write $f_n(x)$ as follows:

$$f_n(x) = \Sigma_{i=1}^{n2^n} I_{[\frac{(i-1)}{2^n}, \frac{i}{2^n})}(f(x))\frac{(i-1)}{2^n} + I_{[n,\infty)}f(x)$$

Which is the combination of indicator functions, f, and $f(x) = n$ which are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable functions. Hence, by corollary 2.14 [1] is a $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable function.

(ii) Define $f_n(x)$ and $f_{n+1}(x)$ as

$$f_n(x) = \min\left\{ n,\ \frac{\lfloor 2^n f(x) \rfloor}{2^n} \right\}, \qquad f_{n+1}(x) = \min\left\{ n+1,\ \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}} \right\}.$$

We show that $f_{n+1}(x) \geq f_n(x)$ for all $x \in \Omega$ by considering three cases.

(1) $f(x) \geq n+1$**:**

$$f_n(x) = n \quad \text{and} \quad f_{n+1}(x) = n+1,$$

hence $f_{n+1}(x) > f_n(x)$.

(2) $n \leq f(x) < n+1$**:** In this range, $\min\{n, \cdot\} = n$, so

$$f_n(x) = n \quad \text{and} \quad f_{n+1}(x) = \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}}.$$

Since $f(x) \geq n$, we have $\lfloor 2^{n+1} f(x) \rfloor \geq \lfloor 2^{n+1} n \rfloor = 2^{n+1} n$, and therefore

$$f_{n+1}(x) = \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}} \geq \frac{2^{n+1} n}{2^{n+1}} = n = f_n(x).$$

(3) $f(x) < n$**:** Then

$$f_n(x) = \frac{\lfloor 2^n f(x) \rfloor}{2^n}, \qquad f_{n+1}(x) = \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}}.$$

Note that $\lfloor 2^{n+1} f(x) \rfloor \geq 2\lfloor 2^n f(x) \rfloor$, so

$$f_{n+1}(x) = \frac{\lfloor 2^{n+1} f(x) \rfloor}{2^{n+1}} \geq \frac{2\lfloor 2^n f(x) \rfloor}{2^{n+1}} = \frac{\lfloor 2^n f(x) \rfloor}{2^n} = f_n(x).$$

Thus, in all cases $f_{n+1}(x) \geq f_n(x)$, showing that $\{f_n(x)\}$ is an increasing sequence.

(iii) Define $f_n(x)$ as

$$f_n(x) = \min\left\{ n,\ \frac{\lfloor 2^n f(x) \rfloor}{2^n} \right\}.$$

Then, for any $n \in \mathbb{N}$, we can write

$$\frac{\lfloor 2^n f(x) \rfloor}{2^n} \ \leq\ f(x) \ <\ \frac{\lfloor 2^n f(x) \rfloor}{2^n} + \frac{1}{2^n},$$

hence

$$0 \ \leq\ f(x) - \frac{\lfloor 2^n f(x) \rfloor}{2^n} \ <\ \frac{1}{2^n}.$$

Therefore, for all $n \geq N_1$,

$$0 \ \leq\ f(x) - f_n(x) \ <\ \frac{1}{2^n},$$

so $f_n(x) \to f(x)$ as $n \to \infty$.

Extra Problem B

If $f, g \in L_1(\Omega, \mathcal{F}, \mu)$, show that

$$\min(f, g) \in L_1(\Omega, \mathcal{F}, \mu),$$

and

$$\min\left(\int f\, d\mu, \int g\, d\mu\right) \geq \int \min(f, g)\, d\mu.$$

Answer

First, note that for any real numbers a and b,

$$\min(a, b) = \tfrac{1}{2}\big(a + b - |a - b|\big).$$

Hence, for measurable functions f and g,

$$\min(f, g) = \tfrac{1}{2}\big(f + g - |f - g|\big).$$

Since $f, g \in L_1(\Omega, \mathcal{F}, \mu)$ and $|f - g| \in L_1(\Omega, \mathcal{F}, \mu)$ (because L_1 is closed under linear combinations and absolute values), it follows that $\min(f, g)$ is also integrable. Therefore,

$$\min(f, g) \in L_1(\Omega, \mathcal{F}, \mu).$$

For the inequality, note that $\min(f, g) \leq f$ and $\min(f, g) \leq g$. Integrating both inequalities yields

$$\int \min(f, g)\, d\mu \leq \int f\, d\mu \quad \text{and} \quad \int \min(f, g)\, d\mu \leq \int g\, d\mu.$$

Thus,

$$\int \min(f, g)\, d\mu \leq \min\left(\int f\, d\mu, \int g\, d\mu\right),$$

Disclaimer

In Extra Problem B, the expression $\min(f, g) = \frac{1}{2}\big(f + g - |f - g|\big)$ was identified using ChatGPT. Throughout the whole homework, I teamed up with Sreejit Roy on thinking about problems.

References

[1] Krishna B Athreya and Soumendra N Lahiri. *Measure theory and probability theory.* Springer, 2006.