

## I. BACKGROUND

- **Functional data are ubiquitous.**

Modern sensors yield curves, images, and surfaces observed over time/space. Functional PCA (FPCA) summarizes such data with a few principal functions for interpretation and modeling (Ramsay & Silverman, 2005).

- **Extensions exist but are isolated.**

- ▶ *Smoothed FPCA*: roughness penalties produce smoother, less noisy components (Huang, Shen, & Buja, 2008; Silverman, 1996).
- ▶ *Sparse FPCA*: sparsity zeros out unimportant regions, improving interpretability (Nie & Cao, 2020; Shen & Huang, 2008).
- ▶ *Multivariate FPCA (MFPCA)*: captures shared variation across multiple functional variables (Happ & Greven, 2018).
- ▶ *Two-way functional data* (e.g., time  $\times$  space): require structure in both domains.

- **Limitations.**

Classical FPCA is noise-sensitive and can yield rough, dense patterns; many methods address either **smoothness** or **sparsity**, or are limited to univariate data.

- **Motivation.**

Develop a regularized FPCA framework that (i) handles **multivariate** and **two-way** functional structures, (ii) imposes **smoothness** (noise reduction) and **sparsity** (feature selection) *simultaneously* on scores and loadings, and (iii) yields low-rank, interpretable, and stable components. Recent work points in this direction but leaves room for a unified treatment and broader applicability (Hagbin, Zhao, & Maadooliat, 2025).

## II. METHODOLOGY

1) **Multivariate FPCA**: Concatenate  $p$  functional variables, where the  $i$ -th variable is observed on  $m_i$  grid points, into  $\mathbf{X} = [X_1 \ X_2 \cdots X_p] \in \mathbb{R}^{n \times M}$ , where  $M = \sum_{i=1}^p m_i$ :

$$\mathbf{X} = \begin{bmatrix} x_{11}(t_{11}) & \cdots & x_{11}(t_{1m_1}) & \cdots & x_{1p}(t_{p1}) & \cdots & x_{1p}(t_{pm_p}) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1}(t_{11}) & \cdots & x_{n1}(t_{1m_1}) & \cdots & x_{np}(t_{p1}) & \cdots & x_{np}(t_{pm_p}) \end{bmatrix} \quad (1)$$

We estimate a rank-one structure with penalties:

$$\min_{u,v} \|\mathbf{X} - uv^\top\|_F^2 + \alpha v^\top \Omega v + p_\gamma(v), \quad (2)$$

where  $\Omega = \text{diag}(\Omega_1, \dots, \Omega_p)$  encodes **roughness** and  $p_\gamma(\cdot)$  induces **sparsity** (soft, hard, or SCAD) (Huang et al., 2008; Nie & Cao, 2020; Shen & Huang, 2008).

2) **Sequential Power Algorithm**: Let  $S(\alpha) = (I + \alpha\Omega)^{-1}$ . Iterate:

1. **Initialize**:  $v$  via rank-one SVD of  $\mathbf{X}$ .
2. **Repeat**:
  - $u \leftarrow \mathbf{X}v$
  - $v \leftarrow S(\alpha) h_\gamma(\mathbf{X}^\top u)$
  - $v \leftarrow v / \|v\|$

**Tuning:**

Choose  $\gamma$  by  $K$ -fold cross-validation.

Choose  $\alpha$  by generalized cross-validation (GCV):  $\text{GCV}(\alpha) = \frac{\|(I - S(\alpha))(\mathbf{X}^\top u)\|_F^2 / M}{(1 - \frac{1}{M} \text{tr } S(\alpha))^2}$ .

## III. TWO-WAY REGULARIZED MFPCA

- **Two-way functional data:**

Two-way functional data consist of a data matrix whose row and column domains are both structured. Classical FPCA focuses on one domain and penalizes only one set of components, often ignoring structure in the second direction.

- **Framework & Penalty:**

$$\min_{u,v} \|\mathbf{X} - uv^\top\|_F^2 + \sum_{j=1}^J \mathcal{P}_j^{[\theta]}(u, v) \quad (3)$$

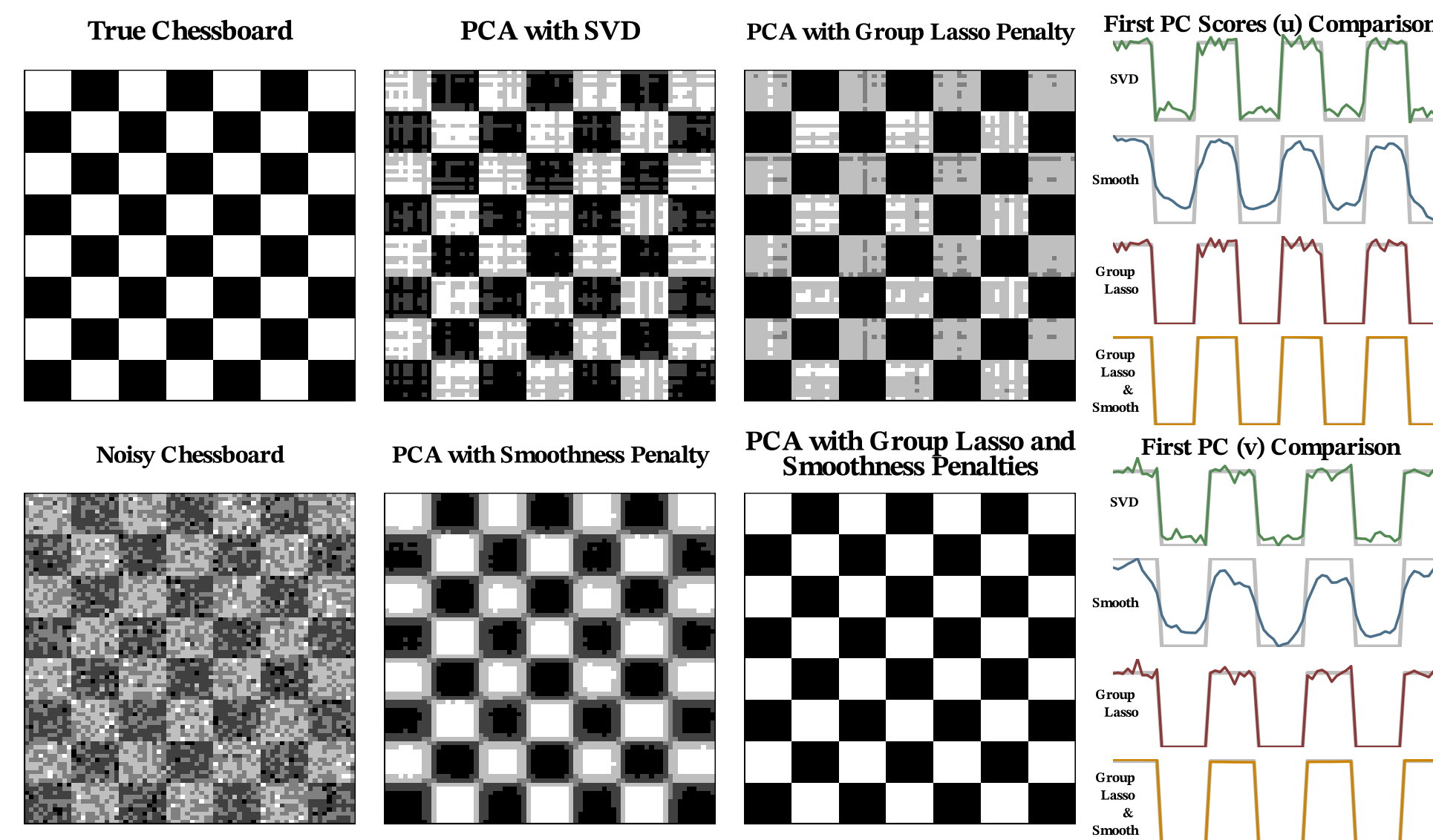
- where  $J$  is the number of penalty components, and  $\theta$  is the vector of all tuning parameters. The composite penalty  $\sum_{j=1}^J \mathcal{P}_j^{(\theta)}(u, v)$  lets us mix regularizers, e.g., **smoothness** with  $\theta = (\alpha_u, \alpha_v)$  and **sparsity** with  $\theta = (\gamma_u, \gamma_v)$  (controlling sparsity), and can include other structures as needed.

- **Sequential Power Algorithm:**

1. Initialize  $u, v$  using rank-one SVD of  $\mathbf{X}$ .
2. Update  $u$  with smoothing and sparsity transformations:  
 $u \leftarrow S_u^{[\alpha_u]} h_u^{[\gamma_u]}(\mathbf{X}v)$
3. Update  $v$  similarly:  
 $v \leftarrow S_v^{[\alpha_v]} h_v^{[\gamma_v]}(\mathbf{X}^\top u)$
4. Normalize  $v$  and deflate  $\mathbf{X}$  to extract further components.

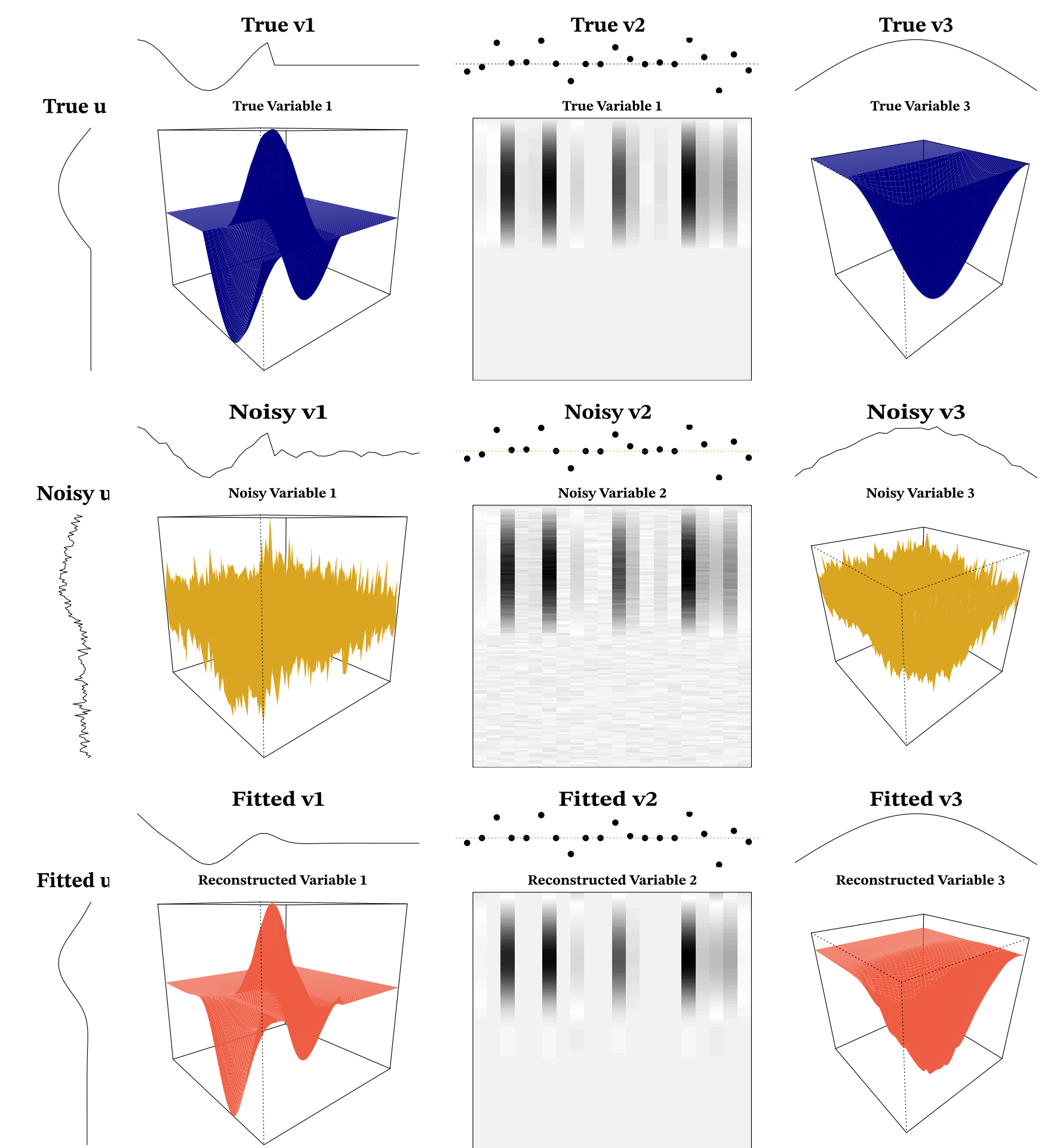
- **Tuning parameters:**

- ▶  $\alpha_u, \alpha_v \rightarrow$  **smoothness** of  $u$  and  $v$ .
- ▶  $\gamma_u, \gamma_v \rightarrow$  **sparsity** of  $u$  and  $v$ .
- ▶ **Conditional tuning strategy:**
  1. Start with no penalties.
  2. Tune sparsity parameters via CV.
  3. Tune smoothness parameters via GCV.
  4. Alternate until convergence.



## IV. CONCLUSION

- **Comprehensive Framework:** ReMPCA extends PCA to functional data, combining smoothness (denoising) and sparsity (feature selection) for structured, low-rank PCs.
- **Methodology:** Penalized SVD with **roughness** and **sparsity** penalties applies regularization to both scores ( $u$ ) and loadings ( $v$ ), tuned via GCV and CV.
- **Results:** Two-way regularization improves reconstruction and interpretability, outperforming one-way methods across simulated and real datasets.
- **Software:** Implemented in the ReMPCA R package with tuning, and visualization tools.
- **Future Work:** Extend to hybrid scalar–functional–image data, nonlinear kernels, and applications in neuroimaging, medicine, and environmental science.



## V. REFERENCES

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