MATH 6410 Foundations of Probability Theory, Homework 2

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App. A: Problem 29

Let $\mathcal{S} = C[0,1]$ and define

$$d_p(f,g) \equiv \Big(\int_0^1 |f(t) - g(t)|^p \, dt \Big)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and

$$d_\infty(f,g) = \sup\{\, |f(t) - g(t)| : t \in [0,1] \,\}.$$

(a) Let $f(x) \equiv 1$. Define

$$f_n(t) \equiv \begin{cases} 1, & 0 \le t \le 1 - \dfrac{1}{n}, \\ n(1-t), & 1 - \dfrac{1}{n} \le t \le 1. \end{cases}$$

Show that $d_p(f_n, f) \to 0$ for $1 \le p < \infty$ but $d_\infty(f_n, f) \not\to 0$.

(b) Fix $f \in C[0,1]$. Let

$$g_n(t) = \begin{cases} f(t), & 0 \le t \le 1 - \dfrac{1}{n}, \\ f\big(1 - \frac{1}{n}\big) + \big(f(1) - f(1 - \frac{1}{n})\big)\, n\big(t + \frac{1}{n} - 1\big), & 1 - \dfrac{1}{n} \le t \le 1. \end{cases}$$

Show that $d_p(g_n, f) \to 0$ for all $1 \le p \le \infty$.

Answer

(a)
- To show that $d_p(f_n, f) \to 0$ for $1 \le p < \infty$, we must prove that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$,

  $$d_p(f_n, f) < \epsilon, \quad t \in [0,1].$$

  We first show that the choice of N only needs to guarantee the inequality on the interval $[\,1 - \frac{1}{N}, 1\,]$. For $t \in [0, 1 - \frac{1}{N}]$, any such choice of N will automatically satisfy the

inequality.
Suppose $t \in [0,\, 1-\frac{1}{N}]$. Then $f_n(t) = 1$, so

$$\begin{aligned}\left(\int_0^{1-\frac{1}{N}} |f_n(t) - f(t)|^p \, dt\right)^{1/p} &= \left(\int_0^{1-\frac{1}{N}} |1-1|^p \, dt\right)^{1/p} \\ &= \left(\int_0^{1-\frac{1}{N}} 0 \, dt\right)^{1/p} \\ &= (0)^{1/p} = 0.\end{aligned}$$

Now, consider $t \in [1-\frac{1}{n},\, 1]$ and let $\epsilon > 0$. To show that $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_p(f_n, f) < \epsilon,$$

it suffices to compute

$$\left(\int_{1-\frac{1}{n}}^{1} |f_n(t) - f(t)|^p \, dt\right)^{1/p} = \left(\int_{1-\frac{1}{n}}^{1} |n(1-t) - 1|^p \, dt\right)^{1/p}.$$

Set $s = n(1-t)$, so $ds = -n\, dt$ and $dt = -\frac{1}{n}\, ds$. When $t = 1 - \frac{1}{n}$, we have $s = 1$; when $t = 1$, we have $s = 0$. Thus,

$$\begin{aligned}\int_{1-\frac{1}{n}}^{1} |n(1-t) - 1|^p \, dt = \int_{s=1}^{0} |s-1|^p \, \frac{-ds}{n} &= \frac{1}{n} \int_0^1 (1-s)^p \, ds \\ = \frac{1}{n} \cdot \frac{1}{p+1} &= \frac{1}{n(p+1)}.\end{aligned}$$

Therefore,

$$\left(\int_{1-\frac{1}{n}}^{1} |f_n(t) - f(t)|^p \, dt\right)^{1/p} = \left(\tfrac{1}{n(p+1)}\right)^{1/p}.$$

Given $\epsilon > 0$, choose

$$N \in \mathbb{N} \;\text{ such that }\; N > \frac{1}{(p+1)\epsilon^{\,p}}.$$

Then for all $n \geq N$,

$$\left(\tfrac{1}{n(p+1)}\right)^{1/p} < \epsilon,$$

which proves $d_p(f_n, f) \to 0$.

- To show that $d_\infty(f_n, f) \not\to 0$, assume the contrary. Thus, let $\epsilon_0 > 0$. Then there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

  $$d_\infty(f_n, f) < \epsilon_0.$$

  In particular, we have

  $$\begin{aligned}& |d_\infty(f_n, f)| < \epsilon_0, \quad \forall n \geq N_0, \\ \Leftrightarrow\ & \sup\{\, |f_n(t) - f(t)| : t \in [0,1] \,\} < \epsilon_0, \quad \forall n \geq N_0, \\ \Rightarrow\ & \sup\{\, |f_n(t) - f(t)| : t \in [1-\tfrac{1}{n}, 1] \,\} < \epsilon_0, \quad \forall n \geq N_0, \\ \Rightarrow\ & \sup\{\, |n(1-t) - 1| : t \in [1-\tfrac{1}{n}, 1] \,\} < \epsilon_0, \quad \forall n \geq N_0.\end{aligned}$$

We will show that

$$\sup\{\, |n(1-t)-1| : t \in [1-\tfrac{1}{n}, 1] \,\} = 1, \quad \forall n \geq N_0.$$

To do so, we need to establish two facts: 1. $|n(1-t)-1| \leq 1$ for all $t \in [1-\frac{1}{n}, 1]$ and $n \geq N_0$, and 2. to show that 1 is the least upper bound, we must prove that for any $\delta > 0$, there exists $t_0 \in [1-\frac{1}{n}, 1]$ such that

$$|n(1-t_0)-1| > 1-\delta,$$

regardless of the choice of n.
Since $n(1-t)$ is continuous with respect to t, we evaluate it at the endpoints of the interval $[1-\frac{1}{n}, 1]$:
(a) At $t = 1-\frac{1}{n}$:

$$\begin{aligned} |n(1-t)-1| &= \left|n\big(1-(1-\tfrac{1}{n})\big)-1\right| \\ &= |1-1| = 0. \end{aligned}$$

(b) At $t = 1$:

$$|n(1-t)-1| = |n(1-1)-1| = |-1| = 1.$$

Hence,

$$|n(1-t)-1| \in [0,1], \quad t \in [1-\tfrac{1}{n}, 1].$$

By taking $t_0 = 1$, we obtain

$$|n(1-t_0)-1| = |n(1-1)-1| = 1,$$

which is greater than $1-\delta$ for any $\delta > 0$. Therefore,

$$\sup\{\, |n(1-t)-1| : t \in [1-\tfrac{1}{n}, 1] \,\} = 1, \quad \forall n \geq N_0.$$

Thus, $d_\infty(f_n, f) \to 1$, regardless of the choice of n, as long as $t \in [1-\frac{1}{n}, 1]$.

(b) We will consider $d_p(g_n, f)$ for $1 \leq p < \infty$ and $d_\infty(f_n, f)$ separately.

- To show $d_p(g_n, f) \to 0$ for $1 \leq p < \infty$, we first show that the choice of N only needs to guarantee the inequality on the interval $[\,1-\frac{1}{N},\, 1\,]$. For $t \in [0,\, 1-\frac{1}{N}]$, any such choice of N will automatically satisfy the inequality.
Suppose $t \in [0,\, 1-\frac{1}{N}]$. Then $g_n(t) = f_(t)$, so

$$\begin{aligned} \left(\int_0^{1-\frac{1}{N}} |g_n(t)-f(t)|^p\,dt\right)^{1/p} &= \left(\int_0^{1-\frac{1}{N}} |f(t)-f(t)|^p\,dt\right)^{1/p} \\ &= \left(\int_0^{1-\frac{1}{N}} 0\,dt\right)^{1/p} \\ &= (0)^{1/p} = 0. \end{aligned}$$

Now, consider $t \in [1-\frac{1}{n}, 1]$ and let $\epsilon > 0$. To show that $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_p(g_n, f) < \epsilon,$$

it suffices to compute

$$\left(\int_{1-\frac{1}{n}}^{1} |g_n(t) - f(t)|^p \, dt \right)^{1/p} = \left(\int_{1-\frac{1}{n}}^{1} \left| f(1-\tfrac{1}{n}) + \big(f(1) - f(1-\tfrac{1}{n})\big)\, n\big(t+\tfrac{1}{n}-1\big) - f(t) \right|^p dt \right)^{1/p}$$

Now, set

$$a := f\big(1 - \tfrac{1}{n}\big), \qquad b := f(1),$$

and define

$$s := n\Big(t + \tfrac{1}{n} - 1\Big) \in [0,1], \qquad t = 1 - \tfrac{1}{n} + \tfrac{s}{n}, \qquad dt = \tfrac{1}{n}\, ds.$$

Then

$$g_n(t) = (1-s)a + sb,$$

and let $x := 1 - \frac{1}{n} + \frac{s}{n}$, then:

$$\int_{1-\frac{1}{n}}^{1} |g_n(t) - f(t)|^p \, dt = \frac{1}{n} \int_0^1 \big|(1-s)a + sb - f(x)\big|^p \, ds.$$

Since $(1-s)$ and s are convex weights, we have

$$|(1-s)a + sb - f(x)| \leq (1-s)\,|a - f(x)| + s\,|b - f(x)|.$$

Moreover, because x lies within distance $\frac{1}{n}$ of both $1 - \frac{1}{n}$ and 1,

$$|a - f(x)| \leq \omega_f\Big(\tfrac{1}{n}\Big), \qquad |b - f(x)| \leq \omega_f\Big(\tfrac{1}{n}\Big),$$

where

$$\omega_f(\delta) := \sup_{|u-v| \leq \delta} |f(u) - f(v)|$$

is the modulus of continuity of f. Hence,

$$|(1-s)a + sb - f(x)| \;\leq\; \omega_f\Big(\tfrac{1}{n}\Big).$$

Plugging this into the integral gives

$$\int_{1-\frac{1}{n}}^{1} |g_n(t) - f(t)|^p \, dt \leq \frac{1}{n} \int_0^1 \Big(\omega_f\Big(\tfrac{1}{n}\Big)\Big)^p \, ds = \frac{1}{n} \Big(\omega_f\Big(\tfrac{1}{n}\Big)\Big)^p.$$

Therefore,

$$\left(\int_{1-\frac{1}{n}}^{1} |g_n(t) - f(t)|^p \, dt \right)^{1/p} \leq \frac{1}{n^{1/p}}\, \omega_f\Big(\tfrac{1}{n}\Big) \xrightarrow[n\to\infty]{} 0.$$

- To show that $d_\infty(g_n, f) \to 0$, consider first $t \in [0,\, 1 - \frac{1}{N}]$. For a given $\epsilon > 0$, we must show that there exists $N \in \mathbb{N}$ such that

$$d_\infty(g_n, f) = \sup\{\, |g_n(t) - f(t)| : t \in [0,\, 1 - \tfrac{1}{N}],\ n \geq N\,\} < \epsilon.$$

But on this interval we have $g_n(t) = f(t)$, so the condition is equivalent to

$$\sup\{\, |f(t) - f(t)| : t \in [0,\, 1 - \tfrac{1}{N}],\ n \geq N\,\} < \epsilon.$$

Clearly the supremum is zero, and thus the inequality is satisfied for any choice of N on $[0,\, 1 - \frac{1}{N}]$.
Now, consider $t \in [1 - \frac{1}{n},\, 1]$ and let $\epsilon > 0$. To show that $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_p(g_n, f) < \epsilon,$$

We have to show

$$\begin{aligned} &d_\infty(g_n, f) = \sup\{\, |g_n(t) - f(t)| : t \in [1 - \tfrac{1}{n},\, 1],\ n \geq N\,\} < \epsilon \\ \implies &\sup\{\, |f\big(1 - \tfrac{1}{n}\big) + \big(f(1) - f(1 - \tfrac{1}{n})\big)\, n\big(t + \tfrac{1}{n} - 1\big) - f(t)| : t \in [1 - \tfrac{1}{n},\, 1],\ n \geq N\,\} < \epsilon \end{aligned}$$

Note, $g_n(t)$ can be written as:

$$g_n(t) = f\big(1 - \tfrac{1}{n}\big) + \frac{t - (1 - \frac{1}{n})}{1 - (1 - \frac{1}{n})}\big(f(1) - f(1 - \tfrac{1}{n})\big)$$

Which represents the linear interpolation of the function $f(t)$ between two points of $f(1)$ and $f(1 - \frac{1}{n})$. For any $t \in [1 - \frac{1}{n}, 1]$, this gives the pointwise bound

$$|g_n(t) - f(t)| \leq \max\Big\{\, |f(1) - f(t)|,\ |f(1 - \tfrac{1}{n}) - f(t)|\,\Big\}.$$

Hence

$$\sup_{t \in [1 - \frac{1}{n}, 1]} |g_n(t) - f(t)| \ \leq\ \max\Big\{ \sup_{t \in [1 - \frac{1}{n}, 1]} |f(1) - f(t)|,\quad \sup_{t \in [1 - \frac{1}{n}, 1]} |f(1 - \tfrac{1}{n}) - f(t)|\,\Big\}.$$

Fix $\epsilon > 0$. By continuity of f at 1, there exists $\delta > 0$ such that $|t - 1| < \delta$ implies $|f(t) - f(1)| < \epsilon/2$. Choose N so large that $1/n < \delta$ for all $n \geq N$ and also $|f(1 - \frac{1}{n}) - f(1)| < \epsilon/2$ for all $n \geq N$. Then for every $n \geq N$ and every $t \in [1 - \frac{1}{n}, 1]$,

$$|f(1) - f(t)| < \tfrac{\epsilon}{2}, \qquad |f(1 - \tfrac{1}{n}) - f(t)| \leq |f(1 - \tfrac{1}{n}) - f(1)| + |f(1) - f(t)| < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

Therefore,

$$\sup_{t \in [1 - \frac{1}{n}, 1]} |g_n(t) - f(t)| < \epsilon \quad \text{for all } n \geq N,$$

which proves $d_p(g_n, f) \to 0$.

App. A: Problem 34

Show that unions of open sets are open and intersection of any two open sets is open. Give an example to show that the intersection of an infinite number of open sets need not be open.

Answer

Consider the open sets A_i. To show that $\bigcup_{i=1}^{\infty} A_i$ is open, we need to prove that for any $x \in \bigcup_{i=1}^{\infty} A_i$, there exists an open ball $B(x, \epsilon)$ contained in $\bigcup_{i=1}^{\infty} A_i$.

$$x \in \bigcup_{i=1}^{\infty} A_i \quad \Longrightarrow \quad \exists N \in \mathbb{N} \text{ such that } x \in A_N.$$

Since A_N is open, there exists $\epsilon > 0$ such that

$$B(x, \epsilon) \subseteq A_N.$$

But $A_N \subseteq \bigcup_{i=1}^{\infty} A_i$, hence

$$B(x, \epsilon) \subseteq \bigcup_{i=1}^{\infty} A_i.$$

Therefore, $\bigcup_{i=1}^{\infty} A_i$ is open.

Now suppose A_1 and A_2 are open. To show that $A_1 \cap A_2$ is open, we must prove that for any $x \in A_1 \cap A_2$, there exists an open ball around x that is contained entirely in $A_1 \cap A_2$.

Let $x \in A_1 \cap A_2$. Then $x \in A_1$ and $x \in A_2$. Since A_1 is open, there exists $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq A_1$. Similarly, since A_2 is open, there exists $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq A_2$.

Now set $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Then

$$B(x, \epsilon) \subseteq B(x, \epsilon_1) \cap B(x, \epsilon_2).$$

Hence

$$B(x, \epsilon) \subseteq A_1 \cap A_2.$$

Therefore, $A_1 \cap A_2$ is open.

Finally, consider the sets

$$A_n = \left(1 - \tfrac{1}{n},\, 1 + \tfrac{1}{n}\right).$$

We claim that

$$\bigcap_{n=1}^{\infty} A_n = \{1\}.$$

Since $\{1\}$ is a singleton and therefore closed, this gives an example where the intersection of infinitely many open sets need not be open.

Proof. Let $x \neq 1$. Suppose for contradiction that $x \in \bigcap_{n=1}^{\infty} A_n$. Then $x \in A_1 = (0, 2)$, so either $0 < x < 1$ or $1 < x < 2$.

1. $0 < x < 1$. Choose

   $$n = \left\lceil \tfrac{1}{1-x} \right\rceil.$$

   Then $n > \frac{1}{1-x}$, which implies $\frac{1}{n} < 1 - x$. Hence

   $$1 - \tfrac{1}{n} > x.$$

   Thus $x \notin A_n$.

2. $1 < x < 2$. Choose
$$n = \left\lceil \tfrac{1}{x-1} \right\rceil.$$
Then $n > \frac{1}{x-1}$, which implies $\frac{1}{n} < x - 1$. Hence
$$1 + \tfrac{1}{n} < x.$$
Thus $x \notin A_n$.

In either case, $x \notin A_n$ for some n, so $x \notin \bigcap_{n=1}^{\infty} A_n$.

Therefore the only point in the intersection is $x = 1$, and so
$$\bigcap_{n=1}^{\infty} A_n = \{1\}.$$

□

App. A: Problem 36

Let $f_n(x) = x^n$ and $g(x) \equiv 0$ on $\mathbb{R}$. Then $\{f_n\}_{n\geq 1}$ converges pointwise to g on $(-1, 1)$, uniformly on $[a, b]$ for $-1 < a < b < 1$, but not uniformly on $(0, 1)$.

Answer

To show $\{f_n\}_{n\geq 1}$ converges pointwise to g on $(-1, 1)$, we must show
$$g(x) = \lim_{n\to\infty} f_n(x), \quad x \in (-1, 1).$$
Let $\epsilon > 0$ and fix $x \in (-1, 1)$.

If $x = 0$, choose any $N \in \mathbb{N}$. Then, for all $n \geq N$,
$$|f_n(x) - g(x)| = |0 - 0| = 0 < \epsilon.$$

If $x \neq 0$, choose $N > \dfrac{\ln(\epsilon)}{\ln(|x|)}$ (By the Archimedean property we know such N must exist). Then, for all $n \geq N$,
$$|f_n(x) - g(x)| = |x^n - 0| = |x|^n.$$
To ensure $|x|^n < \epsilon$, note that for $0 < |x| < 1$,
$$\begin{aligned} |x|^n < \epsilon &\implies \ln(|x|^n) = n\ln(|x|) < \ln(\epsilon) \\ &\iff n > \frac{\ln(\epsilon)}{\ln(|x|)}, \quad \text{since } \ln(|x|) < 0. \end{aligned}$$

To show $\{f_n\}_{n\geq 1}$ converges *uniformly* to g on $[a, b]$ (with $-1 < a < b < 1$), we must show:
$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ \forall x \in [a, b],\ |f_n(x) - g(x)| < \varepsilon.$$

Let $\varepsilon > 0$. We have previously shown that if $x = 0$, then $f_n(x) \equiv g(x)$ for all n, so it suffices to consider $x \in [a, b]$ with $x \neq 0$.
For $x \neq 0$,

$$|f_n(x) - g(x)| = |x^n - 0| = |x|^n.$$

To ensure $|x|^n < \varepsilon$, note that for $0 < |x| < 1$,

$$|x|^n < \varepsilon \iff n\ln(|x|) < \ln(\varepsilon) \iff n > \frac{\ln(\varepsilon)}{\ln(|x|)},$$

since $\ln(|x|) < 0$.
If $\varepsilon \geq 1$, then because $|x|^n < 1 \leq \varepsilon$, any $N \in \mathbb{N}$ works. Thus it suffices to handle $\varepsilon \in (0, 1)$. In that case $\ln(\varepsilon) < 0$. Let

$$M := \max\{|a|, |b|\} \in (0, 1).$$

Then for all $x \in [a, b]$, $|x| \leq M$, hence $\ln(|x|) \leq \ln(M) < 0$. With $\ln(\varepsilon) < 0$, this gives

$$\frac{\ln(\varepsilon)}{\ln(|x|)} \leq \frac{\ln(\varepsilon)}{\ln(M)}.$$

Therefore, choosing

$$N := \left\lceil \frac{\ln(\varepsilon)}{\ln(M)} \right\rceil + 1$$

ensures $N > \dfrac{\ln(\varepsilon)}{\ln(M)} \geq \dfrac{\ln(\varepsilon)}{\ln(|x|)}$ for all $x \in [a, b]$, and hence $|x|^n < \varepsilon$ for all $n \geq N$ and all $x \in [a, b]$.
Thus, for every $\varepsilon > 0$ there exists N such that $\sup_{x\in[a,b]} |f_n(x) - g(x)| < \varepsilon$ for all $n \geq N$, i.e., $f_n \to g$ uniformly on $[a, b]$.
To see that the convergence is not uniform on $(0, 1)$, assume the contrary. Then

$$\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ \forall\, x \in (0, 1),\ |f_n(x) - g(x)| < \varepsilon.$$

Let $0 < \varepsilon_0 < 1$. By the assumption of uniform convergence,

$$\exists\, N \in \mathbb{N} \text{ such that } \forall\, n \geq N,\ \forall\, x \in (0, 1),\ |f_n(x)| < \varepsilon_0,$$

i.e.,

$$\forall\, n \geq N,\ \forall\, x \in (0, 1),\ |x|^n < \varepsilon_0.$$

Taking logs (valid since $0 < |x| < 1$ and $\varepsilon_0 \in (0, 1)$),

$$\forall\, n \geq N,\ \forall\, x \in (0, 1),\ n\ln|x| < \ln\varepsilon_0 \quad \iff \quad \ln|x| < \frac{\ln\varepsilon_0}{n} \leq \frac{\ln\varepsilon_0}{N}.$$

Because $\ln\varepsilon_0 < 0$ and ln maps $(0, 1)$ onto $(-\infty, 0)$ continuously and monotonically, we can choose

$$r \in \left(\tfrac{\ln\varepsilon_0}{N}, 0\right) \quad \text{and set} \quad x_0 := e^{\,r} \in (0, 1).$$

Then $\ln x_0 = r > \frac{\ln\varepsilon_0}{N}$, so for $n = N$ we have

$$N\ln x_0 > \ln\varepsilon_0 \quad \Longrightarrow \quad x_0^N > \varepsilon_0,$$

which contradicts the assumed bound for $n = N$ and $x = x_0$. Hence the convergence is not uniform on $(0, 1)$.

Ch. 1: Problem 2

Let Ω be a finite set, i.e., the number of elements in Ω is finite. Let $\mathcal{F} \subset \mathcal{P}(\Omega)$ be an algebra. Show that $\mathcal{F}$ is a σ-algebra.

Answer.

Since $|\Omega| < \infty$, $|\mathcal{P}(\Omega)| < \infty$. Let $|\mathcal{F}| = N \in \mathbb{N}$. Enumerate the elements of $\mathcal{F}$ by $A_i \in \mathcal{F}$, $1 \leq i \leq N$. Define

$$B_1 := A_1, \qquad B_n := B_{n-1} \cup A_n \quad (2 \leq n \leq N).$$

Since $\mathcal{F}$ is an algebra, $B_1 \in \mathcal{F}$. Assuming $B_k \in \mathcal{F}$ for some $1 \leq k < N$, we have $A_{k+1} \in \mathcal{F}$, and hence $B_{k+1} = B_k \cup A_{k+1} \in \mathcal{F}$. Setting $k = N - 1$ gives $B_N \in \mathcal{F}$, i.e.,

$$((((A_1 \cup A_2) \cup A_3) \cdots) \cup A_N) \in \mathcal{F},$$

equivalently,

$$\bigcup_{i=1}^{N} A_i \in \mathcal{F}.$$

Therefore $\mathcal{F}$ is a σ-algebra.

Ch. 1: Problem 4

Let $\{\mathcal{F}_\theta : \theta \in \Theta\}$ be a family of σ-algebras on Ω. Show that $\mathcal{G} \equiv \bigcap_{\theta \in \Theta} \mathcal{F}_\theta$ is also a σ-algebra.

Answer.

To show that $\mathcal{G}$ is a σ-algebra, we must verify:

(i) $\Omega \in \mathcal{G}$.

(ii) $A \in \mathcal{G} \implies A^c \in \mathcal{G}$.

(iii) If $A_n \in \mathcal{G}$ for $n \geq 1$, then $\displaystyle\bigcup_{n \geq 1} A_n \in \mathcal{G}$.

Proof. (i) Since each $\mathcal{F}_\theta$ is a σ-algebra, $\Omega \in \mathcal{F}_\theta$ for all $\theta \in \Theta$. Hence $\Omega \in \bigcap_{\theta \in \Theta} \mathcal{F}_\theta \equiv \mathcal{G}$.

(ii) Suppose $A \in \mathcal{G}$. Then $A \in \mathcal{F}_\theta$ for all $\theta \in \Theta$. Since each $\mathcal{F}_\theta$ is a σ-algebra, $A^c \in \mathcal{F}_\theta$ for all θ, hence $A^c \in \bigcap_{\theta \in \Theta} \mathcal{F}_\theta \equiv \mathcal{G}$.

(iii) Define $U_N := \bigcup_{n=1}^{N} A_n$. By induction on N, using that $\mathcal{G}$ is closed under finite unions, we have $U_N \in \mathcal{G}$ for all $N \geq 1$. Then for each $\theta \in \Theta$, $U_N \in \mathcal{F}_\theta$ for all N, and since $\mathcal{F}_\theta$ is a σ-algebra,

$$\bigcup_{n \geq 1} A_n \;=\; \bigcup_{N \geq 1} U_N \;\in\; \mathcal{F}_\theta.$$

Thus $\bigcup_{n \geq 1} A_n \in \bigcap_{\theta \in \Theta} \mathcal{F}_\theta \equiv \mathcal{G}$.

$\square$

Ch. 1: Problem 6

Let Ω be a nonempty set and let $\mathcal{A} \equiv \{A_i : i \in \mathbb{N}\}$ be a partition of Ω, i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $\bigcup_{n\geq 1} A_n = \Omega$. Let $\mathcal{F} = \{\bigcup_{i\in J} A_i : J \subset \mathbb{N}\}$ where, for $J = \emptyset$, $\bigcup_{i\in J} A_i = \emptyset$. Show that $\mathcal{F}$ is a σ-algebra.

Answer

In this question, we will assume $J \subseteq \mathbb{N}$.
To show that $\mathcal{F}$ is a σ-algebra, we must verify:

(i) $\Omega \in \mathcal{F}$.

(ii) $B \in \mathcal{F} \implies B^c \in \mathcal{F}$.

(iii) If $B_n \in \mathcal{F}$ for $n \geq 1$, then $\displaystyle\bigcup_{n\geq 1} B_n \in \mathcal{F}$.

Proof. (i) Since $\Omega = \bigcup_{n\geq 1} A_n$, taking $J = \mathbb{N}$ gives $\Omega = \bigcup_{i\in J} A_i \in \mathcal{F}$.

(ii) Let $B = \bigcup_{i\in J} A_i \in \mathcal{F}$ with $J \subseteq \mathbb{N}$. Set $J^c := \mathbb{N} \setminus J$. Because $\{A_i\}$ partitions Ω,

$$B^c \;=\; \Omega \setminus \bigcup_{i\in J} A_i \;=\; \bigcup_{i\in J^c} A_i \in \mathcal{F}.$$

(iii) Suppose $J = \mathbb{N}$. Then, considering each $B_n \in \mathcal{F}$ as $B_n = \bigcup_{i\in J_n} A_i$, we have

$$\bigcup_{n\geq 1} B_n = \bigcup_{n\geq 1}\left(\bigcup_{i\in J_n} A_i\right) \subseteq \bigcup_{n\geq 1}\left(\bigcup_{i\in J} A_i\right) = \bigcup_{i\in J} A_i = \Omega \in \mathcal{F}.$$

□

Problem A

Let Ω be an uncountable set (e.g., $\mathbb{R}$) and let $\mathcal{C}$ be the collection of all singleton sets, i.e.

$$\mathcal{C} = \big\{\{\omega\} : \omega \in \Omega\big\}.$$

Find the smallest σ-algebra generated by $\mathcal{C}$. Justify your answer.

Answer

The following set with respect to Ω is $\sigma\langle\mathcal{C}\rangle$:

$$\mathcal{F} = \{\, A \subset \Omega \mid A \text{ is countable or } A^c \text{ is co-countable}\,\}.$$

To show $\sigma\langle\mathcal{C}\rangle = \mathcal{F}$, we prove both inclusions: $\sigma\langle\mathcal{C}\rangle \subset \mathcal{F}$ and $\mathcal{F} \subset \sigma\langle\mathcal{C}\rangle$.

1. $\sigma\langle\mathcal{C}\rangle \subset \mathcal{F}$. Since each singleton set is countable, it follows that $\mathcal{C} \subset \mathcal{F}$. Moreover, we have previously shown that $\mathcal{F}$ is a σ-algebra (In class, example 1.1.4). Therefore, the smallest σ-algebra generated by $\mathcal{C}$, namely $\sigma\langle\mathcal{C}\rangle$, must be contained in $\mathcal{F}$.

2. $\mathcal{F} \subset \sigma\langle\mathcal{C}\rangle$. Let $A \in \mathcal{F}$. Then either A is countable or A^c is countable, leaving two cases:

   (a) If A is countable, we can write A as a countable union of singletons:

   $$A = \bigcup_{n \geq 1} \{a_n\}.$$

   Since $\sigma\langle\mathcal{C}\rangle$ contains $\mathcal{C}$ and is closed under countable unions, it follows that $A \in \sigma\langle\mathcal{C}\rangle$.

   (b) If A^c is countable, then by the same reasoning $A^c \in \sigma\langle\mathcal{C}\rangle$. Since $\sigma\langle\mathcal{C}\rangle$ is closed under complementation, we conclude that $A = (A^c)^c \in \sigma\langle\mathcal{C}\rangle$.

Disclaimer

I received assistance from ChatGPT (OpenAI's GPT-5 model) in polishing the exposition and organizing parts of the solution to Question 1 in LaTeX. The mathematical reasoning and core steps of the proof are my own.