MATH 6410 Foundations of Probability Theory, Homework 3

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Ch. 1: Problem 3

(a) Show that $\mathcal{F}_6$ in Example 1.1.4 is a σ-algebra.

(b) Define μ on $\mathcal{F}_6$ by

$$\mu(A) = \begin{cases} 1, & \text{if } A \text{ is uncountable,} \\ 0, & \text{if } A \text{ is countable.} \end{cases}$$

Is μ a measure on $\mathcal{F}_6$? Justify your answer.

Answer

Recall that

$$\mathcal{F}_6 = \{A \subset \Omega : A \text{ is countable or } A^c \text{ is countable}\}.$$

(a) $\mathcal{F}_6$ is a σ-algebra

To prove that $\mathcal{F}_6$ is a σ-algebra, we check the three properties:

(i) $\Omega \in \mathcal{F}_6$: Since $\Omega^c = \varnothing$, which is countable, we have $\Omega \in \mathcal{F}_6$. Similarly, $\varnothing \in \mathcal{F}_6$.

(ii) Closed under complements: Suppose $A \in \mathcal{F}_6$. Two cases would occur:

- If A is countable, then A^c has uncountable complement (namely A), so $A^c \in \mathcal{F}_6$.
- If A^c is countable, then clearly $A^c \in \mathcal{F}_6$.

(iii) Closed under countable unions: Suppose $A_n \in \mathcal{F}_6$ for $n \geq 1$. Two cases:

- If each A_n is countable, then $\bigcup_{n\geq 1} A_n$ is a countable union of countable sets, hence countable, so it belongs to $\mathcal{F}_6$.
- If there exists n_0 such that $A_{n_0}^c$ is countable, then

$$\bigcup_{n\geq 1} A_n = \left(\bigcap_{n\geq 1} A_n^c\right)^c \subseteq (A_{n_0}^c)^c.$$

Since $A_{n_0}^c$ is countable, $\bigcap_{n\geq 1} A_n^c$ is countable, and so $\bigcup_{n\geq 1} A_n \in \mathcal{F}_6$.

Thus, $\mathcal{F}_6$ is a σ-algebra.

(b) Is μ a measure on $\mathcal{F}_6$?

We check the measure axioms:

(i) $\mu(\varnothing) = 0$: Since $\varnothing$ is countable, $\mu(\varnothing) = 0$.

(ii) Non-negativity: For all $A \in \mathcal{F}_6$, $\mu(A) \in \{0, 1\} \subseteq [0, \infty)$.

(iii) Countable additivity: Suppose $\{A_n\}_{n\geq 1}$ is a disjoint collection in $\mathcal{F}_6$ with $\bigcup_{n\geq 1} A_n \in \mathcal{F}_6$. Two cases:

  - If every A_n is countable, then each $\mu(A_n) = 0$, and their union is countable, so

$$\mu\left(\bigcup_{n\geq 1} A_n\right) = 0 = \sum_{n=1}^{\infty} \mu(A_n).$$

  - If $\{A_n\}_{n\geq 1} \subset \mathcal{F}_6$ are pairwise disjoint, then at most one A_n is uncountable.

    Proof. Suppose for contradiction that there exist $n_1 \neq n_2$ with $A_{n_1}, A_{n_2} \in \mathcal{F}_6$, both uncountable, and $A_{n_1} \cap A_{n_2} = \varnothing$. Since A_{n_1} and A_{n_2} are uncountable and belong to $\mathcal{F}_6$, their complements $A_{n_1}^c$ and $A_{n_2}^c$ must be countable. But

$$A_{n_1} \cap A_{n_2} = \varnothing \quad \Longrightarrow \quad A_{n_1}^c \cup A_{n_2}^c = \Omega.$$

    The right-hand side is a union of two countable sets, hence countable. However $A_{n_1}, A_{n_2} \subseteq \Omega$ and are uncontainable which contradicts that Ω is countable. Therefore, among pairwise disjoint sets in $\mathcal{F}_6$, at most one is uncountable. $\square$

    Now, let $n_0 \in \mathbb{N}$ such that A_{n_0} is uncountable. Hence, every A_n with $n \neq n_0$ is countable. A countable union of countable sets is countable, so

$$\bigcup_{n\geq 1} A_n = A_{n_0} \cup \bigcup_{n\neq n_0} A_n$$

    is the union of an uncountable set and a countable set, hence uncountable. Therefore,

$$\mu\Bigl(\bigcup_{n\geq 1} A_n\Bigr) = 1 = \mu(A_{n_0}) + \sum_{n\neq n_0} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Hence, μ satisfies the measure axioms and is a valid measure on $\mathcal{F}_6$.

Ch. 1: Problem 8

Let $\Omega \equiv \{1, 2, \ldots\} = \mathbb{N}$ and

$$A_i \equiv \{j : j \in \mathbb{N},\, j \geq i\}, \quad i \in \mathbb{N}.$$

Show that $\sigma\langle\mathcal{A}\rangle = \mathcal{P}(\Omega)$ where $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$.

Answer

For all $n \in \mathbb{N}$, we have $A_{n+1} = A_n \setminus \{n\}$, hence $A_{n+1} \subseteq A_n$. Also, $\mathcal{A}$ is a collection of subsets of $\mathbb{N}$, so $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is the largest σ-algebra on Ω, we have $\sigma\langle\mathcal{A}\rangle \subseteq \mathcal{P}(\Omega)$. Therefore, to show $\sigma\langle\mathcal{A}\rangle = \mathcal{P}(\Omega)$, it suffices to prove $\mathcal{P}(\Omega) \subseteq \sigma\langle\mathcal{A}\rangle$.
Let $B \in \mathcal{P}(\Omega)$. Since $\Omega = \mathbb{N}$ is countable, every subset B is countable and can be written as a union of singletons:

$$B = \bigcup_{b \in B} \{b\}.$$

Thus it is enough to show $\{b\} \in \sigma\langle\mathcal{A}\rangle$ for each $b \in B$; then, using closure under countable unions, $B \in \sigma\langle\mathcal{A}\rangle$ follows.
Fix $b \in \mathbb{N}$. Note that $A_b, A_{b+1} \in \sigma\langle\mathcal{A}\rangle$ and

$$A_b^c = \{1, 2, \ldots, b-1\}.$$

Hence

$$A_b^c \cup A_{b+1} = \{1, 2, \ldots, b-1\} \cup \{b+1, b+2, \ldots\} = \mathbb{N} \setminus \{b\}.$$

Taking complements,

$$\{b\} = \left(A_b^c \cup A_{b+1}\right)^c \in \sigma\langle\mathcal{A}\rangle.$$

Therefore $\{b\} \in \sigma\langle\mathcal{A}\rangle$ for all $b \in B$, and by countable unions $B \in \sigma\langle\mathcal{A}\rangle$. This shows $\mathcal{P}(\Omega) \subseteq \sigma\langle\mathcal{A}\rangle$, and thus $\sigma\langle\mathcal{A}\rangle = \mathcal{P}(\Omega)$.

Ch. 1: Problem 10

Show that in Example 1.1.6, $\mathcal{O}_j \subset \sigma\langle\mathcal{O}_i\rangle$ for all $1 \le i, j \le 4$.

Answer

Recall that

$$\begin{aligned}
\mathcal{O}_1 &= \{(a_1, b_1) \times \cdots \times (a_k, b_k) : -\infty \le a_i < b_i \le \infty,\ 1 \le i \le k\}; \\
\mathcal{O}_2 &= \{(-\infty, x_1) \times \cdots \times (-\infty, x_k) : x_1, \ldots, x_k \in \mathbb{R}\}; \\
\mathcal{O}_3 &= \{(a_1, b_1) \times \cdots \times (a_k, b_k) : a_i, b_i \in \mathbb{Q},\ a_i < b_i,\ 1 \le i \le k\}; \\
\mathcal{O}_4 &= \{(-\infty, x_1) \times \cdots \times (-\infty, x_k) : x_1, \ldots, x_k \in \mathbb{Q}\}.
\end{aligned}$$

We will show $\sigma\langle\mathcal{O}_i\rangle = \mathcal{B}(\mathbb{R}^k)$ for all $1 \le i \le 4$. Then, knowing that $\mathcal{O}_i \subset \sigma\langle\mathcal{O}_i\rangle = \mathcal{B}(\mathbb{R}^k) = \sigma\langle\mathcal{O}_j\rangle$ for all $1 \le i, j \le 4$ proves the proposition.

Proof. (1) $\sigma\langle\mathcal{O}_1\rangle = \mathcal{B}(\mathbb{R}^k)$. The family $\mathcal{O}_1$ (finite products of open intervals, endpoints allowed $\pm\infty$) is a base for the usual topology on $\mathbb{R}^k$, hence the σ-algebra it generates equals the Borel σ-algebra.

(2) $\sigma\langle\mathcal{O}_2\rangle = \mathcal{B}(\mathbb{R}^k)$. First, $\mathcal{O}_2 \subset \mathcal{B}(\mathbb{R}^k)$ (each set is open), so $\sigma\langle\mathcal{O}_2\rangle \subset \mathcal{B}(\mathbb{R}^k)$. Conversely, for $a < b$ in $\mathbb{R}$,

$$(a, b) = (-\infty, b) \cap \Big(\bigcap_{n \ge 1} (-\infty, a + \tfrac{1}{n})\Big)^c,$$

so $(a,b) \in \sigma\langle\{(-\infty, x) : x \in \mathbb{R}\}\rangle$ in $\mathbb{R}$. In $\mathbb{R}^k$,

$$(a_1, b_1) \times \cdots \times (a_k, b_k) = \bigcap_{i=1}^{k} \Big(\mathbb{R}^{i-1} \times (a_i, b_i) \times \mathbb{R}^{k-i}\Big),$$

hence every open rectangle lies in $\sigma\langle\mathcal{O}_2\rangle$, so $\mathcal{B}(\mathbb{R}^k) \subset \sigma\langle\mathcal{O}_2\rangle$.

(3) $\sigma\langle\mathcal{O}_3\rangle = \mathcal{B}(\mathbb{R}^k)$. We have $\mathcal{O}_3 \subset \mathcal{O}_1$, so $\sigma\langle\mathcal{O}_3\rangle \subset \sigma\langle\mathcal{O}_1\rangle = \mathcal{B}(\mathbb{R}^k)$. For $\mathcal{B}(\mathbb{R}^k) \subset \sigma\langle\mathcal{O}_3\rangle$, any open rectangle

$$(a_1, b_1) \times \cdots \times (a_k, b_k) = \bigcup_{\substack{q_i, r_i \in \mathbb{Q} \\ a_i < q_i < r_i < b_i}} (q_1, r_1) \times \cdots \times (q_k, r_k),$$

a *countable* union of sets in $\mathcal{O}_3$. Thus $\mathcal{O}_1 \subset \sigma\langle\mathcal{O}_3\rangle$ and $\mathcal{B}(\mathbb{R}^k) = \sigma\langle\mathcal{O}_1\rangle \subset \sigma\langle\mathcal{O}_3\rangle$.

(4) $\sigma\langle\mathcal{O}_4\rangle = \mathcal{B}(\mathbb{R}^k)$. Clearly $\mathcal{O}_4 \subset \mathcal{O}_2$, so $\sigma\langle\mathcal{O}_4\rangle \subset \sigma\langle\mathcal{O}_2\rangle = \mathcal{B}(\mathbb{R}^k)$. Conversely, for each $x \in \mathbb{R}$,

$$(-\infty, x) = \bigcup_{\substack{q \in \mathbb{Q} \\ q < x}} (-\infty, q),$$

hence $\mathcal{O}_2 \subset \sigma\langle\mathcal{O}_4\rangle$ and thus $\mathcal{B}(\mathbb{R}^k) = \sigma\langle\mathcal{O}_2\rangle \subset \sigma\langle\mathcal{O}_4\rangle$.

Therefore $\sigma\langle\mathcal{O}_i\rangle = \mathcal{B}(\mathbb{R}^k)$ for all $i = 1, 2, 3, 4$, and hence $\mathcal{O}_j \subset \sigma\langle\mathcal{O}_i\rangle$ for all i, j. □

Ch. 1: Problem 18

Let $\Omega \equiv \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and

$$A_n = \{j : j \in \mathbb{N},\, j \geq n\}, \quad n \in \mathbb{N}.$$

Let μ be the counting measure on $(\Omega, \mathcal{F})$. Verify that

$$\lim_{n\to\infty} \mu(A_n) \;\neq\; \mu\Big(\bigcap_{n\geq 1} A_n\Big).$$

Answer

- Fix $j \in \mathbb{N}$. For all $n \geq j+1$, we have $j \notin A_n$. Hence $j \notin \bigcap_{n\geq 1} A_n$. Since j was arbitrary, $\bigcap_{n\geq 1} A_n = \varnothing$, and therefore $\mu\Big(\bigcap_{n\geq 1} A_n\Big) = 0$.
- Each A_n is infinite, so $\mu(A_n) = \infty$ for all $n \in \mathbb{N}$, and thus $\lim_{n\to\infty} \mu(A_n) = \infty$.

Therefore,

$$\lim_{n\to\infty} \mu(A_n) \;\neq\; \mu\left(\bigcap_{n\geq 1} A_n\right).$$

Ch. 1: Problem 19

Let Ω be a nonempty set and let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be a semialgebra. Define

$$\mathcal{A}(\mathcal{C}) \equiv \Big\{ A : A = \bigcup_{i=1}^{k} B_i,\ B_i \in \mathcal{C},\ i = 1, 2, \ldots, k,\ k \in \mathbb{N} \Big\}.$$

(a) Show that $\mathcal{A}(\mathcal{C})$ is the smallest algebra containing $\mathcal{C}$.

(b) Show also that $\sigma\langle \mathcal{C} \rangle = \sigma\langle \mathcal{A}(\mathcal{C}) \rangle$.

Answer

(a) We will show $\mathcal{A}(\mathcal{C})$ is (1) an algebra, (2) contains $\mathcal{C}$, and (3) is the smallest such algebra.

(1.) To show $\mathcal{A}(\mathcal{C})$ is an algebra we verify: (1.1) $\Omega \in \mathcal{A}(\mathcal{C})$; (1.2) if $A, A' \in \mathcal{A}(\mathcal{C})$ then $A \cup A' \in \mathcal{A}(\mathcal{C})$; (1.3) if $A \in \mathcal{A}(\mathcal{C})$ then $A^c \in \mathcal{A}(\mathcal{C})$.

(1.1) Since $\mathcal{C}$ is a semialgebra, $\Omega \in \mathcal{C}$. Taking $k = 1$ and $B_1 = \Omega$, we have $\bigcup_{i=1}^{1} B_i = \Omega \in \mathcal{A}(\mathcal{C})$.

(1.2) If $A = \bigcup_{i=1}^{k} B_i$ with each $B_i \in \mathcal{C}$ and $A' = \bigcup_{i=1}^{k'} B_i'$ with each $B_i' \in \mathcal{C}$, then

$$A \cup A' \;=\; \Big(\bigcup_{i=1}^{k} B_i\Big) \cup \Big(\bigcup_{i=1}^{k'} B_i'\Big) \;=\; \bigcup \_i = 1^k B_i \;\cup\; \bigcup \_i = 1^{k'} B_i',$$

which is a finite union of members of $\mathcal{C}$; hence $A \cup A' \in \mathcal{A}(\mathcal{C})$.

(1.3) If $A = \bigcup_{i=1}^{k} B_i$ with $B_i \in \mathcal{C}$, then

$$A^c \;=\; \Big(\bigcup_{i=1}^{k} B_i\Big)^c \;=\; \bigcap_{i=1}^{k} B_i^c.$$

Because $\mathcal{C}$ is a semialgebra, for each i we can write

$$B_i^c \;=\; \bigcup_{r=1}^{m_i} E_{i,r} \quad \text{with } E_{i,r} \in \mathcal{C} \text{ pairwise disjoint (for fixed } i\text{)}.$$

Hence, by distributing intersections over finite unions,

$$A^c \;=\; \bigcap_{i=1}^{k} \Big(\bigcup_{r=1}^{m_i} E_{i,r}\Big) \;=\; \bigcup_{(r_1,\ldots,r_k) \in \prod_{i=1}^{k} \{1,\ldots,m_i\}} \; \bigcap_{i=1}^{k} E_{i,r_i}.$$

Since $\mathcal{C}$ is a semialgebra, it is closed under finite intersections, so each $\bigcap_{i=1}^{k} E_{i,r_i} \in \mathcal{C}$. Therefore A^c is a finite union of members of $\mathcal{C}$, i.e., $A^c \in \mathcal{A}(\mathcal{C})$.

(2.) If $A \in \mathcal{C}$, take $k = 1$ and $B_1 = A$ to get $A \in \mathcal{A}(\mathcal{C})$. Thus $\mathcal{C} \subseteq \mathcal{A}(\mathcal{C})$.

(3.) Let $\mathcal{B}$ be any algebra with $\mathcal{C} \subseteq \mathcal{B}$. Because $\mathcal{B}$ is closed under finite unions, every set of the form $\bigcup_{i=1}^{k} B_i$ with $B_i \in \mathcal{C}$ belongs to $\mathcal{B}$. Hence $\mathcal{A}(\mathcal{C}) \subseteq \mathcal{B}$. Therefore $\mathcal{A}(\mathcal{C})$ is the smallest algebra containing $\mathcal{C}$.

(b) We show (1) $\mathcal{C} \subseteq \sigma\langle \mathcal{A}(\mathcal{C})\rangle$ and (2) $\mathcal{A}(\mathcal{C}) \subseteq \sigma\langle \mathcal{C}\rangle$.

(1.) By (a.2), $\mathcal{C} \subseteq \mathcal{A}(\mathcal{C})$, and clearly $\mathcal{A}(\mathcal{C}) \subseteq \sigma\langle \mathcal{A}(\mathcal{C})\rangle$. Hence $\mathcal{C} \subseteq \sigma\langle \mathcal{A}(\mathcal{C})\rangle$, so $\sigma\langle \mathcal{C}\rangle \subseteq \sigma\langle \mathcal{A}(\mathcal{C})\rangle$.

(2.) If $A = \bigcup_{i=1}^{k} B_i \in \mathcal{A}(\mathcal{C})$ with each $B_i \in \mathcal{C}$, then $B_i \in \sigma\langle \mathcal{C}\rangle$ for all i, and because $\sigma\langle \mathcal{C}\rangle$ is a σ-algebra (hence closed under finite unions), we get $A \in \sigma\langle \mathcal{C}\rangle$. Thus $\mathcal{A}(\mathcal{C}) \subseteq \sigma\langle \mathcal{C}\rangle$, which implies $\sigma\langle \mathcal{A}(\mathcal{C})\rangle \subseteq \sigma\langle \mathcal{C}\rangle$.

Combining (1) and (2), $\sigma\langle \mathcal{C}\rangle = \sigma\langle \mathcal{A}(\mathcal{C})\rangle$.

Ch. 1: Problem 20

Let μ^* be as in (3.1) of Section 1.3. Verify (3.4)–(3.6).

Hint: Fix $0 < \epsilon < \infty$. If $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$, then find, for each n, a cover $\{A_{nj}\}_{j\geq 1} \subset \mathcal{C}$ such that

$$\mu^*(A_n) \ \leq \ \sum_{j=1}^{\infty} \mu(A_{nj}) + \frac{\epsilon}{2^n}.$$

Answer

Recall a set function μ on a semialgebra $\mathcal{C}$, taking values in $\overline{\mathbb{R}}_+ \equiv [0, \infty]$, is called a *measure* if (i) $\mu(\varnothing) = 0$ and (ii) for any sequence of sets $\{A_n\}_{n\geq 1} \subset \mathcal{C}$ with $\bigcup_{n\geq 1} A_n \in \mathcal{C}$, and $A_i \cap A_j = \varnothing$ for $i \neq j$, we have

$$\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Also, given this measure μ on $\mathcal{C}$, the outer measure induced by μ is the set function μ^*, defined on $\mathcal{P}(\Omega)$, as

$$\mu^*(A) \equiv \inf\left\{ \sum_{n=1}^{\infty} \mu(A_n) \ : \ \{A_n\}_{n\geq 1} \subset \mathcal{C},\ A \subset \bigcup_{n\geq 1} A_n \right\}.$$

And (3.4)–(3.6) are:

$$\mu^*(\varnothing) = 0, \tag{3.4}$$

$$A \subset B \ \Rightarrow \ \mu^*(A) \leq \mu^*(B), \tag{3.5}$$

and for any $\{A_n\}_{n\geq 1} \subset \mathcal{P}(\Omega)$,

$$\mu^*\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n). \tag{3.6}$$

We will prove each of them respectively:

Proof. (3.4) Since $\mu^* \geq 0$, it suffices to show $\mu^*(\varnothing) \leq 0$. Cover $\varnothing$ by the sequence $A_n = \varnothing \in \mathcal{C}$ for all n. Then $\varnothing \subset \bigcup_{n\geq 1} A_n$ and $\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(\varnothing) = 0$. Taking the infimum, $\mu^*(\varnothing) \leq 0$, hence $\mu^*(\varnothing) = 0$.

(3.5) If $A \subset B$ and $\{B_n\}_{n\geq 1} \subset \mathcal{C}$ covers B, then it also covers A: $A \subset B \subset \bigcup_{n\geq 1} B_n$. Therefore $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu(B_n)$. Taking the infimum over all covers of B gives $\mu^*(A) \leq \mu^*(B)$.

(3.6) If some $\mu^*(A_n) = \infty$, the inequality is trivial. Otherwise fix $\varepsilon > 0$. For each n choose a cover $\{A_{nj}\}_{j\ge1} \subset \mathcal{C}$ of A_n such that

$$\sum_{j=1}^{\infty} \mu(A_{nj}) \;\le\; \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

Then $\bigcup_{n\ge1}\bigcup_{j\ge1} A_{nj}$ covers $\bigcup_{n\ge1} A_n$. Reindex $\{A_{nj}\}$ as a single sequence $\{B_m\}_{m\ge1} \subset \mathcal{C}$ to fit the definition of μ^*, and obtain

$$\mu^*\left(\bigcup_{n\ge1} A_n\right) \;\le\; \sum_{m=1}^{\infty} \mu(B_m) \;=\; \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \mu(A_{nj}) \;\le\; \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\mu^*\left(\bigcup_{n\ge1} A_n\right) \;\le\; \sum_{n=1}^{\infty} \mu^*(A_n).$$

□

Ch. 1: Problem 24

Verify that $\mathcal{C}_2$, defined in (3.11), is a semialgebra.

Answer

Recall

$$\mathcal{C}_2 \equiv \{\, I_1 \times I_2 : I_1, I_2 \in \mathcal{C}_1 \,\},$$

where (as in (3.7))

$$\mathcal{C} \equiv \big\{(a,b] : -\infty \le a \le b < \infty\big\} \;\cup\; \big\{(a,\infty) : -\infty \le a < \infty\big\}.$$

To show $\mathcal{C}_2$ is a semialgebra, we verify: (1) $\varnothing \in \mathcal{C}_2$; (2) if $A, B \in \mathcal{C}_2$ then $A \cap B \in \mathcal{C}_2$; (3) for each $A \in \mathcal{C}_2$ there exist disjoint $B_1, \ldots, B_k \in \mathcal{C}_2$ with $A^c = \bigcup_{i=1}^k B_i$.

(1) Take $I_1 = (a_1, b_1]$ with $a_1 = b_1$. Then $I_1 = (a_1, a_1] = \varnothing$, hence $I_1 \times I_2 = \varnothing \in \mathcal{C}_2$ (for any I_2).

(2) Let $A = I_{A1} \times I_{A2}$ and $B = I_{B1} \times I_{B2}$ with $I_{A1}, I_{A2}, I_{B1}, I_{B2} \in \mathcal{C}$. Then

$$A \cap B = (I_{A1} \cap I_{B1}) \times (I_{A2} \cap I_{B2}).$$

In one dimension, the intersection of two sets from $\mathcal{C}$ is again in $\mathcal{C}$ (or empty):

$$\begin{aligned}
(a_1, b_1] \cap (a_2, b_2] &= \big(\max\{a_1, a_2\},\, \min\{b_1, b_2\}\big] \text{ (possibly empty)},\\
(a_1, \infty) \cap (a_2, \infty) &= (\max\{a_1, a_2\}, \infty),\\
(a_1, \infty) \cap (a_2, b_2] &= \big(\max\{a_1, a_2\},\, b_2\big] \text{ (possibly empty)}.
\end{aligned}$$

Hence each factor $I_{A\ell} \cap I_{B\ell} \in \mathcal{C}$ (or is $\varnothing$), so $A \cap B \in \mathcal{C}_2$ (or $= \varnothing$).

(3) Let $A = I_1 \times I_2$ with $I_1, I_2 \in \mathcal{C}$. In one dimension, for $I \in \mathcal{C}$,

$$I^c = \begin{cases} (-\infty, a] \ \cup \ (b, \infty), & \text{if } I = (a, b], \\ (-\infty, a], & \text{if } I = (a, \infty), \end{cases}$$

which is a union of at most two disjoint members of $\mathcal{C}$. Write $I_1^c = C_1^{(1)} \cup C_1^{(2)}$ and $I_2^c = C_2^{(1)} \cup C_2^{(2)}$, where some pieces may be empty and the unions are disjoint.
Using the product structure,

$$\begin{aligned} A^c &= (I_1 \times I_2)^c \\ &= (I_1^c \times \mathbb{R}) \ \cup \ (\mathbb{R} \times I_2^c) \\ &= (I_1^c \times I_2) \ \cup \ (I_1 \times I_2^c) \ \cup \ (I_1^c \times I_2^c). \end{aligned}$$

These three pieces are pairwise disjoint. Substituting the decompositions of I_1^c and I_2^c,

$$A^c = \bigcup_{r=1}^{2} \big(C_1^{(r)} \times I_2\big) \ \cup \ \bigcup_{s=1}^{2} \big(I_1 \times C_2^{(s)}\big) \ \cup \ \bigcup_{r=1}^{2} \bigcup_{s=1}^{2} \big(C_1^{(r)} \times C_2^{(s)}\big),$$

a finite union (at most 8) of pairwise disjoint rectangles, each in $\mathcal{C}_2$. Hence A^c is a finite disjoint union of members of $\mathcal{C}_2$.

Therefore, $\mathcal{C}_2$ is a semialgebra.

Last problem:

Consider the following function:

$$F(x) = x\, I_{\{0<x<1\}} + I_{\{x \geq 1\}}, \qquad x \in \mathbb{R}.$$

(i) Is this $F(\cdot)$ non-decreasing, right-continuous? Check.

(ii) Define suitably a Lebesgue–Stieltjes measure μ_F on a suitable semialgebra $\mathcal{C}$. You do not have to prove $\mathcal{C}$ is a semialgebra or that μ_F is a measure.

(iii) Prove that $\sigma\langle\mathcal{C}\rangle \subseteq \mathcal{B}(\mathbb{R})$. (You may use the fact that $\mathcal{B}(\mathbb{R}) = \sigma\langle\text{open sets}\rangle = \sigma\langle\mathcal{O}_1\rangle = \sigma\langle\mathcal{O}_2\rangle = \sigma\langle\mathcal{O}_3\rangle = \sigma\langle\mathcal{O}_4\rangle$ as in 1.10.)

(iv) Argue that there exists a measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that extends μ_F defined above. (You do not need to prove this step—just state a result for your answer.)

(v) Is this extension unique? (i.e., if μ' is another extension of μ_F, then is it true that $\mu(A) = \mu'(A)$ for all $A \in \mathcal{B}(\mathbb{R})$?) Justify your answer (no proof needed). What is the "common name" for this measure μ?

Answer

Consider $F(x)$ in the following form:

$$F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases}$$

(i) Let $x_1 \le x_2$. Split by where x_1, x_2 fall.

- If $x_2 \le 0$, then $F(x_1) = F(x_2) = 0$.
- If $x_1 \le 0 < x_2 < 1$, then $F(x_1) = 0 \le x_2 = F(x_2)$.
- If $x_1 \le 0$ and $x_2 \ge 1$, then $F(x_1) = 0 \le 1 = F(x_2)$.
- If $0 < x_1 \le x_2 < 1$, then $F(x_1) = x_1 \le x_2 = F(x_2)$.
- If $0 < x_1 < 1 \le x_2$, then $F(x_1) = x_1 \le 1 = F(x_2)$.
- If $1 \le x_1 \le x_2$, then $F(x_1) = F(x_2) = 1$.

In all cases $F(x_1) \le F(x_2)$, so F is non-decreasing.

For Right-continuity, on each open piece $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$, F is continuous. We have to show the right-continuity for $x = 1$:

$$\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} 1 = 1 = F(1).$$

(ii) Let

$$\mathcal{C} \;=\; \{(a, b] : -\infty \le a < b \le \infty\} \;\cup\; \{(a, \infty) : -\infty < a < \infty\}.$$

Define the Lebesgue–Stieltjes measure induced by F on $\mathcal{C}$ by

$$\mu_F\big((a, b]\big) \;:=\; F(b) - F(a), \qquad \mu_F\big((a, \infty)\big) \;:=\; \lim_{x \to \infty} F(x) - F(a) = 1 - F(a).$$

(iii) Each generator in $\mathcal{C}$ is Borel. Thus,

$$(a, b] \;=\; (-\infty, b] \cap (a, \infty), \qquad (-\infty, b] \;=\; \bigcap_{n \ge 1} (-\infty, b + 1/n),$$

and (a, ∞) is open. Hence $\mathcal{C} \subset \sigma\langle \mathcal{O}_2 \rangle = \mathcal{B}(\mathbb{R})$, so $\sigma\langle \mathcal{C} \rangle \subseteq \mathcal{B}(\mathbb{R})$.

(iv) By the Carathéodory Extension (Measure Extension) Theorem, the measure μ_F on the semialgebra $\mathcal{C}$ extends to a measure on $\sigma\langle \mathcal{C} \rangle$; combined with (iii), this gives a measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ extending μ_F.

(v) Yes. since μ_F is σ-finite ($\mu_F(\mathbb{R}) = 1$), the Carathéodory extension is unique on $\sigma\langle \mathcal{C} \rangle$ (and hence on $\mathcal{B}(\mathbb{R})$ here). The resulting Borel measure is the Lebesgue–Stieltjes measure of F, which in this case is the Uniform$(0, 1)$ probability measure:

$$\mu(A) = \lambda\big(A \cap (0, 1]\big) \quad \text{for Borel } A,$$

i.e., Lebesgue measure restricted to $(0, 1]$.

Disclaimer

I used ChatGPT to polish my answer to Question 24. I had a rough initial idea that didn't rely much on the fact that $\mathcal{C}$ is a semi-algebra. ChatGPT incorporated that fact, which made my proof much easier. For Question 20, Jaime helped me understand how to approach the problem.