Waveform transformations

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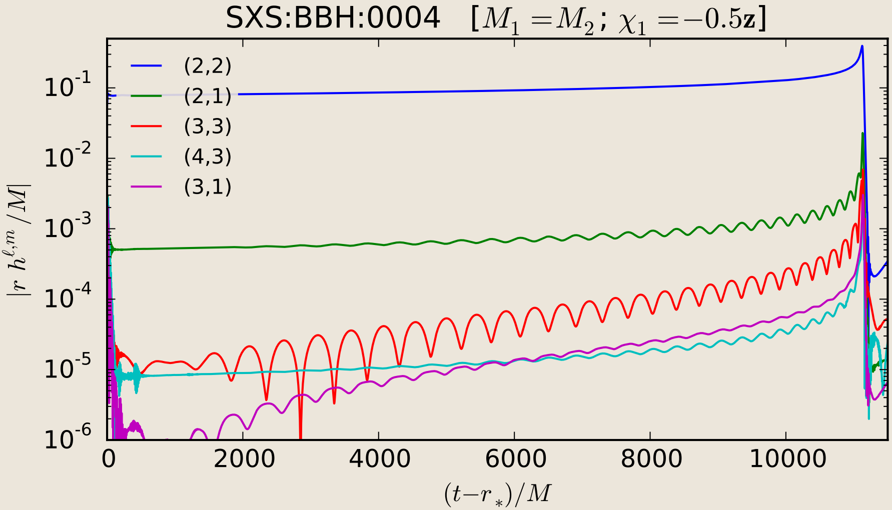
May 19, 2015

Outline

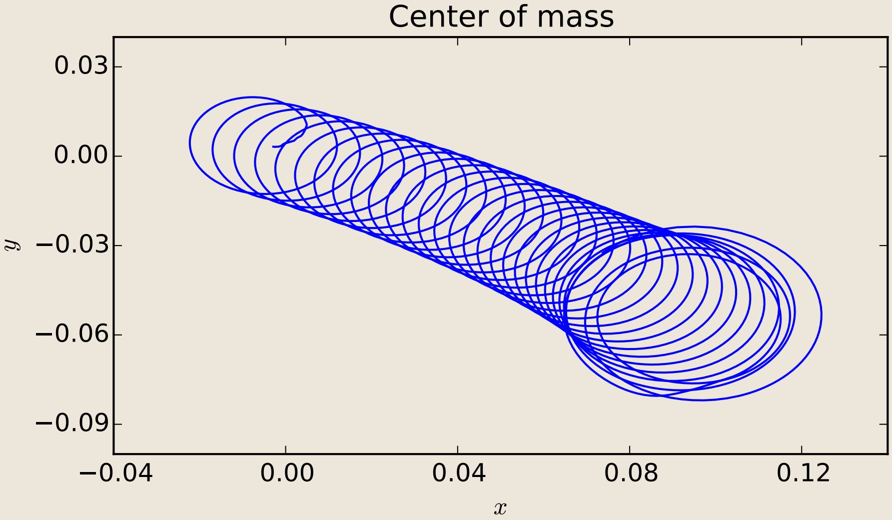
- Motivation
  - Problems in the waveforms
  - Center-of-mass drifts
  - Cleaning up waveforms
- Asymptotic symmetry transformations
  - Ambiguities in waveforms
  - Coordinates on $\mathscr{I}^+$
  - The BMS group
  - Spin-weighted functions
  - Implementing transformations
- Conclusions

Motivation

A waveform mystery



The center of mass



Correcting the center of mass

$$\min_{\boldsymbol{x}_0, \boldsymbol{v}_0} \int_{t_i}^{t_f} \left| \boldsymbol{x}_{\mathrm{CoM}}(t) - (\boldsymbol{x}_0 + \boldsymbol{v}_0\, t) \right|^2 dt$$

Correcting the center of mass

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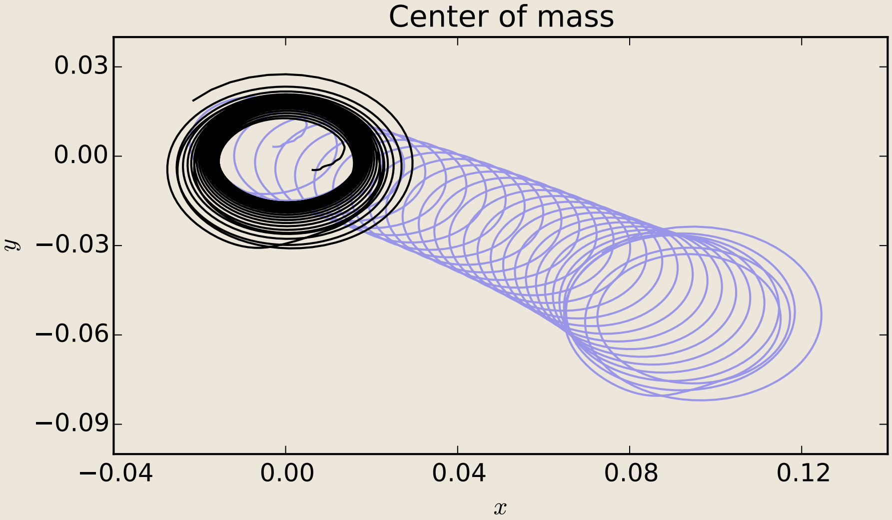
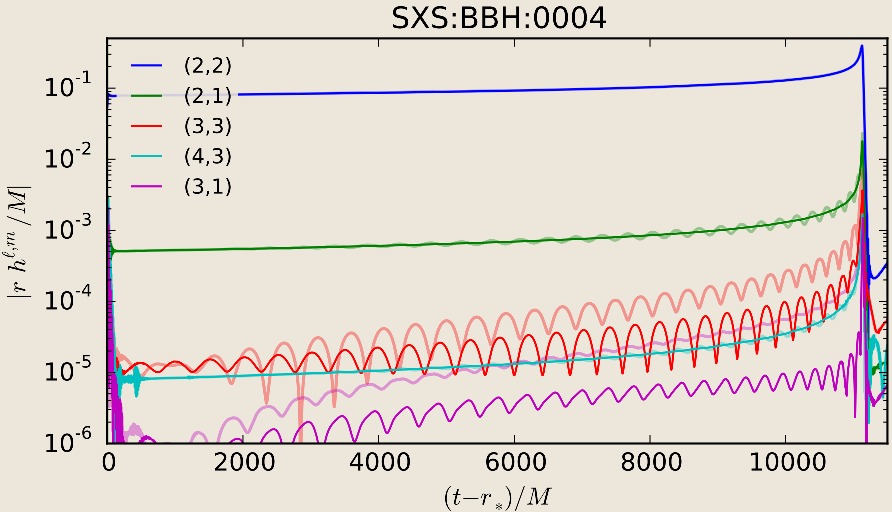
$$\boldsymbol{x}_0 = \frac{4(t_f^2 + t_f t_i + t_i^2) \int \boldsymbol{x}_{\mathrm{CoM}}(t)\, dt - 6(t_f + t_i) \int \boldsymbol{x}_{\mathrm{CoM}}(t)\, t\, dt}{(t_f - t_i)^3}$$

$$\boldsymbol{v}_0 = \frac{12 \int \boldsymbol{x}_{\mathrm{CoM}}(t)\, t\, dt - 6 \int \boldsymbol{x}_{\mathrm{CoM}}(t)\, dt}{(t_f - t_i)^3}$$

Corrected center of mass



Corrected waveform



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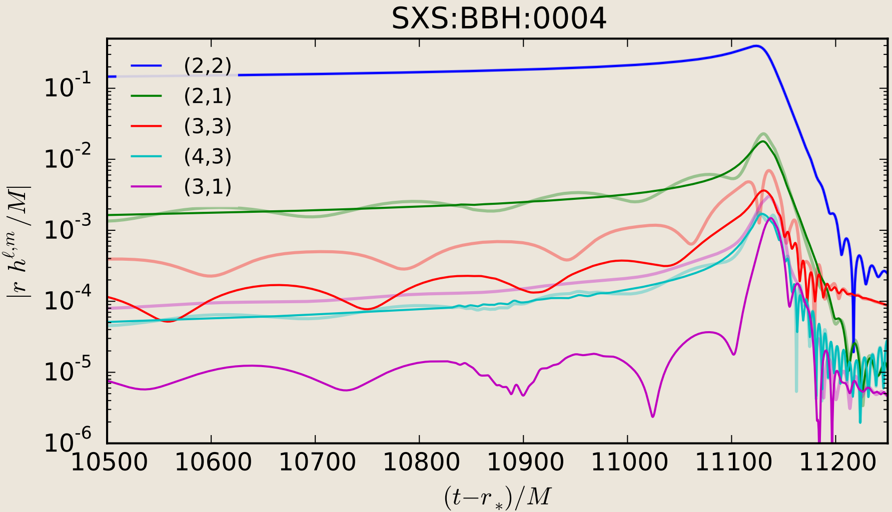
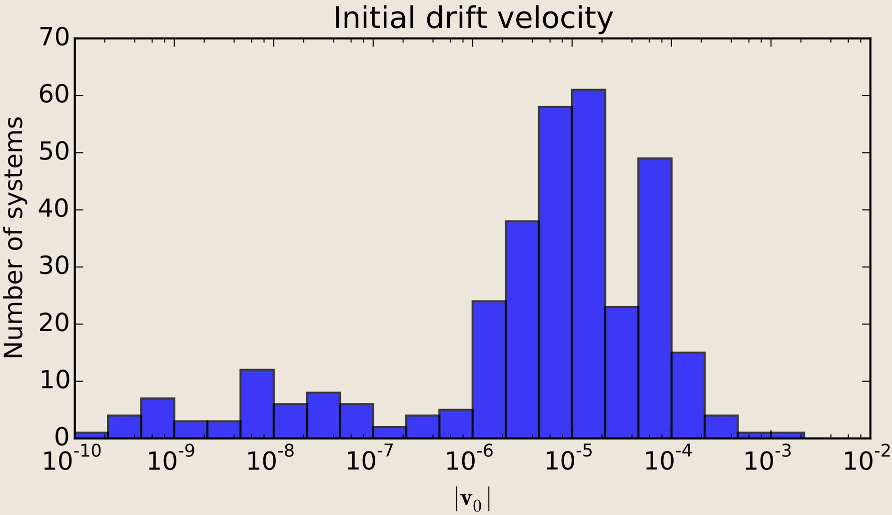


Correcting the catalog



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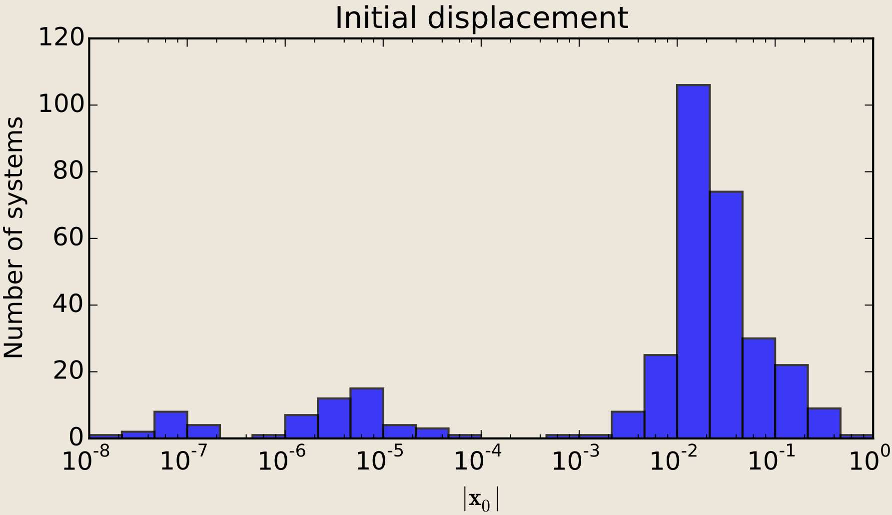

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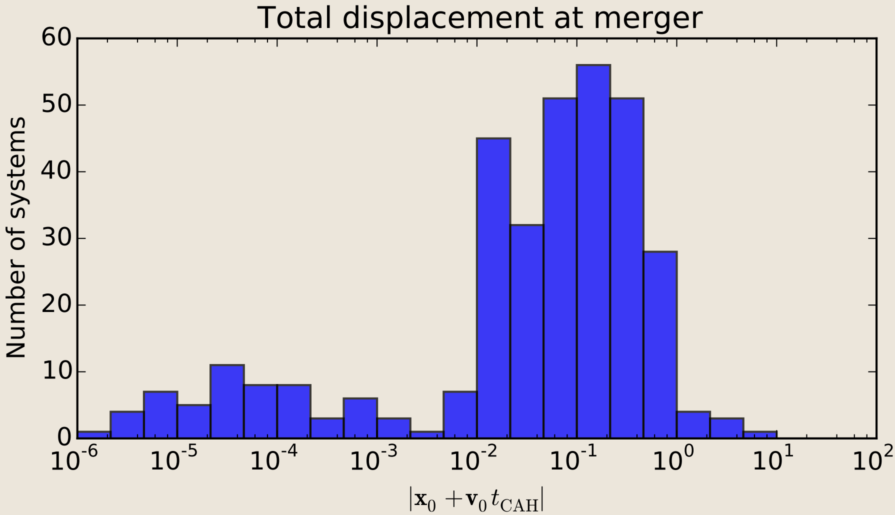

Correcting the initial data

Ask Sergei!

Asymptotic symmetries

Standard ambiguities

- Time translation
- Phase rotation

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- General rotation

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Poincaré group?

Standard ambiguities

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Poincaré group?
Diffeomorphism group?!

Isolated system

Asymptotic flatness

- Radial coordinate r
- "Regularity" of manifold as $r \to \infty$

Bondi coordinates

- Null coordinate u
- Angular coordinates $x^A = (\theta, \phi)$
- Metric:

$$ds^2 = \frac{V}{r} e^{2\beta}\, du^2 - 2e^{2\beta}\, du\, dr \\ + r^2\, h_{AB}\, (dx^A - U^A\, du)\, (dx^B - U^B\, du)$$

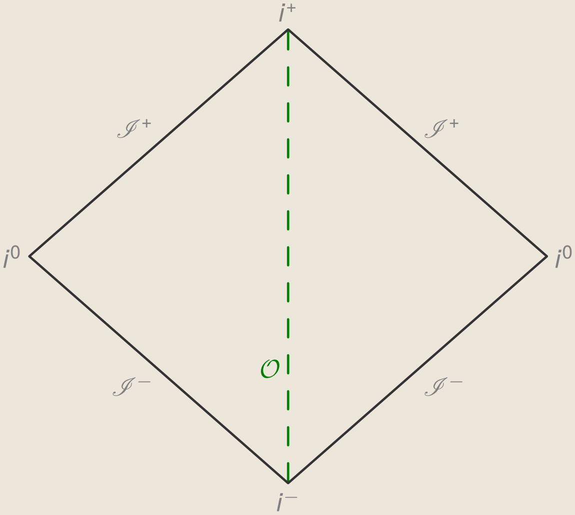
General symmetry group

- Boost
- General rotation
- Supertranslation

Bondi-Metzner-Sachs (BMS) group

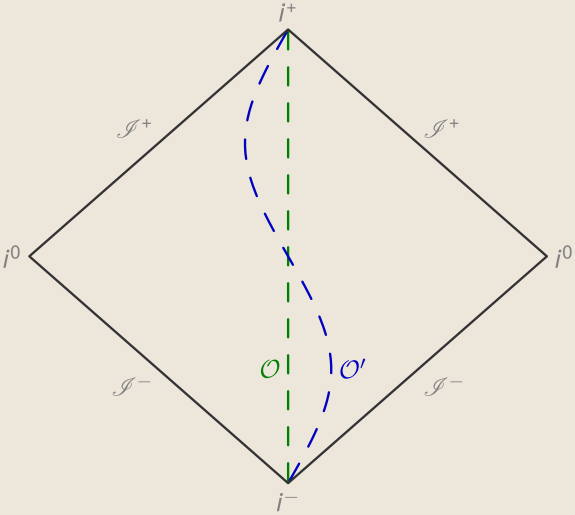
Coordinates on $\mathscr{I}^+$

Penrose diagram



Coordinates on $\mathscr{I}^+$

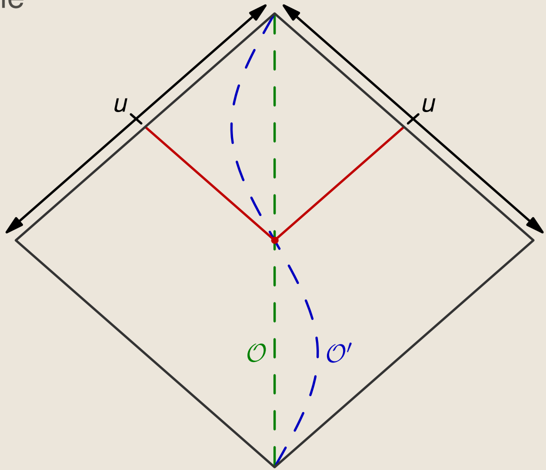
Boosted observer



Coordinates on $\mathscr{I}^+$

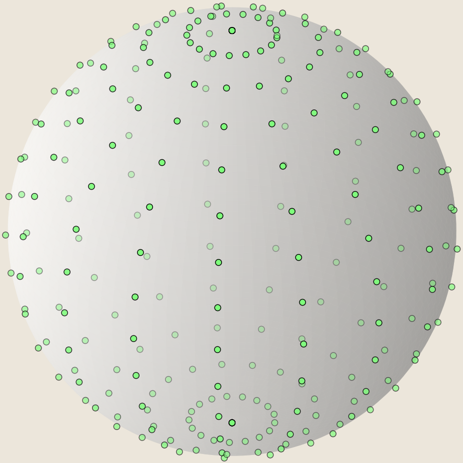
Local time $\rightarrow$ retarded time

$u = t - r$



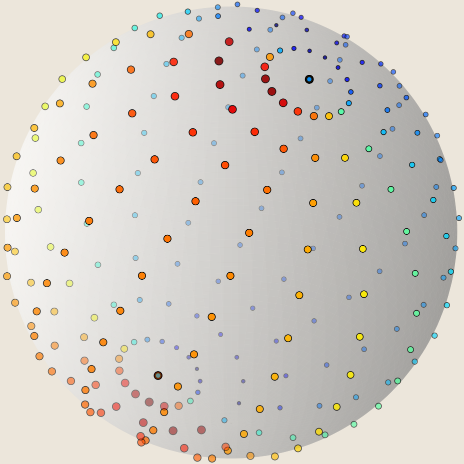
Coordinates on $\mathscr{I}^+$

Sphere in $\mathcal{O}'$



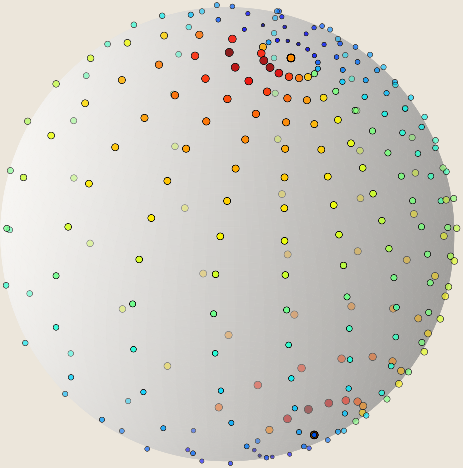
Coordinates on $\mathscr{I}^+$

Points as seen in $\mathcal{O}$ related by rotation



Coordinates on $\mathscr{I}^+$

Points as seen in $\mathscr{O}$ related by boost



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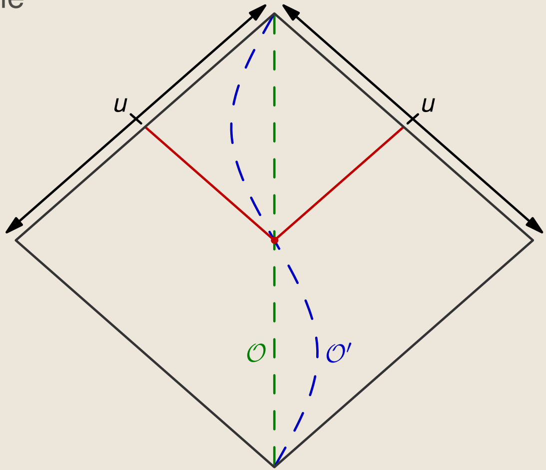
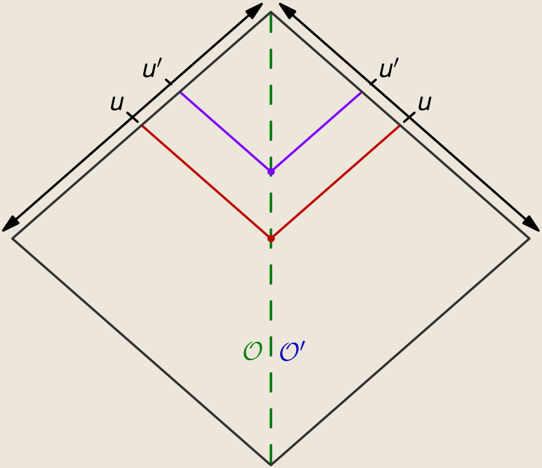


Coordinates on $\mathscr{I}^+$

Time translation

$t \mapsto t + \delta t$

$u \mapsto u + \delta t$

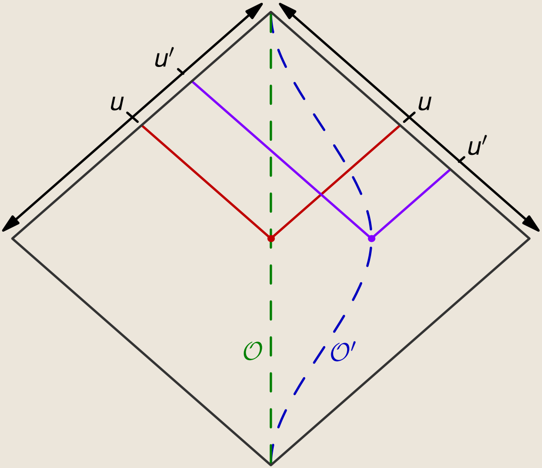


Coordinates on $\mathscr{I}^+$

Space translation

$\boldsymbol{x} \mapsto \boldsymbol{x} + \delta\boldsymbol{x}$

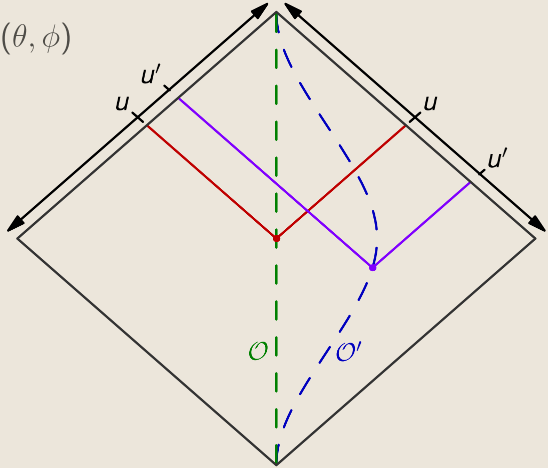
$u \mapsto u - \delta\boldsymbol{x} \cdot \hat{\boldsymbol{r}}$



Coordinates on $\mathscr{I}^+$

Spacetime translation

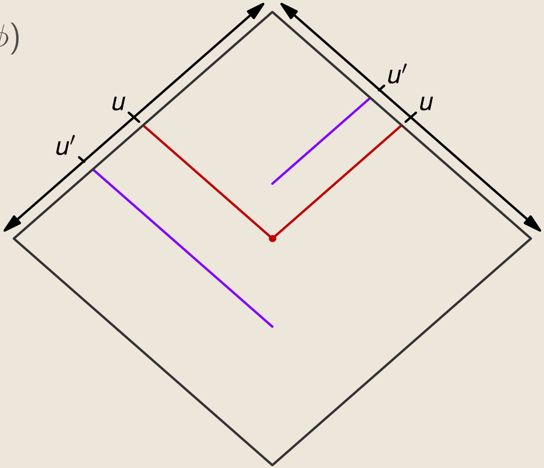
$u \mapsto u + \sum_{\ell=0,1;m} \alpha_{\ell,m} Y_{\ell,m}(\theta,\phi)$



Coordinates on $\mathscr{I}^+$

Supertranslation

$u \mapsto u + \sum_{\ell,m} \alpha_{\ell,m} Y_{\ell,m}(\theta, \phi)$



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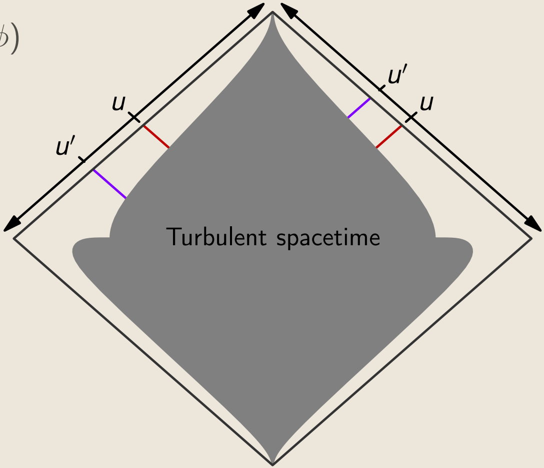

BMS coordinate transformations

$$
\begin{aligned}
\theta &\mapsto \theta' \\
\phi &\mapsto \phi' \\
u &\mapsto K(\theta', \phi')\,[u - \alpha(\theta', \phi')]
\end{aligned}
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$$u' \approx \gamma\,[u - (\boldsymbol{x}_0 + u\,\boldsymbol{v}_0) \cdot \hat{\boldsymbol{r}} - \tilde{\alpha}(\theta', \phi')]$$

BMS waveform transformations

$$r\,h \mapsto \frac{e^{-2i\lambda}}{K}\left[r\,h - \bar{\eth}^2\alpha\right]$$

$$r^2\,\sigma \mapsto \frac{e^{2i\lambda}}{K}\left[r^2\,\sigma - \eth^2\alpha\right]$$

$$c \mapsto \frac{e^{2i\lambda}}{K}\left[c - \eth^2\alpha\right]$$

$$\text{News} \sim \frac{\partial c}{\partial u} \mapsto \frac{e^{2i\lambda}}{K^2}\left[\frac{\partial c}{\partial u}\right]$$

$$r\,\Psi_4 \mapsto \frac{e^{-2i\lambda}}{K^3}\left[r\,\Psi_4'\right]$$

BMS waveform transformations

$$r\,h \mapsto \frac{e^{-2i\lambda}}{K}\left[r\,h - \bar{\eth}^2\alpha\right]$$

BMS waveform transformations

$$r\,h(u,\theta,\phi) \mapsto \frac{e^{-2i\lambda(\theta',\phi')}}{K(\theta',\phi')}\left[r\,h(u',\theta',\phi') - \bar{\eth}^2\alpha(\theta',\phi')\right]$$

Spin-weighted functions (SWFs)

$$\Psi_4(\theta, \phi) = C_{abcd}(\theta, \phi)\, \bar{m}^a\, n^b\, \bar{m}^c\, n^d$$

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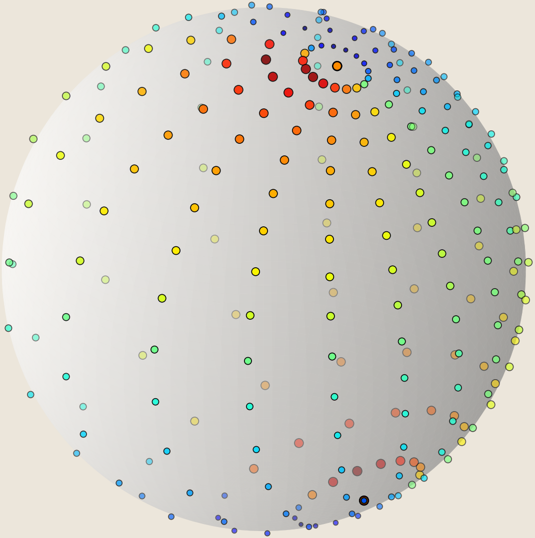
$$R : (\theta, \phi) \mapsto (\theta', \phi')$$

$$\Psi_4'(\theta', \phi') = \Psi_4(\theta, \phi)\, e^{-2i\lambda(\theta', \phi', R)}$$

Spin-weighted functions (SWFs)



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Might be tempted to think

$$\mathrm{SWF} : S^2 \times S^1 \to \mathbb{C}$$

Turns out

$$\mathrm{SWF} : S^3 \to \mathbb{C}$$

This is the Hopf bundle:

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$$

Spin-weighted functions (SWFs)

SWFs are functions on the spin group

Spin(3) = SU(2) = group of unit quaternions

$$\hat{\boldsymbol{r}} = R\,\boldsymbol{z}\,R^{-1}$$

$$\boldsymbol{l} = R\,\frac{\boldsymbol{t}+\boldsymbol{z}}{\sqrt{2}}\,R^{-1}$$

$$\boldsymbol{n} = R\,\frac{\boldsymbol{t}-\boldsymbol{z}}{\sqrt{2}}\,R^{-1}$$

$$\boldsymbol{m} = R\,\frac{\boldsymbol{x}+i\,\boldsymbol{y}}{\sqrt{2}}\,R^{-1}$$

$$\bar{\boldsymbol{m}} = R\,\frac{\boldsymbol{x}-i\,\boldsymbol{y}}{\sqrt{2}}\,R^{-1}$$

Spin-weighted spherical harmonics

$$ {}_{-2}Y_{\ell,m}(\theta,\phi) \longrightarrow {}_{-2}Y_{\ell,m}(R) = \sqrt{\frac{2\ell+1}{4\pi}}\,\mathfrak{D}^{(\ell)}_{2,m}(R) $$

BMS waveform transformations

$$r\,h(u,\theta,\phi) \mapsto \frac{e^{-2i\lambda(\theta',\phi')}}{K(\theta',\phi')}\left[r\,h(u',\theta',\phi') - \bar{\eth}^2\alpha(\theta',\phi')\right]$$

BMS waveform transformations

$$r\,h(u,R) \mapsto \frac{1}{K(R')}\left[r\,h(u',R') - \bar{\eth}^2\alpha(R')\right]$$

Rotor of a boost

$$\Theta' = \arccos\left[\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{r}}_{\theta',\phi'}\right]$$

$$\Theta = 2\,\arctan\left[\sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{\Theta'}{2}\right]$$

$$R_B = \exp\left[\frac{\Theta-\Theta'}{2}\,\frac{\boldsymbol{v}\times\boldsymbol{r}_{\theta',\phi'}}{\left|\boldsymbol{v}\times\boldsymbol{r}_{\theta',\phi'}\right|}\right]$$

$$R' = R_B\,R_F\,e^{\phi'\,\boldsymbol{z}/2}\,e^{\theta'\,\boldsymbol{y}/2}$$

Implementing BMS transformations

for all desired times slices u'_i **do**

for all (θ'_j, ϕ'_j) on equiangular grid **do**

$$R'_j \leftarrow R_B\, e^{\phi'_j \boldsymbol{z}/2}\, e^{\theta'_j \boldsymbol{y}/2}$$

$$u_i \leftarrow \frac{u'_i}{K(R')} + \alpha(R')$$

for all u_k “near” u_i **do**

$$r'h'(u_k, R'_j) \leftarrow \frac{r\,h(u_k, R'_j) - \bar{\eth}^2 \alpha(R'_j)}{K(R'_j)}$$

$$r'h'(u'_i, R'_j) \leftarrow \texttt{interp}\Big([r'h'(u_k, R'_j)],\ u_i\Big)$$

$$r'h'_{\ell,m}(u'_i) \leftarrow \texttt{spinsfast}\Big([r'h'(u'_i, R'_j)]\Big)$$

Conclusions

- No such thing as invariant waveform
- No preferred reference frame
- When comparing waveforms, frames must agree

Need BMS transformations for

- Accurate PN waveforms
- PN–NR comparisons
- Hybrid waveforms
- Making sense of ringdowns