Verification of Equilibrium Point Stability for Linearization of an Aircraft Model

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***Abstract*-Unmanned Air Vehicles (UAVs) are of vital importance in future aviation industry. A UAV is flown by an onboard controller which controls aircraft movements during flight. For convenient design of linear controllers, a stable operating point (equilibrium point) is required against which the aircraft model can be linearized. This paper aims at developing a technique for stability verification of such an operating point. For a nonlinear mathematical model of a UAV, its equilibrium point has been determined using an optimization scheme. The non-linear model is then linearized using Jacobian based numerical linearization technique. Due to insignificant values of coupling terms, the aircraft linear model is decoupled into longitudinal and lateral sub-models. The response of nonlinear model is compared to that of linearized sub-models by applying identical inputs to the both model types. Simulation results for both the cases are presented. It is concluded that this work presents a convenient technique to verify the stability of equilibrium point, leading to successful controller design.**

Keywords- Equilibrium point; Stability; Linearization; UAV; Flight controller

I. Introduction

For the future of aviation being unmanned, aerospace vehicles are leading the race in shifting controls from manned to autonomous. Unmanned Air Vehicles (UAVs) have huge demand when it comes to applications ranging from aerial inspection, target tracking or fighting war against terrorism [5]. Autonomously controlled aerial vehicles have many applications in the civil sector as well. These include weather forecasting, relief activities, area mapping, earth surveys and many more. UAVs not only gave a breakthrough against the barrier of human pilot limitations but also have eliminated the pilot risk. These are the first option when the missions have to be carried out in a sharp terrain or onto a dangerous foe.

UAVs are more prone to failures because of their default characteristic of being pilotless air vehicles. By 1998, the peace time attrition rate for the Pioneer UAV was 17 times greater than that for manned aircraft [1]. By early 2002, out of 65 Predators which had been built, 35% had crashed. These 23 downed predators included 14 which failed due to mechanical problems or human error [8]. At the same time there are a high number of cases where the real cause of failure is unknown. Having many advantages over the conventional aircraft, the overall reliability of UAV is still low and with such high failure rate, the usage of UAVs for operations can be limited [14]. This increased failure rate can be reduced to a large extent by developing a more reliable control system.

Before a UAV takes its physical form, a great amount of time and effort is invested on its modeling and computer based simulations. The success or failure of the final product is directly related to the fidelity level of the simulation. The dynamic modeling is the core of simulations. Control systems to be applied on a UAV are mostly developed through the same system's model [6]. This mathematical model of UAV is composed of various sub models, primarily including Aerodynamic model, Inertia model and Propulsion model. The response of UAV to any input is dependent upon these sub models which then give an inter-related effect.

Output of UAV to various commands or any external disturbance (e-g wind gust) can be simulated through its modeling using differential equations (Equations of Motion) [3]. However since the complete UAV dynamic model is a complex system consisting of fully coupled non-linear equation of motions, it is difficult to directly use the same model for the analysis and design purpose or to develop controllers for autonomous flight. Therefore a linear model is used which is conceived around an operating point. This operating point can be a stable equilibrium point or an unstable one. Also the sensitivity of the linear model controller depends upon the operating point. If the model is linearized around an unstable operating point, then a small perturbation can lead to a nonlinear uncontrolled response [13]. An extraordinary input will be required to bring the model back to its stable condition. Thus, in order to linearize the model, a stable operating point is required .Our work presents a technique to verify the stability of a UAV's operating point, often called the trim point [9].

II. Platform UAV - Aerosonde

Aerosonde is a small sized fixed wing UAV originally designed for meteorological purposes over remote areas and oceans. It was the first UAV to cross Atlantic Ocean [12] and is now being used by the researchers due to ease of availability of its mathematical model.

Mathematical model for Aersonde is a standard 6 DOF model used for fixed wing aircraft that is built upon nonlinear 6 DOF body axis equations. "Aerosim" blockset for Matlab/Simulink is used in order to simulate this model [11]. The model is further made up of different sub models. The complete model is shown in figure (1).

The submodels are explained as following:

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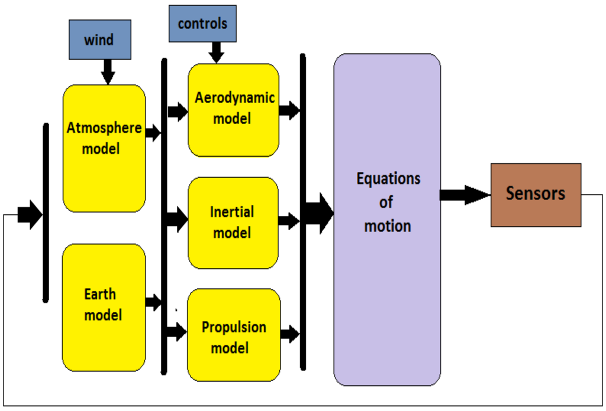

Figure 1. Nonlinear Modeling Architecture

A. *Aerodynamic Model*

Aerodynamic sub-model requires aerodynamic forces and moments for control surface loading/deflections. The aerodynamic forces are lift, drag and side forces in wind axis which are further converted into body axes forces using direction cosine matrix. The aerodynamic moments are roll, pitch and yaw moments. These forces and moments are then utilized for calculating total moment and total acceleration. The aerodynamic model is described in (1).

$$\begin{aligned} &Drag: D = qSC_D \\ &Lift: L = qSC_L \\ &Side\ Force: C = qSC_s \\ &Roll\ moment: l_w = qSbC_l \\ &Pitch\ moment: m_w = qScC_m \\ &Yaw\ moment: n_w = qSbC_n \end{aligned} \tag{1}$$

Where all the parameters are defined in the list of symbols given in the appendix .

B. *Propulsion Model*

Propulsion model is built to simulate the aircraft's engine and propeller. The propeller used here is a fixed pitch propeller. This sub-model provides propeller thrust force and torque. The model is given in (2).

$$\begin{aligned} F_p &= \frac{4}{p^2} rR^4\omega^2 C_T \\ M_p &= -\frac{4}{p^3} rR^5\omega^2 C_P \end{aligned} \tag{2}$$

Where

$$Thrust\ Coefficient \quad C_T = \frac{T}{r n^2 D^4}$$

$$Power\ Coefficient \quad Cp = \frac{P}{r n^3 D^5}$$

$$Advance\ Power\ ratio\ \frac{v}{nD} = \frac{v}{nD}$$

$$Propeller\ Efficiency\ \eta = \frac{v}{nD}\frac{C_T}{Cp}$$

The piston engine model uses known engine parameters like fuel flow and engine power as functions of altitude. The outputs of this model are instantaneous airflow, fuel flow, fuel consumption, MAP (manifold air pressure) for current throttle setting and current altitude, engine power and torque generated due to engine shaft.

C. *Atmospheric Model*

Atmospheric model simulates the external environmental conditions and is used to model the various operating altitudes of the aircraft and study its effects on the aircraft performance due to varying density and pressure of air. It provides the air parameters (static pressure, temperature and density) at current altitude of the aircraft. Turbulence sub-model utilizes Von Kerman [3] turbulence model and provides wind turbulence velocity and acceleration in body axis. Wind-Shear model caters for the effects of wind on the aircraft (moments produced in aircraft due to wind). The model is given in (3)

$$\begin{aligned} q_{wind} &= \frac{1}{u_{aircraft}} \cdot \frac{d_{w_{wind}}}{dt} \\ r_{wind} &= \frac{1}{u_{aircraft}} \cdot \frac{d_{v_{wind}}}{dt} \end{aligned} \tag{3}$$

Wind Forces and Moments block models the forces and moments applied on the aircraft body due to wind. Forces along the 3 axes are given in (4).

$$\begin{aligned} F_{x_{wind}} &= -m_{aircraft} \cdot (\dot{u}_{wind} + q_{w_{wind}} - rv_{wind}) \\ F_{y_{wind}} &= -m_{aircraft} \cdot (\dot{v}_{wind} - pw_{wind} + ru_{wind}) \\ F_{z_{wind}} &= -m_{aircraft} \cdot (\dot{w}_{wind} + pw_{wind} - qu_{wind}) \end{aligned} \tag{4}$$

Moments around the 3 axes are given in (5).

$$\begin{aligned} M_{x_{wind}} &= \frac{1}{c_4{}^2 - c_3 c_9} . (c_9 A - c_4 C) \\ M_{y_{wind}} &= -\frac{1}{c_7} . B \\ M_{z_{wind}} &= \frac{1}{c_4{}^2 - c_3 c_9} . (c_4 A - c_3 C) \end{aligned} \tag{5}$$

Where, A, B and C are:-

$$\begin{aligned} A &= \dot{p}_{wind} - (c_1 r_{wind} + c_2 p_{wind}) \cdot q_{wind} \\ B &= \dot{q}_{wind} - c_5 p_{wind} r_{wind} + c_6 (p_{wind}{}^2 - r_{wind}{}^2) \\ C &= \dot{r}_{wind} - (c_8 p_{wind} - c_2 r_{wind}) \cdot q_{wind} \end{aligned}$$

D. Inertial Model

Inertial model is used to specify the aircraft CG (center of gravity), mass, moments of inertia and products of inertia considering the fuel quantity and aircraft mass. The CG location is a function of fuel quantity as per its consumption during the flight.

E. Earth Gravitational Model

Earth model simulates performance of the UAV in earth's environment. It uses the mathematical models WGS-84, EGM-96 and WMM-2000 to calculate earth radius and gravity, altitude from sea-level and earth magnetic field components at current aircraft position. WGS-84 model calculates local earth radius and the gravity at current aircraft position. The model is given in (6)

$$
\begin{aligned}
R_{meridian} &= \frac{r_e(1-e^2)}{\sqrt[3]{(1-e^2\sin^2 q)}} \\
R_{normal} &= \frac{r_e}{\sqrt{(1-e^2\sin^2 f)}} \\
R_{equiv} &= \sqrt{R_{meridian}R_{normal}} \\
g &= g_{wgs_0}\frac{1+g_{wgs_1}\sin^2 f}{(1-e^2\sin^2 f)^{\frac{1}{2}}}
\end{aligned}
\tag{6}
$$

where WGS-84 coefficients are:

Equatorial radius r_e = 6378137 m

First eccentricity e = 0.081819

Gravity at equator g_{wgs_0} = 9.78 m/s^2

Gravity formula constant g_{wgs_1} = 0.00193

ϕ is latitude at current aircraft position

EGM-96 model calculates the aircraft altitude above mean sea level using EGM-96 undulation model. Ground Detection block computes aircraft altitude above ground level and sets a flag if this altitude is zero. WMM-2000 model uses U.S Department of Defense world magnetic model 2000 and computes the earth magnetic field component at the current location of the aircraft.

F. 6-DOF Body Axis Equations-of-Motion

These equations compute total accelerations, total moments, inertia and parameters related to aircraft position. 6 DOF nonlinear equations in body-axis are presented below.

1) Force Equations: Force equations require angular rates, body axis velocities and a gravity vector in body frame. Generic form of the force equations are shown in (7).

$$
\begin{aligned}
m\dot{U} &= RV - QW - g_D\sin\theta + (X_A + X_T) \\
m\dot{V} &= -RU + PW + g_D\sin\phi\cos\theta + (Y_A + Y_T) \\
m\dot{W} &= QU - PV + g_D\cos\phi\cos\theta + (Z_A + Z_T)
\end{aligned}
\tag{7}
$$

Where, m is total mass of the aircraft, U V and W are velocities in x y z direction respectively, P Q R are angular rates, $g_D$is gravity constant, θ and ϕ are pitch and roll angle respectively, $X_A Y_A$ Z_A are sum of aerodynamic forces in x y and z direction respectively and $X_T Y_T$ Z_T are thrust force in x y and z direction respectively.

2) Moment Equations: Moment equations give net angular moments around the three axes. The generic form of moment equation is given in (8).

$$
\begin{aligned}
&\Gamma\dot{P} = J_{xz}[J_x - J_y - J_z]PQ - [J_z - (J_z - J_y) + J^2{}_{xz}]QR + J_z l + J_{xz} n \\
&J_y\dot{Q} = (J_z - J_x)PR - J_{xz}(P^2 - R^2) + m \\
&\Gamma\dot{R} = [(J_x - J_y)J_x + J^2{}_{xz}]PQ - J_{xz}[J_x - J_y - J_z]QR + J_{xz} l + J_x n \\
&where \\
&\Gamma = J_x J_z - J^2{}_{xz}
\end{aligned}
\tag{8}
$$

3) Kinematic Equations: The attitude of the aircraft can be determined using the angular rates P, Q and R as computed from the above equations. P, Q and R indicate the roll rate, pitch rate and yaw rate respectively. The equations are given in (9).

$$
\begin{aligned}
\dot{\phi} &= P + \tan(\theta)(Q\sin(\phi) + R\cos(\phi)) \\
\dot{\theta} &= Q\cos(\phi) - R\sin(\phi) \\
\dot{\Psi} &= (Q\sin(\phi) + R\cos(\phi))/\cos(\theta)
\end{aligned}
\tag{9}
$$

4) Navigation Equations. Navigation equations use body axis velocities computed from the force equations. The generic form of these equations is given in (10).

$$
\begin{aligned}
\dot{P}_N &= Uc\theta c\psi + V(-c\phi s\psi + s\phi s\theta c\psi) + W(s\phi s\psi + c\phi s\theta c\psi) \\
\dot{P}_E &= Uc\theta s\psi + V(c\phi c\psi + s\phi s\theta s\psi) + W(s\phi c\psi + c\phi s\theta c\psi) \\
\dot{h} &= Us\theta - Vs\phi c\theta - Wc\phi c\theta
\end{aligned}
\tag{10}
$$

III. DETERMINATION OF TRIM POINT

Operating point or in case of an aircraft, the trim point, plays a vital role in UAV's linearization. Trim point is defined as "The control settings for which net forces and the moments around the center of gravity are all equal to zero, the aircraft is said to be flying in trim state, that simply is the state of static equilibrium"[4]. It is the operating point at which algebraic sum of all forces and thus resulting torques acting on aircraft is equal to zero or the states' derivatives (in the state space) are zero and UAV maintains a steady flight. This steady flight can be straight and level flight or a steady turn generally termed as coordinated turn where the altitude loss is avoided. In this work, trim conditions for straight and level flight have been calculated using the Nelder-Mead simplex (direct search) optimization algorithm . For a set of given initial flight conditions, a cost function is generated in which high weightage is assigned to the states significantly affecting the longitudinal stability of UAV, e.g. the linear velocities and angular rates. Rest all states were assigned low weightages. After applying the operating limits, trim conditions are calculated from Matlab. At a velocity of approx 45 knots and an altitude of 3300 ft, the following trimming results were computed.

Inputs
Aileron (δa): -0.492 deg
Elevator(δe): -8.191 deg
Rudder(δr): -0.057 deg
Throttle(δt):0.5698 (56.98%)

States
Velocity in x-axis (u): 22.93 m/s
Velocity in y-axis (v): 0.01 m/s
Velocity in z-axis (w): 1.73 m/s
Roll Rate (p): 0.00 deg/s
Pitch Rate (q): 0.00 deg/s
Yaw Rate (r): 0.00 deg/s
Engine RPM: 4835RPM (≈70% RPM)

IV. Linearization Around Trim Point

For a system to be linear it should hold true the property of superposition- that includes additivity and homogeneity. Most of the physical systems are nonlinear in nature. An aircraft model is a nonlinear system and linearization is required for designing of a linear controller. According to small perturbation theory [5], if a linear system is asymptotically stable, its stability persists from linear to non-linear domain. The non-linear model can be linearized numerically or analytically using Jacobian method[6]. The model is first expressed in the form of implicit state equation:

$$f(x,\dot{x},u)=0$$

Where u is the input vector and x is the state vector of system with $\dot{x}$ as its derivative. The system is to be linearized about the trim conditions expressed by $[u_0, x_0]$. In equation form the linear model can be written as:

$$\dot{x}=Ax+Bu$$

Where

$$A=\begin{pmatrix} \nabla_x f_1 \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \nabla_x f_{12} \end{pmatrix}_{\substack{U=Uo \\ X=Xo}} \qquad B=\begin{pmatrix} \nabla_U f_1 \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ \nabla_U f_{12} \end{pmatrix}_{\substack{U=Uo \\ X=Xo}}$$

A= System matrix B= Input matrix

and

$$\nabla_x f_i \equiv \left[\frac{\partial f_i}{\partial x_1} \frac{\partial f_i}{\partial x_2} \ldots\ldots \frac{\partial f_i}{\partial x_n} \right]$$

f_i for i=1,2,3......,12 are the 12 equations of motion.

$\partial$is partial derivative

Aerosim blockset was used to linearize Aerosonde model. As pointed out, for a stable trim point there will be minimum coupling between longitudinal and lateral dynamics. Thus the cross-diagonal terms of the system matrix 'A' contained negligible terms [4], hence the model was easily decoupled to extract longitudinal and lateral sub-models to be analyzed separately for their respective stability. Analysis of the model showed that 'u', the velocity along body x-axis, 'q' pitch rate,'θ' pitch angle and 'h' the altitude played a major role in affecting the UAVs longitudinal motion. The longitudinal model extracted from the linear models is given in (11):

$$\begin{bmatrix} \dot{u} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} Xu & Xw & -W_0 & -g_0\cos q_0 \\ Zu & Zw & U_0 & -g_0\sin q_0 \\ Mu & Mw & Mq & 0 \\ 0 & 0 & M_q & 1 \end{bmatrix} \begin{bmatrix} u \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} X_{de} & X_{dt} \\ Z_{de} & Z_{dt} \\ M_{de} & M_{dt} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta e \\ \delta t \end{bmatrix} \qquad (11)$$

In this case matrix [u q θ h]' is the decoupled longitudinal state vector. As for inputs, δe is elevator deflection angle in radians and δt is the throttle input in percentage advance. Similarly, the state space representation of lateral sub-model is given in (12):

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} Y_v & -U_0 & V_0 & -g_0\cos q \\ N_v & N_r & N_p & 0 \\ L_v & L_r & L_p & 0 \\ 0 & \tan q_0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ \phi \\ \psi \end{bmatrix} + \begin{bmatrix} Y_{da} & Y_{dr} \\ N_{da} & N_{dr} \\ L_{da} & L_{dr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \qquad (12)$$

The decoupled lateral state vector is [β p φψ]' where β is the sideslip angle, p is the roll rate, ψ is the heading angle and φ is the bank angle. As for inputs, δa is aileron and δr is rudder deflection in radians. For both the longitudinal and lateral sub-models, the state matrix comprises of the so called stability derivatives and input matrix houses the control derivatives. The concise notation used here is taken from [9].

V. Simulation And Results

After the decoupled linear models have been acquired, open loop simulation can be performed. In order to perform the analysis and verify the results, step input was applied to both linear and nonlinear models with same initial conditions. Figure (2) to figure (5) show the response from longitudinal model. A unit step deflection of elevator was applied as the input and pitch rate step response is recorded in figure (2), step response for pitch angle is given in figure (3) and step response for altitude is shown in figure (4). For unit step input of throttle the step response for airspeed is presented in figure (5). Similarly, unit step input was applied to the lateral model, and the response was recorded in terms of roll rate, roll angle and yaw angle. Against a unit step deflection of aileron, step response for roll rate is given in figure (6) and step response for roll angle is given in figure (7). Finally, step response for yaw angle is given in figure (8) once applied with a unit step input deflection at rudder.

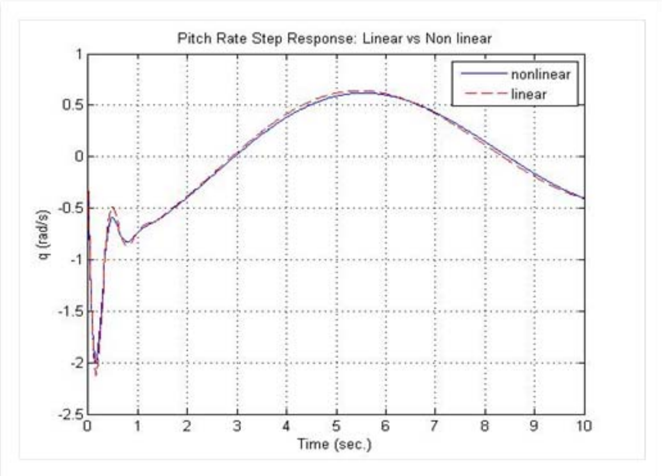

Figure 2. Step Responses for Pitch Rate

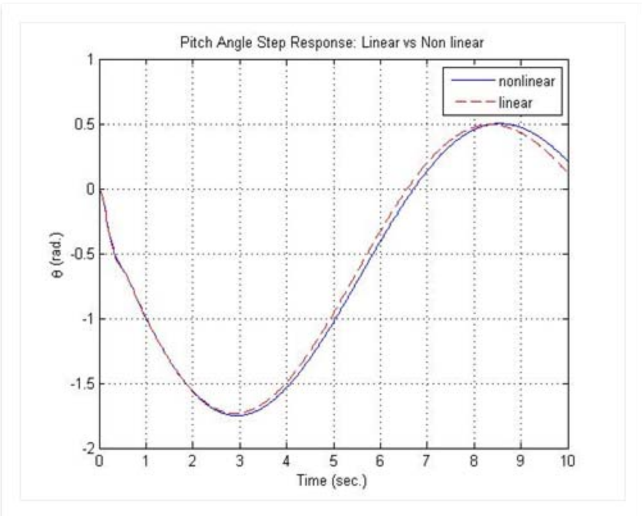

Figure 3. Step Responses for Pitch Angle

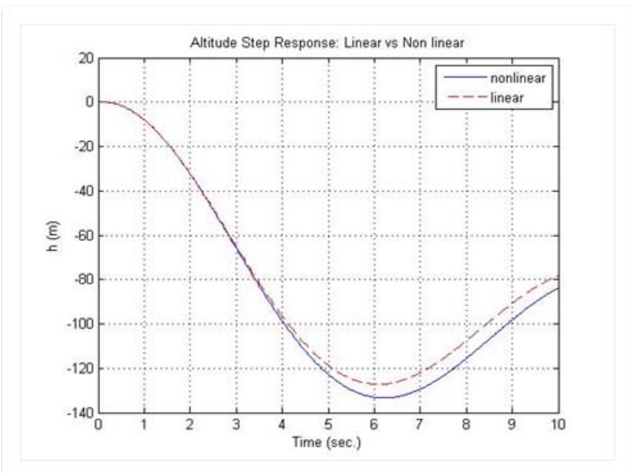

Figure 4. Step Responses for Altitude

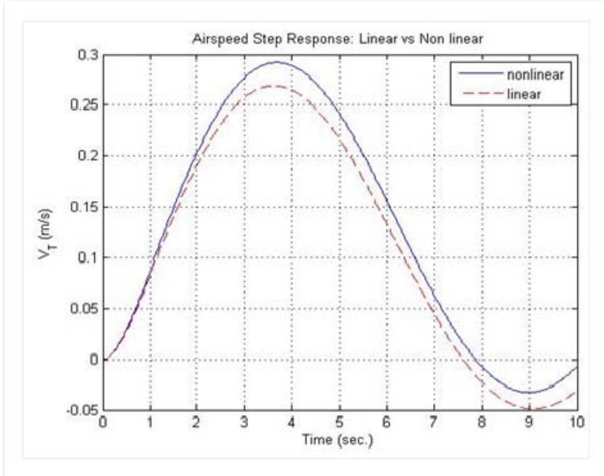

Figure 5. Step Responses for Airspeed

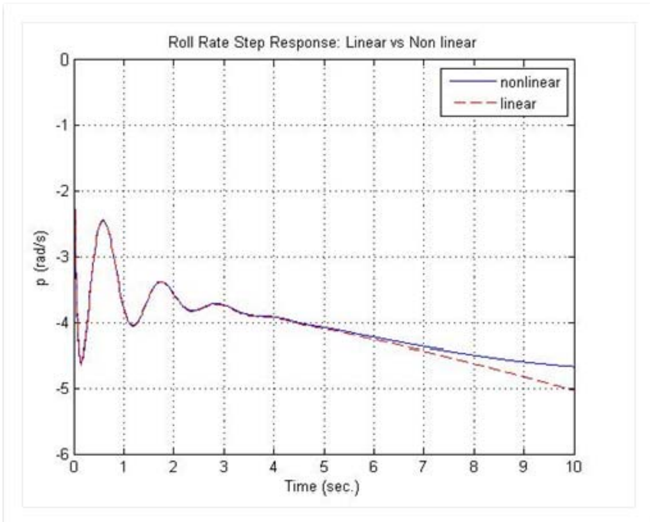

Figure 6. Step Responses for Roll Rate

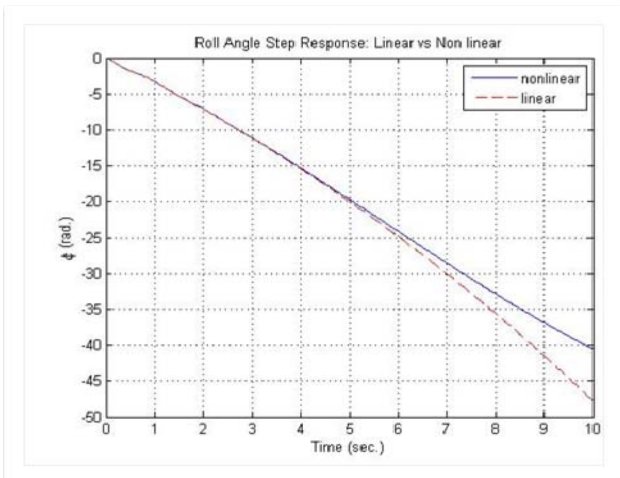

Figure 7. Step Responses for Roll Angle

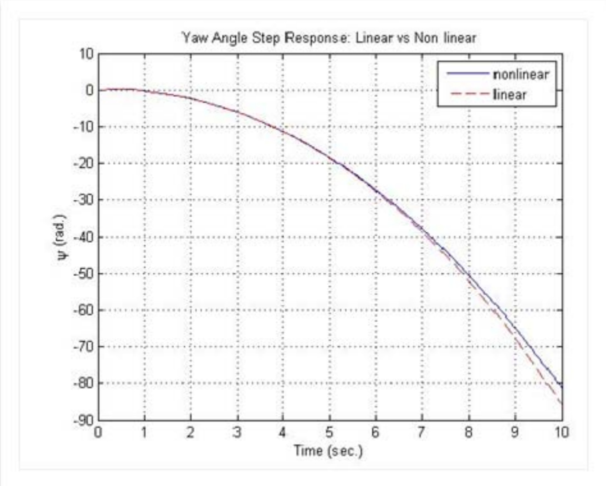

Figure 8. Step Responses for Yaw Angle

The above shown results indicate that the response of linear decoupled model closely follows the response of non-linear model. Only a qualitative comparison is sufficient here to determine that outputs from both model types are in close agreement. The identical responses indicate the similarity of functioning of both models to the same given step input at the trim point. The slight deviation in the lateral model results arises due to an unstable spiral mode of Aerosonde UAV, and not due to any instability of the trim point [15].

VI. CONCLUSION

For linearization of a nonlinear model, the selection of operating point is critical, as it directly affects the performance of the subsequently designed controller. We have presented a technique for stability verification of the trim point. An optimization algorithm has been used to determine the trim point for a nonlinear UAV model. Around this trim point, the linearization of non-linear model was carried out and the obtained linear model was decoupled into lateral and longitudinal sub models. By applying same inputs to the nonlinear and linear models, their responses were compared and found to be in close unison, indicating the stability of the equilibrium point. This work shows a suitable method to verify the stability of aircraft trim point, that may further lead to successful design of a linear flight controller.

Future directions based on this work, can be the design of first order and second order linear controllers using the linearized models, longitudinal and lateral both. The gains for these controllers can be fine tuned for control of the nonlinear model of the UAV.

APPENDIX

List of Symbols

u	Velocity in x-axis
v	Velocity in y-axis
w	Velocity in z-axis
p	Roll rate
q	Pitch rate
r	Yaw rate
φ	Roll angle
θ	Pitch angle
ψ	Yaw angle
δa	Aileron input
δr	Rudder input
δe	Elevator input
δt	Throttle input
C_D	Coefficient of drag
C_L	Coefficient of lift
Cc	Coefficient of side-force
C_l	Coefficient of roll moment
C_m	Coefficient of pitch moment
C_n	Coefficient of yaw moment
S	Surface area of wing
c	Chord length of the wing
b	Wing span
q	Dynamic pressure
Fp	Force due to propeller
Mp	Moment due to propeller
ω	Propeller rpm
C_T	Coefficient of thrust
T	Total thrust
C_P	Coefficient of power
P	Total power
R	Radius of propeller
r_e	Equatorial radius
e	First eccentricity
g_{wgs}	Gravity at equator

REFERENCES

[1] DoD and U.S. Military, "UAV Reliability Study", Progressive Management, 2010.

[2] W. F. Phillips, Mechanics of Flight, 2nd ed., Wiley & Sons. 2010.

[3] H. Bryan, Stability in Aviation: An introduction to dynamical stability as applied to the motions of aeroplanes, Macmillan and Co. ltd., 1911.

[4] B. L. Stevens and F. L. Lewis, Aircraft Control and Simulation, 2nd ed., Wiley Interscience, 1992.

[5] J. M. Sullivan, "Evolution or Revolution ? The rise of UAVs", IEEE Technology and Society Magazine, Vol.25 No. 3, pp 43-49, 2006.

[6] J. Roskam, Airplane Flight Dynamics and Automatic Flight Controls, 3rd ed., DARCO, 2003.

[7] W. M. Debusk "Unmanned Aerial Vehicle Systems for Disaster Relief: Tornado Alley", AIAA Infotech@ Aerospace Conference, AIAA-2010-3506, Atlanta, GA, 2010.

[8] R. Laurenzo ," Combat losses account for most downed predators," Defense Week May,2002.

[9] R. C. Nelson, Flight Stability and Automatic Control, 2nd ed., McGraw-Hill. 1998.

[10] J. H. He, "Linearized perturbation technique and its application in non-linear oscillator", College of Science, Shanghai Donghua University, China.

[11] M. V. Cook, Flight Dynamics Principles, 3rd ed., Elsevier, 2007.

[12] The MathWorks, Inc., Optimization Toolbox User's Guide, Version 3.0.1. The MathWorks, Inc., 2005

[13] Unmanned dynamics, LLC, Aerosim Blockset User's Guide, Version 1.2

[14] M. Mohammadi, A. M. Shahri and Z. Boroujeni, "Modeling and adaptive tracking control of a quadrotor UAV", International Journal of Intelligent Mechatronics and Robotics (IJMIR), vol. 2, pp. 58-81, 2012.

[15] M. Ahsan, H. Rafique and Z. Abbas, "Heading Control of a Fixed Wing UAV using Alternate Control Surfaces", IEEE International Multi Topic Conference (INMIC) , Islamabad, Pakistan, 2012