

# MATH 6410 Foundations of Probability Theory, Homework 6

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## Ch. 2: Problem 28

Let  $\mu$  be the Lebesgue measure on  $([-1, 1], \mathcal{B}([-1, 1]))$ . For  $n \geq 1$ , define

$$f_n(x) = n \mathbf{1}_{(0, n^{-1})}(x) - n \mathbf{1}_{(-n^{-1}, 0)}(x) \quad \text{and} \quad f(x) \equiv 0 \text{ for } x \in [-1, 1].$$

Show that  $f_n \rightarrow f$  a.e. ( $\mu$ ) and  $\int f_n d\mu \rightarrow \int f d\mu$  but  $\{f_n\}_{n \geq 1}$  is not UI.

### Answer

To show  $f_n \rightarrow f$  a.e. ( $\mu$ ) we have to show  $\exists B \in \mathcal{B}([-1, 1])$  s.t.  $\mu(B) = 0$ , and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \quad \text{for all } \omega \in B^c$$

Consider  $B = \emptyset$  and  $B^c = \Omega$ . We have to show for a fixed  $\omega \in B^c = \Omega$ , and a given  $\epsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$  such that

$$|f_n(\omega) - f(\omega)| = |f_n(\omega)| < \epsilon \quad \forall n \geq n_0 \in \mathbb{N}$$

if  $\omega = 0$ , then  $f_n(0) = 0$  and  $f_n(0) \rightarrow f$  a.e. ( $\mu$ ). Now, consider  $\omega \neq 0$ , then

$$\begin{aligned} |f_n(\omega)| < \epsilon &\iff |n \mathbf{1}_{(0, n^{-1})}(\omega) - n \mathbf{1}_{(-n^{-1}, 0)}(\omega)| < \epsilon \\ &\iff \omega \notin \left(-\frac{1}{n}, 0\right) \cup \left(0, \frac{1}{n}\right) \\ &\iff |\omega| > \frac{1}{n} \\ &\iff \frac{1}{|\omega|} < n \end{aligned}$$

Hence, set  $n_0 = \lceil \frac{1}{|\omega|} \rceil + 1$ .

To show  $\int f_n d\mu \rightarrow \int f d\mu$  we have to show  $\forall \epsilon > 0$ , exists  $n_0 \in \mathbb{N}$  such that

$$\left| \int_{\Omega} f_n d\mu - \int_{\Omega} f d\mu \right| < \epsilon \quad \text{for all } n \geq n_0 \in \mathbb{N}$$

since  $f \equiv 0$ ,  $\int_{\Omega} f d\mu = \int_{\Omega} 0 d\mu = 0$ . Additionally,

$$\begin{aligned}
\int_{\Omega} f_n d\mu &= \int_{[-1,1]} f_n d\mu \\
&= \int_{[-1,1]} (n \mathbf{1}_{(0, \frac{1}{n})}(x) - n \mathbf{1}_{(-\frac{1}{n}, 0)}(x)) d\mu \\
&= \int_{[-1,1]} n \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - \int_{[-1,1]} n \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= \int_{(0, \frac{1}{n})} n \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - \int_{(-\frac{1}{n}, 0)} n \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= n \int_{(0, \frac{1}{n})} \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - n \int_{(-\frac{1}{n}, 0)} \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= n \cdot \mu((0, \frac{1}{n})) - n \mu((-\frac{1}{n}, 0)) \\
&= n \cdot (\frac{1}{n}) - n \cdot (\frac{1}{n}) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

Hence,  $\int f_n d\mu \rightarrow \int f d\mu$ .

To show  $\{f_n\}_{n \geq 1}$  is not UI, we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \neq 0$$

Fix  $t > 0$ . Then

$$\{|f_n| > t\} = \begin{cases} (-\frac{1}{n}, 0) \cup (0, \frac{1}{n}) & \text{if } n > t \\ \emptyset & \text{if } n \leq t \end{cases}$$

Thus, considering  $n > t$

$$\begin{aligned}
\int_{\{|f_n| > t\}} |f_n| d\mu &= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |f_n| d\mu \\
&= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |n \mathbf{1}_{(0, n^{-1})}(x) - n \mathbf{1}_{(-n^{-1}, 0)}(x)| d\mu \\
&= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |n \mathbf{1}_{(0, n^{-1}) \cup (-n^{-1}, 0)}(x)| d\mu \\
&= n \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |\mathbf{1}_{(0, n^{-1}) \cup (-n^{-1}, 0)}(x)| d\mu \\
&= n \mu((-\frac{1}{n}, 0) \cup (0, \frac{1}{n})) \\
&= n \cdot \frac{2}{n} \\
&= 2
\end{aligned}$$

Hence, as  $t \rightarrow \infty$ ,  $\sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = 2$ .

## Ch. 2: Problem 30

For  $n \geq 1$ , let  $f_n(x) = n^{-1/2} \mathbf{1}_{(0,n)}(x)$ ,  $x \in \mathbb{R}$ , and let  $f(x) = 0$ ,  $x \in \mathbb{R}$ . Let  $m$  denote the Lebesgue measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Show that  $f_n \rightarrow f$  a.e. ( $m$ ) and  $\{f_n\}_{n \geq 1}$  is UI, but  $\int f_n dm \not\rightarrow \int f dm$ .

### Answer

To show  $f_n \rightarrow f$  a.e. ( $m$ ) we have to show exists  $B \in \mathcal{F}$ , such that  $m(B) = 0$  and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega), \quad \forall \omega \in B^c$$

Let  $\epsilon > 0$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } |f_n(\omega) - f(\omega)| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } |f_n(\omega)| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } \left| \frac{1}{\sqrt{n}} \mathbf{1}_{(0,n)}(\omega) \right| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } \left| \frac{1}{\sqrt{n}} \right| < \epsilon, \quad \forall n \geq n_0 \quad (\text{since } \left| \frac{1}{\sqrt{n}} \mathbf{1}_{(0,n)}(\omega) \right| \leq \frac{1}{\sqrt{n}}) \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } \frac{1}{\sqrt{n}} < \epsilon \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } \frac{1}{n} < \epsilon^2 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t } n > \frac{1}{\epsilon^2} \end{aligned}$$

Hence, consider  $\lceil \frac{1}{\epsilon^2} \rceil + 1 = n_0$

To show  $\{f_n\}_{n \geq 1}$  is UI we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = 0$$

Fix  $t$ . Then

$$\{|f_n| > t\} = \begin{cases} (0, n) & \text{if } \frac{1}{\sqrt{n}} > t \\ \emptyset & \text{if } \frac{1}{\sqrt{n}} \leq t \end{cases}$$

If  $t > 1$  then,  $\frac{1}{\sqrt{n}} \leq t$ , and thus

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\emptyset} |f_n| d\mu$$

However,  $m(\emptyset) = 0$ , hence  $\int_{\emptyset} |f_n| d\mu = 0$ .

Also,  $\forall t$ , if  $n \geq \frac{1}{t^2}$ , then:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{(0,n)} |f_n| d\mu &= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{(0,n)} |n^{-1/2} \mathbf{1}_{(0,n)}(x)| d\mu \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \int_{(0,n)} |\mathbf{1}_{(0,n)}(x)| d\mu \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \mu((0,n)) \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \sqrt{n} \\
&\leq \lim_{t \rightarrow \infty} \frac{1}{t} = 0
\end{aligned}$$

Thus

$$\sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \leq \frac{1}{t} \xrightarrow[t \rightarrow \infty]{} 0$$

which also shows  $(f_n)$  is uniformly integrable.

To show  $\int f_n dm \not\rightarrow \int f dm$  we know

$$\begin{aligned}
\int_{\Omega} f_n dm &= \int_{\mathbb{R}} f_n dm \\
&= \int_{\mathbb{R}} n^{-1/2} \mathbf{1}_{(0,n)}(x) dm \\
&= \int_{(0,n)} n^{-1/2} \mathbf{1}_{(0,n)}(x) dm \\
&= \int_{(0,n)} n^{-1/2} dm \\
&= n^{-1/2} m((0,n)) \\
&= \sqrt{n}
\end{aligned}$$

however

$$\int f dm = \int 0 dm = 0. \int dm = 0$$

Hence,  $\int f_n dm \not\rightarrow \int f dm$ .

## Ch. 2: Problem 36

- (a) Let  $\{f_n\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$  such that  $f_n \rightarrow 0$  in  $L^1(\mu)$ . Show that  $\{f_n\}_{n \geq 1}$  is UI.
- (b) Let  $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$ ,  $0 < p < \infty$ , with  $\mu(\Omega) < \infty$ , such that  $\{|f_n|^p\}_{n \geq 1}$  is UI and  $f_n \xrightarrow{m} f$ . Show that  $f \in L^p(\mu)$  and  $f_n \rightarrow f$  in  $L^p(\mu)$ .

## Answer

(a) Since  $\{f_n\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$  such that  $f_n \rightarrow 0$  in  $L^1(\mu)$ , then  $\int |f_n| d\mu < \infty$ , and

$$\lim_{n \rightarrow \infty} \int |f_n| d\mu = 0.$$

We will prove  $\{|f_n|^p\}_{n \geq 1}$  is UI by contradiction. Thus, assume the contrary, in which  $\{|f_n|^p\}_{n \geq 1}$  isn't UI, which means assuming

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \neq 0$$

Set

$$S(t) := \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu$$

then  $\lim_{t \rightarrow \infty} S(t) \neq 0$ , means  $\exists \epsilon > 0$ , and a sequence  $t_k \rightarrow \infty$ , such that

$$S(t_k) \geq \epsilon, \quad \forall k$$

which means, for each  $k$ , there exists an index  $n_k \in \mathbb{N}$ , such that

$$\int_{\{|f_{n_k}| > t_k\}} |f_{n_k}| d\mu \geq \epsilon$$

and

$$\int |f_{n_k}| d\mu \geq \int_{\{|f_{n_k}| > t_k\}} |f_{n_k}| d\mu \geq \epsilon$$

Therefore, for the sequence of  $\{n_k\}$

$$\lim_{k \rightarrow \infty} \int |f_{n_k}| d\mu \geq \epsilon$$

which contradicts the fact that

$$\lim_{n \rightarrow \infty} \int |f_n| d\mu = 0.$$

Hence,

$$\lim_{t \rightarrow \infty} S(t) = 0$$

(b) To show that  $f \in L^p(\mu)$  and  $f_n \rightarrow f$  in  $L^p(\mu)$ , we have to show i)  $\int |f|^p d\mu < \infty$ , ii)  $\int |f_n|^p d\mu < \infty, \quad \forall n \geq 1$ , and iii)  $\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0$

(i)  $\int |f|^p d\mu < \infty$ : Since  $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$ , we have  $\{|f_n|^p\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$ . Moreover,  $f_n \xrightarrow{m} f$  implies (by Theorem 2.5.2 in [1]) that there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  a.e. ( $\mu$ ). Since  $\{|f_n|^p\}$  is uniformly integrable, applying Theorem 2.5.10 in [1] to  $\{|f_{n_k}|^p\}$  yields  $|f|^p \in L^1(\Omega, \mathcal{F}, \mu)$ . Hence,  $\int |f|^p d\mu < \infty$ , i.e.,  $f \in L^p(\Omega, \mathcal{F}, \mu)$ .

(ii)  $\int |f_n|^p d\mu < \infty, \quad \forall n \geq 1$ : since  $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$ , this condition is satisfied.

(iii)  $\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0$ :

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu &= \lim_{n \rightarrow \infty} \int_{\{|f_n - f| < \epsilon\}} |f_n - f|^p d\mu + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \quad (\text{for a } \epsilon > 0) \\
&< \lim_{n \rightarrow \infty} \int_{\Omega} \epsilon^p d\mu + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \\
&= \epsilon^p \mu(\Omega) + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \\
&\leq \epsilon^p \mu(\Omega) + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} 2^{p-1}(|f_n|^p + |f|^p) d\mu \quad (\text{by } |a - b|^p \leq 2^{p-1}(|a|^p + |b|^p)) \\
&= \epsilon^p \mu(\Omega) + 2^{p-1} \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu + 2^{p-1} \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu
\end{aligned}$$

Since  $f_n \rightarrow f$  in measure, we have  $\mu(\{|f_n - f| \geq \epsilon\}) \rightarrow 0$ . Moreover,  $\{|f_n|^p\}$  is uniformly integrable and  $|f|^p \in L^1(\mu)$ . Hence, for every  $\gamma > 0$ , there exist  $\delta, \delta' > 0$  such that

$$\mu(A) < \delta \implies \sup_{n \in \mathbb{N}} \int_A |f_n|^p d\mu < \gamma, \quad \mu(A) < \delta' \implies \int_A |f|^p d\mu < \gamma.$$

Choose  $N$  such that  $\mu(\{|f_n - f| \geq \epsilon\}) < \min\{\delta, \delta'\}$  for all  $n \geq N$ . Then, for  $n \geq N$ ,

$$\int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu < \gamma, \quad \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu < \gamma.$$

Thus,

$$\begin{aligned}
\int |f_n - f|^p d\mu &\leq \epsilon^p \mu(\Omega) + 2^{p-1} \int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu + 2^{p-1} \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu \\
&\leq \epsilon^p \mu(\Omega) + 2^p \gamma \quad \text{for all } n \geq N.
\end{aligned}$$

Taking  $\limsup_{n \rightarrow \infty}$  and then letting  $\gamma \downarrow 0$  gives

$$\limsup_{n \rightarrow \infty} \int |f_n - f|^p d\mu \leq \epsilon^p \mu(\Omega).$$

Finally, letting  $\epsilon \downarrow 0$  yields

$$\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0.$$

## Ch. 2: Problem 42

Let  $(\Omega_i, \mathcal{F}_i)$ ,  $i = 1, 2$  be two measurable spaces. Let  $f : \Omega_1 \rightarrow \Omega_2$  be  $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ -measurable,  $h : \Omega_2 \rightarrow \mathbb{R}$  be  $\langle \mathcal{F}_2, \mathcal{B}(\mathbb{R}) \rangle$ -measurable, and  $\mu_1$  be a measure on  $(\Omega_1, \mathcal{F}_1)$ . Show that  $g \equiv h \circ f$ , i.e.,  $g(\omega) \equiv h(f(\omega))$  for  $\omega \in \Omega_1$  is in  $L^1(\mu_1)$  iff  $h(\cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$  where  $\mu_2 = \mu_1 f^{-1}$  iff  $I(\cdot) \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_3 \equiv \mu_2 h^{-1})$  where  $I(\cdot)$  is the identity function in  $\mathbb{R}$ , i.e.,  $I(x) \equiv x$  for all  $x \in \mathbb{R}$ , and also that

$$\int_{\Omega_1} g d\mu_1 = \int_{\Omega_2} h d\mu_2 = \int_{\mathbb{R}} x d\mu_3.$$

## Answer

For any  $A \in \mathcal{F}_2$ ,

$$\int_{\Omega_1} \mathbf{1}_A \circ f \, d\mu_1 = \mu_1(f^{-1}(A)) = \mu_2(A) = \int_{\Omega_2} \mathbf{1}_A \, d\mu_2.$$

By linearity, the equality holds for all nonnegative simple functions  $h = \sum_i a_i \mathbf{1}_{A_i}$  and  $g = h \circ f = \sum_i a_i (\mathbf{1}_{A_i} \circ f)$ . If  $h_k \uparrow h$  is a sequence of simple functions, then by MCT [1],

$$\int_{\Omega_1} (h_k \circ f) \, d\mu_1 \uparrow \int_{\Omega_1} (h \circ f) \, d\mu_1, \quad \int_{\Omega_2} h_k \, d\mu_2 \uparrow \int_{\Omega_2} h \, d\mu_2,$$

and therefore,

$$\int_{\Omega_1} (h \circ f) \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2 \quad \text{for all nonnegative measurable } h.$$

For a general integrable function  $h$ , write  $h = h^+ - h^-$  and use linearity. Hence,

$$\int_{\Omega_1} |h \circ f| \, d\mu_1 = \int_{\Omega_2} |h| \, d\mu_2.$$

Thus  $g = h \circ f \in L^1(\mu_1)$  if and only if  $h \in L^1(\mu_2)$ , and whenever this holds,

$$\int_{\Omega_1} g \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2. \tag{1}$$

Now, let  $I : \mathbb{R} \rightarrow \mathbb{R}$  be the identity function  $I(x) = x$ . Applying (1) again with the measurable map  $h : (\Omega_2, \mathcal{F}_2) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  and the function  $I$  on  $\mathbb{R}$  gives:

$$I \in L^1(\mu_3) \iff h \in L^1(\mu_2), \quad \text{and} \quad \int_{\Omega_2} h \, d\mu_2 = \int_{\mathbb{R}} I \, d\mu_3 = \int_{\mathbb{R}} x \, d\mu_3.$$

Combining this with (1) gives

$$\int_{\Omega_1} g \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2 = \int_{\mathbb{R}} x \, d\mu_3,$$

and therefore:

$$g \in L^1(\Omega_1, \mathcal{F}_1, \mu_1) \iff h \in L^1(\Omega_2, \mathcal{F}_2, \mu_2) \iff I \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_3).$$

□

## Extra Problem A

Let  $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}([0, 1]), m)$ , where  $m$  denotes the Lebesgue measure. Define the random variables  $X, \{X_n\}$  as follows: for each  $\omega \in \Omega$ ,

$$X(\omega) = 1, \quad X_1(\omega) = 0, \quad X_n(\omega) = 1 + (n-1)\mathbf{1}_{\{[0, 1/n]\}}(\omega), \quad n \geq 2.$$

- (a) Does  $X_n \rightarrow X$  everywhere as  $n \rightarrow \infty$ ?
- (b) Does  $X_n \rightarrow X$  a.s. ( $P$ ) as  $n \rightarrow \infty$ ?
- (c) Does  $X_n \rightarrow X$  in probability as  $n \rightarrow \infty$ ?
- (d) Does  $X_n \rightarrow X$  in  $L^p$  as  $n \rightarrow \infty$ ? Argue for different values of  $p \in (0, \infty)$ .
- (e) Is  $\{X_n\}$  a uniformly integrable sequence of random variables?

## Answer

(a) To show  $X_n \rightarrow X$  everywhere as  $n \rightarrow \infty$ , we have to show

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in \Omega.$$

To satisfy this condition, we will show that for all  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$|X_n(\omega) - X(\omega)| < \epsilon \quad \text{for all } n \geq n_0.$$

Without loss of generality, we propose  $n_0 \geq 2$ . So, consider  $n \geq 2$ :

$$\begin{aligned} |X_n(\omega) - X(\omega)| < \epsilon &\iff |1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| < \epsilon \\ &\iff |(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| < \epsilon. \end{aligned}$$

Now, for a fixed  $\omega > 0$ , consider  $n \geq \lceil \frac{1}{\omega} \rceil + 1$ ; then  $\frac{1}{n} < \omega$ , and thus  $\mathbf{1}_{[0, 1/n]}(\omega) = 0$ . Therefore, choosing  $n_0 = \lceil \frac{1}{\omega} \rceil + 1$  results in

$$|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| = 0 < \epsilon.$$

Hence,  $\{X_n\}_{n \geq 1}$  converges to  $X$  pointwise.

(b) To show  $X_n \rightarrow X$  a.s. (P) as  $n \rightarrow \infty$ , we must find  $B \in \mathcal{B}([0, 1])$  with  $m(B) = 0$  such that

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in B^c.$$

Let  $B = \{0\}$ , so  $m(B) = 0$  and  $B^c = (0, 1]$ . For  $\omega \in B^c$  and any  $\epsilon > 0$ , choose  $n_0 \in \mathbb{N}$  with  $n_0 \geq 2$  and  $n_0 > \frac{1}{\omega}$ . Then for all  $n \geq n_0$  we have  $\frac{1}{n} < \omega$ , hence  $\mathbf{1}_{[0, 1/n]}(\omega) = 0$ , and

$$|X_n(\omega) - X(\omega)| = |1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| = |(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| = 0 < \epsilon.$$

Thus  $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$  for all  $\omega \in B^c$ , and therefore  $X_n \rightarrow X$  a.s. (P).

(c) To show  $X_n \rightarrow X$  in probability as  $n \rightarrow \infty$ , we have to show

$$\lim_{n \rightarrow \infty} m(\{|X_n - X| > \epsilon\}) = 0.$$

Let  $\epsilon > 0$ , and without loss of generality, consider  $n > 2$ , since it does not affect the limit as  $n \rightarrow \infty$ . Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} m(\{|X_n - X| > \epsilon\}) &= \lim_{n \rightarrow \infty} m(\{|1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| > \epsilon\}) \\ &= \lim_{n \rightarrow \infty} m(\{|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| > \epsilon\}). \end{aligned}$$

Using the same reasoning as in part (a), for a fixed  $\omega > 0$ , consider  $n \geq \lceil \frac{1}{\omega} \rceil + 1$ ; then  $\frac{1}{n} < \omega$ , and thus  $\mathbf{1}_{[0, 1/n]}(\omega) = 0$ . Therefore, choosing  $n_0 = \lceil \frac{1}{\omega} \rceil + 1$  results in

$$\lim_{n \rightarrow \infty} m(\{|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| > \epsilon\}) = \lim_{n \rightarrow \infty} m(\{0 > \epsilon\}) = \lim_{n \rightarrow \infty} m(\emptyset) = 0.$$

Hence,  $X_n \rightarrow X$  in probability.

- (d) To show  $X_n \rightarrow X$  in  $L^p$  as  $n \rightarrow \infty$  for  $p \in (0, \infty)$ , we must show  $\int |X_n|^p dm < \infty$  for all  $n \geq 1$ ,  $\int |X|^p dm < \infty$ , and

$$\lim_{n \rightarrow \infty} \int |X_n - X|^p dm = 0.$$

First,

$$\int |X|^p dm = \int_{[0,1]} 1^p dm = 1 < \infty.$$

Next, for  $n \geq 2$  we have  $X_n = n$  on  $[0, 1/n]$  and  $X_n = 1$  on  $(1/n, 1]$ , hence

$$\begin{aligned} \int_{[0,1]} |X_n|^p dm &= \int_{[0,1/n]} n^p dm + \int_{(1/n,1]} 1^p dm \\ &= n^p \cdot \frac{1}{n} + \left(1 - \frac{1}{n}\right) = n^{p-1} + 1 - \frac{1}{n} < \infty. \end{aligned}$$

Finally, since  $X_n - X = (n-1)\mathbf{1}_{[0,1/n]}$ , we get

$$\int |X_n - X|^p dm = \int_{[0,1/n]} (n-1)^p dm = (n-1)^p \cdot \frac{1}{n} \sim n^{p-1}.$$

Therefore:

- If  $0 < p < 1$ , then  $n^{p-1} \rightarrow 0$ , so  $\int |X_n - X|^p dm \rightarrow 0$  and  $X_n \rightarrow X$  in the  $L^p$ .
- If  $p = 1$ , then  $\int |X_n - X| dm = \frac{n-1}{n} \rightarrow 1 \neq 0$ ; no  $L^1$  convergence.
- If  $p > 1$ , then  $\int |X_n - X|^p dm \sim n^{p-1} \rightarrow \infty$ ; no  $L^p$  convergence.

Hence,  $X_n \rightarrow X$  in  $L^p$  holds exactly for  $0 < p < 1$ , and fails for  $p \geq 1$ .

- (e) To show  $\{X_n\}$  a uniformly integrable sequence of random variables, we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = 0$$

Fix  $t$ . Then

$$\{|X_n| > t\} = \begin{cases} [0, 1] & \text{if } t < 1 \\ [0, \frac{1}{n}] & \text{if } 1 \leq t \leq n \\ \emptyset & \text{if } n < t \end{cases}$$

Therefore, for a fixed  $t$

$$\int_{\{|X_n| > t\}} |X_n| dm = \begin{cases} \int_{[0, \frac{1}{n}]} |X_n| dm + \int_{[\frac{1}{n}, 1]} |X_n| dm = 1 + (1 - \frac{1}{n}) = 2 - \frac{1}{n} & \text{if } t < 1 \\ \int_{[0, \frac{1}{n}]} |X_n| dm = 1 & \text{if } 1 \leq t \leq n \\ \int_{\emptyset} |X_n| dm = 0 & \text{if } n < t \end{cases}$$

Hence

$$\sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = \begin{cases} 2 & \text{if } t < 1 \\ 1 & \text{if } 1 \leq t \end{cases}$$

and so

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = 1 \neq 0$$

Therefore,  $\{X_n\}$  isn't uniformly integrable sequence of random variables.

## Disclaimer

In question 36-b, the argument which

$$\mu(A) < \delta \implies \sup_{n \in \mathbb{N}} \int_A |f_n|^p d\mu < \gamma, \quad \mu(A) < \delta' \implies \int_A |f|^p d\mu < \gamma.$$

was achieved with the help of ChatGPT. Throughout the whole homework, I teamed up with Sreejit Roy on thinking about problems.

## References

- [1] Krishna B Athreya and Soumendra N Lahiri. *Measure theory and probability theory*. Springer, 2006.