MATH 6410 Foundations of Probability Theory, Homework 4

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Ch. 2: Problem 1

Let Ω_i, $i = 1, 2$, be two nonempty sets and let $T : \Omega_1 \to \Omega_2$ be a map. For any collection $\{A_\alpha : \alpha \in I\}$ of subsets of Ω_2, show that

$$T^{-1}\left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I} T^{-1}(A_\alpha), \qquad T^{-1}\left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I} T^{-1}(A_\alpha).$$

Further, show that $(T^{-1}(A))^c = T^{-1}(A^c)$ for all $A \subset \Omega_2$ (*de Morgan's laws*).

Answer

1. To show $T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big) = \bigcup_{\alpha\in I} T^{-1}(A_\alpha)$, we will show $T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big) \subseteq \bigcup_{\alpha\in I} T^{-1}(A_\alpha)$ and $\bigcup_{\alpha\in I} T^{-1}(A_\alpha) \subseteq T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big)$.

   (a) $T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big) \subseteq \bigcup_{\alpha\in I} T^{-1}(A_\alpha)$.
   Consider $\omega_0 \in T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big)$. Then

$$\begin{aligned}
\omega_0 \in T^{-1}\left(\bigcup_{\alpha\in I} A_\alpha\right) &\implies T(\omega_0) \in \bigcup_{\alpha\in I} A_\alpha \\
&\implies \exists\, \alpha_0 \in I \text{ such that } T(\omega_0) \in A_{\alpha_0} \\
&\implies \omega_0 \in T^{-1}(A_{\alpha_0}) \\
&\implies \omega_0 \in \bigcup_{\alpha\in I} T^{-1}(A_\alpha).
\end{aligned}$$

   Hence,

$$T^{-1}\left(\bigcup_{\alpha\in I} A_\alpha\right) \subseteq \bigcup_{\alpha\in I} T^{-1}(A_\alpha).$$

   (b) $\bigcup_{\alpha\in I} T^{-1}(A_\alpha) \subseteq T^{-1}\big(\bigcup_{\alpha\in I} A_\alpha\big)$

Consider $\omega_0 \in \bigcup_{\alpha \in I} T^{-1}(A_\alpha)$. Then

$$\begin{aligned}
\omega_0 \in \bigcup_{\alpha \in I} T^{-1}(A_\alpha) &\implies \exists\, \alpha_0 \in I \text{ such that } \omega_0 \in T^{-1}(A_{\alpha_0}) \\
&\implies \exists\, \alpha_0 \in I \text{ such that } T(\omega_0) \in A_{\alpha_0} \\
&\implies T(\omega_0) \in \bigcup_{\alpha \in I} A_\alpha \\
&\implies \omega_0 \in T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right).
\end{aligned}$$

Hence,

$$\bigcup_{\alpha \in I} T^{-1}(A_\alpha) \subseteq T^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right).$$

2. To show $T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big) = \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$, we will show $T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big) \subseteq \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$ and $\bigcap_{\alpha \in I} T^{-1}(A_\alpha) \subseteq T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big)$.

   (a) $T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big) \subseteq \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$
   Consider $\omega_0 \in T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big)$. Then

$$\begin{aligned}
\omega_0 \in T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) &\implies T(\omega_0) \in \bigcap_{\alpha \in I} A_\alpha \\
&\implies \forall\, \alpha \in I,\ T(\omega_0) \in A_\alpha \\
&\implies \forall\, \alpha \in I,\ \omega_0 \in T^{-1}(A_\alpha) \\
&\implies \omega_0 \in \bigcap_{\alpha \in I} T^{-1}(A_\alpha).
\end{aligned}$$

   Hence,

$$T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) \subseteq \bigcap_{\alpha \in I} T^{-1}(A_\alpha).$$

   (b) $\bigcap_{\alpha \in I} T^{-1}(A_\alpha) \subseteq T^{-1}\big(\bigcap_{\alpha \in I} A_\alpha\big)$
   Consider $\omega_0 \in \bigcap_{\alpha \in I} T^{-1}(A_\alpha)$. Then

$$\begin{aligned}
\omega_0 \in \bigcap_{\alpha \in I} T^{-1}(A_\alpha) &\implies \forall\, \alpha \in I,\ \omega_0 \in T^{-1}(A_\alpha) \\
&\implies \forall\, \alpha \in I,\ T(\omega_0) \in A_\alpha \\
&\implies T(\omega_0) \in \bigcap_{\alpha \in I} A_\alpha \\
&\implies \omega_0 \in T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right).
\end{aligned}$$

Hence,

$$\bigcap_{\alpha \in I} T^{-1}(A_\alpha) \subseteq T^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right).$$

3. To show $(T^{-1}(A))^c = T^{-1}(A^c)$ for all $A \subset \Omega_2$, we will show $(T^{-1}(A))^c \subseteq T^{-1}(A^c)$ and $T^{-1}(A^c) \subseteq (T^{-1}(A))^c$.

   (a) $(T^{-1}(A))^c \subseteq T^{-1}(A^c)$
   Consider $\omega_0 \in (T^{-1}(A))^c$. Then

   $$\begin{aligned} \omega_0 \in (T^{-1}(A))^c &\implies \omega_0 \notin T^{-1}(A) \\ &\implies T(\omega_0) \notin A \\ &\implies T(\omega_0) \in A^c \\ &\implies \omega_0 \in T^{-1}(A^c) \end{aligned}$$

   Hence,

   $$(T^{-1}(A))^c \subseteq T^{-1}(A^c).$$

   (b) $T^{-1}(A^c) \subseteq (T^{-1}(A))^c$
   Consider $\omega_0 \in T^{-1}(A^c)$. Then

   $$\begin{aligned} \omega_0 \in T^{-1}(A^c) &\implies T(\omega_0) \in A^c \\ &\implies T(\omega_0) \notin A \\ &\implies \omega_0 \notin T^{-1}(A) \\ &\implies \omega_0 \in (T^{-1}(A))^c \end{aligned}$$

   Therefore,

   $$T^{-1}(A^c) \subseteq (T^{-1}(A))^c.$$

Ch. 2: Problem 3

Let $f, g : \Omega \to \mathbb{R}$ be $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable. Define

$$h(\omega) = \frac{f(\omega)}{g(\omega)} I\big(g(\omega) \neq 0\big), \qquad \omega \in \Omega.$$

Verify that h is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.

Answer

Let $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable, and define $A = \{\omega \in \Omega : g(\omega) \neq 0\}$. Since $A = g^{-1}(\mathbb{R} \setminus \{0\}) \in \mathcal{F}$, the indicator function $I(g(\omega) \neq 0) : \Omega \to \mathbb{R}$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.
Now, consider the function $\phi : (\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\phi(t) = 1/t$. Since ϕ is continuous on its domain, it is Borel-measurable (Proposition 2.1.2 [1]). Hence, the composition $\xi = \phi \circ g|_A : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable (Proposition 2.1.1 [1]).
Note that $\xi(\omega) = 1/g(\omega)$ for $\omega \in A$. Therefore, $\frac{1}{g(\omega)} I(g(\omega) \neq 0) = \xi(\omega)\, I(g(\omega) \neq 0)$ is the product of two measurable functions from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Since $f(\omega)$ is also measurable, it follows from Proposition 2.1.3 (iii) in [1] that $h(\omega) = f(\omega)\, \frac{1}{g(\omega)}\, I(g(\omega) \neq 0)$ is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R})\rangle$-measurable.

Ch. 2: Problem 6

Let X_i, $i = 1, 2, 3$, be random variables on a probability space $(\Omega, \mathcal{F}, P)$. Consider the random equation (in $t \in \mathbb{R}$)

$$X_1(\omega)t^2 + X_2(\omega)t + X_3(\omega) = 0. \tag{1}$$

(a) Let $A \equiv \{\omega \in \Omega : \text{Equation (1) has two distinct real roots}\}$. Show that $A \in \mathcal{F}$.

(b) Let $T_1(\omega)$ and $T_2(\omega)$ denote the two roots of (1) on A. Define

$$f_i(\omega) = \begin{cases} T_i(\omega), & \omega \in A, \\ 0, & \omega \in A^c, \end{cases} \qquad i = 1, 2.$$

Show that (f_1, f_2) is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}^2) \rangle$-measurable.

Answer

(a) Since $A \equiv \{\omega \in \Omega : \text{Equation (1) has two distinct real roots}\}$, any $\omega \in A$ must satisfy

$$X_2(\omega)^2 - 4X_1(\omega)X_3(\omega) > 0 \quad \text{and} \quad X_1(\omega) \neq 0.$$

Let $\Delta(\omega) := X_2(\omega)^2 - 4X_1(\omega)X_3(\omega)$. Because X_i are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable and sums/products/compositions with continuous maps preserve measurability, Δ is measurable. Hence

$$\{\Delta > 0\} = \Delta^{-1}((0, \infty)) \in \mathcal{F} \quad \text{and} \quad \{X_1 \neq 0\} = X_1^{-1}(\mathbb{R} \setminus \{0\}) \in \mathcal{F}.$$

Therefore

$$A = \{\Delta > 0\} \cap \{X_1 \neq 0\} \in \mathcal{F}.$$

(b) We will show each of f_1 and f_2 are $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}) \rangle$-measurable. Then by proposition 2.1.3 *(i)* in [1] (f_1, f_2) is $\langle \mathcal{F}, \mathcal{B}(\mathbb{R}^2) \rangle$-measurable.
Please note,

$$f_i = T_i \, \mathbf{1}_A + 0 \cdot \mathbf{1}_{A^c},$$

where $A \in \mathcal{F}$ and T_i is measurable on A. Since $\mathbf{1}_A$ and $\mathbf{1}_{A^c}$ are measurable, and are closed under sums and products of measurable functions, to show that that f_i is measurable on Ω, we have to show

$$T_i(\omega) = \frac{-X_2(\omega) \pm \sqrt{X_2(\omega)^2 - 4X_1(\omega)X_3(\omega)}}{2X_1(\omega)}$$

is measurable on A. This follows from the facts that sums, products, and compositions with continuous functions preserve measurability. We also proved in the last question that the division of a measurable function by a non-zero measurable function can be made measurable. Hence, once T_i is shown to be measurable, the measurability of f_i follows directly as a simple consequence of the closure properties of measurable functions.

Problem A

Let $X : \Omega_1 \to \Omega_2$ be a $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$-measurable transformation. Consider the collection (of subsets of Ω_1) defined by

$$\sigma\langle X \rangle = \{X^{-1}(B) : B \in \mathcal{F}_2\}.$$

Then show that:

(a) $\sigma\langle X\rangle$ is a σ-algebra.

(b) $\sigma\langle X\rangle$ is the *smallest* σ-algebra that makes X measurable, i.e., if X is $\langle\mathcal{G},\mathcal{F}_2\rangle$-measurable, then $\sigma\langle X\rangle\subset\mathcal{G}$.

Answer

(a) To show $\sigma\langle X\rangle$ is a σ-algebra, we have to show *(i)* $\emptyset\in\sigma\langle X\rangle$, and $\sigma\langle X\rangle$ is closed under *(ii)* complement and *(iii)* countable union.

(i)

$$\begin{aligned}\emptyset\in\sigma\langle X\rangle &\iff \emptyset\in\{X^{-1}(B):B\in\mathcal{F}_2\}\\ &\iff \exists B_0\in\mathcal{F}_2 \text{ such that } \emptyset=X^{-1}(B_0)\\ &\iff \exists B_0\in\mathcal{F}_2 \text{ such that } \emptyset\equiv\{\omega\in\Omega_1\mid X(\omega)\in B_0\}.\end{aligned}$$

Consider $B_0=\emptyset$, which is valid since $\mathcal{F}_2$ is a σ-algebra. Then

$$X^{-1}(\emptyset)=\{\omega\in\Omega_1:X(\omega)\in\emptyset\}=\emptyset.$$

Hence, $\emptyset=X^{-1}(\emptyset)\in\sigma\langle X\rangle$.

(ii)

$$\begin{aligned}A\in\sigma\langle X\rangle &\iff A\in\{X^{-1}(B):B\in\mathcal{F}_2\}\\ &\iff \exists B_0\in\mathcal{F}_2 \text{ such that } A=X^{-1}(B_0)\\ &\iff \exists B_0\in\mathcal{F}_2 \text{ such that } A\equiv\{\omega\in\Omega_1\mid X(\omega)\in B_0\}.\end{aligned}$$

Then,

$$\begin{aligned}A^c&\equiv(\{\omega\in\Omega_1\mid X(\omega)\in B_0\})^c\\ &=\{\omega\in\Omega_1\mid X(\omega)\notin B_0\}\\ &=\{\omega\in\Omega_1\mid X(\omega)\in B_0^c\}\\ &=X^{-1}(B_0^c).\end{aligned}$$

Since $\mathcal{F}_2$ is a σ-algebra, $B_0^c\in\mathcal{F}_2$. Hence, $A^c\in\{X^{-1}(B):B\in\mathcal{F}_2\}$ and therefore $A^c\in\sigma\langle X\rangle$.

(iii)

$$\begin{aligned}A_1,A_2,\dots\in\sigma\langle X\rangle &\iff \exists B_1,B_2,\dots\in\mathcal{F}_2 \text{ such that } A_i=X^{-1}(B_i)\\ &\iff \exists B_1,B_2,\dots\in\mathcal{F}_2 \text{ such that } A_i\equiv\{\omega\in\Omega_1\mid X(\omega)\in B_i\}.\end{aligned}$$

Now, to calculate $\bigcup_{i\geq1}A_i$:

$$\begin{aligned}\bigcup_{i\geq1}A_i&=\bigcup_{i\geq1}\{\omega\in\Omega_1\mid X(\omega)\in B_i\}\\ &=\{\omega\in\Omega_1\mid X(\omega)\in\bigcup_{i\geq1}B_i\}\\ &=X^{-1}\left(\bigcup_{i\geq1}B_i\right).\end{aligned}$$

Since $\mathcal{F}_2$ is a σ-algebra, $\bigcup_{i\geq 1} B_i \in \mathcal{F}_2$. Thus, $X^{-1}\Big(\bigcup_{i\geq 1} B_i\Big) \in \sigma\langle X\rangle$, which means $\bigcup_{i\geq 1} A_i \in \sigma\langle X\rangle$.

(b) Suppose X is $\langle \mathcal{G}, \mathcal{F}_2\rangle$-measurable. Then for every $B \in \mathcal{F}_2$ we have $X^{-1}(B) \in \mathcal{G}$. Hence

$$\sigma\langle X\rangle = \{X^{-1}(B) : B \in \mathcal{F}_2\} \subseteq \mathcal{G}.$$

Since this holds for every σ-algebra $\mathcal{G}$ making X measurable, $\sigma\langle X\rangle$ is the smallest such σ-algebra.

Problem B

Give an example of a function f that is *not* $\langle \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\rangle$-measurable.

Answer

We know that there exist an element in $\mathcal{P}(\mathbb{R})$ such that it's not in $\mathcal{B}(\mathbb{R})$, meaning it's not Lebesgue measurable [1, 2]. Suppose we call this set collection of sets of real numbers $\mathcal{V}$. Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \mathbf{1}_{\mathcal{V}(x)} = \begin{cases} 1, & x \in \mathcal{V}, \\ 0, & x \notin \mathcal{V}. \end{cases}$$

Then $\{1\}$ is a Borel set in $\mathbb{R}$, but

$$f^{-1}(\{1\}) = \mathcal{V}$$

is not a Borel set. Therefore f is not $\langle \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})\rangle$-measurable.

Disclaimer

In my answers, I have referred to certain propositions, by which I mean the propositions mentioned in the reference [1]

References

[1] Krishna B Athreya and Soumendra N Lahiri. *Measure theory and probability theory.* Springer, 2006.

[2] Halsey Lawrence Royden and Patrick Fitzpatrick. *Real analysis*, volume 32. Macmillan New York, 1988.