Zonal Mean Angular Momentum Budget in Spherical Pressure Coordinates from Reanalysis datasets

Momme Hell, SIO

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1 Introduction

The atmospheric momentum budget is derived in spherical pressure coordinates and then written such that one can be calculated the instantenous budget from reanalysis data.

The relation of the angular momentum (AM) budget, the zonal momentum equation and eddy-fluxes can be found for example in *Peixoto et al.* [chapter 11, 2008]. A shorter introduction of the governing equations is given *Gill* [chapter 4.12, and 13.10, 1982] as well as *Andrews et al.* [1987] or even *Holton* [1975].

Section 2 of this script introduces the AM-equation, section 3 and 4 introduce the vertical and zonal mean due, and section 5 explains how these are applied in pressure coordinates. Section 6 explains how to this calculation is apply to ERA5 data and section 7 outlines two limits that seam to be dominant in the data.

2 The governing Angular Momentum equation

The Angular Momentum (AM) is defined as

$$M = M_\Omega + M_r \ = \ \Omega r^2 \cos^2\phi + ur\cos\phi, \tag{2.1}$$

where Ω is the earth rotation rate, u the zonal wind, and r the mean earth radius. Errors due to variations in r are generally small, though largest in the stratosphere (about 1% following *Gill* [1982, chapter 4.12, p.93]. M_Ω is the Angular Momentum of the Atmosphere as it would be in solid rotation with Earth, and M_r is the movement relative to the earth rotation.

The balance equation for angular momentum per unit volume in the rotating frame with longitude λ, latitude ϕ and height z is

$$\rho\frac{DM}{Dt} = -\frac{\partial\, p}{\partial\lambda} + \rho\ F_\lambda\ r\cos\phi, \tag{2.2}$$

where p is the pressure, ρ is the density, F_λ the frictional torques [eq. 11.4, *Peixoto et al.*, 2008]. The total derivative is

$$\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + u\ \frac{\partial}{r\cos\phi\ \partial\lambda}(\cdot) + v\ \frac{\partial}{r\ \partial\phi}(\cdot) + w\ \frac{\partial}{\partial z}(\cdot). \tag{2.3}$$

The divergence operator of a vector field $\boldsymbol{a}$ is

$$div(\boldsymbol{a}) = \frac{\partial}{\partial z}a_z + \frac{2a_z}{r} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(a_\phi\cos\phi) + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}a_\lambda, \tag{2.4}$$

In pressure coordinates this becomes

$$div(\boldsymbol{a}) = -g\frac{\partial}{\partial p}\rho a_z + \frac{2a_z}{r} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(a_\phi\cos\phi) + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}a_\lambda. \tag{2.5}$$

Separating the total AM into its relative M_r and absolute M_Ω components leads to a balance equation of the absolute AM changes due to advection of planetary vorticity

$$\rho\frac{DM_\Omega}{Dt} = -\rho\ v\ f\ r\cos\phi + \rho f'\ w\ r\ \cos\phi, \tag{2.6}$$

eq:abs_momentum_change

with

$$f = 2\Omega\sin\phi \quad \text{and} \quad f' = 2\Omega\cos\phi.$$

The terms on the right hand side of (2.6) are *sinks* of absolute AM and consequently *sources* of relative AM. The balance equation of *relative* AM is then

$$\frac{D\ M_r}{Dt} = -\frac{1}{\rho}\frac{\partial\ p}{\partial\lambda} + F_\lambda\ r\cos\phi + v\ f\ r\cos\phi - f'\ w\ r\ \cos\phi, \tag{2.7}$$

(eq. 11.5 in *Peixoto et al.* [2008]). We use the Boussinesq approximation and rewrite this equation in spherical pressure-coordinates (similar to *Peixoto et al.* [2008] eq. 11.6)

$$\frac{\partial\ M_r}{\partial t} = -\ \mathrm{div}(M_r\boldsymbol{u}) - \frac{\partial\ \phi(p)}{\partial\lambda} +\ F_\lambda\ r\cos\phi +\ v\ f\ r\cos\phi - f'\ w\ r\ \cos\phi, \tag{2.8}$$

eq:rel_momentum_conservation

or, in z-coordinates

$$\frac{\partial\ M_r}{\partial t} = -\ \mathrm{div}(M_r\boldsymbol{u}) - \frac{1}{\rho_0}\frac{\partial\ p(z)}{\partial\lambda} + F_\lambda\ r\cos\phi + v\ f\ r\cos\phi - f'\ w\ r\ \cos\phi,$$

The last term of the RHS is small due to the aspect ratio of the atmosphere.

3 Vertical integral

The following simplifications are performed in pressure coordinates, since most atmospheric models operate in this coordinate system. The vertical integral in pressure coordinates is defined as

$$\langle \cdot \rangle = \int_0^{p_s} (\cdot) \frac{dp}{g}, \quad \texttt{kg/m}^2, \tag{3.1}$$

with g as the acceleration due to gravity and p_s as the surface pressure.

- The vertical integral of the divergence term is then

$$\langle div(M_r \boldsymbol{u}) \rangle = - \underbrace{\int_0^{p_s} \frac{\partial}{\partial p}(M_r w) dp}_{\text{=0, no velocity at the wall}} + \underbrace{\frac{2}{r} \int_0^{p_s} \frac{M_r w}{g} dp}_{\text{=0, if w is continuos in the vertical}} \tag{3.2}$$

$$+ \frac{1}{r\cos\phi}\frac{\partial}{\partial \phi} \cos\phi \, \langle M_r v \rangle + \frac{1}{r\cos\phi}\frac{\partial}{\partial \lambda} \langle M_r u \rangle \tag{3.3}$$

- The vertical integral of the frictional torques can be written as

$$\langle F_\lambda \rangle = \int_0^{p_s} F_\lambda \frac{dp}{g} = \underbrace{F_\lambda|_{p=p_s(\lambda,\phi)}}_{=\mathcal{F}_{sfc}(\phi,\lambda)} - \underbrace{[F_\lambda]|_{p=0}}_{\text{=0, at the top of the atmosphere}} . \tag{3.4}$$

  the remaining term $\mathcal{F}_{sfc}$ is the total stress at the surface. In pressure coordinates, that is where pressure level intersects the surface.

  The atmosphere loses AM to the surface by two boundary layer processes: (a) by turbulent momentum fluxes F_{tur} or (b) by exciting internal gravity waves F_{wave} that add westward momentum (that is negative AM) to the atmosphere. These waves are thought to propagate as linear waves in the atmosphere until they break and deposit their negative AM locally. This term is larger over large mountain ranges.

- The vertical integral of the pressure term gives

$$\left\langle \frac{\partial \phi}{\partial \lambda} \right\rangle = \int_0^{p_s} \frac{\partial}{\partial \lambda}\phi(p) \; dp = \frac{\partial}{\partial_\lambda} \langle \phi(p) \rangle \, , \tag{3.5}$$

  which is only zero if $\langle \phi(p) \rangle$ is continuous.

- The $f'w$ term in (2.8) is zero in the vertical integral, because the velocity has to be zero at the wall (in z-coordinates).

Eq. (2.8) is then

$$\begin{aligned}\frac{\partial\ \langle M_r\rangle}{\partial t} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}\ \cos\phi\,\langle M_r v\rangle + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\,\langle M_r u\rangle \\ = -\frac{\partial\ \langle\phi\rangle}{\partial\lambda} +\ \mathcal{F}_{sfc}\ r\cos\phi + \langle v\rangle\ \ f\ \ r\cos\phi, \quad \mathtt{m}^2/\mathtt{s}^2.\end{aligned} \tag{3.6}$$

eq:vert_int

4 Zonal mean

The zonal mean is defined as

$$[\ \cdot\ ] = \frac{1}{2\pi}\int_0^{2\pi}\quad d\lambda. \tag{4.1}$$

The zonal mean of eq.(3.6) is then

$$\begin{aligned}\frac{\partial\ \langle[M_r]\rangle}{\partial t} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}\ \cos\phi\,\langle[M_r v]\rangle + \underbrace{\left[\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\,\langle M_r u\rangle\right]}_{\text{=0, comment 1}} \\ = -\underbrace{\left[g\frac{\partial\ \langle z\rangle}{\partial\lambda}\right]}_{\text{comment 2}} + [\mathcal{F}_{sfc}]\ r\cos\phi + \underbrace{\langle[v]\rangle\ \ f\ \ r\cos\phi}_{\text{comment 3}}\end{aligned} \tag{4.2}$$

eq:vert_int_zm

comment 1 The zonal mean of the zonal derivative is zero, because the domain is periodic and $\langle M_r u\rangle$ is continuously differentiable.

comment 2 This term is not zero, because surfaces of geopotential height can intersect the Earth's surface. Therefore, $\partial_\lambda z$ is not necessarily continuously differentiable, and its zonal mean is not always zero (fundamental theorem on calculus). Physically, the pressure difference between sides of mountain ridges leads to a net pressure force acting on the mountain pushing in the direction of less pressure [*Peixoto et al.*, 2008, chapter11.1.14].

comment 3 The vertical integral of the zonal mean meridional winds $\langle[v]\rangle$ is zero on "long" time scales, because there should be no net mass flux across latitude in the atmosphere. However, since we have not yet introduced time means, this term can play a role. It can act as a storage term for momentum.

The resulting equation is then

$$\frac{\partial \ \langle [M_r] \rangle}{\partial t} - \ f \ \langle [v] \rangle \ r\cos\phi = -\left[\frac{\partial \ \langle \Phi \rangle}{\partial \lambda}\right] \ + \ [\mathcal{F}]_{sfc} \ r\cos\phi - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi} \ \cos\phi \, \langle [M_r v] \rangle \, .$$

Rewritten in terms of the zonal-wind momentum equation using $M_r = r\cos\phi \ u$ leads to

$$\frac{\partial \ \langle [u] \rangle}{\partial t} - f \, \langle [v] \rangle = \ -\left[\frac{1}{r\cos\phi}\frac{\partial \ \langle \Phi \rangle}{\partial \lambda}\right] \ + \ [\mathcal{F}]_{sfc} - \frac{1}{r^2\cos^2\phi}\frac{\partial}{\partial\phi} \ r\cos^2\phi \, \langle [uv] \rangle \, . \ \frac{\mathtt{N}}{\mathtt{m}^2} = \mathtt{Pa} \tag{4.3}$$

eq:balance

This resembles the vertical integral of the TEM momentum equation, with the addition of mountain torque [*Andrews and McIntyre*, 1976; *Edmon et al.*, 1980].

5 Vertical integrals, zonal means in pressure coordinates, and Boer Bata

Because pressure coordinates have a kinematic lower boundary condition (i.e the pressure levels intersect the surface), we need to keep track of the number of valid point when calculating budgets. Following *Boer* [1982] and *Wills and Schneider* [2015], the vertical integral is defined as

$$\langle \ \cdot \ \rangle^{\beta} = \int_0^{p_0} \ H_\beta \ (\ \cdot \ ) \ \frac{dp}{g} = \langle H_\beta \ \cdot \ \rangle \, , \quad \mathtt{kg/m}^2. \tag{5.1}$$

beta_int

Note that p_0 is the lowest pressure level that exists in the model, while p_s is the surface pressure that can be above or below p_0 at each grid point. Neither p_0 nor p_s is the same as the mean sea level pressure (MSLP), which is the equivalent pressure at the (geopotential) surface and not the pressure at the interface between land/ocean and atmosphere. For the ocean-atmosphere interface – where the ocean surface is approximately the same as the geoid – p_s is similar to MSLP, while p_s and p_0 are an instantaneous model output and MSLP often a mean over the output time step.

The zonal mean is

$$[\ \cdot\ ]_R = \frac{1}{2\pi} \ \int_0^{2\pi} \ \rho_\beta \ (\ \cdot \ ) \ d\lambda \bigg/ \frac{1}{2\pi}\int_0^{2\pi} \rho_\beta \ d\lambda = [\rho_\beta \cdot\ ]/[\rho_\beta], \ \text{ the averaging operator has no units} \tag{5.2}$$

where the R subscript or superscript denotes the "representative" average, i.e. an average per valid grid point and per unit mass. The weighting density ρ_β is defined as

$$\rho_\beta = p_s \ H_\beta,$$

where H_β it the Boer-beta function [*Boer*, 1982], a Heaviside function of the form

$$H_\beta(p-p_s,\lambda,\phi,t)=\begin{cases}1, & p-p_s>0\\ 0, & p-p_s\le 0.\end{cases} \tag{5.3}$$

This function returns zero when the $p(\lambda,\phi,t)$ is below the surface, and 1 otherwise.

The vertically integrated, zonal and time-mean AM equation (4.3) is then

$$\begin{aligned}\frac{\partial}{\partial t}\ \langle[\rho_\beta\ u]\rangle^\beta - f\,\langle[\rho_\beta\ v]\rangle^\beta \\ = -\left[\frac{\rho_\beta}{r\cos\phi}\frac{\partial\ \langle\Phi\rangle^\beta}{\partial\lambda}\right] + [\rho_\beta\ \mathcal{F}]_{sfc} - \frac{1}{r^2\cos^2\phi}\frac{\partial}{\partial\phi}\ r\cos^2\phi\,\langle[\rho_\beta\ uv]\rangle^\beta.\end{aligned} \tag{5.4}$$

eq:balance_rb

5.1 Instantaneous fields

From now on, we deal with two averages for each term in (5.4), one that is the representative average $[\cdot]_R$, which has the right physical units (Pascal). Its use is to display the data. However, for a variable x with units of Pascal, the terms that are used in the budget equation have the form of $\langle[rb\ x]\rangle^\beta$ and units of Pa^2. Informed by *Boer* [Table 1, 1982] we obtain:

- the representative averages of the linear variables u, v follow the form

$$[x]_R(\phi,t,p)=[x\ \rho_\beta]/[\rho_\beta], \tag{5.5}$$

  and its vertical integral

$$\langle[x]_R\rangle^{[\beta]}=\langle[H_\beta]\ [x]_R\rangle_0^{p_0}. \tag{5.6}$$

  The terms in the budget equation (5.4) $\langle[\rho_\beta\ u]\rangle^\beta$ and $\langle[\rho_\beta\ v]\rangle^\beta$ follow

$$\langle[\rho_\beta\ X]\rangle^\beta=[\langle\rho_\beta\ X\rangle^\beta]. \tag{5.7}$$

- The eddy term is decomposed as

$$\begin{aligned}\langle[\rho_\beta\ u\ v]\rangle^\beta(\phi,t) &= \langle[\rho_\beta]\ [u]_R\ [v]_R\rangle^\beta\ +\ \langle[\rho_\beta\ u^*v^*]\rangle^\beta,\\ &= \langle[\rho_\beta]\ [u]_R\ [v]_R\rangle^\beta\ +\ \langle[\rho_\beta]\ [u^*v^*]_R\rangle^\beta,\end{aligned} \tag{5.8}$$

  with

$$u^*=u-[u]_R,$$

  and then consequently

$$[u^*v^*]_R=[u^*v^*\rho_\beta]/[\rho_\beta],$$

The representative averages of the mean- and eddy-terms are

$$\langle [u\ v]_R \rangle^{\beta} = \frac{\langle [\rho_\beta]\ [u]_R\ [v]_R \rangle^{[\beta]}}{\langle [\rho_\beta] \rangle^{[\beta]}} + \frac{\langle [\rho_\beta]\ [u^* v^*]_R \rangle^{[\beta]}}{\langle [\rho_\beta] \rangle^{[\beta]}}, \tag{5.9}$$

calculated as

$$\langle [u\ v]_R \rangle^{\beta} = \langle [u]_R\ [v]_R \rangle^{[\beta]} + \langle [u^* v^*]_R \rangle^{[\beta]}, \tag{5.10}$$

where $(\ \cdot\ )^{[\beta]}$ indicates the use of the zonal mean Heaviside function $[H_\beta]$ instead of H_β in the integral of (5.1).

- The surface term is simpler because at the surface $\partial_p \rho_\beta = p_s\ \delta(p - p_s) = p_s$, such that we just have to take into account the surface pressure weighting

$$\begin{aligned} [\rho_\beta\ \mathcal{F}_{sfc}] &= [p_s\ \mathcal{F}_{sfc}] && \texttt{Pa}^2, \\ [\mathcal{F}_{sfc}]_R &= [p_s\ \mathcal{F}_{sfc}]/[p_s] && \texttt{Pa}. \end{aligned}$$

The surface drag is split in two components,

$$\mathcal{F}_{sfc} = \mathcal{F}_{tur} + \mathcal{F}_{wave}, \tag{5.11}$$

where $\mathcal{F}_{tur}$ is the surface drag due to turbulent stress and $\mathcal{F}_{wave}$ the surface stress due to the generation of internal wave drag that is induced by surface winds flowing over rough topography. The wave drag is small, such that, even though technically a source/sink term, it is treated as a correction term in the budget equation.

- Following section 3 in *Boer* [1982], the mountain torque term can be rewritten as

$$\begin{aligned}
\left[\frac{\rho_\beta}{r\cos\phi}\frac{\partial\ \langle\Phi\rangle^{\beta}}{\partial\lambda}\right] &= \left[\frac{\rho_\beta}{r\cos\phi}\frac{\partial}{\partial\lambda}\int_0^{p_0} H_\beta\ \Phi\ \frac{dp}{g}\right] \\
&= \left[\frac{\rho_\beta}{r\cos\phi}\int_0^{p_0}\left(\frac{\partial}{\partial\lambda}H_\beta\ \Phi - \delta\Phi\ \frac{\partial\ p_s}{\partial\lambda}\right)\frac{dp}{g}\right] \\
&= \left[\frac{\rho_\beta}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\int_0^{p_0} H_\beta\ \Phi\frac{dp}{g}\right] - \int_0^{p_0}\left[\Phi_s\ \frac{1}{r\cos\phi}\ \frac{\partial\ p_s}{\partial\lambda}\right][H_\beta]\frac{dp}{g} \\
&= [\rho_\beta]\left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\int_0^{p_0} H_\beta\ \Phi\frac{dp}{g}\right]_R - \left[\frac{\Phi_s}{g}\ \frac{1}{r\cos\phi}\ \frac{\partial\ p_s}{\partial\lambda}\right][p_s] \\
&= [\rho_\beta]\left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\ \langle\Phi\rangle^{\beta}\right]_R - \ldots \\
&= [\rho_\beta]\left\langle\left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\,(H_\beta\ \Phi)\right]_R\right\rangle^{[\beta]} - \ldots \\
&= \left\langle[\rho_\beta]\left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\,(H_\beta\ \Phi)\right]_R\right\rangle^{[\beta]} - \left[\frac{\Phi_s}{g}\ \frac{1}{r\cos\phi}\ \frac{\partial\ p_s}{\partial\lambda}\right][p_s]\ \ \texttt{Pa}^2
\end{aligned}$$

The first term on the righthand side might cancel, but the second will not, since it is nonlinear. The representative means are derived by dividing by $[p_s]$ as seen above.

5.2 Time averages

The time average of the instantaneous zonal-mean vertical integrals is defined as the surface pressure weighted time mean

$$\overline{(\ \cdot\ )}^{\sharp}_{R} = \overline{(p_s\ \cdot)}/\bar{p}_s, \tag{5.12}$$

where the overbar is the conventional time mean. The time mean of the above terms is then

$$\begin{aligned}
\overline{[X]}^{\sharp}_{R}(\phi, p) &= \frac{\overline{[p_s]\ [X\ \rho_\beta]/[\rho_\beta]}}{[\bar{p}_s]},\\
\overline{\langle[\rho_\beta\ X]\rangle^{\beta}}^{\sharp} &= \overline{\langle[\rho_\beta][X]_R\rangle_0^{p_0}}^{\sharp},\\
&\ldots
\end{aligned}$$

The budget equation is then

$$\begin{aligned}
&\overline{\frac{\partial}{\partial t}\ \langle[\rho_\beta]\ [u]_R\rangle^{[\beta]}}^{\sharp} - \overline{f\,\langle[\rho_\beta]\ [v]_R\rangle^{[\beta]}}^{\sharp} - \overline{[p_s\ \mathcal{F}_{tur}]}^{\sharp} - \overline{[p_s\ \mathcal{F}_{wave}]}^{\sharp}\\
&\overline{\left\langle[\rho_\beta]\left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\,(H_\beta\ \Phi)\right]_R\right\rangle^{[\beta]}}^{\sharp} - \overline{[p_s]\left[\ \frac{\Phi_s}{g}\ \frac{1}{r\cos\phi}\ \frac{\partial\ p_s}{\partial\lambda}\ \right]}^{\sharp}\\
&\qquad = -\frac{1}{r^2\cos^2\phi\ [\bar{p}_s]}\frac{\partial}{\partial\phi}\ [\bar{p}_s] r\cos^2\phi\ \overline{\langle[\rho_\beta]\ [u]_R\ [v]_R\rangle^{[\beta]}}^{\sharp}\\
&\qquad\quad - \frac{1}{r^2\cos^2\phi\ [\bar{p}_s]}\frac{\partial}{\partial\phi}\ [\bar{p}_s] r\cos^2\phi\ \overline{\langle[\rho_\beta\ u^* v^*]_R\rangle^{[\beta]}}^{\sharp}\\
&\qquad\quad + \mathcal{R},
\end{aligned} \tag{5.13}$$

eq:balance_rb_tm

with ϕ_s being the surface geopotential height and p_s being the surface pressure. Both torque terms are non-zero, either because they are discontinuous or non-linear. $\mathcal{R}$ is the residual of the equation.

Notes:

- The zonal derivatives are not explicitly weighted by surface pressure, while the meridional derivatives are. When no pressure weighting is applied in the zonal derivatives of surface pressure and $\frac{\partial}{\partial\lambda}\,(H_\beta\ \Phi)$ the budget closes better.
- The zonal means of surface variables do not carry $[\cdot]_R$ sub-scripts, because they are not really representative averages.

6 Computation with ERA5

The terms of (5.13) are calculated from hourly ERA5 data [*European Centre For Medium-Range Weather Forecasts*, 2017]. General comments:

- The daily files are computed from hourly fields. First we calculate the zonal mean and then the vertical integrals, and then time mean.
- The tendency term on the left-hand side is computed as the central finite difference between the hourly fields of the vertically integrated zonal-mean zonal wind. For the daily mean estimates, the tendency term is biased due to the missing time steps before and after the daily files. This leads to small discontinuities in the long-term time tendency term because the analysis is performed on daily fields. This bias could be corrected by recalculating the tendency in a post processing step.
- The meridional gradients are calculated using 2nd-order central differencing.

6.1 Variable names and conventions

Instantaneous variables on pressure levels:

`table:pressure_var`

Variable Name	Units	key	ERA5 id
U component of wind	$m\ s^{-1}$	u	131
V component of wind	$m\ s^{-1}$	v	132
Geopotential	$m^2\ s^{-2}$	z	129

Surface variables:

`table:srf_var`

Variable Name	Units	key	ERA5 id
instantaneous surface pressure	Pa	sp	134
instantaneous geopotential	$m^2\ s^{-2}$	z	129
instantaneous eastward turbulent surface stress	$N\ m^{-2}$	iews	229
mean eastward gravity wave surface stress [1]	$N\ m^{-2}$	megwss	235045

[1] The gravity wave stress is a one-hour time mean variable. It represents the momentum that is lost by gravity wave drag over one hour, rather then the momentum at the time step of that hour, as all other variables do. In addition, the gravity wave drag dissipation of the model is only saved for the sum of x-and y-direction. With the variables stored in the ERA5 data base one could write down a balance equation for the wave drag components as

$$(\partial_t u)_{wave} + (\partial_t u)_{wave} = F^x_{wave} + F^y_{wave} - \mathcal{D}_{wave}$$

However ERA5 only provides the total wind fields, which are the result of all tendencies in the equations. To estimate $(\partial_t u)_{wave}$ for closing this equation one would have to solve the meridional momentum equation in a similar fashion to the zonal momentum equation. We simply assume that

$$(\partial_t u)_{wave} \approx F^x_{wave}.$$

This assumption and the use of 1-hour time means instead of the instantaneous tendency is likely the largest source of errors in this calculation.

6.2 Topographic drag segments

The vertical integral of the terms induced by topography are separately saved by continental area. This can be used to infer how much momentum is "lost" over each continental mountain range. We separate the glob into three analysis domains: Europe/Africa (18° E to 60° E), and Asia (60° E to 180° E), the Americas (180° W to 18° E). The sum of the segments reproduces the zonal mean value stored in the vertically integrated budget. The representative means for each segment are calculated as

$$[x]_{rep,j} = \begin{cases} [x\ \rho_{\beta,j}]/[\rho_\beta], & [\rho_\beta] \ \neq 0 \\ [x], & [\rho_\beta] \ = 0. \end{cases} \tag{6.1}$$

For surface variables

$$[x]^j_{rep} = [x\ p^j_s]/[p_s], \tag{6.2}$$

and the budget mean is then

$$[x]^j = [x]^j_{rep}\ [\rho_\beta] \tag{6.3}$$

$$[x]^j = [x]^j_{rep}\ [p_s], \tag{6.4}$$

such that

$$[x]_{rep} = \frac{N_j}{N} \sum_j [x]^j_{rep}. \tag{6.5}$$

(The N_j/N weighting might be an artefact of how xarray calculates the zonal mean).

6.3 Saved fields and example results

The variables are save as instantaneous zonal mean fields with and without weighting (4.3), excluding the tendency term. From these we calculate the instantaneous (hourly) vertically integrated budget equation (5.13) (Fig. 1 and 2) and finally a daily mean (Fig. 3).

In addition, we save instantaneous fields of $[u]_R, [v]_R, [H_\beta]_R$ and $[p_s]$. This allows us to convert between representative and budget means and to correct for discontinuities in the tendency term in a post processing step.

7 Simple limits

The vertical integrated momentum budget (5.13) is rewritten now now neglecting all complicated averaging and weightings as

$$
\begin{aligned}
&\overline{\frac{\partial}{\partial t}\ \langle [u] \rangle} - \overline{f\,\langle [v]_R \rangle} \\
&+\ \overline{\left\langle \left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\,(\ \Phi) \right] \right\rangle} - \overline{\left[\frac{\Phi_s}{g}\ \frac{1}{r\cos\phi}\ \frac{\partial\, p_s}{\partial\lambda} \right]} \\
&+\ \frac{1}{r^2\cos^2\phi}\,\frac{\partial}{\partial\phi}\ r\cos^2\phi\ \left(\overline{\langle [u]\ [v] \rangle} + \overline{\langle [u^* v^*] \rangle} \right) - \overline{[\mathcal{F}_{wave}]} \\
&=\ +\ \overline{[\mathcal{F}_{tur}]} + \mathcal{R},
\end{aligned}
\tag{7.1}
$$

Everything on the left hand side of this equation is called "LHS" in the store files. Over the Southern Ocean and for a long time mean, one expect the tendency term, mountain torque and gravity wave to be small. The equation reduces to

$$
\overline{f\,\langle [v]_R \rangle}\ +\ \frac{1}{r^2\cos^2\phi}\,\frac{\partial}{\partial\phi}\ r\cos^2\phi\ \left(\overline{\langle [u]\ [v] \rangle} + \overline{\langle [u^* v^*] \rangle} \right)\ \approx\ \overline{[\mathcal{F}_{tur}]}. \tag{7.2}
$$

In the presents of topography the total mountain drag, or, better said, the drag that the atmosphere loses to the topography, has to be added in the lonhg-term mean

$$
\begin{aligned}
&\frac{1}{r^2\cos^2\phi}\,\frac{\partial}{\partial\phi}\ r\cos^2\phi\ \left(\overline{\langle [u]\ [v] \rangle} + \overline{\langle [u^* v^*] \rangle} \right) -\ \overline{[\mathcal{F}_{tur}]} \\
&= -\ \overline{\left\langle \left[\frac{1}{r\cos\phi}\ \frac{\partial}{\partial\lambda}\,(\Phi) \right] \right\rangle} + \overline{\left[\frac{\Phi_s}{g}\ \frac{1}{r\cos\phi}\ \frac{\partial\, p_s}{\partial\lambda} \right]} + \overline{[\mathcal{F}_{wave}]} + \mathcal{R}.
\end{aligned}
\tag{7.3}
$$

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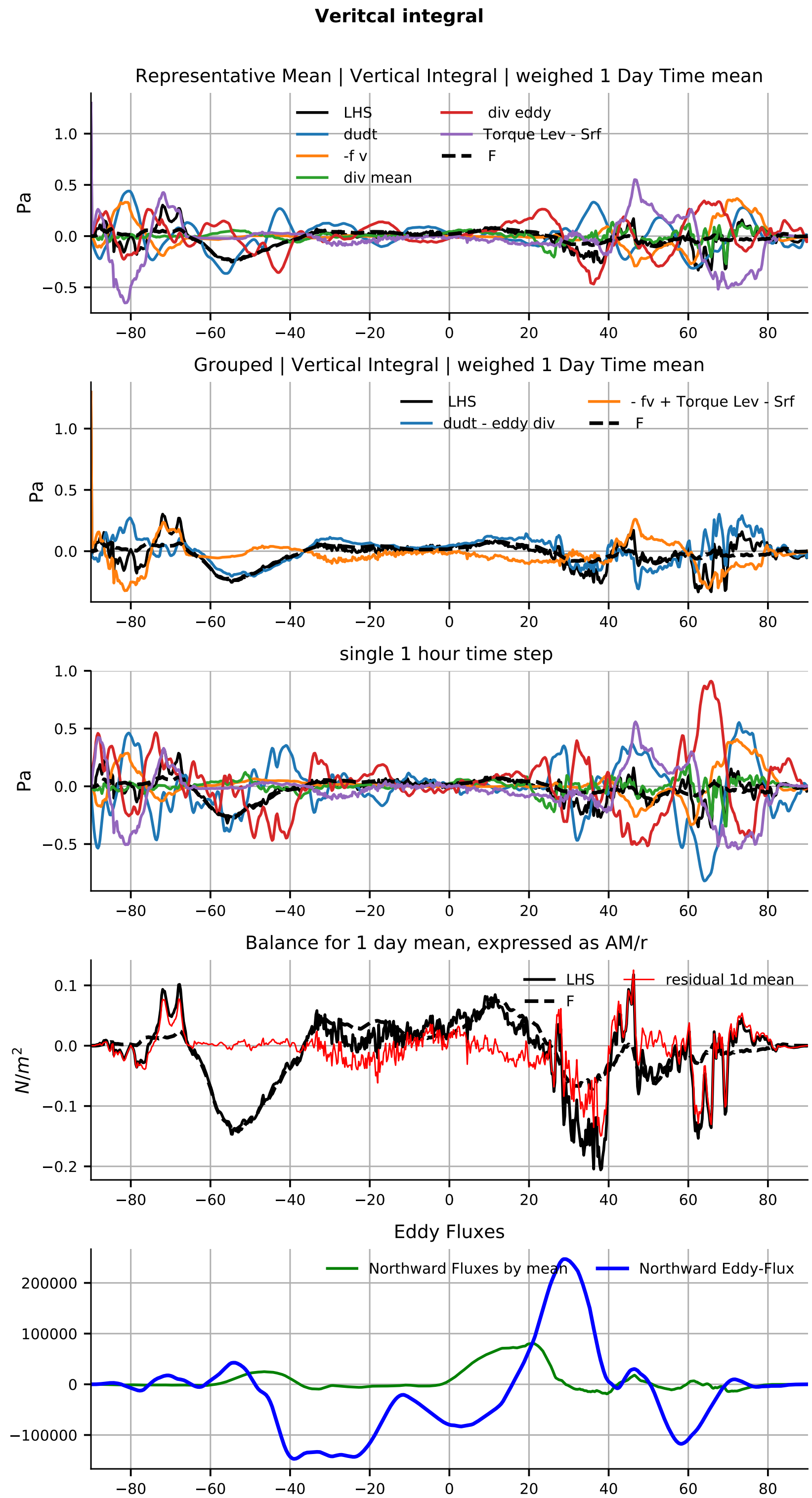

Figure 1: Example the budget calculation of one day (2000-01-18). fig:AM_example1

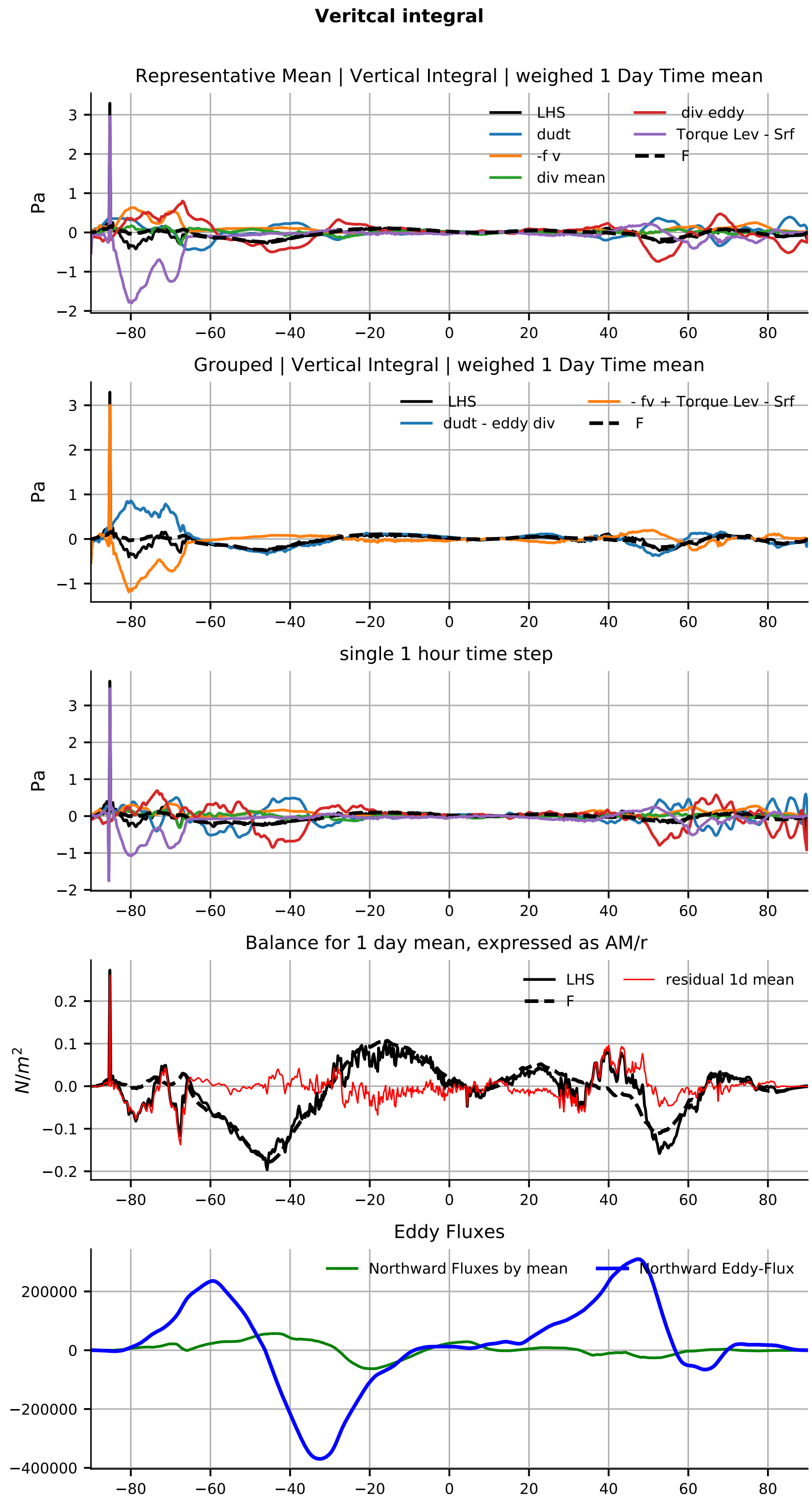

Figure 2: Example the budget calculation of one day (2005-09-28). fig:AM_example2

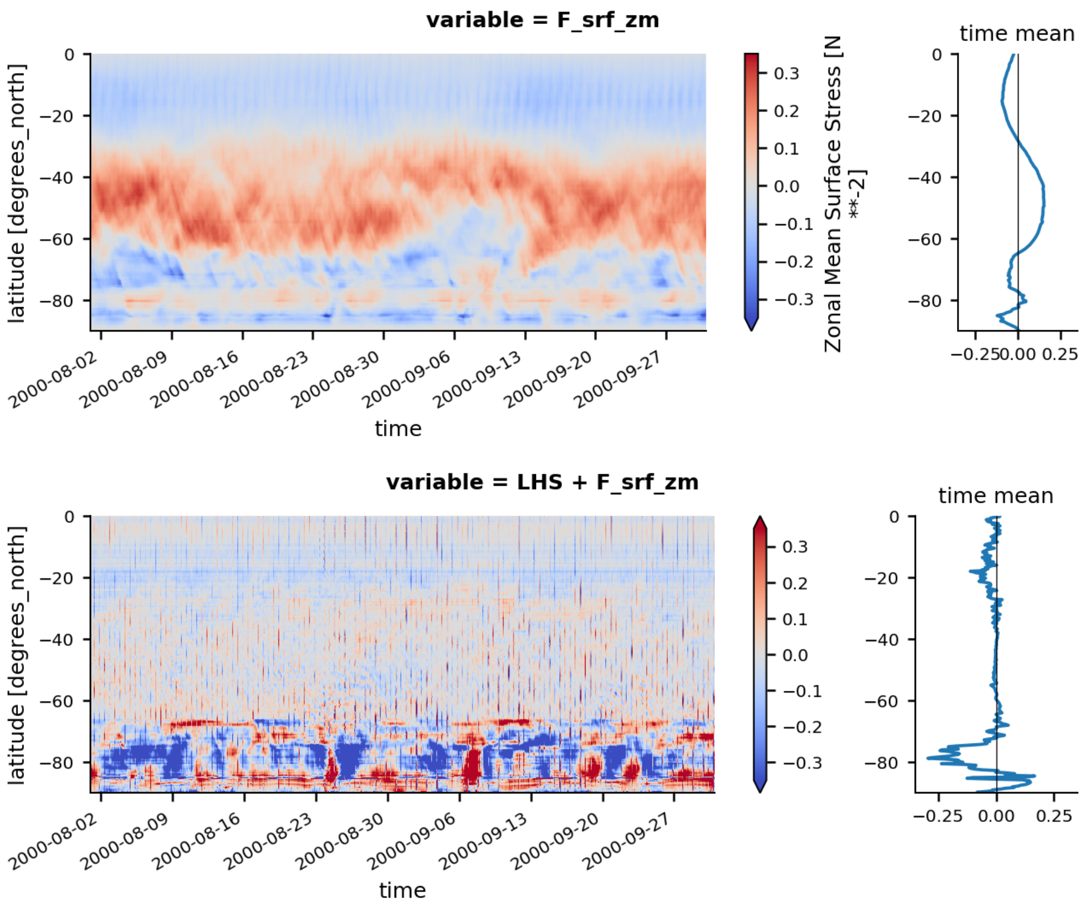

Figure 3: Zonal mean surface stress over the Southern Hemisphere and the residual of the budget calculation over the course of 2 months.. `fig:AM_example3`