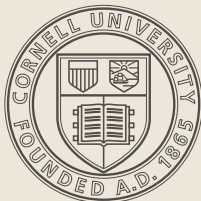


# Waveform transformations

Mike Boyle

May 19, 2015

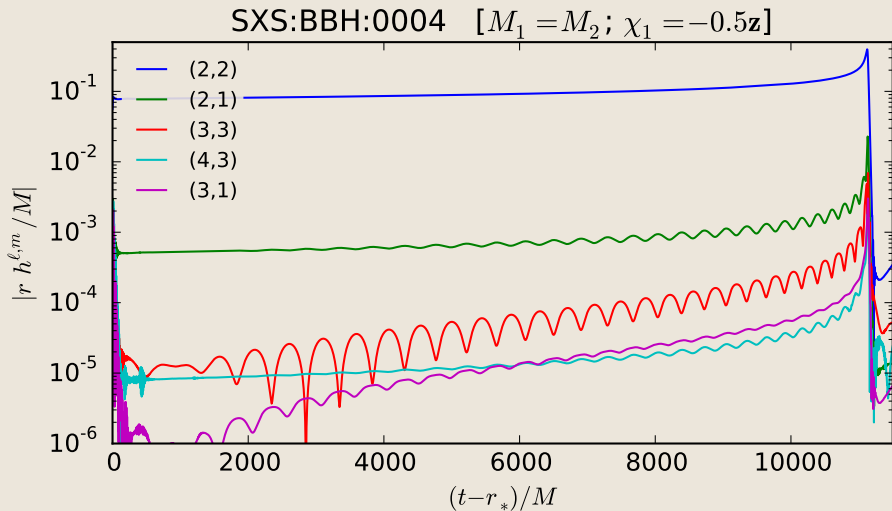


- ▶ Motivation
  - ▶ Problems in the waveforms
  - ▶ Center-of-mass drifts
  - ▶ Cleaning up waveforms
- ▶ Asymptotic symmetry transformations
  - ▶ Ambiguities in waveforms
  - ▶ Coordinates on  $\mathcal{I}^+$
  - ▶ The BMS group
  - ▶ Spin-weighted functions
  - ▶ Implementing transformations
- ▶ Conclusions

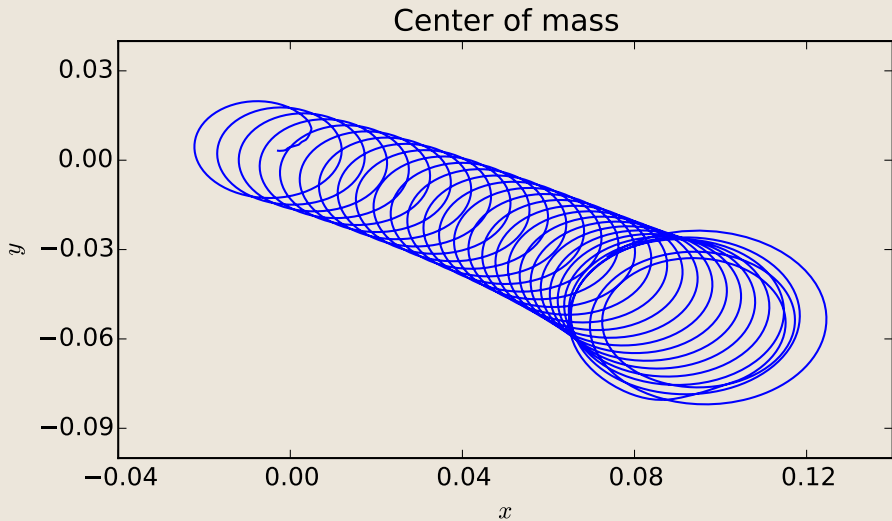


# Motivation

# A waveform mystery



# The center of mass



# Correcting the center of mass



$$\min_{\mathbf{x}_0, \mathbf{v}_0} \int_{t_i}^{t_f} |\mathbf{x}_{\text{CoM}}(t) - (\mathbf{x}_0 + \mathbf{v}_0 t)|^2 dt$$

# Correcting the center of mass

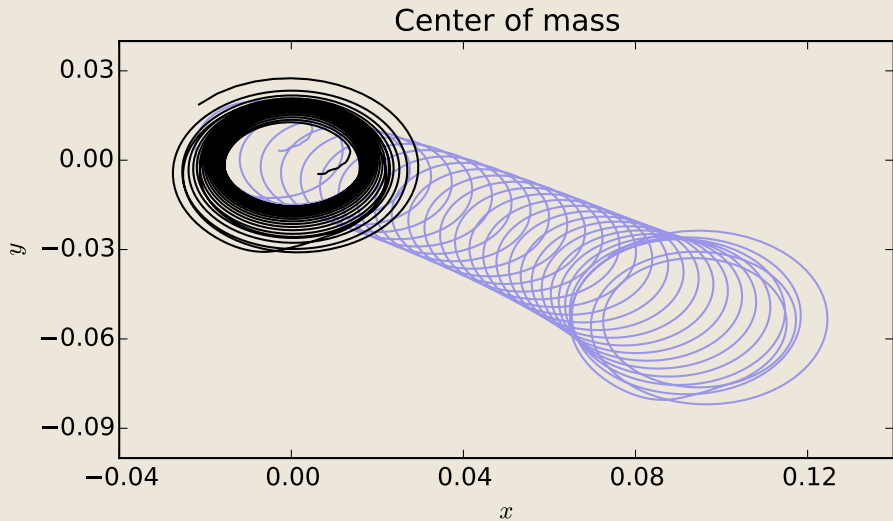


$$\min_{\mathbf{x}_0, \mathbf{v}_0} \int_{t_i}^{t_f} |\mathbf{x}_{\text{CoM}}(t) - (\mathbf{x}_0 + \mathbf{v}_0 t)|^2 dt$$

$$\mathbf{x}_0 = \frac{4(t_f^2 + t_f t_i + t_i^2) \int \mathbf{x}_{\text{CoM}}(t) dt - 6(t_f + t_i) \int \mathbf{x}_{\text{CoM}}(t) t dt}{(t_f - t_i)^3}$$

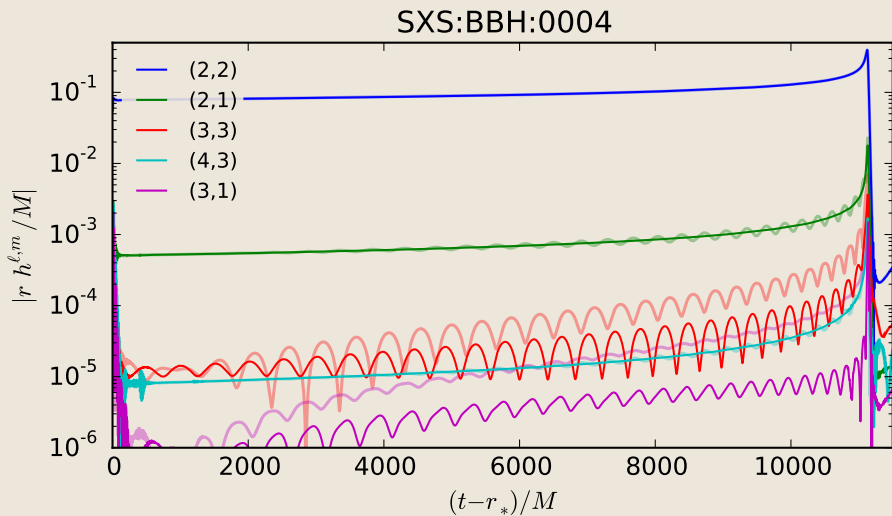
$$\mathbf{v}_0 = \frac{12 \int \mathbf{x}_{\text{CoM}}(t) t dt - 6 \int \mathbf{x}_{\text{CoM}}(t) dt}{(t_f - t_i)^3}$$

# Corrected center of mass

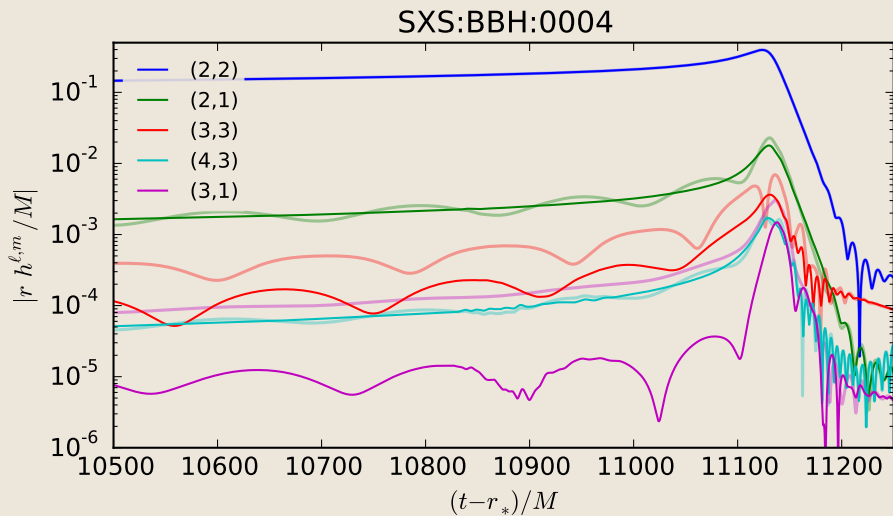




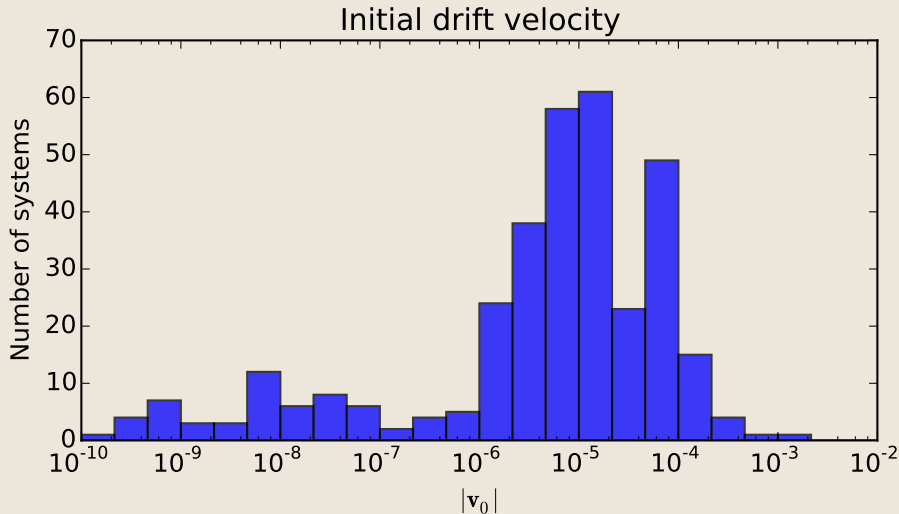
# Corrected waveform



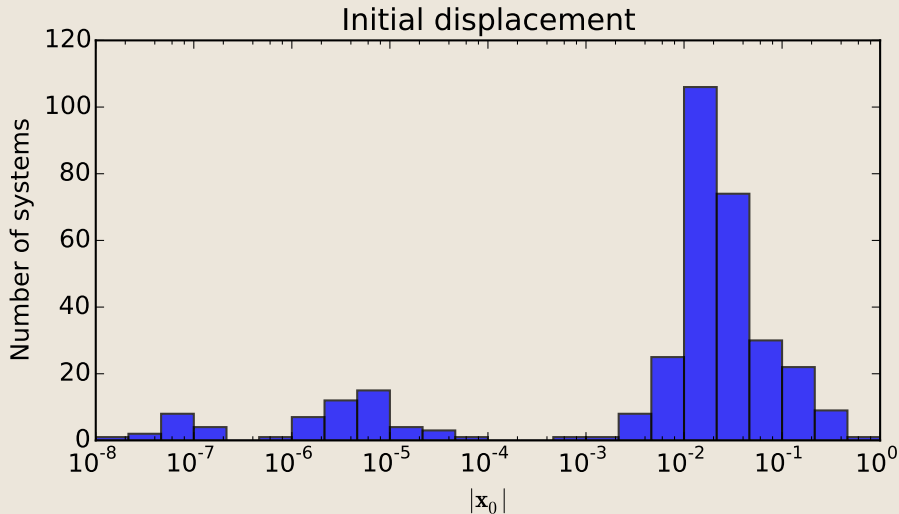
# Corrected waveform



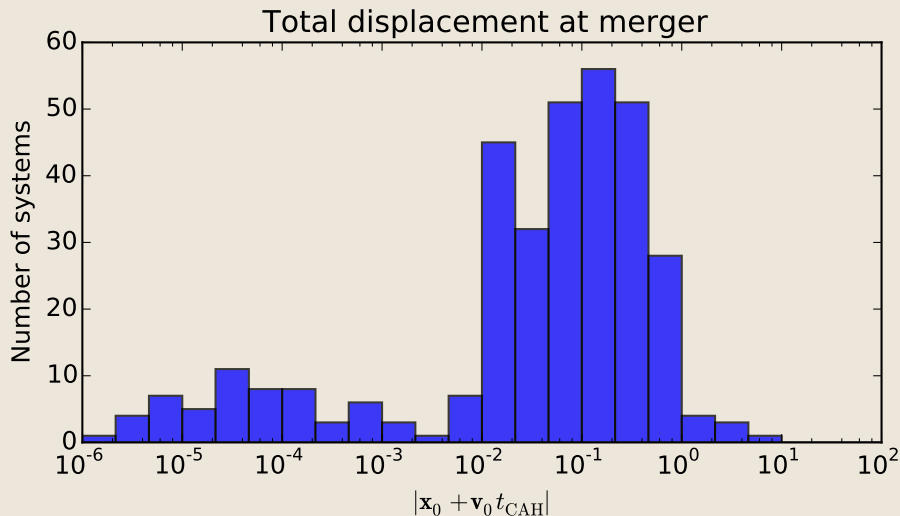
# Correcting the catalog



# Correcting the catalog



# Correcting the catalog



# Correcting the initial data



Ask Sergei!



# Asymptotic symmetries

# Standard ambiguities



- ▶ Time translation
- ▶ Phase rotation



# Standard ambiguities



- ▶ Time translation
- ▶ Phase rotation
- ▶ General rotation

# Standard ambiguities



- ▶ Time translation
- ▶ Space translation
- ▶ Phase rotation
- ▶ General rotation

# Standard ambiguities



- ▶ Time translation
- ▶ Space translation
- ▶ Phase rotation
- ▶ General rotation
- ▶ Boost

# Standard ambiguities



- ▶ Time translation
- ▶ Space translation
- ▶ Phase rotation
- ▶ General rotation
- ▶ Boost

Poincaré group?

# Standard ambiguities



- ▶ Time translation
- ▶ Space translation
- ▶ Phase rotation
- ▶ General rotation
- ▶ Boost

Poincaré group?  
Diffeomorphism group?!



# Isolated system

## Asymptotic flatness

- ▶ Radial coordinate  $r$
- ▶ “Regularity” of manifold as  $r \rightarrow \infty$

## Bondi coordinates

- ▶ Null coordinate  $u$
- ▶ Angular coordinates  $x^A = (\theta, \phi)$
- ▶ Metric:

$$ds^2 = \frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr \\ + r^2 h_{AB} (dx^A - U^A du) (dx^B - U^B du)$$

# General symmetry group



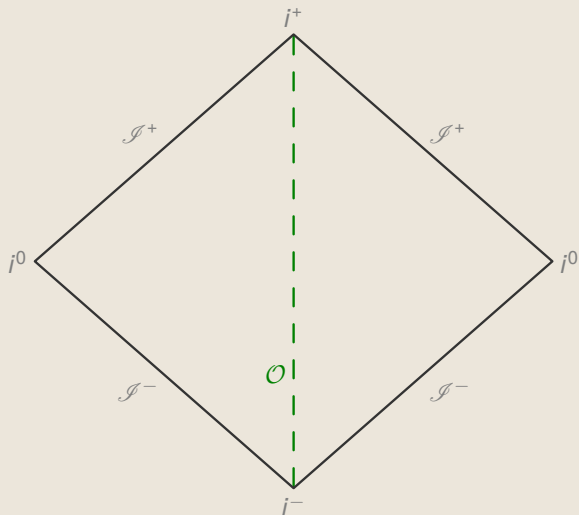
- ▶ Boost
- ▶ General rotation
- ▶ Supertranslation

Bondi-Metzner-Sachs (BMS) group

# Coordinates on $\mathcal{I}^+$



## Penrose diagram

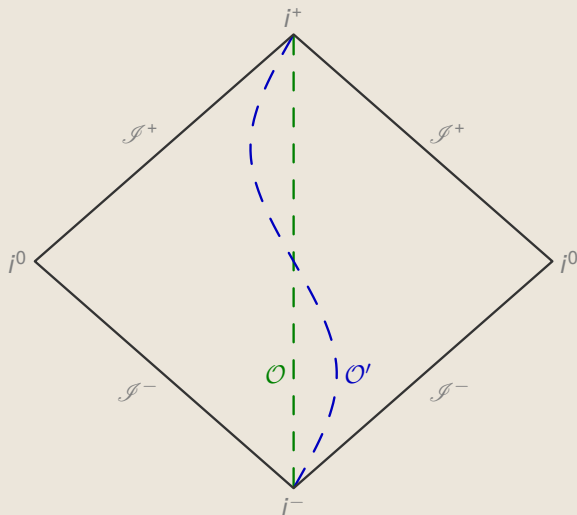




# Coordinates on $\mathcal{I}^+$



Boosted observer

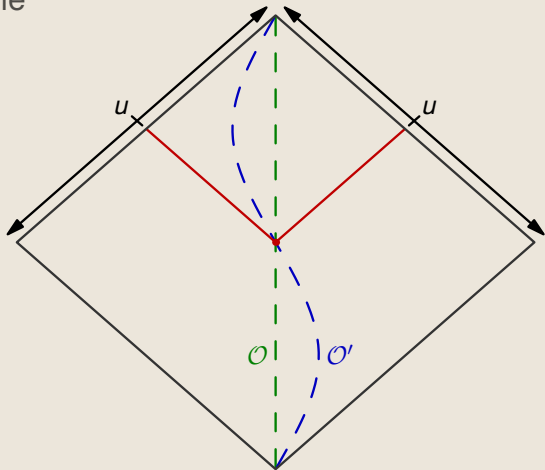


# Coordinates on $\mathcal{I}^+$



Local time  $\rightarrow$  retarded time

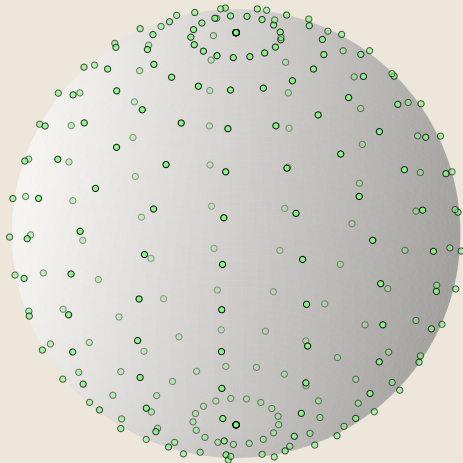
$$u = t - r$$



# Coordinates on $\mathcal{I}^+$



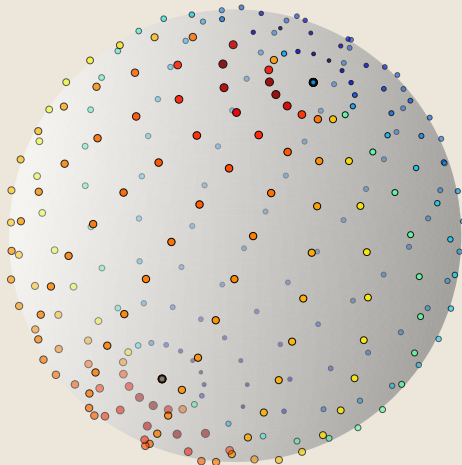
Sphere in  $\mathcal{O}'$



# Coordinates on $\mathcal{I}^+$



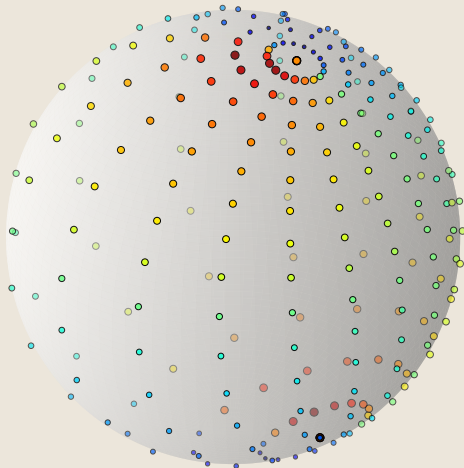
Points as seen in  $\mathcal{O}$   
related by rotation



# Coordinates on $\mathcal{I}^+$



Points as seen in  $\mathcal{O}$   
related by boost

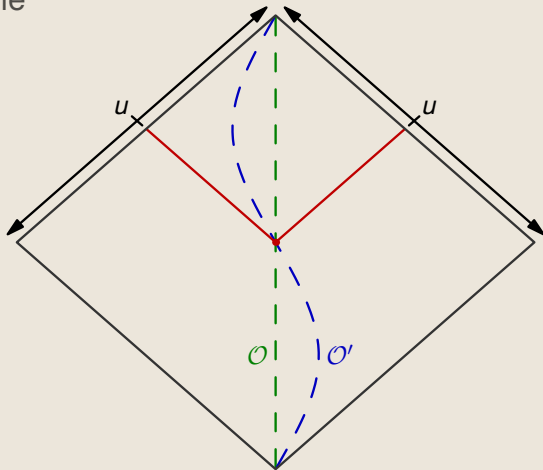


# Coordinates on $\mathcal{I}^+$



Local time  $\rightarrow$  retarded time

$$u = t - r$$



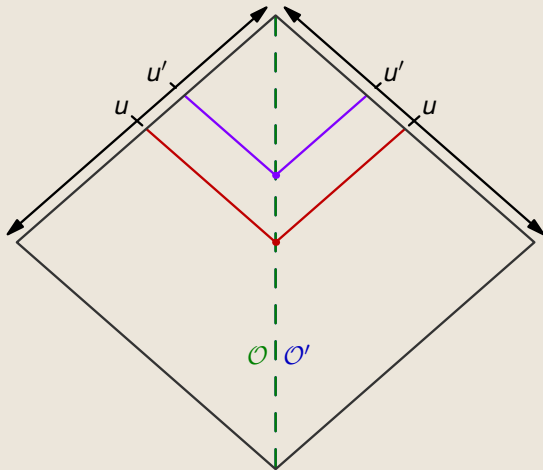
# Coordinates on $\mathcal{I}^+$



Time translation

$$t \mapsto t + \delta t$$

$$u \mapsto u + \delta t$$



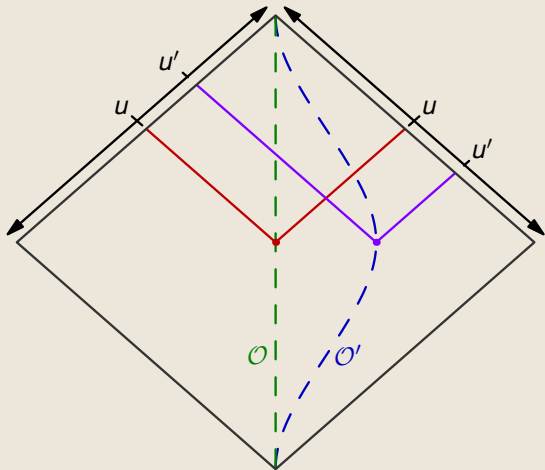
# Coordinates on $\mathcal{I}^+$



Space translation

$$\mathbf{x} \mapsto \mathbf{x} + \delta \mathbf{x}$$

$$u \mapsto u - \delta \mathbf{x} \cdot \hat{\mathbf{r}}$$



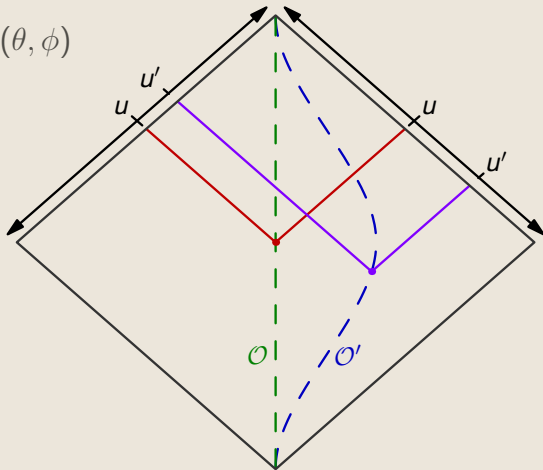


# Coordinates on $\mathcal{I}^+$



Spacetime translation

$$u \mapsto u + \sum_{\ell=0,1;m} \alpha_{\ell,m} Y_{\ell,m}(\theta, \phi)$$

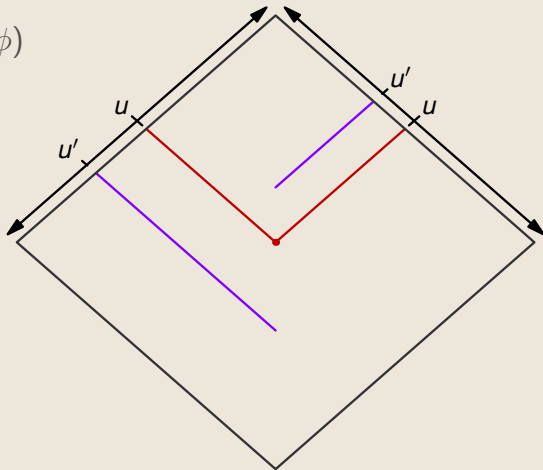


# Coordinates on $\mathcal{I}^+$



Supertranslation

$$u \mapsto u + \sum_{\ell,m} \alpha_{\ell,m} Y_{\ell,m}(\theta, \phi)$$

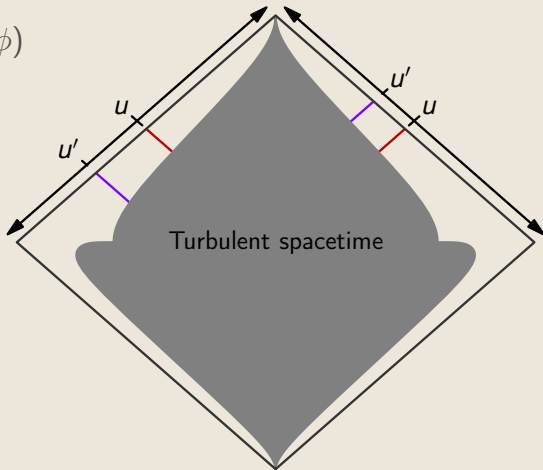


# Coordinates on $\mathcal{I}^+$



Supertranslation

$$u \mapsto u + \sum_{\ell,m} \alpha_{\ell,m} Y_{\ell,m}(\theta, \phi)$$



# BMS coordinate transformations



$$\theta \mapsto \theta'$$

$$\phi \mapsto \phi'$$

$$u \mapsto K(\theta', \phi') [u - \alpha(\theta', \phi')]$$

# BMS coordinate transformations



$$\theta \mapsto \theta'$$

$$\phi \mapsto \phi'$$

$$u \mapsto K(\theta', \phi') [u - \alpha(\theta', \phi')]$$

$$u' \approx \gamma [u - (\mathbf{x}_0 + u \mathbf{v}_0) \cdot \hat{\mathbf{r}} - \tilde{\alpha}(\theta', \phi')]$$

# BMS waveform transformations



$$r h \mapsto \frac{e^{-2i\lambda}}{K} [r h - \bar{\partial}^2 \alpha]$$

$$r^2 \sigma \mapsto \frac{e^{2i\lambda}}{K} [r^2 \sigma - \bar{\partial}^2 \alpha]$$

$$c \mapsto \frac{e^{2i\lambda}}{K} [c - \bar{\partial}^2 \alpha]$$

$$\text{News} \sim \frac{\partial c}{\partial u} \mapsto \frac{e^{2i\lambda}}{K^2} \left[ \frac{\partial c}{\partial u} \right]$$

$$r \Psi_4 \mapsto \frac{e^{-2i\lambda}}{K^3} [r \Psi'_4]$$

# BMS waveform transformations



$$r h \mapsto \frac{e^{-2i\lambda}}{K} [r h - \bar{\sigma}^2 \alpha]$$

# BMS waveform transformations



$$r h(u, \theta, \phi) \mapsto \frac{e^{-2i\lambda(\theta', \phi')}}{K(\theta', \phi')} [r h(u', \theta', \phi') - \bar{\partial}^2 \alpha(\theta', \phi')]$$



# Spin-weighted functions (SWFs)



$$\Psi_4(\theta, \phi) = C_{abcd}(\theta, \phi) \bar{m}^a n^b \bar{m}^c n^d$$

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# Spin-weighted functions (SWFs)



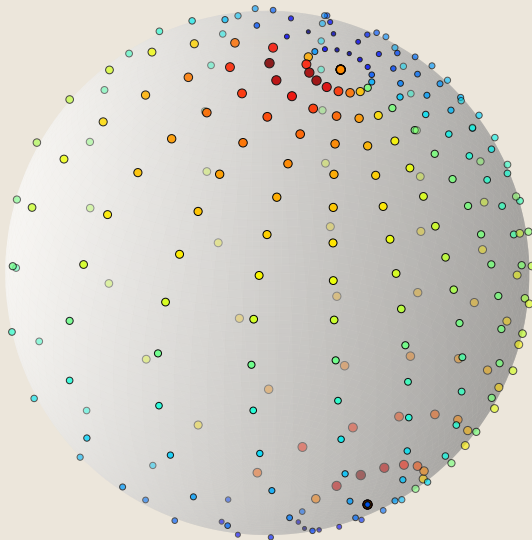
$$\Psi_4(\theta, \phi) = C_{abcd}(\theta, \phi) \bar{m}^a n^b \bar{m}^c n^d$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}(\theta^a - i \phi^a)$$

$$R : (\theta, \phi) \mapsto (\theta', \phi')$$

$$\Psi'_4(\theta', \phi') = \Psi_4(\theta, \phi) e^{-2i\lambda(\theta', \phi', R)}$$

# Spin-weighted functions (SWFs)



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SWFs *are not* functions on the sphere;  
also need alignment of tangent space

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SWFs *are not* functions on the sphere;  
also need alignment of tangent space

Might be tempted to think

$$\psi_4 : S^2 \times S^1 \rightarrow \mathbb{C}$$

Turns out

$$\psi_4 : S^3 \rightarrow \mathbb{C}$$

This is the Hopf bundle:

$$S^1 \hookrightarrow S^3 \twoheadrightarrow S^2$$

# Spin-weighted functions (SWFs)



SWFs are functions on the spin group

$\text{Spin}(3) = \text{SU}(2)$  = group of unit quaternions

$$\hat{r} = R \mathbf{z} R^{-1}$$

$$l = R \frac{\mathbf{t} + \mathbf{z}}{\sqrt{2}} R^{-1}$$

$$n = R \frac{\mathbf{t} - \mathbf{z}}{\sqrt{2}} R^{-1}$$

$$m = R \frac{\mathbf{x} + i \mathbf{y}}{\sqrt{2}} R^{-1}$$

$$\bar{m} = R \frac{\mathbf{x} - i \mathbf{y}}{\sqrt{2}} R^{-1}$$



# BMS waveform transformations



$$r h(u, \theta, \phi) \mapsto \frac{e^{-2i\lambda(\theta', \phi')}}{K(\theta', \phi')} [r h(u', \theta', \phi') - \bar{\partial}^2 \alpha(\theta', \phi')]$$

# BMS waveform transformations



$$r h(u, R) \mapsto \frac{1}{K(R')} [r h(u', R') - \bar{\partial}^2 \alpha(R')]$$

# Rotor of a boost



$$\Theta' = \arccos [\hat{\mathbf{v}} \cdot \hat{\mathbf{r}}_{\theta',\phi'}]$$

$$\Theta = 2 \arctan \left[ \sqrt{\frac{1-\beta}{1+\beta}} \tan \frac{\Theta'}{2} \right]$$

$$R_B = \exp \left[ \frac{\Theta - \Theta'}{2} \frac{\mathbf{v} \times \mathbf{r}_{\theta',\phi'}}{|\mathbf{v} \times \mathbf{r}_{\theta',\phi'}|} \right]$$

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$$R' = R_B e^{\phi' z/2} e^{\theta' y/2}$$

# Implementing BMS transformations



**for all** desired times slices  $u'_i$  **do**

**for all**  $(\theta'_j, \phi'_j)$  on equiangular grid **do**

$$R'_j \leftarrow R_B e^{\phi'_j z/2} e^{\theta'_j y/2}$$

$$u_i \leftarrow \frac{u'_i}{K(R')} + \alpha(R')$$

**for all**  $u_k$  “near”  $u_i$  **do**

$$r'h'(u_k, R'_j) \leftarrow \frac{r h(u_k, R'_j) - \bar{\delta}^2 \alpha(R'_j)}{K(R'_j)}$$

$$r'h'(u'_i, R'_j) \leftarrow \text{interp}([r'h'(u_k, R'_j)], u_i)$$

$$r'h'_{\ell,m}(u'_i) \leftarrow \text{spinsfast}([r'h'(u'_i, R'_j)])$$

# Conclusions



- ▶ No such thing as invariant waveform
- ▶ No preferred reference frame
- ▶ When comparing waveforms, frames must agree

Need BMS transformations for

- ▶ Accurate PN waveforms
- ▶ PN–NR comparisons
- ▶ Hybrid waveforms
- ▶ Making sense of ringdowns