

MATH 6410 Foundations of Probability Theory, Homework 6

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Ch. 2: Problem 28

Let μ be the Lebesgue measure on $([-1, 1], \mathcal{B}([-1, 1]))$. For $n \geq 1$, define

$$f_n(x) = n \mathbf{1}_{(0, n^{-1})}(x) - n \mathbf{1}_{(-n^{-1}, 0)}(x) \quad \text{and} \quad f(x) \equiv 0 \text{ for } x \in [-1, 1].$$

Show that $f_n \rightarrow f$ a.e. (μ) and $\int f_n d\mu \rightarrow \int f d\mu$ but $\{f_n\}_{n \geq 1}$ is not UI.

Answer

To show $f_n \rightarrow f$ a.e. (μ) we have to show $\exists B \in \mathcal{B}([-1, 1])$ s.t. $\mu(B) = 0$, and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \quad \text{for all } \omega \in B^c$$

Consider $B = \emptyset$ and $B^c = \Omega$. We have to show for a fixed $\omega \in B^c = \Omega$, and a given $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$|f_n(\omega) - f(\omega)| = |f_n(\omega)| < \epsilon \quad \forall n \geq n_0 \in \mathbb{N}$$

if $\omega = 0$, then $f_n(0) = 0$ and $f_n(0) \rightarrow f$ a.e. (μ) . Now, consider $\omega \neq 0$, then

$$\begin{aligned} |f_n(\omega)| < \epsilon &\iff |n \mathbf{1}_{(0, n^{-1})}(\omega) - n \mathbf{1}_{(-n^{-1}, 0)}(\omega)| < \epsilon \\ &\iff \omega \notin (-\frac{1}{n}, 0) \cup (0, \frac{1}{n}) \\ &\iff |\omega| > \frac{1}{n} \\ &\iff \frac{1}{|\omega|} < n \end{aligned}$$

Hence, set $n_0 = \lceil \frac{1}{|\omega|} \rceil + 1$.

To show $\int f_n d\mu \rightarrow \int f d\mu$ we have to show $\forall \epsilon > 0$, exists $n_0 \in \mathbb{N}$ such that

$$|\int_{\Omega} f_n d\mu - \int_{\Omega} f d\mu| < \epsilon \quad \text{for all } n \geq n_0 \in \mathbb{N}$$

since $f \equiv 0$, $\int_{\Omega} f d\mu = \int_{\Omega} 0 d\mu = 0$. Additionally,

$$\begin{aligned}
\int_{\Omega} f_n d\mu &= \int_{[-1,1]} f_n d\mu \\
&= \int_{[-1,1]} (n \mathbf{1}_{(0, \frac{1}{n})}(x) - n \mathbf{1}_{(-\frac{1}{n}, 0)}(x)) d\mu \\
&= \int_{[-1,1]} n \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - \int_{[-1,1]} n \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= \int_{(0, \frac{1}{n})} n \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - \int_{(-\frac{1}{n}, 0)} n \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= n \int_{(0, \frac{1}{n})} \mathbf{1}_{(0, \frac{1}{n})}(x) d\mu - n \int_{(-\frac{1}{n}, 0)} \mathbf{1}_{(-\frac{1}{n}, 0)}(x) d\mu \\
&= n \cdot \mu((0, \frac{1}{n})) - n \mu((-\frac{1}{n}, 0)) \\
&= n \cdot (\frac{1}{n}) - n \cdot (\frac{1}{n}) \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

Hence, $\int f_n d\mu \rightarrow \int f d\mu$.

To show $\{f_n\}_{n \geq 1}$ is not UI, we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \neq 0$$

Fix $t > 0$. Then

$$\{|f_n| > t\} = \begin{cases} (-\frac{1}{n}, 0) \cup (0, \frac{1}{n}) & \text{if } n > t \\ \emptyset & \text{if } n \leq t \end{cases}$$

Thus, considering $n > t$

$$\begin{aligned}
\int_{\{|f_n| > t\}} |f_n| d\mu &= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |f_n| d\mu \\
&= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |n \mathbf{1}_{(0, n^{-1})}(x) - n \mathbf{1}_{(-n^{-1}, 0)}(x)| d\mu \\
&= \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |n \mathbf{1}_{(0, n^{-1}) \cup (-n^{-1}, 0)}(x)| d\mu \\
&= n \int_{\{(-\frac{1}{n}, 0) \cup (0, \frac{1}{n})\}} |\mathbf{1}_{(0, n^{-1}) \cup (-n^{-1}, 0)}(x)| d\mu \\
&= n \mu((-\frac{1}{n}, 0) \cup (0, \frac{1}{n})) \\
&= n \cdot \frac{2}{n} \\
&= 2
\end{aligned}$$

Hence, as $t \rightarrow \infty$, $\sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = 2$.

Ch. 2: Problem 30

For $n \geq 1$, let $f_n(x) = n^{-1/2} \mathbf{1}_{(0,n)}(x)$, $x \in \mathbb{R}$, and let $f(x) = 0$, $x \in \mathbb{R}$. Let m denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that $f_n \rightarrow f$ a.e. (m) and $\{f_n\}_{n \geq 1}$ is UI, but $\int f_n dm \not\rightarrow \int f dm$.

Answer

To show $f_n \rightarrow f$ a.e. (m) we have to show exists $B \in \mathcal{F}$, such that $m(B) = 0$ and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega), \quad \forall \omega \in B^c$$

Let $\epsilon > 0$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } |f_n(\omega) - f(\omega)| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } |f_n(\omega)| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } \left| \frac{1}{\sqrt{n}} \mathbf{1}_{(0,n)}(\omega) \right| < \epsilon, \quad \forall n \geq n_0 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } \left| \frac{1}{\sqrt{n}} \right| < \epsilon, \quad \forall n \geq n_0 \quad (\text{since } \left| \frac{1}{\sqrt{n}} \mathbf{1}_{(0,n)}(\omega) \right| \leq \frac{1}{\sqrt{n}}.) \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } \frac{1}{\sqrt{n}} < \epsilon \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } \frac{1}{n} < \epsilon^2 \\ &\iff \exists n_0 \in \mathbb{N}, \text{ s.t. } n > \frac{1}{\epsilon^2} \end{aligned}$$

Hence, consider $\lceil \frac{1}{\epsilon^2} \rceil + 1 = n_0$

To show $\{f_n\}_{n \geq 1}$ is UI we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = 0$$

Fix t . Then

$$\{|f_n| > t\} = \begin{cases} (0, n) & \text{if } \frac{1}{\sqrt{n}} > t \\ \emptyset & \text{if } \frac{1}{\sqrt{n}} \leq t \end{cases}$$

If $t > 1$ then, $\frac{1}{\sqrt{n}} \leq t$, and thus

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu = \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\emptyset} |f_n| d\mu$$

However, $m(\emptyset) = 0$, hence $\int_{\emptyset} |f_n| d\mu = 0$.

Also, $\forall t$, if $n \geq \frac{1}{t^2}$, then:

$$\begin{aligned}
\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{(0,n)} |f_n| d\mu &= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{(0,n)} |n^{-1/2} \mathbf{1}_{(0,n)}(x)| d\mu \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \int_{(0,n)} |\mathbf{1}_{(0,n)}(x)| d\mu \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \frac{1}{\sqrt{n}} \mu((0,n)) \\
&= \lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \sqrt{n} \\
&\leq \lim_{t \rightarrow \infty} \frac{1}{t} = 0
\end{aligned}$$

Thus

$$\sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \leq \frac{1}{t} \xrightarrow{t \rightarrow \infty} 0$$

which also shows (f_n) is uniformly integrable.

To show $\int f_n dm \not\rightarrow \int f dm$ we know

$$\begin{aligned}
\int_{\Omega} f_n dm &= \int_{\mathbb{R}} f_n dm \\
&= \int_{\mathbb{R}} n^{-1/2} \mathbf{1}_{(0,n)}(x) dm \\
&= \int_{(0,n)} n^{-1/2} \mathbf{1}_{(0,n)}(x) dm \\
&= \int_{(0,n)} n^{-1/2} dm \\
&= n^{-1/2} m((0,n)) \\
&= \sqrt{n}
\end{aligned}$$

however

$$\int f dm = \int 0 dm = 0. \quad \int dm = 0$$

Hence, $\int f_n dm \not\rightarrow \int f dm$.

Ch. 2: Problem 36

- (a) Let $\{f_n\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$ such that $f_n \rightarrow 0$ in $L^1(\mu)$. Show that $\{f_n\}_{n \geq 1}$ is UI.
- (b) Let $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$, $0 < p < \infty$, with $\mu(\Omega) < \infty$, such that $\{|f_n|^p\}_{n \geq 1}$ is UI and $f_n \xrightarrow{m} f$. Show that $f \in L^p(\mu)$ and $f_n \rightarrow f$ in $L^p(\mu)$.

Answer

(a) Since $\{f_n\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$ such that $f_n \rightarrow 0$ in $L^1(\mu)$, then $\int |f_n| d\mu < \infty$, and

$$\lim_{n \rightarrow \infty} \int |f_n| d\mu = 0.$$

We will prove $\{|f_n|^p\}_{n \geq 1}$ is UI by contradiction. Thus, assume the contrary, in which $\{|f_n|^p\}_{n \geq 1}$ isn't UI, which means assuming

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu \neq 0$$

Set

$$S(t) := \sup_{n \in \mathbb{N}} \int_{\{|f_n| > t\}} |f_n| d\mu$$

then $\lim_{t \rightarrow \infty} S(t) \neq 0$, means $\exists \epsilon > 0$, and a sequence $t_k \rightarrow \infty$, such that

$$S(t_k) \geq \epsilon, \quad \forall k$$

which means, for each k , there exists an index $n_k \in \mathbb{N}$, such that

$$\int_{\{|f_{n_k}| > t_k\}} |f_{n_k}| d\mu \geq \epsilon$$

and

$$\int |f_{n_k}| d\mu \geq \int_{\{|f_{n_k}| > t_k\}} |f_{n_k}| d\mu \geq \epsilon$$

Therefore, for the sequence of $\{n_k\}$

$$\lim_{k \rightarrow \infty} \int |f_{n_k}| d\mu \geq \epsilon$$

which contradicts the fact that

$$\lim_{n \rightarrow \infty} \int |f_n| d\mu = 0.$$

Hence,

$$\lim_{t \rightarrow \infty} S(t) = 0$$

(b) To show that $f \in L^p(\mu)$ and $f_n \rightarrow f$ in $L^p(\mu)$, we have to show i) $\int |f|^p d\mu < \infty$, ii) $\int |f_n|^p d\mu < \infty$, $\forall n \geq 1$, and iii) $\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0$

(i) $\int |f|^p d\mu < \infty$: Since $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$, we have $\{|f_n|^p\}_{n \geq 1} \subset L^1(\Omega, \mathcal{F}, \mu)$. Moreover, $f_n \xrightarrow{m} f$ implies (by Theorem 2.5.2 in [1]) that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow f$ a.e. (μ). Since $\{|f_n|^p\}$ is uniformly integrable, applying Theorem 2.5.10 in [1] to $\{|f_{n_k}|^p\}$ yields $|f|^p \in L^1(\Omega, \mathcal{F}, \mu)$. Hence, $\int |f|^p d\mu < \infty$, i.e., $f \in L^p(\Omega, \mathcal{F}, \mu)$.

(ii) $\int |f_n|^p d\mu < \infty$, $\forall n \geq 1$: since $\{f_n\}_{n \geq 1} \subset L^p(\Omega, \mathcal{F}, \mu)$, this condition is satisfied.

(iii) $\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0$:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu &= \lim_{n \rightarrow \infty} \int_{\{|f_n - f| < \epsilon\}} |f_n - f|^p d\mu + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \quad (\text{for a } \epsilon > 0) \\
&< \lim_{n \rightarrow \infty} \int_{\Omega} \epsilon^p d\mu + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \\
&= \epsilon^p \mu(\Omega) + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n - f|^p d\mu \\
&\leq \epsilon^p \mu(\Omega) + \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} 2^{p-1} (|f_n|^p + |f|^p) d\mu \quad (\text{by } |a - b|^p \leq 2^{p-1} (|a|^p + |b|^p)) \\
&= \epsilon^p \mu(\Omega) + 2^{p-1} \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu + 2^{p-1} \lim_{n \rightarrow \infty} \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu
\end{aligned}$$

Since $f_n \rightarrow f$ in measure, we have $\mu(\{|f_n - f| \geq \epsilon\}) \rightarrow 0$. Moreover, $\{|f_n|^p\}$ is uniformly integrable and $|f|^p \in L^1(\mu)$. Hence, for every $\gamma > 0$, there exist $\delta, \delta' > 0$ such that

$$\mu(A) < \delta \implies \sup_{n \in \mathbb{N}} \int_A |f_n|^p d\mu < \gamma, \quad \mu(A) < \delta' \implies \int_A |f|^p d\mu < \gamma.$$

Choose N such that $\mu(\{|f_n - f| \geq \epsilon\}) < \min\{\delta, \delta'\}$ for all $n \geq N$. Then, for $n \geq N$,

$$\int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu < \gamma, \quad \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu < \gamma.$$

Thus,

$$\begin{aligned}
\int |f_n - f|^p d\mu &\leq \epsilon^p \mu(\Omega) + 2^{p-1} \int_{\{|f_n - f| \geq \epsilon\}} |f_n|^p d\mu + 2^{p-1} \int_{\{|f_n - f| \geq \epsilon\}} |f|^p d\mu \\
&\leq \epsilon^p \mu(\Omega) + 2^p \gamma \quad \text{for all } n \geq N.
\end{aligned}$$

Taking $\limsup_{n \rightarrow \infty}$ and then letting $\gamma \downarrow 0$ gives

$$\limsup_{n \rightarrow \infty} \int |f_n - f|^p d\mu \leq \epsilon^p \mu(\Omega).$$

Finally, letting $\epsilon \downarrow 0$ yields

$$\lim_{n \rightarrow \infty} \int |f_n - f|^p d\mu = 0.$$

Ch. 2: Problem 42

Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$ be two measurable spaces. Let $f : \Omega_1 \rightarrow \Omega_2$ be $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ -measurable, $h : \Omega_2 \rightarrow \mathbb{R}$ be $\langle \mathcal{F}_2, \mathcal{B}(\mathbb{R}) \rangle$ -measurable, and μ_1 be a measure on $(\Omega_1, \mathcal{F}_1)$. Show that $g \equiv h \circ f$, i.e., $g(\omega) \equiv h(f(\omega))$ for $\omega \in \Omega_1$ is in $L^1(\mu_1)$ iff $h(\cdot) \in L^1(\Omega_2, \mathcal{F}_2, \mu_2)$ where $\mu_2 = \mu_1 f^{-1}$ iff $I(\cdot) \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_3 \equiv \mu_2 h^{-1})$ where $I(\cdot)$ is the identity function in \mathbb{R} , i.e., $I(x) \equiv x$ for all $x \in \mathbb{R}$, and also that

$$\int_{\Omega_1} g d\mu_1 = \int_{\Omega_2} h d\mu_2 = \int_{\mathbb{R}} x d\mu_3.$$

Answer

For any $A \in \mathcal{F}_2$,

$$\int_{\Omega_1} \mathbf{1}_A \circ f \, d\mu_1 = \mu_1(f^{-1}(A)) = \mu_2(A) = \int_{\Omega_2} \mathbf{1}_A \, d\mu_2.$$

By linearity, the equality holds for all nonnegative simple functions $h = \sum_i a_i \mathbf{1}_{A_i}$ and $g = h \circ f = \sum_i a_i (\mathbf{1}_{A_i} \circ f)$. If $h_k \uparrow h$ is a sequence of simple functions, then by MCT [1],

$$\int_{\Omega_1} (h_k \circ f) \, d\mu_1 \uparrow \int_{\Omega_1} (h \circ f) \, d\mu_1, \quad \int_{\Omega_2} h_k \, d\mu_2 \uparrow \int_{\Omega_2} h \, d\mu_2,$$

and therefore,

$$\int_{\Omega_1} (h \circ f) \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2 \quad \text{for all nonnegative measurable } h.$$

For a general integrable function h , write $h = h^+ - h^-$ and use linearity. Hence,

$$\int_{\Omega_1} |h \circ f| \, d\mu_1 = \int_{\Omega_2} |h| \, d\mu_2.$$

Thus $g = h \circ f \in L^1(\mu_1)$ if and only if $h \in L^1(\mu_2)$, and whenever this holds,

$$\int_{\Omega_1} g \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2. \tag{1}$$

Now, let $I : \mathbb{R} \rightarrow \mathbb{R}$ be the identity function $I(x) = x$. Applying (1) again with the measurable map $h : (\Omega_2, \mathcal{F}_2) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and the function I on \mathbb{R} gives:

$$I \in L^1(\mu_3) \quad \text{iff} \quad h \in L^1(\mu_2), \quad \text{and} \quad \int_{\Omega_2} h \, d\mu_2 = \int_{\mathbb{R}} I \, d\mu_3 = \int_{\mathbb{R}} x \, d\mu_3.$$

Combining this with (1) gives

$$\int_{\Omega_1} g \, d\mu_1 = \int_{\Omega_2} h \, d\mu_2 = \int_{\mathbb{R}} x \, d\mu_3,$$

and therefore:

$$g \in L^1(\Omega_1, \mathcal{F}_1, \mu_1) \iff h \in L^1(\Omega_2, \mathcal{F}_2, \mu_2) \iff I \in L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_3).$$

□

Extra Problem A

Let $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}([0, 1]), m)$, where m denotes the Lebesgue measure. Define the random variables $X, \{X_n\}$ as follows: for each $\omega \in \Omega$,

$$X(\omega) = 1, \quad X_1(\omega) = 0, \quad X_n(\omega) = 1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega), \quad n \geq 2.$$

- Does $X_n \rightarrow X$ *everywhere* as $n \rightarrow \infty$?
- Does $X_n \rightarrow X$ a.s. (P) as $n \rightarrow \infty$?
- Does $X_n \rightarrow X$ in probability as $n \rightarrow \infty$?
- Does $X_n \rightarrow X$ in L^p as $n \rightarrow \infty$? Argue for different values of $p \in (0, \infty)$.
- Is $\{X_n\}$ a uniformly integrable sequence of random variables?

Answer

(a) To show $X_n \rightarrow X$ *everywhere* as $n \rightarrow \infty$, we have to show

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in \Omega.$$

To satisfy this condition, we will show that for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|X_n(\omega) - X(\omega)| < \epsilon \quad \text{for all } n \geq n_0.$$

Without loss of generality, we propose $n_0 \geq 2$. So, consider $n \geq 2$:

$$\begin{aligned} |X_n(\omega) - X(\omega)| < \epsilon &\iff |1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| < \epsilon \\ &\iff |(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| < \epsilon. \end{aligned}$$

Now, for a fixed $\omega > 0$, consider $n \geq \lceil \frac{1}{\omega} \rceil + 1$; then $\frac{1}{n} < \omega$, and thus $\mathbf{1}_{[0, 1/n]}(\omega) = 0$. Therefore, choosing $n_0 = \lceil \frac{1}{\omega} \rceil + 1$ results in

$$|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| = 0 < \epsilon.$$

Hence, $\{X_n\}_{n \geq 1}$ converges to X pointwise.

(b) To show $X_n \rightarrow X$ a.s. (P) as $n \rightarrow \infty$, we must find $B \in \mathcal{B}([0, 1])$ with $m(B) = 0$ such that

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in B^c.$$

Let $B = \{0\}$, so $m(B) = 0$ and $B^c = (0, 1]$. For $\omega \in B^c$ and any $\epsilon > 0$, choose $n_0 \in \mathbb{N}$ with $n_0 \geq 2$ and $n_0 > \frac{1}{\omega}$. Then for all $n \geq n_0$ we have $\frac{1}{n} < \omega$, hence $\mathbf{1}_{[0, 1/n]}(\omega) = 0$, and

$$|X_n(\omega) - X(\omega)| = |1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| = |(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| = 0 < \epsilon.$$

Thus $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$ for all $\omega \in B^c$, and therefore $X_n \rightarrow X$ a.s. (P).

(c) To show $X_n \rightarrow X$ in probability as $n \rightarrow \infty$, we have to show

$$\lim_{n \rightarrow \infty} m(\{|X_n - X| > \epsilon\}) = 0.$$

Let $\epsilon > 0$, and without loss of generality, consider $n > 2$, since it does not affect the limit as $n \rightarrow \infty$. Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} m(\{|X_n - X| > \epsilon\}) &= \lim_{n \rightarrow \infty} m(\{|1 + (n-1)\mathbf{1}_{[0, 1/n]}(\omega) - 1| > \epsilon\}) \\ &= \lim_{n \rightarrow \infty} m(\{|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| > \epsilon\}). \end{aligned}$$

Using the same reasoning as in part (a), for a fixed $\omega > 0$, consider $n \geq \lceil \frac{1}{\omega} \rceil + 1$; then $\frac{1}{n} < \omega$, and thus $\mathbf{1}_{[0, 1/n]}(\omega) = 0$. Therefore, choosing $n_0 = \lceil \frac{1}{\omega} \rceil + 1$ results in

$$\lim_{n \rightarrow \infty} m(\{|(n-1)\mathbf{1}_{[0, 1/n]}(\omega)| > \epsilon\}) = \lim_{n \rightarrow \infty} m(\{0 > \epsilon\}) = \lim_{n \rightarrow \infty} m(\emptyset) = 0.$$

Hence, $X_n \rightarrow X$ in probability.

- (d) To show $X_n \rightarrow X$ in L^p as $n \rightarrow \infty$ for $p \in (0, \infty)$, we must show $\int |X_n|^p dm < \infty$ for all $n \geq 1$, $\int |X|^p dm < \infty$, and

$$\lim_{n \rightarrow \infty} \int |X_n - X|^p dm = 0.$$

First,

$$\int |X|^p dm = \int_{[0,1]} 1^p dm = 1 < \infty.$$

Next, for $n \geq 2$ we have $X_n = n$ on $[0, 1/n]$ and $X_n = 1$ on $(1/n, 1]$, hence

$$\begin{aligned} \int_{[0,1]} |X_n|^p dm &= \int_{[0,1/n]} n^p dm + \int_{(1/n,1]} 1^p dm \\ &= n^p \cdot \frac{1}{n} + \left(1 - \frac{1}{n}\right) = n^{p-1} + 1 - \frac{1}{n} < \infty. \end{aligned}$$

Finally, since $X_n - X = (n-1)\mathbf{1}_{[0, 1/n]}$, we get

$$\int |X_n - X|^p dm = \int_{[0,1/n]} (n-1)^p dm = (n-1)^p \cdot \frac{1}{n} \sim n^{p-1}.$$

Therefore:

- If $0 < p < 1$, then $n^{p-1} \rightarrow 0$, so $\int |X_n - X|^p dm \rightarrow 0$ and $X_n \rightarrow X$ in the L^p .
- If $p = 1$, then $\int |X_n - X| dm = \frac{n-1}{n} \rightarrow 1 \neq 0$; no L^1 convergence.
- If $p > 1$, then $\int |X_n - X|^p dm \sim n^{p-1} \rightarrow \infty$; no L^p convergence.

Hence, $X_n \rightarrow X$ in L^p holds exactly for $0 < p < 1$, and fails for $p \geq 1$.

- (e) To show $\{X_n\}$ a uniformly integrable sequence of random variables, we have to show

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = 0$$

Fix t . Then

$$\{|X_n| > t\} = \begin{cases} [0, 1] & \text{if } t < 1 \\ [0, \frac{1}{n}] & \text{if } 1 \leq t \leq n \\ \emptyset & \text{if } n < t \end{cases}$$

Therefore, for a fixed t

$$\int_{\{|X_n| > t\}} |X_n| dm = \begin{cases} \int_{[0, \frac{1}{n}]} |X_n| dm + \int_{[\frac{1}{n}, 1]} |X_n| dm = 1 + (1 - \frac{1}{n}) = 2 - \frac{1}{n} & \text{if } t < 1 \\ \int_{[0, \frac{1}{n}]} |X_n| dm = 1 & \text{if } 1 \leq t \leq n \\ \int_{\emptyset} |X_n| dm = 0 & \text{if } n < t \end{cases}$$

Hence

$$\sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = \begin{cases} 2 & \text{if } t < 1 \\ 1 & \text{if } 1 \leq t \end{cases}$$

and so

$$\lim_{t \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{\{|X_n| > t\}} |X_n| dm = 1 \neq 0$$

Therefore, $\{X_n\}$ isn't uniformly integrable sequence of random variables.

Disclaimer

In question 36-b, the argument which

$$\mu(A) < \delta \implies \sup_{n \in \mathbb{N}} \int_A |f_n|^p d\mu < \gamma, \quad \mu(A) < \delta' \implies \int_A |f|^p d\mu < \gamma.$$

was achieved with the help of ChatGPT. Throughout the whole homework, I teamed up with Sreejit Roy on thinking about problems.

References

- [1] Krishna B Athreya and Soumendra N Lahiri. *Measure theory and probability theory*. Springer, 2006.